VIBRATIONAL CHARACTERISTICS OF BUILDING FRAMES

by

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1943

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1943

Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

from the

Massachusetts Institute of Technology

1946

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Department of Civil and Sanitary Engineering, Feb.25,1946

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Thesis
1946

Cambridge, Massachusetts
February 25, 1946

Professor George W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Sir:

Submitted herewith is the thesis entitled "VIBRATIONAL CHARACTERISTICS OF BUILDING FRAMES." In partial fulfillment for the degree of Master of Science in Civil Engineering from the Massachusetts Institute of Technology,

Respectfully yours,

Signature redacted

Naim Joseph Dloomy

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Mohamed Haba

ACKNOWLEDGMENT

The authors wish to express their sincere appreciation to Professor C. H. NORRIS of the Massachusetts Institute of Technology who suggested this thesis and the methods of approach, and to Mr. Charles F. Peck, Jr., who assisted in getting the technical knowledge for handling the apparatus.

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1. SUMMARY

A. Object:

In many structural problems the vibrational characteristics are very important for the practical design. The theoretical solution of these problems is too tedious and moreover many assumptions are made that are not always close to the truth. Examples of this type of problems are the two dimensional and three dimensional building frames. For this type of problem and many others, we can make an experimental solution from which we can use the experimental results for the practical design.

The object of this thesis is to extend the solution of the two dimensional frame buildings to the three dimensional case, and compare the experimental frequencies and amplitude with the theoretical ones.

B. Scope:

As the three dimensional model was the one in which the authors were interested, they started in building two similar two dimensional frames which had stiffness within the range of the instrument. These models represent typical building frames.

The models were made of steel and were three story single bay frames. The three dimensional model was built from these two similar bents by welding girders between them. All the girders and the columns were prismatic and all the joints were welded to make them perfectly rigid. The floors

were attached to the four corners of the building by means of screws and were made of 3/16 inch steel plates. The supports were made very heavy and rigid.

C. Method:

All the models were vibrated by means of a pulsating force which was provided by permanent magnet speaker which was properly designed for this purpose. An electromagnetic type of a pick up was used and connected to a cathod ray oscilloscope on which the amplitudes were recorded. For determining the natural frequencies, the resonance of the model with that of the loud speaker were observed. A detailed discussion of the instruments may be found in the thesis done by Messrs. Shih-Ying Lee and Maleo L.P.Go. at MIT. 1943.

D. Theoretical Solution:

The frequencies and shapes of the normal modes of vibration were computed by conventional methods assuming that the mass of each story of the frame was concentrated at the level of the floors. The necessary stiffness factors were computed by the slope - deflection method assuming that the two dimensional model and the four sides of the three dimensional model were three story one bay planer building frames with rigid joints.

E. Discussion of Results and Conclusions:

The experimental results obtained from the two dimensional models agree with the theoretical results, as far as the sense of the amplitudes are concerned. In magnitude, however, these results were not in agreement, probably due to the fact that the theoretical assumption was, that the masses were concentrated at the floor level.

The authers believe that this difference of magnitude is due to this assumption only. As far as the frequencies are concerned, all the experimental results were in very close agreement with the theoretical ones. So it would be correct to say that the experimental results are more reliable than the theoretical ones, because it represents the actual conditions of the model.

In comparing the experimental with the theoretical results, for the three dimensional model, it shows that the experimental results are closer to the theoretical, especially in the symmetrical modes, which proves the explanation given above, that the difference in the amplitudes is due to the assumption that the mass of the columns are concentrated at the floor level, and the heavier the floor, as compared to the mass of the columns, the closer the agreement between the experimental and theoretical results.

It is probably that another type of theoretical assumption would lead to a more exact solution, but it would be very complicated and tedious to calculate.

11. INTRODUCTION

Vibration of framed buildings is due to many factors. The most important of which nowadays are the development of heavy machinery, which is installed in framed buildings. The effect of earthquakes, and lately the necessity of constructing air raid shelters and buildings which withstand explosions.

For these reasons engineers are looking for practical methods to be used in solving the vibration problems of framed buildings.

In such studies, the characteristics of the models of vibration are very important.

The object of this thesis is to propose the use of models of the framed buildings to determine their vibrational characteristics.

A steel model was built up to represent a three story framed building, which is properly supported. Then by using a vibrator, the model was ~~being~~ vibrated at a certain frequency which is furnished by an oscillator. This oscillator was tuned at one of the natural frequencies of the model, and that particular mode was excited much more than the others.

Two pick-ups were used to pick up the vibration of the model and it was recorded on a cathod-ray oscilloscope. When the natural mode was excited, a loop of good size appeared on the screen of the cathod-ray oscilloscope. A Vertical variable reading could be taken from the movable pick-up and constant horizontal reading for the fixed pick-up. By taking the reading of the joints, the shape of the mode could be attained.

The model should be constructed in such a manner that its natural vibration is going to be within the range of the equipment used for the experiment. And the size of the different members of the model has to bear a certain ratio to the prototype, so that the investigator can interpret the experimental results from the model to its prototype. The construction of the supports should be in such a manner that it represent the actual conditions.

III. THEORETICAL SOLUTION.

The authers wish to explain briefly the following, in order to facilitate the understanding of the vibration properties .

I. Vibration;

Vibration in its general sense means a motion, which can be either periodic i.e, a motion which repeats itself in all its particulars after a certain interval of time(T) which is called the period of vibration, or nonperiodic.

If the displacement in the x direction is plotted against the time (t) it will form a curve of considerable complication. Fig. I. show the vibration curve of the first mode of a simply supported beam. Which is a sine curve.

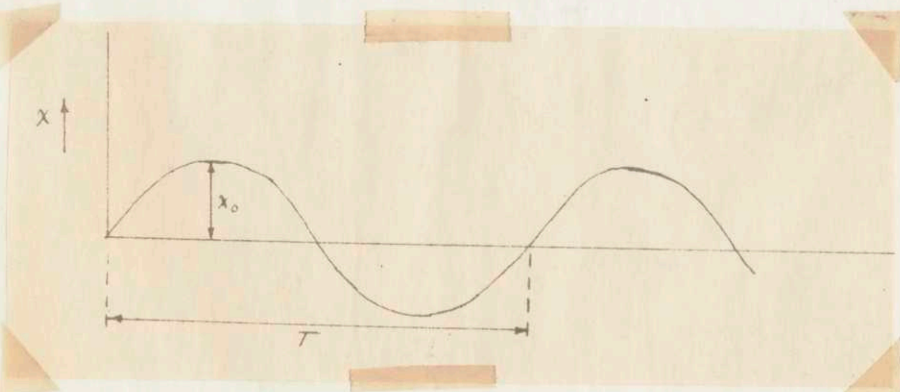


Fig. I.

2.Degrees of freedom.

A mechanical system is said to have one degree of freedom if its geometrical position can be expressed at any instant by one number only. As an example, take a position of moving piston inside a cylinder,its position could be specified at any time by giving its distance from the cylinder end. And so it has one degree of freedom. A weight suspended from a spring in such a manner that it is constrained in guides to move in a vertical direction only (up and down) is the classical single-degree-of-freedom vibration. If it takes (n) numbers to specify the position of a mechanical system, then that system is said to have (n) degrees of freedom.

A disk moving in its plane without any restraint has three degrees of freedom. The x, and y, displacement of the center of gravity , and the angular rotation about an axis passing through the center of gravity and perpendicular to the plane of the disk.

An elastic structure such as a beam, has an infinite number of degrees of freedom because the definition of its position requires one to enumerate the ordinate of the deflection curve for every point on the beam, of which there are an infinite number. But in some cases, the mass of the beam may be concideredto have a negligible effect on the analysis of its motion . Under those conditions, the beam may be considered to have a finite number of degrees of freedom which is determined by the other conditions of the problem.

3. Differential equation of motion of a single-degree of- freedom.

Consider the case of a mass (M) suspended from a rigid ceiling by means of a spring, as shown in Fig. (2) the stiffness of the spring is denoted by its (spring constant), (k). Which is the number of pounds required to expand this spring one inch. In this problem the friction is neglected.

Let an external force $(P_o) \sin \omega t$ act on the mass.

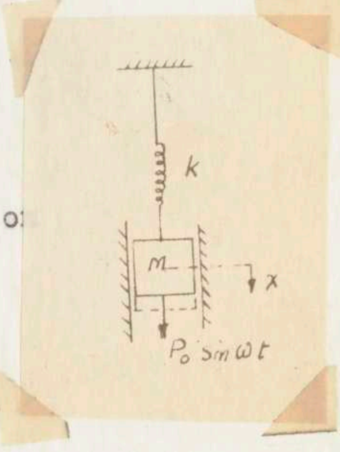
The problem consists in calculating this motion of the mass (M) due to the applied force.

Let (x) be the distance between any instantaneous position of the mass during its motion and the equilibrium position . To find (x) as a function of (M) using Newton second law of motion.

$F = M a$, where (F) is the force ,

(M) is the mass,

and (a) is the acceleration of that mass.



Consider all the forces on the mass to be positive when acting downward, and negative when acting upward.

Fig. (2)

Because the spring follows Hook's Law of proportionality between force and extension, then the spring force will be (k x) in pounds because the spring has expanded (x) inches . This force is negative because the spring pulls upwards while the displacement is downwards. Thus (- k x) is the force of the spring. The force on the spring is acting downwards in this case, therefore it is ($+ P_0 \sin\omega t$) but we have,

$$F = M a$$

$$M a = -k x + P_0 \sin\omega t$$

and $a = \frac{d^2x}{dt^2}$

$$M \times \frac{d^2x}{dt^2} = -k x + P_0 \sin\omega t$$

and $M \frac{d^2x}{dt^2} + k x = P_0 \sin\omega t$..................(1)

This equation is known as the differential equation of motion of a single-degree-of freedom system. The effect of gravity is omitted in this equation since the displacement (x) was measured from the static equilibrium position of the body.

4. Free Vibration

In case there is not any external or impressed force, then ($P_0 \sin\omega t$) will be equal to zero.

$M a + k x = 0$..............................(2)

This case is called the free vibration .

From equation (2) we have ,

$$a = -\frac{k x}{M} = -\frac{k \cdot x}{M}$$

Hence the force

Hence the free vibration is that which takes place without any external or impressed forces, and will be either constant, or a function of time (t).

5. A Natural or Normal Mode of Vibration

A normal mode is a free vibration which is not only periodic, but also has the characteristic, that at any instant, the displacement of each and every point on the structure is the same proportional part of the maximum displacement of that point during the vibration.

In the case of a beam, a natural or normal mode of vibration could be represented by the equation:

$$y = \left[F(t)\right] \left[u(x)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where (y) is the deflection at any point, F(t) is a periodic function of time, and u(x) is a function of the distance along the span.

Equation (3) specifies that at any instant, the shape of the displacement curve is similar to that at any other instant.

Fig. (3) gives a clear idea about the natural mode of vibration.

$$\frac{b_1}{b_2} = \frac{a_1}{a_2}$$

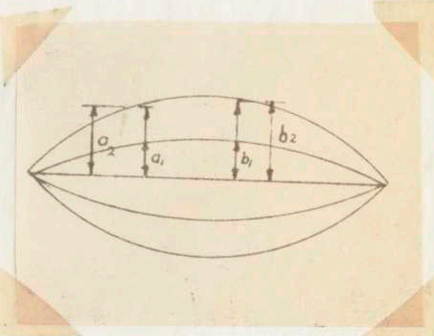


Fig. (3)

As a structure vibrates in one of its natural modes, the motion repeats itself periodically with a frequency which is called the Natural frequency of that mode. Since each mode has its own particular natural frequency, there will be (n) natural frequencies for a structure having (n) degrees of freedom. From that, it is shown that a structure has a number of normal modes of vibration equal to the number of degrees of freedom of that structure.

6. Solution of The Differential Equation of Free Vibration.

From equation (2) we have:

$$M\,a + k\,x = 0$$

By inspection, the solution is:

$$x = c_1 \;\mathrm{Sin}\; t\sqrt{\frac{k}{M}} + c_2 \cos t\sqrt{\frac{k}{M}} \;\ldots\ldots\ldots\ldots (3)$$

where c_1 and c_2 are arbitrary constants. This equation can be verified as follows:

$$\frac{d^2x}{dt^2} = -c_1 \frac{k}{M} \sin t\sqrt{\frac{k}{M}} - c_2 \frac{k}{M} \cos t\sqrt{\frac{k}{M}}$$

$$\therefore M\frac{d^2x}{dt^2} + kx = -c_1\, k \sin t\sqrt{\frac{k}{M}} - c_2\, k \cos t\sqrt{\frac{k}{M}}$$

$$+ c_1\, k \sin t\sqrt{\frac{k}{M}} + c_2\, k \cos t\sqrt{\frac{k}{M}} = 0$$

If we assume that the body is pulled down a distance equal to (x_0) from its statical equilibrium and then released without any initial velocity, then:

$$\text{at } t = 0, \quad x = x_0, \quad \text{and } \frac{dx}{dt} = 0$$

If the first condition is substituted in equation (3) then we have:

$$x_0 = c_1 \cdot 0 + c_2 \cdot 1$$

and therefore $c_2 = x_0$

To substitute for the second condition, differentiate equation (3).

$$\frac{dx}{dt} = c_1\sqrt{\frac{k}{M}}.\cos.t.\sqrt{\frac{k}{M}} - c_2.\sqrt{\frac{k}{M}}.\sin.t.\sqrt{\frac{k}{M}}$$

Substituting the same condition in this equation, you have:

$$0 = c_1 \cdot \sqrt{\frac{k}{M}} \cdot 1 - c_2 \sqrt{\frac{k}{M}} \cdot 0$$

$$\therefore \quad c_1 \cdot \sqrt{\frac{k}{M}} = 0, \quad \text{or } c_1 = 0$$

Substituting back those results in equation (3), we have:

$$x = x_0 \cos \cdot t \cdot \sqrt{\frac{k}{M}}$$

One cycle of this vibration occurs when $t \cdot \sqrt{\frac{k}{M}}$ varies through 360° or 2π radians, if we denote the time of one cycle by (T), as in Fig. (4), we have:

$$T \cdot \sqrt{\frac{k}{M}} = 2\pi \qquad \therefore \quad T = 2\pi\sqrt{\frac{M}{k}} \ldots\ldots\ldots\ldots (4)$$

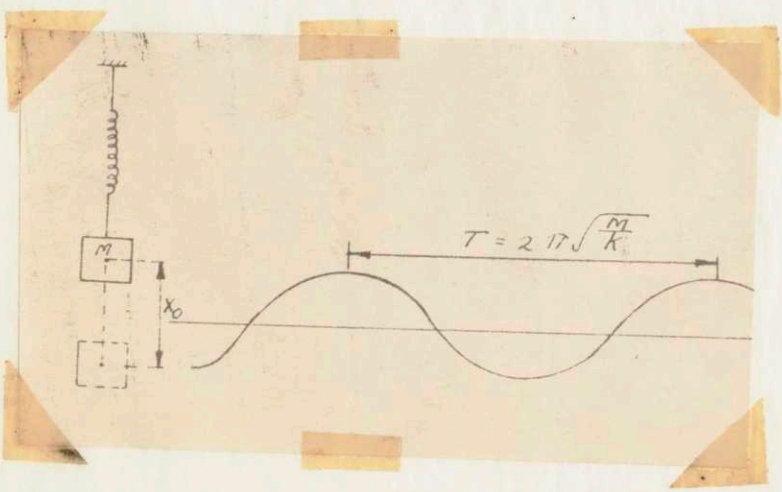


Fig. (4)

It is customary to denote the term $\frac{k}{M}$ by (ω_n) which is called the "natural circular frequency". This value (ω_n) is the angular velocity of the rotating vector which represents the vibrating motion.

The reciprocal of (T), or the natural frequency (f_n) may be defined by the following equations:

$$f_n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

7. Theoretical Solution of the Two Dimensional Frames.

If an exact solution is to be carried out for a frame, we should consider that the masses are distributed all the way along the columns and girders, which means that the frame will have an infinite number of degrees of freedom. For common types of buildings, the mass of the floor is large compared to the mass of the columns, so it is legitimate to assume that the masses are concentrated at the floor level. The system is reduced , therefore to a weightless frame acted upon by concentrated masses, one mass at each floor. The degree of freedom of such a frame is equal to the number of floors.

Fig. (5a) shows such a building frame, having three floors and , therefore, three degrees of freedom.

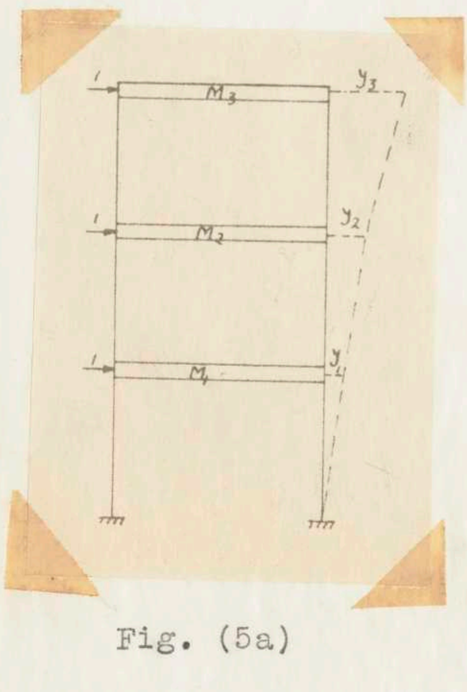


Fig. (5a)

This solution presumably leads to good results as long as the mass of the floor is large compared to the mass of the columns. The motion of these masses during a normal mode of vibration may be represented by the following equations:-

$$y_n = a_n \cdot \sin . 2 \frac{\pi}{T} . t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which

(y_n), is the deflection of the mass (M),

(a_n), is the amplitude of the same mass,

The characteristic shapes of the modes of the structure depend on the elastic and internal properties of the structure which are functions of the dimensions , the conditions of the supports , and the physical properties of the material of the structure.

From equation (5) the acceleration of the mass (M) is given by:-

$$\frac{d^2 y_n}{dt^2} = -a_n . \frac{4\pi^2}{T^2} . \text{Sin } \frac{2\pi}{T} . t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Denoting by (M_n) the mass of the $n^{th.}$ floor, then the inertia force on this mass is

$$P_n = -M_n \cdot \frac{d^2 y_n}{dt^2} . M_n . \frac{4\pi^2}{T^2} . a_n . \sin . \frac{2\pi}{T} . t \ldots\ldots\ldots\ldots (7)$$

Substituting the value of (y_n) from equation (5), you have:-

$$P_n = \left[M_n \frac{4\pi^2}{T^2} \right] . y_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Which means that the inertia force is directly proportional to the displacement.

Substituting the value of $\frac{T^2}{4\pi^2} = K$ in equation (8) we have:-

$$y_n = \frac{P_n}{M_n} \cdot K \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

To find the total deflection at each floor caused by the inertia forces, the unit deflection caused at each floor by a unit force applied at each floor should be calculated.

Denoting δ_{np} the deflection caused at floor (n) by a unit force acting at point (p), then we obtain for the displacements of the various masses:-

$$y_1 = \delta_{11} P_1 + \delta_{12} P_2 + \delta_{13} P_3 \ldots\ldots + \delta_{1n} P_n$$

$$y_2 = \delta_{21} P_1 + \delta_{22} P_2 + \delta_{23} P_3 \ldots\ldots + \delta_{2n} P_n$$

$$y_3 = \delta_{31} P_1 + \delta_{32} P_2 + \delta_{33} P_3 \ldots\ldots + \delta_{3n} P_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_n = \delta_{n1} P_1 + \delta_{n2} P_2 + \delta_{n3} P_3 \ldots\ldots + \delta_{nn} P_n \ldots\ldots(10)$$

Substituting the values of (y_n) in equation (9) we have:-

$$\frac{K}{M_1} P_1 = P_1 \delta_{11} + P_2 \delta_{12} + P_3 \delta_{13} \ldots\ldots + P_n \delta_{1n}$$

$$\frac{K}{M_2} P_2 = P_1 \delta_{21} + P_2 \delta_{22} + P_3 \delta_{23} \ldots\ldots + P_n \delta_{2n}$$

$$\frac{K}{M_3} P_3 = P_1 \delta_{31} + P_2 \delta_{32} + P_3 \delta_{33} \ldots\ldots + P_n \delta_{3n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{K}{M_n} P_n = P_1 \delta_{n1} + P_2 \delta_{n2} + P_3 \delta_{n3} \ldots\ldots + P_n \delta_{nn} \ldots\ldots(11)$$

Transfering the left side of the equation to the right will give:-

$$P_1(\delta_{11}-\frac{K}{M_1})+P_2\delta_{12}+P_3\delta_{13}\cdot\cdot\cdot\cdot+P_n\delta_{1n}=0$$

$$P_1\delta_{21}+P_2(\delta_{22}-\frac{K}{M_2})+P_3\delta_{23}\cdot\cdot\cdot\cdot+P_n\delta_{2n}=0$$

$$P_1\delta_{31}+P_2\delta_{32}+P_3(\delta_{33}-\frac{K}{M_3})\cdot\cdot\cdot+P_n\delta_{3n}=0$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$P_1\delta_{n1}+P_2\delta_{n2}+P_3\delta_{n3}\cdot\cdot\cdot+P_n(\delta_{nn}-\frac{K}{M_n})=0 \quad\cdots\cdots(12)$$

These equations are known as the system of (n) homogeneous equations with (n) unknowns. In order for the unknowns to have values other than zero, the determinant of the coefficients must be equal to zero.

Letting the values of $\frac{K}{M}=K'$ Where in this case all the masses are the same, we have:-

$$\begin{vmatrix} \delta_{11}-K' & \delta_{12} & \delta_{13} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22}-K' & \delta_{23} & \cdots & \delta_{2n} \\ \delta_{31} & \delta_{32} & \delta_{33}-K' & \cdots & \delta_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \cdots & \delta_{nn}-K' \end{vmatrix}=0$$

Solving this determinant leads to an equation of the (n^{th}) degree with (n) real roots.

In our case, where we have three story frame building, this determinant will be expanded to a cubical equation in terms of (K') and the three values of (K') will be found.

Using the relation between (T) and (K)

$$T = 2\pi\sqrt{K} = 2\pi\sqrt{K'M} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I4)$$

and substituting (K) for each mode back in equation(I2) we can obtain the relative values of (P_n) and therefore the relative values of amplitudes could also be attained.

8.THEORETICAL SOLUTION OF THE THREE DIMENSIONAL MODEL.

The solution of the three dimensional building is somewhat more complicated than a two dimensional frame. Fig.(5b) represent the three dimensional model which is composed of two symmetrical bents (Y^1)&(Y^2) and the two unsymmetrical ones (X^1)&(X^2).

If a force is applied to this structure three types of displacement may occur.

1. Horizontal(x)direction,
2. Vertical (y)direction,
3. Tortional.

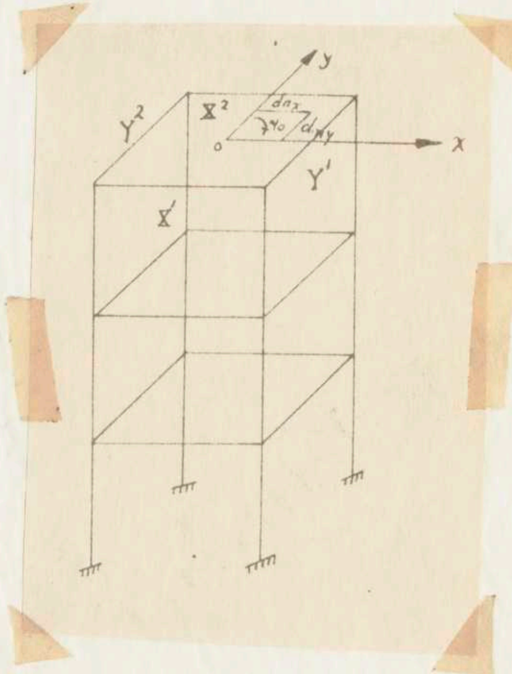


Fig. (5b)

Selecting an origin (o) at the centre of the ($f^{th.}$) floor, and letting (d_{xf})&(d_{yf}) represent the displacement of point(o) in (x)&(y) directions respectively, and (α_{of}) represent the rotation of the floor about an axis passing through point (o) and perpendicular to the floor surface.

(x) being plus to the right,

(y) being plus when up,

(α) being plus when it is clockwise.

Now let,

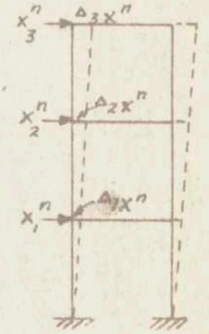
(x_f^n) represent the shear with which the $(f^{th.})$ floor acts on the (X^n) bent.

(y_f^n) represent the shear with which the $(f^{th.})$ floor acts on the (Y^n) bent.

(Δ_{fX^n}) represent the total deflection of the $(f^{th.})$ floor of the (X^n) bent.

(Δ_{fY^n}) represent the total deflection of the (f^{th}) floor of the (Y^n) bent.

Hence,



Bent X^n

$$\Delta_{1X^n} = x_1^n\,\delta_{11}^{X^n} + x_2^n\,\delta_{12}^{X^n} \ldots\ldots + x_f^n\,\delta_{1f}^{X^n}$$

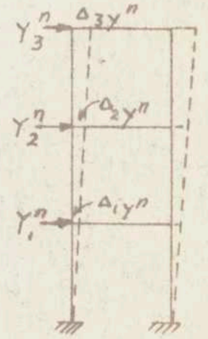
$$\Delta_{2X^n} = x_1^n\,\delta_{21}^{X^n} + x_2^n\,\delta_{22}^{X^n} \ldots\ldots + x_f^n\,\delta_{2f}^{X^n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta_{fX^n} = x_1^n\,\delta_{f1}^{X^n} + x_2^n\,\delta_{f2}^{X^n} \ldots\ldots + x_f^n\,\delta_{ff}^{X^n} \quad \ldots (15a)$$

And,



Bent Y^n

$$\Delta_{1Y^n} = y_1^n\,\delta_{11}^{Y^n} + y_2^n\,\delta_{12}^{Y^n} \ldots\ldots + y_f^n\,\delta_{1f}^{Y^n}$$

$$\Delta_{2Y^n} = y_1^n\,\delta_{21}^{Y^n} + y_2^n\,\delta_{22}^{Y^n} \ldots\ldots + y_f^n\,\delta_{2f}^{Y^n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta_{fY^n} = y_1^n\,\delta_{f1}^{Y^n} + Y_2^n\,\delta_{f2}^{Y^n} \ldots\ldots + y_f^n\,\delta_{ff}^{Y^n} \quad \ldots (15b)$$

Where,

(δ_{11}^{Xn}) is the deflection at the (X^{n}) bent of the first floor caused by a unit load at the first floor.

(δ_{12}^{Yn}) is the deflection at the (Y^{n}) bent of the first floor caused by a unit load at the second floor.

(δ_{fn}^{Xn}) is the deflection at the (X^{n}) bent of the (f^{th}) floor caused by a unit load at the (n^{th}) floor.

Solving equatios (I5a) & (I5b) simultaneously for the values of (x_1^n, x_2^n, x_f^n) & (y_1^n, y_2^n, y_f^n) in terms of $(\Delta_1 X^n, \Delta_2 X^n, \Delta_{3X}^{n})$ and $(\Delta_1 Y^n, \Delta_2 Y^n, \Delta_{fY}^{n})$ respectively will get.

$$X_1^n = (X_{11}^n)\Delta_1 X^n + (X_{12}^n)\Delta_2 X^n \cdots + (X_{1f}^n)\Delta_f X^n$$

$$X_2^n = (X_{21}^n)\Delta_1 X^n + (X_{22}^n)\Delta_2 X^n \cdots + (X_{2f}^n)\Delta_f X^n$$

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$$X_f^n = (X_{f1}^n)\Delta_1 X^n + (X_{f2}^n)\Delta_2 X^n \cdots + (X_{ff}^n)\Delta_f X^n \quad \ldots (16a)$$

and,

$$Y_1^n = (Y_{11}^n)\Delta_1 Y^n + (Y_{12}^n)\Delta_2 Y^n \cdots + (Y_{1f}^n)\Delta_f Y^n$$

$$Y_2^n = (Y_{21}^n)\Delta_1 Y^n + (Y_{22}^n)\Delta_2 Y^n \cdots + (Y_{2f}^n)\Delta_f Y^n$$

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$$Y_f^n = (Y_{f1}^n)\Delta_1 Y^n + (Y_{f2}^n)\Delta_2 Y^n \cdots + (Y_{ff}^n)\Delta_f Y^n \quad \ldots (16b)$$

But,

$$\Delta_{1X^n} = d_{x1} + \alpha_{01} \times y_x$$

$$\Delta_{2X^n} = d_{x2} + \alpha_{02} \cdot y_x$$

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$$\Delta_{fX^n} = d_{xf} + \alpha_{0f} \cdot y_x \qquad \text{I7a}$$

$$\Delta_{1Y^n} = d_{y1} - \alpha_{01} \cdot x_y$$

$$\Delta_{2Y^n} = d_{y2} - \alpha_{02} \cdot x_y$$

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$$\Delta_{fY^n} = d_{yf} - \alpha_{0f} \cdot x_y \qquad \text{I7b}$$

Where (y_x) and (x_y) are the distances from the origin (O) to the centre of the frame in the (x) and (y) directions respectively.

Substituting the values of equations (I7a)&(I7b) in equations (I6a)&(I6b) respectively, will get.

$$x_1^n = \left[(x_{11}^n) d_{x1} + (x_{12}^n) d_{x2} + \ldots + (x_{1f}^n) d_{xf} \right]$$
$$+ y_x \left[(x_{11}^n) \alpha_{01} + (x_{12}^n) \alpha_{02} \cdots + (x_{1f}^n) \alpha_{0f} \right]$$

$$X_2^n = \left[(X_{21}^n) d_{x1} + (X_{22}^n) d_{x2} + \cdot \cdot \cdot + (X_{2f}^n) d_{xf} \right]$$

$$+ Y_x \left[(X_{21}^n) \alpha_{01} + (X_{22}^n) \alpha_{02} + \cdot \cdot \cdot + (X_{2f}^n) \alpha_{0f} \right]$$

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$$X_f^n = \left[(X_{f1}^n) d_{x1} + (X_{f2}^n) d_{xf} + \cdot \cdot \cdot + (X_{ff}^n) d_{xf} \right]$$

$$+ Y_x \left[(X_{f1}^n) \alpha_{01} + (X_{f2}^n) \alpha_{02} + \cdot \cdot \cdot + (X_{ff}^n)(\alpha_{0f} \right] \quad \ldots\ldots (18a)$$

also,

$$Y_1^n = \left[(Y_{11}^n) d_{y1} + (Y_{12}^n) d_{y2} + \cdot \cdot \cdot + (Y_{1f}^n) d_{yf} \right]$$

$$- X_y \left[(Y_{11}^n) \alpha_{01} + (Y_{12}^n) \alpha_{02} + \cdot \cdot \cdot + (Y_{1f}^n) \alpha_{0f} \right]$$

$$Y_2^n = \left[(Y_{21}^n) d_{y1} + (Y_{22}^n) d_{y2} + \cdot \cdot \cdot + (Y_{2f}^n) d_{yf} \right]$$

$$- X_y \left[(Y_{21}^n) \alpha_{01} + (Y_{22}^n) \alpha_{02} + \cdot \cdot \cdot + (Y_{2f}^n) \alpha_{0f} \right]$$

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$$Y_f^n = \left[(Y_{f1}^n) d_{y1} + (Y_{f2}^n) d_{y2} + \cdot \cdot \cdot + (Y_{ff}^n) d_{yf} \right]$$

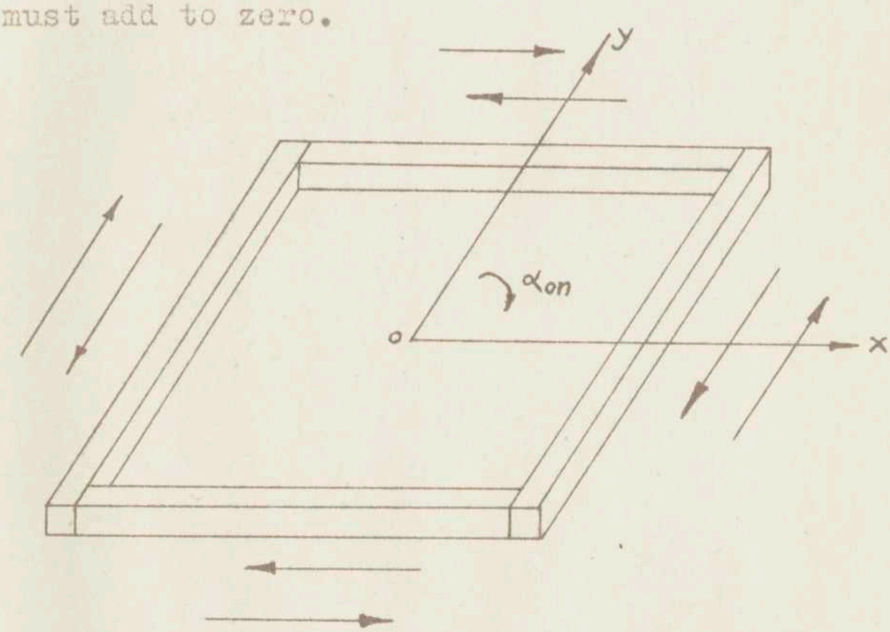
$$- X_y \left[(Y_{f1}^n) \alpha_{01} + (Y_{f2}^n) \alpha_{02} + \cdot \cdot \cdot + (Y_{ff}^n) \alpha_{0f} \right] \ldots\ldots (18b)$$

In order to have an equilibirium condition in each floor the following static equations should hold.

$\sum F_x = 0$ Summation of all the forces in the x direction must add to zero.

$\sum F_y = 0$ Summation of all the forces in the y direction must add to zero.

$\sum M_o = 0$ Summation of all the moments around an axis through the origin and perpendicular to the floor must add to zero.



For $\sum Fx = 0$ will **have,**

$$- M_{tf} \frac{d^2(d_{xf})}{dt^2} - \sum X_f = 0 \quad \text{.................(I9)}$$

For $\sum F_y = 0$ will have,

$$- M_{tf} \frac{d^2(d_{yf})}{dt^2} - \sum Y_f = 0 \quad \text{.................(20)}$$

For $\sum M_o = 0$ will have,

$$- I_{tf} \frac{d^2(\alpha_{of})}{dt^2} - \sum X_f \cdot Y_x + \sum Y_f \cdot X_y = 0 \text{...(2I)}$$

Where,

M_{tf} is the total mass of the fth floor plus the weight of the girders and the columns attached to that floor.

I_{tf} is the total mass moment of inertia of the total mass, assumed to be distributed over that floor.

Σx_f is the summation of the forces acting in the (x) direction at the fth floor.

Σy_f is the summation ofthe forces acting in the (y) direction at the fth floor.

Hense, there will be three equatios of this type for each floor.

Let,

$$d_{xf} = D_{xf} \sin \frac{2\pi t}{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Then,

$$\frac{d^2(d_{xf})}{dt^2} = -D_{xf}\left(\frac{2\pi}{T}\right)^2 \sin \frac{2\pi}{T} t \;,\ldots\ldots\ldots\ldots (23)$$

also,

$$d_{yf} = D_{yf} \sin \frac{2\pi}{T} t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

$$\frac{d^2(d_{yf})}{dt^2} = -D_{yf} \frac{(2\pi)^2}{T} \sin \frac{2\pi}{T} t \quad \ldots\ldots\ldots\ldots (25)$$

and,

$$\alpha_{of} = A_{of} \sin \frac{2\pi}{T} t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

$$\frac{d(\alpha_{of})}{dt^2} = -A_{of}\left(\frac{2\pi}{T}\right)^2 \sin \frac{2\pi}{T} t \quad \ldots\ldots\ldots\ldots (27)$$

If we let $K = \left(\frac{2\pi}{T}\right)^2$ equations (23, 24, & 27) will be, after substituting the values of d_{xf} , d_{yf} and α_{of}

$$\frac{d^2(d_{xf})}{dt^2} = -K\, d_{xf} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

$$\frac{d^2(d_{yf})}{dt^2} = -K\, d_{yf} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

$$\frac{d^2(\alpha_{of})}{dt^2} = -K\, \alpha_{of} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Substituting the values from equations (28,29,&30) in equations (19,20,& 21) respectively will get.

$$M_{tf}\, d_{xf}\, K - \Sigma\, x_f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

$$M_{tf}\, d_{yf}\, K - \Sigma\, y_f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

$$I_f\, \alpha_{of}\, K - \Sigma\, x_f \cdot y_x + \Sigma\, y_f \cdot x_y = 0 \quad \ldots\ldots\ldots\ldots\ldots (33)$$

Substituting the values of (x_f) from equation (18a) in equation (31) will get.

I st floor.

$$-\left[(\Sigma x_{11}^n) - M_{tf} K\right] d_{x1} - (\Sigma x_{12}^n)\, d_{x2} - \cdot\cdot\cdot - (\Sigma x_{1f}^n)\, d_{xf}$$

$$- \Sigma (x_{11}^n \cdot y_x)\, \alpha_{o1} - \Sigma (x_{12}^n \cdot y_x)\, \alpha_{o2} \cdot\cdot\cdot - \Sigma (x_{1f}^n \cdot y_x)\alpha_{of} = 0 \quad \cdots\cdots (34).$$

2nd floor.

$$-\Sigma (x_{21}^n) d_{x1} - \left[(\Sigma x_{22}^n) - M_{tf} K\right] d_{x2} - \cdots\cdots - (\Sigma x_{2f}^n)\, d_{xf}$$

$$- \Sigma (x_{21}^n \cdot y_x)\, \alpha_{o1} - \Sigma (x_{22}^n \cdot y_x)\, \alpha_{o2} - \cdots\cdots - \Sigma (x_{2f}^n \cdot y_x)\alpha_{of} = 0 \quad \cdots\cdots (35)$$

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f th floor.

$$- \sum (x_{f1}^{n}) d_{x1} - \sum (x_{f2}^{n}) d_{x2} - \cdots\cdots - \left[\sum x_{ff}^{n} - M_{tf} K \right] d_{xf}$$

$$- \sum (x_{f1}^{n} \cdot Y_x) \alpha_{01} - \sum (x_{f2}^{n} \cdot Y_x) \alpha_{02} - \cdots\cdots - \sum (x_{ff}^{n} \cdot Y_x) \alpha_{0f} = 0 \quad \cdots\cdots(36)$$

Substituting the values of (Y_f) from equation (18b) in equation(32) will get.

1st. floor.

$$- \left[(\sum Y_{11}^{n}) - M_{tf} K \right] d_{Y1} - (\sum Y_{12}^{n}) d_{Y2} - \cdots\cdots - (\sum Y_{1f}^{n}) d_{Yf}$$

$$+ \sum (Y_{11}^{n} \cdot x_y) \alpha_{01} + \sum (Y_{12}^{n} \cdot x_y) \alpha_{02} + \cdots\cdots + \sum (Y_{ff}^{n} \cdot x_y) \alpha_{0f} = 0 \quad \cdots\cdots(37)$$

2 nd. floor.

$$- (\sum Y_{21}^{n}) d_{Y1} - \left[(\sum Y_{22}) - M_{tf} K \right] d_{Y2} - \cdots\cdots - (\sum Y_{2f}^{n}) d_{Yf}$$

$$+ \sum (Y_{21}^{n} \cdot x_y) \alpha_{01} + \sum (Y_{22}^{n} \cdot x_y) \alpha_{02} + \cdots\cdots + \sum (Y_{2f}^{n} \cdot x_y) \alpha_{0f} = 0 \quad \cdots\cdots(38)$$

. . . .

. . . .

f th.floor.

$$- (\sum Y_{f1}^{n}) d_{Y1} - (\sum Y_{f2}) d_{Y2} - \cdots - \left[(\sum Y_{ff}^{n} - M_{tf} K \right] d_{Yf}$$

$$+ \sum (Y_{f1} \cdot x_y) \alpha_{01} + \sum (Y_{f2}^{n} \cdot x_y) \alpha_{02} + \cdots\cdots + \sum (Y_{ff}^{n} \cdot x_y) \alpha_{0f} = 0 \quad \cdots\cdots(39)$$

Substituting the values of x_f & y_f from equations (18a&18b) in equation (33)will get.

I st. floor.

$$- \sum(x_{11}^n y_x) d_{x1} - \sum(x_{12}^n y_x) d_{x2} + \sum(y_{11}^n x_y) d_{y1} + \sum(y_{12}^n x_y) d_{y2}$$

$$- \cdots\cdots - \sum(x_{1f}^n y_x) d_{xf} + \sum(y_{1f}^n x_y) d_{yf} - y_x \sum(x_{11}^n y_x) \alpha_{01}$$

$$- y_x \sum(x_{12}^n y_x) \alpha_{02} - x_y \sum(y_{11}^n x_y) \alpha_{01} - x_y \sum(y_{12}^n x_y) \alpha_{02} - \cdots\cdots$$

$$- y_x \sum(x_{1f}^n y_x) \alpha_{0f} - x_y \sum(y_{1f}^n x_y) \alpha_{0f} + I_{tf} \cdot K \cdot \alpha_{01} = 0 \quad \cdots\cdots (40)$$

2nd. floor.

$$- \sum(x_{21} y_x) d_{x1} - \sum(x_{22}^n y_x) d_{x2} + \sum(y_{21}^n x_y) d_{y1} + \sum(y_{22}^n x_y) d_{y2}$$

$$- \cdots\cdots - \sum(x_{2f}^n x_y) d_{xf} + \sum(y_{2f}^n x_y) d_{yf} - y_x \sum(x_{21}^n y_x) \alpha_{01}$$

$$- y_x \sum(x_{22} y_x) \alpha_{02} - x_y \sum(y_{21}^n x_y) \alpha_{01} - x_y \sum(y_{22}^n x_y) \alpha_{02} \cdots\cdots$$

$$- y_x \sum(x_{2f}^n y_x) \alpha_{0f} - x_y \sum(y_{2f}^n x_y) \alpha_{0f} + I_{tf} \cdot K \cdot \alpha_{02} = 0 \quad \cdots\cdots (41)$$

. . . .

. . . .

f th. floor.

$$-\sum(x_{f1}^{n}y_x)\,d_{x1}-\sum(x_{f2}\,y_x)\,d_{x2}+\sum(y_{f1}^{n}x_y)\,d_{y1}+\sum(y_{f2}^{n}x_y)\,d_{y2}$$

$$-\cdots\cdots-\sum(x_{ff}^{n}y_x)\,d_{xf}+\sum(y_{ff}^{n}x_y)\,d_{y_f}-y_x\sum(x_{f1}^{n}y_x)\alpha_{o1}$$

$$-y_x\sum(x_{f2}^{n}y_x)\alpha_{o2}-x_y\sum(y_{f1}^{n}x_y)\alpha_{o1}-x_y\sum(y_{f2}^{n}x_y)\alpha_{o2}\cdots\cdots$$

$$-y_x\sum(x_{ff}^{n}y_x)\alpha_{of}-x_y\sum(y_{ff}^{n}x_y)\alpha_{of}+I_{tf}\cdot K\cdot\alpha_{of}=0\quad\cdots\cdots(42)$$

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Stodola's Method of Iteration provides an expedient method which was made use of in the solutions of these equations. An outline of the Iteration procedure will be presented as follows:

Assuming values for,

$$d_{x1}\ ,\ d_{x2}\ \cdots\cdots\ d_{xn}$$

$$d_{y1}\ ,\ d_{y2}\ \cdots\cdots\ d_{yn}$$

$$\alpha_{o1}\ ,\ \alpha_{o2}\ \cdots\cdots\ \alpha_{on}$$

And trying these values as it is explained in the appendix until the solution is carried to convergence.

After one mode has been determined, one equation may be eliminated from the group of equations used by using the orthogonality relationship, which is:-

$$+ d_{x1}^{m} \cdot d_{x1}^{m-1} \cdot M_{tf} + d_{x2}^{m} \cdot d_{x2}^{m-1} \cdot M_{tf} \cdot \cdot \cdot \cdot \cdot \cdot + d_{xn}^{m} \cdot d_{xn}^{m-1} \cdot M_{tf}$$

$$+ d_{y1}^{m} \cdot d_{y1}^{m-1} \cdot M_{tf} + d_{y2}^{m} \cdot d_{y2}^{m-1} \cdot M_{tf} \cdot \cdot \cdot \cdot \cdot \cdot + d_{yn}^{m} \cdot d_{yn}^{m-1} \cdot M_{tf}$$

$$+ \alpha_{o1}^{m} \cdot \alpha_{o1}^{m-1} \cdot I_{tf} + \alpha_{o2}^{m} \cdot \alpha_{o2}^{m-1} \cdot I_{tf} \cdot \cdot \cdot \cdot \cdot \cdot + \alpha_{on}^{m} \cdot \alpha_{on}^{m-1} \cdot I_{tf} = 0$$

In which the exponent (m) refers to the mode.

Another assumption will be made and the solution will be carried to convergence as before for the d_x's , d_y's and α_o's . The orthogonality relationship will again facilitate the elimination of this mode from further assumptions.

The frequencies corresponding to each mode are determined by substituting the proper values in the following equation..

$$K' = K M_{tf}$$

$$\text{and} \quad K = \left(\frac{2\pi}{T}\right)^2$$

$$\therefore T = 2\pi \sqrt{\frac{M_{tf}}{K'}}$$

IV. EXPERIMENTAL SOLUTION.

A. A pparatus.

Fig. (6) shows the apparatus arranged with the Model diagramatically.

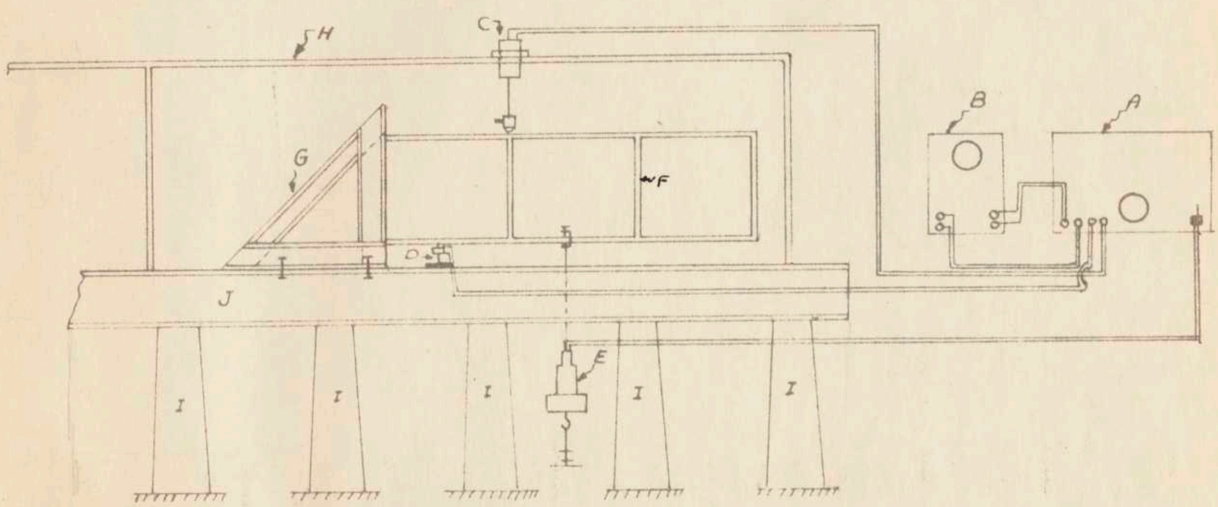


Fig. (6)
Arrangement of model and apparatus.

A. Oscillator.
B. Oscilloscope.
C. Movable pick-up.
D. Fixed pick-up.
E. Vibrator.
F. Model.
G. Supports.
H. Sliding frame.
I. Concrete piers.
J. Continuous channels.

1) Oscillator

An oscillator was used to produce a pulsating current which drives a vibrator. The maximum energy output is around 12 watts. The frequency ranges from 20 to 4000 cycles per second. The output voltage and current are a sine function of time.

2) Oscilloscope

The oscilloscope used was a 3-inch Dumont Cathode-Ray oscilloscope. A description of the principle of the oscilloscope may be found in thesis of Mr. Shih-Ying Lee at M.I.T. 1943.

3) Pick-ups

The pick ups used were of the electromagnetic type. The stationary pick-up was rigidly connected very close to one of the supports.

The movable pick up was mounted on a sliding frame in such a manner so that it could be used at any point on the model.

4) Vibrator

A. A permanent magnet speaker was used for the vibrator.

B. The Models

As it has been mentioned before that the main purpose of the thesis was to compare the theoretical and experimental results found by experiment for model of a three dimensional building frame, The whole structure was built by welding the girders to those two side frames and the floors were attached by means of screws to the four corners of each floor.

All of the members were prismatic and made of steel. The columns were continuous and the girders were welded in order to accomplish a complete rigidity. Model No. (1) which represents the two symetrical side bents, was a three story frame. The full dimensions of which are found in Fig. (7).

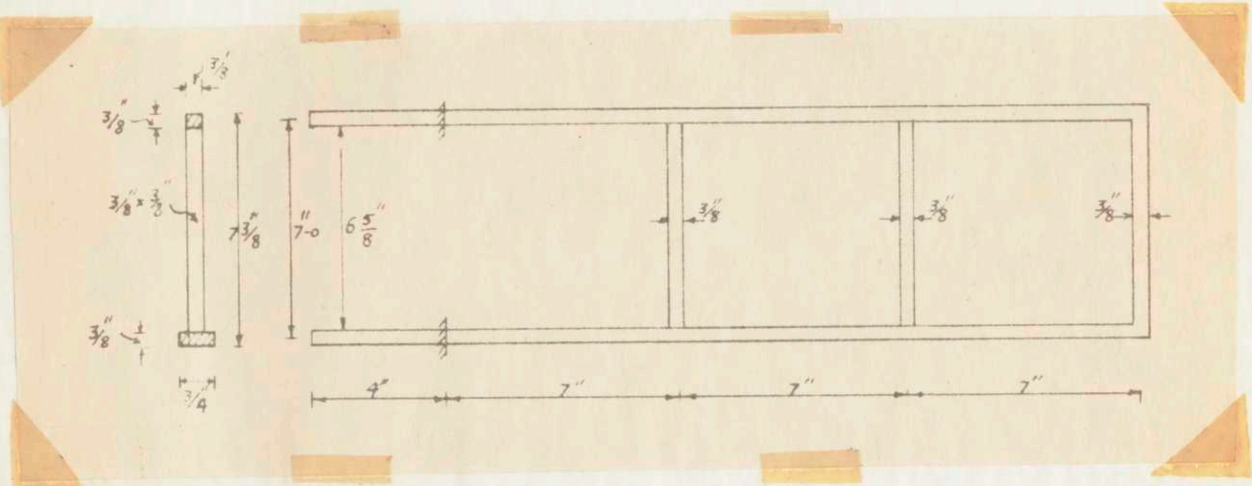


Fig. (7)

Model represents side Y^1 & Y^2 of the three dimensional model.

To assure rigidity at the support connections, a surplus of (4") was provided at the lower end of each column. These ends were attached to a base made of 3" channels. A detail drawing of which is found in Fig. (8).

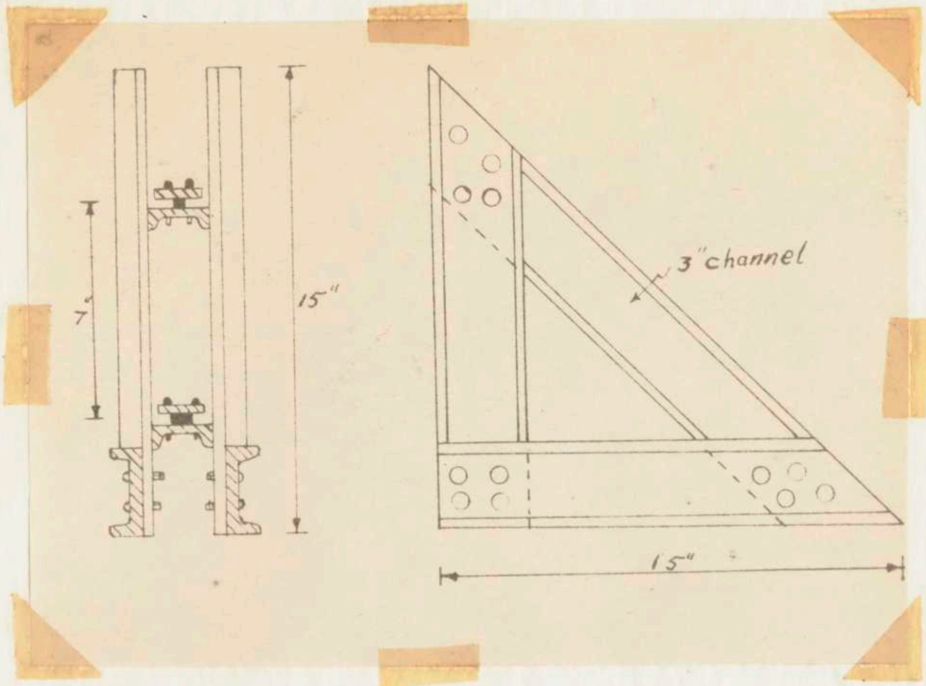


Fig. (8)

Supporting frame of the two side bents.

The ends of the columns were screwed to this comparatively heavy base, and this base was mounted on a heavy continuous channels which are fixed to a concrete piers. The base was attached to these channels by means of heavy clamps.

All the models were mounted horizontally so that when the pick-up will slide on a horizontal bar, all the vertical readings can be taken very easily. To take any horizontal reading, the end of the pick up is attached to a rod at 90° to the vertical and so with the same set up of the models any reading could be taken.

Fig. No. (10) shows the set up of the whole building. A detail drawing of the building model is found in Fig. No (11) The same base was used for the building model by attaching

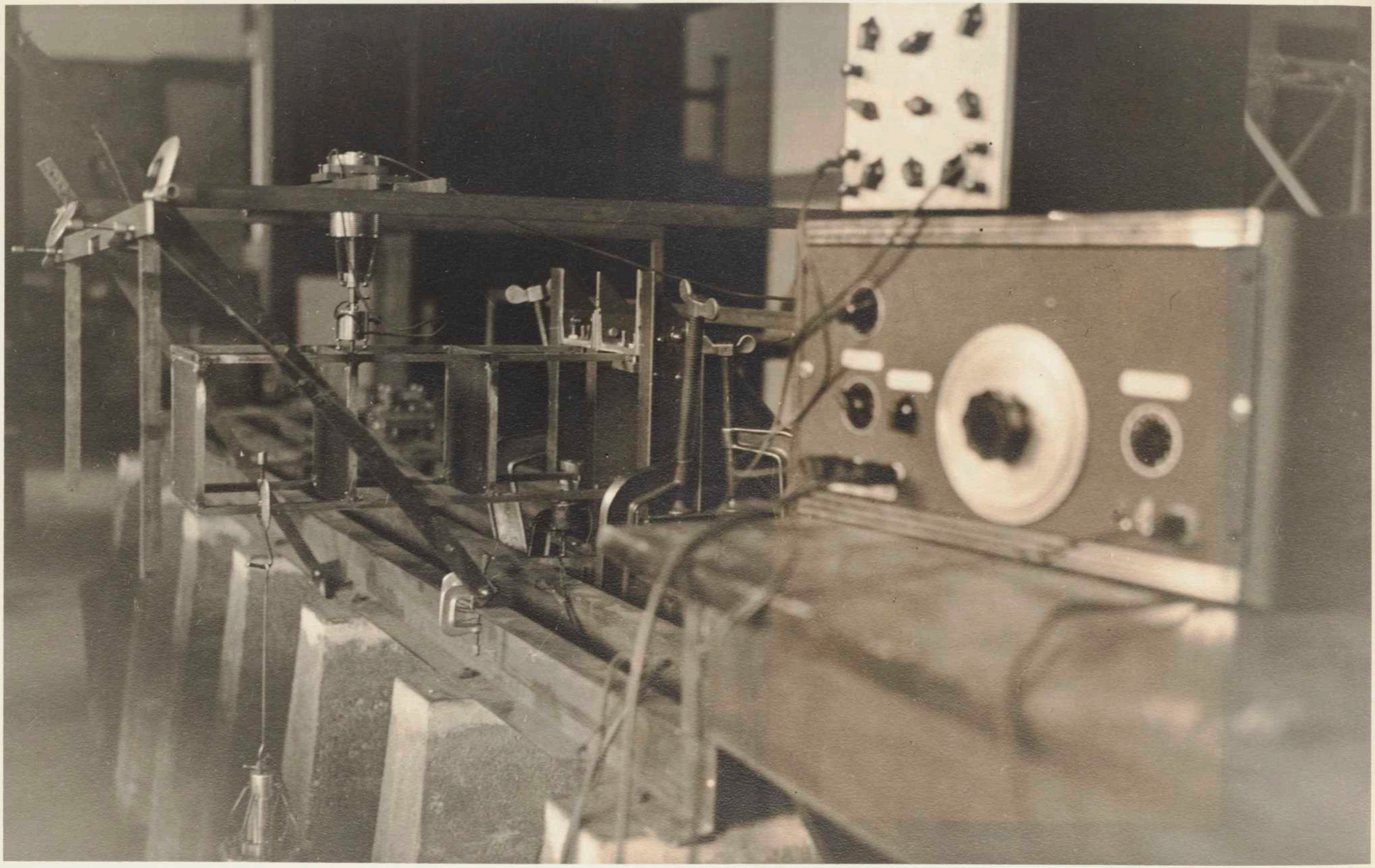
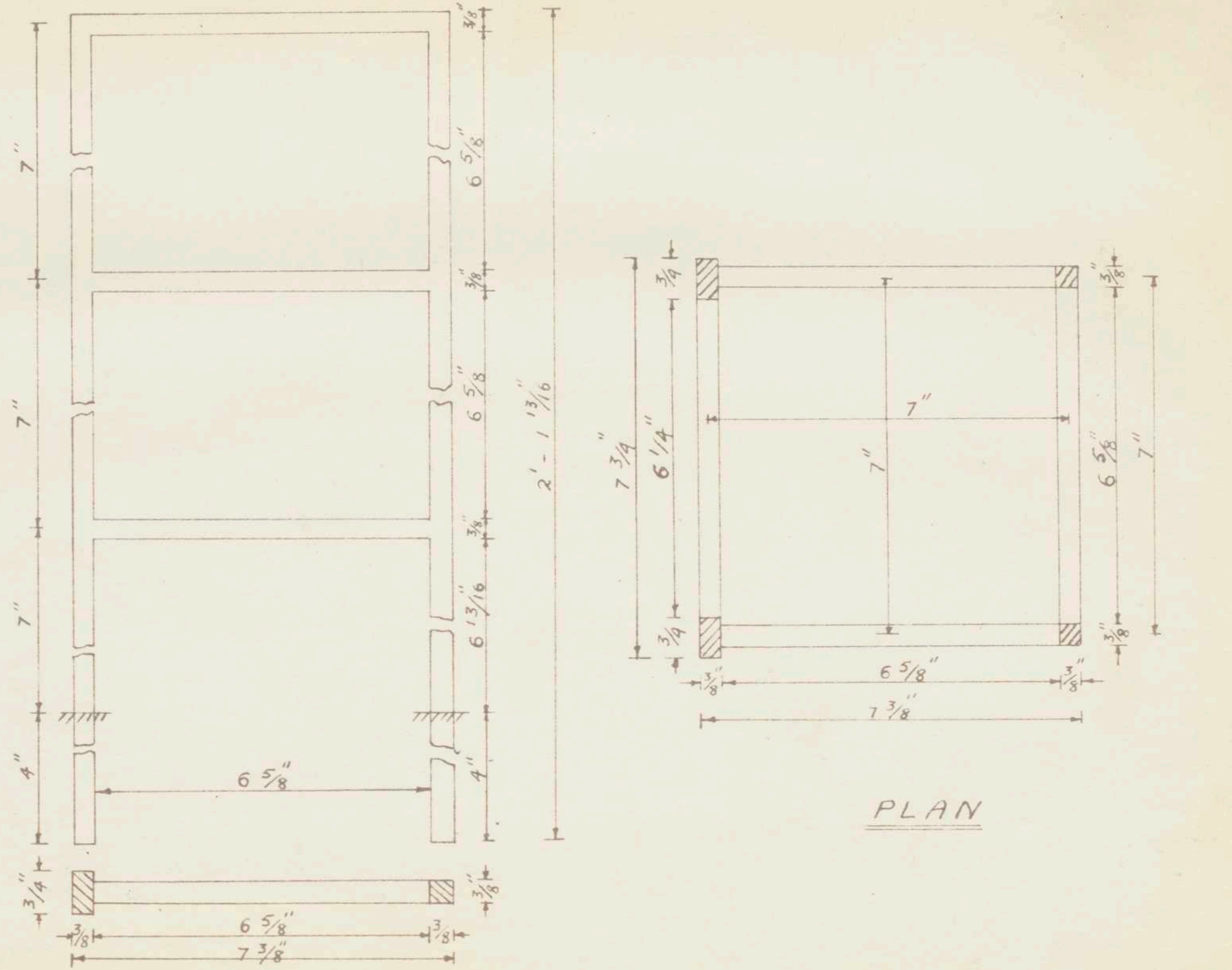


Fig. (10)

Three Dimensional Model-set up.



(Fig) 11. DETAIL DRAWING OF THREE DIMENSIONAL MODEL another side of the triangular base to it. A full detail drawing of which is found in Fig. (12)

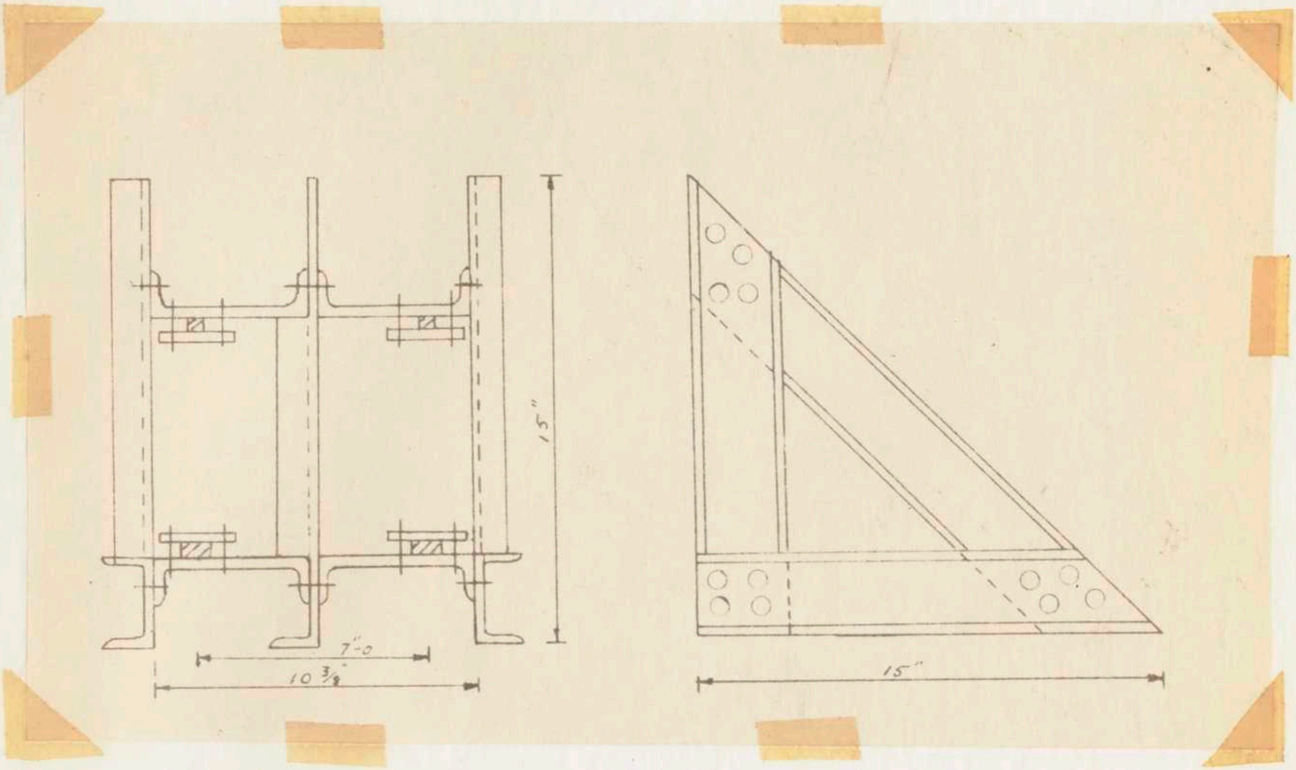


Fig. (12)

Supporting frame of the three dimensional model.

C. Procedure.

a) Calibration of oscillator.

The oscillator was calibrated with steady power source of (60) cycles per second. The output of the oscillator is connected to the right set of plates of the oscilloscope and the 60 cycle power source is connected to the left set of plates. When the frequency of the output voltage of the oscillator has simple ratio with respect to the source frequency, a certain curve is formed on the screen of the cathode ray tube. The shapes of some of those curves are shown in Fig. (13)

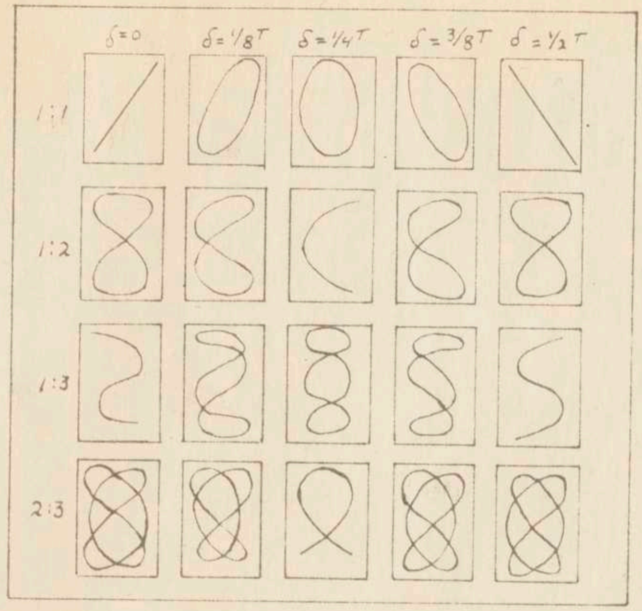


Fig. (13)

Shape of curves on the cathode-ray oscilloscope screen.

For a high frequency the loops are hard to be counted. An outlined proceedure is found in the thesis of Mr. Shih-Ying Lee at MIT 1943.

b) Measurement of the Natural frequencies of the Models.

In measuring the natural frequencies, the movable pick-up was set at a point which had a large amplitude of vibration. Then the oscillator was tuned until a maximum reading was obtained at the oscilloscope screen. The resonance frequency can be easily recognized when the loop which is obtained at the Cathode-ray screen is clearly cut. In many cases it is almost a perfect ellipse. These loops stay still and do not oscilate. However, this is not enough for the recognition of the order of the mode, because higher modes sometimes give the same amplitude as a lower one with the same power input. And very often it was necessary to move up the movable pick on many points so that the mode will show its order.

It is also necessary to let the oscillator be warmed up for a time of fifteen minutes before using it because this is the approximate time that it takes to start furnishing a steady frequency. The Calibration was made under the same condition for this reason also. Care should be taken to turn the frequency dial very slowly from one direction only.

c) Amplitude measurement.

The amplitudes of vibration were measured at the floor levels only because it was sufficient to reveal the mode shape. The movable pick up was made to touch the model very slightly and then moved back very slowly until it releases the contact. This reading should stay constant for that type of mode.

Care was taken in pointing the pick-up plunger at the center line of the columns to assure that the model was in resonance at the time the amplitude readings were taken.

The horizontal component of the loop in the screen was checked for every reading and maintained to be constant throughout the readings of the mode under consideration. The amplitude readings of the vertical component of the loop was registered at the desired points.

V. DISCUSSION OF RESULTS AND CONCLUSIONS.

In comparing the theoretical results with the experimental ones, for the three dimensional model, shown on pages (71, 72, 73), two kinds of modes are observed, symmetrical and unsymmetrical modes. The comparison of the symmetrical modes shows that the frequencies obtained by experiment check pretty well with the symmetrical ones, and the small percentage error is about the same for all the modes, with slightly higher discrepancies in the higher modes, which is probably due to the fact that the calibration curves for the oscillator are much steeper for higher frequencies than for lower ones, and values cannot be read as accurately.

The amplitudes check pretty well also, as far as the shapes are concerned. In magnitude however, the slight discrepancy that appeared is merely due to the assumptions made. The most important of which is assuming that the masses of the columns are concentrated at the floor level. These results would be nearer to the theoretical ones if the masses of the floors are large as compared to the masses of the columns.

In the symmetrical modes, where the model has no vibration in the (x) direction, and no rotation, a good check may be furnished by comparing the theoretical frequencies and amplitudes with the experimental ones of the two dimensional bents.

The experimental results of the two dimensional bents show that their experimental amplitudes (in magnitudes only), are not as near to the theoretical as the symmetrical modes for the three dimensional model, which proves that the effect of the mass of the columns in the two dimensional frame is more than the effect of the three dimensional one.

In considering the case of the unsymmetrical modes, it shows that the frequencies are a little bit higher in the experimental results than in the theoretical ones, Many factors enter in the explanation of this phenomenan. One of these factors is due to the assumption made which states that the masses of the girders and the columns are distributed over the floor. This assumption does not affect the symmetrical modes, because there is no rotation, but it has a greater effect on the unsymmetrical ones.

Another effect is that the maximum amplitude occurs when the vibration is in the (x) direction, a rocker was used to transmit the vibration to the model. The addition of the rocker tends to increase the frequency, when another smaller piece of steel was used as a rocker, it showed a decrease in the frequency.

The percentage error in the frequencies of the higher modes is higher than that in the low modes, the explanation of which is the same as given for the symmetrical modes.

The frequency of the third unsymmetrical mode could not be taken because it occurs either at the top of the low scale, or at the bottom of the medium scale, and the two scales do not overlap.

The fifth unsymmetrical mode could be heard and the frequency was registered, but it was not possible to pick up the amplitude because the instrument could not supply enough power to vibrate the model enough, so that the amplitude could be read on the oscillograph screen.

The frequency of the sixth unsymmetrical mode could not be taken, because it is a very high mode, and enough power could not be supplied to the speaker to cause vibration.

The amplitudes for the first two unsymmetrical modes gave good shapes and were nearer to the theoretical solution than the higher modes. The reason for this discrepancy in the higher modes is the same as that which was given for the symmetrical modes, and that is, in the higher modes, it needs higher power to vibrate the model, a nd then it distorts the shape of the amplitude.

In performing such an experiment, special care should be taken in fixing the model to its base, because any small vibration in the base will cause inconsistencies in the readings of the amplitudes and frequencies.

Four extra modes appeared in this experiment which were believed to be true modes, but it was finally found that it was due to some vibration at the base, which could not be eliminated, although many trials were made to make it as rigid as possible. This vibration could be eliminated easily by making the base heavier and more rigid.

In mounting the speaker, part of the vibration is lost through the hook attached to the model, so it is necessary to place the hook of the speaker right over the screw which is attached to the model. In order not to lose much of the vibration.

Special care should be taken in reading the horizontal amplitude, because any small increment in the distance between the pick-up and the model, will cause a bigger difference in the reading. There is no device in the instrument for making the distance between the pick-up and the model as small as possible, and keeping it constant through all the readings of that mode.

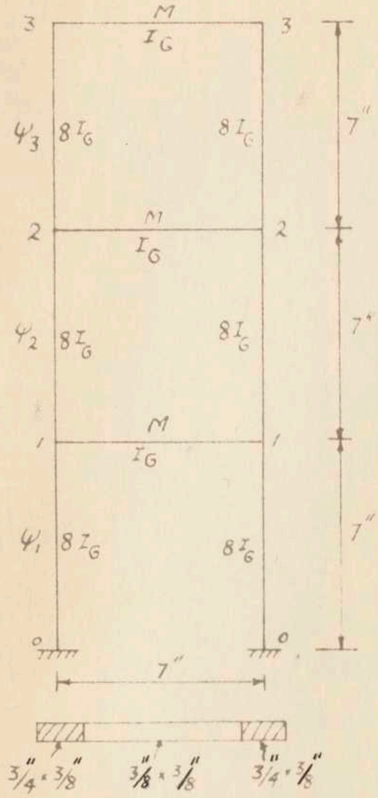
The best place for mounting the speaker is at the point where the maximum amplitude is expected. In dealing with symmetrical modes , it is advisable to attach the speaker at the middle of the girders joining the two symmetrical bents to eliminate the effect of the unsymmetrical modes, on the symmetrical ones.

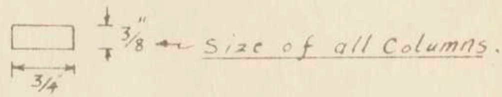
As a conclusion for this experiment, it can be said that the experimental solution for vibration in building frames is a good solution, if special instruments are used by well trained persons, in taking the readings. After getting rid of the above mentioned difficulties, excellent results can be attained.

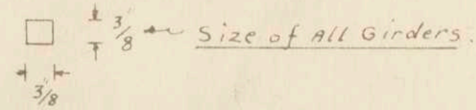
VI. APPENDIX.

BENT X'

Analytical Solution.







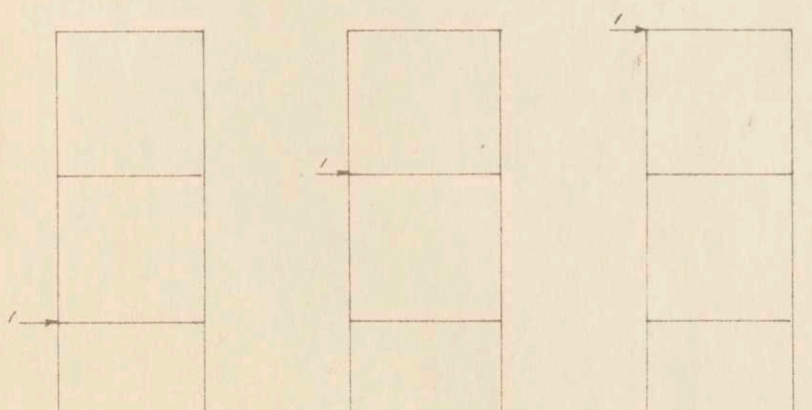
K_c = Col. stiffness factor

K_G = Girder " "

$I_c = 8\,I_G$

$K_c = 8\,K_G$

Loading Conditions.



(1) (2) (3)

Using the Slope Deflection Equations, all the end moments for the members of this frame, can be expressed as follows.

$$M_{01} = 16 EK_G (\theta_1 - 3\psi_1)$$
$$M_{10} = 16 EK_G (2\theta_1 - 3\psi_1)$$
$$M_{11} = 2EK_G (\theta_1 + 2\theta_1 - 0)$$
$$M_{12} = 16 EK_G (2\theta_1 + \theta_2 - 3\psi_2)$$
$$M_{21} = 16 EK_G (\theta_1 + 2\theta_2 - 3\psi_2)$$
$$M_{22} = 2EK_G (2\theta_2 + \theta_2 - 0)$$

$$M_{23} = 16 EK_G (2\theta_2 + \theta_3 - 3\psi_3)$$
$$M_{32} = 16 EK_G (\theta_2 + 2\theta_3 - 3\psi_3)$$
$$M_{33} = 2EK_G (2\theta_3 + \theta_3 - 0)$$

Due to symmetry we have six unknowns. $(\theta_1, \theta_2, \theta_3)$ & (ψ_1, ψ_2, ψ_3)
Using $\Sigma M = 0$ for Each joint you get the following.

$$(1)\quad M_{10} + M_{12} + M_{11} = 0$$
$$(2)\quad M_{21} + M_{23} + M_{22} = 0$$
$$(3)\quad M_{32} + M_{33} = 0$$

Using the shear equations for the three stories you get.

$$(4)\quad 2M_{32} + 2M_{23} + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$$
$$(5)\quad 2M_{21} + 2M_{12} + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$$
$$(6)\quad 2M_{10} + 2M_{01} + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$$

Substituting the expressions for the End moments in these six equations, we obtain six simultaneous equations with six unknowns $(\theta_1, \theta_2, \theta_3)$ & (ψ_1, ψ_2, ψ_3).

$$(1)\quad EK_G (35\theta_1 + 8\theta_2 - 24\psi_1 - 24\psi_2) = 0$$
$$(2)\quad EK_G (8\theta_1 + 35\theta_2 + 8\theta_3 - 24\psi_2 - 24\psi_3 = 0$$
$$(3)\quad EK_G (8\theta_2 + 19\theta_3 - 24\psi_3) = 0$$
$$(4)\quad EK_G (96\theta_2 + 96\theta_3 - 192\psi_3) + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$$
$$(5)\quad EK_G (96\theta_1 + 96\theta_2 - 192\psi_2) + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$$
$$(6)\quad EK_G (96\theta_1 - 192\psi_1) + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$$

Slope Deflection Solution for Deflection equations

equation	$EK_G\psi_1$	$EK_G\psi_2$	$EK_G\psi_3$	$EK_G\theta_1$	$EK_G\theta_2$	$EK_G\theta_3$	(1)	(2)	(3)
1	-24	-24	0	+35	+8	0	0	0	0
2	0	-24	-24	+8	+35	+8	0	0	0
3	0	0	-24	0	+8	+19	0	0	0
4	0	0	-192	0	+96	+96	0	0	+7
5	0	-192	0	+96	+96	0	0	+7	+7
6	-192	0	0	+96	0	0	+7	+7	+7

Solving the above set of equations simultaneously, we obtain the following results.

Solution

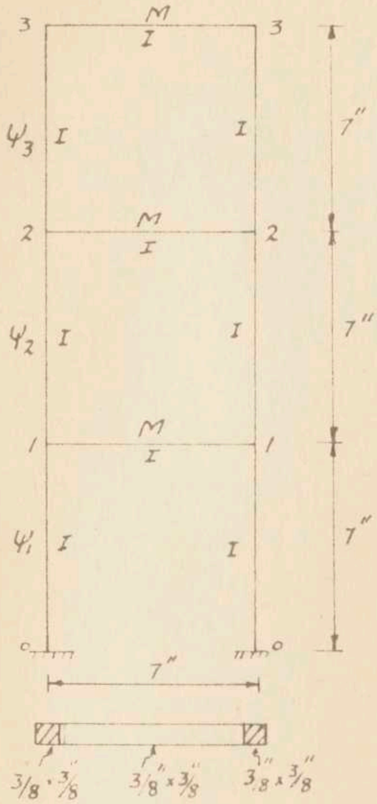
	Loading Conditions.		
	(1)	(2)	(3)
$EK\psi_1$	+0.084200	+0.153855	+0.188292
$EK\psi_2$	+0.069656	+0.257947	+0.387083
$EK\psi_3$	+0.034437	+0.163573	+0.411346
$EI\psi_1$	+0.589400	+1.076985	+1.318044
$EI\psi_2$	+0.487592	+1.805629	+2.709581
$EI\psi_3$	+0.241059	+1.145011	+2.879422

Deflections

	Loading Conditions		
	(1)	(2)	(3)
$EI\delta_1$	+4.125800	+7.538895	+9.226308
$EI\delta_2$	+7.538944	+20.178298	+28.193375
$EI\delta_3$	+9.226357	+28.193375	+48.349329

BENT X^{2}

Analytical Solution



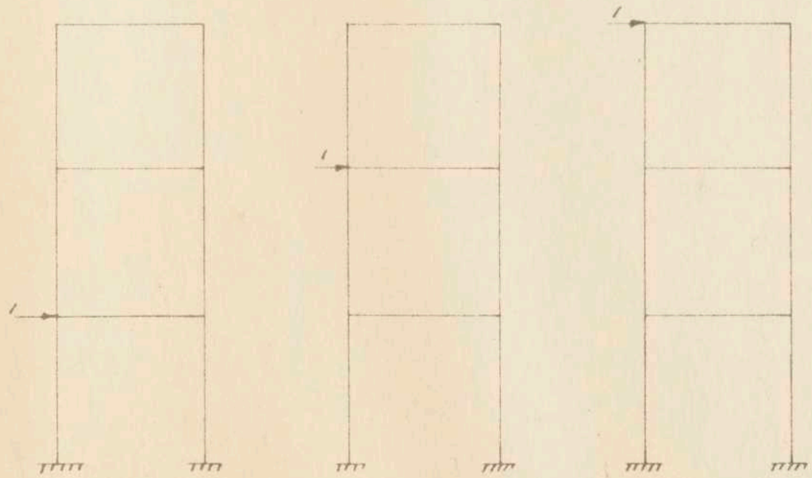
3/8" ← Size of All members.

3/8

$I_c = I_G = I$

$K_c = K_G = K$ = Col. stiffness factor
= Girder " "

Loading Conditions.



(1) (2) (3)

Using the slope Deflection equations, all the end moments for the members of this frame. Canbe expressed as follows.

$$M_{01} = 2EK(\theta_1 - 3\psi_1) \qquad M_{12} = 2EK(2\theta_1 + \theta_2 - 3\psi_2)$$
$$M_{10} = 2EK(2\theta_1 - 3\psi_1) \qquad M_{21} = 2EK(\theta_1 + 2\theta_2 - 3\psi_2)$$
$$M_{11} = 2EK(2\theta_1 + \theta_1 - 0) \qquad M_{22} = 2EK(2\theta_2 + \theta_2 - 0)$$

$$M_{23} = 2EK(2\theta_2 + \theta_3 - 3\psi_3)$$
$$M_{32} = 2EK(\theta_2 + 2\theta_3 - 3\psi_3)$$
$$M_{33} = 2EK(2\theta_3 + \theta_3 - 0)$$

Due to Symmetry we have six unknowns. $(\theta_1, \theta_2, \theta_3)$ & (ψ_1, ψ_2, ψ_3)

Using $\Sigma M = 0$ for each joint you get the following.

$$(1)\ M_{10} + M_{12} + M_{11} = 0$$
$$(2)\ M_{21} + M_{23} + M_{22} = 0$$
$$(3)\ M_{32} + M_{33} = 0$$

using the shear equations for the three stories you get.

$$(4)\ 2M_{32} + 2M_{23} + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$$
$$(5)\ 2M_{21} + 2M_{12} + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$$
$$(6)\ 2M_{01} + 2M_{10} + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$$

Substituting the expressions for the end moments in these six equations, we obtain six simultaneous equations with six unknowns $(\theta_1, \theta_2, \theta_3)$ & (ψ_1, ψ_2, ψ_3).

$$(1)\ EK(14\theta_1 + 2\theta_2 - 6\psi_1 - 6\psi_2) = 0$$
$$(2)\ EK(2\theta_1 + 14\theta_2 + 2\theta_3 - 6\psi_2 - 6\psi_3) = 0$$
$$(3)\ EK(2\theta_2 + 10\theta_3 - \psi_3) = 0$$
$$(4)\ EK(12\theta_2 + 12\theta_3 - 24\psi_3) + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$$
$$(5)\ EK(12\theta_1 + 12\theta_2 - 24\psi_2) + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$$
$$(6)\ EK(12\theta_1 - 24\psi_1) + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$$

Slope Deflection Solution for Deflection equations

equation	$EK\psi_1$	$EK\psi_2$	$EK\psi_3$	$EK\theta_1$	$EK\theta_2$	$EK\theta_3$	(1)	(2)	(3)
1	-6	-6	0	+14	+2	0	0	0	0
2	0	-6	-6	+2	+14	+2	0	0	0
3	0	0	-6	0	+2	+10	0	0	0
4	0	0	-24	0	+12	+12	0	0	+7
5	0	-24	0	+12	+12	0	0	+7	+7
6	-24	0	0	+12	0	0	+7	+7	+7

Solving the above set of equations simultaneously, we obtain the following results.

Solution.

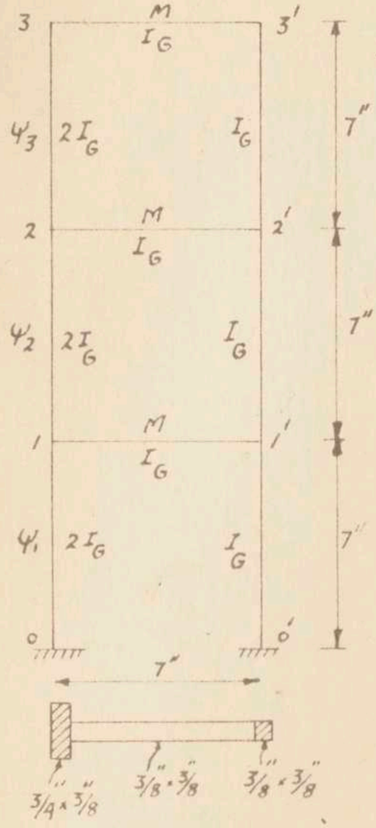
	Loading Conditions.		
	(1)	(2)	(3)
$EK\psi_1$	+0.402810	+0.528100	+0.544265
$EK\psi_2$	+0.125290	+0.669570	+0.815053
$EK\psi_3$	+0.016168	+0.161680	+0.726139
$EI\psi_1$	+2.819670	+3.696700	+3.809855
$EI\psi_2$	+0.877030	+4.686990	+5.705371
$EI\psi_3$	+0.113176	+1.131760	+5.082973

Deflections

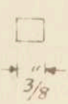
	Loading Conditions		
	(1)	(2)	(3)
$EI\delta_1$	+19.737690	+25.876900	+26.668985
$EI\delta_2$	+25.876900	+58.685830	+66.606582
$EI\delta_3$	+26.669132	+66.608150	+102.187393

BENT Y^1 & Y^2

Analytical Solution



3/8" ← Size of Left Columns.



3/8" ← Size of Girders & Left Col's.

$I_{L.C.}$ = I Left Col.

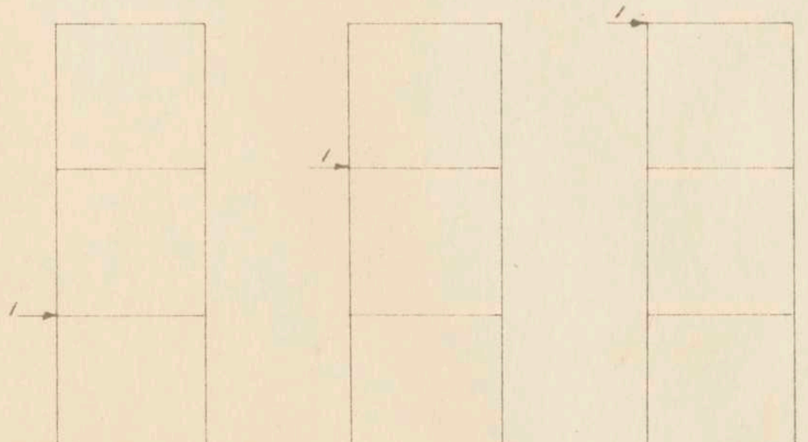
$I_{R.C}$ = I Right Col

I_G = I Girder = I

$I_{L.C} = 2I_{R.C.} = 2I_G = 2I$

$K_{L.C} = 2K_{R.C} = 2K_G = 2K$

Loading Conditions.



(1) (2) (3)

Using the slope Deflection equations, all the end moments for the members of this frame, can be expressed as follows.

$M_{01} = 4EK(\theta_1 - 3\psi_1)$ $\qquad$ $M_{0'1'} = 2EK(\theta_{1'} - 3\psi_1)$

$M_{10} = 4EK(2\theta_1 - 3\psi_1)$ $\qquad$ $M_{1'0'} = 2EK(2\theta_{1'} - 3\psi_1)$

$M_{11'} = 2EK(2\theta_1 + \theta_{1'} - 0)$ $\qquad$ $M_{1'1} = 2EK(2\theta_{1'} + \theta_1 - 0)$

$M_{12} = 4EK(2\theta_1 + \theta_2 - 3\psi_2)$ $\qquad$ $M_{1'2'} = 2EK(2\theta_{1'} + \theta_{2'} - 3\psi_2)$

$M_{21} = 4EK(\theta_1 + 2\theta_2 - 3\psi_2)$ $\qquad$ $M_{2'1'} = 2EK(\theta_{1'} + 2\theta_{2'} - 3\psi_2)$

$M_{22'} = 2EK(2\theta_2 + \theta_{2'} - 0)$ $\qquad$ $M_{2'2} = 2EK(2\theta_{2'} + \theta_2 - 0)$

$M_{23} = 4EK(2\theta_2 + \theta_3 - 3\psi_3)$ $\qquad$ $M_{2'3'} = 2EK(2\theta_{2'} + \theta_{3'} - 3\psi_3)$

$M_{32} = 4EK(\theta_2 + 2\theta_3 - 3\psi_3)$ $\qquad$ $M_{3'2'} = 2EK(\theta_{2'} + 2\theta_{3'} - 3\psi_3)$

$M_{33'} = 2EK(2\theta_3 + \theta_{3'} - 0)$ $\qquad$ $M_{3'3} = 2EK(2\theta_{3'} + \theta_{3'} - 0)$

Using the $\Sigma M = 0$ at every joint you get.

(1) $M_{10} + M_{12} + M_{11'} = 0$ $\qquad$ (4) $M_{1'1} + M_{1'2'} + M_{1'0'} = 0$

(2) $M_{21} + M_{23} + M_{22'} = 0$ $\qquad$ (5) $M_{2'3'} + M_{2'1'} + M_{2'2} = 0$

(3) $M_{32} + M_{33'} = 0$ $\qquad$ (6) $M_{3'3} + M_{3'2'} = 0$

Using the shear equations for the three stories you get.

(7) $M_{01} + M_{10} + M_{1'0'} + M_{0'1'} + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$

(8) $M_{21} + M_{12} + M_{2'1'} + M_{1'2'} + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$

(9) $M_{32} + M_{23} + M_{3'2'} + M_{2'3'} + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$

Substituting the expressions for the end moments in the nine equations we obtain nine simultaneous equations with nine unknowns. $(\theta_1, \theta_2, \theta_3)$, $(\theta_{1'}, \theta_{2'}, \theta_{3'})$ & (ψ_1, ψ_2, ψ_3).

(1) $EK(10\theta_1 + 2\theta_2 + \theta_{1'} - 6\psi_1 - 6\psi_2) = 0$

(2) $EK(2\theta_1 + 10\theta_2 + \theta_3 + \theta_{2'} - 6\psi_2 - 6\psi_3) = 0$

(3) $EK(2\theta_2 + 6\theta_3 + \theta_{3'} - 6\psi_3) = 0$

(4) $EK(\theta_1 + 6\theta_{1'} + \theta_{2'} - 3\psi_1 - 3\psi_2) = 0$

(5) $EK(\theta_2 + \theta_{1'} + 6\theta_{2'} + \theta_{3'} - 3\psi_2 - 3\psi_3) = 0$

(6) $EK(\theta_3 + \theta_{2'} + 4\theta_{3'} - 3\psi_3) = 0$

(7) $EK(12\theta_1 + 6\theta_{1'} - 36\psi_1 + \begin{Bmatrix} 7 \\ 7 \\ 7 \end{Bmatrix} = 0$

(8) $EK(12\theta_1 + 12\theta_2 + 6\theta_{1'} + 6\theta_{2'} - 36\psi_2 + \begin{Bmatrix} 0 \\ 7 \\ 7 \end{Bmatrix} = 0$

(9) $EK(12\theta_2 + 12\theta_3 + 6\theta_{2'} + 6\theta_{3'} - 36\psi_3 + \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = 0$

Slope Deflection Solution for Deflection equations

equ.	$EK\theta_1$	$EK\theta_2$	$EK\theta_3$	$EK\theta_1'$	$EK\theta_2'$	$EK\theta_3'$	$EK\psi_1$	$EK\psi_2$	$EK\psi_3$	(1)	(2)	(3)
1	+10	+2	0	+1	0	0	-6	-6	0	0	0	0
2	+2	+10	+2	0	+1	0	0	-6	-6	0	0	0
3	0	+2	+6	0	0	+1	0	0	-6	0	0	0
4	+1	0	0	+6	+1	0	-3	-3	0	0	0	0
5	0	+1	0	+1	+6	+1	0	-3	-3	0	0	0
6	0	0	+1	0	+1	+4	0	0	-3	0	0	0
7	+12	0	0	+6	0	0	-36	0	0	+7	+7	+7
8	+12	+12	0	+6	+6	0	0	-36	0	0	+7	+7
9	0	+12	+12	0	+6	+6	0	0	-36	0	0	+7

Solving the above set of equations simultaneously, we obtain the following Results.

Solution.

	Loading Conditions.		
	(1)	(2)	(3)
$EK\psi_1$	+0.298115	+0.419551	+0.441191
$EK\psi_2$	+0.121436	+0.562449	+0.712917
$EK\psi_3$	+0.021640	+0.172109	+0.646804
$EI\psi_1$	+2.086805	+2.936857	+3.088337
$EI\psi_2$	+0.850052	+3.937143	+4.990419
$EI\psi_3$	+0.151480	+1.204763	+4.527628

Deflections

	Loading Conditions.		
	(1)	(2)	(3)
$EI\delta_1$	+14.607635	+20.557999	+21.618359
$EI\delta_2$	+20.557999	+48.118000	+56.551292
$EI\delta_3$	+21.618359	+56.551341	+88.244688

Substituting the deflection values in th determinant, we get.

$$\begin{vmatrix} (14.607635-K') & +20.557999 & +21.618359 \\ 20.557999 & +(48.118000-K') & +56.551292 \\ 21.618359 & +56.551341 & +(88.244688-K') \end{vmatrix} = 0$$

Solving this determinant we have the following equation.

$$K'^3 - 150.970323\,K'^2 + 2150.058104\,K' - 5793.500955 = 0$$

For which the roots are obtained

$K' = 135.407923 \qquad K' = 11.99560 \qquad K' = 3.566799$

These K' values substituted in the equation $T = 2\pi\sqrt{\frac{K'M}{EI}}$ with the proper values for M and I gives the following results.

Mode	Frequency cycle/sec.
First	65.2
Second	219.05
Third	401.72

To find the relative amplitudes for each mode substitute the values of K' in the original equations and letting P_3 to be unity and knowing that

$$\frac{P_m}{P_n} = \frac{M_m}{M_n}\cdot\frac{y_m}{y_n} = \frac{M_m}{M_n}\cdot\frac{a_m}{a_n}, \text{ but } M_m = M_n = \text{Constant}$$

Then $\dfrac{P_m}{P_n} = \dfrac{a_m}{a_n}$

The relative amplitudes obtained for each mode are:—

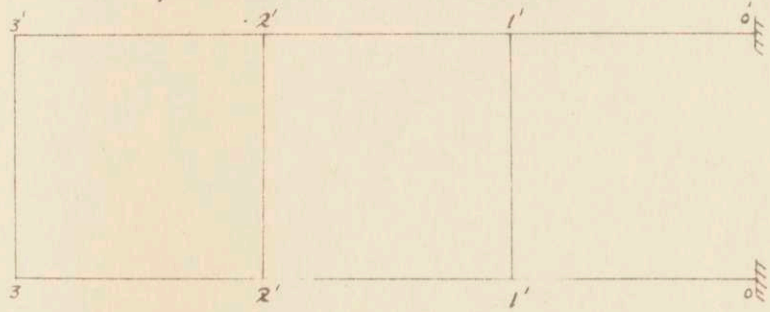
Mode		First	Second	Third
amplitude	a_1	+.301	− 1.190	+ 2.780
	a_2	+.715	− 0.902	− 2.550
	a_3	+1.000	+ 1.000	+ 1.000

Experimental Solution.

The following table show the average results of five different runs. for bent [Y']

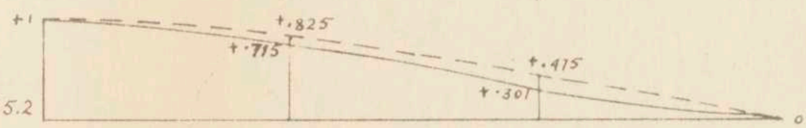
Mode		First	Second	Third.
Frequency		62.0	215	370
Amplitude	a_1	+.475	−0.900	+0.460
	a_2	+.825	−0.407	−1.020
	a_3	+1.000	+1.000	+1.000

Comparison of results



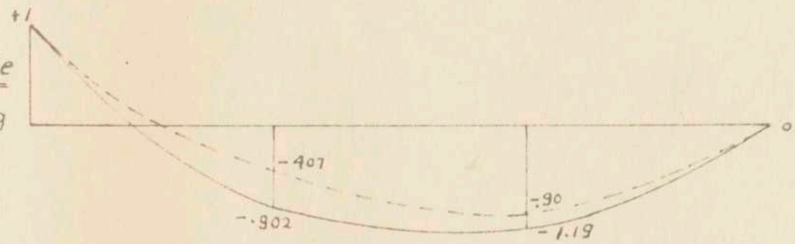
First Mode.

$f_{exp.} = 62$ $f_{the.} = 65.2$



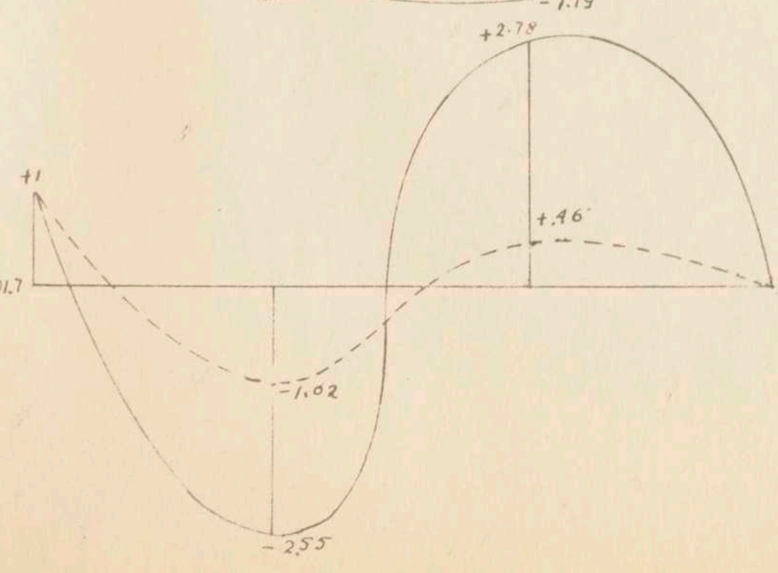
Second Mode

$f_{exp.} = 215$ $f_{the.} = 219$



Third Mode.

$f_{exp.} = 370$ $f_{the.} = 401.7$

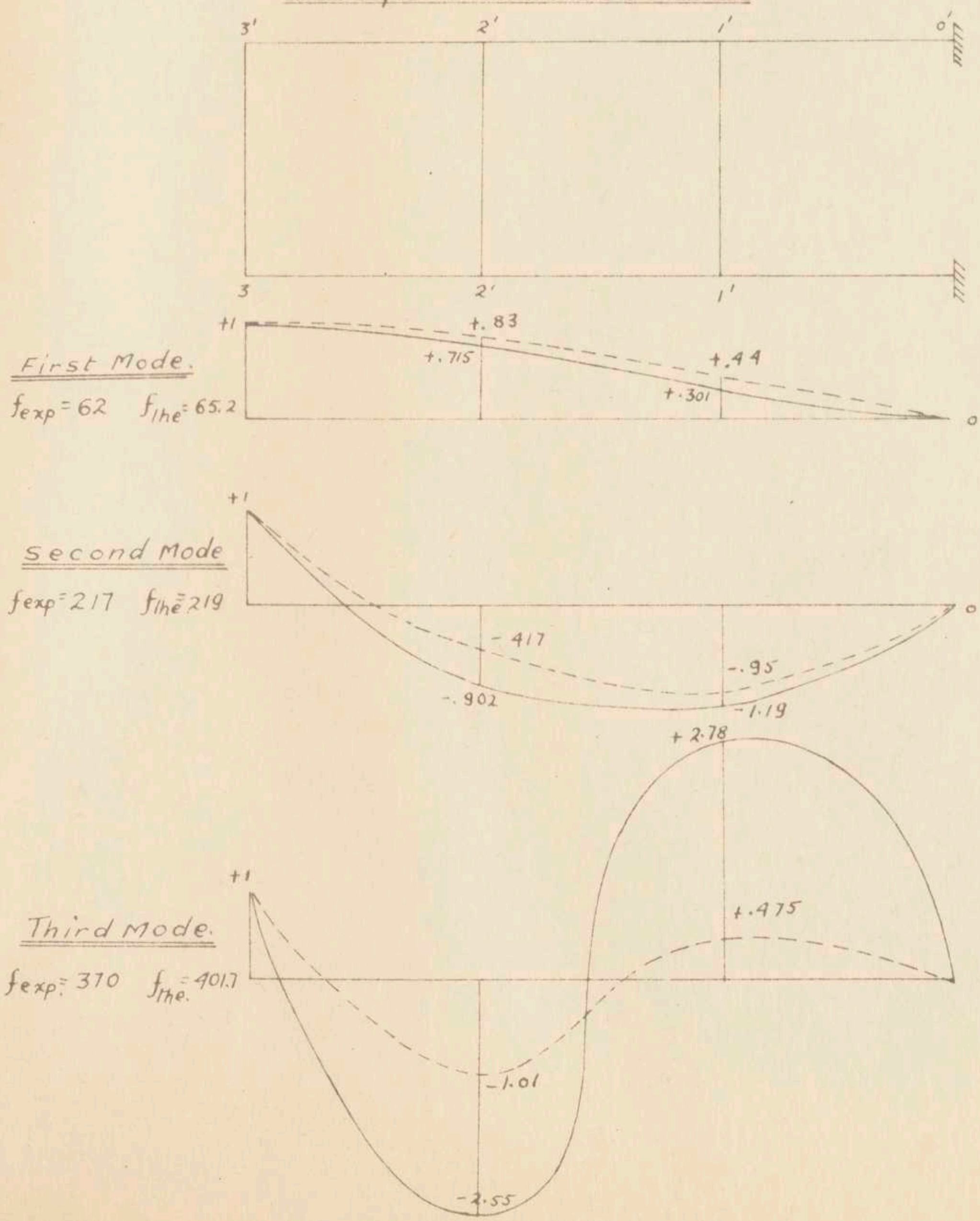


Experimental Solution.

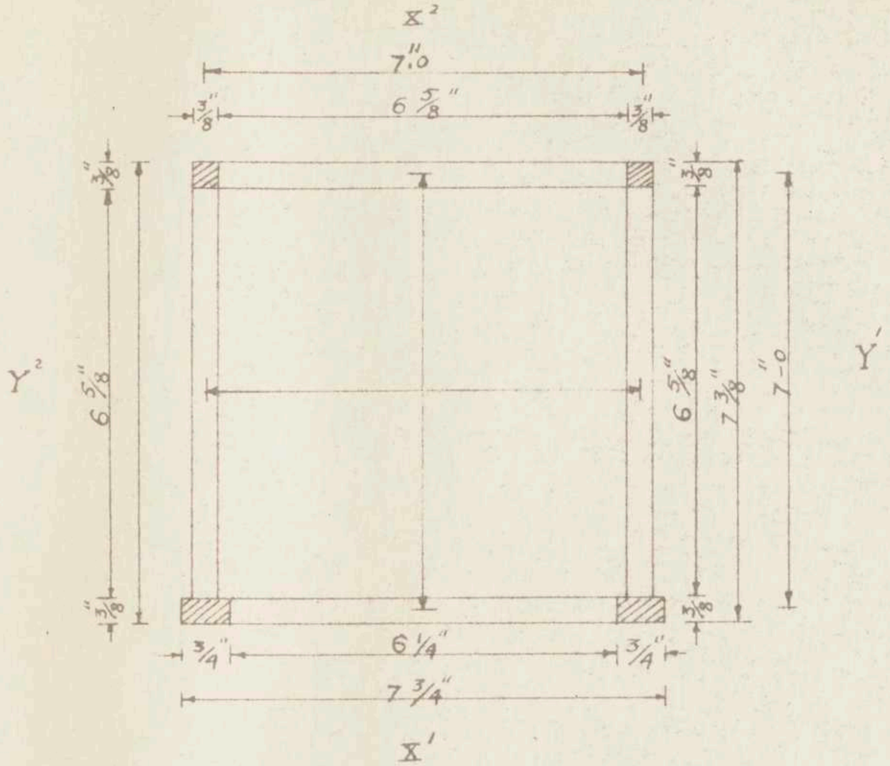
The following Table show the average results of five different runs for bent[Y^2]

Mode		First	Second	Third
Frequency		62.0	217	370
Amplitude	a_1	+ .440	−0.950	+ 0.475
	a_2	+ .830	−0.417	−1.010
	a_3	+1.000	+1.000	+1.000

Comparison of results



Analytical Solution of The Three Dimensional Model



(1) side X'

substituting the proper values in the set of equations (15 a) we get. for side X'

equation	x_1'	x_2'	x_3'	$\Delta_{1X'}$	$\Delta_{2X'}$	$\Delta_{3X'}$	
(1)	+4.125800	+7.538895	+9.226308	−1			=0
(2)	+7.538949	+20.178298	+28.193375		−1		=0
(3)	+9.226357	+28.193375	+48.349329			−1	=0

Solving These equations simultaneously we get the set of (16a)

	$\Delta_{1X'}$	$\Delta_{2X'}$	$\Delta_{3X'}$
x_1' =	+0.894146	−0.516378	+0.130483
x_2' =	−0.516385	+0.565724	−0.231344
x_3' =	+0.130486	−0.231345	+0.130684

(2) side X^2

substituting the proper values in the set of equations (15a) for side X^2 we get

equation	x_1^2	x_2^2	x_3^3	Δ_{1X^2}	Δ_{2X^2}	Δ_{3X^2}	
(1)	+19.737690	+25.876900	+26.668985	−1			= 0
(2)	+25.876900	+58.685830	+66.606582		−1		= 0
(3)	+26.669132	+66.608150	+102.187393			−1	= 0

solving these equations simultaneously we get the set of (16a)

	Δ_{1X^2}	Δ_{2X^2}	Δ_{3X^2}
x_1^2 =	+0.124176	+0.069070	+0.012612
x_2^2 =	−0.069070	+0.103905	−0.049700
x_3^2 =	+0.012614	+0.049702	+0.038890

(3) sides Y^1 & Y^2

Substituting the proper values in the set of equations (15b) we get the following, the same values are correct for side Y^2 also.

equation	y_1'	y_2'	y_3'	$\Delta_{1Y'}$	$\Delta_{2Y'}$	$\Delta_{3Y'}$	
(1)	+14.607635	+20.557999	+21.618359	−1			= 0
(2)	+20.557999	+48.118000	+56.551292		−1		= 0
(3)	+21.618359	+56.551391	+88.244688			−1	= 0

Solving these equations simultaneously we get set of equations (16b) the same values are correct for side Y^2 also

	$\Delta_{1Y'}$	$\Delta_{2Y'}$	$\Delta_{3Y'}$
Y_1' =	+0.180911	−0.102111	+0.021118
Y_2' =	−0.102113	+0.141830	−0.065876
Y_3' =	+0.021119	−0.065876	+0.048375

Substituting the proper values in equations (34, 35, 36); (37, 38, 39); and (40, 41, 42) we get the following set of equations.

Eq.	d_{x_1}	d_{x_2}	d_{x_3}	d_{y_1}	d_{y_2}	d_{y_3}	α_{01}	α_{02}	α_{03}	
1	$-1.018322+MK_{tf}$	+0.585448	-0.143095	0	0	0	+2.694895	-1.565578	+0.412549	= 0
2	+0.585455	$-0.669629+MK_{tf}$	+0.281044	0	0	0	-1.565603	+1.616367	-0.635754	= 0
3	-0.143100	+0.281047	$-0.169574+MK_{tf}$	0	0	0	+0.412552	-0.635751	+0.321279	= 0
4	0	0	0	$-0.361822+MK_{tf}$	+0.204222	-0.042236	0	0	0	= 0
5	0	0	0	+0.204226	$-0.283660+MK_{tf}$	+0.131752	0	0	0	= 0
6	0	0	0	-0.042238	+0.131752	$-0.096750+MK_{tf}$	0	0	0	= 0
7	+2.694895	-1.565578	+0.412549	0	0	0	$-12.474495+IK_{tf}$	+7.171738	-1.752914	= 0
8	-1.565603	+1.616367	-0.635754	0	0	0	+7.171824	$-8.202955+IK_{tf}$	+3.442789	= 0
9	+0.412552	-0.635751	+0.321279	0	0	0	-1.752975	+3.442826	$-2.077282+IK_{tf}$	= 0

To Keep the same Constant throught the nine equations substitute the value of $I = \frac{b^2+d^2}{12} M_{tf}$ where $b = 2x$; $d = 2y$ then let $M_{tf} K = K'$ we have the following equations

Eq.	d_{x_1}	d_{x_2}	d_{x_3}	d_{y_1}	d_{y_2}	d_{y_3}	α_{01}	α_{02}	α_{03}
1	$-1.018322+K'$	$+0.585448$	-0.143095	0	0	0	$+2.694895$	-1.565578	$+0.412549$
2	$+0.585455$	$-0.669629+K'$	$+0.281044$	0	0	0	-1.565603	$+1.616367$	-0.635754
3	-0.143100	$+0.281047$	$-0.169574+K'$	0	0	0	$+0.412552$	-0.635751	$+0.321279$
4	0	0	0	$-0.361822+K'$	$+0.204222$	-0.042236	0	0	0
5	0	0	0	$+0.204226$	$-0.283660+K'$	$+0.131752$	0	0	0
6	0	0	0	-0.042238	$+0.131752$	$-0.096750+K'$	0	0	0
7	$+0.329987$	-0.191703	$+0.050516$	0	0	0	$-1.527483+K'$	$+0.878172$	-0.214643
8	-0.191706	$+0.197922$	-0.077847	0	0	0	$+0.878183$	$-1.009443+K'$	$+0.421566$
9	$+0.050517$	-0.077847	$+0.039340$	0	0	0	-0.214650	$+0.421571$	$-0.254361+K'$

By Looking to these equations, we see that it can be broken down into two sets of equations. One with three and the other with six equations.

If this system is to have solutions different from zero th determinant of the coefficients must be zero.

Solving the determinant of the middle three leads to an equation of the 3rd. degree with three real roots for K'

$$\begin{vmatrix} (-0.361822+K') & +0.204222 & -0.042236 \\ +0.204226 & (-0.283660+K') & +0.131752 \\ -0.042238 & +0.131752 & (-0.096750+K') \end{vmatrix} = 0$$

$$K'^3 - 0.742232\,K'^2 + 0.104234\,K' - 0.001380 = 0$$

Solving for K' will get.

$K' = 0.560730$; $K' = 0.166751$; $K' = 0.014751$

Substituting the values of K' in the middle three equations and assuming $d_{y_3} = 1$ and solving them simultaneously will get.

$K' = 0.560730$	$K' = 0.166751$	$K' = 0.014751$
$d_{y_1} = -2.884338$	$d_{y_1} = -1.162254$	$d_{y_1} = +0.301203$
$d_{y_2} = +2.600519$	$d_{y_2} = -0.903365$	$d_{y_2} = +0.718702$
$d_{y_3} = -1.000000$	$d_{y_3} = +1.000000$	$d_{y_3} = +1.000000$

The second set of homogeneous equations (First & last three), can be solved by Stodola's Method of Iteration as follows:-

	d_{x1}	d_{x2}	d_{x3}	α_{01}	α_{02}	α_{03}
(1)	−1.018322	+0.585448	−0.143095	+2.694895	−1.565578	+0.412549
(2)	+0.585455	−0.669629	+0.281044	−1.565603	+1.616367	−0.635754
(3)	−0.143100	+0.281047	−0.169574	+0.412552	−0.635751	+0.321279
(7)	+0.329987	−0.191703	+0.050516	−1.527483	+0.878172	−0.214643
(8)	−0.191706	+0.197922	−0.077847	+0.878183	−1.004443	+0.421566
(9)	+0.050517	−0.077847	+0.039340	−0.214650	+0.421571	−0.254361

Assume values for:-

$d_{x1}=-1$	$d_{x2}=+.615$	$d_{x3}=-.205$	$\alpha_{01}=+.480$	$\alpha_{02}=-.326$	$\alpha_{03}=+.100$	
+1.018	+.395	+.029	+1.294	+.510	+.041	$=+3.287=K'$, $d_{x1}=+1$
−.585	−.452	−.058	−.751	−.527	−.064	$=-2.437$, $d_{x2}=-.741$
+.143	+.190	+.035	+.198	+.207	+.032	$=+0.805$, $d_{x3}=+.245$
−.330	−.130	−.010	−.733	−.286	−.021	$=-1.510$, $\alpha_{01}=-.459$
+.192	+.133	+.016	+.421	+.327	+.042	$=+1.131$, $\alpha_{02}=+.344$
−.051	−.053	−.008	−.103	−.137	−.025	$=-0.377$, $\alpha_{03}=-.115$

Carrying the Solution to Convergence:-

Last trial

d_{x1}	d_{x2}	d_{x3}	α_{01}	α_{02}	α_{03}		
+1.018322	+.443723	+.036503	+1.230341	+.548012	+.049039	= + 3.325940 = K'	$d_{x1} = +1.0$
−.585455	−.507526	−.071694	−.714768	−.565790	−.075571	= − 2.520804	$d_{x2} = -0.757922$
+.143100	+.213011	+.043258	+.188349	+.222537	+.038190	= + 0.848445	$d_{x3} = +0.255099$
−.329987	−.145296	−.012887	−.697365	−.307394	−.025514	= − 1.518443	$\alpha_{01} = -0.456546$
+.191706	+.150009	+.019859	+.400930	+.351593	+.050111	= + 1.164208	$\alpha_{02} = +0.350039$
−.050517	−.059002	−.010036	−.097997	−.147566	−.030236	= − 0.395354	$\alpha_{03} = -0.118870$

the orthogonality relationship gives the following equation between the d's for the second through the sixth mode.

$$d_{x1} = +.757922\, d_{x2} - .255099\, d_{x3} + 3.728459\,\alpha_{01} - 2.858652\,\alpha_{02} + .970772\,\alpha_{03} \quad \ldots (a)$$

substituting the value of (d_{x1}) in equations (2, 3, 7, 8 and 9) will get.

d_{x2}	d_{x3}	α_{01}	α_{02}	α_{03}		
−.225900	+.131695	+.617242	−.057245	−.067411	= 0	... (2a)
+.172588	−.133069	−.120990	−.226678	+.182362	= 0	... (3a)
+.058401	−.033663	−.297140	−.065146	+.105699	= 0	... (7a)
+.052624	−.028943	+.163415	−.456422	+.235463	= 0	... (8a)
−.039559	+.026453	−.026399	+.277160	−.205321	= 0	... (9a)

Assuming new values for (d_{x2} , d_{x3} , α_{01} , α_{02} and α_{03}) and carrying the solution to convergence, will get:-

$K' = 0.645707$ and $d_{x2} = +1$, $d_{x3} = -0.866245$,

$\alpha_{01} = -0.496478$, $\alpha_{02} = -0.559238$, $\alpha_{03} = +0.464169$

Substituting the values of (d_{x2} , d_{x3} , α_{01} , α_{02} and α_{03}) in equation (a) will get :-

$$d_{x1} = +1.177071$$

Making use of the orthogonality equation, will get:-

$$d_{x1} = -0.849566\, d_{x2} + 0.735933\, d_{x3} + 3.444627\, \alpha_{01} + 3.880063\, \alpha_{02} - 3.220463\, \alpha_{03} \quad \ldots\ldots\ldots (b)$$

Solving equations (a) and (b) for the value of (d_{x2}) after eliminating (d_{x1}) will get :-

$$d_{x2} = +0.616510 d_{x3} - 0.176569\, \alpha_{01} + 4.192078\, \alpha_{02} - 2.607320\, \alpha_{03} \ldots (c)$$

Substituting the values of (d_{x2}) from equation (c) in equations (3a , 7a , 8a , and 9a) will get:-

d_{x3}	α_{01}	α_{02}	α_{03}		
−0.026667	−0.151464	+0.496824	−0.267630	= 0	...(3b)
+0.002342	−0.307452	+0.179676	−0.046571	= 0	...(7b)
+0.003500	+0.154123	−0.235818	+0.098255	= 0	...(8b)
+0.002064	−0.019314	+0.111326	−0.102178	= 0	...(9b)

Assuming new values for (d_{x3} , α_{01} , α_{02} and α_{03}) and carrying the solution to convergence, will get :-

$K' = 0.465315$, and

$d_{x3} = +1$, $\alpha_{01} = +0.686922$, $\alpha_{02} = -0.563,769$, $\alpha_{03} = +0.203684$

substituting the values of d_{x3} , α_{01} , α_{02} , and α_{03} in equation (c) will get:-

$$d_{x2} = -2.399212$$

To find the value of (d_{x1}), the values of d_{x2} , d_{x3} , α_{01} , α_{02} , and α_{03} should be substituted in either equation (a) or (b) . which will give:-

$$d_{x1} = +2.296996$$

Making use of the orthogonality equation we have:-

$$d_{x1} = +1.044500\, d_{x2} - 0.435351\, d_{x3} - 2.442261\, \alpha_{01} + 2.004407\, \alpha_{02} - 0.724171\, \alpha_{03} \quad \ldots\ldots\ldots \text{(d)}$$

solving equations (a, b, and d) simultaneously for the value of (α_{01}) after eliminating (d_{x1} , and d_{x2}) will get:-

$$\alpha_{01} = -0.000574\, d_{x3} + 0.974780\, \alpha_{02} - 0.392544\, \alpha_{03} \quad \ldots\ldots \text{(e)}$$

Substituting the value of (α_{01}) from equation (e) in in equations (3b , 8b and 9b) will get:-

d_{x3}	α_{02}	α_{03}		
-0.026580	$+0.349180$	-0.208174	$=0$	.... (3c)
$+0.003412$	-0.085582	$+0.037755$	$=0$	.... (8c)
$+0.002075$	$+0.092499$	$+0.094596$	$=0$	.... (9c)

To solve for the values of K' Let:-

$$\begin{vmatrix} -0.026580+K' & +0.349180 & -0.208174 \\ +0.003412 & -0.085582+K' & +0.037755 \\ +0.002075 & +0.092499 & +0.094596+K' \end{vmatrix} = 0$$

Expanding this determinant we have the following cubical equation

$$K'^3 - 0.206758 K'^2 + 0.008634 K' - 0.000085 = 0$$

The three real roots of K' are:-

$K' = 0.154410$; $K' = 0.037793$; $K' = 0.014555$

Substituting the value of $K' = 0.154410$ in equations (3c, 8c, and 9c), assuming the value of $d_{x3} = +1$ solving any two of these equations simultaneously and using the third one for a check will get:-

$K' = 0.154410$; $d_{x1} = +1$; $\alpha_{02} = -0.201243$; $\alpha_{03} = +0.276504$

For the other two values of K' substituting in the same manner by assuming $\alpha_{03} = 1$ will get:-

$K' = 0.037793$; $\alpha_{03} = +1$; $\alpha_{02} = +0.656159$; $d_{x3} = -1.875146$

$K' = 0.014555$; $\alpha_{03} = +1$; $\alpha_{02} = +0.759076$; $d_{x3} = +4.736225$

The values of (α_{o1}) for the three values of (K') could be found by substituting the values of $(\alpha_{o3}, \alpha_{o2},$ and $d_{x3})$ for each value of (K') in equation (e)

For $K' = 0.154410$; $\alpha_{o1} = -0.305282$

For $K' = 0.037793$; $\alpha_{o1} = +0.248143$

and For $K' = 0.014555$; $\alpha_{o1} = +0.344669$

substituting the values of $(d_{x3}, \alpha_{o1}, \alpha_{o2}$ and $\alpha_{o3})$ for each value of (K') in equation (c) will get the corresponding values of (d_{x2})

For $K' = 0.154410$; $d_{x2} = -0.894147$

" $K' = 0.037793$; $d_{x2} = -1.056510$

and For $K' = 0.014555$; $d_{x2} = +3.433858$

Substituting the values of $(d_{x2}, d_{x3}, \alpha_{o1}, \alpha_{o2},$ and $\alpha_{o3})$ for each value of (K') in equation (a) will get the corresponding values of (d_{x1})

For $K' = 0.154410$; $d_{x1} = -1.227318$

For $K' = 0.037793$; $d_{x1} = -0.302171$

and For $K' = 0.014555$; $d_{x1} = +1.480313$

Finding the values of K' we can find the frequency :-
we have $K = \left(\frac{2\pi}{T}\right)^2$ where $K = \underline{\underline{K' m_n}}$ $f = \frac{1}{2\pi}\sqrt{\frac{K' EI \cdot n^{*}}{M}}$. Substituting the proper values of E and I we have the following values of Frequencies with their corresponding amplitude values, for each mode.

Mode →	9 th.	8 th.	7 th.	6 th.	5 th.	4 th.	3 rd.	2 nd.	1 st.
K' =	3.325940	0.645704	0.560730	0.465315	0.166751	0.154410	0.037793	0.014751	0.014555
f =	590	262	245	223.5	133.5	128.2	62.8	39.7	39.4
d_{x1} =	+1.0	+1.177071		+2.296996		-1.227318	-0.302171		+1.480313
d_{x2} =	-0.757922	+1.0		-2.399212		-0.894147	-1.056510		+3.433858
d_{x3} =	+0.255099	-0.866245		+1.0		+1.0	-1.875146		+4.736225
d_{y1} =			+2.884338		-1.162254			+0.301203	
d_{y2} =			-2.600519		-0.903365			+0.718702	
d_{y3} =			+1.0		+1.0			+1.0	
α_{01} =	-0.456546	-0.496478		+0.686922		-0.305282	+0.248143		+0.344669
α_{02} =	+0.350039	-0.559238		-0.563769		-0.201243	+0.656159		+0.759076
α_{03} =	-0.118870	+0.464169		+0.203684		+0.276504	+1.00		+1.0
Mode →	9 th.	8 th.	7 th.	6 th.	5 th.	4 th.	3 rd	2 nd.	1 st.

Substituting the values of :-

$$\Delta_{fX^n} = d_{xf} + \alpha_{of} \cdot y_x$$

$$\Delta_{fX'} = d_{xf} + \alpha_{of} \cdot y_x = d_{xf} - \alpha_{of} \cdot 3.5''$$

$$\Delta_{fX^2} = d_{xf} + \alpha_{of} \cdot y_x = d_{xf} + \alpha_{of} \cdot 3.5''$$

$$\Delta_{fY^n} = d_{yf} - \alpha_{of} \cdot x_y$$

$$\Delta_{fY'} = d_{yf} - \alpha_{of} \cdot x_y = d_{yf} - \alpha_{of} \cdot 3.5''$$

$$\Delta_{fY^2} = d_{yf} + \alpha_{of} \cdot x_y = d_{yf} + \alpha_{of} \cdot 3.5''$$

we have the following values.

Mode →	1 st.	f = 39.4		
Amplitude	$\Delta X'$	ΔX^2	$\Delta Y'$	ΔY^2
1st. story	+0.021	+0.326	-0.147	+0.147
2nd. "	+0.094	+0.740	-0.323	+0.323
3rd. "	+0.150	+1.000	-0.425	+0.425
Mode →	2nd.	f = 39.4		
1st. story			+0.301	+0.301
2nd. "			+0.719	+0.301
3rd. "			+1.000	+1.000
Mode →	3rd.	f = 62.8		
1st. story	-0.217	+0.105	-0.162	+0.162
2nd. "	-0.625	+0.231	-0.428	+0.428
3rd. "	-1.000	+0.312	-0.651	+0.651
Mode →	4th.	f = 128.2		
1st. story	-0.069	-1.000	+0.465	-0.465
2nd. "	-0.082	-0.697	+0.307	-0.307
3rd. "	+0.014	+0.858	-0.422	+0.422

Mode →	5 th.	$f = 133.5$		
Amplitude	Δ_{X^1}	Δ_{X^2}	Δ_{Y^1}	Δ_{Y^2}
1st. story			-1.000	-1.000
2nd. "			-0.877	-0.877
3rd. "			+0.877	+0.877
Mode →	6 th.	$f = 223.5$		
1st. story	-0.023	+1.000	-0.512	+0.512
2nd. story	-0.091	-0.929	+0.419	-0.419
3rd. story	+0.061	+0.364	-0.151	+0.151
Mode →	7 th.	$f = 245$		
1st. story			-1.000	-1.000
2nd. "			+0.902	+0.902
3rd. "			-0.347	-0.347
Mode →	8 th.	$f = 262$		
1st. story	+0.986	-0.202	+0.588	-0.588
2nd. "	+1.000	-0.324	+0.622	-0.622
3rd. "	-0.842	+0.256	-0.549	+0.549
Mode →	9 th.	$f = 590$		
1st. story	+1.000	-0.230	+0.615	-0.615
2nd. "	-0.764	+0.173	-0.472	+0.472
3rd. "	+0.258	-0.062	+0.160	-0.160

Experimental Solution.

The following table show the average results of four different runs. for the Three Dimensional model.

Mode →	1st.		f_{exp} = 53	
Amplitude	$\Delta_{X'}$	Δ_{X^2}	$\Delta_{Y'}$	Δ_{Y^2}
1st. story	+ 0.050	+ 0.450	− 0.200	+ 0.200
2nd. "	+ 0.100	+ 0.800	− 0.350	+ 0.350
3rd. "	+ 0.200	+ 1.000	− 0.400	+ 0.400
Mode →	2nd.		f_{exp} = 39	
1st. story			+ 0.333	+ 0.333
2nd. "			+ 0.665	+ 0.665
3rd. "			+ 1.000	+ 1.000
Mode →	3rd.		f_{exp} = 77	
1st. story	− 0.410	+ 0.410	− 0.410	+ 0.410
2nd. "	− 0.740	+ 0.590	− 0.680	+ 0.680
3rd. "	− 1.000	+ 0.545	− 0.770	+ 0.770
Mode →	4th.		f_{exp} = ?.	
1st. story	Cannot get	Bcause it	occur either	on top of
2nd. "	Law scale or	Bottom of	Medium scale	and the
3rd. "	two Scales	do not	over lap.	
Mode →	5th.		f_{exp} = 130	
1st. story			− 1.000	− 1.000
2nd. "			− 0.850	− 0.850
3rd. "			+ 0.900	+ 0.900
Mode →	6th.		f_{exp} = 282	
1st. story	+ 0.375	+ 1.000	− 0.375	+ 0.375
2nd. "	− 0.125	− 1.000	+ 0.750	− 0.750
3rd. "	0.0	+ 0.750	− 0.500	+ 0.500
Mode →	7th.		f_{exp} = 236	
1st. story			− 1.000	− 1.000
2nd. "			+ 1.420	+ 1.420
3rd. "			− 0.715	− 0.715
Mode →	8th.		f_{exp} = 368	
1st. Story	could	not pick	the	shape.
2nd. "				
3rd. "				
Mode →	9th.		f_{exp} = ?.	
1st. story				
2nd. "				
3rd. "				

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