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Polynomial Tau-Functions of the n-th Sawada–Kotera Hierarchy

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Abstract: We give a review of the B-type Kadomtsev–Petviashvili (BKP) hierarchy and find all polynomial tau-functions of the n-th reduced BKP hierarchy (=n-th Sawada–Kotera hierarchy). The name comes from the fact that, for $n = 3$, the simplest equation of the hierarchy is the famous Sawada–Kotera equation.

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1. Introduction

The three most famous hierarchies of Lax equations on one function u are the Korteweg–de Vries (KdV) hierarchy, the Kaup–Kupershmidt hierarchy, and the Sawada–Kotera hierarchy. The Lax operators are, respectively,

$$\mathcal{L} = \partial^2 + u, \tag{1}$$

$$\mathcal{L} = \partial^3 + u\partial + \frac{1}{2}u', \tag{2}$$

$$\mathcal{L} = \partial^3 + u\partial. \tag{3}$$

Let $t = (t_1, t_2, t_3, \ldots)$ and $\tilde{t} = (t_1, t_3, t_5, \ldots)$. Recall that the Kadomtsev–Petviashvili (KP) hierarchy is the following hierarchy of Lax equations on the pseudodifferential operator $L(t, \partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots$ in $\partial = \frac{\partial}{\partial t_1}$ [1]:

$$\frac{\partial L(t,\partial)}{\partial t_k} = \left[\left(L(t,\partial)^k\right)_{\geq 0}, L(t,\partial)\right], \quad k = 1, 2, \ldots. \tag{4}$$

The KdV hierarchy is the second reduced KP hierarchy, meaning that one imposes the following constraint on $L(t, \partial)$:

$$\mathcal{L}(t,\partial) = L(t,\partial)^2 \quad \text{is a differential operator.} \tag{5}$$

In this case, the operator $\mathcal{L}$ is defined by (1), with $u(t) = 2u_1(t)$, and the KP hierarchy (4) reduces to the KdV hierarchy

$$\frac{\partial \mathcal{L}(t,\partial)}{\partial t_k} = \left[\left(\mathcal{L}(t,\partial)^{\frac{k}{2}}\right)_{\geq 0}, \mathcal{L}(t,\partial)\right], \quad k = 1, 3, 5, \ldots. \tag{6}$$

For even k, this equation is trivial; for $k = 3$, Equation (6) is the KdV equation [2].

Recall that in order to construct solutions of the KP hierarchy and the reduced KP hierarchies, one introduces the tau-function $\tau(t)$, defined by [1,2]:

$$L(t,\partial) = P(t,\partial) \circ \partial \circ P(t,\partial)^{-1}, \tag{7}$$

where $P(t,\partial)$ is a pseudodifferential operator, with the symbol

$$P(t,z) = \frac{1}{\tau(t)} \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}\right)(\tau(t)). \tag{8}$$

The tau-function has a geometric meaning as a point on an infinite-dimensional Grassmannian, and in [1], Sato showed that all Schur polynomials are tau-functions of the KP hierarchy. Recently, all polynomial tau-functions of the KP hierarchy and its n-reductions have been constructed in [3] (see also [4]).

The CKP hierarchy (KP hierarchy of type C) can be constructed by making use of the KP hierarchy, and assuming the additional constraint $L(\tilde{t},\partial)^* = -L(\tilde{t},\partial)$ (see, e.g., [5] for details). Its 3-reduction is defined by the constraint that $\mathcal{L}(\tilde{t},\partial) = L(\tilde{t},\partial)^3$ is a differential operator, and the corresponding hierarchy is

$$\frac{\partial \mathcal{L}(\tilde{t},\partial)}{\partial t_k} = \left[\left(\mathcal{L}(\tilde{t},\partial)^{\frac{k}{3}}\right)_{\geq 0}, \mathcal{L}(\tilde{t},\partial)\right], \quad k = 1,3,5,\ldots, k \notin 3\mathbb{Z}, \tag{9}$$

where $\mathcal{L}$ is given by (2). For $k = 5$, we obtain the Kaup–Kupershmidt equation, the simplest non-trivial equation in this hierarchy. All polynomial tau-functions of (9) (and all n reductions of the CKP hierarchy) have been constructed in [5].

In the present paper, we construct all polynomial tau-functions of the n-reduced BKP hierarchies (KP hierarchy of type B). These are hierarchies of Lax equations on the differential operator

$$\mathcal{L}(\tilde{t},\partial) = \partial^n + u_{n-2}(\tilde{t})\partial^{n-2} + \cdots + u_1(\tilde{t})\partial, \tag{10}$$

satisfying the constraint

$$\mathcal{L}(\tilde{t},\partial)^* = (-1)^n \partial^{-1} \mathcal{L}(\tilde{t},\partial)\partial. \tag{11}$$

The n-th reduced BKP hierarchy is

$$\frac{\partial \mathcal{L}(\tilde{t},\partial)}{\partial t_k} = \left[\left(\mathcal{L}(\tilde{t},\partial)^{\frac{k}{n}}\right)_{\geq 0}, \mathcal{L}(\tilde{t},\partial)\right], \quad k = 1,3,5,\ldots, k \notin n\mathbb{Z}. \tag{12}$$

We call it the n-th Sawada–Kotera hierarchy, since, for $n = 3$, $\mathcal{L}$ is given by (3), and for $k = 5$, Equation (12) is the Sawada–Kotera equation [6] (see Equation (33)).

In the present paper, using the description of polynomial tau-functions of the BKP hierarchy [4,7] (see Theorem 1), we find all polynomial tau-functions for the n-th Sawada–Kotera hierarchies (see Theorem 2), and, in particular, for the Sawada–Kotera hierarchy (see Corollary 1).

2. The BKP Hierarchy and Its Polynomial Tau-Functions

In this section, we recall the construction of the BKP hierarchy [8] and description of its polynomial tau-functions from [4,7].

Following Date, Jimbo, Kashiwara, and Miwa [8] (see also [7] for details), we introduce the BKP hierarchy in terms of the so-called twisted neutral fermions ϕ_i, $i \in \mathbb{Z}$, which are generators of a Clifford algebra over $\mathbb{C}$, satisfying the following anti-commutation relation:

$$\phi_i\phi_j + \phi_j\phi_i = (-1)^i \delta_{i,-j}. \tag{13}$$

Consider the right (resp., left) irreducible module $F = F_r$ (resp., F_l) over this algebra by the following action on the vacuum vector $|0\rangle$ (resp., $\langle 0|$):

$$\phi_0|0\rangle = \frac{1}{\sqrt{2}}|0\rangle, \quad \phi_j|0\rangle = 0 \qquad \left(\text{resp., } \langle 0|\phi_0 = \frac{1}{\sqrt{2}}\langle 0|, \quad \langle 0|\phi_{-j} = 0\right), \quad \text{for } j > 0. \tag{14}$$

The quadratic elements
$$\phi_j\phi_k - \phi_k\phi_j \text{ for } j,k \in \mathbb{Z},\ j > k,$$
form a basis of the infinite-dimensional Lie algebra $so_{\infty,odd}$ over $\mathbb{C}$. Let $SO_{\infty,odd}$ be the corresponding Lie group. We proved in ([9], Theorem 1.2a) that a non-zero element $\tau \in F$ lies in this Lie group orbit of the vacuum vector $|0\rangle$ if and only if it satisfies the BKP hierarchy in the fermionic picture, i.e., the following equation in $F \otimes F$:

$$\sum_{j\in\mathbb{Z}} (-1)^j \phi_j\tau \otimes \phi_{-j}\tau = \frac{1}{2}\tau \otimes \tau. \tag{15}$$

Non-zero elements of F, satisfying (15), are called tau-functions of the BKP hierarchy in the fermionic picture.

The group $SO_{\infty,odd}$ consists of elements G leaving the symmetric bilinear form

$$(\phi_j, \phi_k) = (-1)^j \delta_{j,-k} \tag{16}$$

on F invariant, i.e.,
$$(G\phi_j, G\phi_k) = (\phi_j, \phi_k). \tag{17}$$
Stated differently,

$$G\phi_k G^{-1} = \sum_{j\in\mathbb{Z}} a_{jk}\phi_k \text{ (finite sum) with } \sum_{j\in\mathbb{Z}} (-1)^j a_{jk} a_{-j\ell} = (-1)^k \delta_{k,-\ell}. \tag{18}$$

The group orbit of the vacuum vector is the disjoint union of Schubert cells (see Section 3 of [7] for details). These cells are parametrized by the strict partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, with $\lambda_1 > \lambda_2 > \cdot \cdots > \lambda_k > 0$. Namely, the cell, attached to the partition λ is
$$C_\lambda = \{v_1 v_2 \cdots v_{k-1} v_k |0\rangle \,|\, v_j = \sum_{i \geq -\lambda_j} a_{ij}\phi_i \text{ (finite sum) with } a_{-\lambda_j,j} \neq 0\}. \tag{19}$$
An element $\tau \in C_\lambda$ corresponds to the following point in the maximal isotropic Grassmannian (i.e., a maximal isotropic subspace of $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}\phi_j$):

$$\operatorname{Ann}\tau = \{v \in V \,|\, v v_1 v_2 \cdots v_{k-1} v_k |0\rangle = 0\}. \tag{20}$$

For instance, $\operatorname{Ann}|0\rangle = \operatorname{span}\{\phi_1, \phi_2, \ldots\}$.

Using the bosonization of Equation (15), one obtains a hierarchy of differential equations on τ ([4,7,8,10], Section 3). This bosonization is an isomorphism σ between the spin module F and the polynomial algebra $B = \mathbb{C}[\tilde{t}] = \mathbb{C}[t_1, t_3, t_5, \ldots]$. Explicitly, we introduce the twisted neutral fermionic field

$$\phi(z) = \sum_{j\in\mathbb{Z}} \phi_j z^{-j},$$

and the bosonic field

$$\alpha(z) = \sum_{j\in\mathbb{Z}} \alpha_{2j+1} z^{-2j-1} = \frac{1}{2} : \phi(z)\phi(-z) :,$$

where the normal ordering $:\ :$ is defined by

$$: \phi_j\phi_k := \phi_j\phi_k - \langle 0|\phi_j\phi_k|0\rangle;$$

equivalently: $:\phi_j\phi_k := \phi_j\phi_k$ if $j \le k$ and $= -\phi_k\phi_j$ if $j > k$, except when $j = k = 0$, then $:\phi_0\phi_0 := 0$. The operators α_j satisfy the commutation relations of the Heisenberg Lie algebra

$$[\alpha_j, \alpha_k] = \frac{j}{2}\delta_{j-k}, \quad \alpha_i|0\rangle = \langle 0|\alpha_{-i} = 0, \quad \text{for } i > 0, \tag{21}$$

and its representation on F is irreducible ([9], Theorem 3.2). Using this, we obtain a vector space isomorphism $\sigma : F \to B$, uniquely defined by the following relations:

$$\sigma(|0\rangle) = 1, \quad \sigma\alpha_j\sigma^{-1} = \frac{\partial}{\partial t_j}, \quad \sigma\alpha_{-j}\sigma^{-1} = \frac{j}{2}t_j, \quad \text{for } j > 0 \text{ odd}. \tag{22}$$

Explicitly ([9], Section 3.2):

$$\sigma\phi(z)\sigma^{-1} = \frac{1}{\sqrt{2}} \exp\sum_{j=1}^{\infty} t_{2j-1}z^{2j-1} \exp\sum_{j=1}^{\infty} -2\frac{\partial}{\partial t_{2j-1}}\frac{z^{-2j+1}}{2j-1}. \tag{23}$$

Since (15) can be rewritten as

$$\mathrm{Res}_z\, \phi(z)\tau \otimes \phi(-z)\tau\frac{dz}{z} = \frac{1}{2}\tau \otimes \tau,$$

under the isomorphism σ, Equation (15) turns into:

$$\mathrm{Res}_z e^{\sum_{j=1}^{\infty}(t_{2j-1}-t'_{2j-1})z^{2j-1}} e^{\sum_{j=1}^{\infty} 2\left(\frac{\partial}{\partial t'_{2j-1}} - \frac{\partial}{\partial t_{2j-1}}\right)\frac{z^{-2j+1}}{2j-1}} \tau(\tilde{t})\tau(\tilde{t}')\frac{dz}{z} = \tau(\tilde{t})\tau(\tilde{t}'), \tag{24}$$

where $\tilde{t} = (t_1, t_3, t_5, \ldots)$ and $\tilde{t}' = (t'_1, t'_3, t'_5, \ldots)$. Therefore, $\tau(\tilde{t})$ is the vacuum expectation value

$$\tau(\tilde{t}) = \sigma\tau\sigma^{-1} = \langle 0|e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}}\tau. \tag{25}$$

Furthermore, by making a change of variables, as in [10] (p. 972), viz. $t_{2k-1} = x_{2k-1} - y_{2k-1}$ and $t'_{2k-1} = x'_{2k-1} - y'_{2k-1}$, and using the elementary Schur polynomials $s_j(r)$, which are defined by

$$\exp\sum_{k=1}^{\infty} r_k z^k = \sum_{j=0}^{\infty} s_j(r)z^j, \tag{26}$$

we can rewrite (24), where we assume $x_{2k} = y_{2k} = 0$:

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y})s_j(2\tilde{\partial}_y)\tau(\tilde{x} - \tilde{y})\tau(\tilde{x} + \tilde{y}) = 0, \tag{27}$$

where $\tilde{y} = (y_1, 0, y_3, 0, \ldots)$ and $\tilde{\partial}_y = (\frac{\partial}{\partial y_1}, 0, \frac{1}{3}\frac{\partial}{\partial y_3}, 0, \frac{1}{5}\frac{\partial}{\partial y_5}, \ldots)$. Using Taylor's formula, we thus obtain the BKP hierarchy of Hirota bilinear equations [10] (p. 972):

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y})s_j(2\tilde{\partial}_u) \exp\sum_{j=1}^{\infty} y_{2j-1}\frac{\partial}{\partial u_{2j-1}}\tau(\tilde{x} - \tilde{u})\tau(\tilde{x} + \tilde{u})\Big|_{\tilde{u}=0} = 0. \tag{28}$$

Using the notation $p(D)f \cdot g = p(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots)f(\tilde{x} + \tilde{u})g(\tilde{x} - \tilde{u})\Big|_{\tilde{u}=0}$, this turns into

$$\sum_{j=1}^{\infty} s_j(-2\tilde{y})s_j(2\tilde{D})e^{\sum_{j=1}^{\infty} y_{2j-1}D_{2j-1}}\tau \cdot \tau = 0. \tag{29}$$

The simplest equation in this hierarchy is ([10], Appendix 3):

$$(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau \cdot \tau = 0. \tag{30}$$

If we assume that our tau-function does not depend on t_3, then this gives

$$(D_1^6 + 9D_1D_5)\tau \cdot \tau = 0. \tag{31}$$

Letting $x = t_1$, $t = \frac{1}{9}t_5$, and

$$u(x,t) = 2\frac{\partial^2 \log \tau(x,t)}{\partial x^2}, \tag{32}$$

and viewing the remaining t_j as parameters, Equation (31) turns into the famous Sawada–Kotera equation [6]:

$$u_t + 15(uu_{xxx} + u_x u_{xx} + 3u^2 u_x) + u_{xxxxx} = 0. \tag{33}$$

Another approach is by using the wave function; see [8] (p. 345),

$$\begin{aligned} w(\tilde{t}, z) &= \frac{1}{\tau(t)} \exp \sum_{j=1}^{\infty} t_{2j-1} z^{2j-1} \exp - \sum_{j=1}^{\infty} 2\frac{\partial}{\partial t_{2j-1}} \frac{z^{-2j+1}}{2j-1} \tau(t) \\ &= P(\tilde{t}, z) e^{\sum_{j=1}^{\infty} (t_{2j-1}) z^{2j-1}}, \end{aligned} \tag{34}$$

where $P(\tilde{t}, z) = 1 + \sum_{j=1}^{\infty} p_j(\tilde{t}) z^{-j}$, and, in particular,

$$p_1(\tilde{t}) = -2\frac{\partial \log \tau(\tilde{t})}{\partial t_1}. \tag{35}$$

Letting $P(\tilde{t}, \partial)$ be the pseudodifferential operator in $\partial = \frac{\partial}{\partial t_1}$ with the symbol $P(\tilde{t}, z)$, Equation (24) turns into

$$\mathrm{Res}_z P(\tilde{t}, \partial) P(\tilde{t}', \partial') e^{\sum_{j=1}^{\infty} (t_{2j-1} - t'_{2j-1}) z^{2j-1}} \frac{dz}{z} = 1. \tag{36}$$

Now, using the fundamental lemma, Lemma 1.1 of [2] or Lemma 4.1 of [11], we deduce from (36):

$$\begin{aligned} &P(\tilde{t}, \partial)\partial^{-1} P(\tilde{t}, \partial)^* \partial = 1, \\ &\frac{\partial P(\tilde{t}, \partial)}{\partial t_{2j-1}} = -(P(\tilde{t}, \partial)\partial^{2j-1} P(\tilde{t}, \partial)^{-1}\partial^{-1})_{<0} \partial P(\tilde{t}, \partial), \quad j = 1, 2, \ldots. \end{aligned} \tag{37}$$

Next, introducing the Lax operator

$$L(\tilde{t}, \partial) = P(\tilde{t}, \partial)\partial P(\tilde{t}, \partial)^{-1} = \partial + u_1(\tilde{t})\partial^{-1} + u_2(\tilde{t})\partial^{-2} + \cdots,$$

we deduce from (37) that L satisfies [8]

$$\begin{aligned} &L(\tilde{t}, \partial)^* = -\partial^{-1} L(\tilde{t}, \partial)\partial, \\ &\frac{\partial L(\tilde{t}, \partial)}{\partial t_{2j-1}} = \left[\left(L(\tilde{t}, \partial) \right)^{2j-1} \right)_{\geq 0}, L(\tilde{t}, \partial) \right], \quad j = 1, 2, \ldots. \end{aligned} \tag{38}$$

Note that, since $u_1(\tilde{t}) = -\frac{\partial p_1(\tilde{t})}{\partial t_1}$ and the fact that $p_1(t)$ is given by (35), we find that

$$u_1(\tilde{t}) = 2\frac{\partial^2 \log \tau(\tilde{t})}{\partial t_1^2}, \tag{39}$$

which explains the choice (32) of $u(x,t)$ to obtain the Sawada–Kotera equation from the Hirota bilinear Equation (31).

To obtain the second equation of (38), we use (37) and the first equation of (38), which is equivalent (see [8], (p. 356)) to the fact that $L(\tilde{t},\partial)^{2j-1}$, for $j = 1, 2, \ldots$, has zero constant term. Let us prove that the first equation of (38) indeed implies this fact. We have

$$L^k\partial^{-1} = (-\partial^{-1}L^*\partial)^k\partial^{-1} = (-1)^k\partial^{-1}L^{*k} = (-1)^{k+1}(L^k\partial^{-1})^*.$$

Now, using the fact that the constant term of L^k is equal to

$$\mathrm{Res}_\partial L^k\partial^{-1} = -\mathrm{Res}_\partial (L^k\partial^{-1})^* = \mathrm{Res}_\partial (-1)^{k+1}(L^k\partial^{-1})^* = (-1)^k \mathrm{Res}_\partial L^k\partial^{-1},$$

we find that the constant term of L^k is zero whenever k is odd.

Remark 1. *Note that this also means that we can replace the second equation of* (38) *by, cf. [12],*

$$\frac{\partial L(\tilde{t},\partial)}{\partial t_{2j-1}} = \left[\left(L(\tilde{t},\partial))^{2j-1}\right)_{\geq 1}, L(\tilde{t},\partial)\right], \quad j = 1, 2, \ldots.$$

In the formulation of Kupershmidt [12], this means that L satisfies not only the KP equation for the odd times, but also his formulation of the modified KP hierachy (only for the odd times).

Next, we describe polynomial tau-functions $\tau(t_1, t_3, \ldots)$ of the BKP hierarchy obtained in ([7], Theorem 6) (see also [4]). For that, given integers a and b, $a > b \geq 0$, let

$$\begin{aligned} \chi_{a,b}(t,t') &= \frac{1}{2}s_a(t')s_b(t) + \sum_{j=1}^{b}(-1)^j s_{a+j}(t')s_{b-j}(t), \\ \chi_{b,a}(t,t') &= -\chi_{a,b}(t,t'), \quad \chi_{a,a}(t,t') = 0, \end{aligned} \tag{40}$$

and let $\chi_{a,b}(t,t') = 0$ if $b < 0$. Then

Theorem 1 ([7], Theorem 6)**.** *All polynomial tau-functions of the BKP hierarchy* (24)*, up to a scalar multiple, are equal to*

$$\tau_\lambda(\tilde{t}) = Pf\left(\chi_{\lambda_i,\lambda_j}(\tilde{t}+c_i, \tilde{t}+c_j)\right)_{1\leq i,j\leq 2n}, \tag{41}$$

where Pf is the Pfaffiann of a skew-symmetric matrix, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ is an extended strict partition, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \geq 0$, $\tilde{t} = (t_1, 0, t_3, 0, \ldots)$, $c_i = (c_{1i}, c_{2i}, c_{3i}, \ldots)$, $c_{ij} \in \mathbb{C}$.

Remark 2. *The connection between the set of strict partitions and the extended strict partitions is as follows. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a strict partition and k is even, then this partition is equal to the extended strict partition λ. However, if k is odd, the Pfaffian of a $k \times k$ anti-symmetric matrix is equal to* 0*, hence, in that case, we extend λ by the element* 0*, i.e., the corresponding extended strict partition is then $(\lambda_1, \lambda_2, \ldots, \lambda_k, 0)$.*

3. The *n*-th Sawada–Kotera Hierarchy and Its Polynomial Tau-Functions

As we have seen in Section 2, a necessary condition for a tau-function to give a solution of the Sawada–Kotera equation is that $\frac{\partial\tau(\tilde{t})}{\partial t_3} = 0$. This means that the tau-function lies in a smaller group orbit of the vacuum vector $|0\rangle$. Instead of the $SO_{\infty,odd}$ orbit of the vacuum vector $|0\rangle$, we consider the twisted loop group $G_3^{(2)}$, corresponding to the affine Lie algebra $sl_3^{(2)}$, to obtain the 3-reduced BKP hierarchy [8]. More generally (see also [8]), when $n = 2k+1 > 1$ is odd, the $2k+1$-reduced hierarchy is related to the twisted loop group $G_{2k+1}^{(2)}$ corresponding to the twisted affine Lie algebra $sl_{2k+1}^{(2)}$. When $n = 2k > 2$ is even, one has the twisted loop group $G_{2k}^{(2)}$ corresponding to the affine Lie algebra $so_{2k}^{(2)}$ [9,13] (see [14] (Chapter 7) for the construction of these Lie algebras). Elements G in this twisted loop group not only satisfy (18), which implies $\sum_{j\in\mathbb{Z}}(-1)^j a_{kj}a_{\ell,-j} = (-1)^k\delta_{k,-\ell}$, but also the n-periodicity condition $a_{i+n,j+n} = a_{ij}$. This means that these group elements also commute with the operator

$$\sum_{i\in\mathbb{Z}}(-1)^{pn-i}\phi_i\otimes\phi_{pn-i}, \quad \text{for } p = 1,2,3,\ldots,$$

namely

$$\begin{aligned}(G\otimes G)\sum_{i\in\mathbb{Z}}(-1)^{pn-i}\phi_i\otimes\phi_{pn-i} &= \sum_{i\in\mathbb{Z}}(-1)^{pn-i}G\phi_iG^{-1}\otimes G\phi_{pn-i}G^{-1}G \\ &= \sum_{i,j,k\in\mathbb{Z}}(-1)^{pn-i}a_{ji}a_{k,pn-i}\phi_jG\otimes\phi_k\,G \\ &= \sum_{i,j,k\in\mathbb{Z}}(-1)^{pn-i}a_{ji}a_{k-pn,-i}\phi_jG\otimes\phi_k\,G \\ &= \sum_{j\in\mathbb{Z}}(-1)^{pn-j}\phi_jG\otimes\phi_{pn-j}G \\ &= \sum_{j\in\mathbb{Z}}(-1)^{pn-j}\phi_j\otimes\phi_{pn-j}(G\otimes G).\end{aligned}$$

Since $\sum_{i\in\mathbb{Z}}(-1)^{pn-i}\phi_i|0\rangle\otimes\phi_{pn-i}|0\rangle = 0$, we find that the elements τ in the orbit of the vacuum vector of this twisted loop not only satisfy (15), but also satisfy the conditions

$$\sum_{j\in\mathbb{Z}}(-1)^{pn-j}\phi_j\tau\otimes\phi_{pn-j}\tau = 0, \quad p = 1,2,\ldots. \tag{42}$$

This means that $\tau(\tilde{t}) = \sigma(\tau)$ not only satisfies (24), but also the conditions

$$\mathrm{Res}_z z^{pn-1}e^{\sum_{j=1}^{\infty}(t_{2j-1}-t'_{2j-1})z^{2j-1}}e^{\sum_{j=1}^{\infty}2\left(\frac{\partial}{\partial t'_{2j-1}}-\frac{\partial}{\partial t_{2j-1}}\right)\frac{z^{-2j+1}}{2j-1}}\tau(\tilde{t})\tau(\tilde{t}')dz = 0, \quad p = 1,2,\ldots. \tag{43}$$

From (43), one deduces, using the fundamental Lemma ([11], Lemma 4.1) and the first equation of (37), that

$$(P(\tilde{t},\partial)\partial^{pn-1}P(\tilde{t},\partial)^*)_{<0} = (P(\tilde{t},\partial)\partial^{pn}P(\tilde{t},\partial)^{-1}\partial^{-1})_{<0} = 0.$$

Thus, the Lax operator $L(\tilde{t},\partial)$ satisfies

$$(L(\tilde{t},\partial)^{pn})_{\leq 0} = 0, \quad p = 1,2,\ldots. \tag{44}$$

Hence, $\mathcal{L}(\tilde{t},\partial) = L(\tilde{t},\partial)^n$ is a monic differential operator with zero constant term. Moreover, $\mathcal{L}(\tilde{t},\partial)$ is equal to (10), and, by the first formula of (37), we have the relation (11).

Now, if n is odd, one can use the the Sato–Wilson equation, i.e., the second equation of (37), to find that

$$\frac{\partial P(\tilde{t},\partial)}{\partial t_{(2j-1)n}} = 0, \quad j = 1,2,\ldots.$$

From this we find that the tau-function satisfies

$$\frac{\partial\tau(\tilde{t})}{\partial t_{(2j-1)n}} = \lambda_j\tau(\tilde{t}), \quad \lambda_j\in\mathbb{C}, \quad \text{for } j = 1,2,\ldots. \tag{45}$$

Since we consider only polynomial tau-functions, we find that for odd n:

$$\frac{\partial\tau(\tilde{t})}{\partial t_{(2j-1)n}} = 0, \quad j = 1,2,\ldots. \tag{46}$$

If n is even, there is no such restriction, because the Sato–Wilson Equation (37) is only defined for odd flows. However, the additional Equation (43) still holds and gives additional constraints on the tau-function.

Proposition 1. *For n odd, Equation (46) for $j = 1$ and the BKP hierarchy (24) on $\tau(\tilde{t})$ are equivalent to (24) and (43).*

Proof. We only have to show that (46) for $j = 1$ and (24) imply (43). For this, differentiate (24) by t_n and use (46); this gives Equation (43) for $p = 1$. Next, differentiate (43) for $p = 1$ again by t_n and use again (46); this gives (43) for $p = 2$, etc. □

Remark 3. *If n is odd, Proposition 1 gives that a polynomial BKP tau-function is n-th Sawada–Kotera tau-function if and only if τ satisfies $\frac{\partial \tau}{\partial t_n} = 0$.*

Since L satisfies the BKP hierarchy, $\mathcal{L} = L^n$ also satisfies the BKP hierarchy. For $n = 3$, assuming the constraint that $\mathcal{L}$ is a differential operator, $\mathcal{L}$ is given by (3) and $\frac{\partial \mathcal{L}}{\partial t_3} = [(\mathcal{L}^{\frac{5}{3}})_{\geq 0}, \mathcal{L}]$ is the Sawada–Kotera equation (33). This leads to the following definition.

Definition 1. *Let $\mathcal{L} = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_1\partial$ be a differential operator, satisfying (11). The system of Lax equations*

$$\frac{\partial \mathcal{L}(\tilde{t}, \partial)}{\partial t_{2j-1}} = \left[\left(\mathcal{L}(\tilde{t}, \partial)^{\frac{2j-1}{n}} \right)_{\geq 0}, \mathcal{L}(\tilde{t}, \partial) \right], \quad j = 1, 2, \ldots, \tag{47}$$

is called the n-th reduced BKP hierarchy or the n-th Sawada–Kotera hierarchy. For $n = 3$, it is called the Sawada–Kotera hierarchy.

The geometric meaning of Equation (42) is that the space $\operatorname{Ann} \tau$ is invariant under the shift Λ_n, where

$$\Lambda_n(\phi_i) = \phi_{i+n}. \tag{48}$$

As in the $SO_{\infty,odd}$ case, all polynomial tau-functions in this n-reduced case lie in some Schubert cell. Such a Schubert cell has a "lowest" element w_λ, for λ a certain strict partition. This element can be obtained from the vacuum vector by the action of the Weyl group corresponding to $G_n^{(2)}$. The element

$$w_\lambda = \phi_{-\lambda_1}\phi_{-\lambda_2} \cdots \phi_{-\lambda_k}|0\rangle \tag{49}$$

lies in the Weyl group orbit of $|0\rangle$, corresponding to $SO_{\infty,odd}$, (see [15]), however, not all such elements lie in the Weyl group orbit of $|0\rangle$ for $G_n^{(2)}$. For this, consider

$$\operatorname{Ann} w_\lambda = \operatorname{span}\{\phi_{-\lambda_1}, \phi_{-\lambda_2}, \ldots, \phi_{-\lambda_k}\} \oplus \operatorname{span}\{\phi_i | \, i > 0, i \neq \lambda_j, \; j = 1, \ldots, k\}. \tag{50}$$

The element w_λ lies in the $G_n^{(2)}$ Weyl group orbit if and only if $\operatorname{Ann} w_\lambda$ is invariant under the action of Λ_n, which means that the ($\lambda_1 + 1$ shifted) set

$$V_\lambda = \{\lambda_1 + \lambda_i + 1 |, \; i = 1, \ldots, k\} \cup \{\lambda_1 - j + 1 | \, 0 < j < \lambda_1, \; j \neq \lambda_i \text{ for } i = 1, \ldots, k\}, \tag{51}$$

must satisfy the $-n$ shift condition, i.e.,

$$\text{if } \mu_j \in V_\lambda, \text{ then } \mu_j - n \in V_\lambda \text{ or } \mu_j - n \leq 0. \tag{52}$$

Only the elements w_λ, for which the corresponding V_λ satisfies condition (52), lie in the $G_n^{(2)}$ group orbit.

Example 1. *(a) For $n = 2$, the only strict partition λ that satisfies condition (52) is $\lambda = \emptyset$.*
(b) For $n = 3$, the only strict partitions λ that satisfy condition (52) are

$$(3m+1, 3m-2, 3m-5, \ldots, 4, 1) \text{ and } (3m+2, 3m-1, 3m-4, \ldots, 5, 2), \quad m \in \mathbb{Z}_{\geq 0}.$$

Remark 4. *Note that (52) means that λ is a strict partition that is the union of strict partitions $(nm + a_i, n(m-1) + a_i, \ldots, n + a_i, a_i)$, with $1 \leq a_i < n$ and $1 \leq i < n$, such that $a_j - n \neq -a_\ell$. In other words, $a_j + a_\ell \neq n$. Hence there are at most $\left[\frac{n}{2} - 1\right]$ such a_i.*

To a strict partition λ that satisfies condition (52), the corresponding Schubert cell is then obtained through the action on a w_λ by an upper-triangular matrix in the group $G_n^{(2)}$. This produces, up to a constant factor, elements

$$v_\lambda = v_1 v_2 \cdots v_k |0\rangle, \text{ where } v_j = \phi_{-\lambda_j} + \sum_{i \geq 1-\lambda_j} a_{ij}\phi_i \text{ (finite sum)}, \tag{53}$$

and

$$(v_j, v_\ell) = 0, \text{ for } j, \ell = 1, \ldots, k, \text{ and if } \lambda_i = \lambda_j - n, \text{ then } v_i = \Lambda_n(v_j). \tag{54}$$

We first express the constants a_{ij} in terms of other constants by letting

$$a_{ij} = s_{i+\lambda_j}(c_{\overline{\lambda}_j}), \text{ where the } s_i \text{ are elementary Schur polynomials.}$$

Here, we use that

$$1 + \sum_{i=1-\lambda_j}^{\infty} a_{ij} z^{i+\lambda_j} = \exp\left(\sum_{k=1}^{\infty} c_{k,\overline{\lambda}_j} z^k\right),$$

hence, for every $\overline{\lambda}_j$, one can recursively obtain the $c_{k,\overline{\lambda}_j}$. Since $a_{ij} = 0$ for $i >> 0$, one only has a finite number of $c_{k,\overline{\lambda}_j}$. Thus,

$$v_j = \phi_{-\lambda_j} + \sum_{i > -\lambda_j} s_{i+\lambda_j}(c_{\overline{\lambda}_j})\phi_i, \tag{55}$$

where $c_{\overline{\lambda}_j} = (c_{1,\overline{\lambda}_j}, c_{2,\overline{\lambda}_j}, c_{3,\overline{\lambda}_j}, \ldots)$. Here, $\overline{\lambda}_j = \lambda_j \mod n$, which means that there are at most $\left[\frac{n}{2} - 1\right]$ of such infinite series of constants (see Remark 4) and the v_j satisfy the condition

$$\text{if } \lambda_i = \lambda_j - n, \text{ then } v_i = \Lambda_n(v_j). \tag{56}$$

We can now use the isomorphism σ to calculate the bosonization of elements v_λ. For this, we use formula (25) and apply this to v_λ (which is given by (53) with v_j given by (55)). Now, using (22) and (23) and the fact that

$$e^{\sum_{j=1}^{\infty} t_{2j-1} \frac{\partial}{\partial s_{2j-1}}} e^{\sum_{j=1}^{\infty} s_{2j-1} z^{2j-1}} = e^{\sum_{j=1}^{\infty} (t_{2j-1} + s_{j-1}) z^{2j-1}} e^{\sum_{j=1}^{\infty} t_{2j-1} \frac{\partial}{\partial s_{2j-1}}}$$

we find that

$$e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} \phi(z) e^{-\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} = e^{\sum_{j=1}^{\infty} t_{2j-1} z^{2j-1}} \phi(z).$$

Thus, using (55), we find that

$$
\begin{aligned}
v_j(\tilde{t}) :=& e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} v_j e^{-\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} \\
=& e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} (\phi_{-\lambda_j} + \sum_{i\geq 1-\lambda_j} s_{i+\lambda_j}(c_{\overline{\lambda}_j})\phi_i) e^{-\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} \\
=& e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} \mathrm{Res} \sum_{\ell=0}^{\infty} s_\ell(c_{\overline{\lambda}_j}) z^{\ell-\lambda_j}\phi(z) e^{-\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}} \frac{dz}{z} \\
=& \mathrm{Res} \sum_{\ell=0}^{\infty} s_\ell(c_{\overline{\lambda}_j}) z^{\ell-\lambda_j}\phi(z) e^{\sum_{j=1}^{\infty} t_{2j-1}z^{2j-1}} \frac{dz}{z} \\
=& \mathrm{Res}\, z^{-\lambda_j} e^{\sum_{i=1}^{\infty} c_{i,\overline{\lambda}_j} z^i + \sum_{j=1}^{\infty} t_{2j-1}z^{2j-1}} \phi(z)\frac{dz}{z} \\
=& \mathrm{Res}\, z^{-\lambda_j} \sum_{k=0}^{\infty} s_k(\tilde{t} + c_{\overline{\lambda}_j}) z^k \sum_{i\in\mathbb{Z}} \phi_i z^{-i} \frac{dz}{z} \\
=& \phi_{-\lambda_j} + \sum_{i\geq 1-\lambda_j} s_{i+\lambda_j}(\tilde{t} + c_{\overline{\lambda}_j})\phi_i.
\end{aligned}
$$

Since $e^{\sum_{j=1}^{\infty} t_{2j-1}\alpha_{2j-1}}|0\rangle = 0$, we find that the corresponding tau-function is equal to the vacuum expectation value

$$\tau(\tilde{t}) = \langle 0|v_1(\tilde{t})v_2(\tilde{t})\cdots v_k(\tilde{t})|0\rangle. \tag{57}$$

If $k = 2m$, then this is the Pfaffian of a $2m \times 2m$ skew-symmetric matrix. If $k = 2m - 1$, we use the fact that

$$\langle 0|v_1(\tilde{t})v_2(\tilde{t})\cdots v_k(\tilde{t})|0\rangle = 2\langle 0|v_1(\tilde{t})v_2(\tilde{t})\cdots v_k(\tilde{t})\phi_0|0\rangle$$

and again we find a Pfaffian. We thus arrive at the main theorem.

Theorem 2. *All polynomial tau-functions of the n-th Sawada–Kotera hierarchy are, up to a scalar factor, equal to the Pfaffian*

$$\tau_\lambda(\tilde{t}) = Pf\left(\chi_{\lambda_i,\lambda_j}(\tilde{t} + c_{\overline{\lambda}_i}, \tilde{t} + c_{\overline{\lambda}_j})\right)_{1\leq i,j\leq 2m}, \tag{58}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m})$, $m = 0, 1, \ldots$, is an extended strict partition, which satisfies the $-n$ shift condition (52) for (51). The polynomials $\chi_{a,b}$ are given by (40). Here, as before, $\tilde{t} = (t_1, 0, t_3, 0, \ldots)$, and $c_{\overline{\lambda}_i} = (c_{1,\overline{\lambda}_i}, c_{2,\overline{\lambda}_i}, c_{3,\overline{\lambda}_j}, \ldots)$ are arbitrary constants, where we replace, recursively, for all $j = 1, 2, \ldots 2m$ (respectively, for all $j = 1, 2, \ldots 2m - 1$), when $\lambda_{2m} \neq 0$ (resp., $\lambda_{2m} = 0$), the constants $c_{\lambda_j+\lambda_\ell,\overline{\lambda}_j}$, for $j \leq \ell \leq 2m$ (resp., $j \leq \ell < 2m$) as follows:

(1) *If $\overline{\lambda}_j \neq \overline{\lambda}_\ell$, then*

$$
\begin{aligned}
&c_{\lambda_j+\lambda_\ell,\overline{\lambda}_j} = -(-1)^{\lambda_j+\lambda_\ell} \times \\
&\quad s_{\lambda_j+\lambda_\ell}(c_{1,\overline{\lambda}_\ell} - c_{1,\overline{\lambda}_1}, c_{2,\overline{\lambda}_\ell} + c_{2,\overline{\lambda}_1}, \ldots, c_{\lambda_j+\lambda_\ell-1,\overline{\lambda}_\ell} + (-1)^{\lambda_j+\lambda_\ell-1} c_{\lambda_j+\lambda_\ell-1,\overline{\lambda}_j}, c_{\lambda_j+\lambda_\ell,\overline{\lambda}_\ell}).
\end{aligned} \tag{59}
$$

(2) *If $\overline{\lambda}_j = \overline{\lambda}_\ell$ and $\lambda_j + \lambda_\ell$ is even, then*

$$c_{\lambda_j+\lambda_\ell,\overline{\lambda}_j} = -\frac{1}{2} s_{\frac{\lambda_j+\lambda_\ell}{2}}(2c_{2,\overline{\lambda}_j}, 2c_{4,\overline{\lambda}_j}, \cdots, 2c_{\lambda_j+\lambda_\ell-2}, 0). \tag{60}$$

Proof. We still need to use the fact that all vectors v_i, for $i = 1, \dots, k$, form an isotropic subspace. So, assume that $1 \le j \le \ell \le k$ and that v_j and v_ℓ are given by (55). Then

$$\begin{aligned} 0 = (v_j, v_\ell) =& (-1)^{\lambda_j} \sum_{i=0}^{\lambda_j+\lambda_\ell} (-1)^i s_i(c_{\overline{\lambda}_j}) s_{\lambda_j+\lambda_\ell-i}(c_{\overline{\lambda}_\ell}) \\ =& (-1)^{\lambda_j} s_{\lambda_j+\lambda_\ell}(c_{i,\overline{\lambda}_\ell} + (-1)^i c_{i,\overline{\lambda}_j}). \end{aligned} \tag{61}$$

Here, we have used the fact that the coefficient of z^m of

$$e^{\sum_{i=1}^{\infty} x_i z^i + y^i(-z)^i} = e^{\sum_{i=1}^{\infty} x_i z^i} e^{\sum_{i=1}^{\infty} y^i(-z)^i}$$

is equal to $s_m(x_i + (-1)^i y_i) = \sum_{j=0}^{m} (-1)^{m-j} s_j(x) s_{m-j}(y)$.

So, we need to investigate condition (61). Here we have two possibilities, viz. $\overline{\lambda}_j \ne \overline{\lambda}_\ell$ and $\overline{\lambda}_j = \overline{\lambda}_\ell$. If $\overline{\lambda}_j \ne \overline{\lambda}_\ell$, then using the fact that $s_i(x) = x_i +$ terms not containing x_i, we find (59). If $\overline{\lambda}_j = \overline{\lambda}_\ell$, then notice that $s_{\lambda_j+\lambda_\ell}$ only depends on the $c_{i,\overline{\lambda}_j}$ with i even. Thus, all elementary Schur polynomials s_{2i+1} in only the even variables are equal to zero. This means that if $\lambda_j + \lambda_\ell$ is odd, there is no restriction on the constants, but if $\lambda_j + \lambda_\ell$ is even, we find that

$$c_{\lambda_j+\lambda_\ell,\overline{\lambda}_j} = -\frac{1}{2} s_{\lambda_j+\lambda_\ell}(0, 2c_{2,\overline{\lambda}_j}, 0, 2c_{4,\overline{\lambda}_j}, 0, 2c_{6,\overline{\lambda}_j}, \dots, 0, 2c_{\lambda_j+\lambda_\ell-2,\overline{\lambda}_j}, 0, 0).$$

Note that this restriction coincides with (60). □

Remark 5. *Since $\chi_{\lambda_j,\lambda_\ell}$ is given by (40), the constant $c_{2\lambda_1,\overline{\lambda}_1}$ does not appear in (58) and the substitution (60) for $c_{2\lambda_1,\overline{\lambda}_1}$ is void.*

For $n = 3$, see Example 1(b), we only have one infinite series of constants $c_{i,\overline{\lambda}_1}$, which means that we only have the substitutions (60). We therefore find:

Corollary 1. *All polynomial tau-functions of the Sawada–Kotera hierarchy are, up to a non-zero constant factor,*

$$\tau_\lambda(\tilde{t}) = Pf\Big(\chi_{\lambda_i,\lambda_j}(\tilde{t} + c, \tilde{t} + c)\Big)_{1 \le i,j \le 2m}, \tag{62}$$

where λ is one of the following extended strict partitions:

1. $(6m+1, 6m-2, 6m-5, \dots, 4, 1)$;
2. $(6m-2, 6m-5, 6m-8, \dots, 4, 1, 0)$;
3. $(6m+2, 6m-1, 6m-4, \dots, 5, 2)$;
4. $(6m-1, 6m-4, 6m-7, \dots, 5, 2, 0)$,

where $m = 0, 1, \dots$, and $c = (c_1, c_2, c_3, \dots)$ are arbitrary constants in which we substitute recursively, $c_2 = 0$, and $c_8, c_{14}, c_{20}, \dots, c_{12m-4}$ in case 1; $c_2 = 0$, and $c_8, c_{14}, c_{20}, \dots, c_{12m-10}$ in case 2; $c_4, c_{10}, c_{16}, \dots, c_{12m-2}$ in case 3; and $c_4, c_{10}, c_{16}, \dots, c_{12m-8}$ in case 4, respectively, by the following formula

$$c_{2k} = -\frac{1}{2} s_k(2c_2, 2c_4, \dots 2c_{2k-2}, 0) \quad \text{for } k > 1. \tag{63}$$

Remark 6. *If we choose $c = \tilde{c} = (c_1, 0, c_3, 0, c_5, \dots)$, then the corresponding Sawada–Kotera tau-function $\tau_\lambda(\tilde{t})$ is equal, up to a non-zero constant factor, to a Schur Q-function $Q_\lambda(\tilde{t} + \tilde{c})$ (cf. [15]).*

Example 2. *All the Sawada–Kotera tau-functions related to the partition $\lambda = (5,2)$ are given, up to a multiplicative constant, by*

$$c_1^7 + 1120c_7 + 7c_1^6 t_1 - 280c_2^3 t_1 - 280c_5 t_1^2 + 140c_2^2 t_1^3 + t_1^7 + +35c_1^3(2c_2 + t_1^2)^2 + \\ + 7c_1 t_1^5(2c_2 + 3t_1^2) + 35c_1^4(2c_2 t_1 + t_1^3) - 280t_1^2 t_5 - 7c_1^2(40c_5 - 60c_2^2 t_1 - 20c_2 t_1^3 - 3t_1^5 + 40t_5) + \\ + 14c_2(120c_5 + t_1^5 + 120t_5) - 7c_1(40c_2^3 - 60c_2^2 t_1^2 - 10c_2 t_1^4 + t_1(80c_5 - t_1^5 + 80t_5)) + 1120t_7,$$

with $c_i \in \mathbb{C}$, arbitrary. (We have eliminated c_4 by the substitution $c_4 = -c_2^2$; all the other constants that do not appear, disappear automatically).

4. Conclusions

Three classes of solutions of soliton equations have been extensively studied in the literature: rational solutions, soliton solutions, and theta function solutions. Soliton solutions are constructed by making use of vertex operators [2,6,10,11,13,14,16,17]. Theta function solutions are obtained by Krichever's method [18,19]. Rational solutions are obtained by constructing polynomial tau-functions using the group transformations method, introduced by Sato [1]. In the present paper, we describe all polynomial tau-functions for the n-th Sawada–Kotera hierarchy, and, in particular, for the Sawada–Kotera equation.

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