THE LQG/LTR METHOD AND DISCRETE-TIME CONTROL SYSTEMS

by

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B.S., Massachusetts Institute of Technology
(1977)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

MASTER OF SCIENCE

IN MECHANICAL ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1984

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Submitted to the Department of Mechanical Engineering, June 1984, in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering

ABSTRACT

The main question in applying the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR) design methodology to discrete-time systems is whether asymptotic recovery can be achieved in discrete-time. The answer is no. The loop recovery error matrix is developed as a tool for studying this problem. Existing quadratic optimization approaches are examined both as eigenvalue/eigenvector placement techniques and as matrix norm minimizations. The discrete optimal quadratic regulator (DLQR) is derived as the corresponding discrete-time norm minimization. The modal properties of the DLQR solution are shown to be qualitatively similar to the continuous-time results. An expression for the error resulting from the DLQR approach is derived and shown to be dependent on sampling time. A sampled data example of controlling a CH-47 helicopter illustrates this sampling time dependence.

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ACKNOWLEDGEMENTS

I would like to thank my thesis supervisor, Prof. Michael Athans, for his confidence in me in agreeing to undertake this project, for his fine technical guidance, and for his enormously elastic patience.

I am also grateful to Homayoon Kazerooni for the many technical discussions we have had in the past year. Many of the ideas contained in Chapter 3 were born in those discussions and I am certain they had more than one parent.

Finally, I give my deepest thanks to Jennifer Lee, who caught me when I stumbled, scolded me when I was lazy, believed in me when I'd stopped believing myself, and without whose constant and loving support not one word of this document would have ever seen print.

This research was conducted at the MIT Laboratory for Information and Decision Systems with financial support from the National Sciences Foundation under grant NSF/ECS-8210960 and by the NASA Ames and Langley Research Centers under grants NASA/NGL-22-009-124 and NASA/NAG2-297.

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CHAPTER 1: INTRODUCTION

1.1 Background

The linear-quadratic-Gaussian optimization procedure with loop transfer recovery (LQG/LTR) has recently emerged as a powerful design tool for linear, multivariable control systems [1],[2],[3].

The LQG/LTR methodology consists of three steps [1],[2],[3]:

Step 1: Formulate all system performance specifications and stability robustness requirements as limitations on the singular values of the open loop transfer function matrix obtained by breaking the control loop at the plant output;

Step 2: Design a Kalman-Bucy filter (KBF) whose loop transfer function meets the system specs;

Step 3: Recover (i.e. approximate) this KBF loop shape using a quadratic optimal regulator (LQR) adjusted according to the Kwakernaak sensitivity recovery procedure [4].

(The dual to this procedure may also be used. In that case, the loop breaking point for system specs is at the plant input, an LQR design meets these specs, and a KBF, adjusted according to the Doyle and Stein procedure [5], recovers the LQR loop shape).

The resulting compensator has the structure of a full-order observer followed by a state feedback gain matrix. If the observer gain matrix is the solution to an optimal filtering problem, as described in Step (2), then the observer is a KBF with all the attendant desirable loop properties guaranteed by the Kalman frequency domain equality [2]. Hence, the LQG in the name LQG/LTR.

However, any matrix that stabilizes the observer error dynamics and produces a desirable loop shape may be substituted in Step (2). In that case, the term model based compensator with loop transfer recovery (MBC/LTR) more appropriately describes the result.

To date, this MBC/LTR methodology has been developed only for continuous-time systems. Can the technique be extended to discrete-time systems, as well?

The use of the frequency domain for discrete SISO (single input, single output) system design is already well established (see, for example [6],[7]). By using singular values, discrete SISO frequency domain concepts may be generalized to the MIMO (multi-input, multi-output) case, just as in continuous-time. Step (1) of the methodology, therefore, may be readily applied in discrete-time.

As for Step (2), unlike their continuous-time counterparts, discrete Kalman filters provide no significant robustness properties [8]. They do, however, guarantee stability and offer a methodical way of adjusting the loop bandwidth. Thus, Step (2), or an appropriate MBC substitute, may also be readily applied in discrete-time.

The real question in discrete-time MBC/LTR concerns Step (3). In continuous-time, the actual loop shape may be made to approach the target loop shape pointwise in frequency ("asymptotic recovery"). Can a similar procedure be found for discrete-time systems?

1.2 Outline and Scope

The purpose of this thesis is to answer that question and the answer is no.

To arrive at this result we first address the question, "How does continuous-time LTR work?" Chapter 2 presents the mathematical statement of the LTR problem for continuous systems. In Chapter 3, we introduce the loop recovery error matrix and derive necessary and sufficient conditions for exact loop recovery (Theorem 3.2). We then interpret these results in terms of pole/zero cancellations (Theorem 3.7). In general, exact recovery is impossible (Theorem 3.8), but we present these results in order to improve our understanding of the mechanism by which asymptotic recovery occurs.

Chapter 4 discusses the basic issues behind approximate recovery. We present the Kwakernaak asymptotic recovery procedure from two perspectives: as an eigenvalue/eigenvector placement technique (Theorem 4.3) and as a matrix norm minimization (Theorem 4.4).

Chapter 5 presents the mathematical statement of the discrete-time LTR problem. The telling difference from continuous-time is the stability requirement: poles inside the unit circle for discrete systems, versus poles in the left half-plane for continuous ones.

Apart from the stability issue, the discrete, exact recovery question is identical to the continuous one. Chapter 6, therefore, merely summarizes the exact recovery results analogous to those of Chapter 3.

The stability issue, however, makes discrete, asymptotic recovery impossible. Chapter 7 discusses this issue and suggests a matrix norm minimization approach analogous to the Kwakernaak procedure as described in Chapter 4. We, thus, derive the discrete LQR problem (DLQR) as a means of approximating discrete LTR (Theorem 7.1). Again paralleling the development of Chapter 4, we examine the modal properties of the DLQR solution (Theorem 7.2). We see that qualitatively the DLQR approach and Kwakernaak's approach are similar, but that DLQR produces a non-zero recovery error. An expression for this error is then presented (Theorem 7.3) from which we see that recovery error is a function of sampling time for sampled data systems.

Chapter 8 presents some examples to illustrate the key concepts. A SISO example shows how exact recovery works. We then give a MIMO, sampled data example to show the dependence of recovery error on sampling time.

Chapter 9 concludes the thesis with a summary and suggestions for further research.

1.3 Research Contributions

The main contributions of this thesis are:

(1) An equation for the loop recovery error matrix that is useful for examining proposed recovery schemes;

(2) An analysis of the Kwakernaak recovery procedure in light of the above equation;

(3) The demonstration that discrete asymptotic recovery is impossible;

(4) The derivation of the discrete LQR recovery approach as a matrix norm minimization analogous to the Kwakernaak procedure;

(5) The derivation of an expression for the recovery error resulting from the DLQR approach.

CHAPTER 2: CONTINUOUS–TIME PROBLEM DEFINITION

2.1 Introduction

In order to gain insight into the problem of LTR for discrete-time systems, we first examine the problem for continuous-time systems. This chapter presents the mathematical statement of the continuous LTR problem.

2.2 Problem Statement

Figure 2.1 shows the structure of an LQG/LTR control system. We assume a minimal realization, $(\underline{A},\underline{B},\underline{C})$ of the given plant transfer function matrix, $\underline{G}(s)$. That is

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) + \underline{B}\ \underline{d}_I(t) \tag{2.1}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) + \underline{d}_o(t) \tag{2.2}$$

where

$\underline{x}(t) \in R^n \triangleq$ plant state

$\underline{u}(t) \in R^m \triangleq$ control input

$\underline{d}_I(t) \in R^m \triangleq$ input disturbance

$\underline{y}(t) \in R^m \triangleq$ plant output

$\underline{d}_o(t) \in R^m \triangleq$ output disturbance

and $\underline{A},\underline{B},\underline{C}$ are appropriately dimensioned constant matrices. We further assume, for convenience,

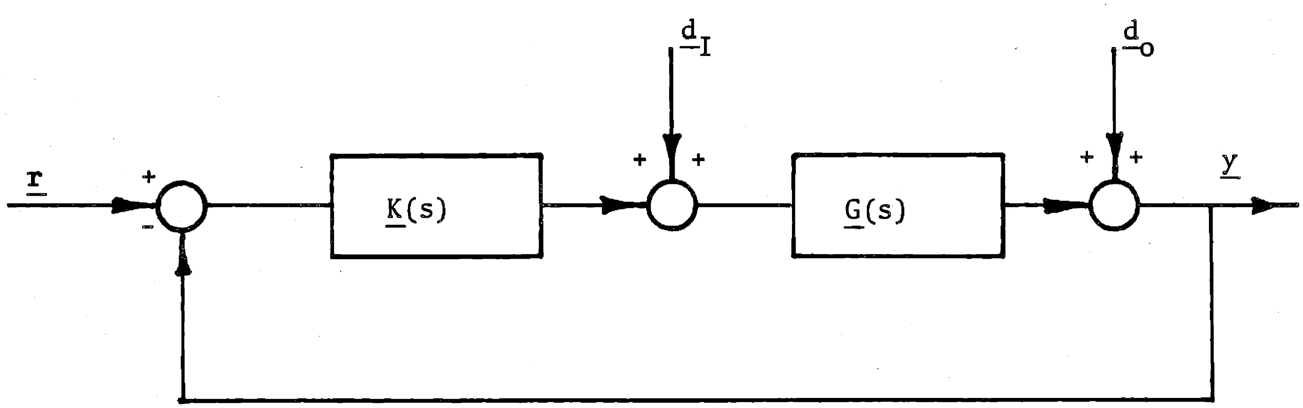

Figure 2.1: LQG/LTR Control System.

$$\text{rank}(\underline{B}) = \text{rank}(\underline{C}) = m \ . \tag{2.3}$$

With these definitions we see that

$$\underline{G}(s) = \underline{C}\ \underline{\Phi}(s)\underline{B} \tag{2.4}$$

where

$$\underline{\Phi}(s) \triangleq (s\underline{I}-\underline{A})^{-1} \tag{2.5}$$

The compensator has the structure of a full-order KBF:

$$\dot{\hat{\underline{x}}}(t) = \underline{A}\ \hat{\underline{x}}(t) + \underline{B}\ \underline{u}(t) - \underline{H}[\underline{C}\ \hat{\underline{x}}(t)-\underline{y}(t)+\underline{r}(t)] \tag{2.6}$$

$$\underline{u}(t) = -\underline{G}\ \hat{\underline{x}}(t) \tag{2.7}$$

where

$\hat{\underline{x}}(t) \in R^n \triangleq$ filter state

$\underline{r}(t) \in R^m \triangleq$ reference input

and $\underline{G}$ and $\underline{H}$ are the LQR and KBF gain matrices, respectively. We see that

$$\underline{K}(s) = \underline{G}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} \ . \tag{2.8}$$

Because of the special structure of the compensator, the closed loop eigenvalues are those of the matrices $\underline{A}-\underline{B}\ \underline{G}$ and $\underline{A}-\underline{H}\ \underline{C}$. Define λ_i, $1\leq i\leq n$, to be the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ and μ_i, $1\leq i\leq n$, the eigenvalues of $\underline{A}-\underline{H}\ \underline{C}$. With these definitions we may pose the following problems:

Problem 1: LTR at Plant Output (LTRO)

Given $\underline{H}$ such that

(i) $\text{Re}(\mu_i) < 0$, for all $1 \leq i \leq n$;

(ii) The singular values of $\underline{C}\ \underline{\Phi}(j\omega)\underline{H}$ meet all design requirements;

find $\underline{G}$ such that

(iii) $\text{Re}(\lambda_i) < 0$, for all $1 \leq i \leq n$;

(iv) $\underline{G}(j\omega)\underline{K}(j\omega) \approx \underline{C}\ \underline{\Phi}(j\omega)\underline{H}$, for all $\omega \in \Omega$,

where Ω is the set of all $0 \leq \omega < \infty$ for which $\underline{G}(j\omega)\underline{K}(j\omega)$ and $\underline{C}\ \underline{\Phi}(j\omega)\underline{H}$ are well-defined (i.e. all required inverses exist).

Problem 2: LTR at Plant Input (LTRI)

Given $\underline{G}$ such that

(i) $\text{Re}(\lambda_i) < 0$, for all $1 \leq i \leq n$;

(ii) The singular values of $\underline{G}\ \underline{\Phi}(j\omega)\underline{B}$ meet all design requirements;

find $\underline{H}$ such that

(iii) $\text{Re}(\mu_i) < 0$, for all $1 \leq i \leq n$;

(iv) $\underline{K}(j\omega)\underline{G}(j\omega) \approx \underline{G}\ \underline{\Phi}(j\omega)\underline{B}$, for all $\omega \in \Omega'$,

where Ω' is the set of all $0 \leq \omega < \infty$ for which $\underline{K}(j\omega)\underline{G}(j\omega)$ and $\underline{G}\ \underline{\Phi}(j\omega)\underline{B}$ are well defined.

Remarks

(1) In Problem 1, the matrix H is given so that the target full state design shown in Figure 2.2 meets the system specifications for command following, disturbance rejection and robustness to uncertainties. In this case, all disturbances and plant uncertainties are referenced to the plant output.

Figure 2.3 shows the analogous target full state design for Problem 2. Here the primary mission is disturbance rejection. All disturbances and plant uncertainties are modelled as occurring at the plant input.

(2) Problems 1 and 2 are dual in the same sense that the LQR and KBF problems are dual. Therefore, in the remainder of this thesis we prove only those theorems which address LTR at the plant output (Problem 1). We merely state the analogous results for LTR at the plant input (Problem 2) and appeal to the principle of duality for proof.

2.3 Concluding Remarks

The MBC structure reduces compensator design to the selection of two matrices- G and H. In Problem 1, for example, the tacit assumption is that most of the creative effort goes into designing H. Success hinges, therefore, on the existence of a routine procedure for designing G. How does one go about finding such a procedure? Chapter 3 lays the groundwork for the answer.

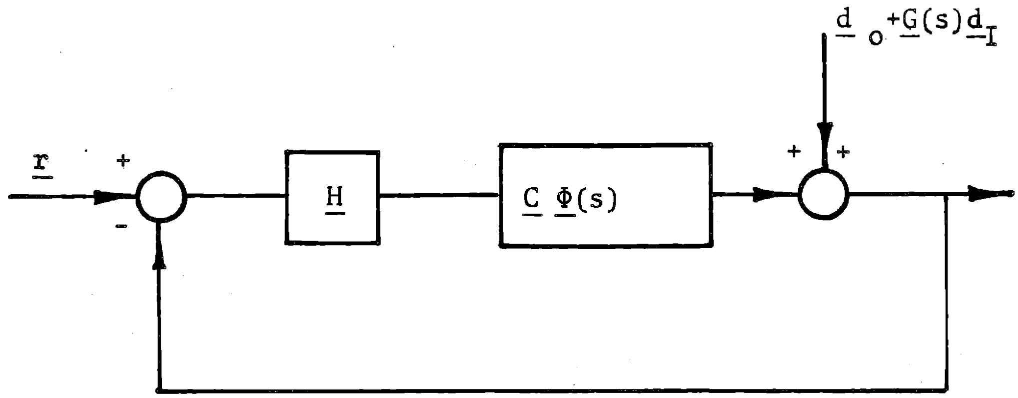

Figure 2.2: Target Full State Design for LTR at Plant Output.

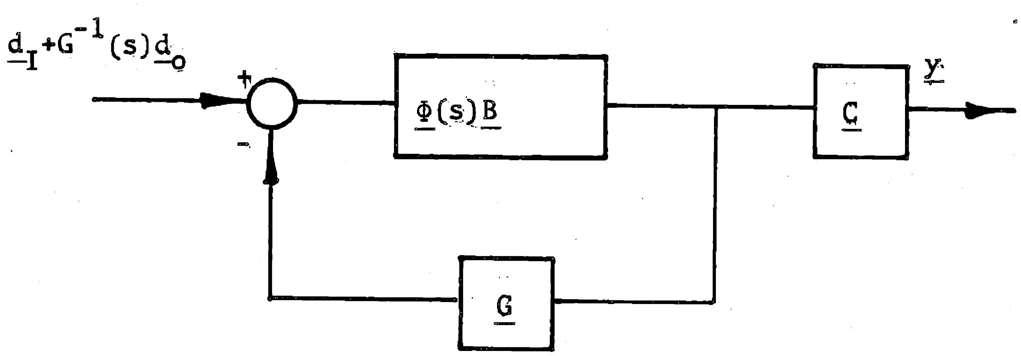

Figure 2.3: Target Full State Design for LTR at Plant Input.

CHAPTER 3: EXACT RECOVERY IN CONTINUOUS TIME

3.1 Introduction

In their seminal 1979 paper [5], Doyle and Stein present the following sufficient condition for exact loop recovery at the plant input:

$$\underline{H}[\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}]^{-1} = \underline{B}[\underline{C}\ \underline{\Phi}(s)\underline{B}]^{-1}\ . \tag{3.1}$$

Since (3.1) is equivalent to $\underline{B}=\underline{0}$, this condition can only be met asymptotically. Doyle and Stein therefore suggest that if $\underline{H}$ can be parametrized by a scalar variable q such that as $q\to\infty$

$$\frac{\underline{H}(q)}{q} \to \underline{B}\ \underline{W}\ , \tag{3.2}$$

for any non-singular $\underline{W}$, then Equation (3.1) holds asymptotically and recovery occurs in the limit. For every value of q along the way, though, the matrix $\underline{A}-\underline{H}(q)\underline{C}$ must be stable. Before the reader even has time to wonder at how unlikely it would be to stumble across just such a parametrization, Doyle and Stein uncover their modified KBF design approach which has exactly the right properties [5].

Equations (3.1) and (3.2) offer a perspective on the recovery problem that is ideally suited for explaining the Doyle/Stein recovery procedure, but rather poorly suited as a starting point for searching for new recovery schemes. In this chapter, therefore, we also consider the problem of exact LTR (i.e.

$$\underline{G}(j\omega)\underline{K}(j\omega) = \underline{C}\ \underline{\Phi}(j\omega)\underline{H}, \qquad \text{for all} \quad \omega\in\Omega \tag{3.3}$$

or, dually,

$$\underline{K}(j\omega)\underline{G}(j\omega) = \underline{G}\ \underline{\Phi}(j\omega)\underline{B}, \quad \text{for all} \quad \omega\epsilon\Omega') \tag{3.4}$$

in order to establish a more useful perspective from which to view the discrete-time recovery problem.

Section 3.2 exposes the structure of the loop recovery error matrix and derives equivalent conditions for exact recovery at the plant output. Section 3.3 shows how these conditions imply certain pole/zero cancellations that support the intuitive idea of how recovery should work. Section 3.4 reinforces the notion that exact recovery is, in general, impossible, but suggests a design criterion for those cases when it is impossible. Finally, Section 3.5 states the corresponding results for recovery at the plant input.

3.2 Equivalent Conditions

Lemma 3.1 and Theorem 3.2 introduce the output recovery error matrix, $E_o(s)$, as a tool for investigating LTR.

Lemma 3.1: Let

$$\underline{E}_o(s) \triangleq \underline{C}\ \underline{\Phi}(s)\underline{H} - \underline{G}(s)\underline{K}(s) \ . \tag{3.5}$$

Then

$$\underline{E}_o(s) = [\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}][\underline{I}+\underline{M}_o(s)]^{-1}\underline{M}_o(s) \tag{3.6}$$

where

$$\underline{M}_o(s) \triangleq \underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} \tag{3.7}$$

Proof:

$$\underline{E}_o(s) \triangleq \underline{C}\ \underline{\Phi}(s)\underline{H}-\underline{G}(s)\underline{K}(s) \tag{3.8}$$

$$= \underline{C}\ \underline{\Phi}(s)\underline{H}-\underline{C}\ \underline{\Phi}(s)\underline{B}\ \underline{G}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} \tag{3.9}$$

$$= \underline{C}\ \underline{\Phi}(s)[\underline{I}-\underline{B}\ \underline{G}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}]\underline{H} \tag{3.10}$$

$$= \underline{C}\ \underline{\Phi}(s)(s\underline{I}-\underline{A}+\underline{H}\ \underline{C})(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} \tag{3.11}$$

$$= [\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}]\underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} \tag{3.12}$$

$$= [\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}]\underline{C}[\underline{I}+(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{C}]^{-1}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} \tag{3.13}$$

$$= [\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}][\underline{I}+\underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}]^{-1}\underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} \tag{3.14}$$

$$= [\underline{I}+\underline{C}\ \underline{\Phi}(s)\underline{H}][\underline{I}+\underline{M}_o(s)]^{-1}\underline{M}_o(s) \tag{3.15}$$

where we get from (3.12) to (3.13) using the identity

$$(\underline{A}+\underline{B})^{-1} = (\underline{I}+\underline{A}^{-1}\underline{B})^{-1}\underline{A}^{-1} \tag{3.16}$$

and from (3.13) to (3.14) with the very handy identity

$$\underline{G}(\underline{I}+\underline{H}\ \underline{G})^{-1} = (\underline{I}+\underline{G}\ \underline{H})^{-1}\underline{G}\ . \tag{3.17}$$

Theorem 3.2: Let $\underline{A}-\underline{B}\ \underline{G}$ be a non-defective[1] matrix with right eigenvectors $\underline{u}_i$, $1\leq i\leq n$, and corresponding left eigenvectors $\underline{v}_i$, $1\leq i\leq n$. Then, with $\underline{E}_o(s)$

---

[1] A matrix is said to be non-defective if it has a complete set of independent eigenvectors.

and $\underline{M}_o(s)$ as defined in Lemma 3.1 the following statements are equivalent:

(i) $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$

(ii) $\underline{M}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$

(iii) $\underline{C}\,\underline{u}_i = \underline{0}$ or $\underline{v}_i^H\,\underline{H} = \underline{0}$,[2] for all $1 \leq i \leq n$

Proof: We show the equivalence of all three statements by proving

Part (a): (i) if and only if (ii)

followed by

Part (b): (ii) if and only if (iii)

Part (a): By Lemma 3.1 we may write

$$\underline{E}_o(j\omega) = [\underline{I}+\underline{C}\,\underline{\Phi}(j\omega)\underline{H}]\,[\underline{I}+\underline{M}_o(j\omega)]^{-1}\underline{M}_o(j\omega) \quad \text{for all} \quad \omega \in \Omega \,. \tag{3.18}$$

(Necessity) If $\underline{M}_o(j\omega) = \underline{0}$ then (3.18) implies $\underline{E}_o(j\omega) = \underline{0}$.

(Sufficiency) Assume $\underline{E}_o(j\omega) = \underline{0}$. Now, $\underline{A}-\underline{H}\,\underline{C}$ stable implies $[\underline{I}+\underline{C}\,\underline{\Phi}(j\omega)\underline{H}]$ is non-singular for all $\omega \in \Omega$. Therefore, (3.18) implies $\underline{M}_o(j\omega) = \underline{0}$.

Part (b): Let λ_i, $1 \leq i \leq n$ be the eigenvalues of $\underline{A}-\underline{B}\,\underline{G}$ and let $\underline{\Lambda} = \text{diag}(\lambda_1,\ldots,\lambda_n)$. Define

$$\underline{U} \triangleq [\underline{u}_1,\ldots,\underline{u}_n]; \quad \underline{V} \triangleq [\underline{v}_1,\ldots,\underline{v}_n] \tag{3.19}$$

with $\underline{u}_i$, $\underline{v}_i$ scaled so that

$$\underline{U}\,\underline{V}^H = \underline{V}^H\underline{U} = I \,. \tag{3.20}$$

[2]The notation $\underline{A}^H$ indicates the Hermitian (i.e. complex conjugate transpose of matrix $\underline{A}$).

It follows that

$$\underline{A}-\underline{B}\ \underline{G} = \underline{U}\ \underline{\Lambda}\ \underline{V}^H \ . \tag{3.21}$$

We now rewrite the definition of $\underline{M}_o(s)$ in matrix residue form:

$$\underline{M}_o(s) = \underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} \tag{3.22}$$

$$= \underline{C}(s\underline{U}\ \underline{V}^H-\underline{U}\ \underline{\Lambda}\ \underline{V}^H)^{-1}\ \underline{H} \tag{3.23}$$

$$= \underline{C}\ \underline{U}(s\underline{I}-\underline{\Lambda})^{-1}\underline{V}^H\underline{H} \tag{3.24}$$

$$= \sum_{i=1}^{n} \frac{\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H}}{s-\lambda_i} \tag{3.25}$$

(Necessity) Assume (iii) holds. Then (3.25) implies that $\underline{M}_o(s) = \underline{0}$ for all s.

(Sufficiency) Assume $\underline{M}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$. Then (3.25) implies

$$\underline{0} = \sum_{i=1}^{n} \frac{\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H}}{j\omega-\lambda_i} \quad \text{for all } \omega\in\Omega \ . \tag{3.26}$$

which implies each term of the sum must vanish individually. Therefore,

$$\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H} = \underline{0} \qquad \text{for all } 1\leq i\leq n \ . \tag{3.27}$$

Let $\underline{\alpha}_i$, $\underline{\beta}_i \in R^m$ such that $\underline{\alpha}_i \triangleq \underline{C}\ \underline{u}_i$ and $\underline{\beta}_i^H \triangleq \underline{v}_i^H\underline{H}$. Then (3.27) implies $\underline{\alpha}_i\ \underline{\beta}_i^H = \underline{0}$ which implies either $\underline{\alpha}_i = \underline{0}$ or $\underline{\beta}_i = \underline{0}$ (or both) for all $i\leq i\leq n$. Therefore, (iii) is true.

Remarks:

(1) Statements (i) and (ii) are equivalent even if $\underline{A}-\underline{B}\ \underline{G}$ is defective. In that case, of course, the modal expansion is not defined so (iii) cannot be asserted.

(2) If the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ are distinct, then (iii) is equivalent to the statement:

(iv) Every mode of $\underline{A}-\underline{B}\ \underline{G}$ is either unobservable from $\underline{C}$ or uncontrollable from $\underline{H}$ (or both).

If the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ are repeated, but $\underline{A}-\underline{B}\ \underline{G}$ is still non-defective, then (iii) is more restrictive than (iv). That is (iii) is a sufficient condition for (iv) but not a necessary one.

3.3 Pole/Zero Cancellation

In general, the loop transfer function, $\underline{G}(s)\underline{K}(s)$, has 2n poles. The target loop transfer function, $\underline{C}\ \underline{\Phi}(s)\underline{H}$, has only n poles - the same n poles as the plant $\underline{G}(s) = \underline{C}\ \underline{\Phi}(s)\underline{B}$. Intuitively, then, exact recovery requires the cancellation of the n extraneous poles of the compensator. Lemmas 3.3 through 3.6 and Theorem 3.7 demonstrate this cancellation.

Definitions: Let λ be an eigenvalue of $\underline{A}-\underline{B}\ \underline{G}$ with corresponding eigenvector $\underline{\beta} \in R^m$, $\underline{\beta} \neq 0$, such that $\underline{C}\ \underline{\beta} = \underline{0}$. Let μ be an eigenvalue of $\underline{A}-\underline{B}\ \underline{G}$ with corresponding left eigenvector $\underline{\gamma} \in R^m$, $\underline{\gamma} \neq 0$, such that $\underline{\gamma}^H\underline{H} = 0$.

Lemma 3.3: λ is an eigenvalue of $\underline{K}(s)$ with corresponding eigenvector $\underline{\beta}$.

Proof: By definition,

$$(\underline{A}-\underline{B}\ \underline{G})\underline{\beta} = \lambda\underline{\beta}\ . \tag{3.28}$$

Then

$$\underline{0} = \underline{C}\ \underline{\beta} = \underline{H}\ \underline{C}\beta \tag{3.29}$$

implies

$$(\underline{A}-\underline{B}\ \underline{G}-\underline{H}\ \underline{C})\underline{\beta} = \lambda\underline{\beta}\ . \tag{3.30}$$

Lemma 3.4: There exists a number, ξ, such that ξ is a transmission zero of $\underline{G}(s)$ and $\xi = \lambda$.

Proof: Rewrite (3.28) as

$$\underline{\beta} = -(\lambda\underline{I}-\underline{A})^{-1}\underline{B}\ \underline{G}\ \underline{\beta} \neq 0 \tag{3.31}$$

Now

$$0 = \underline{C}\ \underline{\beta} = -\underline{C}(\lambda\underline{I}-\underline{A})^{-1}\underline{B}\ \underline{G}\ \underline{\beta} \tag{3.32}$$

and (1) implies $\underline{G}\ \underline{\beta} \neq 0$. Therefore

$$\det[\underline{C}(\lambda\underline{I}-\underline{A})^{-1}\underline{B}] = 0 \tag{3.33}$$

and λ is a transmission zero of G(s).

Lemma 3.5: The eigenvalue of $\underline{K}(s)$, λ, cancels the transmission zero of $\underline{G}(s)$, ξ.

Proof: For multivariable systems Lemmas 3.3 and 3.4 are not enough to prove pole/zero cancellation. Loss of either controllability or observability is also required. We therefore show that the mode corresponding to λ in the product $\underline{G}(s)\underline{K}(s)$ is unobservable.

Define the state equations for $\underline{G}(s)\underline{K}(s)$ by:

$$\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} = \begin{bmatrix} \underline{A} & -\underline{B}\ \underline{G} \\ \underline{0} & \underline{A}-\underline{B}\ \underline{G}-\underline{H}\ \underline{C} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ -\underline{H} \end{bmatrix} \underline{Y} \qquad (3.34)$$

$$\underline{y} = [\underline{C}\quad \underline{0}] \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \qquad (3.35)$$

Since λ is a pole of $\underline{G}(s)\underline{K}(s)$ we may write

$$\begin{bmatrix} \lambda\underline{I}-\underline{A} & \underline{B}\ \underline{G} \\ \underline{0} & \lambda\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C} \end{bmatrix} \begin{bmatrix} \underline{\alpha} \\ \hat{\underline{\alpha}} \end{bmatrix} = \underline{0} \qquad (3.36)$$

for some eigenvector comprised of $\underline{\alpha}$, $\hat{\underline{\alpha}} \in R^n$. We expand (3.36) as

$$(\lambda\underline{I}-\underline{A})\underline{\alpha} + \underline{B}\ \underline{G}\ \hat{\underline{\alpha}} = \underline{0} \qquad (3.37)$$

$$(\lambda\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})\hat{\underline{\alpha}} = \underline{0} \qquad (3.38)$$

From Lemma 3.3 and (3.38) we deduce $\hat{\underline{\alpha}} = \underline{\beta}$. Then (3.37) implies

$$\underline{\alpha} = -(\lambda\underline{I}-\underline{A})^{-1}\underline{B}\ \underline{G}\ \hat{\underline{\alpha}} \tag{3.39}$$

$$= -(\lambda\underline{I}-\underline{A})^{-1}\underline{B}\ \underline{G}\ \underline{\beta} \tag{3.40}$$

$$= \underline{\beta} \text{ (see proof of Lemma 3.4)} \tag{3.41}$$

We see, therefore, that

$$[\underline{C} \quad \underline{0}] \begin{bmatrix} \underline{\alpha} \\ \hat{\underline{\alpha}} \end{bmatrix} = \underline{C}\ \underline{\alpha} = \underline{C}\ \underline{\beta} = \underline{0}\ , \tag{3.42}$$

so that the mode corresponding to λ is unobservable and cancels ξ.

Lemma 3.6: μ is an eigenvalue of $\underline{K}(s)$ with corresponding left eigenvector $\underline{\gamma}$, and μ cancels a zero of $\underline{K}(s)$.

Proof: We prove cancellation by showing the mode corresponding to μ is uncontrollable in $\underline{K}(s)$. By definition,

$$\underline{\gamma}^H(\underline{A}-\underline{B}\ \underline{G}) = \mu\underline{\gamma}^H \tag{3.43}$$

Then

$$\underline{0} = \underline{\gamma}^H\underline{H} = \underline{\gamma}^H\underline{H}\ \underline{C} \tag{3.44}$$

implies

$$\underline{\gamma}^H(\underline{A}-\underline{B}\ \underline{G}-\underline{H}\ \underline{C}) = \mu\underline{\gamma}^H \tag{3.45}$$

so that $\underline{\gamma}^H$ is a left eigenvector of $\underline{A}-\underline{B}\ \underline{G}-\underline{H}\ \underline{C}$. Therefore,

$$\underline{K}(s) = \underline{G}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} \tag{3.46}$$

and $\underline{\gamma}^H\underline{H}=0$ imply the mode corresponding to μ is uncontrollable.

Theorem 3.7: If $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega\epsilon\Omega$, and $\underline{A}-\underline{B}\ \underline{G}$ is non-defective, then every eigenvalue of $\underline{K}(s)$ cancels either a zero of $\underline{G}(s)$ or a zero of $\underline{K}(s)$.

Proof: Assume $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega\epsilon\Omega$, and that $\underline{A}-\underline{B}\ \underline{G}$ is non-defective. If $\underline{u}_i$ and $\underline{v}_i$, $1\underline{<}i\underline{<}n$ are the right and left eigenvectors of $\underline{A}-\underline{B}\ \underline{G}$, then Theorem 3.2 implies that for every eigenvalue λ_i, $1\underline{<}i\underline{<}n$, either $\underline{C}\ \underline{u}_i = \underline{0}$ or $\underline{v}_i^H\underline{H} = \underline{0}$. Lemma 3.5 and 3.6, therefore, imply the required cancellations.

Remarks:

(1) We see from Theorem 3.7 that in order to achieve exact recovery the compensator must effect a partial inversion of the plant. That is, the poles of the plant are left untouched (they are the desired open loop poles), while the plant zeroes are cancelled by compensator poles and the compensator zeroes provide the proper loop shaping.

(2) If the plant has right-half-plane zeroes, then exact recovery conflicts with the requirement of asymptotic stability since compensator poles must cancel those zeroes.

3.4 Exactly Recoverable Loop Shapes

Exact loop transfer recovery is possible only for a restricted class of $\underline{H}$ matrices. Theorem 3.8 defines this class and suggests a design criterion for $\underline{G}$ to achieve exact recovery.

Theorem 3.8: Let $\underline{A}-\underline{B}\ \underline{G}$ be a non-defective matrix with right eigenvectors $\underline{u}_i$, $1\leq i\leq n$, such that $\underline{C}\ \underline{u}_i = \underline{0}$ for all $1\leq i\leq n-m$.
Then the following statements are equivalent:

(i) $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$

(ii) $\underline{C}\ \underline{H} = 0$

Proof: Define the corresponding left eigenvectors $\underline{v}_i$, $1\leq i\leq n$, and matrices $\underline{U}$ and $\underline{V}$ such that

$$\underline{U} \triangleq [\underline{u}_1,\ldots,\underline{u}_n]; \quad \underline{V} \triangleq [\underline{v}_1,\ldots,\underline{v}_n] \tag{3.47}$$

scaled so that

$$\underline{U}\ \underline{V}^H = \underline{V}^H\underline{U} = \underline{I}\ . \tag{3.48}$$

(Sufficiency) Assume $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$. Then Theorem 3.2 implies

$$\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H} = \underline{0} \quad \text{for all} \quad 1\leq i\leq n \tag{3.49}$$

Therefore,

$$\underline{0} = \sum_{i=1}^{n} \underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H} \tag{3.50}$$

$$= \underline{C}\left[\sum_{i=1}^{n} \underline{u}_i\underline{v}_i^H\right]\underline{H} \tag{3.51}$$

$$= \underline{C}\ \underline{U}\ \underline{V}^H\underline{H} \tag{3.52}$$

$$= \underline{C}\ \underline{H} \tag{3.53}$$

(Necessity) Assume $\underline{C}\,\underline{H} = \underline{0}$. Partition $\underline{U}$ such that

$$\underline{U} = [\underline{U}_1\ \underline{U}_2] \tag{3.54}$$

where

$$\underline{U}_1 = [\underline{u}_1\ \ldots \underline{u}_{n-m}];\quad \underline{U}_2 = [\underline{u}_{n-m+1} \ldots\ \underline{u}_n] \tag{3.55}$$

so that $\underline{C}\,\underline{u}_i = \underline{0}$ for all $1\leq i\leq n-m$ implies $\underline{C}\,\underline{U}_1 = \underline{0}$.
Partition $\underline{V}$, conformable with $\underline{U}$, as

$$\underline{V} = [\underline{V}_1\ \underline{V}_2] \tag{3.56}$$

where

$$\underline{V}_1 = [\underline{v}_1 \ldots \underline{v}_{n-m}];\quad \underline{V}_2 = [\underline{v}_{n-m+1} \ldots \underline{v}_n]\ . \tag{3.57}$$

Then,

$$\underline{0} = \underline{C}\,\underline{H} \tag{3.58}$$

$$= \underline{C}\,\underline{U}\,\underline{V}^H\underline{H} \tag{3.59}$$

$$= \underline{C}\,\underline{U}_1\underline{V}_1^H\underline{H} + \underline{C}\,\underline{U}_2\underline{V}_2^H\underline{H} \tag{3.60}$$

$$= \underline{C}\,\underline{U}_2\underline{V}_2^H\underline{H} \tag{3.61}$$

Now, since rank $(\underline{C})=m$, the vectors $\underline{u}_i$, $1\leq i\leq n-m$ are a basis for the null space of $\underline{C}$. The remaining eigenvectors $\underline{u}_i$, $n-m+1\leq i\leq n$, are independent and are not in the null space of $\underline{C}$ which implies $\underline{C}\,\underline{U}_2$ is nonsingular. Equation (3.61) therefore yields $\underline{V}_2^H\underline{H} = \underline{0}$ which implies $\underline{v}_i^H\underline{H} = \underline{0}$ for all $n-m+1\leq i\leq n$. By Theorem 3.2, then, $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega\in\Omega$.

Remarks

(1) Condition (ii) is necessary for exact recovery even without the condition $\underline{C}\,\underline{u}_i = \underline{0}$, $1 \leq i \leq n-m$. For arbitrary $\underline{H}$, therefore, exact recovery is impossible. However, should we be lucky enough to have $\underline{C}\,\underline{H} = \underline{0}$, the conditions of the theorem present a design criterion for $\underline{G}$. That is, we must choose $\underline{G}$ so that $\underline{A}-\underline{B}\,\underline{G}$ is non-defective with n-m eigenvectors in the null space of $\underline{C}$. The left eigenvectors of the remaining m modes are automatically in the left null space of $\underline{H}$ thus satisfying condition (iii) of Theorem 3.2.

(2) How severe a restriction is $\underline{C}\,\underline{H} = \underline{0}$? We answer this question by considering the SISO case. Suppose the plant is realized in observable canonical form. That is

$$\underline{A} = \begin{bmatrix} 0 & 0 & & & -a_o \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \cdot & & \vdots \\ \vdots & 0 & & \cdot & \vdots \\ \vdots & \vdots & & \cdot & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tag{3.62}$$

$$\underline{C} = [0\ 0\ \cdots\cdots\cdots\ 0\ 1] \tag{3.63}$$

$$\underline{H} = \begin{bmatrix} h_o \\ \vdots \\ h_{n-2} \\ h_{n-1} \end{bmatrix} \tag{3.64}$$

The zeroes of $\underline{C}\ \underline{\Phi}(s)\underline{H}$ are the roots of the polynomial

$$h_{n-1}s^{n-1}+\ldots+h_o = 0 \tag{3.65}$$

If we impose the restriction (ii) we have

$$0 = \underline{C}\ \underline{H} = h_{n-1} \tag{3.66}$$

so the numerator of $\underline{C}\ \underline{\Phi}(s)\underline{H}$ is constrained to have order less than or equal to n-2. At high frequencies, therefore, the slope of $\underline{C}\ \underline{\Phi}(j\omega)\underline{H}$ is at least as steep as -40 db/decade. Thus, any $\underline{H}$ which is a solution to a KBF problem is disqualified since every KBF loop rolls off with a slope of only -20 db/decade.

(3) We see from Theorem 3.7 that in order to achieve exact recovery we must "hide" all of the poles of $\underline{K}(s)$ underneath zeroes of $\underline{G}(s)\underline{K}(s)$. This hiding "uses up" zeroes that might otherwise be used to shape the loop. The requirement $\underline{C}\ \underline{H} = \underline{0}$ ensures that the remaining number of zeroes is adequate to reproduce the desired loop shape, $\underline{C}\ \underline{\Phi}(j\omega)\underline{H}$.

For example, consider the case where rank$(\underline{C}\ \underline{B})$ = rank$(\underline{G}\ \underline{H})$=m. The total number of zeroes of $\underline{G}(s)\underline{K}(s)$ is then 2(n-m) which is the maximum any cascade of two mxm, nth order systems may have. If we use up n zeroes as hiding places, then n-2m zeroes are left for shaping the loop. Suppose rank$(\underline{C}\ \underline{H})$=$r\geq 0$. Then the number of zeroes of $\underline{C}\ \underline{\Phi}(s)\underline{H}$ may be as high as n-2m+r. But with only n-2m zeroes left for shaping the loop we may find ourselves r zeroes short. By requiring r=0 (in other words, $\underline{C}\ \underline{H} = \underline{0}$) we preclude this possibility.

3.5 Summary of LTRI Results

We present eight lemmas and theorems, analogous to those of Sections 3.2 through 3.4, for the problem of exact LTR at the plant input.

Lemma 3.9: Let

$$\underline{E}_I(s) \triangleq \underline{G}\ \underline{\Phi}(s)\underline{B} - \underline{K}(s)\underline{G}(s) \tag{3.67}$$

Then

$$\underline{E}_I(s) = \underline{M}_I(s)[\underline{I}+\underline{M}_I(s)]^{-1}[\underline{I}+\underline{G}\ \underline{\Phi}(s)\underline{B}] \tag{3.68}$$

where

$$\underline{M}_I(s) \triangleq \underline{G}(s\underline{I}-\underline{A}+\underline{H}\ \underline{C})^{-1}\underline{B} \tag{3.69}$$

Proof: Dual of proof of Lemma 3.1.

Theorem 3.10: Let $\underline{A}-\underline{H}\ \underline{C}$ be a non-defective matrix with right eigenvectors $\underline{u}_i$, $1\leq i\leq n$, and corresponding left eigenvectors $\underline{v}_i$, $1\leq i\leq n$. Then, with $\underline{E}_I(s)$ and $\underline{M}_I(s)$ as defined in Lemma 3.9 the following statements are equivalent:

(i) $\underline{E}_I(j\omega) = \underline{0}$ for all $\omega\in\Omega'$

(ii) $\underline{M}_I(j\omega) = \underline{0}$ for all $\omega\in\Omega'$

(iii) $\underline{G}\ \underline{u}_i = \underline{0}$ or $\underline{v}_i^H\underline{B} = \underline{0}$, for all $1\leq i\leq n$

Proof: Dual to proof of Theorem 3.2.

Definitions: Let λ be an eigenvalue of $\underline{A}$-$\underline{H}$ $\underline{C}$ with corresponding eigenvector $\underline{\beta} \in R^m$, $\underline{\beta} \neq 0$, such that $\underline{G}$ $\underline{\beta} = 0$. Let μ be an eigenvalue of $\underline{A}$ - $\underline{H}$ $\underline{C}$ with corresponding left eigenvector $\underline{\gamma} \in R^m$, $\underline{\gamma} \neq 0$, such that $\underline{\gamma}^H\underline{B} = 0$.

Lemma 3.11: μ is an eigenvalue of $\underline{K}(s)$ with corresponding left eigenvector $\underline{\gamma}$.

Lemma 3.12: There exists a number, ζ, such that ζ is a transmission zero of $\underline{G}(s)$ and $\zeta = \mu$.

Lemma 3.13: The eigenvalue of $\underline{K}(s)$, μ, cancels the transmission zero of $\underline{G}(s)$, ζ.

Lemma 3.14: λ is an eigenvalue of $\underline{K}(s)$ with corresponding eigenvector $\underline{\beta}$, and λ cancels a zero of $\underline{K}(s)$.

Proof: Dual to proofs of Lemma 3.3 through 3.6.

Theorem 3.15: If $\underline{E}_I(j\omega) = \underline{0}$ for all $\omega \in \Omega'$, and $\underline{A}$-$\underline{H}$ $\underline{C}$ is non-defective, then every eigenvalue of $\underline{K}(s)$ cancels either a zero of $\underline{G}(s)$ or a zero of $\underline{K}(s)$.

Proof: Dual to proof of Theorem 3.7.

Theorem 3.16: Let $\underline{A}$-$\underline{H}$ $\underline{C}$ be a non-defective matrix with left eigenvectors $\underline{v}_i$, $1 \leq i \leq n$ such that $\underline{v}_i^H\underline{B} = \underline{0}$ for all $1 \leq i \leq n-m$. Then the following statements are equivalent:

(i) $\underline{E}_I(j\omega) = \underline{0}$, for all $\omega \in \Omega'$

(ii) $\underline{G}$ $\underline{B} = \underline{0}$.

Proof: Dual to proof of Theorem 3.8.

3.6 Concluding Remarks

The chief purpose in discussing exact recovery has been to expose the basic mechanism by which recovery occurs. We have seen that for arbitrary $\underline{H}$, exact output recovery is impossible. The next chapter explores the question of approximate recovery.

CHAPTER 4: APPROXIMATE RECOVERY IN CONTINUOUS TIME

4.1 Introduction

This chapter discusses approximations to loop transfer recovery for those loop shapes which are not exactly recoverable. Section 4.2 discusses the basic issues in terms of the matrix, $\underline{M}_o(s)$, introduced in Chapter 3. Section 4.3 presents Kwakernaak's method of asymptotic recovery; Section 4.4 re-discovers this method by seeking to minimize an L_2[1]-norm of $\underline{M}_o(s)$. Dual results for the LTRI problems are presented in Section 4.5.

4.2 Basic Issues

This section discusses the basic issues in approximating LTR for continuous-time systems. Theorems 4.1 and 4.2 establish some useful background material.

Theorem 4.1: If $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$, then

$$\phi_K(s) = \phi_R(s) \text{ for all } s$$

where

$$\phi_K(s) \triangleq \det(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})$$

$$\triangleq \text{characteristic polynomial of the compensator}$$

$$\phi_R(s) \triangleq \det(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})$$

$$\triangleq \text{characteristic polynomial of the regulator dynamics}$$

[1] L_2 denotes the space of square-integrable functions.

Proof: Assume $\underline{E}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$.

Then, by Theorem 3.2 $\underline{M}_o(j\omega) = \underline{0}$ for all $\omega \in \Omega$. Therefore,

$$\underline{I} = \underline{I} + \underline{M}_o(j\omega) \quad \text{for all } \omega \in \Omega. \tag{4.1}$$

Taking determinants on both sides yields

$$1 = \det[\underline{I}+\underline{M}_o(j\omega)] \tag{4.2}$$

$$= \det[\underline{I}+\underline{C}(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}] \tag{4.3}$$

$$= \det[\underline{I}+(j\omega I-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{C}] \tag{4.4}$$

$$= \det[(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})] \tag{4.5}$$

$$= \det(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})/\det(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G}) \tag{4.6}$$

$$= \phi_K(j\omega)/\phi_R(j\omega) \quad \text{for all } \omega \in \Omega \tag{4.7}$$

Therefore, $\phi_K(s) = \phi_R(s)$ for all s.

Theorem 4.2: If m=1, then $E_o(j\omega) = 0$, for all $\omega \in \Omega$, if and only if $\phi_K(s) = \phi_R(s)$, for all s.

Proof: (Sufficiency) Assume $E_o(j\omega) = 0$ for all $\omega \in \Omega$. Then, by Theorem 4.1, $\phi_K(s) = \phi_R(s)$.

(Necessity) Assume $\phi_K(s) = \phi_R(s)$, for all s.

Define the following polynomials.

$\phi_G(s) \triangleq \det(s\underline{I}-A)$

$\triangleq$ characteristic polynomial of the plant

$$\phi_F(s) \triangleq \det(s\underline{I}-\underline{A}+\underline{H}\ \underline{C})$$

$\triangleq$ characteristic polynomial of the KBF error dynamics,

and recall the identities:

$$1 + \underline{C}(s\underline{I}-\underline{A})^{-1}\underline{H} = \phi_F(s)/\phi_G(s) \tag{4.8}$$

$$1 + \underline{C}(s\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} = \phi_K(s)/\phi_R(s) \quad . \tag{4.9}$$

By Lemma 3.1,

$$E_o(s) = [1+\underline{C}\ \underline{\Phi}(s)\underline{H}][1+M_o(s)]^{-1}M_o(s) \tag{4.10}$$

$$= \frac{\phi_F(s)}{\phi_G(s)} \left[\frac{\phi_K(s)}{\phi_R(s)}\right]^{-1} \left[\frac{\phi_K(s)}{\phi_R(s)} \ -1\right] \tag{4.11}$$

$$= 0 \tag{4.12}$$

since we have assumed $\phi_K(s) = \phi_R(s)$.

Remarks

(1) As in Theorem 3.2, define the matrix residue expansion of $\underline{M}_o(s)$:

$$\underline{M}_o(s) = \sum_{i=1}^{n} \frac{\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H}}{s-\lambda_i} \tag{4.13}$$

where λ_i is an eigenvalue of $\underline{A}-\underline{B}\ \underline{G}$ and $\underline{u}_i$, $\underline{v}_i$ the corresponding right and left eigenvectors.

It is clear from Lemma 3.1 that to make $\underline{E}_o(s)$ small one must make $\underline{M}_o(s)$ small.

Equation 4.13 suggests three mechanisms for reducing the size of $\underline{M}_o(s)$:

(1) Choose $\underline{G}$ so that $\underline{C}\ \underline{u}_i = \underline{0}$ or $\underline{v}_i^H\ \underline{H} = \underline{0}$ for all $1 \le i \le n$;

(2) Choose $\underline{G}$ so that $\lambda_i < 0$ and $|\lambda_i| >> 1$, for all $1 \le i \le n$;

(3) Choose $\underline{G}$ so that $\underline{C}\ \underline{u}_i \approx \underline{0}$, for all $1 \le i \le n-m$, and $\lambda_i < 0$, $|\lambda_i| >> 1$ for $n-m+1 \le i \le n$.

If it were possible to implement the first mechanism, we would achieve exact recovery. Since, in this chapter, we are concerned with approximate recovery, we assume $\underline{C}\ \underline{\Phi}(j\omega)\underline{H}$ is not exactly recoverable so the first option is not available.

Doyle and Stein [5] have shown that, in general, the second mechanism does not ensure adequate recovery. For scalar systems, the reason for this failure is clear. Rewriting (4.11) as

$$E_o(s) = \frac{\phi_F(s)}{\phi_G(s)} \left[1 - \frac{\phi_R(s)}{\phi_K(s)}\right] \tag{4.14}$$

shows that the recovery error is proportional to the term

$$\left[1 - \frac{\phi_R(s)}{\phi_K(s)}\right] \tag{4.15}$$

The λ_i's are the roots of $\phi_R(s)$, but choosing them according to scheme (2) does nothing to guarantee the smallness of (4.15).

For multivariable systems, the necessity of $\phi_K(s) = \phi_R(s)$ for exact recovery suggests a similar explanation for the failure of scheme (2), although the relationship among $\underline{E}_o(s)$, $\phi_K(s)$ and $\phi_R(s)$ is less apparent. It is apparent, however, on re-examing (4.13), that the magnitude of the residue, $\underline{C}\ \underline{u}_i\underline{v}_i^H\underline{H}$, is as important as the magnitude of the pole. We may well imagine a design procedure in which successive iterations increase the magnitude of the poles, yet simultaneously increase the magnitude of the residues. Such a procedure would fail to decrease $\underline{M}_o(s)$ and therefore fail to recover the desired loop shape.

Scheme (3) is a compromise between schemes (1) and (2). Again, one must be careful to keep the residues small as the m poles are made large. We see in the next section that a systematic procedure exists for implementing scheme (3).

4.3 Kwakernaak's Method

The accepted approach to approximating LTR is generally referred to as Kwakernaak's sensitivity recovery procedure [3]. The gain matrix, $\underline{G}$, is found as the solution to an LQR problem with state weighting matrix $\underline{C}'\underline{C}$ in the limit as the control weighting approaches zero. We therefore seek

$$\min_{u(t)} J = \int_0^\infty [\underline{x}'(t)\underline{C}'\underline{C}\ \underline{x}(t) + \rho\underline{u}'(t)\underline{u}(t)]dt \tag{4.16}$$

subject to

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t); \quad \underline{x}(0) = \underline{x}_o \tag{4.17}$$

The well-known solution is

$$\underline{u}(t) = -\underline{G}\ \underline{x}(t) \tag{4.18}$$

$$\underline{G} = \underline{B}'\underline{K}/\rho \tag{4.19}$$

where $\underline{K} = \underline{K}' \geq \underline{0}$ satisfies

$$\underline{0} = -\underline{K}\ \underline{A} - \underline{A}'\underline{K} - \underline{C}'\underline{C} + \underline{K}\ \underline{B}\ \underline{B}'\underline{K}/\rho \tag{4.20}$$

Since $(\underline{A},\underline{B},\underline{C})$ is given as a minimal realization we have $(\underline{A},\underline{B})$ controllable and $(\underline{A},\underline{C})$ observable so that $\underline{A}-\underline{B}\ \underline{G}$ is guaranteed stable.

Doyle and Stein [1] have shown that as $\rho \to 0$ the loop transmission $\underline{G}(s)\underline{K}(s) \to \underline{C}\ \underline{\Phi}(s)\underline{H}$ pointwise in s. The following theorem describes this approximation in the terms outlined in Section 4.2.

<u>Definitions</u>: Let λ_i, $\underline{u}_i$, $1 \leq i \leq n$, be the eigenvalues and eigenvectors of $\underline{A}-\underline{B}\ \underline{G}$. Let ζ_i be the i-th transmission zero of $\underline{G}(s)$. Let α_i, $1 \leq i \leq n$ be the eigenvalues of $\underline{A}$.

<u>Theorem 4.3</u>: Let

(i) $\text{rank}(\underline{C}\ \underline{B}) = m$

(ii) $\zeta_i \neq \alpha_j$, for all $1 \leq i \leq n-m$, $1 \leq j \leq n$

(iii) $\text{Re}(\zeta_i) < 0$, for all $1 \leq i \leq n-m$

(iv) If $i \neq j$ then $\zeta_i \neq \zeta_j$, for all $1 \leq i,j \leq n-m$.

If $\underline{G}$ is the solution to the optimal control problem (4.16), (4.17), in the limit as $\rho \to 0$, then

$$\underline{C}\ \underline{u}_i \to \underline{0}, \quad \text{for all } 1 \leq i \leq n-m \tag{4.21}$$

and

$$\lambda_i < 0, |\lambda_i| \to \infty, \quad \text{for all } n-m+1 \leq i \leq n \tag{4.22}$$

Proof: See Harvey and Stein [9], pp. 378-380.

Remarks

(1) According to Theorem 4.3, the Kwakernaak recovery method implements the approximation scheme (3) described in Section 4.2 - n-m eigenvectors are placed in the nullspace of $\underline{C}$ while the eigenvalues of the remaining modes approach infinity in m first order Butterworth patterns.

(2) How do we know whether, at the m asymptotically infinite modes, the residues remain finite? We somewhat skirt this issue by noting that the residues in question must be well behaved since Doyle and Stein [1] prove asymptotic recovery without reference to the modal properties of $\underline{A}-\underline{B}\ \underline{G}$.

4.4 L_2-norm Minimization

With appropriately defined norm, the Kwakernaak procedure chooses $\underline{G}$ to minimize the norm of $\underline{M}_o(j\omega)$.

Definitions: Let the L_2-norm of a complex, mxm matrix $\underline{A}(j\omega)$ be defined as

$$||\underline{A}(j\omega)||_2 \triangleq \max_{\underline{v}} \int_0^\infty [\underline{A}(j\omega)\underline{v}]^H[\underline{A}(j\omega)\underline{v}]d\omega \ ; \ \underline{v}\in R^m, \ ||\underline{v}||=1 \tag{4.23}$$

Theorem 4.4: Let the transmission zeroes of $\underline{G}(s)$ all have negative real parts. If $\underline{G}$ is the solution to the optimal control problem (4.16), (4.17) in the limit as $\rho\to 0$, then $\underline{G}$ minimizes $||\underline{M}_o(j\omega)||_2$.

Proof: By definition

$$||\underline{M}_o(j\omega)||_2 = \max_{\underline{v}} \int_0^\infty [\underline{M}_o(j\omega)v]^H[\underline{M}_o(j\omega)\underline{v}]d\omega \;;\quad ||\underline{v}||=1 \tag{4.24}$$

$$= \int_0^\infty [\underline{C}(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{v}_o]^H[\underline{C}(j\omega\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{v}_o]d\omega \tag{4.25}$$

where $\underline{v}_o \in R^m$ maximizes the integral. Let $\underline{x}_o = \underline{H}\ \underline{v}_o$. Then, by Parseval's Theorem [10],

$$||\underline{M}_o(j\omega)||_2 = \pi \int_0^\infty \{\underline{C}\ \exp[(\underline{A}-\underline{B}\ \underline{G})t]\underline{x}_o\}' \{\underline{C}\ \exp[(\underline{A}-\underline{B}\ \underline{G})t]\underline{x}_o\}dt \tag{4.26}$$

Let

$$\underline{x}(t) = \exp[(\underline{A}-\underline{B}\ \underline{G})t]\underline{x}_o \tag{4.27}$$

so that

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t);\ \underline{x}_o = \underline{x}(0) \tag{4.28}$$

$$\underline{u}(t) = -\underline{G}\ \underline{x}(t) \tag{4.29}$$

then

$$||\underline{M}_o(j\omega)||_2 = \pi \int_0^\infty [\underline{C}\ \underline{x}(t)]'[\underline{C}\ \underline{x}(t)]dt\ . \tag{4.30}$$

Now, with

$$J(\rho) \triangleq \int_0^\infty \underline{x}'(t)\underline{C}'\underline{C}\ \underline{x}(t) + \rho\underline{u}'(t)\underline{u}(t)dt \tag{4.31}$$

Kwakernaak [4] has shown that

$$\lim_{\rho \to 0} J(\rho) = \int_0^\infty \underline{x}'(t)\underline{C}'\underline{C}\ \underline{x}(t)dt \tag{4.32}$$

provided $\underline{G}(s)$ has no right-half-plane zeroes. Therefore

$$||\underline{M}_o(j\omega)||_2 = \pi \lim_{\rho \to 0} J(\rho) \tag{4.33}$$

and $\underline{G}$ minimizes $||\underline{M}_o(j\omega)||_2$.

Remarks

(1) Although we cannot always ensure $\underline{M}_o(j\omega)=\underline{0}$, if $\underline{G}(s)$ has no RHP zeroes we may find a minimum norm solution. If $\underline{G}(s)$ has RHP zeroes, then this minimum norm interpretation requires revision.

(2) The role of Theorem 4.4 in this paper is mainly to motivate the use of the discrete LQR (DLQR) problem as a means of approximating LTR for discrete-time systems. We see in Section 7.2 that the DLQR also minimizes $||\underline{M}_o(j\omega)||_2$ with a slight change in definition of the norm.

4.5 Summary of LTRI Results

The following four theorems are the duals of those presented in Section 4.2 through 4.4.

Theorem 4.5: If $\underline{E}_I(j\omega) = \underline{0}$ for all $\omega \in \Omega'$,

then $\phi_K(s) = \phi_F(s)$ for all s,

where

$$\phi_K(s) \triangleq \det(s\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})$$

$$\phi_F(s) \triangleq \det(s\underline{I}-\underline{A}+\underline{H}\ \underline{C}).$$

Theorem 4.6: If m=1, then $E_I(j\omega) = 0$, for all $\omega \in \Omega'$, if and only if $\phi_K(s) = \phi_R(s)$, for all s.

Doyle/Stein Robustness Recovery Method [5]

Definitions: Let λ_i, $\underline{v}_i$, $1 \le i \le n$ be the eigenvalues and left eigenvectors of $\underline{A}-\underline{H}\,\underline{C}$. Let ζ_i be the i-th zero of $\underline{G}(s)$. Let α_i, $1 \le i \le n$, be the eigenvalues of $\underline{A}$.

Theorem 4.7: Let

(i) $\text{rank}(\underline{C}\,\underline{B}) = m$

(ii) $\zeta_i \ne \alpha_j$, $1 \le i \le n-m$, $1 \le j \le n$

(iii) $\text{Re}(\zeta_i) < 0$, $1 \le i \le n-m$

(iv) $i \ne j$ implies $\zeta_i \ne \zeta_j$, $1 \le i,j \le n-m$.

If $\underline{\Sigma} = \underline{\Sigma}' \ge \underline{0}$ satisfies

$$\underline{0} = \underline{A}\,\underline{\Sigma} + \underline{\Sigma}\,\underline{A}' + \underline{B}\,\underline{B}' - \underline{\Sigma}\,\underline{C}'\underline{C}\,\underline{\Sigma}/\mu \tag{4.34}$$

in the limit as $\mu \to 0$, and

$$\underline{H} = \underline{\Sigma}\,\underline{C}'/\mu \tag{4.35}$$

then

$$\underline{v}_i^H\underline{B} \to \underline{0}, \quad \text{for all } 1 \le i \le n-m \tag{4.36}$$

and

$$\lambda_i < 0, \ |\lambda_i| \to \infty \quad \text{for all } n-m+1 \le i \le n \tag{4.37}$$

Theorem 4.8: Let the transmission zeroes of $\underline{G}(s)$ all have negative real parts. If $\underline{\Sigma} = \underline{\Sigma}' \geq \underline{0}$ satisfies (4.34) in the limit as $\mu \to 0$, then $\underline{H} = \underline{\Sigma}\, \underline{C}'/\mu$ minimizes $||\underline{M}_I(j\omega)||_2$.

Proof: Dual to proofs of Theorems 4.1, 4.2, 4.3, and 4.4.

4.6 Concluding Remarks

We have identified the basic issues in approximate recovery in terms of the matrix $\underline{M}_o(s)$ (dually, $\underline{M}_I(s)$). We have also seen how the Kwakernaak recovery scheme may be viewed from two perspectives - as an eigenvalue/eigenvector placement procedure and as a minimization of the norm of $\underline{M}_o$. The second perspective will be used in discrete-time to motivate the analogous discrete LQR approach to approximate recovery.

CHAPTER 5: DISCRETE-TIME PROBLEM DEFINITION

5.1 Introduction

This chapter presents the formal statement of the loop transfer recovery problem for discrete-time, linear, MIMO control systems. We consider only the MBC structure that results in strictly proper compensators (i.e. no direct feedthrough in the compensator output equation).

Section 5.2 contains a brief review of discrete frequency response concepts and establishes the notation to be used. Section 5.3 presents the actual LTR problem.

5.2 Discrete Frequency Response

As in common practice, we define the discrete transfer function (sometimes called the pulse transfer function) of a linear, time-invariant discrete-time system to be the z-transform of the system's unit pulse response. The z-transform, F(z), of a causal discrete signal, f(nT), where T is the sampling time, is defined by

$$F(z) \triangleq \sum_{k=0}^{\infty} f(kT)z^{-k} \tag{5.1}$$

Like its continuous-time counterpart, the discrete transfer function is a handy tool for calculating steady state responses to sinusoidal inputs. Consider a discrete time plant with transfer function G(z) (pulse response g(nT)) excited by a sampled sinusoid $u(nT) = e^{j\omega_o nT}$. Its output y(nT), is found by discrete convolution to be [7]:

$$y(nT) = \sum_{k=0}^{\infty} g(kT)\, e^{j\omega_o(nT-kT)} \tag{5.2}$$

$$= e^{j\omega_o nT} \sum_{k=0}^{\infty} g(kT)\, e^{-j\omega_o kT} \tag{5.3}$$

$$= e^{j\omega_o nT}\, G(z)\Big|_{z=e^{j\omega_o T}} \tag{5.4}$$

which is a sinusoid of the same frequency as the input but scaled in amplitude by $|G(e^{j\omega_o T})|$ and shifted in phase by arg $G(e^{j\omega_o T})$. We define the discrete frequency response, $G^*(j\omega)$:

$$G^*(j\omega) \triangleq G(z)\Big|_{z=e^{j\omega T}} \tag{5.5}$$

from which we see that a system is completely characterized by specifying $G^*(j\omega)$ for all $0 \leq \omega \leq \pi/T$. We use this means of specifying discrete-time system performance in the remainder of this paper.

5.3 Problem Statement

Figure 5.1 shows the structure of a discrete-time LQG/LTR control system. We assume a minimal realization, $(\underline{A},\underline{B},\underline{C})$ of the given plant transfer function matrix, $\underline{G}(z)$. That is

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) + \underline{B}\,\underline{u}(k) + \underline{B}\,\underline{d}_I(k) \tag{5.6}$$

$$\underline{y}(k) = \underline{C}\,\underline{x}(k) + \underline{d}_o(k) \tag{5.7}$$

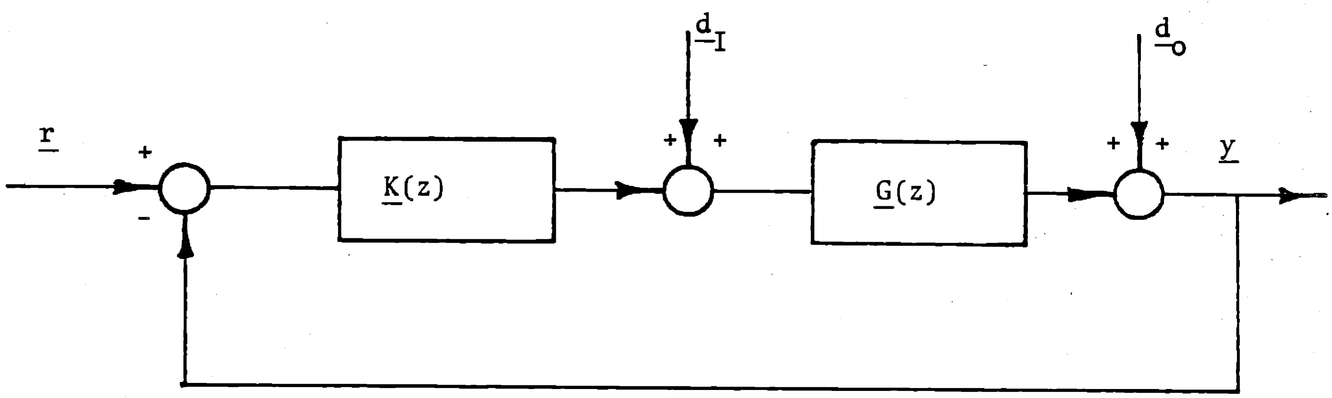

Figure 5.1: Discrete LQG/LTR Control System.

where the argument kT has been abbreviated as k and

$\underline{x}(k) \in R^n \triangleq$ plant state

$\underline{u}(k) \in R^m \triangleq$ control input

$\underline{d}_I(k) \in R^m \triangleq$ input disturbance

$\underline{y}(k) \in R^m \triangleq$ plant output

$\underline{d}_o(k) \in R^m \triangleq$ output disturbance

and $\underline{A}, \underline{B}, \underline{C}$ are appropriately dimensioned constant matrices. We further assume, for convenience,

$$\text{rank}(\underline{B}) = \text{rank}(\underline{C}) = m \quad . \tag{5.8}$$

With these definitions we see that

$$\underline{G}(z) = \underline{C}\ \underline{\Phi}(z)\underline{B} \tag{5.9}$$

where

$$\underline{\Phi}(z) \triangleq (z\underline{I}-\underline{A})^{-1} \tag{5.10}$$

The compensator has the structure of a full-order KBF:

$$\hat{\underline{x}}(k+1) = \underline{A}\ \hat{\underline{x}}(k) + \underline{B}\ \underline{u}(k) - \underline{H}[\underline{C}\ \hat{\underline{x}}(k) - \underline{y}(k) + \underline{r}(k)] \tag{5.11}$$

$$\underline{u}(k) = -\underline{G}\ \hat{\underline{x}}(k) \tag{5.12}$$

where

$\hat{\underline{x}}(k) \in R^n \triangleq$ filter state

$\underline{r}(k) \in R^m \triangleq$ reference input and $\underline{G}$ and $\underline{H}$ are the LQR and KBF gain matrices, respectively. We see that

$$\underline{K}(z) = \underline{G}(z\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})^{-1}\underline{H} . \tag{5.13}$$

Again, because of the special structure of the compensator, the closed loop eigenvalues are those of the matrices $\underline{A}-\underline{B}\ \underline{G}$ and $\underline{A}-\underline{H}\ \underline{C}$. Define λ_i, $1\leq i\leq n$, to be the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ and μ_i, $1\leq i\leq n$, the eigenvalues of $\underline{A}-\underline{H}\ \underline{C}$. With these definitions we may pose the following problems:

Problem 3: Discrete LTR at Plant Output (DLTRO)

Given $\underline{H}$ such that

(i) $|\mu_i|< 1$, for all $1\leq i\leq n$;

(ii) The singular values of $\underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$ meet all design requirements;

find $\underline{G}$ such that

(iii) $|\lambda_i|< 1$, for all $1\leq i\leq n$;

(iv) $\underline{G}^*(j\omega)\underline{K}^*(j\omega) \approx \underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$, for all $\omega\in\Omega_D$,

where Ω_D is the set of all $0\leq\omega\leq\pi/T$ for which $\underline{G}^*(j\omega)\underline{K}^*(j\omega)$ and $\underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$ are well-defined.

Problem 4: Discrete LTR at Plant Input (DLTRI)

Given $\underline{G}$ such that

(i) $|\lambda_i|< 1$, for all $1\leq i\leq n$;

(ii) The singular values of $\underline{G}\ \underline{\Phi}^*(j\omega)\underline{B}$ meet all design requirements;

find $\underline{H}$ such that

(iii) $|\mu_i| < 1$, for all $1 \leq i \leq n$;

(iv) $\underline{K}^*(j\omega)\underline{G}^*(j\omega) \approx \underline{G}\ \underline{\Phi}^*(j\omega)\underline{B}$, for all $\omega \in \Omega_D'$,

where Ω_D' is the set of all $0 \leq \omega \leq \pi/T$ for which $\underline{K}^*(j\omega)\underline{G}^*(j\omega)$ and $\underline{G}\ \underline{\Phi}^*(j\omega)\underline{B}$ are well defined.

CHAPTER 6: EXACT RECOVERY IN DISCRETE TIME

6.1 Introduction

This chapter summarizes the discrete-time theorems concerning exact LTR, analogous to the theorems presented in Chapter 3. By exact LTR we mean

$$\underline{G}^*(j\omega)\underline{K}^*(j\omega) = \underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}, \quad \text{for all } \omega\in\Omega_D\ , \tag{6.1}$$

or, dually,

$$\underline{K}^*(j\omega)\underline{G}^*(j\omega) = \underline{G}\ \underline{\Phi}^*(j\omega)\underline{B} \quad \text{for all } \omega\in\Omega_D' \tag{6.2}$$

6.2 Summary of DLTRO Results

Lemma 6.1: Let

$$\underline{E}_o(z) \triangleq \underline{C}\ \underline{\Phi}(z)\underline{H} - \underline{G}(z)\underline{K}(z)\ . \tag{6.3}$$

Then

$$\underline{E}_o(z) = [\underline{I}+\underline{C}\ \underline{\Phi}(z)\underline{H}]\,[\underline{I}+\underline{M}_o(z)]^{-1}\underline{M}_o(z) \tag{6.4}$$

where

$$\underline{M}_o(z) \triangleq \underline{C}(z\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}$$

Theorem 6.2: Let $\underline{A}-\underline{B}\ \underline{G}$ be a non-defective matrix with right eigenvectors $\underline{u}_i$, $1\leq i\leq n$, and corresponding left eigenvectors $\underline{v}_i$, $1\leq i\leq n$. Then, with $\underline{E}_o(z)$ and $\underline{M}_o(z)$ as defined in Lemma 6.1 the following statements are equivalent:

(i) $\underline{E}_o^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D$

(ii) $\underline{M}_o^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D$

(iii) $\underline{C}\ \underline{u}_i = \underline{0}$ or $\underline{v}_i^H \underline{H} = \underline{0}$, for all $1 \leq i \leq n$

Definitions: Let λ be an eigenvalue of $\underline{A}-\underline{B}\ \underline{G}$ with corresponding eigenvector $\underline{\beta} \in R^m$, $\underline{\beta} \neq 0$, such that $\underline{C}\ \underline{\beta} = \underline{0}$. Let μ be an eigenvalue of $\underline{A}-\underline{B}\ \underline{G}$ with corresponding left eigenvector $\underline{\gamma} \in R^m$, $\underline{\gamma} \neq 0$, such that $\gamma^H \underline{H} = 0$.

Lemma 6.3: λ is an eigenvalue of $\underline{K}(z)$ with corresponding eigenvector $\underline{\beta}$.

Lemma 6.4: There exists a number, ζ, such that ζ is a transmission zero of $\underline{G}(z)$ and $\zeta = \lambda$.

Lemma 6.5: The eigenvalue of $\underline{K}(z)$, λ, cancels the transmission zero of $\underline{G}(z)$, ζ.

Lemma 6.6: μ is an eigenvalue of $\underline{K}(z)$ with corresponding left eigenvector $\underline{\gamma}$, and μ cancels a zero of $\underline{K}(z)$.

Theorem 6.7: If $\underline{E}_o^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D$, and $\underline{A} - \underline{B}\ \underline{G}$ is non-defective, then every eigenvalue of $\underline{K}(z)$ cancels either a zero of $\underline{G}(z)$ or a zero of $\underline{K}(z)$.

Theorem 6.8: Let $\underline{A} - \underline{B}\ \underline{G}$ be a non-defective matrix with right eigenvectors $\underline{u}_i$ $1 \leq i \leq n$, such that $\underline{C}\ \underline{u}_i = \underline{0}$ for all $1 \leq i \leq n-m$. Then the following statements are equivalent:

(i) $\underline{E}_o^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D$

(ii) $\underline{C}\ \underline{H} = 0$.

Theorem 6.9: If $\underline{E}_o^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D$, then

$$\phi_K(z) \triangleq \phi_R(z) \quad \text{for all } z$$

where

$$\phi_K(z) \triangleq \det(z\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})$$

$$\triangleq \text{characteristic polynomial of the compensator}$$

$$\phi_R(z) \triangleq \det(z\underline{I}-\underline{A}+\underline{B}\ \underline{G})$$

$$\triangleq \text{characteristic polynomial of the regulator dynamics.}$$

Theorem 6.10: If m=1, then $\underline{E}_o^*(j\omega) = 0$, for all $\omega \in \Omega_D$, if and only if $\phi_K(z) = \phi_R(z)$, for all z.

Proof: The proofs of Lemmas 6.1, 6.2-6.6 and Theorems 6.2, 6.7-6.10 are identical to the proofs of Lemmas 3.1, 3.3-3.6 and Theorems 3.2, 3.7, 3.8, 4.1 and 4.2 if the following substitutions are made:

(i) z for s

(ii) $\underline{E}_o^*(j\omega)$ for $\underline{E}_o(j\omega)$

(iii) $\underline{M}_o^*(j\omega)$ for $\underline{M}_o(j\omega)$

(iv) $\underline{G}^*(j\omega)$ for $\underline{G}(j\omega)$

(v) $\underline{K}^*(j\omega)$ for $\underline{K}(j\omega)$

(vi) Ω_D for Ω

6.3 Summary of DLTRI Results

Lemma 6.11: Let

$$\underline{E}_I(z) \triangleq \underline{G}\ \underline{\Phi}(z)\underline{B} - \underline{K}(z)\underline{G}(z) \tag{6.6}$$

Then

$$\underline{E}_I(z) \triangleq \underline{M}_I(z)\,[\underline{I}+\underline{M}_I(z)]^{-1}[\underline{I}+\underline{G}\ \underline{\Phi}(z)\underline{B}] \tag{6.7}$$

where

$$\underline{M}_I(z) \triangleq \underline{G}(z\underline{I}-\underline{A}+\underline{H}\ \underline{C})^{-1}\underline{B} \tag{6.8}$$

Theorem 6.12: Let $\underline{A}$-$\underline{H}\ \underline{C}$ be a non-defective matrix with right eigenvectors $\underline{u}_i$, $1 \leq i \leq n$, and corresponding left eigenvectors $\underline{v}_i$, $1 \leq i \leq n$. Then with $\underline{E}_I(z)$ and $\underline{M}_I(z)$ as defined in Lemma 3.9 the following statements are equivalent:

(i) $\underline{E}_I^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D'$

(ii) $\underline{M}_I^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D'$

(iii) $\underline{G}\ \underline{u}_i = \underline{0}$ or $\underline{v}_i^H\underline{B} = \underline{0}$; for all $1 \leq i \leq n$

Definitions: Let λ be an eigenvalue of $\underline{A}$-$\underline{H}\ \underline{C}$ with corresponding eigenvector $\underline{\beta} \in R^m$, $\underline{\beta} \neq 0$, such that $\underline{G}\ \underline{\beta} = \underline{0}$. Let μ be an eigenvalue of $\underline{A}$-$\underline{H}\ \underline{C}$ with corresponding left eigenvector $\underline{\gamma} \in R^m$, $\underline{\gamma} \neq 0$, such that $\underline{\gamma}^H\underline{B} = 0$.

Lemma 6.13: μ is an eigenvalue of $\underline{K}(z)$ with corresponding left eigenvector $\underline{\gamma}$.

Lemma 6.14: There exists a number, ζ, such that ζ is a transmission zero of $\underline{G}(z)$ and $\zeta = \mu$.

Lemma 6.15: The eigenvalue of $\underline{K}(z)$, μ, cancels the transmission zero of $\underline{G}(z)$, ζ.

Lemma 6.16: λ is an eigenvalue of $\underline{K}(z)$ with corresponding eigenvector $\underline{\beta}$, and λ cancels a zero of $\underline{K}(z)$.

Theorem 6.17: If $\underline{E}_I^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D'$, and $\underline{A}$-$\underline{H}$ $\underline{C}$ is non-defective, then every eigenvalue of $\underline{K}(z)$ cancels either a zero of $\underline{G}(z)$ or a zero of $\underline{K}(z)$.

Theorem 6.18: Let $\underline{A}$-$\underline{H}$ $\underline{C}$ be a non-defective matrix with left eigenvectors $\underline{v}_i$, $1 \leq i \leq n$ such that $\underline{v}_i^H \underline{B} = \underline{0}$ for all $1 \leq i \leq n-m$. Then the following statements are equivalent:

(i) $\underline{E}_I^*(j\omega) = \underline{0}$, for all $\omega \in \Omega_D'$

(ii) $\underline{G}\ \underline{B} = \underline{0}$

Theorem 6.19: If $\underline{E}_I^*(j\omega) = \underline{0}$ for all $\omega \in \Omega_D'$, then

$$\phi_K(z) = \phi_F(z) \quad \text{for all } z,$$

where

$$\phi_K(z) \triangleq \det(z\underline{I}-\underline{A}+\underline{B}\ \underline{G}+\underline{H}\ \underline{C})$$

$$\phi_F(z) \triangleq \det(z\underline{I}-\underline{A}+\underline{H}\ \underline{C}).$$

Theorem 6.20: If m=1, then $E_I^*(j\omega) = 0$, for all $\omega \in \Omega_D'$, if and only if $\phi_K(z) = \phi_F(z)$, for all z.

Proof: The proofs of Lemmas 6.11, 6.13-6.16 and Theorems 6.12, 6.17-6.20 are dual to the proofs of Lemmas 6.1, 6.3-6.6 and Theorems 6.2, 6.7-6.10.

6.4 Concluding Remarks

As in Chapter 3, we have presented these exact recovery results only to set the stage for the discussion of approximate recovery that follows.

CHAPTER 7: APPROXIMATE RECOVERY IN DISCRETE TIME

7.1: Introduction

Consider the matrix residue expansion of $\underline{M}_o(z)$:

$$\underline{M}_o(z) = \sum_{i=1}^{n} \frac{\underline{C}\,\underline{u}_i\underline{v}_i^H\underline{H}}{z-\lambda_i} \tag{7.1}$$

where λ_i, $1\leq i\leq n$ are the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ and $\underline{u}_i$, $\underline{v}_i$ the corresponding right and left eigenvectors appropriately scaled. By Lemma 6.1, it is apparent that approximate recovery entails reducing the size of $\underline{M}_o(z)$.

Unlike the continuous time case, however, we do not have the option of increasing the magnitudes of the poles since stability requires $|\lambda_i|< 1$ for all $1\leq i\leq n$. Any form of asymptotic recovery is, therefore, impossible.

Recall, however, the interpretation of Kwakernaak's method as a minimization of the norm of $\underline{M}_o(z)$. Section 7.2 poses the analogous discrete domain minimization problem to derive the discrete LQR (DLQR) problem. Section 7.3 discusses the solution to this problem.

7.2 L_2-norm Minimization

Discrete frequency response functions are periodic and, therefore, not actually elements of L_2. We skirt this issue by modifying the limits of integration in the definition of the norm so that only one period of the function contributes.

Definition: Let the discrete 2-norm of a complex, mxm matrix $\underline{A}(z)$ be defined as

$$||\underline{A}(z)||_{2D} \triangleq \max_{\underline{v}} \int_0^{2\pi/T} [\underline{A}^*(j\omega)\underline{v}]^H[\underline{A}^*(j\omega)\underline{v}]d\omega; \quad \underline{v}\in R^m, \ ||\underline{v}||=1 \tag{7.2}$$

Theorem 7.1: $\underline{G}$ minimizes $||\underline{M}_o(z)||_{2D}$ if and only $\underline{G}$ is a solution to the DLQR problem

$$\min_{\underline{u}(k)} J = \sum_{k=0}^{\infty} \underline{x}'(k)\underline{C}'\underline{C}\ \underline{x}(k) \tag{7.3}$$

subject to

$$\underline{x}(k+1) = \underline{A}\ \underline{x}(k) + \underline{B}\ \underline{u}(k); \ \underline{x}(0) = \underline{x}_o \tag{7.4}$$

where

$$\underline{u}(k) = -\underline{G}\ \underline{x}(k) \tag{7.5}$$

Proof: By definition

$$||\underline{M}_o(z)||_{2D} = \max_{\underline{v}} \int_0^{2\pi/T} [\underline{M}_o^*(j\omega)\underline{v}]^H[\underline{M}_o^*(j\omega)\underline{v}]d\omega; \quad ||\underline{v}||=1 \tag{7.6}$$

$$= \int_0^{\infty} [\underline{C}(e^{j\omega T}\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{v}_o]^H[\underline{C}(e^{j\omega T}\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H}\ \underline{v}_o]d\omega \tag{7.7}$$

where $\underline{v}_o \in R^m$ maximizes the integral. Let $\underline{x}_o = \underline{H}\ \underline{v}_o$. Then, by Parseval's Theorem [11],

$$||\underline{M}_o(z)||_{2D} = \frac{2\pi}{T} \sum_{k=0}^{\infty} [\underline{C}(\underline{A}-\underline{B}\ \underline{G})^k \underline{x}_o]'[\underline{C}(\underline{A}-\underline{B}\ \underline{G})^k \underline{x}_o] \ . \tag{7.8}$$

Let

$$\underline{x}(k) = (\underline{A}-\underline{B}\ \underline{G})^k \underline{x}_o \tag{7.9}$$

so that

$$\underline{x}(k+1) = \underline{A}\ \underline{x}(k) + \underline{B}\ \underline{u}(k);\ \underline{x}(0) = \underline{x}_o \tag{7.10}$$

$$\underline{u}(k) = -\underline{G}\ \underline{x}(k)\ . \tag{7.11}$$

Then

$$||\underline{M}_o(z)||_{2D} = \frac{2\pi}{T} \sum_{k=0}^{\infty} [\underline{C}\ \underline{x}(k)]'[\underline{C}\ \underline{x}(k)]\ . \tag{7.12}$$

Therefore, $\underline{G}$ minimizes $||\underline{M}_o(z)||_{2D}$ if and only if $\underline{G}$ is a solution to the DLQR problem (7.3)-(7.5).

7.3 The DLQR Problem

Consider the optimization problem

$$\min_{u(k)} \ J = \sum_{k=0}^{\infty} \underline{x}'(k)\underline{C}'\underline{C}\ \underline{x}(k) + \rho\underline{u}'(k)\underline{u}(k) \tag{7.13}$$

subject to

$$\underline{x}(k+1) = \underline{A}\ \underline{x}(k) + \underline{B}\ \underline{u}(k)\ . \tag{7.14}$$

The well known solution is [12]:

$$\underline{u}(k) = -\underline{G}\ \underline{x}(k) \tag{7.15}$$

$$\underline{G} = (\rho\underline{I}+\underline{B}'\underline{K}\ \underline{B})^{-1}\underline{B}'\underline{K}\ \underline{A} \tag{7.16}$$

where $\underline{K} = \underline{K}' \geq \underline{0}$ satisfies

$$\underline{K} = \underline{A}'\underline{K}\ \underline{A} + \underline{C}'\underline{C} - \underline{A}'\underline{K}\ \underline{B}(\rho\underline{I}+\underline{B}'\underline{K}\ \underline{B})^{-1}\underline{B}'\underline{K}\ \underline{A}\ . \tag{7.17}$$

Unlike the continuous time LQR, the inverse of the control weighting is not required here. We may let $\rho=0$, therefore, to yield

$$\underline{G} = (\underline{B}'\underline{K}\ \underline{B})^{-1}\underline{B}'\underline{K}\ \underline{A} \tag{7.18}$$

$$\underline{K} = \underline{A}'\underline{K}\ \underline{A} + \underline{C}'\underline{C} - \underline{A}'\underline{K}\ \underline{B}(\underline{B}'\underline{K}\ \underline{B})^{-1}\underline{B}'\underline{K}\ \underline{A}\ . \tag{7.19}$$

It may be verified by substitution in (7.19) that, if rank($\underline{C}\ \underline{B}$)=m[1], then

$$\underline{K} = \underline{C}'\underline{C} \tag{7.20}$$

is a solution, which implies

$$\underline{G} = (\underline{C}\ \underline{B})^{-1}\underline{C}\ \underline{A}\ . \tag{7.21}$$

The following theorem exposes the nature of this solution.

<u>Theorem 7.2</u>: Let λ_i, $1\leq i\leq n$ be the eigenvalues of $\underline{A}-\underline{B}\ \underline{G}$ and $\underline{u}_i$, $1\leq i\leq n$ corresponding eigenvectors. If $\underline{G} = (\underline{C}\ \underline{B})^{-1}\underline{C}\ \underline{A}$ and $\underline{A}-\underline{B}\ \underline{G}$ is non-defective, then

---

[1] If rank($\underline{C}\ \underline{B}$)< m a closed form solution to (7.19) is not known. However, we conjecture that the modal properties of the solution would be similar to those expressed in Theorem 7.2.

(i) $\underline{C}\,\underline{u}_i = \underline{0}$, $1 \leq i \leq n-m$

(ii) $\lambda_i = 0$, $n-m+1 \leq i \leq n$

Proof: Assume $\underline{G} = (\underline{C}\,\underline{B})^{-1}\underline{C}\,\underline{A}$. Then

$$\underline{C}(\underline{A}-\underline{B}\,\underline{G}) = \underline{C}[\underline{A}-\underline{B}(\underline{C}\,\underline{B})^{-1}\underline{C}\,\underline{A}] \tag{7.22}$$

$$= \underline{C}\,\underline{A}-\underline{C}\,\underline{A} \tag{7.23}$$

$$= \underline{0} \tag{7.24}$$

Since rank($\underline{C}$)=m, the rows of $\underline{C}$ are m independent left eigenvectors of $\underline{A}-\underline{B}\,\underline{G}$ corresponding to m zero eigenvalues. We label these zero eigenvalues λ_i, $n-m+1 \leq i \leq n$ and (ii) is proved.

Define the modal matrix $\underline{U}$, its inverse $\underline{V}^H$, and $\underline{\Lambda} = \text{diag}(\lambda_1 \ldots \lambda_n)$ so that

$$\underline{V}^H(\underline{A}-\underline{B}\,\underline{G})\underline{U} = \underline{\Lambda} \quad . \tag{7.25}$$

Now $\underline{V}^H$ may be partitioned.

$$\underline{V}^H = \begin{bmatrix} \underline{V}_1^H \\ \\ \underline{C} \end{bmatrix} \tag{7.26}$$

where rank(V_1)=n-m. Matrix $\underline{U}$ may be partitioned conformally as

$$\underline{U} = [\underline{U}_1 \ \underline{U}_2] \ , \tag{7.27}$$

with rank($\underline{U}_1$)=n-m, and scaled so that

$$\underline{I} = \underline{V}^H \underline{U} = \begin{bmatrix} \underline{V}_1^H \underline{U}_1 & \underline{V}_1^H \underline{U}_2 \\ \underline{C}\ \underline{U}_1 & \underline{C}\ \underline{U}_2 \end{bmatrix} . \tag{7.28}$$

Therefore $\underline{C}\ \underline{U}_1 = 0$ which implies

$$\underline{C}\ \underline{u}_i = 0, \quad \text{for all } 1 \leq i \leq n-m \tag{7.29}$$

and (i) is proved.

Remarks

(1) Theorem 7.2 is the discrete-time analog of Theorem 4.3. As in the continuous-time case, n-m eigenvectors of $\underline{A}-\underline{B}\ \underline{G}$ find their way into the nullspace of $\underline{C}$. Unlike the continuous-time case, however, the eigenvalues of the other m modes remain finite, but note that they wind up at the origin - the fastest any discrete-time mode can be.

(2) According to (i) and Lemma 6.4, n-m poles are equal to plant zeroes. The requirement of stability then restricts this approach to minimum phase plants.

7.4 Discrete Recovery Error

By Theorem 7.2 we see that the DLQR solution places n-m closed loop system poles at the plant zeroes and the remaining m poles at the origin. These m poles at the origin are the source of the finite recovery error resulting from the DLQR approach. This error is quantified in the following theorem.

Theorem 7.3: If

$$\underline{G} = (\underline{C}\ \underline{B})^{-1}\underline{C}\ \underline{A} \tag{7.30}$$

then

$$\underline{E}_o(z) = [\underline{I}+\underline{C}\ \underline{\Phi}(z)\underline{H}]\underline{C}\ \underline{H}[z\underline{I}+\underline{C}\ \underline{H}]^{-1} \tag{7.31}$$

Proof: By definition of $\underline{M}_o(z)$ and using the Laurent expansion we have:

$$\underline{M}_o(z) = \underline{C}(z\underline{I}-\underline{A}+\underline{B}\ \underline{G})^{-1}\underline{H} \tag{7.32}$$

$$= \underline{C}\ z^{-1}\sum_{i=0}^{\infty} (\underline{A}-\underline{B}\ \underline{G})^{i}z^{-i}\underline{H} \tag{7.33}$$

$$= \underline{C}\ \underline{H}/z \tag{7.34}$$

where (7.34) follows from

$$\underline{C}(\underline{A}-\underline{B}\ \underline{G}) = \underline{0}\ , \tag{7.35}$$

By Lemma 6.1 and (7.34), then,

$$\underline{E}_o(z) = [\underline{I}+\underline{C}\ \underline{\Phi}(z)\underline{H}]\underline{C}\ \underline{H}[z\ \underline{I}+\underline{C}\ \underline{H}]^{-1}\ . \tag{7.36}$$

Remarks

(1) Equation (7.36) demonstrates again the dependence of the recovery error on the product $\underline{C}\ \underline{H}$. We see that if indeed $\underline{C}\ \underline{H} = \underline{0}$ then (7.30) is the gain that achieves exact recovery.

(2) The SISO case of Equation (7.36) reveals the fact that recovery error is a function of sampling time.

Consider a SISO plant realized in observable canonical form (see Equations (3.62)-(3.64)). The characteristic polynomial of the matrix $\underline{A}$-$\underline{H}$ $\underline{C}$ can be written as:

$$0 = s^n + (a_{n-1}+h_{n-1})s^{n-1}+\ldots+(a_o+h_o) \tag{7.37}$$

Suppose λ_i, $1\leq i\leq n$ are the eigenvalues of $\underline{A}$, and μ_i, $1\leq i\leq n$ are the eigenvalues of $\underline{A}$-$\underline{H}$ $\underline{C}$. For any n^{th} order monic polynomial, the coefficient of the $(n-1)^{th}$ order term is equal to the negative sum of the roots of the polynomial. Therefore,

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i \tag{7.38}$$

and

$$a_{n-1} + h_{n-1} = -\sum_{i=1}^{n} \mu_i \tag{7.39}$$

Combining these two yields:

$$h_{n-1} = \sum_{i=1}^{n} \lambda_i-\mu_i \tag{7.40}$$

$$= \sum_{i=1}^{n} \mathrm{Re}(\lambda_i)-\mathrm{Re}(\mu_i) \tag{7.41}$$

since the eigenvalues occur in complex conjugate pairs. We may then say that h_{n-1} is equal to the "net leftward movement in the z-plane of the dominant system poles." Noting that $h_{n-1} = \underline{C}\,\underline{H}$ we see that the error (7.36) is an increasing function of h_{n-1} for fixed z. The dependence of h_{n-1} on sampling time is best illustrated by a simple example.

Suppose we have a continuous time plant,

$$G(s) = \frac{5}{s+5}, \tag{7.42}$$

that we wish to control with a piecewise-constant input (sampled data system). Further, suppose we know that all our specs will be met if we only move the pole to s=-10. We may translate this spec into a discrete pole location problem, but the translation depends on sampling time, T.

Suppose T=.01. Then the open loop pole is at:

$$\lambda = e^{-5(.01)} = 0.95 \tag{7.43}$$

The desired closed loop pole is:

$$\mu = e^{-10(.01)} = 0.90 \tag{7.44}$$

So that

$$\underline{C}\,\underline{H} = h_{n-1} = \lambda - \mu = 0.05. \tag{7.45}$$

Suppose now that T=.05. Then the open loop pole is:

$$\lambda = e^{-5(.05)} = 0.78 \tag{7.46}$$

and the desired closed loop pole is:

$$\mu = e^{-10(.05)} = 0.61 \tag{7.47}$$

so that

$$\underline{C}\,\underline{H} = h_{n-1} = \lambda - \mu = 0.17 \tag{7.48}$$

We see that the same pole re-location in the s-plane corresponds to different pole movement in the z-plane depending on T, and that the poles move further if the sampling time is longer. Thus the recovery error is an increasing function of the sampling time. (We expect this, of course, since we know that as $T \to 0$ we should approach the continuous-time case of asymptotic recovery). Section 8.3 demonstrates this sampling time dependence in a MIMO example.

7.5 Summary of DLTRI Results

Theorem 7.4: $\underline{H}$ minimizes $||\underline{M}_I(z)||_{2D}$ if and only if $\underline{H}$ is a solution to the DLQR problem

$$\min_{\underline{u}(k)} J = \sum_{k=0}^{\infty} \underline{x}'(k)\underline{B}\ \underline{B}'\underline{x}(k) \tag{7.49}$$

subject to

$$\underline{x}(k+1) = \underline{A}'\underline{x}(k) + \underline{C}'\underline{u}(k); \quad \underline{x}(0) = \underline{x}_o \tag{7.50}$$

where

$$\underline{u}(k) = -\underline{H}'\underline{x}(k) \tag{7.51}$$

Theorem 7.5: Let λ_i, $1\le i\le n$ be eigenvalues of $\underline{A}$-$\underline{H}\ \underline{C}$ and $\underline{v}_i$, $1\le i\le n$ corresponding left eigenvectors. If $\underline{H} = \underline{A}\ \underline{B}(\underline{C}\ \underline{B})^{-1}$ and $\underline{A}$-$\underline{H}\ \underline{C}$ is non-defective, then

(i) $\underline{v}_i^H\underline{B} = \underline{0}$, $1\le i\le n-m$

(ii) $\lambda_i=0$, $n-m+1\le i\le n$

Theorem 7.6: If

$$\underline{H} = \underline{A}\ \underline{B}(\underline{C}\ \underline{B})^{-1} \tag{7.52}$$

then

$$\underline{E}_I(z) = \underline{G}\ \underline{B}[z\underline{I}+\underline{G}\ \underline{B}]^{-1}[\underline{I}+\underline{G}\ \underline{\Phi}(z)\underline{B}] \tag{7.53}$$

Proofs: Proofs of Theorems 7.4 through 7.6 and dual to those of Theorems 7.1 through 7.3.

7.6 Concluding Remarks

We have shown that asymptotic recovery is impossible. The DLQR approach represents "the best you can do" (in the quadratic minimization sense) to reduce the recovery error for arbitrary $\underline{H}$ (dually, $\underline{G}$). We see, in the next chapter, some numerical examples of how good "the best" really is.

CHAPTER 8: EXAMPLES

8.1: Introduction

This chapter presents some examples of discrete-time control system design using MBC/LTR. Section 8.2 uses a purely hypothetical SISO plant to illustrate an exact recovery. Section 8.3 uses a model of a CH47 helicopter [1] to illustrate approximate recovery in the MIMO case and the dependence of recovery error on sampling time.

8.2: SISO Exact Recovery

Consider the SISO plant:

$$G(z) = \frac{z-0.8}{(z-1.0)(z-0.9)} \tag{8.1}$$

We obtain the observable canonical realization:

$$\underline{A} = \begin{bmatrix} 0 & -0.9 \\ 1 & 1.9 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} -0.8 \\ 1 \end{bmatrix} \tag{8.2}$$

$$\underline{C} = [0 \quad 1]$$

Suppose we are given the full state design matrix $\underline{H}$:

$$\underline{H} = \begin{bmatrix} 0.0025 \\ 0 \end{bmatrix} \tag{8.3}$$

which places both eigenvalues of $\underline{A}-\underline{H}\,\underline{C}$ at z=0.95. We note that we have

(i) rank$(\underline{C}\,\underline{B})=1$

(ii) $\underline{C}\,\underline{H} = 0$

(iii) $\underline{C}\,\underline{\Phi}(z)\underline{B}$ minimum phase

so we expect to be able to recover exactly the desired loop shape

$$\underline{C}\,\underline{\Phi}(z)\underline{H} = \frac{.0025}{(z-1.0)(z-0.9)} \tag{8.4}$$

We implicitly solve the discrete LQR problem

$$\min_{\underline{u}(k)} \; J = \sum_{k=0}^{\infty} \underline{x}'(k)\underline{C}'\underline{C}\,\underline{x}(k) \tag{8.5}$$

by choosing

$$\underline{G} = (\underline{C}\,\underline{B})^{-1}\underline{C}\,\underline{A} = [1.0 \quad 1.9] \; . \tag{8.6}$$

The eigenvalues of

$$\underline{A}-\underline{B}\,\underline{G} = \begin{bmatrix} 0.8 & 0.62 \\ 0 & 0 \end{bmatrix} \tag{8.7}$$

are at z= 0.8, 0.0 - one at the plant zero and one at the origin. The compensator is given by

$$K(z) = \underline{G}(z\underline{I}-\underline{A}+\underline{B}\,\underline{G}+\underline{H}\,\underline{C})^{-1}\underline{H} \tag{8.8}$$

$$= [1.0 \quad 1.9] \begin{bmatrix} z-0.8 & -0.6175 \\ 0 & z \end{bmatrix}^{-1} \begin{bmatrix} 0.0025 \\ 0 \end{bmatrix} \tag{8.9}$$

$$= \frac{0.0025z}{z(z-0.8)} \tag{8.10}$$

$$= \frac{0.0025}{z-0.8} \tag{8.11}$$

So the actual loop transmission is

$$G(z)K(z) = \frac{z-0.8}{(z-1.0)(z-0.9)} \quad \frac{0.0025}{(z-0.8)} \tag{8.12}$$

$$= \frac{0.0025}{(z-1.0)(z-0.9)} \tag{8.13}$$

and the recovery is exact.

8.3 MIMO Approximate Recovery

A continuous time model of a CH47 tandem rotor helicopter at 40 knot airspeed is [1]:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix} \underline{u}(t) \tag{8.14}$$

$$\underline{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix} \underline{x}(t) \tag{8.15}$$

where the outputs are vertical velocity and pitch attitude.

We assume a simpling time

$$T = 0.01 \text{ sec} \quad . \tag{8.16}$$

If the input, $\underline{u}(t)$, is piecewise constant, that is,

$$\underline{u}(t) = u(kT), \quad kT \leq t < (k+1)T \qquad k=0,1,\ldots. \tag{8.17}$$

then the system obeys the difference equation,

$$\underline{x}(k+1) = \begin{bmatrix} 0.9998 & 0.0001 & 0.0222 & -0.3198 \\ -0.0014 & 1.0044 & -0.0144 & -0.3005 \\ 0.0000 & 0.0002 & 0.9842 & 0.0119 \\ 0.0000 & 0.0000 & 0.0099 & 1.0001 \end{bmatrix} \underline{x}(t)$$

$$+ \begin{bmatrix} 0.0014 & -0.0012 \\ 0.0036 & -0.0862 \\ 0.0035 & 0.0001 \\ 0.0000 & 0.0000 \end{bmatrix} \underline{u}(k) \tag{8.18}$$

$$\underline{y}(k) = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 57.3000 \end{bmatrix} \underline{x}(k) \tag{8.19}$$

Suppose we are given $\underline{H}$:

$$\underline{H} = \begin{bmatrix} 0.0031 & 0.0023 \\ 0.2421 & -0.0018 \\ 0.0002 & 0.0080 \\ 0.0000 & 0.0017 \end{bmatrix} \tag{8.20}$$

which results in the loop shape, $\underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$ shown in Figure 8.1. We attempt to recover this loop by choosing

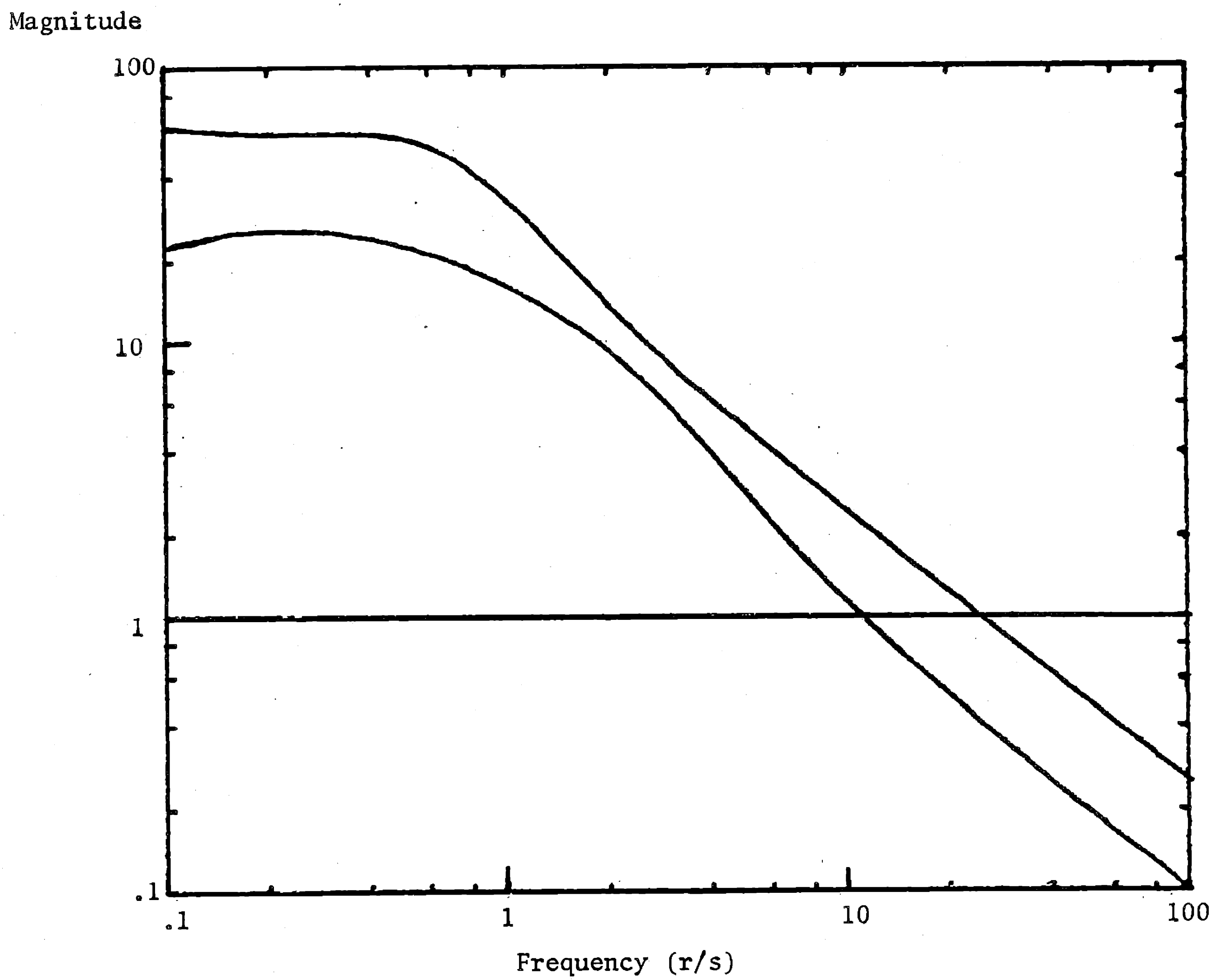

Figure 8.1: Singular Values of $\underline{C}\,\underline{\Phi}^*(j\omega)\underline{H}$ T=0.01.

$$\underline{G} = (\underline{C}\ \underline{B})^{-1}\underline{C}\ \underline{A} = \begin{bmatrix} 0 & 1 & 564 & 5.6\text{x}10^4 \\ 0 & -12 & 24 & 2350 \end{bmatrix}$$

The singular values of $\underline{G}^*(j\omega)\underline{K}^*(j\omega)$ are plotted along with the target singular values of $\underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$ (Fig. 8.2). As predicted in Section 7.3 the recovery error is non-zero.

We repeat the above procedure changing the sampling time to

$$T = 0.05 \text{ sec} . \tag{8.22}$$

With

$$\underline{H} = \begin{bmatrix} 0.0096 & 0.0023 \\ 0.7423 & -0.0165 \\ 0.0011 & 0.0311 \\ 0.0000 & 0.0084 \end{bmatrix} \tag{8.23}$$

the loop shape of Figure 8.3 results.

Choosing

$$\underline{G} = (\underline{C}\ \underline{B})^{-1}\underline{C}\ \underline{A} = \begin{bmatrix} 0 & 1 & 113 & 2350 \\ 0 & -2.3 & 4.7 & 97.4 \end{bmatrix} \tag{8.24}$$

produces the loop shape of Figure 8.4. Both $\underline{H}$ matrices were chosen to produce roughly the same open loop crossover frequency (~ 10 r/s). As expected, the recovery error is greater with the larger sampling time.

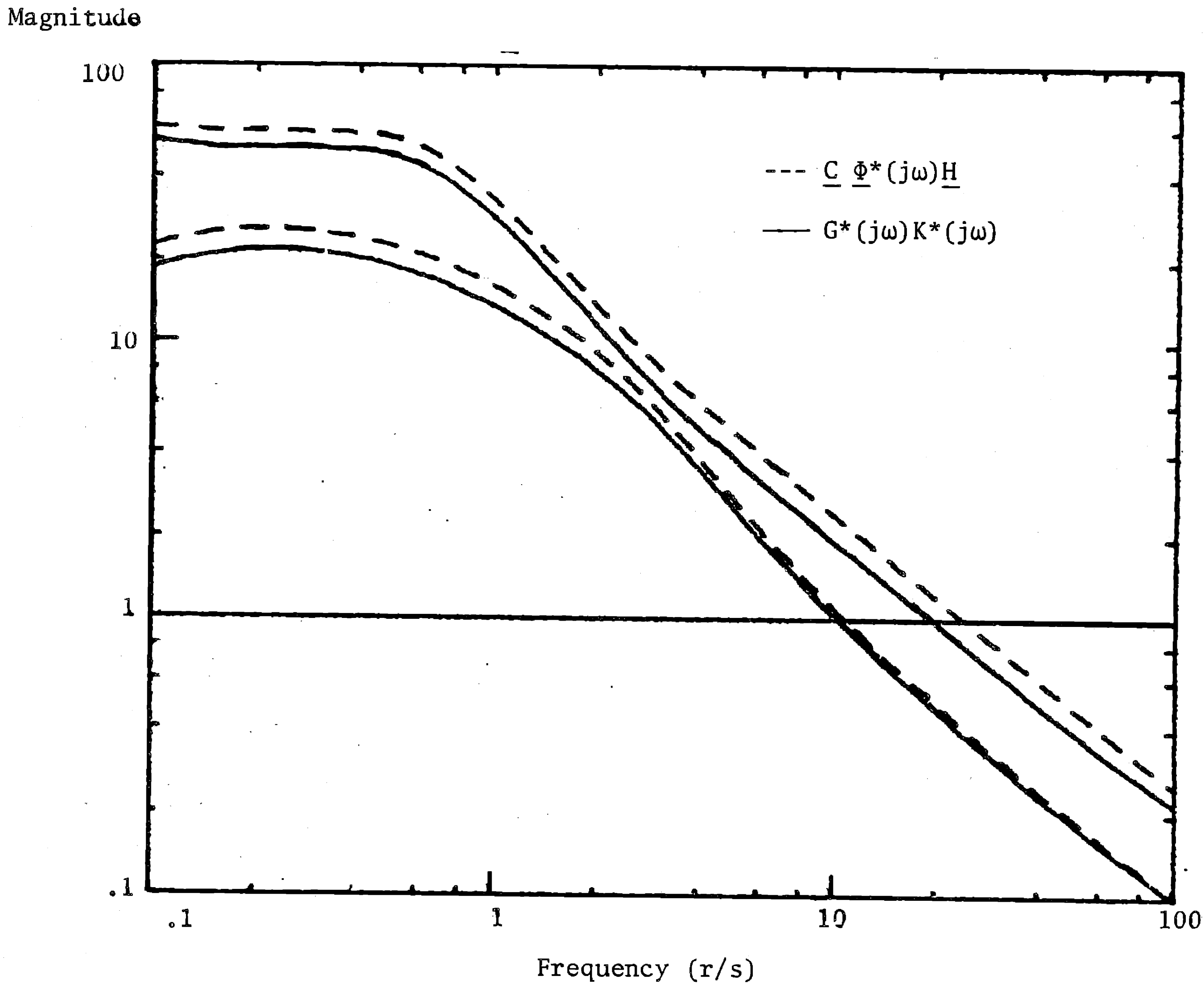

Figure 8.2: Loop Recovery T=0.01.

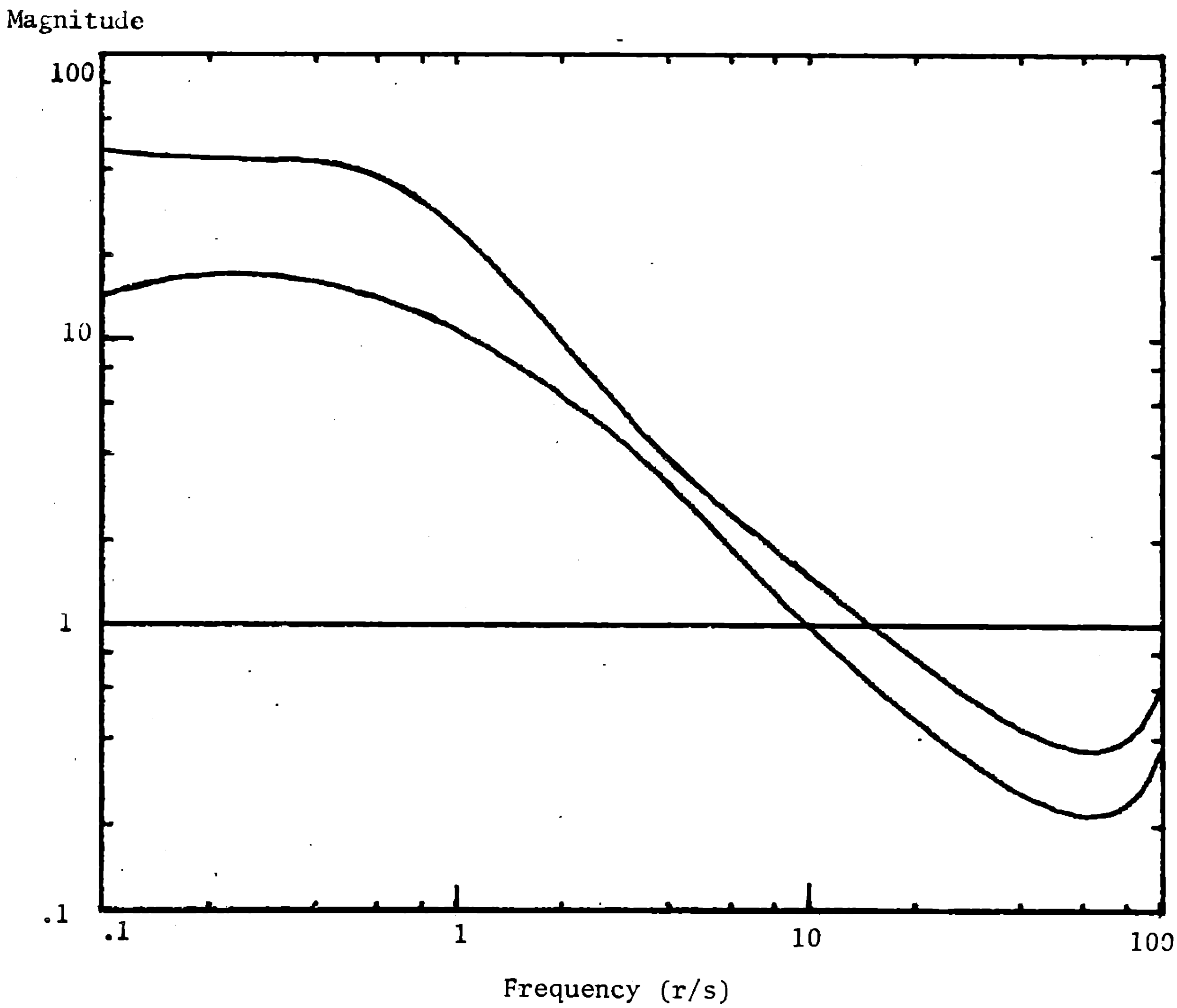

Figure 8.3: Singular Values of $\underline{C}\ \underline{\Phi}^*(j\omega)\underline{H}$
T=0.05

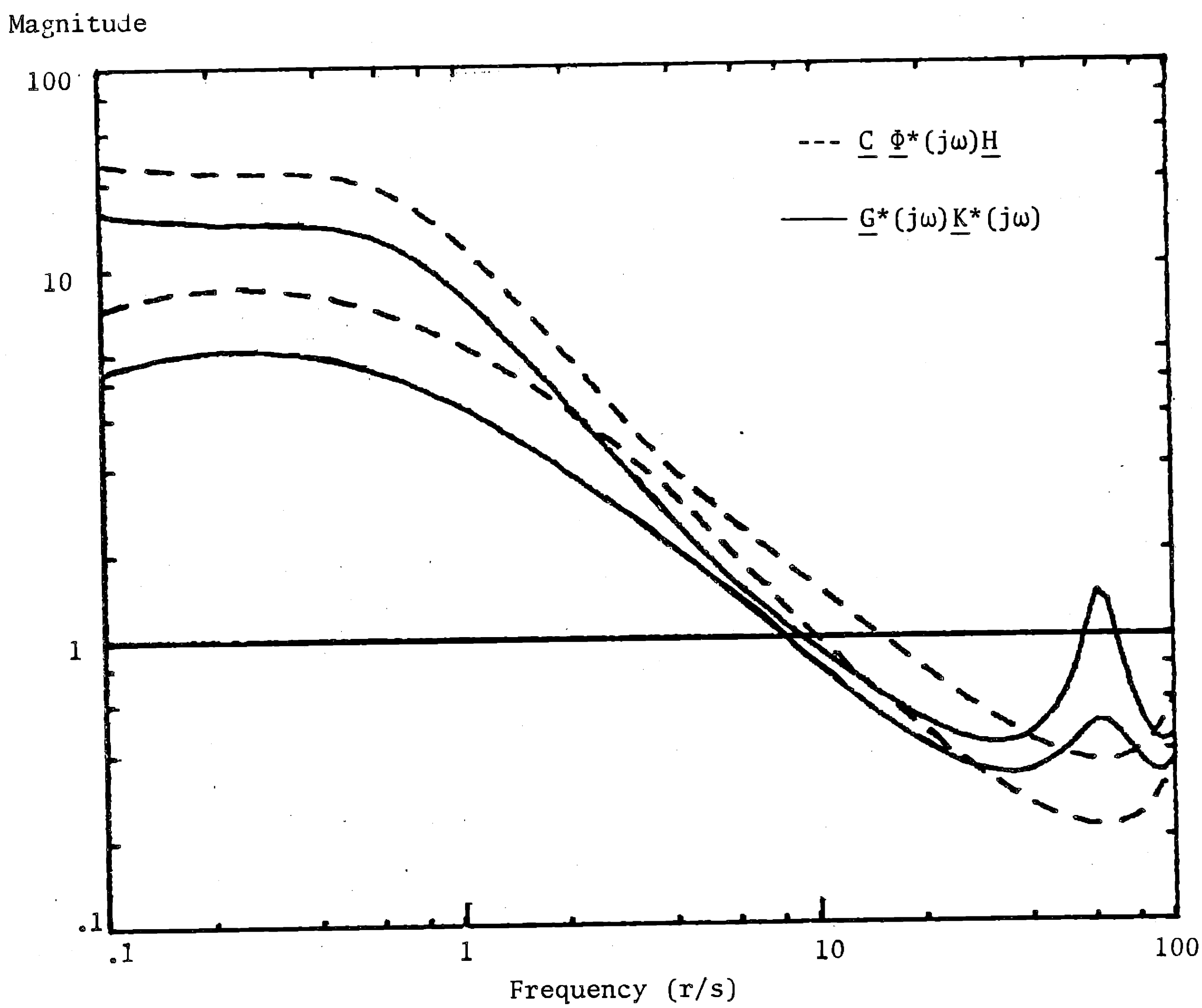

Figure 8.4: Loop Recovery T=0.05.

CHAPTER 9: CONCLUSION

9.1 Summary

We have reached the following conclusions concerning discrete-time LTR:

(1) Asymptotic recovery is impossible;

(2) Recovery using DLQR is qualitatively similar to Kwakernaak's continuous-time recovery-both techniques minimize a norm of $\underline{M}_o$; both make n-m modes of $\underline{A}$-$\underline{B}$ $\underline{G}$ unobservable from $\underline{C}$ and make the remaining modes as fast as possible; and

(3) The recovery error using DLQR is non-zero and a function of sampling time for sampled-data systems.

Whether the MBC/LTR approach is attractive in light of these conclusions is a matter for individual designers to decide for themselves.

9.2 Suggestions for Future Research

(1) The compensators in this thesis are all strictly proper. Since exact recovery depends on successfully hiding all of the compensator poles, the use of non-strictly proper (though, of course, still proper) compensators seems promising in applications where such a compensator can be implemented.

(2) Since exact recovery with a strictly proper compensator requires $\underline{C}\ \underline{H} = \underline{0}$, research into full-state design techniques that comply with this constraint may prove worthwhile.

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