TABLE OF SPHERE PACKING DENSITY BOUNDS

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Table 1 gives the best packing densities known for congruent spheres in Euclidean spaces of dimensions 1 through 48 and 56, 64, and 72, along with the best upper bounds known for the optimal packing density. It was originally based on Tables I.1(a) and I.1(b) in [10] and the Nebe-Sloane Catalogue of Lattices.

The columns of the table list the following information:

n	Number of dimensions.
Center density	Number of spheres per unit volume in space, assuming unit radius.
#	Number of spheres per unit cell in an underlying (Bravais) lattice.
Density	Packing density, i.e., $\pi^{n/2}/\Gamma(n/2+1)$ times the center density.
Upper bound	Best upper bound known for the optimal packing density, rounded up.
Ratio	Ratio of the upper bound to the known density, rounded up.
References	References for the sphere packing and the upper bound.

There are also three supplementary files:

`densitybounds.txt`	Plain text file containing the upper and lower density bounds from this table.
`packings.txt`	Explicit descriptions of packings in up to 38 dimensions as Gram matrices for lattices together with translation vectors (if needed to form a periodic packing).
`rigorous.txt`	Exact rational coefficients needed to verify the bounds quoted from [1] using linear programming bounds.

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Table 1. Sphere packing density bounds.

n	Center density	#	Density	Upper bound	Ratio	References
1	2^{-1}	1	1		1	
2	$2^{-1} \cdot 3^{-1/2}$	1	0.9068996821171089 . . .		1	[26, 27, 12]
3	$2^{-5/2}$	1	0.7404804896930610 . . .		1	[13, 14]
4	2^{-3}	1	0.6168502750680849 . . .	0.6361073321551329	1.032	[15, 7]
5	$2^{-7/2}$	1	0.4652576133092586 . . .	0.5126451306253027	1.102	[16, 7]
6	$2^{-3} \cdot 3^{-1/2}$	1	0.3729475455820649 . . .	0.4103032818801865	1.101	[16, 7]
7	2^{-4}	1	0.2952978731457125 . . .	0.3211471056675559	1.088	[16, 7]
8	2^{-4}	1	0.2536695079010480 . . .		1	[16, 30]
9	$2^{-9/2}$	1	0.1457748758081711 . . .	0.1911204152968963	1.312	[5, 7]
10	$2^{-7} \cdot 5$	40	0.0996157828077088 . . .	0.1434100871082547	1.440	[3, 7]
11	$2^{-8} \cdot 3^{2}$	72	0.0662380270098011 . . .	0.1067252934567631	1.612	[20, 7]
12	3^{-3}	1	0.0494541766242440 . . .	0.0797117710668987	1.612	[11, 7]
13	$2^{-8} \cdot 3^{2}$	72	0.0320142921603497 . . .	0.0601644380983860	1.880	[20, 7]
14	$2^{-4} \cdot 3^{-1/2}$	1	0.0216240960824471 . . .	0.0450612211935181	2.084	[18, 7]
15	$2^{-9/2}$	1	0.0168575706567626 . . .	0.0337564432797899	2.003	[2, 7]
16	2^{-4}	1	0.0147081643974308 . . .	0.0249944093845237	1.700	[2, 7]
17	2^{-4}	1	0.0088113191823211 . . .	0.0184640903350649	2.096	[18, 1]
18	$2^{-18} \cdot 3^{9}$	512	0.0061678981253312 . . .	0.0134853404450862	2.187	[4, 1]
19	$2^{-7/2}$	1	0.0041208062797686 . . .	0.0098179551395438	2.383	[18, 1]
20	$2^{-31} \cdot 7^{10}$	4	0.0033945814107126 . . .	0.0071270536033763	2.100	[28, 1]
21	$2^{-5/2}$	1	0.0024658847115024 . . .	0.0051596603948176	2.093	[18, 1]
22	$2^{-23} \cdot 3^{-21/2} \cdot 11^{11}$	3	0.0024510340441211 . . .	0.0037259419689206	1.521	[9, 1]
23	2^{-1}	1	0.0019053281934260 . . .	0.0026842798864291	1.409	[19, 1]
24	1	1	0.0019295743094039 . . .		1	[19, 6]
25	$2^{-1/2}$	1	0.0006772120097731 . . .	0.0013841907222857	2.044	[8, 1]
26	$3^{-1/2}$	1	0.0002692200504338 . . .	0.0009910238892216	3.682	[8, 1]
27	$2^{-1/2}$	16384	0.0001575943907278 . . .	0.0007082297958617	4.495	[29, 1]
28	1	8192	0.0001046381049248 . . .	0.0005052542161057	4.829	[29, 1]
29	$2^{-1/2}$	32768	0.0000341446469074 . . .	0.0003598581852089	10.54	[29, 1]
30	1	8192	0.0000219153534478 . . .	0.0002559028743732	11.68	[29, 1]
31	$2^{-47/2} \cdot 3^{15}$	1	0.0000118377651859 . . .	0.0001817083813917	15.35	[24, 1]
32	$2^{-24} \cdot 3^{16}$	1	0.0000110407493088 . . .	0.0001288432887595	11.67	[24, 1]
33	$2^{-25} \cdot 3^{33/2}$	1	0.0000041406882896 . . .	0.0000912356039023	22.04	Elkies (see [10, p. xx]), [1]
34	$2^{-25} \cdot 3^{33/2}$	1	0.0000017669738891 . . .	0.0000645221967438	36.52	Elkies (see [10, p. xx]), [1]
35	$2^{3/2}$	1	0.0000009461904151 . . .	0.0000455743843107	48.17	[10, p. 234], [1]
36	$2^{18} \cdot 3^{-10}$	1	0.0000006161466094 . . .	0.0000321530553313	52.19	[17, 1]
37	$2^{5/2}$	1	0.0000003213562007 . . .	0.0000226586900106	70.51	[10, p. xxxvii], [1]
38	2^{3}	1	0.0000001835874319 . . .	0.0000159506499105	86.89	[10, p. xxxvii], [1]
39	$2^{-41/2} \cdot 3^{16} \cdot 7^{-1/2}$	1	0.0000001004160423 . . .	0.0000112168687009	111.8	[8, Cor. 8], [1]
40	$2^{-45/2} \cdot 3^{17}$	1	0.0000000784800488 . . .	0.0000078801051697	100.5	[8, Cor. 8], [1]
41	$2^{-43/2} \cdot 3^{17}$	1	0.0000000610716131 . . .	0.0000055306464395	90.57	[8, Cor. 8], [1]
42	$2^{-22} \cdot 3^{18}$	1	0.0000000498110957 . . .	0.0000038780970907	77.86	[8, Cor. 8], [1]
43	$2^{-45/2} \cdot 3^{19}$	1	0.0000000401571902 . . .	0.0000027169074727	67.66	[8, Cor. 8], [1]
44	$2^{-43} \cdot 3^{-24} \cdot 17^{22}$	4	0.0000000364088490 . . .	0.0000019017703144	52.24	[9, 1]
45	$2^{-44} \cdot 3^{-24} \cdot 17^{45/2}$	4	0.0000000278915432 . . .	0.0000013300905665	47.69	[9, 1]
46	$3^{-93/2} \cdot 13^{23}$	3	0.0000000286095747 . . .	0.0000009295151556	32.49	[9, 1]
47	$2^{-70} \cdot 3^{-24} \cdot 5^{47/2} \cdot 7^{47/2}$	2	0.0000000221448620 . . .	0.0000006490757338	29.32	[9, 1]
48	$2^{-24} \cdot 3^{24}$	1	0.0000000231782953 . . .	0.0000004529067791	19.55	[20, 1]
56	$2^{-28} \cdot 5^{28} \cdot 29^{-4}$	1	0.0000000000535530 . . .	0.0000000248393710	463.9	[25, 1]
64	3^{16}	1	0.0000000000013260 . . .	0.0000000013129981	990.2	[21, 22, 1]
72	2^{36}	1	0.0000000000001458 . . .	0.0000000000673580	461.8	[23, 1]

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