SIMPLIFYING DYNAMIC MODELS BY RETAINING SELECTED BEHAVIOR MODES

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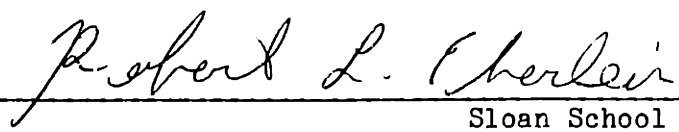
Signature of Author 
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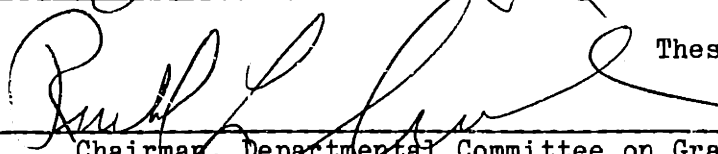
Accepted by 
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ABSTRACT

Many useful dynamic models developed in economics and other areas are very large. The size of these models makes them expensive to use and difficult to understand. When only certain behavior modes generated by the large models are of interest it may be possible to develop simplified models that contain much of the information in the original model with respect to these behavior modes. This information takes the form of interactions among the variables of the model, and the causes of these interactions.

Attention is restricted to linear time invariant systems of difference equations, though most of the results apply without alteration to differential equations. It is assumed that the behavior modes of interest have been identified. The major obstacle to the development of useful simplified models is to retain the interpretability of the simplified model. To do this requires that the variables of the simplified model correspond in some obvious manner to the variables of the original model. This requirement is used in deriving methods for choosing the variables of the original model most important in generating the behavior modes of interest.

The relationship of the output of the simplified models developed to the output of large models is explored. Because the simplified models only retain elements of the dynamics of the original model information is lost, and the character of the output is changed. This difference is attributable to the behavior modes of the sytem that are not considered to be of interest. The possiblities for the estimation of the simplified models based on the output of the original model are explored. The results are negative; in most cases the development of simplified models that retain behavior modes requires complete knowledge of the original model.

The appliciation of the techniques developed is made to a medium-size econometric model. The modes of interest are the business cycle frequency oscillatory modes that the econometric model generates. The results of this application indicate that prices are important in generating these modes in the model considered. The application also gives information on the relative merits of the simplification techniques developed.

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Finally, I would like to thank two small cities, Ushuaia and Inuvik, and the great distance that separates them, without whose existence I would have lacked the motivation for the timely completion of this thesis.

Contents

Notation

Conventions

Vectors are written as lower case underscored characters such as $\underline{x}$, $\underline{y}$, $\underline{\xi}$. All vectors are column vectors.

Subvectors will be subscripted with either a 1 or a 2 as in $\underline{x}_1$ and $\underline{x}_2$.

Elements of vectors will be subscripted with an index variable as in $\underline{\xi}_i$, for the i'th element of $\underline{\xi}$.

In distinguishing between vectors with the same symbol superscript index characters or numbers will be used as in $\underline{1}^i$ and $\underline{f}^1$.

Matrices are written as upper case underscored characters such as $\underline{A}$, $\underline{B}$, $\underline{\Lambda}$.

Zero matrices and vectors will be written as $\underline{0}$ and identity matrices will be written as $\underline{I}$; the dimension should be obvious from usage.

Partitioned matrices will be written encased in " ¦ ¦ " .

Determinants of matrices will be written as Det($\underline{\text{MATRIX}}$) .

The trace of a matrix will be written as Trace($\underline{\text{MATRIX}}$) .

The transfer function of a system will be written as $\hat{\underline{G}}(z)$.

The square root of -1 will be written as j .

The expectations operator will be given by E and will operate over the terms enclosed in square brackets "[ ]" .

Variables covered by a tilde ($\tilde{\underline{A}}$, $\tilde{\underline{B}}$ etc) are the simplified model counterparts to the variable written without the tilde.

The generalized inverse of a matrix $\underline{F}$ will be written as $\underline{F}^-$.

The symbols listed below are reserved for the purposes noted. The basic data-generating process or the "true system" will be given by

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{B}\underline{u}_t + \underline{D}\underline{e}_t$$

or writing this as a partioned matrix we have

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{B}_1 \\ \underline{B}_2 \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{D}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \ .$$

$\underline{x}$ — The state variable. It is assumed throughout that $\underline{x}$ is interpretable in terms of physically and economically meaningful concepts.

$\underline{u}$ — A variable of exogenous inputs into the system. The use of the current $\underline{u}$ is consistent with standard econometric notation.

$\underline{e}$ — A white noise process.

$\underline{A}$ — The NxN dynamics matrix of the system.

$\underline{B}$ — The NxP matrix determining the response of the system to exogenous inputs.

$\underline{D}$ — The NxM matrix determing the response of the system to different noise inputs.

$\underline{x}_1, \underline{x}_2$ - the subvectors of the $\underline{x}$ vector made up of the first n elements and the second N-n elements of $\underline{x}$ respectively.

$A_{11}, A_{12}, A_{21}, A_{22}$ - the submatrices of the matrix $\underline{A}$. $\underline{A}_{11}$ is nxn, $\underline{A}_{12}$ is nx(N-n), A_{21} is (N-n)xn and $\underline{A}_{22}$ is (N-n)x(N-n).

$\underline{B}_1, \underline{B}_2$ - the submatrices of $\underline{B}$ including all columns, $\underline{B}_1$ is nxP and $\underline{B}_2$ is (N-n)xP .

$\underline{D}_1, \underline{D}_2$ - the submatrices of $\underline{D}$ including all columns, $\underline{D}_1$ is nxM and $\underline{D}_2$ is (N-n)xM .

λ_i - the i'th eigenvalue of the matrix $\underline{A}$

$\underline{\Lambda}$ - the modal matrix of $\underline{A}$. (The NxN diagonal matrix with the eigenvalues of $\underline{A}$ on the diagonal).

$\underline{\Lambda}_{11}$, - the mxm submatrix matrix $\underline{\Lambda}$ with the eigenvalues of interest along the diagonal and zeroes elsewhere

$\underline{R}$ - the NxN matrix with columns given by the right eigenvectors of the matrix $\underline{A}$. The eigenvectors are normalized to be of length one.

$\begin{vmatrix} \underline{R}_{11} & \underline{R}_{12} \\ \underline{R}_{21} & \underline{R}_{22} \end{vmatrix}$ - a partition for $\underline{R}$. $\underline{R}_{11}$ is nxm, $\underline{R}_{12}$ is nx(N-m), $\underline{R}_{21}$ is (N-n)xm and $\underline{R}_{22}$ is (N-n)x(N-m) with $m \leq n$.

$\begin{vmatrix} \underline{R}_1 & \underline{R}_2 \end{vmatrix}$ - a different partition for $\underline{R}$. $\underline{R}_1$ is Nxm and $\underline{R}_2$ is Nx(N-m)

$\underline{r}^k$ - the k'th right eigenvector of $\underline{A}$ (or column of $\underline{R}$) .

$\underline{L}$ - the NxN matrix with rows given by the left eigenvectors of $\underline{A}$. The eigenvectors are normalized so that $\underline{LR} = \underline{RL} = \underline{I}$. This is possible as long as $\underline{A}$ is nondefective.

$\begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \\ \underline{L}_{21} & \underline{L}_{22} \end{vmatrix}$ - a partition for $\underline{L}$. $\underline{L}_{11}$ is mxn, $\underline{L}_{12}$ is mx(N-n), $\underline{L}_{21}$ is (N-m)xn and $\underline{L}_{22}$ is (N-m)x(N-n) with $m \leq n$.

$\begin{vmatrix} \underline{L}_1 \\ \underline{L}_2 \end{vmatrix}$ - a different partition for $\underline{L}$. $\underline{L}_1$ is mxN and $\underline{L}_2$ is (N-m)xN

$\underline{l}^k$ - the k'th left eigenvector of $\underline{A}$ (or transposed row of $\underline{L}$).

$\underline{\Omega}$ - the covariance matrix for the noise terms exciting the data generating process.

$\underline{\xi}$ - the vector of transformed states that results when $\underline{x}$ is premultiplied by $\underline{L}$

$\underline{\Gamma}(k)$ - the state covariance matrix $\underline{\Gamma}(k) = E\left[\underline{x}_t \underline{x}_{t-k}^T\right]$

$\underline{\Psi}(z)$ - at $(z=e^{j\theta})$ the power spectrum of the state variables

0 Models and Small Models

Introduction

This thesis considers the use of small models as representatives of large models. Attention is restricted to systems of linear, timeinvariant, first-order, stochastic difference equations. The emphasis is on simplifications intended to maintain internally generated dynamics. The relationships of such simplified models to available data and the consequences of and possibilities for estimation of the smaller models are considered. Because almost all models are simplified representations of complex processes, the material developed can be useful in the evaluation and understanding of all models.

0-A The Problem

It is assumed throughout that the available data are generated by a large system of linear, time-invariant, stochastic difference equations. This "system" or "data-generating process" will be consistently represented by

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{B}\underline{u}_t + \underline{D}\underline{e}_t \ . \qquad (0.1)$$

In this equation, $\underline{x}$ is a state vector of N components, and it is assumed that some or all of the components of $\underline{x}$ are observed and are of interest.

The P component vector $\underline{u}$ represents exogenous variables which enter and influence the system. The M component vector $\underline{e}$ represents an error process with expectation $\underline{0}$, variance $\underline{\Omega}$ and no correlation over time. The matrix $\underline{A}$ (NxN) is referred to as the dynamics matrix and determines the effect of previous state values on the evolution of the system. The matrices $\underline{B}$ (NxP) and D (NxM) determine the influence of the exogenous variables and noise on the system.

The "system" as given in equation 0.1 will be taken as given and it will be assumed that the "system" is precisely what we want to understand and simplify. Treating the "system" as exactly correct is a useful device for developing the theory. However, it needs to be recognized that every model or "system" is an approximate method of representing the processes of interest. In order to stress this point the word "system" will always appear in quotation marks when it refers to equation 0.1. The "system" will also be referred to as the original model or the full model.

The "system" given in equation 0.1 is likely to be very large and difficult to understand. In simplifying the model we will go from such a large model to a smaller model of the same form. We define the size of the simplified model in terms of the number of endogenous variables that it contains.[†] Analogously to equation 0.1, the small simplified model will be

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[†]Different definitions of size are of course available. Blaszcynski (1982) reviews the literature in this area.

given by

$$\tilde{\underline{x}}_t = \tilde{\underline{A}}\underline{x}_{t-1} + \tilde{\underline{B}}\underline{u}_t + \tilde{\underline{D}}\underline{e}_t \cdot \qquad (0.2)$$

The tilde is used to distinguish elements of the simplified model from those of the original system; otherwise equations 0.1 and 0.2 are the same. The state vector ($\tilde{\underline{x}}$) is of length n for the simplified model with n<N. The exogenous input and noise vectors are the same, and the $\tilde{\underline{B}}$ and $\tilde{\underline{D}}$ matrices are dimensioned accordingly ($\tilde{\underline{B}}$ is nxP and $\tilde{\underline{D}}$ is nxM).

The simplified model is influenced by the same noise and exogenous variables as the system. The simplification essentially reduces only the number of states to be considered, though, in practice, the influence of some of the exogenous variables would likely be small. That is, columns of the $\tilde{\underline{B}}$ and $\tilde{\underline{D}}$ matrices would be approximately zero.

Such a simplified model is desirable for a number of reasons. It is easier to understand, better suited for communication, tractable for the purposes of policy development, less costly to simulate and simpler to estimate. However, the simple model is by nature an approximation. Elements of the more complex "system" must be ignored, and accuracy will therefore be sacrificed. The nature of the inaccuracies that arise and the their interdependence have to be considered in developing a simple model.

Because simple models cannot be the same as more complex systems it is necessary to choose the characteristics of the system to be maintained in developing a simple model. The choice of a simplified model will, in

general, depend on the characteristics of concern. One characteristic of a system is an observed pattern of behavior which the system generates. For example, one pattern of behavior which has long been of interest to economists is a periodic, 3 to 10 year fluctuation in employment and production in industrialized capitalist economies. This is commonly referred to as the business cycle (Hicks 1950). Suppose we had a large but very accurate of model of world economic interactions. The model might generate a number of different patterns of behavior including, for example, growth and business cycles. We could use a simplification of that model to understand the mechanisms underlying business cycles. A different simplification might be used to understand the causes of growth.

Let us take as given a good, but complex, model that displays patterns of behavior of interest to us. The questions regarding the model that most naturally arise are

1) Is there a systematic way of understanding and explaining the sources of the patterns of behavior produced by the model?

2) Can we build a smaller model that is consistent with our explanation and can be used to communicate the explanation?

3) If we use a smaller model to develop policies that are seen as beneficial, then what effect do these policies have when implemented on the complex model?

4) Is it possible to estimate the parameters of this simplified model without reference to the larger model?

It will become clear as the theory is developed that these questions are quite closely related.

O-B An Approach

In order to make headway in addressing the above questions it is necessary to have a clearly defined and restricted approach. We have chosen to concentrate on the behavior associated with certain eigenvalues of the homogeneous "system."[†] This approach follows a fairly rich history of model reduction based on modal characteristics and is closest in spirit a technique termed selective modal analysis developed by Perez (1981).

In motivating the approach, we concentrate on the first two questions given above, which are closely related. Understanding and explaining a complex model is closely related to developing a simplified model. An explanation of the interactions driving behavior is at some level a model, and if the explanation is to be easily understood then the model must be simple.

We will first give a definition of "the behavior generated by a system." In this definition we motivate from a heuristic point of view the use of eigenvalues as characterizing the dynamics of a system. The manner in which we define the behavior generated by a system is useful in discussing what it means for a simplified model to be easily interpretable.

Behavior of a "System"

In the spirit of model simplification as a means of understanding we will begin our definition by considering an extremely simple "system".

[†]The homogeneous "system" is given by equation 0.1 with $\underline{B}$ and $\underline{D}$ both $\underline{0}$. The homogeneous "system" will give the behavior over time that the "system" would display if no exogenous variables were to influence it.

This "system" has a scalar state variable, and for this reason it will be referred to as the SCALAR model. In this simple case it is easy to define the behavior patterns that the "system" generates and to identify the qualities of the "system" that give rise to these behavior patterns. We can then generalize this simple case.

Suppose that the scalar variable ζ is of interest to us and ζ_t is given by

$$\zeta_t = \rho\zeta_{t-1} + \underline{\beta u}_t + \varepsilon_t \qquad (0.3)$$

with $\underline{u}$ some set of exogenous variables and ε a noise term.[†] The behavior generated by this SCALAR model is the change in behavior of ζ relative to $\underline{u}$. If the output of the system were always the same as some combination of the exogenous variables $\underline{u}$ then the system would not be generating any behavior. If, on the other hand, the exogenous variables were constant and ζ was growing, then the system would be generating growth behavior.

For this equation it is clear that the only way the behavior of ζ can differ from that of the exogenous inputs is through its own past values. The effect of ζ's past values on ζ is given by the number ρ. If the number ρ is zero then ζ is just a linear combination of the exogenous variables and the SCALAR model generates no behavior. If the number ρ is real and between 0 and 1 then ζ displays monotone adjustment to changes in the

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[†]The constants ζ, ρ, $\underline{\beta}$ and ε are allowed to take complex values. The vector $\underline{u}$ is assumed real.

inputs. If ρ is complex or real and negative then ζ displays oscillation in its real component. If $|\rho|$ is larger than 1 then ζ will display growth. The growth will be monotone if ρ is a positive real number and will consist of expanding oscilations otherwise; in either case the effect of the exogenous inputs eventually becomes unimportant.

It is relatively easy to get to the heart of the SCALAR model and we can generalize this. What we would like to do is take a complicated "system" such as that given in equation 0.1 and rewrite it as a number of SCALAR models. This can always be done. Every system of the form of equation 0.1 can be transformed into N equations of the form of 0.3 as long as the $\underline{A}$ matrix is nondefective.† The transformation will yield N new states and the dynamics will be determined in terms of these states rather than the the original states $\underline{x}$. Each new state will have dynamics determined by a SCALAR model. Fortunately it will always be possible to move backward from the transformed states to the original states. That is, it is possible to add up the output of the N SCALAR models in order to find the value of any state x_{it}. Moreover, the way in which we add up the states of the SCALAR models does not change over time.

Let us review our approach to defining the behavior generated by a "system." We begin with a large model of the form

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†A square matrix $\underline{A}$ of dimension N is nondefective if its eigenvectors form a basis for N-dimensional space. A sufficient but not necessary condition for this is that $\underline{A}$ have distinct eigenvalues (Strang 1980, chapter 5). It will be assumed throughout this thesis that the $\underline{A}$ matrix is nondefective, though many of the results extend to the case in which $\underline{A}$ is defective (Davison, 1967).

$$\underline{x}_t = \underline{\underline{A}}\underline{x}_{t-1} + \underline{\underline{B}}\underline{u}_t + \underline{\underline{D}}\underline{e}_t \ , \tag{0.4}$$

which can be transformed into N SCALAR models of the form

$$\xi_{jt} = \lambda \xi_{jt-1} + \underline{\beta}_j \underline{u}_{jt} + \epsilon_{jt} \ . \tag{0.5}$$

And for any state x_i the equation

$$x_{it} = \underline{r}_i \underline{\xi}_t \tag{0.6}$$

gives the value of the state at time t. In the above equation $\underline{\xi}_t$ is a vector composed of the outputs of the SCALAR models of equation 0.5. Equation 0.6 is another example of a SCALAR model, but it has $\rho=0$ and, therefore, does not generate any behavior. From this we conclude that the behavior the "system" generates is contained in the N equations of the form 0.3.

Each variable of the transformed model moves according to the value of one of the eigenvalues. Since the original states are simple sums of the transformed states the original model exhibits behavior composed of sums of the behavior determined by the eigenvalues. Hence, the eigenvalues tell us what patterns of behavior not contained in the input of a system can be displayed by the output of a "system."

Interpreting a Simplified Model

The above discussion shows in what sense the eigenvalues of the matrix A can be said to describe the dynamics of the system. A simplifed model that contains some of the eigenvalues of the original system preserves a subset of the internally generated dynamics of the "system." Certainly the SCALAR models are of this type. Each preserves one of the internally generated modes of behavior. The problem with the SCALAR models is that they are not easily interpreted. It is true that it is always possible to transform the "system" to arrive at a model in the form of equation 0.5. The transformation required, however, is quite complicated, and, though it is easy to understand the SCALAR models which result, it is not easy to understand how they relate to the original "system."

We would like to find simplified models that can be easily related to the original system. The most obvious definition of this, and the one we will follow in this thesis, is to require that the states of the simplified model correspond to states of the original "system". Suppose that the simplified model can be transformed into n SCALAR models and that these SCALAR models are a subset of those of the original "system." If this can be done, the simplified model maintains some of the internally generated dynamics of the "system." Further, if the transformations are similar for the original model and the simplified model, then the simplified model's states can be easily interpreted in terms of the states of the "system." We will consider this point in somewhat more detail and rigor in chapter 3.

To summarize, useful simplified models should maintain the dynamic characteristics of interest in the "system". The state vector of the

simplified model ($\tilde{\underline{x}}$) should also be interpretable in terms of meaningful economic and physical concepts. These two properties will ensure that the simplified model will fulfill the goals of model simplification. The development of models that have these properties (approximately) is the major theme of this thesis. The extent to which such models fail to generate dynamics and have interpretable states is, in some sense, a measure of the distance of the simplified model from the system. This is subject is discussed in chapter 3.

In simplifying, accuracy is lost. For the type of simplification that we are considering, one important aspect of accuracy is the ability of the model to correlate with the data. The original model, if it is entirely correct, will be in accord with the data. In the case of Gaussian noise this means that the least squares estimation of the model will yield unbiased and efficient estimates of the model's parameters. The fact that a simplified model is not in accord with the data means that the standard approaches to model estimation and hypothesis testing are not exactly correct. The relationship of the simplified models to the data will be discussed at length. One question to be considered is the possibility of using relatively simple estimation techniques to determine the parameters of the simplified models which maintain dynamic characteristics. It is shown that simplfied models of the type considered are not amenable to simple estimation techniques.

0-C A Simple Macroeconomic "System"

In this section we will present a model which will be used as an example for the simplification techniques to be developed. The model is a

simple, Keynesian, aggregate macroeconomic model and will be referred to as the MACRO model. This model will be taken as the "system," and we will develop simplified models based on it. The analysis of the MACRO model is useful in understanding the theory of model simplification. In chapter 6 we will consider the simplification of a medium sized econometric model.

The MACRO model is based on the multiplier-accelerator model of Samuelson (1939), in combination with a model developed in Tobin (1975). The model differs somewhat from that originally presented by Samuelson in that, rather than using lags of income, the model uses exponential averages (Granger and Newbold 1977, Levenbach and Cleary 1982). This makes the dynamics of the model less dependent on the choice of the time period (Low 1980). This is important. The response to a stimulus after one month can be expected to be different from the response after one quarter. In a theoretical model, there is no fundamental time period, hence we would like our results to remain the same whether weeks, months or quarters are chosen. The basic time period for the MACRO model is a month.

The consumption equation for the model is taken to be of the form

$$C_t = a_1 YP_t + a_2 K_t - a_3 (R_t - PIE_t) + e_{1t} , \qquad (0.7)$$

where YP represents permanent income, K is the capital stock, which is a measure of non human wealth, R is the interest rate and PIE the expected inflation rate. The equation for consumption is given in constant dollars. The formulation is similar to that in Modigliani (1975). The assumption is made that higher real interest rates have a negative impact on consumption.

The equation for determing permanent income, following Friedman (1956) and Sargent (1979, chapter 12), is given by

$$YP_t = YP_{t-1} + a_4(Y_{t-1} - YP_{t-1}) + e_{2t} . \qquad (0.8)$$

Income Y is defined according to the national accounts identity

$$Y_t = C_t + I_t + G_t \qquad (0.9)$$

with C representing consumption, I investment and G government expenditure.

Following Samuelson (1939) the equations representing investment are formulated on the assumption of a constant capital output ratio. Capital is adjusted toward a target capital stock, with target capital stock given by expected income multiplied by the target capital output ratio. Investment is given by

$$I_t = a_5K_t + a_6(a_7YS_t - a_8(R_t - PIE_t) - K_t) + e_{3t} \qquad (0.10)$$

with K representing the capital stock and YS smoothed income. The first term in this expression is meant to capture the required depreciation replacement of capital. The term $a_7YS - a_8(R - PIE)$ gives the target capital stock. This is decreasing with the expected real interest rate, which is given by (R - PIE). The coefficient a_6 gives the speed with which the capital stock is adjusted toward its target value. Smoothed income is given by

$$YS_t = YS_{t-1} + a_9(Y_{t-1} - YS_{t-1}) + e_{4t} \cdot \qquad (0.11)$$

It is assumed that consumption demand is always met through the over- and underutilization of the existing capital stock. The capital stock does not, therefore, directly influence production. However, the investment equation (0.10) keeps the economy near a constant capital output ratio and is therefore consistent with a constant returns to scale production function, demand effectively determining the capital stock.

The capital stock is decreased by depreciation and increased by investment. Depreciation is assumed to be a constant proportion of the existing capital stock. The capital stock is thus given by

$$K_t = (1-a_5)K_{t-1} + I_{t-1} \cdot \qquad (0.12)$$

In the above equation, a_5 represents the depreciation rate on capital and has previously appeared in the investment equation.

The price equation is an inflation-augmented Phillips curve. The price variable (P) can best be thought of as the logarithm of price. Therefore, the first difference in P represents the percentage rate of change in price. Price enters into the consumption and investment equations only through expected inflation. Therefore, representing only the log of price presents no problem in these equations. Taking P as the log of price is very convenient in the specification of the money demand equation which is discussed below.

The price equation is given by

$$P_t = P_{t-1} + a_{10}(YS_{t-1} + a_{11}(Rt\text{-}1 - PIEt\text{-}1) - b_1) + a_{12}PIE_{t-1} + e_{5t}\,, \tag{0.13}$$

with P representing the log of price and b_1 the potential level of output. Note that the expected inflation term is specified with a coefficient (a_{12}) but that this coefficient will normally be taken to be 1. The real interest rate enters into the price equation because a lower real interest rate will raise the level of production that can profitably be maintained given a fixed aggregate labor supply.

We take M as as the log of money and assume that it is exogenously determined by the government and its agencies. The assumption is made that the money market clears at all times and this equilibrium condition is used to specify the interest rate as

$$R_t = a_{13}(P_t - M_t) + b_2 + e_{6t} \tag{0.14}$$

with b_2 a constant term that represents the log of the velocity of money.

Both YS and YP represent, to some extent, expectations. The expectations are clearly adaptive and are based on the past values of the variables of interest. The third equation which is an expectations equation, is that for expected inflation and is again adaptive. The expected inflation is given by the formula

$$PIE_t = PIE_{t-1} + a_{14}(PI_{t-1} - PIE_{t-1}) + e_{7t}\,, \tag{0.15}$$

with PI the measured inflation rate, given by

$$PI_t = a_{15} (P_t - AP_t) \tag{0.16}$$

with

$$AP_t = AP_{t-1} + a_{15}(P_{t-1} - AP_{t-1}) \ . \tag{0.17}$$

The variable AP represents an average or historical price level against which the current price level can be compared in order to determine inflation.

There are two exogenous variables in the model, government expenditure and the money supply. The government expenditure is in constant dollars and is one determinant of income. The log of the money supply enters into the interest rate equation. For simplicity, the error terms entering the equations are assumed to be uncorrelated among themselves and over time. The parameters of the model as well as the numerical values of the various matrices will be given in chapter 3.

Linearity

The above model has been presented as a linear model. This has been done for obvious reasons, but the model presented is clearly an approximation. The interest rate term in the equations for consumption and investment (0.7 and 0.10) has entered linearly. The effect of the interest rate could be more accurately represented as being proportional to the levels of income and consumption. The change in the log of price (0.13) could also be more accurately represented as being proportional to the percentage deviation of smoothed income from its normal value. The linear

relationships used will not differ very much from the mentioned nonlinear relationships if income does not deviate too far from its reference value. The interest rate equation (0.14) is also an approximate representation, and it would be more natural to take the log of the interest rate as depending on the log of real money supply. Linearity is one of many ways in which the "system" is an approximation.

Behavior

The output of the model when excited by a small amount of noise is given in figure 0-1. The model displays oscillations with a period of approximately 70 months. This is the pattern of behavior that is of interest. Simplification of the MACRO model will be discussed in chapters 3 and 4. For now we use the MACRO model to illustrate the close

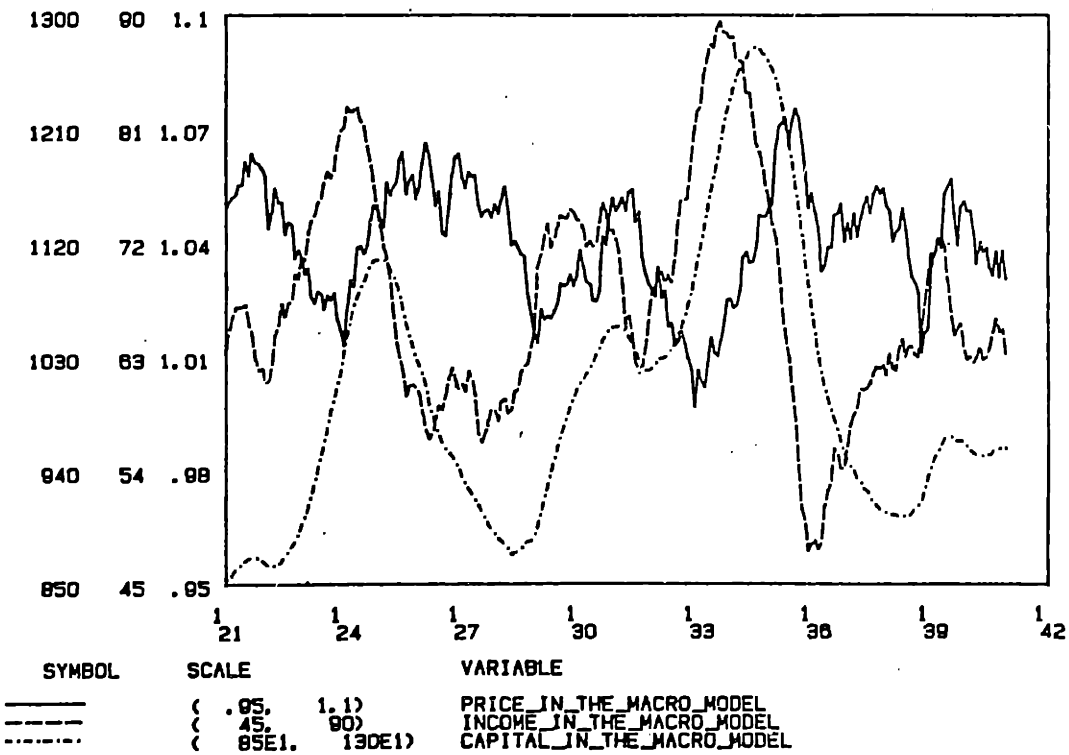

Figure 0-1: Simulation of the MACRO Model with Noise Entering

relationship between simplification and understanding. In order to understand a complex model, it is necessary to develop some sort of simple model.

For example, in the model developed, the following verbal explanation of the observed behavior pattern can be given.

> When there is a large discrepancy between desired and actual capital, investment is high. High investment means high income and high capital accumulation. But income (and therefore smoothed income) begins to fall off as soon as investment does, whereas the capital stock increases as long as investment is positive. When net investment equals zero, the capital stock will be at a maximum, but both income and smoothed income will be falling. Lower smoothed income will lower investment, which will cause the capital stock to decrease. When investment (and therefore income) reaches its minimum, the capital stock will be decreasing very rapidly and will do so until net investment rises to zero. At this time, income will have been rising and eventually exceed smoothed income. Rising smoothed income will cause increased investment. And we are back where we started.

The above description of the interactions leading to oscillation can be represented by a simple model. The levels in this model are smoothed income and capital, and the equations for the model are

$$\tilde{C}_t = \tilde{a}_1 \tilde{YS}_t + \tilde{a}_2 \tilde{K}_t + \tilde{e}_{1t} \tag{0.18}$$

$$\tilde{I}_t = \tilde{a}_5 \tilde{K}_t + \tilde{a}_6(\tilde{a}_7 \tilde{YS}_t - \tilde{K}_t) + \tilde{e}_{3t} \tag{0.19}$$

$$\tilde{Y}_t = \tilde{C}_t + \tilde{I}_t + \tilde{G}_t \tag{0.20}$$

$$\tilde{YS}_t = \tilde{YS}_{t-1} + \tilde{a}_9(\tilde{Y}_{t-1} - \tilde{YS}_{t-1}) + \tilde{e}_{4t} \, . \tag{0.21}$$

$$\tilde{K}_t = (1-\tilde{a}_5)\tilde{K}_{t-1} + \tilde{I}_{t-1} \, . \tag{0.22}$$

Smoothed income (YS) is serving as a proxy for permanent income in the

consumption equation (0.18). Other than this the equations of the MACRO model have been altered only by removing the excluded states. The resulting simplified model is essentially the multiplier-accelerator model of Samuelson (1939).

This is a simplified model and is intended to capture the "essence" of the dynamics observed in figure 0-1. Does it do so? And if so, what methods are available to derive such a simple model? These questions will be considered at length.

0-D Generalizations and Restrictions

The theory in this thesis is aimed largely at the development of simple models from more complex models. Large, complete models are not available for most economic applications. One of the major motivations for this thesis is to gain insight into the relationship between models in use and the systems that they represent. The recognition that models are only approximations can be useful in deciding what characteristics it is necessary for a model to possess.

The theory is developed for linear time invariant models. The mathematics developed are strictly applicable only to models that are linear and time invariant. This class of models, though large, is hardly complete. However, many existing dynamic models have dynamics that can be reasonably represented by linear models, and this theory can therefore be reasonably applied to these models. In addition, many of the conceptual issues are of direct relevance to much larger classes of systems. Nonetheless, the development of this type of theory for nonlinear models

remains a major research area.

0-E Organization of the Thesis

This thesis consists of seven chapters in addition to this introduction. In the first chapter the literature on dynamic model simplification techniques is considered. We concentrate on tying together threads from the economics and engineering literature in a fairly unified manner. In the second chapter the problem of estimation of dynamic models is first reviewed. The Kullback-Leibler information criterion is then introduced and used to discuss implications of the standard estimation approach for simplified models.

In chapter 3 the approach to model simplification used in this thesis is developed. This approach is very similar to that of Perez (1981), and therefore the results of Perez are presented first. A somewhat more general framework for considering the simplification problem is then introduced, and used in order to develop measures of the nearness of a simplified model to the original system. The implications of the simplification considered for problems in dynamic model estimation are considered in chapter 4. The problems associated with applying the methods to simultaneous equation models are considered in chapter 5. Methods for dealing with nondynamic equations, which normally form a part of a dynamic model, are also developed.

The theory developed can be applied to existing large models. The resulting small models can aid in the understanding and refinement of the large models. The theory developed is applied to a large econometric model

(The Michigan Quarterly Econometric Model) with concentration on business-cycle dynamics that the model is capable of generating endogenously. Two simplifications of the model are developed and compared. Estimation of a simplified model is also performed and the results compared with the simplified models obtained from the larger model.

The results of the research are summarized in chapter 7.

1 Structural Model Order-Reduction Techniques

Introduction

In this chapter we will review dynamic model simplification techniques based on the structural properties of the model. There has been a substantial amount of work done in this area in both economics and engineering. In reviewing the literature, we concentrate on simplification techniques intended to maintain elements of the dynamic structure (eigenvalues and vectors) of the original system. Other simplification techniques closely related to this approach, notably perturbation and aggregation techniques, will also be discussed. For a broader overview of the literature in model reduction the reader is referred to Genesio and Milanese (1976) and Sandell et al. (1978).

The concentration in this chapter will be on simplification based on the characteristics of a large but known model. This is to be contrasted with simplification based on the input and output characteristics of a system. In chapter 2 we will consider the input and output characteristics of a system and how these can be applied to model simplification.

An existing model is necessary in order to apply the simplification techniques considered in this chapter. The existing model will often be referred to as the "system." The existing large model itself can be no more than an approximate representation of the underlying processes of concern. To stress this fact the word "system", when it refers to the larger underlying model, will always appear in quotation marks.

Model simplification techniques are also important because they give us a basis for judging models against a "system" which is presumed to exist but has not been formalized. It is possible to determined some of the characteristics that the "system" must have in order to make existing models valid simplifications. This is a form of validation for existing models and, more generally, existing approaches to modeling. It is this aspect of simplification that has most often been the focus of the economic literature reviewed in this chapter.

The purpose of this chapter is not to document the development of a line of thought, but to identify commonalities in the various approaches to model simplification. We will try to provide a unifying framework for a number of model simplification techniques. Material both from the economics literature and from the control theory literature will be developed together. The chapter is organized so as to bring out as much as possible how closely interrelated the various techniques are. Consequently, the material will not be presented in chronological order. Most of the theory is applicable to both differential and difference equation systems. The mathematics for the two are almost identical and we

restrict our attention to the difference equation case except when there is need to do otherwise.

1-A Aggregation

Model simplification and aggregation have long been issues of concern to economists. The early works in this area are in economics. As the early aggregate macroeconomic models were developed there was concern with this issue (Klein 1946a, 1946b, May 1946, Pu 1946). Early work on aggregation was concerned with the question of how micro and macro relationships are related. The question, simply stated, is this: given a relationship which holds at the level of an individual such as

$$y_i = f(x_i) \ , \tag{1.1}$$

when can the same relationship be expected to apply in the aggregate? Specifically, under what conditions does the formula

$$\Sigma_{i=1}^{n} y_i = f(\Sigma_{i=1}^{n} x_i) \tag{1.2}$$

hold?

Equations 1.1 and 1.2 represent the simplest case in the study of the correspondence between macro relationships and micro relationships. For the simple example given, f must be linear for equation 1.2 to hold. More general formulations of the relationship between micro and macro structures yield slightly less restrictive results (Theil 1954, Stoker 1980, 1981). A related issue is the use of cross-sectional data to model aggregate

relationships (Kuh 1959, Kuh and Welsch 1976, Stoker 1982). Aggregation issues of this type are still an area of current research interest, but it should be noted that many of the results in this area are negative. That is, except under very special circumstances much of the aggregation commonly done in economics is not valid.

A closely related problem to that of micro-macro aggregation is the aggregation of commodities and industries in input-output analysis. The need for aggregation in input-output analysis arises for two reasons. First the greater the level of detail in the analysis the larger the cost of gathering and recording data. Second, the larger the input-output matrix the more the analysis (especialy inverting the matrix inversion) costs. The cost of matrix inversion is not as great a consideration with the computers presently available but was a major motivating force in earlier work.

The problem of aggregation was recognized and discussed by the early workers in input-output analysis. Holzman (1953) discusses at length the rationale behind the 97 industry classification of the American economy then in use. The essential criterion given for combining two industries is that they have the same input mix requirements. This requires that the columns of the input-output matrix associated with the two industries be proportional (identical when written in share form).

In further work on this issue by Hatanaka (1952), Fei (1956), Ara (1959) and Morimoto (1970) these criteria are generalized and restated in mathematical terms. In the more general case it is supposed that the

"true" input-output relationship is given by

$$\underline{z} = \underline{W}\underline{z} + \underline{f} \ , \tag{1.3}$$

with $\underline{z}$ the gross demand vector, $\underline{W}$ the input output matrix, and $\underline{f}$ final demand. We can consider the simplified form of equation 1.3 of

$$\underline{S}\underline{z} = \underline{\tilde{W}}\underline{S}\underline{z} + \underline{S}\underline{f} \ , \tag{1.4}$$

with $\underline{S}$ an aggregation matrix.† Since $\underline{f}$ is arbitrary and equation 1.3 is assumed to have a solution, it follows that $\underline{z}$ is arbitrary. Premultiplying equation 1.3 by $\underline{S}$ and subtracting this from 1.4 yields the aggregation condition

$$\underline{S}\underline{W} = \underline{\tilde{W}}\underline{S} \ . \tag{1.5}$$

The interpretation of this aggregation condition in the input-output context is that each member of a group of industries has the same derived demand for the products of every industry group (Ara 1959).

Conditions for exact aggregation, not suprisingly, are rarely fulfilled. In fact, Hatanaka's (1952) major conclusion was that aggregation was unlikely ever to be valid. Conditions for approximate aggregation (i.e., equation 1.5 almost holds) are much more common.

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†In the literature on input-output analysis it is assumed that $\underline{S}$ is a matrix that consists of only 0's and 1's. Thus, the matrix $\underline{S}$ simply adds up the contribution of different industries. This restriction does not alter the mathematics.

Aggregation algorithms for existing input-output matrices have been considered in Fisher (1969, chapter 3) and Theil (1967, chapter 9). These minimize some measure of the error in equation 1.5.

Dynamic Aggregation

The work in the area of input-output analysis is very closely related to the notion of aggregation in linear dynamic systems as developed by Aoki (1968, 1971, 1978). The dynamic system to be considered is given by

$$\underline{x}_t = \underline{Ax}_{t-1} + \underline{Bu}_t \tag{1.6}$$

$$\underline{y}_t = \underline{Cx}_t \quad , \tag{1.7}$$

with $\underline{x}$ a vector of state variables, A an NxN matrix, $\underline{C}$ a kxN matrix ($k \leq N$) and $\underline{y}$ an aggregation of the state vector.† The term $\underline{u}$ represents an exogenous input, and the NxN matrix $\underline{B}$ represents the influence of that exogenous input on the system. If possible, it is desirable to represent the system of equations 1.6 and 1.7 by a simpler dynamic system such as

$$\tilde{\underline{y}}_t = \tilde{\underline{A}}\tilde{\underline{y}}_{t-1} + \tilde{\underline{B}}\tilde{\underline{u}}_t \quad , \tag{1.8}$$

with the tilde denoting analogues to the variables in equations 1.6 and 1.7. Exact aggregation is possible when the output of the dynamic system

---

†Equation 1.7 is often an observation equation so that $\underline{y}$ is the vector of observed variables. In such a situation it is necessary to consider simplified systems of an order greater than that implied by $\underline{y}$. In this case the notation becomes even more burdensome, but the essential results do not change.

of equation 1.8 will match $\underline{y}_t$ as defined in equation 1.7 exactly.

The conditions for exact aggregation given by Aoki are that

$$\underline{\tilde{A}}\underline{C} = \underline{C}\underline{A} \tag{1.9}$$

and

$$\underline{\tilde{B}} = \underline{C}\underline{B} \ . \tag{1.10}$$

If it is possible to find an $\underline{\tilde{A}}$ that satisfies 1.9 then the resulting simplified model of 1.8 will behave precisely as the more complicated system of equations 1.6 and 1.7 does. Equation 1.9 is precisely the same as equation 1.5 after the change in notation. The requirements for exact aggregation in the dynamic model are precisely those in the input output framework.† The conditions for exact aggregation imply, among other things, that the eigenvalues of the matrix $\underline{\tilde{A}}$ will be a subset of the eigenvalues of the matrix $\underline{A}$.

The aggregation approach to model simplificaton is interesting in its own right, and it also helps provide a framework for much of the later discussion. The fact that the eigenvalues of the simplified model are a subset of those of the full model is important in this respect. Many model simplification techniques are designed to preserve some of the eigenvalues of the original system. Simplification techniques of this type can be represented as aggregations, the aggregation matrix taking a form specific to the simplification technique.

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†It should be noted that though Aoki (1968) does reference the literature on aggregation in economics, he does not refer to any work that deals with aggregation in input-output analysis.

1-B Modal Preservation Techniques

One of the characteristics of an exact aggregate model as discussed in the previous section is that it maintains some of the eigenvalues of the original system. The eigenvalues of a system determine the dynamic character of the system as described in section 0-B. Briefly, the eigenvalues of a system can be used in order to separate the system's different behavior modes. The terms "eigenvalues" and "modes" will be used interchangeably in the discussion.

One approach to the simplification of models is to see that the simplified models maintain the "dominant" modes, or eigenvalues, of a system. Davison (1966), Gopal and Mehta (1982), Litz (1980) and Marshall (1966) all consider model simplification from this perspective. Dominant modes have to be chosen carefully, especially in relationship to the output that the reduced-order model is intended to generate (Gopal and Mehta 1982). In addition to relating the modes to the states and also to the driving exogenous variables, it is possible to discard the fast, stable modes (Mahmoud and Singh 1982). The two approaches are, of course, closely related.

Regardless of the choice of modes, the algebra is very similar. Given a dynamic system as represented by equation 1.6, it is possible to generate the variable $\underline{\xi}$ according to the transformation

$$\underline{\xi}_t = \underline{L}\underline{x}_t \ , \tag{1.11}$$

where the matrix $\underline{L}$ has rows given by the left eigenvectors of $\underline{A}$, so that

$$\underline{\xi}_t = \underline{\Lambda}\underline{\xi}_{t-1} + \underline{LBu}_t \ . \tag{1.12}$$

In equation 1.12 the matrix $\underline{\Lambda}$ is diagonal so that the system may be written variable by varible as

$$(\underline{\xi}_t)_i = \lambda_i(\xi_{t-1})_i + (\underline{LBu}_t)_i \ . \tag{1.13}$$

Because the dynamics can be separated as in equation 1.13, it is possible to treat the different elements of the $\underline{\xi}$ vector independently.

Davison (1966) ignores the elements of the $\underline{\xi}$ vector not associated with the modes of interest, effectively assuming them to be zero. The vector $\underline{\xi}$ of equation 1.11 is transformed by setting its last (N-n) elements to zero. The reduced-order model then becomes

$$\underline{\tilde{x}}_t = \underline{\tilde{A}}\underline{\tilde{x}}_{t-1} + \underline{\tilde{B}}\,\underline{u}_t \tag{1.14}$$

with

$$\underline{\tilde{A}} = \underline{R}_{11}\underline{\Lambda}_{11}\underline{R}_{11}^{-1} \tag{1.15}$$

and

$$\underline{\tilde{B}} = \underline{R}_{11}\ |\underline{L}_{11}\ \underline{L}_{12}|\ \underline{B} \tag{1.16}$$

After some debate with Chidambara (Chidambara 1967, Davison 1967), Davison proposed a slightly modified form for going from the simplified model to the states represented. Using equation 1.14 to describe the dynamics of $\underline{\tilde{x}}$, Chidambara (1967) defined the best approximation to the states to be

$$\hat{\underline{x}}_{1t} = \underline{\tilde{x}}_t + \{ \,|\underline{I} \;\; 0|(\underline{I} - \underline{A})^{-1} - (I - \underline{\tilde{A}})^{-1}\underline{R}_{11}|\underline{L}_{11} \;\; \underline{L}_{12}| \,\} \, \underline{B} \, \underline{u}_t \,, \quad (1.17)$$

which maintains the steady-state response to a constant input, $\underline{u}_t$.

In a tack similar to Davison's, Marshall (1966) developed a reduced-order model based on the assumption that the excluded dynamics are very fast and stable. Marshall assumes that the influence of the fast modes on the rate of change in $\underline{\tilde{x}}$ can be incorporated into the influence of the exogenous variables. The resulting model is very similar to that given in equation 1.14 except that in this case $\underline{\tilde{B}}$ is given by

$$\underline{\tilde{B}} = |\,(\underline{I} - \underline{\tilde{A}}) \;\; \underline{0}\,|\,(\underline{I} - \underline{A})^{-1} \, \underline{B} \quad (1.18)$$

or

$$\underline{\tilde{B}} = \underline{R}_{11} \,|\underline{L}_{11} \;\; \underline{L}_{12}|\, \underline{B} + \underline{R}_{11}(\underline{I}-\underline{\Lambda}_{11})\underline{R}_{11}^{-1}\underline{R}_{12}(\underline{I}-\underline{\Lambda}_{22}^{-1})|\underline{L}_{21} \;\; \underline{L}_{22}|\underline{B} \,. \quad (1.19)$$

This is very similar to the model proposed by Chidambera (1967).

The model given by equations 1.15 and 1.19 maintains the steady-state response of the original model, but the modes are not excited in the same way. Davison (1967, p. 121) gives an example where the response of the model described by equation 1.18 is strikingly different from that of the system, while the response of the model given by equations 1.14 to 1.16 is nearly the same.

1-C Singular Perturbation

There is a model simplification technique, closely related to the the techniques of Davison and Marshall, known as singular perturbation.

Kokotovic, O'Malley and Sannuti (1976) offer an excellent survey on the singular perturbation technique. The singular perturbation technique has a slightly different appearance in the discrete-time representation than in the continuous-time representation. The continuous-time version will be developed first and used to derive a discrete-time analog that leads very nicely back to the economics literature.

Continuous-Time Models

The model to be considered is a continuous time dynamic model of the form

$$\dot{\underline{s}} = \underline{E}\underline{s} + \underline{H}\underline{u} \ , \tag{1.20}$$

with the $\dot{}$ denoting a time derivative. The singular perturbation approach requires that some of the states have dynamics that are very quick relative to those of other states. Mathematically this means that the system can be written in partitioned form as

$$\begin{bmatrix} \dot{\underline{s}}_1 \\ \varepsilon\dot{\underline{s}}_2 \end{bmatrix} = \begin{bmatrix} \underline{F}_{11} & \underline{F}_{12} \\ \underline{F}_{21} & \underline{F}_{22} \end{bmatrix} \begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \end{bmatrix} + \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2' \end{bmatrix} \underline{u} \ , \tag{1.21}$$

with ε , the perturbation parameter, assumed to be very small. The submatrices in the $\underline{F}$ matrix are assumed to be of the same order of magnitude, and this determines the value that ε actually takes. In order for equation 1.21 to be satisfied, it is necessary for the derivative $\dot{\underline{s}}_2$ to be very large or the right-hand side $(\underline{F}_{21}\underline{s}_1 + \underline{F}_{22}\underline{s}_2 + \underline{H}_2'\underline{u})$ to be very small. If the dynamics of $\underline{s}_2$ are stable, $\underline{s}_2$ will very quickly converge to its equilibrium value. Given the slow rate of change of $\underline{s}_1$ relative to $\underline{s}_2$,

the dynamics of $\underline{s}_2$ are dominated by the submatrix $\underline{F}_{22}$.

If the matrix $\underline{F}_{22}$ has eigenvalues with negative real parts (the condition for stability in the continuous-time model), then the second row of 1.21 may be rewritten by setting ε to zero. This is referred to as a singular perturbation because setting ε to zero effectively changes the model from one of order N to one of order $n<N$. The equations for the reduced-order model are

$$\dot{\underline{s}}_1 = (\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21})\,\underline{s}_1 + (\underline{H}_1 - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{H}_2')\,\underline{u} \qquad (1.22)$$

and

$$s_2 = -\underline{F}_{22}^{-1}\underline{F}_{21}\underline{s}_1 - \underline{F}_{22}^{-1}\underline{H}_2'\,\underline{u} \quad . \qquad (1.23)$$

The dynamics of the states are determined in equation 1.22, which determines the evolution of the $\underline{s}_1$ states over time. The second set of states are determined, at any point in time, by the static equation 1.23. The value at time t_o of $\underline{s}_2$ affects future values of $\underline{s}$ only through the influence on $\underline{s}_1$. The relationship of the singular perturbation model to discrete-time approximations of continuous-time models is discussed in chapter 5. A discrete-time analog to the above development is presented below.

Discrete-Time Models

There are a number of ways of dealing with the discrete analog of the continuous-time singular perurbation problem. Phillips (1983) presents a relatively general framework for dealing with the numerical analysis of such problems. We will focus on a particular realization most akin to the continuous time problem as stated above and which relates to the economics

literature.

The essence of the discussion for the continuous-time model is that it is possible to ignore the dynamics of fast moving states. The singular perturbation yields a model in which the previous values of the fast moving states do not matter; the values of the fast states are given very nearly by the values of the slower states and the exogenous varibles. This property is possessed by difference equations of the form

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \epsilon\underline{A}_{12} \\ \underline{A}_{21} & \epsilon\underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{B}_1 \\ \underline{B}_2 \end{vmatrix} \underline{u}_t \tag{1.24}$$

with ϵ a small perturbation parameter. On the basis of equation 1.23 we further assume that the dynamics matrix has the property that

$$\underline{A}_{21} = \underline{K}\,\underline{A}_{11} \ , \tag{1.25}$$

with $\underline{K}$ some (N-n) x n matrix. Setting ϵ to zero again yields a system of lower dynamic order. The equations that describe the resulting model are

$$\underline{x}_{1t} = \underline{A}_{11}\underline{x}_{1t-1} + \underline{B}_1\underline{u}_t \tag{1.26}$$

and

$$\underline{x}_{2t} = \underline{A}_{21}\underline{x}_{1t-1} + \underline{B}_2\underline{u}_t \tag{1.27}$$

or

$$\underline{x}_{2t} = \underline{K}\underline{x}_{1t} + (\underline{B}_2 - \underline{K}\,\underline{B}_1)\,\underline{u}_t \ . \tag{1.28}$$

This result, though not terribly striking, is useful for the comparison of singular perturbations to other model simplification

techniques. We will first discuss the relationship of singular perturbations to the simplification techniques based on dominant modes considered in section 1-B. Then, we will consider singular perturbations relative to the ideas of weak coupling.

The singular perturbation where the dynamics matrix is diagonal is of special interest. If we denote this dynamics matrix by $\underline{\Lambda}$ then equations 1.26 and 1.27 become (replacing $\underline{x}$ by $\underline{\xi}$ and $\underline{B}_1$ by $|\underline{L}_{11}\ \underline{L}_{12}|\underline{B}$)

$$\underline{\xi}_{1t} = \underline{\Lambda}_{11}\underline{\xi}_{1t-1} + |\underline{L}_{11}\ \underline{L}_{12}|\ \underline{B}\ \underline{u}_t \tag{1.29}$$

and

$$\underline{\xi}_{2t} = |\underline{L}_{21}\ \underline{L}_{22}|\ \underline{B}\ \underline{u}_t\ . \tag{1.30}$$

This is precisely the same as the simplified model which would be derived using the model simplification techniques of Davison (1966). This can be easily seen by premultiplying equation 1.14 by $\underline{\tilde{R}}^{-1}$.

The singular perurbation simplification and the simplification based on dominant modes are equivalent on a more general level. A more exact statement of this, as well as a proof of the assertion, is included in the appendix. It is clear that the two approaches have to be similar in the following sense. In the singular perturbations problem it is the quick adjustment that distinguishes state variables, which is tantamount to saying that the fast state variables are heavily influenced by fast modes. Therefore looking only at the slower modes will approximately give us the slow states. Singular perturbations work only because the two sets of modes can be assigned to the two sets of states. This is not necessarily

true in the problem as posed by Davison (1966).

It is possible to apply singular singular perturbation simplification technique to nonlinear models. Application of the singular perturbation technique requires that the states be separable into those with fast dynamics and those with slow dynamics. If this can be done in a nonlinear model then the technique of singular perturbation can be applied. The two sets of states characterize the dynamics much as eigenvalues characterize the dynamics of a linear system. Thus, if the states are separable into fast and slow, it is possible to classify the "behavior modes" of the nonlinear system in a tractable manner.

1-D Weak Coupling and Near Decomposability

The discrete-time singular perturbation model is quite similar to a model introduced by Simon and Ando (1961). This is a weakly coupled model in which states interact quite strongly within groups but only weakly between groups. The equation for this is

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \epsilon\underline{A}_{12} \\ \epsilon\underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} \tag{1.31}$$

with ϵ again a small parameter. Setting ϵ to zero produces two decoupled systems, and for ϵ small this will be an accurate representation of the system's dynamics. Specifically, Simon and Ando showed that the dynamics associated with all but the largest roots of the two submatrices $\underline{A}_{11}$ and $\underline{A}_{22}$ may be accurately represented by setting ϵ to zero.

The longer-term dynamics may depend on the values of the off-diagonal

elements. Following Ando and Simon, if $\underline{A}_{11}$ and $\underline{A}_{22}$ are stable matrices, then with a constant input the longer-term dynamics can be recaptured by aggregating all the elements of $\underline{x}_1$ and all those of $\underline{x}_2$. The resulting simple system is

$$\begin{vmatrix} y_{1t} \\ y_{2t} \end{vmatrix} = \begin{vmatrix} \lambda_1^1 & a_{12} \\ a_{21} & \lambda_1^2 \end{vmatrix} \begin{vmatrix} y_{1t-1} \\ y_{2t-1} \end{vmatrix} \tag{1.32}$$

with

$$y_{1t} = (\underline{l}_1^1)^T \underline{x}_{1t} \tag{1.33}$$

$$a_{12} = (\underline{l}_1^1)^T \underline{A}_{12} \underline{r}_1^1 \tag{1.34}$$

$$y_{2t} = (\underline{l}_1^2)^T \underline{x}_{1t} \tag{1.35}$$

$$a_{12} = (\underline{l}_1^2)^T \underline{A}_{12} \underline{r}_1^2 \tag{1.36}$$

Where $\underline{r}_1^1$ and $\underline{l}_1^1$ are the right and left eigenvectors associated with the largest root (λ_1^1) of $\underline{A}_{11}$, and $\underline{r}_1^2$ and $\underline{l}_1^2$ are the right and left eigenvectors associated with the largest root (λ_1^2) of $\underline{A}_{22}$. The aggregated states can be used to reconstruct the original states from the relative proportions of the states in the right eigenvectors $\underline{r}_1^1$ and $\underline{r}_1^2$.

Ando and Fisher (1963) considered the case in which the upper right hand block of the $\underline{A}$ matrix is arbitrary and the lower left-hand block is small. In this case the influence of the first component of the state vector on the second component is small. In such a situation it is possible to consider the short-term dynamics of the second portion of the state variable independently of those of the first. The interactions among

the partitioned state variables can then be captured by the largest root of $\underline{A}_{22}$.

The simplification techniques of Ando, Fisher and Simon are closely related to the dominant mode techniques of Davison and Marshall. The dominant modes for the Ando-Simon, Ando-Fisher model are those associated with the largest eigenvalues of each block. That the resulting simplified models are quite similar is a consequence of the continuity of the mapping from the $\underline{A}$ matrix to the roots and vectors of that matrix (assuming distinct eigenvalues). The dominant mode technique is somewhat more general in the sense that it does not require near decomposability of the $\underline{A}$ matrix. However, this generality is not as great as it might at first seem. This is because the degree to which the reduced-order model can represent the dynamics of states, or aggregates of states, will depend on the near decomposability. This will be discussed in chapter 3.

1-E Similar Approaches

A closely related approach is the subspace projection technique (Mahmoud and Singh 1981, section 4.3). The basic system as given in equation 1.6 is simplified by projecting the state onto a subspace. The error arising from simplification is restricted to have no components on this subspace, which is again the exact aggregation condition. The choice of the subspace is then made so that the simplified model is optimal according to some quadratic loss criterion.

Obinata and Inooka (1976) consider the problem of choosing a small model that in some sense minimizes the error resulting from the

approximation. This minimization is done for a simplified model of fixed size, and the results will depend on the choice of a loss function. In their development Obinat and Inooka focus on the error that arises from the input-output form of the model and not the error arising from initial conditions. They show that the minimizaton of such an error is equivalent to the minimization of a matrix which has $\underline{\tilde{A}C} - \underline{CA}$ (using the notation of equation 1.9) as a component. Thus, this technique yields a reduced-order model which is in some sense optimal. This approach can be interpreted as a minimization of the aggregation error, since the condition for perfect aggregation is that $\underline{\tilde{A}C} - \underline{CA}$ be zero. This is very similar to the approach taken by Fisher (1969) and Theil (1967), which was discussed earlier. Eitleberg (1981, 1982) has extended these results and tied this technique in with the singular perturbation technique.

Another approach to minimizing the model error is to consider solely the input-output characteristics of the system. This approach is closely related to the stochastic estimation of reduced-order models, which will be discussed in the next chapter. The major difference in this approach is that it assumes knowledge of the system. Consequently, it is possible to develop the simplified model from this knowledege, rather than from the input-output characteristics of the system. The approach is similar to the one considered above except that the entire error, including that arising from initial conditions, is minimized. Solutions to this problem are dependent on the input exciting the system; the response to impulse excitations has been considered extensively (Wilson 1970, Mahmmoud and Singh 1981).

If the optimizations for "full error optimizaton" and "equation error optimization" are done over long time periods and for stable models, it can be shown that the results converge. The reason for this is that the equation error and full error methods differ only in that equation error ignores the initial conditions, which become unimportant after a sufficiently long time. If the model is stationary and a white noise term is considered as the input, the simplification processe will converge to the quasi-maximum likelihood model considered in chapter 2.

1-F Summary

The economic and engineering literature on the problem of model simplification has been reviewed. The approaches to the model simplification problem considered have all been based on the assumption that the "system" is known. The approaches differ from both a mathematical and a conceptual point of view but it is the similarities that are remarkable. Weak coupling and distinct dynamics are the two fundamental concepts which flow through the literature. In chapter 3 we will approach the model simplification problem with a different focus, but in a manner consistent with the techniques considered in this chapter.

2 Information and Model Simplification

Introduction

In the last chapter we considered the development of simplified models under the assumption that the "system" generating the phenomena of interest is known. The techniques considered retain elements of the physical and behavioral structure underlying the model. Such techniques are not directly applicable to situations in which the model parameters need to be estimated. One goal of this thesis, to which chapter 4 is devoted, is to relate parameter estimation to model simplifications like those considered in chapter 1. To do this it is necessary to review how estimation relates to simplified models.

We begin with a brief summary of the most commonly used approaches to parameter estimation and statistical model validation. For fairly obvious reasons, some of these techniques are not suitable for models explicitly recognized as simplifications. The notion of information is then

introduced as a basis for justifying the most commonly used parameter estimation techniques. The relatively small body of literature on the use of information measure for the explicit simplification of models is then considered. Finally, the implications of the commonly employed Kullback-Leibler information criterion for the class of models considered in this thesis are derived.

2-A Standard Estimation Techniques

Consider the estimation of the parameters of a model of the form

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{B}\underline{u}_t + \underline{D}\underline{e}_t \tag{2.1}$$

and

$$\underline{y}_t = \underline{C}\underline{x}_t + \underline{F}\underline{u}_t + \underline{G}\underline{\varepsilon}_t \tag{2.2}$$

with $\underline{y}_t$ representing the observed variables. The parameters of such a model will include elements of the $\underline{A}$, $\underline{B}$, $\underline{D}$, $\underline{C}$, $\underline{F}$, $\underline{G}$, $\underline{\Omega} = E[\underline{e}_t\underline{e}_t^T]$ and $\underline{\Xi} = E[\underline{\varepsilon}_t\underline{\varepsilon}_t^T]$ matrices. It is normally assumed that $\underline{e}$ and $\underline{\varepsilon}$ are serially uncorrelated (white noise) errors and are not correlated with one another or with $\underline{x}$ or $\underline{u}$ at any time.

In the simplest case, the matrix $\underline{C}$ is the identity matrix, and both $\underline{F}$ and $\underline{G}$ are zero, so that the model may be rendered by equation 2.1 alone. If the error terms are serially uncorrelated then the vector $\underline{x}_{t-1}$ can be treated as predetermined. Generalized least squares of the system of equations, taking account of the contemporaneous covariance of the error $\underline{e}$, will yield consistent and asymptotically efficient estimators (Johnston 1972, Theil 1971).

In another commonly considered case the error term displays first-order autocorrelation, given by

$$\underline{e}_t = \underline{R}\underline{e}_{t-1} + \underline{\mu}_t \qquad (2.3)$$

with the noise term $\underline{\mu}_t$ assumed to be white noise and uncorrelated with all other variables. If the error term evolves according to equation 2.3, the simple least squares estimates will be inconsistent, as the error term will be correlated with $\underline{x}_{t-1}$. Consistent estimation of the coeffecient matrices, $\underline{A}$ and $\underline{B}$, is possible by correcting for autocorrelation. The method is analogous to the single equation methods considered in Johnston (1971, pp. 317-320), and there are some relatively weak identification conditions (see also Fair 1970).

If the $\underline{C}$ matrix is not the identity matrix, but rather a kxN matrix with k<N, the estimation problem is somewhat more complicated. Because the states are not known, it is necessary to derive estimates of both the states and the unknown parameters simultaneously. The state estimation problem, given all the parameters of the model, has a recursive solution known as the Kalman filter (Kalman 1960, Kalman and Bucy 1961). The estimation of the parameters of the system can then be done iteratively in conjunction with this state estimation technique (Mehra 1974, Peterson 1975. Schweppe 1973, 1976). In order to carry out such estimation it is necessary to place a large number of restrictions on the parameters to be estimated. If the error term $\underline{e}$ displays serial correlation as in equation 2.3 this can easily be corrected for by including this equation with an extended state vector. If the error term $\underline{\varepsilon}$ displays serial correlation,

the problem is somewhat more difficult (Sage and Melsa 1971).

The above parameter estimation techniques are related to the issue of state estimation with simplified models, which issue arises separately from the parameter estimation problem in some engineering applications where simplified models can be derived without estimation. Aoki and Huddle (1959), Asher (1976), Gelb (1974), and Miller and Mukundan (1982) consider the derivation of state estimators given the knowledge that the model is incorrect. In order to derive such state estimators it is necessary to know the relationship between the simplified model and the "system" from which it derives.

Misspecification

Regardless of the number of states observed, the estimation techniques assume that the model is correct up to an error term which is essentially unrelated to any explanatory variables. This holds true for a great deal of the hypothesis testing that occurs in such models as well. Certainly the standard econometric tests are based on a null hypothesis that the model is correct (Berndt and Savin 1977, Engle 1982, Harvey and Collier 1977, Hausman 1978, Sargan 1977). The tests of time series models are usually for the whiteness of the noise (Godfrey 1979, Pierce 1972, Poskitt and Tremayne 1982). Since a simplified model is by necessity simpler than the data-generating process, even the best simplified model will be misspecified. An important question arises about the nature of that misspecification and what results when simplified models are estimated under the assumption that they are correct.

It has long been recognized that models are approximations, but the strategy has been to treat these approximations as exact. Fisher (1961) noted that if the approximation is reasonably good, then this can be a very rewarding strategy. More recently White (1982), and Domowitz and White (1982) have considered some of the implications of misspecification in econometric models and have proposed methods to get around problems that arise. The basic approach is still to treat the model as correctly specified, however. There is, beyond simple practicality, some theoretical justification for this in terms of the distance from the real model relative to what appears to be a fairly natural norm. This norm can be derived as a specific information measure.

2-B Information

In this section we will consider the definition of information and the use of information as a criterion in the selection of a simplified model. Information is a fairly broad concept and can be used as a unifying framework for a great deal of estimation theory (Kullback 1959, Larimore 1983). Information theory has also been used in relation to the problem of model simplification, and this will be discussed. The use of information in the estimation of simplified models is potentially an important avenue of research, and we will discuss this in chapter 4 relative to the simplified models developed in chapter 3.

In considering information in the context of model simplification, we consider the information content of a model relative to a "system." The best model relative to an information measure is the one with the highest relative information content. The information content of a model is a

function of the difference between the information of an observation relative to the system and the information of an observation relative to the model. The information associated with an observation is, at a heuristic level, the amount that the observation surprises us (Theil 1967). Of course, in order to implement this definition of a good simplified model it is necessary to define information, and there can be more than one definition of information.

Theil (1967, chapter 9) has considered the use of an information criterion in the definition of the optimal aggregation of an input-output model. This work was done in an essentially deterministic framework but is close in spirit to the more recent information-based techniques for model simplification and estimation. Below we will concentrate on a specific definition of the information of an observation. We will first introduce this information measure, then review how this has been used in estimation of models and in the simplification of models. In section 2-C we will present the implications that such a definition of information has in the context of stable models that are excited only by noise.

The measure of information that we consider is the Kullback-Leibler information also simply called the Kullback information (Kullback and Leibler 1951, Kullback 1959). Consider the following measure:

$$KLI = E_{\underline{\theta}} \left[\log\{f(\underline{y},\underline{\hat{\theta}})\} - \log\{f(\underline{y},\underline{\theta})\} \right], \tag{2.4}$$

where $\underline{\theta}$ represents the parameter determining the distribution f, and $\underline{y}$ represents the observables. The information measure as defined in equation

2.4 can be thought of as a measure of the distance between two distributions. If the parameter $\underline{\theta}$ is fixed, equation 2.4 can be used as a metric for the distance between the two distributions $\underline{\theta}$ and $\underline{\hat{\theta}}$ (Baram 1976). When $\underline{\hat{\theta}} = \underline{\theta}$, equation 2.4 gives the Kullback-Leibler information. The Kullback-Leibler information is always less than or equal to 0 and will be 0 only when $\underline{\hat{\theta}} = \underline{\theta}$. Thus, this information measure can be thought of as a measure of the distance of the vector $\underline{\hat{\theta}}$ to the true distribution.

The proof of the consistency of maximum likelihood as given in Wald (1949) makes use of this measure of information. Also, if the true parameter is not contained within the parameter space of the estimators, then the maximum likelihood estimator will converge asymptotically to the paremeter which maximizes the the Kullback-Leibler information (White 1982).

The true distribution for use in calculating the information is not generally available to the modeler. Consequently, it is necessary to find an approximate method for the minimization of the Kullback-Leibler information. For a model with a constant number of coefficients simple maximum likelihood will achieve this, but if the number of coefficients, is allowed to vary, this is not true. Akaike (1973, 1974a, 1974b, 1976) has shown that it is possible to account approximately for the number of independently adjustable parameters in order to achieve a statistic valid over models of different sizes. Chow (1979, 1981) offers an alternative approximate technique for maximizing the Kullback-Leibler information number.

If the assumption is made that the error entering the model is Gaussian, then the maximization of the Kullback-Leibler information number is the same as the minimization of the expectation of the mean-square one-step prediction error. The mean square error is comprised of specification error, coefficient error and unsystematic error (the unsysematic error is the only error in a correctly specified model at the actual parameter values). For finite samples the trade-off between the coefficient error and the specification error can be important. But since the sum of squares will always be lower for larger models, simply comparing mean-square regression error will not be informative. For work related to this the reader is also referred to Baram (1976), Baram and Sandell (1978), Chipman (1975, 1976, 1983), Fair (1979, 1980) and Sawa (1978).

The concept of information can also be applied to the simplification problem with existing models. This approach is similar to that of Theil (1967), mentioned earlier, but is based on a somewhat different definition of information. In Baram (1976), Baram and Be'eri (1981) the Kullback-Leibler information is considered as a basis for simplification of models. Anderson, Moore and Hawkes (1978) consider a similar problem and show that estimates thus obtained also minimize a certain measure of the distance of the power spectrum of the simplified model from that of the system.

2-C The Quasi Maximum Likelihood Limit

In the above discussion we have considered some of the approaches that are normally taken in the estimation of parameters of simplified models. In this section we consider the consequences of implementing such

estimators for models which are simplifications of linear, time invariant dynamic "systems." In this section we first introduce the notation for a restricted form of the more general "system" given in equation 2.1. Using this "system" we determine which simplified model will result when it is assumed that the simplified model is actually correct.

We assume that the "system" is given by

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{D}\underline{e}_t \ , \tag{2.5}$$

which can be written in partitioned form as

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{D}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \ , \tag{2.6}$$

where $\underline{e}_t$ is a white noise error term.† There are no exogenous inputs into this system with the exception of the error terms. Such a system is referred to as autonomous because its time path does not depend on any deterministic outside influences. We will also assume that the "system" is stable (has eigenvalues less than one in absolute value). This will ensure that the state variables $\underline{x}$ will be stationary processes (Cramer and Leadbetter 1967). These restrictive assumptions allow for a more complete discussion.

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†We do not assume that the error term has a Gaussian distribution. However, the likelihood function we consider will be correct only if the error is Gaussian. The result is essentially one for quadratic loss functions.

The simplified model will be of the same form as equation 2.5 but with fewer state variables. Specifically, the simplified model will take the form

$$\underline{x}_{1t} = \hat{\underline{A}}\underline{x}_{1t-1} + \hat{e}_t \ , \tag{2.7}$$

where the matrix $\hat{\underline{A}}$ is of dimension nxn. We use the notation $\hat{\underline{A}}$ to distinguish this simplifed model from the $\tilde{\underline{A}}$'s that were discussed in the previous chapter and are based on the structure of the "system."

The assumption that the model given in equation 2.7 is correct represents a restriction on the parameter space of the larger model given in equation 2.6. The matrix $\underline{A}_{12}$ of equation 2.6 is assumed to be $\underline{0}$. However, a further assumption is needed in order to apply the previously mentioned result, that the maximum likelihood estimator converges to the point in parameter space which maximizes the Kullback-Leibler information number. It is necessary to assume that the covariance matrix $\underline{D\Omega D}^T$ of the error $\underline{e}$ of the original "system" is block diagonal.

The contribution of an observation to the log-likelihood function for the model as defined in equation 2.5 is given by (excluding the constant term)

$$- 1/2\log\{\mathrm{Det}(\underline{D\Omega D}^T)\} - 1/2(\underline{x}_t - \underline{A}\underline{x}_{t-1})^T \{\underline{D\Omega D}^T\}^{-1} (\underline{x}_t - \underline{A}\underline{x}_{t-1}) \ . \tag{2.8}$$

Assuming the model given in equation 2.7 is correct, the contribution to the quasi-log-likelihood of a single observation (again excluding the

constant term) is given by

$$-1/2\log\{\mathrm{Det}(\underline{\hat{D}\hat{\Omega}\hat{D}}^{T})\} - 1/2(\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1})^{T} \{\underline{\hat{D}\hat{\Omega}\hat{D}}^{T}\}^{-1} (\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1}) . \quad (2.9)$$

Unless the matrix $\underline{D\Omega D}^{T}$ is block diagonal, it is clear that the maximization of 2.9 (summed over observations) is not the same as the maximization of the actual likelihood function using the restricted number of explanatory variables.

When the number $\underline{D\Omega D}^{T}$ matrix is block diagonal, it is possible to maximize the likelihood function by first maximizing the expression in equation 2.5 and then maximizing a similar expression in the excluded states. It is not actually necessary, of course, to carry out this second maximization; it is sufficient that it can be done. This is a slightly different way of stating the well known result that $\underline{x}_1$ has to be a sufficient statistic for the original system to allow its consideration in isolation from $\underline{x}_2$ (Cox and Hinkley, 1974 section 2.2).

The obvious way to obtain estimates of the reduced-order coefficient matrix is to assume that the reduced-order model is correctly specified and perform maximum likelihood. What the above discussion has shown is that this is equivalent to doing maximum likelihood on the complete model subject to the restriction that the error covariance matrix is block diagonal and that A12 is zero. Consequently, we can determine the asymptotic limit of the quasi-maximium likelihood estimator by applying the result on the Kullback-Leibler information number stated above.

In this case it is straightforward to calculate the limit of the quasi-maximum likelihood estimator directly. We assume that the error $\underline{e}$ satisfies the necessary regularity conditions for the manipulations that follow.† Appealing to the ergodic theorem (Doob 1953, Cramer and Leadbetter 1967), the sample log likelihood will converge to the expected value of the incremental log likelihood. The expectation of the quasi log-likelihood equation for the reduced-order model is given by

$$E\left[-1/2\log\{\mathrm{Det}(\underline{\hat{D}\hat{\Omega}\hat{D}}^T)\} - 1/2(\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1})^T \{\underline{\hat{D}\hat{\Omega}\hat{D}}^T\}^{-1} (\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1})\right]. \tag{2.10}$$

The maximization of this with respect to the i,j element of $\hat{\underline{A}}$ will yield, for all (i,j), the first-order conditions

$$E\left[(\underline{x}_{1t-1})_j (\underline{e}^i)^T (\underline{\hat{D}\hat{\Omega}\hat{D}}^T)^{-1} (\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1})\right] = 0 , \tag{2.11}$$

with $\underline{e}^i$ representing the vector with a 1 in the i'th position and 0's elsewhere. Rearranging the above equation and combining the equation for all (i,j), we get the matrix equation

$$E\left[(\underline{\hat{D}\hat{\Omega}\hat{D}}^T)^{-1}(\underline{x}_{1t} - \underline{\hat{A}x}_{1t-1})\underline{x}_{1t-1}^T\right] = \underline{0} , \tag{2.12}$$

which is the standard orthogonality condition on the equation errors relative to the explanatory variables. Inserting the equation for $\underline{x}_{1t}$

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†Because the system given in equation 2.5 was assumed to be stable, it can be shown that the output vector will retain the regularities of the vector $\underline{e}$. In order to appeal to the ergodic theorem we require strict stationarity which will be guaranteed if the fourth moments of $\underline{e}$ are bounded.

(2.6) in the above equation and leaving off $(\hat{\underline{D}}\hat{\underline{\Omega}}\hat{\underline{D}}^T)^{-1}$ gives

$$E[\ \{(\underline{A}_{11} - \hat{\underline{A}})\underline{x}_{1t-1} + \underline{A}_{12}\underline{x}_{2t-1}\}\ \underline{x}_{1t-1}^T\ ] = 0\ . \tag{2.13}$$

Rearranging this equation gives

$$\hat{\underline{A}} = \underline{A}_{11} + \underline{A}_{12}\ E[\ \underline{x}_{2t}\underline{x}_{1t}^T]\ E[\ \underline{x}_{1t}\underline{x}_{1t}^T\ ]^{-1} \tag{2.14}$$

or

$$\hat{\underline{A}} = \underline{A}_{11} + \underline{A}_{12}\ \underline{\Gamma}_{21}\ \underline{\Gamma}_{11}^{-1} \tag{2.15}$$

with $\underline{\Gamma}$ the covariance for the states $\underline{x}$. This is similar to the result for the standard regression model with excluded variables (Sawa 1978).

The above result shows what the quasi-maximum likelihood estimate of a simplified model based on data generated by an undriven dynamic system converges to asymptotically. The resulting model will depend on both the $\underline{A}$ matrix and the characteristics of the noise driving the system. The properties of the resulting model will be considered further in chapter 4.

The results are somehat more general than they at first appear, because it is possible to consider the $\underline{x}_2$ vector as being composed partly of exogenous variables. That is, as long as the exogenous variables can be represented by a strictly stationary stochastic process they can be incorporated in the result. Inclusion of other kinds of exogenous variables presents difficulties.

2-D Summary

For practical purposes the literature on the estimation of models, simplified or not, assumes correct specification. While it is recognized that models are not generally correctly specified, few methods for making use of this information have been developed. In this chapter we briefly introduced the notion of the information of a simplified model relative to the "system." This notion of information can be important in that it provides a unifying framework for the evaluation of a model. A specific type of information, the Kullback-Leibler information, has been considered. The Kullback-Liebler information criterion is the most commonly employed information criterion in both estimation and simplification. The result of applying this criterion to an autonomous linear model has been determined. The simplified model that results depends on both the A matrix and the characteristics of the noise entering the "system."

3 Simplification Based on Selected Behavior Modes

Introduction

In the previous discussion we have reviewed a number of methods for arriving at simplified models. Model simplification methods based on the dynamic characteristics of the "system" are all closely related. In this chapter we will approach the model simplification problem from a different perspective, but in a manner consistent with the literature discussed in chapter 1. In this chapter we consider the development of simplified models based on the retention of specific behavior modes. This chapter is devoted to the problem of deriving a simplified model from an existing dynamic model. In the next chapter we will discuss this approach to model simplification in the context of estimation.

The philosophy of model simplification in this chapter is the same as that of Perez (1981), Perez, Schweppe and Verghese (1982a, 1982b) and Perez et al. (1983). In these works this approach has been termed selective modal analysis, because it is based on the selection of modes and the

development of simplified models based on these modes. This is a generalization of simplified models based on dominant behavior modes (Davison 1966, Litz 1980, Marshall 1966), where dominant is normally taken to mean the least damped (Gopal and Mehta 1982). The dynamic behavior patterns of interest will not necessarily coincide with those which are the least stable, thus a more general framework is helpful.

The discussion in this chapter is largely self-contained. We begin by defining what is meant by a model and a simplified model and then consider which characteristics of the simplified model to retain. Following this, the theory of selective modal analysis is reviewed for difference equations; this review also serves to develop some of the notation for the discussion that follows. Building on this, some generalizations of the selective modal analysis approach are considered. The simplification techniques considered here are based on the assumption that the states to be retained in the simplified model have been chosen. As will be seen, however, the techniques of simplification are helpful in discussing the methods of determining the states to be chosen.

The chapter closes with an example using the small macroeconomic model that was presented in the introductory chapter. In this example the full methodology of finding states, developing a simplified model, and interpreting that model is considered. In the example we make use of all the material developed in chapter 3. Nonetheless, the consideration of the example can be useful at many points in the discussion.

3-A Characteristics of Simplified Models

Models are representations of processes of concern. All models are simplifications and even the most detailed model can do no more than approximate the processes of interest. Taking an existing model as an exact representation of some "system" serves to make the discussion of the simplification process easier and does not arise from any contention that large models are exact. To emphasize this fact the word "system," when used to refer to the larger model, will always appear in quotation marks. Understanding the relationship of simplified models to the models from which they derive can help us understand what all models are telling us about reality.

The properties desirable in a simplified model and, therefore, the best methods for model simplification will depend on purpose. The focus of this research is on retaining the internally generated dynamics of interest with easily interpretable models. There are two strong motivations for this. The first is that a simplified model can serve as a useful vehicle for understanding a more complicated model and, therefore, the processes that model is designed to represent. The second motivation stems from a desire to find policy or control rules based on the simplified model that can be easily implemented in the larger model.

The characteristics of the simplified model and the criteria for choosing it will depend on the purpose of the modeling. Much of the work on stochastic models has been based on the criterion of predictive accuracy (Akaike 1973, 1974a, Chow 1979, Fair 1979, 1980, Sawa 1978). The work done with deterministic models has concentrated on dynamic model responses

and the development of control laws. Eitelberg (1981), (1982), Litz (1980) and Obinata and Inooka (1976) concentrate on the response of the models to different inputs. Gopal and Mehta (1982), Hickin and Sinha (1975, 1980), Rao and Lamba (1975) and Rogers (1971) are concerned with using reduced order models for controller design. The retention of a physical interpretation of the reduced-order model is closely related to the idea of controller design and is emphasized by Aoki (1968) and Litz (1980).

The above criteria are interrelated in a number of ways which will become clearer as we procede. Though the criteria overlap, there are also trade-offs among them. We have identified two characteristics which the simplified models should retain: the modes of interest and easy interpretation. These concepts need to be carefully defined in order to be used in the development of simplified models. The definition of behavior modes is straightforward for linear models. In developing this definition we will follow the heuristic discussion of the introduction. The interpretability of the model is more difficult to define and will be discussed again after selective modal analysis has been reviewed.

We deal only with the simplification of linear, time invariant dynamic models. This restriction is required to maintain tractability but limits the direct applicability of the material discussed. Nonetheless the techniques discussed in this thesis can be useful for nonlinear models. The theory of linear models can often be applied to nonlinear models with quite good results.

The theory of simplification discussed in this chapter is based on the

assumption that there is a given large model which is to be simplified. The large model, which will also be referred to as the "system" or "data-generating process" is given by

$$\underline{x}_t = \underline{\underline{A}}\underline{x}_{t-1} + \underline{\underline{B}}\underline{u}_t + \underline{\underline{D}}\underline{e}_t \qquad (3.1)$$

with $\underline{x}_t$ the state vector, $\underline{u}_t$ a vector of exogenous variables and $\underline{e}_t$ a white noise term. The matrix $\underline{A}$ is NxN, $\underline{B}$ is NxP and $\underline{D}$ is NxM; the vectors $\underline{x}$, $\underline{u}$ and $\underline{e}$ are dimensioned accordingly. The matrix $\underline{A}$ is known as the dynamics matrix; it is through this matrix that past values of the states influence future values. The influence of past values of the state on future values determines the character of the internally generated dynamics. The exogenous variables might impart dynamics not generated by the interactions of the states to the state variables, but the exogenous variables would have to be analyzed to understand these.

Let us define a simplified model as a model that has a reduced number of state variables. We write the simplified model as

$$\underline{\tilde{x}}_t = \underline{\underline{\tilde{A}}}\underline{\tilde{x}}_{t-1} + \underline{\underline{\tilde{B}}}\underline{u}_t + \underline{\underline{\tilde{D}}}\underline{e}_t \ , \qquad (3.2)$$

with the tilde used to denote the fact that this is a simplified model. The matrix $\underline{\tilde{A}}$ is nxn with $n<N$. It may also be possible to eliminate some of the components of the exogenous input vector, though this aspect of simplification will not be treated in any detail. The goal is to have the simplified model maintain certain internally generated dynamics of interest and be easily interpretable.

We have stated that the simplified model is intended to maintain the internally generated dynamics. In order to summarize the dynamic character of the "system," we consider a transformation of the state vector. Premultiplying 3.1 by the matrix of left eigenvectors $\underline{L}$ [†] and noting that $\underline{RL} = \underline{I}$, where $\underline{R}$ is the matrix of right eigenvalues, we have, following Porter and Crossely (1972)

$$\underline{Lx}_t = \underline{LA}(\underline{RL})\underline{x}_{t-1} + \underline{LBu}_t + \underline{LDe}_t \tag{3.3}$$

or

$$\underline{\xi}_t = \underline{\Lambda\xi}_{t-1} + \underline{LBu}_t + \underline{LDe}_t \tag{3.4}$$

where $\underline{\xi}$ refers to the transformed state $\underline{Lx}$, and $\underline{\Lambda}$ is a diagonal matrix with the eigenvalues of $\underline{A}$ along the diagonal. The elements of $\underline{\xi}$ satisfy the equation

$$\underline{\xi}_{it} = \lambda_i \underline{\xi}_{it-1} + (\underline{LBu}_t)_i + (\underline{LDe}_t)_i \quad , \tag{3.5}$$

so that the dynamics of the transformed state are given by the eigenvalues λ_i. The time path of the state vector $\underline{x}$ will be given by a linear combination of the right eigenvectors with the time varying weight for the k'th right eigenvector $\underline{r}^k$ given by $\underline{\xi}_k$. That is,

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[†]The following convention will be conformed to throughout. The right eigenvectors written as column vectors normalized to length 1 will be combined into the matrix $\underline{R}$. The matrix of left eigenvectors written as row vectors will be combined into the matrix $\underline{L}$ and will be normalized so that $\underline{RL} = \underline{I}$.

$$\underline{x}_t = \Sigma_{k=1}^{N} \{ \underline{\xi}_{kt} \underline{r}^k \} \quad . \tag{3.6}$$

The vectors $\underline{r}^k$ for k = 1,2,.. N are constant over time in the above expression. It is in this manner that the dynamics of the original model are given by the eigenvalues of the $\underline{A}$ matrix.

There are two important elements of the transformation to modal form described above. The first is that the modes are all decoupled. What happens in one mode does not influence the behavior of any of the other modes. The second important feature of the above transformation is its invertability. Given the transformed states at time t it is possible to recover the original states at time t.

The simplified model can be transformed in the same way as the larger model to yield its internally generated dynamics. The requirement that the dynamics of interest be retained is a requirement that some of the eigenvalues of the original model be retained in the simplified model.

In addition to retaining the relevant dynamics, a simplified model should be easy to interpret. The requirement that the model be interpretable is a requirement that the states of the simplified model ($\underline{\tilde{x}}$) be associated with physically or economically meaningful variables. If the states of the original model are meaningful in this manner then interpretability is the requirement that the simplified states can be associated with some of the original states or their aggregations.

We will concentrate on the case in which the state vector of the

simplified model can be interpreted as a subset of the state vector of the original model. Equation 3.1 can be written in partitioned form as

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{B}_1 \\ \underline{B}_2 \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{D}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \quad , \tag{3.7}$$

with $\underline{x}_1$ of length n, $\underline{x}_2$ of length (N-n) and with the matrices partitioned accordingly. The requirement of intrepretability can be expressed as the requirement that $\underline{x}_1$ correspond to $\tilde{\underline{x}}$.

Roughly speaking, for $\underline{x}_1$ to correspond to $\tilde{\underline{x}}$ the interactions among the states must be approximately the same in the "system" and the simplified model. It is always possible to ensure this by simply ignoring the other states; of course such a simplified model will not retain the dynamics of interest. It is more useful to consider interpretability in conjunction with the retention of the dynamics of interest. For this, the transformation to modal form must be approximately the same for the "system" and the model, and the interactions among the states must be approximately the same.

3-B Selective Modal Analysis

The basic problem selective modal analysis addresses can be stated very simply. Given a large dynamic model and one or more modes of interest, how can a simplified model retaining those modes and a subset of the original states be derived. Perez (1981), Perez, Schweppe and Verghese (1982a, 1982b) and Perez et al. (1983) have considered two different

aspects of this question. The first is the determination of the characteristics that the simplified model will possess. The second aspect is the development of an algorithm for deriving the simplified model without requiring full knowledge of the eigenvalues and vectors of the large model. The characteristics of the simplified model are the most relevant for our purpose.

The selective modal analysis approach begins with the assumption that the modes of interest have been chosen. We want to consider a partition such as that given in equation 3.7; the choice of the partition will be considered in further discussion. The partitioned equations are represented in a block diagram in Figure 3-1. The first subset of the states $\underline{x}_1$ appears in the upper half of the figure. The lower half of the figure represents the interactions of the second subset of the states among themselves and with the first set of states. The goal of the simplification process is to replace the lower half of the block diagram by a nondynamic subsystem as in figure 3-2. The nondynamic gain $\underline{M}$ represents the elements of feedback that we lose in the simplification process, and it should be relatively small if the states are to be similar to those of the original system.

The equation for $\underline{x}_1$ for the "system" is given by

$$\underline{x}_{1t} = \underline{A}_{11}\underline{x}_{1t-1} + \underline{A}_{12}\underline{x}_{2t-1} + \underline{B}_1\underline{u}_t + \underline{D}_1\underline{e}_t \tag{3.8}$$

and we would like to approximate this by

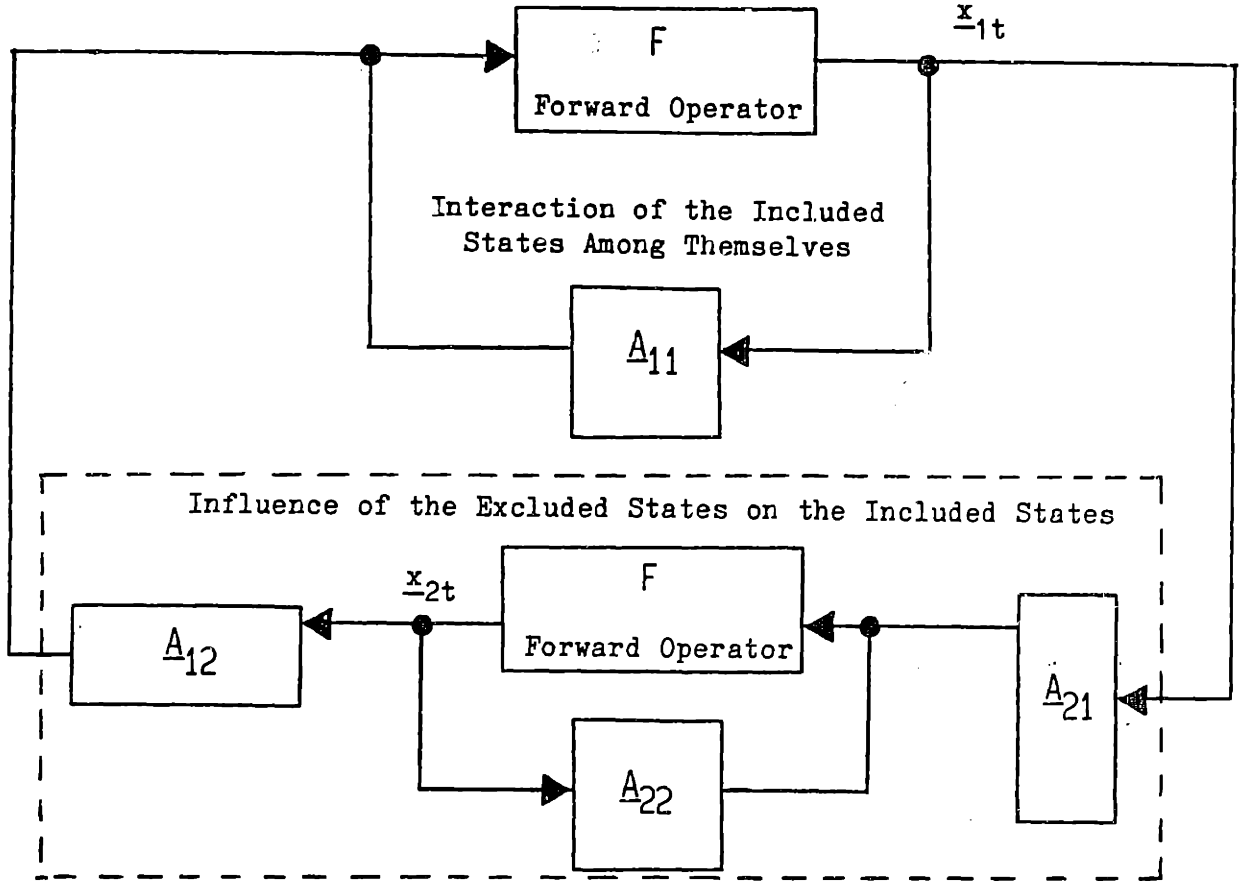

Figure 3-1 Block Diagram Representing the Influence of the Excluded States on the Included States

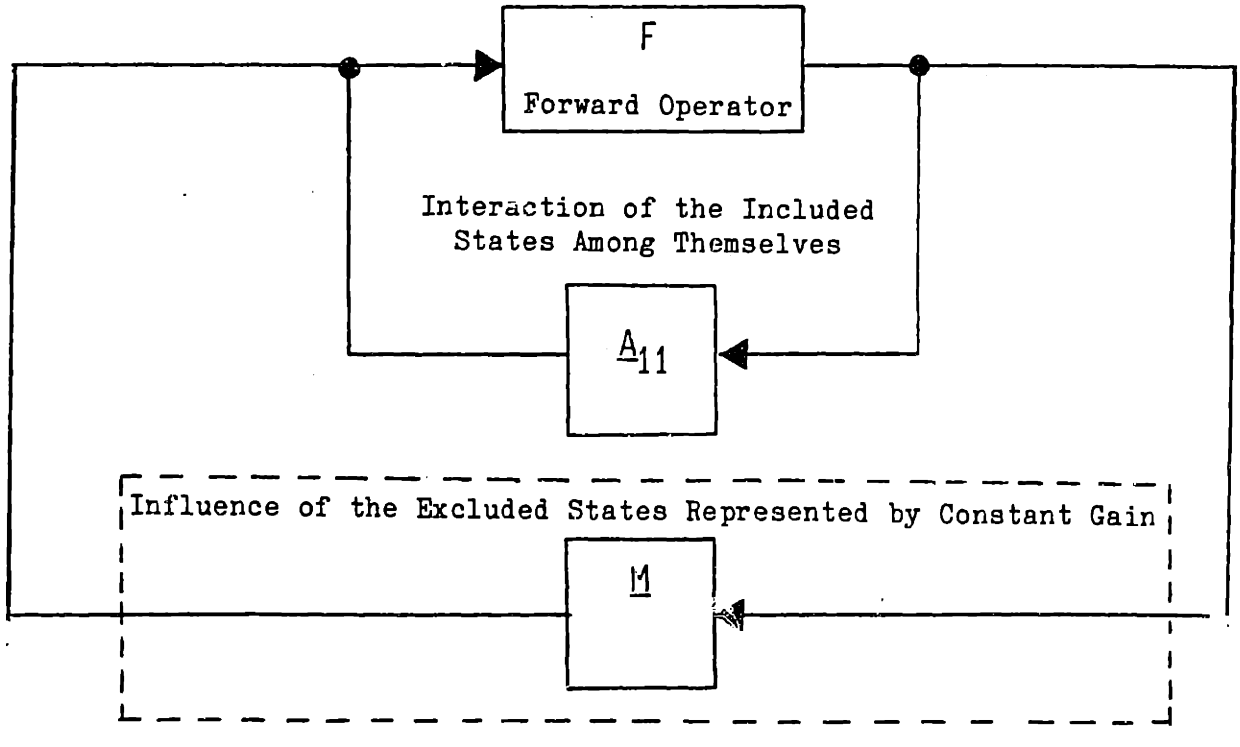

Figure 3-2 Block Diagram With the Influence of the Excluded States Represented by a Constant Gain

$$\underline{x}_{1t} = \underline{A}_{11}\underline{x}_{1t-1} + \underline{M}\,\underline{x}_{1t-1} + \underline{\tilde{B}u}_t + \underline{\tilde{D}e}_t \quad . \tag{3.9}$$

Equation 3.9 could be made to hold exactly if we could find an $\underline{M}$ such that

$$\underline{Mx}_1 = \underline{A}_{12}\underline{x}_2 \quad . \tag{3.10}$$

Because $\underline{x}_1$ and $\underline{x}_2$ are in general linearly independent, this equation will not have a solution valid for all t. It is for this reason that the simplified model will have to be an approximation. Basing the simplification on selected modes is one way to restrict the range over which $\underline{x}$ varies (that is, of restricting $\underline{x}_1$ and $\underline{x}_2$ to be linearly dependent). Without loss of generality, let the first m modes be taken as the modes of interest. One way to restrict our attention to the modes of interest is by requiring that the homogeneous dynamics of the simplified model match those of the "system" when the initial conditions take the form

$$\underline{x}_o = \Sigma_{k=1}^{m} \underline{\xi}_{ko}\, \underline{r}^k \quad , \tag{3.11}$$

where $\underline{\xi}_{ko}$ represents the initial level of the k'th mode. The state vector $\underline{x}_t$ can be written as a linear combination of all of the right eigenvectors. Equation 3.11 restricts our attention to a certain part of the $\underline{x}_t$ vector.

When initial conditions take the form of equation 3.11, it is possible to derive a simplified model that will give the same homogeneous response. The eigenvectors for $\underline{A}$ will satisfy the equation

$$\begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{R}_{11} \\ \underline{R}_{21} \end{vmatrix} = \begin{vmatrix} \underline{R}_{11} \\ \underline{R}_{21} \end{vmatrix} \underline{\Lambda}_{11} \tag{3.12}$$

and the lower blocks of this equation can be rewritten to yield

$$\underline{A}_{21}\underline{R}_{11} + \underline{A}_{22}\underline{R}_{21} = \underline{R}_{21}\,\underline{\Lambda}_{11} \,. \tag{3.13}$$

By concentrating on a single right vector ($\underline{r}^k$) we can write

$$\underline{A}_{21}\underline{r}_1^k + \underline{A}_{22}\underline{r}_2^k = \lambda_k\,\underline{r}_2^k \tag{3.14}$$

which, after rearrangement, gives

$$\underline{r}_2^k = \{(\lambda_k\underline{I} - \underline{A}_{22})^{-1}\underline{A}_{21}\}\underline{r}_1^k \,. \tag{3.15}$$

For the special case in which only one mode is of interest we see that a natural solution for $\underline{M}$ satisfying equation 3.10 is given by

$$\underline{M} = \underline{A}_{12}\{(\lambda_k\underline{I} - \underline{A}_{22})^{-1}\underline{A}_{21}\} \,. \tag{3.16}$$

The solution for $\underline{M}$ given above for a single mode of interest has a nice interpretation in that it is the transfer function of the lower block in Figure 3-1 evaluated at $z = \underline{\lambda}_k$. The transfer function represents the response of the second block at a given generalized frequency to excitation at that same generalized frequency. Referring to figure 3-1, the transfer function tells us how much of the behavior mode of interest gets put out by the second block, when the second block only sees the behavior mode of

interest.

For the single mode of interest we define the simplified model to be

$$\tilde{\underline{x}}_t = (\underline{A}_{11} + \underline{A}_{12}\{(\lambda_k \underline{I} - \underline{A}_{22})^{-1}\underline{A}_{21}\})\,\tilde{\underline{x}}_{t-1} + \tilde{\underline{B}}\underline{u}_t + \tilde{\underline{D}}\underline{e}_t \,. \tag{3.17}$$

The determinants of $\tilde{\underline{B}}$ and $\tilde{\underline{D}}$ will be discussed further on.

This simplified model defined in equation 3.17 preserves the entries for the included states of both the left and right eigenvector associated with the mode of interest. The construction was designed to preserve the right eigenvector. Noting that

$$(\underline{l}_1^k)^T \underline{A}_{12} + (\underline{l}_2^k)^T \underline{A}_{22} = (\underline{l}_2^k)^T(\lambda_i) \tag{3.18}$$

or

$$(\underline{l}_2^k)^T = (\underline{l}_1^k)^T \underline{A}_{12}\{(\lambda_k \underline{I} - \underline{A}_{22})^{-1}\} \tag{3.19}$$

or

$$(\underline{l}_1^k)^T \underline{A}_{11} + (\underline{l}_1^k)^T \underline{A}_{12}\{(\lambda_k \underline{I} - \underline{A}_{22})^{-1}\underline{A}_{21}\} = \lambda_k (\underline{l}_1^k)^T \tag{3.20}$$

we see that the left eigenvector is also preserved. Intuitively, the simplified model preserves the influence of the mode on the state (by preserving the right vector) and the influence of the states on the mode (by preserving the left vector).

The above solution for $\underline{M}$ for the single mode of interest will not be applicable if there is more than one mode of interest; in this case it is necessary to generalize the definition of $\underline{M}$. If $\underline{x}$ can be written as a

combination of the modes of interest then equation 3.10 will be satisfied as long as

$$\underline{M}\underline{r}_1^k = \underline{A}_{12}\{(\lambda_k \underline{I} - \underline{A}_{22})^{-1}\underline{A}_{21}\}\underline{r}_1^k \quad \text{for } k = 1,2, \ldots m . \tag{3.21}$$

In the situation where more than one mode is of interest, there will generally not be a solution for the matrix $\underline{M}$ that preserves both the right and left eigenvectors. However, as long as the number of included modes is less then or equal to the number of included states, an $\underline{M}$ satisfying equation 3.21 exists.† The requirement that the number of modes of interest be less than or equal to the number of state variables retained is essentially a restriction on the space over which the states can vary.

The simplifed model that results for a given selection of the modes of interest and states tc be retained will generally not be unique. Any $\underline{M}$ satisfying

$$\underline{M} = \underline{A}_{12} \left|\underline{r}_2^1 \; \underline{r}_2^2 \ldots \underline{r}_2^m\right| \left|\underline{r}_1^1 \; \underline{r}_1^2 \ldots \underline{r}_1^m\right|^- \tag{3.22}$$

with $^-$ denoting a left inverse (Strang 1980, Ben-Israel 1974) will retain the right eigenvectors of the modes of interest. The matrix $\underline{M}$ is not unique unless the number of modes is equal to the number of states. In order to derive a simplified model with $\underline{A}_{11}$ and $\underline{\tilde{A}}$ as close as possible to

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†For this it is necessary also to assume that $\underline{r}_1^1, \underline{r}_1^2, \ldots \underline{r}_1^m$ is of full column rank. This requires that the included states be active in the included modes in distinct ways. It is always possible and desirable to partition the states so that this is true for nondefective $\underline{A}$ matrices.

one another, a left inverse that minimizes some measure of the norm of $\underline{M}$ can be chosen.

The simplified model can preserve an important element of the structure of the original system. If all the state variables that influence a given state variable retained in the simplified model are also included in the simplified model then the simplifed model will preserve the influence of the included states on the given state. That is, if a row of the $\underline{A}_{12}$ matrix is $\underline{0}$ then there will be a $\underline{M}$ matrix which has the corresponding row zero.† The result follows from the observation that right multiplication preserves row zeros. Therefore the zero row of $\underline{A}_{12}$ will manifest itself in $\underline{M}$ as defined in equation 3.22.

An implication of the above result is that any change in the influence of the included states on a given included state can be attributed to the influence of at least one excluded state. This excluded state must be influenced through some feedback loop by an included state. If the excluded state were not influenced directly or indirectly by the included states then it would be decoupled from the rest of the "system." The sign of the $\underline{M}$ matrix entry attributable to the excluded feedback loop cannot, however, be determined by the nature of the feedback involved. That is, the $\underline{M}$ matrix may have a negative entry even though only positive feedback links have been excluded. This is because the $\underline{M}$ matrix, in part, captures

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† There will also exist matrices $\underline{M}$ for with nonzero entries in the row corresponding zero to the zero row. Any row vector that is perpendicular to $\underline{R}_{11}$ can be added to any row of $\underline{M}$ without violating equation 3.21. There is no apparent benefit from putting the nonzero rows in the $\underline{M}$ matrix.

the dynamic character of the excluded states. If an excluded state is out of phase with an included state for the mode of interest then the effect of the excluded state on the gain can be opposite that expected.

A simple example can make this clear. Consider the simple "system" given by

$$\begin{vmatrix} x_{1t} \\ x_{2t} \end{vmatrix} = \begin{vmatrix} -1 & \epsilon \\ 1 & 0 \end{vmatrix} \begin{vmatrix} x_{1t-1} \\ x_{2t-1} \end{vmatrix} \tag{3.23}$$

with ϵ small. The eigenvalues for this system are given approximately by -1 and ϵ. The feedback connection from x_1 to x_2 is positive (1) as is the feedback connection from x_2 to x_1 (ϵ). However, when the model is simplified, and the -1 root and first state are retained, the resulting value of $\underline{M}$ is approximately $-\epsilon$. The reason for this is that x_1 in the current period is on the opposite side of zero from x_1 in the previous period. Thus $\underline{M}$ captures the effect of the relative dynamic phasing of x_1 and x_2. This fact, though simple, can be useful in interpreting the reasons for the magnitude and sign of the entries in the matrix $\underline{M}$.

Choosing the States

In the above discussion it has been assumed that the partition of the states is given. In practice the choice of states to be retained is probably the most difficult step in the development of a simplified model. The better the choice of states, the smaller the additional gain ($\underline{M}$) required to retain the modes of interest. Of course, the more states the model includes the smaller the required gain ($\underline{M}$) as well. There is a trade-off between the changes required in the existing feedback linkages

and the size of the simplified model. But, for a given size of model, it is desirable to pick the set of states whose feedback connections are the most important in the determination of the modes of interest.

Selective modal analysis introduces the notion of a participation factor. The participation factor for the i'th state in the k'th mode is defined by

$$p_{ik} = (\underline{l}^k)_i (r^k)_i \ , \tag{3.24}$$

and is a complex number for modes associated with complex roots. Writing the state as a linear combination of the right vectors, as in equation 3.6, we see that the i'th entry of the right vector gives the level of activity of the i'th state in the k'th mode, essentially how much the state changes when the level of the mode changes. The i'th entry in the left vector gives the contribution of the i'th state to the k'th mode. We take $\underline{\xi}_{kt}$ as a measure of the level of the k'th mode at time t. Writing

$$\underline{\xi}_{kt} = \Sigma_{i=1}^{N} (\underline{l}^k)_i (\underline{x})_{it} \tag{3.25}$$

we see that $(\underline{l}^k)_i$ determines how much the k'th mode changes when the i'th state changes.

In isolation the contribution of a state to the determination of a mode $(\underline{l}^k)_i$ and the level of activity of a state in a given mode $(\underline{r}^k)_i$ can be somewhat misleading measures and are subject to problems in scaling. Changing the units of measurement for the state variables will affect the

values of the entries in the right and left eigenvectors. For example, if the units of measurement of x_i were changed from dollars to thousands of dollars then $(\underline{r}^k)_i$ would decrease by a factor of 1000 and $(\underline{l}^k)_i$ would increase by a factor of 1000. However, the product $(\underline{l}^k)_i(\underline{r}^k)_i$ is not affected by scaling.† The product tells us how much the state is activated by and contributing to the mode and hence is called the participation factor.

Participation factors are dimensionless and independent of both the choice of units for the state variables and the normalization of the eigenvectors. Participation factors sum to one across states for a given mode and across modes for a given state. The participation matrix is defined as the matrix with the (i,k) entry given by the participation factor of the i'th state in the k'th mode. The composition of a row of the participation matrix is a rough indication of the importance of different modes in a given state, and the composition of a column indicates the importance of states for a given mode. Participation factors cannot completely capture the feedback from one mode into a state, from that state to a different mode, from the different mode to a second state, and from the second state to the original mode; it is possible that consideration of the participation factors alone would not indicate either of the states as important. An example appears in appendix A3. This is why participation factors are only rough indications of the importance of modes in states and vice versa. More complete ways of measuring the importance of states in modes will be discussed later in this chapter.

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†It is assumed that the left eigenvector is normalized so that the inner product of right and left vectors is 1. The normalization of the right vectors does not affect the participation factors.

3-C Extensions of Selective Modal Analysis

We have given an overview of the elements of selective modal analysis fundamental to the discussion that will follow. There are a number of useful modifications and refinements to the basic selective modal analysis theory. It is possible to consider models that preserve the left instead of the right eigenvectors and hence retain the contributions of the included states to the included modes. Consideration needs to be given to the influence of noise and exogenous variables on the models. Finally, it is shown that the simplified models do not introduce complex coefficients as long as modes arising from complex eigenvalues are always represented by conjugate pairs of eigenvalues in the simplified model.

Reduction Preserving the Left Eigenstructure

The simplified models considered so far have all preserved some elements of some of the right-hand eigenvectors. We can interpret this as the preservation of the influence of the modes of interest on the included states. It is also possible to consider simplifications that preserve the left-hand eigenvectors. This can be interpreted as preserving the contribution of the included states to the modes of interest.

We concentrate on the contribution of the first n states to the modes of interest. Again we would like to find a gain matrix $\underline{M}$ that can replace the less relevant dynamic subsystem but still maintain the essential components of the dynamics of interest. We again consider the homogeneous response of the model. However, rather then restricting the space over which the initial conditions vary, we consider the effect of only the included states on the modes of interest. For an arbitrary set of initial

conditions we would like to choose a simplified model so that the included states have the same effect on the modes of interest in the original and simplified models.

We use the vector $\underline{\xi}$ to measure the levels of the modes in the model. The modes of interest are therefore given by

$$\underline{\xi}_{1t} = \underline{L}_{11}\underline{x}_{1t} + \underline{L}_{12}\underline{x}_{2t} \cdot \tag{3.26}$$

The levels of the modes of interest are determined by the values of both the included and excluded states. Changing the initial value of $\underline{x}_1$ will change the later values of the modes of interest by changing later values of $\underline{x}_1$ and $\underline{x}_2$. This is also true of the initial value of $\underline{x}_2$. We ignore the influence of the initial values of $\underline{x}_2$, or, equivalently, assume $\underline{x}_{2o}=\underline{0}$.

At the initial time the contribution of the included states to the modes of interest is

$$\underline{\xi}_{1o} = \underline{L}_{11}\underline{x}_{1o} \cdot \tag{3.27}$$

The initial values for the included states influence both the included states and the excluded states in later periods. In the first period the effect of the included states on the modes of interest is given by

$$\underline{L}_{11}\underline{A}_{11}\underline{x}_{1o} + \underline{L}_{12}\underline{A}_{21}\underline{x}_{1o} \cdot \tag{3.28}$$

But because $\left| \underline{L}_{11}\ \underline{L}_{12} \right|$ is made up of left eigenvectors for $\underline{A}$ we have

$$(\underline{L}_{11}\underline{A}_{11} + \underline{L}_{12}\underline{A}_{21})\underline{x}_{1o} = \underline{A}_{11}\underline{L}_{11}\underline{x}_{1o} \cdot \tag{3.29}$$

For the effect of the included states on the modes of interest to be retained in the simplified model we therefore require that

$$\underline{L}_{11}(\underline{A}_{11} + \underline{M}) = \underline{L}_{11}\underline{A}_{11} + \underline{L}_{12}\underline{A}_{21} \tag{3.30}$$

or

$$\underline{L}_{11}\underline{M} = \underline{L}_{12}\underline{A}_{21} \ , \tag{3.31}$$

and if this holds, the effect of the initial value of $\underline{x}_1$ on the modes of interest will be the same in all periods. From equations 3.31 the solution for $\underline{M}$ is

$$\underline{M} = \underline{L}_{11}^{-}\ \underline{L}_{12}\ \underline{A}_{21} \ , \tag{3.32}$$

with $\underline{L}_{11}^{-}$ the right inverse for $\underline{L}_{11}$. Equatione 3.32 is an analogue of the expression for $\underline{M}$ obtained for the simplified model based on the right eigenvectors.

For an arbitrary set of initial conditions the simplified model based on the left vectors preserves the effect of the included states on the mode of interest. If any initial value for a state of interest is changed, the change in the value of the modes of interest will be the same in the original and simplified models. On the other hand, if the initial value of any of the excluded states is changed the modes of interest may be changed in the original model, but will not be changed in the simplified model.

When only one mode is of interest it is possible to choose $\underline{M}$ to be the transfer function as in equation 3.16 thus, as previously noted, preserving the right and left eigenvector associated with the mode of interest.

There is a result for the preservation of the feedback structure in the simplified model based on the left eigenvectors analogous to the one given for the simplfied model based on the right eigenvectors. In the left eigenvector case any included state that does not influence any of the excluded states will not have its influence on the included states changed. That is, if a column of the $\underline{A}_{21}$ matrix is zero, then the corresponding column of $\underline{M}$ will be zero. This follows from preservation of zero columns on left multiplication. The reduced-order model preserves the structure of the transmission of dynamics from one state to another.

As for the case of the result on the right eigenvectors there is a useful corollary of the above result. Whenever the influence of an included state on another included state is changed in the simplified model, this change can be attributed to the influence, in the original model, of the first included state on at least one excluded state. And again the change in the influence reflects both the importance of the excluded feedback and the relative phasing over the behavior mode of interest of the included and excluded variables.

The simplified model based on the left-hand eigenvectors is similar to that based on the right-hand eigenvectors but has a slightly different emphasis. If there were no feedback in the "system" between the excluded and included states, the two models would be the same. The difference

between the two models is one measure of how good the simplified models are. This will be commented on further in the section on the quality of the simplified models.

Response to Exogenous Variables

The reduced-order model is intended to maintain the internally generated dynamics of interest. In addition, consideration must be given the effect of the inputs into the system on the excitation of these modes. If the driving variable $\underline{u}$ is such that it has a great deal of impact on one of the modes of interest, then it will be important to capture this effect. On the other hand, if the driving variable does not excite the modes of interest appreciably, then its influence on the reduced order model should be small. Similar comments apply to the effect of the white noise input $\underline{e}$. If certain modes of the system are highly excited by the noise entering, then a good simplified model will also reflect this.

The exogenous variables will excite the mode of interest in the "system," and it is desirable to have them excite the modes of interest in the simplified model in a similar manner. Essentially, when the "system" and the simplified model are transformed to diagonal form, we would like to have the same influence of the exogenous variables on the modes of interest.[†] The simplifed model can always be transformed to modal form by

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[†]The normalization chosen for the eigenvectors will determine the magnitudes, but not the dynamics, of the transformed variables $\underline{\xi}$. This is because postmultiplying the right eigenvector matrix by any nonsingular diagonal matrix $\underline{V}$ will give another right eigenvector matrix. We will assume in the discusssion which follows that the left vectors of the simplifed models are normalized so that $\underline{\xi}_{ko} = \underline{\xi}_{ko}$.

premultiplying by $\tilde{\underline{L}}$ to yield

$$\underline{\xi}_t = \underline{\Lambda}_{11}\underline{\xi}_{t-1} + \tilde{\underline{\underline{L}}}\underline{\underline{B}}\underline{u}_t + \tilde{\underline{\underline{L}}}\underline{\underline{D}}\underline{e}_t \tag{3.33}$$

analogously to equation 3.4. The preservation of the excitement of the modes of interest in the reduced-order model thus requires that

$$\tilde{\underline{L}}_1\tilde{\underline{B}} = (\underline{\underline{L}}\underline{\underline{B}})_1 = \underline{L}_1\underline{B} \tag{3.34}$$

and

$$\tilde{\underline{L}}_1\tilde{\underline{D}} = (\underline{\underline{L}}\underline{\underline{D}})_1 = \underline{L}_1\underline{D} , \tag{3.35}$$

where the subscript 1 refers to the first m rows of the subscripted matrices. Defining

$$\tilde{\underline{B}} = \tilde{\underline{R}}_1\underline{L}_1\underline{B} = \tilde{\underline{R}}_1\left|\underline{L}_{11}\ \underline{L}_{12}\right|\underline{B} \tag{3.36}$$

and

$$\tilde{\underline{D}} = \tilde{\underline{R}}_1\underline{L}_1\underline{D} = \tilde{\underline{R}}_1\left|\underline{L}_{11}\ \underline{L}_{12}\right|\underline{D} \tag{3.37}$$

will satisfy equations 3.34 and 3.35. It would also be possible to use other right inverses of $\tilde{\underline{L}}_1$ in place of $\tilde{\underline{R}}_1$, though this does not seem like a useful generalization.

If the left-hand eigenvectors are maintained in the simplified model and the number of retained modes equals the number of retained states, then the effect of the exogenous variables has a nice interpretation. In this case the equations for $\tilde{\underline{B}}$ and $\tilde{\underline{D}}$ become

$$\tilde{\underline{B}} = \underline{B}_1 + \underline{L}_{11}^{-1} \underline{L}_{12} \underline{B}_2 \tag{3.38}$$

and

$$\tilde{\underline{D}} = \underline{D}_1 + \underline{L}_{11}^{-1} \underline{L}_{12} \underline{D}_2 \cdot \tag{3.39}$$

In the above equations the influence of the exogenous variables on the included variables is the original influence plus an effect which is attributable to the excluded states. The exogenous variables influence the excluded states, which in turn excite the modes of interest. The product of the two influences tells us how much to adjust the $\tilde{\underline{B}}$ matrix away from $\underline{B}_1$.

Steady-State Properties

The preservation of the steady-state gain of the model has been a subject of some discussion (Chidambara 1967, Davison 1967). The steady-state gain is the response of the model output to a constant input vector $\underline{u}_t = \underline{u}_o$ (we assume no influence of noise). The gain is a linear map from a constant input vector to a fixed, long-run response. It gives the equilibrium effect of a change in the exogenous variables. The steady-state gain is defined only if the model is stable and is therefore not completely general. When transformed to diagonal form, the full model and the simplified model, with $\tilde{\underline{B}}$ as defined above, are affected in the same way by the exogenous variables. Therefore we have

$$\tilde{\underline{\xi}}_{ss} = \underline{\xi}_{1ss} = (\underline{I} - \underline{\Lambda})_{11}^{-1} \left| \underline{L}_{11} \; \underline{L}_{12} \right| \underline{Bu}_{ss} \cdot \tag{3.40}$$

The values for the transformed states will be the same in steady state for the "system" and the simplified model. This does not imply that the values

of the state variables will be the same. The values of the included states in the full system are given by

$$\underline{x}_{1ss} = \underline{R}_{11}(\underline{I} - \underline{\Lambda})^{-1}_{11} \left|\underline{L}_{11}\ \underline{L}_{12}\right| \underline{Bu}_{ss} + \underline{R}_{12}(\underline{I} - \underline{\Lambda})^{-1}_{22} \left|\underline{L}_{21}\ \underline{L}_{22}\right| \underline{Bu}_{ss} \quad (3.41)$$

whereas for the reduced-order model they are given by

$$\underline{\tilde{x}}_{ss} = \underline{\tilde{R}}_{1}(\underline{I} - \underline{\Lambda})^{-1}_{11} \left|\underline{L}_{11}\ \underline{L}_{12}\right| \underline{Bu}_{ss} \ . \quad (3.42)$$

The steady-state gain of the simplified model will not generally match that of the entire system. It will not, in general, be possible to find an $\underline{\tilde{A}}$ which will preserve simultaneously the the influence of the exogenous variables on the modes of interest and the steady-state response of the included states.

The reduced-order model based on the right-hand eigenvectors is of special interest. Its steady-state value differs from the that of the system by

$$\underline{R}_{12}(\underline{I} - \underline{\Lambda})^{-1}_{22} \left|\underline{L}_{21}\ \underline{L}_{22}\right| \underline{Bu}_{ss} \quad (3.43)$$

which is the influence of the excluded modes. This is zero if the excluded modes are not controllable (see Chen 1970 or Porter and Crossely 1972).

The steady-state gain can be one of the most striking ways in which a simplified model differs from the original "system." For the simplification based on the right vectors, from equation 3.43, the

difference in the steady-state gains is entirely attributable to excluded modes. A simplified model which excludes these modes cannot be expected to retain the steady-state response of the system without badly approximating the influence of the exogenous variables on the modes of interest. The steady-state response can be best thought of as a scaling problem. The transformed states can always be rescaled to give the correct steady-state response. Specifically, it is possible to consider[†]

$$\underline{\hat{x}}_t = \underline{\tilde{x}}_t + \underline{R}_{12}(\underline{I} - \underline{\Lambda})^{-1}_{22} \left| \underline{L}_{21} \; \underline{L}_{22} \right| \underline{Bu}_t \tag{3.44}$$

as a scaled state that preserves the steady state gain. This is essentially the solution proposed by Davison (1967).

Equation 3.44 depends only on the excluded modes. Thus, even if the "system" is unstable it may be possible to corect for the "steady-state gain" error. To do this requires that the unstable modes all be retained in the simplified model. In this case, equation 3.44 is well defined and the time path of $\underline{\tilde{x}}$ will give a good approximation to the time path of $\underline{x}_1$.

Real Coefficients

One important criterion in the choice of the model order-reduction technique under consideration is that the resulting model be easily interpreted. The reduced-order model should be closely related to our

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[†] We will assume that the right eigenvectors are retained. A similar but more cumbersome expression exists for the $\underline{\tilde{x}}$ associated with retaining the left vectors. However, for the case in which the "system" is unstable, the disussion is only valid for the simplification based on the right vectors.

understanding of the relevant economic and physical processes. This means, among other things, that it should have real coefficients. It would be difficult, for example, to relate marginal propensity to consume of .6+.3j to standard consumer theory. Because the eigenvalues and eigenvectors are, in general, complex numbers, we need to clarify the conditions required for the $\underline{\tilde{A}}$, $\underline{\tilde{B}}$ and $\underline{\tilde{D}}$ matrices to be real.

There will be a realization of the simplified model with all the matrices real if, whenever a mode represented by a complex root in the model is retained, the mode represented by the conjugate of that complex root and the associated vector are also retained. By the associated vector we simply mean that if $\underline{\tilde{A}}$ is restricted to have a right vector $\underline{\tilde{r}}^k$ for eigenvalue λ_k then $\underline{\tilde{A}}$ must be allowed to have a right vector $\underline{\bar{\tilde{r}}}^k$ for $\bar{\lambda}_k$. Similarily for left vectors. Because the simplified model will not necessarily be unique, only the existence of a real $\underline{\tilde{A}}$ (and a means of computing it) can be guaranteed. This result is a generalization of a result from Perez, Schweppe and Verghese (1982a).

The proof that this is true is straightforward and is given in detail in the appendix. The result follows essentially from the preservation of column conjugation when a matrix is right multiplied by a real matrix. This is true for the $\underline{M}$ matrix as well as the $\underline{\tilde{B}}$ and $\underline{\tilde{D}}$ matrices.

3-D The Quality of the Simplified Model

In the previous discussion we have stressed the fact that the simplified model should be easily interpretable and give good information

about policy changes. Hence, we consider two approaches to evaluating how well the simplifed model can perform: the interpretability of the model and the response of the original and simplifed model to similar policy changes. The quality of a simplified model will depend heavily on the choice of the states to be retained. In the discussion, we will consider in detail the question of how to choose the states to be retained.

In developing a simplified model it is normally necessary to make approximations. A simplified model can hardly be an exact representation of a more complex model. However, in order to determine the extent to which the simplification process requires approximation, it is necessary to have a precise definition of what approximation is. In order to do this we will characterize the ideal situation in which the states of the simplified model correspond exactly to those of the original system. The idealized situation will rarely, if ever, obtain. But this is to be expected, for the ideal situation is simply a benchmark against which various simplifications can be compared.

In the introduction to this thesis we suggested that if the "system" and the simplified model could be transformed in the same manner and yield the same result then the simplified model would be easily interpretable in terms of the "system." The transformation that we consider is the transformation to diagonal form given in equations 3.3 and 3.4. There is a problem in applying this transformation to both the original model and the simplified model, because the two will not be of the same dimension. However, it is possible to augment the transformation of the simplified model so that it can be applied to the original "system." This can be done

by assuming that no change is made to the states not included in the simplified model.

Exact Simplification

One of the most important and interesting features of the transformation of a dynamic model to diagonal form is that the transformation is invertible. The original system can be recovered from the diagonal form, and statements about the diagonal form will therefore also apply to the original system. We will consider only invertible transformations of simplified models and their application to the "system." The simplified model is represented in equation 3.2. Its transformation is accomplished by premultiplying the model by a nonsingular matrix. Because the simplified model is designed to retain m of the dynamic modes, the transformation matrix we will consider is

$$\underline{T} = \begin{bmatrix} \tilde{\underline{L}}_1 \\ \underline{K} \end{bmatrix} = \begin{bmatrix} (\tilde{\underline{1}}^1)^T \\ (\tilde{\underline{1}}^2)^T \\ \vdots \\ (\tilde{\underline{1}}^m)^T \\ \underline{K} \end{bmatrix} \tag{3.45}$$

with $\underline{K}$ any (n-m)xn matrix, which makes $\underline{T}$ nonsingular. The original "system" is transformed by

$$\begin{bmatrix} \underline{\zeta}_{1t} \\ \underline{\zeta}_{2t} \end{bmatrix} = \begin{bmatrix} \underline{T} & \underline{0} \\ \underline{0} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{bmatrix} \tag{3.46}$$

to yield

$$\begin{vmatrix} \underline{Tx}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{TA}_{11}\underline{T}^{-1} & \underline{TA}_{12} \\ \underline{A}_{21}\underline{T}^{-1} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{Tx}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{TB}_1 \\ \underline{B}_2 \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{TD}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \quad . \quad (3.47)$$

The transformed system 3.47 should correspond to the transformation of the simplified model, which takes the form

$$\underline{T\tilde{x}}_t = \underline{\tilde{TAT}}^{-1} \underline{T\tilde{x}}_{t-1} + \underline{\tilde{TB}}_1\underline{u}_t + \underline{\tilde{TD}}_1\underline{e}_t \quad (3.48)$$

For the transformation to have the same effect on the original and simplified models we require that the transformation alter the interactions among the included states in the same way for the original and simplified models. That is, we require that

$$\underline{TA}_{11}\underline{T}^{-1} = \underline{\tilde{TAT}}^{-1} \quad \text{or} \quad \underline{A}_{11} = \underline{\tilde{A}} \; . \quad (3.49)$$

The off-diagonal blocks are not as straightforward.

For the simplified model only the original set of states $\underline{x}_1$ is included. This suggests that we should require that the off-diagonal blocks ($\underline{TA}_{12}$ and $\underline{TA}_{21}$) be $\underline{0}$ for exact simplification. This is somewhat too strong in that we are concentrating on the internally generated dynamics, and these are generated by feedback. If either of the off-diagonal blocks is zero then there is no feedback between the first and second sets of states. Because the transformation matrix $\underline{T}$ is nonsingular, this is equivalent to the requirement that either $\underline{A}_{12}$ or $\underline{A}_{21}$ be $\underline{0}$. Selecting n states such that $\underline{A}_{12}=\underline{0}$ or $\underline{A}_{21}=\underline{0}$ is termed exact simplification.

The conditions for what we call "exact simplification" are severe. They require complete decomposability of the dynamics matrix to a block triangular matrix. If exact simplification obtains, then the number of modes maintained will equal the number of states maintained. Not all of the modes will necessarily be of interest.

In developing simplified models in which the number of retained modes and the number of retained states are equal, the full transformation $\underline{T}$ is given by $\underline{\tilde{L}}$. In considering cases in which the number of retained modes is less than the number of retained states, a choice for $\underline{K}$ in equation 3.45 will have to be made. Rather than considering all possible $\underline{K}$ it is more practical to utilize the noninvertible transformation which results from using the right and left eigenvectors of the simplified model. The use of a noninvertible transformation is one way the approximate nature of the model simplification process manifests itself. We transform the simplifed model by

$$\underline{\tilde{L}}_1\underline{\tilde{x}}_t = \underline{\tilde{L}}_1\underline{\tilde{A}}\underline{\tilde{R}}_1\underline{\tilde{L}}_1\underline{\tilde{x}}_{t-1} + \underline{\tilde{L}}_1\underline{\tilde{B}}\underline{u}_t + \underline{\tilde{L}}_1\underline{\tilde{D}}\underline{e}_t \tag{3.50}$$

and, analogously, transform the original model by

$$\begin{vmatrix} \underline{\tilde{L}}_1\underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{\tilde{L}}_1\underline{A}_{11}\underline{\tilde{R}}_1 & \underline{\tilde{L}}_1\underline{A}_{12} \\ \underline{A}_{21}\underline{\tilde{R}}_1 & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{\tilde{L}}_1\underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{\tilde{L}}_1\underline{B}_1 \\ \underline{B}_2 \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{\tilde{L}}_1\underline{D}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \quad . \tag{3.51}$$

The distance of the simplification from an exact simplification for the modes of interest can then be aproximated by consideration of

$$\tilde{\underline{L}}_1 \underline{A}_{11} \tilde{\underline{R}}_1 - \underline{\Lambda}_{11} \ , \tag{3.52}$$

which we would like to be $\underline{0}$, and of

$$\tilde{\underline{L}}_1 \underline{A}_{12} \quad \text{and} \quad \underline{A}_{21} \tilde{\underline{R}}_1 \ , \tag{3.53}$$

one of which we would like to be $\underline{0}$.

The above conditions are stated in terms of the eigenvectors of the simplified model. But the construction of the simplified model is intended to preserve, or nearly preserve, the eigenvectors of the large model, and therefore we can substitute $\underline{L}_{11}$ and $\underline{R}_{11}$ for $\tilde{\underline{L}}_1$ and $\tilde{\underline{R}}_1$.

Scaling

The conditions we have derived for judging the quality of a model are very difficult to use as criteria for choosing states. The conditions are stated in terms of matrix products, which makes searching over all possible sets of included states prohibitively expensive. In addition, the numerical values of the matrix products are subject to the effects of scaling of the state variables and normalization of the eigenvectors. That is, if the units of the first variable were to be redefined, the matrix products would change. Similarily, if the left, instead of the right, eigenvectors were normalized to length one, then the values would also change. We need to develop straightforward criteria for the choice of states to be included in a simplified model which will not be sensitive to the scaling problem.

The scaling problems arises in two ways. The first is by the choice of units of measurement of the states, and the second is the choice of normalization for the vectors. We will call the associated scaling matrices $\underline{S}$ (for states) and $\underline{V}$ (for vectors), with both matrices diagonal. The scaled state variable $\underline{x}_{scaled}$ is defined by

$$\underline{x}_{scaled} = \underline{Sx} \ , \tag{3.54}$$

and for this scaled state the corresponding dynamics matrix will be given by

$$\underline{A}_{scaled} = \underline{S}\ \underline{A}\ \underline{S}^{-1} \ . \tag{3.55}$$

Scaling affects the matrix products considered above in the following manner:

$$\underline{R}_{scaled} = \underline{SRV}^{-1} \ , \tag{3.56}$$

$$\underline{L}_{scaled} = \underline{VLS}^{-1} \ , \tag{3.57}$$

$$\underline{L}_{11}\underline{A}_{11}\underline{R}_{11scaled} = \underline{V}_{11}\underline{L}_{11}\underline{A}_{11}\underline{R}_{11}\underline{V}_{11}^{-1} \ , \tag{3.58}$$

$$\underline{L}_{11}\underline{A}_{12scaled} = \underline{V}_{11}\underline{L}_{11}\underline{A}_{12}\underline{S}_{22}^{-1} \tag{3.59}$$

and

$$\underline{A}_{21}\underline{R}_{11scaled} = \underline{S}_{22}\underline{A}_{21}\underline{R}_{11}\underline{V}_{11}^{-1} \ . \tag{3.60}$$

From equation 3.58 only the normalization of the vectors affects the $\underline{L}_{11}\underline{A}_{11}\underline{R}_{11}$ matrix. Because this matrix is premultiplied by $\underline{V}_{11}$ and postmultiplied by $\underline{V}_{11}^{-1}$, the diagonal elements will not be affected by scaling.

For all the off-diagonal elements of $\underline{L}_{11}\underline{A}_{11}\underline{R}_{11}$ and $\underline{L}_{11}\underline{A}_{12}$ and $\underline{A}_{21}\underline{R}_{11}$ the issues of scaling cannot be ignored. The off-diagonal matrices offer special problems because they are scaled on one side by the scaling factor for the states and on the other side by the normalization for the vectors. It would be possible to overcome this to some extent by considering products such as $\underline{L}_{11}\underline{A}_{12}\underline{R}_{21}$ for $\underline{L}_{11}\underline{A}_{12}$, but this requires matrix multiplications and was not pursued further.

It is useful to consider the case in which the number of included modes is one. In this case the $\underline{L}_{11}$ and $\underline{R}_{11}$ matrices are row and column vectors. The condition on the upper diagonal term is that $(\underline{l}_1^k)^T\underline{A}_{11}(\underline{r}_1^k)=\lambda_k$, which may be rewritten as[†]

$$\text{Trace}(\underline{A}_{11}(\underline{r}^k)(\underline{l}^k)^T) = \lambda_k \tag{3.61}$$

or

$$\Sigma_{i=1}^{n} \{ \Sigma_{j=1}^{n} (a_{ij} (l^k)_i (r^k)_j) /\lambda_k \} = 1 . \tag{3.62}$$

The term $(a_{ij} (l^k)_i (r^k)_j) /\lambda_k$ is the elasticity of the k'th eigenvalue with respect to changes in the (i,j) entry of the $\underline{A}$ matrix. The requirement given in equation 3.63 is tells us that the sum of the eigenvalue elasticities over all the feedback links of the included states should be one. The consideration of eigenvalue elastivities is a technique used extensively in Kuh (1983 see also CCREMS 1983, chapter 2). Some useful results on eigenvalue elasticities are given in appendix A3.

[†]This is also equivalent to the condition that the sum of the participation factors, as they were previously defined, be one.

The above condition is stated for the included states, but the off-diagonal elasticities and those for the excluded states can also be useful. If the original model is exactly simplifiable, then the eigenvalue elasticity with respect to a_{ij} for $i>n$ or $j>n$ will be zero. This follows directly from the observation that if $\underline{A}_{12}=\underline{0}$ then $\underline{R}_{12}=\underline{0}$ and $\underline{L}_{12}=\underline{0}$. Thus for $i>n$ we have $\underline{l}_i^k=0$. Whenever $j>n$ and $i\leq n$ we have $a_{ij}=0$. Similar logic applies if $\underline{A}_{21}=\underline{0}$. Graphically,

$$\underline{A}_{12}=\underline{0} \rightarrow \begin{vmatrix} xxxx & \underline{A}_{12}=\underline{0} \\ \underline{l}_2^k=0 & \underline{l}_2^k=\underline{0} \end{vmatrix} \quad \text{and} \quad \underline{A}_{21}=\underline{0} \rightarrow \begin{vmatrix} xxxx & \underline{r}_2^k=\underline{0} \\ \underline{A}_{21}=\underline{0} & \underline{r}_2^k=\underline{0} \end{vmatrix}, \tag{3.63}$$

for k = 1,2, . . n and with the (i,j) entry in the above matrices corresponding to the elasticity of the k'th eigenvalues with respect to a_{ij}.

The elasticities also have implications for the degree to which the model is simplifiable. Suppose that the eigenvalue elasticities associated with the excluded states are all zero. If the right eigenvector for the mode of interest has a nonzero entry for an excluded state, the left eigenvector will have zero entries for the excluded state and all states that are directly influenced by the excluded state. This follows by observing that if $\underline{r}_i^k \neq 0$ for some $i \in (n+1,N)$ then from the fact that

$$a_{ji}\underline{r}_i^k\underline{l}_j^k = 0 \tag{3.64}$$

we have that

$$\lambda_k \underline{1}_i^k = (\underline{1}_1^k \underline{A}_{21} + \underline{1}_2^k \underline{A}_{22})_i = \underline{0} \tag{3.65}$$

and

$$a_{ji} \neq 0 \rightarrow \underline{1}_j^k = 0 \quad . \tag{3.66}$$

An analogous result holds with respect to non zero entries in the left eigenvector.

The big advantage of the elasticity criterion is that it is easily applied if an eigenvalue decomposition of the "system" has been performed. There is no matrix multiplication involved in choosing the states. The states can be selected simply by calculating the entire elasticity matrix and removing states that have only small elasticities associated with their feedback links. The elasticities are independent of both the scaling of the states and the normalization of the eigenvectors. In addition, the definition of a large eigenvalue elasticity does not depend on the magnitude of the eigenvalue, since the sum across all elements is normalized to be one.

3-E Effects of Policy Changes

In concentrating on the endogenously generated dynamics, it is appropriate to consider the effects of feedback policies. Feedback policies based on the simplified model can be implemented on the original system. If the states are easily interpretable, then we would simply like to implement the policy developed for the simplified model on the "system." If the effects of the policy are the same for both the system and the model then the model is, in one sense, a good simplified model (Forrester and Senge 1978).

When implemented on the system, policies will affect both the modes of behavior retained in the simplified model and the remaining modes. It is clear that in the simplified model only the modes of interest will be affected. In the development, we will concentrate on the derivatives of the eigenvalues for small changes in the parameters. Similar results are obtained by considering policies based on the modal form of the simplified model (that is, policies which do not change the eigenvectors of the simplified model).

The same policy implemented on both the "system" and simplified model should have approximately the same effect. We consider the effect of a change in terms of the derivatives of eigenvalues with respect to elements of the $\underline{A}$ matrix. A simplified model is exactly correct if the derivatives of all eigenvalues with respect to $\underline{\tilde{A}}$ are the same as the derivatives with respect to $\underline{A}_{11}$. With all eigenvalues referring to those of the original "system," both those of interest and those not of interest. For the simplified model it is assumed that changes will have no effect on the excluded modes. The above definition is the same as the definition of "exactly simplifiable" given in the previous discussion, as will be shown below.

The definition of an exactly correct simplified model given above requires that the number of states and modes is equal, since the effects of parameter changes on modes in the simplified model must be considered in comparison to the effects on modes in the "system." It will be assumed therefore that $\underline{L}_{11}$ and $\underline{R}_{11}$ are square matrices. Since the derivative of

the k'th eigenvalue with respect to the (i,j) element of the $\underline{A}$ matrix is given by

$$\frac{\partial}{\partial a_{ij}} \lambda_k = (\underline{l}^k)_i (\underline{r}^k)_j \tag{3.67}$$

the eigenvectors can be used to compare derivatives. For the simplified model to be exactly correct according to the definition given above, we require that

$$(\underline{r}_1^k)(\underline{l}_1^k)^T = (\tilde{\underline{r}}_k)(\tilde{\underline{l}}_k)^T , \quad \text{for } k = 1 \text{ to } n \tag{3.68}$$

and

$$(\underline{r}_1^k)(\underline{l}_1^k)^T = \underline{0} , \quad \text{for } k = n+1 \text{ to } N . \tag{3.69}$$

Because the matrices $\underline{R}_{11}$ and $\underline{L}_{11}$ are assumed to be nonsingular, the requirement of equation 3.68 can be rewritten as

$$\underline{R}_{11} = \tilde{\underline{R}} \quad \text{and} \quad \underline{L}_{11} = \tilde{\underline{L}} , \tag{3.70}$$

with appropriate normalization. The normalization is required because the $\underline{R}_{21}$ submatrix will, in general, be nonzero, and the columns of the $\underline{R}_{11}$ matrix will therefore have a smaller norm than those of $\tilde{\underline{R}}$.

The requirements of equations 3.68 and 3.69 can be met exactly only if, for all excluded roots, the included states do not have nonzero entries in both the right and left eigenvectors associated with the modes of interest. If the included states enter into the right eigenvectors associated with the excluded roots then the included states will be

influenced by the excluded modes. However, this influence will affect the included modes only if the excitation of the included states feeds back to the excluded states. If there are no nonzero entries for the included states in the left eigenvectors of the excluded modes then this will not occur. Similarily, if the entries for the included states in the right eigenvectors of the excluded modes are zero then the excluded modes will not excite the included states. The condition for exact simplification is then given by

$$\underline{R}_{12} = \underline{0} \text{ or } \underline{L}_{21} = \underline{0}, \tag{3.71}$$

and since $\underline{RL} = \underline{I}$ this also gives

$$\underline{L}_{12} = \underline{0} \text{ or } \underline{R}_{21} = \underline{0}, \tag{3.72}$$

and

$$\underline{R}_{11}\underline{L}_{11} = \underline{I} \, . \tag{3.73}$$

To summarize, based on the derivatives of the eigenvalues the conditions that we require for exactly correct simplification are

$$\underline{L}_{11}\underline{R}_{11} = \underline{I} \tag{3.74}$$

and

$$\underline{L}_{11}\underline{R}_{12} = \underline{0} \text{ or } \underline{L}_{21}\underline{R}_{11} = \underline{0} \; ; \tag{3.75}$$

equivalent to these conditions are the conditions

$$\underline{R}_{11}\underline{L}_{11} = \underline{I} \tag{3.76}$$

and

$$\underline{R}_{11}\underline{L}_{12} = \underline{0} \text{ or } \underline{R}_{21}\underline{L}_{11} = \underline{0} \, . \tag{3.77}$$

In a similar manner to that considered in the discussion of physical interpretability, we can restrict our attention to the selection of a limited number of modes. In this case we apply for the limited number of modes the criterion given in the above equations. Equation 3.73 needs modification to read that $\underline{L}_{11}\underline{R}_{11}$ is diagonal and has Trace m.

From the above criterion we again need to derive practical techniques for finding the important states. The issues are the same as they were in the case where we considered the interpretability of the simplified model. For equations 3.74 and 3.75 the resulting matrices are scaled by the normalization of the eigenvectors. For a single root, equation 3.74 collapses to the sum of the participation factors for the included states. However, a number of additional conditions are added by the requiremets of 3.75. For equations 3.76 and 3.77 the resulting matrices are scaled by the scaling matrix for state measurement.

Generalized Participation Factors

We consider two measures of the importance of a state in a mode derived from the above conditions. The first is the generalized participation participation factor, so called because it is a generalization of the participation factor as it was previously defined. The generalized participation factor is defined to be

$$\underline{R}_1\underline{L}_1 \, . \tag{3.78}$$

For a single mode the generalized participation factor has the participation factors of the states in that mode along the diagonal. For multiple modes the generalized participation factor has the sums of the participation factors over the modes of interest on the diagonal. If conjugate modes are included, then the generalized participation factor will be a real matrix.

In judging the importance of a set of states in a set of modes, we consider the difference

$$\underline{R}_1\underline{L}_1 - \begin{vmatrix} \underline{I}_n & \underline{0} \\ \underline{0} & \underline{0} \end{vmatrix} \tag{3.79}$$

where the states of interest have been permuted to represent the first n rows. Because this permutation can be carried out after the matrix product $\underline{R}_1\underline{L}_1$ has been calculated, it does not represent a computational burden. The off-diagonal elements of equation 3.79 are all subject to the effect of scaling. Two obvious ways to correct for this are to scale the entries by the elements of the $\underline{A}$ matrix or the standard errors of these elements (multiply the (i,j) entry by a_{ji}). The outcome of scaling the entries of $\underline{R}_1\underline{L}_1$ by the elements of the $\underline{A}$ matrix will be referred to as the $\underline{A}$ weighted generalized participation factors.

The $\underline{A}$ weighted generalized participation factors are the sum over the modes of interest of the eigenvalue elasticities weighted by the eigenvalues. If the model is exactly simplifiable then the $\underline{A}$ weighted generalized participation factor will have nonzero entries only on the

diagonal associated with the included states.

Cross-Modal Participation Factors

The other technique for selecting the states is based on equations 3.74 and 3.75 and is called the cross-modal participation factor. Again this is a measure that is analogous to a participation factor, but measures the participation of a state in exciting one mode which has the potential to excite other states that may be important in the mode of interest. The matrix products given in equations 3.74 and 3.75 are not separable in the same way that was used above. In order to implement these it is necessary to consider them essentially component by component.

The left-vector-based cross-modal participation factor (LCMPF) is calculated by considering the product of the left vector of interest with all the right vectors. The (i,p) element of the left-vector cross-modal participation factor for mode k is given by

$$LCMPF_{ip} = \underline{l}_i^k \underline{r}_i^p \ . \tag{3.80}$$

The right-vector-based cross-modal participation factor (RCMPF) is calculated by considering the product of the right vector of interest with all the left vectors. The (i,p) element of the right-vector cross-modal participation factor for mode k is given by

$$RCMPF_{ip} = \underline{r}_i^k \underline{l}_i^p \ . \tag{3.81}$$

The cross-modal participation factors are sensitive to the

normalization chosen for the eigenvectors. Using the notation of equations 3.54 to 3.59 we have

$$LCMPF_{scaled} = v_k \; LCMPF \; \underline{V}^{-1} \tag{3.82}$$

and

$$RCMPF_{scaled} = 1/v_k \; RCMPF \; \underline{V} \; , \tag{3.83}$$

with v_k the k'th diagonal element of $\underline{V}$. Different choices of the normalization can yield different results. There does not seem to be any straightforward way to overcome this problem. Further discussion of the cross-modal participation factors is contained in apendix A3.

3-F Simplifying a Small Macroeconomic Model

In figure 3-3 the equations of the simple macroeconomic model presented in the introduction are repeated. This model will be referred to as the MACRO model. The matrices associated with the MACRO model are presented in figures 3-4A and 3-4B. In these matrices the nondynamic equations for consumption, investment, the inflation rate, the interest rate and income have been removed. These variables have been expressed in terms of the included states in arriving at the matrices presented. It is for this reason that some of the errors entering into the model are subscripted t-1. Because the errors in the equations were assumed to be uncorrelated, the error sequence entering will still be white noise.†

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†If there were correlation among the error terms, it would be necessary to extend the model in order to include this correlation. This could be done by constructing states made up of the lagged values of the error terms. This has been avoided for reasons of simplicity.

Model Equations for Simple Macoreconomic Model

$$AP_t = AP_{t-1} + a_{15}(P_{t-1} - AP_{t-1})$$

$$C_t = a_1 YP_t + a_2 K_t - a_3(R_t - PIE_t) + e_{1t}$$

$$I_t = a_5 K_t + a_6(a_7 YS_t - a_8(R_t - PIE_t) - K_t) + e_{3t}$$

$$K_t = (1-a_5)K_{t-1} + I_{t-1}$$

$$P_t = P_{t-1} + a_{10}(YS_{t-1} + a1_1(Rt-1 - PIEt-1) - b_1) + a_{12}PIE_{t-1} + e_{5t}$$

$$PI_t = a_{15}(P_t - AP_t)$$

$$PIE_t = PI_{t-1} + a_{14}(PI_{t-1} - PIE_{t-1}) + e_{7t}$$

$$R_t = a_13(P_t - M_t) + b_2 + e_{6t}$$

$$Y_t = C_t + I_t + G_t$$

$$YP_t = YP_{t-1} + a_4(Y_{t-1} - YP_{t-1}) + e_{2t}$$

$$YS_t = YS_{t-1} + a_9(Y_{t-1} - YS_{t-1}) + e_{4t}$$

Definitions

Endogenous

AP - Average Price (log of dollars)
C - Consumption (constant dollars/month)
I - Investment (constant dollars/month)
K - Capital (constant dollars)
P - Price (log of dollars)
PI - Price Inflation (/month)
PIE - Price Inflation Expected (/month)
R - Interest Rate (/month)
Y - Income (constant dollars/month)
YP - Permanent Income (constant dollars/month)
YS - Smoothed Income (constant dollars/month)

Exogenous

G - Government Expenditure (constant dollars/month)
M - Money Supply (log of dollars)

Coefficient		(units)
a_1	.3	(dimensionless)
a_2	.01	(/month)
a_3	1000.	(constant dollars)
a_4	.025	(/month)
a_5	.005	(/month)
a_6	.04	(/month)
a_7	18.	(months)
a_8	25000.	(constant dollars)

Coefficient		(units)
a_9	.2	(/month)
a_{10}	.00012	(log of $ / constant dollar)
a_{11}	0.0	(constant dollars)
a_{12}	1.	(month)
a_{13}	.2	(/month/log of dollars)
a_{14}	.05	(/month)
a_{15}	.1	(/month)
b_1	68.	(constant dollars)
b_2	0.	(/month)

Figure 3-3 Equations for the MACRO Model

Dynamics matrix

	AP_{t-1}	K_{t-1}	P_{t-1}	PIE_{t-1}	YP_{t-1}	YS_{t-1}
AP_t	$1-a_{15}$	0	a_{15}	0	0	0
K_t	0	$1-a_6$	$-a_6a_8a_{13}$	a_6a_8	0	a_6a_7
P_t	0	0	$1+a_{10}a_{11}a_{13}$	$-a_{10}a_{11}$	0	$-a_{10}$
PIE_t	$-a_{14}a_{15}$	0	$a_{14}a_{15}$	$1-a_{14}$	0	0
YP_t	0	a_4^* $a_2+a_5-a_6$	$-a_4a_{13}^*$ $a_3+a_6a_8$	a_4^* $a_3+a_6a_8$	$1-a_4+a_4^*$ a_1	a_4^* a_6a_7
YS_t	0	a_9^* $a_2+a_5-a_6$	$-a_9{}^3{}_{13}^*$ $a_3+a_6a_8$	a_9^* $a_3+a_6a_8$	a_9^* a_1	$1-a_9+a_9^*$ a_6a_7

The B Matrix†

	G_{t-1}	M_{t-1}
AP_t	0	0
K_t	0	$a_6a_8a_{13}$
P_t	0	-a10a11a13
PIE_t	0	0
YP_t	a_4	$a_4a_{13}^T$
YS_t	a_9	$a_9a_{13}^T$

$T = (a_3+a_6a_8)$

The D Matrix

	e_{1t-1}	e_{2t}	e_{3t-1}	e_{4t}	e_{5t}	e_{6t-1}	e_{7t}
AP_t	0	0	0	0	0	0	0
K_t	0	0	1	0	0	$-a_6a_8$	0
P_t	0	0	0	0	1	$a_{10}a_{11}$	0
PIE_t	0	0	0	0	0	0	1
YP_t	a_4	1	a_4	0	0	$-a_4T$	0
YS_t	a_9	0	a_9	1	0	$-a_9T$	0

$T = (a_3+a_6a_8)$

†The B matrix will also include the effect from the constants. b_1 places a $-a_{10}$ in the row corresponding to price. b_2 is assumed 0.

Figure 3-4A The Matrices for the MACRO Model

Dynamics matrix

	AP_{t-1}	K_{t-1}	P_{t-1}	PIE_{t-1}	YP_{t-1}	YS_{t-1}
AP_t	.9	0	.1	0	0	0
K_t	0	.96	-200	1000	0	.72
Pt	0	0	1.0	1.0	0	.0001
PIE_t	-.005	0	.005	.95	0	0
YPt	0	-.000625	-10.0	50.0	.9825	.018
YSt	0	-.005	-80	400	.06	.944

The B Matrix†

	G_{t-1}	M_{t-1}
AP_t	0	0
K_t	0	200
P_t	0	0
PIE_t	0	0
YP_t	.025	10
YS_t	.2	80

The D Matrix
(All e's with variance 1)

	e_{1t-1}	e_{2t}	e_{3t-1}	e_{4t}	e_{5t}	e_{6t-1}	e_{7t}
AP_t	0	0	0	0	0	0	0
K_t	0	0	1	0	0	-.02	0
P_t	0	0	0	0	.005	0	0
PIE_t	0	0	0	0	0	0	.00004
YP_t	.025	.55	.025	0	0	-.001	0
YS_t	.2	0	.2	.8	0	-.008	0

†The B matrix will also include the effect from the constants. b_1 places a -.0001 in the row corresponding to price. b_2 is assumed 0.

Figure 3-4B Numerical Values for the Matrices of the MACRO Model.

The eigenvalues of the MACRO model are given in figure 3-5. All the eigenvalues are less than 1 in magnitude, and the model is therefore stable. In the data plotted in figure 0-1 we observed apparent oscillations of six years' duration. We are centering on this oscillatory behavior as the mode of interest. This mode corresponds to the second eigenvalue of the MACRO model, which has magnitude .97 and implies a period of 74 months or about 6 years. The other modes in the MACRO model correspond to two real roots and a second complex root that implies oscillation of 243 months or approximately 20 years.

Root Index	Real Part	Imaginary Part	Magnitude	Period (Months)
1	.979	0	.979	-
2	.971	.0823	.974	74.3
3	.967	.0250	.967	243
4	.883	0	.883	-

Figure 3-5 The Characteristic Roots[†] of the MACRO Model

In determining the important states, we first consider the generalized participation factor. We write this as the product $(\underline{R}_1\underline{L}_1)$, so that if the (i,j) element of the generalized participation factor is multiplied by a_{ji} the result will be independent of the choice of units for the state vector. The unscaled and scaled values for the generalized participation factor are given in figure 3-6, where it is clear that the issues of scaling are of some consequence. The diagonal elements are all of order 1, but the off-diagonal elements get as large as 18,000.

[†]The equivalent continuous time derivatives with time measured in months are given by -.0212, -.0263 $\pm$ .0846j, -.0336 $\pm$ .0259j and -.124.

Unscaled

	AP	K	P	PIE	YP	YS
AP	0.335	0	0.0469	-13.0	-0.0004	-0.0011
K	4280.	0.572	7150.	18100.	-8.29	11.3
P	0.643	0	0.607	-10.3	-0.0011	-0.0002
PIE	0.0285	0	0.0219	-0.593	0	0
YP	109.	0.0144	200.	973.	-0.219	0.355
YS	-344.	-0.050	-19.6	14000.	0.434	1.30

Scaled

	AP	K	P	PIE	YP	YS
AP	0.301	0.	0.	0.0643	0.	0.
K	0.	0.549	0.	0.	0.00518	-0.0554
P	0.0643	-0.0180	0.607	-0.0515	0.0105	0.0142
PIE	0.	0.00401	0.0220	-0.564	-0.00222	-0.0109
YP	0.	0.	0.	0.	-0.215	0.0213
YS	0.	-0.0363	-0.00234	0.	0.00783	1.22

Figure 3-6 The Generalized Participation Factor

If the generalized participation factor matrix is scaled by the elements of the A matrix, then the values are all much more moderate. The diagonal elements dominate when the generalized participation factors are weighted, in this manner and this is in part attributable to the magnitude of the on-diagonal entries in the A matrix. The largest diagonal element is for smoothed income (YS). This is followed by the elements for price (P), expected inflation (PIE) and capital (K) which are approximately one half the magnitude. The off-diagonal links that appear to be important are those for the links between capital (K) and smoothed income (YS), price (P) and average price (AP), average price (AP) and expected inflation (PIE), and finally smoothed income (YS) and capital (K).

The cross-modal participation factors are reported in figure 3-7. It is clear from this figure that the model is a strongly coupled model. No clear patterns emerge from the cross-modal participation factors for this model. Though capital, smoothed income, and price do seem to dominate, there are also some important entries for permanent income, expected inflation and average price. The price term appears to be somewhat less important than the capital term again. All values are reported as magnitudes, and there is hence only one entry per root.

Left-Vector-Based Cross-Modal Participation Factor

	Root 1	Root 2	Root 3	Root 4
AP	0.816	0.298	0.178	0.876
K	0.431	0.446	0.448	0.443
P	0.751	0.379	0.148	0.176
PIE	0.665	0.333	0.265	1.66
YP	1.48	0.158	0.136	0.0924
YS	0.328	0.761	0.235	1.14

Right-Vector-Based Cross-Modal Participation Factor

	Root 1	Root 2	Root 3	Root 4
AP	0.412	0.298	0.453	1.80
K	0.488	0.446	0.418	0.764
P	0.225	0.379	0.166	1.00
PIE	0.333	0.333	0.333	0.318
YP	0.893	0.158	0.144	0.225
YS	0.204	0.761	0.247	1.30

Figure 3-7 Cross-Modal Participation Factors

We have seen the general pattern that the important variables seem to be smoothed income (YS), capital (K) and price (P). We will consider the simplified model based on smoothed income and capital, which seems to be somewhat more important than price. We could also consider including all

three variables with or without an extra mode. We will restrict ourselves to the single simplified model in the interest of clarity and brevity. However, it is worth noting that the simplified model arrived at is not necessarily unique. If the "system" is not nearly decomposable, and the MACRO model is not, then the final simplification chosen will require a good deal of approximation, and it is possible that other simplifications with different interpretations might be equally valid.

In discussing the example, we will use the numerical values of the elements of the $\underline{A}$ matrix. The upper block of the $\underline{A}$ matrix with capital written first and smoothed income second is given by

$$\underline{A}_{11} = \begin{vmatrix} .96 & .72 \\ -.005 & .944 \end{vmatrix} . \tag{3.84}$$

The M matrices that are added to this in deriving the simplified model are given for the right- and left-vector-based models respectively by

$$\underline{M}_{right} = \begin{vmatrix} -.0136 & .122 \\ -.00374 & .0509 \end{vmatrix} \quad \underline{M}_{left} = \begin{vmatrix} .00748 & .680 \\ .814E\text{-}4 & .0299 \end{vmatrix} . \tag{3.85}$$

The $\tilde{\underline{A}}$ matrices as well as the transformed $\tilde{\underline{B}}$ and $\tilde{\underline{D}}$ matrices for the simplified models are shown in figure 3-8.

For both models the effect from smoothed income to capital investment is increased and this is due to the simplification of the impact of price on the investment decision. For the model based on the left-hand vectors this increase is quite large. The magnitude and sign of the effect are

Model Based on Right Vectors

$\tilde{A}$

	K_{t-1}	YS_{t-1}
K_t	.947	.842
YS_t	-.00874	.995

$\tilde{B}$

	Gt-1	Mt-1	
K_t	2.01	918	-.858
YS_t	.271	98.1	.00233

$\tilde{D}$

	e_{1t-1}	e_{2t}	e_{3t-1}	e_{4t}	e_{5t}	e_{6t-1}	e_{7t}
K_t	2.01	-4.56	2.58	8.87	35.8	-.0919	0.7259
YS_t	.271	.239	.220	1.04	-.0978	0.0098	.561

Model Based on Left Vectors

$\tilde{A}$

	K_{t-1}	YS_{t-1}
K_t	.968	1.38
YS_t	-.00492	.974

$\tilde{B}$

	Gt-1	Mt-1	
K_t	-.299	80.3	-.875
YS_t	.197	78.7	-.0320

$\tilde{D}$

	e_{1t-1}	e_{2t}	e_{3t-1}	e_{4t}	e_{5t}	e_{6t-1}	e_{7t}
K_t	-.299	-6.58	.701	0	36.5	-.0080	-4.05
YS_t	.197	-.0716	.197	.800	1.34	-.0079	.274

Figure 3-8 The Matrices for the SIMPLE Models

interesting. At first glance it appears that the increase in income will cause an increase in price, which will raise interest rates and thereby lower investment. However, prices lag smoothed income so that while smoothed income is falling, prices are often rising, which further lowers investment. The relative timing of the two variables in the mode of interest changes what would seem to be the obvious effect from prices. This is a good illustration of the importance of timing in determining the matrix $\underline{M}$.

The simplified model based on the right eigenvectors shows a large change in the effect of capital on income, almost doubling the value in the original model. The effect of capital on income arises in the original model because the higher level of capital decreases investment. This lowers both smoothed and permanent income. The changed income affects price which further changes investment and ultimately income.

For the simplified model based on the left vectors there is little change in the effect of capital on income. This is, in part, a reflection of the tendency of this model to preserve the effects of the included variables. Capital influences price only through YS, and feedback through price cannot therefore alter this feedback link for the left vector based model. What change there is is due to the influence of capital on YP through investment.

The diagonal entries in the $\underline{M}$ matrices arise for the same reasons as the off-diagonal entries. For $\underline{M}_{right}$ the changes are attributable to the impact of price on the two variables. For $\underline{M}_{left}$ the changes are

attributable to the influence of capital on permanent income for the (1,1) entry and the influence of smoothed income on price for the (2,2) entry. We can sum up the compositions of the $\underline{M}$ matrices graphically as

$$\underline{M}_{right} = \begin{vmatrix} P & P \\ P & P \end{vmatrix} \quad M_{left} = \begin{vmatrix} YP & P \\ YP & P \end{vmatrix} . \tag{3.86}$$

In short, price seems to be an important excluded variable.

The Simplified Model From First Principles

Many models start as simplifications. One of the important advantages of the simplification techniques we are considering is that they allow the development of a model which could in theory be derived from first principles. The above simplification can be considered as such a simplification. We briefly present the justification for the simplified model along the lines that the MACRO model was presented.

Income in the SIMPLE model is given by the national incomes identities

$$\tilde{Y}_t = \tilde{C}_t + \tilde{I}_t + \tilde{G}_t , \tag{3.87}$$

with consumption determined by

$$\tilde{C}_t = \tilde{a}_1 \tilde{YS}_t + \tilde{a}_2 \tilde{K}_t + \tilde{e}_{1t} \tag{3.88}$$

where $\tilde{YS}$ is a proxy for permanent income. Investment is given by a stock adjustment toward a constant capital output ratio according to the equation

$$\tilde{I}_t = \tilde{a}_5\tilde{K}_t + \tilde{a}_6(\tilde{a}_7\tilde{YS}_t - \tilde{K}_t) + \tilde{e}_{3t}, \tag{3.89}$$

where smoothed income is determined by

$$\tilde{YS}_t = \tilde{YS}_{t-1} + \tilde{a}_9(\tilde{Y}_{t-1} - \tilde{YS}_{t-1}) + \tilde{e}_{4t} . \tag{3.90}$$

Finally, capital is increased by investment and depreciates proportionally to its own level according to

$$\tilde{K}_t = (1-\tilde{a}_5)\tilde{K}_{t-1} + \tilde{I}_{t-1} . \tag{3.91}$$

We can again combine these equations to yield a state transition matrix of

	$\tilde{K}_{t-1}$	$\tilde{YS}_{t-1}$
$\tilde{K}_{t-1}$	$1-\tilde{a}_6$	$\tilde{a}_6\tilde{a}_7$
$\tilde{YS}_{t-1}$	$\tilde{a}_9(\tilde{a}_5-\tilde{a}_6+\tilde{a}_2)$	$1-\tilde{a}_9+\tilde{a}_9(\tilde{a}_1+\tilde{a}_6\tilde{a}_7)$

. (3.92)

This is the same as the associated block of the full state transition matrix except for the addition of the direct effect of smoothed income on consumption. It is not possible to regain the values for all of the coefficients of the original model. We can, however, get these values for $\tilde{a}_6$ and $\tilde{a}_7$:

original	$a_6 = .04$	$a_7 = 18.0$	(3.93)
right	$\tilde{a}_6 = .0535$	$\tilde{a}_7 = 15.7$	(3.94)
left	$\tilde{a}_6 = .0325$	$\tilde{a}_7 = 42.4$.	(3.95)

In the simplified model based on the right eigenvectors the speed with which the capital stock is corrected is increased substantially, while the implied capital output ratio decreases slightly. In the model based on the left eigenvectors the speed with which the capital stock adjusts decreases, but the target capital output ratio increases substantially.

The results for the MACRO model are of some interest in their own right. The SIMPLE model that we have derived from the MACRO model has different parameters than the original model. The parameters are changed as a consequence of feedback links that the simplified model does not take into account. The SIMPLE model is very similar to the multiplier-accelerator model of Samuelson (1939) and Hicks (1950) which was part of the basis of the MACRO model. This model has been criticized as an explanation of the business cycle because the parameters required to produce oscillations of business-cycle frequencies are unrealistic (Evans 1969, Low 1980). However, the reason that the parameters have to be unrealistic is that important feedback, mostly through prices, has been removed.

The quality of the SIMPLE model as a simplification of the MACRO model is open to question and will be further discussed below. The simplification does raise one important point, however. The parameters of a simplified model will generally be different from the parameters of the original model. The difference is a result of the approximation required in going to the simplified model. The magnitude of the difference is, in part, a factor of the degree of approximation. What degree of

approximation is acceptable will clearly depend on purpose. But if the model is only to be only a rough approximation, it is not clear how much weight should be put on accurate or even reasonable parameters.

Exogenous Variables and Steady-State Gain

Both the money supply and government expenditure influence smoothed income in the MACRO model, government expenditure through its effect on income and the money supply through its effect on the interest rate and investment. Only the money supply influences capital directly, again through its effect on the interest rate. In the simplified model both government expenditure and the money supply directly influence both of the included variables. The values for the $\underline{B}$ and $\underline{\tilde{B}}$ matrices were given in figures 3-4B and 3-8 respectively. In figure figure 3-8 the entry corresponding to the exogenous variable b_1 is also included (the b_1 series is constant over time at 68).

The steady-state gain for the models is calculated according to the formula $(\underline{I} - \underline{A})^{-1}\underline{B}$, where we have included the constant b_1(=68) as an element of the exogenous variables. The steady-state gains of the MACRO and SIMPLE models are given in figure 3-9. The gains of the simplified models are substantially different from those of the MACRO model. In fact, given the same values of the exogenous variables as were used for the little model (G=30, M=1, b_1=68), the equilibrium capital stocks for the right- and left-vector-based models are 12700 and 15300 compared with 1030 for the MACRO model. The equilibrium income levels are -311 and 353 compared with 68 for the MACRO model.

MACRO Model

	G	M	b_1
K	-18.2	.763E-3	28.8
YP	0	0	1

Model based on RIGHT Vectors

	G	M	b_1
K	31.1	11400	-.311
YP	-.404	-363	.999

Model Based on LEFT Vectors

G	M	b_1
34.5	14500.	-8.83
1.03	283	.427

Figure 3-9 Gain Matrices for the MACRO and SIMPLE Models

The values for the steady-state gains for these simplified models are substantially different from those of the MACRO model. We have seen that this difference in the steady-state gains between the "system" and it simplifications can be interpreted as the influence of the excluded modes on the gain. For the simulations we report, we will add in this difference to make the results comparable in scale.

It is also possible to define the $\underline{\tilde{B}}$ matrices so that the simplified models retain the steady state gain of the original model. In this case the $\underline{\tilde{B}}$ matrices which satisfy this condition are uniquely determined. These are presented in Figure 3-10. The $\underline{\tilde{B}}$ matrices which preserve the steady state gain have much smaller magnitude entries than do those that preserve the influence of the exogenous variables on the modes of interest.

Model based on Right Vectors

	G	M	b_1
K	-.974	.409E-4	.541
YS	-.159	.671E-6	.231

Model Based on Left Vectors

	G	M	b_1
	-.591	.248E-4	-.540
	-.089	-.375E-5	.153

Figure 3-10 $\underline{\tilde{B}}$ Matrices That Preserve the Model Equilibrium

Policies on the SIMPLE Models

One of the bases for the simplification criterion that we used was the retention of the impacts of feedback policies. We consider implementing policies through the determination of money supply and government expenditure via a feedback law, as well as through the alteration of the model's feedback links. The purpose of doing this is to get some feel for the difference in response of the full MACRO model and the SIMPLE model.

In determining the criteria for choosing a simplified model we considered the effect on the simplified model and the "system" of changing an element of the feedback structure. In figure 3-11 the effect on the modes of the MACRO model of changing the feedback links between capital and smoothed income is considered. The comparison shows that the effects are similar in the original and simplified models. The effect on the period for the link from smoothed income to smoothed income changes substantially. This is due in part to the fact that the period elasticities will sum to zero while those for YS and K in the MACRO model do not. The diagonal entries for the included states also appear to be important in determining the period of the fluctuation of the mode associated with the third root.

Root	$K_{t-1} \rightarrow K_t$	$K_{t-1} \rightarrow YS_t$	$YS_{t-1} \rightarrow K_t$	$YS_{t-1} \rightarrow YS_t$
1 (magnitude)	.0457 (0)	-.978E-3 (0)	-.00156 (0)	0.00838 (0)
2 (magnitude)	.309	-.0262	-.0162	.659
right	(.496)	(.00388)	(.00388)	(.496)
left	(.496)	(.00356)	(.00356)	(.496)
2 (period)	-3.69	-.33	-.348	-3.91
right	(-1.2)	(-.54)	(-.54)	(2.28)
left	(.269)	(-.498)	(-.498)	(.727)
3 (magnitude)	.0969 (0)	.0347 (0)	.029 (0)	.0118 (0)
3 (period)	44 (0)	-.0806 (0)	-.0845 (0)	7.94 (0)
4 (magnitude)	.138 (0)	-.0120 (0)	-.0159 (0)	-.3473 (0)

Figure 3-11 The Elasticity of the Eigenvalues With Respect to Links Between the Retained States. Those for Simplified Models Appear in Parentheses. Magnitude elasticities sum to 1 and period elasticities sum to zero.

The other policy test we consider is the actual implementation of a feedback policy in the original and simpified models. This will allow us to evaluate the performance of the simplification at a level which is not as local as the above analysis and also give us information about the effect of exogenous variables. The feedback rule we consider is a countercyclical government expenditure policy.[†] When smoothed income is

[†]In implementing this rule we are really changing the structure of the problem. The exogenous variables are becoming endogenous variables, and there are well-recognized problems in considering the original model to be in some sense structural (Lucas 1976). However, the analysis that this approach allows is informative about model simplification, and we therefore proceed with it.

high the government decreases expenditure, and when smoothed income is low the government increases expenditure. This simple policy is useful for illustrating what happens when we go from a simplified model to the original system. It was chosen in order to make the effect on the model readily visible and is not meant to be an optimal or even necessarily desirable policy.

In particular, we consider the simple and rather extreme policy

$$\underline{G}_t = \bar{G} - .5\ (YS_t - Y\bar{S})\ , \tag{3.96}$$

where the bars denote some constant values. The government adjusts spending in response to endogenous changes in perceived total demand. We ignore any lags that might be unavoidable in the actual implementation of such a policy. Given this simple policy, the $\underline{\tilde{A}}$ matrices are altered to become

$$\underline{\tilde{A}}_{right} = \begin{vmatrix} .947 & -.163 \\ -.00874 & .860 \end{vmatrix} \qquad \underline{\tilde{A}}_{left} = \begin{vmatrix} .978 & 1.53 \\ -.0049 & .875 \end{vmatrix} \tag{3.97}$$

This policy decreases the magnitude of the root of interest in both models. For the model based on the right-hand eigenvectors the root of interest breaks into two real roots with values .96 and .85. For the simplified model based on the left-hand eigenvectors the root is decreased in magnitude to .92 but remains oscillatory with a slightly increased period of 78 months.

When the same policy is implemented on the full system the results are

similar. The magnitude of the root of interest goes from .98 to .88, and the period increases by a small amount to 76 months. The other oscillatory root increases in magnitude from .97 to .98, the period changing from 243 to 203 months. The smallest root changes from .88 to .94 and the other root is left approximately the same.

The same policy can be implemented on the simplified models with the $\tilde{\underline{B}}$ martrices adjusted so as to preserve the steady state gain as given in figure 3-10. In both cases the policies make the simplified models thus defined less stable, and the model based on the right-hand eigenvalues actually becomes unstable. $\tilde{\underline{A}}_{left}$ is altered to have a root with a magnitude of .997 and a period of 72 months, and $\tilde{\underline{A}}_{right}$ is altered to have a root with a magnitude of 1.01 and a period of 73 months. The $\tilde{\underline{B}}$ matrices that preserve the steady state gain do not give a good indication of the impact of exogenous varibles on the modes of interest.

Behavior of the SIMPLE Models

We have derived the simplfied models on the basis of the feedback structure of the MACRO model. In the above discussion we have considered a number of the properties of the simplified model. In this section we will consider the responses of the full and simplified models to idealized inputs and noise entering the models.

For the noise driven simulations the exogenous variables were held constant, and the plots begin 100 months into the simulation in order to eliminate the effects of transients. The deterministic responses are considered to a 10% step increase in both government expenditure and money.

The variables of the simplified models are adjusted by a constant factor given by the difference in the steady state gain between the SIMPLE models and the MACRO model. Because this gain is not adjusted in response to the step increases in money and government expenditure, the responses of the simplifed models do show steady-state differences from those of the MACRO model.

The noise run for the MACRO and simplified models is given in figures 3-12 and 3-13. The output seems to match reasonably well for the various models. The time path of capital in the simplified models is much more jagged than it is for the MACRO model. There seems to be a somewhat increased importance of the influence of the noise inputs in determining the behavior of the simplified models. The timing of the peaks and troughs in the capital stock also seems to be somewhat different for the simplified

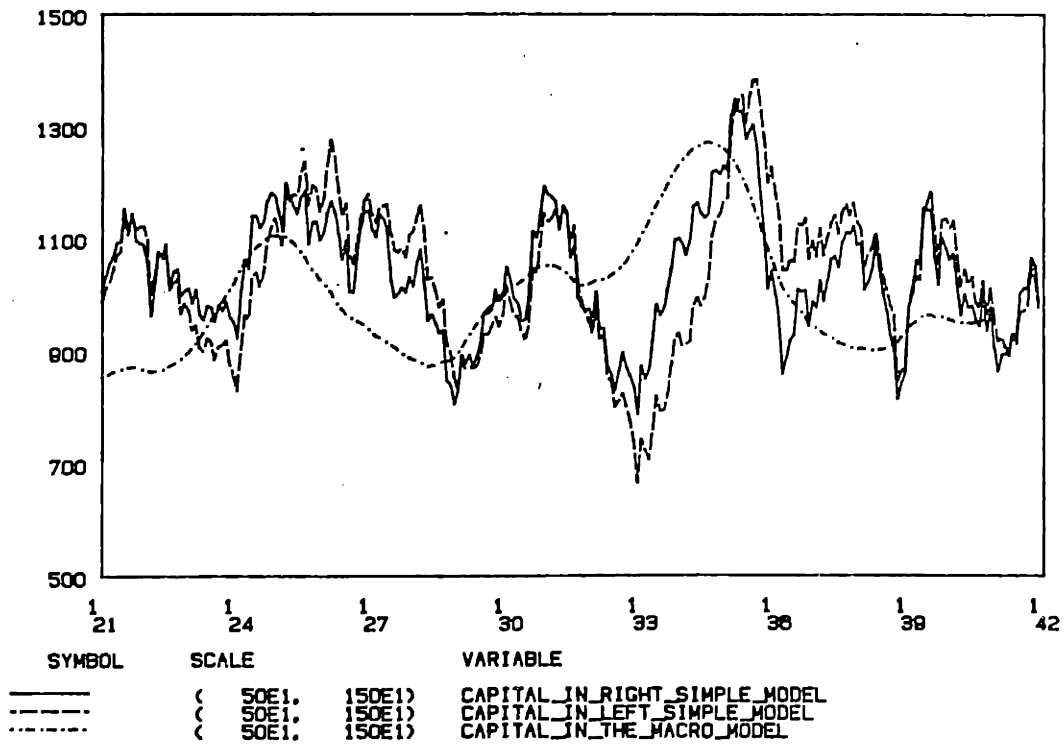


Figure 3-12: Capital Stock Behavior with Entering Noise

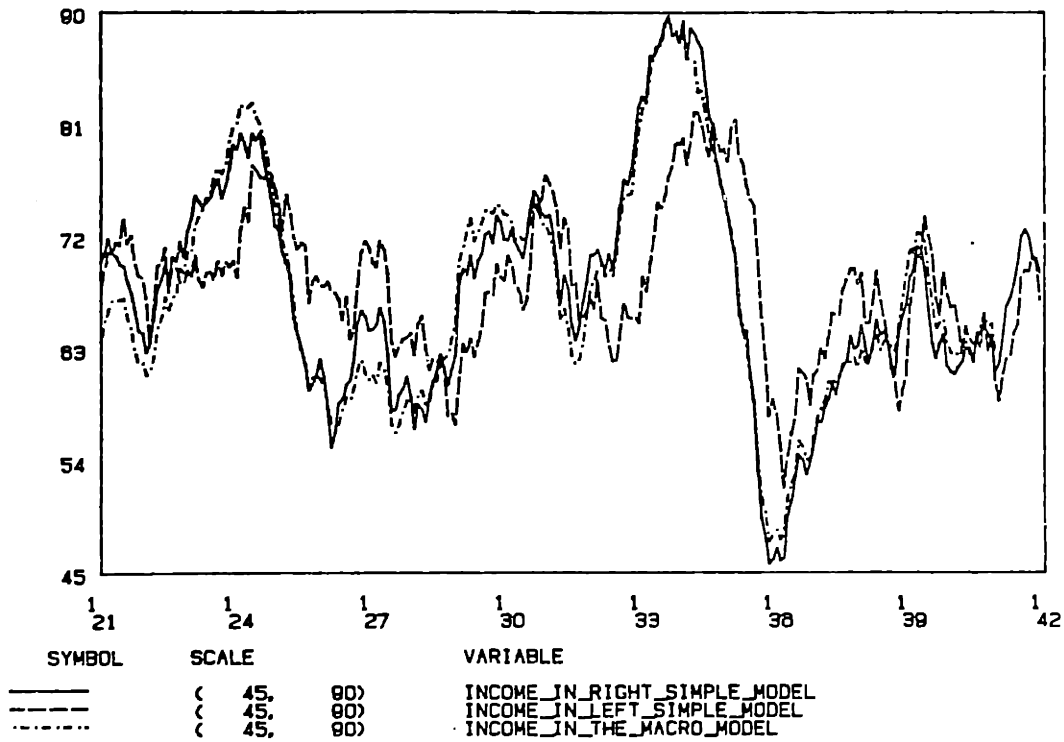


Figure 3-13: Smoothed Income Behavior with Entering Noise

and MACRO models. This is not so true for smoothed income for which the output of the simplified models matches that of the MACRO model reasonably well.

The response of the MACRO model and simplified models to a step increase in government expenditure is given in figures 3-14 and 3-15. The fact that the steady-state gain is not retained in the simplifed model is quite clear in this simulation, especially for the capital stock. The capital stock in the MACRO model is actually decreased, through an increase in the interest rate, by the increase in government expenditure. In the simplified models the capital stocks are both increased. Disregarding the difference in the steady-state gain, the models all respond in a similar fashion to the change. The capital stock first shows an increase and then

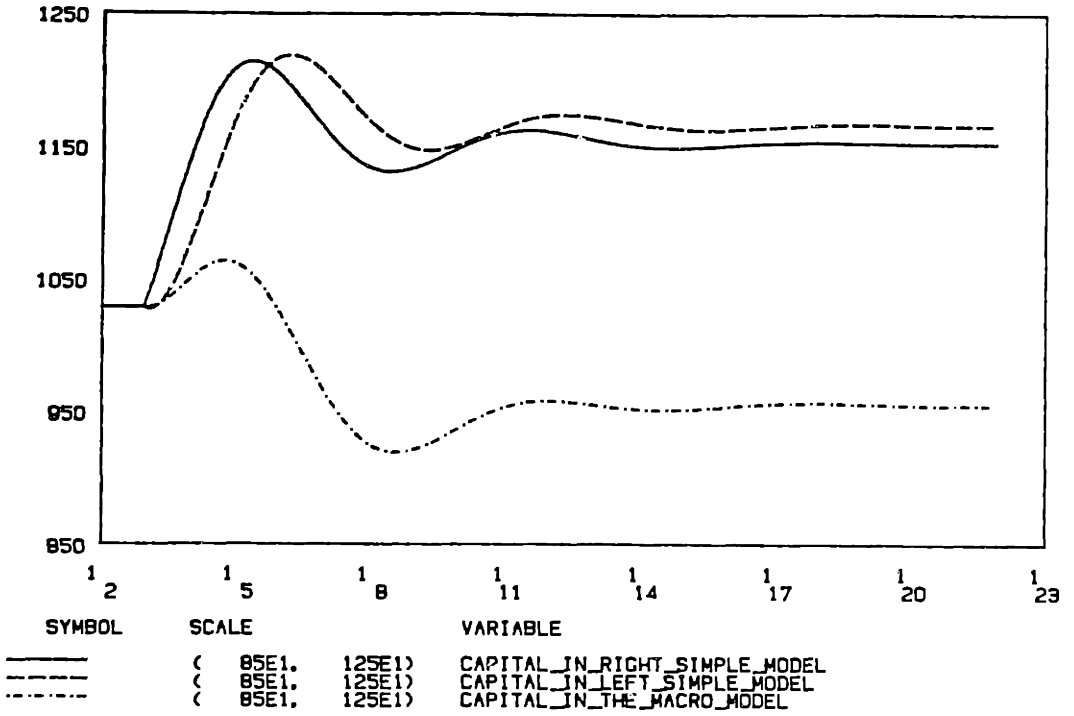


Figure 3-14: Response of Capital to a Step Increase in Government Expenditure

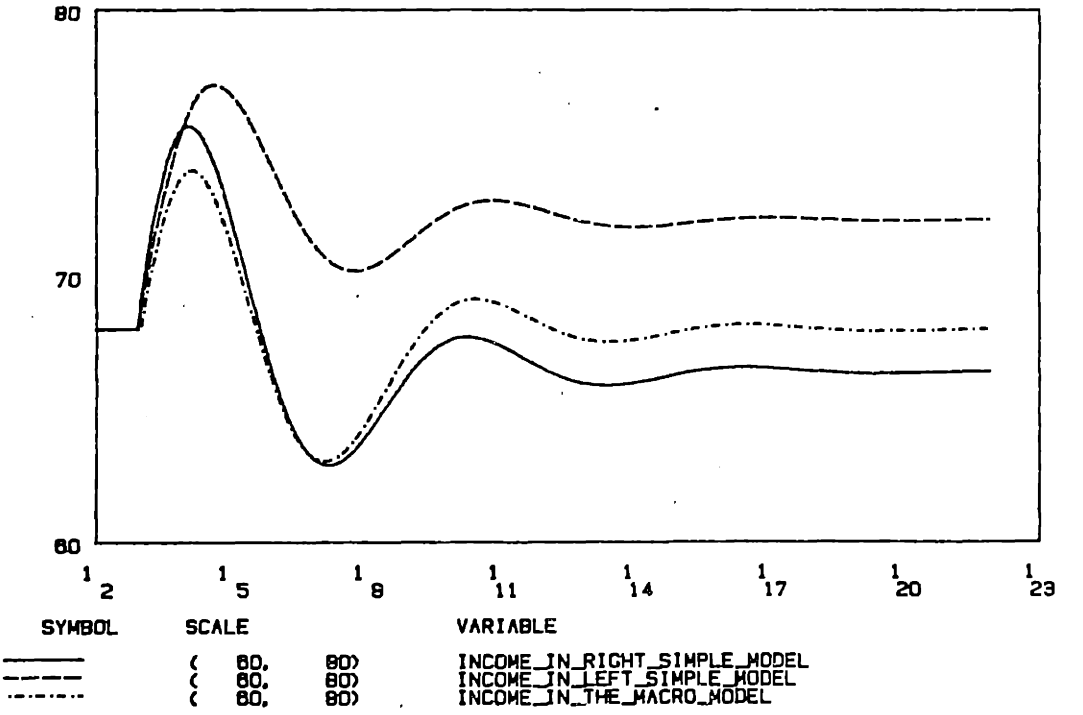


Figure 3-15: Response of Smoothed Income to a Step Increase in Government Expenditure

falls, as does income. The mode of interest is apparent in all of the variables.

The change in the money supply has very similar attibutes to the change in government expenditure as can be seen in figures 3-16 and 3-17. In this case the MACRO model returns to its initial real equilibrium, the money stock simply increasing the price level. The simplified models both go to a higher level of capital, and income goes to a higher level for the model based on the left vectors and a lower level for the model based on the right vectors. Again all variables follow essentially the same time path, showing first an increase and then a decrease. The capital stock in the simplified models appears to react somewhat more strongly.

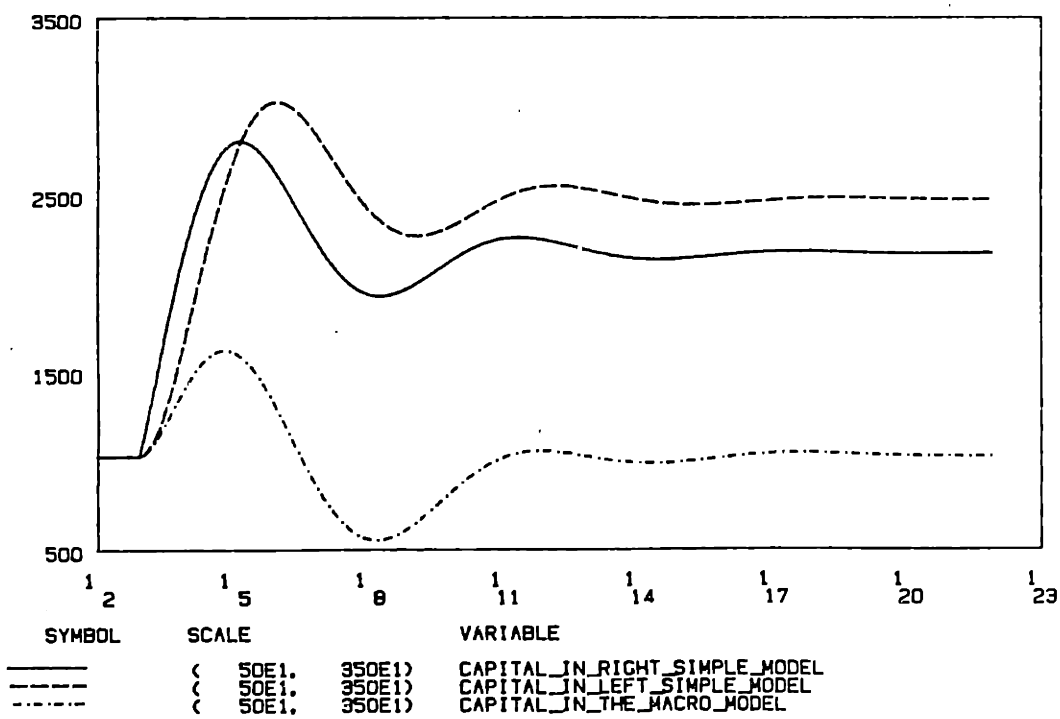


Figure 3-16. Response of Capital to a Step Increase in the Money Supply

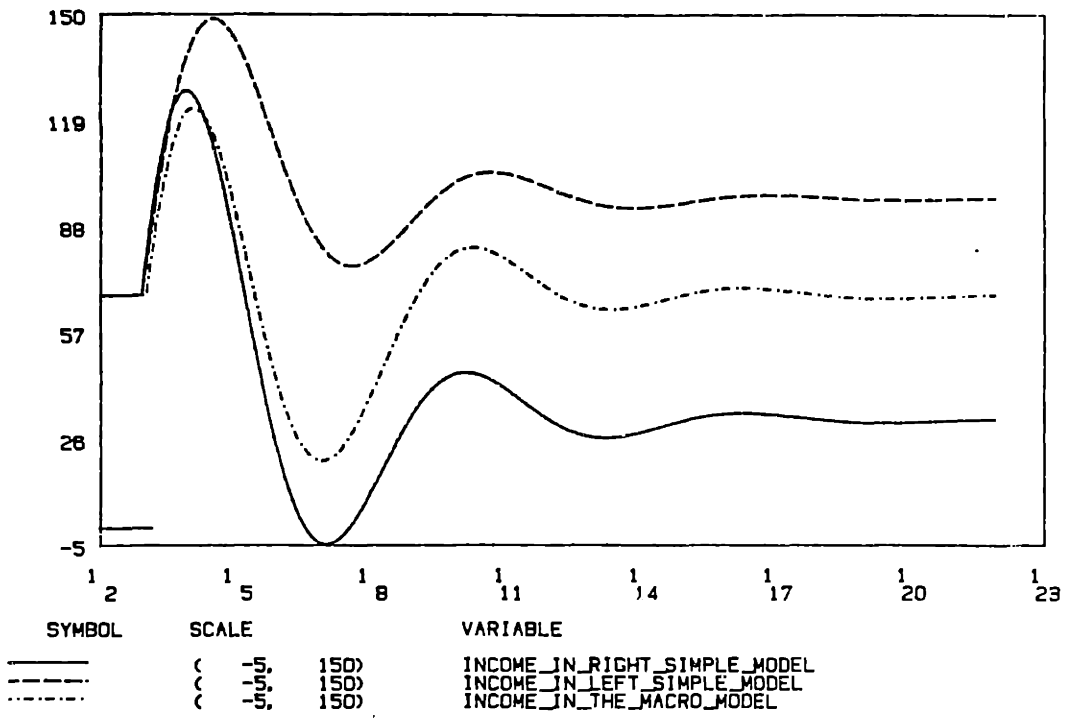


Figure 3-17: Response of Smoothed Income to a Step Increase in the Money Supply

In general the behavior of the MACRO model and that of its simplifications match quite well. The steady-state responses of the MACRO model and its simplifications differ significantly. In general, the most noticeable differences betwween the models seem to be attributable to the more direct influence of the exogenous variables and noise on the simplified models.

3-G Summary

We have considered the problem of developing simplified models that retain some of the dynamic characteristics of the original system. We approach this problem in a manner that is very much akin to selective modal analysis as developed by Perez (1981). We first reviewed the selective modal analysis approach and then offered some useful extensions of this approach. Following this the question of how good a simplified model is relative to the original system was considered. Very restrictive conditions for exact simplification were given as a benchmark against which the quality of a simplified model can be judged. From this perspective different measures of the importance of a state in a mode of interest were developed. An extended example served to illustrate how the different tools and simplifications can be applied.

4 Simplified Models and Data

Introduction

In the previous chapter we considered the development of simplified models that preserved certain aspects of the "system's" dynamics and could be easily interpreted. In order to do this we assumed that we had complete knowledge of the "system." In this chapter we consider issues of statistical analysis in the simplified model. These include the characteristics of the simplified model output relative to the output of the "system" and the requirements for specifying simplified models in a manner that is amenable to estimation. In this regard it is important to determine how much knowledge of the "system" is necessary for such easily implemented estimators.

The simplified models that we considered in the previous chapter were designed to be easily interpretable and to retain the dynamic modes of interest. In this chapter we will continue to focus on simplifications of

this type. The simplifications we have considered have not made appeal to the data which the "system" generates, and the properties of the simplified model with respect to these data need to be determined. These properties can help determine when and by what techniques the simplified models can be usefully estimated.

This chapter begins with a consideration of the properties of the simplified model. We concentrate on the second-order moments of the states and residuals for the "system" and the simplified models. For much of this discussion it is necessary to assume the data generating process is stationary. In the second part of this chapter we consider transformations of the data which could be used to estimate a simplified model. These transformations are then used to gain insights into the problems of model identification relative to the observability and controllability of a system. In the final section we consider the estimation of the simplified models based on the output of the system. The results of that section suggest that it is difficult to estimate simplified models of the type considered. Fairly complete knowledge of the system is required to arrive at simplified models of the type we have considered.

Autonomous Models

Equation 3.1 defines the linear model of general interest. For much of the discussion that follows we consider the autonomous linear model given by

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{D}\underline{e}_t \tag{4.1}$$

or, writing this as a partioned matrix,

$$\begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{D}_1 \\ \underline{D}_2 \end{vmatrix} \underline{e}_t \ . \tag{4.2}$$

This system is a restricted case of equation 3.1 and is called autonomous because there are no exogenous variables driving it. The autonomous system has certain advantages in discussing the characteristics of reduced-order models. Much of the statistical literature on time series models is based on models like 4.1. The consideration of only autonomous models is not as strong a restriction as it might first appear because it may be possible to include the exogenous variables among the states of the "system." This can always be done if the exogenous variable can be represented as a stationary, stochastic process (Hannan 1970).

It will be assumed, unless otherwise stated, that the autonomous model is stable, which requires that all of the eigenvalues of $\underline{A}$ have a modulus less than one. The imposition of stability guarantees that the $\underline{x}_t$ series will have a finite variance. This makes it possible to talk about power spectra and probability limits without first detrending the data.

4-A Properties of the Simplified Models

The model simplificatcns that we have considered have been based on the structure of the internally generated dynamics of the "system." We briefly considered the output of the "system" when we discussed the steady state gains of the "system" and its simplifications. In this section we will look at the characteristics of data generated by simplified models and

how these relate to those of the data generated by the "system." The characteristics we concentrate on are the correlational properties of the states among themselves and over time. We consider three types of correlations: correlations among the variables generated by the simple model, correlations that result when the simplified model is used for one-step prediction on the actual data, and correlations between the states and these prediction errors. A convenient and useful way to consider correlations is through the power spectra of the different variables.

Power Spectra for the Simplified Model

The power spectrum is one of the important output characteristics of a dynamic system (Bendat and Piersol 1980, Granger and Hatanaka 1964, Hannan 1970, Harvey 1981). Consideration of the output spectrum is only valid for stable models, so the discussion will not apply to all models. The insights that the discussion produce are, however, of general interest. Included in consideration of the power spectrum is the state covariance, which is given by integrating the power spectrum from $-\pi$ to π.

It is assumed in this section that the simplified model has eigenvalues which are a subset of the eigenvalues of the original system. This simplifies the discussion considerably and does not alter the general applicability. The choice of vectors to be maintained in the simplified model is not restricted.

The power spectrum for the basic data-generating process is given by $\underline{\Psi}(z)$ evaluated at $z = e^{j\theta}$, with $\underline{\Psi}(z)$ given by (Hannan 1970)

$$\underline{\Psi}(z) = (z\underline{I} - \underline{A})^{-1}\underline{D\Omega D}^{T}(\frac{1}{z}\,\underline{I} - \underline{A})^{-1T} \ . \tag{4.3}$$

$\underline{\Omega}$ is the covariance matrix of the entering noise. The portion of the spectrum of interest for the simplified model is the upper block, which is associated with the included states. The upper block of the power spectrum is given by

$$\underline{\Psi}_{11}(z) = \left| \underline{I}\ \underline{O} \right| (z\underline{I} - \underline{A})^{-1}\ \underline{D\Omega D}^{T}\ (\frac{1}{z}\,\underline{I} - \underline{A})^{-1T} \left| \begin{matrix} \underline{I} \\ \underline{O} \end{matrix} \right| . \tag{4.4}$$

In order to see the effect that the reduction in order has on the output spectrum, it is useful to consider the output spectrum of the original model in a slightly modified form. Substituting the modal decomposition of the $\underline{A}$ matrix in equation 4.3, we have

$$\underline{\Psi}(z) = (z\underline{I} - \underline{R\Lambda L})^{-1}\underline{D\Omega D}^{T}(\frac{1}{z}\,\underline{I} - \underline{R\Lambda L})^{-1T} \tag{4.5}$$

$$= [\underline{R}(z\underline{I} - \underline{\Lambda})\underline{L}]^{-1}\ \underline{D\Omega D}^{T}\ [\underline{R}(\frac{1}{z}\,\underline{I} - \underline{\Lambda})\underline{L}]^{-1T} \tag{4.6}$$

$$= \underline{L}^{-1}(z\underline{I} - \underline{\Lambda})^{-1}\underline{R}^{-1}\ \underline{D\Omega D}^{T}\ \underline{R}^{-1T}(\frac{1}{z}\,\underline{I} - \underline{\Lambda})^{-1}\underline{L}^{-1T} \tag{4.7}$$

$$= \underline{R}(z\underline{I} - \underline{\Lambda})^{-1}\underline{L}\ \underline{D\Omega D}^{T}\ \underline{L}^{T}(\frac{1}{z}\,\underline{I} - \underline{\Lambda})^{-1}\underline{R}^{T} \tag{4.8}$$

$$= \underline{R}\ \underline{\Phi}(z)\ \underline{R}^{T} \ , \tag{4.9}$$

with

$$\underline{\Phi}(z) = (z\underline{I} - \underline{\Lambda})^{-1}\underline{L}\ \underline{D\Omega D}^{T}\ \underline{L}^{T}(\frac{1}{z}\,\underline{I} - \underline{\Lambda})^{-1} \ . \tag{4.10}$$

$\underline{\Phi}(z)$ is the power spectrum of the modes of the model. That is, if the

states were transformed so that they could be written with a diagonal dynamics matrix as in equation 3.4, then $\underline{\Phi}(z)$ would give the power spectrum of the transformed states $\underline{\xi}$.

The power spectrum of the included states corresponds to the upper left-hand block of the above expression. Writing the upper block in terms of the blocks of $\underline{R}$ and $\underline{\Phi}(z)$, we have

$$\underline{\Psi}(z)_{11} = \underline{R}_{11}\underline{\Phi}_{11}(z)\underline{R}_{11}^T + \underline{R}_{12}\underline{\Phi}_{21}(z)\underline{R}_{11}^T + \underline{R}_{12}\underline{\Phi}_{21}(z)\underline{R}_{21}^T + \underline{R}_{12}\underline{\Phi}_{22}(z)\underline{R}_{21}^T \quad . \quad (4.11)$$

The simplified model has an output power spectrum of

$$\underline{\tilde{\Psi}}(z) = (z\underline{I} - \underline{\tilde{A}})^{-1}\underline{\tilde{D}\tilde{\Omega}\tilde{D}}^T(\frac{1}{z}\underline{I} - \underline{\tilde{A}})^{-1T} \quad . \quad (4.12)$$

Substituting in the definitions of the different right-hand variables, we note that $\underline{\tilde{\Omega}}$ is given by

$$\underline{\tilde{\Omega}} = \underline{\tilde{R}}\left|\begin{array}{cc|c|c|c} \underline{I} & \underline{O} & \underline{LD\Omega D}^T\underline{L}^T & \begin{array}{c}\underline{I}\\ \hline \underline{O}\end{array} & \underline{\tilde{R}}^T \end{array}\right. \quad . \quad (4.13)$$

Thus, we can write

$$\underline{\tilde{\Psi}}(z) = (zI - \underline{\tilde{A}})^{-1}\,\underline{\tilde{R}}\left|\begin{array}{cc|c|c} \underline{I} & \underline{O} & \underline{LD\Omega D}^T\underline{L}^T & \begin{array}{c}\underline{I}\\ \hline \underline{O}\end{array} \end{array}\right.\underline{\tilde{R}}^T(\frac{1}{z}\underline{I} - \underline{\tilde{A}})^{-1T} \quad . \quad (4.14)$$

Noting that $\underline{\tilde{R}}$ is matrix of right eigenvectors of $\underline{\tilde{A}}$ we have

$$(z\underline{I} - \underline{\tilde{A}})^{-1}\underline{\tilde{R}} = [\underline{\tilde{R}}^{-1}(z\underline{I} - \underline{\tilde{A}})]^{-1} \tag{4.15}$$

$$= [(z\underline{I} - \underline{\Lambda}_{11})\underline{\tilde{R}}^{-1}\ ]^{-1} \tag{4.16}$$

$$= \underline{\tilde{R}}(z\underline{I} - \underline{\Lambda}_{11})^{-1}\ . \tag{4.17}$$

Substituting this into equation 4.14 gives

$$\underline{\tilde{\Psi}}(z) = \underline{\tilde{R}}(zI - \underline{\Lambda}_{11})^{-1} \left| \underline{I}\ \underline{0} \right| \underline{LD\Omega D^T L^T} \left| \begin{matrix} \underline{I} \\ \underline{0} \end{matrix} \right| (\tfrac{1}{z}\underline{I} - \underline{\tilde{A}})^{-1}\underline{\tilde{R}}^T\ . \tag{4.18}$$

it follows that

$$\underline{\tilde{\Psi}}(z) = \underline{\tilde{R}} \left| I\ 0 \right| \underline{\Phi} \left| \begin{matrix} \underline{I} \\ 0 \end{matrix} \right| \underline{\tilde{R}}^T \tag{4.19}$$

$$= \underline{\tilde{R}}\ \underline{\Phi}_{11}(z)\ \underline{\tilde{R}}^T\ . \tag{4.20}$$

Comparison of equation 4.20 with 4.11 shows that the power spectrum of the simplified model does not capture the contributions to the power spectrum attributable to the covariance of the excluded modes with the included modes, or to the variance in the excluded modes. For the reduced-order model based on the right-hand eigenvectors the difference in the power spectra of the system and reduced order model is given by

$$\underline{R}_{12}\underline{\Phi}_{21}(z)\underline{R}_{11}^T + \underline{R}_{11}\underline{\Phi}_{12}(z)\underline{R}_{12}^T + \underline{R}_{12}\underline{\Phi}_{22}(z)\underline{R}_{12}^T\ . \tag{4.21}$$

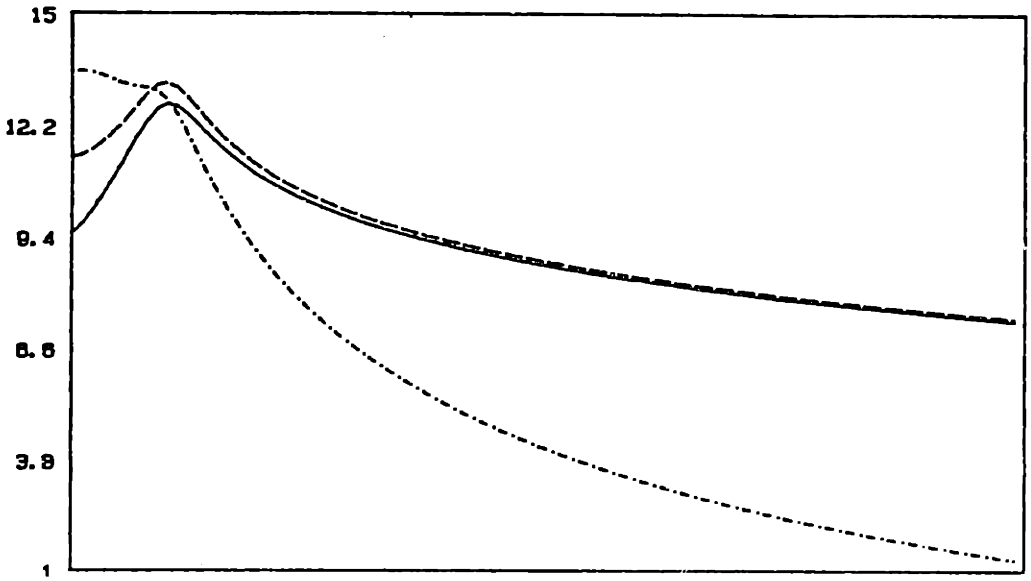
It is possible to consider a choice of the simplified that minimizes a measure of the discrepancy between the power spectrum of the system and that of the reduced-order model; the quasi maximum likelihood estimator discussed in chapter 2 is one such simplification (Anderson, Moore and Hawkes 1978).

The matrix given in equation 4.21 is not necessarily a positive definite matrix. The existence of additional modes can decrease or increase the variance associated with a given state. Whether the variance increases or decreases will be determined in large part by the structure of the covariance matrix $\underline{\Omega}$ and the $\underline{D}$ matrix. For example, if the $\underline{A}$, $\underline{D}$ and $\underline{\Omega}$ matrices are given by

$$\underline{A} = \begin{vmatrix} .8 & .1 \\ 0 & .9 \end{vmatrix} \quad \underline{D} = \underline{I} \quad \underline{\Omega} = \begin{vmatrix} 1.25 & -1 \\ -1 & 1.25 \end{vmatrix} , \tag{4.22}$$

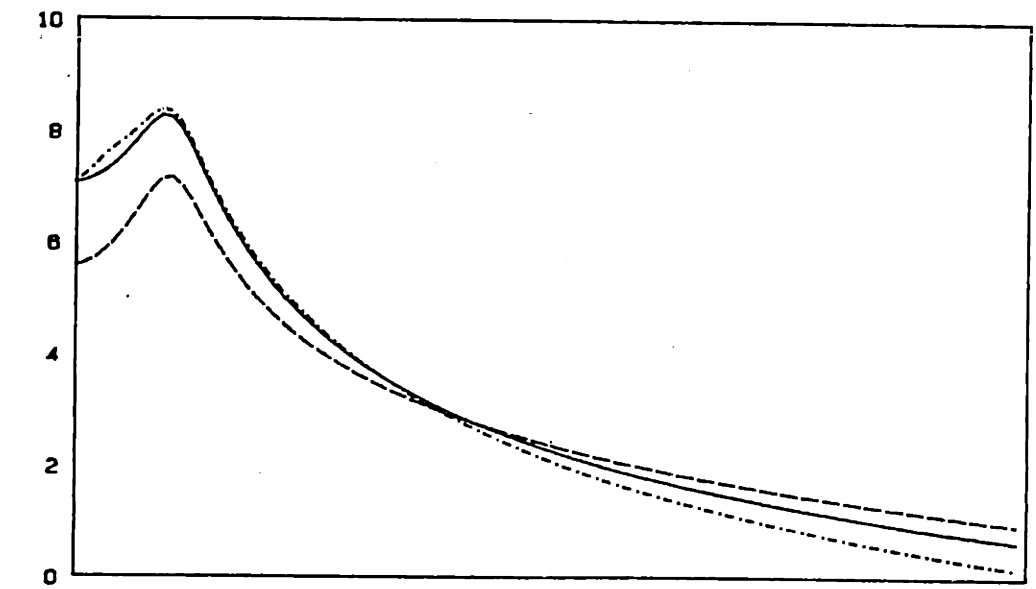
then the simplified model based on the first states and retaining the .8 root will have a state variance of 12, whereas the state variance for the first state in the original model is 3. In this example the included and excluded modes retain the effect of the noise, but when the first state is increased by the noise the second state is likely to be decreased. In the next period the second state pulls back the first. This ultimately decreases the variance of the first state. The same effects can also be apparent over frequency bands in the power spectrum.

In figures 4-1 and 4-2 the log power spectra for the MACRO model, which was considered in chapter 3, are given. The diagonal elements associated with capital (K) are plotted in figure 4-1. The power spectrum



SYMBOL	SCALE	VARIABLE
———	(1. 15)	CAPITAL_IN_RIGHT_SIMPLE_MODEL
– – – –	(1. 15)	CAPITAL_IN_LEFT_SIMPLE_MODEL
–·–·–	(1. 15)	CAPITAL_IN_THE_MACRO_MODEL

Figure 4-1: Log of Power Spectra for Capital (K) in the MACRO and SIMPLE Models



SYMBOL	SCALE	VARIABLE
———	(0. 10)	INCOME_IN_RIGHT_SIMPLE_MODEL
– – – –	(0. 10)	INCOME_IN_LEFT_SIMPLE_MODEL
–·–·–	(0. 10)	INCOME_IN_THE_MACRO_MODEL

Figure 4-2: Log of Power Spectra for Smoothed Income (YS) in the MACRO and SIMPLE Models

for capital in the MACRO model is shaped differently than it is for the simplified models. The power spectrum for capital in the MACRO model shows a great deal of power at very low frequencies, this attibutable to the largest real root. In the simplifed model the power at the low frequencies is considerably reduced, and a pronounced peak occurs at a frequency of .013 per month or a period of approximately 72 months. At the lower frequencies the power for capital from the simplified models is greater than that for the MACRO model. This is attributable to the increased direct influence of the noise series in the simplified models. Because the noise series has a flat spectrum, it dominates the state covariance matrix at the higher frequencies.

In figure 4-2 the log power spectra for smoothed income in the different models is plotted. The power for smoothed income shows a pronunced peak at the frequency of interest in all the models. The value of the power for the simplified models based on the right vectors matches that of the MACRO model nearly perfectly at the lower frequencies. That for the simplified model based on the left vectors shows somewhat less power at the low frequencies. At the higher frequencies the simplified models again show greater power than the MACRO model, though the difference between the MACRO and simplified models is not as great as it was for capital.

The Residual Spectrum

An alternative to consideration of the reduced-order model output spectrum is to consider the power spectrum of the residuals when the reduced-order model is used to explain time series data for the included

states. This approach is considered as a method of testing the adequacy of a model for dealing with oscillatory modes in Eberlein (1983). Again in this section we need to assume that the "system" is stable.†

The residuals generated when the reduced-order model is used to predict $\underline{x}_{1t}$ from $\underline{x}_{1t-1}$ are given by

$$\underline{\tilde{e}}_t = \underline{x}_{1t} - \underline{\tilde{A}}\underline{x}_{1t-1} \tag{4.23}$$

$$= (\underline{A}_{11} - \underline{\tilde{A}})\underline{x}_{1t-1} + \underline{A}_{12}\underline{x}_{2t-1} + \underline{e}_{1t} \tag{4.24}$$

$$= \underline{A}_{12}\underline{x}_{2t-1} - \underline{M}\underline{x}_{1t-1} + \underline{e}_{1t} \cdot \tag{4.25}$$

The residuals have a correlation function $\underline{T}(k)$, which is given by

(4.26)

$$\underline{T}(k) = E[\, \underline{M}\underline{x}_{1t-1}\underline{x}_{1t-k-1}^T\underline{M}^T - \underline{M}\underline{x}_{1t-1}\underline{x}_{2t-k-1}^T\underline{A}_{12}^T$$

$$- \underline{A}_{12}\underline{x}_{2t-1}\underline{x}_{1t-k-1}^T\underline{M}^T + \underline{A}_{12}\underline{x}_{2t-1}\underline{x}_{2t-k-1}^T\underline{A}_{12}^T + \underline{A}_{12}\underline{x}_{2t-1}\underline{e}_{1t-k}^T$$

$$- \underline{M}\underline{x}_{1t-1}\underline{e}_{1t-k}^T - \underline{e}_{1t}\underline{x}_{1t-k-1}^T\underline{M}^T + \underline{e}_{1t}\underline{x}_{2t-k-1}^T\underline{A}_{12}^T + \underline{e}_{1t}\underline{e}_{1t-k}^T \,] \cdot$$

This can be rewritten in terms of the autocorrelation function of the $\underline{x}$ series. Taking the z transform of the autocorrelation series and solving some algebra, we see that the power spectrum of the residual is given by

---

†It would also suffice for the unstable modes to be exactly simplifiable as this was defined in chapter 3. In this case the simplified model must contain all of the unstable modes.

$$\underline{\Psi}_{\hat{e}}(z) = \underline{M}\underline{\Psi}_{11}(z)\underline{M}^T + \underline{A}_{12}\underline{\Psi}_{22}(z)\underline{A}_{12}^T \tag{4.27}$$

$$- \underline{A}_{12}\underline{\Psi}_{21}(z)\underline{M}^T - \underline{M}\underline{\Psi}_{12}(z)\underline{A}_{12}^T$$

$$+ \left|-\underline{M} \quad \underline{A}_{12}\right| \underline{A}^{-1}(z\underline{I} - \underline{A})^{-1} \begin{vmatrix} \underline{\Omega}_{11} \\ \underline{\Omega}_{21} \end{vmatrix}$$

$$+ \left|\underline{\Omega}_{11} \quad \underline{\Omega}_{12}\right| (\tfrac{1}{z}\underline{I} - \underline{A})^{-1}(\underline{A}^{-1})^T \begin{vmatrix} (-\underline{M})^T \\ \underline{A}_{12}^T \end{vmatrix} + \underline{\Omega}_{11} ,$$

which is difficult to simplify much further. However, it is possible to glean some information from equation 4.27.

The last two terms arise from the correlation of the noise term $\underline{e}_{1t}$ with values of the $\underline{x}$ series that follow it. Because these terms contain only one transfer function term, we can expect that their variation over frequencies will be dominated by the variation of the other terms. This is especially true in the neighborhood of the roots of $\underline{A}$.† Concentrating on the first terms we see that the magnitude of the power spectrum depends on the difference between $\underline{A}_{11}$ and $\tilde{\underline{A}}$ and the magnitude of $\underline{A}_{12}$. Even if $\underline{A}_{12}$ is relatively large, if $\underline{A}_{21}$ is relatively small, so that $\underline{M}$ will be quite small, the power spectrum of the residuals can be approximated by

$$\underline{A}_{12}\underline{\Psi}_{22}(z)\underline{A}_{12}^T + \underline{\Omega}_{11} . \tag{4.28}$$

---

†In particular, this should be true at a value of z with $|z| = 1$ which maximizes some quadratic form on the power spectrum.

The fact that $\underline{A}_{21}$ is small guarantees that the spectral power inherent in $\underline{A}_{22}$ will dominate the power spectrum of the residuals.

This result can be stated with somewhat less notation. Suppose that the "system" is close to being exactly simplifiable as this was defined in chapter 3 ($\underline{A}_{12}=\underline{0}$ or $\underline{A}_{21}=\underline{0}$). If the system has this structure then the errors generated using a simplified representation of the model will have a power spectrum dominated by the white noise series $\underline{e}$ and the dynamics associated with the excluded states. The case in which $\underline{A}_{21}$ only is small is of special interest. The appearance of modes generated by $\underline{A}_{22}$ in the residuals is not a call for concern. Thus, if the modes are distinct, evaluation of the residual power spectrum within the frequency bands of interest is a justified model diagnostic.

The power spectra for the residuals when the simplified versions of the MACRO model are used on the output of the MACRO model are plotted in figures 4-3 and 4-4. Figure 4-3 shows the power spectrum of the diagonal entry for capital. The residual spectrum is considerably smaller than the original output spectrum. The residual spectrum for the model based on the left-hand eigenvectors shows a peak at approximately 72 months. The residual spectrum for the model based on the right eigenvectors shows a continual decline, which is expected since this is approximately the shape of the spectra of the excluded modes.

The same patterns hold in the residual spectra for smoothed income as shown in figure 4-4. In this case, however, the decrease in the residual sprectra as the frequency increases is more pronounced. The model based on

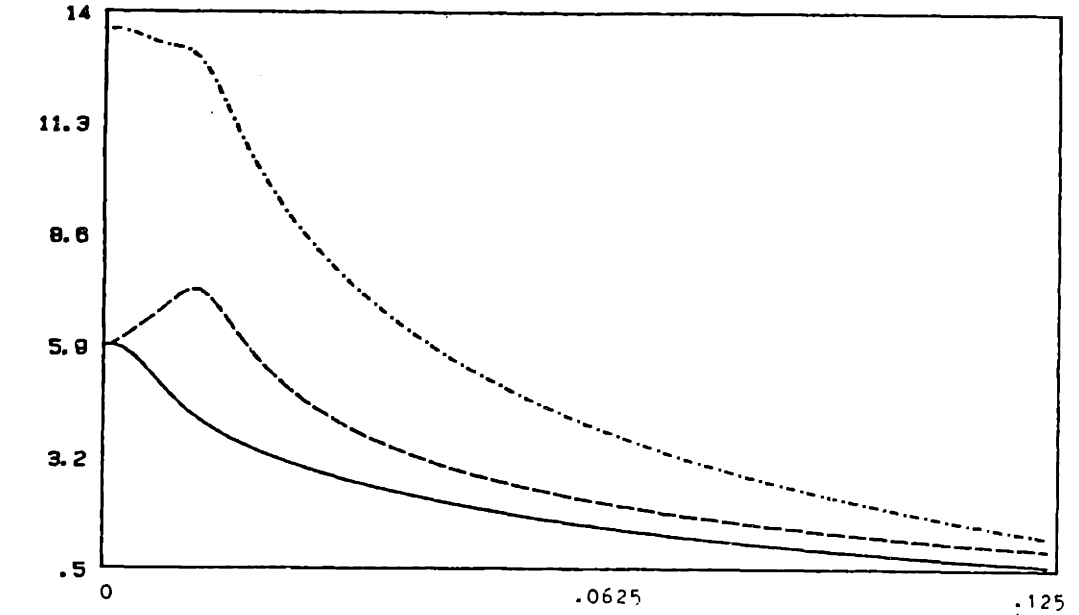


Figure 4-3: Log of Power Spectra Residuals for Capital (K) in SIMPLE Models

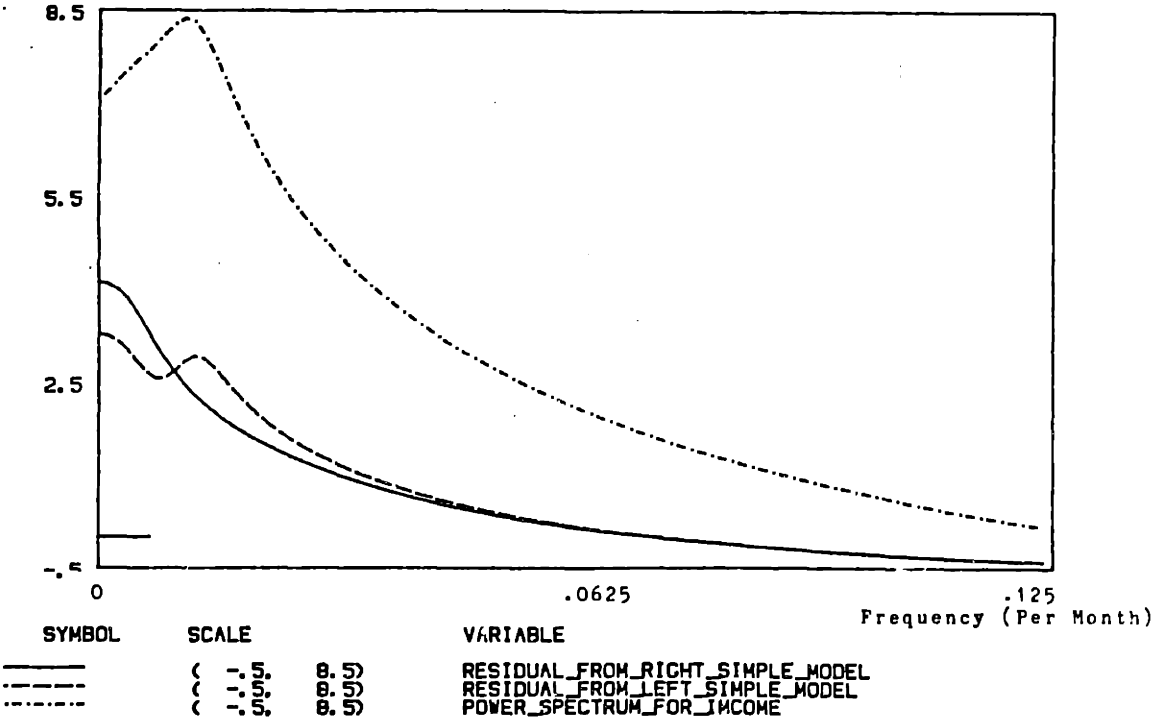


Figure 4-4: Log of Power Spectra Residuals for Smoothed Income (YS) in SIMPLE Models

the left vectors again shows a peak at approximately 72 months.

Residual State Cross-spectrum

When the simplified model is used to generate one-step prediction errors with output of the data generating process, the errors will have components of all the state variables. The error when the simplified model is used to generate one-step prediction errors is given by

$$\underline{\tilde{e}}_t = \underline{A}_{12}\underline{x}_{2t-1} - \underline{M}\underline{x}_{1t-1} \cdot \tag{4.29}$$

The covariance of the error at t with the state variables at t-k is given by

$$E[\ \underline{x}_t\ \underline{\tilde{e}}_{t-k}^T\ ] = \begin{vmatrix} \underline{\Gamma}_{12}(k+1)\underline{A}_{12}^T - \underline{\Gamma}_{11}(k+1)\underline{M}^T \\ \underline{\Gamma}_{22}(k+1)\underline{A}_{12}^T - \underline{\Gamma}_{21}(k+1)\underline{M}^T \end{vmatrix} . \tag{4.30}$$

Taking the z transform of this the cross-spectrum between the residuals and the explanatory variables is given by

$$\underline{\Psi}_{x\tilde{e}}(z) = \frac{1}{z} \begin{vmatrix} \underline{\Psi}_{12}(z)\ \underline{A}_{12}^T - \underline{\Psi}_{11}(z)\ \underline{M}^T \\ \underline{\Psi}_{22}(z)\ \underline{A}_{12}^T - \underline{\Psi}_{21}(z)\ \underline{M}^T \end{vmatrix} , \tag{4.31}$$

evaluated at $z = e^{j\theta}$ as usual.

The cross-spectrum of the residuals and the states will normally be nonzero and will contain many of the elements of the original spectrum of the states. The cross spectra for the MACRO model's states and residuals have much the same character as the spectra we have seen. The

simplification based on the left vectors continues to show a peak in power at approximately 72 months.

The various data relevant to the variances of the MACRO model are presented in figure 4-5. The data are a useful summary of the various points that have been discussed. The variance matrix for the noise entering the simplified models is substantially different from that entering the MACRO model. The amplification lost in the model reduction process is in part made up in the altered variance. The variances of the states of the simplified models are lower than in the MACRO model, though of a comparable magnitude.

The variances of the residuals based on the simplified models are substantially lower than the variances of the states, and the variance for the error in capital is also lower than the variance of the noise exciting capital in the simplified models. This is due to the smooth nature of capital in the MACRO model relative to the simplified models, which was observed in the previous chapter. The covariances between the included states and the residuals are quite high, and the correlation coefficients are all in the neighborhood of .5 to .8.

4-B Data Transformations

In this section we will consider some data transformations useful in the consideration of simplified models. In particular, we derive a transformation of the "system" output that yields data consistent with the simplified model. That is, the simplified model, if excited by the same noise and exogenous variable series as the "system," would generate the

Covariance Matrix for the Noise Entering the MACRO Model

	AP	K	P	PIE	YP	YS
AP	0	0	0	0	0	0
K		1.00	0	0	.0250	.200
P			.250E-4	0	0	0
PIE				.16E-8	0	0
YP					.304	.0100
YS						.720

Covariance for the MACRO Model

	AP	K	P	PIE	YP	YS
AP	.00023	-1.95	.00028	.647E-5	-.0621	-.192
K		31000	-.892	-.00715	1020	1410
P			.00039	.943E-5	-.0284	-.188
PIE				.348E-6	.000252	-.00604
YP					41.1	48.7
YS						171

Covariance Matrices for the Noise Entering the Simplified Models

Model Based on Right Vectors

	K	YS
K	1390	6.1
YS		1.5

Model Based on Left Vectors

	K	YS
K	1370	49.5
YS		2.52

Covariance Matrices for the States of the Simplified Models

Model Based on Right Vectors

	K	YS
K	15400	70.8
YS		149

Model Based on Left Vectors

	K	YS
K	24800	42.1
YS		52.8

Covariance Matrices for the Residuals Using the Simplified Models to Explain the Data Generated by the MACRO Model

Model Based on Right Vectors

	K	YS
K	12.8	5.34
YS		2.81

Model Based on Left Vectors

	K	YS
K	58.2	-2.98
YS		2.29

Covariance of the Residuals With $\underline{x}_1$ Using the Simplified Models to Explain the Data Generated by the MACRO Model

Model Based on Right Vectors

	K	YS
K	435	180
YS	31.4	13.4

Model Based on Left Vectors

	K	YS
K	-1030	93.2
YS	-87.6	12.0

Figure 4-5 Variances and Covariances for the MACRO Model and its Simplifications

transformed data.

The interpretation of the results depends on the derivation used for the simplified models. Two cases of special interest are those in which the right and left eigenvectors are preserved with the number of retained modes equal to the number of retained states. We allow for the more general specification of the "system" and do not require stability of the $\underline{A}$ matrix for the results in this section.

As always, let the underlying system or data-generating process be given by

$$\underline{x}_t = \underline{A}\underline{x}_{t-1} + \underline{B}\underline{u}_t + \underline{D}\underline{e}_t \ . \tag{4.32}$$

Consider the transformation of the output series defined by

$$\hat{\underline{x}}_t = \tilde{\underline{R}}_1 \, | \, \underline{L}_{11} \ \underline{L}_{12} \, | \, \begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} \tag{4.33}$$

with $\tilde{\underline{R}}_1$ the right eigenvector matrix of the simplified model associated with the modes of interest. We will assume that the simplfied model has been constructed with initial conditions, the $\tilde{\underline{B}}$ matrix and the $\tilde{\underline{D}}$ matrix all defined in a manner analogous to $\hat{\underline{x}}$; that is,

$$\tilde{\underline{x}}_o = \hat{\underline{x}}_o \ , \quad \tilde{\underline{B}} = \tilde{\underline{R}}_1 \, | \, \underline{L}_{11} \ \underline{L}_{12} \, | \, \begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} \tag{4.34}$$

and

$$\tilde{\underline{D}} = \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} . \tag{4.35}$$

If these conditions hold, then the output path of the simplified model exactly matches that of the transformed states $\underline{\hat{x}}$.

To show this we note that $\underline{\tilde{x}}_o = \underline{\hat{x}}_o$ by equation 4.34 and assuming that

$$\underline{\tilde{x}}_{t-1} = \underline{\hat{x}}_{t-1} \tag{4.36}$$

we have

$$\underline{\tilde{x}}_t = \tilde{\underline{A}}\underline{\hat{x}}_{t-1} + \tilde{\underline{B}}\underline{u}_t + \tilde{\underline{D}}\underline{e}_t \tag{4.37}$$

$$= \tilde{\underline{A}}\,\tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \{ \underline{Bu}_t + \underline{De}_t \} \tag{4.38}$$

$$= \tilde{\underline{R}}_1 \underline{A}_{11} \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \{ \underline{Bu}_t + \underline{De}_t \} \tag{4.39}$$

$$= \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \underline{A} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \{ \underline{Bu}_t + \underline{De}_t \} \tag{4.40}$$

$$= \tilde{\underline{R}}_1 \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} \tag{4.41}$$

$$= \underline{\hat{x}}_t . \tag{4.42}$$

The simplified model is in accord with the data that are modified according to equation 4.33. That is, there would be no difference between data generated by the simplified model and the series $\underline{\hat{x}}$. The simplified model will not be uniquely recoverable from the transformed states unless the number of retained modes matches the number of retained states. This is intuitively clear from the fact that there are degrees of freedom in the choice of $\underline{M}$ unless this is the case. This lack of identification[†] (or identifiability) results from the transformation of the states, which lowers the dimension of the space over which they vary. This will be discussed further in section 4-C.

For the simplified model based on the right eigenvectors the above results can be interpreted in terms of restricting the space over which $\underline{x}$ is allowed to vary. Suppose that the output series $\underline{x}_t$ is transformed according to the following rule

$$\underline{\hat{x}}_t = \begin{vmatrix} \underline{R}_{11} \\ \underline{R}_{21} \end{vmatrix} \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \underline{x}_t \,. \tag{4.43}$$

This transformation is not, strictly speaking, a projection, since the transformation matrix is not symmetric (Strang 1976). However, the matrix is idempotent, and the result of the transformation can be written as a combination of the right eigenvectors associated with the included modes.

[†]The term identification will be used in the econometric sense. That is, a model is identified if it possible to recover the unknown elements of the model uniquely from sufficiently large amounts of data. We will use estimation to refer to the actual determination of the parameters; whenever state estimation is discussed it will be referred to explicitly as such.

It is therefore reasonable to think of this transformation as similar to a projection of the vector $\underline{x}_t$ onto the space spanned by the right eigenvectors associated with the included modes.

The reduced-order model developed using the left eigenvectors has a slightly different realization. Assuming that the number of modes and states are equal, then, by analogy to 4.43, we can consider projecting the state vector onto its first n components via

$$\underline{\hat{x}}_t = \begin{vmatrix} L_{11}^{-1} \end{vmatrix} \begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix} \underline{x}_t \tag{4.44}$$

$$= \underline{x}_{1t} + L_{11}^{-1}\underline{L}_{12}\underline{x}_{2t} \, . \tag{4.45}$$

The advantage that the retention of the left-hand eigenvectors has is that the transformed states are generated as the sum of the included states and a contribution from the excluded states.

The size of the contribution from the excluded states is not an indication of the quality of the simplified model as this was discussed in the previous chapter. Rather, the size is an indication of how important excluded states and modes are in determining directly or indirectly the time path of the included states. Even if there is no feedback from the included states to the excluded, states the transformation can show a strong influence from the included states. This is similar to the issue of the steady-state gain as it was discussed in chapter 3.

The projection given in equation 4.45 has a very nice intuitive

interpretation. The level of activity of a mode at time t can be determined by premultiplying the state vector by the left eigenvector associated with that mode. In so doing there are contributions from each state. In order to maintain all these contributions in a smaller state vector containing only n entries, it is necessary to distribute the contributions of the excluded states among the included states. The most obvious way to do this is according to the share of the different modes that the included states already account for. Thus, if the excluded states contribute a great deal to the presence of a mode in which the i'th state variable is important, the i'th state variable will change a great deal.

The major problem with the notion of estimation via the projection given in equation 4.33 is that this projection requires full-system knowledge. The prerequisite for such knowledge is that the original full system be treated as known, presumably via some sort of estimation. The difficulty and expense of such full-system estimation is one of the reasons for considering reduced-order models. Thus, though the idea of projecting the state variable onto a lower dimension subspace is intuitively very appealing, it has only limited empirical applicability.

The projection matrices for the simplifications of the MACRO model are given in figure 4-6. The projections involve a substantial alteration of the variables retained. Though much of the apearance of the matrices is attributable to scaling, the alterations are still quite large. This is, of course, in line with our findings for the steady-state gain in chapter 3, the steady state gain is altered substantially in the simplified models. The projections capture this change in the steady-state gain. That is, if

the "system" were to start out in equilibrium and the $\underline{x}_0$ were transformed according to the equations discussed above, the difference between the original and transformed variables would be precisely the difference in the steady-state gain. This suggests one way of dealing with unstable models, namely to rescale the variables in the simplified models according to the difference between the transformed and actual states at some point in time. This is equivalent to what was done with the MACRO model in chapter 3.

Simplified Model Based on the Right Vectors

	AP	K	P	PIE	YP	YS
$\tilde{K}$	4280	.572	7150	18100	-8.29	11.1
$\tilde{YS}$	-344	-.0505	-19.6	14000	.435	1.30

Simplified Model Based on the Left Vectors

	AP	K	P	PIE	YP	YS
$\tilde{K}$	7200	1.0	7300	-101000	-12.0	0
$\tilde{YS}$	15.0	0	.268	6860	-.13	1.0

Figure 4-6 Projection Matrices for the MACRO Model

The result we have given above states that the transformed variables will have the same time path as the variables generated by the simplified models. It follows that the comparison of the MACRO model output and that of the simplified models is equivalent to the comparison of the transformed and the actual data. That is, if the data for the MACRO model were transformed in the manner outlined above the resulting transformed variables would follow the time path of the variables from the simple model

simulations exactly. Those data are plotted in figures 3-12, and 3-13.

4-C Controllability and Observability

In the above discussion we have seen that the simplified model will have the correct input and output characteristics with respect to a transformation of the "system" output. There are some situations in which the "system" output may be invariant to such transformations. In order to investigate this we will add the measurement equation

$$\underline{y}_t = \underline{C}\underline{x}_t \tag{4.46}$$

to the standard "system" equations. The vector $\underline{y}$ represents the series of observations that we get from the "system" and will normally be of lower dimension than the state variable $\underline{x}$.† With the addition of $\underline{y}$ there are two important properties of the system that need to be considered: how much can be discovered about the behavior of the states from the observation of $\underline{y}$ and how much the behavior of the states can be affected by the exogenous variables. We discuss these in turn and use the ideas of model simplification we have developed to relate the two.

Controllability

A model is said to be controllable if the inputs can be varied so as to get the model to an arbitrary point in state space in a finite amount of time (Chen 1970, chapter 5). Following Porter and Crossely (1972,

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†Measurement error in the equation for $\underline{y}$ has been omitted for simplicity. Its inclusion would not change the results.

chapter 4) we note that the controllability of the dynamic system by the inputs $\underline{u}$ is equivalent to the requirement that[†]

$$(\underline{1}^k)^T \underline{B} \neq \underline{0} \tag{4.47}$$

for all k. The similar condition on the noise can be written conveniently as

$$(\underline{1}^k)^T \underline{D\Omega D}^T (\underline{1}^k) \neq \underline{0} \tag{4.48}$$

for all k. If equations 4.47 and 4.48 both fail to hold for some k then that mode is not "excitable." With the exception of the initial conditions no influence on the "system" will cause the behavior mode to appear. If the mode is stable, its appearance in the output will be limited to the initial observations. That is, as the number of observations goes to infinity, it adds nothing to our knowledge of stable but unexcited modes. If the mode is unstable it will appear as a deterministic pattern in the data, which is sufficient to allow identification.

The ideas of model simplification that have been considered can add some understanding to why the "system" is not identified. We can assume without loss of generality that the N'th mode is excited by neither the noise nor the exogenous variables. The simplified model based on the right eigenvectors and including only the first N-1 modes and all states[††] will

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[†]We assume that the eigenvalues of the system are distinct.

[††]It is not necessary to include all the states, of course, but to do so is convenient in this case.

be correct for the transformation of the state given by equation 4.43. However, the difference between the transformation of the state variable and the original state variable is given by

$$\underline{r}^N(\underline{l}^N)^T\underline{x}_t \ , \tag{4.49}$$

which is $\underline{0}$ if $\underline{x}_t$ does not contain the N'th mode. Any simplified model based on the first N-1 modes and preserving the right eigenvectors will give an exact simplified model.

Since the number of retained states exceeds the number of retained modes, the simplification will not be unique. In particular, any M satisfying the equation

$$\underline{M} = \underline{v}\ (\underline{l}^N)^T \tag{4.50}$$

with $\underline{v}$ an arbitrary vector can be added to the $\underline{A}$ matrix and preserve the right eigenvectors associated with the modes of interest.

The lack of certain modes inherent in the "system" but not the data will generate perfect collinearity in the data. This follows from the fact that the $\underline{x}$ vector can always be represented as a linear combination of the first N-1 right eigenvectors. Because the right and left eigenvectors are orthogonal,

$$(\underline{l}^N)^T\ \underline{x}_t = \underline{0} \quad \text{for } t = 0,1, \ . \ . \ . \ , \tag{4.51}$$

so that if the system were estimated using least squares the $(\underline{X}^T\underline{X})$ matrix would not be invertible. There are, of course, good methods for detecting the presence of multicolinearity (Belsley, Kuh, and Welsch 1980) that can prove useful if this seems to be a problem.

What the above argument has shown is the way in which the lack of excited modes can manifest itself as multicollinearity on the data and a consequent lack of identification of the coefficients of the "system." This is a special case of what can always be a problem in the estimation of a system. The cause of the trouble in this case is not that the "system" itself is degenerate, but rather that there are elements of the dynamics of the "system" that do not manifest themselves in the data.

Observability

The above discussion is applicable to a situation in which certain of the modes are not excited by either the noise or the exogenous input. There is a corresponding problem when the full state vector is not observed. If observations are available only for a limited number of states and their linear combinations then it may be possible for the "system" to generate patterns of behavior that do not manifest themselves in the available data. A "system" is called observable if it is possible after a finite amount of time to determine the values of all the states at time 0 from the system output (Chen 1974). If a "system" is not observable, there will be different trajectories in state space that generate the same output stream.

If a "system" is not observable, it is not possible to find the best

estimates of the state variables (Anderson and Moore 1978, chapter 4). Since the state estimates are necessary in order to estimate the parameters of the model, it follows that these parameters cannot be estimated if the "system" is not observable. This requirement of observability is very similar to the requirement that all the modes be excited. The similarity becomes clearest when we consider the observation equation in terms of a transformation of the states for a simplified model.

The conditions for observability can be stated in terms of the $\underline{C}$ matrix of equation 4.46 as the requirement that (Porter and Crossely 1972)

$$\underline{C}(\underline{r}^k) \neq \underline{0} \tag{4.52}$$

for all k. We can think of the transformation equation on the included states as an observation equation on these states, with the observation matrix given by

$$\underline{C} = \tilde{\underline{R}}_1 \; \vdots \; \underline{L}_{11} \; \underline{L}_{12} \; \vdots \; . \tag{4.53}$$

This definition of $\underline{C}$ will not satisfy equation 4.52 for any of the excluded modes. These modes are not observable in the transformed data.

In the more general case suppose that $\underline{C}$ is given and has the property that

$$\underline{C}\,\underline{r}^k = \underline{0} \;, \quad k = n+1, n+2, \; . \; . \; N \; . \tag{4.54}$$

In this case the column space of the $\underline{C}$ matrix is necessarily spanned by the first n left eigenvectors, so that $\underline{C}$ may be written in the form

$$\underline{C} = \underline{C}' \, | \, \underline{L}_{11} \, \underline{L}_{12} \, | \, . \tag{4.55}$$

Postmultiplying equation 4.55 by any right inverse of the left vectors and then those left vectors, we have

$$\underline{C} \, | \underline{L}_{11} \, \underline{L}_{12} |^{-} \, | \underline{L}_{11} \, \underline{L}_{12} | = \underline{C}' | \underline{L}_{11} \, \underline{L}_{12} | | \underline{L}_{11} \, \underline{L}_{12} |^{-} \, | \underline{L}_{11} \, \underline{L}_{12} | \tag{4.56}$$

$$= \underline{C}' \, | \underline{L}_{11} \, \underline{L}_{12} | \tag{4.57}$$

$$= \underline{C} \, . \tag{4.58}$$

But the series

$$\underline{\hat{x}}_t = | \underline{L}_{11} \, \underline{L}_{12} |^{-} \, | \underline{L}_{11} \, \underline{L}_{12} | \, \underline{x}_t \tag{4.59}$$

would be generated by the simplified model, which retains the modes associated with the $\underline{L}_{11}$ vectors. The observations made are equivalent to observations generated by

$$\underline{\hat{y}}_t = \underline{C} \, \underline{\hat{x}}_t \, , \tag{4.60}$$

and it follows that the model that can be identified using the observations will be a simplified model.

The choice of the left inverse for $\begin{vmatrix} \underline{L}_{11} & \underline{L}_{12} \end{vmatrix}$ in equation 4.56 was arbitrary, and it appears that the model identified will depend on it. The reason for this is that in order to estimate the model it is necessary to reduce the $\underline{C}$ matrix to dimension n from N. This reduction will be arbitrary but must preserve the rank of $\underline{C}$. The choice of the reduction will determine the character of the simplified model. In many cases the $\underline{C}$ is a matrix of the form

$$\underline{C} = \begin{vmatrix} \underline{I} & \underline{0} \end{vmatrix} , \tag{4.61}$$

and there is little, if any, choice on how to reduce $\underline{C}$. If a $\underline{C}$ matrix in the form of equation 4.61 satisfies eqution 4.56 it is because $\underline{L}_{12}$ is equal to $\underline{0}$. In general, the satisfaction of equation 4.56 is more likely to be a restriction on the nature of the "system's" dynamics than a restriction on $\underline{C}$.

The above discussion has been made without reference to a priori restrictions that might apply to a model. It may be possible to achieve identification through the use of such restrictions even if the "system" is not controllable and observable.

4-D Identification and Estimation

In this section we consider the issues of identification and estimation of the simplified model from available data. This problem has already been discussed to some extent in the section on transforming the data for the original model. In this section we consider the problem using the actual "system" time series. The problem is first discussed for

exactly simplifiable models where the issues are quite straightforward. The construction of instrumental variables for the estimation of the simplifed model is then considered. Finally it is shown that there is no way of identifying the parameters of the simplified model without appeal to the larger "system" from which the simplified model arises.

Exactly Simplifiable Models

There is one circumstance in which it is clearly possible to arrive at a consistent estimator. If $\underline{A}_{12}$ is zero then the dynamics of the included states can be exactly represented by the simplified model. Equation-by-equation estimation will yield consistent estimates, and system-generalized least squares will yield consistent and efficient estimates. Similarily, with limited and noisy observations, state space engineering identification techniques will yield consistent estimators.

The alternative condition for perfect simplification is that $\underline{A}_{21}=\underline{0}$. In this case the reduced-order model will again have a dynamics matrix given by $\underline{A}_{11}$, though $\underline{\tilde{B}}$ will not generally be equal to $\underline{B}_1$. If $\underline{A}_{21}$ is zero then the dynamics generated by the included states do not influence the excluded states. Thus, though the included states will contain dynamics not generated by the simplified model, these dynamics are not affected by interactions among the included and excluded states. The excluded states can usefully be taken as exogenous to the reduced-order model[†]

[†]Strictly speaking, the requirement that the covariance matrix of the noise ($\underline{D\Omega D}^T$) be block diagonal would also have to be imposed (Engle, Hendry and Richard 1983).

The estimation of the reduced-order model is not a trivial matter when $\underline{A}_{21}$ is zero but $\underline{A}_{12}$ is not. The model can also be thought of as a model in the form

$$\underline{x}_{1t} = \underline{A}_{11}\underline{x}_{1t-1} + \underline{\tilde{B}u}_t + \underline{\ell}_t + \underline{e}_{1t} \tag{4.62}$$

with $\underline{\ell}_t$ a vector autoregressive-moving-average error term. This model is close to the actual model, with the exception that the error term cannot be influenced by the $\underline{u}$ series. Consistent estimation of equations of the type 4.62 has been considered in the literature before (Fair 1971, Sargan 1978).

The situation in which $\underline{A}_{21}$ is zero is also a situation in which a very nice interpretation can be given to the use of autocorrelation correction in the residuals. This autocorrelation correction can be thought of as a black box representation of intentionally excluded and not terribly relevant dynamics. With the lagged dependent variables in equations of the form of 4.62, the estimates resulting from using an autoregressive correction would differ from those of the noncorrected form by approximately $\underline{A}_{12}\underline{\Gamma}_{12}\underline{\Gamma}_{11}^{-1}$. The complexity of this type of estimator will depend on how simple a representation of the less relevant dynamics will suffice. In many situations it is to be expected that a representation of the excluded modes of order less than n might suffice.

As we have noted before, the conditions for exact simplification are very restrictive, requiring that $\underline{A}_{12}=\underline{0}$ or $\underline{A}_{21}=\underline{0}$. However, if either of these submatrices is small, the discussion will still apply approximately.

Instruments

In section 3-B we considered a projection of the state variables which makes the simplified models exact. An alternative approach to this problem is to try to find variables uncorrelated with the errors that the simplified model makes with respect to the included states. We have seen that the errors and state variables are highly correlated. The technique of instrumental variables (Johnston 1972, Theil 1971) finds a set of variables uncorrelated with the errors that enter the equation, but correlated with the right-hand side variables.

The error term that the simplified model generates when the data are applied to it is given by

$$\underline{x}_{1t} - \hat{\underline{x}}_{1t} = \underline{A}_{12}\underline{x}_{2t-1} - \underline{Mx}_{1t-1} \cdot \tag{4.63}$$

Consider an instrument written in terms of the current values of the states as

$$\hat{\underline{y}}_t = \underline{W}_1\underline{x}_{1t} + \underline{W}_2\underline{x}_{2t} \cdot \tag{4.64}$$

For this to be a valid instrumnent we require that

$$E\left[\ \hat{\underline{y}}_{t-1}\ (\underline{A}_{12}\underline{x}_{2t-1} - \underline{Mx}_{1t-1})^T\ \right] = \underline{0} \tag{4.65}$$

or

$$W_1\ (\underline{\Gamma}_{12}\underline{A}_{12}^T - \underline{\Gamma}_{11}\underline{M}^T) + \underline{W}_2(\underline{\Gamma}_{22}\underline{A}_{12}^T - \underline{\Gamma}_{21}\underline{M}^T) = \underline{0} \cdot \tag{4.66}$$

The other condition required for $\hat{\underline{y}}$ to be a valid instrument is that

$$E\left[\underline{\hat{y}}_t \underline{x}_{1t}\right] = \underline{W}_1 \underline{\Gamma}_{11} + \underline{W}_2 \underline{\Gamma}_{21} \tag{4.67}$$

be of full column rank.

It will normally be possible to find a solution to equations 4.66 and 4.67. The resulting instrument could then be used in order to estimate the simplified model. Unfortunately, in order to determine the solution to equation 4.66 given, for example, $\underline{W}_1$, it is necessary to know the value of $\underline{M}$. Knowledge of this this value essentially requires a model simplification.

It is also possible to consider instruments based on linear combinations of the included states. In this case we write the instrument as

$$\underline{\tilde{y}}_t = \Sigma_k \, p_k \underline{x}_{1t-k} \tag{4.68}$$

so that the requirements for the instrument take the form

$$\Sigma_k \, p_k \{\underline{\Gamma}_{12}(-k)\underline{A}_{12}^T + \underline{\Gamma}_{11}(-k)\underline{M}^T\} = 0 \tag{4.69}$$

and that

$$\Sigma_k \, p_k \{\underline{\Gamma}_{11}(k)\} \tag{4.70}$$

be of full rank.

It will not normally be possible to find a small number of p_k's that satisfy the above equations. The number of degrees of freedom available is not as great as it was in the case previously considered. Of course, it is possible to combine the two transformations into a more general set of instruments. In any case, generation of the instruments would require knowledge of the actual values of the $\underline{A}_{12}$ and $\underline{M}$ matrices.

No Simple Estimator

The most important result of this section is a negative result about the inability to identify a simplified model without reference to the complete "system." It is clear that there will always be a number of "systems" that give rise to any given simplified model. In this manner, any specific type of simplified model can be thought of as determining an equivalence class of "systems." For example, simplified models that preserve the right eigenvectors for n modes and the first n states would define an equivalence relation on all models in the form of equation 4.1. The results in this section tell us what equivalence classes of simplified models can be distinguished by the data without explicitly referring to the original "system."

Specifically, if $\underline{e}_t$ in equation 4.1 is Gaussian, then, using only the prediction error given by

$$\hat{\underline{e}}_{1t} = \underline{x}_{1t} - \hat{\underline{A}}\underline{x}_{1t-1} , \tag{4.71}$$

it is not possible to identify the parameters of simplified models of the

type discussed in chapter 3. We first define the characteristics of a model that determine the characteristics of the error given in equation 4.71. We then show that these characteristics are not sufficient to determine what simplified models the application of the methods of chapter 3 to the "system" would generate.

The output of a model of the type given in equation 4.1 with $\underline{e}_t$ distributed normally with mean zero and variance $\underline{\Omega}$ will have a Gaussian distribution with mean zero and variance $\underline{\Gamma}$ given by the solution to (Anderson and Moore 1978)

$$\underline{\Gamma} = \underline{A}\,\underline{\Gamma}\,\underline{A}^T + \underline{\Omega} \ . \tag{4.72}$$

We will assume that $\underline{\Gamma}$ is a positive definite matrix. The information that is available from the one-step prediction error of equation 4.71 will be contained in the joint distribution of $\underline{x}_{1t}$ and $\underline{x}_{1t-1}$. This will be normal and is characterized by the second moments

$$\underline{\Gamma}_{11} \tag{4.73}$$

and

$$\underline{A}_{11}\,\underline{\Gamma}_{11} + \underline{A}_{12}\,\underline{\Gamma}_{21} \ . \tag{4.74}$$

The quantities in the above equations characterize the information that is available from the one-step prediction error.

We first show that given an arbitrary model it is possible to find a decoupled model ($\underline{A}_{12}$ and $\underline{A}_{21}$ both zero) for which the distribution of $\underline{x}_{1t}$

and $\underline{x}_{1t-1}$ matches that of the original model. To do this we set

$$\hat{\underline{A}} = (\underline{A}_{11}\ \underline{\Gamma}_{11} + \underline{A}_{12}\ \underline{\Gamma}_{21})\ \underline{\Gamma}_{11}^{-1} \tag{4.75}$$

and

$$\hat{\underline{\Omega}} = \underline{\Omega}_{11} + \underline{A}_{12}\ (\underline{\Gamma}_{22} - \underline{\Gamma}_{21}\underline{\Gamma}_{11}^{-1}\underline{\Gamma}_{12})\ \underline{A}_{12}^{T} . \tag{4.76}$$

The variance $\hat{\underline{\Omega}}$ is positive semidefinite. This follows from the observation that $\underline{\Omega}_{11}$ is positive semidefinite and that $(\underline{\Gamma}_{22} - \underline{\Gamma}_{21}\underline{\Gamma}_{11}^{-1}\underline{\Gamma}_{12})$ is the inverse of the lower right-hand block of $\underline{\Gamma}^{-1}$ and is therefore positive definite. The equation determining the variance of the model defined by equations 4.75 and 4.76 is given by

$$\hat{\underline{\Gamma}} = \hat{\underline{A}}\ \hat{\underline{\Gamma}}\ \hat{\underline{A}}^{T} + \hat{\underline{\Omega}} . \tag{4.77}$$

Substituting the definitions of $\hat{\underline{A}}$ and $\hat{\underline{\Omega}}$, we can check to see that $\underline{\Gamma}_{11}$ is a solution to equation 4.77. We have

$$\underline{\Gamma}_{11} \stackrel{?}{=} (\underline{A}_{11}\ \underline{\Gamma}_{11} + \underline{A}_{12}\ \underline{\Gamma}_{21})\ \underline{\Gamma}_{11}^{-1}\ \hat{\underline{\Gamma}}\ (\underline{\Gamma}_{11}^{-1})^{T}\ (\underline{A}_{11}\ \underline{\Gamma}_{11} + \underline{A}_{12}\ \underline{\Gamma}_{21})^{T} + \tag{4.78}$$

$$\underline{\Omega}_{11} + \underline{A}_{12}\ (\underline{\Gamma}_{22} - \underline{\Gamma}_{21}\underline{\Gamma}_{11}^{-1}\underline{\Gamma}_{12})\ \underline{A}_{12}^{T} .$$

After some substitution the right hand side of the above equation can be rewritten as

$$\left|\underline{A}_{11} \quad \underline{A}_{12}\right| \left|\begin{matrix} \underline{\Gamma}_{11} & \underline{\Gamma}_{12} \\ \underline{\Gamma}_{21} & \underline{\Gamma}_{22} \end{matrix}\right| \left|\begin{matrix} \underline{A}_{11}^{T} \\ \underline{A}_{12}^{T} \end{matrix}\right| + \underline{\Omega}_{11} , \tag{4.79}$$

which is just the upper left-hand block of the variance equation 4.72 for

$\underline{x}$. This verifies the identity in equation 4.78.

In chapter 2 we showed that if we assume a simplified model to be correct, the maximum likelihood estimator would converge to

$$\hat{\underline{A}} = \underline{A}_{11} + \underline{A}_{12}\,\underline{\Gamma}_{12}\,\underline{\Gamma}_{11}^{-1} \cdot \tag{4.80}$$

What the above result shows is that any useful estimate based only on the joint distribution of $\underline{x}_{1t}$ and $\underline{x}_{1t-1}$ will also converge to this value. This follows because we have shown that a model that actually is correct and has an $\underline{A}$ matrix given by equation 4.80 will generate the same joint distribution as the "system." Thus, an estimator that gives the correct $\underline{A}$ matrix when there is a correct $\underline{A}$ matrix will give an $\hat{\underline{A}}$ as in equation 4.80 for all other models.

The above result defines the equivalence classes it is possible to generate from the selected data. What we show next is that the equivalence class thus defined bears little relation to the equivalence classes it would be possible to define using simplified models of the type considered in chapter 3. Models with the same eigenvalues can generate data which will yield different values for the $\hat{\underline{A}}$ matrix. This follows from the observation that if $\underline{A}_{21}$, is zero changing $\underline{A}_{12}$ from 0 to some nonzero matrix will change $\hat{\underline{A}}$ but not alter $\tilde{\underline{A}}$ (however defined).

Conversely, models with different eigenvalues can generate the same simplified models. Another way to state this, since $\hat{\underline{A}}$ can derive from a model with $\underline{0}$ off-diagonal entries, is that $\hat{\underline{A}}$ does not retain the

eigenvalues of $\underline{A}$. To see this, we consider a model of the form

$$\underline{A} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{0} & \underline{0} \end{vmatrix} \quad \text{and} \quad \underline{D} = \underline{I} \ . \tag{4.81}$$

In this case $\underline{\Gamma}_{11}$ satisfies the equation

$$\underline{\Gamma}_{11} = \underline{A}_{11}\underline{\Gamma}_{11}\underline{A}_{11}^{T} + \underline{\Omega}_{11} + \underline{A}_{12}\underline{\Omega}_{21}\underline{A}_{12}^{T} \ , \tag{4.82}$$

and we can write

$$\hat{\underline{A}} = \underline{A}_{11} + \underline{A}_{12}\underline{L}_{21}\underline{\Gamma}_{11}^{-1} \ . \tag{4.83}$$

If $\underline{\Omega}$ is positive definite then a small change in $\underline{A}_{11}$ can be compensated by adjusting $\underline{\Omega}_{11}$ to yield the same value for $\underline{\Gamma}_{11}$. We then adjust $\underline{A}_{12}$ and $\underline{\Omega}_{21}$ to regain the original value of $\hat{\underline{A}}$ using equation 4.80. By equation 4.72 we can adjust $\underline{\Omega}_{11}$ so that $\underline{\Gamma}_{11}$ remains invariant. Because $\underline{\Omega}$ was initially positive definite none of these adjustments will alter this.

The above argument seems somewhat burdensome in light of the fact that we really only require one example of a situation in which $\hat{\underline{A}}$ does not retain the eigenvalues, and these are easy to come by. For example, in the MACRO model the variance-based simpification is given by

$$\begin{vmatrix} .955 & .943 \\ -.0050 & 1.0346 \end{vmatrix} \ . \tag{4.84}$$

and has complex conjugate eigenvalues of magnitude .996 that imply

oscillation of 112 months. However, the argument serves to illustrate the importance of the characteristics of the noise terms in determining the value of $\underline{\hat{A}}$. Changing the nature of the exciting noise can have a large impact on what the nature of the data-based simplified model will be. These simplified models will always be stable, but it is not possible to say a great deal more.

The result is similar in spirit to the indistinguishability between a regression coefficient and an autoregressive term in an equation with only a lagged dependent variable on the right-hand side (Pindyck and Rubinfeld, section 7.7). The extraneous process and the process of interest are difficult to distinguish. This is the situation we are faced with here as well. The attributes of the "system" which are extraneous to the determination of behavior modes, notably the influence of noise, determine the characteristics of the data.†

This result is quite important in that it severely restricts the types of simplified models that can be derived using information theoretic criteria in the way they are commonly used (Anderson, Moore and Hawkes 1978, Baram and Be'eri 1981, Baram and Sandell 1978). The characteristics of a model we have concentrated on do not play an important role in the determination of the simplified models derived in this manner. Simplified models based on the minimization of an information criterion give a great deal of weight to the nature of the inputs into the model. This is not

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†This can also be interpreted as a result about errors in variables. We have shown above that the error term is correlated with the included state variables. The included states do not have the necessary information to yield valid instruments.

surprising, but it does limit the situations in which applications of such models will give good results.

Because the result we derived above is negative, the fact that it is for a stationary autonomous model is not restrictive. In the more general case the same problems arise, and the result will continue to hold. One important restriction is that of the normality of the noise sequence exciting the model. If the noise entering a model is not normally distributed, then the higher-order moments characterizing the error distribution may suffice for identification. This result is analogous to the one that holds in simultaneous equation econometric models, where noise that is not distributed normally may identify an otherwise unidentified model (Hausman 1981).

The main impact that the result has for the derivation of estimators is the elimination of the possibiity that any simple estimator of the reduced-order model exists. It is normally possible to define estimators through the use of a time history of values of $\underline{x}_1$, that will consistently identify the entire system (Mehra 1974, Peterson 1975, Schweppe 1973, 1976). Unfortunately, identification of the whole system is a very expensive process, and one of the reasons for developing simplified models is to avoid generating the entire system.

4-E Summary

In this chapter we have considered the relationship of the simplified models developed in chapter 3 to the output of the original model. The power spectrum of the simplified models relative to the original model has been considered. In addition, the characteristics of the error generated when the simplified models are used in conjunction with the output of the original system have been determined. The simplified models have been shown to be consistent with a transformation of the output of the original "system." The relationship that this transformation has with the observability and controllability of the original model has been discussed. The most important result of this chapter is the negative result about the lack of simple estimators for the simplified models. The derivation of simplified models of the type considered in chapter 3 requires essentially full knowledge of the dynamic system.

5 Simplification and Simultaneous Equation Models

Introduction

The theory we have developed has dealt strictly with state space difference equation models. Many models in use have a number of nondynamic equations and very often simultaneous equations determining the state variables. In this chapter we consider some of the implications that these types of equations have for the tools developed. In particular, we will consider the interpretation of simultaneous equations as arising from singular perturbation simplifications of continuous-time models. This discussion serves as a bridge between the theory developed and standard simultaneous equation models and also allows us to consider some of the transformations that are helpful in applying the tools developed to econometric models.

We begin by introducing the notation of simultaneous equation models. We show that the model simplification techniques considered cannot generally make use of the structural restrictions on simultaneous equation

models. We briefly review the notion of simultaneity and then present singular perturbation model simplification as a source of simultaneity. We show that this is in accord with the idea of placing equilibrium restrictions on an economic model. Viewing simultaneity as arising from a singular perturbation simplification has interesting implications for the structure of simultaneous equation systems which bear on the question of whether structure is retained in the simplified models. Finally, we present some useful transformations for models with nondynamic equations.

5-A Simultaneous Equation Models

The simplification techniques considered preserve certain structural characteristics of the "system." Specifically, for simplification based on the right vectors, if rows of $\underline{A}_{12}$ are zero then the corresponding row of $\underline{\tilde{A}}$ is the same as that of $\underline{A}_{11}$. an analogous result holds for simplifications based on the left vectors. In econometric models, however, the structural characteristics of the dynamic model may not match those of the simultaneous system. We consider why this is and its consequences for the model simplification techniques considered here.

Any simultaneous system of equations such as

$$\underline{Q}\underline{x}_t = \underline{S}\underline{x}_{t-1} + \underline{T}\underline{u}_t + \underline{e}_t \tag{5.1}$$

can be written in the form of the dynamic equations we have been considering by premultiplying by $\underline{Q}^{-1}$ to yield

$$\underline{x}_t = \underline{Q}^{-1}\underline{S}\underline{x}_{t-1} + \underline{Q}^{-1}\underline{T}\underline{u}_t + \underline{Q}^{-1}\underline{e}_t \, . \tag{5.2}$$

Unless $\underline{Q}$ has special properties, it is not possible to determine the effects of restrictions arising from economic theory on the transition matrix $\underline{Q}^{-1}\underline{S}$. More specifically, linear restrictions on the coefficients of the $\underline{Q}$, $\underline{S}$ and $\underline{T}$ matices correspond to complicated nonlinear restrictions on the $\underline{Q}^{-1}\underline{S}$ and $\underline{Q}^{-1}\underline{T}$ matrices. It is partly for this reason that equations of the form of 5.1 are often called structural, whereas equations of the form of 5.2 are often called reduced-form equations.

The model order-reduction techniques that have been considered are applicable to equation 5.2. The simplification techniques have been shown in chapter 3 to preserve some of the structural characteristics of the "system." Simplifications based on the right eigenvectors preserve the direct influence of the included states on a given included state. If no excluded state directly influences the given state, then the simplification will not change the influence of any included state on the given state. Simplification based on the left vectors preserves the direct influence of a given included state on the included states. If the given state does not directly influence any excluded state then the simplification will not change the influence of the given state on the included states.

The model simplifications considered preserve some of the structural restrictions on the dynamics matrix $\underline{A}$. However, the dynamics matrix $\underline{A}$, following equation 5.2, is given by $\underline{Q}^{-1}\underline{S}$ for a simultaneous model. The simplifications we have considered do not preserve the restrictions on the simplified model in the same manner that these are imposed on the simultaneous system. This is cause for some concern, because a

parsimonious model, when written in simultaneous equation form, may, to all appearances, lose this parsimony when written according to equation 5.2. The simplified model will be based on the original model in the form of equation 5.2, and therefore may have a larger number of parameters than the model it was based on.

Structural restrictions of the matrices of equation 5.1 are not easily transferable to the simplifed model. The simplified model is given by

$$\underline{\tilde{A}} = \underline{A}_{11} + \underline{A}_{12}\underline{R}_2\underline{R}_1^{-} \quad . \tag{5.3}$$

Rewriting this in terms of the partitions of the $\underline{Q}$ and $\underline{S}$ matrices,[†] we have

$$\begin{aligned}\underline{\tilde{A}} = {} & \{\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}\}^{-1}\underline{S}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\{\underline{Q}_{22} - \underline{Q}_{21}\underline{Q}_{11}^{-1}\underline{Q}_{12}\}^{-1}\underline{S}_{21} + \\ & \{\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}\}^{-1}\underline{S}_{12}\underline{R}_2\underline{R}_1^{-} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\{\underline{Q}_{22} - \underline{Q}_{21}\underline{Q}_{11}^{-1}\underline{Q}_{12}\}^{-1}\underline{S}_{22}\underline{R}_2\underline{R}_1^{-} \quad .\end{aligned} \tag{5.4}$$

It is possible to premultiply equation 5.4 by any nonsinugular matrix and recover some of the original elements of the structure. One matrix which suggests itself is the upper left-hand block of the $\underline{Q}$ inverse matrix. This premultiplication yields a dynamic system of the form

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[†]In order to construct the partitioned inverse of the $\underline{Q}$ matrix it may be necessary to reorder the equations. This will not change the nature of the restrictions or the $\underline{A}$ matrix for the model.

$$\{\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}\}\tilde{\underline{x}}_t = [\underline{S}_{11} + \underline{S}_{12}\underline{R}_2\underline{R}_1^{-}$$

$$- \{\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}\}\{\underline{Q}_{12}\underline{Q}_{22}^{-1}\{\underline{Q}_{22} - \underline{Q}_{21}\underline{Q}_{11}^{-1}\underline{Q}_{12}\}^{-1}(\underline{S}_{21} + \underline{S}_{22}\underline{R}_2\underline{R}_1^{-})]\tilde{\underline{x}}_{t-1} \tag{5.5}$$

$$+ \{\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}\}(\tilde{\underline{Bu}}_t + \tilde{\underline{De}}_t)$$

In this equation a number of submatrices of the original structural submatrices do appear. Unfortunately, the other elements of the structural matrices are also included in a nonlinear fashion. The original structural restrictions on the simplified model again manifest themselves as nonlinear restrictions. The same applies to restrictions which might be placed on the coefficients of the exogenous variables.

The realization of restrictions in a simplified model is similar to the realization of restrictions in a reduced-form model. Straightforward restrictions on the structure of a simultaneous model become, through the process of matrix inversion and multiplication, complicated nonlinear restrictions. The simplification theory considered cannot be used to yield a simplified simultaneous equation model with restrictions analogous to those placed on the original model.

In general, restrictions on the simultaneous "system" will not have a simple correspondence to restrictions of the simplified model. There are certain types of simultaneous equation models, however, for which the structural restrictions correspond almost directly to restrictions on the reduced-order model. In these situations the simplified models will retain

some of the structural restrictions. In the next sections we explore some of the causes of simultaneity in economic models and consider their implications for the structure of the simultaneities. The discussion shows that for simultaneities arising from equilibrium conditions the reduced-form model maintains many of the structural restrictions. The discussion also invites the notion of simultaneity as a result of a singular perturbation of a continuous-time dynamic system. Singular perturbation is a dynamic model simplification technique which was presented briefly in chapter 1 and will be discussed again below.

5-B The Nature of Simultaneities

To understand the relationship of the model order-reduction techniques to simultaneous equation models it is useful to consider the sources of simultaneities in econometric models. The use of simultaneous equation models in econometrics has sparked a great deal of controversy over the years. The debate has centered on whether it makes sense to consider models with bidirectional contemporaneous causality (Bentzel and Hansen 1954, Strotz 1960, Strotz and Wold 1960, Wold 1960).

The problem of measuring and defining causality is itself a very difficult one, and there has been an ample literature on this topic; Geweke (1982), Granger and Newbold (1977), Newbold (1982), Simon (1953) and Sims (1972) all discuss the notion of causality and its empirical application. A very important point in all the discussions is that the concept of causality is well specified only in the context of a mathematical model.

For certain types of models causality is a concrete notion. For the simultaneous equation model of the type given in equation 5.1 there is contemporaneous causality between two elements of $\underline{x}$ if there is a row of the $\underline{Q}$ matrix with nonzero entries in positions corresponding to those elements. If the matrix $\underline{Q}$ cannot be written (after permutation) as a triangular matrix then there is bidirectional contemporaneous causality. If the matrix $\underline{Q}$ can be written as a triangular matrix, it is then possible to solve for the elements of $\underline{x}$ recursively from predetermined variables.

For continuous-time models the situation is somewhat different. A causal-continuous time system of linear differential equations can be written as

$$\dot{\underline{s}} = \underline{Es} + \underline{Hu} + \underline{e} \tag{5.6}$$

with $\underline{s}$ denoting the states, $\dot{\underline{s}}$ their time derivatives, $\underline{u}$ an exogenous variable and $\underline{e}$ a continuous-time, stochastic process. The i'th variable is said to cause the j'th variable in this system if the (j,i) element of the matrix $\underline{E}$ is nonzero. The effect of the singular perturbation is to introduce simultaneous casuality, and this will be discussed below.

A number of works have considered simultaneity as arising from nonsimultaneous models. Koopmans (1950) considers some of the problems associated with continuous-time models and their estimation. Strotz (1960) takes the notion of a continuous-time model somewhat further and presents the view that discrete-time simultaneous equation models have their basis in sampled continuous-time models. Fisher (1970) considers the

representation of simultaneous models as identifying inherently nonsimultaneous interactions that occur on a time scale faster than the observations. Fisher dealt primarily with the nature of the averaging process as it determines the results of estimation. Senge (1976) considered some extensions of Fisher's analysis.

On a slightly different tack Bergstrom (1976) and, more recently, Gandolfo (1981) have also considered simultaneous equation systems as approximations to continuous-time systems. The approach taken in these works is different from that described in the previous paragraph in that there is no attempt to distinguish between separate time scales. The assumption is made that the continuous-time model is given by equation 5.6. The model is then estimated in the form[†]

$$(\underline{s}_t - \underline{s}_{t-\delta}) = \delta\{\underline{E}(\underline{s}_t + \underline{s}_{t-\delta})/2 + \underline{H}(\underline{u}_t + \underline{u}_{t-\delta})/2\} \ , \qquad (5.7)$$

on the assumption that the averaging will yield more accurate estimates. The resulting system is a simultaneous equation system.

In going from continuous- to discrete-time models, there are different problems associated with different sampling procedures. There are cases where it is more accurate to assume that observations are made on an integration over time, such as

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[†]The treatment of the driving variable is more difficult than the representation made here. The equations given are simply meant to convey the flavor of the approach.

$$\underline{o}_t = \int_{t-\delta}^{t+\delta} \underline{c}\, dt \ , \tag{5.8}$$

with $\underline{o}_t$ some available observation and $\underline{c}$ the underlying continuous variable. This problem is considered in Brewer (1973) and Gandolfo (1981). We will assume that the actual values of the states are observed, but at discrete intervals.

5-C Singular Perturbation and Simultaneity

We first present the singular perturbation technique of model simplification and show that the resulting simplified model contains a set of simultaneous equations. The discrete-time approximation to the singularly perturbed model is then presented for further discussion.

We assume that equation 5.6 can be used to describe the evolution of the economic states. The singular perturbation approach requires that these equations be partitionable, as

$$\begin{vmatrix} \dot{\underline{s}}_1 \\ \varepsilon\dot{\underline{s}}_2 \end{vmatrix} = \begin{vmatrix} \underline{F}_{11} & \underline{F}_{12} \\ \underline{F}_{21} & \underline{F}_{22} \end{vmatrix} \begin{vmatrix} \underline{s}_1 \\ \underline{s}_2 \end{vmatrix} + \begin{vmatrix} \underline{H}_1 \\ \underline{H}_2' \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{e}_1 \\ \underline{e}_2' \end{vmatrix} . \tag{5.9}$$

with ε a small parameter chosen so that $\underline{F}_{21}$ and $\underline{F}_{22}$ are of the same order of magnitude as $\underline{F}_{11}$ and $\underline{F}_{12}$. ε is referred to as the perturbation parameter, and setting it to zero will yield the singular perturbation simplification. Under the assumption that the $\underline{F}_{22}$ submatrix is stable (its eigenvalues all have negative real parts for a continuous-time system), the system in equation 5.9 may be approximated by assuming that the second set of states is near its equilibrium value given the first set of states

(Kokotovic, O'Malley and Sannuti 1976). That is, we assume that $\varepsilon \dot{\underline{s}}_2 = \underline{\tilde{e}}$, where $\underline{\tilde{e}}$ is an error term assumed to have properties similar to $\underline{e}_2'$.† Under these assumptions we write the singular perturbation form of the equations determining the second set of states as

$$\underline{F}_{21}\underline{s}_{1t} + \underline{F}_{22}\underline{s}_{2t} + \underline{H}_2'\underline{u}_t = \underline{\tilde{e}}_2 \cdot \tag{5.10}$$

This is clearly a simultaneous equation model.

From the above derivations there are some interesting observations that can be made on causality, as we previously discussed. In the continuous-time system the i'th state is said to have a direct causal influence on the j'th state if the i'th state influences the rate of change of the j'th. One of the effects of a singular perturbation on a continuous-time model is to introduce contemporaneous causality, which retains the characteristics of the causality in the continuous-time model. If, in the continuous-time model, there is bidirectional causality between two of the fast states, this will manifest itself as bidirectional contemporaneous causality in the singularly perturbed model. In this framework simultaneous causality is the result of nonsimultaneous causality in an underlying continuous system.

The states $\underline{s}_1$ in equation 5.10 may be taken as predetermined in the singularly perturbed model. The reason for this is that the $\underline{s}_1$ states are

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†We assume that the error $\underline{\tilde{e}}$ is a white noise error term, though it would be more accurate to allow $\underline{\tilde{e}}$ a correlation function commensurate with the dynamics of $\underline{s}_2$. This would not change the results.

assumed to evolve relatively slowly. Observation of the second set of states at intervals of duration δ will yield a discrete-time simultaneous equation system. The first set of states is now assumed to evolve according to the continuous-time equation

$$\dot{\underline{s}}_1 = \underline{F}_{11}\underline{s}_1 + \underline{F}_{12}\underline{s}_2 + \underline{H}_1\underline{u} + \underline{e}_1 , \tag{5.11}$$

which, on substituting the solution to equation 5.10 for $\underline{s}_2$, yields

$$\dot{\underline{s}}_1 = (\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21})\underline{s}_1 + (\underline{H}_1 - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{H}_2')\underline{u} + \hat{\underline{e}}_1 . \tag{5.12}$$

Since the first states are assumed to have much slower dynamics, it is more likely that these dynamics can be determined from the data. As we have mentioned, there are a number of methods for going from the above continuous system to a discrete-time system (Bergstrom 1976, Gandolfo 1981). For our purposes it is most convenient to assume that the continuous-time equation 5.12 can be reasonably approximated by replacing time derivatives with differences as in

$$\underline{s}_{1t} = \underline{s}_{1t-\delta} + \delta(\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21})\underline{s}_{1t-\delta} + \delta(\underline{H}_1 - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{H}_2')\underline{u}_{t-\delta} + \ddot{\underline{e}}_{1t} . \tag{5.13}$$

For the above equation to be a good approximation, we need δ to be small relative to the dynamics of $\underline{s}_1$ but large relative to the dynamics of $\underline{s}_2$.

For notational convenience and consistency with previous chapters, we will replace the time interval δ by 1 in the remainder of the discussion. We can combine equations 5.9 and 5.10 in order to obtain the system given

by

$$\begin{bmatrix} \underline{I} & \underline{0} \\ \underline{F}_{21} & \underline{F}_{22} \end{bmatrix} \begin{bmatrix} \underline{s}_{1t} \\ \underline{s}_{2t} \end{bmatrix} = \begin{bmatrix} \underline{I}+\underline{F}_{11} & \underline{F}_{12} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{s}_{1t-1} \\ \underline{s}_{2t-1} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ -\underline{H}_2' \end{bmatrix} \underline{u}_t + \begin{bmatrix} \underline{H}_1 \\ \underline{0} \end{bmatrix} \underline{u}_{t-1} + \underline{er}_t, \tag{5.14}$$

with er_t representing the two error terms $\underline{\hat{e}}_{1t}$ and $\underline{\tilde{e}}_{2t}$. Combining the lagged exogenous variables into the state variables (CCREMS 1983, chapter 2), this set of equations can be transformed to

$$\begin{bmatrix} \underline{I} & \underline{0} & \underline{0} \\ \underline{F}_{21} & \underline{F}_{22} & \underline{0} \\ \underline{0} & \underline{0} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{s}_{1t} \\ \underline{s}_{2t} \\ \underline{lu}_t \end{bmatrix} = \begin{bmatrix} \underline{I}+\underline{F}_{11} & \underline{F}_{12} & \underline{H}_1 \\ \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{s}_{1t-1} \\ \underline{s}_{2t-1} \\ \underline{lu}_{t-1} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ -\underline{H}_2' \\ \underline{I} \end{bmatrix} \underline{u}_t + \begin{bmatrix} \underline{er}_t \\ \underline{0} \end{bmatrix}. \tag{5.15}$$

These equations can be premultiplied by the inverse of the leftmost matrix to yield the state space representation that we have been using.

5-D Equilibrium Restrictions

In this section we consider simultaneous equations as manifestations of equilibrium restrictions and relate this interpretation to singular perturbation model simplification. The approach is close in spirit to the ideas put forward by Strotz (1960). The results indicate that this interpretation of the source of simultaneous equations gives a special structure to simultaneous equation models.

The consideration of the nature of simultaneous equations in economic models must be based on the aspects of economic theory that give rise to these simultaneities. The most important sources of simultaneities in economics are equilibrium requirements. In each market of an economic

system the equilibrium price must be such that supply and demand are in balance. Underlying this equilibrium view of the world is a notion that disequilibrium situations are quickly and dynamically corrected. Specification of the disequilibrium mechanisms is very difficult but general assumptions about their characteristics can easily be made (Arrow and Hahn 1971, Fisher 1983, Samuelson 1943). Essentially, we assume disequilibrium situations are rectified very quickly to produce an equilibrium (or near equilibrium).

This notion of adjustment is very much in line with our development of the singular perturbations simplification. We assume that the underlying economic model can be represented by a system of differential equations in the form of equation 5.9 and that the singular perturbation of the system can be performed accordingly. The fast states consist of the variables which are assumed to be in equilibrium. The interest rate, for example, could be assumed to adjust very quickly in order to clear the money market, with money moving relatively slowly.

The equations resulting from a singular perturbation (5.11) look very much like a standard simultaneous equation model. Two things need consideration in these equations. The first is whether the singular perturbation derivation of the simultaneous equation system yields equations which are the same as those that would be arrived at by the imposition of the equilibrium conditions of the theory. Second, what implications do the above results have for dynamic analysis of simultaneous equations models? A simple example will make this clear. The example is quite similar to one presented in Samuelson (1943, chapter 9) but has a

different focus.

A Simple Example

Consider the modeling of supply and demand in a single market. Suppose that in the continuous-time framework supply is given by

$$q^s = \alpha p + \beta c \ , \tag{5.16}$$

with p representing price and c a cost variable. Demand is given by

$$q^d = \kappa p + \gamma y \ , \tag{5.17}$$

with y representing income. The equilibrium condition for this market can be summarized by the equations

$$p = \frac{\gamma y - \beta c}{\alpha - \kappa} \tag{5.18}$$

and

$$q = \frac{\alpha\gamma y - \kappa\beta c}{\alpha - \kappa} \ . \tag{5.19}$$

Now let us postulate that price and quantity adjust according to the following rules:

$$\dot{p} = \phi \ (q^d - q^s) \tag{5.20}$$

and

$$\dot{q} = \theta \ (\ (q^d + q^s)/2 - q) \ , \tag{5.21}$$

with ϕ and θ both constant. These equations could conceivably be imbedded

in a larger model with the income and cost determined endogenously. Price and quantity are assumed to be fast, stable variables. The following equation describes the dynamics (for ϕ and θ positive, stability is guaranteed as long as $\kappa - \alpha < 0$):

$$\begin{vmatrix} (1/\phi)\dot{p} \\ (1/\theta)\dot{q} \end{vmatrix} = \begin{vmatrix} \kappa - \alpha & 0 \\ (\alpha+\kappa)/2 & -1 \end{vmatrix} \begin{vmatrix} p \\ q \end{vmatrix} + \begin{vmatrix} \gamma & -\beta \\ \gamma & \beta \end{vmatrix} \begin{vmatrix} y \\ c \end{vmatrix} . \tag{5.22}$$

If we assume that ϕ and θ are both large, the dynamics of equation 5.18 will be very fast. Setting the left-hand side equal to zero, we get the singular perturbation conditions

$$\begin{vmatrix} p \\ q \end{vmatrix} = 1/(\alpha-\kappa) \begin{vmatrix} 1 & 0 \\ (\alpha+\kappa)/2 & \alpha-\kappa \end{vmatrix} \begin{vmatrix} \gamma & -\beta \\ \gamma & \beta \end{vmatrix} \begin{vmatrix} y \\ c \end{vmatrix} , \tag{5.23}$$

which on solving, are the same as the equilibrium conditions given in equations 5.18 and 5.19. This is indicative of the nature of the simultaneous estimate. Though it is possible to estimate α, β, κ and γ, it is not possible to estimate ϕ and θ or their relative magnitudes. In general it is not possible to recover the original dynamics given the simultaneous estimation. However, it is clear that the equilibrium restrictions for this model are equivalent to the singular perturbation form.

Structure Imposed by Singular Perturbations

To generalize, the use of singular perturbations does indeed yield equilibrium conditions that are consistent with the equilibrium conditions often imposed in economic models. In addition, it is not generally

possible to reconstruct the dynamic model from the simultaneous form of equation 5.15.

Premultiplying the second row in equation 5.15 by any diagonal matrix with all positive entries will yield a system in which all the zero and sign restrictions implicit in $\underline{F}_{21}$ and $\underline{F}_{22}$ hold. Thus, the premultiplication does not alter the equilibrium conditions, but it does alter the dynamics. There are a host of underlying dynamic models that will yield the same equilibrium conditions. Observation of equilibrium conditions alone cannot be used to discriminate among these models. The fast dynamics are inherently unidentifiable. More specifically, the relative magnitudes of the different eigenvalues associated with the fast dynamics cannot be determined.

The above analysis was carried out for a linear dynamic model. However the results also hold for nonlinear, underlying models that yield linear equilibrium conditions. Singular perturbations can also be carried out on nonlinear models, though it is somewhat more difficult to check the conditions required for stability of the fast dynamics.

The more central issue of this chapter concerns the implications of simultaneity for the theory developed. The dynamics matrix describing the internally generated dynamics of the singularly perturbed system is clearly not of full rank. We consider equation 5.15 after premultiplying by the inverse of the leftmost matrix. The inverse of this matrix is given by

$$\begin{vmatrix} \underline{I} & \underline{0} & \underline{0} \\ -\underline{F}_{22}^{-1}\underline{F}_{21} & \underline{F}_{22}^{-1} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{vmatrix} . \tag{5.24}$$

Premultiplication of equation 5.15 by this matrix yields the following dynamic system:

$$\begin{vmatrix} \underline{s}_{1t} \\ \underline{s}_{2t} \\ \underline{lu}_t \end{vmatrix} = \begin{vmatrix} \underline{I}+\underline{F}_{11} & \underline{F}_{12} & \underline{H}_1 \\ -\underline{F}_{22}^{-1}\underline{F}_{21}(\underline{I}+\underline{F}_{11}) & -\underline{F}_{22}^{-1}\underline{F}_{21}\underline{F}_{12} & -\underline{F}_{22}^{-1}\underline{F}_{21}\underline{H}_1 \\ \underline{0} & \underline{0} & \underline{I} \end{vmatrix} \begin{vmatrix} \underline{s}_{1t-1} \\ \underline{s}_{2t-1} \\ \underline{lu}_{t-1} \end{vmatrix} + \tag{5.25}$$

$$\begin{vmatrix} \underline{0} \\ \underline{F}_{22}^{-1}\underline{H}'_2 \\ \underline{I} \end{vmatrix} \underline{u}_t + \begin{vmatrix} \underline{e}_t \\ \underline{0} \end{vmatrix}$$

The nonzero eigenvalues of the state transition matrix for this system are the same as the eigenvalues for the transition matrix of equation 5.12.† The reduced-order model is meant to match the dynamics of the system as the speed of the fast dynamics goes to infinity (as ε goes to 0). Therefore, the eigenvalues of the transition matrix given in 5.25 do not match those of the original system. However, the stability characteristics will be the same under certain conditions.††

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†The eigenvalues differ by one, since the continuous dynamic model is being approximated by a discrete dynamic model.

††Specifically, if the eigenvalues of $\underline{F}_{22}$ and of $(\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21})$ have negative real parts, then, if ε is less than some ε, the original system matrix $\underline{E}$ will be stable (Vidyasagar 1978 section 4.3).

All the nontrivial dynamics in 5.25 are contained in the first set of rows of equation 5.15. This suggests that, given the approach we have taken to the approximation of slower dynamics, simultaneities that arise from requirements of equilibrium should be of the form

$$\underline{F}_{22}\underline{s}_{2t} = \underline{F}_{21}\,\underline{s}_{1t} - \underline{F}_{21}\,\underline{u}_t + \tilde{\underline{e}}_t \,, \tag{5.26}$$

with no lagged endogenous variables appearing on the right-hand side ($\underline{s}_{1t}$ is taken as predetermined). Equation 5.26 does not tell us anything about the dynamics of the full system.

The model-order-reduction techniques considered in chapter 3 can be directly applied to equations of the form of equation 5.25. In such a fully recursive block of equations no simultaneities arise, and the coefficients of the equation correspond directly to the coefficients that determine the dynamics. In this case economic restrictions on the structural equations manifest themselves directly in the dynamics matrix. Thus the model-reduction techniques do maintain structural restrictions.

Accurate Estimation as a Source of Simultaneity

We have also mentioned that systems of simultaneous equations can arise in the design of better estimators of sampled continuous-time systems. In this case the model simplifications that we have considered can be performed directly on the original continuous-time system. All of the mathematics of simplification carry over exactly in the determination of the simplifed dynamics matrix. The restrictions that exist on the original dynamics matrix will be imposed on the simplified dynamics matrix.

5-E Nondynamic Equations

Our discussion has indicated that simultaneities arising from equilibrium restrictions produce models with certain structure. Specifically, variables involved in simultaneities can be written in terms of only current values of the state variables. It is also quite common for economic models to have a number of nondynamic identities. For example, in the MACRO model, which we considered in chapters 3 and 4, income was given by the equation

$$Y_t = C_t + I_t + G_t \ . \tag{5.27}$$

In analyzing the example we substituted the above expression for Y wherever Y occurred and did not include Y as a state variable.

A variable defined by a nondynamic equation does not add to the dynamic complexity of a model. The components of a nondynamically defined variable do enter the dynamic structure, and it is the feedback through the components that determines the dynamic character of the model. Consideration of the components yields the information that is used to determine what the important feedback channels are. By including the nondynamically defined variables, accuracy is lost, because the effect of the components of a nondynamically defined variable is partly captured by that variable.

The exclusion of the nondynamic equations can be done through substitution, as it was in the example presented. It is also possible, and

in some circumstances more convenient, to include the nondynamically defined variables in calculating the eigenvalues for the model and then correct for this inclusion. If there are p nondynamic equations in a simultantous equation model, the model can be written in the form

$$\begin{vmatrix} \underline{Q}_{11} & \underline{Q}_{12} \\ \underline{Q}_{21} & \underline{Q}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t} \\ \underline{x}_{2t} \end{vmatrix} = \begin{vmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{0} & \underline{0} \end{vmatrix} \begin{vmatrix} \underline{x}_{1t-1} \\ \underline{x}_{2t-1} \end{vmatrix} + \begin{vmatrix} \underline{T}_1 \\ \underline{T}_2 \end{vmatrix} \underline{u}_t + \underline{\varepsilon}_t \qquad (5.28)$$

with $\underline{x}_2$ of dimension p. It is possible to rearrange the columns of the above equations (that is, permute the states) without having any effect on the structure of the model. Since the $\underline{Q}$ matrix is assumed nonsingular, it is possible to choose $\underline{Q}_{22}$ to be nonsingular. The permutation which will normally accomplish this is to put the variables defined by nondynamic equations into $\underline{x}_2$; however, other permutations might also work.

There are two ways to remove the effect of the nondynamic variables in equation 5.28. The first is to substitute

$$\underline{x}_{2t} = -\underline{Q}_{22}^{-1}\underline{Q}_{21}\underline{x}_{1t} + \underline{Q}_{22}^{-1}\underline{T}_2\underline{u}_t + \underline{Q}_{22}^{-1}\underline{\varepsilon}_{2t} \qquad (5.29)$$

for $\underline{x}_2$, yielding the smaller model

$$(\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})\,\underline{x}_{1t} = (\underline{S}_{11} - \underline{S}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})\,\underline{x}_{1t-1} + \qquad (5.30)$$

$$(\underline{T}_1 + \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{T}_2)\underline{u}_t + \underline{S}_{12}\underline{T}_2\underline{u}_{t-1} + \hat{\underline{\varepsilon}}_t.$$

This yields the dynamics matrix

$$(\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})^{-1} (\underline{S}_{11} - \underline{S}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21}) \cdot \tag{5.31}$$

The second way to remove the effect of the nondynamic equations is to solve for the eigenvectors of the larger system, and then alter these. The $\underline{A}$ matrix for the simultaneous equation model can be written, using the partitioned inverse of $\underline{Q}$, as

$$\begin{vmatrix} (\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})^{-1}\underline{S}_{11} & (\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})^{-1}\underline{S}_{12} \\ -\underline{Q}_{21}\underline{Q}_{11}^{-1}(\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})^{-1}\underline{S}_{11} & -\underline{Q}_{21}\underline{Q}_{11}^{-1}(\underline{Q}_{11} - \underline{Q}_{12}\underline{Q}_{22}^{-1}\underline{Q}_{21})^{-1}\underline{S}_{11} \end{vmatrix} . \tag{5.32}$$

It is straightforward to show that if $\underline{v}$ is a right eigenvector of the matrix given in equation 5.31 associated with a nonzero root, then $\underline{v}_1$ is a right eigenvector of the matrix given in equation 5.32 and is associated with the same root. If $\underline{w}$ is a left eigenvector of the matrix given in equation 5.30 associated with a nonzero root, then $\delta_1 - \underline{w}_2\underline{Q}_{21}\underline{Q}_{11}^{-1}$ is a left eigenvector of the matrix given in equation 5.31 and is associated with the same nonzero root.

With these expressions for the vectors of the two matrices we can move from one representation to another without trouble.

5-F Summary

In this chapter we have considered dynamic model simplification relative to simultaneous equation models. The structural restrictions in a simultaneous equation model are not generally maintained in the simplified models developed in chapter 3. However, simultaneities that arise from equilibrium conditions do have structural restrictions that can be maintained in the simplified models. We have shown this by considering simultaneities as arising from equilibrium conditions in an underlying continuous-time model. This was done using the singular perturbation simplification of a model and interpreting it in terms of simultaneous equations. Finally, we derived some useful identities which are satisfied by the eigenvectors of a model with nondynamically defined variables such as might arise from simultaneity.

6 Simplifying the Michigan Quarterly Econometric Model

Introduction

In this chapter we will apply some of the techniques that have been considered in the previous chapters to a medium-size econometric model. The model chosen for this work is the model developed by the University of Michigan research seminar in quantitative economics. For the simplification, we concentrate on business-cycle frequency oscillatory modes. The application of the simplification techniques considered in this thesis serves to give some insights into the structures in the model that generate the modes of interest. In addition, the application allows for the evaluation of the different techniques in a useful setting.

This chapter begins with an overview of the econometric model to which the simplifications are applied. This overview is necessarily brief, and other references for further discussion of the model are given. Following

this overview we define the modes of interest and determine the states of interest. The different approaches to measuring the importance of states in generating the behavior modes of interest are considered, and the features of the different approaches that become apparent in this application are summarized. Given the choice of states, two simplified models are developed, one is based on only the modes of interest, and the second incorporates other, closely related, modes of the model. The structure of these simplifications and the features of MQEM that give rise to this structure are then discussed.

The simplified models are simulated for comparison with the response of the full model.† Following this, the transformations of the state variables required for the simplified model to be in accord with the data, as discussed in chapter 4, are considered. These transformations alter the data substantially, and the transformed data are compared with the original data in terms of their general characteristics. The transformed data are then used in the estimation of the simplified model. The simplified model is also estimated using historical data for the endogenous variables.

6-A The MQEM Model

The model will be refered to as the Michgan Quarterly Econometric Model or simply MQEM. The model is a "medium-size" econometric model consisting of 63 behavioral equations and 99 identities. Of the

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†The simulations we consider are all deterministic. Deterministic simulation is a useful means of comparing one model's behavior to another's, given a set of inputs. The output of these deterministic simulations are not compared to the historical values of the variables (Hendry and Richard 1982).

identities, 45 define percentage rates of changes in quantities and prices. These variables are used only to monitor the model and do not influence other endogenous variables. Of the remaining 54 identities all but 9 are defined solely in terms of current variables, and therefore the variables defined in these equations are not considered as separate states.

The model is broken down into 6 sectors-- a price sector, a productivity and employment sector, a consumption and investment sector, an income flows sector, a monetary sector and an output composition sector. Ninety-six exogenous variables influence the model, and 40 of these are dummy variables. Of the remaining variables, 24 are discretionary variables of governments and their agencies (primarily federal). There are 7 exogenous price variables in the model, these are average prices for imports, exports, gasoline, cars, farm products and raw materials. Figure 6-1 gives a rough idea of the breakdown of the feedback between the sectors of the model. The figure is based on the strucure of the $\underline{A}$ matrix for the linearized model. For a more complete description of the model the reader is referred to Belton, Hymans and Lown (1981) and Hymans and Shapiro (1974).

The MQEM model is a nonlinear, simultaneous equation model. In analyzing the model, we use a linearized version of the original nonlinear model. The linearization was performed by evaluating the partial derivatives of the nonlinear model for the second quarter of 1973. The values of the partial derivatives of the nonlinear model depend on the values of the exogenous and endogenous variables at the time of

Sector Being Influenced	Sector Influencing: Prices	Consumption and Investment	Poductivity and Employment	Income Flows	Monetary	Output Composition
Prices key variables	strong PPNF JCMH	-	medium REM QMHT	-	weak RAAA	weak JIPM JCAP
Consumption and Investment k.v.	strong PC PCS PCN PPNF JCMH UCEAVEQ	strong GNP72 GNPAVEQ	-	strong YPWS YPERM72 YD72	medium RAAA	weak JIPM JCAP
Productivity and Employment k.v.	-	strong GNP72	strong RUM RUC QMHT7	-	-	weak JIPM JCAP
Income Flows key variables	medium JCMH	medium GNP72	medium RUM RUC QMHT7	strong YPWS YP YUNB	weak RG5	-
Monetary key variables	strong PPNF	-	-	-	strong RAAA RTB M1BPLUS	-
Output Composition key variables	-	-	strong CS72 IPDQ72 IBFNC72	-	-	weak JIPM JCAP FSMF72

Variable Definitions

CS72 - Consumption of services in constant dollars
FSMF72 - Final sales of manufactured goods
GNP72 - Gross national product in constant dollars
GNPAVEQ - Average of GNP72
IBFNC72 - Non-residential investment in structures
IPDQ72 - Investment of producers in durables
JCAP - Index for industrial production capacity
JCMH - wage index
JIPM - Index for industrial production rate
M1BPLUS - Money supply
PC - Consumer price index
PCN - Nondurables price index
PCS - Services price index
PPNF - Business price deflator
QMHT - Trend growth in productivity
RAAA - Corporate bond interest rate
RG5 - 5 year government bond rate
RTB - Treasury bill interest rate
REM - Employment rate
RUC - Unemployment rate for all workers
RUM - Males over 20 unemployment
UCEAVEQ - Average cost of capital relative to labor
YD72 - Disposable income
YP - Personal income
YPWS - Wages and salaries
YUNB - Unemployment benefits

Figure 6-1 Overview of the Feedback Structure in MQEM

linearization. Abnormal values of the endogenous and exogenous variables could therefore have an impact on the structure of the linearized model. In order to avoid this possibility the historical values of the variables were smoothed prior to linearization. The linearization and the resulting calculation of the eigenvalues and vectors is that used by Kuh (1983) and is discussed in more detail there.

When linearized and put into state space form, MQEM has 113 additional endogenous variables, which must be constructed to accommodate lags of greater than one period. This would raise the dimension of the model to 275, but, as we have mentioned, 45 of the variables are used only for monitoring purposes, and 45 are defined in terms of current variables only. This lowers the effective dimension of the model to 185. Of the 185 potentially nonzero roots, eigenvectors were calculated for the largest 123, the smallest of these had a magnitude of .33. For a more complete description of the overall dynamic character of the model the reader is referred to Kuh (1983). For consideration of the computational issues involved in the partial root-vector decomposition of the model the reader is referred to CCREMS (1983).

6-B Modes and Variables

In this section we will introduce the modes of interest and then describe the determination of the states that were included in the simplified models. The simplifications we consider are all simplifications of the linearized version of the existing model. They help explain the source of the modes in the existing model. Determining the source of the modes can be used in judging the quality of the original model. If the

sources of the behavior generated by the model are not reasonable then the model model is failing to inform us about some aspect of reality. This will be commented on briefly, but the focus in this chapter is the relationship of the simplified models to the model from which they derive.

Modes

The modes of interest for the simpification are oscillatory modes of business-cycle frequencies. Business cycles can be defined broadly as periodic deviations from trend in many of the aggregate measurements of economic activity and performance. A great deal has been written on the business cycle (for example, Haberler 1936, Hicks 1950, Metzler 1941, Mitchell 1951 and Samuelson 1939), and determining the underlying causes of business cycles is still a matter of some interest [Lucas (1981) has a general discussion of some of the more recent work]. It is of some interest to apply the techniques developed in previous chapters to investigate what MQEM has to tell us about business-cycle movements.

We consider the roots of MQEM which are complex and imply fluctuations with a periodicity of between 2 and 10 years. This is the range over which "a business cycle" normally occurs. According to the U.S. Bureau of Economic Analysis, the shortest business cycle was 1.5, years ending in January 1920, and the longest was 9.5, years ending in December of 1969 (BCD 1982). Ten roots of MQEM have implied periods in this range. The specific roots are shown in figure 6-2. For these roots the shortest implied fluctuations are of 9.6 quarters or slightly under 2.5 years in duration. The longest implied fluctuations are of 42.6 quarters or slightly over 10.5 years in duration. All of the roots considered have

magnitude less than one and are therefore associated with stable behavior modes.

Root Index[†]	Real Part	Complex Part	Magnitude	Period (Quarters)
45	.738	.281	.790	17.3
47	.764	.113	.772	42.6
52	.561	.429	.706	9.62
53	.597	.354	.694	11.74
61	.595	.192	.626	20.1

Figure 6-2 The Eigenvalues of the Modes of Interest[††]

Choosing the States

In determining the states to be retained in the simplified model the various techniques discussed in chapter 3 were considered. We first briefly review these techniques and their motivation and then summarize the results of their application. This summary provides some indication of the strengths and weaknesses of the different techniques.

Generalized Participation Factors

The generalized participation factor is a matrix generalization of the scalar participation factors used in Perez (1981). The Generalized participation factor is defined as

$$\begin{vmatrix} \underline{R}_{11}\underline{L}_{11} & \underline{R}_{11}\underline{L}_{12} \\ \underline{R}_{21}\underline{L}_{11} & \underline{R}_{21}\underline{L}_{12} \end{vmatrix} . \qquad (6.1)$$

[†]The roots of MQEM have been indexed by magnitude. The root index is included to indicate the positions of the roots of interest relative to the other roots of the model.

[††]For the reader inclined to think in continuous time, the equivalent eigenvalues, with time still measured in quarters, are given by $-.23 \pm .36j$, $-.26 \pm .15j$, $-.35 \pm .65j$, $-.36 \pm .53j$ and $-.47 \pm .31$

If the generalized participation factor takes the form

$$\begin{vmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{0} \end{vmatrix}, \tag{6.2}$$

the model is what is called "exactly simplifiable". The diagonal elements of the generalized participation factors are the sums across the included modes of the scalar participation factors.

The off-diagonal elements of the generalized participation factors are subject to the influence of scaling the states variables. To overcome this, the factors can be scaled by the elements of the $\underline{A}$ matrix. To do this, the (i,j) element of the generalized participation factor is multiplied by the (j,i) element of the $\underline{A}$ matrix. In the results reported, the generalized participation factors were broken into two components. The diagonal elements were considered without any scaling, which is the same as considering the sums of participation factors. The off-diagonal elements were weighted by the elements of the $\underline{A}$ matrix according to the method given.

Cross-Modal Participation Factors

The cross-modal participation factors are meant to capture feedback channels that opperate through more than one mode. As an example of such a feedback channel consider the case in which a state excites a mode, which excites a second state, which excites a second mode, which excites the original state. The scalar participation factors for the two states may be zero in such a situation, but the states may nonetheless be important in the feedback. The cross-modal participation factor is defined on the basis

of the right or left eigenvector for the mode of interest. The right-vector-based cross-modal participation factors for a mode k are given by

$$\underline{r}_i^k \underline{l}_i^p \quad \text{for } p = 1,2, \ldots N . \tag{6.3}$$

There is a similar expression for the left-vector-based cross-modal participation factor.

The cross-modal participation factor gives information both about the states that seem to be important and about additional modes that seem to be coupled with the mode of interest. The cross-modal participation factors arise from the consideration of the requirements for exact simplification. The cross-modal participation factors will be changed by the normalization of the vectors that is used. In our calculations the vectors against which the vector of interest was compared were normalized length 1. For example, in the right-based cross-modal participation factors as given in equation 6.3, $\underline{l}^p$ would be normalized to length 1 for all p. The requirements for exact simplification are that either the right-based or the-left based cross-modal participation factors for all the modes of interest be zero for all the excluded states.

Eigenvalue Elasticities

The eigenvalue elasticities are simply the percentage change in an eigenvalue caused by a one-percent change in an entry of the $\underline{A}$ matrix. The eigenvalue elasticities are similar to the generalized participation factor but are defined for a single mode. We considered the elasticity of response of the magnitude of the eigenvalue and the elasticity of response

of the period associated with the eigenvalue (CCREMS 1983, also appendix A3).

Applying the Tools

In figure 6-3 the results of the application of the various tools are described. Next to each state variable are written the symbols corresponding to the method indicating that the state variable is important. The underlined state variables are those that were eventually chosen for inclusion in the simplified models. The variables enclosed in curly brackets ({}) are construct variables defined by nondynamic equations. These variables were considered only in the evaluation of the root sensitivities.

None of the methods selects out a small number of the states. This is an indication of the fact that a large number of states are involved in generating the modes of interest. MQEM is some distance from being an exactly smplifiable model with respect to the modes of interest. Though this makes it difficult to develop a simplified model that does not require a great deal of approximation, it does yield good information about how the different selection tools perform empirically.

The most striking pattern arises in the consideration of the cross-modal participation factors. When the cross-modal participation factor for one mode indicates that a state is important, then it is likely that the cross-modal participation factor for the other modes will also indicate that the state is important. This same pattern holds true with respect to the other modes which the cross-modal participation factors indicate to be

Variable	Indications of Importance
AUTOS	++@@!*$#####
CDAN72	#####
CDFE72	$
CDO72	$$
CN72	$$$
CS72	$$$
GDEBTP	$$$
{GNP}	+@
GNPAVEQ	
{GNP72}	++@@
HOUSES	####
IBFNC72	@@$$##
{IBF72}	+
IINV72	!$##
IPDO72	++@@@$$$$####
IPDQ72	+@@@@!**$$$##
IRC72	+++@@@$####
JCMH	@$$$$$
JIPM	!##
M1BPLUS	+!*$$#
M2PLUS	!*$$$###
M72	!$$$###
{PC}	+
PCDO	####
PCN	@@$$$####
PCS	++@!**$$$$$##
PG	####
PINC	####
PIPDAG	####
PIPDO	####
PIPDQ	+$###

Variable	Indications of Importance
PPNF	+@@@!*$$$$$#####
QHT1	!*
QMHT	!*
RAAA	$
RCD	!####
RCP	####
RPPERM	@!
RUG	####
RUM	!*$$$####
SINV72	!$$
TCF	#####
TCSL	####
TIBSL	#####
TP	#####
TSIF	$#####
UCEAVEQ	@!
UCKPDQ	!
YCP	#####
YOL	$
YPDIV	$$#
YPERM72	+@!*
YPWS	@@@!*$$#####
YUNB	$#####
AUTOS(-1)	#####
CDAN72(-1)	#####
GDEBTP(-1)	$$$
GNPAVEQ(-1)	+*
GNPAVEQ(-2)	++@!**#
GNP72(-1)	+!$###
GNP72(-2)	+!###
GNP72(-3)	##
GNP72(-4)	#

Symbol	Value	Meaning
+	$\frac{a_{ij}}{\lvert\lambda\rvert}\frac{\partial \lvert\lambda\rvert}{\partial a_{ij}}$	Eigenvalue Elasticity Magnitude Sensitivity
@	$\frac{a_{ij}}{\text{period}}\frac{\partial\ \text{period}}{\partial a_{ij}}$	Eigenvalue Elasticity Period Sensitivity
!	$\Sigma_{k=1}^{n}\ \underline{l}_i^k \underline{r}_i^k$	Diagonal Elements of Generalized Participation Factors
*	$\Sigma_{k=1}^{n}\ a_{ij}\underline{l}_i^k \underline{r}_j^k$	A Weighted Off-Diagonal Elements of Generalized Participation Factors
$	$l_i^k r_i^p \quad p = 1, \ldots N$	Left-Vector-Based Cross-Modal Participation Factor
#	$\underline{r}_i^k \underline{l}_i^p \quad p = 1, \ldots N$	Right-Vector-Based Cross-Modal Participation Factor

Variable	Indications of Importance
GTRP(-1)	$
HOUSES(-1)	#####
HOUSES(-2)	#####
IBF(-1)	!
IBF72(-1)	**
IBF72(-2)	**
IBF72(-3)	*
IBF72(-4)	!
IRC72(-1)	+!#####
JCMH(-1)	@$$$
JCMH(-2)	@$$
JCMH(-3)	@$
JCMH(-4)	+@@
M1BPLUS(-1)	!*
M2PLUS(-1)	!*$$#
PC(-1)	$$$##
PC(-2)	+@$$
PC(-3)	+!$$
PC(-4)	+!
PC(-5)	+!*
PC(-6)	+!**
PC(-7)	+!*
PC(-8)	+!
PCDFE(-1)	#
PCDO(-1)	####
PCS(-1)	+@!**$$$$##
PINC(-1)	##
PIPDAG(-1)	##
PIPDO(-1)	####
PIPDQ(-1)	$$####
PPNF(-1)	+@@!*$$$$###
PPNF(-2)	+@!**$$$##
PPNF(-3)	!*$
PPNF(-4)	$
QHT1(-1)	!**
QHT1(-2)	+!**
QHT1(-3)	+!**
QHT1(-4)	+!**
QHT1(-5)	+!**
QHT1(-6)	+!**
QMHT(-1)	!**
QMHT(-2)	!**
QMHT(-3)	*
RPPERM(-1)	@!
RUM(-1)	!**$$$#
UCEAVEQ(-1)	@!
UCEAVEQ(-2)	@!
UCEAVEQ(-3)	++@@@!*
UCEAVEQ(-4)	++@@@@!*$##
UCKPDQ(-1)	!
UCKPDQ(-2)	!
UCKPDQ(-3)	!
YCBT(-1)	#####
YOL(-1)	$##
YPERM72(-1)	+@!*$##
YT72(-1)	!

Symbol	Value	Meaning
+	$\frac{a_{ij}}{\lvert\lambda\rvert} \frac{\partial \lvert\lambda\rvert}{\partial a_{ij}}$	Eigenvalue Elasticity Magnitude Sensitivity
@	$\frac{a_{ij}}{\text{period}} \frac{\partial \text{ period}}{\partial a_{ij}}$	Eigenvalue Elasticity Period Sensitivity
!	$\Sigma_{k=1}^{n} \underline{l}_i^k \underline{r}_i^k$	Diagonal Elements of Generalized Participation Factors
*	$\Sigma_{k=1}^{n} a_{ij} \underline{l}_i^k \underline{r}_j^k$	A Weighted Off-Diagonal Elements of Generalized Participation Factors
$	$l_i^k r_i^p \quad p = 1, \ldots N$	Left-Vector-Based Cross-Modal Participation Factor
#	$\underline{r}_i^k \underline{l}_i^p \quad p = 1, \ldots N$	Right-Vector-Based Cross-Modal Participation Factor

Figure 6-3 Measurement of the Important States

important relative to the modes of interest. The reason for this is that the results are dominated by the matrix to which the vector for the mode of interest is being applied. For example, in comparing the left vector for mode 45 to the matrix of right vectors, it seems to be the matrix of right vectors which dominates the results.

This pattern in the cross-modal participation factors is an indication of the coupling of the model with respect to the modes of interest. The right and left vectors for the modes of interest have a large number of entries that are quite similar in magnitude. As a result the variation of the entries in the vectors of interest is far exceeded by the variation of the entries in the other vectors in the model. This conclusion seems to be borne out by the fact that the vector of interest is rarely indicated to be of importance in the cross-modal particiation factor.

It should be remembered that the cross-modal participation factors are being used to check whether one of two conditions holds. Roughly speaking, a state is indicated as important only if it shows up in both the left- and right-vector-based cross-modal participation factors. Thus, the indications of the cross-modal participation factors are not so uninformative as might at first appear to be the case. The number of cases in which both the left- and right-vector cross-modal participation factors indicate that a state is important is much smaller than for either measure in isolation.

The other important pattern that seems to emerge is that the eigenvalue elasticities indicate states that are also indicated by the

generalized participation factors. This is not surprising, for the generalized participation factors are sums of the eigenvalue elasticities. However, the observation does indicate that the result of including more than one mode in the search for the states of interest does not exclude potentially important states. This could happen if the eigenvalue elasticities were of opposite sign and canceled. This seldom happens, and it is noteworthy that indications of the off-diagonal entries of the eigenvalue elasticities often correspond to indications of the on-diagonal entries of the generalized participation factors.

The results we have achieved and the relative merits of the various measures of the importance of a state are specific to this appliciation. Application to different models could conceivably yield far different patterns. If the model these tools are applied to is exactly simplifiable, the results will all converge. MQEM is not exactly simplifiable and the different methods yield different results. The different methods do, however, overlap.

The States

The final choice of states was made on the basis of the combined indications of the various methods. The most weight was given to the indications of the generalized participation factors and the least to the cross-modal participation factors. The 15 states chosen and their definitions appear in figure 6-4. Most the states chosen were indicated to be important from many perspectives as can be seen in figue 6-3. The exceptions to this are imports (M72), the unemployment rate (RUM) and the average relative cost of capital (UCEAVEQ). These were chosen primarily on

the basis of the indications of the generalized participation factors, which suggested they were quite important.

AUTOS - Retail sales of new automobiles (Millions of cars per year)

IPDQ72 - Real nonresidential investment in producers' durables (Billions of 1972 dollars per year)

$GNPAVEQ_{-2}$ - Six quarter moving average of GNP (Billions of 1972 dollars per year, lagged twice)

M1BPLUS - Money (M1B) plus all savings deposits at depository institutions, such as savings banks, not included in M1B (Billions of current dollars)

M72 - Imports (Billions of 1972 dollars per year)

PCS - Price deflator for consumption of services (index, 1972 = 100)
PCS_{-1} (lagged once)

PPNF - Deflator for business nonfarm GNP (index, 1972 = 100)
$PPNF_{-1}$ (lagged once)
$PPNF_{-2}$ (lagged twice)

RUM - Unemployment rate for males 20 years of age and over (percentage)
RUM_{-1} (lagged once)

$UCEAVEQ_{-4}$ - The user cost of capital relative to the average wage rate (dollars per capital unit per year relative to index of dollars per labor unit per year, lagged 4 quarters)

$YPERM72_{-1}$ - Permanent income (Billions of 1972 dollars per year, lagged once)

YPWS - Private wages and salaries (Billions of current dollars per year)

Figure 6-4 States included in the Simplified Models

Price variables seem to play an important role in the modes of interest. Six of the 15 selected variables ($PCS_{0,-1}$, $PPNF_{0,-1,-2}$ $UCEAVEQ_{-1}$) are price-related variables. This is a much larger proportion than for the model as a whole, in which fewer than 20% of the variables and approximately 25% of the estimated equations are for prices. The quantity variables are not concentrated in any particular sector, but cover a broad spectrum of the model's quantity variables. The consumption and investment sector of the model is represented by new automobile sales (AUTOS), investment in producers durables (IPDQ72), average GNP ($GNPAVEQ_{-1}$) and imports (M72). All are measured in real terms. The productivity and employment sector is represented by the unemployment rate and its lagged values ($RUM_{0,-1}$). The income flows sector is represented by permanent income ($YPERM72_{-1}$) and private wages and salaries (YPWS). And the monetary sector is represented by the money supply (M1BPLUS), which might also be considered a price variable.

Historical data for the variables chosen are plotted in figure 6-5, 6-6, 6-7 and 6-8. Both the consumer price index for services (PCS) and the business price deflator (PPNF) show a fairly steady growth over the sample period. The average user cost of capital relative to wage shows somewhat more movement and a general decline. The decline is attributable to the decrease in the real rate of interest, which is measured using PPNF to calculate the inflation rate. The quantities of automobile sales (AUTOS), investment (IPDQ72) and imports (M72), figure 6-6, show much more evidence of the modes we are investigating. This is true again of the rate of unemployment (RUM), and, to a lesser extent, the average gross national product (GNPAVEQ) and imports (M72). The latter two series show a definite

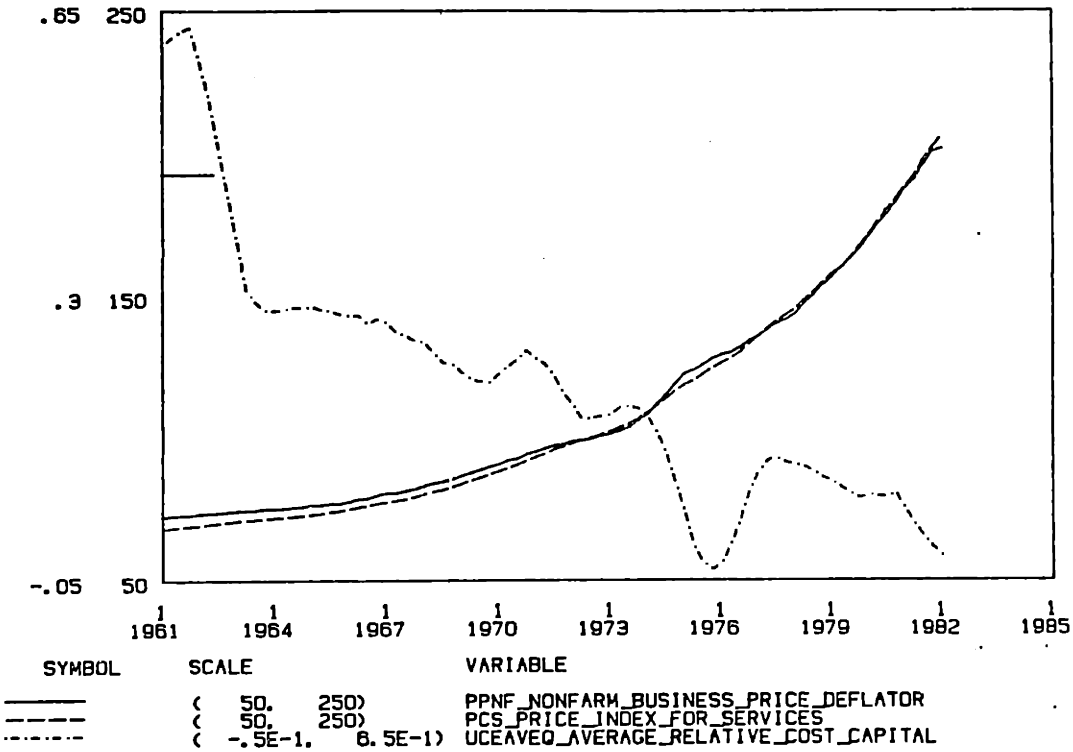

Figure 6-5: Historical Data for Prices

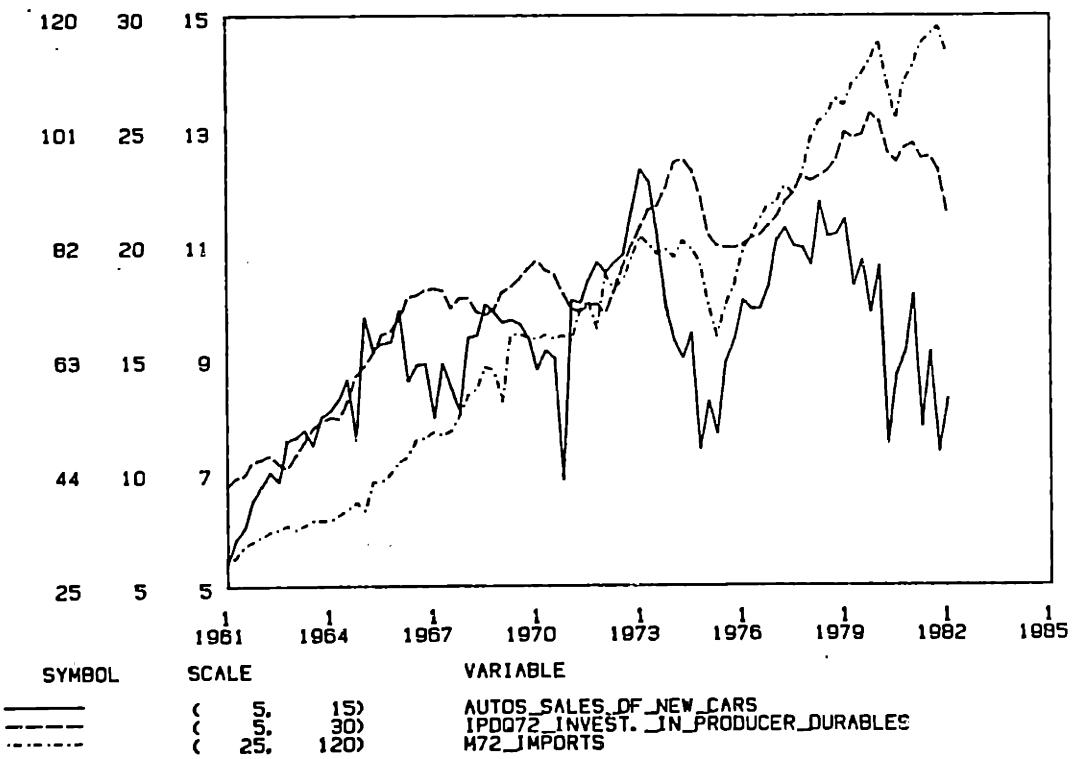

Figure 6-6: Historical Data for Consumption and Investment Variables

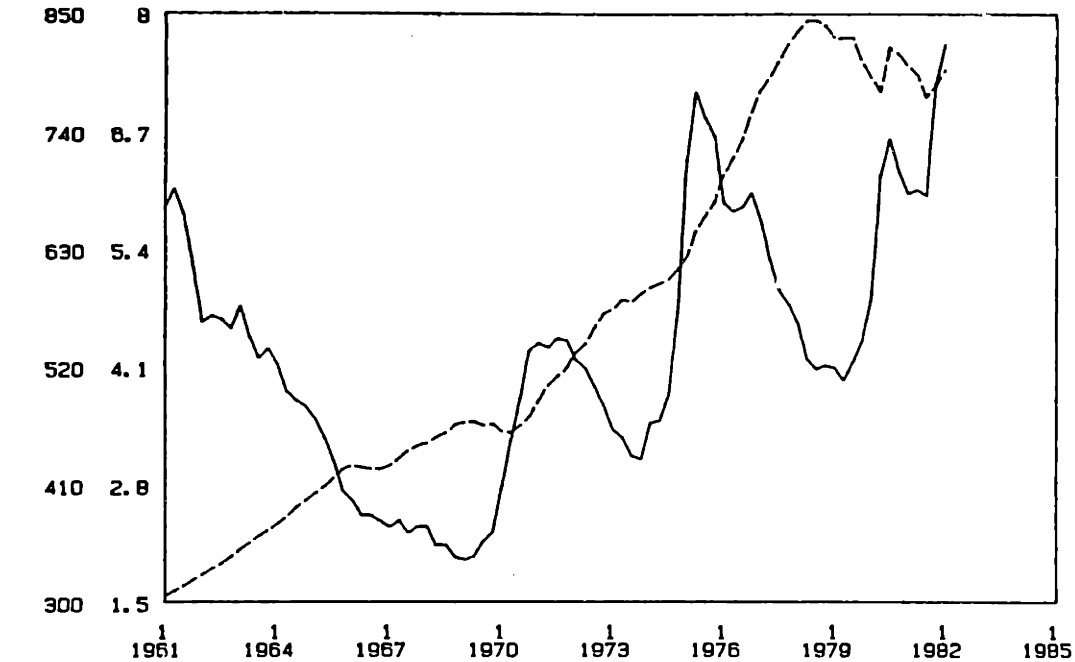

Figure 6-7: Historical Data for Selected Variables

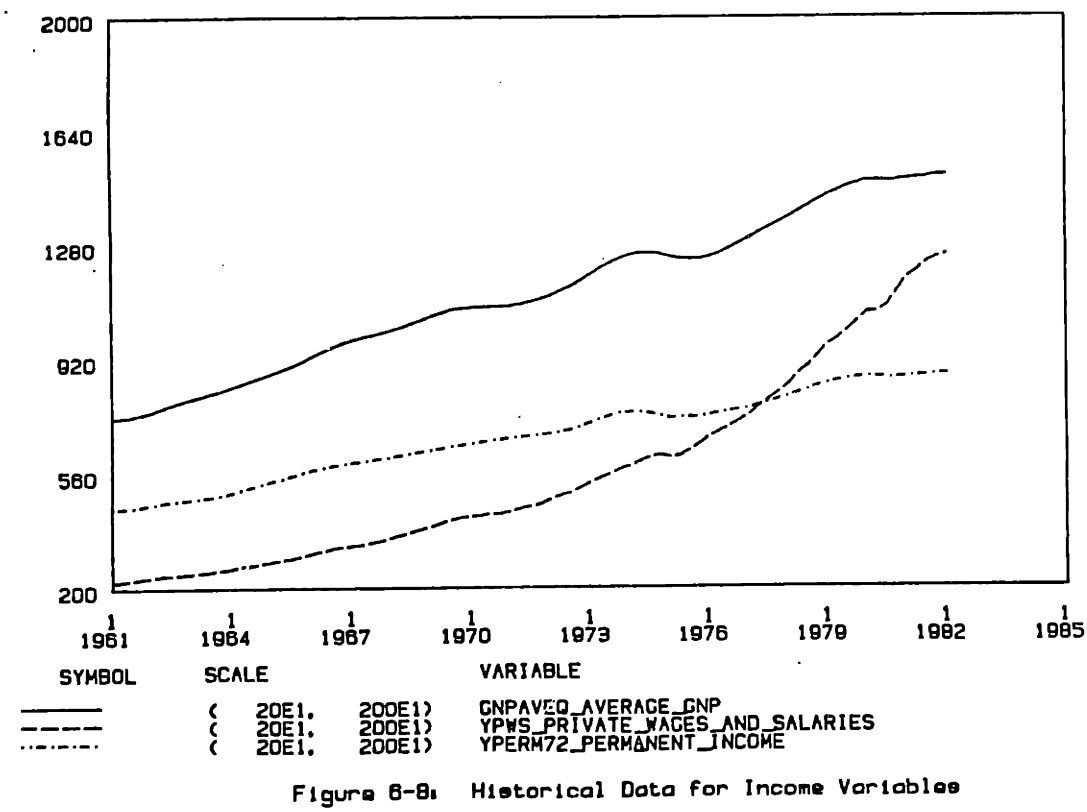

Figure 6-8: Historical Data for Income Variables

growth trend over time (figures 6-7 and 6-8). The income terms both for payments of wages and salaries (YPWS) and for permanent income (YPERM72) are much smoother and again show growth over time (figure 6-8).

The quantity variables included definitely show more evidence of the influence of the modes of interest on their behavior. The behavior of the prices seems to be dominated by an inflationary mode. Removal of some of the other modes in the price series will bring out fluctuations in prices over the business-cycle more clearly. Removing other modes is often done by differencing, but the type of transformation considered in chapter 4 can also be applied, and this will be considered later.

Additional Modes

Two simplifications of the model are considered. One retains only the modes of interest and will be referred to as the ten-mode model. The second model incorporates related modes as well and will be referred to as the fifteen-mode model. In order to generate the fifteen-mode model it was necessary to choose an additional set of modes that were important to the included states. The modes chosen also seemed to have the excluded states as important determinants. The additional number of modes selected was retricted to five in order to achieve a model which had the same number of modes and states. The additional roots that were included are given in figure 6-9.

Two of the additional modes included in the fifteen-mode model are not stable modes. This can potentially obscure the dynamic modes that are of interest. That is, the inclusion of an unstable mode can potentially

Root Index	Real Part	Complex Part	Magnitude	Period (Quarters)
2	1.027	0	1.027	-
23	1.0	0	1.0	-
32	.997	0	.997	-
33	.996	0	.996	-
35	.964	0	.964	-

Figure 6-9 Additional Roots Included in Fifteen-mode Model[†]

impair our understanding of the mechanisms generating the modes of interest. At the same time, in judging the quality of the simpified model it is possible that including the mode that is not of interest is crucial to the useful comparison of the simplified and large models.

The ten-mode simplified model was calculated using the Moore-Penrose generalized inverse (Ben-Israel 1974) of the included right vectors. A simplifed model calculated in this manner will not, generally, contain only modes of the original model. The additional modes added to the ten-mode model are all stable. Three are real (.955, .694 and -.377), and one is complex with a magnitude of .369 and implying oscillations of 35 quarters.

6-C The Simplified Models

We consider two simplifications based on the business-cycle modes, the ten-mode model and the fifteen-mode model. Only the modes of interest were

[†]The continuous-time equivalents of the additional modes are .027, 0, -.001, -.004 and -.04.

retained in the ten-mode model and additional modes of MQEM were included in the fifteen-mode model. Both simplifications preserve the right-hand eigenvectors. Because of the large size of the simplified models, we do not include the actual parameter values but restrict our discussion to the basic feedback structure. The matrices for the simplified models are included in appendix A6.

Structural Overview

In overviewing the structure of the simplified models we will use the format used in the presentation of MQEM. The format is useful in giving an indication of the feedback structure that is important in generating the modes of interest. In particular, the changes in the feedback structure from the full model to the simplified models give an indication of which feedback loops distinguish the modes of interest from the general feedback structure.

Figure 6-10 shows a sector by sector breakdown of the feedback links in the ten-mode model. In determining the importance of the different links, we considered scaling of the $\underline{\tilde{A}}$ matrix by the means and variances of the retained states and exercised a good deal of judgment. Figures 6-1 and 6-10 show some of the differences between MQEM and the ten-mode simplified model. The links enclosed in dashed boxes have changed one ranking, those enclosed in solid boxes, two rankings, and those enclosed in bold solid boxes, three rankings; the plus and minus signs indicate whether the links are of more or less importance than they were in the full model.

The largest changes in the ten-mode model (figure 6-10) are for the

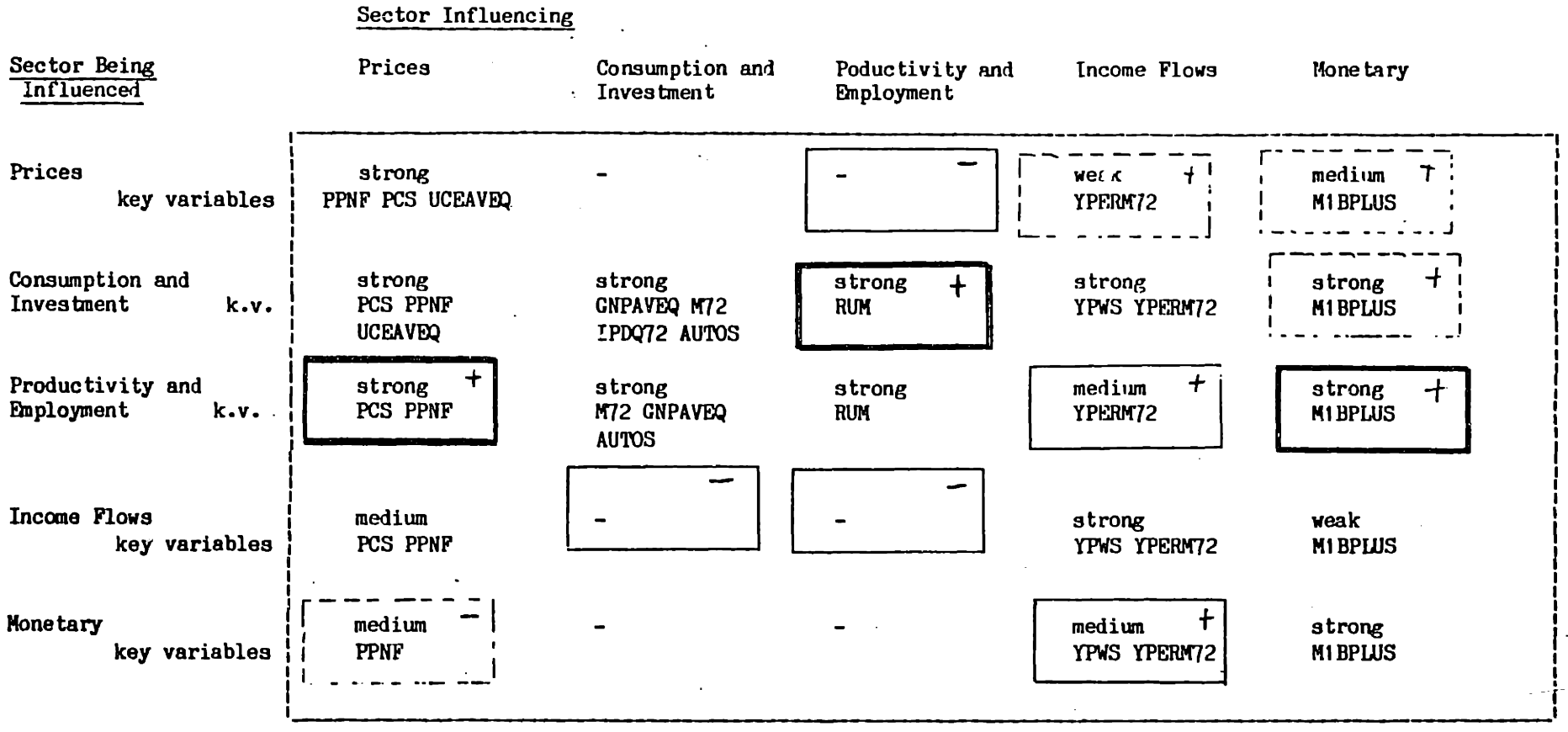

Figure 6-10 Feedback Structure of Simplified Model Which Retains 10 Modes

effects of prices on employment, employment on consumption and money on employment. These three links have gone from zero to strong. Income flows have become substantially more important in the determination of employment and money, and income flows are affected less strongly by consumption and employment. The changes in the importance of the income variables may be attributable to a change in scaling.

The analogous diagram for the simplified model based on the 15 modes is given in figure 6-11; it is very similar. The notable difference is the importance of money to employment and productivity, which is zero for the fifteen-mode model and strong for the ten-mode model. The income flows sector also seems to have had less change relative to the original model than for the ten-mode model. For both models the price variables continue to influence all of the other variables strongly. This is not surprising given the large number of price variables included.

Causes of Oscillations

The simplification process is intended in part to make the mechanisms generating the dynamics of interest understandable. The simplified models seem to indicate that three mechanisms are at work in the generation of the modes of interest. The first is a price and quantity adjustment process similar to the cobweb model (Ezekiel 1938, Mansfield 1970, p. 241) but based more on demand response than on supply response. The second is similar to the first but is related to the markets for factors of production. The third is an investment and income process having characteristics similar to the multiplier-accelerator model of Samuelson (1939), which was discussed in the MACRO model example of chapter 3.

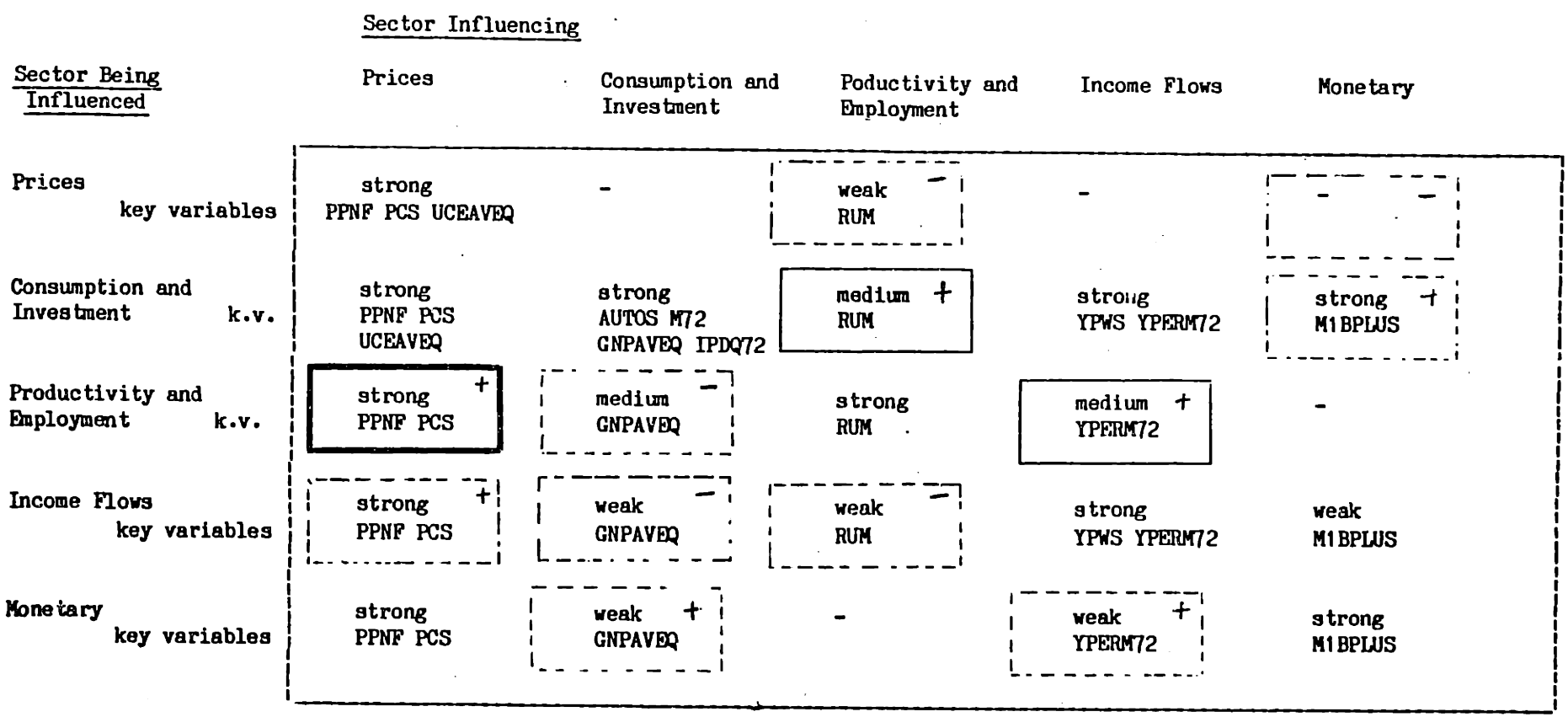

Figure 6-11 Feedback Structure of Simplified Model Which Retains 15 Modes

It has long been recognized that the reaction of prices to demand and demand to prices can generate a wide variety of behavior (Scarf 1960). The traditional cobweb model generates fluctuations by having prices this year determine next year's production, which determines prices next year. If prices are high this year, production next year will be higher and prices lower. This will cause lower production next year, which will mean higher prices, thereby returning the system to disequilibrium conditions similar to those of the first year and setting the stage for another fluctuation. The standard diagram for this process, which reveals the source of the name, is shown in figure 6-12.

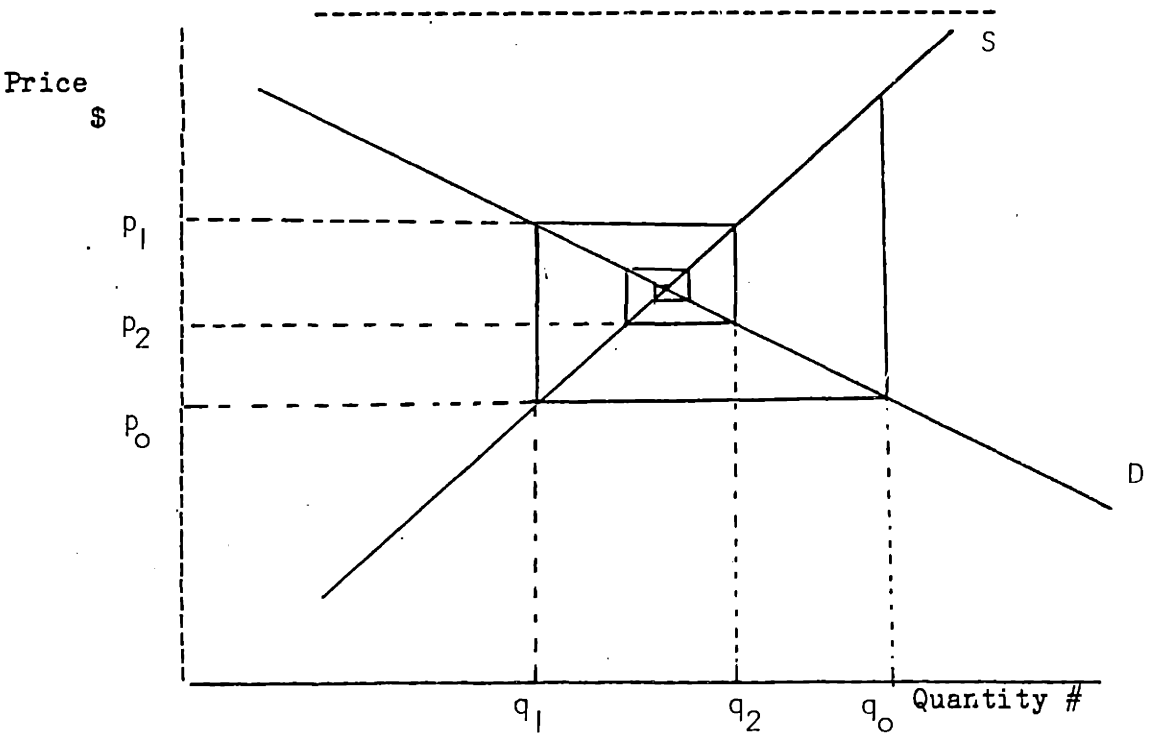

Figure 6-12 Simple Cobweb Diagram

A very similar process to the cobweb process seems to be active in MQEM. In MQEM, however, it is the response of demand rather than supply that appears to be responsible for generating the behavior modes of

interest. In MQEM both price and demand are determined over time rather than at any one instant in time. The mechanisms for adjustment in MQEM are working in the right direction; increased demand increases prices, which lowers demand. But the delays involved are sufficient to generate the oscillatory behavior modes. This process can be represented by the extremely simple causal-loop or influence diagram (Richardson and Pugh 1982) shown in figure 6-13.

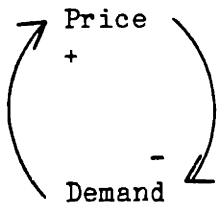

Figure 6-13 Simple Feedbak Structure Causing Oscillations

The second mechanism important in generating the behavior modes of interest is somewhat more quantity-based, although price effects are still important. As the demand for production decreases, the demand for the factors of production decreases. This causes a decrease in employment, which decreases demand. The decrease in employment causes a decrease in the wage rate relative to the cost of capital (UCEAVEQ). At the lower wage rate more labor is hired, and this increases final demand. Figure 6-14 gives a causal-loop diagram of these mechanisms.

There is one final mechanism that is almost entirely quantity-oriented. The effect of investment (IPDQ72) on average GNP (GNPAVEQ) and

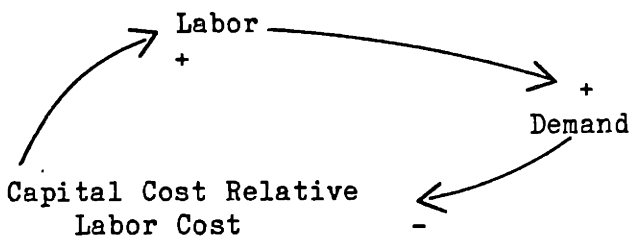

Figure 6-14 Simple Feedbak Structure Involving Costs

then the feedback to investment. This is essentially the classical accelerator of Samuelson (1939), which we discussed at some length in the MACRO model used as an example in chapter 3.

In MQEM prices seem to play a very important role in determining the modes of interest. Two of the three mechanisms which seem to be active in generating the oscillatory modes of interest involve price.

6-D Simulation

The homogenous response of the simplified models will be the same as that for the full linear model when the initial conditions are a linear combination of the included right eigenvectors. When this simulation is performed the responses match for about the first 2 years. Thereafter, the unstable modes in the full linearized model drive the model away from equilibrium. Both of the simplied models go very quickly to equilibrium, though the fifteen-mode model does begin to show some drift from this equilibrium after 15 years. In this section we consider the homogeneous response to a more general set of initial conditions. The initial

conditions chosen were simply the deviation from trend of the endogenous variables in second quarter of 1973. The exogenous variables were all set to 0 for the homogeneous response simulations.

The responses of three variables have been plotted-- new automobile sales (AUTOS), imports (M72) and the nonfarm business price deflator (PPNF). Figure 6-15 shows the time path of automobile sales (AUTOS) from the given initial conditions. The plots for autos all show the modes of interest in the first few years. The intitial response of autos in the full model is an increase, auto sales then fall off and recover. The simpified models do not show the initial increase; automobile sales begin the simulations in decline and then recover. In the ten-mode model auto

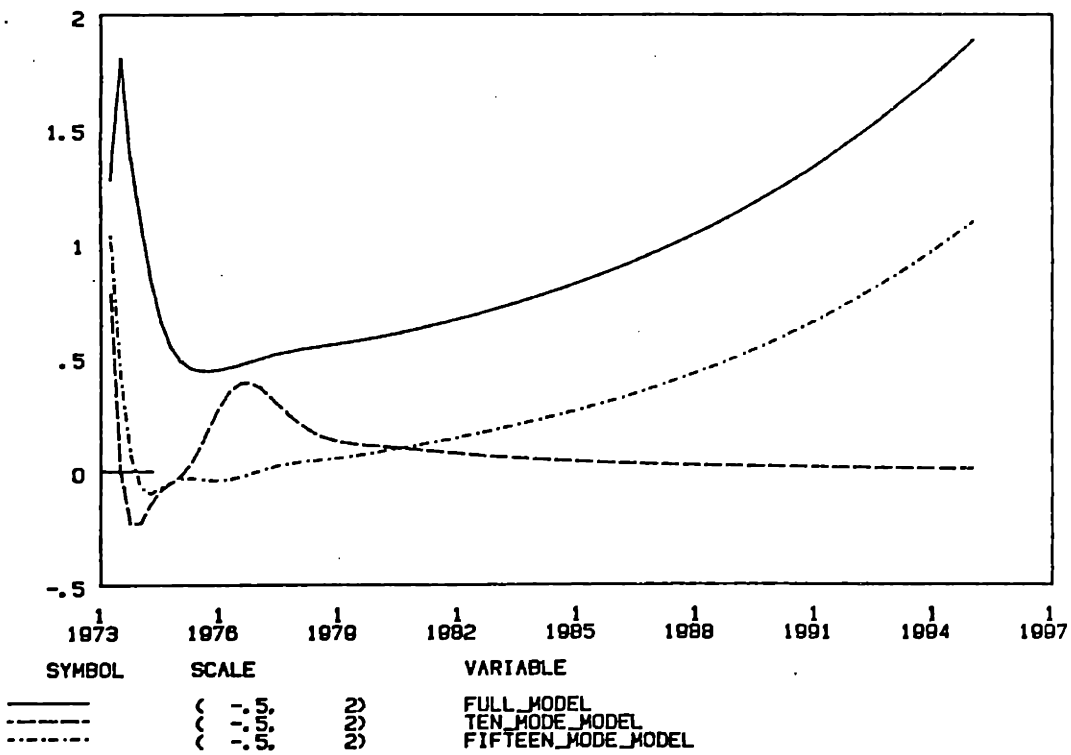

Figure 6-15: The Homogeneous Response of Automobile Sales

sales show a substantial increase after decreasing and finally fall back to equilibrium.

In both the full linearized model and the fifteen-mode model auto sales show growth after the intial response. This is due to the unstable roots in the full model and the unstable root included in the fifteen-mode model. The ten-mode model does not have any unstable roots and therefore does not generate this growth. This is not a concern in itself. However, the recovery of auto sales is much stronger in the ten-mode model than it is in either the fifteen-mode model or the full model. It appears that the interaction of the aditional modes in the fifteen-mode model causes the behavior with respect to the modes of interest to differ from that of the ten-mode model.

Imports (M72) are plotted in figure 6-16 and show less movement than auto sales did. The modes of interest are not really apparent in the full linear model or in the fifteen-mode model. However, imports in the ten-mode model show somewhat more movement, which may be attributable to the modes of interest. Again it appears that the additional modes included in the fifteen-mode model interact strongly enough with the modes of interest to affect the behavior influenced by the modes of interest strongly. Almost the same comments apply to the business price deflator (PPNF), which is plotted in figure 6-17. Price in the ten-mode model shows behavior characteristic of the modes of interest, but this behavior is masked in the full and fifteen-mode models. For this price variable, however, there is also a distinct difference between the full model and the fifteen-mode model. The fifteen-mode model shows growth in PPNF

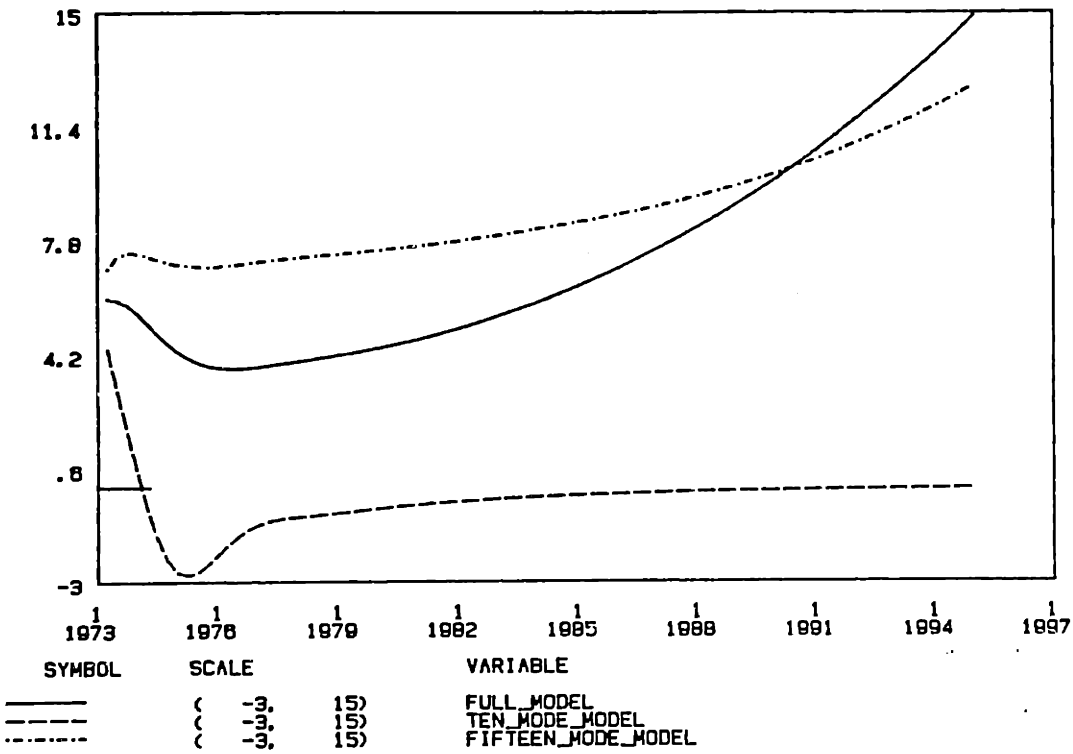

Figure 6-16: The Homogeneous Response of Imports

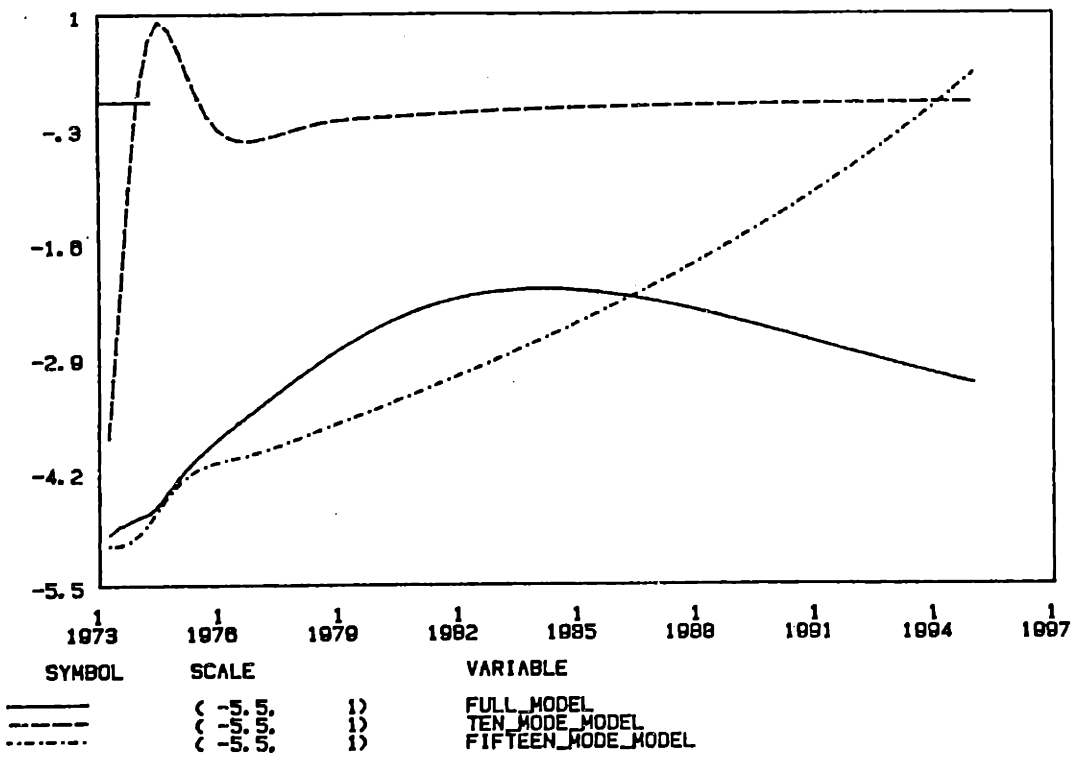

Figure 6-17: The Homogeneous Response of the Business Price Deflator (PPNF)

attributable to the unstable mode that was included. In the full model this behavior is not apparent, and price actually shows a decline. This is the same sort of difference that we have seen between the ten-mode model and the fifteen-mode model in the other simulations.

The historical simulation of the full linearized model does not seem to display the modes of interest very well. Consequently, it is difficult to draw conclusions about how well the simplified models are displaying the modes of interest from these simulations. In figure 6-18 the unemployment rate (RUM) is plotted for the full and simplified models. The response of the linearized model to the exogenous inputs is not very accurate itself and actually becomes negative after 1976. However, the important thing to

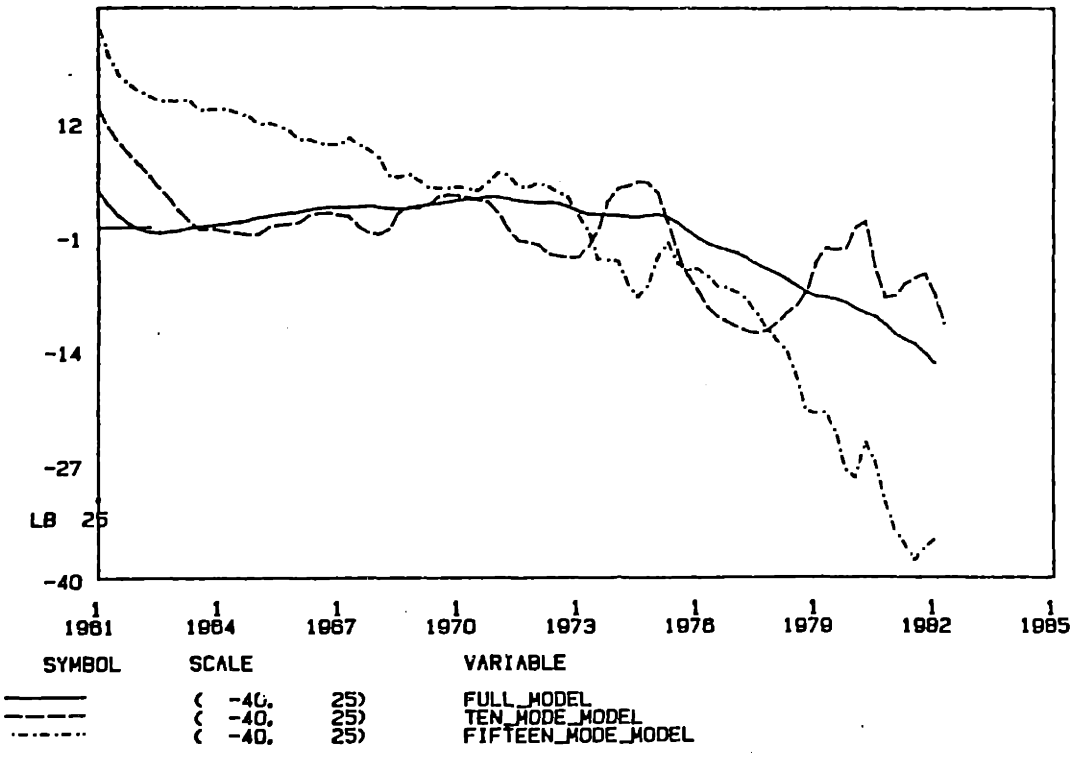

Figure 6-18: The Simulation Response Of Unemployment (RUM) to Exogenous Inputs

note in this plot is that the two simplified models do show the behavior modes of interest and the ten-mode model shows it very strongly. The fact that the ten-mode model shows this is not so surprising since the exogenous variables input a linear combination of the right eigenvectors associated with the modes of interest at each time (recall that $\underline{\tilde{B}} = \underline{\tilde{R}}_1 | \underline{L}_{11}\ \underline{L}_{12} | \underline{B}$).

The above discussion has indicated that ignoring modes that are closely related to the states of interest can change the simulation results in an undesirable way. Specifically, the degree to which the simplified model displays the mode of interest does not seem to agree with what happens in the full model. Given that the goal of the simplification process is not to match the homogeneous response patterns of the full model, this is not necessarily bad. To the extent that the mode of interest is simply being obscured by the other included modes we have, lost information by including these modes. However, to the extent that the additional modes are coupled with the modes of interest (through the retained states), the fifteen-mode simplified model is a better simplified model.

6-E Estimation

In chapter 4 we introduced a transformation of the data that yields a data series consistent with the simplified model. This was also presented as a way to detrend data. In this section we briefly review this transfomation and consider the effect that it has on the data. The results of estimating the simplified model using both the transformed and the original data are then considered.

In section 4-B we showed that it is possible to transform the output from the standard linear "system" according to the formula

$$\underline{\hat{x}}_t = \underline{\tilde{R}}_1 \, |\underline{L}_{11} \; \underline{L}_{12}| \; \underline{x}_t \tag{6.4}$$

and have the transformed data $\underline{\hat{x}}$ be in accord with the simplified model. What this means is that if the same exogenous input and noise series drive the "system" and the simplified model then the output of the "system," when transformed by equation 6.4, will match the output of the simplified model exactly. The matrix $\underline{\tilde{R}}_1$ represents the right eigenvectors of the simplified model associated with the modes of interest. The matrix $|\underline{L}_{11} \; \underline{L}_{12}|$ represents the left eigenvectors of the "system" associated with the modes of interest.

We stated that the above transformation could be thought of as a method of detrending the data since, it will effectively remove the behavior associated with all but the included modes. If the "system" is unstable, transforming its output by equation 6.4 with only stable modes included should yield data that have only the trend of the exogenous variables. Note also that unless the number of states is equal to the number of modes the transformed data will be perfectly collinear.

We have performed the above transformation on the historical time series of the endogenous variables used in MQEM. Note that in this case we are not taking the output of the model, but rather the output of some process for which MQEM is meant to be a useful representation. Thus the

transformed data cannot be expected to have precisely the same characteristics as transformed data of the actual model output. Consideration of the data transformed according to equation 6.4 is therefore a test of the validity of MQEM with respect to the modes of interest. The same holds true of estimation.

In figures 6-19, 6-20 and 6-21 we plot the actual and transformed values for three variables. The values have been adjusted to have the same means. Recalling the discussion of the small macroeconomic model in chapter 3, we note that this adjustment is necessary because of the differences in steady-state gain. For the unstable model we are considering, there is no steady-state gain, but it is still useful to have the data on the same scale.

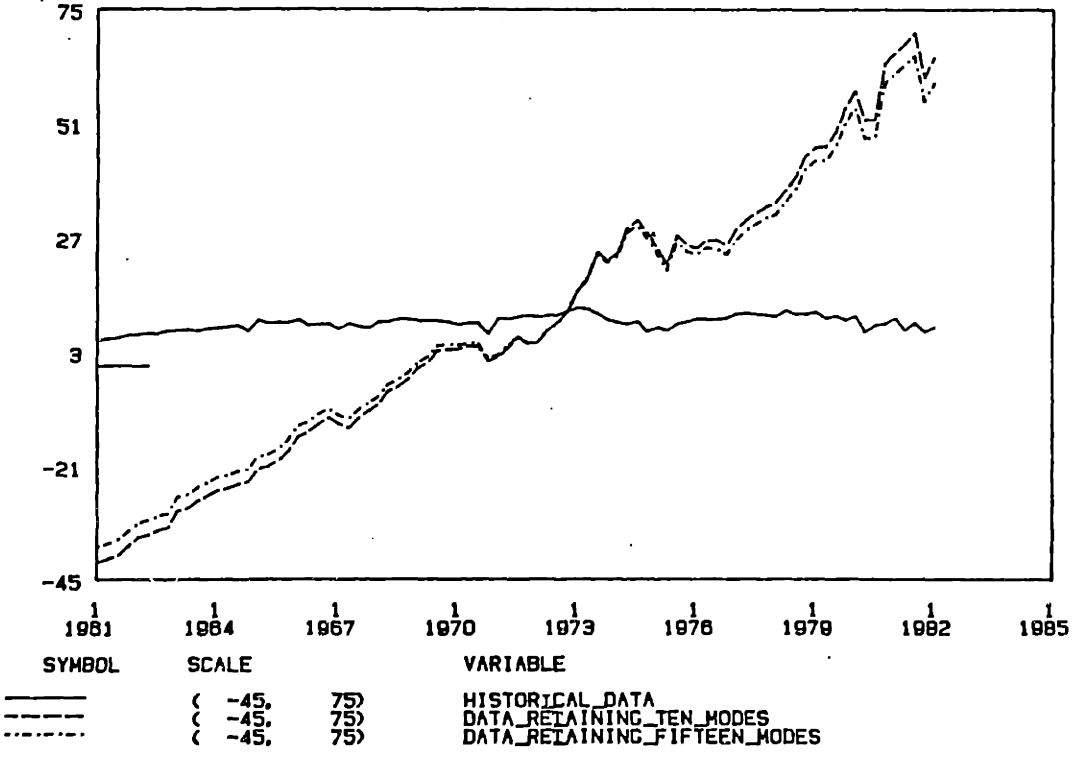

Figure 6-19: The Historical and Transformed Data for Automobile Sales (AUTOS)

In figure 6-19 the values for automobile sales (AUTOS) have been plotted. The transformed data show a substantially larger degree of variation than do the actual data. The business-cycle modes are present in the transformed data but are not strong. The business-cycle modes that are present do not match those for the actual data. This is likely a reflection on the degree of approximation in the simplied models. In the transformed values for imports (M72) given in figure 6-20, the series seem to have the same qualitative characteritics. All of the series show fluctuations over the business cycle. Again the transformed data fluctuate more strongly than the actual series do. For the business price deflator (PPNF) shown in figure 6-21 the results are somewhat different. The projection based on the 15 modes show business cycle fluctuations much more strongly than does price, but it also tracks the original series reasonably

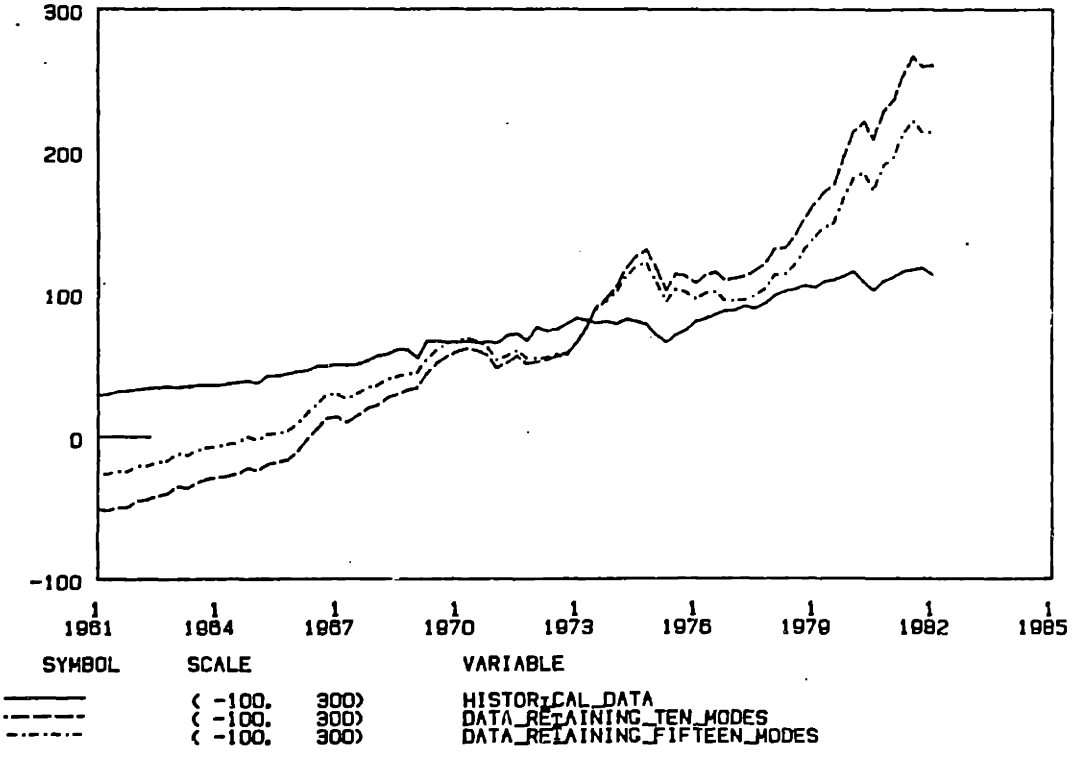

Figure 6-20: The Historical and Transformed Data for Imports (M72)

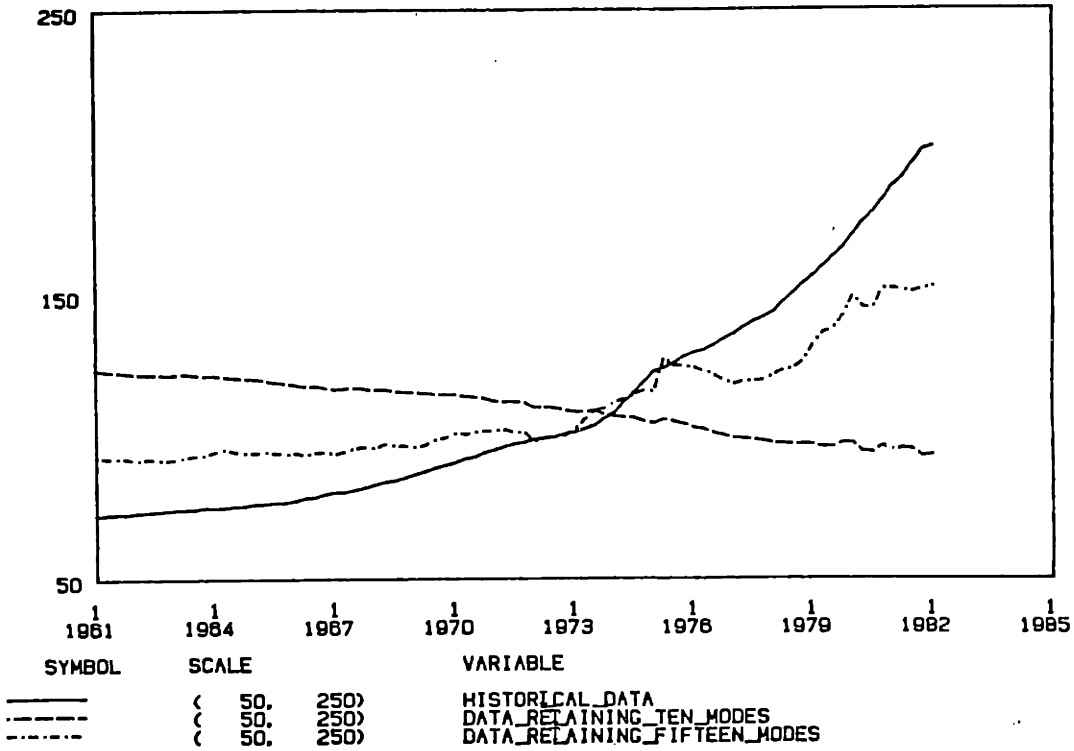

Figure 6-21: The Historical and Transformed Data for the Business Price Deflator (PPNF)

well. On the other hand, the transformed data based on only the 10 modes of interest shows a decline and very little fluctuation.

Estimation

The estimation of the simplified models can be done with a number of levels of restrictions imposed. We first consider estimation with the transformed data plotted above. Only the fifteen-mode model will be considered in this section. The data transformed for the ten-mode model would not be sufficient to identify the parameters of the model as was discussed in chapter 4.

As has been noted before, the simplified model is influenced by all of the exogenous variables, though a number of these influences may be unimportant. Unfortunately, in the estimation of the model it is necessary

to restrict the number of exogenous variables that enter the model. If this were not done, the number of variables entering the equations would far exceed the available number of observations. In order to allow estimation we restricted the $\underset{\sim}{B}$ matrix. Two restrictions were considered: first, the restriction that the $\underset{\sim}{B}$ matrix be the same as the $\underset{\sim}{B}$ matrix derived from the original model and, second that the column dependencies in the estimated $\underset{\sim}{B}$ matrix be the same as those in the fifteen-mode model. This second restriction can be implemented by using the transformed exogenous variables on the right-hand side of the estimating equations. A constant term was also included in the regressions.

The results of the estimation will again be summarized in terms of the general feedback structure of the model that we estimated. We do this following the format of the discussion of the simplified models. Figure 6-22 shows an overview diagram of the simplified model estimated on the transformed data and with the $\underset{\sim}{\hat{B}}$ matrix restricted to equal $\underset{\sim}{B}$ for the fifteen-mode model. The differences from the fifteen mode simplified model are marked with the previously defined boxes. The most important feature of the estimated model is the more uniform nature of the feedback relationships relative to the simplified models we have considered previously. Prices are still very important, but the other variables seem to be playing an important role as well.

The estimated model has 5 pairs of complex eigenvalues and 5 real eigenvalues. The largest real eigenvalue is 1.23, impyling a very large rate of growth. The model has one other root larger than one (1.05). The complex modes have periods ranging from 9 quarters to 55 quarters.

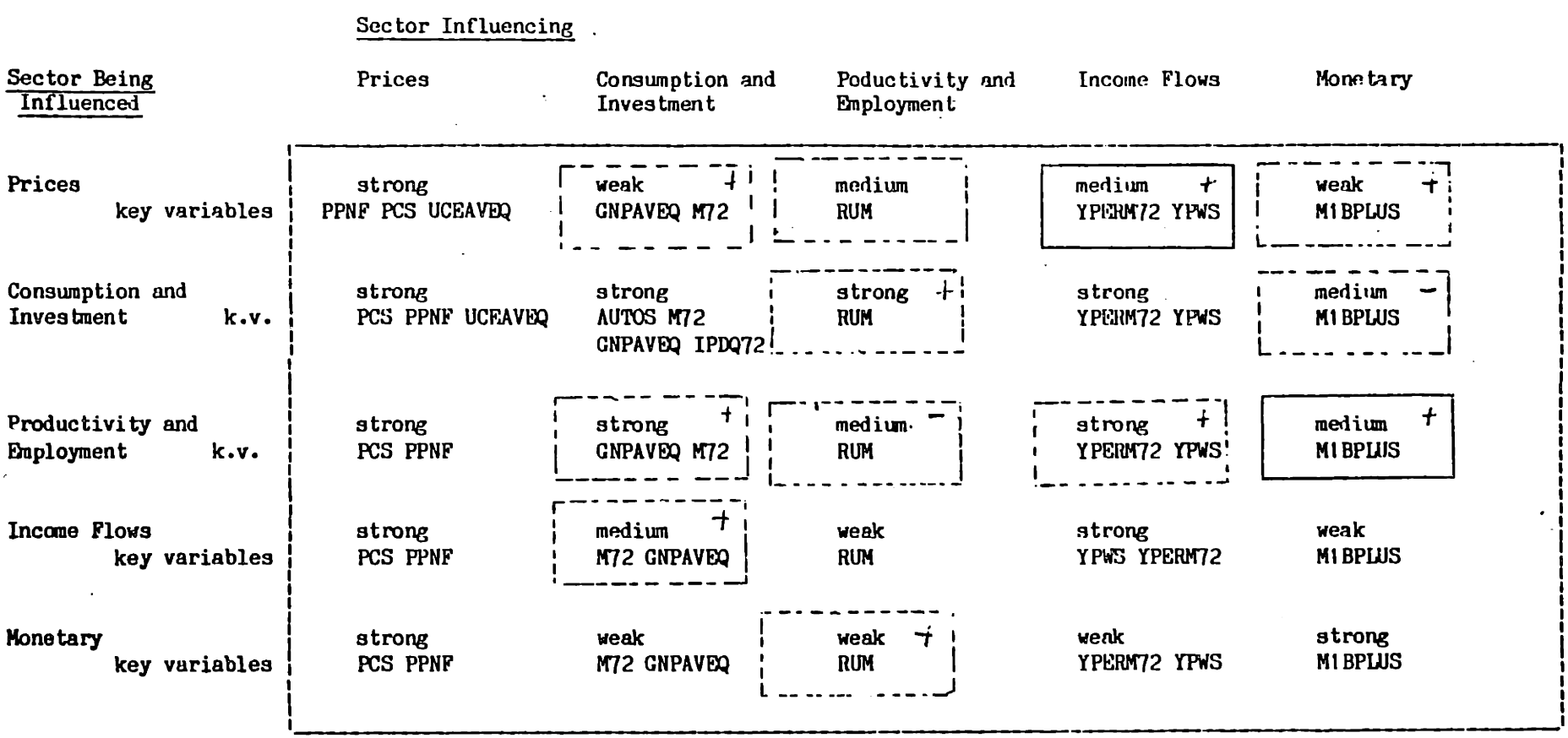

Figure 6-22 Feedback Structure of Model Based on Transformed Data With the $\bar{B}$ Matrix Restricted

When the $\hat{B}$ matrix is restricted to have only the same column dependencies as $\tilde{B}$ of the fifteen-mode model, the results do not change substantially. The importance of the consumption and investment variables in the determination of price is increased somewhat, as is the importance of unemployment overall. The money supply also seems to have a stronger influence on investment and consumption. The estimated model has 7 complex roots with periods of 5, 9, 16, 17, 26, 34 and 103 quarters. All of the modes are stable.

For the estimation based on the historical data the transformed exogenous variables were used to avoid the unrestricted estimation of the $\hat{B}$ matrix. When the historical values for the endogenous variables are used for the estimation, the resulting $\hat{A}$ matrix has a structure similar to that obtained by estimation with the transformed data.† The overview of the structure for the case in which only the column dependencies of the $\hat{B}$ matrix are required is given in figure 6-23. The changes are measured relative to the matrix based on estimation using the transformed data. The productivity and unemployment variables play a less important role, as do the income flows variables. The resulting $\hat{A}$ matrix, when the $\hat{B}$ matrix is restricted to having only the same column dependencies, has six complex eigenvalue pairs and three real roots. The two largest roots are real and are 1.12 and 1.06, implying a large degree of instability. Of the complex roots the four largest range in magnitude from .679 to 1.01 and imply

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†The regression results using the historical data and the restriction that $\hat{B} = \tilde{B}$ were of very poor quality. The residuals generally had higher variance than the left-hand variables, and the model which resulted had one eigenvalue of -9.

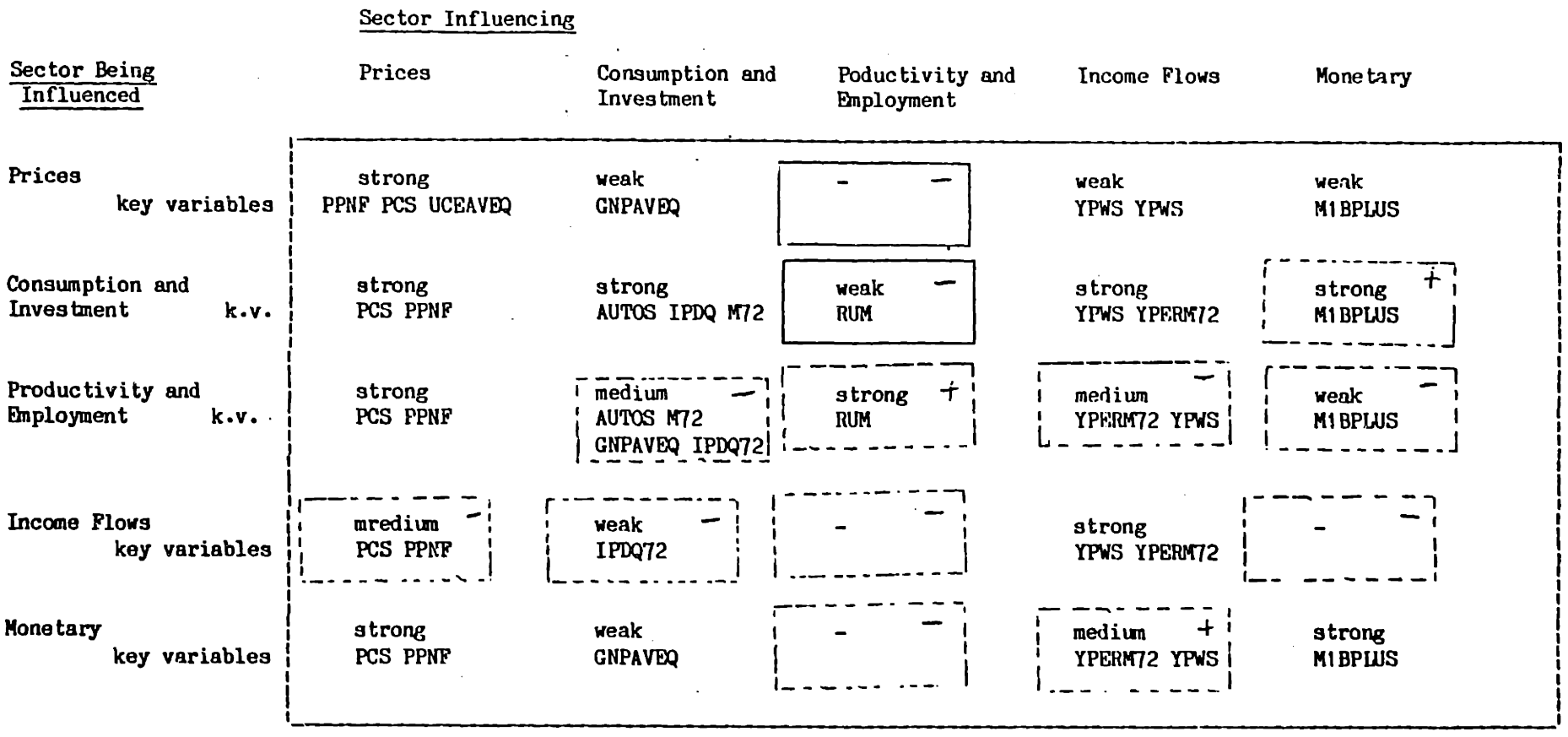

Figure 6-23 Feedback Structure of Model Based on Unrestricted Estimation Using Historical Data

periods ranging from 15 to 51 quarters.

According to the theory that was developed in chapter 4, estimation on the transformed data should yield the fifteen-mode simplified model if the full linear model corresponded to the "data-generating process." The estimated model based on the transformed data is different from the fifteen-mode simplified model but does have some similarities in its dynamic structure. Given the relatively small amount of data relative to the size of the simplified model, this conclusion must necessarily remain weak.

6-F Summary

In this chapter we have considered the application of the tools developed in this thesis to a medium-size econometric model. In order to apply the tools, it was necessary to linearize the econometric model. Generalized participation factors, cross-modal participation factors and eigenvalue elasticities were all used in the determination of the important states. The different tools yield different results, and this is in part attributable to the nature of both the model considered and the modes chosen. The MQEM model is not an exactly simplifiable model, and though this makes the simplified models developed very approximate, it also makes the comparison of the tools more useful.

Two simplified models were developed. One based only on the the modes of interest and another that incorporated additional modes from the original model. The two models were then compared on the basis of their simulation responses. The model based only on the modes of interest showed

these modes much better in the simulations. However, there is some indication that this was because the ten-mode model was not representing the feedback structure of the states of interest as accurately.

The data transformation discussed in chapter 4 was applied to the historical data for the endogenous variables. The transformed data were then used in estimation of a simplified model. The resulting simplified model had an different feedback structure from the fifteen-mode model, but did have a number of eigenvalues similar to those of interest. The model was also estimated using the actual data for the endogenous variables. The results were similar to those for the estimation based on the transformed data.

7 Conclusions

Following the organization of this thesis we now review the issues that have been addressed and the results of the research. The problem and the general line of enquiry will first be stated. Sections 7-B and 7-C are then devoted to the results of chapters 3 and 4. The results of chapters 5 and 6 are are discussed in section 7-D. In section 7-E areas for further research are discussed.

7-A Overview of Model Simplification

We have considered the problem of simplifying a large model based on the dynamic characteristics of the large model. In particular, we have considered simplifications that preserve specific eigenvalues of the original model. We have restricted out attention to linear time-invariant systems of difference equations. The motivating problems for the research have been these:

1) Finding techniques for the development of simplified models that can be used to determine and convey the characteristics of a complex "system" generating behavior modes of interest.

2) Developing simplified models that can be used in formulating easily implementable policies for the "system."

3) Understanding how the behavior of the simplified models relates to the behavior of the original "system."

A simplification is defined as a model which has fewer states than the original "system." The simplified model is intended to capture selected behavior modes, and cannot generate all the behavior modes generated by the original system. Thus, a model that displays growth and oscillation could be simplified to a model that displays only oscillation or only growth. It is always possible to do this, but it is necessary to determine the most useful way to do so. Models that give the best understanding will necessarily need to be easily interpretable in terms of physically or economically meaningful concepts. Given our assumption that the "system" is interpretable in this manner, we have required that the simplified model be based on states corresponding to a subset of the states of the "system."

Determining whether the states of a simplified model correspond to the states of the original "system" is not an easy task and requires a

substantial amount of notation. Essentially though, a model can be said to be exactly simplifiable if the modes of interest can be generated by including a set of states that have no feedback relationships with the other states of the "system." Such a condition is very restrictive and can hardly be expected to be fulfilled in practice. However, this definition is a reasonable means of developing tools for the choice of the important states in the generation of the behavior modes of interest.

The simplified models developed on the basis of the characteristics of the "system" need to be defined in terms of their relationships to the behavior of the states of the original system. The simplification will yield a model with behavior that is different from the behavior of the original system. The ways in which the data will fail to be in accord with the data of the original "system" need to be determined. In addition, the possibility of developing simplified models based on observations of the output of the "system" is considered.

We have defined the basic nature of the problem considered in this thesis. In the next sections we will summarize the results of the research.

7-B The Structure of Model Simplification

In chapter 3 we considered at length the question of how to develop simplified models that retain both the behavior modes of interest and interpretability. After reviewing the relevant results from the theory of selective modal analysis (Perez 1981, Perez et al. 1983) some generalizations were developed.

Simplifications that have been considered in the literature have normally been based on the retention of the right eigenvectors of the original model. Simplified models can also be based on the left eigenvectors. A simplified model based on the right eigenvectors maintains the activity of the included states associated with the modes of interest. A simplification based on the left eigenvectors maintains the influence of a given state on the modes of interest. The simplified model based on the left eigenvectors allows for the influence of exogenous variables in an interesting matter. Consideration of a small macroeconomic model did not reveal the use of the right or left eigenvectors as clearly superior.

The influence of the exogenous variables on the simplified models has been considered. The simplified models were chosen so that the exogenous variables would have the same influence on the modes of interest in both the full and simplified models. This choice for modeling the influence of the exogenous variables was evaluated in terms of the implications it has for the steady-state gain of the simplified model. The simplified models defined in this manner will not maintain the steady-state gain. It was shown, however, that the change in the steady-state gain is essentially attributable to the influence of excluded modes. Since the simplified models are not trying to capture all of the modes, this error does not seem to be unreasonable. Consideration of a small macroeconomic model showed that maintaining the steady-state gain can be detrimental to the evaluation of the influence of the exogenous variables on the simplified model's behavior.

The more general problem of defining the quality of a simplified model was then considered. In defining the quality of a simplified model, we concentrated on its interpretability. We considered this in terms of transformations of the simplified and original models. If the two models could be transformed in the same manner and yield the same result, the simplified model was termed an "exact simplification" of the original model. Because the simplified model will be of a lower dimension than the original "system" it was necessary to define what the same transformation meant.

The conditions for exact simplification were that the modes of interest be generated by a simplified model with included states that are not influenced by the excluded states, or do not influence the excluded states. This requires block diagonality of the dynamics matrix after appropriate permutation. The conditions for exact simplification are extreme and would rarely, if ever, be expected to be fulfilled. The notion of an exact simplification is very useful, however, as a tool for judging how good a simplified model is. In particular, we use the notion of exact simplification to develop tools that can be used to choose the states to be included in a simplified model.

A similar approach to judging the quality of a simplified model is the consideration of the results of changing feedback links in both the original model and the simplified model. The change in an eigenvalue caused by changing a given feedback link should be the same in both models. The excluded eigenvalues are not included in the simplified model and are therefore assumed not to be changed when elements of the simplified model's

structure are changed. To match this, the effect on the excluded eigenvalues of changing the feedback links among the included states in the original "system" should be zero.

The requirement that changing the feedback links among the included states have the same effect on both the original "system" and the simplified model is severe. It was shown that this requirement is the same as the conditions for perfect simplification previously discussed. The conditions can again be used in order to develop tools for choosing the states to be included in a model.

Three major tools for determining what states to include in a simplified model were discussed, the generalized participation factors, the cross-modal participation factors and the eigenvalue elasticities. The generalized participation factors essentially measure the extent to which the derivatives of eigenvalues, with respect to the feedback links among the included states, differ between the original and simplified models. The cross-modal participation factors measure one modes interaction with another. And the eigenvalue elasticities measure to what extent the states either generate or are affected by a mode in a manner that will further affect the states of interest. These different tools were applied in chapter 5.

7-C Simplifications and Data

In chapter 4 the qualities of the simplified model relative to the data generated by the original system were investigated. First, the

covariance characteristics were considered. Then it was shown that it was possible to transform the output of the original "system" to be consistent with the simplified model. Following this, issues in estimation were considered.

The power spectrum, and therefore the covariance structure, of the simplified models will in general be different from that of the original "system." It was shown that this difference is attributable to the influence of the excluded modes and the cross spectra between the included and excluded modes. Because the simplified models are intended to capture only the modes of interest, these characteristics of the simplified models relative to the "system" are to be expected. The power spectrum for a small macroeconomic model was considered and clearly showed these attributes. The excluded roots of the small macroeconomic model considered have a heavy influence on the power spectrum of capital, which was not apparent in the simplified models.

Consideration was also given to the power spectrum of the residuals that are generated when the simplified model is used in conjunction with the output of the original "system." The power spectrum will be dominated by the excluded modes if the model is nearly exactly simplifiable, but otherwise it will show characteristics of the modes of interest. It was also shown that the cross spectra between the included states and the previously discussed residuals will have a similar character to the power spectra of the included states. For the small macroeconomic model used as an example the residuals derived from the model based on the left eigenvectors showed more power at the frequencies of interest. This result

was the only real qualitative difference found between the simplified model based on the left vectors and the simplified model based on the right vectors.

It is possible to transform the output of the original model in order to make the simplified model agree with the data. What this means is that if the original model and the simplified model were driven by the same inputs, then the output of the simplified model would be the same as the transformed output of the original "system." The transformations require full information on the eigenvectors of the original model and for this reason are not extremely useful for empirical application. However, the transformations considered do yield some interesting insights into the nature of estimation in the face of a dynamic system which is either not controllable or not observable.

The results on the estimation of the simplified models were all very negative. While it is possible to construct valid instruments for the purpose of estimation, to do so requires complete knowledge of the system. In addition, it was shown that using only information on the one-step prediction error that a model makes will give the same value of parameters as would be achieved if we assumed that the simplified model were correct in estimating it. This is a relatively strong result in that the use of any information theoretic measure of the quality of a model relative to the output of the "system" will yield the same estimate. The results were all asymptotic in nature. The question of the best procedure for estimation given limited quantities of data was not addressed. The development of a simplified model that maintains the eigenvalues of the "system" essentially

requires complete knowledge of the "system."

7-D Applying the Tools

In chapter 5 we considered some of the problems associated with the application of the tools developed to simultaneous equation models. The structural restrictions that might be imposed on a simultaneous equation model cannot be easily imposed on the simplified models, and this results in a "less simple" simple model. However, simultaneity can itself be considered the result of a simplification, and when it is, the resulting model structure does not cause problems in the simplification process. Some useful transformations for dealing with nondynamic equations were also considered.

In chapter 6 we considered a simplification of an existing econometric model, the Michigan Quarterly Econometric Model (MQEM). The is a nonlinear simultaneous equation model. A linearized version of the model was used in the simplifications. In simplifying MQEM, we focused on the business-cycle frequency modes. There were five pairs of these in MQEM with implied periods ranging from two to ten years. The modes involved a large number of variables. This meant that the simplified models required a substantial amount of approximation. The strong coupling, however, also allowed us to compare the different tools for evaluating the importance of a state in generating the modes of interest.

When different tools were applied in the determination of the important states, different results were achieved. The major implication

of the work done seemed to be that for a mode which is reasonably heavily coupled with all the states the cross-modal participation factors reflect the nature of other modes in the model rather than that of the mode of interest. The other quality of the tools that was observed was that the generalized participation factors and the eigenvalue elasticities gave similar results.

Once the states had been chosen, two simplified models were developed. For the first model, only the modes of interest were explicitly included. For the second simplified model, additional modes from MQEM which seemed to be important in the feedback structure of the chosen states were included. Two of the added modes were not stable modes, while all the additional modes that resulted in the first model were stable. The homogeneous responses of the simplified models and the full model were compared and there were, of course, differences. The results indicated that modes which are heavily coupled with the included states have a significant influence of the way the modes of interest are excited. This suggests that it is useful and important to consider other modes that are closely related to the modes of interest in the development of the simplified models.

Three basic feedback mechanisms that generate the modes of interest in MQEM were identified. The first is a negative feedback loop involving price and demand. Higher price lowers demand, which lowers price. The delays involved in the transmission of these influences generate oscillatory behavior modes in MQEM. The second mechanism is the same except that it holds with respect to the derived demand for labor. The third mechanism is a quantity adjustment mechanism involving investment in

producer durables. The simplified models are approximate, and a great number of feedback loops from the full model were left out, or proxied by more direct links in the simplified models.

The historical data for the endogenous variables retained in the simplified model were transformed according to the method mentioned above. If the linear version of MQEM were exactly correct, then the data transformed in this manner would be entirely consistent with the simplified models. The data transformed in this manner do show somewhat more of the modes of interest than do the original data, and this is especially true for the price variables. However, many of the series display substantially different behavior with respect to the business-cycle modes, and this is likely a reflection on the degree of approximation involved in the simplifications.

The simplified model was estimated using both the transformed and the untransformed historical data. The results were similar and showed a relatively more homogeneous feedback structure than that of the simplified models. If the original linear model were exactly correct, then the estimates based on the transformed data would converge to the simplified model asymptotically. Given the large size of the simplified models relative to the available data, it was difficult to draw any conclusions in this case.

7-E Future Research

The first and most obvious area for future research is in application. In this thesis we have developed a number of tools that seem to be potentially useful in the simplification and understanding of large models. The application we have considered has brought out some of the strengths and weaknesses of the methods considered, but many of these results are very likely specific to this application. Application of the tools in other situations would be informative as to their general usefulness. Moreover, the simplification of models yields results that are of interest in themselves.

The relationship of the simplified models to the data was considered in chapter 4. The results were substantially negative with respect to our ability to generate simple models with reference only to available data. The issues of controllability and observability and their relationship to model simplification seem to be of considerable interest, especially in the estimation of models from small numbers of observed variables. The theory of model simplification can give a useful perspective on this issue.

Simplified models are potentially very useful in the statistical validation of larger models. In chapter 4 we showed a transformation of the output of the original "system" which would make the simplified model correct with respect to the transformed data. Having transformed the data in this manner, it is possible to estimate the simplified models. It is always possible to do this by transforming the data by, for example, any left vector. The use of a simplified model with interpretable states has the advantage that the discrepancy between the data and the model can be

interpreted. That is, the influence of one state on another can often be given some physical meaning, so that a discrepancy in a feedback link can be useful in refining the larger model.

The development of tools such as we have considered for nonlinear models would be invaluable. The similarity of the singular perturbation simplification techniques and the modal simplification techniques suggest that this is possible. The singular perturbation simplification is valid in both linear and nonlinear models. The reason for this is essentially that the fast states in the singularly perturbed model can be grouped together under one "modal" heading. If it is possible to develop similar grouping of states in more general nonlinear models, it is likely that some headway can be made in this problem.

Appendix A1

The singular perturbation problem (Kokotovic, O'Malley and Sannuti 1976) is a method for seperating fast and slow dynamics. There is a sense in which this is equivalent to the model order reduction problem with dominant eigenvalues as done by Davison (1967). In this section we make that relationship exact.

The following equation will be assumed to give the dynamics of the system,

$$\dot{\underline{s}} = \underline{E}\underline{s} + \underline{u} \tag{A1.1}$$

where the dynamics matrix $\underline{E}$ can be written in partitioned form as

$$\underline{E} = \begin{vmatrix} \underline{F}_{11} & \underline{F}_{12} \\ 1/\epsilon\ \underline{F}_{21} & 1/\epsilon\ \underline{F}_{22} \end{vmatrix} \tag{A1.2}$$

with ϵ , the perturbation parameter, assumed very small.

If ϵ is sufficiently small then the singular perturbation simplfication of the system given in equation A1.1 and the simplification based on dominant modes will be very close. This is a consequence of the fact that for ϵ sufficiently small the smaller eigenvalues of $\underline{E}$ will be almost exactly those of $\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21}$ (Vidyasagar 1978, section 4.3). Thus the limiting case of a singular perturbation model simplification is a modal preservation model simplification.

The eigenvalues of the matrix $\underline{E}$ can be partitioned into the slow and

fast dynamics. This follows from the fact that (N-n) of the eigenvalues of $\underline{E}$ are of order $1/\epsilon$, while n of the eigenvalues are of constant order (Vidyasagar 1978, section 4.3). We take the eigenvalues of constant order as the dominant modes. And let $\underline{R}_1$ represent the right eigenvectors associated with these modes. Premultiplication of the matrix $\underline{E}$ by ϵ will not change the eigenvectors of the matrix. Thus $\underline{R}_1$ satisfies

$$\underline{F}_{21}\underline{R}_{11} + \underline{F}_{22}\underline{R}_{21} = \epsilon\, \underline{R}_{21}\underline{\Lambda}_{11} \tag{A1.3}$$

or premultiplying by the inverse of $\underline{F}_{22}$, which is assumed to exist, we have

$$\underline{F}_{22}^{-1}\underline{F}_{21}\underline{R}_{11} = -\underline{R}_{21} + \epsilon\, \underline{R}_{21}\underline{\Lambda}_{11} \ . \tag{A1.4}$$

Premultiplying by $\underline{F}_{12}$ and postmultiplying by the inverse of $\underline{R}_{11}$ we have

$$-\underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21} = \underline{F}_{12}\underline{R}_{21}\underline{R}_{11}^{-1} - \epsilon\, \underline{F}_{12}\underline{R}_{21}\underline{\Lambda}_{11}\underline{R}_{11}^{-1} \ . \tag{A1.5}$$

Substituting the identity

$$\underline{F}_{11}\underline{R}_{11} + \underline{F}_{12}\underline{R}_{21} = \underline{R}_{11}\underline{\Lambda}_{11} \tag{A1.6}$$

into the last term of equation A1.5 we have

$$-\underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21} = \underline{F}_{12}\underline{R}_{21}\underline{R}_{11}^{-1} - e\,\{(\underline{R}_{11}\underline{\Lambda}_{11} - \underline{F}_{11}\underline{R}_{11})\underline{\Lambda}_{11}\underline{R}_{11}^{-1}\} \ . \tag{A1.7}$$

The term in curly brackets in the above expression involves a similarity transformation of $\underline{\Lambda}_{11}$ and a similarity transformation of $\underline{\Lambda}_{11}^2$ and is

therefore of bounded magnitude. Because this term is premultiplied by ε it is clear that

$$\lim_{\varepsilon \to 0} \underline{F}_{12}\underline{R}_{21}\underline{R}_{11}^{-1} = -\underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21} \tag{A1.8}$$

The singular perturbation simplification of the model given in equation A1.2 has a dynamics matrix given by

$$\underline{F}_{11} - \underline{F}_{12}\underline{F}_{22}^{-1}\underline{F}_{21} \ . \tag{A1.9}$$

The dynamics matrix resulting from modal simplification is given by

$$\underline{F}_{11} + \underline{F}_{12}\underline{R}_{21}\underline{R}_{11}^{-1} \ . \tag{A1.10}$$

From equation A1.8 it is clear that the singular perturbation and modal simplification are equivalent in the limit.

The above result shows that singular perturbation simplification is in some sense a special case of modal simplification. In the case of singular perturbation it is possible to very clearly associate the different modes with different states. This association is not required in the modal simplification case, and it is in this sense that modal simplification is more general. The singular perturbation approach has the very strong advantage that it can be applied to nonlinear models. While this is not true for modal simplification, the close relationship between the two simplification techniques does suggest some potential for developing modal simplification techniques for nonlinear models.

Appendix A3

In this appendix we will present some discussion of the topics covered in chapter 3. First, an example that shows how the consideration of scalar participation factors alone can be misleading in the determination of the states important in a mode. Included in this example is a consideration of cross-modal participation factors. We then show that the inclusion of complex conjugate eigenvalues results in real simplified models. Finally, we present some useful results on eigenvalue elasticities.

Participation Factors and Cross-Modal Coupling

The scalar participation factors do not account for the possibility that states can be involved in a mode essentially through other modes. If a state is has zero entries in both the left and right eigenvectors for a given mode then the state cannot be involved in the mode. However, if the state has a zero entry in the left, or the right eigenvector this is not necessarily grounds to dismiss the state.

Consider a three level model with an $\underline{A}$ matrix given by

$$\begin{vmatrix} -1 & 3 & 3 \\ 2 & -1 & -2 \\ -2 & 3 & 4 \end{vmatrix} . \qquad \text{(A3.1)}$$

The matrix has eigenvalues of 2, 1 and -1 and the eigenvector matrices are given by

$$\underline{R} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \qquad \underline{L} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{vmatrix}. \tag{A3.2}$$

and the participation matrix is given by

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}. \tag{A3.3}$$

with the columns coresponding to the modes in the order 2,1,-1.

From the participation matrix it appears that the last state is solely responsible for both the eigenvalue 2 and the eigenvalue 1. And yet it is clear that the third states feedback with itself (4) is not generating either of these modes.

The example is clearly contrived, but it does illustrate an important point. The fact that a state does not have a nonzero entry in either a right or a left vector does not mean that the state is not important to the mode. The second state in the above example is influenced by the level of the second mode and influences the first mode. The first state, on the other hand, influences the level of the second mode, and is influenced by the first mode. And the impact is, apparently, quite important.

It is interesting to note the result of applying the other tools considered to the above model. If we concentrate on a single mode at a time the generalized participation factors are given by

$$2:\ \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}\quad 1:\ \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{vmatrix}\quad -1:\ \begin{vmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}\ . \qquad \text{(A3.4)}$$

When weighted by the elements of the A matrix the basic form of these matrices does not change. No matter how these matrices are added up we do not arrive at something indicating a nearly exactly simplifiable mode.

The cross model participation factors for the first root are given by

$$\text{right-vector-based}\ \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{vmatrix}\quad \text{left-vector-based}\ \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \qquad \text{(A3.5)}$$

And in these the first state and the second state do show up. The fact that both the matrices have more then one nonzero row indicates that we cannot just look at a single mode and a single state.

Real Coefficients

If whenever a complex root is included in the reduced-order model its conjugate is also included the resulting reduced order-model can be made to have real coefficients.

In order to show this we need the following two lemmas:

Lemma: (A3.6)

If the matrix $\underline{F}$ can be written as

$$\underline{F} = \left| \underline{f}^1 \; \bar{\underline{f}}^1 \; \underline{f}^2 \; \bar{\underline{f}}^2 \; \ldots \; \underline{f}^k \; \bar{\underline{f}}^k \; \underline{f}^{k+1} \; \underline{f}^{k+2} \; \ldots \; \underline{f}^n \right| , \tag{A3.7}$$

with the $^-$ denoting conjugation and the vectors $\underline{f}^{k+1} \ldots \underline{f}^n$ being real, and the matrix $\underline{G}$ can be written as

$$G = \begin{vmatrix} (\underline{g}^1)^T \\ (\underline{g}^{-1})^T \\ (\underline{g}^2)^T \\ (\underline{g}^{-2})^T \\ \vdots \\ (\underline{g}^k)^T \\ (\underline{g}^{-k})^T \\ (\underline{g}^{k+1})^T \\ \vdots \\ (\underline{g}^n)^T \end{vmatrix} \tag{A3.8}$$

with $\underline{g}^{k+1} \ldots \underline{g}^n$ real, then the product $\underline{FG}$ will be real.

The proof of this follows from writing out the definition of the product

$$\underline{FG} = \mid \underline{f}^1(\underline{g}^1)^T + \bar{\underline{f}}^1(\bar{\underline{g}}^1)^T + \ldots + \underline{f}^k(\underline{g}^k)^T + \bar{\underline{f}}^k(\bar{\underline{g}}^k)^T + \underline{f}^{k+1}(\underline{g}^{k+1})^T + \ldots + \underline{f}^n(\underline{g}^n)^T \mid \tag{A3.9}$$

$$= \mid \bar{\underline{f}}^1(\bar{\underline{g}}^1)^T + \underline{f}^1(\underline{g}^1)^T + \ldots + \bar{\underline{f}}\underline{k}(\bar{\underline{g}}^k)^T + \underline{f}^k(\underline{g}^k)^T + \underline{f}^{k+1}(\underline{g}^{k+1})^T + \ldots + \underline{f}^n(\underline{g}^n)^T \mid \tag{A3.10}$$

$$= \underline{\bar{F}\bar{G}} \quad . \tag{A3.11}$$

Lemma: (A3.12)

If the matrix $\underline{G}$ can be written as in equation A3.9 and in addition has a right inverse, then at least one right inverse $\underline{G}^-$ can be written as in equation A3.8

Suppose we write $\underline{G}^-$ in the form

$$\underline{G}^{-1} = \mid \underline{f}^1\ \underline{f}^{1'}\ \underline{f}^2\ \underline{f}^{2'} \ldots \underline{f}^k\ \underline{f}^{k'}\ \underline{f}^{k+1} \ldots \underline{f}^n \mid . \tag{A3.13}$$

Then from the fact that the product $\underline{G}(\underline{G}^-) = \underline{I}$ it follows that

$$(\bar{\underline{g}}^1)^T\underline{\bar{f}}^1 = \overline{((\underline{g}^1)^T\underline{f}^1)} = 1 = (\bar{\underline{g}}^1)^T\underline{f}^{1'} \tag{A3.14}$$

$$(\underline{g}^1)^T\underline{\bar{f}}^1 = \overline{(\bar{\underline{g}}^1)^T\underline{f}^1} = 0 = (\underline{g}^1)^T\underline{f}^{1'} \tag{A3.15}$$

$$(\underline{g}^2)^T\underline{\bar{f}}^1 = \overline{(\bar{\underline{g}}^2)^T\underline{f}^1} = 0 = (\underline{g}^2)^T\underline{f}^{1'} \ldots \quad . \tag{A3.16}$$

Combining these equations we get

$$\underline{G\bar{f}}^1 = \underline{Gf}^{1'} \ , \tag{A3.17}$$

it follows that using $\underline{\bar{f}}^1 = \underline{f}^{1'}$ will also yield a right inverse of G. The same holds true for $\underline{f}^{2'}$ . . . $\underline{f}^{k'}$ yielding the desired result.

Of course all that has just been done for the right inverse applies equally as well to the left inverse. Now we can consider the different elements of the simplified models considered. For the $\underline{\tilde{A}}$ matrix we have

$$\underline{M} = \underline{A}_{12}\underline{R}_{21}\underline{R}_{11}^{-} \tag{A3.18}$$

or

$$\underline{M} = \underline{L}_{11}^{-}\underline{L}_{12}\underline{A}_{21} \ , \tag{A3.19}$$

in both cases applying the lemmas to the two complex matrices, we see that the result can always be made to be real. Similarily for the $\underline{\tilde{B}}$ and $\underline{\tilde{D}}$ matrices the lemma A3.7 tells us that the matrix

$$\underline{\tilde{R}}_1 \ | \ \underline{L}_{11} \ \underline{L}_{12} \ | \tag{A3.20}$$

will be real as long as $\underline{\tilde{R}}_1$ is chosen to have conjugate columns in accordance with those of $\underline{L}_1$. And Lemma A3.13 guarantees that such a choice of $\underline{\tilde{R}}_1$ exists for the model based on the left vectors.

It can also be shown that if the Moore-Penrose generalized inverse is used then the resulting model will be real.

Eigenvalue Elasticities

In this section we sill present some simple but useful manipulative facts about eigenvalues elasticities and their relation to participation factors. Much of the information in the root sensitivities can be summarized using participation factors. The participation factor for the i'th state in the k'th mode is simply the product of the i'th element of the k'th right eigenvector with the i'th element of the k'th left eigenvector. To understand the meaning of this we begin with the definition of root sensitivity.

The first root sensitivity to be considered is the elasticity of the magnitude of the k'th root with respect to the (i,j) element of the $\underline{A}$ matrix. The magnitude of a root is given by

$$(\lambda_k \bar{\lambda}_k)^{.5} \tag{A3.21}$$

where λ_k represents the root and the overbar denotes complex conjugation. The derivative of this with respect to the (i,j) element of $\underline{A}$ is therefore given by

$$\frac{1}{(\lambda_k \bar{\lambda}_k)^{.5}} \text{ real } (\bar{\lambda}_k \frac{\partial}{\partial a_{ij}} \lambda_k) \tag{A3.22}$$

$$= \frac{1}{(\lambda_k \bar{\lambda}_k)^{.5}} \text{ real } (\bar{\lambda} (\underline{l}^k)_i (\underline{r}^k)_j) \tag{A3.23}$$

with (real) denoting the the operator taking the real part of a complex

number. ($\underline{l}^k$ and $\underline{r}^k$ are the right and left eigenvectors normalized to have an inner product of 1.)

The elasticity of the magnitude with respect to the (i,j) element of $\underline{A}$ is therefore given by

$$\eta(|\lambda_k|, a_{ij}) = \frac{a_{ij}}{(\lambda_k \bar{\lambda}_k)} \text{ real } (\bar{\lambda}_k (\underline{l}^k)_i (\underline{r}^k)_j) . \quad \text{(A3.24)}$$

Summing these across rows and noting that we can interchange the order of the summation operator and the real operator (both of which are linear) we have

$$\Sigma_{i=1}^N \eta(|\lambda_k|, a_{ij}) = \frac{1}{(\lambda_k \bar{\lambda}_k)} \text{ real } (\Sigma_{i-1}^N \{\bar{\lambda}_k (\underline{l}^k)_i a_{ij} (\underline{r}^k)_j\}) \quad \text{(A3.24)}$$

$$= \frac{1}{(\lambda_k \bar{\lambda}_k)} \text{ real } (\bar{\lambda}_k \lambda_k (\underline{l}^k)_j (\underline{r}^k)_j) \quad \text{(A3.25)}$$

$$= \text{real } ((\underline{l}^k)_j (\underline{r}^k)_j) . \quad \text{(A3.26)}$$

The second equality following because $\underline{l}^k$ is a left eigenvector for $\underline{A}$ and the last equality following becasue $\bar{\lambda}\lambda$ is real and therefore can be taken outside the real operator. The column sum by precisely the same argument is given by

$$\Sigma_{j-1}^N \eta(|\lambda_k|, a_{ij}) = \text{real } ((\underline{l}^k)_i (\underline{r}^k)_i) . \quad \text{(A3.27)}$$

The elasticities of root magnitude with respect to the coefficients in the $\underline{A}$ matrix have equal sums across rows and columns and are equal to the participation factors. The participation factor can, thus, be interpreted as the elasticity of response of the root magnitude to a change of all elements in the i'th row (or column) of the $\underline{A}$ matrix. That is, if every entry in the i'th column of $\underline{A}$ were increased by 1%, then the magnitude of the k'th root would change by 1% times the real part of the participation factor. Changing every element in the i'th column is essentially changing the importance of the i'th state in determining the behavior of the system. It follows that the participation factors can be interpreted as the importance of the i'th states in determining the k'th mode.

The sum across states of the participation factors as defined above is one. Such a sum represents the elasticity of response of the magnitude of the roots with respect to an equal percentage change in all the elements of the $\underline{A}$ matrix. Such a change in a difference equation setting will decrease uniformly the stability of the roots. The sum across all roots is also one. This is because the dynamics of every state can be decomposed into the effects of various roots. Since summing the effects of these roots up will always return the original dynamics, the sum across roots is equivalent to the effect of a state on itself. This, logically, should be one.

The participation factors with respect to the period of the i'th root are derived in a similar manner. The elasticity of the period implied by the k'th root with respect to the (i,j) element of $\underline{A}$ is given by

$$\frac{a_{ij}}{\frac{2\pi}{\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right)}} \frac{\partial}{\partial a_{ij}} \frac{2\pi}{\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right)} \tag{A3.28}$$

$$= \frac{1}{\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right) \text{real}(\lambda)^2} \cos^2\left(\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right)\right) \text{imag}\left(\bar{\lambda}_k (\underline{l}^k)_i (\underline{r}^k)_j\right) \cdot \tag{A3.29}$$

We can again sum this across columns or rows as we did above. The sums will be the same and yield the following expression

$$\text{PERPART}(k,i) = \frac{\lambda_k \bar{\lambda}_k}{\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right) \text{real}(\lambda)^2} \cos^2\left(\tan^{-1}\left(\frac{\text{imag}(\lambda)}{\text{real}(\lambda)}\right)\right) \text{imag}\left((\underline{l}^k)_i (\underline{r}^k)_i\right) \tag{A3.30}$$

PERPART(k,i) can be interpretted as the elasticity of the period associated with a root with respect to a change in all elements of a row or column. The sum over all states of perpart is zero. This is expected since the sum over all states measures the effect of changing all the entries in the $\underline{A}$ matrix by some percentage. Such an alteration is the same as multiplying the $\underline{A}$ matrix by a constant, and this will affect the magnitude of all roots but not any of the periods. The sum across conjugate roots is generally nonzero and does not appear to be easily interpretable.

Appendix A6

The various simplified model $\underset{\sim}{A}$ matrices are reported in this appendix. The matrices are 15 by 15 and they are written with the names of the state variables above and next to the elements of the $\underset{\sim}{A}$ matrix.

$\underset{\sim}{A}$ Matrix for Ten-Mode Model

	AUTOS(-1)	IPD072(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	0.65149	-0.0133	-0.01385	0.01277	0.02239
IPD072	-0.56608	1.07652	0.4429	0.08852	-0.08788
M1BPLUS	0.26148	0.21063	0.82981	-0.27863	-0.38537
M72	0.04403	0.06121	0.10866	0.60542	-0.05499
PCS	-0.01222	-0.01245	-0.01241	0.01099	1.40795
PPNF	-0.11565	-0.0167	0.02772	-0.02547	0.0518
RUM	0.0749	-0.01943	-0.02194	0.04932	-0.0323
YPWS	-0.27316	0.20753	0.2637	0.07641	0.24491
GNPAVEQ(-2)	9.81554	-1.46679	-2.17031	4.11267	-7.92245
PCS(-1)	0.	0.	0.	0.	1.
PPNF(-1)	0.	0.	0.	0.	0.
PPNF(-2)	0.	0.	0.	0.	0.
RUM(-1)	0.	0.	0.	0.	0.
UCEAVEQ(-4)	0.18629	-0.07627	-0.09135	0.02581	-0.06238
YPERM72(-1)	0.13449	-0.10716	-0.1098	-0.2015	-0.86578

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	0.09687	-0.48501	0.04242	0.00075	-0.009
IPD072	0.89704	-0.16336	-0.26352	-0.06242	-1.00418
M1BPLUS	-0.36579	0.02646	-0.07779	-0.01436	-0.3734
M72	0.07859	-0.12708	-0.02148	-0.00315	-0.1511
PCS	0.00645	-0.00796	0.00845	0.00107	-0.52235
PPNF	1.06693	0.00728	-0.00281	0.00009	-0.08569
RUM	-0.00163	0.94981	-0.00017	0.00164	0.08934
YPWS	0.57705	-0.57761	0.58185	-0.01606	-0.29677
GNPAVEQ(-2)	-7.72097	-6.31567	-1.84307	0.55349	6.40448
PCS(-1)	0.	0.	0.	0.	0.
PPNF(-1)	1.	0.	0.	0.	0.
PPNF(-2)	0.	0.	0.	0.	0.
RUM(-1)	0.	1.	0.	0.	0.
UCEAVEQ(-4)	-0.2157	-0.03621	0.02457	0.00655	0.22276
YPERM72(-1)	-0.69085	0.05204	0.67613	-0.01046	-1.20793

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	0.02021	-0.08633	0.4152	-0.03288	-0.02082
IPD072	-0.21808	-1.21061	0.373	5.25582	0.05927
M1BPLUS	2.46708	-0.98264	0.06504	0.59918	0.0371
M72	-0.04588	-0.18221	-0.02123	0.11644	0.01324
PCS	-0.01153	-0.02388	-0.00047	-0.01881	-0.01017
PPNF	-0.1414	-0.27096	0.1274	-0.02624	0.00613
RUM	0.02911	-0.00833	-0.19137	-0.12905	-0.00544
YPWS	-0.13814	-0.84177	0.53811	1.52159	-0.09613
GNPAVEQ(-2)	1.64186	4.95323	-17.8969	-0.77557	-0.59328
PCS(-1)	0.	0.	0.	0.	0.
PPNF(-1)	0.	0.	0.	0.	0.
PPNF(-2)	1.	0.	0.	0.	0.
RUM(-1)	0.	0.	0.	0.	0.
UCEAVEQ(-4)	0.05156	0.23945	-0.2345	0.07026	-0.01518
YPERM72(-1)	-0.23438	0.49034	0.08908	0.06426	0.71761

A Matrix for Fifteen Mode Model

	AUTOS(-1)	IPDQ72(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	0.65949	0.01763	0.00744	0.0542	0.09495
IPDQ72	1.20233	0.5327	-0.02129	-0.3437	-2.55189
M1BPLUS	-0.09553	0.28907	0.98057	-0.19481	-1.67281
M72	1.11109	-0.04018	-0.0047	0.77271	-2.1225
PCS	-0.0471	0.0232	0.00292	-0.00229	1.41551
PPNF	-0.00244	0.02273	-0.00033	0.00036	-0.07737
RUM	0.04164	-0.01216	0.00042	0.00364	0.09619
YPWS	0.36549	-0.30495	-0.00703	-0.46018	-5.01547
GNPAVEQ(-2)	16.7186	-0.3768	-0.09429	0.36889	-44.9999
PCS(-1)	-0.	-0.	0.	-0.	1.
PPNF(-1)	-0.	-0.	0.	-0.	0.
PPNF(-2)	0.	0.	-0.	0.	-0.
RUM(-1)	0.	0.	-0.	0.	-0.
UCEAVEQ(-4)	-0.025	0.01982	0.00228	0.05847	-0.02705
YPERM72(-1)	1.25701	0.66006	0.10627	0.51739	-0.48014

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	0.18191	-0.15125	0.02218	-0.0037	-0.13256
IPDQ72	2.40647	-5.79117	-0.04985	0.03428	2.20526
M1BPLUS	-1.6138	-0.26058	-0.03603	-0.03817	0.06455
M72	1.01262	-2.6071	-0.00363	0.01265	1.497
PCS	0.56965	0.58748	0.01149	-0.0026	-0.58377
PPNF	2.4425	0.43097	0.00055	-0.00267	-0.00481
RUM	-0.02194	1.43116	-0.00082	0.00223	-0.01096
YPWS	-0.41095	-8.15901	1.0237	0.04911	4.25656
GNPAVEQ(-2)	29.0416	25.2456	0.00473	0.5451	28.8907
PCS(-1)	-0.	0.	0.	0.	-0.
PPNF(-1)	1.	0.	0.	0.	-0.
PPNF(-2)	0.	-0.	-0.	-0.	0.
RUM(-1)	0.	1.	-0.	-0.	0
UCEAVEQ(-4)	-0.11304	1.15537	0.00805	-0.00876	-0.06967
YPERM72(-1)	10.083	9.69307	0.35582	-0.09306	-2.69245

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	-0.0257	-0.13288	0.16595	0.24846	-0.024
IPDQ72	-4.09865	1.90976	5.68783	-1.22812	0.06643
M1BPLUS	5.43461	-2.78901	0.92889	2.14443	0.0838
M72	-1.21306	0.59136	2.75641	-1.01909	0.01681
PCS	-0.75048	0.28327	-0.49891	0.20637	-0.00737
PPNF	-2.10004	0.70655	-0.37862	0.15626	0.00322
RUM	0.02392	-0.0526	-0.4869	-0.20039	-0.00354
YPWS	1.64417	-0.83825	8.20359	-2.72932	0.03182
GNPAVEQ(-2)	-37.5534	18.3299	-17.7581	-5.67444	0.50035
PCS(-1)	0.	-0.	-0.	-0.	-0.
PPNF(-1)	0.	-0.	-0.	-0.	-0.
PPNF(-2)	1.	0.	0.	0.	0.
RUM(-1)	-0.	0.	0.	0.	0.
UCEAVEQ(-4)	0.30407	-0.13351	-1.06851	1.04381	-0.00591
YPERM72(-1)	-14.531	6.35592	-7.84721	5.15023	0.64906

$\hat{A}$ Matrix From Estimation on Transformed Data with $\hat{B}=\tilde{B}$

	AUTOS(-1)	IPDQ72(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	3.61184	0.31475	-0.00508	1.86452	-8.74455
IPDQ72	-8.4276	-1.19704	0.048	-7.2522	27.9707
M1BPLUS	21.4446	2.05398	0.92205	8.54717	-63.7964
M72	1.33371	1.84243	0.07652	4.03545	-3.36781
PCS	0.17651	0.30161	0.03077	0.71185	-1.08626
PPNF	-0.17581	0.5382	0.0378	0.79383	-2.17019
RUM	-1.60861	-0.25548	0.00638	-0.92933	4.75289
YPWS	6.36263	3.68343	0.23809	8.37989	-33.9277
GNPAVEQ(-2)	109.41	65.1855	1.72237	166.01	-290.545
PCS(-1)	0.43587	0.24386	0.02706	0.74067	-2.04463
PPNF(-1)	-0.24123	0.48541	0.03427	0.70565	-1.95941
PPNF(-2)	-0.34424	0.46121	0.02819	0.60769	-1.66725
RUM(-1)	-2.36482	-0.48148	0.00226	-1.63968	6.78311
UCEAVEQ(-4)	1.94513	0.69756	0.00941	2.3127	-5.41979
YPERM72(-1)	16.2463	10.0149	0.61407	22.4783	-53.811

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	-1.72048	-3.69862	-0.07839	-0.08932	4.9095
IPDQ72	23.3725	-10.4287	0.39452	0.44937	-13.5281
M1BPLUS	-42.6985	7.62343	-0.4995	-0.29939	41.0774
M72	-4.24436	26.2744	0.21814	-0.32468	-3.14952
PCS	0.48051	3.39866	0.11523	-0.04864	0.04513
PPNF	0.41616	9.3072	0.16233	-0.0747	-0.29641
RUM	1.23878	1.3455	0.02817	0.05007	-2.42208
YPWS	-0.62107	35.4103	1.84791	-0.61199	9.63425
GNPAVEQ(-2)	-152.588	691.052	1.65755	-13.17	2.98569
PCS(-1)	0.38655	1.59845	0.10111	-0.04152	1.06124
PPNF(-1)	-1.45657	8.67726	0.15191	-0.06613	-0.24839
PPNF(-2)	-3.23452	8.62598	0.13738	-0.06087	-0.31414
RUM(-1)	2.20722	-0.90035	0.03441	0.09822	-3.13746
UCEAVEQ(-4)	-2.58148	4.56144	-0.03768	-0.16796	1.56549
YPERM72(-1)	1.92894	117.903	1.64341	-1.80865	4.07046

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	6.69395	-2.60635	5.95169	5.07686	0.08486
IPDQ72	-45.7619	13.8919	0.93735	-23.474	-0.4959
M1BPLUS	86.16	-30.3438	15.9578	8.90591	0.83065
M72	10.7308	-2.52271	-22.0841	19.6884	-0.11786
PCS	3.35298	-2.49823	-2.7317	3.58	-0.149
PPNF	5.15817	-3.04619	-8.23856	5.0243	-0.18684
RUM	-3.82998	1.19051	-1.93199	-2.87393	-0.04529
YPWS	34.0288	-18.282	-23.691	41.4539	-0.94553
GNPAVEQ(-2)	446.274	-113.704	-519.914	826.219	1.4488
PCS(-1)	3.71383	-2.78865	-0.95049	3.15962	-0.13969
PPNF(-1)	7.55355	-3.7378	-7.79052	4.43399	-0.17989
PPNF(-2)	9.18214	-3.74502	-7.91839	4.00695	-0.16514
RUM(-1)	-6.46088	2.05081	-0.54177	-5.65115	-0.06084
UCEAVEQ(-4)	7.22372	-2.20828	-2.37689	10.5737	0.06064
YPERM72(-1)	52.1935	-23.112	-86.0567	112.87[illegible]	-0.45866

$\tilde{A}$ Matrix From Estimation on Transformed Data With $\hat{B}$ Only Restricited to Retain Column Dependencies

	AUTOS(-1)	IPD072(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	0.82694	0.50193	0.04595	1.49408	-2.15999
IPD072	7.86758	-2.50359	-0.2342	-4.19769	-12.578
M1BPLUS	-5.59405	4.5879	0.97514	5.18506	1.01516
M72	-6.39561	2.61167	0.16704	1.70966	13.7485
PCS	1.33214	0.33767	0.03442	1.25628	-4.73206
PPNF	0.58589	0.7574	0.06578	1.44467	-4.64728
RUM	-0.4927	-0.42007	-0.02054	-0.95683	1.76245
YPWS	7.77153	4.46599	0.4424	12.2425	-44.1589
GNPAVEQ(-2)	-257.385	92.7494	5.62565	49.5231	504.565
PCS(-1)	1.87922	0.2293	0.02201	1.32332	-6.27026
PPNF(-1)	0.35515	0.75809	0.0719	1.41885	-4.27354
PPNF(-2)	0.0068	0.81589	0.07956	1.40418	-3.80368
RUM(-1)	0.14809	-0.74217	-0.04174	-1.28927	0.40863
UCEAVEQ(-4)	-2.67449	1.01551	0.06057	0.91414	4.62063
YPERM72(-1)	-11.078	12.009	0.79731	14.4274	7.11975

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	-3.10951	1.26984	0.07068	-0.1046	-0.44665
IPD072	30.0352	-49.915	-0.55479	0.51827	19.6063
M1BPLUS	-50.1819	25.1483	0.84741	-0.55265	-16.3248
M72	-3.69866	62.6686	0.68267	-0.34957	-16.7803
PCS	0.52836	3.44611	0.12063	-0.054	2.64997
PPNF	0.5536	13.9469	0.25483	-0.09772	0.92064
RUM	2.61285	-0.85829	-0.05332	0.07004	0.41179
YPWS	-12.3763	54.3107	2.17831	-0.72902	15.8267
GNPAVEQ(-2)	-108.699	2149.11	21.0778	-13.5753	-612.212
PCS(-1)	0.08667	-0.4561	0.0798	-0.04197	4.17797
PPNF(-1)	-1.43803	14.6034	0.26619	-0.09697	0.67818
PPNF(-2)	-3.31453	16.5933	0.2861	-0.10333	0.2261
RUM(-1)	3.87368	-7.19608	-0.12065	0.12001	2.35678
UCEAVEQ(-4)	-2.40889	20.9364	0.19669	-0.17227	-6.21136
YPERM72(-1)	-4.26305	215.385	3.0715	-1.78018	-42.2716

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	10.9244	-6.03681	-1.25863	7.47616	-0.13443
IPD072	-65.474	30.7157	52.2817	-35.8287	0.81252
M1BPLUS	120.495	-58.8964	-35.9985	46.3604	-1.42023
M72	7.71902	-2.21728	-61.2172	23.0995	-0.70557
PCS	4.48924	-3.05685	-1.39863	3.58853	-0.19212
PPNF	7.13472	-4.40857	-11.1264	6.46913	-0.34103
RUM	-7.72197	3.70562	1.18368	-4.98345	0.07233
YPWS	64.841	-34.7485	-36.7699	48.1878	-1.75307
GNPAVEQ(-2)	280.912	-101.019	-2148.34	943.865	-22.4422
PCS(-1)	5.42116	-3.51418	2.40546	2.83466	-0.1534
PPNF(-1)	10.1307	-5.50111	-11.8652	6.43937	-0.36007
PPNF(-2)	12.511	-6.03256	-13.8737	6.84162	-0.38477
RUM(-1)	-10.9405	5.26631	7.58415	-8.42046	0.15453
UCEAVEQ(-4)	6.32585	-2.80676	-21.2238	12.158	-0.2348
YPERM72(-1)	58.0134	-30.9036	-200.672	120.455	-2.74324

$\hat{A}$ Matrix From Estimation on Untransformed Data with $\hat{B}=\tilde{B}$

	AUTOS(-1)	IPDQ72(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	0.13931	-0.38112	-0.00529	0.20635	-0.64233
IPDQ72	1.62953	1.36106	0.0742	-0.83662	0.16613
M1BPLUS	3.59996	2.29711	0.74868	2.71095	-8.79468
M72	-0.36506	-0.75267	-0.0133	0.82659	1.71449
PCS	-0.43524	-0.18381	0.00967	-0.10585	2.34801
PPNF	-1.00599	-0.29543	0.01345	-0.15144	2.58846
RUM	0.00688	0.10736	0.00273	-0.05719	0.03792
YPWS	-3.51106	-2.60942	0.11118	-0.86899	8.98691
GNPAVEQ(-2)	2.05027	12.452	-1.31358	24.9422	-26.118
PCS(-1)	-0.19956	-0.14925	0.00522	-0.12117	1.99828
PPNF(-1)	-0.76187	-0.35739	0.00654	-0.12119	2.20242
PPNF(-2)	-0.83041	-0.37394	0.0034	-0.11889	2.46812
RUM(-1)	0.08584	0.05316	0.0157	-0.1823	0.29628
UCEAVEQ(-4)	0.07868	0.1409	-0.02076	0.30723	-0.42059
YPERM72(-1)	-0.31105	0.70252	-0.02279	1.96173	-1.40797

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	0.33539	-0.47454	-0.22006	-0.03318	1.95866
IPDQ72	-2.15196	2.98126	1.16768	0.14164	-7.16038
M1BPLUS	20.7849	5.2991	-2.77849	-0.02765	28.1315
M72	-0.86946	-2.26384	-0.3338	-0.02549	0.73901
PCS	-1.02327	-1.6352	0.04683	-0.00867	-1.83716
PPNF	-0.82803	-2.92708	-0.00401	-0.00881	-2.56526
RUM	0.0096	1.08251	0.09784	0.01381	-0.59935
YPWS	-8.16206	-13.4483	0.95037	-0.19746	-8.96167
GNPAVEQ(-2)	100.94	48.7598	-20.2007	-0.54539	147.211
PCS(-1)	-1.12036	-1.47655	0.05239	-0.00266	-1.45922
PPNF(-1)	-0.7004	-3.05637	-0.04263	-0.0157	-2.07258
PPNF(-2)	-1.76423	-3.32967	-0.0667	-0.02048	-2.12918
RUM(-1)	-0.56784	1.02029	0.20678	0.02111	-1.53654
UCEAVEQ(-4)	1.21493	0.56119	-0.26014	-0.01954	1.93827
YPERM72(-1)	6.80099	3.76846	-1.42992	-0.11165	9.76191

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	-0.58267	0.23114	-1.32903	-8.35991	0.0285
IPDQ72	1.95017	-0.66352	5.01575	32.3095	-0.15644
M1BPLUS	-24.7748	4.86796	-30.6456	-100.142	-0.34461
M72	-0.63935	1.4759	-0.84601	-8.74004	0.08208
PCS	1.131	-0.02987	1.61446	2.24516	0.03611
PPNF	1.89811	-0.11996	2.44984	2.42765	0.06618
RUM	0.15372	-0.20274	0.50022	2.53938	-0.01285
YPWS	5.76873	1.84205	10.4666	9.27752	0.4867
GNPAVEQ(-2)	-110.337	18.4161	-188.952	-751.645	-3.52391
PCS(-1)	1.22552	-0.10993	1.47495	2.77752	0.02175
PPNF(-1)	1.83499	0.02222	2.29392	0.77526	0.07463
PPNF(-2)	2.87618	0.03088	2.38862	-0.08486	0.08562
RUM(-1)	0.61482	-0.16004	1.4268	6.35689	-0.0045
UCEAVEQ(-4)	-1.2941	0.18967	-2.37876	-8.41354	-0.0452
YPERM72(-1)	-9.29589	2.49777	-13.7574	-70.1942	0.62841

$\tilde{A}$ Matrix From Estimation on Untransformed Data With $\hat{B}$ Only Restricited to Retain Column Dependencies

	AUTOS(-1)	IPDQ72(-1)	M1BPLUS(-1)	M72(-1)	PCS(-1)
AUTOS	0.28062	-0.41206	0.00826	0.0135	-0.20967
IPDQ72	0.02993	0.74029	0.01558	0.02816	0.21408
M1BPLUS	-1.35314	-1.13358	1.00823	-0.09723	-2.75406
M72	0.2931	-0.70548	-0.01817	0.56684	1.14216
PCS	-0.19728	0.03197	0.00537	0.04663	1.05842
PPNF	-0.28483	0.03623	0.01384	-0.01417	-0.36344
RUM	-0.15471	0.09499	0.00081	0.03479	-0.27789
YPWS	0.80325	-2.13741	0.0479	-0.19993	1.01559
GNPAVEQ(-2)	-0.14636	0.42045	0.10725	0.26888	3.27452
PCS(-1)	-0.	-0.	0.	-0.	1.
PPNF(-1)	-0.	-0.	0.	-0.	-0.
PPNF(-2)	0.	-0.	0.	-0.	-0.
RUM(-1)	0.	-0.	0.	-0.	-0.
UCEAVEQ(-4)	-0.0059	0.00306	-0.00015	-0.00156	0.02353
YPERM72(-1)	0.1054	0.36937	0.05741	0.0392	-1.01097

	PPNF(-1)	RUM(-1)	YPWS(-1)	GNPAVEQ(-3)	PCS(-2)
AUTOS	-0.26059	-1.00734	-0.0383	-0.01759	0.18419
IPDQ72	0.03706	0.01637	0.01645	0.00613	-0.49481
M1BPLUS	-1.68289	-0.22155	-0.18808	-0.18416	4.91031
M72	-1.25784	-3.0443	-0.09688	-0.0564	-0.30181
PCS	0.21555	0.07753	-0.01369	-0.00499	-0.07228
PPNF	0.93122	0.34	0.01635	0.00677	0.32694
RUM	0.25379	1.0147	-0.00974	0.00276	0.27863
YPWS	-1.81544	-4.10344	0.76549	-0.00472	0.4382
GNPAVEQ(-2)	-2.16639	1.9154	0.12662	0.81278	-3.41844
PCS(-1)	-0.	0.	0.	-0.	0.
PPNF(-1)	1.	0.	0.	-0.	0.
PPNF(-2)	-0.	0.	0.	-0.	0.
RUM(-1)	-0.	1.	0.	-0.	0.
UCEAVEQ(-4)	-0.00502	-0.00434	0.00061	-0.00078	-0.00872
YPERM72(-1)	-1.1465	1.74789	0.25441	0.00298	0.04199

	PPNF(-2)	PPNF(-3)	RUM(-2)	UCEAVEQ(-5)	YPERM72(-2)
AUTOS	0.15477	0.07406	0.2585	-0.91473	0.03368
IPDQ72	0.0527	-0.0355	0.31054	-0.00683	-0.00411
M1BPLUS	-0.02321	-0.4482	0.91891	-24.7046	0.25632
M72	0.57246	0.39418	1.21019	-1.06425	0.12756
PCS	-0.21269	0.05312	-0.10626	-0.62453	-0.0011
PPNF	-0.03353	-0.04533	-0.2262	1.61346	-0.01339
RUM	-0.10716	-0.04839	-0.17093	-0.06162	-0.00582
YPWS	-0.29654	1.00344	1.06649	4.95522	-0.00421
GNPAVEQ(-2)	0.52526	-0.2957	-2.70511	2.52631	0.06911
PCS(-1)	-0.	0.	-0.	0.	0.
PPNF(-1)	-0.	0.	-0.	0.	0.
PPNF(-2)	1.	0.	-0.	0.	0.
RUM(-1)	-0.	0.	0.	0.	-0.
UCEAVEQ(-4)	-0.00478	-0.00015	0.00497	0.94419	0.00166
YPERM72(-1)	-0.10812	0.13368	0.8384	-13.3988	0.89236

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