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Defining Relations for Minimal Unitary Quantum Affine W-Algebras

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Abstract: We prove that any unitary highest weight module over a universal minimal quantum affine W-algebra at non-critical level descends to its simple quotient. We find the defining relations of the unitary simple minimal quantum affine W-algebras and the list of all their irreducible positive energy modules. We also classify all irreducible highest weight modules for the simple affine vertex algebras in the cases when the associated simple minimal W-algebra is unitary.

1. Introduction

Vertex algebras have been playing an increasingly important role in quantum physics (see e.g. [BML15,BMR19,CKLW,Ch19,Beem20] and references therein). Some of the most relevant to physics among them are unitary vertex algebras.

The problem of unitarity of highest weight representations of infinite-dimensional Lie superalgebras has been a hot topic in Mathematics and Physics in the 1980s and 1990s. Unitary highest weight representations have been classified for many infinite-dimensional Lie algebras and superalgebras, including affine Lie algebras [K], Virasoro algebra [FQS85-1,GKO85], N=1, 2, 3 and 4 superconformal algebras [FQS85-2,FQS85-1,FQS86,KR,KW85,KT85,BFK86,GKO86,DPYZ86,ET87, ET88-1,ET88-2,KS89,M90,FST,Ad01,CLRW]. All these Lie (super)algebras arise naturally in the context of a special class of vertex algebras, called quantum affine W-algebras, which are obtained by quantum Hamiltonian reduction from affine vertex algebras.

Quantum affine W-algebras $W_k(\mathfrak{g}, x, f)$, are simple vertex algebras constructed in [KRW03,KW04], starting from a datum $(\mathfrak{g}, x, f)$ and $k \in \mathbb{C}$. Here $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is a basic Lie superalgebra, i.e. $\mathfrak{g}$ is simple, its even part $\mathfrak{g}_{\bar 0}$ is a reductive Lie algebra and $\mathfrak{g}$ carries an even invariant non-degenerate supersymmetric bilinear form $(.|.)$, x is an ad–diagonalizable element of $\mathfrak{g}_{\bar 0}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, $f \in \mathfrak{g}_{\bar 0}$ is such that $[x, f] = -f$

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and the eigenvalues of $ad\, x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive, and k is non-critical, i.e. $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Recall that $W_k(\mathfrak{g}, x, f)$ is the unique simple quotient of the universal W-algebra, denoted by $W^k(\mathfrak{g}, x, f)$, which is freely strongly generated by elements labeled by a basis of the centralizer of f in $\mathfrak{g}$ [KW04].

In the paper [KMP23] we focused on *minimal* data $(\mathfrak{g}, x, f)$, i.e. the cases when f is an even minimal nilpotent element. In this case x and f are contained in a (unique) subalgebra $\mathfrak{s}$, isomorphic to $sl(2)$. The Virasoro, Neveu–Schwarz and $N = 2$ algebras cover all minimal W-algebras (associated with $\mathfrak{g} = sl(2)$, $spo(2|1)$, $sl(2|1)$ respectively), such that the 0-eigenspace $\mathfrak{g}_0$ of $ad\, x$ is abelian: their unitarity, and that of their irreducible highest weight modules, has been studied in the papers quoted above. In [KMP23] we dealt with the cases when $\mathfrak{g}_0$ is not abelian, and we found all non-critical $k \in \mathbb{C}$ for which the simple minimal W-algebras $W_k(\mathfrak{g}, x, f)$, denoted henceforth by $W_k^{\min}(\mathfrak{g})$, are unitary, along with the (partly conjectural) classification of unitary highest weight modules over their universal covers $W^k_{\min}(\mathfrak{g})$. We call this set of values of k the *unitarity range*. Recall from [AKMPP18] that a level k is said to be *collapsing* if $W_k^{\min}(\mathfrak{g})$ is the simple affine vertex algebra attached to the centralizer $\mathfrak{g}^\natural$ in $\mathfrak{g}_0$ of $\mathfrak{s}$. If k is collapsing, then the unitarity of $W_k^{\min}(\mathfrak{g})$ reduces to the unitarity of an affine vertex algebras, which is well-understood. Requiring that the unitary range contains non-collapsing levels imposes severe restrictions on the Lie superalgebra $\mathfrak{g}$, namely $\mathfrak{g}$ must be one of the algebras listed in Table 1. It turns out that the unitarity range (described explicitly in Table 1) is precisely the set of levels for which the affine vertex subalgebra of $W_k^{\min}(\mathfrak{g})$ generated by $\mathfrak{g}^\natural$ is integrable when viewed as a $\widehat{\mathfrak{g}}^\natural$-module.

The analysis of the unitarity of $W_k^{\min}(\mathfrak{g})$ developed in [KMP23] led to the more general problem of classifying unitary highest weight modules over $W_k^{\min}(\mathfrak{g})$; these results, regarding modules in the Neveu–Schwarz sector, are summarized in Sect. 2.

In particular, we gave a mathematically rigorous proofs of the classification for the $N = 3$ and $N = 4$ superconformal algebras, due to Miki [M90] and Eguchi-Taormina [ET87,ET88-1,ET88-2], respectively.

It was conjectured in [KMP23] that actually any unitary highest weight $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$. In the present paper we prove this conjecture (Theorem 5.1 and Corollary 5.2). The proof is based on the classification of irreducible highest weight modules over the simple affine vertex algebras $V_k(\mathfrak{g})$ for these k (Theorem 3.1). We also find generators for the maximal ideal I^k of $W^k_{\min}(\mathfrak{g})$, so that $W_k^{\min}(\mathfrak{g}) = W^k_{\min}(\mathfrak{g})/I^k$ (Theorem 4.1). Finally, for all non-critical $k \in \mathbb{C}$ in the unitarity range, we classify all irreducible highest weight modules over $W_k^{\min}(\mathfrak{g})$ (Theorem 6.1), using the Zhu algebra method. It follows from this classification that all these vertex algebras have infinitely many irreducible modules, and therefore are not rational, unless k is a collapsing level. We also prove that for $W_k^{\min}(\mathfrak{g})$ all irreducible positive energy modules are highest weight modules (Theorem 6.9), whereas this statement does not hold for $V_k(\mathfrak{g})$ (Remark 3.3). In the final section, we show examples of positive energy modules over $V_k(\mathfrak{g})$ which are not highest weight and we outline an approach to more general representations of $W_k^{\min}(\mathfrak{g})$, showing the existence of non-positive energy modules.

In the present paper we keep notation and terminology of [KMP23]; in particular, $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of positive and non-negative integer numbers, respectively. The base field is $\mathbb{C}$.

2. Setup

2.1. Minimal W-algebras. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple finite-dimensional Lie superalgebra over $\mathbb{C}$ with a reductive even part $\mathfrak{g}_{\bar{0}}$ and a non-degenerate even supersymmetric invariant bilinear form $(.|.)$. They are classified in [K77]. Let $\mathfrak{s} = \{e, x, f\} \subset \mathfrak{g}_{\bar{0}}$ be a *minimal* $sl(2)$-triple, i.e. $[x, e] = e$, $[x, f] = -f$, $[e, f] = x$ and the $ad\, x$ gradation of $\mathfrak{g}$ has the form

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \quad \text{where } \mathfrak{g}_j = 0 \text{ for } |j| > 1, \mathfrak{g}_1 = \mathbb{C}e, \ \mathfrak{g}_{-1} = \mathbb{C}f. \tag{2.1}$$

Denote by $\mathfrak{g}^\natural$ the centralizer of $\mathfrak{s}$ in $\mathfrak{g}_0$. Let $W^k_{\min}(\mathfrak{g})$ be the associated to $\mathfrak{s}$ and $k \in \mathbb{C}$ universal quantum affine W-algebra [KW04]. This vertex algebra is strongly generated by fields $J^{\{a\}}$, $a \in \mathfrak{g}^\natural$, $G^{\{v\}}$, $v \in \mathfrak{g}_{-1/2}$, L of respective conformal weight $1, 3/2, 2$ and λ-brackets explicitly given in [KW04]. We normalize the bilinear form $(.|.)$ by the condition $(x|x) = \frac{1}{2}$, and denote by $h^\vee$ the corresponding dual Coxeter number. We shall assume throughout the paper that $k \neq -h^\vee$, i.e. k is non-critical. Then the vertex algebra $W^k_{\min}(\mathfrak{g})$ has a unique maximal ideal I^k, and we let $W_k^{\min}(\mathfrak{g}) = W^k_{\min}(\mathfrak{g})/I^k$ be the corresponding simple vertex algebra.

2.2. Unitary conformal vertex algebras. Let V be a conformal vertex algebra with conformal vector $L = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. Let ϕ be a conjugate linear involution of V. A Hermitian form $H(. \, , .)$ on V is called *ϕ–invariant* if, for all $a \in V$, one has [KMP22]

$$H(v, Y(a, z)u) = H(Y(A(z)a, z^{-1})v, u), \quad u, v \in V. \tag{2.2}$$

Here the linear map $A(z) : V \to V((z))$ is defined by

$$A(z) = e^{zL_1} z^{-2L_0} g, \tag{2.3}$$

where

$$g(a) = (-1)^{L_0+2L_0^2}\phi(a), \quad a \in V. \tag{2.4}$$

We say that V is *unitary* if there is a conjugate linear involution ϕ of V and a ϕ-invariant Hermitian form on V which is positive definite.

2.3. Unitary minimal W-algebras and unitary highest weight modules. In classifying the unitary minimal W-algebras $W_k^{\min}(\mathfrak{g})$, one first considers the levels k when $W_k^{\min}(\mathfrak{g})$ is either $\mathbb{C}$ or an affine vertex algebra. Such levels are called *collapsing levels*.

The triples $(\mathfrak{g}, \mathfrak{s}, k)$ such that $W_k^{\min}(\mathfrak{g}) = \mathbb{C}$ are described in Proposition 3.4 of [AKMPP18]. Corollary 7.12 of [KMP22] provides the list of triples $(\mathfrak{g}, \mathfrak{s}, k)$ such that $W_k^{\min}(\mathfrak{g})$ is an affine unitary vertex algebra.

Turning to the non collapsing levels, it is proven in [KMP22, Proposition 7.9] that, if $W_k^{\min}(\mathfrak{g})$ is unitary and k is not a collapsing level, then the parity of $\mathfrak{g}$ is compatible with the $ad\, x$–gradation, i.e. the parity of the whole subspace $\mathfrak{g}_j$ is $2j \mod 2$. It follows from [KRW03, KW04] that for each basic simple Lie superalgebra $\mathfrak{g}$ there is at most one minimal datum $(\mathfrak{g}, \mathfrak{s})$ that is compatible with parity, and the complete list of the $\mathfrak{g}$ which admit such a datum is as follows:

$$\begin{gathered} sl(2|m) \text{ for } m \geq 3, \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0, \\ osp(4|m) \text{ for } m > 2 \text{ even}, \quad D(2, 1; a), \quad F(4), \quad G(3). \end{gathered} \tag{2.5}$$

Table 1. Unitarity range, $\mathfrak{g}^\natural$, $M_i(k)$, and $h^\vee$

$\mathfrak{g}$	$psl(2\|2)$	$spo(2\|3)$	$spo(2\|m), m > 4$	$D(2,1;\frac{m}{n}),\ m,n \in \mathbb{N}$	$F(4)$	$G(3)$
$-k$	$\mathbb{N}+1$	$\frac{1}{4}(\mathbb{N}+2)$	$\frac{1}{2}(\mathbb{N}+1)$	$\frac{mn}{m+n}\mathbb{N},\ (m,n)=1$	$\frac{2}{3}(\mathbb{N}+1)$	$\frac{3}{4}(\mathbb{N}+1)$
$\mathfrak{g}^\natural$	$sl(2)$	$sl(2)$	$so(m)$	$sl(2)\oplus sl(2)$	$so(7)$	G_2
$-M_i(k)$	$k+1$	$4k+2$	$2k+1$	$\frac{m+n}{n}k+1,\ \frac{m+n}{m}k+1$	$\frac{3}{2}k+1$	$\frac{4}{3}k+1$
$h^\vee$	0	$\frac{1}{2}$	$2-\frac{1}{2}m$	0	-2	$-\frac{3}{2}$

Remark 2.1. Recall from [K77] that in the case of $D(2,1;a)$, the parameter a ranges over $\mathbb{C}\setminus\{0,-1\}$ (if $a=0,-1$ the superalgebra is not simple); moreover the symmetric group $\mathcal{S}_3$ generated by the tranformations $a\mapsto -1-a$, $a\mapsto 1/a$ induces Lie superalgebra isomorphisms.

The even part $\mathfrak{g}_{\bar 0}$ of $\mathfrak{g}$ in these cases is $\mathfrak{g}^\natural\oplus\mathfrak{s}$ with $\mathfrak{g}^\natural$ a reductive Lie algebra. In Propositions 7.1. and 7.2 of [KMP23] it is proven that the conjugate linear involutions of $W_k^{\min}(\mathfrak{g})$ that fix the Virasoro vector L are in one-to-one correspondence with the conjugate linear involutions ϕ of $\mathfrak{g}$ that fix pointwise the triple $\{e,x,f\}$. It is easy to see that, in order to have $W_k^{\min}(\mathfrak{g})$ unitary, $\phi_{|\mathfrak{g}^\natural}$ must be the conjugation corresponding to a compact real form. Such a conjugate linear involution is called an *almost compact involution* and Proposition 3.2 of [KMP23] shows that an almost compact involution ϕ_{ac} exists in all the cases listed in (2.5). Since the cases $\mathfrak{g}=spo(2|m)$, $m=0,1$, and 2, correspond to the well understood cases of the simple Virasoro, Neveu–Schwarz and $N=2$ vertex algebras, we exclude these $\mathfrak{g}$ from consideration. For the other cases, Theorem 1.4 of [KMP23] lists all the pairs $(\mathfrak{g},k)$ for which the ϕ_{ac}-invariant hermitian form on $W_k^{\min}(\mathfrak{g})$ is positive definite, making $W_k^{\min}(\mathfrak{g})$ a unitary vertex algebra. We say that k belongs to the unitary range if k is not collapsing and $W_k^{\min}(\mathfrak{g})$ is a unitary vertex algebra.

We have [KMP23]

$$\mathfrak{g}^\natural=\bigoplus_{i\in S}\mathfrak{g}_i^\natural \text{ and } \mathfrak{g}_0=\mathfrak{g}^\natural\oplus\mathbb{C}x, \tag{2.6}$$

where $\mathfrak{g}_i^\natural$ are simple summands of $\mathfrak{g}^\natural$, and either $S=\{1\}$ or $S=\{1,2\}$; the latter happens only for $D(2,1;\frac{m}{n})$. Let $\mathfrak{h}^\natural$ be a Cartan subalgebra of $\mathfrak{g}^\natural$, then $\mathfrak{h}=\mathfrak{h}^\natural+\mathbb{C}x$ is a Cartan subalgebra of $\mathfrak{g}_0$ and of $\mathfrak{g}$. Let $\theta\in\mathfrak{h}^*$ be the root of e, so that $(\theta|\theta)=2$, and denote by $\theta_i\in(\mathfrak{h}^\natural)^*$ the highest root of $\mathfrak{g}_i^\natural$. Denote by $M_i(k)$ the levels of the affine Lie algebras $\widehat{\mathfrak{g}}_i^\natural$ in $W^k_{\min}(\mathfrak{g})$. Then we have for $i\in S$ [KMP23]

$$M_i(k)=\frac{2k}{(\theta_i|\theta_i)}+\chi_i, \tag{2.7}$$

$$\chi_i=-\xi(\theta_i^\vee), \tag{2.8}$$

where $\xi\in(\mathfrak{h}^\natural)^*$ is the highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$ (this module is irreducible, except for $\mathfrak{g}=psl(2|2)$, when its two irreducible components turn out to be isomorphic). In [KMP23] it is shown that $\chi_i=-2$ when $\mathfrak{g}=spo(2|3)$, and $\chi_i=-1$ otherwise.

In Table 1 we describe the unitary range in all cases, along with the subalgebras $\mathfrak{g}^\natural$, the numbers $M_i(k)$, and the dual Coxeter numbers $h^\vee$.

We also make a specific choice for a set of simple roots Π of $\mathfrak{g}$ in each case. In Table 2 we list our choice of $\Pi=\{\alpha_1,\dots\}$ of $\mathfrak{g}$, ordered from left to right, as well as the invariant bilinear form $(.|.)$ on $\mathfrak{h}^*$, and the root θ.

Table 2. Simple roots Π, invariant form $(.|.)$, and the highest root θ of $\mathfrak{g}$

$\mathfrak{g}$	Π	$(.\|.)$	θ
$psl(2\|2)$	$\{\epsilon_1-\delta_1,\delta_1-\delta_2,\delta_2-\epsilon_2\}$	$(\epsilon_i\|\epsilon_j)=\delta_{i,j}=-(\delta_i\|\delta_j)$ $(\epsilon_i\|\delta_j)=0$	$\epsilon_1-\epsilon_2$
$spo(2\|2m+1), m\geq 1$	$\{\delta_1-\epsilon_1,\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m,\epsilon_m\}$	$(\epsilon_i\|\epsilon_j)=-\frac{1}{2}\delta_{i,j}$, $(\delta_1\|\delta_1)=\frac{1}{2}$, $(\epsilon_i\|\delta_1)=0$	$2\delta_1$
$spo(2\|2m), m\geq 3$	$\{\delta_1-\epsilon_1,\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m,\epsilon_{m-1}+\epsilon_m\}$	$(\epsilon_i\|\epsilon_j)=-\frac{1}{2}\delta_{i,j}$, $(\delta_1\|\delta_1)=\frac{1}{2}$, $(\epsilon_i\|\delta_1)=0$	$2\delta_1$
$D(2,1;a)$	$\{\epsilon_1-\epsilon_2-\epsilon_3,2\epsilon_2,2\epsilon_3\}$	$(\epsilon_1\|\epsilon_1)=\frac{1}{2}$, $(\epsilon_2\|\epsilon_2)=\frac{-1}{2(1+a)}$, $(\epsilon_3\|\epsilon_3)=\frac{-a}{2(1+a)}$ $(\epsilon_1\|\epsilon_2)=(\epsilon_1\|\epsilon_3)=(\epsilon_2\|\epsilon_3)=0$	$2\epsilon_1$
$F(4)$	$\{\frac{1}{2}(\delta_1-\epsilon_1-\epsilon_2-\epsilon_3),\epsilon_3,\epsilon_2-\epsilon_3,\epsilon_1-\epsilon_2\}$	$(\epsilon_i\|\epsilon_j)=-\frac{2}{3}\delta_{i,j}$, $(\delta_1\|\delta_1)=2$ $(\epsilon_i\|\delta_1)=0$	δ_1
$G(3)$	$\{\delta_1+\epsilon_3,\epsilon_1,\epsilon_2-\epsilon_1\}$	$(\epsilon_i\|\epsilon_j)=\frac{1-3\delta_{i,j}}{4}$, $(\delta_1\|\delta_1)=\frac{1}{2}$ $(\epsilon_i\|\delta_1)=0$, $\epsilon_1+\epsilon_2+\epsilon_3=0$	$2\delta_1$

Let $\Delta^\natural$ be the set of roots of $(\mathfrak{g}^\natural,\mathfrak{h}^\natural)$. We made our choice of Π so that $\Pi^\natural=\Pi\cap\Delta^\natural$ is a set of simple roots for $\mathfrak{g}^\natural$. Write $\mathfrak{g}^\natural=\mathfrak{n}_-^\natural\oplus\mathfrak{h}^\natural\oplus\mathfrak{n}_+^\natural$ for the triangular decomposition of $\mathfrak{g}^\natural$ corresponding to choosing $\Pi^\natural$ as a set of simple roots. Note that α_1 is an isotropic root, and that we have

$$(\alpha_1)_{|\mathfrak{h}^\natural}=-\xi. \tag{2.9}$$

We parametrize the highest weight modules for $W^k_{\min}(\mathfrak{g})$ following Section 7 of [KW04]. For $\nu\in(\mathfrak{h}^\natural)^*$ and $\ell_0\in\mathbb{C}$, let $L^W(\nu,\ell_0)$ denote the irreducible highest weight $W^k_{\min}(\mathfrak{g})$–module with highest weight (ν,ℓ_0) and highest weight vector v_{ν,ℓ_0}. This means that one has

$$\begin{aligned}
&J_0^{\{h\}}v_{\nu,\ell_0}=\nu(h)v_{\nu,\ell_0}\text{ for }h\in\mathfrak{h}^\natural,\quad L_0v_{\nu,\ell_0}=\ell_0v_{\nu,\ell_0},\\
&J_n^{\{u\}}v_{\nu,\ell_0}=G_n^{\{v\}}v_{\nu,\ell_0}=L_nv_{\nu,\ell_0}=0\text{ for }n>0,\ u\in\mathfrak{g}^\natural,\ v\in\mathfrak{g}_{-1/2},\\
&J_0^{\{u\}}v_{\nu,\ell_0}=0\text{ for }u\in\mathfrak{n}_+^\natural.
\end{aligned}$$

Let $P^+\subset(\mathfrak{h}^\natural)^*$ be the set of dominant integral weights for $\mathfrak{g}^\natural$ and let

$$P_k^+=\left\{\nu\in P^+\mid\nu(\theta_i^\vee)\leq M_i(k)\text{ for all }i\geq 1\right\}. \tag{2.10}$$

Definition 2.2. An element $\nu\in P_k^+$ is called an *extremal weight* if $\nu+\xi$ doesn't lie in P_k^+. Equivalently, ν is extremal if

$$\nu(\theta_i^\vee)>M_i(k)+\chi_i\text{ for some }i\in S. \tag{2.11}$$

For $\nu \in P^+$, introduce the following number [KMP23]

$$A(k,\nu) = \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu) - k - 1), \tag{2.12}$$

where $2\rho^\natural$ is the sum of positive roots of $\mathfrak{g}^\natural$. The main result of [KMP23] is the following theorem.

Theorem 2.3. *Let $L^W(\nu, \ell_0)$ be an irreducible highest weight $W^k_{\min}(\mathfrak{g})$–module for $\mathfrak{g} = psl(2|2)$, $spo(2|m)$ with $m \geq 3$, $D(2,1;a)$, $F(4)$ or $G(3)$.*

(1) *This module can be unitary only if the following conditions hold:*
   (a) *$M_i(k)$ are non-negative integers,*
   (b) *$\nu(\theta_i^\vee) \leq M_i(k)$ for all i,*
   (c)

$$\ell_0 \geq A(k,\nu), \tag{2.13}$$

   and equality holds in (2.13) if $\nu(\theta_i^\vee) > M_i(k) + \chi_i$ for $i = 1$ or 2.

(2) *This module is unitary if the following conditions hold:*
   (a) *$M_i(k) + \chi_i \in \mathbb{Z}_+$ for all i,*
   (b) *$\nu(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all i (i.e. ν is not extremal),*
   (c) *inequality (2.13) holds.*

Applying this theorem to $L^W(0,0) = W_k^{\min}(\mathfrak{g})$ one recovers the unitarity range displayed in Table 1.

2.4. A technical result. Let Δ be the set of roots of $\mathfrak{g}$ and let $\widehat{\Pi} = \{\alpha_0 = \delta - \theta, \alpha_1, \ldots\}$ be a set of simple roots of the affine Lie superalgebra $\widehat{\mathfrak{g}}$. For an isotropic root $\beta \in \widehat{\Pi}$, we denote by $r_\beta(\widehat{\Pi})$ the set of simple roots in the set $\widehat{\Delta}$ of roots of $\widehat{\mathfrak{g}}$ obtained by the corresponding odd reflection. We denote by x_α a root vector of $\widehat{\mathfrak{g}}$, attached to $\alpha \in \widehat{\Delta}$, and by $w.$ the shifted action of $\widehat{W}$: $w.\lambda = w(\lambda + \widehat{\rho}) - \widehat{\rho}$.

Lemma 2.4. [KMP23, Lemma 11.3] *Let $\widehat{\Pi}'$ be a set of simple roots for $\widehat{\Delta}$. Let M be a $\widehat{\mathfrak{g}}$–module and assume that $m \in M$ is a singular vector with respect to $\widehat{\Pi}'$. If $\alpha \in \widehat{\Pi}'$ is an isotropic root and $x_{-\alpha}m \neq 0$, then $x_{-\alpha}m$ is a singular vector with respect to $r_\alpha(\widehat{\Pi}')$.*

3. Classification of Irreducible Highest Weight Representations of $V_k(\mathfrak{g})$ in the Unitary Range

Denote by $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ the universal and simple affine vertex algebras of level k associated to $\mathfrak{g}$. Denote by $L(\lambda)$ the irreducible highest weight $V^k(\mathfrak{g})$-module of highest weight λ. For $h \in \mathbb{C}$ and $\nu \in (\mathfrak{h}^\natural)^*$, set

$$\widehat{\nu}_h = k\Lambda_0 + h\theta + \nu. \tag{3.1}$$

Note that every highest weight module for $V^k(\mathfrak{g})$ has highest weight $\widehat{\nu}_h$ for some $\nu \in (\mathfrak{h}^\natural)^*$ and $h \in \mathbb{C}$.

Theorem 3.1. *Let k be in the unitary range. Then, up to isomorphism, the irreducible highest weight $V_k(\mathfrak{g})$-modules are as follows:*

(1) *$L(\widehat{\nu}_h)$ with $\nu \in P^+_k$ non-extremal and $h \in \mathbb{C}$ arbitrary;*

(2) $L(\widehat{\nu}_h)$ *with* ν *extremal and* h *from the set* $E_{k,\nu}=\{(\xi|\nu),k+1-(\xi|\nu)\}$.

Lemma 3.2. *Let* k *be in the unitary range. Then* $V_k(\mathfrak{g})$ *is integrable for* $\widehat{\mathfrak{g}}^\natural$.

Proof of Lemma 3.2. It is enough to check that $(x_{\theta_i})_{(-1)}^N\mathbf{1}=0$ for some $N\in\mathbb{N}$ and all i. Using Table 2 one easily checks that $\theta-\alpha_1$ is a root of $\mathfrak{g}$. Since $(x_{\alpha_1})_{(0)}\mathbf{1}=0$, $\mathbf{1}$ is a singular vector in $V^k(\mathfrak{g})$ also for $r_{\alpha_1}(\widehat{\Pi})$. Since $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is nonzero in $V^k(\mathfrak{g})$, it follows from Lemma 2.4 that $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is a singular vector in $V^k(\mathfrak{g})$ for $\widehat{\Pi}'=r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$. Note that, since $k\neq 0$,

$$\begin{aligned}(x_{-\theta+\alpha_1})_{(1)}(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}&=[(x_{-\theta+\alpha_1})_{(1)},(x_{\theta-\alpha_1})_{(-1)}]\mathbf{1}=((h_{\theta-\alpha_1})_{(0)}+k(x_{-\theta+\alpha_1}|x_{\theta-\alpha_1}))\mathbf{1}\\&=k(x_{-\theta+\alpha_1}|x_{\theta-\alpha_1})\mathbf{1}=\gamma\mathbf{1},\quad \gamma\neq 0.\end{aligned}$$

In particular, $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ generates $V^k(\mathfrak{g})$.

Let $\Lambda'=k\Lambda_0-\alpha_0-\alpha_1=k\Lambda_0-\delta+\theta-\alpha_1$ be the weight of $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$. By a case-wise verification one shows that the roots $\eta_i:=\delta-\theta_i$ are in $\widehat{\Pi}'$ and, by (2.7) and (2.8),

$$(\Lambda'|\eta_i^\vee)=\frac{2}{(\theta_i|\theta_i)}k+(\alpha_1|\theta_i^\vee)=M_i(k)-\chi_i-(\xi|\theta_i^\vee)=M_i(k).$$

Hence the vector $(x_{\theta_i})_{(-1)}^{M_i(k)+1}(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is singular in $V^k(\mathfrak{g})$ for $\widehat{\Pi}'$. Since the vector $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ generates $V^k(\mathfrak{g})$, it follows that $V_k(\mathfrak{g})$ is $\widehat{\mathfrak{g}}^\natural$–integrable. □

Proof of Theorem 3.1. In [GK15, Sections 4 and 6] the characters of highest weight $\widehat{\mathfrak{g}}$-modules with highest weight $k\Lambda_0$ have been computed using only their integrability with respect to $\widehat{\mathfrak{g}}^\natural$, which implies that such modules are irreducible. It was deduced from this in [GS18, Theorem 5.3.1] that, if $V_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}^\natural$-module, then the $V^k(\mathfrak{g})$–modules, which are integrable over $\widehat{\mathfrak{g}}^\natural$, descend to $V_k(\mathfrak{g})$.

By Lemma 3.2, we are left with proving that the modules listed in (1), (2) are $\widehat{\mathfrak{g}}^\natural$–integrable. Let v be a highest weight vector for $L(\widehat{\nu}_h)$.

Case (1) ν is not extremal, $h\in\mathbb{C}$. If $(\widehat{\nu}_h|\alpha_1)\neq 0$ and $(\widehat{\nu}_h-\alpha_1|\alpha_0+\alpha_1)\neq 0$, then $(x_{\theta-\alpha_1})_{(-1)}(x_{-\alpha_1})_{(0)}v$ is a singular vector with respect to the set of simple roots $r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$. Moreover, by (2.7), (2.8), and (2.9) we have for $i\in S$

$$\begin{aligned}m_i:=(\widehat{\nu}_h-2\alpha_1-\alpha_0|\eta_i^\vee)&=M_i(k)-\chi_i+2(\alpha_1|\theta_i^\vee)-(\nu|\theta_i^\vee)\\&=M_i(k)-\chi_i-2(\xi|\theta_i^\vee)-(\nu|\theta_i^\vee)\\&=M_i(k)+\chi_i-(\nu|\theta_i^\vee)\in\mathbb{Z}_+\end{aligned}\tag{3.2}$$

by (2.11). Since $\eta_i\in r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$, we see that, in $L(\widehat{\nu}_h)$,

$$(x_{\theta_i})_{(-1)}^{m_i+1}(x_{\theta-\alpha_1})_{(-1)}(x_{-\alpha_1})_{(0)}v=0,$$

hence $L(\widehat{\nu}_h)$ is $\widehat{\mathfrak{g}}^\natural$–integrable.

If $(\widehat{\nu}_h|\alpha_1)=0$, since $(\widehat{\nu}_h|\alpha_0+\alpha_1)=(\widehat{\nu}_h|\alpha_0)\neq 0$, we have that $(x_{\theta-\alpha_1})_{(-1)}v$ is a singular vector for $r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$. Moreover,

$$(\widehat{\nu}_h-\alpha_1-\alpha_0|\eta_i^\vee)=M_i(k)-(\nu|\theta_i^\vee)\in\mathbb{Z}_+\tag{3.3}$$

since $\nu\in P_k^+$. We can therefore conclude as above that $L(\widehat{\nu}_h)$ is $\widehat{\mathfrak{g}}^\natural$–integrable.

Finally, if $(\widehat{\nu}_h|\alpha_1) \neq 0$ and $(\widehat{\nu}_h - \alpha_1|\alpha_0+\alpha_1) = 0$, then $(x_{-\alpha_1})_{(0)}v$ is a singular vector for $r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$. Moreover,

$$(\widehat{\nu}_h - \alpha_1|\eta_i^\vee) = M_i(k) - (\nu|\theta_i^\vee) \in \mathbb{Z}_+, \tag{3.4}$$

and we conclude as in the previous case.

Case (2) ν is extremal, $h \in E_{k,\nu}$. If $h = (\xi|\nu)$, then $(\widehat{\nu}_h|\alpha_1) = 0$. We can then conclude as in the non-extremal case. If $h = k+1-(\xi|\nu)$, we may assume that $(\widehat{\nu}_h|\alpha_1) \neq 0$. Then, since $(\theta|\alpha_1) = 1$, we have

$$\begin{aligned}(\widehat{\nu}_h - \alpha_1|\alpha_0+\alpha_1) &= (k\Lambda_0 + (k+1-(\xi|\nu))\theta + \nu - \alpha_1|\alpha_0+\alpha_1)\\ &= k - 2(k+1-(\xi|\nu)) + 1 + (k+1-(\xi|\nu)) + (\nu|\alpha_1) = 0.\end{aligned}$$

We can therefore conclude as in the non-extremal case.

We are left with proving that any irreducible highest weight module for $V_k(\mathfrak{g})$ is of the form (1) or (2). Let $L(\widehat{\nu}_h)$ be an irreducible highest weight $V_k(\mathfrak{g})$-module. We prove that, necessarily, $\nu \in P_k^+$. Indeed, the action of $\mathfrak{g}^\natural$ on a highest weight vector v should be locally finite, so that $\nu \in P^+$. If $h \notin E_{k,\nu}$ then $(\widehat{\nu}_h|\alpha_1) \neq 0$ and $(\widehat{\nu}_h - \alpha_1|\alpha_0+\alpha_1) \neq 0$. It follows that $(x_{\theta-\alpha_1})_{(-1)}(x_{-\alpha_1})_{(0)}v$ is a highest weight vector with respect to the set of simple roots $r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$ of highest weight $\Lambda' = \widehat{\nu}_h - \alpha_0 - 2\alpha_1$. If $L(\widehat{\nu}_h)$ is integrable with respect to $\widehat{\mathfrak{g}}^\natural$, the computation done in (3.2) shows that m_i should be a non-negative integer, hence

$$(\nu|\theta_i^\vee) \leq M_i(k) + \chi_i \leq M_i(k).$$

It follows that $\nu \in P_k^+$ and it is not extremal, i.e. $L(\widehat{\nu}_h)$ is of type (1).

If $h = (\xi|\nu)$ (resp. $h = k+1-(\xi|\nu)$), then $(\widehat{\nu}_h|\alpha_1) = 0$ (resp. $(\widehat{\nu}_h|\alpha_0+\alpha_1) = 0$) and as in (3.3) (resp. (3.4)) we get that $\nu \in P_k^+$. In particular, $L(\widehat{\nu}_h)$ is of type (1) if ν is not extremal and of type (2) if ν is extremal. □

Remark 3.3. All modules listed in Theorem 3.1 are of positive energy (the definition is recalled in Sect. 6), but there might exist positive energy $V_k(\mathfrak{g})$–modules outside of this list. In § 7.1 we present arguments for this claim in the case $\mathfrak{g} = spo(2|3)$, $k = -m/4$, and $m \geq 4$ even.

One of the referees pointed out that in [GS18, 5.6.4] the authors give an example of an irreducible positive energy module which is not a highest weight module. Similar examples, with $\mathfrak{g}$ as in Table 1 and k in the unitarity range, are given in [GS18, 5.6.6] (note that in [GS18] a normalization of the bilinear form different from ours is used).

Let us explain what happens in our situation. If M is a positive energy module for a vertex algebra V, we set $M_{top} = Zhu(M)$, where $Zhu(-)$ is the Zhu functor between positive energy modules and modules for the Zhu algebra $A(V)$. We start with the $V_k(\mathfrak{g})$–module $L(\widehat{\nu}_h)$ with $\nu \in P_k^+$ non-extremal and $h \in \mathbb{C}$ arbitrary, from Theorem 3.1. Then $L(\widehat{\nu}_h)_{top}$ contains an irreducible $\mathfrak{g}_{\bar{0}}$–submodule $E = U(\mathfrak{g}_{\bar{0}}).v$, where v is the highest weight vector of $L(\widehat{\nu}_h)$. Next we consider the Kac module E_1, which is a suitable quotient of $Ind_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} E$. As mentioned in [GS18], if all $M_i(k)$ are large enough, we get that E_1 is a module for Zhu's algebra $A(V_k(\mathfrak{g}))$. Using Zhu's functor we conclude the following:

- there exists a positive energy $V_k(\mathfrak{g})$–module $\widetilde{L}(\widehat{\nu}_h)$ such that $\widetilde{L}(\widehat{\nu}_h)_{top} = E_1$.
- the module $\widetilde{L}(\widehat{\nu}_h)$ is indecomposable and has an irreducible subquotient isomorphic to $L(\widehat{\nu}_h)$.

Hence non-highest weight positive energy modules for $V_k(\mathfrak{g})$ exist. But still these modules are weight modules with finite-dimensional weight spaces.

Let us mention one important consequence. Take $h \in \mathbb{C}$ such that E is finite-dimensional, irreducible module and assume that $M_i(k)$ are large enough. Then E_1 is an indecomposable finite-dimensional module for Zhu's algebra $A(V_k(\mathfrak{g}))$, and therefore the category KL_k [AMP22] is not semisimple. In the case $\mathfrak{g} = spo(2|3)$, one can show that the category $KL_{-m/4}(\mathfrak{g})$ is not semisimple for $m \geq 4$.

4. Explicit Description of the Maximal Ideal of $W^k_{\min}(\mathfrak{g})$

Let, as before, $\Pi = \{\alpha_1, \ldots\}$ be the set of simple roots for $\mathfrak{g}$ given in Table 2, and I^k be the maximal ideal of $W^k_{\min}(\mathfrak{g})$. Also denote by J^k the maximal ideal of $V^k(\mathfrak{g})$. Set

$$v_i = (J^{\{x_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1}, \quad i \in S. \tag{4.1}$$

If an irreducible highest weight $W^k_{\min}(\mathfrak{g})$–module $L^W(\nu, \ell_0)$ is unitary, then, restricted to the affine subalgebra $V^{\beta_k}(\mathfrak{g}^\natural)$ (see [KMP23, (7.4), (7.5)] for the definition of this subalgebra), it is unitary, hence a direct sum of irreducible integrable highest weight $\widehat{\mathfrak{g}}^\natural$–modules of levels $M_i(k),\ i \in S$. But it is well-known [FZ92,KWang92] that all these modules descend to the simple affine vertex algebra $V_{\beta_k}(\mathfrak{g}^\natural)$, and are annihilated by the elements v_i. In particular, applying this argument to $W_k^{\min}(\mathfrak{g}) = L^W(0,0)$, we deduce that

$$v_i \in I^k. \tag{4.2}$$

Theorem 4.1. *Let k be in the unitary range. The maximal ideal I^k is generated by the singular vector*

$$\widetilde{v}_1 = (J^{\{x_{\theta_1}\}}_{(-1)})^{M_1(k)-1} G^{\{x_{-\alpha_1}\}}_{(-1)}\mathbf{1} \tag{4.3}$$

if $\mathfrak{g} = spo(2|3)$ and by the singular vectors $v_i,\ i \in S$ (cf. (4.1)) in the other cases.

We split the proof according to whether $\mathfrak{g} \neq spo(2|3)$ or $\mathfrak{g} = spo(2|3)$.

4.1. $\mathfrak{g} \neq spo(2|3)$. In this case $\chi_i = -1,\ i \in S$.

Lemma 4.2. *Set $\eta_i = \delta - \theta_i,\ i \in S$. One has*

$$\dim V^k(\mathfrak{g})_{k\Lambda_0 - j\eta_i} = 1 \textit{ for all } j \geq 0, \tag{4.4}$$

$$\dim V_k(\mathfrak{g})_{k\Lambda_0 - j\eta_i} = 1 \textit{ if and only if } j \leq s, \tag{4.5}$$

where

$$s = (k\Lambda_0 | \eta_i^\vee). \tag{4.6}$$

In particular, for each $M \in \mathbb{Z}_{\geq 0}$, the vector $(x_{-\theta+\alpha_1})_{(1)}(x_{\theta_i})^M_{(-1)}(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is a multiple of $(x_{\theta_i})^M_{(-1)}\mathbf{1}$.

Proof. Any basic Lie superalgebra $\mathfrak{g}$ different form $spo(2|3)$ admits a set of simple roots Π'' with the property that θ_i is the maximal root, hence the root $\eta_i = \delta - \theta_i$ lies in $\widehat{\Pi}''$. Since $V^k(\mathfrak{g})$ is the vacuum module, $x_\alpha \mathbf{1} = 0$ for any root $\alpha \in \Delta$, in particular $x_\alpha \mathbf{1} = 0$ for any $\alpha \in \Pi''$. Since $(x_{-\theta_i})_{(1)}\mathbf{1} = 0$, $\mathbf{1}$ is a singular vector also for $\widehat{\Pi}''$. Since η_i is simple in $\widehat{\Pi}''$, it is clear that $V^k(\mathfrak{g})_{k\Lambda_0 - j\eta_i} \subset \mathbb{C}(x_{\theta_i})^j_{(-1)}\mathbf{1}$, and indeed $\dim V^k(\mathfrak{g})_{k\Lambda_0 - j\eta_i} = 1$, since $(x_{\theta_i})^j_{(-1)}\mathbf{1} \neq 0$ in $V^k(\mathfrak{g})$. This proves (4.4). By Lemma 3.2, $V_k(\mathfrak{g})$ is integrable for $\widehat{\mathfrak{g}}^\natural$, in particular, $(x_{\theta_i})^j_{(-1)}\mathbf{1} = 0$ if and only if $j > (k\Lambda_0|\eta_i^\vee)$, hence (4.5) holds. The final claim follows from (4.4) by comparing weights. □

Introduce the following vectors in $V^k(\mathfrak{g})$, where $i \in S$:

$$w_i = (x_{\theta_i})^{M_i(k)+1}_{(-1)}(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}, \tag{4.7}$$

$$s_i = (x_{\alpha_1})_{(0)} w_i, \tag{4.8}$$

$$u_i = (x_{-\theta+\alpha_1})_{(1)} w_i. \tag{4.9}$$

Proposition 4.3. *The vectors s_i are singular in the universal affine vertex algebra $V^k(\mathfrak{g})$ and generate J^k.*

Proof. Recall that $\alpha_0 = \delta - \theta$. Since $(x_{\alpha_1})_{(0)}\mathbf{1} = 0$, $\mathbf{1}$ is singular also for $r_{\alpha_1}(\widehat{\Pi})$. Since $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is nonzero, it follows from Lemma 2.4 that $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$ is singular in $V^k(\mathfrak{g})$ for $\widehat{\Pi}' = r_{\alpha_0+\alpha_1} r_{\alpha_1}(\widehat{\Pi})$. Let $\Lambda' = k\Lambda_0 - \delta + \theta - \alpha_1$ be the weight of $(x_{\theta-\alpha_1})_{(-1)}\mathbf{1}$.

Since $\delta - \theta_i$ is in $\widehat{\Pi}'$ and, by (2.7) and (2.8),

$$(\Lambda'|(\delta-\theta_i)^\vee) = \frac{2}{(\theta_i|\theta_i)}k + (\alpha_1|\theta_i^\vee) = M_i(k) - \chi_i - (\xi|\theta_i^\vee) = M_i(k),$$

we see that w_i is singular for $\widehat{\Pi}'$. Since $V^k(\mathfrak{g})/\langle w_i \rangle$ is integrable with respect to $\widehat{\mathfrak{g}}^\natural$, it is irreducible, because the computation of its character formula in [GK15] did not use irreducibility, but only integrability. Hence the vectors w_i generate the maximal proper ideal of $V^k(\mathfrak{g})$.

The weight of u_i is $k\Lambda_0 - s'\eta_i$, where

$$s' := 1 + (\Lambda'|\eta_i^\vee) = 1 + (k\Lambda_0|\eta_i^\vee) - (\alpha_0 + \alpha_1|\eta_i^\vee) = 1 + s + (\alpha_1, \theta_i^\vee),$$

and s is defined in (4.6). Since $(\alpha_1|\theta_i^\vee) \leq -1$, we have $s' \leq (k\Lambda_0|\eta_i^\vee)$, so $(J^k)_{k\Lambda_0 - s'\eta_i} = 0$ by Lemma 4.2 (2). Since $w_i \in J^k$, we have $u_i \in J^k$, so $u_i = 0$.

We now observe that the fact that $u_i = 0$ implies that w_i is a singular vector for $r_{-\alpha_0-\alpha_1}(\widehat{\Pi}') = r_{\alpha_1}(\widehat{\Pi})$. Indeed, if $\beta \neq \alpha_0+\alpha_1$ is a simple root for $r_{-\alpha_0-\alpha_1}(\widehat{\Pi}')$, then it is a positive root for $\widehat{\Pi}'$, hence, since w_i is singular for $\widehat{\Pi}'$, $(x_\beta)_{(0)}(x_{\theta_i})^{M_i(k)+1}_{(-1)}(x_{\theta-\alpha_1})_{(-1)}\mathbf{1} = 0$. Observe that, since $\alpha_0 = \delta - \theta$, $u_i = 0$ simply means that

$$(x_{-\theta+\alpha_1})_{(1)} w_i = 0.$$

Having shown that w_i are singular for $r_{\alpha_1}(\widehat{\Pi})$, it follows that the s_i are either zero or singular vectors for $\widehat{\Pi}$.

Let $\Lambda_i''' = k\Lambda_0 - (M_i(k)+1)(\delta - \theta_i) - \delta + \theta - \alpha_1$ be the weight of w_i. Since $k \leq 0$ (see Table 1), we have

$$(\Lambda_i'''|\alpha_1) = ((M_i(k)+1)\theta_i|\alpha_1) + 1 = -k + 1 > 0,$$

so,

$$(x_{-\alpha_1})_{(0)}s_i = c(-k+1)w_i - (x_{\alpha_1})_{(0)}(x_{-\alpha_1})_{(0)}w_i, \text{ where } c \in \mathbb{C}.$$

Since w_i are singular vectors for $r_{\alpha_1}(\widehat{\Pi})$, we see that

$$(x_{-\alpha_1})_{(0)}w_i = 0.$$

It follows that $(x_{-\alpha_1})_{(0)}s_i$ are nonzero multiples of w_i, hence the s_i generate the maximal proper ideal of $V^k(\mathfrak{g})$, since the w_i do. □

Proof of Theorem 4.1 *for* $\mathfrak{g} \neq spo(2|3)$.

Since $L(k\Lambda_0) = V^k(\mathfrak{g})/J^k$ is irreducible, H_0 is exact on category $\mathcal{O}$ and $H_0(L(k\Lambda_0)) \neq 0$, by [Ar05], we have that the $W^k_{\min}(\mathfrak{g})$-module $H_0(V^k(\mathfrak{g}))/H_0(J^k)$ irreducible, so $I^k = H_0(J^k)$. Assume first that $|S| = 1$. From Proposition 4.3, we deduce that $I^k = H_0(J^k)$ is a highest weight module. Due to [KRW03, (2.19)] its highest weight is

$$((M_1(k)+1)\theta_1, M_1(k)+1). \tag{4.10}$$

By (4.2), v_1 lies in I^k; moreover, it has weight (4.10), hence it is a highest weight vector. In particular it is singular and generates I^k. When $|S| = 2$, $\mathfrak{g}$ is of type $D(2,1;\frac{m}{n})$ and $k = \frac{mn}{m+n}q$, $q \in \mathbb{N}$. $H_0(J^k)$ is the sum of two highest weight modules of weights $(mq\theta_1, mq)$, $(nq\theta_2, nq)$. By Remark 2.1, we can assume $m > n$; then v_2 is singular, by the above comparing weight argument. We should prove that v_1 is not in the submodule generated by v_2. Otherwise, we can reach v_1 applying to v_2 a combination of operators $L_m, m < 0$, $J^{\{u\}}_r, u \in \mathfrak{g}^\natural, r \leq 0, G^{\{v\}}_s$, $v \in \mathfrak{g}_{-1/2}, s < 0$. We can clearly assume that the v's appearing are root vectors; let β_v be the corresponding root, so that $\eta_v := (\beta_v)_{|\mathfrak{h}^\natural}$ is the corresponding weight. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ with α_1 odd. Then $\eta_v \in \pm\{\frac{\alpha_2-\alpha_3}{2}, \frac{\alpha_2+\alpha_3}{2}\}$. Define the weight of $G^{\{v\}}_s$ to be the pair $(\eta_v, -s)$; likewise, if u is a root vector of $\mathfrak{g}^\natural$, and η_u is the corresponding root, define the weight of $J^{\{u\}}_r$ as $(\eta_u, -r)$. Finally, declare that the weight of L_n is $(0, -n)$. Let Δ_W be the set of weights. Note that any element of Δ_W is a positive integral linear combination of $\Pi_W = \{(-\alpha_2, 0), (-\alpha_3, 0), (\frac{\alpha_2+\alpha_3}{2}, \frac{1}{2})\}$. It follows that

$$(mq\alpha_2, mq) - (nq\alpha_3, nq) = a_1(-\alpha_2, 0) + a_2(-\alpha_3, 0) + a_3(\tfrac{\alpha_2+\alpha_3}{2}, \tfrac{1}{2}), \quad a_i \in \mathbb{Z}_+,$$

so that $a_1 = -nq, a_2 = mq, a_3 = 2(m-n)q$. Since a_1 is negative, we have the required contradiction.

4.2. $\mathfrak{g} = spo(2|3)$. Introduce the following vector

$$r_1 = (x_{\alpha_1})_{(0)}u_1 = (x_{\alpha_1})_{(0)}(x_{-\theta+\alpha_1})_{(1)}w_1. \tag{4.11}$$

Proposition 4.4. *The vector r_1 is singular in the universal affine vertex algebra $V^k(\mathfrak{g})$ and generates J^k.*

Proof. As in the proof of Proposition 4.3, the vector w_1 generates the maximal proper ideal of $V^k(\mathfrak{g})$. Let μ be weight of w_1. An explicit calculation shows that $(\mu|\alpha_0+\alpha_1) = -k - \frac{1}{2} \neq 0$, hence $u_1 \neq 0$, and it is singular for $r_{\alpha_1}\widehat{\Pi}$. So r_1 is either 0 or singular for $\widehat{\Pi}$. The first possibility does not occur, since

$$(k\Lambda_0 - (M_1(k)+1)(\delta-\theta_1)|\alpha_1) = \tfrac{1}{2}(M_1(k)+1) \neq 0.$$

The claim follows. □

Proof of Theorem 4.1 for $\mathfrak{g} = spo(2|3)$.

Arguing as in the previous subsection, by Proposition 4.4 it follows that the maximal ideal in $W^k_{\min}(\mathfrak{g})$ generated by a singular vector v of weight $(2(m-2)\omega_1, m-3/2)$, where ω_1 is the fundamental weight for $\mathfrak{sl}_2$ and $m = M_1(k)+2$. We observe that also the weight of $\widetilde{v}_1$ (cf. (4.3)) is $(2(m-2)\omega_1, m-3/2)$. Moreover, using (4.2) and the relations

$$(G^{\{x_{-\alpha_1}\}})_{(0)}\widetilde{v}_1 = c_1 v_1, \quad (G^{\{x_{-\xi}\}})_{(1)} v_1 = c_2\widetilde{v}_1, \text{ where } c_1, c_2 \in \mathbb{C}\setminus\{0\},$$

we see that $\widetilde{v}_1 \in I^k$, hence it is a multiple of v, so it is singular and generates I^k. □

5. Descending from $W^k_{\min}(\mathfrak{g})$ to $W_k^{\min}(\mathfrak{g})$

Let H_0 be the quantum Hamiltonian reduction functor from the category $\mathcal{O}_k$ of $\widehat{\mathfrak{g}}$-modules of level k to the category of $W^k_{\min}(\mathfrak{g})$-modules [KW04]. By [Ar05], it is exact. As in (3.1), for $\nu \in P^+_k$ and $h \in \mathbb{C}$ let $\widehat{\nu}_h = k\Lambda_0 + h\theta + \nu$. By [Ar05,KW04], $H_0(L(\widehat{\nu}_h)) = 0$ if $\widehat{\nu}_h(\alpha_0^\vee) = k - 2h \in \mathbb{Z}_{\geq 0}$, and $H_0(L(\widehat{\nu}_h) = L^W(\nu, \ell_0(h))$ if $\widehat{\nu}_h(\alpha_0^\vee) = k - 2h \notin \mathbb{Z}_{\geq 0}$, where

$$\ell_0(h) = \frac{(\widehat{\nu}_h|\widehat{\nu}_h + 2\widehat{\rho})}{2(k+h^\vee)} - h, \tag{5.1}$$

and $L^W(\nu, \ell_0)$ is the irreducible highest weight $W^k_{\min}(\mathfrak{g})$-module with highest weight (ν, ℓ_0) [KMP23,KW04].

Theorem 5.1. *Let k be in the unitary range. Then all irreducible highest weight* $W^k_{\min}(\mathfrak{g})$*–modules* $L^W(\nu, \ell_0)$ *with* $\ell_0 \in \mathbb{C}$ *when* $\nu \in P^+_k$ *is not extremal, and* $\ell_0 = A(k, \nu)$ *otherwise, descend to* $W_k^{\min}(\mathfrak{g})$.

Proof. If $k - 2h \in \mathbb{Z}_{\geq 0}$, then, by [KMP23, Lemma 11.8], for $h' = k + 1 - h$ we have $\ell_0 := \ell_0(h) = \ell_0(h')$. Since $k - 2h' \notin \mathbb{Z}_+$ we conclude that $H_0(L(\widehat{\nu}_{h'})) = L^W(\nu, \ell_0)$. So

$$\text{for each } \ell_0 \text{ there is } \tilde{h} \in \mathbb{C} \text{ such that } L^W(\nu, \ell_0) = H_0(L(\widehat{\nu}_{\tilde{h}})). \tag{5.2}$$

By Theorem 3.1, if ν is not extremal, then $L(\widehat{\nu}_{\tilde{h}})$ is a $V_k(\mathfrak{g})$–module, hence $L^W(\nu, \ell_0(\tilde{h}))$ is a $W_k^{\min}(\mathfrak{g})$-module. Note that $h \in E_{k,\nu}$ if and only if $\ell_0(h) = A(k, \nu)$. It follows from Theorem 3.1 that, if ν is extremal, then $L(\widehat{\nu}_{\tilde{h}})$ is a $V_k(\mathfrak{g})$–module, hence $L^W(\nu, A(k, \nu))$ is a $W_k^{\min}(\mathfrak{g})$-module. □

A simple application of Theorem 5.1 is the proof of Conjecture 4 in [KMP23].

Corollary 5.2. *Any unitary* $W^k_{\min}(\mathfrak{g})$*–module* $L^W(\nu, \ell_0)$ *descends to* $W_k^{\min}(\mathfrak{g})$.

Proof. By Theorem 1.3 (1) of [KMP23] (see Theorem 2.3 (1)), the unitary representations of $W^k_{\min}(\mathfrak{g})$ occur as representations listed in Theorem 5.1. □

Remark 5.3. Conjecture 2 from [KMP23] that all $W_k^{\min}(\mathfrak{g})$-modules $L^W(\nu, A(k, \nu))$ for extremal ν are unitary is still an open problem, except for $\mathfrak{g} = spo(2|3)$ and $psl(2|2)$. We can prove this statement also when $\mathfrak{g} = spo(2|n)$ and $k = -1$.

6. Classification of Irreducible Highest Weight Representations of $W_k^{\min}(\mathfrak{g})$ in the Unitarity Range

The main result of this section is the following theorem.

Theorem 6.1. *Let k be in the unitary range. The modules appearing in Theorem 5.1 form the complete list of inequivalent irreducible highest weight representations of* $W_k^{\min}(\mathfrak{g})$.

Remark 6.2. Combining (5.2) and Theorem 6.1, we have proven that the irreducible highest weight modules of $W_k^{\min}(\mathfrak{g})$ are precisely the non-zero images of the irreducible $V_k(\mathfrak{g})$ under Hamiltonian reduction.

We need to recall some well-known facts about Zhu algebras in the super case [KWang92].
Let $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$ be a conformal vertex algebra, graded by the eigenspaces of L_0, with the parity $p(V_n) \equiv 2n \mod 2$. For $a \in V_n$ we write $\deg a = n$. Define bilinear maps $* : V \times V \to V$, $\circ : V \times V \to V$, setting

$$a * b = \begin{cases} Res_z\left(Y(a,z)\frac{(z+1)^{\deg a}}{z}b\right) & \text{if } a, b \in V_{\bar{0}},\\ 0 & \text{if } a \text{ or } b \in V_{\bar{1}}.\end{cases}$$

$$a \circ b = \begin{cases} Res_z\left(Y(a,z)\frac{(z+1)^{\deg a}}{z^2}b\right) & \text{for } a \in V_{\bar{0}},\\ Res_z\left(Y(a,z)\frac{(z+1)^{\deg a-\frac{1}{2}}}{z^2}b\right) & \text{for } a \in V_{\bar{1}}.\end{cases}$$

Denote by $O(V) \subset V$ the $\mathbb{C}$-span of elements of the form $a \circ b$, and by $Zhu(V)$ the quotient space $V/O(V)$. Then $Zhu(V)$ is an associative algebra. Recall from [FZ92, (3.1.12)] or [KWang92, Lemma 1.1.(3)] that if $[a]$ denotes the class of an element $a \in V$ in $Zhu(V)$, then for all $a, b \in V_{\bar{0}}$ the following relations hold:

$$[a_{(-1)}\mathbf{1}] * [b] = [(a_{(-1)} + a_{(0)})b], \quad [b] * [a_{(-1)}\mathbf{1}] = [a_{(-1)}b]. \tag{6.1}$$

Recall that a module M over a conformal vertex algebra is called a *positive energy* module if L_0 is diagonalizable on M and all its eigenvalues lie in $h+\mathbb{R}_{\geq 0}$ for some $h \in \mathbb{C}$:

$$M = \bigoplus_{j\in h+\mathbb{R}_{\geq 0}} M_j, \quad M_h \neq \{0\}.$$

The subspace M_h is called the *top component* of M.

Recall [FZ92, KWang92] that there is one-to-one correspondence between irreducible positive energy V–modules and irreducible modules over the Zhu algebra $Zhu(V)$, which associates to a V-module M the $Zhu(V)$-module M_h. Namely, to $Y^M(a,z) = \sum_j a_j^M z^{-j-\deg a}$ one associates $a_0^M{}_{|M_h}$. By the above construction of the Zhu algebra, it follows from [KW04, Theorem 7.1] that

$$Zhu(W^k_{\min}(\mathfrak{g})) \simeq \mathbb{C}[L] \otimes U(\mathfrak{g}^\natural). \tag{6.2}$$

Under the correspondence between irreducible positive energy $W^k_{\min}(\mathfrak{g})$-modules and irreducible modules over its Zhu algebra, the module $L^W(\nu, h)$ goes to the irreducible highest weight $\mathfrak{g}^\natural$-module with highest weight ν on which L acts by the scalar h, which we denote by $V(\nu, h)$. It follows from (6.2) that

$$Zhu(W_k^{\min}(\mathfrak{g})) \simeq \left(\mathbb{C}[L] \otimes U(\mathfrak{g}^\natural)\right)/J(\mathfrak{g}), \tag{6.3}$$

where $J(\mathfrak{g})$ is a 2-sided ideal of the associative algebra $\mathbb{C}[L] \otimes U(\mathfrak{g}^\natural)$. So any non-zero element in $J(\mathfrak{g})$ imposes a condition on the highest weight (ν, h) of the $Zhu(W_k^{\min}(\mathfrak{g}))$-module $V(\nu, h)$.

Proof of Theorem 6.1: the case $\mathfrak{g} = spo(2|3)$.

First we present a proof in the case $\mathfrak{g} = spo(2|3)$, which gives a motivation for the proof in the general case.

Let $k = -\frac{m}{4}$, where $m \in \mathbb{Z}_{\geq 3}$, so that $M_1(k) = m - 2$ and $P_k^+ = \{j\omega_1 \mid 0 \leq j \leq m-2\}$ (here ω_1 is the fundamental weight for $\mathfrak{g}^\natural \cong sl_2$). Set $\mathcal{W}^k = W^k_{\min}(spo(2|3))$, $\mathcal{W}_k = W_k^{\min}(spo(2|3))$.

Denote by $L_k[j, q]$ the irreducible highest weight $\mathcal{W}_k$-module of level k generated by a highest weight vector $v_{j,q}$, such that for $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} &L_n v_{j,q} = q\delta_{n,0} v_{j,q}, \ G^+_{(n+1/2)} v_{j,q} = G^-_{(n+1/2)} v_{j,q} = G^0_{(n+1/2)} v_{j,q} = 0 \\ &J^0_{(n)} v_{j,q} = j\delta_{n,0} v_{j,q}, \ J^+_{(n)} v_{j,q} = J^-_{(n+1)} v_{j,q} = 0. \end{aligned}$$

Here we use notation for the generators of $\mathcal{W}_k$ as in [KW04, Section 8.5]. Note that $U(sl(2))v_{j,q} = V(j\omega_1)$ is the irreducible highest weight $sl(2)$–module with highest weight $j\omega_1$.

Lemma 6.3. *Set* $\Omega = -\frac{m-2}{4}([L] + \frac{[J^0]}{4}) + \frac{1}{8}[J^+] * [J^-]$. *Then*

$$\Omega * [J^-]^{m-3} \in J(\mathfrak{g}). \tag{6.4}$$

Proof. By Theorem 4.1, we have that $(J^-)^{m-3}_{(-1)} G^- \in I^k$, hence $[G^+_{(0)}(J^-)^{m-3}_{(-1)} G^-] = 0$. Since G^+_0 acts as a derivation, using (6.1) we get that

$$\begin{aligned} 0 &= [(G^+)_{(0)}(J^-)^{m-3}_{(-1)} G^-] = [(G^0)_{(-1)}(J^-)^{m-4}_{(-1)} G^-] + \cdots + \\ &[(J^-)^{m-5}_{(-1)}(G^0)_{(-1)}(J^-)_{(-1)} G^-] + [(J^-)^{m-4}_{(-1)}(G^+)_{(0)} G^-] \\ &= (m-3)[(J^-)^{m-4}_{(-1)}(G^0)_{(-1)} G^-] + \tfrac{(m-3)(m-4)}{2}[(J^-)^{m-5}_{(-1)}(G^-)_{(-2)} G^-] \\ &\quad + [(J^-)^{m-4}_{(-1)}(G^+)_{(0)} G^-] \\ &= (m-3)[(G^0)_{(-1)} G^+] * [J^-]^{m-4} + \tfrac{(m-3)(m-4)}{2}[(G^-)_{(-2)} G^-] * [J^-]^{m-5} \\ &\quad + [(G^+)_{(0)} G^-] * [J^-]^{m-3}. \end{aligned} \tag{6.5}$$

Using λ–bracket formulas from [KW04, Section 8.5] we get that the following relations hold in $\mathcal{W}^k$:

$$\begin{aligned} (G^+)_{(0)} G^- &= \tfrac{2k+1}{2} L + \tfrac{1}{8} : J^- J^+ : + \tfrac{k+1}{4} \partial J^0, \\ (G^0)_{(0)} G^- &= \tfrac{1}{8} : J^- J^0 : - \tfrac{k+1}{2} \partial J^-, \\ (G^-)_{(0)} G^- &= -\tfrac{1}{4} : J^- J^- : . \end{aligned}$$

This implies, in $Zhu(\mathcal{W}^k)$,

$$[G^+_{(0)}G^-] = \tfrac{2k+1}{2}([L] + \tfrac{1}{4(2k+1)}[J^+] * [J^-] - \tfrac{k+1}{2(2k+1)}[J^0]), \tag{6.6}$$

$$[G^0_{(-1)}G^-] = -[(G^0)_{(0)}G^-] = -\tfrac{1}{8}[J^0] * [J^-] - \tfrac{k+1}{2}[J^-] = -\tfrac{1}{8}([J^0] + 4(k+1)) * [J^-], \tag{6.7}$$

$$[G^-_{(-2)}G^-] = -[(G^-)_{(-1)}G^-] = [(G^-)_{(0)}G^-] = -\tfrac{1}{4}[J^-] * [J^-]. \tag{6.8}$$

Substituting (6.6), (6.7), (6.8) into (6.5) and collecting $[J^-]^{m-3}$, we get (6.4), where

$$\begin{aligned}\Omega &= \tfrac{2k+1}{2}[L] + \tfrac{1}{8}[J^+] * [J^-] - \tfrac{k+1}{4}[J^0] - \tfrac{(m-3)(m-4)}{8} - \tfrac{m-3}{8}([J^0] + 4(k+1)).\\ &= -\tfrac{m-2}{4}[L] + \tfrac{1}{8}[J^+] * [J^-] + \tfrac{m-4}{16}[J^0] - \tfrac{(m-3)(m-4)}{8} - \tfrac{m-3}{8}([J^0] - m + 4)\\ &= -\tfrac{m-2}{4}[L] - \tfrac{m-2}{16}[J^0] + \tfrac{1}{8}[J^+] * [J^-].\end{aligned}$$

□

Proposition 6.4. *Let $L_k[j, q]$ be an irreducible highest weight $\mathcal{W}_k$–module. Then it is isomorphic to exactly one of the following modules:*

- $L_k[j, q]$ *with* $0 \le j \le m-4$ *and* $q \in \mathbb{C}$*;*
- $L_k[m-3, \frac{m-3}{4}]$*;*
- $L_k[m-2, \frac{m-2}{4}]$.

Proof. First, $j \in \{0, \dots, m-2\}$ and the top component is $L_k[j, q]_{top} = V(j\omega_1) \otimes \mathbb{C}_q$, where $V(j\omega_1)$ is the $(j+1)$–dimensional irreducible $sl(2)$–module with highest weight $j\omega_1$, and $\mathbb{C}_q$ the 1-dimensional $\mathbb{C}[L]$–module on which L_0–acts as multiplication with $q \in \mathbb{C}$. If $0 \le j \le m-4$, then $[J^-]^{m-3}$ acts trivially on $L_k[j, q]_{top}$ for each $q \in \mathbb{C}$, hence the same holds for the action of $[G^+{}_{(0)}G^-] * [J^-]^{m-3}$. If $m-3 \le j \le m-2$, then $(J^-)^{m-3}_{(0)}$ acts non-trivially on $L_k[j, q]_{top}$. Hence there exists $w \in L_k[j, q]_{top}$ such that $w' = (J^-)^{m-3}_{(0)} w$ is a lowest weight vector for $sl(2)$, i.e.

$$(J^0)_{(0)} w' = -j w', \ (J^-)_{(0)} w' = 0.$$

Then we have, by Lemma 6.3:

$$\begin{aligned} 0 &= (\Omega * [J^-]^{m-3}) w\\ &= \left(-\tfrac{m-2}{4}(L_0 + \tfrac{1}{4}(J^0)_{(0)}) + \tfrac{1}{8}(J^+)_{(0)}(J^-)_{(0)}\right)(J^-)^{m-3}_{(0)} w\\ &= \left(-\tfrac{m-2}{4}(L_0 + \tfrac{1}{4}(J^0)_{(0)}) + \tfrac{1}{8}(J^+)_{(0)}(J^-)_{(0)}\right) w'\\ &= -\tfrac{m-2}{4}(q - \tfrac{j}{4}) w', \end{aligned}$$

hence, when $j = m-3$ or $m-2$, we have that $q = \frac{j}{4}$. □

Since the modules appearing in Theorem 6.1 in case of $\mathfrak{g} = spo(2|3)$ are exactly those listed in Proposition 6.4, Theorem 6.1 is proved in this case.
Proof of Theorem 6.1: *the case* $\mathfrak{g} \neq spo(2|3)$.

We first illustrate the strategy of the proof in the case $\mathfrak{g} = psl(2|2)$. We use notation of [KW04, Section 8.4].

Set $\mathcal{W}^k_{\min} = W^k_{\min}(psl(2|2))$ and $\mathcal{W}^{\min}_k = W^{\min}_k(psl(2|2))$.

Let $m = -k - 1 = M_1(k) \in \mathbb{Z}_{>0}$. Then $V_k(sl(2))$ is a vertex subalgebra of $\mathcal{W}_k^{\min}$ generated by $J^\pm, J^0$. The odd generators of conformal weight $3/2$ are $G^\pm, \overline{G}^\pm$. By Theorem 4.1 the maximal ideal I^k is generated by the singular vector $(J^+)_{(-1)}^{m+1}\mathbf{1}$. Then $(G^-)_{(0)}(\overline{G}^-)_{(0)}(J^+_{(-1)})^{m+1}\mathbf{1} \in I^k$. By the λ-bracket formulas given in [KW04, Section 8.4] we have

$$(G^-)_{(0)}\overline{G}^-_{(0)}(J^+_{(-1)})^{m+1}\mathbf{1} = (m+1)(G^-)_{(0)}(J^+_{(-1)})^m\overline{G}^+$$
$$= -(m+1)m(J^+_{(-1)})^{m-1}(G^+)_{(-1)}\overline{G}^+ + (m+1)(J^+_{(-1)})^m(G^-)_{(0)}\overline{G}^+. \tag{6.9}$$

Using the definition of the Zhu algebra $Zhu(\mathcal{W}^k_{\min})$, we have that for $G = G^\pm$ or $G = \overline{G}^\pm$ and each $v \in \mathcal{W}^k_{\min}$:

$$\mathrm{Res}_z \frac{(z+1)^{\deg G - 1/2}}{z} G(z)v = (G_{(-1)} + G_{(0)})v = 0 \text{ in } Zhu(\mathcal{W}^k_{\min}). \tag{6.10}$$

Using (6.10) and the λ-bracket formulas given in [KW04, Section 8.4], we get

$$[(G^+)_{(-1)}\overline{G}^+] = -[(G^+)_{(0)}\overline{G}^+] = -[(J^+)_{(-2)}\mathbf{1}] = [J^+], \tag{6.11}$$
$$[(G^-)_{(0)}\overline{G}^+] = [L] - \tfrac{1}{2}[(J^0)_{(-2)}\mathbf{1}] = [L] + \tfrac{1}{2}[J^0], \tag{6.12}$$

Using (6.9), (6.11),(6.12) and (6.1), we find

$$[(G^-)_{(0)}(\overline{G}^-)_{(0)}(J^+_{(-1)})^{m+1}\mathbf{1}] = -m(m+1)[J^+]^m + (m+1)([L] + \tfrac{1}{2}[J^0]) * [J^+]^m$$
$$= (m+1)\left([L] + \tfrac{1}{2}[J^0] - m\right) * [J^+]^m$$

so that, setting $\Omega = [L] + \frac{1}{2}[J^0] - m$, we have from (6.10)

$$\Omega * [J^+]^m \in J(\mathfrak{g}). \tag{6.13}$$

Proposition 6.5. *Let $L_k^{N4}[j, \ell_0]$ be an irreducible highest weight $\mathcal{W}_k^{\min}$–module. Then it is isomorphic to exactly one of the following modules:*

- $L_k^{N4}[j, \ell_0]$ *with* $0 \le j \le m-1$ *and* $\ell_0 \in \mathbb{C}$;
- $L_k^{N4}[m, \frac{m}{2}]$.

Proof. Let $v[j, \ell_0]$ be a highest weight vector for $L_k^{N4}[j, \ell_0]$. Then $j \in \{0, \dots, m\}$ and the top component is $L_k^{N4}[j, \ell_0]_{top} = V(j\omega_1) \otimes \mathbb{C}_{\ell_0}$, where $V(j\omega_1)$ is the $(j+1)$–dimensional irreducible $sl(2)$–module with highest weight $j\omega_1$, and $\mathbb{C}_{\ell_0}$ the 1-dimensional $\mathbb{C}[L]$–module on which $L(0)$–acts as multiplication by $\ell_0 \in \mathbb{C}$. If $0 \le j \le m-1$, then $\Omega * [J^+]^m$ acts trivially on $L_k^{N4}[j, \ell_0]_{top}$ for each $\ell_0 \in \mathbb{C}$.

If $j = m$, then $(J^+_{(0)})^m$ acts non-trivially on $L_k^{N4}[j, \ell_0]_{top}$. Then there exists $w \in L_k^{N4}[j, \ell_0]_{top}$ such that $(J^+_{(0)})^m w = v[j, \ell_0]$ is a highest weight vector for $sl(2)$. We get by (6.10)

$$0 = (\Omega * [J^+]^m)w$$
$$= (L_{(0)} + \tfrac{1}{2}(J^0_{(0)}) = -m(J^+_{(0)})^m w$$
$$= (\ell_0 - m/2)\, v[j, \ell_0].$$

This implies that for $j = m$ we need to have $\ell_0 = \frac{m}{2}$. □

Table 3. Root data

$\mathfrak{g}$	$spo(2\|3)$	$psl(2\|2)$	$spo(2\|m), m>4$	$D(2,1;\frac{m}{n})$	$F(4)$	$G(3)$
γ_1	$\delta_1+\epsilon_1$	$\epsilon_1-\delta_2$	$\delta_1+\epsilon_1$	$\epsilon_1+\epsilon_2-\epsilon_3, \epsilon_1+\epsilon_2+\epsilon_3$	$\frac{1}{2}(\delta_1+\epsilon_1+\epsilon_2-\epsilon_3)$	$\delta_1-\epsilon_3$
γ_2	δ_1	$\delta_1-\epsilon_2$	$\delta_1+\epsilon_2$	$\epsilon_1+\epsilon_2+\epsilon_3, \epsilon_1-\epsilon_2+\epsilon_3$	$\frac{1}{2}(\delta_1+\epsilon_1+\epsilon_2+\epsilon_3)$	$\delta_1+\epsilon_2$
θ_i	ϵ_1	$\delta_1-\delta_2$	$\epsilon_1+\epsilon_2$	$2\epsilon_2, 2\epsilon_3$	$\epsilon_1+\epsilon_2$	$\epsilon_2-\epsilon_3$
θ	$2\delta_1$	$\epsilon_1-\epsilon_2$	$2\delta_1$	$2\epsilon_1$	δ_1	$2\delta_1$

Proposition 6.5 proves, in particular, Theorem 6.1 for $\mathfrak{g} = psl(2|2)$. We now deal with the general case $\mathfrak{g} \neq spo(2|3)$. We shall see that a relation similar to (6.13) holds in $Zhu(W^k_{\min}(\mathfrak{g}))$. We do not need a very precise expression for Ω: what is really relevant is the fact that the action of Ω gives a relation which is linear in ℓ_0.

We start by observing that there exist two odd positive roots γ_1, γ_2 such that

$$\theta - \gamma_1 - \gamma_2 = -\theta_i, \ i \in S. \tag{6.14}$$

This fact can be verified directly by looking at Table 3.

Using (6.14) and the explicit expression for $[G^{\{u\}}{}_\lambda G^{\{v\}}]$ given in [AKMPP18, (1.1)] we find

$$G^{\{x_{\theta_i-\gamma_1}\}}{}_{(0)}G^{\{x_{\theta_i-\gamma_2}\}} = \sum_{s=1}^{\dim \mathfrak{g}_{1/2}} :J^{\{[x_{\theta_i-\gamma_1},u^s]^\natural\}}J^{\{[u_s,x_{\theta_i-\gamma_2}]^\natural\}}: +2c_1(k+1)\partial J^{\{x_{\theta_i}\}}, \tag{6.15}$$

$$G^{\{x_{-\gamma_1}\}}{}_{(0)}G^{\{x_{\theta_i-\gamma_2}\}} = -2(k+h^\vee)c_2L + c_2\sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{a^\alpha\}}J^{\{a_\alpha\}}: + \sum_{s=1}^{\dim \mathfrak{g}_{1/2}} :J^{\{[x_{-\gamma_1},u^s]^\natural\}}J^{\{[u_s,x_{\theta_i-\gamma_2}]^\natural\}}: +2(k+1)c_3\partial J^{\{h^\natural_{-\theta_i+\gamma_2}\}}, \tag{6.16}$$

where c_1, c_2, c_3 are constants independent of k. Here $a^\natural$ denotes the orthogonal projection of $a \in \mathfrak{g}$ to $\mathfrak{g}^\natural$ and $\{u_s\}$, $\{u^s\}$ are basis of $\mathfrak{g}_{1/2}$ dual with respect to the bilinear form

$$\langle u, v\rangle = (x_{-\theta}|[u,v]). \tag{6.17}$$

Set $x_{\theta_i-\gamma_j} = [x_{-\gamma_j}, x_{\theta_i}]$, $j = 1,2$. Recall that, by [KW04, Theorem 2.1 (e)], if $u \in \mathfrak{g}_{-1/2}$ and $a \in \mathfrak{g}^\natural$, then

$$[J^{\{a\}}_{(-1)}, G^{\{u\}}_{(0)}] = G^{\{[a,u]\}}_{(-1)}. \tag{6.18}$$

Lemma 6.6. *If $\mathfrak{g} \neq spo(2|3)$, $N \in \mathbb{Z}_{\geq 1}$ and $M \in \mathbb{Z}_{\geq 2}$, then, for $i \in S$ we have*

$$[G^{\{x_{-\gamma_j}\}}_{(0)}, (J^{\{x_{\theta_i}\}}_{(-1)})^N] = -N(J^{\{x_{\theta_i}\}}_{(-1)})^{N-1}G^{\{x_{\theta_i-\gamma_j}\}}_{(-1)}, \quad j=1,2. \tag{6.19}$$

$$\begin{aligned}(G^{\{x_{-\gamma_1}\}})_{(0)}(G^{\{x_{-\gamma_2}\}})_{(0)}(J^{\{x_{\theta_i}\}}_{(-1)})^M\mathbf{1} \\ = M\Big((M-1)(J^{\{x_{\theta_i}\}}_{(-1)})^{M-2}(G^{\{x_{\theta_i-\gamma_1}\}})_{(-1)}(G^{\{x_{\theta_i-\gamma_2}\}})_{(-1)}\mathbf{1} \\ - (J^{\{x_{\theta_i}\}}_{(-1)})^{M-1}(G^{\{x_{-\gamma_1}\}})_{(0)}(G^{\{x_{\theta_i-\gamma_2}\}})\Big).\end{aligned} \tag{6.20}$$

Proof. We prove (6.19) by induction on N, with base $N=1$ given by (6.18). If $N>1$ we have

$$[G^{\{x_{-\gamma_j}\}}_{(0)}, (J^{\{x_{\theta_i}\}}_{(-1)})^N] = [G^{\{x_{-\gamma_j}\}}_{(0)}, (J^{\{x_{\theta_i}\}}_{(-1)})^{N-1}](J^{\{x_{\theta_i}\}}_{(-1)}) + (J^{\{x_{\theta_i}\}}_{(-1)})^{N-1}[G^{\{x_{-\gamma_j}\}}_{(0)}, (J^{\{x_{\theta_i}\}}_{(-1)})]$$
$$= (1-N)(J^{\{x_{\theta_i}\}}_{(-1)})^{N-2}G^{\{x_{\theta_i-\gamma_j}\}}_{(-1)}(J^{\{x_{\theta_i}\}}_{(-1)}) - (J^{\{x_{\theta_i}\}}_{(-1)})^{N-1}G^{\{x_{\theta_i-\gamma_j}\}}_{(-1)}$$
$$= -N(J^{\{x_{\theta_i}\}}_{(-1)})^{N-1}G^{\{x_{\theta_i-\gamma_j}\}}_{(-1)}.$$

In the final equality we have used that $2\theta_i - \gamma_j \notin \Delta$ for $\mathfrak{g} \neq spo(2|3)$.
We now prove (6.20):

$$\begin{aligned}(G^{\{x_{-\gamma_1}\}})_{(0)}(G^{\{x_{-\gamma_2}\}})_{(0)}(J^{\{x_{\theta_i}\}}_{(-1)})^M\mathbf{1} &= -M(G^{\{x_{-\gamma_1}\}})_{(0)}(J^{\{x_{\theta_i}\}}_{(-1)})^{M-1}(G^{\{x_{\theta_i-\gamma_2}\}})_{(-1)}\mathbf{1}\\ &= M(M-1)(J^{\{x_{\theta_i}\}}_{(-1)})^{M-2}(G^{\{x_{\theta_i-\gamma_1}\}})_{(-1)}(G^{\{x_{\theta_i-\gamma_2}\}})_{(-1)}\mathbf{1}\\ &\quad - M(J^{\{x_{\theta_i}\}}_{(-1)})^{M-1}(G^{\{x_{-\gamma_1}\}})_{(0)}(G^{\{x_{\theta_i-\gamma_2}\}})\mathbf{1}.\end{aligned}$$

□

By Theorem 4.1, if $M = M_i(k)+1$, then the element displayed in (6.20) lies in I^k, hence its projection to $Zhu(W_k^{\min}(\mathfrak{g}))$ lies in $J(\mathfrak{g})$. Next we calculate explicitly this projection. First remark that, by (6.10),

$$[(G^{\{x_{\theta_i-\gamma_1}\}})_{(-1)}(G^{\{x_{\theta_i-\gamma_2}\}})_{(-1)}\mathbf{1}] = -[(G^{\{x_{\theta_i-\gamma_1}\}})_{(0)}(G^{\{x_{\theta_i-\gamma_2}\}})]. \tag{6.21}$$

Substituting (6.21) into (6.20) and using (6.15), (6.16), we get

$$\begin{aligned}&\frac{1}{M_i(k)+1}[(G^{\{-\gamma_1\}})_{(0)}(G^{\{-\gamma_2\}})_{(0)}(J^{\{x_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1}] =\\ &-M_i(k)\left(\sum_{\gamma=1}^{\dim\mathfrak{g}_{1/2}}[J^{\{[u_s,x_{\theta_i-\gamma_2}]^\natural\}}]*[J^{\{[x_{\theta_i-\gamma_1},u^s]^\natural\}}]*[J^{\{x_{\theta_i}\}}]^{M_i(k)-1} - 2(k+1)c_1[J^{\{x_{\theta_i}\}}]^{M_i(k)}\right)\\ &+c_2\left(-(k+h^\vee)[L]+\tfrac{1}{2}\sum_{\alpha=1}^{\dim\mathfrak{g}^\natural}[J^{\{a_\alpha\}}][J^{\{a^\alpha\}}]\right)*[J^{\{x_{\theta_i}\}}]^{M_i(k)}+\\ &-\left(\sum_{s=1}^{\dim\mathfrak{g}_{1/2}}[J^{\{[u_s,x_{\theta_i-\gamma_2}]^\natural\}}]*[J^{\{[x_{-\gamma_1},u^s]^\natural\}}]+2(k+1)c_3[\partial J^{\{h^\natural_{-\theta_i+\gamma_2}\}}]\right)*[J^{\{x_{\theta_i}\}}]^{M_i(k)}.\end{aligned} \tag{6.22}$$

To finish the calculation we need the following fact.

Lemma 6.7. *For suitable constants d_1, d_2, d_3, independent of k, we have*

$$\begin{aligned}\sum_{s=1}^{\dim\mathfrak{g}_{1/2}}[J^{\{[u_s,x_{\theta_i-\gamma_2}]^\natural\}}]*[J^{\{[x_{\theta_i-\gamma_1},u^s]^\natural\}}] &= d_1\,[J^{\{h^\natural_{-\theta_i+\gamma_2}\}}]*[J^{\{x_{\theta_i}\}}]+d_2\,[J^{\{x_{\theta_i}\}}]*[J^{\{h^\natural_{-\theta_i+\gamma_1}\}}]\\ &= (d_1\,[J^{\{h^\natural_{-\theta_i+\gamma_2}\}}]+d_2\,[J^{\{h^\natural_{-\theta_i+\gamma_1}\}}]+d_3)*[J^{\{x_{\theta_i}\}}].\end{aligned} \tag{6.23}$$

Proof. We can assume that u_s, u^s are root vectors. A direct inspection shows that $[u_s, x_{\theta_i-\gamma_2}]^\natural$, $[x_{\theta_i-\gamma_1}, u^s]^\natural$ are both non-zero only if $u_s \in \mathfrak{g}_\alpha$, $u^s \in \mathfrak{g}_\beta$ with either $-\alpha = \theta_i - \gamma_2$ and $\beta = \gamma_1$ or $\alpha = \gamma_2$ and $-\beta = \theta_i - \gamma_1$. □

Set

$$\Omega_i = (-M_i(k)(d_1\,[J^{\{h^\natural_{-\theta_i+\gamma_2}\}}] + d_2\,[J^{\{h^\natural_{-\theta_i+\gamma_1}\}}] + d_3) - c_2(k+h^\vee)[L] + \tfrac{1}{2}c_2 \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} [J^{\{a_\alpha\}}][J^{\{a^\alpha\}}] + \sum_{s=1}^{\dim \mathfrak{g}_{1/2}} [J^{\{[u_s, x_{\theta_i-\gamma_2}]^\natural\}}] * [J^{\{[x_{-\gamma_1}, u^s]^\natural\}}] - 2(k+1)c_3[J^{\{h^\natural_{-\theta_i+\gamma_2}\}}]). \tag{6.24}$$

Note that all the constants involved in (6.24) do not depend on k; moreover, by (6.14),

$$c_2 = (x_\theta | [x_{\theta_i-\gamma_1}, x_{\theta_i-\gamma_2}]) \neq 0. \tag{6.25}$$

Substituting (6.23) in (6.22) we obtain

Proposition 6.8. *In $Zhu(W^k_{\min}(\mathfrak{g}))$, we have*

$$\Omega_i * [J^{\{x_{\theta_i}\}}]^{M_i(k)} \in J(\mathfrak{g}),\ i \in S. \tag{6.26}$$

Proof of Theorem 6.1. We have to prove that if $L^W(\nu, \ell_0)$ is a irreducible highest weight $W_k^{\min}(\mathfrak{g})$-module, then the pair (ν, ℓ_0) is among those listed in the statement. Note that there is a non-zero vertex algebra homomorphism $\Theta : V^{M_i(k)}(\mathfrak{g}_i^\natural) \to W_k^{\min}(\mathfrak{g})$. Since $W_k^{\min}(\mathfrak{g})$ is unitary, we have that $\Theta(V^{M_i(k)}(\mathfrak{g}_i^\natural)) = V_{M_i(k)}(\mathfrak{g}^\natural)$. In particular, $L^W(\nu, \ell_0)$ is integrable as a $V_{M_i(k)}(\mathfrak{g}_i^\natural)$-module, hence $\nu \in P_k^+$. If ν is not extremal, we are done. Assume therefore that ν is extremal. The action of the Zhu algebra on the top component $V(\nu, \ell_0)$ of $L^W(\nu, \ell_0)$ is given by the action of the elements $[J^{\{a\}}]$ on the irreducible finite-dimensional $\mathfrak{g}^\natural$-module $V(\nu, \ell_0)$ of highest weight ν, while $[L]$ acts as the multiplication by ℓ_0. Consider the $sl(2)$-triple $\{x_{\theta_i}, h_{\theta_i}, x_{-\theta_i}\}$. Since ν is extremal, $\nu(\theta_i^\vee) = M_i(k)$ for some $i \in S$. By $sl(2)$-theory applied to $U(sl(2))v$ where $v \in V(\nu, \ell_0)$ is a highest weight vector, we see that v is a non-zero multiple of $x_{\theta_i}^{M_i(k)} v'$, with $v' = x_{-\theta_i}^{M_i(k)} v$. It follows that

$$0 = \Omega_i [J^{\{x_{\theta_i}\}}]^{M_i(k)} v' = \Gamma_i(\ell_0)[J^{\{x_{\theta_i}\}}]^{M_i(k)} v', \tag{6.27}$$

where, by (6.24), $\Gamma_i(\ell_0) = -c_2(k+h^\vee)\ell_0 + b$ and $b \in \mathbb{C}$ is independent of ℓ_0, and $c_2 \neq 0$ by (6.25). Since, by construction, $[J^{\{x_{\theta_i}\}}]^{M_i(k)} v' \neq 0$, we find from (6.27) that $\Gamma_i(\ell_0) = 0$. Since $L^W(\nu, A(k,\nu))$ is a representation of $W_k^{\min}(\mathfrak{g})$, then $\Gamma_i(A(k,\nu)) = 0$. But Γ_i is a linear relation, so $\ell_0 = A(k,\nu)$ is the unique solution of the equation $\Gamma_i(\ell_0) = 0$. □

A further refinement of Theorem 6.1 is the following result.

Theorem 6.9. *Let k be in the unitary range. The irreducible highest weight $W_k^{\min}(\mathfrak{g})$-modules are all the irreducible positive energy representations.*

Proof. Let $\mathcal{V}_k(\mathfrak{g}^\natural) = \bigotimes_{i\in S} V_{M_i(k)}(\mathfrak{g}_i^\natural)$. Since each $V_{M_i(k)}(\mathfrak{g}_i^\natural)$ is a rational vertex algebra, then $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural)) = \bigotimes_i Zhu(V_{M_i(k)}(\mathfrak{g}_i^\natural))$ is a semi-simple finite-dimensional associative algebra.

Let M be an irreducible positive energy $W_k^{\min}(\mathfrak{g})$–module. Then the top component M_{top} is an irreducible module for Zhu's algebra $Zhu(W_k^{\min}(\mathfrak{g}))$. By using the embedding $\mathcal{V}_k(\mathfrak{g}^\natural) \to W_k^{\min}(\mathfrak{g})$ we see that M_{top} is also a $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$–module. Since $[L]$ is a central element in $Zhu(W_k^{\min}(\mathfrak{g}))$, it acts on M_{top} as multiplication by a scalar, hence, by (6.3), M_{top} remains irreducible when restricted to $\mathfrak{g}^\natural$ and therefore it is an irreducible $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$–module. Since $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$ is a semi-simple finite-dimensional associative algebra, we conclude that M_{top} is finite-dimensional. This implies that M_{top} contains a highest weight $\mathcal{V}_k(\mathfrak{g}^\natural)$–vector w, which is then a highest weight vector for the action of $W_k^{\min}(\mathfrak{g})$. Therefore $W_k^{\min}(\mathfrak{g})w$ is a highest weight submodule of M. Irreducibility of M implies that $M = W_k^{\min}(\mathfrak{g})w$ is a highest weight $W_k^{\min}(\mathfrak{g})$–module. □

7. On Positive Energy Modules Over $V_k(\mathfrak{g})$ and $W_k^{\min}(\mathfrak{g})$

By Theorem 6.9, the irreducible highest weight $W_k^{\min}(\mathfrak{g})$-modules are all the irreducible positive energy representations. In this section we show, by contrast, that $V_k(\mathfrak{g})$ admits positive energy modules which are not highest weight, and we show that $W_k^{\min}(\mathfrak{g})$ admits non-positive energy modules.

The $V_k(\mathfrak{g})$–modules which we construct belong to class of modules called *relaxed* highest weight modules. These modules appear in [AdM95,FST,LMRS,R], in the context of the representation theory of $V_k(\mathfrak{sl}_2)$; later they have been systematically studied in [KR1,KR2,KaRW] for higher rank cases. We will show the existence of such modules for $V_k(\mathfrak{g})$ for some values of k belonging to the unitary range. We explore a free field realization which enables us to construct relaxed highest weight modules from relaxed highest weight modules over the Weyl vertex algebra (also called $\beta\gamma$ ghost vertex algebra). These relaxed modules were previously studied in [RW,AW22,AdP19].

On the contrary, the vertex algebra $W_k^{\min}(\mathfrak{g})$ does not have non-highest weight positive energy modules. The non-highest weight relaxed highest weight $V_k(\mathfrak{g})$–modules are either mapped to zero by quantum Hamiltonian reduction, so they don't contribute to $W_k^{\min}(\mathfrak{g})$–modules; or they are mapped to highest weight $W_k^{\min}(\mathfrak{g})$–modules. This connection deserves a more detailed analysis in the future.

We shall see below that $W_k^{\min}(\mathfrak{g})$ contains non-positive energy modules having all infinite dimensional weight spaces.

7.1. Positive energy modules over $V_k(\mathfrak{g})$ which are not highest weight. We follow the notation and results of [AMPP20, Section 6].

Consider the superspace $\mathbb{C}^{m|2n}$ equipped with the standard supersymmetric form $\langle\cdot,\cdot\rangle_{m|2n}$ given in [K77]. Let $V = \Pi\mathbb{C}^{m|2n}$, where Π is the parity reversing functor. Let $M_{(m|2n)}$ be the universal vertex algebra generated by V with λ–bracket

$$[v_\lambda w] = \langle w, v\rangle. \tag{7.1}$$

Let $\{e_i\}$ be the standard basis of V and let $\{e^i\}$ be its dual basis with respect to $\langle\cdot,\cdot\rangle$ (i. e. $\langle e_i, e^j\rangle = \delta_{ij}$). In this basis the λ-brackets are given by

$$[e_h{}_\lambda e_{m-k+1}] = \delta_{hk}, \quad [e_{m+i}{}_\lambda e_{m+2n-j+1}] = -\delta_{ij}, \quad [e_{m+n+i}{}_\lambda e_{m+n-j+1}] = \delta_{ij},$$

for $h, k = 1, \dots m,\ i, j = 1 \dots, n$.

The vertex algebra $M_{(m|n)}$ is called the Weyl–Clifford vertex algebra and, in the physics terminology, the $\beta\gamma bc$ system (cf. [FMS86]). In the case $m = 0$ (resp. $n = 0$), we have the Weyl vertex algebra $M_{(n)} := M_{(0,2n)}$, (resp. the Clifford vertex algebra $F_{(m)} := M_{(m,0)}$). Clearly, we have the isomorphism:

$$M_{(m|2n)} \cong F_{(m)} \otimes M_{(n)}.$$

It was proved in [AdP19, AW22, RW] that the Weyl vertex algebra $M_{(n)}$ has a remarkable family of irreducible positive energy modules, which we denote by $\widetilde{U_n}(\mathbf{a})$, whose top components are isomorphic to

$$U_n(\mathbf{a}) = x_1^{a_1} \cdots x_n^{a_n} \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}], \quad \text{for } \mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{C} \setminus \mathbb{Z})^n.$$

It was proved in [AMPP20, Section 6] that the simple affine vertex algebra $V_{-1/2}(spo(2r|s))$ is realized as a fixed point subalgebra $M^+_{(s|2r)}$ of $M_{(s|2r)}$. In particular, for $\mathfrak{g} = spo(2|3)$, we have

$$V_{-1/2}(\mathfrak{g}) = M^+_{(3|2)}.$$

From Proposition 4.4 it follows that

$$V_{-m/4}(\mathfrak{g}) = \frac{V^{-m/4}(\mathfrak{g})}{V^{-m/4}(\mathfrak{g})((x_{\theta_1})_{(-1)})^{m+1}\mathbf{1}}.$$

This easily implies that

$$V_{-m/4}(\mathfrak{g}) \hookrightarrow V_{-1/2}(\mathfrak{g})^{\otimes \frac{m}{2}} \hookrightarrow M_{(3|2)}^{\otimes \frac{m}{2}} \cong M_{(\frac{3m}{2}|m)}.$$

In this way, we get a non-zero vertex algebra homomorphism $\Phi^{(m)} : V_{-m/4}(\mathfrak{g}) \to M_{(\frac{3m}{2}|m)}$ for each $m \geq 4$ even. Now using positive energy modules $\widetilde{U_m}(\mathbf{a})$ for $M_{(m)}$, we construct $M_{(\frac{3m}{2}|m)}$–modules $F_{(\frac{3m}{2})} \otimes \widetilde{U_m}(\mathbf{a})$, which by restriction become $V_{-m/4}(\mathfrak{g})$–modules. Since their top components are not highest weight $\mathfrak{g}$–modules, we have constructed a family of non highest weight, positive energy $V_{-m/4}(\mathfrak{g})$–modules. Interestingly, these modules are still integrable for $\widehat{\mathfrak{g}}^{\natural}$. We shall present details and irreducibility analysis in our forthcoming publications.

7.2. Non-positive energy modules over $W_k^{min}(\mathfrak{g})$. For a conformal vertex algebra V let $\mathcal{E}(V)$ be the category of all (weak) V–modules, on which L_0 is diagonalizable, and let $\mathcal{E}^+(V)$ be the category of positive energy modules in $\mathcal{E}(V)$. Set $\mathcal{E}_k = \mathcal{E}(W_k^{min}(\mathfrak{g}))$, $\mathcal{E}_k^+ = \mathcal{E}^+(W_k^{min}(\mathfrak{g}))$, where k is from the unitary range for $\mathfrak{g}$.

A $W_k^{min}(\mathfrak{g})$–module M is called a weight $W_k^{min}(\mathfrak{g})$–module if it is a weight module for $\widehat{\mathfrak{g}}^{\natural}$.

Let $\mathcal{E}_k^{fin}$ be the subcategory of $\mathcal{E}_k$ consisting of weight modules with finite multiplicities.

Theorem 6.9 shows that each irreducible module in $\mathcal{E}_k^+$ belongs to the category $\mathcal{E}_k^{fin}$. One very interesting question is to see if there are weight representations of $W_k^{min}(\mathfrak{g})$ outside of the category $\mathcal{E}_k^+$. We believe that there are no such modules in $\mathcal{E}_k^{fin}$:

Conjecture 7.1. *The irreducible highest weight $W_k^{\min}(\mathfrak{g})$-modules are all the irreducible representations in the category $\mathcal{E}_k^{fin}$.*

However, weight modules which are not in $\mathcal{E}_k^+$ do exist. In order to see this, we use Kac–Wakimoto free field realization of $W^k_{min}(\mathfrak{g})$ [KW04, Theorem 5.2], which gives a vertex algebra homomorphism

$$\Psi : W^k_{\min}(\mathfrak{g}) \to \mathcal{H} \otimes \mathbb{V}^k(\mathfrak{g}^{\natural}) \otimes F(\mathfrak{g}_{1/2}),$$

where

- $\mathcal{H}$ is the Heisenberg vertex algebra of rank 1 generated by a field a such that $[a_\lambda a] = \lambda$;
- $\mathbb{V}^k(\mathfrak{g}^\natural) = \bigotimes_{i\in S} V^{M_i(k)+\chi_i}(\mathfrak{g}_i^\natural)$;
- $F(\mathfrak{g}_{1/2})$ is a fermionic vertex algebra.

Lemma 7.2. *There is a embedding of vertex algebras* $\Phi : \bigotimes_{i\in S} V_{-\chi_i}(\mathfrak{g}_i^\natural) \to F(\mathfrak{g}_{1/2})$ *uniquely determined by*

$$b \mapsto \tfrac{1}{2}\sum_{\alpha\in A} :\Phi^\alpha\Phi_{[u_\alpha,b]}:, \quad b\in\mathfrak{g}^\natural. \tag{7.2}$$

Proof. We start with Kac–Wakimoto free field realization [KW04, Theorem 5.1]

$$\Psi' : W^k_{\min}(\mathfrak{g}) \to \mathcal{V} = V^1(\mathbb{C}a)\otimes V^{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2}),$$

where

$$\alpha_k(a,b) = \delta_{i,j}(M_i(k)+\chi_i)(a|b)_i^\natural \text{ for } a\in\mathfrak{g}_i^\natural, b\in\mathfrak{g}_j^\natural,\ i,j\geq 0.$$

Choose $\mu\in\mathbb{C}$ such that the $\mathcal{V}$-invariant Hermitian form on $M(\mu)\otimes V^{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$ is invariant for the action of $W^k_{\min}(\mathfrak{g})$. Since $\Psi'(J^{\{b\}})\in V^{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$, it follows that the map $J^{\{b\}}\mapsto \Psi'(J^{\{b\}})_{(-1)}(v_\mu\otimes\mathbf{1}\otimes\mathbf{1})$, $b\in\mathfrak{g}_i^\natural$, extends to a map

$$\bigotimes_{i\in S} V^{M_i(k)}(\mathfrak{g}_i^\natural) \to \mathbb{C}v_\mu\otimes V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2}) \tag{7.3}$$

that, identifying $\mathbb{C}v_\mu\otimes V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$ with $V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$ in the obvious way, is a vertex algebra homomorphism. Now plug $\alpha_i(k)=0$, i.e. $M_i(k)=-\chi_i$, into (7.3). We get the homomorphism $\Phi' : \bigotimes_i V^{-\chi_i}(\mathfrak{g}_i^\natural)\to F(\mathfrak{g}_{1/2})$ explicitly given by (7.2). Since $F(\mathfrak{g}_{1/2})$ is unitary, it is completely reducible, hence the image of Φ' is simple. Therefore Φ' descends to Φ. □

The map Ψ' induces a map $\Psi : W^k_{\min}(\mathfrak{g})\to V^1(\mathbb{C}a)\otimes V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$. We first prove that the image of Ψ is simple.

Proposition 7.3. *Assume that* $M_i(k)+\chi_i\in\mathbb{Z}_{\geq 0}$ *for all* i. *Then* $\Psi(W^k_{\min}(\mathfrak{g})) = W_k^{\min}(\mathfrak{g})$.

Proof. As shown in the proof of Theorem 4.1, the vectors v_i generate the maximal ideal in $W^k_{\min}(\mathfrak{g})$ (although v_1 is not a singular vector when $\mathfrak{g}=spo(2|3)$). Hence it suffices to check that in $V^1(\mathbb{C}a)\otimes V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$ we have:

$$(J^{\{x_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1} = 0 \quad i\geq 1.$$

From Kac–Wakimoto free field realization and Lemma 7.2 we get that the map $b\mapsto b+\frac{1}{2}\sum_{\alpha\in A}:\Phi^\alpha\Phi_{[u_\alpha,b]}:$ induces a homomorphism

$$\Theta := \Phi\circ\Psi_{|V^{M_i(k)}} : V^{M_i(k)}(\mathfrak{g}_i^\natural)\to V_{M_i(k)+\chi_i}(\mathfrak{g}_i^\natural)\otimes V_{-\chi_i}(\mathfrak{g}_i^\natural)\subset V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2}).$$

Since $M_i(k)+\chi_i\in\mathbb{Z}_{\geq 0}$, the vertex algebra $V_{\alpha_k}(\mathfrak{g}^\natural)\otimes F(\mathfrak{g}_{1/2})$ is unitary, hence it is completely reducible, so $Im\,\Theta$ is simple and in turn $(J^{\{x_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1}=0$. □

We have therefore proved the following result.

Theorem 7.4. *If k is non-collapsing and lies in the unitary range, then Ψ induces a non-trivial homomorphism of vertex algebras*

$$\widetilde{\Psi} : W_k^{\min}(\mathfrak{g}) \to \mathcal{H} \otimes \mathbb{V}_k(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2}),$$

where $\mathbb{V}_k(\mathfrak{g}^\natural) = \bigotimes_{i \in S} V_{M_i(k)+\chi_i}(\mathfrak{g}_i^\natural)$.

Since $\mathbb{V}_k(\mathfrak{g}^\natural)$ and $F(\mathfrak{g}_{1/2})$ are rational vertex algebras, they don't have non-positive energy modules. But the Heisenberg vertex algebra $\mathcal{H}$ is not rational and it has very rich representation theory:

(fin) The category of $\mathcal{H}$–modules with finite-dimensional weight spaces $\mathcal{E}^{fin}(\mathcal{H})$ is semi-simple and every irreducible $\mathcal{H}$–module in that category is of highest weight. The category $\mathcal{E}^{fin}(\mathcal{H})$ concides with the category of modules in $\mathcal{E}^+(\mathcal{H})$ of finite length.

(inf) The category $\mathcal{E}(\mathcal{H})$ admits irreducible $\mathcal{H}$–modules with infinite-dimensional weight spaces [FGM14].

Remark 7.5. Using Theorem 7.4 and the representation theory of the Heisenberg vertex algebra $\mathcal{H}$ one can prove the following facts: assume that E_1 is an irreducible module in $\mathcal{E}(\mathcal{H})$. Let E_2 be a $\mathbb{V}_k(\mathfrak{g}^\natural)$-module. Then we have:

(1) $E_1 \otimes E_2 \otimes F(\mathfrak{g}_{1/2})$ lies in $\mathcal{E}_k$. It lies in the category $\mathcal{E}_k^+$ if and only if E_1 lies in the category $\mathcal{E}^+(\mathcal{H})$.
(2) If E_1 is a weight $\mathcal{H}$-module with infinite-dimensional weight spaces then $E_1 \otimes E_2 \otimes F(\mathfrak{g}_{1/2})$ is a non-positive energy weight $W_k^{min}(\mathfrak{g})$–module with infinite-dimensional weight spaces.
(3) If E_1 is a non-weight $\mathcal{H}$-module, then $E_1 \otimes E_2 \otimes F(\mathfrak{g}_{1/2})$ is a non-weight $W_k^{min}(\mathfrak{g})$-module. In particular, we can construct analogs of non-weight $V_k(\mathfrak{g})$–modules from Remark 3.3.

Further details on applications of Theorem 7.4 and Remark 7.5 will appear elsewhere.

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