A STUDY OF FISH TYPE MOTION AND PROPULSION SYSTEMS

by

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ABSTRACT

A theoretical study has been made of the thrust and efficiency of fish-type propulsion systems. Fish are modelled as long, round three-dimensional flexible cylinders in an incompressible irrotational fluid. The assumed swimming motion consists of plane waves progressing axially along the cylinder with exponentially growing amplitude and constant wave number and phase velocity.

The velocity field surrounding the cylinder is determined by finding solutions to La Place's equation subject to appropriate geometrical surface-compatibility conditions. The requirement that energy be radiated from the fish into the environment is satisfied by using Hankel functions of the second type (with complex arguments) as solutions. Pressures are determined from the time-dependent Bernoulli equation. Work is then found by pressure-times-area calculations, and the integration of squared Hankel functions is required for the determination of thrust by momentum methods. All results are presented in closed-form expressions.

In addition to expressions for velocity, work, thrust and efficiency, the following topics are discussed: boundary conditions for a cylinder whose dimensions vary with axial distance; fluid-fish force interactions; three-dimensional virtual mass; and first-law derivations for thrust. Conclusions consider the dependence of swimming efficiency on wave number, and on phase velocity.

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TABLE OF SYMBOLS

(Including only symbols used throughout this work and not locally used forms).

Symbol	Definition
$h(x,t)$	Displacement of the fish centerline from the x axis
x, r, θ	Three dimensional polar coordinates
V_r, V_x, V_θ	Components of the velocity field
$\bar{u}$	Stream velocity
u'	Stream-wise perturbation velocity
F	$= F(r,\theta,x,t) = 0$ equation of the fish surface.
r^*	Usually the perpendicular distance between the fish center line and surface
r_s	r, evaluated at the surface
$D(\,)/Dt$	The substantive time derivative
$(V_r)_s$	The radial velocity at the surface
p	Complete or total pressure
p_{con}	The constant part of the total pressure
p'	$= p_{lin}$, the linearized perturbation pressure.
n	= 1, descriptive of the separable angular variable
ω	The angular frequency, a real number
k	Wavenumber, a complex number
J_n	Bessel function of the first kind of order n
Y_n	Bessel function of the second kind of order n
H_n^1	Bessel function of the third kind of order n; Hankel type 1 function
H_n^2	Besselffunction of the third kind of order n; Hankel type 2 function.

C	$= C_{wk}$, the field constant
ϕ	Velocity Potential
M	Virtual mass (two dimensional)
$\dot{W}$	Rate of doing work
T	Thrust
α	Dimensionless frequency = $wr^*/\bar{u}$
Z	$= kr^*$
D	Drag
B	Amplitude of $h(x,t)$
α	$= \omega r^*/\bar{u}$, a dimensionless frequency, (called FREQ or OMEGA in FORTRAN)

SUMMARY

Chapter I descibes the kinematic boundary condition which is satisfied by the fish within a nonviscous fluid. This velocity field is formed from the separable solutions to the La Place' equation in three dimensions, and specialized to satisfy the requirements of the condition that energy be radiated outward from the fish. The pressure is found from the general time-dependent Bernoulli equation. Velocities and pressures are expressed in terms of Hankel functions of the second type. In Chapter I fish are modelled as long cylinders with constant cross section. The problem of the dynamics of a finite fish and of an elliptic cross sectioned fish are sketched in the appendicies.

Chapter II uses the resulting forms of the velocity and pressure fields to calculate the fish's thrust by momentum methods, and work by direct integration of the pressure field at the surface. An expression for efficiency is presented. Work and thrust are then related by energy methods which clarify the differences between this theory and the slender-body theory.

Chapter III presents diagramatically the velocity field and evaluates work and thrust under certain fish movements. Evaluation of these quantities involves integration of squared Hankel functions; this integration is completed in Appendix C. The source programs for evalutation of the

work, thrust and velocities on the IBM7090 are written in Fortran and presented in Appendix D.

The concluding chapter considers the physical interpretation of the graphical results and expands upon the applications of the theory presented in Chapters I and II.

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INTRODUCTION

There has been considerable interest in the past few decades in studying the physics of fish locomotion because of possible applications to efficient propulsion of boats and ships of all types. The study of biomechanisms such as are found in snakes, birds, bats and other animals has led to many practical and productive insights for pure and applied science. However, the study of animal locomotion in fluids has not yet matured sufficiently to contribute to practical mechanical propulsion systems, or for that matter, to the actual biological principles which may be correlated with fluid mechanics.

For at least one half of a century biologists have observed the remarkable behavior of fish. Bainbridge[14*] has noted that the speed of the swimming fish is independent of surface shape, and Gray[11,12] has observed that fish seem to disobey the standard rigid body fluid mechanical drag laws. This anomaly has been popularly called the "fish paradox." In fact, Gray[13] has also observed that amputation of the rear fin of certain fish does not significantly alter their "powers of locomotion." (Amputation did however alter the wave configuration on their backs). Gray[11] also noticed that when fish are placed on a peg board, they arch their backs in such a way as to brace themselves against the pegs

* Numbers placed in such a manner refer to the Bibliography. Bainbridge and other authors mentioned in the Introduction are discussed further beginning on page 4.

to gain forward momentum by reaction.

Gray's and Bainbridge's observations and an experimental paper by Rosen[15] on the vortex motions generated in the wake of fish, caused this author to believe that the actual body configuration in swimming, represented geometrically by wave number, uniquely determines swimming efficiency. It was also believed that the three dimensional virtual mass of the fluid associated with fish motion might somehow be construed to be a variable and to be a factor upon which the efficiency would depend. The former intuition concerning wave number is, it is hoped, substantiated in Chapter II of this paper; the latter statement concerning virtual mass is made plausible in Appendix B.

It appeared wisest to begin a study of fish motion with Lighthill's[2] excellent and conceptually simple paper which is based on the slender-body theory. Instead of extending Lighthill's work which neglects the velocity field details, the next step was chosen to be a description of this field. Thus a linearized surface condition was written and coupled with the general inviscid equation of fluid motion. The solution for the flow field was derived from the three dimensional La Place equation. This solution was written in terms of Hankel functions of the second type to satisfy a directional radiation condition.

To calculate thrust and work, we attempted temporarily to use a two dimensional virtual mass theory, similiar to

Lighthill

Lighthill's approach. In this method we employed a streamwise perturbation. This involved a detailed energy approach following the First Law. However, it soon became apparent that using a perturbation velocity with a two dimensional virtual mass involved contradictory ideas. This method of calculating work etc. was abandoned altogether in favor of a simple and more straight-forward approach.

This latter approach was based upon integration of pressures and momentum considerations. In order to calculate the thrust exactly, we were forced to integrate analytically squared Hankel functions between a finite radius and infinity. It was then found that energy input, thrust and thus efficiency depended upon wave number. These conclusions concerning the correlation between wavenumber, work, thrust and efficiency are not simple to interpret physically, because such relationships are not common to steady state rigid body fluid dynamics. Certain comments concerning thrust and wave number have however been made by van Karman.[13] As far as we know, he was the first to see such relationshps; his comments being confined to two-dimensional cases.

After gaining some confidence in the plausibility of our solutions, the resulting functions of Hankel function with complex arguments were programmed for computation on the IBM 7090, so that they could be graphically plotted.

A good historical review of the fish swimming paradox and of the attempts to harness swimming type movements to produce thrust can be found in Fraize[17]. To conclude this introduction, we shall indicate only the more important contributions to the problem of the swimming flexible fish.

Three theoretical papers only have been written on the three-dimensional model of the fish.[2,9,10]. Lighthill[2] investigates the movements of a flexible finite three dimensional fish. He employs the approximations of the slender body theory which he reviews in reference 3. His analysis is based fundamentally on the assumptions that there is no streamwise perturbation velocity and that the total flow is the sum of the flows due to fluid motions past a stretched straight fish and those due to the two dimensional flow, lateral to the body. Therefore, he assumes that the usual two dimensional concept of virtual mass leads to correct expressions for the fish's thrust, work and efficiency.

Taylor[10] gives a finite amplitude analysis employing empirical flow field data from experiments on long cylinders to calculate lateral forces. Taylor's model is infinitely long and has a constant cross section which is invariant with distance.

The third three dimensional analysis is given by Cummings[9] who is generally neglected in the literature. He uses a completely different approach from either Lighthill or Taylor. Cummings calculates the forces and moments on an elongated body which has time variant motion from a potential field which is caused by a row of sources interior to the body. This method is limited by the fish shapes which can be represented by the source sink method, but is not limited to only slightly non-uniform fields.

There have been several two dimensional theoretical analyses. Wu[8] calculates the thrust, work and efficiency of a plate of finite cord waving in a potential fluid field. He uses a linearized analysis for the velocity field and employsthe Theodorsen functions of an oscilating airfoil. Wu's results contain a discussion of the thrust , work and efficiency of a two dimensional plate in terms of a dimensionless frequency and wavenumber.

Bonthron[10] calculates the two dimensional potential flow due to a hinged surface. Siekman[4] presents a thin plate model of fish and derives thrust, and work by replacing the model by a vortex sheet. His results include diagrams of thrust and work versus dimensionless frequency with wave number as a parameter. He also presents experimental verifications of his work based on studies of a thin waving plate in a canal.

The experimental investigations have been concentrated mainly on trying to understand the actual mechanism of swimming by flow field visualization, and by measurements on frequency and velocity. Nowhere are values of wave length or wave number tabulated.

Rosen[15] has perhaps been most successful in flow field visualization. He employed a thin layer of milk on the bottom of a small shallow tank. The milk became disturbed and followed the fluid motions caused by a fish swimming down the tank. Rosen claims to have seen a trail of vortices in the wake of these fish and proposes the theory that fish are able to rederive energy from the vortex motions of the fluid: that is from the pressure fields of such motions. These vortices are supposedly generated at the head and travel downstream.

Kelly[7] in reviewing his experiments on two dimensional mechanical models mentions that a three dimensional mechanical model is under construction by the Navy.

Gray[1], the accepted biological authority on fish mechanisms, reviews his observations on the speeds, lengths and tail beat frequencies correlated with swimming speeds of actual fish. He also proposes or implies the theory that fish swim by sensing a stationary "peg" structure in the vortices of the fluid against which it may push. This is the origin of the vortex peg theory. Gray also states the so called fish paradox, namely, that fish do not seem to have the energy necessary to overcome the drag on their bodies as calculated from rigid body drag laws.

Bainbridge[14] presents experimental data from several species of fish. He records speed, frequency and mean forward velcoity with length and swimming motion amplitudes as parameters. His graphs indicate linear relationships between frequency and velocity for any one length of fish. He concludes that the observed swimming speed is independent of body shape. In a later paper,[25] Bainbridge discusses his experiments on fish in which wave number varied with the length of fish and where the lateral area presented to the fluid at the fin end varied with time.

In particular, the literature presents nowhere the effect of a (downwash streamwise) perturbation velocity on thrust or work for a three-dimensional model. Thus, there has been no correct or complete expression for the first law of thermodynamics for this case. No paper has been found which describes a three-dimensional boundary condition or describes in any way the relationship or dependence of efficient three dimensional swimming on wave number.

I FISH MOTION COUPLED WITH A PERIODIC FIELD

The first chapter of this paper formulates the problem of describing the movements of a very long fish and the inviscid, incompressible flow field resulting from these movements. When stretched straight, the fish lies along the x axis in a r,θ,x cylindrical coordinate system, and is assumed stationary relative to the fluid which flows parallel to its body at constant velocity $\bar{u}$. The fish is flexible and makes small movements at right angles to the stream. These movements, $h(x,t)$, are displacements of the fish centerline from the x axis and in the $\theta=0,x$ plane.

We then write out the general surface condition relating $h(x,t)$ to the flow field. We solve La Place's equation in cylindrical coordinates . The boundary and surface conditions are then considered in order to determine the exact form of the field. The time dependent Bernoulli equation is then solved for the pressure.

Thus, our mathematical problem is to find suitable expressions for the velocities V_r, V_θ, V_x (in the r, theta and x direction respectively), and the pressure, p, when

$$h(x,t) \text{ is given} \tag{1}$$

for all x between +L and -L, where L is a very large number. We wish to find the velocities and the pressure from the La Place equation (equation (8)) and the time dependent Bernoulli equation (equation (24)). The boundary conditions on the velocity field are that

the velocities are to satisfy the special surface condition (equation (7)), (2a)

the velocities are to be finite at infinity and at the surface, (2b)

the velocities must represent waves which radiate energy or information unidirectionally outward from the fish, (2c)

and these velocities must represent movements at the fish surface which satisfy the conservation of momentum laws. (2d)

Conditions a, b, and c are absolutely essential. For an infinitely long fish, (2d) is mainly of academic interest.

IA. GEOMETRIC COMPATIBILITY

To demonstrate the relationship between the movements of a flexible, inextensible, nonpermeable body and the fluid flow about this body, we consider arbitrary movements $h(x,t)$ of the centerline of this body in the $\theta=0, x$ plane. The movements are perpendicular to V_x, the velocity in the x direction.

$h(x,t)$ can be related to the resulting perturbed field about the body by a general kinematic surface condition. This is a condition on mass continuity at the surface and requires the resultant velocity at the surface to be wholly tangential to it since there can be no flow through the surface. The usual condition on mass continuity can be replaced by a special surface condition.[1p.7] If we describe the surface by some function $F(r,\theta,x,t)$, then the condition that a particle of fluid situated on the surface have no normal velocity relative to this surface is

$$\frac{D(F)}{Dt} = 0 \tag{3}$$

where D/Dt represents the substantial time derivative. Once F is determined, the condition (3) may be explicitly stated.

From Diagram 1 (see page 32) it is easily seen that $\vec{r}_s = \vec{r}^* + \vec{h}$, where $\vec{r}_s$ is the radius vector to the surface of the body, $\vec{r}^*$ is the radius vector to the surface of the body measured from the actual body centerline, and $\vec{h}$ is $h(x,t)$, the displacement of the body centerline from the x axis.

$$F(r,\theta,x,t) = \vec{r}_s - \vec{h} - \vec{r}^* = 0 \tag{4}$$

is then the equation of the surface. The Lamb[1p.7] surface condition can now be written as

$$\frac{DF}{Dt} = \frac{D(\vec{r}_s)}{Dt} - \frac{D\vec{h}}{Dt} - \frac{D\vec{r}^*}{Dt} = 0 \quad . \tag{5}$$

Equation (5) may be further interpreted according to reference 1 as stating that the relative velocity of a particle on the surface is either tangential to the surface or zero. Thus the time rate of change of position* due to the steady state flow velocity at a point on the surface in the field at any time t, is equal to the local velocity caused by the change in position of the surface in the field itself at that point.

Now consider small displacements $h(x,t)$. That is, let $|h(x,t)| \ll |r^*(x,\phi)|$, (see diagram 1). Then by inspecting Diagram 1, the quantities $(\theta - \phi)$ and $\Omega = (\partial h/\partial x)/\sqrt{1+(\partial h/\partial x)^2}$ are both very small when compared to θ say. In addition, if we also assume that $\vec{r}^*$ is only a very weak function of x, then $D\vec{r}^*/Dt$ is almost wholly tangential to the surface. Thus for a fish which makes small motions $h(x,t)$ at right angles to the stream at theta equal to zero, the scaler components of the geometric compatibility condition (5) written in the radial direction are

$$\frac{Dr_s}{Dt} - \frac{Dh}{Dt}\cos\theta = 0 \qquad (6)$$

if r^* is a weak function of x. (See diagram 2.) But since the radial velocity at the surface $(V_r)_s$, is just Dr_s/Dt, and the definition of the substantive derivative is

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{V}\cdot\mathrm{grad}(\ )$$

we have from (6) that

$$(V_r)_s = (\partial h/\partial t + V_x \partial h/\partial x)\cos\theta$$

We consider V_x, the velocity in the x direction to be the sum of a mean stream velocity $\bar{u}$ and a disturbance pertur-

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* for a fluid particle

bation u', which is caused mainly by h(x,t). Then the surface condition becomes

$$(V_r)_s = \left[\partial h/\partial t + (\bar{u} + u')\partial h/\partial x\right]\cos\theta \quad . \tag{6a}$$

But since u' is of the order of h, $\partial h/\partial t$ or $\partial h/\partial x$, we approximate $(\bar{u} + u')\partial h/\partial x$ by $\bar{u}\partial h/\partial x$. Thus, the "linearized" surface condition becomes

$$(V_r)_s = (\partial h/\partial t + \bar{u}\partial h/\partial x)\cos\theta \quad , \tag{7}$$

where we have assumed that

$$r^*(x,\theta) = r^*(\theta),$$

$$|h(x,t)| \ll |r^*| ,$$

and

$$u' = [O]\,\partial h/\partial t \ll \bar{u} \quad .$$

IB SOLUTION FOR THE VELOCITY FIELD

We now propose a velocity field which will at least yield u' and V_r for the completion of equation (7). We consider only periodic movements in x and t. Then, periodic separable solutions to the flow field equations will be developed to satisfy once and for all the surface boundary condition (7).

Since our fluid is incompressible (ρ not a function of pressure) and inviscid, we may consider the field to be irrotational. A velocity potential ϕ may thus be defined[21, p.184] which satisfies the condition of irrotationality. This condition is represented by the so called La Place equation which is written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial x^2} = 0, \qquad (8)$$

in cylindrical coordinates.

Although we may formally proceed to solve (8) by separation of variable techniques, since we are interested in motions which are periodic in x and t, it is convenient to guess a solution of the form **

$$\phi = R\exp(Kx + i\omega t + in\theta) \qquad (9)$$

where R is an undetermined function of r. Substituting (9) into (8) gives

$$\left(r^2 d^2R/dr^2 + rdR/dr + (-n^2)R + K^2r^2R\right)\exp(Kx+in\theta+i\omega t) = 0. \qquad (10)$$

Equation (10) is identified as Bessel's equation.[22] The standard solution to (10) is[22]

$$R = C_1J_n(Kr) + C_2Y_n(Kr) \qquad (11)$$

where J_n and Y_n are Bessel functions of the first and second kinds respectively, and the C's are constants. However, (11) may also be written in the form,

$$R = C_3H_n^{(1)}(Kr) + C_4H_n^{(2)}(Kr) \qquad (12)$$

** Formal proceedures prove the assumed theta variation.

where $H_n^{(1)}$ and $H_n^{(2)}$ are generally referred to as Hankel functions of the first and second kinds respectively, or as Bessel functions of the third kind. These functions are particularly convenient when discussing the propagation direction of waves or of energy. The Hankel functions are generally defined as [22]

$$H_n^1 = J_n + iY_n \tag{13}$$

and

$$H_n^2 = J_n - iY_n{}^{**} \tag{14}$$

Combining (9), (12),(13), and (14), and definitions for the velocity potential we may complete the solution for the field.

$$\phi = \left[C_3H_n^1(Kr) + C_4H_n^2(Kr)\right] \exp(Kx + in\theta + i\omega t) \tag{15a}$$

$$u' = \partial\phi/\partial x$$

$$= K\left[C_3H_n^1(Kr) + C_4H_n^2(Kr)\right] \exp(Kx+in\theta+i\omega t) \tag{15b}$$

$$V_r = \partial\phi/\partial r$$

$$= \left[C_3H_{n-1}^1(Kr)K - nH_n^1(Kr)/r\right] \left(\exp(Kx+in\theta+i\omega t)\right) \tag{15c}$$

$$+\left[C_4\, H_{n-1}^2(Kr)K - nH_n^2(Kr)/r\right]\left(\exp(Kx+in\theta+i\omega t)\right)$$

and

$$V_\theta = \partial\phi/\partial\theta(i/r)$$

$$= \left[C_3H_n^1(Kr) + C_4H_n^2(Kr)\right](in/r)\exp(Kx+in\theta+i\omega t) \tag{15d}$$

since

$$d(y_n)/dr = (Ky_{n-1} - ny_n/r) ,$$

**From this point on we shall always denote $H^{(1)}$ and $H^{(2)}$ by H^1 and H^2.

where

$$y_n = J_n, Y_n, H_n^2, H_n^1 \quad .$$

IB 1. BOUNDARY CONDITIONS

We may now determine the specific forms of the velocity field (15) which are pertinent to the fish problem.

To allow for exponential growth of amplitudes in x, we may define a complex wave number K in equations (15) where

$$K = K_r + iK_i \quad . \tag{16}$$

K_r and K_i are real numbers. The x variation is then of the form

$$\exp(Kx) = \exp(K_r x)\exp(K_i xi). \tag{17}$$

If we let K_r be positive, then waves grow with increasing amplitude in x.

The type of Hankel function H^1 or H^2 used to describe the velocity field or pressure is determined by condition (2c), i.e. by the condition that energy, pressure, convective mass transfer etc. radiate outward along positive r toward infinity. Reflections backwards in the direction of decreasing r are not permitted. The Hankel functions were used originally in order to formulate this condition most simply. This formulation proceeds by noting that for large r[239.83]

$$e^{i\omega t}H_n^1(Kr) \doteq \text{const}(2/Kr)^{\frac{1}{2}}e^{-K_i r}e^{i(K_r r + \omega t)} \qquad (18)$$

and

$$e^{i\omega t}H_n^2(Kr) \doteq \text{const}'(2/Kr)^{\frac{1}{2}}e^{+K_i r}e^{-i(K_r r - \omega t)}. \qquad (19)$$

Since K_r is positive, $\exp(i\omega t)H_n^2$ is the pertinent function because $K_r r - \omega t$ represents a wave of phase velocity ω/K_r moving outward toward infinity. K_i must then be a negative number* Thus, in (13),(14) and (15)

$$C_3 = 0 \qquad (20)$$

since H_n^1 represents a reflective situation.

We next determine n in (15). Since h(x,t) is not a function of theta, we note by inspection that the surface condition (7) will require $C_\theta \exp(in\theta)$** in the expression (15) for V_r to be $\cos 1\theta$. Thus since

$$C_\theta \exp(in\theta) = C_{\theta_r}\cos(n\theta) + iC_{\theta_i}\sin(n\theta),$$
$$n = 1, \text{ and } C_{\theta_i} = \theta. \qquad (21)$$

These conclusions,(21), may be reached alternatively by substituting the complete expression for V_r from (15) into (7), expanding in sines and cosines and simplifying.***

---

* note that all velocities approach zero and $V_x \to \bar{u}$ as $r \to \infty$ in accordance with condition (2b)!

** consider C_θ to be a complex constant which is part of C_3 and C_4.

***There are two interesting ideas related to our study of boundary conditions which will now be noted. Consider the case in which h(x,t) and thus the resultant field vary as $\exp(iK_i x)$ with $K_r = 0$. Then by (19), the

IB 2. FINAL FORM OF SOLUTION

We may now write down the complete solution for the velocity field and linearized pressure. Combining (15) with (16),(19),(20), and (21), we have

$$V_r = \sum_{\omega}\sum_{K} C_{\omega k}\left[H_0^2(Kr) - \frac{1}{Kr}H_1^2(Kr)\right]\cos\theta e^{k_r x} e^{i(K_1 x + \omega t)},$$

$$V_\theta = \sum_{\omega}\sum_{K} C_{\omega k}(i/Kr)\left[H_1^2(Kr)\right]\cos\theta e^{k_r x} e^{i(K_1 x + \omega t)}, \qquad (22)$$

$$u' + \bar{u} = V_x$$

$$= \sum_{\omega}\sum_{K} C_{\omega k} H_1^2(Kr)\cos\theta e^{K_r x} e^{i(K_1 x + \omega t)}$$

where

$$C_k = C_4 K. \qquad (23)$$

---

pertinent solution in H^2 shows that information concerning pressures and amplitudes is radiated in the r direction with infinite velocity since ω/K_r is infinite. On the other hand, if there is a small change in the amplitude of the field represented by $\exp(K_r x)$ as x varies, (i.e. $K_r \neq 0$) then at any x, there will be a finite velocity of propagation of a disturbance. This large but finite velocity is associated with diffusion or convection. It may be of considerable interest when flow field interactions between two neighboring fish are analysed since ω/K_r will then be the velocity with which pressure or mass disturbances (but not energy) will be propagated.

We also note that we have imposed no boundary conditions in x on the solution. This condition might be of some importance in considering finite length fish. Then if the velocities are to be finite at all r, including r = 0, it can be shown that n must be greater than zero. However, in this case, the condition on radiation toward infinity is not possible to satisfy as far as we know. Thus we have chosen to represent the solution of the long fish, which neglects end conditions in x and instead to write the answers in terms of Hankel function which satisfies the radiation condition at infinity. This seems to be the more rigorous and fundamental approach. If the results of these assumptions are experimentally verified, one can only conclude that end effects are not important.

The summations generalize the problem and are justified as solutions themselves since they are simply additive combinations of solutions to entirely linear equations. The actual velocity field is simply the real parts of (22). (22) is represented graphically in Chapter III for very simple motion, $h(x,t)$. In these motions, ω and K have only one value each, and thus we call C_k, "C", the field constant. The field constant is calculated in detail in sectionIIa.

IC. BERNOULLI SOLUTION FOR THE PRESSURE

For momentum calculations of the thrust, terms such as pdA where dA is an area will be of interest. The time averaged pressure multiplied by a time invariant, the area, then will yield non-zero answers only if the second order terms of the pressure are included. It is of interest then to calculate a general expression for pressure. Thus, we investigate the time dependent general Bernoulli equation which is valid for non-viscous irrotational fluids. It is obtained by integrating the Euler equations of motion, and is, for an incompressible fluid

$$(p_{tot} - p_{\infty})/\rho = -\frac{\partial \phi}{\partial t} - \frac{V_x^2 + V_\theta^2 + V_r^2}{2} \quad . \qquad (24)$$

The arbitrary time function usually associated with this equation has been omitted[1 p.19]. (24) holds specifically for the case of no force fields. By using the relationships between velocity potential and velocities, (15), and the definition of V_x, it follows immediately that (24) may be put into the form

$$p_{tot}/\rho = \frac{\partial}{\partial t}(u'/k) + u'\bar{u} - \frac{V_\theta^2 + V_x^2 + V_r^2}{2} - (\bar{u}^2/2 - p_\infty/\rho) \tag{25}$$

It is convenient to break (25) into its first and second order components. Therefore, let us define p_{tot} as

$$p_{tot} = p_{linearized} + p_{2nd} + p_{const} \tag{26}$$

where

$$p_{lin} = \rho\frac{\partial}{\partial t}(u'/K) + u'\bar{u}\rho ,$$

$$p_{2nd} = \rho\frac{-(V_x^2 + V_\theta^2 + V_r^2)}{2} , \tag{27}$$

and

$$\bar{p}_{con} = -\rho(\bar{u}^2/2 - p_\infty/\rho) .$$

The expressions for the velocity (15) and the pressure (25) now allow us to calculate the thrust and work.

II. WORK, THRUST AND EFFICIENCY

There are two well defined methods of calculating the thrust and work produced by a fish. The first is a straight forward momentum calculation for thrust, and, lateral force times local lateral velocity evaluation for the work. The second method is to write the first law of thermodynamics for a control volume and to calculate the associated energy integrals. Although this second method, the energy method, has been used by several authors,[2,10] the terms of the first law have generally not been written out or criticized. In particular, the effects of the streamwise perturbation, u', have been completely neglected in the literature. We shall attempt an interpretation of the first law here only as supplementary information to momentum methods because we feel that the calculation of the different energies involve somewhat arbitrary assumptions concerning the control volume which are not rigorously justifiable.

IIA. MOMENTUM CALCULATION FOR THE THRUST

We now proceed to calculate the thrust by control volume and momentum methods. In IIB, we calculate the work by employing the pressure field.

Newton's Second Law for a control volume fixed in inertial space may be expressed as follows: (see the control volume in Diagram 4a)[20 and 21]

$$T = \oint\!\!\oint_A p\vec{n}_x \cdot \vec{dA} + \oint\!\!\oint_A \rho(\vec{V}\cdot\vec{n}_x)^2(\vec{n}_x\cdot\vec{dA}) + \rho\frac{\partial}{\partial t}\iiint_{Vol}\vec{V}\cdot\vec{n}_x dVol \qquad (28)$$

where T is the thrust exerted by the fish on the fluid in the x direction, and $\vec{n_x}$ is the unit normal vector in that direction. The integral over the volume averages out to zero in time for an incompressible fluid. Thus, if dA, (or dA_1 etc.) is now understood as $\vec{n}_x \cdot \vec{dA}$, the pressure-area integral may be rewritten from (24) as,

$$\oint\!\!\oint_A \left(p_\infty - \rho\frac{\partial\phi}{\partial t} - \rho\frac{(V_\theta^2 + V_r^2 + V_x^2)}{2}\right)dA.$$

If we assume a bar type of fish with constant round cross sections at 2 and 1, then for small motions

$$\iint_{A_1} p\, DA_1 \qquad = \iint_{A_2} p\,dA_2$$

Furthermore, as we shall be concerned only with the time averaged thrust, the time average of $(\partial\phi/\partial t)dA$ is zero from (9) since ϕ is seen to be proportional to $e^{i\omega t}$. $\rho(\partial\phi/\partial t)dA$ has zero value in any case because the integral of the perturbation velocity times area involves the evaluation of $\int_0^{2\pi}\cos\theta d\theta$. Therefore

$$-\oint\!\!\oint p\,dA \;\doteq \left\{\rho/2\int_0^{2\pi}\int_{r*}^{\infty}(V^2 + V_r^2 + V_x^2)\,r\,dr\,d\theta\right\}_2^1$$

$$= \left\{\rho/2\int_0^{2\pi}\int_{r*}^{\infty}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right) r\,dr\,d\theta\right\}_1^2.$$

But by Green's Theorem[1]

$$\rho/2\oint\!\!\oint\!\!\oint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial\theta}\right)^2\right\}dA = \rho/2\oint\left(\phi\,\frac{\partial\phi}{\partial n}dl\right)_{sur} \qquad (29)$$

where dl is now a line integral about the circumference of the fish. (29) is generally known to hold for the reduction of a volume integral to a surface integral. That (29) is correct for the area to line integral reduction, when 1 and 2 are evaluated at x and at x + dx is shown in Appendix A.

For small motions dl is approximately $r^*d\theta$. The

* i.e., r^* is a weak function of θ and of x.

subscript s or sur refers to the quantity evaluated at the surface. From expressions for the real parts of ϕ and V_r, we have on the time average that

$$\oint -p\,dA = \tfrac{1}{4}\rho \oint \mathrm{Re}\left(CH_1^2/Kr^* \exp(Kx+i\omega t+in\theta)\right) \cdot \cdot \mathrm{Re}\left(C(H_0^2 + H_1^2/Kr^*)\exp(Kx+i\omega t+in\theta)\right) r^{*2} d\theta \tag{30}$$

We now proceed to calculate the right hand side of (28). We note that

$$V_x^2 dA = (\bar{u}^2 + 2u'\bar{u} + (u')^2)dA.$$

Upon integration, $\bar{u}^2$ and $\bar{u}u'$ have no consequence for the same reasons as previously noted for $\underset{cons}{p}dA$ and $(\partial\phi/\partial t)dA$. The righthand side of (28) therefore reduces to the real part of

$$\left\{ \int_0^{2\pi}\int_{r^*}^{\infty} (u')^2 dA \right\}_x . \tag{31}$$

For fixed K and ω, the time average of (31) becomes upon substitution from (22)

$$\left\{ \int_0^{2\pi}\int_{r^*}^{\infty} \overline{\left(\mathrm{Re}(CH_1^2 \cos\theta e^{K_r x} \exp(ik_1 x + i\omega t))\right)^2} r\,dr\,d\theta \right\}_x \tag{32}$$

where the bar denotes time averages and where Re denotes the real part of the expression. When (32) is expanded by breaking the exponential into real and imaginary parts, the square of the parenthetical expression involves terms such as $\cos^2(K_1 x+\omega t)$, $\sin^2(K_1 x+\omega t)$ and

$\sin(k_1x + \omega t)\cos(k_1x + \omega t)$. The time average of the latter term is zero, that of the first two is one-half. The square of the parenthetical term thus becomes, after some simplifying

$$\frac{e^{2k_r x}\cos^2\theta}{2}\left[(\mathrm{Re}C)^2 + (\mathrm{Im}C)^2\right]\left[(\mathrm{Im}H_1^2)^2 + (\mathrm{Re}H_1^2)^2\right]$$

or

$$\frac{e^{2k_r x}\cos^2\theta}{2}\left|C^2\right|\left|H_1^{2\,2}\right| \tag{33}$$

where the vertical lines enclosing C and H_1^2 indicate that the magnitudes only of these quantities are considered, and hence only the magnitude of the product of their magnitudes squared. The integral (32) may now be written,

$$\left\{ \int \frac{e^{2k_r x}}{2}\pi\left|C^2\right| \int_{r*}^{\infty} (H_1^2)^2\ r dr \right\}^{\frac{1}{2}} \tag{34}$$

where r* is a weak function of x. It can be easily shown however that since rdr is a real quantity,

$$\left|(H_1^2)^2\right|\ r dr = \left|(H_1^2)^2 r dr\right| \quad . \tag{35}$$

We are thus confronted with the evaluation of the integral

$$\left|\int_{r*}^{\infty}[H_1^2(Kr)]^2 r dr\right| .$$

Since this is not a commonly used integral, it has been evaluated analytically in Appendix C where an exact expression is given for its numerical value. With the results of Appendix C, (34) becomes

$$\left\{(\int e^{2K_r x}\pi r*^2/4)\right\}^{\frac{1}{2}}\left|C^2\right|\left|\frac{2H_1^2(Kr*)H_0^2(Kr*)}{Kr*} - (H_1^2(Kr*))^2 - (H_0^2(Kr*))^2\right| . \tag{36}$$

DETERMINATION OF THE FIELD CONSTANT C.

We now proceed to determine C in (36). We call C the field constant. It is determined by application of the boundary condition (7) for given ω and K. To find actual values for the thrust etc., we now specialize our solution for an explicit $h(x,t)$. Let

$$h(x,t) = B\sin(K_1 x + \omega t)e^{k_r x}$$

where B is a real amplitude. Then, equation (7) becomes with (26)

$$V_{r_s} = e^{k_r x}\cos\theta\Big\{\Big(\mathrm{Re}\big[C(H_o^2 - H_1^2/Lr^*)\big]\Big)\cos(K_1 x + \omega t) - \Big(\mathrm{Im}\big[C(H_o^2 - H_1^2/Kr^*)\big]\Big)\sin(K_1 x + \omega t)$$

$$= Be^{k_r x}\cos\theta\Big[(\omega + K_1\bar{u})\cos(K_1 x + \omega t) + (K_r\bar{u})\sin(K_1 x + \omega t)\Big] \qquad (37)$$

or

$$\mathrm{ReCRe}(H_o^2 - H_1^2/Kr^*) - \mathrm{ImCIm}(H_o^2 - H_1^2/Kr^*) = (\omega + K_1\bar{u})B$$

and

$$-\mathrm{ImCRe}(H_o^2 - H_1^2/Kr^*) - \mathrm{ReCIm}(H_o^2 - H_1^2/Kr^*) = K_r\bar{u}B$$

Squaring both sides and adding leads to

$$|C^2| = \frac{B^2\,\big|(\omega + K_1\bar{u})^2 + (K_r\bar{u})^2\big|}{\big|(H_o^2(Kr^*) - H_1^2(Kr^*)/Kr^*)^2\big|} \quad . \qquad (38)$$

We can now compute the time averaged thrust, $\bar{T}$.

TIME AVERAGED THRUST

From (28), (29), (30), and (31),

$$\bar{T} = \rho \oint \left[(u')^2 - \frac{V_\theta^2 + V_r^2 + (u')^2}{2} \right] dA \tag{39a}$$

$$= \rho \oint (u')^2 dA - \rho \tfrac{1}{2} \oint (\phi \partial\phi/\partial n)_{sur} r^* d\theta \tag{39b}$$

or

$$= B^2 \rho e^{2k_r x} \pi r^{*2}/4 \left[(\omega + K_1 \bar{u})^2 + (K_r \bar{u})^2 \right] (T(1) - P(1)) \tag{39c}$$

where T(1) and P(1) are given in (41) below.* If we evaluate 1 and 2 at x and x+dx as previously mentioned, then

$$\exp(2K_r x)/_x^{x+dx} = 2K_r dx \exp(2K_r x).$$

If we define a dimensionless thrust per unit length, T*, then

$$T^* = \bar{T} \exp(-2K_r x)/(K_r dx \rho B^2 \pi \bar{u}^2)$$

$$= \tfrac{1}{2} \left[(\alpha + Z(2))^2 + Z(1)^2 \right] (T(1) - P(1)) \tag{40}$$

where*

$$T(1) = \frac{\left| \frac{2H_1^2(Z)H_0^2(Z)}{Z} - H_0^2(Z)^2 - H_1^2(Z)^2 \right|}{\left| (H_0^2(Z) - H_1^2(Z)/Z)^2 \right|} \tag{41a}$$

$$P(1) = \mathrm{Re}((H_1^2(Z)/Z)/(H_0^2(Z) - H_1^2(Z)/Z)) \tag{41b}$$

and

$$\alpha = \omega r^*/\bar{u} \tag{41c}$$

$$Z = Kr^* \tag{41d}$$

$$Z(1) = K_r r^* \tag{41e}$$

$$Z(2) = K_1 r^* \tag{41f}$$

T* is rederived in Appendix G by integrating pressures over the surface.

II.B. WORK INPUT RATE

The rate at which work is done by the fish on the fluid is equivalent to the forcewhich the fish exerts on

*T(1) and Z(1) etc. are conveniently defined in (40 and (41) to be compatible with FORTRAN statements in Appendix D.

the fluid in the direction of lateral motion($\theta = 0$) multiplied by the local velocity in that direction. The total work is, on the time average, for small motions

$$\bar{\dot{W}} = \oint_A \overline{p\cos\theta(\partial h/\partial t)dA} = \oint_A (p_{lin} + p_{2nd} + p_{con})(\partial h/\partial t)dA \tag{42}$$

where

$$dA = r^* d\theta dx.$$

The constant part of the pressure, p_{con}, will not appear in the final expression because $p_{con}dA$ is invariant in x if r* is a weak function of x. p_{2nd} will on the time average cancel out since it will be multiplied by the local velocity. (Integration in θ or third order consideration would in any case be sufficient reasons for neglecting this latter term.) p_{lin} may be calculated from the Bernoulli equation(27). Breaking p_{lin} into its component parts, we have

$$p_{lin} = -\cos\theta e^{k_r x}\left(Re\left[\left(\frac{i\omega + \bar{u}K}{K}\right)CH_1^2\right]\cos(K_1 x + \omega t) + -Im\left[\left(\frac{i\omega+\bar{u}K}{K}\right)CH_1^2\right]\sin(K_1 x + \omega t)\right)_{r^*}. \tag{27}$$

If we assume as before that

$$h = Be^{k_r x}\sin(K_1 x + \omega t),$$

or

$$\partial h/\partial t = +Be^{k_r x}\omega\cos(K_1 x + \omega t) ,$$

then, only the cosine part of the first order pressure will survive the time average of $\bar{\dot{W}}$. From (42) and

(27) we have then,

$$\bar{\dot{W}} = \mathrm{Re}\left[\frac{(i\omega + \bar{u}K)}{K}CH_1^2\right]_{r*} r^* \omega B\rho \int_x e^{2K_r x}\overline{\cos^2(K_1 x+\omega t)}\,dx \int_0^{2\pi}\cos^2\theta\, d\theta$$

The time average is taken before the integration is performed. It has been assumed in the above that r* is a weak function of x. Therefore

$$\bar{\dot{W}} = \left\{ \mathrm{Re}\left[\rho\frac{(i\omega + \bar{u}K)}{K}CH_1^2\right]_{r*} \frac{r^*\omega B\pi}{2}\frac{e^{2K_r x}}{2K_r} \right\}_x \tag{43}$$

If we let $\dot{W}$ be evaluated at x and x + dx as previously, then

$$e^{2K_r x}/2K_r \Big/_x^{x+dx} \doteq e^{2K_r x}dx \; . \tag{44}$$

Let us define a dimensionless Work per unit length by

$$W^* = \frac{\bar{\dot{W}}\, e^{-2K_r x}}{K_r dx B^2 \pi\rho\, \bar{u}^3}$$

Then, by employing (43), (44) and (38),*

$$W^* = \frac{1}{2}\frac{\alpha}{Z(1)}\left[i(\alpha + Z(2)) + (Z(1))\right] W(1)$$

where (45)

$$W(1) \doteq \left[\frac{(i\alpha + Z)}{Z}\frac{H_1^2(Z)}{H_0^2(Z) - \frac{H_1^2(Z)}{Z}}\right]$$

and where α, Z, Z(1), and Z(2) are defined by (41 b,c,d,e).

IIC. EFFICIENCY

Define the efficiency of a work producing fish by

$$\eta = \bar{T}\bar{u}/\bar{\dot{W}} = \frac{\text{Energy available for thrust}}{\text{Total Energy Input}} \tag{46}$$

* W* and W(1) are plotted against α in Chapter III.

But it can be easily shown that

$$T^*\bar{u}/W^* = \bar{T}\bar{u}/\bar{W} \ . \tag{47}$$

Therefore*

$$\eta = T^*/W^* \ . \tag{48}$$

IID. ENERGY CONSIDERATIONS

Since the energy approach+ has been used by several authors[2,10] without criticism, we now formulate the first law of thermodynamics for the control volume of Diagram 4. This control volume does not exchange heat with its surroundings. Furthermore, it encompasses no energy sources since the area between 1 and 2 of the control volume is just deep enough beneath the fish surface so that the fish surface never passes out of the control volume. This depth will then be a maximum at $\theta = 90^{\circ}$ or $\theta = 0^{\circ}$ and x. The depth will be a little greater then $h(x,t)$. Yet we consider that the control volume is not so deep as to contain any "sources" of energy within the fish as previously mentioned. These considerations are clearly justified by our previous small amplitude analysis.

We proceed by purely mechanical considerations. The energy equation for the rate of doing work is found by multiplying a generalized form of equation (28) for the thrust, by a small displacement $\vec{Ds}$ of a fluid particle. Since there are no energy sources or heat transfer, the first law of thermodynamics for the incompressible fluid

* η is plotted against α in Chapter III.

\+ an approach to be considered only as supplementary material to IIA as previously mentioned on page 20.

of the control volume of Diagram 4 fixed in inertial space is, on the time average,

$$\overline{\vec{F}\cdot\vec{Ds}} = \overline{\vec{Ds}\cdot\int_F \vec{dF}} = \int_F \overline{\vec{dF}\cdot\vec{Ds}} = \overline{dW} = \oint_A \overline{\vec{V}\cdot\vec{Ds}(\vec{V}\cdot\vec{dA})} \tag{49}$$

since the vector force, $\vec{F}$, which includes both body and surface forces is independent of $\vec{Ds}$. Therefore, expanding (49) into its r, theta and x scaler components,

$$\begin{aligned}\overline{dW} &= \overline{F}_x Ds_x + \overline{F}_r Ds_r + \overline{F}_\theta Ds_\theta \\ &= \oint_A \overline{(V_x Ds_x + V_\theta Ds_\theta + V_r Ds_r)\, V_x dA_x}.\end{aligned} \tag{50}$$

But

$$\begin{aligned}\overline{F}_r Ds_r &= \overline{F}_\theta Ds_\theta = 0 \\ &= \oint_A \overline{V_\theta\, V_x dA_x Ds_\theta} = \oint_A \overline{V_r\, V_x dA_x Ds_r}\end{aligned} \tag{51}$$

since there are no net forces tending to revolve the control volume about the x axis or push it radially outward on the time average. Therefore, dividing both sides of (50) by dt, we get on the time average

$$d\overline{W} = \overline{F_x V_x} = \oint_A \overline{V_x^3}\, dA_x \tag{52}$$

since

$$Ds_x/dt = Ds_x/Dt = Dx/Dt = V_x$$

by definition for the velocity of a fluid particle.

(52) may be rewritten as

$$\begin{aligned}\overline{F_x V_x} &= \overline{T(\bar{u} + u')} = \oint_A \overline{p dA_x(\bar{u} + u')} + \oint_A \overline{V_x^2(\bar{u} + u') dA} \\ \mathrm{I} &= \mathrm{II} + \mathrm{III}\end{aligned} \tag{53}$$

which gives exactly the same $\bar{T}$ as (28).

The terms of (53) may be interpreted as follows: I is the work consumption of the real fluid dynamical drag forces - or the work done by the thrust exerted on the control volume by the fish; III is the rate at which kinetic energy is lost to the wake of the fish from the control volume; and II is the rate at which work is consumed by fluctuating pressures on the perturbation velocity - this term is some times referred to as flow work. Term II is generally neglected in treatments involving the first law because this term's significance really stems from the inclusion in the theory of the perturbed velocity field. Also, generally a $\dot{W}_{in}$ of the fish itself is included in the first law statement. But, as we have just shown, this term must be included only if the specified control volume includes the fish.

III GRAPHICAL REPRESENTATIONS (Diagrams and Graph Results)

Diagrams refered to in the text which relate to derivations are numbered one through four; graphical results are presented in Diagrams five through nine. All results depend upon h, where $h = Be^{k_r x}\sin(K_1 x+\omega t)$ as usual. (B, K_r and K_1 are real numbers).

The perturbed velocity field is plotted in Diagrams 5a and 5b. These results are derived from (22) in dimensionless form. The actual field would require a large constant

$\bar{u}$ to be added in the x direction to all velocities. The area between $r = 0$ and $r = 1$ in Diagrams 5a and 5b is considered to be occupied by the fish.

Thrust in Diagram 6 is determined from expression (G3) of Appendix G; work in Diagram 7 from equation (45); and efficiency in Diagrams 8 and 9 from expression (48) and (G3) and (45).

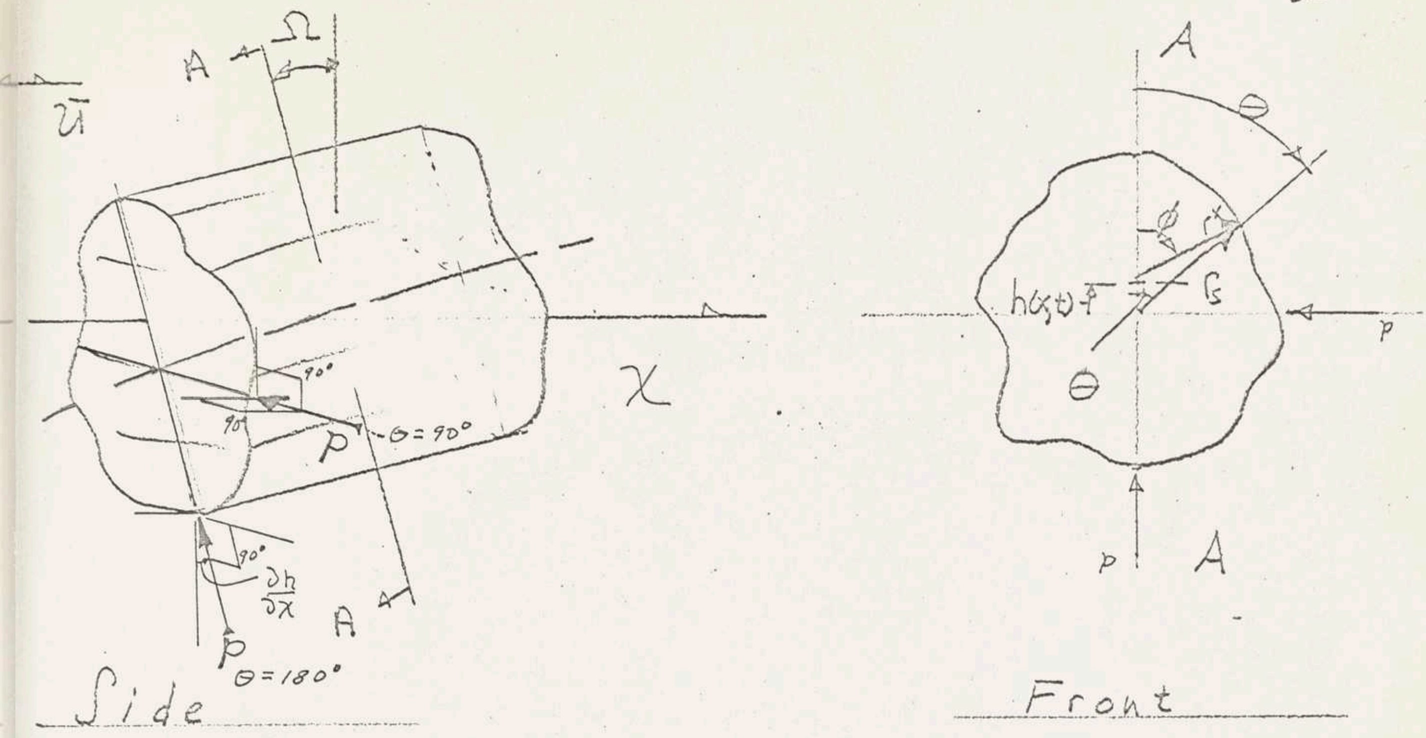


DIAGRAM 1 FISH CROSS SECTION

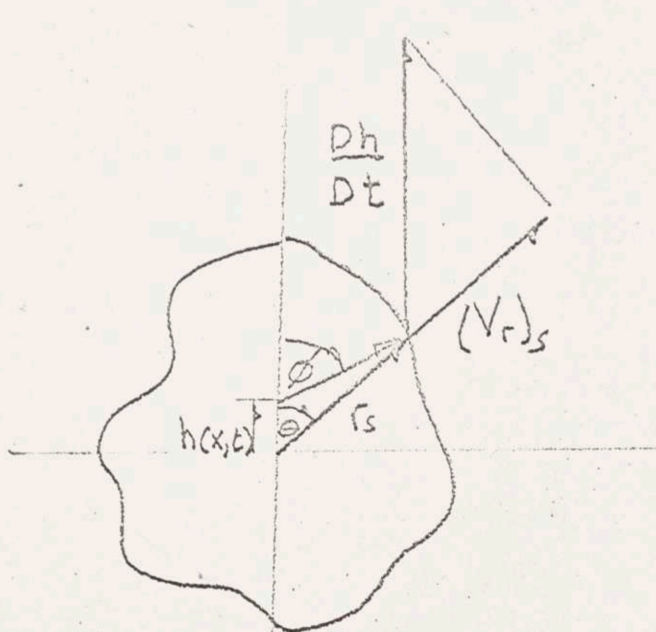


DIAGRAM 2 SURFACE VELOCITIES

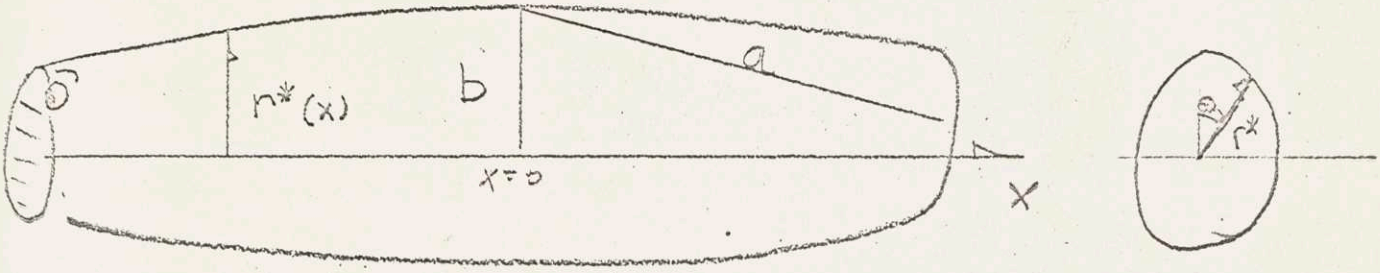


DIAGRAM 3 ELLIPTIC FISH

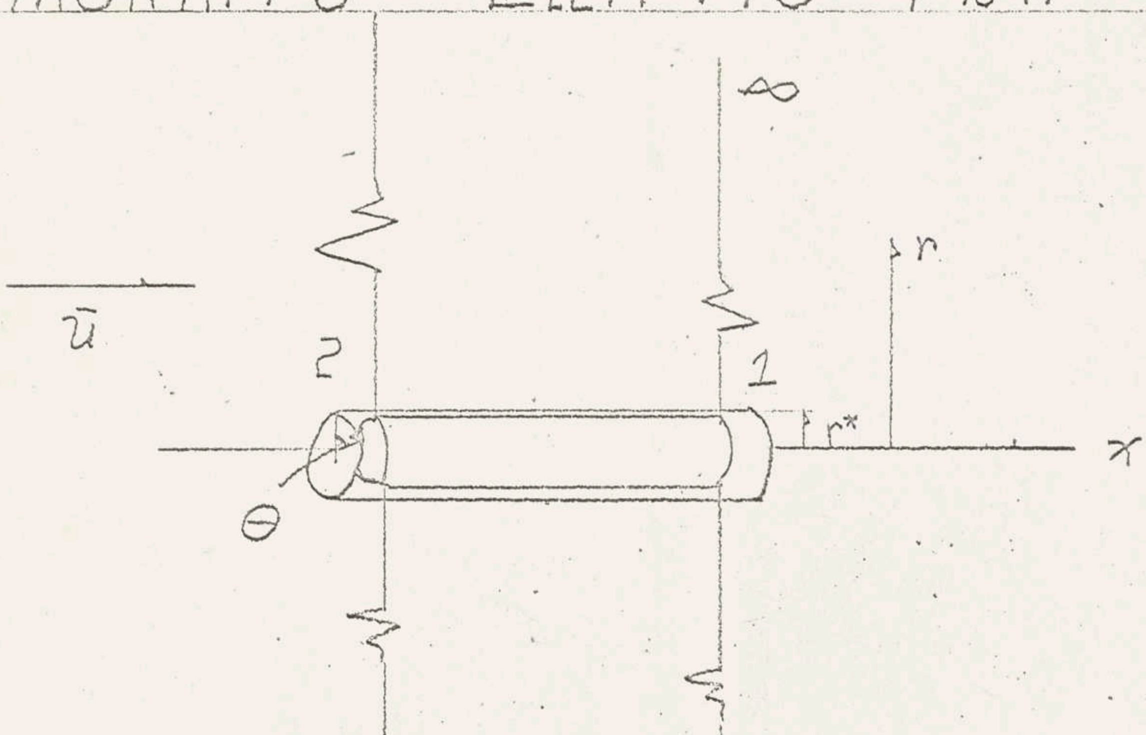


DIAGRAM 4a.

CONTROL VOLUME for THRUST and WORK

(Excluding Volume between $r = 0$ and $r \cong r^*$)

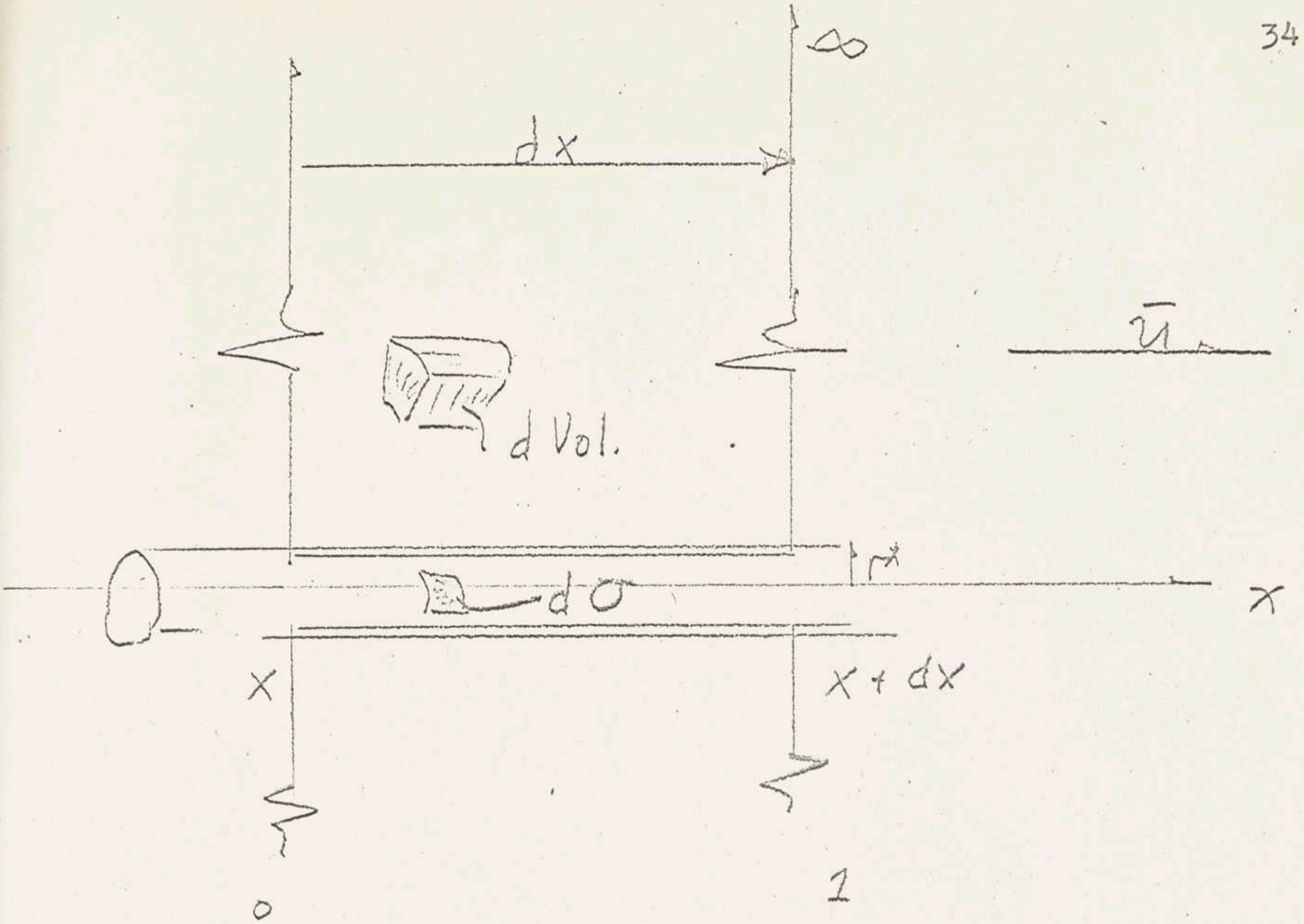


DIAGRAM 4b. ON VIRTUAL MASS

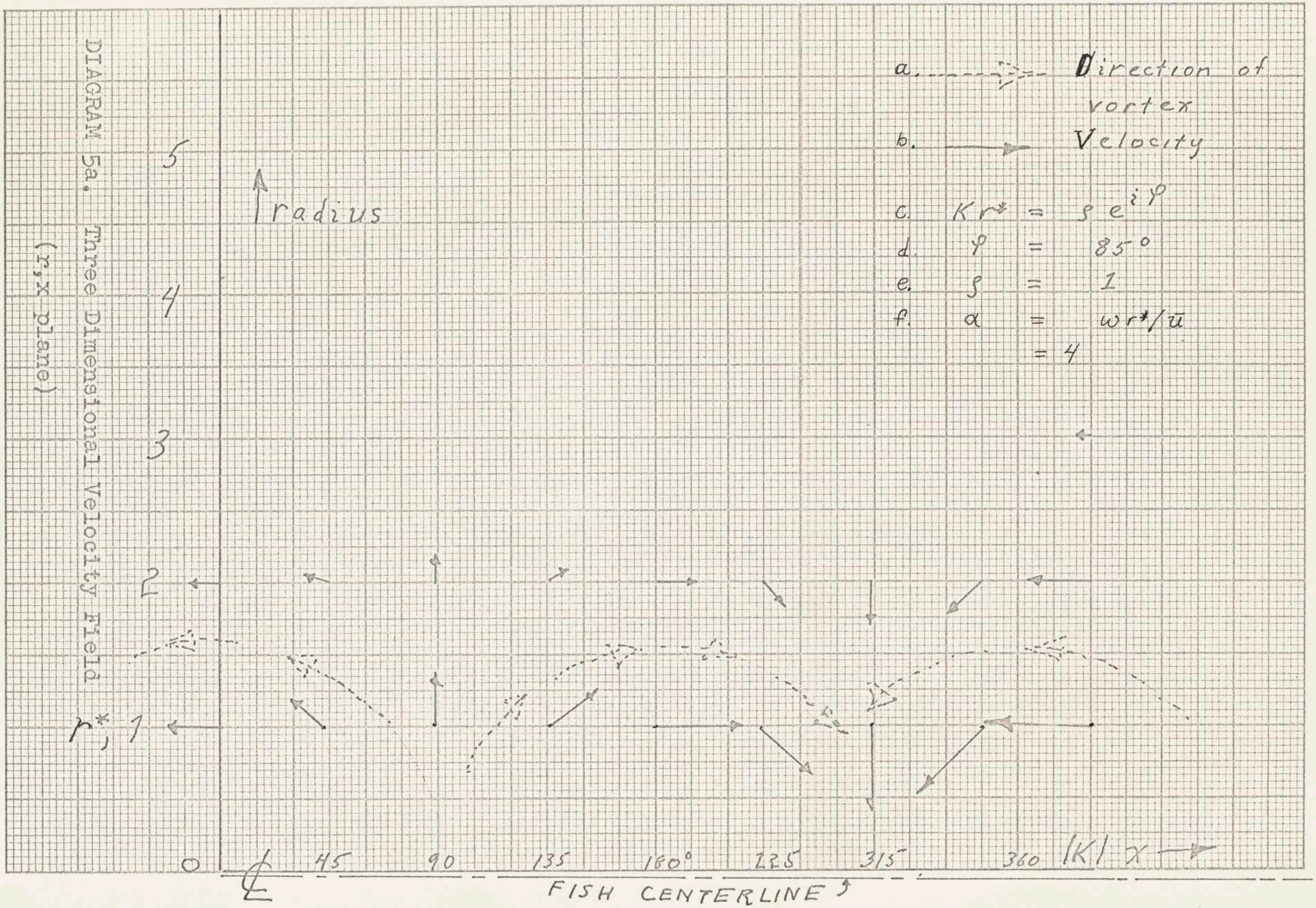


DIAGRAM 5a. Three Dimensional Velocity Field

(r,x plane)

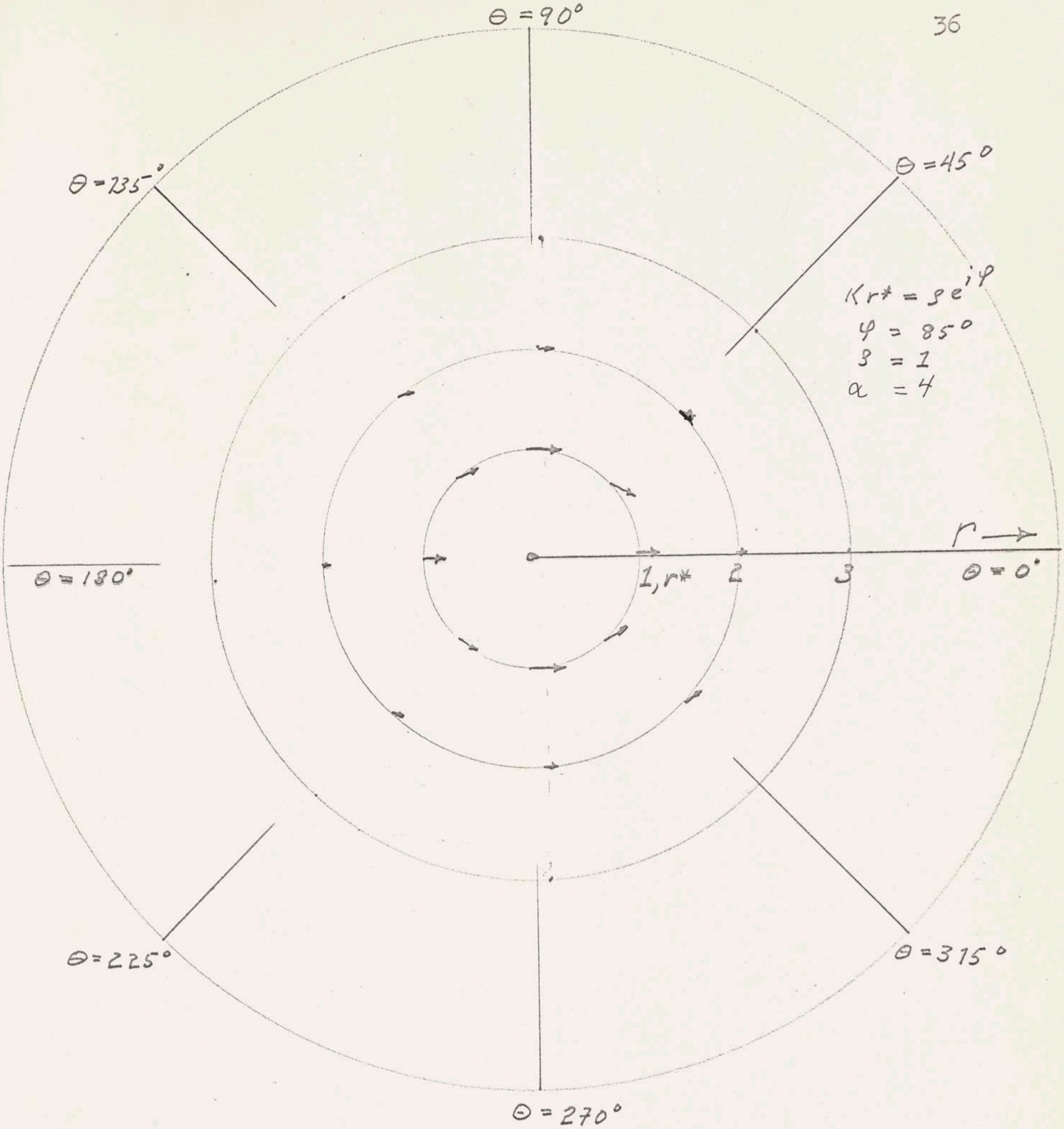


DIAGRAM 5b. Three Dimensional Velocity Field
(r,θ plane)

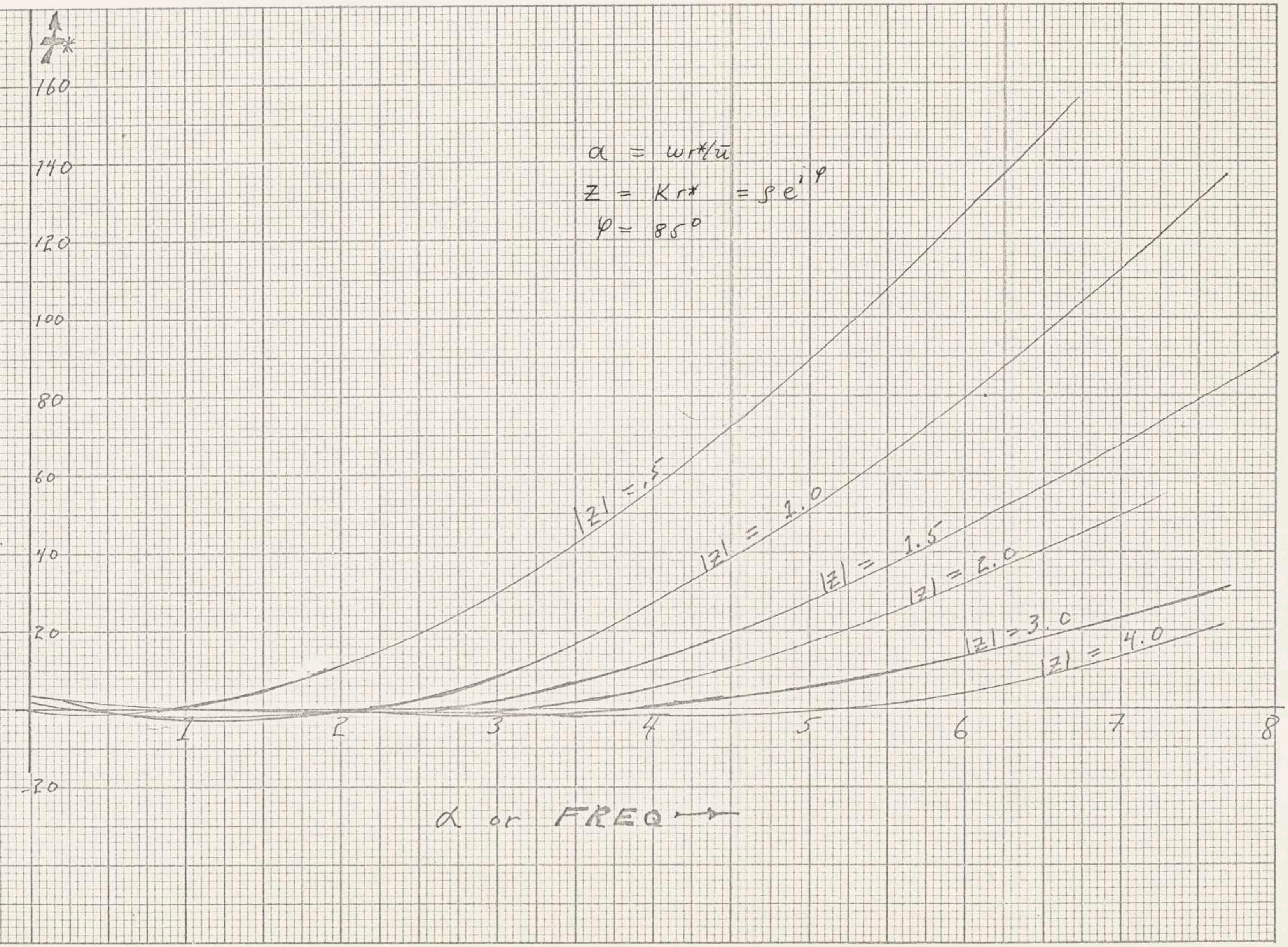


DIAGRAM 6. Thrust versus Frequency with Wavenumber as a Parameter

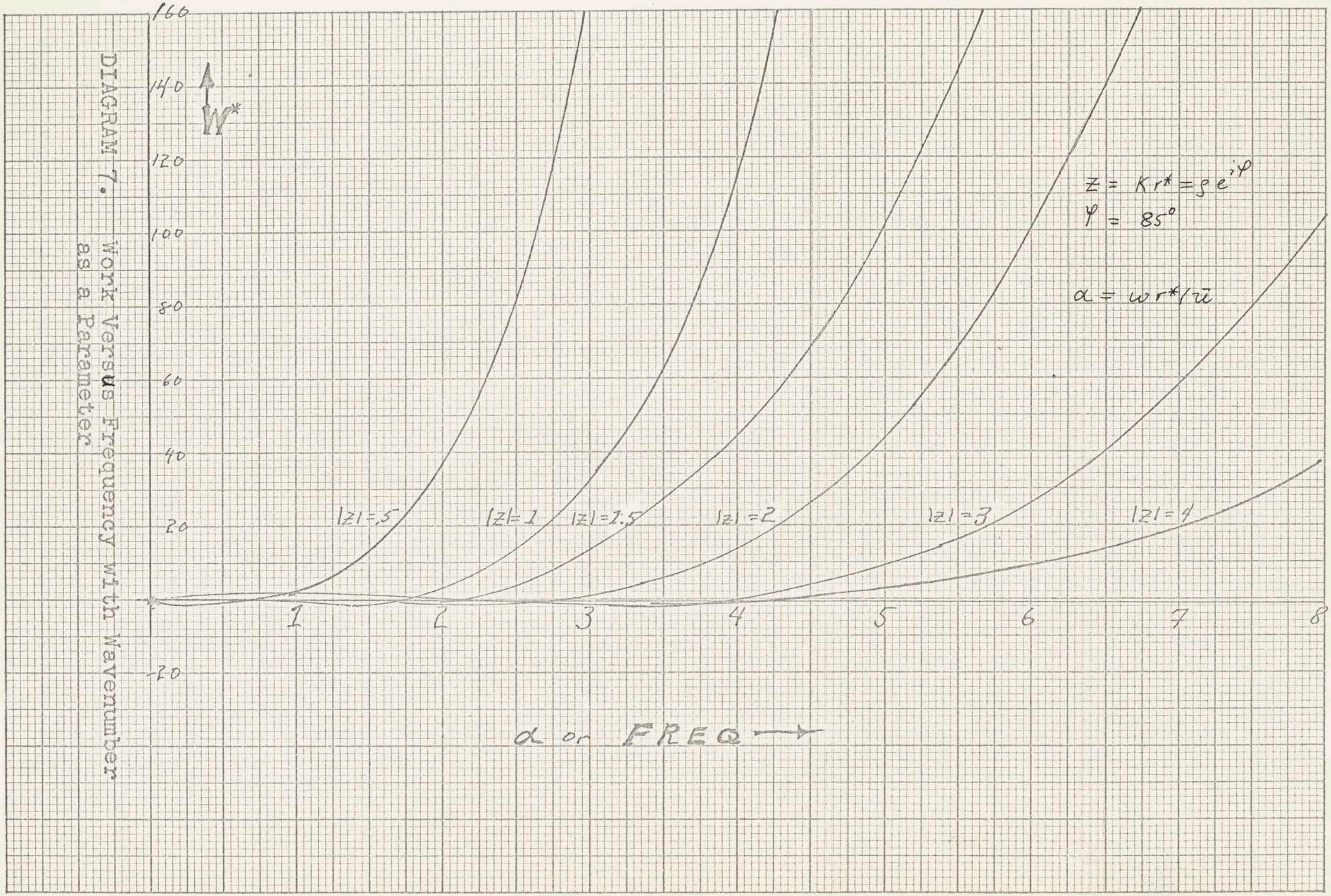


DIAGRAM 7. Work Versus Frequency with Wavenumber as a Parameter

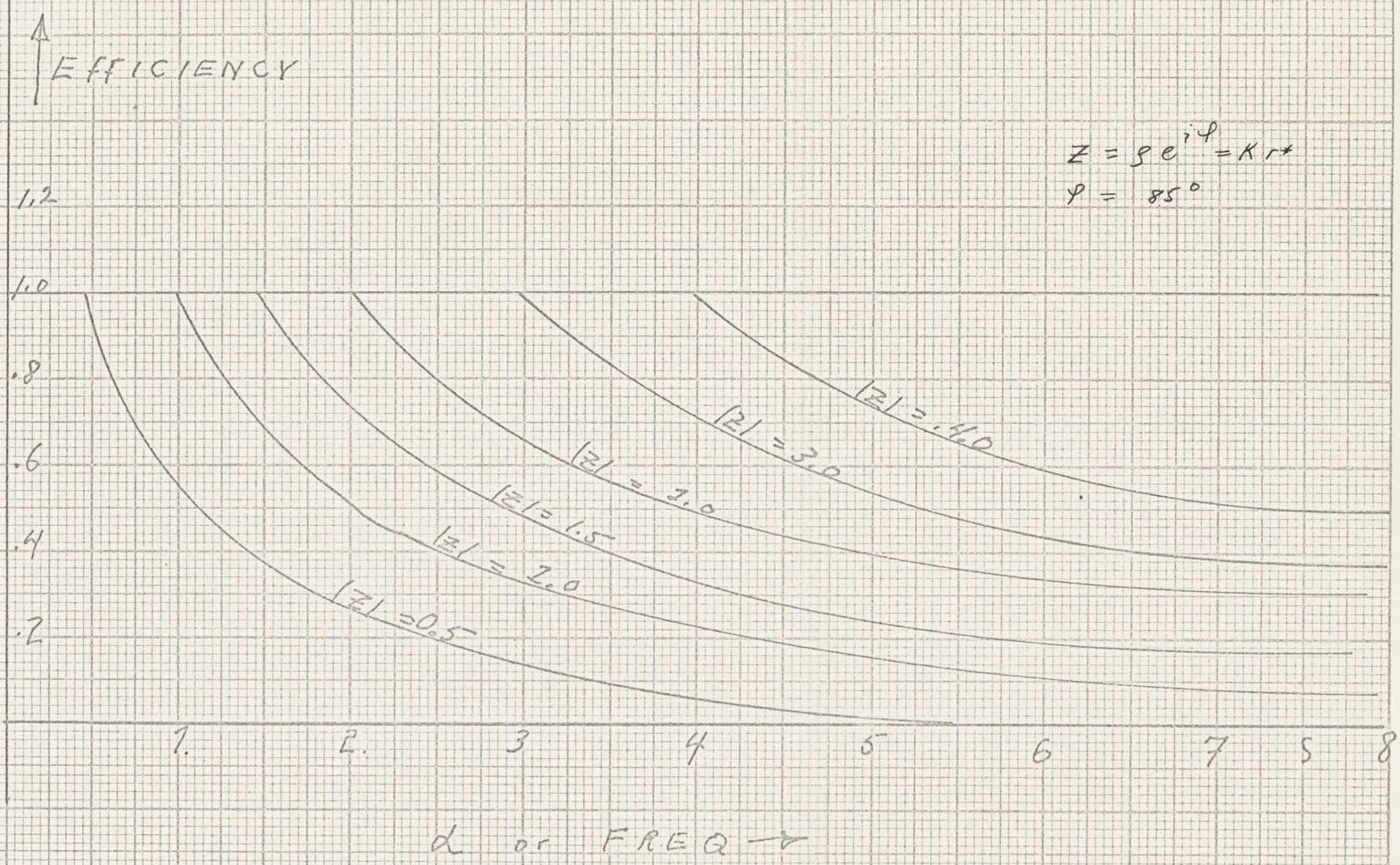
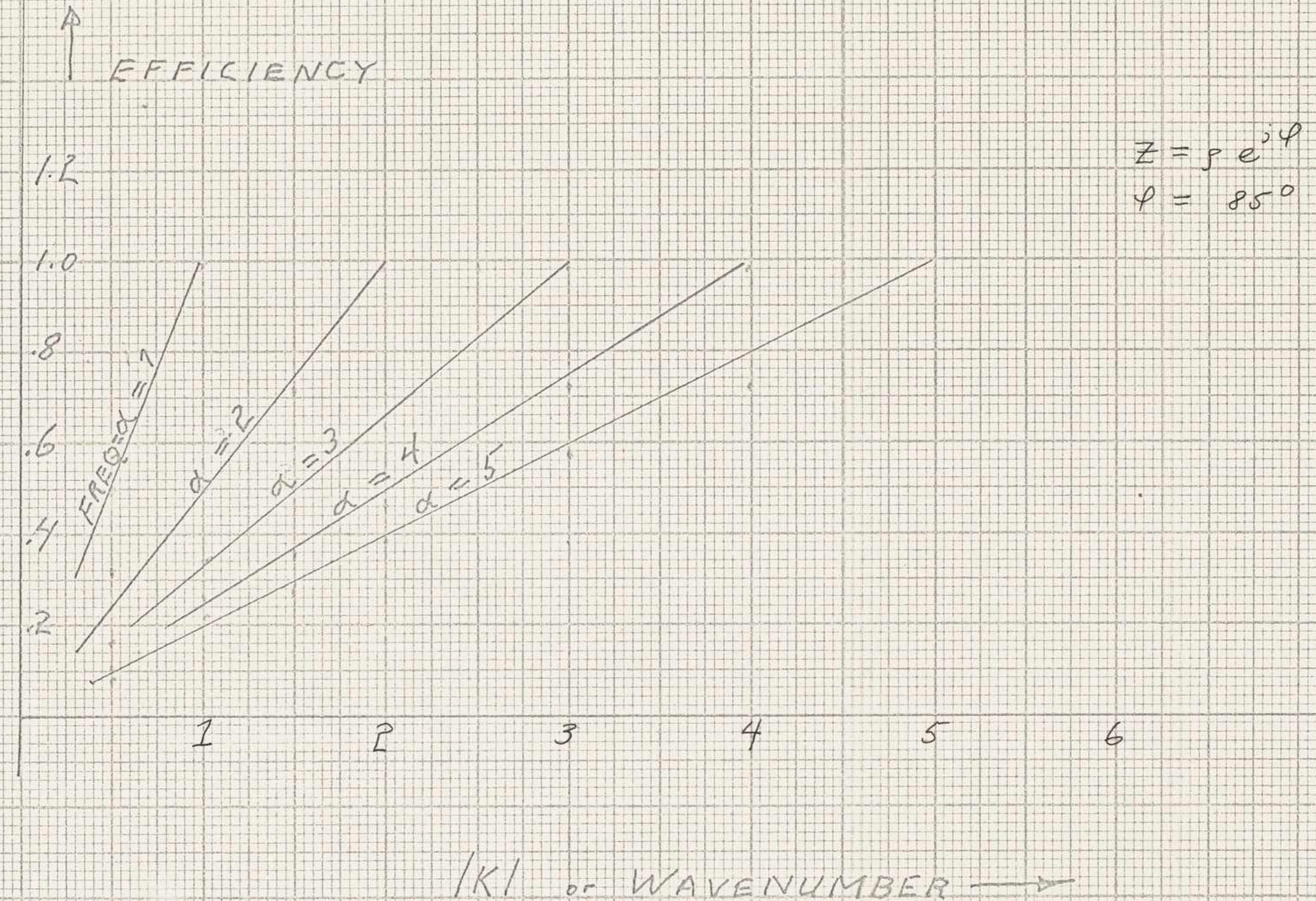


DIAGRAM 8. Efficiency Versus Frequency with Wave-
Number as a Parameter

DIAGRAM 9. Efficiency Versus Wavenumber with Frequency as a Parameter



IV. CONCLUSIONS

Diagram 5 shows that the magnitude of the velocity field dies off rapidly and exponentially with increasing distance from the fish. The two dimensional graph, 5a, of the velocity field "above" the fish, shows that there are cores of vorticity of varying sense distributed along the length of the center of the fish. Although it is necessary to obtain one three dimensional picture of the field to see exactly how the fields of these vortices connect* it now appears as if the vortex cores form a torous like corkscrew field. This field may wind within and around the fish.

We may draw several conclusions from diagrams 6 through 9. For the case where $c/\bar{u} \lesssim 1$, (c is ω/K_1, the phase velocity) T* and W* are very small and are negative in the vicinity of $c/\bar{u} = 1$. (Also for the case in which $c/\bar{u} = 1$, the perturbed velocity field is negligible because C, the field constant is practically 0) Also T* and W* approach zero for any frequency as wavenumber, K, increases. Furthermore, the efficiency defined in (G4) of Appendix G as

$$\eta = c + \text{small correction}$$

$$\eta = T/(T + \text{losses}),$$

is always greater than one. The reason for this is that T and W (the total work put into the fluid) are negative in the

*This research is presently being carried to completion.

$c/\bar{u} \lesssim 1$ region. The losses to the wake etc. however are always positive. Therefore the total work, T + losses, is less than T, and the efficiency is correspondingly greater than one.

For the case of $c/\bar{u} > 1$, efficiency increases linearly with wavenumber and approaches zero for all frequency as wavenumber approaches zero. Efficiency also goes to zero for all wavenumber at large frequency. Furthermore, in the $c/\bar{u} > 1$ region, W* and T* increase almost quadratically with frequency and decrease extremely rapidly with increasing wavenumber for any one frequency. These results all show that swimming can be achieved most efficiently and productively with regard to thrust at as low a frequency and as high a wavenumber as possible without encountering the $c/\bar{u} \lesssim 1$ region. From our graphs, we estimate that $\bar{u}/c$ should be between .75 and .8 for an efficiency of between 80 and 90 percent. This conclusion correlates well with Bainbridge's observations on actual living fish[25] and with Lighthill's[2] results.

It is also noted that our results for T*, W* and efficiency are of exactly the same form as those obtained by Wu[8]. Wu analysed the swimming of a waving plate with oscillating airfoil theory.

We would like to point out that our values for thrust and work are probably accurate in comparison with any finite fish theory. Although the net momentum exchanged in the x direction will be greater for a finite fish than for an infinite fish, the net energy lost to the wake will also be proportionately greater and will cancel the former effect.

The physical plausibility or our results, and the correlations between our results and other work previously mentioned, indicate that an exact study of end conditions will not greatly amplify an understanding of the swimming fish mechanics.***

The present theory would however be improved if it is found possible to match the form of the momentum derivation for thrust with the form of the pressure surface integration. However, this would be difficult, since the momentum derivation employs the second order part of the pressure (i.e. H_1^2 and its derivatives squared), whereas the surface pressure integration uses only the first order part of the pressure (.e. $\bar{u}u_1'$ or H_1^2 alone.).

*** A possible approach to the finite fish case by means of our theory might be to assume that r* varies strongly with x. Then one would assume that r* approaches zero rapidly at the ends of the fish. In this case the boundary condition (E1) and the method of calculating thrust of Appendix G would probably be simplest to employ.

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APPENDIX A. PROOF THAT

$$\oint\!\!\!\oint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}dA \equiv \oint\!\!\!\oint\!\!\!\oint_{x\ Vol}^{x+dx}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right\}dVol$$

IF EVALUATED AT X AND AT X $\pm$ dX

The above equivalence was stated in equation (38). We now show it to be true. In general, Green's Theorem cannot be applied to the integral on the right hand side of the equivalence sign, if reduction to a line integral is desired, unless of course one is dealing with a two dimensional velocity potential.[1p.66] For the fish problem, this would mean no dependence of ϕ upon x in accordance with the assumptions of the slender body theory.

That Green's theorem also applies to (38) under the special condition of evaluation of the areas at x and x + dx is shown in the following.

We wish to evaluate

$$-\oint\!\!\!\oint_A \overline{pdA} = + \frac{\rho}{2}\oint\!\!\!\oint\left\{\left(\frac{\partial\phi}{\partial X}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}dA \qquad \text{(A1)}$$

but

$$\oint\!\!\!\oint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}dA = -\iint_A\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}_x dA_x + \iint_A\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}_{x+dx} dA_{x+dx} \qquad \text{(A2)}$$

if $\nabla\phi$ is continuous by definition. Therefore*

$$\oint\!\!\!\oint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right\}dA = \frac{\partial}{\partial x}\iint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right\}_x dA_x\,dx$$

$$= \frac{\partial}{\partial x}\iiint_{Vol}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right\}_x dVol_x \qquad \text{(A3)}$$

---

* This step was brought to my attention by Prof. Edward Kurtz.

But according to Green's Theorem[1], (A3) is equivalent to

$$\frac{\partial}{\partial x}\int_{\sigma}\left(\phi\frac{\partial\phi}{\partial n}\right)_{\text{surface}} d\sigma \quad \text{or} \quad \frac{\partial}{\partial x}\int_{\sigma}\left(\phi\frac{\partial\phi}{\partial n}\right)_{\text{surface}} dx r^* d\theta \qquad \text{(A4)}$$

which is exactly the same answer as previously obtained in (29). Therefore, we can only conclude that the thesis of Appendix A is correct.

APPENDIX B. ON THREE DIMENSIONAL VIRTUAL MASS

In this section we would like to comment on the differences between the two dimensional virtual mass of a slender body model fish, and that of a non slender body three dimensional fish. We shall consider the virtual mass of a circular cross sectioned cylinder because this special shape facilitates integration and makes it easier to see the difference between the two and three dimensional concept of virtual mass.

The virtual mass of a circular cross section of unit length of a long cylinder subject to a two dimensional flow is [1] $\rho\pi r^{*2}/2$ per unit length, if ρ is the mass density of the water, and r^*, the radius of the cross section.

On the other hand, the virtual mass of a long circular cylinder which has "wiggled" sufficiently to produce a three dimensional (potential) flow is a function of a characteristic x dimension parameter. This may be shown as follows:

The kinetic energy of the volume of fluid (see diagram 46) between x=0 and x=1, is (let dx be a unit length),

$$\text{K.E.} = \tfrac{1}{2}\rho\oint\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial\theta}\frac{1}{r}\right)^2\right]d\mathrm{Vol} \qquad \text{(B1)}$$

$$= \tfrac{1}{2}\oint_\sigma\rho\,\phi\frac{\partial\phi}{\partial n}\,d\sigma = \tfrac{1}{2}\rho\int_0^1\int_0^{2\pi}(r^*d\Theta(V_r)_s^2 r^*)\,dx$$

If r^* is not a function of x, then

$$\text{K.E.} = \int_0^1 (\pi r^{*2}\tfrac{1}{2}\rho(V_r)_s^2)_{\Theta=0}\,dx \qquad \text{(B2)}$$

where V_r is given by (26). But the inside of (B2) is just the two dimensional virtual mass which is not yet multiplied by its x dimension. In two dimensional theory, as in the slender body theory, the bracketed term is not a function of x. But if we integrate (B2), we must consider V_r's x variation of e^{kx}, where k is the complex wave number. (We note that K may itself be a function of x.) (B2) becomes, (the equality holds if k is not a function of x)

$$\text{K.E.} \doteq \left\{\tfrac{1}{2}\rho r^{*2}V_{r_{x=0}}^2\pi\,(e^{k\cdot 1}-1)/k1\right\}_{\substack{\Theta=0\\ r=r^*}}. \qquad \text{(B3)}$$

Thus, we may define a virtual mass for a fish who wiggles periodically in x from (B3) by

$$\text{K.E.} \doteq \left\{\rho\; V_r^2\tfrac{1}{2}\pi r^{*2}(e^{k} - 1)/k\cdot 1\right\}_{\substack{x=0\\ r=r^*}}$$

or

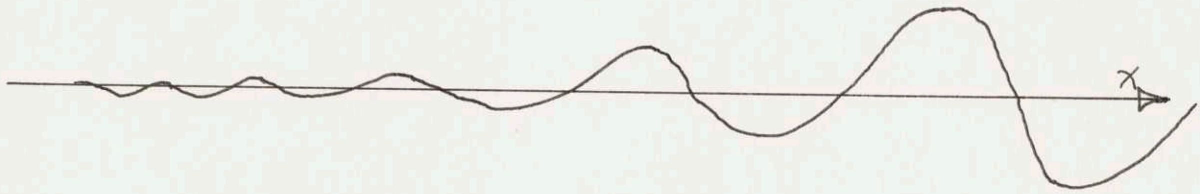
$$M_{3D} = \rho\,\tfrac{1}{2}\,\pi r^{*2}(e^k - 1)/k\cdot 1 \qquad \text{(B4)}$$

per unit length, which may be greater or less than M_{2D} which is $\tfrac{1}{2}\rho\pi r^{*2}$, depending upon whether or not

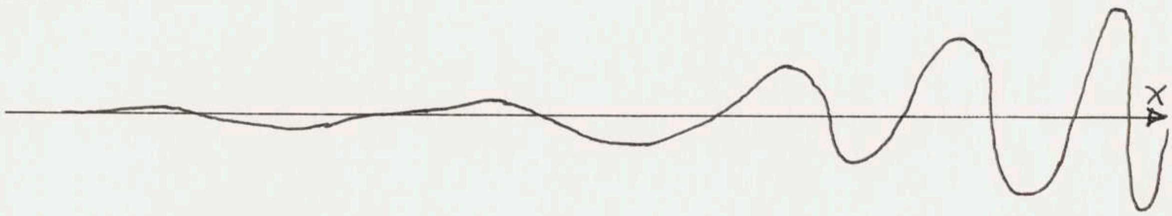
$$(e^k - 1)/k \qquad \text{(B5)}$$

is greater or less than 1. But (B5) is always larger than 1, being close to 1 for small values of k or for very large wavelengths. Thus, for the same lateral velocity at $\theta=0$, $V_{rat\ \theta=0}$, our fish will put more work into the fluid, produce a greater thrust and lose more energy to the wake than the slender bodied fish. But since thrust and work both vary linearly with virtual mass[2,3], the efficiency will be unaffected by this variation.

We might make one further comment. For finite fish who have the tendency or ability to vary their wave number, it would seem as if the natural motion of fish might be

a) 

instead of

b) 

since the former would have a smaller K at the rear and thus a smaller virtual mass and hence less kinetic energy lost to the wake than the latter case b, for the same lateral velocity amplitudes.

APPENDIX C. INTEGRATION OF $\int_a^b (H_n^2(Kr))^2 r\,dr = I$

Consider the equation for U^r,

$$r^2 d^2U^r/dr^2 + r\,dU^r/dr + (K^2r^2 - n^2)U^r = 0 . \qquad (C1)$$

It was noted in (21) that

$$U^r = C_6H_n^1(Kr) + C_7H_n^2(Kr) \quad . \tag{C2}$$

was a solution to (C1). A special solution for U^r satisfying (C1) was found to be

$$U^r = C_7H_1^2(Kr). \tag{C3}$$

Now to find I, multiply (C1) by $2dU^r/dr$, getting

$$2r^2\frac{d^2U^r}{dr^2}\frac{dU^r}{dr} + 2r(dU^r/dr)^2 + 2(K^2r^2-1)U^r dU^r/dr = 0$$

or

$$\frac{d}{dr}(r^2(dU^r/dr)^2 + (K^2r^2 -1)U^{r^2}) - 2K^2rU^{r^2} = 0.$$

Integrating between a and b gives

$$\left(r^2UdU^r/dr)^2 + (Kr^2 -1)U^{r^2}\right) \Big/_a^b = \int_a^b 2K^2U^{r^2}rdr \quad . \tag{C4}$$

But, from (C3), since [22 p.152]

$$U^r = C_7H_1^2(Kr),\ dU^r/dr = C_7KH_1^{2\prime}(Kr).$$

Letting $C_7 = 1$ for convenience,

$$r^2K^2\left(H_1^{2\prime}(Kr)\right)^2 + (K^2r^2-1)(H_1^2(Kr))^2 = 2K^2\int_a^b (H_1^2(Kr))^2 rdr$$

or

$$\int_a^b (H_1^2(Kr))^2 rdr = \tfrac{1}{2}r^2\left((H_1^{2\prime}(Kr))^2 + (1-1/K^2r^2)(H_1^2(Kr))^2\right) \tag{C5}$$

But the recurrence formulas for Bessel's function lead to[23 p.83]

$$\frac{d}{dr}H_1^2(Kr) = -H_1^2(Kr)/Kr + H_o^2(Kr) = H_1^{2\prime}(Kr), \tag{C6}$$

therefore (C5) becomes

$$\int_a^b (H_1^2(Kr))^2 rdr = \tfrac{1}{2}r^2\left(-2H_1^2(Kr)H_o^2(Kr)/Kr + (H_o^2(Kr))^2 + (H_1^2(Kr))^2\right)\Big/_a^b$$

$$= I$$

Since $H^2_{1,0} = J_{1,0} - iY_{1,0}$, a must be $\neq 0$, or else I will be infinite. In addition, since for large Kr, (if $Kr = \rho e^{i\varphi}$,

$$H^2_1(Kr) = H^2_1(\rho e^{i\varphi}) = \frac{e^{\rho \sin\varphi}}{(\frac{1}{2}r)^{\frac{1}{2}}} e^{-i(\rho\cos\varphi + \frac{1}{2}\theta - 3/4)} \qquad (c7)$$

where $Re(Kr) > 0$, $-\pi/2 < \varphi < \pi/2$. We see that φ must be small or between $-\pi/2$ and zero, (that is $K_i < 0$, and $K_r > 0$) if $r > 0$, for $H^2_1(Kr)$ to converge. Then at $r = \infty$, or $\rho = \infty$,

$$r^2 H_1 H_2 / Kr \text{ and} (H^2_0(Kr))^2 r^2 \text{ and } (H^2_1(Kr))^2 r^2$$

all approach zero! Therefore

$$I = \int_{r*}^{\infty} (H^2_1(Kr))^2 r dr = \tfrac{1}{2} r*^2 \Big(2H^2_1(Kr*) H^2_0(Kr*)/(Kr*) - (H^2_0(Kr*))^2 - (H^2_1(Kr*))^2 \Big) \qquad (C8)$$

where

$$Kr* \neq 0,$$

which is the desired evaluation. A similiar type of integration may be found in Mc Lachlan[23], pages 103 to 104 for Bessel's function of the first kind. We believe that the above integration will hold for Bessel functions of the first, second or third types, (the latter are Hankel functions of the first or second kinds) if, appropriate limits of integration are assumed. This fact does not seem to be generally known.

APPENDIX D. FORTRAN PROGRAMS FOR CHAPTER III

The program used to generate values for dimensionless thrust, work and efficiency is presented on page 51 . Thrust is called TSTAR; work, WSTAR; and efficiency EFF. T(1) Z(1) and w(1) are defined in the text on pages 25, 25 and 27 respectively.

The program used to generate the velocity field is presented on page 55.

Both programs calculate the Hankel functions by computing Bessel functions of the first and second types, and of the zeroth and first orders with the aid of subroutine COMBES[24], This subroutine was used in its original form[24] except for modifications in calling LNGAM (a subroutine of COMBES) and a complete revision of COMLOG (also a subroutine of COMBES). The revision of COMLOG was suggested by its author[24] in a personal communication. The resulting accuracy of the new COMBES is estimated at two decimal places.

```
C     CALCULATES WORK AND THRUST FOR FISH PROBLEM
      DIMENSION H1(2),H0(2),Z(2),TEMP1(2),TEMP2(2),TEMP3(2),H0SQ(2),H1SQ
     1(2)      ,Y0(2),       Y1(2),W(2),BJRE(100),BJIM(100),THRUST(2),
     2YRE(50),YIM(50),BOTTOM(2),  TOP(2),WINSID(2),TEMP4(2),
     3FJ0(2),FJ1(2),P(2),C(2),XFRONT(2),XWSTAR(2)
 1    READ 6000, FREQM
      PRINT6000, FREQM
      READ 7000,RZM
      PRINT7000,RZM
      READ 1000,ANGZ
      PRINT1000,ANGZ
      READ 4000, FREQ
 5    FREQ = FREQ + 1.
      RZ = 0.
 10   RZ = RZ + .5
      Z(1)=RZ*COSF(ANGZ)
      Z(2)=RZ*SINF(ANGZ)
      BETA = 0.
      ALPHA = 0.
      N = 1
      CALL COMBES(Z(1),Z(2),ALPHA,BETA,N,BJRE   ,BJIM   ,YRE   ,YIM   )
      FJ0(2)= BJIM(1)
      FJ0(1)= BJRE(1)
      Y0(1) = YRE(1)
      Y0(2) = YIM(1)
      H0(1) =FJ0(1) +  Y0(2)
      H0(2) =FJ0(2) -  Y0(1)
      FJ1(1)= BJRE(2)
      FJ1(2)= BJIM(2)
      Y1(1) = YRE(2)
      Y1(2) = YIM(2)
      H1(1) =FJ1(1) +  Y1(2)
      H1(2) =FJ1(2) -  Y1(1)
      CALL CDIV (TEMP1,1,1,H1,1,1,Z,1,1)
      BOTTOM(1) = H0(1) - TEMP1(1)
      BOTTOM (2) = H0(2) - TEMP1(2)
      WINSID(1) = 1.+ FREQ*Z(2)/((Z(2))**2 + (Z(1))**2)
      WINSID(2) = FREQ*Z(1)/((Z(2))**2 + (Z(1))**2)
      XFRONT(2) = FREQ + Z(2)
      XFRONT(1) = Z(1)
      CALL CDIV(C,1,1,XFRONT,1,1,BOTTOM,1,1)
      CALL CMULT(XWSTAR,1,1,WINSID,1,1,C,1,1)
      CALL  CMULT(W,1,1,XWSTAR,1,1,H1,1,1)
 18   TSTAR = -W(1)*Z(2)/(Z(1)*2.) + W(2)/2.
 180  WSTAR = W(1)*FREQ/(2.*Z(1))
      EFF = TSTAR/(WSTAR)
      PRINT1000,ANGZ
      PRINT 4000,FREQ
      PRINT 8000,RZ
      PRINT 2000,WSTAR,TSTAR
      PRINT 5000,EFF
      PRINT3000,W(1),W(2)
      IF(RZ - RZM) 10,20,20
 20   IF(FREQ - FREQM) 5,1,1
 1000 FORMAT(6H ANGZ=E25.6)
 2000 FORMAT(13H WSTAR,TSTAR=2E25.6)
 3000 FORMAT(11H W(1),W(2)=2E15.6)
 4000 FORMAT(6H FREQ=E25.6)
 5000 FORMAT(5H EFF=E25.6)
```

```
6000 FORMAT(7H FREQM=E25.6)                                        54
7000 FORMAT(5H RZM=E25.6)
8000 FORMAT(4H RZ=E25.6)
     END(1,1,0,0,0,1,1,0,0,1,0,0,0,0,0)
                                                                       63

                                                        TOTAL          63
```

```
C     CALCULATES VELOCITIES FOR FISH FIELD
      DIMENSION H1(2,40), H0(2,40), TEMP1(2,40),FJ0(2,40),FJ1(2,40),
     2      BOTTOM(2,40), WAVENO(2), C(2),VR(2), VTHETA(2), UPRIME(2),
     3AXP(2),Z(2),Y0(2,40),Y1(2,40),BJRE(100),  TEMP(2,40),BJIM(100),
     4YRE(50),YIM(50)
      CONTINUE
 1    READ 1000, K, DELTR
      PRINT 1000, K, DELTR
      READ 2000, OMEGA,DARG,DTHETA
      PRINT2000, OMEGA,DARG,DTHETA
      READ 3000,WAVENO(1),WAVENO(2)
      PRINT3000,WAVENO(1),WAVENO(2)
      R= 1.
      Z(1) = WAVENO(1)*R
      Z(2) = WAVENO(2)*R
      BETA = 0.
      ALPHA = 0.
      N = 1
      CALL COMBES(Z(1),Z(2),ALPHA,BETA,N,BJRE,BJIM,YRE,YIM)
      FJ0(1,J) = BJRE(1)
      FJ0(2,J) = BJIM(1)
      Y0(1,J) = YRE(1)
      Y0(2,J) = YIM(1)
      H0(1,J ) = FJ0(1,J ) + Y0(2,J )
      H0(2,J ) = FJ0(2,J ) - Y0(1,J )
      FJ1(1,J) = BJRE(2)
      FJ1(2,J) =  BJIM(2)
      Y1(1,J) = YRE(2)
      Y1(2,J) = YIM(2)
      H1(1,J ) = FJ1(1,J )+ Y1(2,J)
      H1(2,J ) = FJ1(2,J ) - Y1(1,J )
C  CALCULATE THE CONSTANT PART OF THE FIELD
      CALL CDIV(TEMP1,1,1,H1,1,1,Z,1,1)
      BOTTOM(1,1) = H0(1,1) - TEMP1(1,1)
      BOTTOM(2,1) = H0(2,1) - TEMP1(2,1)
      TEMP1(1) =SQRTF((OMEGA + WAVENO(2))**2 + WAVENO(1)**2)
      TEMP1(2) = 0.
      CALL CDIV(C,1,1,TEMP1,1,1,BOTTOM,1,1)
C  SET UP AN ARRAY AND FILL IT WITH VALUES FOR DIFFERENT  RS
      R = 1.
      DO 200 J=1,K
      Z(1) = WAVENO(1)*R
      Z(2) = WAVENO(2)*R
      CALL COMBES(Z(1),Z(2),ALPHA,BETA,N,BJRE,BJIM,YRE,YIM)
      FJ0(1,J) = BJRE(1)
      FJ0(2,J) = BJIM(1)
      Y0(1,J) = YRE(1)
      Y0(2,J) = YIM(1)
      H0(1,J ) = FJ0(1,J ) + Y0(2,J )
      H0(2,J ) = FJ0(2,J ) - Y0(1,J )
      FJ1(1,J) = BJRE(2)
      FJ1(2,J) =  BJIM(2)
      Y1(1,J) = YRE(2)
      Y1(2,J) = YIM(2)
      H1(1,J ) = FJ1(1,J )+ Y1(2,J)
      H1(2,J ) = FJ1(2,J ) - Y1(1,J )
      CALL CDIV(TEMP1,J,1,H1,J,1,Z,1,1)
      BOTTOM(1,J) = H0(1,J) - TEMP(1,J)
      BOTTOM(2,J) = H0(2,J) - TEMP(2,J)
```

```
      R = R + DELTR
 200  CONTINUE
C     VARY THETA AT FIXED KX (ARG)
      THETA = 0.
      ARG = 0.
      J = 1
 205  AXP(1) = COSF(ARG)
      AXP(2) = SINF(ARG)
      CALL CMULT(VR,1,1,BOTTOM,J,1,AXP,1,1)
      CALL CMULT(VTHETA,1,1,TEMP1,J,1,AXP,1,1)
      CALL CMULT(UPRIME,1,1,H1,J,1,AXP,1,1)
      CALL CMULT(VR,1,1,VR,1,1,C,1,1)
      CALL CMULT(VTHETA,1,1,VTHETA,1,1,C,1,1)
      CALL CMULT(UPRIME,1,1,UPRIME,1,1,C,1,1)
      GROWTH = EXPF(ARG*WAVENO(1)/WAVENO(2))
      DO 210 M = 1,2
      VR(M) = VR(M)*GROWTH
      VTHETA(M) = VTHETA(M)*GROWTH
      UPRIME(M) = UPRIME(M)*GROWTH
 210  CONTINUE
 220  VR(1) =-VR(2)*COSF(THETA)
      VTHETA(2) = -VTHETA(1)*COSF(THETA)
      UPRIME(1) =-UPRIME(2)*COSF(THETA)
      PRINT 4000, J
      PRINT 5000, THETA
      PRINT 7000, ARG
      PRINT 6000,VR(1),VTHETA(2),UPRIME(1)
      IF ( THETA - 3.2) 230,240,240
 230  THETA =  THETA + DTHETA
      GO TO 220
C NOW VARY AXIAL DISTANCE IN X
 240  THETA = 0.
      IF(ARG - 6.3 ) 250,260,260
 250  ARG = ARG + DARG
      GO TO 205
C FINALLY CALL ON ALL THE VALUES KEPT IN THE ARRAY BEFORE 200
 260  ARG = 0.
      J = J + 1
      IF(J-K) 205,205,1
 1000 FORMAT(9H K,DELTR=I3,E15.6)
 2000 FORMAT(19H OMEGA,DARG,DTHETA=3E15.6)
 3000 FORMAT(21H WAVENO(1),WAVENO(2)=2E15.6)
 4000 FORMAT(3H J=I3)
 5000 FORMAT(7H THETA=E15.6)
 6000 FORMAT(18H VR,VTHETA,UPRIME=3E15.6)
 7000 FORMAT(5H ARG=E15.6)
      END
```

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TOTAL 106*

APPENDIX E. FISH WITH VARIANT CROSS SECTION

We now consider the case where r^* is a strong function of x. Let r^* not be a function of θ or ϕ. Since Ω in Diagram 1 is of the order of $\partial h/\partial x$, to a first order approximation r^* may be considered to have derivatives in the x direction only. Thus in (5),

$$Dr^*/Dt \doteq \bar{u}\partial r^*/\partial x \ ,$$

or if (5) is written in the r direction only, then

$$(V_r/\cos\theta - \bar{u}\partial r^*/\partial x)_{r^*} = (\partial h/\partial t + \bar{u}\partial h/\partial x) \qquad \text{(E1)}$$

where we have implied that $\partial r^*/\partial x$ is of the order of V_r or $\partial h/\partial t$. (E1) is then the surface condition for the case in which the cross section of the fish varies with axial distance.

Take the case of an elongated elliptic fish. Let the ellipse have its major axis on the x axis, and minor axis at x = 0, and at $\theta = 90^\circ$ say. (See Diagram 3). Then, if a and b are constants (see Diagram 3),

$$r^*(x) = (a^2 - x^2)^{\frac{1}{2}}xb/a$$

and (E2)

$$\partial r^*/\partial x = (a^2 - x^2)^{\frac{1}{2}}xb/a.$$

APPENDIX F. CONSERVATION OF MOMENTUM

The conservation of momentum condition is not easily incorporated into a long fish theory since all end forces are not determinate. When introduced, such a condition may be incorporated only by assuming that there are no end forces in the fish body or that the fish is finite. But in this case, the flow field used in this paper is not rigorously applicable, but only an approximation to reality. However, for completeness we now briefly discuss Newton's Law on lateral movements.

Lighthill[2] has been, as far as we know, the first and only author to actually write the equation on the dynamic force balance between the fish and the surrounding fluid. He did this by nelecting end conditions and by assuming the so called slender-body theory.

The slender-body theory may employ[2] a two dimensional virtual mass. in order to calculate thrust and work. This is a mass of fluid associated with a cylinder having lateral velocity (or constant axial velocity) only. This mass, multiplied by the lateral velocity squared, represents, supposedly, the entire kinetic energy of three-dimensional fluid movements to a good approximation. This virtual mass is a rigid body two-dimensional concept and is evaluated from the time invariant geometrical body configuaration in question. The slender-body theory virtual mass concept then assumes that the flow past a cross section S(x) of fish, is equivalent to the lateral flow plus the axial flow past an infinitely long cylinder with constant cross section S(x). This assumption holds strictly only if r* is not a function of x or if down-stream conditions have no influence on the upstream flow pattern.* This is equivalent to saying not only that the streamwise perturbation is small, but negligible at all times and at all points. If M(x) is the virtual mass of fluid associated with an infinitely long cylinder of cross section S(x), then Newton's Law may be written as[2]

$$\int_0^1 \rho S(x) \frac{\partial^2 h}{\partial t^2} dx = \int_0^1 M(x) \frac{D(V_r)}{Dt}\bigg|_{\theta=0} dx \qquad \text{(F1)}$$

for the entire fish for all situations in which u' and end conditions are negligible. Analogously the conservation of angular momentum may be written as

$$\int_0^1 x \rho S(x) \frac{\partial^2 h}{\partial t^2} dx = \int_0^1 x M(x) \frac{D(V_r)}{Dt}\bigg|_{\theta=0} dx \quad . \qquad \text{(F2)}$$

If M(x) were a three dimensional virtual mass, one would have greater faith in (F1 and 2). The most accurate method of writing the invariance principles however is to express the lateral force exactly in the O = O direction as

$$\int_0^1 \rho \frac{\partial^2 h}{\partial t^2} S(x) dx = \int_0^1 \int_0^{2\pi} p \cos\theta \, (r^*) dx d\theta \qquad \text{(F3)}$$

where p is the linearized pressure and where r* is an arbitrary function of θ and x. Also we may write analogously

*and visa versa.

to (F2)

$$\int_0^1 \rho\, x \frac{\partial^2 h}{\partial t^2} S(x)dx = \int_0^1 \int_0^{2\pi} xp\cos\theta\,(r^*)dxd\theta. \qquad \text{(F4)}$$

For a given h(x,t), and therefore given ω and k, if (F3) and (F4) are not satisfied, then rigid body reflex motions will result. These motions are similiar to the reflex of a gun due to the momentum deficiency after a bullet is shot. Lighthill has called these rigid body recoils applied to a fish "F(t) + xG(t)." These movements when added to h(x,t) would cause (F3) and(F4) to be satisfied. Lighthill[2] however is concerned with recoils as a correction to (F1) or (F2). Since he considers only a two dimensional virtual mass, such a correction would seem to be unimportant in comparison with the errors due to neglecting the velocity field perturbations. A more correct virtual mass has been described in Appendix B.

APPENDIX G. THRUST DERIVATION BY SURFACE PRESSURE INTEGRATION

A numerically concise and perhaps more physically intuitive method of deriving an expression for thrust is now accomplished by integrating the pressure over the surface.

We wish to find the resultant of the pressure forces exerted by the fluid on a fish cross section in the x direction. Consider a cylindrical cross section of length dx and radius r*. Let this section be displaced a distance, h, from the x axis in the θ = 0 direction as usual. Because of this displacement, the cross section will be inclined by an angle Ω (see Diagram 1) in the $\theta = 0$, x plane. At $\theta = 0$ and $\theta = 180^\circ$ the x component of the pressure will be a maximum or minium and equal to $p\sin\Omega$. At $\theta = 90^\circ$ and 270° the pressure is perpendicular* to the x axis and has no component in that direction (see Diagram 1). Therefore, since $\Omega = \partial h/\partial x(\ 1 + (\partial h/\partial x)^2\ )^{\frac{1}{2}}$, to a good approximation the x component of the pressure is

$$p(\partial h/\partial x)\cos\theta\ . \tag{G1}$$

The force in the x direction exerted by the fluid pressure on a small area of this cross section is correspondingly

$$p(\partial h/\partial x)\cos\theta(r^* dx d\theta). \tag{G2}$$

The total force or the thrust exerted on the elementary cross section of length dx is equivalent to (G2) integrated over θ. To evaluate (G2) we consider only the linear part of the pressure ($p_{lin} = \rho\,\partial\phi/\partial t + \rho\bar{u}u'$) since the second order components (the velocities squared) when multiplied by $\partial h/\partial x$ would become third order quantities. The second order terms would in any case have zero time average or integrate over theta to zero when multiplied by $(\partial h/\partial x)\cos\theta$. From equation (27) on page 26 we have therefore

$$T = \mathrm{Re}\left\{Be^{2k_r x}\rho\left(\frac{i\omega+\bar{u}K}{K}\right)CH_1^2(Kr^*)e^{i(K_1x+\omega t)}\cdot\left[\bar{u}K_1\cos(K_1x+\omega t) + \bar{u}K_r\sin(K_1x+\omega t)\right]\right\}$$

if h is defined as usual. Or, with the definition of T^*, α and z on page 25, the time averaged value of T leads to

*Actually it is the vector normal to the surface or the pressure force which is perpendicular to the x axis

$$T^* = -\tfrac{1}{2}\mathrm{Re}\left\{\frac{i\alpha+Z}{Kr^*}\, CH_1^2(Z)\right\}\frac{Z(2)}{Z(1)} + \tfrac{1}{2}\mathrm{Im}\left\{\frac{i\alpha+Z}{Z}\right\}CH_1^2(Z) \qquad (G3)$$

T^* is plotted against α in Diagram 6 with Z as a parameter.

The efficiency, T^*/W^* may now be written in the very simple form

$$\eta = K_i/\omega + K_r/\omega\left[\mathrm{Im}\left\{\frac{i\omega+K\bar{u}}{K}CH_1^2\right\}\Big/ \mathrm{Re}\left\{\frac{i\omega+K\bar{u}}{K}CH_1^2\right\}\right] \qquad (G4)$$

η = phase velocity + small correction.