TABLE OF KISSING NUMBER BOUNDS

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The kissing problem asks how many spheres can be arranged tangent to a given sphere, if they all have the same size and their interiors cannot overlap. The maximum such number in n dimensions is called the n-dimensional kissing number. Equivalently, we can ask how many points can be arranged on the surface of a sphere such that no two distinct points form an angle of less than 60° with the center of the sphere. See [4] for more information about the kissing problem.

Table 1 shows the best upper and lower bounds known for the kissing numbers in dimensions up to 48 as well as 72. It was originally based on a table of lower bounds that was part of the Nebe-Sloane Catalogue of Lattices, and before that Tables I.2(a) and I.2(b) in [4].

The columns of the table list the following information:

Dimension	Number of dimensions.
Lower bound	The best construction currently known.
Upper bound	The best upper bound currently known.
Ratio	Ratio of the upper and lower bounds, rounded up.
References	References for the construction and the upper bound.

There are also three supplementary files:

`kissingbounds.txt`	Plain text file containing the lower and upper bounds from this table.
`dimensions1-24.txt`	Explicit coordinates for kissing configurations in up to 24 dimensions, including alternate configurations constructed in [3, 12, 13, 23].
`dimensions32-47.txt`	Details of how configurations in dimensions 32 through 47 can be constructed using the approach of [5] and constant weight codes from [2].

In the file `dimensions1-24.txt`, the inner product coefficients are a sequence $c_1, \dots, c_n$ that defines the inner product of two vectors x and y via $\langle x, y\rangle = \sum_{i=1}^{n} c_i x_i y_i$. In many cases this avoids the need for irrational coordinates, since it amounts to rescaling the i-th coordinate by a factor of $\sqrt{c_i}$ with the usual inner product. However, there are cases in which irrational coordinates are still needed, such as 13 dimensions.

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Table 1. Kissing number bounds.

Dimension	Lower bound	Upper bound	Ratio	References
1	2		1	
2	6		1	
3	12		1	[22]
4	24		1	[21, 18]
5	40	44	1.100	[8, 17]
6	72	77	1.084	[8, 10]
7	126	134	1.064	[8, 17]
8	240		1	[8, 15, 20]
9	306	363	1.187	[14, 16]
10	510	553	1.085	[6, 16]
11	592	868	1.467	[6, 9]
12	840	1355	1.614	[14, 9]
13	1154	2064	1.789	[24, 9]
14	1932	3174	1.643	[6, 9]
15	2564	4853	1.893	[14, 9]
16	4320	7320	1.695	[1, 9]
17	5346	10978	2.054	[11, 9]
18	7398	16406	2.218	[11, 9]
19	10668	24417	2.289	[11, 9]
20	17400	36195	2.081	[11, 9]
21	27720	53524	1.931	[11, 9]
22	49896	80810	1.620	[11, 9]
23	93150	122351	1.314	[11, 9]
24	196560		1	[11, 15, 20]
25	197048	265006	1.345	[7, 9]
26	198512	367775	1.853	[7, 9]
27	199976	522212	2.612	[7, 9]
28	204368	752292	3.682	[7, 9]
29	208272	1075991	5.167	[7, 9]
30	219984	1537707	6.991	[7, 9]
31	232874	2213487	9.506	[7, 9]
32	345408	3162316	9.156	[2, 9]
33	360640	4494570	12.47	[2, 9]
34	380868	6422593	16.87	[2, 9]
35	409548	9162403	22.38	[2, 9]
36	484568	13017098	26.87	[2, 9]
37	494312	18498316	37.43	[2, 9]
38	566652	26496684	46.77	[2, 9]
39	755988	37826766	50.04	[2, 9]
40	1064368	53589200	50.35	[2, 9]
41	1170384	76287040	65.19	[2, 9]
42	1250676	108404055	86.68	[2, 9]
43	1745692	153813582	88.12	[2, 9]
44	2948552	220788272	74.89	[5, 9]
45	3047160	316735249	104.0	[2, 9]
46	5318060	441900184	83.10	[2, 9]
47	9741412	621658419	63.82	[2, 9]
48	52416000	867897072	16.56	[14, 9]
72	6218175600	2545617287927	409.4	[19, 9]

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