MOTIONAL TRANSIENTS IN POWER SELSYNS

by

M. M. E. Kaci

Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Electrical Engineering

at the

Massachusetts Institute of Technology

1945

Signature redacted
Signature of Author ....

Signature of Thesis Signature redacted
Supervisor ....

Signature of Chairman, Department Committee on Graduate Students
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ACKNOWLEDGMENT

The author wishes to express his indebtedness to Professor W. V. Lyon who suggested and supervised the present project. Thanks are due to Professor H. E. Edgerton for kindly lending the motion-picture camera which was used in obtaining experimental records. The invaluable help provided by Mr. N. Orbay throughout the laboratory work and in the preparation of the final thesis report is to be mentioned with gratitude.

328 Commonwealth Avenue
Boston 15, Massachusetts
October 22, 1945

Professor G. W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Professor Swett:

In partial fulfillment of the requirements for the Degree of Master of Science, the undersigned respectfully submits this thesis entitled, "Motional Transients in Power Selsyns."

Very truly yours,

Signature redacted

M. M. E. Kaci

C O N T E N T S

INTRODUCTION

Although the steady-state and electrical transient performance of selsyn systems have extensively been studied by several investigators, the motional transients occurring in them have so far received no attention. This negligence may be attributed to the mathematical difficulties encountered in attempting a rigorous analysis of the subject. It is shown in Part III of this thesis that an exact solution of the problem of motional transients in selsyns is, in fact, practically impossible. However it is desirable to possess a means of predicting the motional behavior of a selsyn set when its state of equilibrium is mechanically disturbed. Therefore a convenient approximate analysis must be resorted to. It is the purpose of the present thesis to obtain an approximate solution which will be in satisfactory agreement with the actual conditions. The proposed problem is to study the transient variation of the relative rotor angle of a selsyn system following the sudden application of a constant load.

The fundamental equations of the induction motor and their derivation as presented in Part III are based upon the lectures given by Professor W. V. Lyon in the subject of advanced alternating-current machinery.

PART I

RESULTS AND DISCUSSION

The transient variation of the relative angular displacement between the rotors of a selsyn system upon the sudden application of a constant load is a damped sinusoidal function of time given by

$$\delta = \delta_0 - \delta_1 e^{-\alpha t} \cos kt$$

Constants δ_0 , δ_1 , and k are defined on pages 50 and 51 . An expression for damping constant α is given on page 63 . Due to inevitable simplifying assumptions made in the course of the analysis, the equation given above constitutes only an approximate solution. A comparison between the computed and experimental results shows, however, that the degree of approximation is satisfactory (see pages 5 and 8).

The experimental results consist of nine photographic records taken under different speed and load conditions. The original photographs are shown on pages 14 to 22 . Each of these pages represents one single run. The sequence of the pictures is from the lower to the upper end of each strip. The order of the strips is from left to right. The beginning of the transient period of each run is indicated by a circle placed in front of the corresponding picture. The first three photographic records were taken by the use of a one-piece commutator (see Part II), and therefore the interval of time between two consecutive pictures corresponds to one revolution of the selsyn set. In obtaining the last six records the use was made of a two-piece commutator; hence two consecutive pictures are separated by an interval of time corresponding to one half revolution.

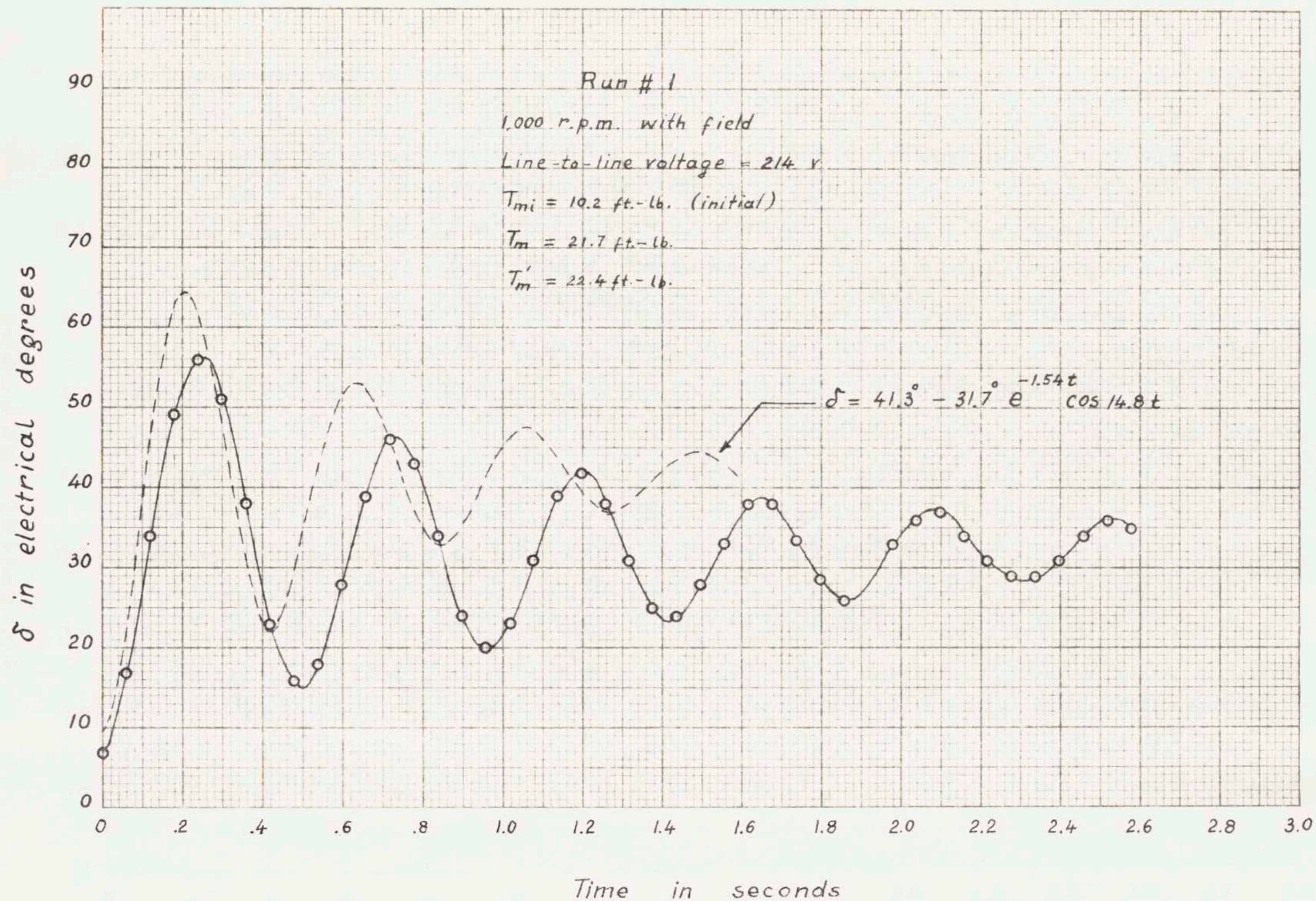
In order to make more explicit the information contained in the photographic records, curves are plotted which show the variation of the relative rotor displacement as a function of time (see pages 5 to 13). The speed and load conditions corresponding to each run are stated on the graphs.

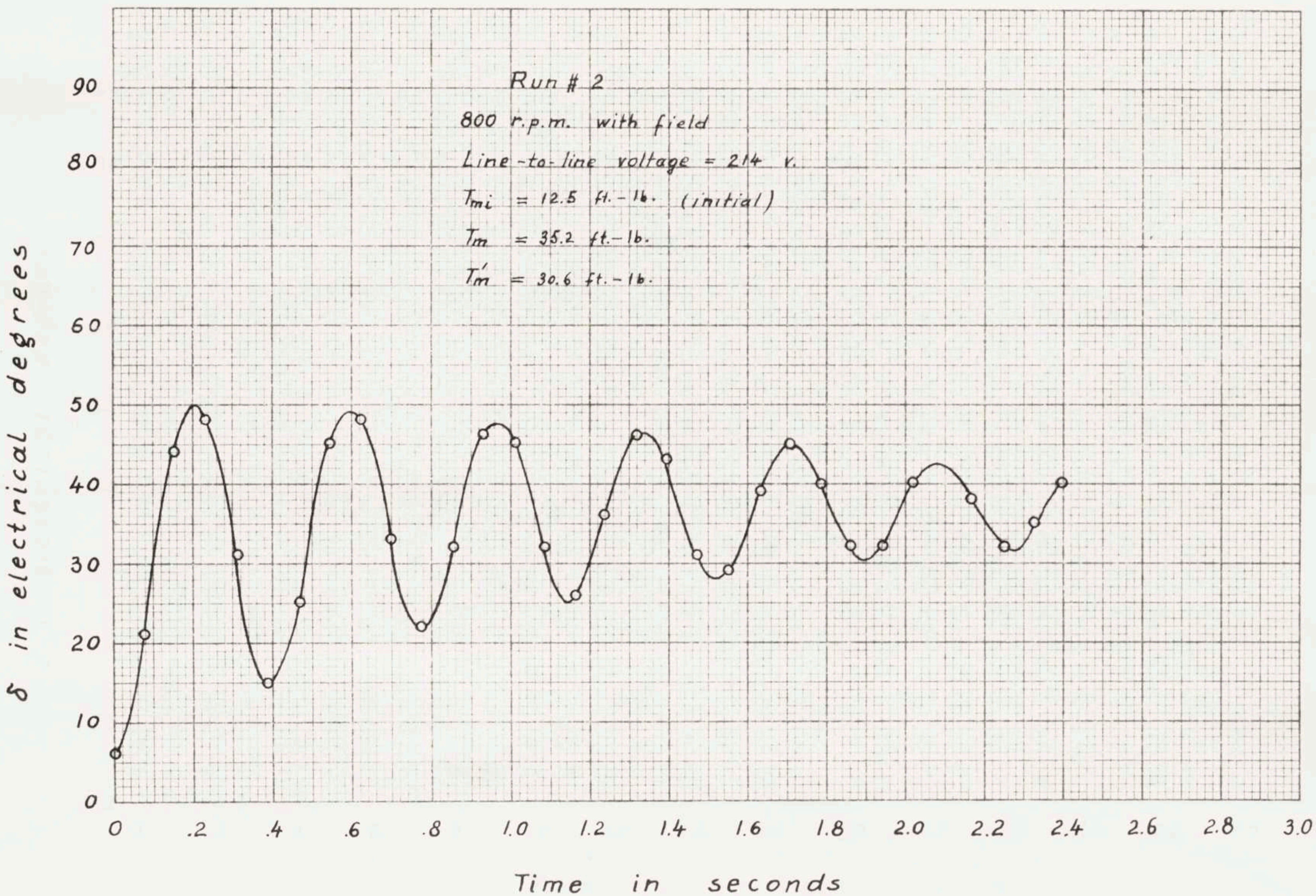
An examination of the experimental curves immediately shows that they are not exact damped sinusoidal curves as the analysis predicts. The median line of each curve has an exponential trend instead of being a horizontal straight line. This discrepancy is due to the fact that, in the analysis, the suddenly applied torques are assumed to stay constant throughout the transient period,whereas, in the actual laboratory experiments, they undergo a slight exponential change.

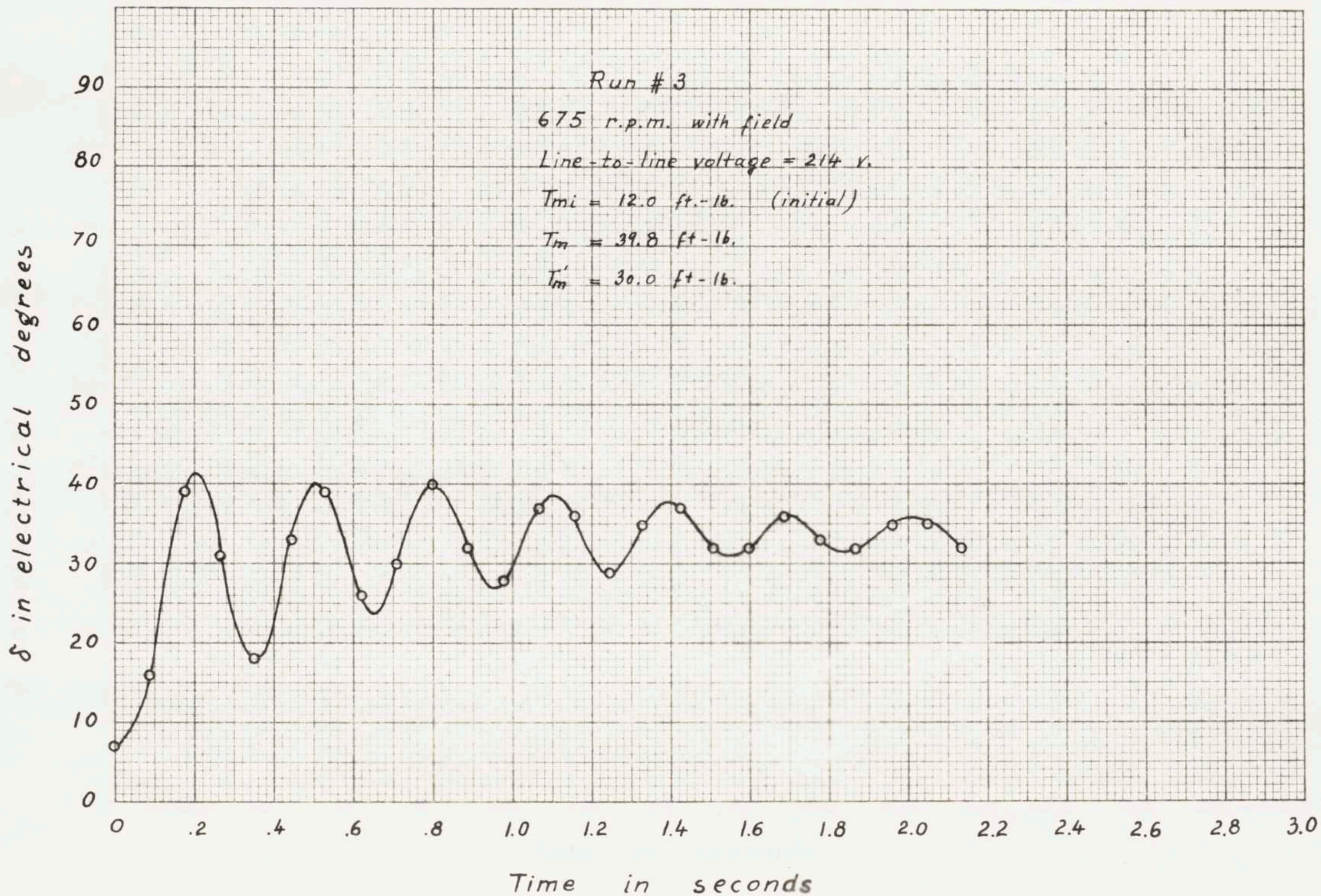
It is seen from the graphs that the period, the amplitude, and the damping of the oscillations are affected by the speed and applied torque. The effect of speed upon the period of oscillations is more pronounced when the selsyns operate in the direction of rotating magnetic field (see graphs on pages 5 , 6 , and 7). As the speed increases the frequency of oscillations becomes smaller. However the speed has practically no effect upon frequency when the system operates against the rotating magnetic field (see the graphs on pages 8 , 9 , and 10). These facts are in agreement with the analytical results. Eq.(42), page 50 , shows that the frequency of oscillations depends on the slip, and the effect of speed is greatest at near synchronism. The speed has a strongly marked effect upon the amplitude of oscillations when the operation is with field. (See the graphs on pages 5 , 6 , and 7 .) As the speed is increased the amplitude of oscillations becomes larger. This also is in conformity with the analytical prediction (see the expression of δ_1 on page 51). The curves on pages 5 , 6 , and 7 show that, when the system runs in the direction of field, larger speeds result in stronger damping. However, when operating against field, the damping is almost independent of speed. The analytic expression of the damping constant given by Eq.(66), page 62 , is in accordance with these observations.

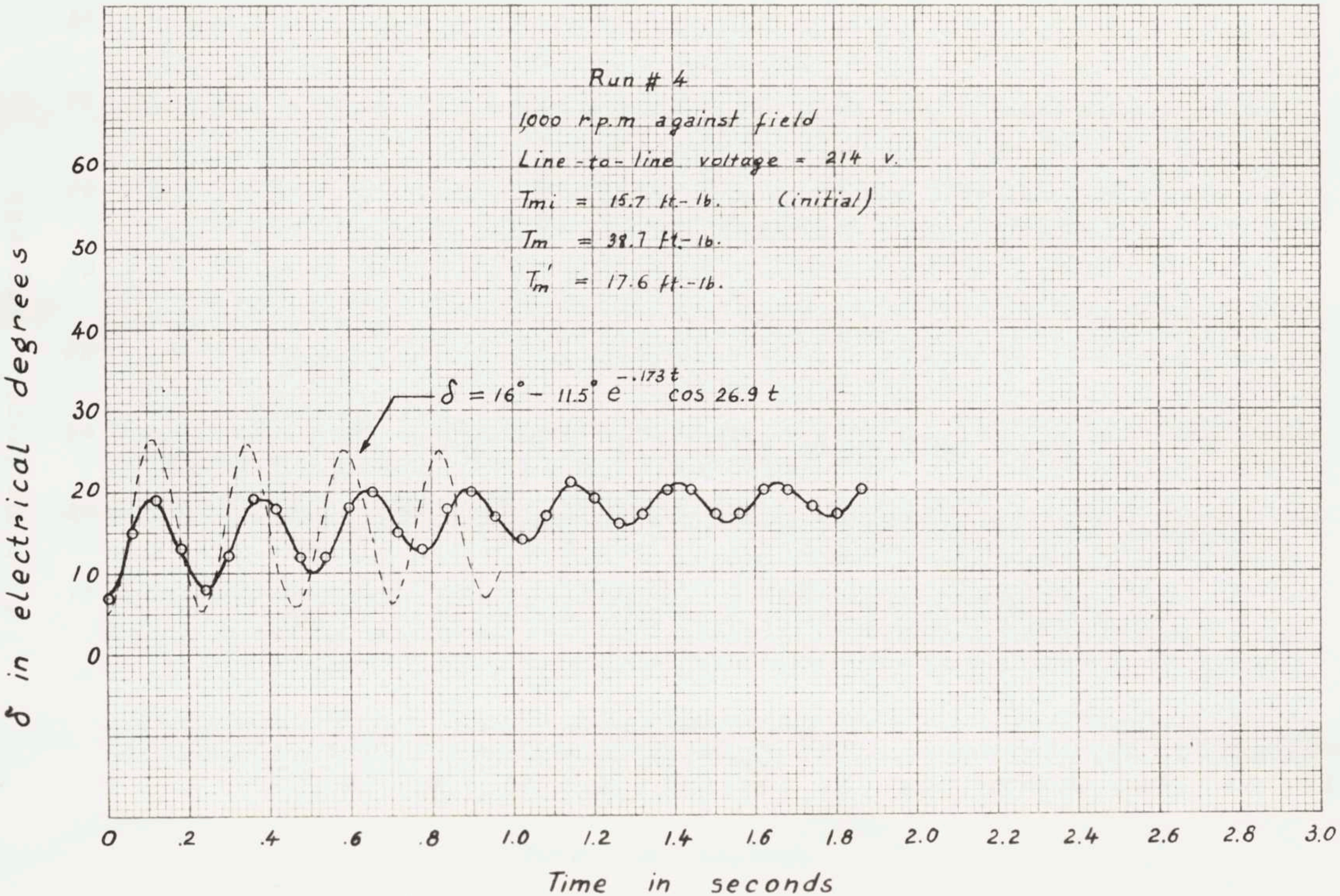
The effect of the magnitude of a suddenly applied torque upon the frequency and amplitude of oscillation may be seen by comparing the curves plotted on pages 7, and 12 . It is seen that the frequency is independent of torque, whereas the amplitude is nearly proportional to it. These facts are in agreement with Eq.(42) and the expression of δ_1 given on page 51 .

Run # 9 which is represented on page 13 shows that if the voltage impressed on the selsyn is reduced, the period and the amplitude of oscillations tend to increase. The analytical expressions referred to in the foregoing paragraph predict the same conclusions.

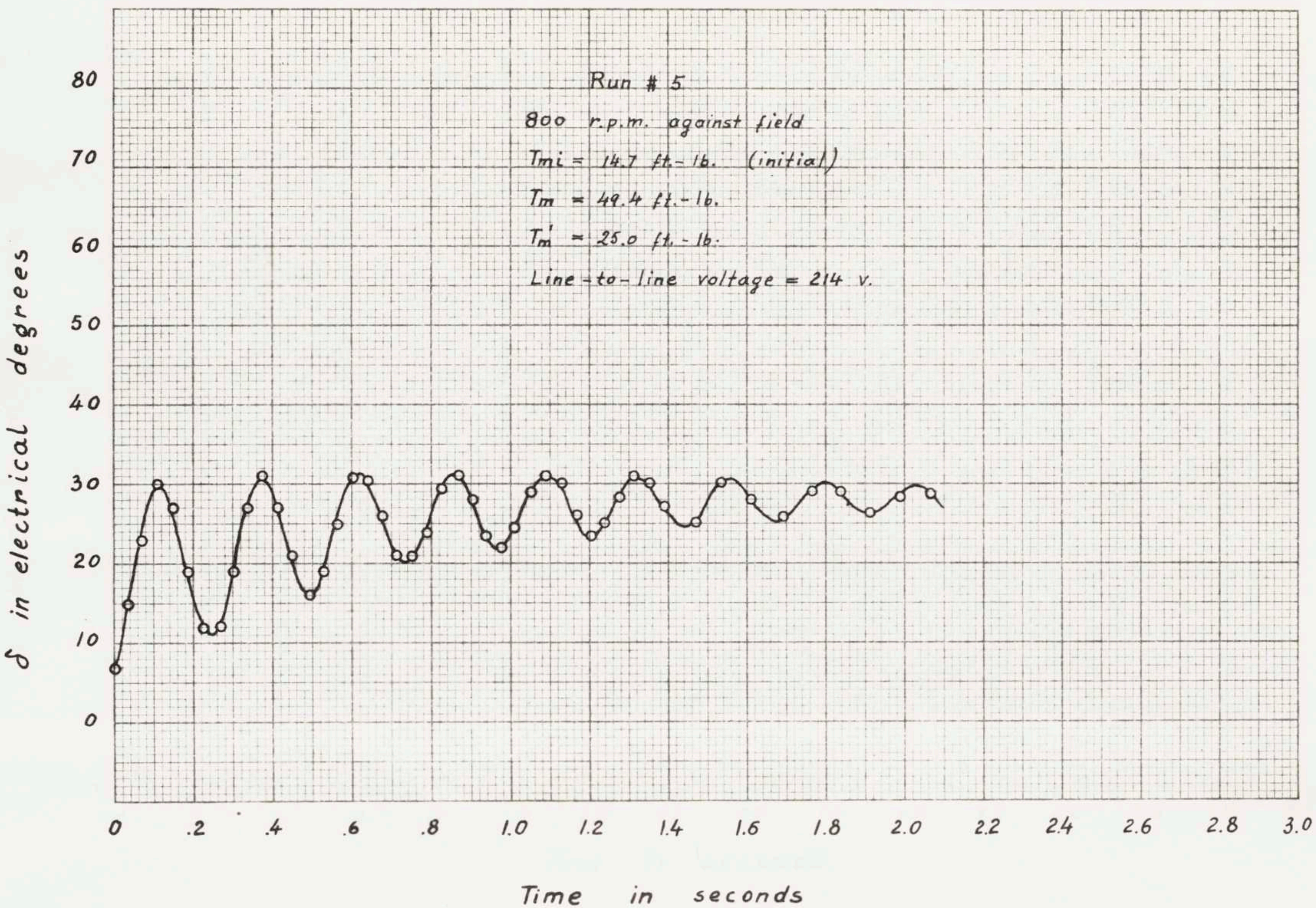

Run # 1
1,000 r.p.m. with field
Line-to-line voltage = 214 v
T_{mi} = 10.2 ft.-lb. (initial)
T_m = 21.7 ft.-lb.
T'_m = 22.4 ft.-lb.
$\delta = 41.3^\circ - 31.7^\circ e^{-1.54t} \cos 14.8t$
δ in electrical degrees
0
10
20
30
40
50
60
70
80
90
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds

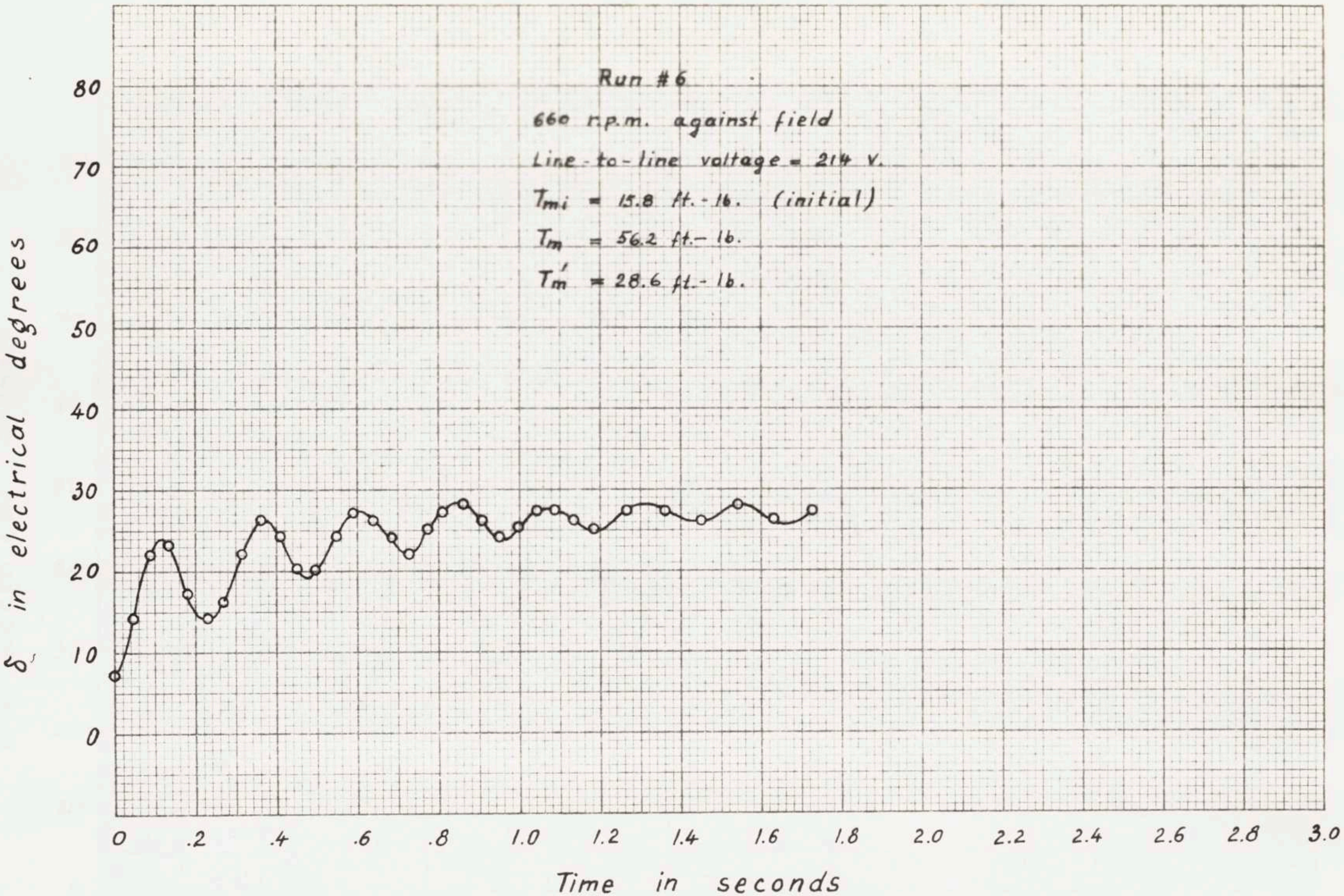

Run # 2
800 r.p.m. with field
Line-to-line voltage = 214 v.
T_{mi} = 12.5 ft.-lb. (initial)
T_m = 35.2 ft.-lb.
T_m' = 30.6 ft.-lb.
δ in electrical degrees
0
10
20
30
40
50
60
70
80
90
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds

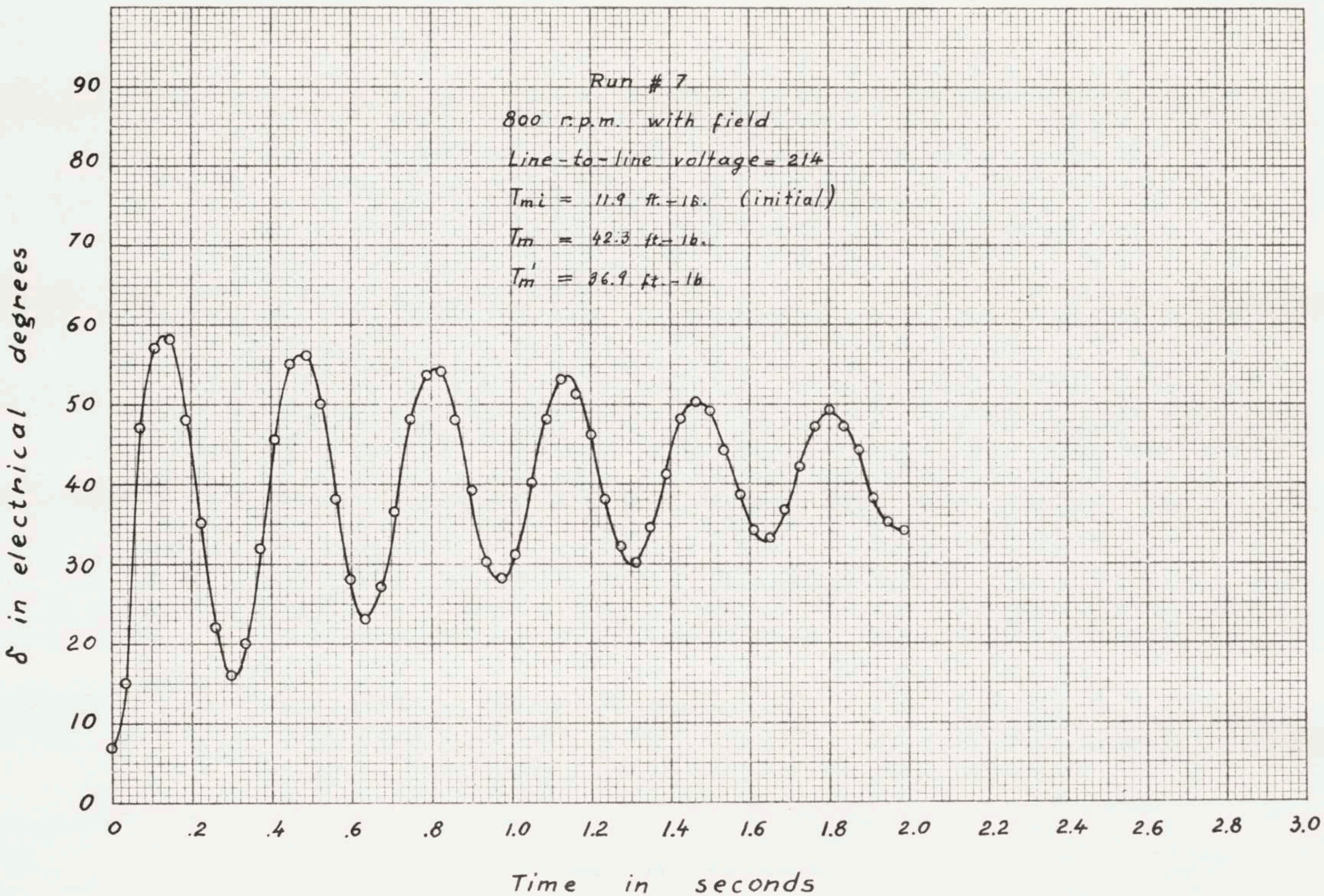

Run # 3
675 r.p.m. with field
Line-to-line voltage = 214 v.
Tmi = 12.0 ft.-lb. (initial)
Tm = 39.8 ft-lb.
T'm = 30.0 ft-lb.
δ in electrical degrees
90
80
70
60
50
40
30
20
10
0
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds

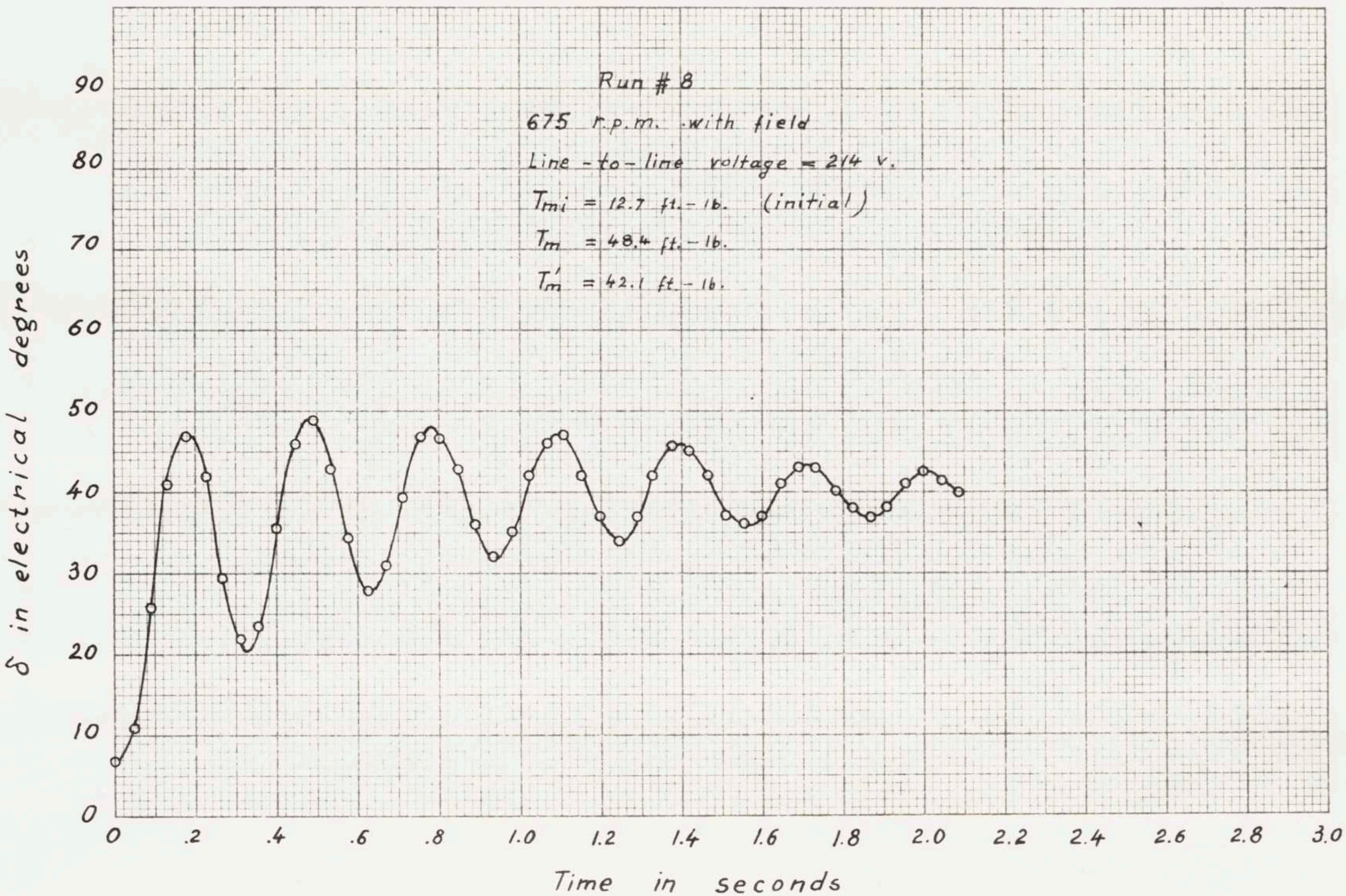


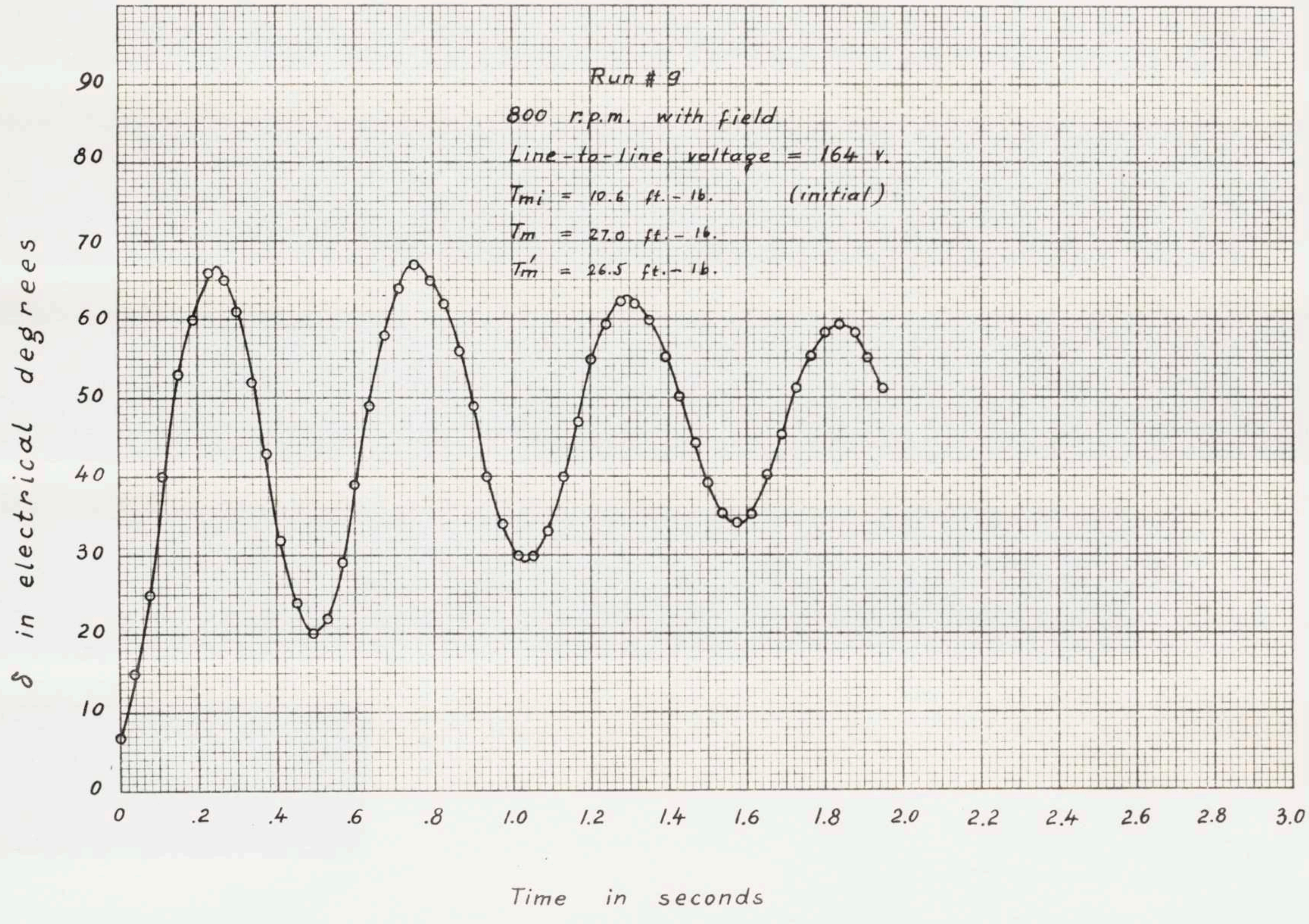
Run # 4
1,000 r.p.m against field
Line-to-line voltage = 214 v.
Tmi = 15.7 ft.-lb. (initial)
Tm = 38.7 ft.-lb.
T'm = 17.6 ft.-lb.
δ = 16° − 11.5° e^(−.173 t) cos 26.9 t
δ in electrical degrees
0
10
20
30
40
50
60
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds

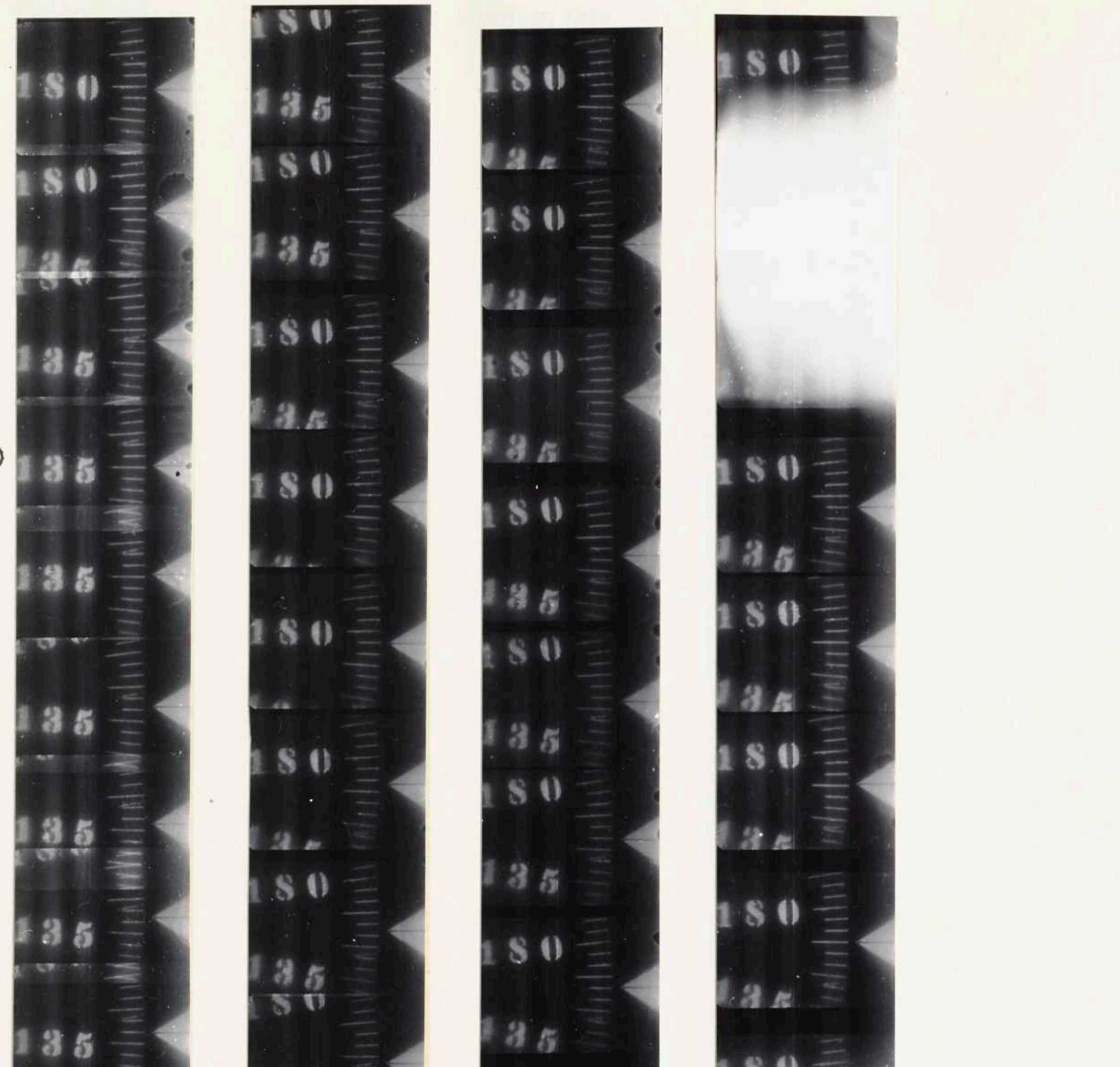

Run # 5
800 r.p.m. against field
Tmi = 14.7 ft.-lb. (initial)
Tm = 49.4 ft.-lb.
Tm' = 25.0 ft.-lb.
Line-to-line voltage = 214 v.
δ in electrical degrees
0
10
20
30
40
50
60
70
80
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds


Run #6
660 r.p.m. against field
Line-to-line voltage = 214 V.
T_{mi} = 15.8 ft.-lb. (initial)
T_m = 56.2 ft.-lb.
T_m' = 28.6 ft.-lb.
δ in electrical degrees
0
10
20
30
40
50
60
70
80
0 .2 .4 .6 .8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8 3.0
Time in seconds

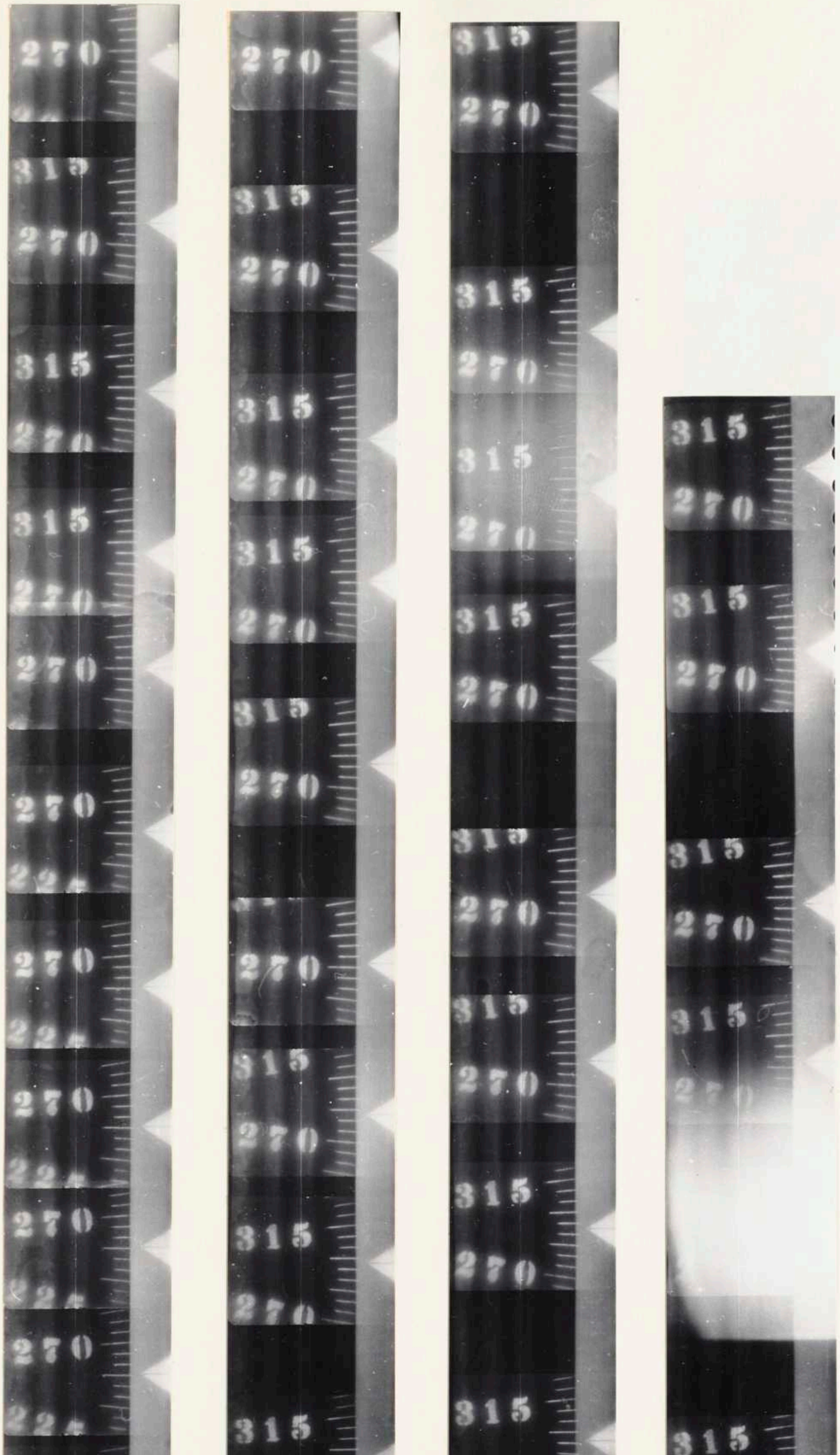

Run # 7
800 r.p.m. with field
Line-to-line voltage = 214
T_{mi} = 11.9 ft.-lb. (initial)
T_m = 42.3 ft.-lb.
T_m' = 36.9 ft.-lb
δ in electrical degrees
Time in seconds
0
10
20
30
40
50
60
70
80
90
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0


Run # 8
675 r.p.m. with field
Line-to-line voltage = 214 v.
T_{mi} = 12.7 ft.-lb. (initial)
T_m = 48.4 ft.-lb.
T_m' = 42.1 ft.-lb.
δ in electrical degrees
0
10
20
30
40
50
60
70
80
90
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds

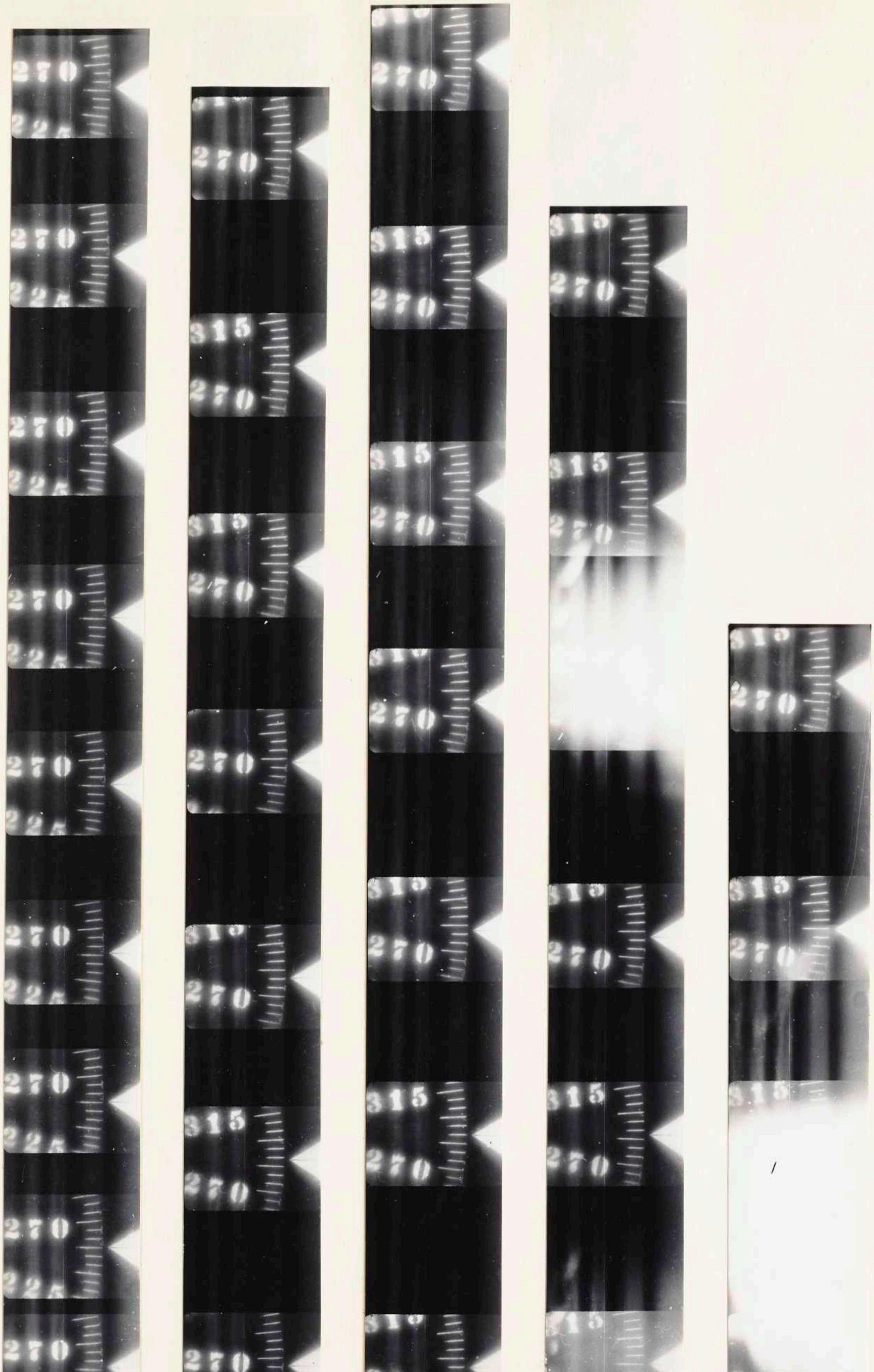

Run # 9
800 r.p.m. with field
Line-to-line voltage = 164 v.
Tmi = 10.6 ft.-lb. (initial)
Tm = 27.0 ft.-lb.
Tm' = 26.5 ft.-lb.
δ in electrical degrees
0
10
20
30
40
50
60
70
80
90
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
Time in seconds



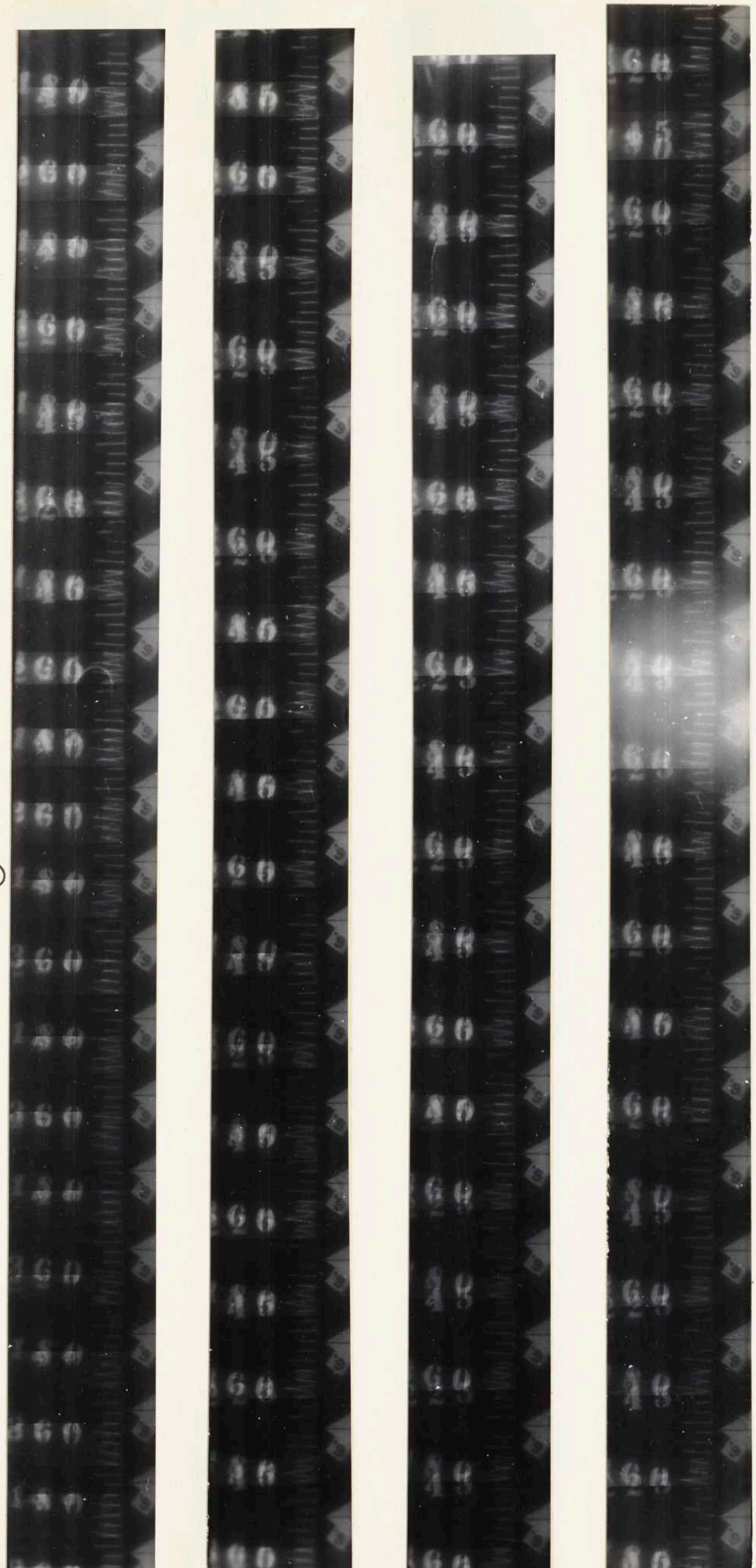
Run #1



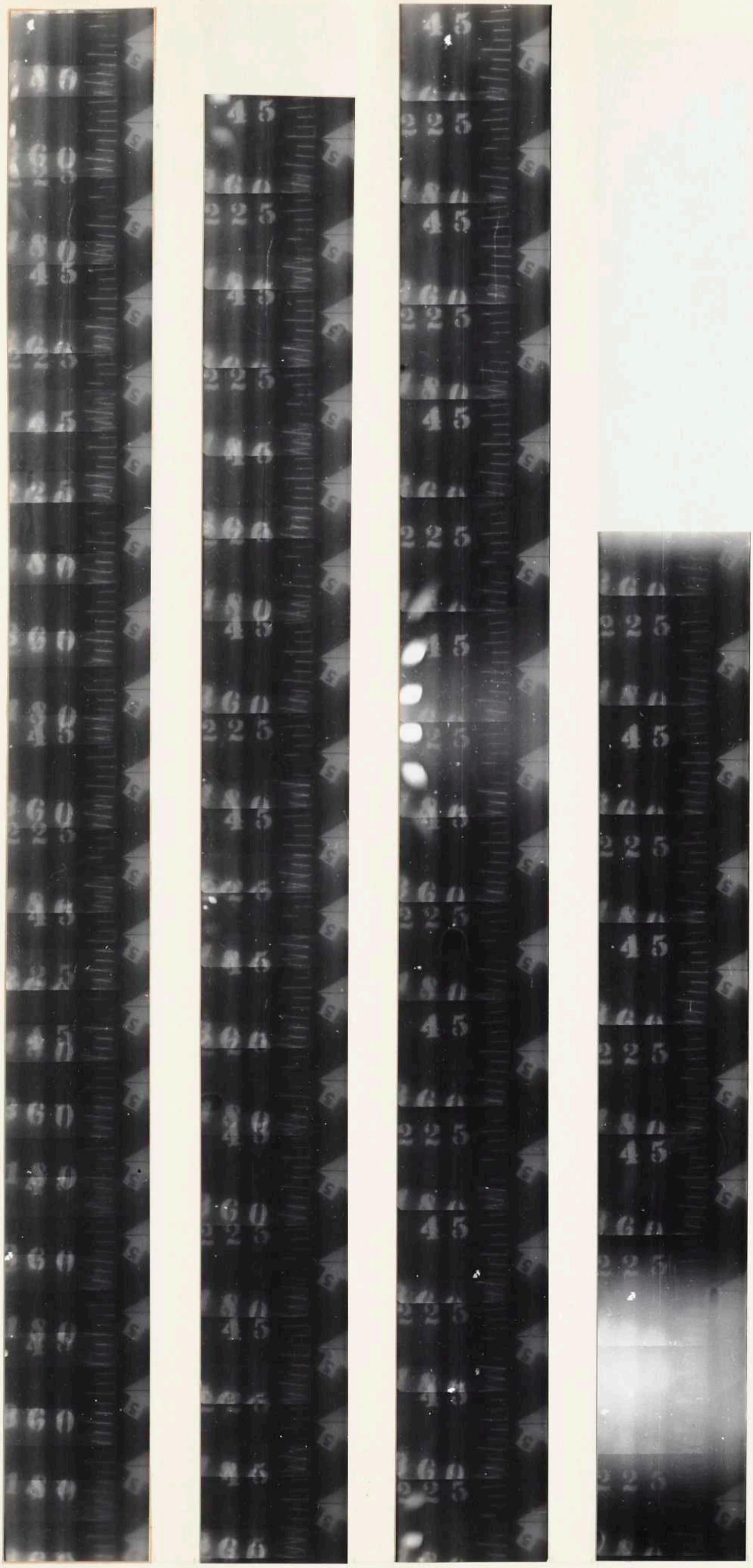
Run #2



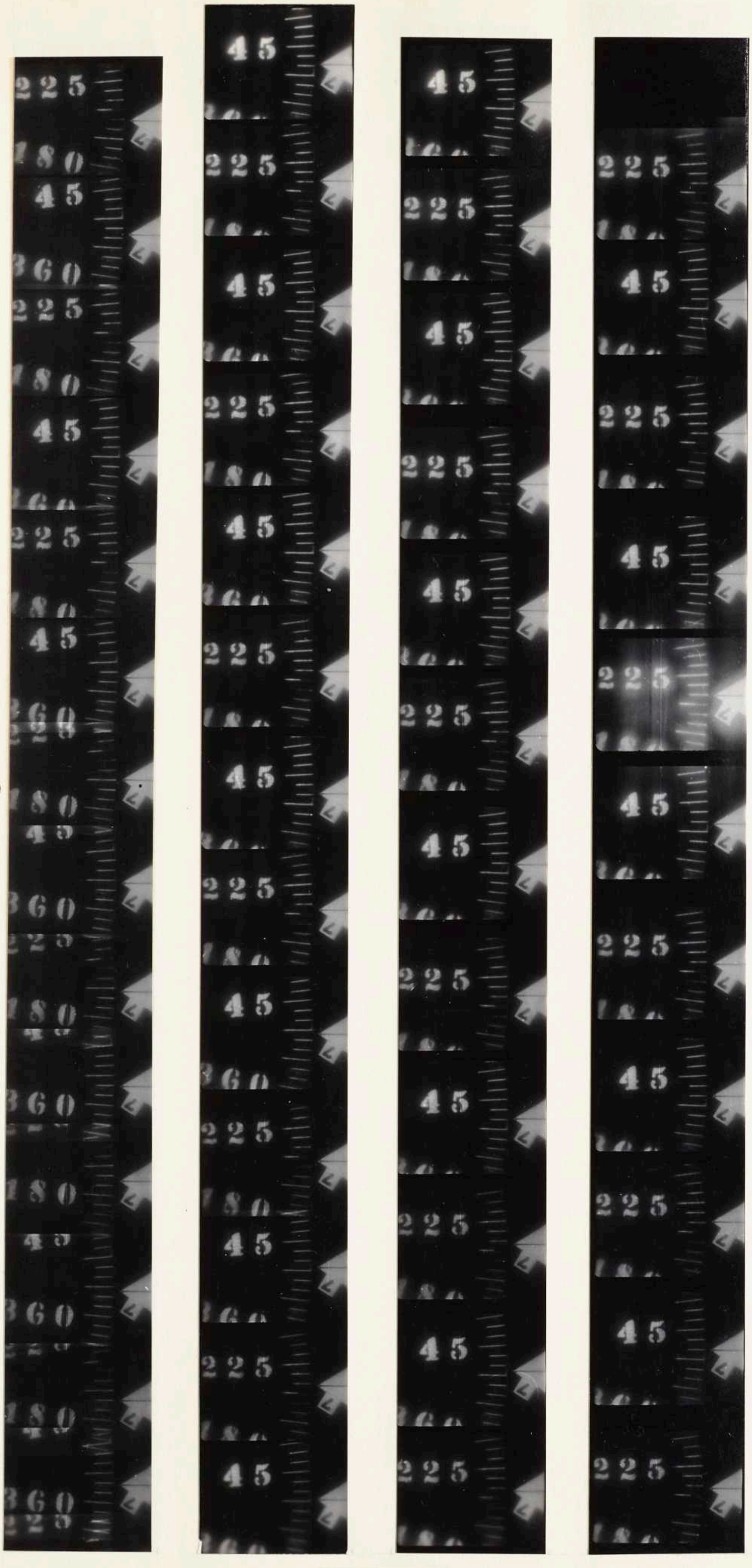
Run #3



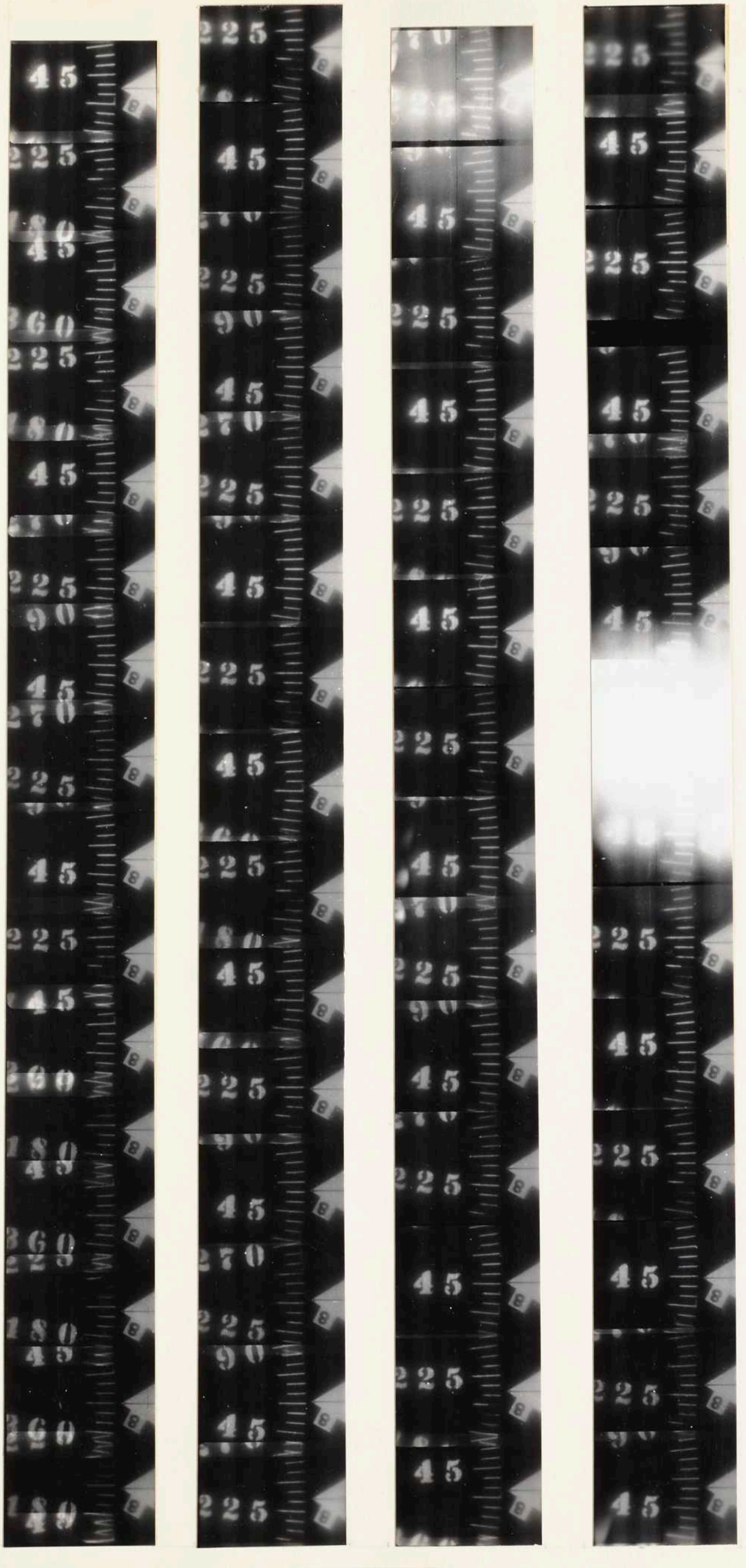
Run #4



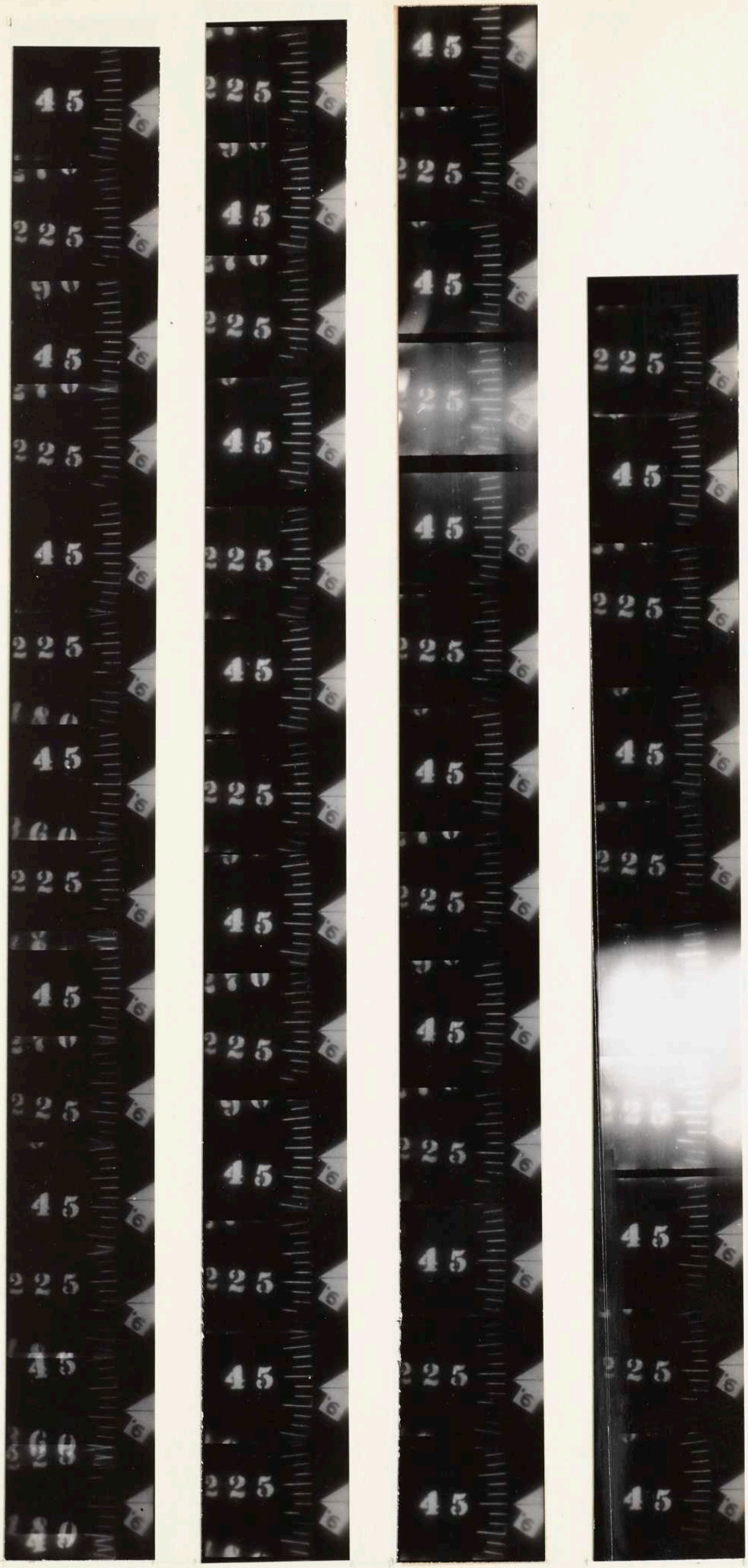
Run #5



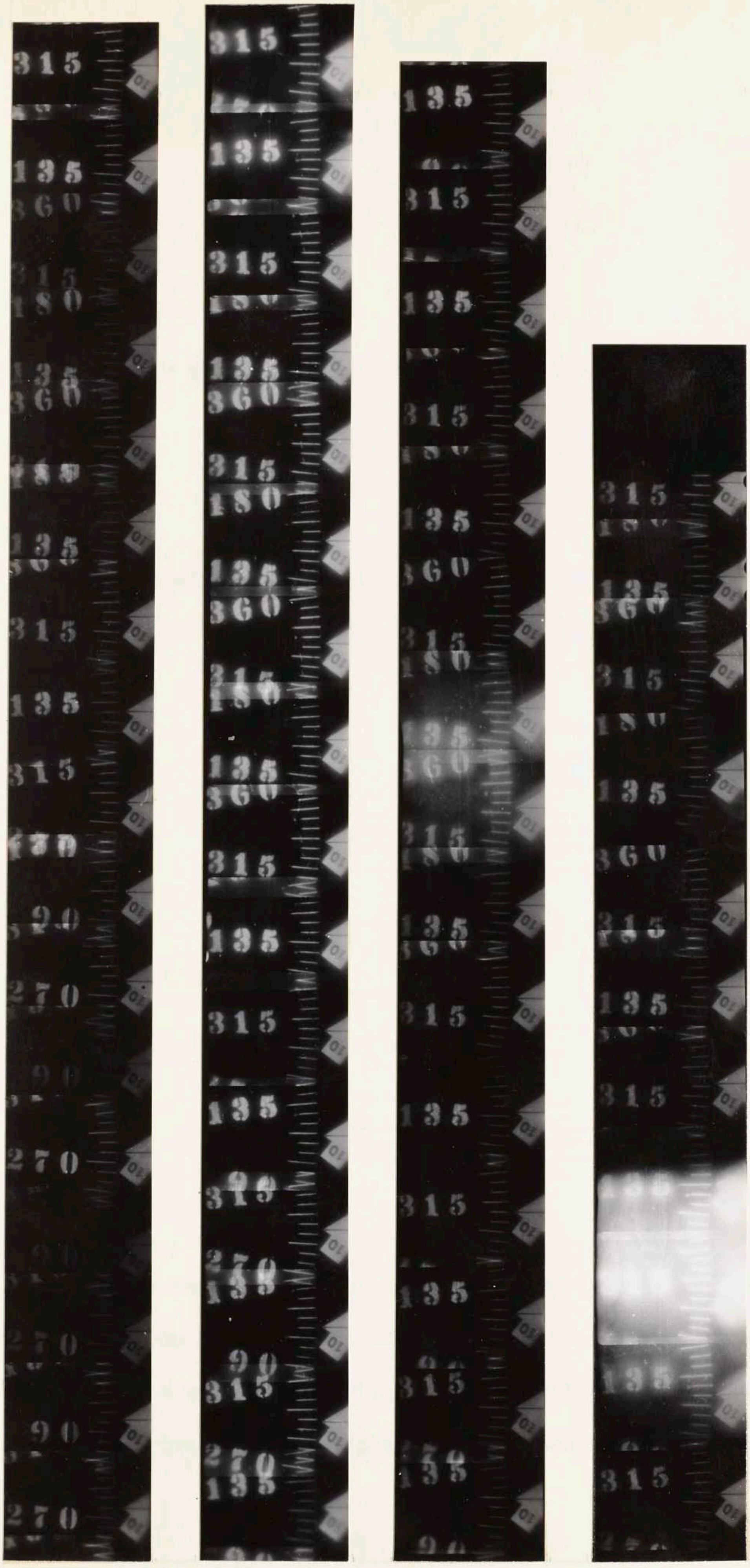
Run #6



Run #7



Run #8



Run #9

PART II

DESCRIPTION OF EXPERIMENTAL PROCEDURE

The experimental part of the present investigation is conducted in accordance with the following diagram and explanation:

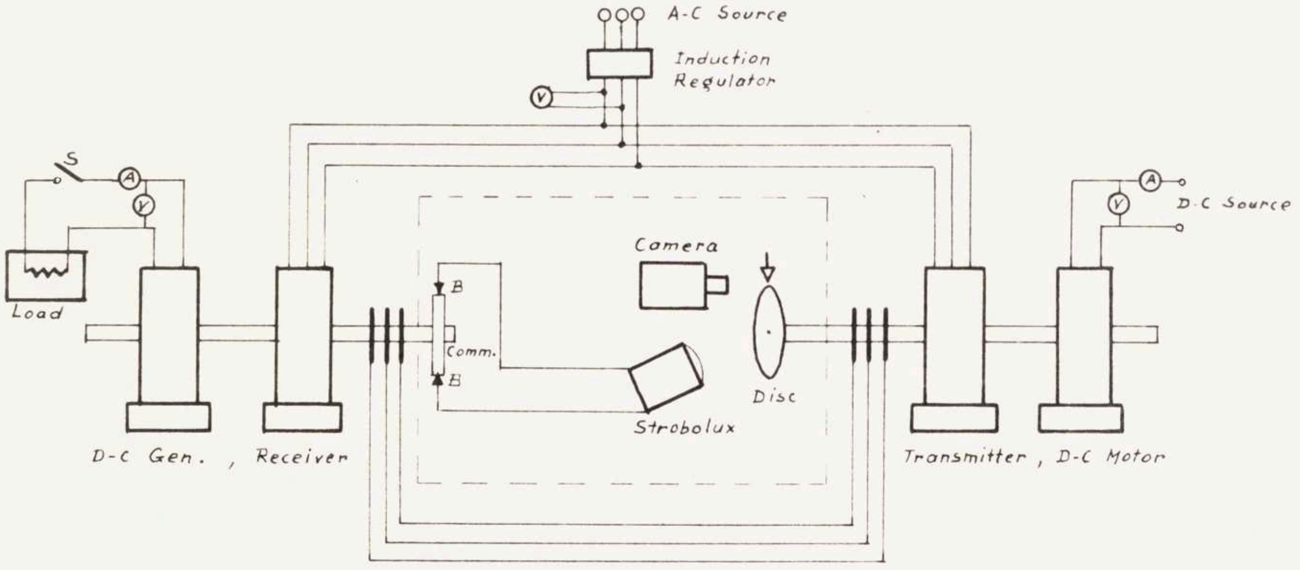


Fig. 6

The complete setup consists of two identical three-phase slip-ring induction motors, two direct-current machines, and equipment (inclosed within dotted rectangle in Fig. 6) serving to record the transient variation of the relative angular displacement between the rotors of the induction machines. The induction motors are electrically interconnected to operate as a selsyn set, and the stators are fed from a balanced three-phase 60 ~ supply through a three-phase induction regulator, the latter providing a convenient means of voltage variation.

One of the d-c machines is operated as a shunt-motor and is mechanically coupled to the transmitter. The purpose of this machine is to drive the transmitter and to supply the constant mechanical torque T_m (mentioned in the analysis) during the transient period. The speed is controlled by the field rheostat of the shunt-motor; and the power input to the latter is measured by means of a voltmeter and an ammeter.

The second d-c machine is mechanically coupled to the receiver and is operated as a separately excited generator, supplying power to a resistive load. The power output of the generator is measured by means of a voltmeter and an ammeter. The purpose of this d-c machine is to simulate a constant mechanical torque applied to the receiver shaft during the transient period. With switch s (see Fig. 6) open, the resistive load is disconnected from the generator, and the selsyn set runs under steady-state no-load condition. The closing of switch s starts the motional transient which is to be investigated.

The principle underlying the operation of the recording equipment is quite simple. Refer to Fig. 6. On the receiver shaft is mounted a special commutator half of whose peripheral surface is conducting and the other half nonconducting. Two stationary brushes which are electrically connected to a "Strobolux" make sliding contact with the commutator. When both brushes come into contact with the conducting part of the commutator surface, the "Strobolux" produces an almost instantaneous flash, which illuminates a disc attached to the shaft of the transmitter. This disc is marked with uniformly spaced divisions, five electrical degrees apart. A stationary pointer placed near its circumference serves as a reference mark. When the selsyn set is in steady-state operation the transmitter and the receiver rotors run at the same speed; and, under the flashes produced by the "Strobolux," the disc appears to be stationary. This is due to the fact that, at steady-state, there is no relative motion between the transmitter and receiver rotors. If, however, the steady operation is disturbed by a sudden application of load, a relative motion occurs between the rotors; and, under the flashes produced by the "Strobolux," the disc appears to be in motion. The apparent motion of the disc is an exact reproduction of the relative motion between the rotors. A moving picture camera placed in front of the disc registers this apparent motion, thus providing an experimental record of the transient variation of the angle δ between the rotors. The camera used for this purpose is of the continuous-motion type, and not the conventional intermittent motion device.

The experimental record produced by the motion-picture camera consists of a series of photographs, each corresponding to one flash of the "Strobolux," and consequently to one revolution of the receiver rotor. Knowing the r.p.m. of the receiver, it is easy to determine the instant corresponding to each picture. Therefore, by reading the angular displacement indicated by the pointer in each picture, and plotting it against time, one obtains an experimental curve of the motional transient of the selsyn set.

* * * * *

P A R T III

T H E O R E T I C A L   A N A L Y S I S

1. Derivation of the Fundamental Equations of an Induction Motor.

(a) Assumptions. In order to obtain reasonably simple differential equations the following assumptions are indispensable:

(1) The winding resistances are constant.

(2) The winding self-inductances are constant.

(3) The mutual inductance between any two windings is proportional to the cosine of their relative electrical angular displacement.

(4) The windings are symmetrical.

The first assumption implies that the effect of frequency upon resistance is negligible. The second assumption neglects the effect of magnetic saturation. The third disregards the presence of possible space harmonics of the air gap flux distribution.

The foregoing are essential assumptions. Some additional ones will be stated below, because they conform with the particular machines used in the experimental part of this thesis. Thus the unnecessary complications due to excessive generality will be avoided.

(5) The machine is a Y-connected 3-phase induction motor.

(6) There is no neutral connection.

(b) The use of symmetrical components. In the study of transient or unbalanced steady-state performance of rotating electrical machinery, the method of symmetrical components proves to be a powerful analytical device. Its use in the analysis of transient phenomena not only reduces the number of differential equations to be considered, but also results in simpler equations.

Let A, B, and C be any three real or complex quantities, either constant or variable. Then writing

$$\begin{aligned} A &= X_0 + X_1 + X_2 \\ B &= X_0 + e^{-j120^\circ}X_1 + e^{j120^\circ}X_2 \qquad (1) \\ C &= X_0 + e^{j120^\circ}X_1 + e^{-j120^\circ}X_2 \end{aligned}$$

and solving the system of linear equations thus obtained yields:

$$\begin{aligned} X_0 &= \frac{1}{3}(A + B + C) \\ X_1 &= \frac{1}{3}(A + e^{j120^\circ}B + e^{-j120^\circ}C) \qquad (2) \\ X_2 &= \frac{1}{3}(A + e^{-j120^\circ}B + e^{j120^\circ}C) \end{aligned}$$

The quantities X_0, X_1, and X_2 are called the symmetrical components of A, B, C.

If $A + B + C = 0$, then $X_0 = 0$ and system (1) becomes:

$$A = X_1 + X_2$$

$$B = e^{-j120^\circ}X_1 + e^{j120^\circ}X_2 \qquad (3)$$

$$C = e^{j120^\circ}X_1 + e^{-j120^\circ}X_2$$

It should be observed that if A, B, C are real, then X_1 and X_2 become conjugate quantities. This fact can readily be seen from Eqs. (2).

(c) Derivation of the fundamental equations. Let Fig.1 represent, at a given instant t, the stator and the rotor of a three-phase induction machine having a wound rotor whose relative position with respect to the stator is a certain function of time $\theta = f(t)$. θ is measured in electrical radians. The machine is in conformity with the assumptions stated on page 27.

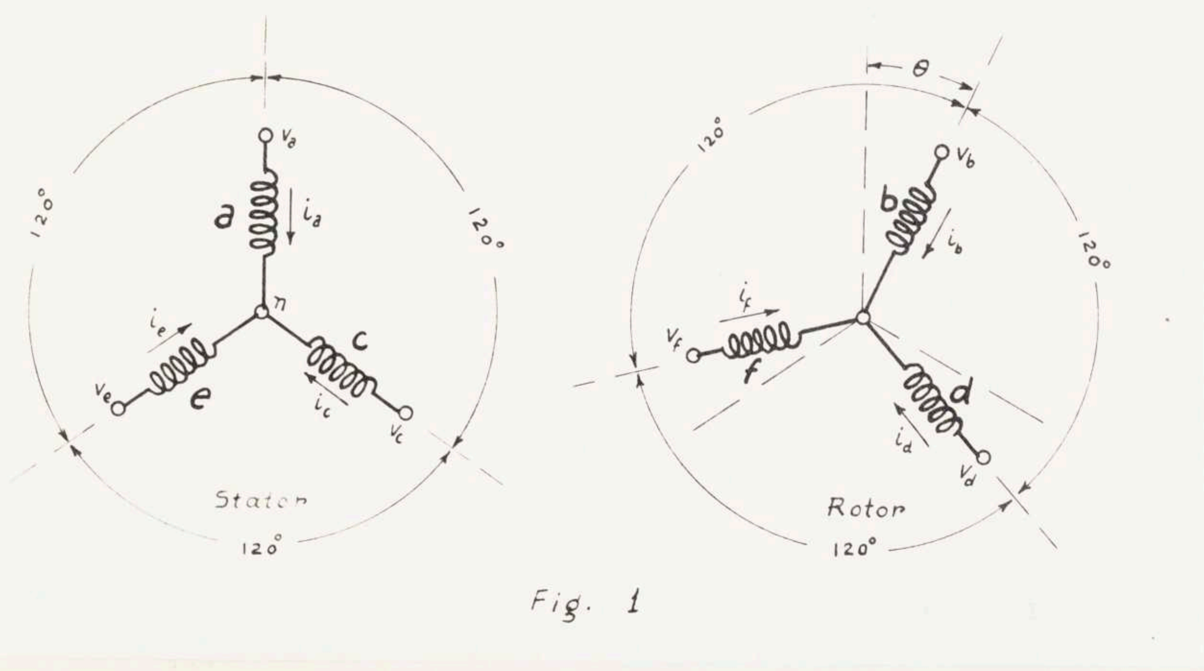


Fig. 1

The letters a, c, e designate the stator windings or phases. The rotor windings are denoted by b, d, f. The stator windings, as well as the rotor windings, are 120 electrical degrees apart. The instantaneous current in each phase is indicated by the letter i, with a subscript showing the particular winding in which it is flowing. The instantaneous terminal-to-neutral potentials are indicated by v_a, v_b, v_c, etc. Let,

r_a = effective resistance of each stator phase,

r_b = effective resistance of each rotor phase,

L_{aa} = self-inductance of each stator phase,

L_{bb} = self-inductance of each rotor phase,

M_a = mutual inductance between any two stator phases,

M_b = mutual inductance between any two rotor phases,

$L_a = L_{aa} - M_a$ = synchronous self-inductance of each stator phase,

$L_b = L_{bb} - M_b$ = synchronous self-inductance of each rotor phase,

M = mutual inductance between a stator and a rotor winding when their axes coincide.

At the instant t the instantaneous mutual inductance between a stator and a rotor winding will be designated by the letter M, with two subscripts referring to the windings in question. Thus M_{ab} will indicate the mutual inductance, at the instant t, between windings a and b. Referring to Fig. 1:

$$M_{ab} = M_{cd} = M_{ef} = M\cos\theta = \frac{M}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$M_{ad} = M_{cf} = M_{eb} = M\cos(\theta + 120^\circ) = \frac{M}{2}\left[e^{j(\theta + 120^\circ)} + e^{-j(\theta + 120^\circ)}\right] \qquad (4)$$

$$M_{af} = M_{cb} = M_{ed} = M\cos(\theta - 120^\circ) = \frac{M}{2}\left[e^{j(\theta - 120^\circ)} + e^{-j(\theta - 120^\circ)}\right]$$

The potential drop across winding a may be written as follows:

$$v_a = r_a i_a + L_{aa} p i_a + M_a p(i_c + i_e) + p(M_{ab} i_b + M_{ad} i_d + M_{af} i_f)$$

where p denotes $\frac{d}{dt}$. Since $i_c + i_e = -i_a$, the first three terms of this equation become:

$$r_a i_a + L_{aa} p i_a - M_a p i_a = (r_a + L_a p) i_a$$

Thus, the potential drops across the stator windings are:

$$v_a = (r_a + L_a p) i_a + p(M_{ab} i_b + M_{ad} i_d + M_{af} i_f) \qquad (5)$$

$$v_c = (r_a + L_a p) i_c + p(M_{af} i_b + M_{ab} i_d + M_{ad} i_f) \qquad (6)$$

$$v_e = (r_a + L_a p) i_e + p(M_{ad} i_b + M_{af} i_d + M_{ab} i_f) \qquad (7)$$

At this stage the introduction of symmetrical components results in considerable simplification. Let,

i_{ao}, i_{a1}, i_{a2} = symmetrical components of i_a, i_c, i_e

i_{bo}, i_{b1}, i_{b2} = symmetrical components of i_b, i_d, i_f

v_{ao}, v_{a1}, v_{a2} = symmetrical components of v_a, v_c, v_e

Then according to Eqs. (2), page 28,

$$i_{ao} = \frac{1}{3}(i_a + i_c + i_e) = 0$$

$$i_{a1} = \frac{1}{3}(i_a + e^{j120^\circ} i_c + e^{-j120^\circ} i_e)$$

$$i_{a2} = \frac{1}{3}(i_a + e^{-j120^\circ} i_c + e^{j120^\circ} i_e)$$

$$i_{bo} = \frac{1}{3}(i_b + i_d + i_f) = 0$$

$$i_{b1} = \frac{1}{3}(i_b + e^{j120^\circ} i_d + e^{-j120^\circ} i_f)$$

$$i_{b2} = \frac{1}{3}(i_b + e^{-j120^\circ} i_d + e^{j120^\circ} i_f)$$

$$v_{ao} = \frac{1}{3}(v_a + v_c + v_e)$$

$$v_{a1} = \frac{1}{3}(v_a + e^{j120^\circ} v_c + e^{-j120^\circ} v_e)$$

$$v_{a2} = \frac{1}{3}(v_a + e^{-j120^\circ} v_c + e^{j120^\circ} v_e)$$

Now adding Eqs. (5), (6), and (7) together, and dividing the resulting equation by three gives:

$$v_{ao} = \frac{1}{3}(r_a + L_a p)(i_a + i_c + i_e) + \frac{1}{3}p(M_{ab} + M_{ad} + M_{af})(i_b + i_d + i_f) = 0$$

because $i_a + i_c + i_e = 0$, and $i_b + i_d + i_f = 0$.

To obtain v_{a1} multiply Eq.(6) by e^{j120° and Eq.(7) by e^{-j120° and add the resulting equations to Eq.(5); then divide by three. (For convenience in writing the equations, e^{j120° and e^{-j120° will henceforth be replaced by a and $\bar{a}$ respectively.)

$$v_{a1} = \tfrac{1}{3}(r_a + L_a p)(i_a + a i_c + \bar{a} i_e) + \tfrac{1}{3}p\Big[M_{ab}(i_b + a i_d + \bar{a} i_f)$$

$$+ M_{ad}(\bar{a} i_b + i_d + a i_f) + M_{af}(a i_b + \bar{a} i_d + i_f)\Big]$$

$$= \tfrac{1}{3}(r_a + L_a p)(i_a + a i_c + \bar{a} i_e) + \tfrac{1}{3}p(M_{ab} + \bar{a}M_{ad} + aM_{af})(i_b + a i_d + \bar{a} i_f)$$

$$= (r_a + L_a p) i_{a1} + \tfrac{3}{2}Mp(e^{j\theta} i_{b1}) \qquad (8)$$

since

$$i_a + a i_c + \bar{a} i_e = 3 i_{a1}$$

$$i_b + a i_d + \bar{a} i_f = 3 i_{b1}$$

$$M_{ab} + \bar{a}M_{ad} + aM_{af} = \tfrac{3}{2}Me^{j\theta}$$

The last relation above can readily be obtained from Eqs.(4).

To obtain v_{a2} it is sufficient to remember that $v_{a2} = \bar{v}_{a1}$, $i_{a2} = \bar{i}_{a1}$, $i_{b2} = \bar{i}_{b1}$. Then automatically,

$$v_{a2} = (r_a + L_a p) i_{a2} + \tfrac{3}{2}Mp(e^{-j\theta} i_{b2})$$

This last equation, however, is not independent of Eq.(8), and therefore should be disregarded as trivial.

Writing the rotor potential drop equations which are similar to Eqs.(5), (6), (7), and following the same procedure as before, one obtains:

$$v_{b1} = (r_b + L_b p)i_{b1} + \frac{3}{2}Mp(e^{-j\theta}i_{a1})$$

Thus, under the assumptions stated on page 27 , the fundamental differential equations of the induction motor are:

$$v_{a1} = Z_a i_{a1} + mp(e^{j\theta}i_{b1})$$

$$v_{b1} = Z_b i_{b1} + mp(e^{-j\theta}i_{a1}) \qquad (9)$$

where,

$$Z_a = r_a + L_a p$$

$$Z_b = r_b + L_b p$$

$$m = \frac{3}{2}M$$

(d) Electromagnetic torque developed by the induction motor. Let Fig. 2 (next page) represent a stationary winding w_1 and a moving winding w_2. The relative displacement β between w_1 and w_2 is a certain function of time, and is measured in electrical radians. At the instant t the windings carry instantaneous currents i_1 and i_2, and the applied voltages are e_1 and e_2.

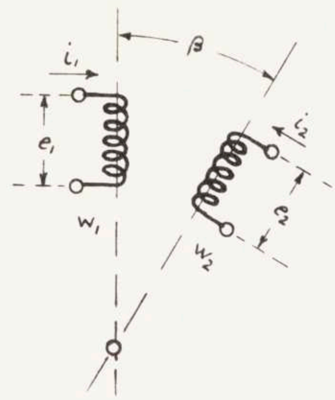


Fig. 2

Let,

r_1, L_1 = resistance and self-inductance of w_1,

r_2, L_2 = resistance and self-inductance of w_2,

M_{12} = mutual inductance between w_1 and w_2 at time t,

T_{12} = electromagnetic torque developed between w_1 and w_2 at time t,

P = number of poles of the windings.

At the instant t the energy stored in the windings is

$$W = \tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 + M_{12} i_1 i_2$$

Also,

$$e_1 = r_1 i_1 + L_1 p i_1 + p(M_{12} i_2)$$

$$e_2 = r_2 i_2 + L_2 p i_2 + p(M_{12} i_1)$$

Suppose, during an infinitesimal interval of time dt, w_2 advances through an angle $d\beta$. During this short interval of time,

energy input to the system $= (e_1 i_1 + e_2 i_2)dt$, energy released in the form of heat = $(r_1 i_1^2 + r_2 i_2^2)dt$,

energy released in the form of work = $\frac{2}{P} T_{12} d\beta$,

change in stored energy = $dW = L_1 i_1 di_1 + L_2 i_2 di_2 + d(M_{12} i_1 i_2)$

According to the principle of conservation of energy,

$$(e_1 i_1 + e_2 i_2)dt = (r_1 i_1^2 + r_2 i_2^2)dt + dW + \frac{2}{P} T_{12} d\beta$$

Substituting the values of e_1, e_2 and dW in this equation and simplifying yields:

$$T_{12} = \frac{P}{2} i_1 i_2 \frac{dM_{12}}{d\beta} \tag{10}$$

Eq.(10) serves as a startimg point for the derivation of the torque developed by the induction motor.

Consider now the induction motor of Fig. 1, page 29. The electromagnetic torque developed between each pair of stator and rotor windings is, according to Eq.(10),

$$T_{ab} = \frac{P}{2} i_a i_b \frac{d}{d\theta} M_{ab}$$

$$T_{cd} = \frac{P}{2} i_c i_d \frac{d}{d\theta} M_{ab}$$

$$T_{ef} = \frac{P}{2} i_e i_f \frac{d}{d\theta} M_{ab}$$

$$T_{ad} = \frac{P}{2} i_a i_d \frac{d}{d\theta} M_{ad}$$

$$T_{cf} = \frac{P}{2} i_c i_f \frac{d}{d\theta} M_{ad}$$

$$T_{eb} = \frac{P}{2} i_e i_b \frac{d}{d\theta} M_{ad}$$

$$T_{af} = \frac{P}{2} i_a i_f \frac{d}{d\theta} M_{af}$$

$$T_{cb} = \frac{P}{2} i_c i_b \frac{d}{d\theta} M_{af}$$

$$T_{ed} = \frac{P}{2} i_e i_d \frac{d}{d\theta} M_{af}$$

The sum of these nine component torques yields the total torque developed by the machine. The use of symmetrical components is again advantageous. If the first three equations are added together and the currents involved are expressed in terms of their symmetrical components according to Eq.(3), page 29, one obtains, upon simplification,

$$T_{ab} + T_{cd} + T_{ef} = \frac{3P}{2} (i_{a2} i_{b1} + i_{a1} i_{b2}) \frac{d}{d\theta} M_{ab} \tag{11}$$

Similar treatment of the next three equations, and then the last three, gives,

$$T_{ad} + T_{cf} + T_{eb} = \frac{3P}{2} (\bar{a} i_{a2} i_{b1} + a i_{a1} i_{b2}) \frac{d}{d\theta} M_{ad} \tag{12}$$

$$T_{af} + T_{cb} + T_{ed} = \frac{3P}{2} (a i_{a2} i_{b1} + \bar{a} i_{a1} i_{b2}) \frac{d}{d\theta} M_{af} \tag{13}$$

Now adding Eqs.(11), (12), (13) together, and denoting the total electromagnetic torque by T_e results in

$$\begin{aligned} T_e &= \frac{3P}{2} i_{a2} i_{b1} \frac{d}{d\theta} (M_{ab} + \bar{a} M_{ad} + a M_{af}) \\ &+ \frac{3P}{2} i_{a1} i_{b2} \frac{d}{d\theta} (M_{ab} + a M_{ad} + \bar{a} M_{af}) \end{aligned} \tag{14}$$

It is easy to see from Eqs.(4), page 31, that:

$$M_{ab} + \bar{a}M_{ad} + aM_{af} = \frac{3}{2} Me^{j\theta},$$

$$M_{ab} + aM_{ad} + \bar{a}M_{af} = \frac{3}{2} Me^{-j\theta}.$$

Taking the derivative of these last expressions with respect to θ, and substituting in Eq.(14) yields

$$T_e = j \frac{9}{4} PM(i_{a2}i_{b1}e^{j\theta} - i_{a1}i_{b2}e^{-j\theta}) \tag{15}$$

This equation can be put into a more convenient form by observing that the two terms in parentheses are conjugate complex quantities. Since the difference of two conjugate complex quantities equals j times twice the imaginary part of the first quantity, (15) may be written:

$$T_e = - \frac{9}{2} PM \mathfrak{I}\left[i_{a2}i_{b1}e^{j\theta}\right]$$

$$= - 3 Pm \mathfrak{I}\left[\bar{i}_{a1}i_{b1}e^{j\theta}\right] \tag{16}$$

where $m = \frac{3}{2} M$, and $\bar{i}_{a1} = i_{a2}$.

2. Fundamental Equations of a Selsyn Set.

(a) General considerations. The fundamental equations of an induction motor as derived in the preceding article are perfectly general under the assumptions stated on page 27. The steady-state or transient performance of the induction machine - when it is electrically, mechanically, or electromechanically coupled to an outside system - can be investigated by means of these equations provided the electrical and mechanical quantities occurring therein are made to conform with the coupling conditions.

When two identical induction motors are combined to form a selsyn set, the simultaneous consideration of their individual equations, modified to satisfy the electrical connections, leads to the fundamental equations of selsyns.

Consider two identical induction motors and suppose that they are interconnected as shown in Fig. 3.

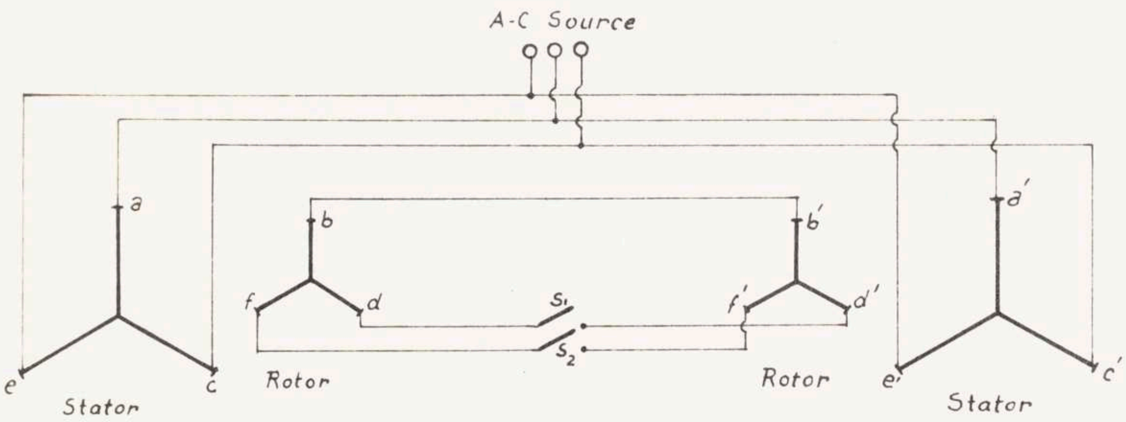


Fig. 3

When the stators are connected to a common a-c source, rotating magnetic fields which are identical are set up in both machines. However, switches s_1 and s_2 being open, no rotor currents exist, and consequently the machines stay stationary. In general there is an alternating potential difference across the open switches s_1 and s_2, due to the difference between the relative position of each rotor with respect to its stator. If, however, one of the rotors is moved to the same relative position as the other, the potential difference across s_1 and s_2 disappears. This being done, suppose that the switches are closed. The machines still remain stationary; but if one of the rotors is now moved to a new position, currents start flowing in the rotor circuits. Consequently the second rotor is acted upon by an electromagnetic torque tending to move it to a new position so that the rotor currents will annul. That is to say, under the influence of this torque the second rotor assumes the same relative position as that of the first rotor. It follows from these considerations that if one of the rotors is driven continuously, the other duplicates the motion. This is the principle underlying the operation of a selsyn set, and Fig. 3 - with switches s_1 and s_2 closed - is the circuit diagram of the set.

The induction machine which is mechanically driven is termed the <u>transmitter</u>, and the other is called the <u>receiver</u>. Ordinarily the transmitter is driven at a constant speed, and the receiver is subjected to a constant mechanical load. Under these circumstances the receiver rotor revolves with the same speed as that of the transmitter; but it lags the latter in the direction of motion by a constant angle δ_o just enough to develop a constant electromagnetic torque which balances the applied mechanical load plus the friction and windage.

The transmitter may be driven either in the direction of the rotating magnetic field or in the opposite direction. In either case, the receiver rotor lags that of the transmitter in the direction of motion.

(b) Fundamental equations. Consider two identical induction machines I and I' which are in conformity with the six assumptions mentioned on page 27. Let:

v_a, v_c, v_e = stator terminal-to-neutral voltages of I,

i_a, i_c, i_e = stator currents of I,

v_b, v_d, v_f = rotor terminal-to-neutral voltages of I,

i_b, i_d, i_f = rotor currents of I,

θ = angular displacement between the rotor and the stator of I,

v'_a, v'_c, v'_e = stator terminal-to-neutral voltages of I',

i'_a, i'_c, i'_e = stator currents of I',

v'_b, v'_d, v'_f = rotor terminal-to-neutral voltages of I',

i'_b, i'_d, i'_f = rotor currents of I',

θ' = angular displacement between the rotor and the stator of I'.

If the two induction machines are interconnected according to Fig. 3, so as to form a selsyn set, the following conditions must be satisfied:

$$v'_a = v_a , \qquad v'_c = v_c , \qquad v'_e = v_e ,$$

$$v'_b = v_b , \qquad v'_d = v_d , \qquad v'_f = v_f ,$$

$$i'_b = -i_b , \qquad i'_d = -i_d , \qquad i'_f = -i_f .$$

In terms of symmetrical components, these relations become:

$$v'_{ao} = v_{ao}\,, \qquad v'_{a1} = v_{a1}\,, \qquad v'_{a2} = v_{a2}\,,$$

$$v'_{bo} = v_{bo}\,, \qquad v'_{b1} = v_{b1}\,, \qquad v'_{b2} = v_{b2}\,,$$

$$i'_{bo} = -i_{bo}\,, \qquad i'_{b1} = -i_{b1}\,, \qquad i'_{b2} = -i_{b2}\,.$$

Writing the fundamental equations of each machine as shown on page 34 , and replacing v'_{a1}, v'_{b1}, i'_{b1} by v_{a1}, v_{b1}, $-i_{b1}$ respectively, yields

$$v_{a1} = Z_a i_{a1} + mp(e^{j\theta} i_{b1}) \qquad (17)$$

Machine I

$$v_{b1} = Z_b i_{b1} + mp(e^{-j\theta} i_{a1}) \qquad (18)$$

$$v_{a1} = Z_a i'_{a1} - mp(e^{j\theta'} i_{b1}) \qquad (19)$$

Machine I'

$$v_{b1} = -Z_b i_{b1} + mp(e^{-j\theta'} i'_{a1}) \qquad (20)$$

Now equate the right-hand sides of Eqs.(18) and (20), and write the resulting equation together with Eqs.(17) and (19) :

$$\begin{aligned} v_{a1} &= Z_a i_{a1} + mp(e^{j\theta} i_{b1}) \\ 0 &= 2Z_b i_{b1} + mp(e^{-j\theta} i_{a1}) - mp(e^{-j\theta'} i'_{a1}) \qquad (21) \\ v_{a1} &= Z_a i'_{a1} - mp(e^{j\theta'} i_{b1}) \end{aligned}$$

Eqs.(21) are the fundamental equations of the selsyn set.

Let T_e and T'_e denote the electromagnetic torque developed by the machines I and I' respectively. Then according to Eq.(16),

$$T_e = -3\,Pm\,\Im\left[\bar{i}_{a1} i_{b1} e^{j\theta}\right] \tag{22}$$

$$T_e' = 3\,Pm\,\Im\left[\bar{i}'_{a1} i_{b1} e^{j\theta'}\right] \tag{23}$$

Throughout the rest of this analysis the machine I will be the transmitter and I' the receiver.

(c) <u>Application to balanced steady-state operation</u>. At steady-state both the transmitter and receiver rotors revolve at a constant angular speed of n electrical radians per second. The receiver rotor is acted upon by a constant mechanical torque and lags the transmitter rotor, in the direction of motion, by a constant angle of δ_o electrical radians. It should be stated here that n and δ_o are algebraic quantities. When the rotors revolve in the direction of the rotating magnetic field n and δ_o are both positive, and when the rotors revolve against the field n and δ_o are both negative. Then, under steady-state conditions,

$$\theta = nt \tag{24}$$

$$\theta' = nt - \delta_o \tag{25}$$

regardless of the direction of rotation. Substituting Eqs.(24) and (25) in Eqs.(21), (22), and (23) yields:

$$\begin{aligned} v_a &= Z_a i_a + mp(e^{jnt} i_b) \\ 0 &= 2Z_b i_b + mp(e^{-jnt} i_a) - mp(e^{-jnt} e^{j\delta_o} i_a') \\ v_a &= Z_a i_a' - mp(e^{-j\delta_o} e^{jnt} i_b) \end{aligned} \tag{26}$$

$$T_e = -\,3\ \mathrm{Pm}\,\mathcal{J}\left[\bar{i}_a e^{jnt} i_b\right] \tag{27}$$

$$T_e' = \ \ 3\ \mathrm{Pm}\,\mathcal{J}\left[e^{-j\delta_0}\,\bar{i}_a' e^{jnt} i_b\right] \tag{28}$$

The indices showing the sequence of the currents and voltages have been dropped here for simplicity. Henceforth currents and voltages without sequence indices will represent positive sequence components. The negative sequence components will be indicated by the conjugates of the corresponding positive sequence components without sequence indices.

If Heaviside's "shifting" principle is used, Eqs.(26) may be reduced to stationary coupled-circuit equations. To this end, multiply the second equation of (26) by e^{jnt}, and using Heaviside's "shifting" principle, shift this factor to the right of the operator in each term. (See Appendix C.). Also multiply both sides of the third equation of (26) by $e^{j\delta_0}$. Upon these manipulations (26) becomes:

$$\begin{aligned} v_a &= Z_a i_a + mp(e^{jnt} i_b) \\ 0 &= 2Z_b'(e^{jnt} i_b) + m(p - jn) i_a - m(p - jn)(e^{j\delta_0} i_a') \\ e^{j\delta_0} v_a &= Z_a(e^{j\delta_0} i_a') - mp(e^{jnt} i_b) \end{aligned} \tag{29}$$

where

$$Z_b' = r_b + L_b(p - jn)$$

The voltages actually impressed upon the stator of each machine are balanced three-phase quantities. If the root-mean-square value and the angular velocity of the line-to-neutral voltage are V and ω respectively, then the positive sequence symmetrical component is

$$v_a = \frac{\sqrt{2}\,V}{3}\left[\cos\omega t + a\cos(\omega t - 120^\circ) + \bar{a}\cos(\omega t + 120^\circ)\right] = \frac{V}{\sqrt{2}}\,e^{j\omega t}$$

This is a vector of constant magnitude, rotating positively with the angular velocity ω electrical radians per second. Therefore, under steady-state conditions, the unknown currents i_a, $e^{jnt}i_b$, and $e^{j\delta_0}\,i_a'$ are likewise vectors of constant magnitude and angular velocity ω . Consequently the differential operator p in Eqs.(29) may be replaced by $j\omega$. Making this substitution and then multiplying the second equation of (29) by $\frac{\omega}{\omega - n}$ yields,

$$\begin{aligned} v_a &= z_a' I_a + j\omega m I_b \\ 0 &= 2z_b' I_b + j\omega m I_a - j\omega m I_a' \\ e^{j\delta_0} v_a &= z_a' I_a' - j\omega m I_b \end{aligned} \qquad (30)$$

where

I_a = steady-state value of i_a ,
I_b = steady-state value of $e^{jnt}i_b$,
I_a' = steady-state value of $e^{j\delta_0}\,i_a'$,
$z_a' = r_a + j\omega L_a$,
$z_b' = \frac{r_b}{s} + j\omega L_b$,
$s = \frac{\omega - n}{\omega}$ = slip .

Eqs.(30) may be written in the following slightly different form:

$$
\begin{aligned}
v_a &= z_a I_a + jx_m(I_a + I_b) \\
0 &= 2z_b I_b + jx_m(I_a + I_b) + jx_m(I_b - I_a') \qquad (31) \\
e^{j\delta_0} v_a &= z_a I_a' + jx_m(I_a' - I_b)
\end{aligned}
$$

where

$$
\begin{aligned}
z_a &= r_a + j\omega(L_a - m) = r_a + jx_a \,, \\
z_b &= \frac{r_b}{s} + j\omega(L_b - m) = \frac{r_b}{s} + jx_b \,, \\
x_m &= j\omega m \quad .
\end{aligned}
$$

If the transformation ratio of each induction motor is 1:1, (which is the case with the machines used in the experimental part of this project) then x_a and x_b are identified as the leakage reactances of the stator and the rotor respectively. Also, x_m is recognized as the magnetizing reactance.

Eqs.(31) immediately suggest the stationary network of Fig. 4, which is the steady-state equivalent circuit of the selsyn set.

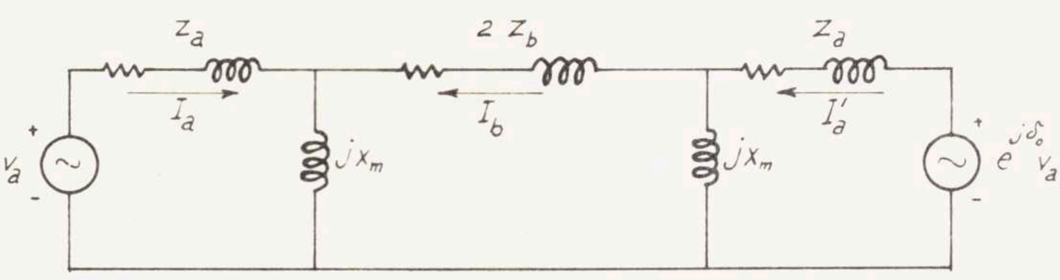


Fig. 4

Currents I_a, I_b, and I_a' may be obtained by either solving Eqs.(31) or by applying "Thévenin's Theorem" to the network of Fig. 4. The solution is:

$$I_a = \frac{1 - B(e^{j\delta_o} - 1)}{A} \frac{v_a}{jx_m} \tag{32}$$

$$I_b = B(e^{j\delta_o} - 1) \frac{v_a}{jx_m} \tag{33}$$

$$I_a' = \frac{1 - B(e^{-j\delta_o} - 1)}{A} \frac{e^{j\delta_o} v_a}{jx_m} \tag{34}$$

where

$$A = \frac{z_a}{jx_m} + 1 \tag{35}$$

$$B = \frac{jx_m}{2(z_a + Az_b)} \tag{36}$$

The steady-state torque developed by each induction machine is obtained by substituting Eqs.(32), (33), and (34) in Eqs.(27) and (28). Thus, upon simplifying,

$$\begin{aligned} T_e &= -3\,\mathrm{Pm}\,\mathfrak{I}\left[\bar{I}_a I_b\right] \\ &= \frac{3\,P|v_a|^2}{2\omega A|z_a + Az_b|^2}\left[A\,\frac{r_b}{s}\,(1 - \cos\delta_o) - (x_a + Ax_b)\sin\delta_o\right] \end{aligned} \tag{37}$$

$$\begin{aligned} T_e' &= 3\,\mathrm{Pm}\,\mathfrak{I}\left[\bar{I}_a' I_b\right] \\ &= \frac{3\,P|v_a|^2}{2\omega A|z_a + Az_b|^2}\left[A\,\frac{r_b}{s}\,(1 - \cos\delta_o) + (x_a + Ax_b)\sin\delta_o\right] \end{aligned} \tag{38}$$

The letter A occurring in (37) and (38) represents the magnitude of the vector defined by Eq.(35). The imaginary part of the vector A is very small compared to its real part, and therefore A may generally be replaced by its magnitude.

3. Analysis of Motional Transients.

(a) General considerations. The problem of motional transients in selsyns is essentially an electromechanical proposition. The occurrence of a sudden change in the existing load conditions affects both the electrical and the mechanical state of the system, thus resulting in electrical as well as motional transients. A mathematically rigorous analysis of these electromechanical transients would require the simultaneous solution of Eqs.(21) in connection with

$$
\begin{aligned}
Jp^2\theta &= T_e + T_m \\
Jp^2\theta' &= T_e' - T_m'
\end{aligned}
$$

where T_e, T_e' are defined by Eqs.(22) and (23), and T_m, T_m' are mechanical torques acting on the transmitter and receiver shafts respectively. A close examination of the system of simultaneous differential equations just mentioned will clearly show that a rigorous method of attack is out of question. The fact that both the currents and the angular rotor displacements are unknown functions of time, renders the mathematical solution of the system practically impossible. Therefore , unless the use is made of a differential analyzer, some simplifying assumption must be made so that at least an approximate mathematical treatment of the problem will be possible.

Consider a selsyn set operating under balanced steady-state condition at no-load, and suppose that a constant mechanical torque is suddenly applied on the receiver shaft. The problem is to determine the ensuing transient variation of the relative angular displacement δ between the transmitter and receiver rotors.

(b) First approximate solution. The fundamental assumption underlying the present analysis may be stated as follows:

(1) Let δ be the instantaneous angular displacement of the rotors at time t. Then the instantaneous electromagnetic torque of each machine at this instant is assumed to be the same as if the selsyn set were operating at steady-state with a constant angular displacement $\delta_o = \delta$.

The assumption implies that the effect of $\frac{d\delta}{dt}$ upon electromagnetic torque is ignored. Therefore the solution obtained through the use of this assumption is void of damping.

The following additional assumptions will be made here:

(2) The angular rotor displacement is small enough to allow the substitution of δ for $\sin\delta$.

(3) The source is an infinite bus.

(4) The mechanical torque acting on each rotor is constant during the motional transient period.

Now let:

T_m = mechanical torque acting on the transmitter rotor,

T_m' = mechanical torque acting on the receiver rotor,

T_e = electromagnetic torque of the transmitter,

T_e' = electromagnetic torque of the receiver,

T_f = friction and windage of the transmitter,

T_f' = friction and windage of the receiver,

J = moment of inertia of the rotating part of each unit.

The following motional differential equations may be written:

$$Jp^2\theta = T_e + T_m - T_f \tag{39}$$

$$Jp^2\theta' = T_e' - T_m' - T_f' \tag{40}$$

where p^2 designates $\frac{d^2}{dt^2}$. Subtracting (40) from (39) and substituting (37) and (38) in the result assumption (1), yields,

$$p^2\delta + k^2\sin\delta = u \tag{41}$$

where

$$k^2 = \frac{3\,P|V_a|^2(x_a + Ax_b)}{\omega JA|z_a + Az_b|^2} \tag{42}$$

$$u = \frac{1}{J}(T_m + T_m' + T_f' - T_f) \tag{43}$$

If $\sin\delta$ is replaced by δ according to assumption (2), Eq.(41) becomes a linear differential equation whose solution is

$$\delta = \delta_0 - \delta_1 \cos kt \tag{44}$$

where

$$\delta_0 = \frac{u}{k^2}$$

$$\delta_1 = \frac{u}{k^2} - \delta_i$$

δ_i = initial angular displacement between the rotors.

Eq.(44) shows that, if damping is neglected, the transient variation of δ , ensuing the sudden application of a constant load, is of the form of sustained sinusoidal oscillations. In the following part the effect of these sustained oscillations upon the electrical behavior of the selsyn set will be investigated in detail. Then it will be possible to determine the damping torque, and consequently, to arrive at a more accurate solution of the problem.

(c) <u>The effect of small sustained oscillations</u>. Consider the selsyn set under sustained small oscillations defined by

$$\delta = \delta_0 - \delta_1 \cos kt$$

and assume, for simplicity, that only the receiver is oscillating. Then,

$$\theta = nt ,$$

$$\theta' = nt - \delta = nt - \delta_0 + \delta_1 \cos kt ;$$

and

$$e^{j\theta} = e^{jnt} ,$$

$$e^{j\theta'} = e^{j(nt - \delta_0)}\, e^{j\delta_1 \cos kt}$$

$$= e^{j(nt - \delta_0)} \left[\cos(\delta_1 \cos kt) + j\sin(\delta_1 \cos kt)\right]$$

$$= e^{j(nt - \delta_0)} \left[1 + j\delta_1 \cos kt\right] \qquad \text{(if } \delta_1 \text{ is small)}$$

$$= e^{j(nt - \delta_0)} \left[1 + j\,\frac{\delta_1}{2}(e^{jkt} + e^{-jkt})\right]$$

Substitute these values of $e^{j\theta}$ and $e^{j\theta'}$ in Eqs.(21):

$$v_a = Z_a i_a + mp(e^{jnt} i_b)$$

$$0 = 2Z_b i_b + mp(e^{-jnt} i_a) - mp\left[e^{-j(nt - \delta_0)}(1 - j\,\frac{\delta_1}{2}\, e^{jkt} - j\,\frac{\delta_1}{2}\, e^{-jkt}) i'_a\right]$$

$$v_a = Z_a i'_a - mp\left[e^{j(nt - \delta_0)}(1 + j\,\frac{\delta_1}{2}\, e^{jkt} + j\,\frac{\delta_1}{2}\, e^{-jkt}) i_b\right] \qquad (45)$$

Let

$$i_a = i_{ao} + \Delta i_a$$
$$i_b = i_{bo} + \Delta i_b \qquad (46)$$
$$i'_a = i'_{ao} + \Delta i'_a$$

where i_{ao}, i_{bo}, i'_{ao} are steady-state currents corresponding to constant speed n and relative rotor displacement δ_0 ; and Δi_a, Δi_b, $\Delta i'_a$ are incremental currents due to the oscillations. The steady-state components of currents satisfy the following equations:

(See the next page.)

$$v_a = Z_a i_{ao} + mp(e^{jnt} i_{bo})$$

$$0 = 2Z_b i_{bo} + mp(e^{-jnt} i_{ao}) - mp\left[e^{-j(nt - \delta_0)} i'_{ao}\right] \qquad (47)$$

$$v_a = Z_a i'_{ao} - mp\left[e^{j(nt - \delta_0)} i_{bo}\right]$$

Substituting (46) into (45) and subtracting from the resulting system the corresponding equations of (47) one obtains,

$$0 = Z_a \Delta i_a + mp(e^{jnt} \Delta i_b)$$

$$0 = 2Z_b \Delta i_b + mp(e^{-jnt} \Delta i_a) - mp(e^{-jnt} e^{j\delta_0} \Delta i'_a)$$

$$+ mp\left[e^{-j(nt - \delta_0)} j \frac{\delta_1}{2}(e^{jkt} + e^{-jkt})(i'_{ao} + \Delta i'_a)\right] \qquad (48)$$

$$0 = Z_a \Delta i'_a - mp\left[e^{j(nt - \delta_0)} \Delta i_b\right]$$

$$- mp\left[e^{j(nt - \delta_0)} j \frac{\delta_1}{2}(e^{jkt} + e^{-jkt})(i_{bo} + \Delta i_b)\right]$$

Now apply the following steps to Eqs.(48):

1. Neglect Δi_b and $\Delta i'_a$ in $(i_{bo} + \Delta i_b)$ and $(i'_{ao} + \Delta i'_a)$.

2. Multiply the second equation by e^{jnt} and shift this factor into the operand of each term by means of Heaviside's "shifting" formula. Then multiply both sides by $\dfrac{p}{p - jn}$.

3. Multiply the third equation by $e^{j\delta_0}$.

The outcome of these manipulations is the following system:

(See the next page.)

$$0 = Z_a \Delta I_a + mp\Delta I_b$$

$$e_{a1} + e_{a2} = 2Z_b'' \Delta I_b + mp\Delta I_a - mp\Delta I_a' \qquad (49)$$

$$e_{b1} + e_{b2} = Z_a \Delta I_a' - mp\Delta I_b$$

where

$$Z_b'' = \frac{p}{p - jn} r_b + pL_b$$

$$\Delta I_a = \Delta i_a$$

$$\Delta I_b = e^{jnt} \Delta i_b$$

$$\Delta I_a' = e^{j\delta_o} \Delta i_a'$$

$$e_{a1} = - mp(j \frac{\delta_1}{2} e^{jkt} I_{ao}')$$

$$e_{a2} = - mp(j \frac{\delta_1}{2} e^{-jkt} I_{ao}')$$

$$e_{b1} = mp(j \frac{\delta_1}{2} e^{jkt} I_{bo})$$

$$e_{b2} = mp(j \frac{\delta_1}{2} e^{-jkt} I_{bo})$$

The quantities e_{a1}, e_{a2}, e_{b1}, and e_{b2} are all known voltages, because the expressions in the brackets are known. [I_{ao}' and I_{bo} are steady-state currents given by Eqs.(34) and (33) respectively.] Observing that the operands of e_{a1} and e_{b1} have an angular velocity $\omega + k$, and those of e_{a2} and e_{b2} an angular velocity $\omega - k$, one obtains:

$$
\begin{aligned}
e_{a1} &= \tfrac{1}{2}\, m(\omega + k)\,\delta_1\, e^{jkt} I'_{ao} \\
e_{a2} &= \tfrac{1}{2}\, m(\omega - k)\,\delta_1\, e^{-jkt} I'_{ao} \\
e_{b1} &= -\tfrac{1}{2}\, m(\omega + k)\,\delta_1\, e^{jkt} I_{bo} \\
e_{b2} &= -\tfrac{1}{2}\, m(\omega - k)\,\delta_1\, e^{-jkt} I_{bo}
\end{aligned}
\qquad (50)
$$

Now consider Eqs.(49). These are the equations of a stationary network containing impressed voltages of two different frequencies. Using the principle of superposition, let

I_{a1}, I_{b1}, I'_{a1} = components of currents due to e_{a1} and e_{b1} ,
I_{a2}, I_{b2}, I'_{a2} = components of currents due to e_{a2} and e_{b2} .

Then, taking e_{a1} and e_{b1} only, one may write

$$
\begin{aligned}
0 &= Z_a \Delta I_{a1} + mp\Delta I_{b1} \\
e_{a1} &= 2Z''_b \Delta I_{b1} + mp\Delta I_{a1} - mp\Delta I'_{a1} \\
e_{b1} &= Z_a \Delta I'_{a1} - mp\Delta I_b
\end{aligned}
\qquad (51)
$$

The angular velocity of applied voltages e_{a1} and e_{b1} is $\omega + k$. Therefore the incremental currents involved in Eqs.(51) must have the same angular velocity. Letting $h = \frac{k}{\omega}$, and replacing p by $j\omega(1 + h)$, Eqs.(51) can be put into the following form:

(See the next page.)

$$0 = z_{a1}\Delta I_{a1} + jx_m(\Delta I_{b1} + \Delta I_{a1})$$

$$E_{a1} = 2z_{b1}\Delta I_{b1} + jx_m(\Delta I_{b1} + \Delta I_{a1}) + jx_m(\Delta I_{b1} - \Delta I'_{a1}) \qquad (52)$$

$$E_{b1} = z_{a1}\Delta I'_{a1} + jx_m(\Delta I'_{a1} - \Delta I_{b1})$$

where

$$z_{a1} = \frac{r_a}{1 + h} + jx_a$$

$$z_{b1} = \frac{r_b}{s + h} + jx_b$$

$$E_{a1} = \frac{e_{a1}}{1 + h} = \frac{1}{2} x_m \delta_1 e^{jkt} I'_{ao}$$

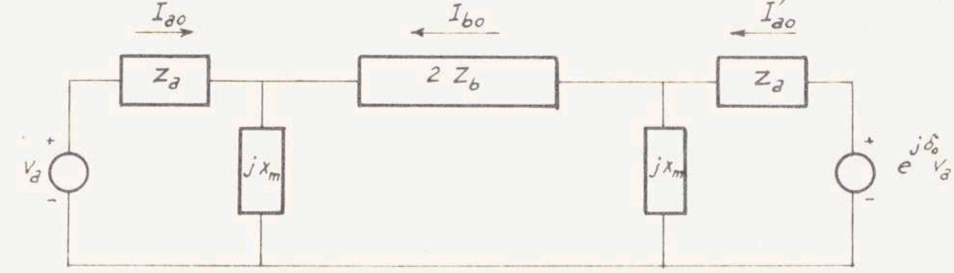
$$E_{b1} = \frac{e_{b1}}{1 + h} = - \frac{1}{2} x_m \delta_1 e^{jkt} I_{bo}$$

Eqs.(52) suggest the equivalent network shown in Fig. 5b. Writing Eqs.(51) in terms of e_{a2}, e_{b2}, ΔI_{a2}, etc.; and following the same procedure as before, one obtains the equivalent network of Fig. 5c. It should be noticed that in this case p is replaced by $j\omega(1 - h)$. Figs. 5b and 5c constitute the hunting networks of the selsyn set. Together with the steady-state equivalent circuit, these networks completely determine the electrical performance during sustained oscillations. To this end the following procedure is to be used:

1. From the equivalent circuit of Fig. 5a determine the steady-state currents I_{ao}, I_{bo}, and I'_{ao}. These are of fundamental frequency f.

2. Substituting I'_{ao} and I_{bo} into the expressions given with Fig. 5b, determine the voltages E_{a1} and E_{b1}. Then, from the network of Fig. 5b,

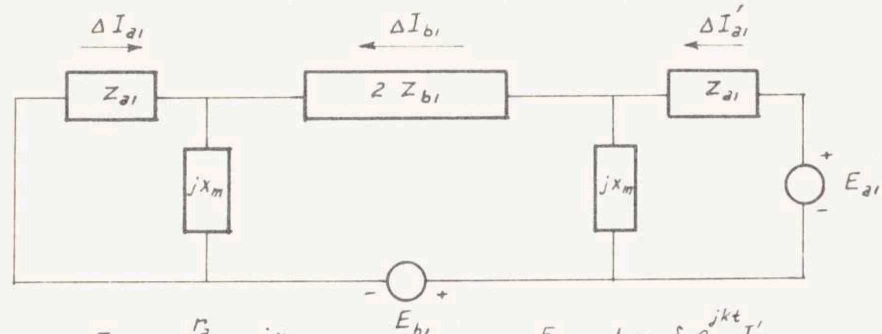
Fig. 5a



$$Z_a = r_a + jx_a$$

$$Z_b = \frac{r_b}{s} + jx_b$$

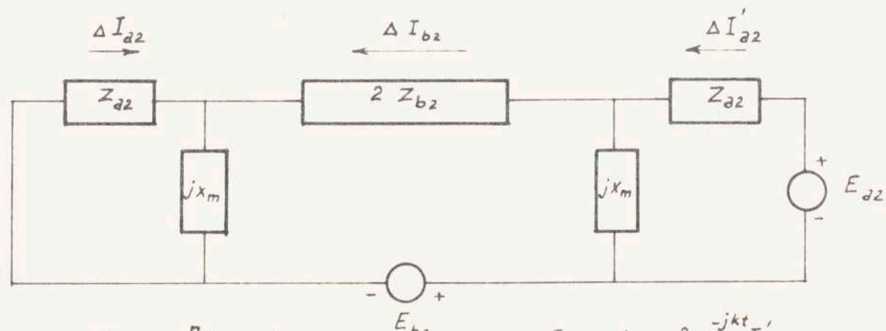
Fig. 5b



$$Z_{a1} = \frac{r_a}{1+h} + jx_a \qquad E_{a1} = \frac{1}{2}x_m\delta_1 e^{jkt} I'_{ao}$$

$$Z_{b1} = \frac{r_b}{s+h} + jx_b \qquad E_{b1} = -\frac{1}{2}x_m\delta_1 e^{jkt} I_{bo}$$

Fig. 5c



$$Z_{a2} = \frac{r_a}{1-h} + jx_a \qquad E_{a2} = \frac{1}{2}x_m\delta_1 e^{-jkt} I'_{ao}$$

$$Z_{b2} = \frac{r_b}{s-h} + jx_b \qquad E_{b2} = -\frac{1}{2}x_m\delta_1 e^{-jkt} I_{bo}$$

determine the hunting currents ΔI_{a1}, ΔI_{b1}, and $\Delta I'_{a1}$. These are of frequency $f(1 + h)$.

3. Repeating the same procedure as in 2, but using the network of Fig.5c, determine ΔI_{a2}, ΔI_{b2}, $\Delta I'_{a2}$. These are of frequency $f(1 - h)$.

4. Add the steady-state and hunting components of each current together to obtain its resultant value under sustained oscillations. Thus, $I_a = I_{ao} + \Delta I_{a1} + \Delta I_{a2}$, and so forth...

The hunting components of currents are given by the following expressions in terms of machine constants and applied source voltage:

$$
\begin{aligned}
\Delta I_{a1} &= -\frac{\delta_1}{2x_m}\frac{B_1}{AA_1}\left[B(AA_1 - 1)(e^{j\delta_0} - 1) - e^{j\delta_0}\right] e^{jkt} v_a \\
\Delta I_{b1} &= \frac{\delta_1}{2x_m}\frac{B_1}{A}\left[B(AA_1 - 1)(e^{j\delta_0} - 1) - e^{j\delta_0}\right] e^{jkt} v_a \\
\Delta I'_{a1} &= \frac{\delta_1}{2x_m}\frac{B_1}{AA_1}\left[B(AA_1 - 1 - \frac{1}{B_1})(e^{j\delta_0} - 1) - (1 + \frac{1}{B_1})e^{j\delta_0}\right] e^{jkt} v_a \\
\Delta I_{a2} &= -\frac{\delta_1}{2x_m}\frac{B_2}{AA_2}\left[B(AA_2 - 1)(e^{j\delta_0} - 1) - e^{j\delta_0}\right] e^{-jkt} v_a \\
\Delta I_{b2} &= \frac{\delta_1}{2x_m}\frac{B_2}{A}\left[B(AA_2 - 1)(e^{j\delta_0} - 1) - e^{j\delta_0}\right] e^{-jkt} v_a \\
\Delta I'_{a2} &= \frac{\delta_1}{2x_m}\frac{B_2}{AA_2}\left[B(AA_2 - 1 - \frac{1}{B_2})(e^{j\delta_0} - 1) - (1 + \frac{1}{B_2})e^{j\delta_0}\right] e^{-jkt} v_a
\end{aligned}
\tag{53}
$$

where

$$A_1 = \frac{z_{a1}}{jx_m} + 1 \, , \qquad\qquad B_1 = \frac{jx_m}{2(z_{a1} + A_1 z_{b1})} \, ,$$

$$A_2 = \frac{z_{a2}}{jx_m} + 1 , \qquad B_2 = \frac{jx_m}{2(z_{a2} + A_2 z_{b2})} .$$

(d) The damping torque due to small sustained oscillations. In the steady-state operation the electromagnetic torque developed by each selsyn motor is a constant quantity. When the selsyn set undergoes small sustained oscillations, however, the torque expression of each machine contains both constant and pulsating components. This is due to the fact that the torque producing currents are of different frequencies. The pulsating part of the electromagnetic torque includes components of fundamental hunting frequency as well as its harmonics.

The part of the fundamental hunting frequency component which is in time quadrature with the sustained oscillations is the damping torque. The oscillations being defined by $\delta_0 - \delta_1 \cos kt$, it follows that the damping torque will be of the form $K \sin kt$. Therefore, to determine the damping torque, first the expression of the total electromagnetic torque must be considered and terms containing other than fundamental hunting frequency components discarded. Then from the remaining part, those terms which are of the form $K \sin kt$ must be singled out.

The electromagnetic torque of each induction machine under sustained oscillations is obtained by substituting the resultant currents into Eqs.(22) and (23), page 43, and remembering that

$$e^{j\theta} = e^{jnt} ,$$

and

$$e^{j\theta'} = e^{j(nt - \delta_o)}\left[1 - j\,\frac{\delta_1}{2}(e^{jkt} + e^{-jkt})\right]$$

Thus:

$$T_e = -\,3\ \mathrm{Pm}\,\mathcal{J}\left[(\bar{I}_{ao} + \Delta\bar{I}_{a1} + \Delta\bar{I}_{a2})(I_{bo} + \Delta I_{b1} + \Delta I_{b2})\right] \qquad (54)$$

$$T_e' = 3\ \mathrm{Pm}\,\mathcal{J}\left[(\bar{I}_{ao}' + \Delta\bar{I}_{a1}' + \Delta\bar{I}_{a2}')(I_{bo} + \Delta I_{b1} + \Delta I_{b2})(1 + j\frac{\delta_1}{2}e^{jkt} + j\frac{\delta_1}{2}e^{-jkt})\right] \qquad (55)$$

What we are interested in is not the seperate electromagnetic torque of each machine, but the difference $T_e - T_e'$. Subtracting (55) from (54), neglecting the products of incremental currents, and discarding the terms which contain other than fundamental hunting frequency, yields

$$(T_e - T_e')_k = -\,3\ \mathrm{Pm}\,\mathcal{J}\Big[(\bar{I}_{ao} + \bar{I}_{ao}')\Delta I_{b1} + (\bar{I}_{ao} + \bar{I}_{ao}')\Delta I_{b2} + (\Delta\bar{I}_{a1} + \Delta\bar{I}_{a1}')I_{bo}$$

$$+ (\Delta\bar{I}_{a2} + \Delta\bar{I}_{a2}')I_{bo} + \bar{I}_{ao}'I_{bo}\,j\,\delta_1 \cos kt\Big] \qquad (56)$$

The last term in the brackets can immediately be disregarded. Using the formulas given on page 58, the rest of the terms contained in Eq.(56) may be written as follows:

$$(\bar{I}_{ao} + \bar{I}_{ao}')\Delta I_{b1} = F_1 B_1 e^{jkt} \qquad (57)$$

$$(\bar{I}_{ao} + \bar{I}_{ao}')\Delta I_{b2} = F_2 B_2 e^{-jkt} \qquad (58)$$

$$(\Delta\bar{I}_{a1} + \Delta\bar{I}_{a1}')I_{bo} = \frac{G}{\bar{A}_1}\, e^{-jkt} \qquad (59)$$

$$(\Delta\bar{I}_{a2} + \Delta\bar{I}_{a2}')I_{bo} = \frac{G}{\bar{A}_2}\, e^{jkt} \qquad (60)$$

where,

$$F_1 = -\tfrac{1}{2}\,\frac{\delta_1|v_a|^2}{x_m^2\,|A|^2}\left[2\,B(AA_1 - 1)\sin\delta_0 + j(1 + e^{j\delta_0})\right] \tag{61}$$

$$F_2 = -\tfrac{1}{2}\,\frac{\delta_1|v_a|^2}{x_m^2\,|A|^2}\left[2\,B(AA_2 - 1)\sin\delta_0 + j(1 + e^{j\delta_0})\right] \tag{62}$$

$$G = \frac{jA\,\delta_1|v_a|^2}{2\,x_m^2|A|^2}\left[\,B(1 - e^{-j\delta_0}) + \left|B(e^{j\delta_0} - 1)\right|^2\right]$$

Since A_1 and A_2 are very nearly equal (computations show that $A_1 = .051\,\underline{/-17.0^\circ} + 1$ and $A_2 = .053\,\underline{/-18.2^\circ} + 1$), expressions (59) and (60) combine to give,

$$\frac{G}{\bar{A}_1}(e^{jkt} + e^{-jkt}) = 2\,\frac{G}{\bar{A}_1}\cos kt$$

which is to be discarded. An examination of (61) and (62) will show that if $A_1 = A_2$, then $F_1 = F_2$. Consequently the sum of (57) and (58) yields,

$$F_1(B_1e^{jkt} + B_2e^{-jkt}) = F_1\left[(B_1 + B_2)\cos kt + j(B_1 - B_2)\sin kt\right]$$

Upon discarding the cosine part of this last expression, the only term left in the right-hand side of Eq.(56) is $jF_1(B_1 - B_2)\sin kt$, and therefore,

$$T_d - T_d' = -\,3\,Pm\,\mathcal{J}\left[j\,F_1(B_1 - B_2)\sin kt\right] \tag{63}$$

where T_d and T_d' denote the damping torque of the transmitter and the receiver respectively. Taking the derivative of δ with respect to time gives

$$p\,\delta = p(\delta_0 - \delta_1\cos kt) = \delta_1 k\sin kt$$

whence,

$$\sin kt = \frac{1}{\delta_1 k}\, p\delta \tag{64}$$

Substituting (64) into (63) yields

$$T_d - T_d' = - cp\delta \tag{65}$$

where c is the damping constant and is given by

$$c = \frac{3\ Pm}{\delta_1 k}\, \mathcal{I}\left[j\, F_1 (B_1 - B_2) \right]$$

Substituting in this expression the previously defined values of F_1, B_1, B_2; and assuming $A_1 = A_2 = A$ and $AA_1 - 1 = \frac{2z_a}{jx_m}$, one obtains:

$$c = \frac{3\ P|v_a|^2}{2\omega^2 |A|^2}\left(r_a + A\, \frac{r_b}{s^2 - h^2}\right) \mathcal{I}\left[\frac{\frac{z_a - Az_b}{z_a + Az_b}\sin\delta_o + j(1 + \cos\delta_o)}{(z_{a1} + Az_{b1})\ (z_{a2} + Az_{b2})} \right] \tag{66}$$

This is the formula which will be employed in computing the damping constant. Eq.(66) shows that the value of the damping constant is affected by the applied voltage, the steady-state relative displacement angle δ_o , and the speed of the selsyn set. It is interesting to note that the amplitude of the sustained oscillations does not affect the value of c, whereas the frequency of oscillations does.

(e) Second approximate solution. In the first approximate solution presented in (b), page 49 , the damping torque was assumed to be negligible. As a result of this assumption the solution was found to be undamped sinusoidal oscillations. Now that the damping torque accompanying such sustained oscillations is known, it will be possible to take it into account, and consequently to obtain an improved approximate solution.

If the damping torque is taken into account, Eqs.(39) and (40), page 50, may be written as

$$Jp^2\theta = T_e + T_d + T_m - T_f \tag{67}$$

$$Jp^2\theta' = T_e' + T_d' - T_m' - T_f' \tag{68}$$

Subtracting (68) from (67), and substituting (65) in the resulting equation, yields

$$p^2\delta + \frac{c}{J}\,p\delta + k^2\sin\delta = u$$

or, replacing $\sin\delta$ by δ,

$$p^2\delta + 2\alpha p\delta + k^2\delta = u \tag{69}$$

where $\alpha = \frac{c}{2J}$, and k^2, u are defined by Eqs.(42) and (43), page 50.

The roots of the determinantal equation of (69) are:

$$\mu_1, \mu_2 = -\alpha \pm\sqrt{\alpha^2 - k^2}$$

Actual computations show that, in general, α is about one percent of k; therefore the roots may be taken as

$$\mu_1 = -\alpha + jk, \qquad \mu_2 = -\alpha - jk$$

Following the conventional procedure of solving linear differential equations; and using the initial conditions $\delta\Big]_{t=o} = \delta_i$ and $p\delta\Big]_{t=o} = 0$, one obtains

$$\delta = \delta_o - \frac{\delta_1}{k} e^{-\alpha t} \mathcal{R}_e\left[j\mu_2 e^{jkt} \right] \qquad (70)$$

where δ_o and δ_1 are as defined on page 51 . Since α is very small compared to k, μ_2 may be taken as $-jk$ and Eq.(70) becomes:

$$\delta = \delta_o - \delta_1 e^{-\alpha t} \cos kt \qquad (71)$$

Eq.(71) shows that the transient variation of the relative angular displacement between the transmitter and receiver rotors is a damped sinusoidal function of time. It should be borne in mind that, in view of the assumptions leading to this result, Eq.(71) represents only an approximate solution.

* * * * *

APPENDIX A

Determination of Machine Constants.

The name-plate data of the machines used in the experimental work are given below:

Transmitter:

Induction motor No. 47C
10 hp, 220 volts, 27.3 amps.
1160 r.p.m., 60 cps, 3-phase

Driving motor:

d-c motor No. 301
10 hp, 230 volts, 37.7 amps.
1250 r.p.m.

Receiver:

Induction motor No. 47D
10 hp, 220 volts, 27.3 amps.
1160 r.p.m., 60 cps, 3-phase

Loading generator:

d-c generator No. 92B
12 kw, 550 volts, 21.8 amps.
1800 r.p.m.

The electrical constants of the induction motors are determined by means of the conventional methods. The data given by the no-load run and blocked-rotor test for machine No. 47C are plotted on pages 68 and 69 respectively. The measured ohmic resistances of the stator and rotor are

$R_a = 0.157$ ohm per phase (stator)

R_b = 0.237 ohm per phase (rotor)

Computations based on the no-load and blocked-rotor data yield,

r_a = 0.178 ohm per phase

r_b = 0.269 ohm per phase

x_a = 0.565 ohm per phase

x_b = 0.565 ohm per phase

x_m = 11.30 ohms per phase

In separating the equivalent effective resistance into its parts, it is assumed that the ratio of r_a and r_b is the same as that of R_a and R_b. The separation of leakage reactances is based on the assumption that x_a and x_b are equal. The ratio of transformation of the induction motor is found to be 1.01, which may be taken as unity for all practical purposes.

To determine the combined friction and windage loss of the induction motor and the associated d-c machine, the latter is freed from all electrical connections and driven by the induction motor. The power input to the induction motor is plotted against terminal voltage on page 70. The friction and windage loss is obtained by extrapolating the curve to zero voltage. It is found to be 820 watts.

The combined moment of inertia of the induction motor and its associated d-c machine is determined by making a retardation run. The d-c machine - freed from all electrical connections - is driven by the induction motor at nearly synchronous speed. Then the latter is suddenly disconnected from its source and several time and speed readings are taken as the machines slow down. The result is plotted on page 71 . As soon as the induction motor is disconnected from its source the machines constitute a purely mechanical system which is acted upon by inertia and friction torques only. Therefore, the equation of motion becomes:

$$J \frac{d\omega}{dt} = - T_f$$

where

J = moment of inertia,

ω = angular velocity of the machines,

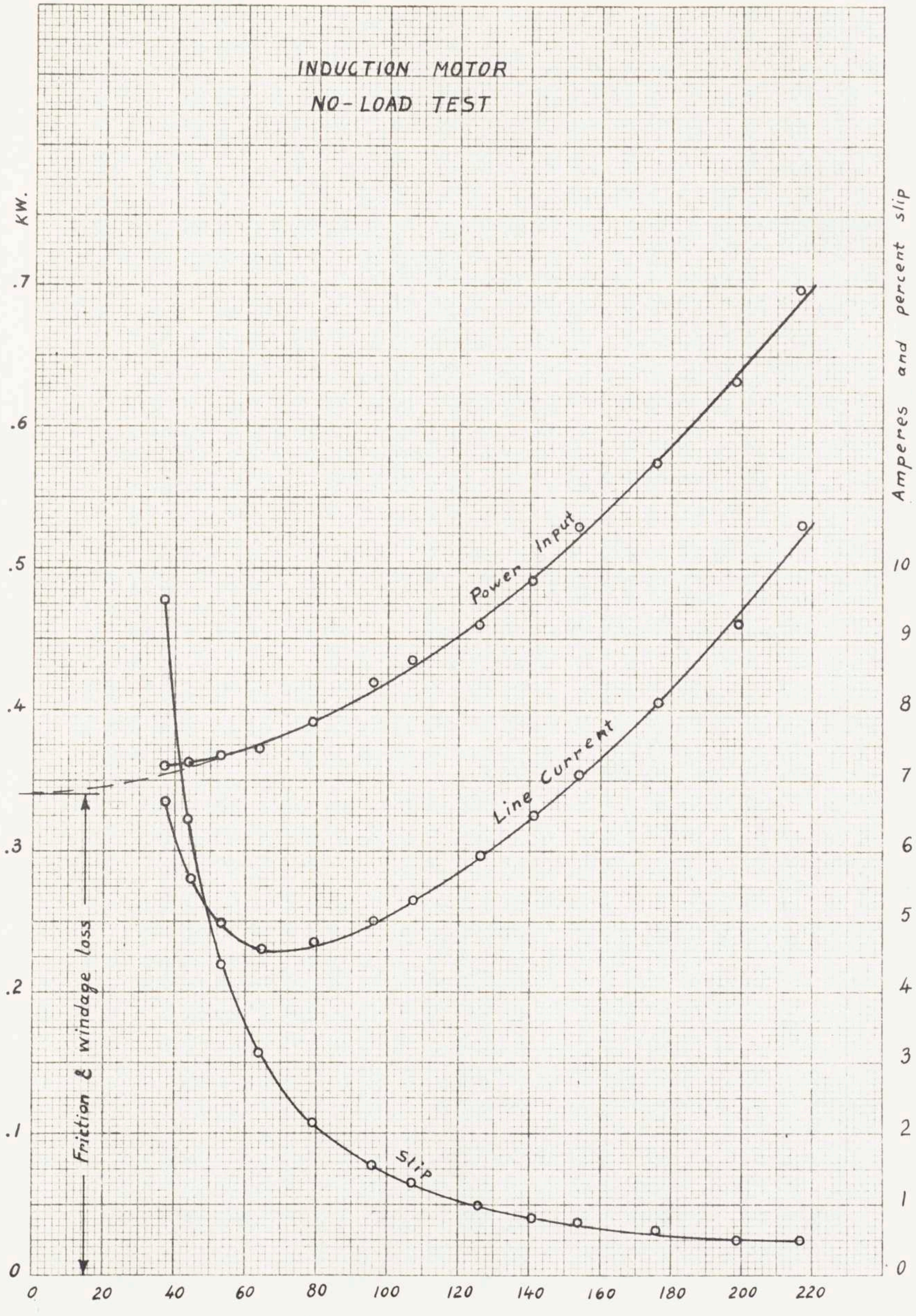
T_f = combined friction and windage torque of the machines.

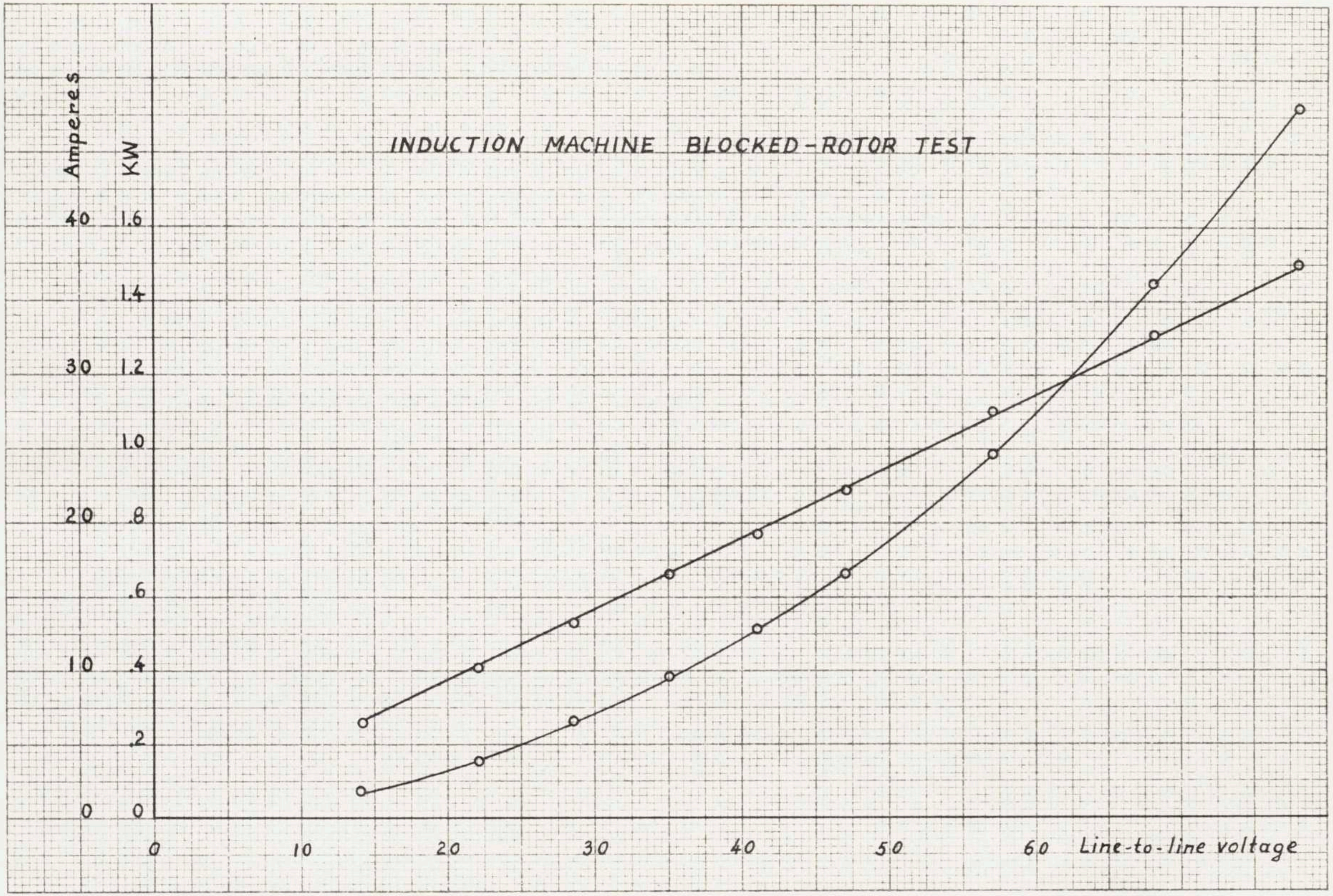
$\frac{d\omega}{dt}$ is the slope of the curve shown on page 71 .

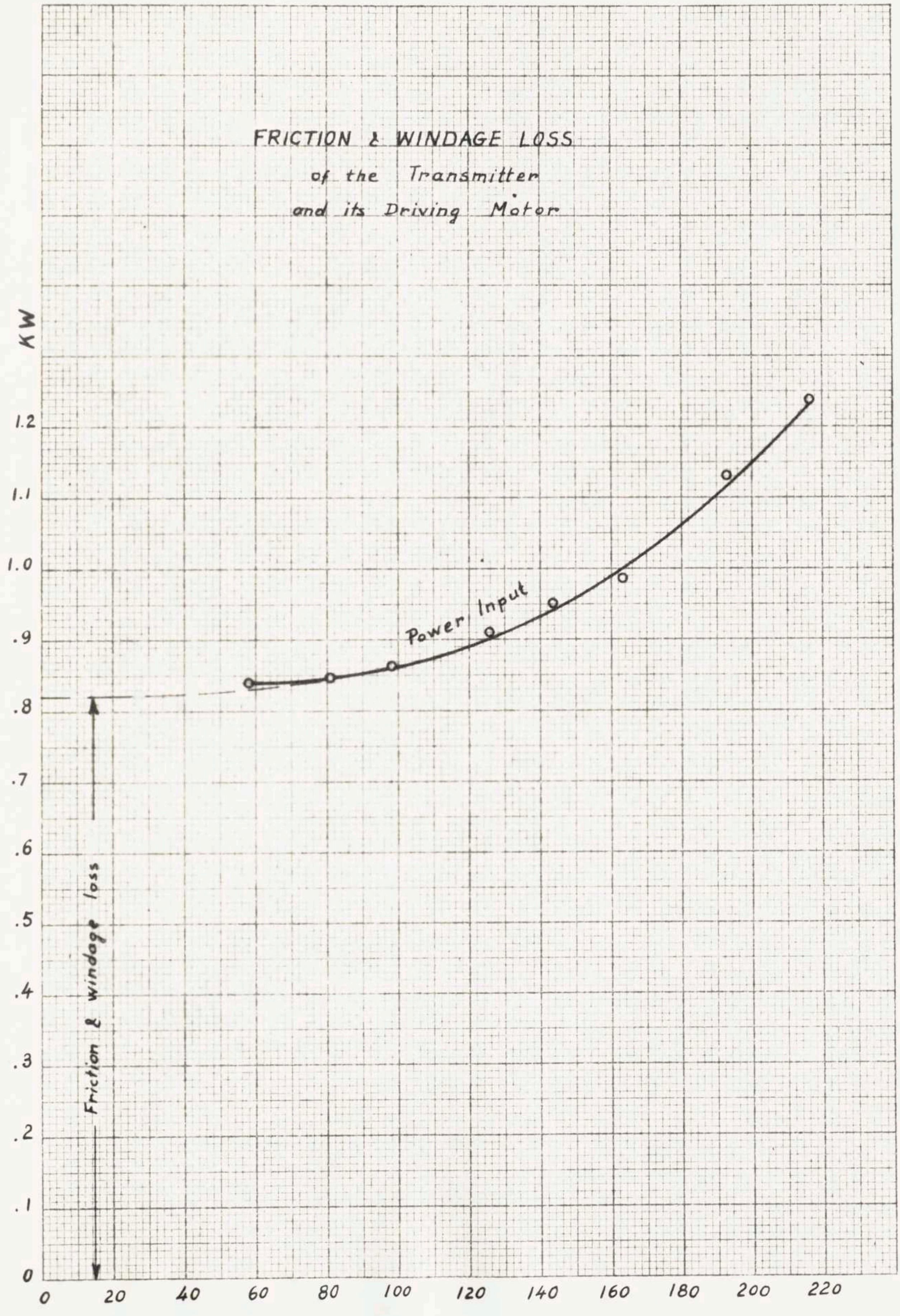
Knowing this slope and T_f corresponding to $t = 0$ it is easy to compute J. Computations give

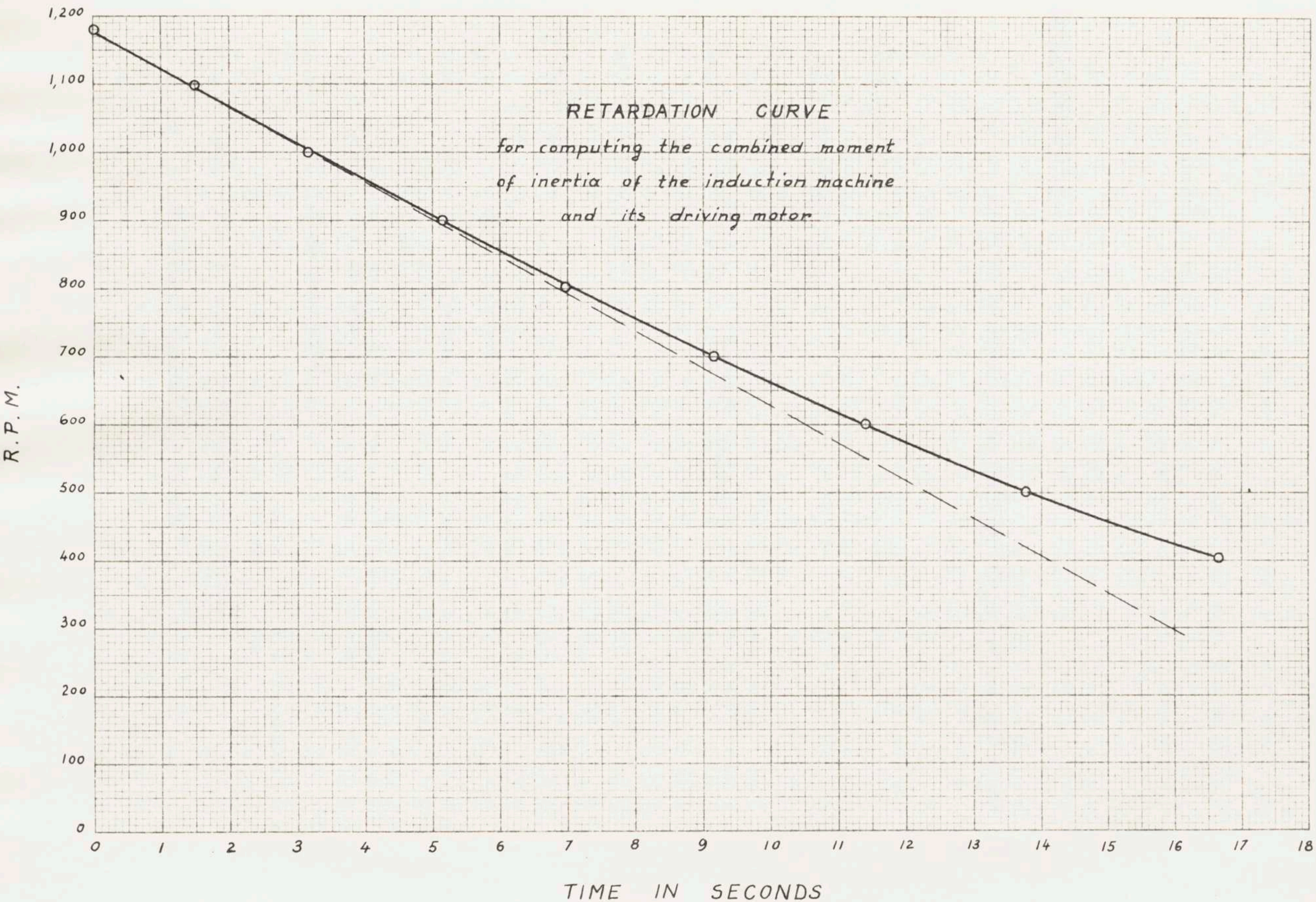
$$J = 0.28 \text{ lb.- ft.- sec.}^2 \text{ per electrical radian}$$

* * * * *


INDUCTION MOTOR
NO-LOAD TEST
KW.
Amperes and percent slip
Power Input
Line Current
Slip
Friction & windage loss
Line - to - line voltage
0
.1
.2
.3
.4
.5
.6
.7
0 20 40 60 80 100 120 140 160 180 200 220
0 1 2 3 4 5 6 7 8 9 10


INDUCTION MACHINE BLOCKED-ROTOR TEST
Amperes
KW
40 1.6
1.4
30 1.2
1.0
20 .8
.6
10 .4
.2
0 0
0
10
20
30
40
50
60
Line-to-line voltage


FRICTION & WINDAGE LOSS
of the Transmitter
and its Driving Motor
KW
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
Power Input
Friction & windage loss
0
20
40
60
80
100
120
140
160
180
200
220
Line-to-line voltage


RETARDATION CURVE
for computing the combined moment of inertia of the induction machine and its driving motor
R. P. M.
1,200
1,100
1,000
900
800
700
600
500
400
300
200
100
0
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
TIME IN SECONDS

APPENDIX B

Sample Computation.

As an illustrative example the solution of Run #1 is presented here.

Operation data:

rpm = 1000, with field

V = 214 volts, line-to-line

T_{mi} = 10.2 ft-lbs. (initial torque)

T_m = 21.7 ft-lbs.

T_m' = 22.4 ft-lbs.

Machine data:

r_a = 0.178 ohm per phase

r_b = 0.269 ohm per phase

x_a = 0.565 ohm per phase

x_b = 0.565 ohm per phase

x_m = 11.30 ohms per phase

P = 6

J = 0.28 lb.- ft.- sec.^{2} per electrical radian

ω = 377 electrical radians per second.

Solution:

$$s = \frac{1}{\omega}(\omega - \pi P \frac{rpm}{60}) = 0.167$$

$$z_a = 0.178 + j\,0.565$$

$$z_b = 1.61 \;\; + j\,0.565$$

$$A = \frac{z_a}{jx_m} + 1 = 1.05$$

$$z_a + Az_b = 1.87 + j\,1.16 = 2.2 \underline{/31.8^o}$$

$$x_a + Ax_b = 1.16$$

$$v_a = \frac{V}{\sqrt{2}\;\sqrt{3}} = 87.4 \quad \text{volts}$$

From the relations given on pages 50 and 51 one obtains:

$$k^2 = \frac{3 \times 6 \times (87.4)^2 \times 1.16}{377 \times 0.28 \times 1.05 \times (2.2)^2} \times \frac{550}{746} = 219$$

$$u = \frac{21.7 + 22.4}{0.28} = 158 \qquad \text{(neglecting } T_f - T_f'\text{)}$$

$$\delta_o = \frac{u}{k^2} = 0.72 \quad \text{electrical radian}$$

$$= \underline{41.3 \text{ electrical degrees}}.$$

$$\delta_i = \frac{T_{mi}}{Jk^2} = \frac{10.2}{0.28 \times 219} = 9.6 \quad \text{electrical degrees.}$$

$$\delta_1 = \delta_o - \delta_i = \underline{31.7 \text{ electrical degrees}}.$$

Now the damping constant will be determined.

$$h = \frac{k}{\omega} = \frac{\sqrt{219}}{377} = 0.04$$

$$z_{a1} = \frac{r_a}{1 + h} + jx_a = 0.171 + j\,0.565$$

$$z_{b1} = \frac{r_b}{s + h} + jx_b = 1.30 + j\,0.565$$

$$z_{a1} + Az_{b1} = 1.54 + j\,1.16 = 1.93\ \underline{/37.0^\circ}$$

$$z_{a2} = \frac{r_a}{1 - h} + jx_a = 0.185 + j\,0.565$$

$$z_{b2} = \frac{r_b}{s - h} + jx_b = 2.12 + j\,0.565$$

$$z_{a2} + Az_{b2} = 2.40 + j\,1.16 = 2.67\ \underline{/25.8^\circ}$$

$$z_a - Az_b = -\,1.52 - j\,0.028 = -\,1.52\ \underline{/1.1^\circ}$$

Substituting the numerical values in Eq.(66), page 62, one obtains $c = 1.168$. Then,

$$\alpha = \frac{c}{2J} = \frac{550 \times 1.168}{746 \times 2 \times 0.28} = 1.54$$

Therefore, according to Eq.(71), the solution is

$$\delta = 41.3^\circ - 31.7^\circ\, e^{-1.54t} \cos 14.8t$$

* * * * *

APPENDIX C

Heaviside's "Shifting" Principle.

If the operand of an operational expression contains a factor $e^{\alpha t}$, then it is possible to shift this factor to the left of the operator by changing the symbol p to $p + \alpha$ throughout the operator. That is,

$$F(p)\left[e^{\alpha t} f(t)\right] = e^{\alpha t} F(p + \alpha)\left[f(t)\right]$$

where

$F(p)$ = operator (function of p)

$f(t)$ = any function of t .

The reverse transformation is also feasible. That is to say, an outside factor $e^{\alpha t}$ may be shifted into the operand by changing p to $p - \alpha$ throughout the operator. Thus:

$$e^{\alpha t} F(p)\left[f(t)\right] = F(p - \alpha)\left[e^{\alpha t} f(t)\right]$$

These two transformations constitute Heaviside's "Shifting" principle. (See reference 4.)

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REFERENCES

1. "The Application of Tensors to the Analysis of Rotating Electrical Machinery." Gabriel Kron (General Electric.)
2. "The Doubly Fed Machine." C. Concordia, S. B. Crary, G. Kron (AIEE Transaction, Vol. 61, pp 286 - 289.)
3. "Equivalent Circuits for the Hunting of Electrical Machinery." G. Kron (AIEE Transaction, Vol. 61, pp 290 - 296.)
4. "Heaviside's Operational Calculus." E. J. Berg (Mc Graw-Hill) page 61.