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Decoherent histories approach to the cosmological measure problem

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Abstract. The method of decoherent histories allows probabilities to be assigned to sequences of quantum events in systems, such as the universe as a whole, where there is no external observer to make measurements. This paper applies the method of decoherent histories to address cosmological questions. Using a series of simple examples, beginning with the harmonic oscillator, we show that systems in a stationary state such as an energy eigenstate or thermal state can exhibit decoherent histories with non-trivial dynamics. We then examine decoherent histories in a universe that undergoes eternal inflation. Decoherent histories that assign probabilities to sequences of events in the vicinity of a timelike geodesic supply a natural cosmological measure. Under reasonable conditions, such sequences of events do not suffer from the presence of unlikely statistical fluctuations that mimic reality.

1. Introduction

The consistent or decoherent histories approach to quantum mechanics is a method for assigning probabilities to sequences of events for a quantum-mechanical system [1-35]. Because it does not rely on the notion of measurement, the decoherent histories approach is useful in theories such as quantum cosmology where probabilities have to be assigned, but no external system is measuring or decohering the system – in this case, the universe itself. The cosmological measure problem addresses the question of how to assign probabilities to events in various cosmological scenarios, e.g., eternal inflation [36-54]. Some apparently reasonable cosmological measures have counter-intuitive consequences, e.g., the notion that we are all just ‘Boltzmann brains’ that arose from a statistical fluctuation. This paper applies the method of decoherent histories to the cosmological measure problem. First, the formalism of decoherent histories is reviewed. Second, the method is applied to simple systems to show that stationary states such as energy eigenstates or thermal states can still exhibit non-trivial decoherent histories, in contradiction to the natural intuition that systems in stationary states ‘do nothing.’ (This result answers a question raised by Boddy *et al.* about whether systems in stationary states can exhibit decoherent histories that fluctuate over time [54].) Decoherent histories are used to show that systems described by a stationary state of quantum jump models or environmentally induced decoherence models can still be thought of as exhibiting non-trivial temporal fluctuations. Third, decoherent histories are applied to eternal inflation [55-56]. Decoherent histories for the sequences of events that occur in the vicinity of a timelike geodesic are shown to give rise to cosmological measures that differ from conventional volume-counting measures, an approach similar to that of Nomura [57-59]. Under conventional assumptions about the potentials in the underlying physics, such



histories give rise to a picture of eternal inflation in which a period of rapid inflation gives rise to a Friedman-Robertson-Walker cosmology in de Sitter space with small cosmological term. After a period long compared with the Hubble time of the de Sitter space, but short compared with the time required to generate a Boltzmann brain, quantum/thermal fluctuations give rise to another period of rapid inflation, and the cycle recommences.

2. Review of decoherent histories

The decoherent histories approach originated with Griffiths [1-3], who called this approach consistent histories, and was independently developed by Omnès [4-7] and by Gell-Mann and Hartle [8-15], who termed this approach decoherent histories. We will adopt the latter nomenclature. Consider a sequence of projective measurements that could be made on a quantum system at times $t_1 < \dots < t_n$. (The extension to generalized measurements will be given below.) The measurement at time t_k has exhaustive and mutually exclusive outcomes α_k . This measurement corresponds to a set of projection operators $\{P_{\alpha_k}^k\}$ in the Heisenberg picture, where $P_{\alpha_k}^k P_{\alpha'_k}^k = \delta_{\alpha_k \alpha'_k} P_{\alpha_k}^k$, and $\sum_{\alpha_k} P_{\alpha_k}^k = I$. A *history* $\tilde{\alpha}$ corresponds to a sequence of outcomes $\tilde{\alpha} = \alpha_1 \dots \alpha_n$. The decoherence functional $D(\tilde{\alpha}, \tilde{\alpha}')$ for initial state ρ_i is defined to be

$$D(\tilde{\alpha}, \tilde{\alpha}') = \text{tr} P_{\alpha_n}^n \dots P_{\alpha_1}^1 \rho_i P_{\alpha'_1}^1 \dots P_{\alpha'_n}^n = \text{tr} P_{\tilde{\alpha}}^\dagger \rho_i P_{\tilde{\alpha}'}. \quad (1)$$

If one performs this sequence of measurements, then the probability for the sequence of events that constitutes the history $\tilde{\alpha}$ are given by the diagonal term of the decoherence functional $p(\tilde{\alpha}) = D(\tilde{\alpha}, \tilde{\alpha})$. Note that these probabilities are non-negative and sum to 1.

The off-diagonal terms in the decoherence function measure the degree of quantum interference between different histories. When these terms are comparable to the corresponding on-diagonal terms, it means that measurements that correspond to earlier events in the history have a strong effect on the probabilities for later events [35]. For example, in the double-slit experiment, the histories that correspond to which slit the particle goes through are coherent: consequently, a measurement that determines through which slit a particle passes has a strong effect on probability of where the particle lands on the screen. Indeed, such a measurement destroys the characteristic interference pattern of the double-slit experiment.

The presence of coherence means that the probabilities $p(\tilde{\alpha})$ fail to obey probability sum rules. For example, let P_1^1, P_2^1 project onto the states that go through slit 1 or slit 2 at time t_1 , and P_x^2 project onto states that land on the screen at point x . The presence of coherence means that

$$\text{tr} P_x^2 \rho_i = \sum_{jj'} \text{tr} P_x^2 P_j^1 \rho P_j^1 \neq \sum_j \text{tr} P_x^2 P_j^1 \rho P_j^1. \quad (2)$$

That is, the probabilities for where the particle lands on the screen in the absence of a measurement of which slit it passed through are different from the probabilities for where the particle lands on the screen given that a measurement has been made. More generally, if

$$|D(\tilde{\alpha}, \tilde{\alpha}')|^2 / D(\tilde{\alpha}, \tilde{\alpha}) D(\tilde{\alpha}', \tilde{\alpha}') \leq \epsilon \ll 1, \quad (3)$$

then the histories are said to be approximately decoherent: such histories obey probability sum rules to accuracy ϵ .

The decoherence functional characterizes the degree to which measurements affect the future behavior of a quantum system. If a system decoheres with respect to a particular set of measurements, then the measurements made in the past have minimal effect on the outcomes of measurements made in the future. If this is so, then we can assign probabilities to the sequence of events corresponding to the measurement outcomes *whether the measurements are actually performed or not*. That is, although they are defined in terms of the mathematical apparatus

of measurements – projection operators or more generally positive operator valued measures (POVMs) – decoherent histories represent a method for assigning probabilities to sequences of events in the absence of measurement. In the words of Griffiths [1-3], decoherent histories refer to sequences of events ‘that we can talk about at the breakfast table.’ In the double slit experiment, we are not allowed to talk about the particle going through either one slit or the other, because to explain the interference pattern on the screen, it must go through both at once.

3. Stationary states support time-dependent decoherent histories

We now apply the method of decoherent histories to show that stationary states can exhibit non-trivial time-dependent decoherent histories.

First look at histories that trivially decohere – the histories for energy eigenstates of a closed physical system. In this case, $P_j^k = |E_j\rangle\langle E_j|$ for energy eigenstates $|E_j\rangle$. Note P_j^k is independent of the time step k . We have $P_j^k P_{j'}^k = \delta_{jj'} P_j$. and $D(\tilde{j}; \tilde{j}') \propto \delta_{\tilde{j}\tilde{j}'}$, independent of the initial state ρ_i . Here the initial projection takes ρ_i to an energy eigenstate, and subsequent projections simply confirm that the system remains in that state: histories of energy eigenstates do not exhibit time-dependent fluctuations.

Histories of other variables do exhibit fluctuations, however. A system can possess complementary sets of decoherent histories. In the harmonic oscillator, for example, even though energy does not fluctuate, phase does. Consider an harmonic oscillator with Hamiltonian $\hbar\omega a^\dagger a = \hbar\omega \sum_{\ell=0}^{\infty} |\ell\rangle\langle\ell|$, where $|\ell\rangle$ is the ℓ th energy eigenstate. For simplicity, restrict attention to the subspace \mathcal{H}_N of states whose energy is less than $N\hbar\omega$. Within this space, we can define phase states $|\phi\rangle = N^{-1/2} \sum_{\ell=0}^{N-1} e^{i\ell\phi} |\ell\rangle$. The phase states evolve in time as $|\phi\rangle \rightarrow |\phi + \omega t\rangle$. The N states $|\phi_j\rangle$ where $\phi_k = 2\pi j/N$ form an orthonormal basis for \mathcal{H}_N . Over time $\Delta t = 2\pi/N\omega$, we have

$$|\phi_j\rangle \rightarrow U_{\Delta t} |\phi_j\rangle = |\phi_{j+1}\rangle, \quad (4)$$

where $U_{\Delta t} = e^{-iH\Delta t/\hbar}$ and $j+1$ is defined modulo N . That is, over time Δt , the states $|\phi_j\rangle$ evolve deterministically into each other.

Suppose that the oscillator starts out in its ground state $|0\rangle$, and consider the histories defined by measurement operators $P_j = |\phi_j\rangle\langle\phi_j|$ spaced at intervals Δt . The decoherence functional is

$$D(j_1 \dots j_n, j'_1 \dots j'_n) = \text{tr} P_{j_n} U_{\Delta t} \dots U_{\Delta t} P_{j_1} |0\rangle\langle 0| P_{j'_1} U_{\Delta t}^\dagger \dots U_{\Delta t}^\dagger P_{j'_n}. \quad (5)$$

Because of the deterministic evolution of the phase states, we have $D(j_1 \dots j_n, j'_1 \dots j'_n) = 0$ unless $j'_n = j_n$, $j_k = j_{k-1} + 1$, and $j'_k = j'_{k-1} + 1$. That is, the off-diagonal terms of $D(j_1 \dots j_n, j'_1 \dots j'_n)$ are all zero, and the on-diagonal terms reflect the deterministic nature of the time evolution. The first projection yields equal probabilities $1/N$ for all phase states $|\phi_k\rangle$, and the subsequent evolution is entirely deterministic. The phase measurement corresponds to decoherent histories, even though the initial state is the ground state.

The set of decoherent histories corresponding to phase state evolution describes a quite different type of behavior from the behavior given by the set of decoherent histories corresponding to energy eigenstates. Described in terms of energy, the system remains static. Described in terms of phase, the system fluctuates. The two types of histories, energy and phase, represent complementary ways of describing the evolution of the same physical system.

As one might expect, histories that mix phase and energy eigenstates fail to decohere. Indeed, histories that begin in an energy eigenstate, progresses through a sequence of phase states, and then end in an energy eigenstate, are fully coherent: it is straightforward to show that the off-diagonal parts of the decoherence functional are of the same size as the on-diagonal parts. In particular, if one starts in the ground state, and then ends in the ground state, phase fluctuations

do not decohere. In general, histories that begin and end in a pure state decohere only if the histories are completely deterministic. When the final projector is a pure state, $P_n = |\phi\rangle\langle\phi|$, we have

$$D(\tilde{\alpha}; \tilde{\alpha}') = \langle\phi|P_{\tilde{\alpha}}|\psi\rangle\langle\psi|P_{\tilde{\alpha}'}^\dagger|\phi\rangle, \quad (6)$$

and

$$|D(\tilde{\alpha}, \tilde{\alpha}')|^2/D(\tilde{\alpha}, \tilde{\alpha})D(\tilde{\alpha}', \tilde{\alpha}') = 1, \quad (7)$$

unless $D(\tilde{\alpha}; \tilde{\alpha}') \propto \delta_{\tilde{\alpha}\tilde{\alpha}'}$: to be decoherent, the histories must be deterministic. So time-dependent histories of probabilistic fluctuations that begin in the vacuum and end in the vacuum are coherent.

By contrast, histories that begin in a mixed state such as a thermal state, and end in an mixed state, can be decoherent. Now consider decoherent histories of thermal states. Such states can either arise from interaction with a reservoir at temperature $T = 1/k_B\beta$, or as subsystems of a larger system that is in a pure state. The latter case arises in gravitational contexts such as Hawking radiation, Unruh radiation, and de Sitter space. The mathematical question of whether or not histories decohere depends only on the thermal form of the state and on the dynamics, not on whether the system is thermal because it is interacting with a reservoir or thermal because it is entangled. A considerable literature shows that stationary, thermal states exhibit non-trivial, temporally fluctuating, decoherent histories [12-35]. The positions of particles that begin in thermal states and that undergo Brownian motion exhibit decoherent histories, as do hydrodynamic variables – the coarse-grained values of quantum field, energies, and particle densities.

4. Decoherent histories and quantum jumps

When the system in question is an open system interacting with its environment, or equivalently a subsystem of a larger system, then the quantum jump picture yields decoherent histories for sequences of projections corresponding to the jump operators [18-22]. The decoherence of histories of quantum jumps allows one to relate the decoherent histories approach to the idea of environmentally induced decoherence. In particular, if the subsystem's dynamics can be described by a Lindblad equation, then the resulting stochastic Schrödinger equation intrinsically gives rise to decoherent histories. So, for example, a subsystem in a stationary thermal state that is a fixed point of the Lindblad equation undergoes decoherent histories described by probabilistic jumping from state to state. Such decoherent histories exist both when the open system is in a thermal state because of its interaction with a thermal environment, and when it is a subsystem of a larger system in a pure state. The automatic existence of decoherent histories corresponding to histories of the stochastic Schrödinger equation and of quantum state diffusion is particularly useful as the jump operators for systems weakly coupled to a Markovian environment represent jumps between energy eigenstates. For such systems, we are allowed to talk at the breakfast table about the system hopping thermally from energy eigenstate to energy eigenstate as described by the Bloch-Redfield equation, even though the system as a whole is a stationary thermal state.

We present here a simple derivation of why systems that evolve according to a Lindblad equation exhibit decoherent histories. In contrast to previous derivations [18-22], which focus on an Ito calculus derivation of the relation between quantum jumps and decoherent histories, the derivation given here is based on environmentally induced decoherence. The Lindblad equation represents the most general infinitesimal completely positive (i.e., legal) time evolution for a quantum system. A general Lindblad equation takes the form

$$\frac{\partial\rho}{\partial t} = -i[H, \rho] - \gamma/2 \sum_j (L_j^\dagger L_j \rho - 2L_j \rho L_j^\dagger + \rho L_j^\dagger L_j). \quad (8)$$

The Lindblad equation for a given system interacting with its environment can be derived by starting with system and environment in the uncorrelated state $\rho_S \otimes \rho_E$, and applying the unitary system-environment time evolution $U(\Delta t)$ over a time Δt equal to the correlation time of the environment. The system evolves to

$$\rho_S(0) \rightarrow \rho_S(\Delta t) = \text{tr}_E \rho U(\Delta t) \rho_S \otimes \rho_E U^\dagger(\Delta t). \quad (9)$$

Expanding to second order in Δt yields the infinitesimal form of the Lindblad equation (8). That is, in addition to being the general infinitesimal form for a completely positive map, the Lindblad equation has a physical interpretation as an approximate infinitesimal time evolution for a system interacting unitarily with an environment whose correlations decay over a characteristic time.

To look at decoherent histories, for simplicity consider the case where there is only one Lindblad operator $L_1 = L$ and $H = 0$: $\partial\rho/\partial t = (-\gamma/2)(L^\dagger L\rho - 2L\rho L^\dagger + \rho L^\dagger L)$. Use the polar decomposition to write $L = UA$, where U is unitary, $U^\dagger = U^{-1}$, and A is Hermitian, $A = A^\dagger$. The infinitesimal dynamics generated by the Lindblad equation over time Δt is equivalent to the following measurement plus feedback procedure:

(1) Make a generalized measurement on the system with POVM operators $M_1 = A^2\gamma\Delta t$ and $M_2 = 1 - A^2\gamma\Delta t$. With probability $p_1 = \gamma\Delta t \text{tr} A^2\rho$ the system goes to the state $\rho_1 = (1/p_1)A\rho A$, and with probability $p_0 = 1 - p_1$ the system goes to the state $\rho_0 = (1/p_0)\sqrt{1 - A^2\gamma\Delta t} \rho \sqrt{1 - A^2\gamma\Delta t} \approx \rho_0$. Because any generalized measurement can be written as a von Neumann measurement on system plus an ancilla [55], we have $\rho_1 = p_1^{-1}P_1\rho \otimes \sigma P_1$, $\rho_0 = p_0^{-1}P_0\rho \otimes \sigma P_0$, for projectors $P_1, P_0 = 1 - P_1$, and ancilla in state σ . (This technique shows how to generalize decoherent histories from projective measurements to generalized measurements [35].)

(2) Now feed back the result of the measurement. If the result of the measurement is the state 1, apply the unitary transformation U . If the result is 0, do nothing. The system is now in the state

$$\begin{aligned} \rho' &= \sqrt{1 - A^2\gamma\Delta t} \rho \sqrt{1 - A^2\gamma\Delta t} + L\rho L^\dagger \gamma\Delta t \\ &= \rho - (\gamma\Delta t/2)(L^\dagger L\rho - 2L\rho L^\dagger + \rho L^\dagger L) + O(\Delta t^2). \end{aligned} \quad (9)$$

Because the Lindblad equation is mathematically equivalent to projective measurement on system plus ancilla followed by unitary feedback, the set of histories of the system plus ancilla corresponding to the projections P_0, P_1 repeated at time intervals Δt are decoherent. (Note that in this picture the unitary time evolutions between projections depend on the previous history of projections, corresponding to the more general model of decoherent histories given by Gell-Mann and Hartle [7-15].) One can think of this demonstration of decoherence as a *derivation* of the stochastic Schrödinger equation or of a quantum jump model. Because the histories are decoherent, we can describe the time evolution of the system in terms of a stochastic process: over time Δt the system goes to the state $\rho_1 = p_1^{-1}L\rho L^\dagger$ with probability $p_1 = \text{tr} L\rho L^\dagger \gamma\Delta t$, or remains in the state $\rho_0 = \sqrt{1 - A^2\gamma\Delta t} \rho \sqrt{1 - A^2\gamma\Delta t}$ with probability $p_0 = 1 - p_1$. The treatment of the general Lindblad equation is essentially the same, except now there are multiple types of quantum jumps that can occur, one type for each Lindblad operator, and the time evolution in between jumps includes the effect of the system Hamiltonian. Note that the histories induced by the Lindblad equation are decoherent for any initial state, including stationary states such as thermal states or energy eigenstates.

This result establishes that an open system whose time evolution is governed by a Lindblad equation can be thought of as undergoing quantum jumps even when it is in a stationary state. The histories corresponding to different sequences of quantum jumps are decoherent.

As an example, consider a harmonic oscillator with Hamiltonian $H = \hbar\omega \sum_{\ell} |\ell\rangle\langle\ell| = \hbar\omega a^{\dagger}a$ as above, interacting linearly with a bath of modes of the electromagnetic field at temperature $T = 1/\beta$. The oscillator obeys the Lindblad equation

$$\frac{\partial\rho}{\partial t} = -i[H, \rho] - \frac{\gamma_+}{2}(aa^{\dagger}\rho - 2a^{\dagger}\rho a + \rho aa^{\dagger}) - \frac{\gamma_-}{2}(a^{\dagger}a\rho - 2a\rho a^{\dagger} + \rho a^{\dagger}a), \quad (10)$$

where $\gamma_+/\gamma_- = e^{-\beta\omega}$. The thermal state $\rho_{th} = (1/Z)e^{-\beta H}$ is a stationary state of the time evolution. In the thermal state, the oscillator exhibits decoherent histories over sequences of energy eigenstates, in which the n th energy eigenstates absorbs photons at a rate $n\gamma_+$ and emits photons at a rate $n\gamma_-$.

4.1. Decoherent histories for open systems over times longer than the relaxation time

Consider an open system that relaxes over time τ to a fixed state ρ_0 of the system's Hamiltonian H , e.g., a thermal state $\rho_0 = (1/Z)e^{-\beta H}$. Suppose that the time intervals between the projectors P_{α_j} are much longer than τ . Then *any* set of histories decoheres (i.e., not merely the jump histories given by the Lindblad operators). The reason is simple: because the system relaxes to the same state independent of the input state, if one waits for $\gg \tau$, the probabilities for measurement are just given by the probabilities for measurement on ρ_0 , independent of whether some measurement was made long ago or not.

4.2. Time-dependent fluctuations in stationary states

The notion that a stationary quantum state does not fluctuate in time seems at first a perfectly reasonable one [54]. However, the time-independent history of a stationary state can also be decomposed as a quantum superposition of time-dependent fluctuating histories. Under a wide variety of circumstances, those histories decohere, and so we are free to describe the time evolution of such systems in terms of those histories. A recent paper [54] suggested the contrary, it is worth discussing briefly the decoherent histories in thermal states, why do Boddy *et al.* decide that decoherent histories are not possible in such states [54]? There are two reasons. First, they use a time-symmetric version of decoherent histories that includes both initial and final states. Second, they use decoherent histories with only one intermediate set of events between those initial and final states. While it is true that such histories do not decohere, it is unclear why one should to restrict one's attention so such histories.

We review their argument. The time-symmetric version of decoherent histories is appropriate when all or part of the universe possesses a final state, as in spatially and temporally compact universes whose state is computed by the Hartle-Hawking imaginary time procedure, or in the Horowitz-Maldacena model of black hole evaporation. It does not seem that such a situation holds in inflationary models, and so it is unclear why this formalism should be applied here.

The decoherence functional $D(\tilde{\alpha}, \tilde{\alpha}')$ for initial state ρ_i and final state ρ_f is defined to be

$$D(\tilde{\alpha}, \tilde{\alpha}') = Z^{-1} \text{tr} \rho_f P_{\alpha_n}^n \dots P_{\alpha_1}^1 \rho_i P_{\alpha'_1}^1 \dots P_{\alpha'_n}^n = Z^{-1} \text{tr} \rho_f P_{\tilde{\alpha}}^{\dagger} \rho_i P_{\tilde{\alpha}'}, \quad (11)$$

where $Z^{-1} = \text{tr} \rho_f \rho_i$. As before, histories decohere if the off-diagonal terms in the decoherence functional are small compared with the on-diagonal ones, given that initial state is ρ_i and the final state is ρ_f . The addition of the final state in the decoherence function is equivalent to adding one additional measurement operator, whose final projection corresponds to a measurement revealing that the system is in ρ_f . Such a measurement can be performed, for example, by adjoining an ancillary system in state ρ_A and performing a projection P on system and ancilla such that $\text{tr}_A P I \otimes \rho_A P = \rho_f$.

Boddy *et al.* consider closed quantum systems and investigate situations where the initial and final state are both stationary states of the system dynamics. They consider histories with a single projection between the initial and final state, for which the decoherence functional is

$$Z^{-1} \text{tr} \rho_f P_{\alpha} \rho_i P_{\alpha'}. \quad (12)$$

Boddy *et al.* note correctly that this decoherence functional is not dependent on the evolution times between the initial state, the projections, and the final state. They conclude (also correctly), that such histories do not exhibit perfect decoherence.

Two questions: first, why use the time-symmetric version of the decoherent histories formalism? Boddy *et al.*'s justification is that they are interested in histories that begin and end in a thermal state, e.g., the state of the fields de Sitter space. When looking at cosmological histories, however, there is no particular reason for making this restriction unless one desires artificially to restrict the set of possible decoherent histories. If one uses the ordinary formulation of decoherent histories, starting from initial states and evolving forward in time, thermal states such as those in de Sitter space can exhibit a wide variety of non-trivial decoherent histories. The second question is simpler: why restrict attention to histories with only one set of events? Suppose that one adds a second set of events, so that the decoherence functional is

$$Z^{-1} \text{tr} \rho_f P_{\alpha_2} P_{\alpha_1} \rho_i P_{\alpha'_1} P_{\alpha'_2}. \quad (13)$$

In this case, because the projection operators in the Heisenberg picture do depend on time, the decoherence functional depends on the time difference between the two sets of events. As noted above, decoherent histories over multiple sets of events also exhibit time dependence even when the initial and final states are stationary.

Boddy *et al.* are correct that the particular set of histories that they investigate do not decohere. It would be a mistake to conclude, however that systems that begin and end in stationary states cannot exhibit decoherent histories. Indeed, a rather trivial counterexample occurs when both ρ_i and ρ_f are the fully mixed stationary state I/d , corresponding to a thermal state with infinite temperature. In this case, the different sets of histories described above for both open and for closed systems naturally decohere. When both open and for closed systems naturally decohere. When ρ_i and ρ_f are thermal states $Z^{-1} e^{-H/kT}$ at finite temperature, then we can project those states onto a typical subspace of dimension $d \propto e^S$, where $S = H/kT - \ln Z$ is the entropy of the thermal state. Once again, these states naturally decohere as in the examples above.

5. Quantum cosmology and decoherent histories

Having established that stationary states do indeed exhibit quantum fluctuations, at least by the criterion of decoherent histories, let's turn to quantum cosmology in models of eternal inflation. In such models, vacuum energy induces an effective cosmological term Λ , causing spacetime locally to resemble de Sitter space. Inertial observers witness an event horizon at distance $\ell = \sqrt{3/\Lambda}$, and detect horizon radiation in a thermal state with temperature $T = 1/2\pi\ell$.

Look at decoherent histories corresponding to measurements made by such an inertial observer as the universe undergoes inflation and settles down into de Sitter space with a small cosmological term [57-59]; after a long time (estimated below) the observer encounters a local thermal fluctuation that gives rise to a region that possesses a high cosmological term and that is large enough to seed another inflationary epoch. Actually to make these measurements, such an inertial observer would have to be a hardy individual, capable of surviving extreme temperatures and curvatures. (Assume, however, that the observer does not fall into a black hole.) The whole point of decoherent histories, however, is to be able to assign probabilities to events whether or not the measurements corresponding to the events are actually performed. An hypothetical

inertial observer suffices to assign decoherent histories. The measurements correspond to coarse grained observations of fields, energy densities, pressure, etc., in the local vicinity of the observer. As noted above in [10-35], such histories generically exhibit approximate decoherence.

Consider decoherent histories that correspond to the inertial observer making coarse-grained measurements of the fields in her vicinity together with the effective cosmological term Λ . The events along any history depend on the value of Λ_0 in the first projection in the history. Suppose for the moment that Λ_0 is large, corresponding to high energy in the false vacuum. The observer initially sees thermal fluctuations in the fields corresponding to de Sitter space at high temperature, and witnesses the surrounding spacetime inflate at a rate $\propto \sqrt{\Lambda_0}$. After a characteristic time-scale τ_0 , the observer enters a region in which false vacuum decays.

That is, in such a history the universe undergoes inflation via the usual scenarios, yielding a universe more or less like our own. If at the end of inflation the cosmological term is non-zero, then the region in the vicinity of the observer will eventually settle down to de Sitter space again with effective cosmological term $\Lambda_1 < \Lambda_0$ and a horizon at distance $\ell_1 = \sqrt{3/\Lambda_1}$. The state within the horizon is thermal with temperature $T_1 = 1/2\pi\ell_1$. As above, this thermal state can exhibit decoherent histories corresponding to fluctuations in local energy density. Eventually, the region in the vicinity of the observer will exhibit a fluctuation that takes it back to a regime with high Λ and will begin inflating again.

To estimate the time τ_1 it takes for rapid inflation to recommence, we look at how long it takes for a thermal fluctuation to generate a re-inflating region. By assumption, the dynamics possesses at least one quantum state for a region of radius $\ell = C\ell_0 = C\sqrt{3/\Lambda_0}$ that undergoes inflation with effective cosmological term Λ_0 , where C is a positive $O(1)$ constant. The energy that has to be collected from the thermal radiation in de Sitter space to attain energy density Λ_0 over a volume ℓ^3 is

$$\Delta E = \ell^3 \Lambda_0 = 3C^3 \ell_0. \quad (14)$$

The de Sitter radiation has temperature $T_1 = 1/2\pi\ell_1$. To collect the energy ΔE within a region of spatial extent ℓ reduces the entropy of the surrounding de Sitter radiation by $\Delta S = \Delta E/T_1$. The probability that a thermal fluctuation at the de Sitter temperature gives rise to a region with the energy density needed to re-inflate is thus $e^{-\Delta S} = e^{-6\pi C^3 \ell_0 \ell_1}$.

It is not enough for the energy required for re-inflation to assemble itself: the fields must also be in the proper false vacuum state. If the energy is assembled in a random state, the overlap with the proper inflating state goes as e^{-S_0} , where $S_0 = \pi\ell_0^2$ is the de Sitter entropy for the fields in the region – i.e., the maximum entropy for the region. The overall thermal probability for a fluctuation that creates the inflating region then goes as

$$e^{-\Delta E/T_1 - S_0} = e^{-6\pi C^3 \ell_0 \ell_1 - \pi\ell_0^2} = e^{-6C^3 \sqrt{S_0 S_1} - S_0}. \quad (15)$$

Here $S_0 = \pi\ell_0^2$ is the entropy of the high energy density de Sitter space with cosmological term Λ_0 , and S_1 is the entropy of the low energy density de Sitter space. Remarkably – given the simple and non-gravitational nature of the argument – equation (15) reproduces (up to the value of the constant C) the Farhi-Guth-Guven formula [56] for the thermal probability of exciting an inflating volume of spatial extent $\approx \ell_0$. Equation (15) shows that the decoherent histories corresponding to a hardy inertial observer reproduce the eternal inflation picture suggested by Albrecht [49].

Equation (15) gives the thermal probability for a fluctuation that can re-ignite inflation. To estimate the time it takes for such a fluctuation to arise, note that in de Sitter space with cosmological term Λ_1 , the characteristic time for fluctuations to arise and decay is $\approx \pi/T = 4\pi^2\ell_1$, yielding a time

$$\tau_1 \approx 4\pi^2\ell_1 e^{(6\pi C^3 \ell_0 \ell_1 + \pi\ell_0^2)}, \quad (16)$$

for the inertial observer to encounter another rapidly inflating region. Note that process of reinflation is much more likely to begin with a small region with large cosmological term, than with a large region with small cosmological term. This observation suggests that reinflation should typically begin at a scale close to the Planck scale.

5.1. Initial state

As in [49] this argument yields an ergodic model of eternal inflation. Inflation takes place at high energy scales; the value of the field rolls downhill, yielding an FRW universe with the usual features; eventually, the presence of a small cosmological term yields a fluctuation that causes inflation to begin again with large cosmological term. By the arguments given above, such histories are generically decoherent. The initial state of the universe as a whole can be taken to be the stationary state given by the ergodic average of the the state of the universe over time.

Note that, although ergodic, the histories seen by an inertial observer are not time-reversal invariant. Every time rapid inflation begins again at high Λ_0 , it supplies a large source of free energy so that the inertial observer sees entropy increasing, consistent with the second law of thermodynamics. This time-asymmetry of the individual histories is nonetheless consistent with time-reversal invariance of the initial, stationary state of the universe *and* of its dynamics. If instead of using the theory of decoherent histories with an initial state, we use decoherent histories that end in a final state, then we can decompose the stationary state of the universe into a superposition of decoherent histories that end in a final state of high Λ_0 , and that evolve backward in time. The histories that correspond to inertial observers with a fixed final state are then just the time reversed version of histories with a fixed initial state: even though they are moving in the opposite direction in time, the time-reversed inertial observers still observe entropy increasing.

6. Re-inflation versus Boltzmann brains

Now compare the probability of the inertial observer encountering another rapidly inflating region with the probability of encountering a thermal fluctuation that mimics some small piece of our universe, e.g., a ‘Boltzmann brain.’ The argument for the probability of recreating an inflating region via a thermal fluctuation is readily generalized to calculating the probability of recreating any system with energy ΔE and entropy $S = S_{max} - \Delta S$, where S_{max} is the maximum entropy for the system confined to the volume in which it is created. The thermal probability of such a fluctuation is $e^{-\Delta E/T_1 - \Delta S}$. Comparing with equation (14) we see that as long as the energy ΔE in the Boltzmann brain is greater than the energy $\approx \ell_0$ required to reignite inflation, then the inertial observer is more likely to encounter brains that arise by the ordinary process of evolution in an FRW universe rather than ones that arise from thermal fluctuations. If ℓ_0 is at the grand unification scale or a shorter length scale, e.g. the Planck scale, then the vast majority of brains encountered by the observer over its infinite history will be the usual kind of brains.

7. Summary

This paper investigated the question of whether stationary states can exhibit non-trivial temporal fluctuations. Viewed through the lens of decoherent histories, the answer is an unqualified Yes. Closed systems in energy eigenstates exhibit decoherent histories with non-trivial temporal fluctuations. Open systems that are subsystems of larger systems exhibit decoherent histories that correspond to the quantum jump or stochastic Schrödinger model. We then investigated decoherent histories that correspond to observations made by an inertial observer in models of eternal inflation. Such histories contain periods of inflation leading to FRW universes with small cosmological term; thermal fluctuations from the cosmological term then reignite inflation. The time scale required to reignite inflation is long compared with the

horizon scale but much shorter than the time required to generate thermal fluctuations that mimic systems that evolved from initial low-entropy states.

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