Diversity and Expertise in Representative Governance

by

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ABSTRACT

Representative democracy is a widespread and essential part of democratic governance. Our understanding of it has been largely shaped by the *triumph of elections* that vest peripheral access to power through episodical polls. How would representative democracy look under different selection rules? In an attempt to reflect on foundational principles on which we could build a renewed case for representative democracy as democratic governance, this thesis explores democratic innovations for selecting representatives and focuses on the interplay between selection mechanisms, epistemic performance, and procedural aspects.

The first chapter investigates the optimal number of voters needed to aggregate votes on a binary issue under majority rule. It takes an *epistemic* view where the issue at hand has a ground truth "correct" outcome and each one of n voters votes correctly with a fixed probability, known as their competence. Assuming that the best experts, i.e., those with the highest competence, can be identified to form an *epistemic congress*, this chapter surprisingly shows that the optimal congress size should be linear in the population size, even with expert decision-making.

The next chapters delve into the concept of liquid democracy, a governance mechanism in which citizens can either vote directly or delegate their votes to others, and examine the epistemic and procedural performances of this approach offering insights from both theoretical and empirical perspectives.

Taking an epistemic view, the second chapter highlights delegation scenarios where liquid democracy is likely to find the ground truth. It treats delegations as a stochastic process akin to well-known processes on random graphs — such as preferential attachment and multi-types branching process — and relate their dynamics to liquid democracy's performance. Along the way, it proves new bounds on the size of the largest component in an infinite Pólya urn process which may be of independent interest.

The third chapter presents empirical experiments designed to compare liquid democracy with direct democracy, the counter-factual. It validates the theoretical findings of the second chapter, providing evidence that delegation mechanisms align with theoretical expectations.

The fourth chapter analyzes delegation dynamics in a real-world setting and explores how liquid democracy functions in scenarios with contentious issues. It reveals insights into the patterns of delegation and the usage of liquid democracy in non-epistemic contexts.

The fifth chapter reflects on lottocracy (selection of representatives at random) and proxy democracy (selection based on self-selection and flexible nominations that determine the relative influence of representatives) as two models to select representatives. It investigates the procedural aspects of both selection mechanisms exploring how inclusion, equality and legitimacy would be realized under lottocracy and proxy democracy.

The sixth chapter, drawing on computational social choice, formulates a unified framework for comparing selection mechanisms. It devises a model in which different selection mechanisms can be formalized and evaluated axiomatically. It classifies selection mechanisms based on whether they are *open-closed*, *flexible-rigid*, and *direct-virtual* and propose the following five axioms: proportionality, diversity, monotonicity, faithfulness, and effectiveness.

Throughout, the thesis intertwines insights from mathematics (social choice theory, applied probability, statistics), political philosophy, and empirical analyses to provide a comprehensive exploration of different facets of representative democracy. The interdisciplinary approach reflects the complexity and richness of democratic governance and calls for continued collaboration across disciplines to tackle its challenges and shape its future.

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Obstacles don't have to stop you. If you run into a wall, don't turn around and give up. Figure out how to climb it, go through it, or work around it.

– Michael Jordan

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Chapter 1

Introduction

Democratic governance is an area of active social scientific inquiry: new forms of citizen representation, participation, and deliberation are being tested and deployed around the world [79, 241]. Defined as a process in which the members of the group being governed have binding authority over the decisions, democratic governance is in inevitable tension with pluralism: on which grounds should group members to be obliged by a decision they may very well disagree with, when the institutional arrangement regards them as free and equal? This legitimacy dilemma further intensifies with the introduction of representative democracy, according to which group members' representatives decide for all. The people's binding authority is henceforth manifested in two phases: the selection stage (where representatives are selected) and the decision stage (where representatives make decisions).

The *triumph of elections* in the eighteenth century amounted to equate representative democracy with elections [161]. It further coincided with the development of theories and symbols for the expression of the people's will through their representatives, voting equality, formal authorization and accountability mechanisms. At the same time, representative democracy, and more often, elections, are increasingly perceived as founded on elitist principles [145] or decried for their oligarchic drift [227].

While some have argued that representation would allow for a greater form of democracy in which motivated and competent citizens are selected by the people through free and fair elections [213] resulting in a division of labor [92], representative democracy is sometimes seen as a lesser form of democracy, a pragmatic approximation of the democratic ideal deemed necessary to accommodate large polity [171]. Concomitantly, pollsters have observed global discontent with the ways in which democracies work [236], historical lows in trust in representative institutions in the United States [128] and plummeting democratic values worldwide [226]. Many scholars have documented democratic recessions [59, 69], democratic discontent [209], democratic decline [8, 56, 149] and lessons, successes and failures of democratic transitions [159, 160]. After Francis Fukuyama's optimistic view that democratization would bring an end to history and a stable equilibrium for governance [90], others came to fear that representative democracies would be merely a blip in history between long-lasting phases of political inequity and illiberalism.

Philosopher John Rawls' answer to the legitimacy paradox was to ensure that power was "exercised in accordance with a constitution the essentials of which all citizens as free and equal may reasonably be expected to endorse in the light of principles and ideals acceptable to their common human reason [199]." However, in the face of rising "frustration with and alienation from the political elite" [241], Rawls' answer might not be sufficient. Instead, we might be on the verge of a socio-political moment where "nations feel tormented by evils so great that the idea of changing their constitutions presents itself to their thoughts" [66].[1] Such a moment is necessarily as scary as it is risky: as Pippa Norris argues, "could there be enough agreement in the room that it wouldn't basically get worse rather than better? And that's the big danger." [2] This, however, is reminiscent of the words famously carved on the

[1] My translation for the French: "les nations se sentent tourmentées de maux si grands que l'idée d'un changement de leur constitution se présente à leur pensée."

[2] Is democracy in decline? Kennedy School professors voice optimism and concerns. https://ash.harvard.edu/democracy-decline-kennedy-school-professors-voice-optimism-and-concerns Last accessed: July 27th 2023

southeast wall of the Jefferson memorial and borrowed from a letter Thomas Jefferson wrote to Samuel Kercheval [126]:

> I am certainly not an advocate for frequent and untried changes in laws and constitutions. I think moderate imperfections had better be borne with; because, when once known, we accommodate ourselves to them, and find practical means of correcting their ill effects. But I know also, that laws and institutions must go hand in hand with the progress of the human mind. As that becomes more developed, more enlightened, as new discoveries are made, new truths disclosed, and manners and opinions change with the change of circumstances, institutions must advance also, and keep pace with the times. We might as well require a man to wear still the coat which fitted him when a boy, as civilized society to remain ever under the regimen of their barbarous ancestors.

We may have the opportunity to provide a moment of fundamental rethinking of the values and contexts on which Western republicanism was built and for all citizens to co-author renewed institutions. The eighteenth-century model of representative democracy was designed for the geographic representation of small, homogeneous, and restricted demos. James Madison explains the ratio of representatives to citizens to be one for thirty thousand in his institutional framework [112], the abbot Sieyès refers to the six million French scattered throughout a vast and disconnected territory. Representative democracies were infamously founded on discrimination against a majority of the population, limiting voting rights to males, property owners, or whites. This framework is both no longer relevant and harmful for historically and newly marginalized communities [89, 170, 196]. How could we ensure that the diversity of large nations is fairly represented in representative democracies and that representatives have the relevant experience and expertise to legitimately decide on behalf of the citizenry?

At the core of these questions is an intuition that representative democracy is not bound to be a lesser form of democracy, but that it fails to embody core democratic values when the latter are primarily thought of as a function of the selection stage and not of the decision stage. [3] Intriguingly, our reasons and symbols for democratic values (such as equality and inclusion) in electoral democracies, following a Schumpeterian account of representative democracy, are often defined at the selection stage: the one-person-one-vote slogan confers equal weight in selecting between a set of candidates and holding them accountable periodically, but it does not guarantee that the set of candidates was democratically or fairly chosen. In such a model of representative governance, democratic values are relegated to a selection problem [148].

1.1 Thesis Objectives

In this thesis, we explore theories of representation where group members have binding authority over the decisions in that they have an *equitable* chance of being selected to hold political offices: moving beyond reasons and symbols for the demos as instrumental in selecting representatives, could we provide reasons and symbols for all citizens to have reasonable and equitable chances of becoming representatives?

Several theoretical and empirical aspects of representation could be discussed as a result of this question such as campaigning rules, places where representative institutions are held, processes through which decisions are made, or selection models (and associated legitimacy) to choose the representatives. Building on recent work in social choice theory [e.g., 51, 83, 102, 129] and political philosophy [e.g., 145, 229, 230], this thesis explores the *epistemic*

[3] The concept of representative democracy remains popular — a Pew poll from 2017 conducted across 38 nations found that a median of 78% of participants believe it to be a good way to govern. [237] However, large shares of citizens from several countries also indicated in 2021 that their political systems needed reform. [238]

dimension[4] and the *procedural dimension*[5] of selection rules for representatives, in an attempt to reflect on foundational principles on which we could build a renewed case for representative democracy as democratic governance.[6]

In this collection of essays, we investigate *epistocracy* (that selects experts), *proxy democracy* (in which citizens either self-select to be representatives or flexibly nominate self-selected citizen(s) through frequent nomination processes; in turn, representatives have a weight equal to the number of citizens they represent, which scales their votes in congress), *liquid democracy* (a variant of proxy voting with area-specific transitive delegations and the possibility to recall delegations at any time) and *lottocracy* (where representatives are drawn randomly). We do not intend to conclude on an ultimate selection model, but rather to explore the ecology of selection rules for representative assemblies. We hold that no such process could provide a definitive, time-invariant or context-independent answer to the problems of representation. Most selection models come with defensible benefits and exploitable failure modes. Exploring various selection rules may allow us to reflect on normative accounts taken for granted and processes more adequate for particular contexts, to distribute the failure modes combining complementary bodies selected for different intent, and to move beyond a single dominant view on how representation is operated in democratic governance.

The thesis does not engage with other important aspects of decision-making that occur once representatives are selected. For instance, the interplay between selection and post-selection treatment of representatives is another area of active research (see, for in-

[4]The epistemic dimension is related to the capacity to make *good* decisions under a decision-making process.

[5]The procedural dimension is related to the intrinsic values of the selection procedure.

[6]In this view of representation, elections are also not inherently problematic, they only become so when the demos has very little oversight on who they have to choose from [148]. Places are experimenting with novel methods for selecting candidates in elections for instance in Nigeria, Georgia, Ghana, or Nevada; see, e.g., [60, 184] and https://www.bosch-stiftung.de/sites/default/files/publications/pdf/2022-11/Exploring_Worldwide_Democratic_Innovations_Long_Report.pdf). We view elections as inducing one mode of representation that comes with strengths and shortcomings, and are interested in other modes of representation achievable through other selection rules.

stance, [142]). Further, following Jürgen Habermas' work on deliberative democracy [109], many others have also studied and deployed democratic innovations that empower public reasons to emerge through cogent discussions and participation of citizens. [7] Expanding our understanding of how those included in these various consultative phases, as citizens-representatives, are selected, given binding power, and held accountable could be complementary to re-designing democratic governance for the twenty-first century [11]. Last, changing our theoretical conception of representation should be accompanied by a theory of change. In particular, institutions may need re-designing to allow all citizens, regardless of their occupation, to be reasonably able to become representatives [8] Campaigning may also require tighter regulation to open races to more people, regardless of wealth or fame.[9] Such reflections are outside of the scope of this work.

1.2 Thesis Outline

The essays assembled in this thesis attempt to expand our imagination regarding the design of representative institutions to shape the future of epistemically and procedurally responsible forms of democratic representation. The first two chapters of the thesis explore epistocracy and liquid democracy from a mathematical perspective, within an established framework in social choice theory that investigates the truth-seeking dimension of decision-making. The next chapter provides results on experimentation with liquid democracy testing the theory

[7]Deliberative polling and mini-publics have diversified the voice of those heard when writing constitutions [182], legalizing abortion [58], or setting the agenda to curb climate change [182]. Some of these initiatives are providing randomly selected citizens binding power over decisions [180]. Participatory budgeting, invented in Brazil in the 80s, further lets citizens allocate public funds through collaborative processes [233]. Participatory planning and multilevel policymaking have spread in Latin America. https://epd.eu/wp-content/uploads/2022/09/case-study-latin-america.pdf Countries got equipped with ministries of the future (Sweden), institutionalized online participatory tools in Taiwan, Nigeria, North Macedonia or South Korea, experimented with versions of proxy democracy in Argentina, and Germany [121].

[8]In that vein, citizen assemblies experiment with periodic reunions during weekends to accommodate various schedules.

[9]See Lessig [148] for more details on that.

previously developed. The fifth chapter provides descriptive metrics for a liquid democracy experiment conducted with subjective questions and a large community. The sixth chapter benchmarks two selection procedures, lottocracy, and proxy democracy, in an attempt to highlight the normative and contingent trade-offs they induce and is particularly interested in how they achieve descriptive representation. The last chapter suggests a novel axiomatic framework to provide mathematical guarantees over procedural aspects of representative democracy.

1.2.1 How Many Representatives Do We Need? The Optimal Size of an Epistemic Congress

Chapter 2 investigates the interplay between expertise and the size of the representative body. Imagine (unrealistically) that we can fill the representative layers of a democratic process with the most expert citizens, assumed to make decisions independently in the probabilistic sense. How many representatives do we need to maximize the probability that a majority of them makes the correct decision?

Following Condorcet's seminal approach, we rely on the *epistemic model* for decision-making in which citizens differentiate between two options where one is assumed to be better than the other. Citizens have a probability of finding the correct outcome (referred to as their expertise) drawn independently from a fixed distribution that governs the votes they cast. [10] We hypothesize that experts, defined as those with higher chances of finding

[10] In more detail, in Condorcet's seminal epistemic approach, n voters cast independent binary votes and assume that one of the outcomes is objectively better and that each voter independently selects the better outcome with probability $p > 1/2$ (that is, if voters have just enough information to be more accurate than an unbiased coin at deciphering the correct outcome). Then the probability that a majority of voters finds the ground truth tends to one as the population increases. This result is known as the Condorcet Jury Theorem. In other words, a majority will asymptotically be better than any fixed number of voters. Now, if we tweak the model to allow some voters to be more accurate than others (each voter i is correct with distinct probability p_i where the p_i are independent identically distributed samples from a fixed distribution), under mild conditions, the above statement generalizes to: a majority vote will asymptotically outperform any fixed number of experts.

the ground truth (or, mathematically, those whose expertise corresponds to the first-order statistics drawn from the expertise distribution) are selected and find the optimal number of experts needed to maximize that a majority of them are correct.

Against a common intuition [221], we find that for a fixed and regular distribution of expertise, a constant fraction of the population is needed to maximize the probability of being correct. In other words, under these modelling assumptions, a linear number of citizens is needed to justify optimal decision-making under majority rule: even in a scenario where experts are chosen, optimality seems to heavily rely on large (hence diverse) representative bodies. This result is striking because it holds even when allowing the top representatives to become arbitrarily accurate, choosing the correct outcome with probabilities approaching one.

We further extend the Condorcet jury theorem (see footnote 10) to cases in which the distribution of expertise varies with the population size: we characterize scenarios in which few experts are more accurate than majority voting with high probability. If the underlying expertise distribution changes with the population size (as it may be if information or education infrastructures are hard to keep at a constant performance level as the population increases) and the rate at which the societal bias towards the ground truth decreases too fast, there exists a regime in which, over time, a sub-linear number of experts submitted to majority voting would beat majority voting amongst all citizens with high probability.

It goes without saying that the hypothesis that experts could be arbitrarily chosen to form the representative body is a mathematical abstraction. While it helps relating modelling assumption to the role of expertise in collective-decision making, it is not realistic or defensible in actual representative democracy.

This work was conducted with Tao Lin and Daniel Halpern and published in the Proceedings of the 36th AAAI Conference on Artificial Intelligence [203].

1.2.2 In Defense of Liquid Democracy

Chapter 3 investigates a novel selection model, *liquid democracy*. Liquid democracy is conceptually situated between direct democracy, in which voters have direct influence over decisions, and representative democracy, where voters choose delegates who represent them for some time. Under liquid democracy, voters have a choice: they can either vote directly on an issue as in direct democracy or delegate their vote to another voter, entrusting them to vote on their behalf. The defining feature of liquid democracy is that these delegations are *transitive*: if voter 1 delegates to voter 2 and voter 2 delegates to voter 3, then voter 3 votes (or delegates) on behalf of all three voters.

Liquid democracy has gained prominence around the world. The most impressive example is that of the German Pirate Party, which adopted the *LiquidFeedback* platform in 2010 [137]. Other political parties, such as the Net Party in Argentina and Flux in Australia, have run on the wily promise that, once elected, their representatives would be essentially controlled by voters through a liquid democracy platform. Companies are also exploring the use of liquid democracy for corporate governance; Google, for example, has run a proof-of-concept experiment [113].

Practitioners, however, recognize that there is a potential flaw in liquid democracy, namely, the possibility of *concentration of power*, in the sense that certain voters amass a relatively large number of delegations, giving them pivotal influence over the final decision.

To understand this problem, we study the paradigm in the *epistemic model* as in Chapter 2, where voters decide on a binary issue for which there is a ground truth. Previous work showed that, under certain assumptions on the delegation model, a few voters may amass such a large influence that liquid democracy becomes less likely to identify the ground truth compared to direct democracy [129]. We quantify the amount of permissible concentration of power and examine more realistic delegation models that ensure that (with high probability) there is a sharp limit on the maximum number of delegations received. Our results demonstrate that the delegation process can be treated as a stochastic process akin to well-known processes on random graphs —such as preferential attachment and multi-types branching process—that are sufficiently bounded for our purposes [73]. As a side result, we prove new bounds on the size of the largest component in an infinite Pólya urn process, which may be of independent interest. Our work suggests that existing and new results in random graph theory may alleviate concerns raised about liquid democracy and bolster the case for the applicability of this emerging paradigm.

This work was conducted with Daniel Halpern, Joe Halpern, Ali Jadbabaie, Elchanan Mossel, and Ariel Procaccia and was published in the Proceedings of the 2023 ACM Conference on Economics and Computation [111].

1.2.3 An Empirical Analysis of Liquid Democracy's Epistemic Performance

In Chapter 4, we test the theoretical results of the third chapter through a series of twelve experiments. Through a matched pair design, we form delegation graphs induced by the delegation process in liquid democracy, and compare the results to direct voting. We estimate the theoretical delegation mechanisms studied in the previous chapter and find that the empirical delegation mechanisms are consistent with those identified theoretically. A higher propensity to delegate is associated with lower expertise, and delegation rates toward representatives tend to increase with their relative expertise, which is consistent with the delegation models identified in Chapter 2.

Taken together, the results of Chapters 3 and 4 uncover theoretical regimes empirically validated in which liquid democracy performs well, bolstering the case that liquid democracy may leverage interpersonal knowledge embedded in social networks to improve decision-making. Importantly, these conclusions are limited to well-connected networks and non-polarising, binary, and factual questions.

This work was conducted with the guidance of Adam Berinsky and discussed with Daniel Halpern, Joe Halpern, Ali Jadbabaie, and Ariel Procaccia. An early version of this work was accepted at the ACM Conference on Equity and Access in Algorithms, Mechanisms, and Optimization 2022 [202].

1.2.4 A Descriptive Analysis of Liquid Democracy's Procedural Performance

Beyond the epistemic dimension of liquid democracy, delegation dynamics in large groups for contentious and potentially polarizing issues remain under-studied in the literature. Chapter 5 presents the results of an experiment run with an academic institution, involving 117 participants each answering 11 questions about the institute's governance. While it has often been mentioned that cycles in the delegation graph and concentration of power are practical obstacles to liquid democracy, we show that in practice those are unlikely: there are very few two-cycles and little evidence of concentration of power. However, we find that a large portion of persons that are delegated to did not participate in the survey, posing another kind of practical concerns.

We also observe that the delegative feature is used as envisioned by liquid democracy advocates [228]: participants tends to choose different representatives for different questions.

Finally, this chapter raises a series of questions in terms of the behavioral and cognitive aspects of individual delegations that require further attention as we explore scaling up novel selection processes for representative democracy.

1.2.5 How to Open Democratic Representation to the Future?

Chapter 6 engages with procedural aspects of selection mechanisms for representative democracy in an attempt to broaden institutions' perspectives. We consider the ecology of selection rules for representative assemblies (such as parliamentary chambers), introducing proxy democracy as a selection rule for representation in open democracies and comparing it to lottocracy.

The chapter also investigates how Hélène Landemore's accounts of democratic, legitimate and descriptive representations [145] are realized under lottocracy and proxy democracy, drawing on political and social choice theories to integrate these traditionally separate fields of study. While proxy democracy opens representative institutions reinforcing the current understanding of representative values, lottocracy cannot be fully justified in that context; this essay builds on recent political theory to characterise appropriate novel grasps on the concept of representation[57, 108, 145].

Last, this chapter identifies a gap in the normative theory of lottocracy that raises a series of questions. Biased self-selection may impair lottocracy's promise to promote descriptive representation: should self-selection be handled by mandates, quotas, or ignored? In the first case, is there a moral duty to serve as a representative or a substantive argument that those in power should not seek it? In the second case, which fairness or equity standards should replace the equality principle? In the third case, why should equality be preferred over diversity?

While democracies historically tend to try out novel procedures that fit a particular normative ideal and evaluate other externalities after the fact, this essay benchmarks lottocracy and proxy democracy, in an attempt to highlight the normative and contingent trade-offs. While both systems outperform electoral alternatives on the dimensions under study, they induce tensions often overlooked. Nonetheless, clarifying the normative compromises seems crucial to address the challenges facing democratic systems and shape the future of epistemically and procedurally responsible forms of representation.

This work was published in the European Journal of Risk Regulation [201].

1.2.6 An Axiomatic View for Representative Democracy

Chapter 7 explores employing concepts and tools from computational social choice to devise a model in which different selection mechanisms can be formalized and compared from a procedural standpoint. As the world's democratic institutions are challenged by dissatisfied citizens, scholars have proposed and analyzed various (innovative) methods to select representative bodies. However, a unified framework to analyze and compare different selection mechanisms is missing, resulting in very few comparative works. This chapter intends to define desirable representation axioms to be conceptualized and evaluated and proposes a unifying mathematical formulation of different selection mechanisms as well as various social-choice-inspired axioms. We classify selection mechanisms based on whether they are *open-closed*, *flexible-rigid*, and *direct-virtual* and propose the following five axioms: proportionality, diversity, monotonicity, faithfulness, and effectiveness.

This research was conducted with Niclas Bohemer, Rachael Colley, Markus Brill, Piotr Faliszewski and Edith Elkind thanks to the workshop, Algorithmic Technology for Democracy, organized by Davide Grossi, Ulrike Hahn, Michael Maes, and Andreas Nitsche at the Lorentz Center [205].

1.3 Reflections on the Methods

The approach taken in this thesis mixes elements of social choice theory, applied probability, statistics, and political philosophy.

The first three chapters of the thesis investigate the epistemic performance of selection models for representative democracy. This account focuses on truth-seeking scenarios, where a group is tasked with organizing to find, with high probability, an unknown true state of the world. While being a common mathematical formulation, this is a notoriously contested model for democracy [187], because it suffers from important objections: democratic governance is not reducible to right or wrong and the quality of a selection rule is not about selecting a representative body with a a priori high probability of voting correctly.

I agree with these objections and do not intend to argue that the epistemic framework model is right for democracy. Democratic decision-making is far more complex and intricate, almost always meddling with value-based considerations and intense deliberation or lobbying that cannot be captured by a right-or-wrong aggregative framework. When tasked to describe patterns under specific and defined hypotheses, the epistemic framework is simply intended as a tool to expand our understanding of a particular aspect of decision-making: most decisions do partially rely on objective dimensions that the epistemic approach intends to capture.

Without imposing a stylized and simplifying mathematical abstraction onto such a complex object as democratic governance, I am interested in counter-intuitive or empirically validated lessons we could gain from studying this dimension of decision-making taken in isolation, and their interplay with other key aspects of democratic decision-making.

Within the truth-seeking framework where rules are evaluated based on their likelihood to find the good answer, experts (defined mathematically as those more likely to find the correct outcome) have an advantage. Interestingly, however, we show that a small epistocracy is not optimal under common assumptions.

On the contrary, we could expect liquid democracy to lead to excessive concentration of power that would limit its likelihood of identifying with high probability, correct outcomes. In turn, we identify theoretical conditions under which transitive delegations allow sufficiently numerous and competent representatives to meet certain guarantees on the quality of the outcome. Next, we confront through a series of experiments in a context where the questions are designed to have a correct outcome. The theory and empirical parts allow us to test whether liquid democracy lives up to its promise to displace the necessary condition of collective intelligence from "knowing about an issue" to "knowing who knows about this issue."

Alone, the epistemic dimension of decision-making is not enough to reason about novel forms of representation. This is why I turned to political philosophy to reflect on procedural aspects of the novel selection rules. The breadth and depth of works that have explored questions of representation in political philosophy is both breath-taking and dating back to ancient philosophers, though it remains unknown to many of the mathematical communities that study democracy from a computational perspective. While, in the last chapter, I propose a unified framework to investigate procedural aspects of selection rules following my readings of political philosophy, I also attempted to contribute to the recent developments in the field regarding selection rules for representative democracy, some of which invoke social choice research. Connecting knowledge, rigor, concepts and intuitions from the different disciplines felt like the exploration of a kaleidoscope's different facets which, taken together, gave a more complete picture of the solution space.

In my humble journey trying to learn both the mathematical and philosophical languages concerned with representative aspects of democracy, I came to appreciate their incredible synergy. I hope that we will keep creating communication channels across fields that are animated by the same problems and that interdisciplinary approaches will contribute to shaping meaningful questions and providing apposite answers.

1.4 Glossary

Below are a few terms defined in the introduction that we will encounter frequently in the thesis.

Epistemic Dimension of Decision-Making: A lens through which examining decision-making processes that is concerned with the processes' ability to identify *correct* or *good* outcomes, where the metric for *correct* or *good* is independent from the procedure.

Epistemic Model: A mathematical abstraction of decision-making proposed by the Marquis de Condorcet in 1785, in which the decision is for a binary outcome and one of the outcomes is assumed to be objectively better. Each voter casts an independent vote parameterized by their expertise, a number between 0 and 1.

Epistocracy: A selection process in which the representatives making the decisions are selected based on their expertise.

Liquid Democracy: A selection process in which citizens either self-select to be representatives or delegate (potentially transitively) their vote to another citizens for area-specific decisions with the possibility dynamically delegate or recall one's vote at any time. In turn, representatives have a weight equal to the number of citizens they (transitively) represent, which scales their votes in the representative assembly.

Lottocracy: A selection process in which representatives are drawn randomly from the population.

Procedural Dimension of Decision-Making: A lens through which examining decision-making processes that is concerned with the procedures' intrinsic values.

Proxy Democracy: A selection process in which citizens either self-select to be representatives or flexibly nominate self-selected citizen(s) through frequent nomination processes. In turn, representatives have a weight equal to the number of citizens they represent, which scales their votes in the representative assembly.

Chapter 2

How Many Representatives Do We Need? The Optimal Size of an Epistemic Congress

However small the Republic may be, the Representatives must be raised to a certain number, in order to guard against the cabals of a few; and however large it may be, they must be divided to certain number, in order to guard against the confusion of a multitude.

– James Madison[1]

Abstract

Aggregating opinions of a collection of agents is a question of interest to a broad array of researchers, ranging from economists to statisticians, computer scientists, philosophers, and political scientists designing democratic institutions. This work investigates the optimal

[1]In Federalist Paper No. 10 [112].

number of agents needed to decide on a binary issue under majority rule. We take an *epistemic* view where the issue at hand has a ground truth "correct" outcome and each one of n voters votes correctly with a fixed probability, known as their competence level or competence. These competencies come from a fixed distribution $\mathcal{D}$. Observing the competencies, we must choose a specific group that will represent the population. Finally, voters sample a decision (either correct or not), and the group is correct as long as more than half the chosen representatives voted correctly. Assuming that we can identify the best experts, i.e., those with the highest competence, to form an *epistemic congress* we find that the optimal congress size should be linear in the population size. This result is striking because it holds even when allowing the top representatives to become arbitrarily accurate, choosing the correct outcome with probabilities approaching 1. We then analyze real-world data, observing that the actual sizes of representative bodies are much smaller than the optimal ones our theoretical results suggest. We conclude by examining under what conditions congresses of sub-optimal sizes would still outperform direct democracy, in which all voters vote. We find that a small congress would beat direct democracy if the rate at which the societal bias towards the ground truth decreases with the population size fast enough, and we quantify the speed needed for constant and polynomial congress sizes.

2.1 Introduction

Modern governments often take the form of a representative democracy, that is, a college of chosen representatives form a congress to make decisions on behalf of the citizenry. Clearly, the performance of the congress depends on the number of representatives, and this optimal number of representatives has been subject to great debate.[2] In the Federalist Paper No. 56, Madison argued that there should be *a representative for every thirty thousand inhabitants*.

[2]See activists at https://thirty-thousand.org who advocate for enlarging the congress.

In response, every ten years between 1785 and 1913, the American congress was enlarged–in aggregate from 65 to 435–adapting to evolving state populations [220]. However, since 1913, this number of representatives has remained constant, bringing the current number of inhabitants per representative well below Madison's objective.

2.1.1 Problem Statement

Quantitative research rationalizing the optimal congress size dates back to at least the 1970s. Taagepera [221] concluded that the number of representatives should be the cube root of the population size. These findings are regarded as seminal [123] and have influenced political decisions and referendums, such as the 2020 Italian referendum to reduce the size of both chambers from 945 to 600 parliamentary [65, 167].

Yet, recent work using machinery from physics and economics revisited these claims and showed that the optimal number should be larger, at least proportional to the square root of the population size [e.g., 12, 167]. In particular, Magdon-Ismail and Xia [157] explored an *epistemic* setup that groups voters into pods of size L, and each pod selects one representative. The authors find that the congress size ought to be linear under this model when voting is cost-less.

Note that our setup for decision-making in a congress on binary issues, applied here in a democratic context, resembles an ensemble of classifiers in machine learning: classifiers are "voters" who predict a binary outcome, and they collectively decide, through a majority rule, on the decision's outcome [157]. To obtain a good ensemble of classifiers, one can measure the accuracy of all classifiers and keep only the most accurate ones. Designers have used these ideas to reduce uncertainty in decisions and increase the classifiers' performance by combining their predictions [191, 208, 239].

We can now reformulate our research question in these terms: how many agents should we include to maximize the accuracy of the decision?

2.1.2 Contributions

Our contributions span multiple directions: through novel proofs techniques, we strengthen the pessimistic results of Magdon-Ismail and Xia [157] for congress under the epistemic approach, finding that even with the ability to identify the most accurate agents to form a congress, the optimal number of representatives remains linear in the size of the population. However, we find that all is not lost for congresses of a more practical nature: we follow this up with comparisons of different sizes and identify conditions for smaller congresses to be more accurate than when the entire society votes.

In the epistemic setting, voters decide on a binary issue and aim at differentiating between the ground truth correct choice, the value 1, and its alternative, 0. Each voter has a competence level in $[0, 1]$ representing the probability they vote correctly. Further, the competence levels of the population are drawn according to some distribution $\mathcal{D}$. We take the idealized view that, given a target size k, we can identify the k most competent voters in society (that is, the first k order statistics from the competence levels $p_1, \ldots, p_n$ where each $p_i \sim \mathcal{D}$) to form the congress. These members then vote on the issue, and the outcome is that of the majority. We first observe that, if voters' competence levels are the expected values of the order statistics from uniform distribution $\mathcal{U}(0, 1)$, the optimal size of congress is between $(3 - 2\sqrt{2})n$ and $\frac{n}{2}$. For arbitrary distributions where the maximum competence level is bounded away from 1 and the inverse cumulative distribution function is Lipschitz continuous, the optimal size is $\Theta(n)$ with more refined bounds based on the distribution.

The assumption that we can identify the k best agents is purposefully idealistic; we give the congress its best shot at favoring smaller sizes by granting it an unrealistically powerful selection procedure. This assumption strengthens our claim: the optimal congress size is linear with the population size *even* under the generous assumption that only the most competent agents represent. Second, our first analysis of the uniform distribution in $[0, 1]$ allows the best agents' competence to converge towards 1 as the population size increases, again providing an unrealistic assumption favoring small congress sizes. Under this extra generous assumption, we prove that the optimal congress size remains, strikingly, linear with the population size — later, we generalize this to more realistic distributions.

We then turn to study real-world data on the sizes of countries' representative bodies. Here, we notice that congresses in the real world are on the order of the cube root of the population size, much smaller than the optimal (linear) size our theoretical results suggest. We then seek to understand when real-world congress sizes can be deemed effective: we identify conditions on the distribution of competence level under which a smaller congress outperforms the majority. If the population is unbiased or biased towards 0, a congress composed of experts with expertise levels above 0.5 trivially outperforms the majority. We further find that, for a population whose average level of competence is biased above 0.5, a relatively small congress can still be better than the majority as long as the bias is small enough. We characterize this threshold for both single-agent and n^r-sized congresses.

2.1.3 Related Work

The use of an epistemic approach, relying on voting to aggregate objective opinions, is well studied in computational social choice [35]. One particularly significant result is the Condorcet Jury Theorem [64, 106], which shows that in the limit, a majority vote by an increasing number of independent voters biased towards the correct outcome will be correct with probability approaching 1. Note that this epistemic setup models legislative decisions or referendums in which one choice is inherently more desirable for society — yet, this *correct* outcome is not known a priori, and agents are trying to uncover it. Subsequent work has studied extensions of the Condorcet Jury Theorem in instances where the voters are inhomogeneous, dependent, or strategic, as summarized in a survey paper by Nitzan and Paroush [181].

The first work about the optimal size of parliaments focused on maximizing parliament's efficiency [221]. For them, maximizing efficiency was equivalent to minimizing the communication time spent on discussions with constituents — the authors ultimately stated that the average time spent talking to the constituents per congress-members should be equal to the time spent talking to the other congress-members. Hence, Taagepera [221] argued that the optimal congress size should follow a "cube root law". Margaritondo [167] revisited this work and found a flaw in the original proof, arguing that the optimal size under this model should instead be $\Theta(\sqrt{n})$. Empirical papers [13, 221] focused on finding the optimal number of representatives have used country data to back up the "square root law" result. Jacobs and Otjes [123], on the other hand, investigate potential causal effects of different congress sizes.

The work of Auriol and Gary-Bobo [12] also aims to derive the optimal number of representatives for a society. However, their model lies in stark contrast to the epistemic one: they assume that voters have preference-based utilities, with an uninformative prior, and the representatives are chosen uniformly at random from society. They too reach the conclusion that the optimal size of congress is proportional to the square root of the population size. Zhao and Peng [245] look at the optimal number of representatives in a social network. They consider a set of nodes representative if together they can reach all other nodes in at most m steps (where $m = \Theta(\log n)$ is an exogenous threshold). The goal is to find the minimum size of such a set. Under a certain class of realistic social networks, they find that the minimum should be proportional to n^γ for some $\frac{1}{3} \leq \gamma \leq \frac{5}{9}$.

Finally, we build upon the work of Magdon-Ismail and Xia [157]. There, the authors consider a model for representative democracy where agents are grouped into K groups of sizes L and each chooses one representative per group. Importantly, the competencies are drawn from a distribution $\mathcal{D}$ only after the agents are grouped. The authors then derive the group size that maximizes the probability that the representatives make the correct decision. They show the optimal group size is constant, so the optimal number of representatives (which is, in the simplest setup, the population size divided by the number of groups) should then be linear in the population size. The fact that the level of competence is drawn after grouping people imposes a trade-off between how accurate the representatives will be and how many representatives (n/L) there are. Indeed, the best agent in each group has a competence level that is the top order statistic of the distribution with L draws. For instance, the top level of competence from a uniform distribution is in expectation $1 - \frac{1}{L+1}$, which gets large only if the number of groups n/L is small. Importantly, the trade-off implied by the model favors large congresses. Yet, one could wonder whether the optimal congress size remains linear if one breaks with this implicit trade-off allowing the highest competencies to become arbitrarily large. This is precisely the gap we fill.

2.2 Model

Let n be the number of voters in the society. Following the epistemic approach, voters need to choose between two options, 0 and 1, where 1 is assumed to be the ground truth. Each voter i is endowed with a level of expertise (or competence) $p_i \in [0, 1]$, which is the probability that she votes "correctly" (i.e., votes for option 1). Depending on the instance, we will sometimes assume that the p_is are sampled from some distribution $\mathcal{D}$ whose support is contained in $[0, 1]$ and other times assume the p_is are deterministic (perhaps also depending on n which will always be clear from context).

Given $p_1, \ldots, p_n$, we sort voters by decreasing competence level, denoted by $p_{(1)} \geq \cdots \geq p_{(n)}$, where $p_{(i)}$ is the competence level of the i^{th} best voter. (Note that, for notational convenience, this is the reverse of normal order statistics.) Let $X_{(1)}, \ldots, X_{(n)}$ be Bernoulli random variables denoting their votes, with $X_{(i)} = 1$ meaning a correct vote for the i^{th} best voter and 0 otherwise; the $X_{(i)}$s are conditionally independent given $p_{(i)}$s, and $\Pr[X_{(i)} = 1 \mid p_{(i)}] = p_{(i)}$.

A congress of size k is composed of the k best voters in society and makes a correct decision when a strict majority is correct, $\sum_{i=1}^{k} X_{(i)} > k/2$.[3] One may envision other rules to select the congress members, for example, the group representatives analyzed by Magdon-Ismail and Xia [157]. Here we take the best k voters, and this can be seen as a best-case scenario for accuracy. Strikingly, as we will show, even under this strong assumption, the optimal number of representatives is already very large, which suggests that the optimal number would even be larger in more realistic scenarios.

Although the assumption that we can sample the k best experts is unrealistic if one thinks at the democratic context, we could argue that finding the k most accurate classifiers in an ensemble is indeed realistic. In any case, this sampling method favors small congresses while allowing the sampled congress to reach the maximal probability of making the correct decision. Our conclusions hence read that *despite* the generous assumptions, a large number of voters are needed to maximize a collective's chance to make a correct decision.

2.3 Optimal Size of an Epistocracy

In this section, we prove theoretical bounds on the optimal size of congress for several natural distributions. We begin by formally stating our problem.

For fixed voter competencies $p_{(1)} \geq \cdots \geq p_{(n)}$, we define $K^\star$ to be the optimal size of congress, the size k that maximizes the probability that the representatives make a correct

[3] A strict rather than weak majority here corresponds to tie-breaking in favor of the incorrect outcome. Tie-breaking in the other direction would not asymptotically change our results.

decision (for convenience breaking ties in favor of an arbitrary odd k[4]). Formally,

$$K^\star \in \arg\max_{1\le k\le n} \left\{ \Pr\left[\sum_{i=1}^{k} X_{(i)} > \frac{k}{2} \;\middle|\; X_{(i)} \sim \mathrm{Bern}(p_{(i)}) \right] \right\}.$$

We note that since $K^\star$ is a function of the voter competencies, if these competencies are random samples, then $K^\star$ is a random variable. However, we sometimes assume for tractability that the competencies match their expectation, that is, $p_{(i)}$ is exactly equal to the expectation of the $(n+1-i)$'th order statistic of n draws from $\mathcal{D}$. In this case, $K^\star$ is a deterministic value for each n.

For fixed voter competencies $p_{(1)} \ge \cdots \ge p_{(n)}$, let $\mathcal{E}_k^j$ be the event that exactly j of the top experts out of k are correct. Our characterization of the optimal size $K^\star$ relies on the following key lemma.

Lemma 1. *For fixed competencies $p_{(1)} \ge \cdots \ge p_{(n)}$, for all odd $k \le n$ with $k = 2\ell+1$,*

- *If $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} < \frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})}$, then $K^\star \ne k$.*
- *If $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} > \frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})}$, then $K^\star \ne k+2$.*

The proof of the lemma involves comparing a congress of some specific size k to one of size $k+2$ (recall that we chose $K^\star$ to be odd, so we may as well restrict ourselves to odd k). Clearly, if the top $k+2$ experts have a higher chance of being correct than k, then k cannot be optimal (and vice-versa). Importantly, this gives us a sufficient condition to rule out certain values of k. For example, if we know that for all $k < c$ the first condition of the lemma holds, then that implies $K^\star \ge c$.

Proof of Lemma 1. For any $k \le n$, let $q_k = \sum_{j=\lfloor k/2\rfloor+1}^{k} \Pr[\mathcal{E}_k^j]$ be the probability that a congress of size k will be correct. We have that $K^\star \in \arg\max_{k\le n} q_k$. Fix $p_{(1)} \ge \cdots \ge p_{(n)}$

[4]Note that there must always be an optimal k that is odd, as for any even k, due to our strict majority constraint, $k-1$ must have overall accuracy at least as high.

and a specific $k = 2\ell + 1$. We will show that $q_{k+2} > q_k$ (resp. $<$) is equivalent to $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} < \frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})}$ (resp. $>$). If $q_{k+2} > q_k$ (resp. $<$), then $K^\star \neq k$ (resp. $k+2$) as that would imply $K^\star$ is not optimal.

Let us now consider $q_{k+2} - q_k$. The only way the two new experts can change the outcome from incorrect to correct is when exactly ℓ of the top k experts were correct (so the majority of k were incorrect), and the two new experts are correct. Conversely, the only scenario in which a correct outcome becomes incorrect is when exactly $\ell + 1$ of the top k experts are correct while the two new experts are incorrect. Since $\mathcal{E}_k^j$ is the event that exactly j of the top k experts out of n are correct, we can formally write the above as

$$\begin{aligned} q_{k+2} - q_k = &\Pr[\mathcal{E}_k^{\ell}] \cdot p_{(k+1)}p_{(k+2)} \\ &- \Pr[\mathcal{E}_k^{\ell+1}] \cdot (1 - p_{(k+1)})(1 - p_{(k+2)}). \end{aligned}$$

Rearranging this yields the two equivalent inequalities previously stated. □

For a set of representatives $S \subseteq [k]$, let $w(S) = \prod_{i \in S} p_{(i)} \cdot \prod_{i \in [k] \setminus S}(1 - p_{(i)})$ be the probability that exactly those in S are correct (and those in $[k] \setminus S$ are incorrect). We then have the following.

Lemma 2. *For each* $\mathcal{E}_k^j$*,*

$$\Pr[\mathcal{E}_k^j] = \frac{1}{k-j} \sum_{\substack{S \subseteq [k] \\ |S| = j+1}} w(S) \sum_{i \in S} \frac{1 - p_{(i)}}{p_{(i)}}.$$

Proof. By the definition of $\mathcal{E}_k^j$, $\Pr[\mathcal{E}_k^j] = \sum_{\substack{S \subseteq [k] \\ |S| = j}} w(S)$. We then note that

$$\sum_{\substack{S \subseteq [k] \\ |S| = j}} w(S) = \frac{1}{k-j} \sum_{\substack{S \subseteq [k] \\ |S| = j+1}} \sum_{i \in S} w(S \setminus \{i\})$$

because when we count the sets S of size j by first selecting a set of size $j+1$ and then removing one of its $j+1$ elements, each set of size j is counted exactly $k-j$ times. Therefore,

$$\begin{aligned}\Pr[\mathcal{E}_k^j] &= \frac{1}{k-j} \sum_{\substack{S \subseteq [k] \\ |S|=j+1}} \sum_{i \in S} w(S \setminus \{i\}) \\ &= \frac{1}{k-j} \sum_{\substack{S \subseteq [k] \\ |S|=j+1}} w(S) \sum_{i \in S} \frac{1-p_{(i)}}{p_{(i)}},\end{aligned}$$

as needed. □

Armed with these lemmas, we can now move to proving bounds on the optimal congress size.

2.3.1 Standard Uniform Distribution

First, we focus on the case where competence levels are drawn from uniform distribution $\mathcal{U}(0,1)$. For tractability, as discussed in the problem statement, we assume that the competence levels are exactly equal to their expectation, i.e., $p_{(i)} = \frac{n+1-i}{n+1}$ (see e.g., Ma [153]). In this case, the competence levels of the top experts approach 1 asymptotically. Note that this is an unrealistic assumption that, again, acts in favor of small congresses. Including it emphasizes the striking nature of the result: Even with top experts becoming arbitrarily accurate and the ability to identify the most accurate members of society, the optimal size of congress remains a constant fraction of the population.

Theorem 1. *Suppose* $p_{(i)} = \frac{n+1-i}{n+1}$. *Then,* $(3-2\sqrt{2}) \cdot n - O(1) \leq K^\star \leq \frac{1}{2} \cdot n + O(1)$.

Proof. Recall that we can focus only on odd k. Fix some odd $k \leq n$ where $k = 2\ell + 1$ for some non-negative integer ℓ. Our goal will be to compare $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]}$ and $\frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})} = \frac{(n-k)(n-k-1)}{(k+1)(k+2)}$ in order to apply Lemma 1.

By Lemma 2 with $j = \ell$ and using the fact that $k - \ell = \ell + 1$,

$$\Pr[\mathcal{E}_k^\ell] = \frac{1}{\ell + 1} \sum_{\substack{S \subseteq [k] \\ |S| = \ell + 1}} w(S) \sum_{i \in S} \frac{i}{n + 1 - i}. \tag{2.1}$$

We begin with the lower bound. Let us consider the inner sum of Equation (2.1). We have that for all S,

$$\sum_{i \in S} \frac{i}{n + 1 - i} \geq \frac{1}{n} \sum_{i \in S} i \geq \frac{1}{n} \sum_{i=1}^{\ell+1} i = \frac{(\ell + 1)(\ell + 2)}{2n}$$

where the first inequality holds because $i \geq 1$ for all i and the second inequality holds because $|S| = \ell + 1$ and $S \subseteq [k]$ hence the minimum it could sum to is that of the smallest $\ell + 1$ positive integers. As this bound is independent of S, we can pull it out of the outer sum to yield

$$\Pr[\mathcal{E}_k^\ell] \geq \frac{\ell + 2}{2n} \sum_{\substack{S \subseteq [k] \\ |S| = \ell + 1}} w(S) = \frac{k + 3}{4n} \cdot \Pr[\mathcal{E}_k^{\ell+1}]$$

where the last inequality holds because $\ell + 2 = \frac{k-1}{2} + 2 = \frac{k+3}{2}$. This allows us to write $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^\ell]} \leq \frac{4n}{k+3}$, so in order to invoke the first item of Lemma 1 to show a specific value of k is not optimal, we need a sufficient condition for k to guarantee

$$\frac{4n}{k + 3} < \frac{(n - k)(n - k - 1)}{(k + 1)(k + 2)}. \tag{2.2}$$

Note that Equation (2.2) is implied by $4n < \frac{(n-k-1)^2}{k+1}$ which we can rearrange to $(k + 1)^2 - 6n(k + 1) + n^2 > 0$. The left hand side of the inequality is a quadratic in $(k + 1)$ with roots at $(3 \pm 2\sqrt{2}) \cdot n$. Since the squared term is positive and hence the quadratic is only non-positive between the two roots, as long as $(k + 1) < (3 - 2\sqrt{2}) \cdot n$, the inequality holds. Along with the first item of Lemma 1, this implies the desired $(3-2\sqrt{2})\cdot n - O(1)$ lower bound.

Next, we will show the upper bound. In the inner summand of Equation (2.1), $i \in [k]$ so $i \le k$, and hence $\frac{i}{n+1-i} \le \frac{k}{n+1-k}$. This yields

$$\begin{aligned}
\Pr[\mathcal{E}_k^\ell] &\le \frac{1}{\ell+1} \sum_{\substack{S\subseteq[k]\\ |S|=\ell+1}} w(S) \sum_{i\in S} \frac{k}{n+1-k} \\
&\le \frac{1}{\ell+1} \sum_{\substack{S\subseteq[k]\\ |S|=\ell+1}} w(S)\cdot|S|\cdot\frac{k}{n+1-k} \\
&= \frac{k}{n+1-k} \sum_{\substack{S\subseteq[k]\\ |S|=\ell+1}} w(S) = \frac{k}{n+1-k}\Pr[\mathcal{E}_k^{\ell+1}].
\end{aligned}$$

Here, we get that $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} \ge \frac{k}{n+1-k}$. As with the lower bound, to invoke the second item of Lemma 1, we need a sufficient condition for

$$\frac{k}{n+1-k} > \frac{(n-k)(n-k-1)}{(k+1)(k+2)}. \tag{2.3}$$

Equation (2.3) is equivalent to

$$k(k+1)(k+2) > (n-k-1)(n-k)(n-k+1).$$

As both sides are the product of three consecutive integers, this will be true as long as $n-k-1 < k$, or equivalently $k+2 > \frac{n}{2}+\frac{3}{2}$. Applying Lemma 1 yields the desired upper bound. □

Hence, we have proved that for competencies equal to the expectation of $\mathcal{U}[0,1]$ order statistics, a constant fraction of the total population is necessary to maximize the probability the representatives make the correct decision. We conjecture that $K^\star$ is in fact close to $n/4$ in this set up (see simulations in Figure 2.1).

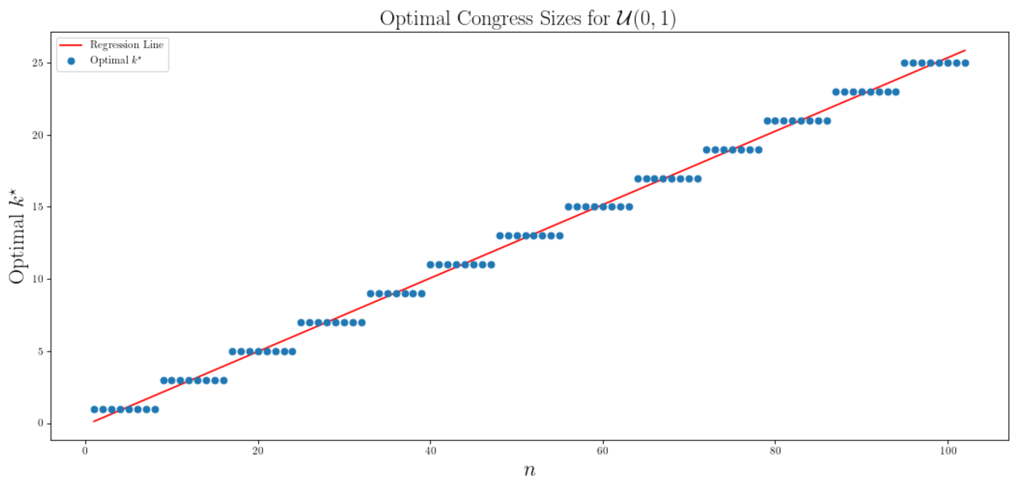


Figure 2.1: Optimal value of k for $\mathcal{U}(0,1)$ competence levels following their expectation. The line of best fit is very close to $n/4$.

2.3.2 Distributions Bounded Away From 1

Next, we consider a broad class of distributions that do not allow for arbitrarily accurate experts. Unlike in the previous section, we do not fix $p_{(i)}$ to be their expectation; instead, they are random draws from $\mathcal{D}$. Under relatively mild conditions, we show that the optimal size $K^\star$ grows linearly in the population size with high probability.

Theorem 2. *Let $\mathcal{D}$ be any continuous distribution supported by $[L, H]$ with cumulative distribution function $F(\cdot)$. If $0 < L < \frac{1}{2} < H < 1$, and $F^{-1}(\cdot)$ is M-Lipschitz continuous with $0 < M < \infty$,*[5] *then, with probability at least $1 - 4e^{-2n\varepsilon^2}$ the competency draws will yield an optimal $K^\star$ such that*

$$c_H n - O(1) \leq K^\star \leq c_L n + O(1)$$

for all n and $\varepsilon > 0$, where $c_H = 1 - F\left(\frac{1}{1+\sqrt{\frac{1-H}{H}}} + M\varepsilon\right)$ and $c_L = 1 - F\left(\frac{1}{1+\sqrt{\frac{1-L}{L}}} - M\varepsilon\right)$.

We remark that $L \geq 0$ is sufficient for the lower bound $c_H n - O(1) \leq K^\star$ to hold and

[5] This condition is satisfied when the PDF of $\mathcal{D}$ is lower bounded by $1/M$, which is satisfied by, e.g., uniform, normal, and beta distributions truncated to $[L, H]$.

vice-versa, $H \leq 1$ is sufficient for the upper bound to hold. Both of these bounds individually hold with probability at least $1 - 2e^{-2n\varepsilon^2}$.

To prove Theorem 2, we will make use of the following well-known concentration inequality.

Lemma 3 (Dvoretzky–Kiefer–Wolfowitz inequality, see e.g., 169)**.** *Let $p_{(1)} \geq \cdots \geq p_{(n)}$ be n sorted i.i.d. draws from $\mathcal{D}$. For every $\varepsilon > 0$,*

$$\Pr\left[\forall i \in [n], \left|F(p_{(i)}) - \frac{n-i}{n}\right| \leq \varepsilon\right] \geq 1 - 2e^{-2n\varepsilon^2}.$$

Lemma 3 implies that, with probability at least $1 - 2e^{-2n\varepsilon^2}$, for every $i \in [n]$,

$$\left|F(p_{(i)}) - \frac{n-i}{n}\right| \leq \varepsilon.$$

Since F^{-1} is assumed to be M-Lipschitz continuous,

$$\left|p_{(i)} - F^{-1}\left(\frac{n-i}{n}\right)\right| \leq M\varepsilon. \tag{2.4}$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. We will show that both the lower bound $c_H n - O(1) \leq K^\star$ and the upper bound $K^\star \leq c_L n + O(1)$ each occur with probability at least $1 - 2e^{-2n\varepsilon^2}$ which, by a union bound, proves the desired probability. As previously mentioned, we will only prove the lower bound here. Fix arbitrary odd k and n with $k \leq n$ where $k = 2\ell + 1$ for some non-negative integer ℓ. We will give sufficient conditions as a function of n and k for which we can apply Lemma 1.

First, by Lemma 2 with $j = \ell$, $\Pr[\mathcal{E}_k^\ell] = \frac{1}{k-\ell} \sum_{\substack{S \subseteq [k] \\ |S| = \ell+1}} w(S) \sum_{i \in S} \frac{1-p_{(i)}}{p_{(i)}}$. Because the support of $\mathcal{D}$ is upper-bounded by H, $p_{(i)} \leq H$ for all i with probability one. So, $\sum_{i \in S} \frac{1-p_{(i)}}{p_{(i)}} \geq$

$(\ell+1)\frac{1-H}{H}$. Noting that $\ell+1 = \frac{k+1}{2} = k-\ell$ and $\Pr[\mathcal{E}_k^{\ell+1}] = \sum_{\substack{S\subseteq[k]\\|S|=\ell+1}} w(S)$, after rearranging we have $\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} \leq \frac{H}{1-H}$. Further, we note that $\frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})} \geq \frac{p_{(k+2)}^2}{(1-p_{(k+2)})^2}$.

Now, if we want to apply the first item of Lemma 1 to show some k is not optimal, it suffices to require that

$$\frac{p_{(k+2)}^2}{(1-p_{(k+2)})^2} > \frac{H}{1-H} \iff p_{(k+2)} > \frac{1}{1+\sqrt{\frac{1-H}{H}}}. \tag{2.5}$$

Relying on Equation (2.4), it holds that $p_{(k+2)} \geq F^{-1}\left(\frac{n-k-2}{n}\right) - M\varepsilon$. If we require

$$F^{-1}\left(\frac{n-k-2}{n}\right) - M\varepsilon > \frac{1}{1+\sqrt{\frac{1-H}{H}}}, \tag{2.6}$$

then Equation (2.5) is satisfied and hence so will the condition of Lemma 1, which implies that such k cannot be optimal. Equation (2.6) gives $\frac{k}{n} \leq 1 - F\left(\frac{1}{1+\sqrt{\frac{1-H}{H}}} + M\varepsilon\right) - \frac{2}{n}$, so

$$\frac{K^\star}{n} \geq 1 - F\left(\frac{1}{1+\sqrt{\frac{1-H}{H}}} + M\varepsilon\right) - \frac{2}{n}.$$

Multiplying by n yields the desired lower bound.

Symmetric to the lower bound, we have that

$$\frac{\Pr[\mathcal{E}_k^{\ell+1}]}{\Pr[\mathcal{E}_k^{\ell}]} \geq \frac{L}{1-L}.$$

Further,

$$\frac{p_{(k+1)}p_{(k+2)}}{(1-p_{(k+1)})(1-p_{(k+2)})} \leq \frac{p_{(k+1)}^2}{(1-p_{(k+1)})^2}.$$

Hence, to prove a certain value $k+2$ is not optimal using Lemma 1, it suffices that

$$\frac{p_{(k+1)}^2}{(1-p_{(k+1)})^2} < \frac{1-L}{L},$$

which is equivalent to

$$p_{(k+1)} < \frac{1}{1+\sqrt{\frac{L}{1-L}}} \tag{2.7}$$

Now, relying on Equation (2.4), it holds that

$$p_{(k+1)} \leq F^{-1}(\frac{n-k-1}{n}) + M\varepsilon.$$

If we require

$$F^{-1}(\frac{n-k-1}{n}) + M\varepsilon < \frac{1}{1+\sqrt{\frac{1-L}{L}}}, \tag{2.8}$$

then Equation (2.7) is satisfied and hence $p_n^{k+2} - p_n^k < 0$, which implies that such k cannot be optimal. Solving Equation (2.8) gives $\frac{k}{n} > 1 - F\left(\frac{1}{1+\sqrt{\frac{1-L}{L}}} - M\varepsilon\right) - \frac{1}{n}$.

Hence, as long as $\frac{k}{n} > 1 - F\left(\frac{1}{1+\sqrt{\frac{1-L}{L}}} - M\varepsilon\right) - \frac{1}{n}$, the condition of Lemma 1 will be satisfied. Multiplying through by n yields the desired upper bound.

$\square$

This proves that for competencies drawn from an arbitrary distribution whose support is bounded away from 1, a constant fraction of the total population is needed to maximize the probability that the representatives make the correct decision on behalf of the entire population.

We illustrate Theorem 2 by distribution $\mathcal{D} = \mathcal{U}(0.1, 0.9)$. Letting $\varepsilon = \sqrt{\frac{\log n}{2n}}$, one can check that $0.186n \leq K^\star \leq 0.813n$ with probability at least $1 - \frac{4}{n}$ for sufficiently large ns.

2.4 When Epistocracy Outperforms Democracy

Our theoretical results from the previous section suggest that the optimal size of a congress should be linear in the size of the population. However, this may not be feasible in many scenarios, and there are other desiderata one must consider in choosing an "optimal" size. Hence, we now turn to *comparing* how well different sizes of congresses perform in the epistemic model.

As a baseline, we will compare the accuracy of a congress to the accuracy of *direct democracy* in which all n members of society vote. This comparison is well-motivated by classic results such as the *Condorcet Jury Theorem* and extensions thereof, which show that the entire society will converge to the correct answer if and only if the competency distribution is *biased* toward the correct answer, that is, $\mathbb{E}_{p\sim\mathcal{D}}[p] > 1/2$. We aim to find bounds on *how* biased this distribution must be for congresses of different sizes to outperform the entire society.

We now state our problem formally. We will be interested in how the cutoff of the bias of the competency distribution varies with n; hence, we will allow the distribution $\mathcal{D}$ to depend on n by having a distribution $\mathcal{D}_n$ for each n. We use F_n and f_n to denote the CDF and PDF of $\mathcal{D}_n$ respectively. Let $\Gamma_n^{\boldsymbol{p}}(k)$ be the gain in probability of correctness by using a congress of size k instead of the entire population, given competence levels $\boldsymbol{p} = (p_{(1)}, \ldots, p_{(n)})$:

$$\begin{aligned}\Gamma_n^{\boldsymbol{p}}(k) = \Pr&\left[\sum_{i=1}^{k} X_{(i)} > \frac{k}{2} \;\middle|\; X_{(i)} \sim \mathrm{Bern}(p_{(i)})\right] \\ &- \Pr\left[\sum_{i=1}^{n} X_{(i)} > \frac{n}{2} \;\middle|\; X_{(i)} \sim \mathrm{Bern}(p_{(i)})\right].\end{aligned}$$

Similar to the definition of $K^\star$, $\Gamma_n^{\boldsymbol{p}}(k)$ is a random variable whose randomness comes from the random draws of $p_i \sim \mathcal{D}_n$. We aim at identifying, for certain values of k, for what kinds of distributions $\mathcal{D}_n$ we have $\Gamma_n^{\boldsymbol{p}}(k) > 0$ with high probability as n grows large.

2.4.1 Dictatorship

First, we consider an extreme case: when can a single voter outperform the entire society? In particular, we identify conditions under which $\Gamma_n^{\boldsymbol{p}}(1) > 0$ or $\Gamma_n^{\boldsymbol{p}}(1) < 0$. We show that if the distributions $\mathcal{D}_n$ put high enough probability mass on competence levels near 1 and its mean $\mathbb{E}_{\mathcal{D}_n}[p]$ is not much larger than 1/2, then $\Gamma_n^{\boldsymbol{p}}(1) > 0$ with high probability as n grows large, and $\Gamma_n^{\boldsymbol{p}}(1) < 0$ on the contrary. The probability mass conditions are satisfied by many natural classes of distributions; we give several examples (e.g., uniform and beta distributions) in Revel et al. [204].

Theorem 3. *Let $k = 1$.*

- *Suppose $\mathbb{E}_{\mathcal{D}_n}[p] \leq \frac{1}{2} + a\sqrt{\frac{\log n}{n}}$ and $f_n(x) \geq \underline{C}(1-x)^{\underline{\beta}-1}$ for $x \in [1-\underline{\delta}, 1]$ for some constants $a, \underline{C}, \underline{\beta}, \underline{\delta} > 0$. If $a < \sqrt{\mathbb{E}_{\mathcal{D}_n}[p(1-p)] \cdot \min\{1, 2/\underline{\beta}\}}$, then, with probability at least $1 - n^{-\Omega(1)}$, $\Gamma_n^{\boldsymbol{p}}(1) > 0$.*

- *Suppose $\mathbb{E}_{\mathcal{D}_n}[p] \geq \frac{1}{2} + a\sqrt{\frac{\log n}{n}}$ and $f_n(x) \leq \overline{C}$ for $x \in [1-\overline{\delta}, 1]$ for some constants $a, \overline{C}, \overline{\delta} > 0$. If $a > \frac{1}{\sqrt{2}}$, then with probability at least $1 - n^{-\Omega(1)}$, $\Gamma_n^{\boldsymbol{p}}(1) < 0$.*

We sketch a proof of the theorem first. When $\mathbb{E}_{\mathcal{D}_n}[p] = \frac{1}{2} + O(\sqrt{\frac{\log n}{n}})$, by Hoeffding's inequality, the entire population makes a correct decision with probability $1 - O(n^{-c_1})$ for some constant c_1, while by our assumption on $\mathcal{D}_n$ the top expert is correct with probability $p_{(1)} = 1 - O(n^{-c_2})$. We identify conditions on $\mathcal{D}_n$ for which $c_1 < c_2$ or $c_1 > c_2$.

For the proof, we will need the following lemmas, the first and third are well-known concentration inequalities, and the second is a standard bound on the standard normal CDF, which we prove here for completeness.

Lemma 4 (Berry-Esseen Theorem)**.** *Let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = \sigma_i^2 > 0$, and $\mathbb{E}[|X_i|^3] = \rho_i < \infty$. Let F_{S_n} be the CDF of $S_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}}$ and Φ be the CDF of the standard normal distribution. Then, there exists an absolute constant C_1 such that*

$$|F_{S_n}(x) - \Phi(x)| \le \frac{C_1}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \max_{1 \le i \le n} \frac{\rho_i}{\sigma_i^2}, \quad \forall x \in \mathbb{R}$$

Lemma 5 (Bounds on standard normal CDF)**.** *Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \mathrm{d}t$ be the CDF of the standard normal distribution. Then we have for any $x > 0$,*

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2} \le \Phi(-x) = 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

Proof. The right inequality is because

$$\begin{aligned}
1 - \Phi(x) &= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t \\
&\le \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \frac{t}{x} e^{-\frac{t^2}{2}} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \left(-e^{-\frac{t^2}{2}} \right) \Bigg|_{t=x}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}.
\end{aligned}$$

The left inequality is because

$$\begin{aligned}
1 - \Phi(x) &= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t \\
&\ge \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \frac{(t^2+1)^2 - 2}{(t^2+1)^2} e^{-\frac{t^2}{2}} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \left(-\frac{t}{t^2+1} e^{-\frac{t^2}{2}} \right) \Bigg|_{t=x}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-\frac{x^2}{2}}.
\end{aligned}$$

$\square$

Lemma 6 (Hoeffding's Inequality)**.** *Let $X_1, \dots, X_n$ be independent random variables bounded by $0 \le X_i \le 1$. Then*

$$\Pr\left[\sum_{i=1}^n X_i \ge \mathbb{E}[\sum_{i=1}^n X_i] + t\right] \le \exp(-\frac{2t^2}{n}),$$

for any $t > 0$. The other direction also holds:

$$\Pr\left[\sum_{i=1}^n X_i \le \mathbb{E}[\sum_{i=1}^n X_i] - t\right] \le \exp(-\frac{2t^2}{n}).$$

Now we prove Theorem 3.

Proof of Theorem 3. To simplify notations we write $\Pr\left[\sum_{i=1}^k X_{(i)} > \frac{k}{2} \mid X_{(i)} \sim \mathrm{Bern}(p_{(i)})\right]$ as $\Pr\left[\sum_{i=1}^k X_{(i)} > \frac{k}{2} \mid \boldsymbol{p}\right]$. Recalling the definition of $\Gamma_n^{\boldsymbol{p}}(k)$, since $\Pr\left[\sum_{i=1}^k X_{(i)} > \frac{k}{2} \mid \boldsymbol{p}\right] = 1 - \Pr\left[\sum_{i=1}^k X_{(i)} \le \frac{k}{2} \mid \boldsymbol{p}\right]$, $\Gamma_n^{\boldsymbol{p}}(k)$ can be equivalently written as

$$\Gamma_n^{\boldsymbol{p}}(k) = \Pr\left[\sum_{i=1}^n X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[\sum_{i=1}^k X_{(i)} \le \frac{k}{2} \;\middle|\; \boldsymbol{p}\right].$$

To show either $\Gamma_n^{\boldsymbol{p}}(k) > 0$ or $\Gamma_n^{\boldsymbol{p}}(k) < 0$, we will compare $\Pr\left[\sum_{i=1}^n X_{(i)} \le \frac{n}{2} \mid \boldsymbol{p}\right]$ with $\Pr\left[\sum_{i=1}^k X_{(i)} \le \frac{k}{2} \mid \boldsymbol{p}\right]$. To do this, we prove the following lemmas:

Lemma 7. *Suppose $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \le \frac{1}{2} + \varepsilon_n$ where $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some constant $a > 0$. Let $\varepsilon = b\sqrt{\frac{\log n}{n}}$ for some constant $b > 0$. Suppose $\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] > \varepsilon$. Let $c = \frac{a+b}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)]-\varepsilon}}$. Then we have: with probability at least $1 - 2n^{-2b^2}$ (over the random draw of $\boldsymbol{p} \sim \mathcal{D}_n$),*

$$\Pr\left[\sum_{i=1}^n X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] \ge \frac{1}{\sqrt{2\pi}} \cdot \frac{c\sqrt{\log n}}{(c^2\log n + 1)} \cdot \frac{1}{n^{c^2/2}} - \frac{C_1}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon}} \frac{1}{\sqrt{n}},$$

where C_1 is the constant in Berry-Esseen theorem (Lemma 4).

Proof. Given $\boldsymbol{p} = (p_{(1)}, \dots, p_{(n)})$, each $X_{(i)}$ independently follows $\mathrm{Bern}(p_{(i)})$. We use Berry-

Esseen theorem (Lemma 4) for $Y_i = X_{(i)} - p_{(i)}$, $i = 1, \dots, n$. Noticing that $\mathbb{E}[Y_i] = 0$, $\sigma_i^2 = \mathbb{E}[Y_i^2] = p_{(i)}(1 - p_{(i)})$, and $\rho_i = \mathbb{E}[|Y_i|^3] = p_{(i)}(1 - p_{(i)})[(1 - p_{(i)})^2 + p_{(i)}^2 \leq \sigma_i^2$, the theorem implies

$$\left| \Pr\left[\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n \sigma_i^2} \leq x\right] - \Phi(x) \right| \leq \frac{C_1}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \max_{1 \leq i \leq n} \frac{\rho_i}{\sigma_i^2} \leq \frac{C_1}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \stackrel{\text{def}}{=} \Delta_1$$

for any $x \in \mathbb{R}$, where $\Phi(x)$ is CDF of the standard normal distribution. Therefore,

$$\begin{aligned} \Pr\left[\sum_{i=1}^n X_{(i)} \leq \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] &= \Pr\left[\sum_{i=1}^n X_{(i)} - \sum_{i=1}^n p_{(i)} \leq \frac{n}{2} - \sum_{i=1}^n p_{(i)} \;\middle|\; \boldsymbol{p}\right] \\ &= \Pr\left[\frac{\sum_{i=1}^n Y_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \leq \frac{\frac{n}{2} - \sum_{i=1}^n p_{(i)}}{\sqrt{\sum_{i=1}^n \sigma_i^2}}\right] \\ &\geq \Phi\left(\frac{\frac{n}{2} - \sum_{i=1}^n p_{(i)}}{\sqrt{\sum_{i=1}^n \sigma_i^2}}\right) - \Delta_1 \end{aligned} \tag{2.9}$$

We note that $\sum_{i=1}^n p_{(i)} = \sum_{i=1}^n p_i$ is the sum of n i.i.d. draws from distribution $\mathcal{D}_n$, with mean $\mathbb{E}[\sum_{i=1}^n p_i] = n\mathbb{E}_{p\sim\mathcal{D}_n}[p]$. By Hoeffding's inequality (Lemma 6), letting $t = n\varepsilon$, we have

$$\sum_{i=1}^n p_i \leq n\mathbb{E}_{p\sim\mathcal{D}_n}[p] + n\varepsilon \tag{2.10}$$

with probability at least $1 - \exp(-\frac{2(n\varepsilon)^2}{n}) = 1 - n^{-2b^2}$. Also, $\sum_{i=1}^n \sigma_i^2 = \sum_{i=1}^n p_i(1 - p_i)$ is the sum of n i.i.d. draws from a distribution, with mean $n\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)]$, so

$$\sum_{i=1}^n \sigma_i^2 \geq n\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - n\varepsilon \tag{2.11}$$

also with probability at least $1 - \exp(-\frac{2(n\varepsilon)^2}{n}) = 1 - n^{2b^2}$. By a union bound, we have with probability at least $1 - 2n^{-2b^2}$, both Equation (2.10) and Equation (2.11) hold, which imply

$$
\begin{aligned}
\Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] &\ge \Phi\left(\frac{\frac{n}{2} - \sum_{i=1}^{n} p_{(i)}}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}}\right) - \Delta_1 \\
&\ge \Phi\left(\frac{\frac{n}{2} - n\mathbb{E}_{p\sim\mathcal{D}_n}[p] - n\varepsilon}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}}\right) - \Delta_1 \\
&\ge \Phi\left(\frac{-n\varepsilon_n - n\varepsilon}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}}\right) - \Delta_1 \\
&\ge \Phi\left(\frac{-n\varepsilon_n - n\varepsilon}{\sqrt{n\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - n\varepsilon}}\right) - \Delta_1 \\
&= \Phi\left(-\sqrt{n}\frac{\varepsilon_n + \varepsilon}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon}}\right) - \Delta_1 \\
&= \Phi\left(-\sqrt{n}\frac{(a+b)\sqrt{\frac{\log n}{n}}}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon}}\right) - \Delta_1 \\
&= \Phi\left(-c\sqrt{\log n}\right) - \Delta_1.
\end{aligned}
$$

Using Lemma 5 with $x = c\sqrt{\log n}$, we get

$$
\Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] \ge \Phi\left(-c\sqrt{\log n}\right) - \Delta_1 \ge \frac{1}{\sqrt{2\pi}}\frac{c\sqrt{\log n}}{c^2\log n + 1}\frac{1}{n^{c^2/2}} - \Delta_1,
$$

concluding the proof. $\square$

Lemma 8. *Suppose $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \ge \frac{1}{2} + \varepsilon_n$ where $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some constant $a > 0$. Let b be a constant with $0 < b < a$. Then we have: with probability at least $1 - n^{-2b^2}$ (over the random draw of $\boldsymbol{p} \sim \mathcal{D}_n$),*

$$
\Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] \le \frac{1}{n^{2(a-b)^2}}.
$$

Proof. We note that $\sum_{i=1}^{n} p_{(i)} = \sum_{i=1}^{n} p_i$ is the sum of n i.i.d. draws from distribution $\mathcal{D}_n$, with mean $\mathbb{E}[\sum_{i=1}^n p_i] = n\mathbb{E}_{p\sim\mathcal{D}_n}[p]$. Let $\varepsilon = b\sqrt{\frac{\log n}{n}} < \varepsilon_n$. By Hoeffding's inequality (Lemma 6), with probability at least $1 - \exp(-\frac{2(n\varepsilon)^2}{n}) = 1 - n^{-2b^2}$, it holds that

$$\sum_{i=1}^n p_i \geq n\mathbb{E}_{p\sim\mathcal{D}_n}[p] - n\varepsilon \geq \frac{n}{2} + n\varepsilon_n - n\varepsilon > \frac{n}{2}.$$

Assuming $\sum_{i=1}^n p_i \geq n\mathbb{E}_{p\sim\mathcal{D}_n}[p] - n\varepsilon$ holds, we consider the conditional probability $\Pr[\sum_{i=1}^n X_{(i)} \leq \frac{n}{2} \mid \boldsymbol{p}]$. Given $\boldsymbol{p}$, $X_{(i)}$'s are independent Bernoulli random variables with means $\mathbb{E}[X_{(i)}] = p_{(i)}$. Hence, by Hoeffding's inequality (Lemma 6),

$$\begin{aligned}
\Pr\left[\sum_{i=1}^n X_{(i)} \leq \frac{n}{2} \,\middle|\, \boldsymbol{p}\right] &\leq \exp\left(-\frac{2(\sum_{i=1}^n p_{(i)} - \frac{n}{2})^2}{n}\right) \\
&\leq \exp\left(-\frac{2(n\varepsilon_n - n\varepsilon)^2}{n}\right) = \exp\left(-2n(\varepsilon_n - \varepsilon)^2\right) = \frac{1}{n^{2(a-b)^2}}.
\end{aligned}$$

□

Lemma 9. *Suppose the PDF of $\mathcal{D}_n$ satisfies $f_n(x) \geq \underline{C}(1-x)^{\underline{\beta}-1}$ for $x \in [1-\underline{\delta}, 1]$ for some constants $\underline{C}, \underline{\beta}, \underline{\delta} > 0$. Then, for sufficiently large n, with probability at least $1 - n^{-d}$ over the random draw of $\boldsymbol{p} \sim \mathcal{D}_n$,*

$$\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] \leq \left(\frac{\underline{\beta} d \log n}{\underline{C} n}\right)^{1/\underline{\beta}}.$$

Proof. We note that $\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] = 1 - p_{(1)}$, so for any $x \in [0,1]$,

$$\begin{aligned}
\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] \leq x] = \Pr[1 - p_{(1)} \leq x] = \Pr[p_{(1)} \geq 1 - x] &= 1 - \Pr[p_{(1)} < 1 - x] \\
&= 1 - \Pr[\max_{1\leq i\leq n} p_i < 1 - x] \\
&= 1 - F_n(1-x)^n.
\end{aligned}$$

We let x be such that $F_n(1-x) = 1 - \frac{d\log n}{n}$, i.e., $x = 1 - F_n^{-1}(1 - \frac{d\log n}{n})$, then $F_n(1-x)^n =$

$(1-\frac{d\log n}{n})^n \le e^{-d\log n} = n^{-d}$. So, with probability at least $1-F_n(1-x)^n \ge 1-n^{-d}$, we have

$$\Pr[X_{(1)}=0 \mid \boldsymbol{p}] \le x = 1-F_n^{-1}\left(1-\frac{d\log n}{n}\right).$$

We then show that $1-F_n^{-1}\left(1-\frac{d\log n}{n}\right) \le \left(\frac{\underline{\beta} d\log n}{\underline{C} n}\right)^{1/\underline{\beta}}$. Define $G(t)=1-F_n(1-t)$ for $t\in[0,1]$. This implies

$$1-F_n^{-1}(1-y)=G^{-1}(y)$$

for any $y\in[0,1]$. We note that for t sufficiently close to 1, $f_n(x)\ge \underline{C}(1-x)^{\underline{\beta}-1}$ for any $x\in[1-t,1]$, implying

$$G(t)=1-F_n(1-t)=\int_{1-t}^1 f_n(x)\mathrm{d}x \ge \int_{1-t}^1 \underline{C}(1-x)^{\underline{\beta}-1}\mathrm{d}x = \int_0^t \underline{C}u^{\underline{\beta}-1}\mathrm{d}u = \frac{\underline{C}}{\underline{\beta}}t^{\underline{\beta}}.$$

Let $\underline{G}(t)=\frac{\underline{C}}{\underline{\beta}}t^{\underline{\beta}}$. We have $G(t)\ge\underline{G}(t)$ and $\underline{G}^{-1}(y)=(\frac{\underline{\beta}}{\underline{C}}y)^{1/\underline{\beta}}$. Since $G(t)\ge\underline{G}(t)$ and $\underline{G}^{-1}(y)$ is increasing in y, we have

$$G(t)\ge\underline{G}(t) \implies \underline{G}^{-1}(G(t))\ge t \implies \underline{G}^{-1}(y)\ge G^{-1}(y).$$

Therefore,

$$1-F_n^{-1}(1-y)=G^{-1}(y)\le\underline{G}^{-1}(y)=(\frac{\underline{\beta}}{\underline{C}}y)^{1/\underline{\beta}}.$$

Letting $y=\frac{d\log n}{n}$, we conclude that

$$\Pr[X_{(1)}=0\mid\boldsymbol{p}]\le 1-F^{-1}\left(1-\frac{d\log n}{n}\right)\le\left(\frac{\underline{\beta} d\log n}{\underline{C} n}\right)^{1/\underline{\beta}}.$$

□

Lemma 10. *Suppose the PDF of $\mathcal{D}_n$ satisfies $f_n(x)\le\overline{C}$ for $x\in[1-\overline{\delta},1]$ for some constants $\overline{C},\overline{\delta}>0$. Then, for sufficiently large n, with probability at least $1-n^{-d}$ over the random* *draw of $\boldsymbol{p} \sim \mathcal{D}_n$,*

$$\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] \geq \frac{1}{\overline{C} n^{d+1}}.$$

Proof. We note that $\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] = 1 - p_{(1)}$, so for any $x \in [0,1]$,

$$\begin{aligned}\Pr[\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] \geq x] = \Pr[1 - p_{(1)} \geq x] = \Pr[p_{(1)} \leq 1 - x] &= \Pr[\max_{1 \leq i \leq n} p_i < 1 - x] \\ &= F_n(1-x)^n.\end{aligned}$$

We let $x = \frac{1}{\overline{C} n^{d+1}}$. Then for sufficiently large n, $x \geq 1 - \overline{\delta}$, and hence $f_n(t) \leq \overline{C}$ for $t \in [1-x, 1]$, which implies

$$1 - F_n(1-x) = \int_{1-x}^{1} f_n(t) \mathrm{d}t \leq \int_{1-x}^{1} \overline{C} \mathrm{d}t = x\overline{C} = \frac{1}{n^{d+1}},$$

or equivalently

$$F_n(1-x) \geq 1 - \frac{1}{n^{d+1}}.$$

Using inequality $(1 - \frac{x}{n})^n \geq 1 - x$ (for $n \geq 1, 0 \leq x \leq n$), we get

$$F_n(1-x)^n \geq \left(1 - \frac{1}{n^{d+1}}\right)^n \geq 1 - \frac{1}{n^d}.$$

Therefore, with probability at least $1 - \frac{1}{n^d}$, $\Pr[X_{(1)} = 0 \mid \boldsymbol{p}] \geq x = \frac{1}{\overline{C} n^{d+1}}$ holds. $\square$

To prove $\Gamma_n^{\boldsymbol{p}}(1) > 0$, we use Lemma 7 and Lemma 9 to get

$$\begin{aligned}\Gamma_n^{\boldsymbol{p}}(1) &= \Pr\left[\sum_{i=1}^{n} X_{(i)} \leq \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[X_{(1)} = 0 \;\middle|\; \boldsymbol{p}\right] \\ &\geq \frac{1}{\sqrt{2\pi}} \frac{c\sqrt{\log n}}{(c^2 \log n + 1)} \frac{1}{n^{c^2/2}} - \frac{C_1}{\sqrt{\mathbb{E}_{p \sim \mathcal{D}_n}[p(1-p)] - \varepsilon}} \frac{1}{\sqrt{n}} - \left(\frac{\underline{\beta} d \log n}{\underline{C} n}\right)^{1/\underline{\beta}}\end{aligned}$$

with probability at least $1 - 2n^{-2b^2} - n^{-d}$, where $c = \frac{a+b}{\sqrt{\mathbb{E}_{p \sim \mathcal{D}_n}[p(1-p)] - \varepsilon}}$, $\mathbb{E}_{p \sim \mathcal{D}_n}[p] \leq \frac{1}{2} + \varepsilon_n$ with

$\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some $a > 0$, and $\varepsilon = b\sqrt{\frac{\log n}{n}}$ for some $b > 0$, and $\underline{C}$ and $\underline{\beta}$ are constants. If $c^2/2$ is a constant such that

$$c^2/2 < \min\left\{1/2, 1/\underline{\beta}\right\},$$

then $\Gamma_n^{\boldsymbol{p}}(1) = O(\frac{1}{n^{c^2/2}}) > 0$ for sufficiently large n. Requiring $c^2/2 < \min\left\{1/2, 1/\underline{\beta}\right\}$ is equivalent to requiring

$$a + b < \sqrt{(\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon) \cdot \min\{1, 2/\underline{\beta}\}},$$

which can be satisfied when a and b are constants such that $a < \sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] \cdot \min\{1, 2/\underline{\beta}\}}$, $0 < b < \sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] \cdot \min\{1, 2/\underline{\beta}\}} - a$, and n is sufficiently large (so $\varepsilon = b\sqrt{\frac{\log n}{n}}$ is sufficiently small).

To prove $\Gamma_n^{\boldsymbol{p}}(1) < 0$, we use Lemma 8 and Lemma 10 to get

$$\begin{aligned}\Gamma_n^{\boldsymbol{p}}(1) &= \Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[X_{(1)} = 0 \;\middle|\; \boldsymbol{p}\right] \\ &\le \frac{1}{n^{2(a-b)^2}} - \frac{1}{\overline{C}n^{d+1}}\end{aligned}$$

with probability at least $1 - n^{-2b^2} - n^{-d}$, where $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \ge \frac{1}{2} + \varepsilon_n$ with $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some constant $a > 0$, with any $b < a$, and $\overline{C}$ is a constant. When

$$2(a-b)^2 > d + 1,$$

we have $\Gamma_n^{\boldsymbol{p}}(1) = -O(\frac{1}{n^{d+1}}) < 0$ for sufficiently large n. The inequality $2(a-b)^2 > d+1$ is satisfied when $a > \frac{1}{\sqrt{2}}$ and b, d are sufficiently close to 0. $\square$

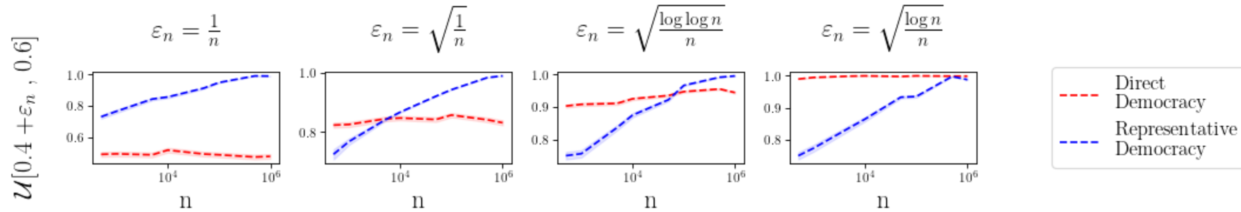


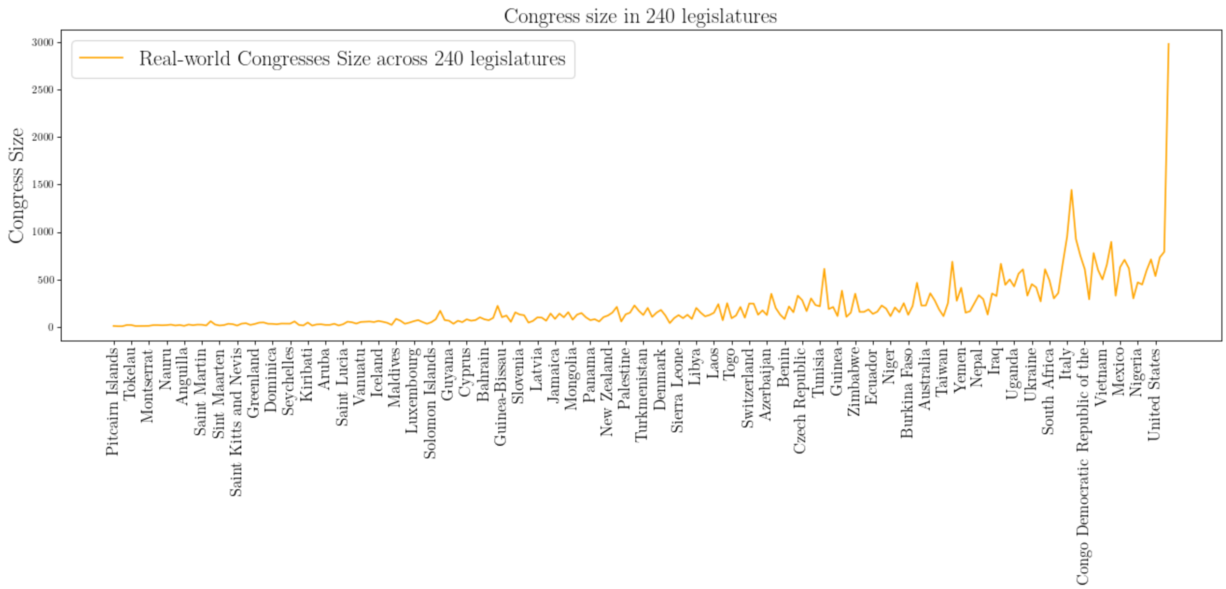
Figure 2.2: Estimates of $\Pr[\sum_{i=1}^{k} X_{(i)} > \frac{k}{2} \mid \boldsymbol{p}]$ (Representative Democracy) and $\Pr[\sum_{i=1}^{n} X_{(i)} > \frac{n}{2} \mid \boldsymbol{p}]$ (Direct Democracy) as a function of the population size for different values of ε_n, with $k = n^{0.36}$ and $\mathcal{D}_n = \mathcal{U}[0.4 + \varepsilon_n, 0.6]$. For large ε_n, the population size needs to reach a critical mass for the congress to outperform direct democracy.

2.4.2 Real-world and Polynomial-sized Congress

We now turn our attention to more practical congress sizes. As discussed in the introduction, prior work has suggested that the size of congress should be near the cube root of the population size. Exploring real-world data for 240 legislatures (the data comes from https://en.wikipedia.org/wiki/List_of_legislatures_by_number_of_members; we considered the number of representatives to be the total number of representatives in both chambers), we re-ran regression analysis of Auriol and Gary-Bobo [12] on the log of the congress sizes of many countries compared to the log of the population size, which yields a slope of 0.36 (with intercept -0.65 and coefficient of determination $R^2 = 0.85$)), suggesting $k = \Theta(n^{0.36})$. See results in Figure 2.3.

Next, we numerically investigate how congresses of this size perform compared to direct democracy with different levels of bias. We consider $k = n^{0.36}$ and $\mathcal{D}_n = \mathcal{U}(L + \varepsilon_n, 1 - L)$ such that $\mathbb{E}_{\mathcal{D}_n}[p] = \frac{1+\varepsilon_n}{2}$. So the society is slightly biased toward the correct answer. We identify sequences $(\varepsilon_n)_{n=1}^{\infty}$ such that a congress of size k outperforms direct democracy for sufficiently large n.

The simulations were run on a MacBook Pro as follows: for a given distribution, we sample n competencies and votes associated with these competencies. We perform two majority votes — with all the voters and with the top k voters. Repeating this operation 1,000



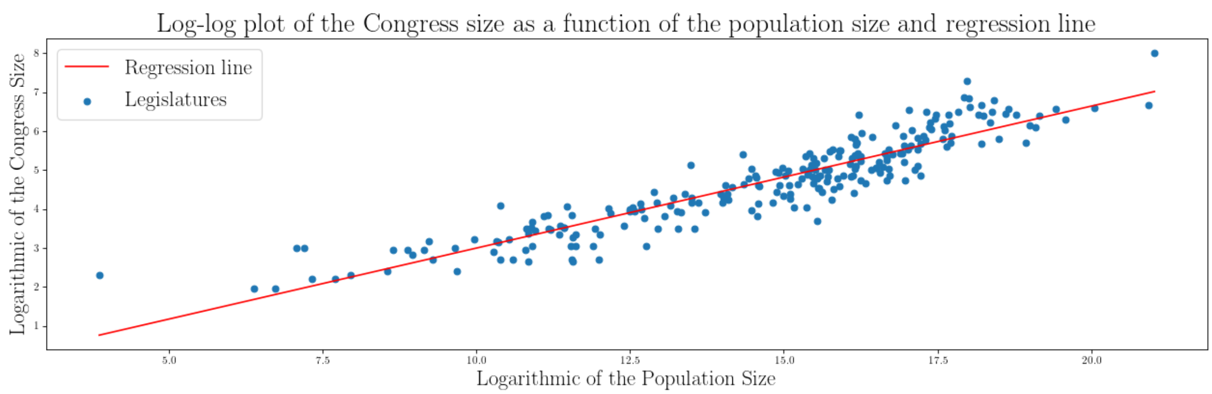


Figure 2.3: Congress sizes in 240 legislatures (top) and log-log plot of the Congress size as a function of the Population size (bottom). The regression line yields $\log k = 0.36 \log n - 0.65$, or $k = cn^{0.36}$, with a coefficient of determination $R^2 = 0.85$. Note that in the top plot, we only show a handful of countries for obvious space constraints. In reality, the United States is not the country with the largest congress (it has 535 congress-members per our computation, merging both chambers).

times, we estimate the probabilities that the majority of all voters (Direct Democracy) and k voters (Representative Democracy) are correct. Figure 2.2 displays the probabilities and 95% confidence intervals for different population sizes, with $L = 0.4$. Additional simulations are located in Figure 2.4.

Unsurprisingly, the larger the bias, the smaller the gain. For $L \leq 0.1$ and a bias of order $\sqrt{\log n/n}$, there is a no gain from relying on the congress, while if the bias is of order $\sqrt{\log\log n/n}$, there is positive gain. Yet, for $L = 0.4$, a bias of order $\sqrt{\log n/n}$ systematically yields a strictly negative gain for $n \leq 10^6$.

Let us now formalize and prove this result for general distributions. If the average competence level of the population, $\mathbb{E}_{\mathcal{D}_n}[p]$, is larger than $\frac{1}{2}$ by a constant margin, then both the entire population and a congress of size n^r will be correct with probabilities that are exponentially close to 1. Hence, again, to make things more interesting, we are concerned with the case where $\mathbb{E}_{\mathcal{D}_n}[p] = \frac{1}{2} + \varepsilon_n$ with $0 < \varepsilon_n < o(1)$. We identify conditions on ε_n, n and $\mathcal{D}_n$ under which $\Gamma_n^{\boldsymbol{p}}(k) > 0$ or $\Gamma_n^{\boldsymbol{p}}(k) < 0$.

Theorem 4. *Let $k = n^r$ for some constant $0 < r < 1$.*

- *Suppose $\mathbb{E}_{\mathcal{D}_n}[p] \leq \frac{1}{2} + a\sqrt{\frac{\log n}{n}}$, and $1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) \geq \frac{k}{n} + \Omega(\sqrt{\frac{\log n}{n}})$ for some constants $a, \alpha > 0$. If $a < \sqrt{\mathbb{E}_{\mathcal{D}_n}[p(1-p)]}$ and $\alpha > \frac{a}{2\sqrt{r \cdot \mathbb{E}_{\mathcal{D}_n}[p(1-p)]}}$, then, with probability at least $1 - n^{-\Omega(1)}$, $\Gamma_n^{\boldsymbol{p}}(k) > 0$.*
- *Suppose $\mathbb{E}_{\mathcal{D}_n}[p] \geq \frac{1}{2} + a\sqrt{\frac{\log n}{n}}$ and $1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) \leq \frac{1}{n^{1+\Omega(1)}}$ for some constants $a, \alpha > 0$. If $\alpha < \frac{1}{2}$ and $a > \sqrt{r}\alpha$, then, with probability at least $1 - n^{-\Omega(1)}$, $\Gamma_n^{\boldsymbol{p}}(k) < 0$.*

Intuitively, in the first item above, the condition on the CDF,

$$1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) \geq \frac{k}{n} + \Omega(\sqrt{\frac{\log n}{n}}),$$

and the condition on α imply that $\mathcal{D}_n$ assigns large enough probability to high competence levels $p > \frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}$, so a congress of size n^r will be composed of competent enough experts and hence will beat the entire population. The conditions in the second item are in the opposite direction.

We remark that the above conditions on the relation between a and α are sharp: for distributions $\mathcal{D}_n$ that are concentrated around $1/2$, we have $\mathbb{E}_{\mathcal{D}_n}[p(1-p)] \approx 1/4$, so the first condition becomes $\alpha > \frac{a}{2\sqrt{r \cdot 1/4}} = \frac{a}{\sqrt{r}}$, or equivalently $a < \sqrt{r}\alpha$, while the second condition is the opposite: $a > \sqrt{r}\alpha$.

Finally, we note that the conditions in Theorem 4 on the distributions $\mathcal{D}_n$ are satisfied by many natural classes of distributions, e.g., beta distributions and normal distributions truncated to $[0,1]$. We identify more examples in Revel et al. [204].

Next, we prove Theorem 4. Similar to the proof of Theorem 3, we write $\Gamma_n^{\boldsymbol{p}}(k)$ as

$$\Gamma_n^{\boldsymbol{p}}(k) = \Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[\sum_{i=1}^{k} X_{(i)} \le \frac{k}{2} \;\middle|\; \boldsymbol{p}\right].$$

To show either $\Gamma_n^{\boldsymbol{p}}(k) > 0$ or $\Gamma_n^{\boldsymbol{p}}(k) < 0$, we will compare $\Pr\left[\sum_{i=1}^{n} X_{(i)} \le \frac{n}{2} \mid \boldsymbol{p}\right]$ with $\Pr\left[\sum_{i=1}^{k} X_{(i)} \le \frac{k}{2} \mid \boldsymbol{p}\right]$.

Lemma 11. *Suppose* $1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) \ge \frac{k}{n} + \varepsilon$ *where* $\varepsilon = b\sqrt{\frac{\log n}{n}}$ *for some constants* $\alpha, b > 0$. *Then, with probability at least* $1 - 2n^{-2b^2}$ *(over the random draw of* $\boldsymbol{p} \sim \mathcal{D}_n$*),*

$$\Pr\left[\sum_{i=1}^{k} X_{(i)} \le \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \le \frac{1}{k^{2\alpha^2}}.$$

Proof. By DKW inequality (Lemma 2.5), with probability at least $1 - 2e^{-2n\varepsilon^2} = 1 - 2n^{-2b^2}$ over the random draw of $\boldsymbol{p} \sim \mathcal{D}_n$, it holds that $|F_n(p_{(i)}) - \frac{n-i}{n}| \le \varepsilon$ for every $i \in [n]$. In particular, for $i = 1, \ldots, k$, we have

$$F_n(p_{(i)}) \ge \frac{n-i}{n} - \varepsilon \ge \frac{n-k}{n} - \varepsilon = 1 - \frac{k}{n} - \varepsilon,$$

This implies

$$1 - F_n(p_{(i)}) \leq \frac{k}{n} + \varepsilon \leq 1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}})$$

and hence

$$p_{(i)} \geq \frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}.$$

Assuming the above inequalities hold, we consider the conditional probability $\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \mid \boldsymbol{p}\right]$. Given $\boldsymbol{p}$, the $X_{(i)}$'s are independent draws from $\mathrm{Bern}(p_{(i)})$ distributions, with means $\mathbb{E}[X_{(i)}] = p_{(i)}$, hence, by Hoeffding's inequality (Lemma 6),

$$\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \leq \exp\left(-\frac{2(\sum_{i=1}^{k} p_{(i)} - \frac{k}{2})^2}{k}\right).$$

Plugging in $p_{(i)} \geq \frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}$, we get

$$\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \leq \exp\left(-\frac{2(\frac{k}{2} + \alpha\sqrt{k\log k} - \frac{k}{2})^2}{k}\right) = \frac{1}{k^{2\alpha^2}}.$$

$\square$

Proof of the first item of Theorem 4. By Lemma 7 and Lemma 11, we have

$$\begin{aligned}\Gamma_n^{\boldsymbol{p}}(k) &= \Pr\left[\sum_{i=1}^{n} X_{(i)} \leq \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \\ &\geq \frac{1}{\sqrt{2\pi}} \frac{c\sqrt{\log n}}{(c^2\log n + 1)} \frac{1}{n^{c^2/2}} - \frac{C_1}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon}} \frac{1}{\sqrt{n}} - \frac{1}{k^{2\alpha^2}}\end{aligned}$$

with probability at least $1 - 4n^{-2b^2}$, where $c = \frac{a+b}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)]-\varepsilon}}$, $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \leq \frac{1}{2} + \varepsilon_n$ with $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some $a > 0$, and $\varepsilon = b\sqrt{\frac{\log n}{n}}$ for some $b > 0$, and α is a constant. Since

$k = n^r$,

$$\Gamma_n^{\boldsymbol{p}}(k) \geq \frac{1}{\sqrt{2\pi}} \frac{c\sqrt{\log n}}{(c^2 \log n + 1)} \frac{1}{n^{c^2/2}} - \frac{C_1}{\sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon}} \frac{1}{\sqrt{n}} - \frac{1}{n^{2r\alpha^2}}$$

When $c^2/2 < 1/2$ and $c^2/2 < 2r\alpha^2$, we have $\Gamma_n^{\boldsymbol{p}}(k) = O(\frac{1}{n^{c^2/2}}) > 0$ for sufficiently large n. The latter requirement $c^2/2 < 2r\alpha^2$ is satisfied when $\alpha > \frac{c}{2\sqrt{r}}$. The former requirement $c^2/2 < 1/2$ is equivalent to $a + b < \sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)] - \varepsilon\}}$, which is satisfied when constants $a < \sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)]\}}$, $0 < b < \sqrt{\mathbb{E}_{p\sim\mathcal{D}_n}[p(1-p)]\}} - a$, and n is sufficiently large. $\square$

Lemma 12. *Suppose* $1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) \leq \frac{1}{n^{1+\Omega(1)}}$ *for some constant* $\alpha > 0$*, and suppose* $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \geq \frac{1}{2} + \varepsilon_n$ *with* $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ *for some constant* $a > 0$*. Then, with probability at least* $1 - \frac{1}{n^{\Omega(1)}}$ *(over the random draw of* $\boldsymbol{p} \sim \mathcal{D}_n$*),*

$$\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{1 - o(1)}{2\alpha\sqrt{\log k}} \cdot \frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}} - \frac{2C_1}{(1 - o(1))\sqrt{k}},$$

where C_1 *is the constant in Berry-Esseen theorem (Lemma 4).*

Proof. Given $p_{(1)}, \ldots, p_{(k)}$, each $X_{(i)}$ independently follows $\mathrm{Bern}(p_{(i)})$. We use Berry-Esseen theorem (Lemma 4) for $Y_i = X_{(i)} - p_{(i)}$, $i = 1, \ldots, k$. Noticing that $\mathbb{E}[Y_i] = p_{(i)}$, $\sigma_i^2 = \mathbb{E}[Y_i^2] = p_{(i)}(1 - p_{(i)})$, and $\rho_i = \mathbb{E}[|Y_i|^3] = p_{(i)}(1 - p_{(i)})[(1 - p_{(i)})^2 + p_{(i)}^2] \leq \sigma_i^2$, the theorem implies

$$\left|\Pr\left[\frac{\sum_{i=1}^k Y_i}{\sum_{i=1}^k \sigma_i^2} \leq x\right] - \Phi(x)\right| \leq \frac{C_1}{\sqrt{\sum_{i=1}^k \sigma_i^2}} \max_{1\leq i\leq k} \frac{\rho_i}{\sigma_i^2} \leq \frac{C_1}{\sqrt{\sum_{i=1}^k \sigma_i^2}}$$

for any $x \in \mathbb{R}$, where $\Phi(x)$ is CDF of the standard normal distribution. Therefore,

$$\begin{aligned}
\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] &= \Pr\left[\sum_{i=1}^{k} X_{(i)} - \sum_{i=1}^{k} p_{(i)} \leq \frac{k}{2} - \sum_{i=1}^{k} p_{(i)} \;\middle|\; \boldsymbol{p}\right] \\
&= \Pr\left[\frac{\sum_{i=1}^{k} Y_i}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}} \leq \frac{\frac{k}{2} - \sum_{i=1}^{k} p_{(i)}}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right] \\
&\geq \Phi\left(\frac{\frac{k}{2} - \sum_{i=1}^{k} p_{(i)}}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right) - \frac{C_1}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}.
\end{aligned}$$

We consider $\sum_{i=1}^{k} p_{(i)}$. By the assumption that $1 - F_n(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) = \Pr_{p_i \sim \mathcal{D}_n}[p_i > \frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}] = \frac{1}{n^{1+\Omega(1)}}$, using a union bound we have with probability at least $1 - n\frac{1}{n^{1+\Omega(1)}} = 1 - \frac{1}{n^{\Omega(1)}}$, all p_i's (for $i = 1, \ldots, n$) satisfy $p_i \leq \frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}$. Hence,

$$\sum_{i=1}^{k} p_{(i)} \leq k(\frac{1}{2} + \alpha\sqrt{\frac{\log k}{k}}) = \frac{k}{2} + \alpha\sqrt{k \log k},$$

which implies

$$\begin{aligned}
\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] &\geq \Phi\left(\frac{\frac{k}{2} - (\frac{k}{2} + \alpha\sqrt{k \log k})}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right) - \frac{C_1}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}} \\
&= \Phi\left(\frac{-\alpha\sqrt{k \log k}}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right) - \frac{C_1}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}} \qquad (2.12)
\end{aligned}$$

We then consider $\sum_{i=1}^{k} \sigma_i^2 = \sum_{i=1}^{k} p_{(i)}(1 - p_{(i)}) = \sum_{i=1}^{k} p_{(i)} - \sum_{i=1}^{k} p_{(i)}^2$. We note that the p_i's (for $i = 1, \ldots, n$) are n i.i.d. random draws from distribution $\mathcal{D}_n$ whose mean is $\mathbb{E}_{p \sim \mathcal{D}_n}[p] \geq \frac{1}{2} + \varepsilon_n$, by Hoeffding's inequality, their average satisfies

$$\frac{1}{n} \sum_{i=1}^{n} p_i \geq \mathbb{E}_{p \sim \mathcal{D}_n}[p] - \varepsilon \geq \frac{1}{2} + \varepsilon_n - \varepsilon,$$

with probability at least $1-\exp(-2n\varepsilon^2)$. We choose $\varepsilon = O(\sqrt{\frac{\log n}{n}})$ so the probability is $1-\frac{1}{n^{\Omega(1)}}$. We also note that $\frac{1}{n}\sum_{i=1}^{n} p_i \le \frac{1}{k}\sum_{i=1}^{k} p_{(i)}$ because $p_{(1)},\ldots,p_{(k)}$ are the k largest values in $p_1,\ldots,p_n$. Therefore,

$$\sum_{i=1}^{k} p_{(i)} \ge \frac{k}{n}\sum_{i=1}^{n} p_i \ge k(\frac{1}{2}+\varepsilon_n-\varepsilon).$$

Moreover, since previously we had $p_i \le \frac{1}{2}+\alpha\sqrt{\frac{\log n}{n}}$ for all $i=1,\ldots,n$, it holds that

$$\sum_{i=1}^{k} p_{(i)}^2 \le k\left(\frac{1}{2}+\alpha\sqrt{\frac{\log n}{n}}\right)^2 = k(\frac{1}{4}+o(1)).$$

Therefore,

$$\sum_{i=1}^{k}\sigma_i^2 = \sum_{i=1}^{k} p_{(i)} - \sum_{i=1}^{k} p_{(i)}^2 \ge k(\frac{1}{2}+\varepsilon_n-\varepsilon) - k(\frac{1}{4}+o(1)) = k(\frac{1}{4}-o(1)).$$

Plugging $\sum_{i=1}^{k}\sigma_i^2 \ge k(\frac{1}{4}-o(1))$ into Equation (2.12), we get

$$\begin{aligned}\Pr\left[\sum_{i=1}^{k} X_{(i)} \le \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] &\ge \Phi\left(\frac{-\alpha\sqrt{k\log k}}{\sqrt{k(\frac{1}{4}-o(1))}}\right) - \frac{C_1}{\sqrt{k(\frac{1}{4}-o(1))}}\\ &= \Phi\left(\frac{-2\alpha\sqrt{\log k}}{1-o(1)}\right) - \frac{2C_1}{(1-o(1))\sqrt{k}}.\end{aligned}$$

Using Lemma 5 with $x = \frac{2\alpha\sqrt{\log k}}{1-o(1)}$, we have

$$\Phi\left(\frac{-2\alpha\sqrt{\log k}}{1-o(1)}\right) \ge \frac{1}{\sqrt{2\pi}}\frac{2\alpha\sqrt{\log k}(1-o(1))}{4\alpha^2\log k+1}e^{-\frac{4\alpha^2\log k}{2(1-o(1))}} = \frac{1}{\sqrt{2\pi}}\frac{1-o(1)}{2\alpha\sqrt{\log k}}\frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}}$$

which implies

$$\Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \geq \frac{1}{\sqrt{2\pi}} \frac{1-o(1)}{2\alpha\sqrt{\log k}} \frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}} - \frac{2C_1}{(1-o(1))\sqrt{k}},$$

concluding the proof. □

Proof of the second item of Theorem 4. To prove $\Gamma_n^{\boldsymbol{p}}(k) < 0$, we use Lemma 8 and Lemma 12 to get

$$\begin{aligned}\Gamma_n^{\boldsymbol{p}}(k) &= \Pr\left[\sum_{i=1}^{n} X_{(i)} \leq \frac{n}{2} \;\middle|\; \boldsymbol{p}\right] - \Pr\left[\sum_{i=1}^{k} X_{(i)} \leq \frac{k}{2} \;\middle|\; \boldsymbol{p}\right] \\ &\leq \frac{1}{n^{2(a-b)^2}} - \frac{1}{\sqrt{2\pi}} \frac{1-o(1)}{2\alpha\sqrt{\log k}} \frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}} + \frac{2C_1}{(1-o(1))\sqrt{k}},\end{aligned}$$

with probability at least $1 - n^{-2b^2} - n^{-\Omega(1)} = 1 - n^{-\Omega(1)}$, where $\mathbb{E}_{p\sim\mathcal{D}_n}[p] \geq \frac{1}{2} + \varepsilon_n$ with $\varepsilon_n = a\sqrt{\frac{\log n}{n}}$ for some $a > 0$, $0 < b < a$, $1 - F_n(1 + \alpha\sqrt{\frac{\log k}{k}}) = \frac{1}{n^{1+\Omega(1)}}$ for some $\alpha > 0$, and C_1 is some constant. Since $k = n^r$, or $n = k^{\frac{1}{r}}$,

$$\Gamma_n^{\boldsymbol{p}}(k) \leq \frac{1}{k^{\frac{2(a-b)^2}{r}}} - \frac{1}{\sqrt{2\pi}} \frac{1-o(1)}{2\alpha\sqrt{\log k}} \frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}} + \frac{2C_1}{(1-o(1))\sqrt{k}},$$

When inequalities $\frac{2\alpha^2}{1-o(1)} < \frac{2(a-b)^2}{r}$ and $\frac{2\alpha^2}{1-o(1)} < \frac{1}{2}$ are satisfied, we have $\Gamma_n^{\boldsymbol{p}}(k) = -O\left(\frac{1}{\sqrt{\log k}} \frac{1}{k^{\frac{2\alpha^2}{1-o(1)}}}\right) < 0$ for sufficiently large n. The former is satisfied when $a > \sqrt{r}\alpha$ and b is sufficiently close to 0. The latter is satisfied when $\alpha < \frac{1}{2}$. □

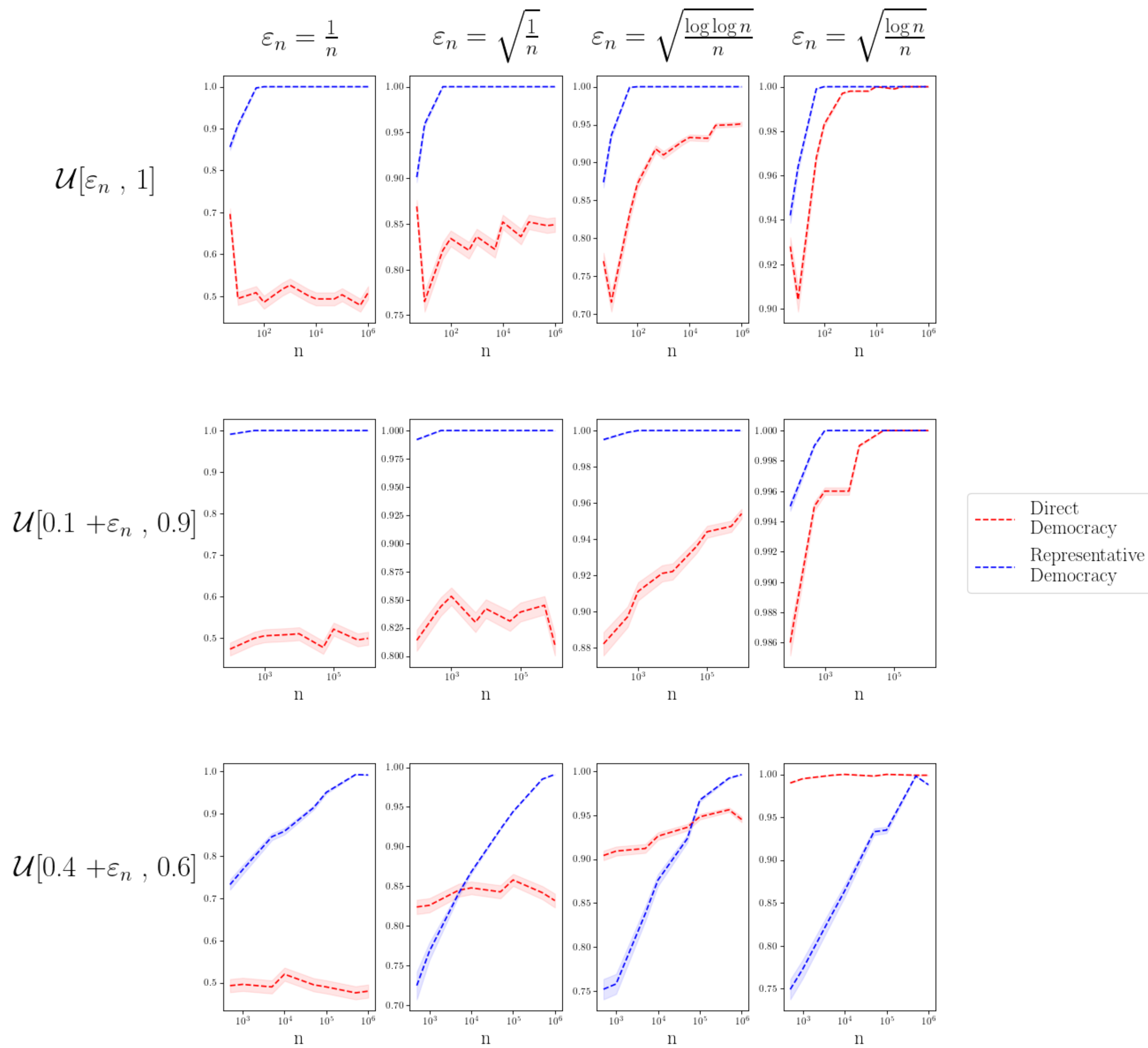


Figure 2.4: Estimates of $\Pr[\sum_{i=1}^{k} X_{(i)} > \frac{k}{2} \mid \boldsymbol{p}]$ (Representative Democracy) and $\Pr[\sum_{i=1}^{n} X_{(i)} > \frac{n}{2} \mid \boldsymbol{p}]$ (Direct Democracy) with 95% confidence intervals as a function of the population size for different values of ε_n, with $k = n^{0.36}$ and $\mathcal{D}_n = \mathcal{U}[0.4 + \varepsilon_n, 0.6]$. For large society biases, the population size needs to reach a critical mass for the congress to outperform direct democracy. Note that $\mathbb{E}[p_i] = \frac{1+\varepsilon_n}{2}$ so ε_n can be thought of as the bias of society towards the correct answer. The top image is for $L = 0$, the middle one is for $L = 0.1$ and the bottom one for $L = 0.4$.

2.5 Discussion

In the epistemic setting, we have proved that under mild conditions, through the lens of an epistemic approach, current congresses are run with a sub-optimal size. However, despite this, it seems that these smaller congresses can still be cogent by at least beating the majority under appropriate conditions.

Current debates about the number of representatives in democracies tend to discuss reductions in size, not increases, as embodied by a 2020 Italian referendum approved reducing congress' size from 945 to 600 [65]. Indeed, even under the assumption that a larger congress would lead to a "correct" answer more often, this is clearly not the only desiderata to consider. Even under the strong assumption that the congress members' votes reflect those of the top experts in society, congress-members are costly for the taxpayers. Beyond this, the legitimacy and representativeness [170] of the institution are constantly under scrutiny. Designing political institutions relying solely on mathematical insights could yield unforeseen negative externalities (did Madison not warn against the *confusion of the multitude*?). Cognitive, sociological, and economic knowledge should be coupled with mathematical analyses to reach a reasonable trade-off rather than optimizing a single factor.

Incorporating a cost analysis, similar to Magdon-Ismail and Xia [157] also seems particularly relevant to quantify the trade-off between the congress accuracy and its costs for the constituents. They find that adding a cost polynomial in the number of representatives and a benefit of choosing the correct outcome polynomial in the number of voters decreases the optimal congress size to $O(\log n)$. Finally, this work supports, to some extent, propositions to constitute assemblies of citizens under liquid democracy [26, 48, 101, 105, 111, 129, 173] that would vote on behalf of the entire population. Indeed, liquid democracy could yield very large citizen assemblies deemed desirable by our findings. Further research on the accuracy of such citizen assemblies could discuss the influence of the voters' weight in the weighted majority's performance.

Chapter 3

In Defense of Liquid Democracy

It can always be taken into the calculation, and counted at a certain figure, a higher figure being assigned to the suffrages of those whose opinion is entitled to greater weight. There is not in this arrangement any thing necessarily invidious to those to whom it assigns the lower degrees of influence. Entire exclusion from a voice in the common concerns is one thing: the concession to others of a more potential voice, on the ground of greater capacity for the management of the joint interests, is another.

– John Stuart Mill[1]

Abstract

The dynamics of random transitive delegations on a graph are of particular interest when viewed through the lens of an emerging voting paradigm known as *liquid democracy*. This paradigm allows voters to choose between directly voting and transitively delegating their votes to other voters, so that those selected cast a vote weighted by the number of delegations

[1]See John Stuart Mill's *Considerations on Government* [171], where he outlines his contentious view on plural voting.

they receive. In the epistemic setting, where voters decide on a binary issue for which there is a ground truth, previous work showed that, under certain assumptions on the delegation model, a few voters may amass such a large amount of influence that liquid democracy is less likely to identify the ground truth than direct voting. We quantify the amount of permissible concentration of power and examine more realistic delegation models, showing they behave well by ensuring that (with high probability) there is a permissible limit on the maximum number of delegations received. Our results demonstrate that the delegation process can be treated as a stochastic process akin to well-known processes on random graphs —such as preferential attachment and multi-types branching process—that are sufficiently bounded for our purposes. Along the way, we prove new bounds on the size of the largest component in an infinite Pólya urn process (a generalization of the preferential attachment model), which may be of independent interest. Our work suggests that existing and new results in random graph theory may alleviate concerns raised about liquid democracy and bolster the case for the applicability of this emerging paradigm.

3.1 Introduction

Liquid democracy is a voting paradigm that is conceptually situated between *direct democracy*, in which voters have direct influence over decisions, and *representative democracy*, where voters choose delegates who represent them for a period of time. Under liquid democracy, voters have a choice: they can either vote directly on an issue like in direct democracy, or delegate their vote to another voter, entrusting them to vote on their behalf. The defining feature of liquid democracy is that these delegations are *transitive*: if voter 1 delegates to voter 2 and voter 2 delegates to voter 3, then voter 3 votes (or delegates) on behalf of all three voters.

In recent years, liquid democracy has gained prominence around the world. The most impressive example is that of the German Pirate Party, which adopted the *LiquidFeedback* platform in 2010 [137]. Other political parties, such as the Net Party in Argentina and Flux in Australia, have run on the wily promise that, once elected, their representatives would be essentially controlled by voters through a liquid democracy platform. Companies are also exploring the use of liquid democracy for corporate governance; Google, for example, has run a proof-of-concept experiment [113].

Practitioners, however, recognize that there is a potential flaw in liquid democracy, namely, the possibility of *concentration of power*, in the sense that certain voters amass a relatively large number of delegations, giving them pivotal influence over the final decision. This scenario seems inherently undemocratic — and it is not a mere thought experiment. Indeed, in the LiquidFeedback platform of the German Pirate Party, a linguistics professor at the University of Bamberg received so many delegations that, as noted by Der Spiegel,[2] his "vote was like a decree."

3.1.1 Problem Statement

Kahng et al. [129] examine liquid democracy's concentration-of-power phenomenon from a theoretical viewpoint and establish a troubling impossibility result in what has been called the *epistemic* setting, that is, one where there is a ground truth.[3] Informally, they demonstrate that, even under the strong assumption that voters delegate only to more "competent" voters, any "local mechanism" satisfying minimal conditions will, in certain instances, be subject to concentration of power, leading to relatively low accuracy. More specifically, Kahng et al. model the problem as a decision problem where voters decide on an issue with two outcomes, $\{0, 1\}$, where 1 is correct (the ground truth) and 0 is incorrect. Each of the voters $i \in \{1, \ldots, n\}$ is characterized by a *competence* $p_i \in [0, 1]$. The binary vote V_i of each voter

[2] http://www.spiegel.de/international/germany/liquid-democracy-web-platform-makes-professor-most-powerful-pirate-a-818683.html

[3] The use of the term "epistemic" in this context is well-established in the social choice literature [151, 189].

i is drawn independently from a Bernoulli distribution, that is, each voter votes correctly with probability p_i. Under direct democracy, the outcome of the election is determined by a majority vote: the correct outcome is selected if and only if more than half of the voters vote for the correct outcome; that is, it is correct if and only if $\sum_{i=1}^{n} V_i \geq n/2$. Under liquid democracy, there exists a set of weights, weight_i for each $i \in [n]$, which represent the number of votes that voter i gathered transitively after delegation (if voter i delegates, then $\mathsf{weight}_i = 0$). The outcome of the election is then determined by a weighted majority; it is correct if and only if $\sum_{i=1}^{n} \mathsf{weight}_i V_i \geq n/2$.

Kahng et al. also introduce the concept of a *delegation mechanism*, which determines whether voters delegate and, if so, to whom they delegate. They are especially interested in *local mechanisms*, where the delegation decision of a voter depends only on their local neighborhood according to an underlying social network. They assume that voters delegate only to those with strictly higher competence, which excludes the possibility of cyclic delegations. To evaluate liquid democracy, Kahng et al. test the intuition that society makes more informed decisions under liquid democracy than under direct democracy (especially given the foregoing assumption about upward delegation). To that end, they define the *gain* of a delegation mechanism to be the difference between the probability the correct outcome is selected under liquid democracy and the probability the correct outcome is selected under direct democracy. A delegation mechanism satisfies *positive gain* if its gain is strictly positive in some cases, and it satisfies *do no harm* if its loss (negative gain) is at most ε for *all* sufficiently large instances. The main result of Kahng et al. is that local mechanisms can never satisfy these two requirements. Caragiannis and Micha [42] further strengthen this negative result by showing that there are instances where local mechanisms perform much worse than either direct democracy or dictatorship (the most extreme concentration of power).

These results undermine the case for liquid democracy: the benefits of delegation appear to be reversed by concentration of power. However, the negative conclusion relies heavily on modeling assumptions and has not been borne empirically [18]. In this chapter, we provide a rebuttal by introducing an arguably more realistic model in which liquid democracy is able to avoid a degree of concentration of power that precludes accurate aggregation, thereby satisfying both do-no-harm and positive-gain (for suitably defined extensions). Crucially, some have experimented with liquid democracy in collaboration with a number of different companies and organizations [202], and — as we explain in more detail later — *the experimental results largely support our theoretical predictions*. In particular, accuracy under liquid democracy is higher than under direct democracy despite some concentration of power, and participants' delegation behavior aligns with one of the theoretical models introduced in this work.

3.1.2 Contributions

Our contributions are threefold. First, building on the work of Kahng et al. [129], we provide a general framework to analyze the dynamics of stochastic network formed through the transitive delegations. Second, we identify large classes of delegation models where liquid democracy performs well in that delegations lead to an increase in the group's expertise while inducing a sufficiently small amount of concentration of power. Third, along the way, we prove new high-probability bounds on the size of the largest component in an infinite Pólya urn process; this result may be of independent interest.

3.1.2.1 Stochastic Delegations

Our point of departure from the existing literature is the way we model delegation in liquid democracy. To emphasize these differences, instead of calling these delegation functions *mechanisms*, we instead call them delegation *models*, as they are intended to capture independent voter behavior rather than prescribing to each voter to whom they must delegate.

Our delegation models are defined by $M = (q, \varphi)$, where $q : [0,1] \to [0,1]$ is a function that maps a voter's competence to the probability they delegate and $\varphi : [0,1]^2 \to \mathbb{R}_{\geq 0}$ maps a pair of competencies to a weight. In this model, each voter i votes directly with probability $1 - q(p_i)$ and, conditioned on delegating with probability $q(p_i)$, delegates to voter $j \neq i$ with probability proportional to $\varphi(p_i, p_j)$. Crucially, a voter does not need to "know" the competence of another voter to decide whether to delegate; rather, the delegation *probabilities* are merely influenced by competence in an abstract way captured by φ. Also, note that delegation cycles are possible, and we take a worst-case approach to dealing with them: If the delegations form a cycle, then all voters in the cycle are assumed to be incorrect (vote 0).[4]

The most significant difference between our model of delegation and that of Kahng et al. [129] is that in our model, each voter has a chance of delegating to any other voter, whereas in their model, an underlying social network restricts delegation options. Our model captures a connected world where, in particular, voters may have heard of experts on various issues even if they do not know them personally. Although our model eschews an explicit social network, it can be seen as embedded into the delegation process, where the probability that i delegates to j takes into account the probability that i is familiar with j in the first place.

Another difference between our model and that of Kahng et al. [129] is that we model the competencies $p_1, \ldots, p_n$ as being sampled independently from a distribution $\mathcal{D}$. While this assumption is made mainly for ease of exposition, it allows us to avoid edge cases and obtain robust results.

3.1.2.2 Delegation Models

Our goal is to identify delegation models that satisfy (probabilistic versions of) positive gain and do no harm. Our first technical contribution, in Section 3.2.1, is the formulation of

[4]In LiquidFeedback, delegation cycles are, in fact, ignored.

general conditions on the model and competence distribution that are sufficient for these properties to hold (Lemma 13). In particular, to achieve the more difficult do no harm property, we present conditions that guarantee the maximum weight $\mathsf{max\text{-}weight}(G_n)$ accumulated by any voter is sub-linear with high probability and that the expected increase in competence post-delegation is at least a positive constant times the population size. These conditions intuitively prevent extreme concentration of power and ensure that the representatives post-delegation are sufficiently better than the entire population to compensate for any concentration of power that does happen.

Although the proof is straightforward, the benefit of this lemma is that it then suffices to identify models and distribution classes that verify these conditions. A delegation model M and a competence distribution $\mathcal{D}$ induce a distribution over delegation instances that generates random graphs in ways that relate to well-known graph processes, which we leverage to analyze our models. Specifically, we introduce three models, all shown to satisfy do no harm and positive gain under *any* continuous distribution over competence levels. The first two models, *upward delegation* and *confidence-based delegation*, can be seen as interesting but somewhat restricted case studies, which demonstrate the robustness of our approach. By contrast, the *general continuous delegation* model is, as the name suggests, quite general. Moreover, it is realistic: its predictions are consistent with the experiments displayed in Chapter 4.

Upward Delegation. In Section 3.3, we consider a model according to which the probability of delegating p is exogenous and constant across competencies, and delegation can only occur towards voters with strictly higher competence. That is, the probability that any voter i delegates is $q(p_i) = p$ and the weight that any voter i puts on another voter j is $\varphi(p_i, p_j) = \mathbb{I}_{\{p_j - p_i > 0\}}$. This model captures the fact that there might be some reluctance to delegate regardless of the voter's competence but does assume that voters act in the interest of society by only delegating to voters that are more competent than they are.

To generate a random graph induced by such a model, one can add a single voter at a time in order of decreasing competence and allow the voter to either not delegate (with probability $1 - p$) and create their own disconnected component, or delegate to the creator of any other component with probability proportional the size of the component. This works because delegating to any voter in the previous components is possible (since they have strictly higher competence) and would result in the votes being concentrated in the originator of that component by transitivity. Such a process is exactly the one that generates a preferential attachment graph with a positive probability of not attaching to the existing components [216]. We can then show that, with high probability, no component grows too large so long as $p < 1$ (see Section 3.1.2.3 for an overview of this step). Further, continuity of the competence distribution ensures that enough lower competence voters delegate to higher competence voters to sufficiently increase the average.

Confidence-Based Delegation. In Section 3.4, we consider a model in which voters delegate with probability decreasing in their competencies and choose someone at random when they delegate. That is, the probability $q(p_i)$ that any voter i delegates is decreasing in p_i and the weight that any voter i gives to any voter j is $\varphi(p_i, p_j) = 1$. In other words, in this model, competence does not affect the probability of receiving delegations, only the probability of delegating.

To generate a random graph induced by such a model, one can begin from a random vertex and study the delegation tree that starts at that vertex. A delegation tree is defined as a branching process, where a node i's "children" are the nodes that delegated to node i. In contrast to classical branching processes, the probability for a child to be born increases as the number of people who already received delegations decreases. Nevertheless, we prove that, with high probability, as long as a delegation tree is no larger than $O(\log n)$, our heterogeneous branching process is dominated by a sub-critical graph branching process [5]. We can then conclude that no component has size larger than $O(\log n)$ with high probability. Next, we show that the expected competence among the voters that do not delegate is strictly higher than the average competence. Finally, given that no voter has weight larger than $O(\log n)$, we prove that a small number of voters end up in cycles with high probability. We can therefore show that the conditions of Lemma 13 are satisfied.

General Continuous Delegation. Finally, we consider a general model in Section 3.5 where the likelihood of delegation is fixed and the weight assigned to each voter when delegating is increasing in their competence. That is, each voter i delegates with probability $q(p_i) = p$ and the weight that voter i places on voter j is $\varphi(p_i, p_j)$, where φ is continuous and increases in its second coordinate. Thus, in this model, the delegation distribution is slightly skewed towards more competent voters. Note that this does not imply that all voters delegate to someone with expected competence higher than their own, simply that they are more likely to delegate to a higher competence than a lower competence voter.

To generate a random graph induced by such a model, we again consider a branching process, but now voters j and k place different weights on i per φ. Therefore, voters have a *type* that governs their delegation behavior; this allows us to define a multi-type branching process with types that are continuous in $[0, 1]$. The major part of the analysis is a proof that, with high probability, as long as the delegation tree is no larger than $O(\log n)$, our heterogeneous branching process is dominated by a sub-critical Poisson multi-type branching process. To do so, we group the competencies into buckets that partition the segment $[0, 1]$ into small enough pieces. We define a new φ' that outputs, for any pair of competencies p_i, p_j, the maximum weight a voter from i's bucket could place on a voter from j's bucket. We can show that such a discrete multi-type branching process is sub-critical and conclude that no component has size larger than $O(\log n)$ with high probability. In a similar fashion to

Confidence-Based Delegation, we also show that there is an expected increase in competence post-delegation.

Consistency with experiments. The experiments conducted by some tested the power of liquid democracy to uncover the truth with respect to questions in areas like general knowledge, popular culture and spatial reasoning [202]. In brief, the (still somewhat preliminary) results suggest that (i) liquid democracy is overall more likely to pinpoint the truth compared to direct democracy, despite concentration of power, (ii) competence is inversely correlated with the chance of delegation, and (iii) the likelihood of delegating to another voter increases with their competence. The results, therefore, support the assumptions and predictions made by the general continuous delegation model. While there are still gaps between the theory and the experiments — as we discuss in Section 3.6 — the experiments indicate that the model is useful in at least *some* practical scenarios.

3.1.2.3 Component Sizes in Infinite Pólya Urn Processes

Lastly, recall that to prove that upward delegation satisfies do no harm, we show that the largest component in an infinite Pólya's urn process is sub-linear with high probability (Lemma 15). We briefly expand on the proof as this result was, to the best of our knowledge, not previously known in the random graph literature, and may be of independent interest.

We begin by focusing on the first t^γ bins (for a suitably chosen γ depending on the attachment probability p) and derive an upper bound on the expected size of these bins. This allows us to use Markov's inequality and union bound over all bins to show that simultaneously all of them are sublinear in size with high probability.

Second, we take care of the remaining bins by observing that each additional bins's growth is isomorphic to a classic Pólya urn process with two bins, whose limiting dynamic follows a Beta distribution. We analyze the rate of convergence, which allows us to give sufficiently strong bounds using Chebyshev's inequality after exactly $t - t^\gamma$ steps, and union bound over all of these bins, concluding that all are sublinear with high probability.

3.1.3 Related work

Our work is part of the field of computational social choice [35] and random graph theory [5, 73]. The most closely related paper is that of Kahng et al. [129], which was discussed in detail above. It is worth noting, though, that they complement their main negative result with a positive one: when the mechanism can restrict the maximum number of delegations (transitively) received by any voter to $o(\sqrt{\log n})$, do no harm and positive gain are satisfied. Imposing such a restriction would require a central planner that monitors and controls delegations. Gölz et al. [101] build on this idea: they study liquid democracy systems where voters may nominate multiple delegates and a central planner chooses a single delegate for each delegator in order to minimize the maximum weight of any voter.

Similarly, Brill and Talmon [37] propose allowing voters to specify ordinal preferences over delegation options and possibly restricting or modifying delegations in a centralized way. Caragiannis and Micha [42], and then Becker et al. [18] also consider central planners; they show that, for given competencies, the problem of choosing among delegation options to maximize the probability of a correct decision is hard to approximate. In any case, implementing these proposals would require a fundamental rethinking of the practice of liquid democracy. By contrast, our positive results show that *decentralized* delegation models are inherently self-regulatory, which supports the effectiveness of the current practice of liquid democracy.

More generally, there has been a significant amount of theoretical research on liquid democracy in recent years. To give a few examples: Green-Armytage [105] studies whether it is rational for voters to delegate their vote from a utilitarian viewpoint; Christoff and Grossi

[48] examine a similar question but in the context of voting on logically interdependent propositions; Bloembergen et al. [25] and Zhang and Grossi [243] study liquid democracy from a game-theoretic viewpoint. On the empirical side, some research has investigated delegation dynamics of liquid democracy while making institutional decisions [113, 137] and a recent one has directly analyzed its epistemic performance Chapter 4, finding generally positive results.

Further afield, liquid democracy is related to another paradigm called *proxy voting*, which dates back to the work of Miller [173]. Proxy voting allows voters to nominate representatives that have been previously declared. Cohensius et al. [51] study utilitarian voters who vote for the representative with the closest platform to theirs; they prove that the outcome of an election with proxy votes yields platforms closer to the median platform of the population than classical representative democracy. Their result provides a different viewpoint on the value of delegation.

As a final note regarding the social choice literature, this work is methodologically embedded in the branch of social choice theory that investigates the accuracy of collective decision-making under the assumption that there exists a, a priori unknown, correct answer that the group tries to uncover [22, 43, 64]. More particularly, it relates to a line of papers focused on decision-making process involving a set of designated representatives [157, 204].

Next, our work builds on the random graph literature, as our delegation processes are related to well-known stochastic graph processes. Upward delegation can be viewed as a generalization of the preferential attachment model where agents do not attach to the existing component(s) with a fixed probability. Classical preferential attachment models assume that a new node attaches to existing node n_0 with probability (parameterized by an *attachment function*) depending on the degree of n_0 [16, 73]. Various properties of the preferential attachment model have been widely studied [28, 86, 206]. Of particular interest, Krapivsky and Redner [140] analyze the distribution of the number of nodes with given transitive in-degree (or in-component size) for different attachment functions.

In our setting, the probability of attaching grows linearly with the degree (see Bhamidi [23] for examples of other attachment functions and Borgs et al. [30] for generalization of properties that depend on the attachment function). Additionally, our nodes may create a new component with fixed probability – a setup introduced by Simon [216] and usually referred to as an infinite Pólya urn process. Others have studied the distribution of degrees [71], the distribution of the number of components with k persons at time t [49] and the conditions for the emergence of infinite components [52]. However, to the best of our knowledge, the existing results do not allow us to conclude any results on the size of the largest component with high probability after a finite amount of time: They either look only at the limiting case, or simply at degree distributions, which are not sufficient for our results. Further, our proof relies on existing work on classical two-urn Pólya process [76, 127, 158, 168, 193].

In the other cases, we prove that our confidence-based and general continuous delegation models are dominated by well-known subcritical binomial [5] and multi-type Poisson branching process [29], which have size at most $O(\log n)$ with high probability.

Finally, our work is in line with an influential tradition leveraging random graph theory to understand complex social [e.g., 80, 99, 100, 124, 175, 198] and economic [e.g., 77, 135, 136, 185] dynamics.

3.2 Model

There is a set of n voters, denoted $[n] = \{1, \ldots, n\}$. We assume voters are making a decision on a binary issue and there is a correct alternative and an incorrect alternative. Each voter i has a *competence level* $p_i \in [0, 1]$ which is the probability that i votes correctly. We denote the vector of competencies by $\vec{p}_n = (p_1, \ldots, p_n)$. When n is clear from the context, we sometimes drop it from the notation.

Delegation graphs A *delegation graph* $G_n = ([n], E)$ on n voters is a directed graph with voters as vertices and a directed edge $(i, j) \in E$ denoting that i delegates their vote to j. Again, if n is clear from context, we occasionally drop it from the notation. The outdegree of a vertex in the delegation graph is at most 1 since each voter can delegate to at most one person. Voters that do not delegate have no outgoing edges. In a delegation graph G_n, the *delegations received* by a voter i, $\mathsf{dels}_i(G_n)$, is defined as the total number of people that (transitively) delegated to i in G_n, (i.e., the total number of ancestors of i in G_n). The *weight* of a voter i, $\mathsf{weight}_i(G_n)$, is $\mathsf{dels}_i(G_n) + 1$ (the number of delegation they received plus their own weight) if i votes directly, and 0 if i delegates. We define $\mathsf{max\text{-}weight}(G_n) = \max_{i \in [n]} \mathsf{weight}_i(G_n)$ to be the largest weight of any voter and define $\mathsf{total\text{-}weight}(G_n) = \sum_{i=1}^{n} \mathsf{weight}_i(G_n)$. Since each vote is counted at most once, we have that $\mathsf{total\text{-}weight}(G_n) \leq n$. However, note that if delegation edges form a cycle, then the weight of the voters on the cycle and voters delegating into the cycle are all set to 0 and hence will not be counted. In particular, this means that $\mathsf{total\text{-}weight}(G_n)$ may be strictly less than n.[5]

Delegation instances We call the tuple $(\vec{p}_n, G_n)$ a *delegation instance*, or simply an instance, on n voters. Let $V_i = 1$ if voter i would vote correctly if i did vote, and $V_i = 0$ otherwise. Fixed competencies $\vec{p}_n$ induce a probability measure $\mathbb{P}_{\vec{p}_n}$ over the n possible binary votes V_i, where $V_i \sim \text{Bern}(p_i)$. Given votes $V_1, \ldots, V_n$, we let X_n^D be the number of correct votes under direct democracy, that is, $X_n^D = \sum_{i=1}^{n} V_i$. We let $X_{G_n}^F$ be the number of correct votes under liquid democracy with delegation graph G_n, that is, $X_{G_n}^F = \sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot V_i$. The probability that direct democracy and liquid democracy are correct are $\mathbb{P}_{\vec{p}_n}[X_n^D > n/2]$ and $\mathbb{P}_{\vec{p}_n}[X_{G_n}^F > n/2]$, respectively.

[5]This is a worst-case approach where cycles can only hurt the performance of liquid democracy, since this assumption is equivalent to assuming that all voters on the cycles vote incorrectly.

Gain of a delegation instance We define the *gain* of an instance as

$$\text{gain}(\vec{p}_n, G_n) = \mathbb{P}_{\vec{p}_n}[X^F_{G_n} > n/2] - \mathbb{P}_{\vec{p}_n}[X^D_n > n/2].$$

In words, it is the difference between the probability that liquid democracy is correct and the probability that majority is correct.

Randomization over delegation instances In general, we assume that both competencies and delegations are chosen randomly. Each voter's competence p_i is sampled i.i.d. from a fixed distribution $\mathcal{D}$ with support contained in $[0, 1]$. Delegations will be chosen according to a *model* M. A model $M = (q, \varphi)$ is composed of two parts. The first $q : [0, 1] \to [0, 1]$ is a function that maps competencies to the probability that the voter delegates. The second $\varphi : [0, 1]^2 \to \mathbb{R}_{\geq 0}$ maps pairs of competencies to a weight. A voter i with competence p_i will choose how to delegate as follows:

- With probability $1 - q(p_i)$ they do not delegate.
- With probability $q(p_i)$, i delegates; i places weight $\varphi(p_i, p_j)$ on each voter $j \neq i$ and randomly sample another voter j to delegate to proportional to these weights. In the degenerate case where $\varphi(p_i, p_j) = 0$ for all $j \neq i$, we assume that i does not delegate.

A competence distribution $\mathcal{D}$, a model M, and a number n of voters induce a probability measure $\mathbb{P}_{\mathcal{D},M,n}$ over all instances $(\vec{p}_n, G_n)$ of size n.

We can now redefine the *do no harm (DNH)* and *positive gain (PG)* properties from Kahng et al. [129] in a probabilistic way.

Definition 1 (Probabilistic Do No Harm)**.** A model M satisfies *probabilistic do no harm* with respect to a class $\mathfrak{D}$ of distributions if, for all distributions $\mathcal{D} \in \mathfrak{D}$ and all $\varepsilon, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbb{P}_{\mathcal{D},M,n}[\text{gain}(\vec{p}_n, G_n) \geq -\varepsilon] > 1 - \delta.$$

Definition 2 (Probabilistic Positive Gain)**.** A model M satisfies *probabilistic positive gain* with respect to a class $\mathfrak{D}$ of distributions if there exists a distribution $\mathcal{D} \in \mathfrak{D}$ such that for all $\varepsilon, \delta > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$,

$$\mathbb{P}_{\mathcal{D},M,n}[\text{gain}(\vec{p}_n, G_n) \geq 1 - \varepsilon] > 1 - \delta.$$

Intuitively, note that positive gain and do no harm relate to the notion of concentration of the weighted sum $\sum_{i=1}^{n} \mathsf{weight}_i V_i$. Indeed, the probability of direct democracy being correct approaches 1 as n increases when the average competence is strictly above a half. As a result, do no harm is satisfied by a delegation model exactly when the probability that liquid democracy is correct also approaches 1. This happens when the expertise post-delegation remains strictly above a half and the weighted sum $\sum_{i=1}^{n} \mathsf{weight}_i V_i$ concentrates. Further, positive gain is verified if there exists a setup where the average group competence is strictly below a half, and the average expertise post-delegation remains strictly above a half and the weighted sum $\sum_{i=1}^{n} \mathsf{weight}_i V_i$ concentrates. In turn, these established benchmarks are directly mapped to existing concerns in social choice theory on the convergence of weighted majorities [110].

3.2.1 Core Lemma

Next, we give a key lemma, which provides sufficient conditions for a model M to satisfy probabilistic do no harm and probabilistic positive gain with respect to a class $\mathfrak{D}$ of distributions. This lemma will form the basis of all of our later results.

Lemma 13. *If M is a model, $\mathfrak{D}$ a class of distributions, n a number of persons, and for all distributions $\mathcal{D} \in \mathfrak{D}$, there is an $\alpha \in (0,1)$ and $C : \mathbb{N} \to \mathbb{N}$ with $C(n) \in o(n)$ such that*

$$\mathbb{P}_{\mathcal{D},M,n}\left[\mathsf{max\text{-}weight}(G_n) \le C(n)\right] = 1 - o(1) \tag{3.1}$$

$$\mathbb{P}_{\mathcal{D},M,n}\left[\sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \sum_{i=1}^{n} p_i \ge 2\alpha n\right] = 1 - o(1), \tag{3.2}$$

then M satisfies probabilistic do no harm. If in addition, there exists a distribution $\mathcal{D} \in \mathfrak{D}$ and an $\alpha \in (0,1)$ such that

$$\mathbb{P}_{\mathcal{D},M,n}\left[\sum_{i=1}^{n} p_i + \alpha n \le n/2 \le \sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \alpha n\right] = 1 - o(1), \tag{3.3}$$

then M satisfies probabilistic positive gain.

In words, condition (3.1) ensures that, as the number of voters grows large, the weighted number of correct votes under liquid democracy will concentrate around its expectation, $\sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i$. Standard concentration results already imply this holds for direct democracy. Condition (3.2) ensures that these expectations are sufficiently separated. So with high probability, liquid democracy will have more correct votes than direct democracy, which is sufficient to guarantee DNH. Finally, Condition (3.3) ensures that in some cases, the expectations for direct and liquid votes will be below and above half the voters, respectively, which after applying concentration means there will likely be a large gain.[6]

Throughout many of the proofs, we will make use of the following well-known concentra-

[6] Note that these are sufficient conditions only. Say $1/4$ of all voters delegate to a single voter i with competence $p_i < 1$. Here, if the remaining voters don't delegate and have average competence $> 2/3$, then, with high probability, even with delegations, the weighted majority will be correct. We could easily strengthen it to include instances like these where a single voter receives a linear number of votes or even other corner cases depending on the interplay between the weights and competence increase. Our result captures succinct, interesting, and general patterns that relate macro metrics and convergence. A more fine-grained approach, and interesting future direction, would be to use the notions of voters' effect or influence, designed by [110] in the case of weighted majority and weighted vote.

tion inequality [119]:

Lemma 14 (Hoeffding's Inequality). *Let $Z_1, \cdots, Z_n$ be independent, bounded random variables with $Z_i \in [a, b]$ for all i, where $-\infty < a \leq b < \infty$. Then*

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i - \mathbb{E}[Z_i] \geq t\right] \leq \exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$

and

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i - \mathbb{E}[Z_i] \leq -t\right] \leq \exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$

for all $t \geq 0$.

Armed with Lemma 14, we now prove Lemma 13.

Proof. We establish the two properties (3.1) and (3.2) separately.

Probabilistic do-no-harm: We first show that a model M that satisfies conditions (3.1) and (3.2) satisfies probabilistic do no harm. Fix an arbitrary competence distribution $\mathcal{D} \in \mathfrak{D}$ and let α and C be such that (3.1) and (3.2) are satisfied. Without loss of generality, suppose that $C(n) \leq n$ for all n, as replacing any larger values of $C(n)$ with n will not affect (3.1) (since $\mathsf{max\text{-}weight}(G_n) \leq n$ for all graphs G_n on n vertices). Fix $\varepsilon, \delta > 0$. We must identify some n_0 such that for all $n \geq n_0$, $\mathbb{P}_{\mathcal{D},M,n}[\text{gain}(\vec{p}_n, G_n) \geq -\varepsilon] > 1 - \delta$.

We will begin by showing there exists $n_1 \in \mathbb{N}$ such that for all instances $(\vec{p}_n, G_n)$ on $n \geq n_1$ voters, if both

$$\mathsf{max\text{-}weight}(G_n) \leq C(n) \text{ and} \tag{3.4}$$

$$\sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \sum_{i=1}^{n} p_i \geq 2\alpha n, \tag{3.5}$$

then

$$\text{gain}(\vec{p}_n, G_n) \geq -\varepsilon. \tag{3.6}$$

Since (3.4) and (3.5) each hold with probability $1-o(1)$ by (3.1) and (3.2), for sufficiently large n, say $n \geq n_2$, they will each occur with probability at least $1 - \delta/2$. Hence, by a union bound, for all $n \geq n_2$, they both occur with probability at least $1 - \delta$. By taking $n_0 = \max(n_1, n_2)$, this implies that probabilistic do no harm is satisfied.

We now prove that, for sufficiently large n, (3.4) and (3.5) imply (3.6). First, we will show that

$$\text{gain}(\vec{p}_n, G_n) \geq -\mathbb{P}_{\vec{p}_n}[X_n^D > X_{G_n}^F]. \tag{3.7}$$

Indeed, we have that

$$\begin{aligned}
\mathbb{P}_{\vec{p}_n}[X_n^D > n/2] &= \mathbb{P}_{\vec{p}_n}[X_n^D > n/2, X_{G_n}^F > n/2] + \mathbb{P}_{\vec{p}_n}[X_n^D > n/2, X_{G_n}^F \leq n/2] \\
&\leq \mathbb{P}_{\vec{p}_n}[X_{G_n}^F > n/2] + \mathbb{P}_{\vec{p}_n}[X_n^D > X_{G_n}^F]
\end{aligned}$$

where the first transition holds by the law of total probability, and the second because the corresponding events are contained in each other. That is,

$$\left\{X_n^D > n/2, X_{G_n}^F > n/2\right\} \subseteq \left\{X_{G_n}^F > n/2\right\}$$

and

$$\left\{X_n^D > n/2, X_{G_n}^F \leq n/2\right\} \subseteq \left\{X_n^D > X_{G_n}^F\right\}.$$

Re-arranging the terms above yields (3.7).

Hence, for our purpose, it suffices to show that (3.4) and (3.5) imply $\mathbb{P}_{\vec{p}_n}\left[X_n^D > X_{G_n}^F\right] \leq \varepsilon$. Intuitively, we will use (3.5) to show the expected value of X_n^D is well below the expected value of $X_{G_n}^F$. Then we will show both X_n^D and $X_{G_n}^F$ concentrate well around their means, where for the latter we will need (3.4). Together, these observations imply that $X^F_{G_n} > X^D_n$ with high probability.

Fix an instance $(\vec{p}_n, G_n)$ on n voters satisfying (3.4) and (3.5). We will show that for sufficiently large n,

$$\mathbb{P}_{\vec{p}_n}\left[X^D_n < \sum_{i=1}^{n} p_i + \alpha n\right] > 1 - \varepsilon/2 \tag{3.8}$$

and

$$\mathbb{P}_{\vec{p}_n}\left[X^F_{G_n} > \sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \alpha n\right] > 1 - \varepsilon/2. \tag{3.9}$$

Note that since (3.5) holds for this instance, $\sum_{i=1}^n p_i + \alpha n \leq \sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i - \alpha n$. Therefore, when both events whose probability is considered in (3.8) and (3.9) hold, $X^D_n \leq X^F_n$. Hence,

$$\mathbb{P}_{\vec{p}_n}[X^D_n \leq X^F_{G_n}] \geq \mathbb{P}_{\vec{p}_n}\left[X^D_n < \sum_{i=1}^{n} p_i + \alpha n, X^F_{G_n} > \sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \alpha n\right] > 1 - \varepsilon$$

where the last inequality holds by a union bound. This implies that $\mathbb{P}_{\vec{p}_n}[X^D_n \leq X^F_{G_n}] < \varepsilon$, as needed.

It remains to be shown that (3.8) and (3.9) hold for sufficiently large n. For (3.8), this follows directly from Hoeffding's inequality (Lemma 14). To prove (3.9), first note that, as shown in Kahng et al. [129],

$$\begin{aligned}
\mathrm{Var}_{\vec{p}_n}\left[X^F_{G_n}\right] &= \sum_{i=1}^{n} \mathsf{weight}_i(G_n)^2 \cdot p_i(1-p_i) \\
&\leq \frac{1}{4} \cdot \sum_{i=1}^{n} \mathsf{weight}_i(G_n)^2 \\
&\leq \frac{1}{4} \cdot \sum_{i=1}^{\lceil n/C(n) \rceil} C(n)^2 \\
&< nC(n) \in o(n^2),
\end{aligned}$$

where the first inequality holds because $p(1-p)$ is upper bounded by $1/4$, the second because $\sum_{i=1}^{n} \mathsf{weight}_i(G_n) \leq n$ with each $\mathsf{weight}_i(G_n) \leq C(n)$ so the value is maximized by setting as many terms to $C(n)$ as possible, and the final inequality holds because $C(n) \leq n$.

Hence, by Chebyshev's inequality,

$$\mathbb{P}_{\vec{p}_n}\left[X_{G_n}^F \leq \mathbb{E}_{\vec{p}_n}\left[X_{G_n}^F\right] - \alpha n\right] \leq \frac{\mathrm{Var}_{\vec{p}_n}[X_{G_n}^F]}{(\alpha n)^2}.$$

This bound is $o(1)$ because the numerator is $o(n^2)$ and the denominators is $\Omega(n^2)$. This implies that for sufficiently large n, it will be strictly less than $\varepsilon/2$, so (3.9) holds.

Probabilistic positive gain: Fix a distribution $\mathcal{D} \in \mathfrak{D}$ and an $\alpha \in (0,1)$ such that (3.3) holds. We want to show that M satisfies probabilistic positive gain. Since $\mathcal{D} \in \mathfrak{D}$, it also satisfies (3.1) for some C. We show below that there exists an n_3 such that all instances $(\vec{p}_n, G_n)$ with $n \geq n_3$ voters satisfying (3.4) for which $\sum_{i=1}^{n} p_i + \alpha n \leq n/2 \leq \sum_{i=1}^{n} \mathsf{weight}_i(G_n) \cdot p_i - \alpha n$, we have that $\mathrm{gain}(\vec{p}_n, G_n) \geq 1 - \varepsilon$. As with the DNH part of the proof, since the events of (3.1) and (3.3) each hold with probability $1 - o(1)$, for sufficiently large n, say $n \geq n_4$, they each occur with probability at least $1 - \delta/2$. Hence, by a union bound, for all $n \geq n_4$, they both occur with probability $1 - \delta$. For $n_0 = \max(n_3, n_4)$, probabilistic positive gain is satisfied.

It remains to show that that if (3.1) and (3.3) hold for a specific instance $(\vec{p}_n, G_n)$, then $\mathrm{gain}(\vec{p}_n, G_n) \geq 1 - \varepsilon$ for sufficiently large n. Since $\mathcal{D} \in \mathfrak{D}$, (3.8) and (3.9) are both satisfied for sufficiently large n. When

$$\sum_{i=1}^{n} p_i + \alpha n \leq n/2 \leq \sum_{i=1}^{n} p_i - \mathsf{weight}_i(G_n) \cdot \alpha n$$

is satisfied as well, we get that $\mathbb{P}_{\vec{p}_n}\left[X_n^D > n/2\right] < \varepsilon/2$ and $\mathbb{P}_{\vec{p}_n}\left[X_{G_n}^L > n/2\right] > 1 - \varepsilon/2$, so $\mathrm{gain}(\vec{p}_n, G_n) > 1 - \varepsilon$ is immediate. $\square$

In the following sections, we investigate natural delegation models and identify conditions such that the models satisfy probabilistic do no harm and probabilistic positive gain. In all instances, we will invoke Lemma 13 after showing that its sufficient conditions are satisfied.

3.3 Strictly Upward Delegation Model

We now turn to the analysis of a simple model that assumes that voters either do not delegate with fixed exogenous probability or delegate to voters that have a competence greater than their own.

Formally, for a fixed $p \in [0, 1]$ we let $M_p^U = (q, \varphi)$ be the model consisting of $q(p_i) = p$ for all $p_i \in [0, 1]$, and $\varphi(p_i, p_j) = \mathbb{I}_{\{p_j > p_i\}}$ for all $i, j \in [n]$. That is, voter i delegates with fixed probability p and puts equal weight on all the more competent voters. In other words, if voter i delegates, then i does so to a more competent voter chosen uniformly at random. Note that a voter with maximal competence will place 0 weight on all other voters, and hence is guaranteed not to delegate. We refer to M_p^U as the *Upward Delegation Model* parameterized by p.

Theorem 5 (Upward Delegation Model). *For all $p \in (0, 1)$, M_p^U satisfies probabilistic do no harm and probabilistic positive gain with respect to the class $\mathfrak{D}^C$ of all continuous distributions.*

The proof of the theorem relies on novel bounds we drive on the largest bin size in an infinite Pólya's urn process [49, 216]. We first formally define the process and present our bound in Lemma 15. A *Pólya's urn process with attachment probability* p begins at time $t = 1$ with one ball in one bin. At each timestep $t > 1$, a new ball arrives. With probability $1 - p$, a new bin is created and the new ball is placed in that bin; with probability p, the ball joins an existing bin, and it does so with probability proportional to the number of balls in the bins, i.e., if there are three bins containing 1, 2, and 3 balls respectively, it joins each with probability $1/6, 2/6$, and $3/6$ respectively. We then have the following.

Lemma 15. *For all $p \in (0,1)$ and $t \geq 1$, let L_t^p be the random variable denoting the maximum number of balls in any bin after running the infinite Pólya's urn process with new-bin probability p for t steps. Then, there exists $\delta < 1$ depending only on p such that for all $T \geq 1$, $\Pr[L_T^p \leq T^\delta] = 1 - o(1)$.*

Proof. Fix the parameter $p \in (0,1)$. Choose γ to be a constant such that $3/4 < \gamma < 1$; note that $p + (1-p)\gamma < p + (1-p) = 1$. Choose δ (for the lemma statement) such that $p + (1-p)\gamma < \delta < 1$. Notice that we can choose γ and δ such that δ is arbitrarily close to $3/4 + p/4$.

Let $B^{(k)}$ denote the k-th bin. Let $U_t^{(k)}$ be the size of $B^{(k)}$ at time t. Since there are at most t bins by time t, notice that $L_t^p = \max(U_t^{(1)}, \ldots, U_t^{(t)})$. In general, our approach will be to analyze bins separately and show that $U_T^{(k)}$ remains below T^δ with high enough probability so that we can union bound over all possible $k \leq T$. That is, we will show

$$\sum_{k=1}^{T} \Pr[U_T^{(k)} > T^\delta] = o(1),$$

which also implies $\Pr[L_T^p > T^\delta] = o(1)$. Hence, it will be useful to consider this process more formally from the perspective of the kth bin, $B^{(k)}$. The kth bin $B^{(k)}$ is "born" at some time $t \geq k$, the kth time in which a ball does not join a preexisting bin, at which point $U_t^{(k)} = 1$ (prior to this, $U_t^{(k)} = 0$). More specifically, the first bin $B^{(k)}$ is guaranteed to be born at time $t = 1$ and for all other $k > 1$, $B^{(k)}$ will be born at time $t \geq k$ with probability $\binom{t-1}{k-1}(1-p)^k p^{t-k}$, although these exact probabilities will be unimportant for our analysis. Once born, we have the following recurrence on $U_t^{(k)}$ describing the probability $B^{(k)}$ will be chosen at time t:

$$U_t^{(k)} = \begin{cases} U_{t-1}^{(k)} + 1 & \text{with probability } \frac{p \cdot U_{t-1}^{(k)}}{t-1} \\ U_{t-1}^{(k)} & \text{with probability } 1 - \frac{p \cdot U_{t-1}^{(k)}}{t-1}. \end{cases}$$

Let $W_t^{(k)}$ be the process for the size of bin that is born at time k. That is, $W_k^{(k)} = 1$, and for $k > t$, $W_t^{(k)}$ follows the exact same recurrence as $U_t^{(k)}$. Note that since the kth bin $B^{(k)}$ can only be born at time k or later, we have that $W_t^{(k)}$ stochastically dominates $U_t^{(k)}$ for all k and t. Hence, it suffices to show that

$$\sum_{k=1}^{T} \Pr[W_T^{(k)} > T^{\delta}] = o(1). \tag{3.10}$$

We split our analysis into two parts: the first consider the first T^{γ} bins, while the second considers the last $T - T^{\gamma}$ bins.

We first show that $\sum_{k=1}^{T^{\gamma}} \mathbb{P}[W_T^{(k)} > T^{\delta}] = o(1)$. Note that the expectation of $W_n^{(k)}$

$$\mathbb{E}[W_n^{(k)}] = \frac{\Gamma(n+p)\Gamma(k)}{\Gamma(p+k)\Gamma(n)} \tag{3.11}$$

for all $k \leq n$, where Γ represents the Gamma function.

$$\mathbb{E}[W_T^{(k)}] = \frac{\Gamma(T+p)\Gamma(k)}{\Gamma(p+k)\Gamma(T)}$$

for all $k \leq T$, where Γ represents the Gamma function. The second was showing

$$\sum_{k=1}^{T^{\gamma}} \mathbb{P}[W_T^{(k)} > n^{\delta}] = o(1).$$

Recall that $W_k^{(k)} = 1$ and we have the following recurrence for all $t > k$:

$$W_t^{(k)} = \begin{cases} W_{t-1}^{(k)} + 1 & \text{with probability } \frac{p \cdot W_{t-1}^{(k)}}{t-1} \\ W_{t-1}^{(k)} & \text{with probability } 1 - \frac{p \cdot W_{t-1}^{(k)}}{t-1}. \end{cases}$$

By the tower property of expectation, for all $t \geq k+1$,

$$\begin{aligned} \mathbb{E}[W_t^{(k)}] &= \mathbb{E}[\mathbb{E}[W_t^{(k)} \mid W_{t-1}^{(k)}]] \\ &= \mathbb{E}[W_{t-1}^{(k)}(1 + \frac{p}{t-1})] \\ &= \mathbb{E}[W_{t-1}^{(k)}](1 + \frac{p}{t-1}). \end{aligned}$$

Thus, by a straightforward induction argument and the fact that $\mathbb{E}[W_k^{(k)}] = 1$,

$$\mathbb{E}[W_T^{(k)}] = \mathbb{E}[W_k^{(k)}] \prod_{i=k}^{T-1} (1 + \frac{p}{i}) = \prod_{i=k}^{T-1} (1 + \frac{p}{i}).$$

Expanding this, we have

$$\begin{aligned} \prod_{i=k}^{T-1} (1 + \frac{p}{i}) &= \prod_{i=k}^{T-1} \frac{i+p}{i} \\ &= \frac{1}{\prod_{i=k}^{T-1} i} \cdot \prod_{i=k}^{T-1} (i+p) \\ &= \frac{(k-1)!}{(T-1)!} \cdot \frac{\prod_{i=0}^{T-1}(i+p)}{\prod_{i=0}^{k-1}(i+p)} \\ &= \frac{(k-1)!}{(T-1)!} \frac{\frac{\Gamma(p+T)}{\Gamma(p)}}{\frac{\Gamma(k+p)}{\Gamma(p)}} \\ &= \frac{\Gamma(T+p)\Gamma(k)}{\Gamma(p+k)\Gamma(T)}, \end{aligned}$$

where the fourth equality holds because $\Gamma(x+1) = x\Gamma(x)$ for all $x \in \mathbb{R}$, and the last uses the fact that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. This proves (3.11).

Using this along with Gautchi's inequality [96], $(t+p-1)^p \leq \frac{\Gamma(p+t)}{\Gamma(t)} \leq (t+p)^p$, to approximate the Γ terms, we can apply Markov's inequality and use algebra to get $\sum_{k=1}^{n^\gamma} \mathbb{P}[W_n^k > n^\delta] = o(1)$.

In detail, we use Markov's inequality to show that for all k,

$$\Pr\left[W_T^{(k)} > T^\delta\right] \leq \frac{\mathbb{E}[W_T^{(k)}]}{T^\delta} = \frac{1}{T^\delta} \cdot \frac{\Gamma(T+p)}{\Gamma(T)} \cdot \frac{\Gamma(k)}{\Gamma(k+p)}.$$

Hence,

$$\sum_{k=1}^{T^\gamma} \Pr[W_T^{(k)} > T^\delta] \leq \sum_{k=1}^{T^\gamma} \frac{1}{T^\delta} \cdot \frac{\Gamma(T+p)}{\Gamma(T)} \cdot \frac{\Gamma(k)}{\Gamma(k+p)} = \frac{1}{T^\delta} \cdot \frac{\Gamma(T+p)}{\Gamma(T)} \cdot \sum_{k=1}^{T^\gamma} \frac{\Gamma(k)}{\Gamma(k+p)}.$$

What remains to be shown is that

$$\frac{1}{T^\delta} \cdot \frac{\Gamma(T+p)}{\Gamma(T)} \cdot \sum_{k=1}^{T^\gamma} \frac{\Gamma(k)}{\Gamma(k+p)} = o(1).$$

To do this, we will use Gautschi's inequality [96] which states that for all $x > 0$, since $p \in (0,1)$,

$$(x+p-1)^p \leq \frac{\Gamma(p+x)}{\Gamma(x)} \leq (x+p)^p$$

We then have that

$$
\begin{aligned}
\frac{1}{T^\delta} \cdot \frac{\Gamma(T+p)}{\Gamma(T)} \cdot \sum_{k=1}^{T^\gamma} \frac{\Gamma(k)}{\Gamma(k+p)} &\leq \frac{(T+p)^p}{T^\delta} \cdot \sum_{k=1}^{T^\gamma} \frac{1}{(k+p-1)^p} \\
&= \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \sum_{k=3}^{T^\gamma} \frac{1}{(k+p-1)^p} \right) \\
&\leq \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \sum_{k=3}^{T^\gamma} \frac{1}{(k-1)^p} \right) \\
&= \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \sum_{k=2}^{T^\gamma-1} \frac{1}{k^p} \right) \\
&\leq \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \sum_{k=2}^{T^\gamma} \frac{1}{k^p} \right) \\
&\leq \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \int_1^{T^\gamma} \frac{1}{x^p}\, dx \right) \\
&= \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{1}{p^p} + \frac{1}{(1+p)^p} + \frac{x^{1-p}}{1-p} \Big|_1^{T^\gamma} \right) \\
&= \frac{(T+p)^p}{T^\delta} \cdot \left(\frac{T^{\gamma(1-p)}}{1-p} + \frac{1}{p^p} + \frac{1}{(1+p)^p} - \frac{1}{1-p} \right).
\end{aligned}
$$

Notice that asymptotically, this upper bound is $O(T^{-\delta+p+\gamma\cdot(1-p)})$. By our choice of δ, $\delta > p + \gamma \cdot (1-p)$, so this implies that it is is $o(1)$, as desired.

Now consider the final $T - T^\gamma$ components. We will prove that $\Pr[W_T^{(T^\gamma+1)} > T^\delta] = o(1/T)$. Since $W_T^{(k)}$ stochastically dominates $W_T^{(k')}$ for all $k' \geq k$, this implies that $\Pr[W_T^{(k)} > T^\delta] = o(1/T)$ for all $k \geq T^\gamma + 1$. Hence,

$$\sum_{k=T^\gamma+1}^{T} \Pr\left[W_T^{(k)} > T^\delta\right] = o(1).$$

To do this, we compare the $W_t^{(T^\gamma+1)}$ process to another process, V_t. We define $V_0 = 1$, and for $t > 0$, take V_t to satisfy the following recurrence:

$$V_t = \begin{cases} V_{t-1} + 1 & \text{with probability } \frac{V_{t-1}}{t+n^\gamma} \\ V_{t-1} & \text{with probability } 1 - \frac{V_{t-1}}{t+n^\gamma}. \end{cases}$$

This is identical to the W recurrence with t shifted down by $n^\gamma+1$ except without the p factor. Hence, $V_{T-T^\gamma+1}$ clearly stochastically dominates $W_T^{(T^\gamma+1)}$. For convenience in calculation, we will instead focus on bounding V_T which itself stochastically dominates $V_{T-T^\gamma+1}$.

Next, note that the V_t process is isomorphic to the following classic Pólya's urn process. We begin with two bins, one with a single ball and the other with n^γ balls. At each time, a new ball is added to one of the two bins with probability proportional to the bin size. The process V_t is isomorphic to the size of the one-ball urn after t steps. Classic results tell us that for fixed starting bin sizes a and b, as the number of steps grows large, the possible proportion of balls in the a-bin follows a Beta(a, b) distribution [76, 127, 158, 168, 193].

The mean and variance of such a Beta distribution would be sufficient to prove our necessary concentration bounds; however, for us, we need results after exactly $T - T^\gamma$ steps, not simply in the limit. Hence, we will be additionally concerned with the speed of convergence to this Beta distribution.

Let $X_T = \frac{V_T}{T}$ and $Z_T \sim \text{Beta}(1, T^\gamma)$. From Janson [125], we know that the rate of convergence is such that, for any $p \geq 1$

$$\ell_p(X_T, Z_T) = \Theta(1/T) \tag{3.12}$$

where ℓ_p is the *minimal L_p metric*, defined as

$$\ell_p(X, Y) = \inf \left\{ \mathbb{E}[|X' - Y'|^p]^{1/p} \mid X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y \right\},$$

which can be thought of as the minimal L_p norm over all possible couplings between X and Y. For our purposes, the only fact about the ℓ_p metric we will need is that $\ell_p(X,0) = \mathbb{E}[|X|^p]^{1/p}$ where 0 is the identically 0 random variable. Since ℓ_p is in fact a metric, the triangle inequality tells us that $\ell_p(0, X_n) \leq \ell_p(0, Z_n) + \ell_p(Z_n, X_n)$, so, combining with (3.12), we have that

$$\mathbb{E}[|X_T|^p]^{1/p} \leq \mathbb{E}[|Z_T|^p]^{1/p} + \Theta(1/T) \tag{3.13}$$

for all $p \geq 1$.

Note that since $Z_T \sim \text{Beta}(1, T^\gamma)$,

$$\mathbb{E}[Z_T] = \frac{1}{1 + T^\gamma} = \Theta(T^{-\gamma})$$

and

$$\text{Var}[Z_T] = \frac{T^\gamma}{(2 + T^\gamma)(1 + T^\gamma)^2} = \Theta(T^{-2\gamma}).$$

Given these results, we are ready to prove that V_T is smaller than T^δ with probability $1 - o(1/T)$. Precisely, we want to show that $\Pr[X_T \geq T^{\delta-1}] = o(1)$. By Chebyshev's inequality,

$$\Pr[X_T \geq T^{\delta-1}] \leq \frac{\text{Var}[X_T]}{(T^{\delta-1} - \mathbb{E}[X_T])^2}.$$

Inequality (3.13) with $p = 1$ along with the fact that X_T and Z_T are always nonnegative implies that $\mathbb{E}[X_T] \leq \mathbb{E}[Z_T] + \Theta(1/T) = O(T^{-\gamma})$. Hence, $T^{\delta-1} - \mathbb{E}[X_T] = \Omega(T^{\delta-1})$ since $\delta - 1 > -1/2 > -\gamma$. We can therefore write:

$$\left(T^{\delta-1} - \mathbb{E}[X_T]\right)^2 = \Omega(T^{-2(\delta-1)}). \tag{3.14}$$

Inequality (3.13) with $p = 2$ implies that $\sqrt{\mathbb{E}[X_T^2]} \leq \sqrt{\mathbb{E}[Z_T^2]} + \Theta(1/T)$. Hence,

$$\begin{aligned}
\mathbb{E}[X_T^2] &\leq (\Theta(1/T) + \sqrt{\mathbb{E}[Z_T^2]})^2 \\
&\leq (\Theta(1/T) + \sqrt{\mathbb{E}[Z_T]^2 + \mathrm{Var}[Z_T]})^2 \\
&\leq (\Theta(1/T) + \sqrt{\Theta(T^{-2\gamma})})^2 \\
&= (\Theta(1/T) + \Theta(T^{-\gamma}))^2 \\
&= \Theta(T^{-\gamma})^2 \\
&= \Theta(T^{-2\gamma}).
\end{aligned}$$

Next, note that $\mathrm{Var}[X_T] \leq \mathbb{E}[X_T^2]$, so

$$\mathrm{Var}[X_T] = O(T^{-2\gamma}) \tag{3.15}$$

as well. Combining (3.14) and (3.15), we have that

$$\Pr[X_T \geq T^{\delta-1}] \leq \frac{\mathrm{Var}[X_T]}{(T^{\delta-1} - \mathbb{E}[X_T])^2} = O\left(T^{-2\gamma+2(1-\delta)}\right).$$

Since $-2\gamma + 2(1-\delta) < 1$, given our assumption that $3/4 < \gamma < \delta$, it follows that $\Pr[X_T \geq T^{\delta-1}] = o(1/T)$, which allows us to conclude that

$$\sum_{k=T^\gamma+1}^{T} \Pr[W_T^{(k)} > T^\delta] = o(1).$$

Since we showed earlier that $\sum_{k=1}^{T^\gamma} \Pr[W_T^{(k)} > T^\delta] = o(1)$, we have that

$$\sum_{k=1}^{T} \Pr[W_T^{(k)} > T^\delta] = o(1),$$

as needed. $\square$

We are now ready to prove the theorem about Upward Delegation.

Proof of Theorem 5. To prove the theorem, we will prove that the Upward Delegation Model with respect to $\mathfrak{D}^C$ satisfies (3.1), (3.2), (3.3), which implies that the model satisfies probabilistic do no harm and positive gain by Lemma 13.

Upward Delegation satisfies (3.1)

To do this, we will simply show that the component sizes in G_n sampled according to $\mathbb{P}_{D,M,n}$ have the same distribution as the bin sizes in a Pólya's urn process with attachment probability p, and hence $\textsf{max-weight}(G_n)$ follows the same distribution as L_n^p. Once we have shown this, (3.1) follows immediately from Lemma 15 as $n^\delta \in o(n)$.

To that end, fix some sampled competencies $\vec{p}_n$. Recall that each entry p_i in $\vec{p}_n$ is sampled i.i.d. from $\mathcal{D}$, a continuous distribution. Hence, almost surely, no two competencies are equal. From now on, we condition on this probability 1 event. Now consider sampling the delegation graph G_n. By the design of the model M_p^U, we can consider a random process for generating G_n that is isomorphic to sampling according to $\mathbb{P}_{D,M,n}$ as follows: first, order the competencies $p_{(1)} > p_{(2)} > \cdots > p_{(n)}$ (note that such strict order is possible by our assumption that all competencies are different) and rename the voters such that voter i has competence $p_{(i)}$; then construct G_n iteratively by adding the voters one at a time in decreasing order of competencies, voter 1 at time 1, voter 2 at time 2, and so on.

We start with the voter with the highest competence, voter 1. By the choice of φ, voter 1 places weight 0 on every other voter and hence by definition does not delegate. This voter forms the first component in the graph G_n, which we call $C^{(1)}$. Then, we add voter 2 who either delegates to voter 1 joining component $C^{(1)}$ with probability p, or starts a new component $C^{(2)}$ with probability $1 - p$. Next, we add voter 3. If $2 \in C^{(1)}$ (that is, if 2 delegated to 1), 3 either delegates to 1 (either directly or through 2 by transitivity) with probability p or she starts a new component $C^{(2)}$. If $2 \in C^{(2)}$, then 3 either delegates to 1 with probability $p/2$ and is added to $C^{(1)}$, or delegates to 2 with probability $p/2$ and is added to $C^{(2)}$, or starts a new component $C^{(3)}$. In general, at time t, if there are k existing components $C^{(1)}, \dots, C^{(k)}$, voter t either joins each component $C(j)$ with probability $\frac{p|C(j)|}{t-1}$ or starts a new component with probability $1-p$. To construct G_n, we run this process for n steps. Notice that this is identical to the Pólya's urn process with bins $B^{(k)}$ and balls replaced with components $C^{(k)}$ and voters being run for n steps, as needed.

Upward Delegation satisfies (3.2)

We will show there exists $\alpha \in (0,1)$ such that $\sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i - \sum_{i=1}^n p_i \geq 2\alpha n$ with high probability, so (3.2) is satisfied. Note that in the present scheme, cycles are impossible, so do need to worry about ignored voters.

Since $\mathcal{D}$ is a continuous distribution, there exists $a < b$ such that $\pi_a := \mathcal{D}[\{p : p < a\}] > 0$ and $\pi_b := \mathcal{D}[\{p : p > b\}] > 0$. Let $N_{a,n}(\vec{p}_n)$ be the number of voters in $\vec{p}_n$ with competence $p_i < a$ and $N_{b,n}(\vec{p}_n)$ be the number of voters with competence $p_i > b$. When we sample competencies, since each is chosen independently, $N_{a,n} \sim \mathrm{Bin}(n, \pi_a)$ and $N_{b,n} \sim \mathrm{Bin}(n, \pi_b)$. By Hoeffding's inequality (Lemma 14) and the union bound, with probability $1 - o(1)$, there will be at least $\pi_a/2 \cdot n$ voters with competence $p_i < a$ and $\pi_b/2 \cdot n$ voters with competence $p_i > b$. Indeed,

$$
\begin{aligned}
\mathcal{D}^n[N_{a,n} > \frac{n\pi_a}{2}, N_{b,n} > \frac{n\pi_b}{2}] &= 1 - \mathcal{D}^n[\{N_{a,n} \leq \frac{n\pi_a}{2}\} \cup \{N_{b,n} \leq \frac{n\pi_b}{2}\}] \\
&\geq 1 - (\mathcal{D}^n[N_{a,n} \leq \frac{n\pi_a}{2}] + \mathcal{D}^n[N_{b,n} \leq \frac{n\pi_b}{2}]) \\
&\geq 1 - \exp(-\frac{n\pi_a^2}{2}) - \exp(-\frac{n\pi_b^2}{2}),
\end{aligned} \tag{3.16}
$$

where the first line comes from De Morgan's law, the second from the union bound, and the last from Heoffding's inequality (Lemma 14).

Conditioned on this occurring, each voter with competence $p_i < a$ has probability at least $p\pi_b/2$ of delegating to a voter with competence at least b. As they each decide to do this independently, the number $N_{ab,n}$ of n voters deciding to do this stochastically dominates a random variable following the $\text{Bin}(\pi_a/2 \cdot n, p \cdot \pi_b/2)$ distribution. We can again apply Hoeffding's inequality to conclude that with probability $1 - o(1)$, at least $\pi_a \cdot \pi_b \cdot p/8 \cdot n$ voters do so. Indeed,

$$\begin{aligned}\mathcal{D}[N_{ab,n} > \frac{np\pi_a\pi_b}{8} \mid N_{a,n} > \frac{n\pi_a}{2}, N_{b,n} > \frac{n\pi_b}{2}] &\geq \mathcal{D}[\text{Bin}(\frac{n\pi_a}{2}, \frac{p\pi_b}{2}) > \frac{np\pi_a\pi_b}{8}] \\ &\geq 1 - \exp(-\frac{np^2\pi_a\pi_b^2}{4}),\end{aligned} \tag{3.17}$$

where the first inequality holds because $N_{ab,n}$ stochastically dominates the corresponding binomial random variable and the second holds by Hoeffding's inequality. Finally, using (3.16) and (3.17), we have

$$\begin{aligned}\mathcal{D}[N_{ab,n} > \frac{np\pi_a\pi_b}{8}] \geq \mathcal{D}[N_{ab,n} > \frac{np\pi_a\pi_b}{8} \mid N_{a,n} > \frac{n\pi_a}{2}, N_{b,n} > \frac{n\pi_b}{2}] \\ \cdot \mathcal{D}[N_{a,n} > \frac{n\pi_a}{2}, N_{b,n} > \frac{n\pi_b}{2}] \\ \geq 1 - o(1).\end{aligned}$$

Under these upward delegation models, delegations can only increase the total competence of all voters. Hence,

$$\sum_{i=1}^{n} \mathsf{dels}_i(G_n) \cdot p_i - \sum_{i=1}^{n} p_i \geq (b-a)N_{ab.n}.$$

Each of these $\pi_a \cdot \pi_b \cdot p/8 \cdot n$ voters results in a competence increase of at least $b - a$. Hence, under these high probability events, the total competence increase is at least $(b-a)\cdot\pi_a\cdot\pi_b\cdot p/8\cdot n$. Indeed, since $\mathcal{D}[N_{ab,n} > \frac{np\pi_a\pi_b}{8}] = 1 - o(1)$, this implies $\mathcal{D}[\sum_{i=1}^{n} \mathsf{dels}_i(G_n) \cdot p_i - \sum_{i=1}^{n} p_i > \frac{np\pi_a\pi_b}{8}] = 1 - o(1)$. By choosing $\alpha = \frac{p\pi_a\pi_b}{8}(b-a)$, we see that there is an $\alpha \cdot n$ increase in competence with high probability, as needed.

We have proved that for any continuous distribution $\mathcal{D}$, and for well-behaved realizations of $\vec{p}_n$ which occur with high probability, the graph generated from the random delegation process yields an increase in the expected sum of the votes of at least $\alpha \cdot n$. We can then conclude that M_p^U satisfies Equation (3.2) with respect to the class of continuous distributions.

Upward Delegation satisfies (3.3)

We now show that there exists a distribution $\mathcal{D}$ such that $\sum_{i=1}^n p_i + \alpha n \leq n/2 \leq \sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i - \alpha n$ with probability $1 - o(1)$ for some $\alpha > 0$. This implies that the model satisfies probabilistic positive gain by Lemma 13, and will conclude the proof.

We take $\mathcal{D}$ to be $\mathcal{D}_\eta$, the uniform distribution $\mathcal{U}[0, 1-2\eta]$ for some small $0 < \eta < p/512$. Let $\alpha = \eta/2$. Clearly, $\mu_{\mathcal{D}_\eta}$, the mean of D_η, is $1/2 - \eta$. Since each $p_i \overset{i.i.d.}{\sim} \mathcal{D}_\eta$, the p_is are bounded independent random variables with mean $1/2 - \eta$, so Hoeffding's inequality directly implies that $\sum_{i=1}^n p_i \leq n/2 - n\eta/2 = n/2 - n\alpha$ with high probability.

Now consider $\mathcal{E}_F$, the event consisting of instances $(\vec{p}_n, G_n)$ such that $\sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i \geq n/2 + n\alpha$. We denote by $\mathcal{E}_D$ the event that $\sum_{i=1}^n p_i \geq n/2 - 3n\eta/2$. The same reasoning as before implies that $\Pr_{\mathcal{D}_\eta, M_p^U, n}(\mathcal{E}_D) = 1 - o(1)$.

Let $a = 1/4 - \eta/2$ and $b = 1/2 - \eta$, so we have that $\pi_a := \mathcal{D}_\eta[p_i < a] = 1/4$ and $\pi_b := \mathcal{D}_\eta[p_i > b] = 1/2$. We proved in the preceding derivation that $\sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i - \sum_{i=1}^n p_i > \frac{np\pi_1\pi_b}{8} = \frac{np}{64}(1-\eta)$ with high probability. Hence, if both this and $\mathcal{E}_D$ occur, which is the case with high probability, by the union bound, it follows that $\sum_{i=1}^n \mathsf{weight}_i(G_n) \cdot p_i > n/2 + n(\frac{p}{128}(1-\eta) - 3\eta/2)$ with high probability.

Since $\eta < \frac{p}{512} < 1/2$, we have that

$$\frac{p}{128}(1-\eta) - 3\eta/2 > \frac{p}{256} - 3\eta/2 > 2\eta - 3\eta/2 = \eta/2 = \alpha,$$

and we can conclude that $\mathcal{E}_F$ occurs with high probability. Hence, M_p^U satisfies Equation (3.3). $\square$

3.4 Confidence-Based Delegation Model

We now explore a model according to which voters delegate with probability that is strictly decreasing in their competence and when they do decide to delegate, they do so by picking a voter uniformly at random. This models the case where voters do not need to know anything about their peers' competencies, but do have some sense of their own competence, and delegate accordingly.

Formally, for any q, let $M_q^C = (q, \varphi^1)$ where $\varphi^1(p_i, p_j) = 1$ for all $i, j \in [n]$. Voter i puts equal weight on all the voters and hence samples one uniformly at random when they delegate. We refer to M_q^C as the Confidence-Based Delegation Model.

Theorem 6 (Confidence-Based Delegation Model)**.** *All models M_q^C with monotonically decreasing q satisfy probabilistic do no harm and probabilistic positive gain with respect to the class $\mathfrak{D}^C$ of all continuous distributions.*

Proof. We show that the Confidence-Based Model satisfy (3.1), (3.2) and (3.3).

Confidence-Based Delegation satisfies (3.1)

Fix some distribution $\mathcal{D} \in \mathfrak{D}^C$. We show there exists $C(n) \in O(\log n)$ such that (3.1) holds.

Note that when sampling an instance $(\vec{p}_n, G_n)$, the probability an arbitrary voter i chooses to delegate is precisely $p := \mathbb{E}_{\mathcal{D}}[q]$. To see this, consider how a voter i chooses whether to delegate: they first sample a competence $p_i \sim \mathcal{D}$ and then sample whether or not to delegate from $\text{Bern}(q(p_i))$. Treating this as a single process, it is clear that the overall probability of choosing to delegate is exactly $\mathbb{E}_{\mathcal{D}}[q]$ by integrating out the competence.

Further, since $\mathcal{D}$ is continuous and q is monotonically decreasing, $p \in (0,1)$. When a voter does decide to delegate, they do so by picking another voter uniformly at random. Hence, we can consider the marginal distribution of delegation graphs directly (ignoring the competencies). We will show that when sampling a delegation graph, for any specific voter i, with probability $1-o(1/n)$, $\mathsf{dels}_i(G_n) \leq C(n)$, which implies $\mathsf{weight}_i(G_n) \leq C(n)$. A union bound over all n voters implies $\mathsf{max\text{-}weight}(G_n) \leq C(n)$ with probability $1 - o(1)$.

To that end, we will describe a branching process similar to the well-known *graph branching process* [5], which has the property that the distribution of its size exactly matches the distribution of $\mathsf{dels}_i(G_n)$ for an arbitrary voter i. We will compare this process to a known graph branching process that has size at most $O(\log n)$ with high probability. We will show our process is sufficiently dominated such that it too has size at most $O(\log n)$ with high probability. The branching process works as follows. Fix our voter i. We sample which other voters end up in i's "delegation tree" (i.e., its ancestors in G_n) dynamically over a sequence of time steps. As is standard for these processes, all voters V will be one of three types, live, dead, or neutral. Dead voters are those whose "children" (i.e., voters who delegate to them) we have already sampled. Live voters are voters who have decided to delegate, but whose children have not yet been sampled. Neutral voters are still in the "pool" and have yet to commit to a delegation. At time zero, i is a live voter, there are no dead voters, and all other voters $V \setminus \{i\}$ are neutral. At each time step, we take some live voter j, sample which of the neutral voters choose to delegate to j, add these voters as live vertices, and update j as dead. The procedure ends when there are no more live vertices, at which point the number of delegations received by i is simply the total number of dead vertices.

Let us now describe this more formally. Following the notation of Alon and Spencer [5], let Z_t denote the number of voters we sample to delegate at time t. Let Y_t be the number of live vertices at time t; we have that $Y_0 = 1$. At time t, we remove one live vertex and add Z_t more, so we have the recursion $Y_t = Y_{t-1} - 1 + Z_t$. We let N_t be the number of neutral vertices at time t. We have that $N_0 = n-1$, and $N_t = N_{t-1} - Z_t$. Note that after t time steps, there are t dead vertices and Y_t live ones, so this is equivalent to $N_t = n-1-t-Y_t$. To sample Z_t, we fix some live voter j and ask how many of the neutral voters chose to delegate to j, conditioned on them not delegating to any of the dead voters. Note that when sampling at this step, there are $t-1$ dead voters and conditioned on the neutral voters not delegating to the dead ones, the probability they delegate to any of the other $n-t$ individuals (not including themselves) is exactly $\frac{p}{n-t}$, equally split between them for a total delegation probability of p. Hence $Z_t \sim \text{Bin}(N_{t-1}, \frac{p}{n-t}) \sim \text{Bin}(n-t-Y_{t-1}, \frac{p}{n-t})$. We denote by $\mathfrak{X}^D_{n,p}$ the random variable that counts the size of this branching process, i.e., the number of time steps until there are no more live vertices. Note that the number of delegations received by any voter has the same distribution as $\mathfrak{X}^D_{n,p}$.

Choose some constant p' such that $p < p' < 1$. We will be comparing the $\mathfrak{X}^D_{n,p}$ to a graph branching process $\mathfrak{X}^G_{n,p'}$. The graph branching process is nearly identical, except the probability each of the neutral vertex joins our component is independent of the number of dead vertices and is simply $\frac{p'}{n}$. In other words, $Z_t \sim \text{Bin}(N_{t-1}, \frac{p'}{n})$. A key result about this branching process is the probability of seeing a component of a certain size ℓ decreases exponentially with ℓ. In other words, there is some constant c such that

$$\mathbb{P}_{\mathcal{D},M_q^C,n}[\mathfrak{X}^G_{n,p'} \leq c\log(n)] = 1 - o(1/n).$$

Take $C(n) = c\log(n)$. Note that as long as t is such that $\frac{p}{n-t} \leq \frac{p'}{n}$, the sampling in the delegation branching process is dominated by the sampling in this graph branching process. Hence, as long as $\frac{p}{n-C(n)} \leq \frac{p'}{n}$, $\mathbb{P}[\mathfrak{X}^D_{n,p} \leq c\log(n)] \geq \mathbb{P}[\mathfrak{X}^G_{n,p'} \leq c\log(n)]$. Since $C(n) \in O(\log n)$, this is true for sufficiently large n, so for such n, $\mathbb{P}[\mathfrak{X}^D_{n,p} \leq c\log(n)] = 1 - o(1/n)$. By a union bound over all n voters, this implies the desired result.

Confidence Based Delegation satisfies (3.2)

Let $\bar{q}$ be such that $\bar{q}(x) = 1 - q(x)$, so $\bar{q}$ represents the probability someone with competence x does not delegate. Notice that $\mathbb{E}_{\mathcal{D}}[\bar{q}]$ is exactly the probability an arbitrary voter will not delegate. Let $q^+(x) = \bar{q}(x)x$ and let

$$\mu^* = \frac{\mathbb{E}_{\mathcal{D}}[q^+]}{\mathbb{E}_{\mathcal{D}}[\bar{q}]}.$$

Expanding the definition, we see that μ^* is exactly the expected value of a voter's competence, conditioned on them not voting. Let $\mu_{\mathcal{D}}$ the mean of the competence distribution $\mathcal{D}$. We first show that $\mu^* > \mu_{\mathcal{D}}$. Indeed, since both x and $\bar{q}(x)$ are strictly increasing functions of x, the Fortuin–Kasteleyn–Ginibre (FKG) inequality [88] tells us that $\mathbb{E}_{\mathcal{D}}[q^+] > \mathbb{E}_{\mathcal{D}}[\bar{q}] \cdot \mathbb{E}_{\mathcal{D}}[x] = \mathbb{E}_{\mathcal{D}}[\bar{q}] \cdot \mu_{\mathcal{D}}$. This implies that the expected competence conditioned on not delegating is strictly higher than the overall expected competence.

Next, we will show that for any constant $\gamma > 0$, with high probability, both $\sum_{i=1}^{n} p_i \leq (\mu + \gamma)n$ and $\sum_{i=1}^{n} \mathsf{weight}_i(G) p_i \geq (\mu^* - \gamma)n$. If we choose $\gamma = (\mu^* - \mu)/3$ and $\alpha = \gamma/2$, it follows that, with high probability,

$$\sum_{i=1}^{n} \mathsf{weight}_i(G) p_i - \sum_{i=1}^{n} p_i \geq 2\alpha n,$$

implying that (3.2) is satisfied.

Since the p_is are bounded independent variables, it follows directly from Heoffding's inequality that $\sum_{i=1}^{n} p_i \leq n(\mu + \gamma)$ with high probability, so we now focus on showing $\sum_{i=1}^{n} \mathsf{weight}_i(G) \cdot p_i \geq (\mu^* - \gamma)n$ with high probability. To do this, we will first show that, with high probability, the delegation graph G satisfies $\mathsf{dels}_i(G) \leq C(n)$ for all i and $\mathsf{total\text{-}weight}(G) \geq n - C(n) \log^2 n$.

We showed in the earlier part of this proof that $\mathsf{dels}_i(G) \leq C(n)$ with high probability. We will now prove that $\mathbb{P}_{\mathcal{D}, M_q^C, n}[\mathsf{total\text{-}weight}(G) \geq n - C(n) \log^2 n \mid \mathsf{dels}_i(G) \leq C(n)] = 1 - o(1)$.

To do this, we will first bound the number of voters that, with high probability, end up in cycles. Fix a voter i and sample i's delegation tree. Voter i will only end up in a cycle if i chooses to delegate to someone in this delegation tree. Since we are conditioning on $\mathsf{dels}_i(G) \leq C(n)$, the maximum size of this tree is $C(n)$. Hence, the total φ weight that voter i places on someone in the tree is at most $C(n)$, while the total weight they place on all voters is $n-1$. Hence, the probability that i delegates to someone in their tree can be at most $p \cdot C(n)/(n-1)$. Since this is true for each voter i, the expected number of voters in cycles is at most $np\frac{C(n)}{(n-1)} \in O(\log n)$. By Markov's inequality, the probability that more than $\log^2 n$ voters are in cycles is at most $np\frac{C(n)}{(n-1)\log^2 n} = O(1/\log n) = o(1)$.

Next, since we have conditioned on $\mathsf{dels}_i(G) \leq C(n)$, no single voter, and in particular no single voter in a cycle, can receive more than $C(n)$ delegations. So conditioned on the high probability event that there are at most $\log^2 n$ voters in cycles, there are at most $C(n)\log^2 n$ voters that delegate to those in cycles. This implies that $\mathsf{total\text{-}weight}(G) \geq n - C(n)\log^2 n + \log^2 n$ with high probability.

We now show that, conditioned on the graph satisfying these properties, the instance $(\vec{p}, G)$ satisfies $\sum_{i=1}^{n} \mathsf{weight}_i(G) \cdot p_i \geq n(\mu^* - \gamma)$ with high probability. Note that the competencies satisfy that those that don't delegate are drawn i.i.d. from the distribution of competencies conditioned on not delegating, which has mean μ^*. Fix an arbitrary graph G satisfying the properties. Suppose M is the set of voters that do not delegate. Note that for each $i \in M$, $\mathsf{weight}_i(G) \leq C(n)$, by assumption. Further $\sum_{i\in M} \mathsf{weight}_i(G) \geq n - C(n)\log^2(n)$. Hence, when we sample the non-delegator p_is, $\mathbb{E}[\sum_{i\in M} \mathsf{weight}_i(G) \cdot p_i] \geq (n - C(n)\log^2(n)) \cdot \mu^*$. Moreover,

$$\mathrm{Var}[\sum_{i\in M} \mathsf{weight}_i(G) \cdot p_i] \leq \sum_{i\in M} \mathsf{weight}_i(G)^2 \leq C(n) \cdot n.$$

This follows from the fact that $\mathrm{Var}[p_i] \leq 1$ and that we have fixed the graph G and hence $\mathsf{weight}_i(G)$ for each i, so these terms can all be viewed as constants. In addition, we know that, for each voter i, $\mathsf{weight}_i(G) \leq C(n)$, and $\sum_{i=1}^n \mathsf{weight}_i(G) \leq n$. Hence, we can directly apply Chebyshev's inequality:

$$\begin{aligned}\mathbb{P}_{\mathcal{D},M_q^C,n}\left[\sum_{i\in M}\mathsf{weight}_i(G)p_i < n(\mu^*-\gamma)\right] &< \frac{\mathrm{Var}[\sum_{i\in M}\mathsf{weight}_i(G)p_i]}{(\mathbb{E}[\sum_{i\in M}\mathsf{weight}_i(G)p_i] - n(\mu^*-\gamma))^2}\\ &\leq \frac{nC(n)}{(\gamma n - C(n)\log^2(n)\mu^*)^2}\\ &\in o(1),\end{aligned}$$

where the final step holds because the numerator is $o(n^2)$ and the denominator is $\Omega(n^2)$. Hence, $\sum_{i\in M}\mathsf{weight}_i(G)p_i \geq n(\mu^*-\gamma)$ with high probability, as needed.

To summarize, we have proved that, conditioned on $\mathsf{dels}_i(G) \leq C(n)$ for all i and $\mathsf{total\text{-}weight}(G) \geq n - C(n)\log^2 n, \sum_{i=1}^n \mathsf{weight}_i(G)\cdot p_i \geq n(\mu^* - \gamma/3)$ occurs with high probability. Given that, conditioned on $\mathsf{dels}_i(G) \leq C(n), \mathsf{total\text{-}weight}(G) \geq n - C(n)\log^2 n$ occurs with high probability and that $\mathsf{dels}_i(G) \leq C(n)$ occurs with high probability, we can conclude by the chain rule that the intersection of these events hold with high probability. Given that the probability of any of this event is greater than the probability of the intersection, we can conclude that $\sum_{i=1}^n \mathsf{weight}_i(G)\cdot p_i \geq n(\mu^*-\gamma/3)$ occurs with probability $1-o(1)$, as desired.

Confidence-Based Delegation satisfies (3.3)

We finally show there exists a distribution $\mathcal{D}$ such that $\sum_{i=1}^n p_i + \alpha n \leq n/2 \leq \sum_{i=1}^n \mathsf{weight}_i(G_n)\cdot p_i - \alpha n$ with probability $1-o(1)$. This implies that the model M_q^C satisfies probabilistic positive gain by Lemma 13.

Using the notation of the analogous proof in Section 3.3, let $\mathcal{D}_\eta = \mathcal{U}[0, 1-2\eta]$ for $\eta \in [0, 1/2)$. Note that as a function of η, $\frac{\mathbb{E}_{\mathcal{D}_\eta}[q^+]}{\mathbb{E}_{\mathcal{D}_\eta}[\bar{q}]}$, the expected competence conditioned on not delegating, is continuous. Moreover, if $\eta = 0$, then

$$\frac{\mathbb{E}_{\mathcal{D}_0}[q^+]}{\mathbb{E}_{\mathcal{D}_0}[\bar{q}]} > \mu_{\mathcal{D}_0} = 1/2.$$

Hence, for sufficiently small $\eta > 0$,

$$\frac{\mathbb{E}_{\mathcal{D}_\eta}[q^+]}{\mathbb{E}_{\mathcal{D}_\eta}[\bar{q}]} > 1/2 > \mu_{\mathcal{D}_\eta}.$$

We choose $\mathcal{D}_\eta$ to be our distribution for this choice of η. As in the previous section, let $\mu^*_{\mathcal{D}_\eta} = \frac{\mathbb{E}_{\mathcal{D}_\eta}[q^+]}{\mathbb{E}[\bar{q}]}$. Note that $\mu_{\mathcal{D}_\eta} = 1/2 - \eta$. Let $\gamma = \min(\frac{1/2 - \mu_{\mathcal{D}_\eta}}{2}, \frac{\mu^*_{\mathcal{D}_\eta} - 1/2}{2})$ and $\alpha = \gamma$. By the earlier argument for (3.2), we have that with high probability

$$\sum_{i=1}^{n} p_i \leq n(\mu + \gamma) \leq n/2 - \alpha n$$

and

$$\sum_{i=1}^{n} \mathsf{weight}_i(G) p_i \geq n(\mu^* - \gamma) \geq n/2 + \alpha n.$$

By the union bound, we have that both occur simultaneously with high probability, so (3.3) holds. $\square$

3.5 Continuous General Delegation Model

Finally, we study a model in which voters delegate with fixed probability, and they do so by picking a voter according to a continuous increasing delegation function. This is a general model in which delegations can either go to more or less competent neighbors but where more competent voters are more likely to be chosen over less competent ones.

Formally, let $M^S_{p,\varphi} = (q^p, \varphi)$ where q^p is a constant function equal to p, that is, $q^p(x) = p$

for all $x \in [0,1]$, and $\varphi(x, y)$ is non-zero, continuous, and increasing in y. We then have the following.

Theorem 7 (Continuous General Delegation Model)**.** *All models $M^S_{p,\varphi}$ with $p \in (0,1)$ and φ that is non-zero, continuous, and increasing in its second coordinate satisfy probabilistic do no harm and probabilistic positive gain with respect to the class $\mathfrak{D}^C$ of all continuous distributions.*

Proof. Fix $M^S_{p,\varphi}$ and $\mathcal{D} \in \mathfrak{D}^C$. Note that since φ is continuous and always positive on the compact set $[0,1]^2$, φ is in fact uniformly continuous and there are bounds $L, U \in \mathbb{R}^+$ such that φ is bounded in the interval $[L, U]$. Additionally, we can assume without loss of generality that for all $x \in [0,1]$, $\mathbb{E}_{\mathcal{D}}[\varphi(x,\cdot)] = 1$. Indeed, $\mathbb{E}_{\mathcal{D}}[\varphi(x,\cdot)]$ is a positive, continuous function of x, so replacing φ by $\varphi'(x,y) = \varphi(x,y)/\mathbb{E}_{\mathcal{D}}[\varphi(x,\cdot)]$ induces the same model and satisfies the desired property.

The Continuous General Delegation Model satisfies (3.1)*.*

Our goal is to show there is some $C(n) \in O(\log n)$ such that, with high probability, no voter receives more than $C(n)$ delegations. To do this, just as in the proof of Theorem 6, we consider a branching process of the delegations received beginning with some voter i. We will show that under minimal conditions on the sampled competencies (which all occur with high probability), this branching process will be dominated by a well-known *subcritical multi-type Poisson branching process* [29], which has size $O(\log n)$ with high probability.

For a fixed competence vector $\vec{p}_n$, the branching process for the number of delegations received by a voter i works as follows. We keep track of three sets of voters: those that are live at time t (L_t), those dead at time t (D_t), and those neutral at time t (N_t). Unlike in the proof of Theorem 6, where it was sufficient to keep track of the number of voters in each category, here we must keep track of the voter identities as well, as they do not all delegate with the same probability. At time zero, the only live voter is voter i and the rest are neutral, so $L_0 = \{i\}$, $D_0 = \emptyset$, and $N_0 = [n] \setminus \{i\}$. As long as there are still live voters, we sample the next set of delegating voters Z_t in time t by choosing some live voter $j \in R_{t-1}$ and sampling its children. Once j's children are sampled, j becomes dead, and j's children become live. All voters that did not delegate and were not delegated to remain neutral. The children are sampled independently; the probability they are included is the probability they delegate to j conditioned on them not delegating to the dead voters in D_{t-1}. For each voter $k \in N_{t-1}$, k will be included with probability

$$p \cdot \frac{\varphi(p_k, p_j)}{\sum_{k' \in [n] \setminus (D_{t-1} \cup \{k\})} \varphi(p_k, p_{k'})}.$$

This is precisely the probability k delegates to j conditioned on them not delegating to any voter in D_{t-1}. We continue this process until there are no more live voters, at which point the number of delegations is simply the number of dead voters, or equivalently, the total number of time steps. We denote by $\mathfrak{X}^D_{\vec{p}_n,i}$ the size of the branching process parameterized by competencies $\vec{p}_n$ and a voter $i \in [n]$.

Our goal will be to compare $\mathfrak{X}^D_{\vec{p}_n,i}$ to the outcome of a well-known multi-type Poisson branching process. In this branching process, there are a fixed finite number k of types of voters.[7] The process itself is parameterized by a $k \times k$ matrix M, where $M_{\tau\tau'}$ is the expected number of children of type τ' a voter of type τ will have. The process is additionally parameterized by the type $\tau \in [k]$ of the starting voter. The random variable Y_t keeps track of the number of live voters of each type; it is a vector of length k, where the τth entry is the number of live voters of type τ. Hence, $Y_0 = e_\tau$, the (basis) vector with a 1 in entry τ and an entry 0 for all other types. We sample children by taking an arbitrary live voter of type τ' (the τ' component in Y_{t-1} must be positive, indicating that there is such a voter), and sampling its children Z_t, which is also a vector of length k, each entry

[7] In the literature, these are often called *particles*, but to be consistent with our other branching processes, we call them voters here.

indicating the number of children of that type. The vector Z_t is sampled such that the τ'' entry is from the $\text{Pois}(M_{\tau'\tau''})$ distribution. That is, children of different types are sampled independently from a Poisson distribution, with the given expected value. We have the recursion $Y_t = Y_{t-1} + Z_t - e_{\tau'}$.

Note that this means that there is no "pool" of voters to choose from; in fact, it is possible for this process to grow unboundedly large (see [5, Section 11.6] for the classical description of the single-type Poisson branching process). Nonetheless, this process will still converge often enough to remain useful. We denote by $\mathfrak{X}^P_{M,\tau}$ the random variable that gives the size of this branching process, parameterized by expected-children matrix M and starting voter type $\tau \in [k]$. Such a branching process is considered *sub-critical* if the largest eigenvalue of M is strictly less than 1 [29]. In such a case, if we begin with voter of any type $\tau \in [k]$, the probability of the branching process surviving ℓ steps decreases exponentially in ℓ. Hence, there is some c such that for all $\tau \in [k]$,

$$\mathbb{P}[\mathfrak{X}^P_{M,\tau} \leq c \log(n)] = 1 - o(1/n).$$

To compare these branching processes, we make a sequence of adjustments to the original branching process that at each step creates a dominating branching process slightly closer in flavor to the multi-type Poisson. In the end, we will be left with a sub-critical multi-type Poisson process that we can bound.

Fix some $\varepsilon > 0$, which is a parameter in all of our steps. Later, we will choose ε to be sufficiently small (specifically, such that $p\frac{(1+\varepsilon)^3}{1-2\varepsilon} < 1$) to ensure that the Poisson branching process is sub-critical. To convert from our delegation branching process to the Poisson branching process, we take a voter's type to be their competence (which completely characterizes their delegation behavior). However, to compare to the Poisson process, there must be a finite number of types. Hence, we partition the interval $[0, 1]$ into B buckets, each of size $1/B$, such that voters in the same bucket delegate and are delegated to "similarly". We choose B large enough such that all points in $[0,1]^2$ within a distance of $\sqrt{2}/B$ of each other differ in φ by at most $L \cdot \varepsilon$. (Recall that the range of φ is in the interval $[L, U]$.) This is possible since φ is uniformly continuous. Further, this implies any points $(x,y),(x',y')$ within a square with side length $1/B$ have the property that $\varphi(x,y) \leq \varphi(x',y') + L \cdot \varepsilon \leq (1+\varepsilon) \cdot \varphi(x',y')$. Note that B depends only on φ and ε, and hence is a constant with respect to the number of voters n.

We say a voter i is of type τ if $\frac{\tau-1}{B} < p_i \leq \frac{\tau}{B}$ for $1 \leq \tau \leq B$ (with a non-strict inequality for $\tau = 1$, so 0 is of type 1). Let $S_\tau = (\frac{\tau-1}{B}, \frac{\tau}{B}]$ be the set of competencies of type τ (except that, in the case that $\tau = 1$, we take S_1 to be the closed interval $[0, \frac{1}{B}]$). Let $\pi_\tau = \mathcal{D}[S_\tau]$ be the probability that a voter has type τ. Since the types form a partition of $[0,1]$, we have that $\sum_{\tau=1}^{B} \pi_\tau = 1$.

For any two types τ, τ', we define

$$\varphi'(\tau, \tau') = \sup_{(x,y) \in S_\tau \times S_{\tau'}} \varphi(x,y).$$[8]

We abuse notation by extending φ' to operate directly on competencies in $[0,1]$ by first converting competencies to types and then applying φ'. Then, φ' has the property that for any $p_i, p_j \in [0,1]$,

$$\varphi(p_i, p_j) \leq \varphi'(p_i, p_j) \leq (1+\varepsilon)\varphi(p_i, p_j).$$

We have that for all τ, if $x \in S_\tau$, then

$$\sum_{\tau'=1}^{B} \varphi'(\tau, \tau')\pi_{\tau'} = \mathbb{E}_{\mathcal{D}}[\varphi'(x, \cdot)] \leq (1+\varepsilon) \cdot \mathbb{E}_{\mathcal{D}}[\varphi(x, \cdot)] = (1+\varepsilon).$$

[8] Note that, because φ is increasing in its second coordinate, one can actually write $\tilde{\varphi}(\tau, \tau') = \sup_{x \in S_\tau} \varphi(x, \frac{\tau'}{B})$.

Hence, we define

$$\tilde{\varphi}(\tau, \tau') = \varphi'(\tau, \tau') \cdot \frac{(1+\varepsilon)}{\sum_{\tau''=1}^{B} \varphi'(\tau, \tau'')\pi_{\tau''}}.$$

We again abuse notation to allow $\tilde{\varphi}$ to operate directly on competencies. We have that $\tilde{\varphi}(x, y) \geq \varphi'(x, y) \geq \varphi(x, y)$ for all competencies $x, y \in [0, 1]$ and further, for all τ, $\sum_{\tau'=1}^{B} \tilde{\varphi}(\tau, \tau')\pi_{\tau'} = 1 + \varepsilon$.

The Poisson branching process we will eventually compare to is one with B types parameterized by the expected-children matrix M, where

$$M_{\tau\tau'} = p\frac{(1+\varepsilon)^2}{1-2\varepsilon}\tilde{\varphi}(\tau, \tau').$$

First, we show that M has largest eigenvalue strictly less than 1 (for our choice of ε), so that the branching process will be subcritical. Indeed, M has only positive entries, so we need only exhibit an eigenvector with all nonnegative entries such that the associated eigenvalue is strictly less than 1. The Perron-Frobenius theorem tells us this eigenvalue must be maximal.

Next, we give details for proving the Poisson process is subcritical, as well as completing the comparison of between the original delegation process and this one. The comparison makes use of the concentration of the number of voters in each bucket.

The eigenvector we consider is $\vec{\pi} = (\pi_1, \ldots, \pi_B)$ (which has nonnegative entries, as each π_τ is a probability). We show it has eigenvalue $p\frac{(1+\varepsilon)^3}{1-2\varepsilon}$, strictly less than 1 due to our choice of ε. We show it has eigenvalue $p\frac{(1+\varepsilon)^3}{1-2\varepsilon}$, strictly less than 1 due to our choice of ε. Indeed, we have that

$$(M\vec{\pi})_\tau = \sum_{\tau'=1}^{B} \pi_\tau \tilde{\varphi}(\tau, \tau')\pi_{\tau'} = \pi_\tau p\frac{(1+\varepsilon)^3}{1-2\varepsilon}$$

by the definition of $\tilde{\varphi}$. Hence, $\vec{\pi}$ is our desired eigenvector.

Since $\mathfrak{X}^P_{M,\tau}$ is sub-critical for all τ, we have that there is some c such that for all $\tau \in [B]$, $\mathbb{P}[\mathfrak{X}^P_{M,\tau} \leq c\log(n)] = 1 - o(1/n)$. We take $C(n) = c\log(n)$.

Now we consider our branching process, $\mathfrak{X}^D_{\vec{p},i}$. To make the comparison, we will need some minimal concentration properties. We first show that the sampled competencies $\vec{p}$ satisfy these properties with high probability, and then show that, conditioned on these properties, the branching process $\mathfrak{X}^D_{\vec{p},i}$ is easily comparable to a Poisson process. The properties are the following:

1. For each voter $i \in [n]$, $\sum_{j \neq i} \varphi(p_i, p_j) \geq (1 - \varepsilon) \cdot n$.

2. For each type $\tau \in [B]$, the number of voters of type τ, $|\{i \mid p_i \in S_\tau\}| \leq (1 + \varepsilon)\pi_\tau n$.

For the first property, fix the competence p_i of a single voter i. Then when sampling the p_js, $\sum_{j \neq i} \varphi(p_i, p_j)$ is the sum $n-1$ independent variables, all in the interval $[L, U]$, with mean 1. Hence, by Hoeffding's inequality, for all competencies c, $\mathcal{D}^n[\sum_{j \neq i} \varphi(p_i, p_j) \geq (1-\varepsilon)n \mid p_i = c] = 1 - o(1/n)$, where the $o(1/n)$ term is independent of c. By the law of total probability, this implies that even when p_i is sampled as well, the $1 - o(1/n)$ bound continues to hold. By a union bound over all n voters, this holds for everybody with probability $1 - o(1)$.

For the second property, note that the number of voters of type τ follows a $\mathrm{Bin}(n, \pi_\tau)$ distribution. A simple application of Hoeffding's inequality implies that for this τ, $|\{i \mid p_i \in S_\tau\}| \leq (1 + \varepsilon)\pi_\tau n$ (note that this holds even in the extreme cases where $\pi_\tau = 0$ or $\pi_\tau = 1$). As the number B of types is fixed and independent of n, a union bound over all B types implies this holds for all τ with probability $1 - o(1)$.

Now fix some voter competencies $\vec{p}$ such that both properties hold. We will first upper bound the probability a voter of type τ delegates to a voter of type τ'. Hence, we can compare our branching process to one with these larger probabilities, and this will only dominate our original process.

To that end, since $|D_{t-1}| = t - 1 \leq t$ (recall that D_{t-1} consists of the dead voters at time

$t-1$), using the first property, we have that for all $i \in [n]$,

$$\sum_{j \in [n] \setminus (D_{t-1} \cup \{i\})} \varphi(p_i, p_j) \geq (1-\varepsilon)n - U \cdot t.$$

Hence, as long as $t \leq \varepsilon n/U$, $\sum_{j \in [n] \setminus (D_{t-1} \cup \{i\})} \varphi(p_i, p_j) \geq (1-2\varepsilon)n$.

Including the fact that $\varphi(p_i, p_j) \leq \tilde{\varphi}(p_i, p_j)$ for all p_i and p_j, we have that for all time steps $t \leq \varepsilon n/U$,

$$p \cdot \frac{\varphi(p_i, p_j)}{\sum_{k' \in [n] \setminus (D_{t-1} \cup \{k\})} \varphi(p_i, p_{k'})} \leq \frac{p}{n} \cdot \frac{\tilde{\varphi}(p_i, p_j)}{(1-2\varepsilon)}.$$

Note that for sufficiently large n, $C(n) \leq \varepsilon n/U$, so from now on we restrict ourselves to such n.

Further, note that by the second property, there will never be more than $(1+\varepsilon)\pi_\tau n$ neutral voters of type τ. Hence, if we take a voter of type τ' at time step $t \leq C(n)$, the number of children it will have of type τ will be stochastically dominated by a $\mathrm{Bin}((1+\varepsilon)\pi_\tau n, \frac{p}{n} \cdot \frac{\tilde{\varphi}(p_i,p_j)}{(1-2\varepsilon)})$, and this is independent for each τ. As n grows large, this distribution approaches a $\mathrm{Pois}(p\frac{(1+\varepsilon)}{1-2\varepsilon}\tilde{\varphi}(\tau,\tau'))$. In particular, this means that for sufficiently large n, it will be stochastically dominated by a $\mathrm{Pois}(p\frac{(1+\varepsilon)^2}{1-2\varepsilon}\tilde{\varphi}(\tau,\tau'))$ distribution (note the extra $(1+\varepsilon)$ factor). Hence, if voter i is of type τ, up to time $t \leq C(n)$, $\mathfrak{X}^D_{\vec{p},i}$ is dominated by $\mathfrak{X}^P_{M,\tau}$, so

$$\mathbb{P}_{\mathcal{D},M^S_{p,\varphi}.n}[\mathfrak{X}^D_{\vec{p},i} \geq C(n)] \geq \mathbb{P}_{\mathcal{D},M^S_{p,\varphi}.n}[\mathfrak{X}^P_{M,\tau} \geq C(n)] = 1 - o(1/n).$$

A union bound over all n voters tells us this is true for all voters simultaneously with probability $1-o(1)$, as needed.

Next, we turn to the proofs of (3.2) and (3.3) that follow a similar structure to Confidence-based, however, they are quite a bit more intricate due to the inter dependencies between competence level and delegation probability.

The Continuous General Delegation Model satisfies (3.2).

To show (3.2) holds, we first show the following.

Let $\mu_{\mathcal{D}}$ be the mean of the competence distribution $\mathcal{D}$. For a fixed x, let $\varphi_x^+(y)$ be the function $\varphi(x,y) \cdot y$. We show that there is some $c > 0$ such that for all $x \in [0,1]$,

$$\mathbb{E}_{\mathcal{D}}[\varphi_x^+] \geq \mu_{\mathcal{D}} + c. \tag{3.18}$$

Indeed, if we view $\mathbb{E}_{\mathcal{D}}[\varphi_x^+]$ as a function of x for $x \in [0,1]$, first note that it is a continuous function on a compact set, and hence it attains its minimum. Further, for all $x \in [0,1]$, since $\varphi(x,y)$ and y are both increasing functions of y, by the FKG inequality [88],

$$\mathbb{E}_{\mathcal{D}}[\varphi^+] > \mathbb{E}_{\mathcal{D}}[\varphi(x,\cdot)] \cdot \mu_{\mathcal{D}} = \mu_D,$$

since, by assumption, $\mathbb{E}_{\mathcal{D}}[\varphi(x,\cdot)] = 1$. Hence, this attained minimum must be strictly larger than μ, implying (3.18).

Since $\varphi(x,y)$ is normalized so that $\mathbb{E}_{y\sim\mathcal{D}}[\varphi(x,y)] = 1$, $\mathbb{E}_{y\sim\mathcal{D}}[\varphi(x,y) \cdot y]$ is the expected competence of the voter to whom someone of competence x delegates to (prior to other competencies being drawn). Hence, (3.18) tells us that "on average", all voters (regardless of competence) tend to delegate to those with competence strictly above the mean. Ideally, we would choose $\alpha \approx c/2$ and hope that some concentration result tells us that the weighted competencies post-delegation will be strictly above $\mu + c/2$ (the mean of all competencies will be close to μ by standard concentration results). However, proving this concentration result is surprisingly subtle, as there are many dependencies between different voter delegations. Indeed, if one voter with high competence and many delegations chooses to delegate "downwards" (that is, to someone with very low competence), this can cancel out all of the "expected" progress we had made thus far. Hence, the rest of this proof involves proving concentration *does* in fact hold. We prove this by breaking up the process of sampling instances into much more manageable pieces, where, in each, as long as nothing goes "too" wrong, concentration will hold.

In particular, we will prove that for all $\gamma > 0$, with high probability,

$$\sum_{i=1}^{n} \mathsf{weight}_i(G) \cdot p_i - \sum_{i=1}^{n} p_i \geq (c(1-p) - \gamma)n. \tag{3.19}$$

Fix such a γ. As in the previous part, fix $\varepsilon > 0$ which will paramaterize our steps. We will later choose ε sufficiently small to get our desired result (precisely ε such that $6\varepsilon + \varepsilon^2 < \gamma$). By choosing $\gamma < c(1-p)$, this value is positive, so we can choose $\alpha = \frac{c(1-p)-\gamma}{2}$ which proves Equation (3.2)

To that end, we define a sequence of six sampling steps that together are equivalent to the standard sampling process with respect to $\mathcal{D}$ and $M^S_{p,\varphi}$. In each step, we will show that with high probability, nothing "goes wrong", and conditioned on nothing going wrong in all these steps, we will get the α improvement that we desire. The six steps are as follows:

1. Sample a set $M \subseteq [n]$ of voters that choose not to delegate. Each voter is included independently with probability p.

2. Sample competencies p_i for $i \in [n] \setminus M$. Each p_i is sampled i.i.d. from $\mathcal{D}$.

3. Sample competencies p_j for $j \in M$. Each p_j is sampled i.i.d. from $\mathcal{D}$.

4. Sample a set $R \subseteq [n] \setminus M$ of delegators that delegate to those in M. Each voter $i \in [n] \setminus M$ is included independently with probability $\frac{\sum_{j \in M} \varphi(p_i, p_j)}{\sum_{j \in [n] \setminus \{i\}} \varphi(p_i, p_j)}$, that is, the total φ weight they put on voters in M divided by the total φ weight they put on all voters.

5. Sample delegations of voters in $[n] \setminus (M \cup R)$. At this point, we are conditioning on such voters delegating, and when they do delegate, they do so to voters in $[n] \setminus M$. Hence, for each $i \in [n] \setminus (M \cup R)$, they delegate to $j \in [n] \setminus (M \cup \{i\})$ with probability

$\frac{\varphi(p_i,p_j)}{\sum_{j'\in[n]\setminus(M\cup\{i\})}\varphi(p_i,p_{j'})}$.

6. Sample delegations of voters in R. At this point, we are conditioning on such voters delegating to those in M. Hence, for each $i \in R$, they choose to delegate to $j \in M$ with probability $\frac{\varphi(p_i,p_j)}{\sum_{j'\in M}\varphi(p_i,p_{j'})}$.

We now analyze each step, describing what could "go wrong". Let $\mathcal{E}_1, \ldots, \mathcal{E}_6$ be the events that nothing goes wrong in each of the corresponding steps. We define these events formally below. Our goal is to show that $\mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}[\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_6] = 1 - o(1)$.

• Let $\mathcal{E}_1$ be the event that $(p-\varepsilon)\cdot n \leq |M| \leq (p+\varepsilon)\cdot n$. Note that M is the sum of n independent Bernouilli random variables with success probability p. It follows directly from a union bound over both variants of Hoeffding's inequality that

$$\mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}[(p-\varepsilon)\cdot n \leq |M| \leq (p+\varepsilon)\cdot n] = 1 - o(1).$$

• Let $\mathcal{E}_2$ be the event that $\sum_{i\in[n]\setminus M} p_i \leq n(\mu+\varepsilon)(1-p+\varepsilon)$. Note that $\sum_{i\in[n]\setminus M} p_i$ is the sum of $n - |M|$ i.i.d. random variables with mean μ. Conditioning on event $\mathcal{E}_1$, $|M|$ is lower bounded by $n(p-\varepsilon)$, implying that $n - |M| \leq n(1-p+\varepsilon)$ as well. It follows from Lemma 14 that

$$\mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}\left[\sum_{i\in[n]\setminus M} p_i \leq n(\mu+\varepsilon)(1-p+\varepsilon) \;\middle|\; \mathcal{E}_1\right] = 1 - o(1)$$

which, combined with $\mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}[\mathcal{E}_1] = 1 - o(1)$, proves that $\mathcal{E}_1 \cap \mathcal{E}_2$ occurs with probability $1 - o(1)$.

• Let $\mathcal{E}_3$ be the event consisting of all instances $(\vec{p}, G)$ such that

$$\frac{\sum_{j\in M}\varphi(p_i,p_j)\cdot p_j}{\sum_{j\in M}\varphi(p_i,p_j)} \geq \frac{(1-\varepsilon)}{(1+\varepsilon)}(\mu + c)$$

for all $i \in [n] \setminus M$.

We show $\mathcal{E}_3$ occurs with high probability conditional on $\mathcal{E}_1$ and $\mathcal{E}_1$ (conditioning on $\mathcal{E}_2$ is unnecessary. but makes the final statement easier). Fix a set of voters M and p_i for $i \in [n] \setminus M$ satisfying $\mathcal{E}_1$ and $\mathcal{E}_2$. For each $i \in [n] \setminus M$, we will show that with probability $1 - o(1/n)$, when we sample the p_js for $j \in M$, they satisfy

$$\sum_{j \in M} \varphi(p_i, p_j) \leq |M|(1 + \varepsilon) \tag{3.20}$$

and

$$\sum_{j \in M} \varphi(p_i, p_j) \cdot p_j \geq |M|(1 - \varepsilon)(\mu + c). \tag{3.21}$$

(3.20) follows from the fact that $\sum_{j \in M} \varphi(p_i, p_j)$ is the sum of $|M|$ bounded independent random variables

with mean $\mathbb{E}_{y \sim \mathcal{D}}[\varphi(p_i, y)] = 1$. By Hoeffding's inequlaity, since $|M|$ is linear in n, $\sum_{j \in M} \varphi(p_i, p_j)$ is at most $|M|(1 + \varepsilon)$ with probability $1 - o(1/n)$.

(3.21) follows from the fact that $\sum_{j \in M} \varphi(p_i, p_j) \cdot p_j$ is also the sum of $|M|$ bounded independent random variables with mean $\mathbb{E}_{\mathcal{D}}[\varphi^+_{p_i}]$. Again, since we have conditioned on $\mathcal{E}_1$, $|M|$ is lower bounded by $(p - \varepsilon)n$, which by Hoeffding's inequality implies that $\sum_{j \in M} \varphi(p_i, p_j)p_j$ is at least $|M|(1 - \varepsilon)\mathbb{E}_{\mathcal{D}}[\varphi_{p_i}]$ with probability $1 - o(1/n)$.

Finally, we can conclude via a union bound that $\frac{\sum_{j \in M} \varphi(p_i, p_j) \cdot p_j}{\sum_{j \in M} \varphi(p_i, p_j)} \geq \frac{(1-\varepsilon)}{(1+\varepsilon)}(\mu + c)$ with probability $1 - o(1/n)$ for any $i \in [n] \setminus M$. Hence, by another union bound over the at most n voters $i \in [n] \setminus M$, $\frac{\sum_{j \in M} \varphi(p_i, p_j) \cdot p_j}{\sum_{j \in M} \varphi(p_i, p_j)} \geq \frac{(1-\varepsilon)}{(1+\varepsilon)}(\mu + c)$ for all $i \in [n] \setminus M$ with high probability.

By the law of total probability, $\mathcal{E}_3$ conditioned on $\mathcal{E}_1$ and $\mathcal{E}_2$ occurs with probability $1 - o(1)$, which proves that $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ occurs with probability $1 - o(1)$ by the chain rule.

• Let $\mathcal{E}_4$ be the entire sample space. Nothing can "go wrong" during this sampling step. So trivially, $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \mathcal{E}_4$ occurs with probability $1 - o(1)$.

- Let $\mathcal{E}_5$ be the event that $\mathsf{dels}_i(G) \leq C(n)$ for all $i \in [n] \setminus M$ and $\mathsf{total\text{-}weight}(G) \geq n - C(n)^2 \log(n)$ in the subgraph G sampled (i.e., with delegations only from voters not in R or M). We will show $\mathcal{E}_5$ occurs with high probability even when we sample a full delegation graph (that is, samples delegations for all voters), which implies it continues to hold even when we sample only some delegations (recall that at this step we have only sampled delegations from voters in $[n] \setminus (M \cup R)$).

The proof of this is very similar to the one in Theorem 6, with one extra step to allow for different φ weights.

It was proved in the previous part of this proof that, for all voters i, we have that $\mathsf{dels}_i(G) \leq C(n)$ with probability $1 - o(1)$ (not conditioned on anything) when we sample entire delegation graphs, so we can safely condition on this fact. We now prove that $\mathbb{P}_{\mathcal{D}, M^S_{p,\varphi}, n}[\mathsf{total\text{-}weight}(G_n) \geq n - O(\log^3 n) \mid \mathsf{dels}_i(G) \leq C(n)] = 1 - o(1)$.

We begin by bounding the number of voters that end up in cycles. Fix some voter i, and let us begin by sampling their delegation tree.

Since we are conditioning on the tree having size at most $C(n)$, the most weight that voter i can place on all of the voters in i's delegation tree is $U \cdot C(n)$. The minimum weight that i can place on all voters is $L(n-1)$. Hence, the probability that i delegates to someone in i's tree conditional the delegation tree having size at most $C(n)$ is at most $p \cdot \frac{U \cdot C(n)}{L \cdot (n-1)}$. Since i was arbitrary, this implies that the expected number of voters in cycles can be at most $n \cdot p \cdot \frac{U \cdot C(n)}{L \cdot (n-1)} \in O(\log n)$.

Applying Markov's inequality just as in the analogous proof in the previous section, the probability that more than $\log^2 n$ voters are in cycles is at most $np\frac{UC(n)}{L(n-1)\log^2 n} = O(1/\log n) = o(1)$. Further, the total number of people that could delegate to voters in cycles is at most $C(n)$ times the number of voters in cycles. Hence, with probability $1 - o(1)$, there are at most $C(n) \cdot \log^2 n$ voters delegating to those in cycles. This implies the desired bound. Hence, we have proved that $\mathbb{P}_{\mathcal{D}, M^S_{p,\varphi}, n}[\mathcal{E}_5] = 1 - o(1)$. Since we have already shown that

$\mathbb{P}_{\mathcal{D},M^S_{\vec{p},\varphi},n}[\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3\cap\mathcal{E}_4] = 1-o(1)$, a union bound implies that $\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3\cap\mathcal{E}_4\cap\mathcal{E}_5$ occurs with probability $1-o(1)$ as well.

- We now consider the sixth step. To define $\mathcal{E}_6$, we need some new notation. Fix competencies $\vec{p}$ and a partial delegation graph G such that $(\vec{p}, G)$ is in the first five events. We define Q_i for $i \in R$ to be the random variable representing the competence of the voter to whom i delegates. Since we know i delegates to a voter in M, note that

$$Q_i(G) = p_j \text{ with probability } \frac{\varphi(p_i, p_j)}{\sum_{j' \in M} \varphi(p_i, p_{j'})} \text{ for all } j \in M.$$

Let $\mathcal{E}_6$ be the event consisting of all instance $(\vec{p}, G)$ such that that $\sum_{i\in R} \mathsf{dels}_i(G)\cdot Q_i(G) \geq \frac{(1-\varepsilon)^2}{1+\varepsilon}(\mu+c)(1-p-2\varepsilon)\cdot n$. We show that $\mathbb{P}_{\mathcal{D},M^S_{\vec{p},\varphi},n}[\mathcal{E}_6 \mid \mathcal{E}_1\cap\cdots\cap\mathcal{E}_5] = 1-o(1)$. This, combined with the the fact $\mathbb{P}_{\mathcal{D},M^S_{\vec{p},\varphi},n}[\mathcal{E}_1\cap\cdots\cap\mathcal{E}_5] = 1-o(1)$ (shown earlier), implies that $\mathbb{P}_{\mathcal{D},M^S_{\vec{p},\varphi},n}[\mathcal{E}_1\cap\cdots\cap\mathcal{E}_6] = 1-o(1)$. It follows from the definition of Q_i that

$$\mathbb{E}[Q_i] = \sum_{j\in M} \frac{\varphi(p_i, p_j)}{\sum_{j'\in M}\varphi(p_i, p_{j'})}\cdot p_j = \frac{\sum_{j\in M}\varphi(p_i,p_j)\cdot p_j}{\sum_{j\in M}\varphi(p_i,p_j)}.$$

By conditioning on $\mathcal{E}_3$, we have that $\mathbb{E}[Q_i] \geq \frac{(1-\varepsilon)}{(1+\varepsilon)}(\mu+c)$ for each $i \in R$. Hence, $\mathbb{E}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i] \geq (n-|M|-C(n)^2\log(n))\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot(\mu+c)$, since we are conditioning on $\mathcal{E}_3$ and $\mathcal{E}_5$. Further, for sufficiently large n, $C(n)^2\log(n)\leq \varepsilon n$; since we are conditioning on $\mathcal{E}_1$, $|M| \leq (p+\varepsilon)n$, so we have that for sufficiently large n,

$$\mathbb{E}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i] \geq (1-p-2\varepsilon)\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot(\mu+c)\cdot n \in \Omega(n).$$

Next, consider $\mathrm{Var}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i]$. Since each Q_i takes on values in $[0,1]$, $\mathrm{Var}[Q_i]\leq 1$. Further, each summand is independent, as each Q_i is independent and we have fixed G, so we can can view $\mathsf{dels}_i(G)$ as a constant. Hence, $\mathrm{Var}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i] \leq \sum_{i\in R}\mathsf{dels}_i(G)^2 \in o(n^2)$

since, for all i, $\mathsf{dels}_i(G) \le C(n) \in O(\log n)$ and $\sum_i \mathsf{dels}_i(G) \le n$. Hence,

$$\begin{aligned}
&\mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i < \frac{(1-\varepsilon)^2}{1+\varepsilon}(\mu+c)(1-p-2\varepsilon)\cdot n]\\
&\le \mathbb{P}_{\mathcal{D},M^S_{p,\varphi},n}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i < (1-\varepsilon)\mathbb{E}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i]]\\
&\le \frac{\mathrm{Var}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i]}{\varepsilon^2\cdot\mathbb{E}[\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i]^2} \in o(1)
\end{aligned}$$

where the second inequality is due to Chebyshev's inequality, which is $o(1)$ because the numerator is $o(n^2)$ and the denominator is $\Omega(n^2)$. This implies the desired result.

Finally, we show that for all instance $(\vec{p}, G) \in \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_6$, (3.2) holds, and hence so does (3.19). We have that $\sum_{i=1}^n w_i(G)\cdot p_i = \sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i(G) + \sum_{j\in M} p_j$, because in G each voter $i \in R$ delegates all of their $\mathsf{dels}_i(G)$ votes to the voter in M with competence $Q_i(G)$. Hence, $\sum_{i=1}^n w_i(G)\cdot p_i - \sum_{i=1}^n p_i = \sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i(G) - \sum_{i\in[n]\setminus(M\cup R)} p_i$. Since $(\vec{p}, G) \in \mathcal{E}_2$, we have that $\sum_{i\in[n]\setminus M} p_i \le n(\mu+\varepsilon)(1-p+\varepsilon)$. Since $(\vec{p}, G) \in \mathcal{E}_6$, we have that $\sum_{i\in R}\mathsf{dels}_i(G)\cdot Q_i(G) \ge \frac{(1-\varepsilon)^2}{1+\varepsilon}(\mu+c)(1-p-2\varepsilon)\cdot n$. Hence, this difference is at least

$$\begin{aligned}
((\mu+c)(1-p-2\varepsilon) - (\mu+\varepsilon)(1-p+\varepsilon))n &\ge (c(1-p) - 3\varepsilon\mu - 2\varepsilon c - (1-p)\varepsilon - \varepsilon^2)n\\
&\ge (c(1-p) - 6\varepsilon - \varepsilon^2)n
\end{aligned}$$

where the second inequality holds because, $c, (1-p), \mu \le 1$. By choosing ε such that $6\varepsilon + \varepsilon^2 \le \gamma$ ($\varepsilon = \min(\gamma/7, 1)$ will do), (3.19) follows.

The Continuous General Delegation Model Satisfies (3.3)

We now show that there exists a distribution $\mathcal{D}$ and $\alpha > 0$ such that $\sum_{i=1}^n p_i + \alpha n \le n/2 \le \sum_{i=1}^n \mathsf{weight}_i(G_n)\cdot p_i - \alpha n$ with probability $1 - o(1)$. This implies that the model $M^S_{p,\varphi}, n$ satisfies probabilistic positive gain by Lemma 13.

As in earlier arguments, let $\mathcal{D}_\eta = \mathcal{U}[0, 1-2\eta]$ for $\eta \in [0, 1/2)$. Note that

$$f(\eta) = \inf_{x \in [0,1]} \left\{ \mathbb{E}_{D_\eta}[\varphi_x^+] \right\} \cdot (1-p) - 3\eta/2$$

is a continuous function of η.

Moreover, $f(0) > 0$. Hence, for sufficiently small $\eta > 0$, $f(\eta) > 0$.

Consider $\mathcal{D}_\eta$ for some $\eta > 0$ such that $f(\eta) > 0$. Let $\alpha = \min(\eta/2, f(\eta)/2)$. Since $\mu_{\mathcal{D}_\eta} = 1/2 - \eta$, by Hoeffding's inequality, $\sum_{i=1}^n p_i \le (1/2 - \eta/2)n \le n/2 - \alpha n$ with high probability.

Next, note that we can choose $c = \inf_{x \in [0,1]} \left\{ \mathbb{E}_{D_\eta}[\varphi_x^+] \right\}$ in order to satisfy (3.18). Hence, by choosing $\gamma = f(\eta)/2$, it follows from (3.19) that

$$\sum_{i=1}^n \mathsf{weight}_i(G) \cdot p_i - \sum_{i=1}^n p_i \ge (c(1-p) - f(\eta)/2)n = (3\eta/2 + f(\eta)/2)n \ge (3\eta/2 + \alpha)n$$

with high probability. Further, by Hoeffding's inequality, $\sum_{i=1}^n p_i \ge (1/2 - 3\eta/2)n$ with high probability, so by the union bound applied to these inequalities,

$$\sum_{i=1}^n \mathsf{weight}_i(G) \cdot p_i \ge n/2 + \alpha n$$

with high probability, as needed.

$\square$

3.6 Discussion

This chapter relies on a set of assumptions and modeling choices that are worth discussing.

First, a prominent feature of our model is that there is no underlying social network, that is, there is no restriction on whom a voter may delegate to. As we explained in Section 3.1, we believe this is realistic. But we can, in fact, extend our results to a model where a directed social network is first sampled, and then a (q, φ)-model is followed. The social network must be sampled such that the neighbors of each voter are chosen uniformly at random, although the number of such neighbors could follow any small-tailed distribution. Intuitively, delegation proportional to weighting the neighbors of i (rather than the entire population) is equivalent to a possibly different weighting over the entire population.[9]

Second, building on Kahng et al. [129], we assume that there exists a true best alternative. Needless to say, this assumption is necessary if we wish to "defend" liquid democracy against their conclusions. But it is also an extremely well-studied assumption that dates back to the 18th century [240]. The existence of a ground truth is easily justified in the contexts of prediction markets or corporate governance, where alternative policies can be measured in terms of concrete metrics like "estimated revenue in five years," and these metrics can be communicated to voters. That said, some decisions inherently rely on other subjective criteria that we do not capture.

Third, again like previous papers [18, 42, 129], we assume that voters vote independently. Admittedly, this is not a realistic assumption; relaxing it, as it was relaxed for the classic Condorcet Jury Theorem [110, 181], is a natural direction for future work.

Fourth, our models do not take strategic behaviors into account. It would be interesting to bridge our work and those capturing game-theoretic issues in liquid democracy [25, 243].

More generally, our work aims to provide a better understanding of a prominent shortcoming of liquid democracy: concentration of power. But there are others. For example, any voter can see the complete delegation graph under current liquid democracy systems — a feature that helps voters make informed delegation decisions (because one's vote can be transitively delegated). This may lead to voter coercion, however, and the tradeoff between

[9]This extension does not carry over to undirected networks, since if voters have a small number of neighbors, we would expect many 2-cycles to form after delegation, which, under the worst-case cycle approach, may not be canceled out by the overall increase in competence.

transparency and security is poorly understood. Nevertheless, there are many reasons to be excited about the potential of liquid democracy [26]. We believe that our results provide another such reason and hope that our techniques will be useful in continuing to build the theoretical and empirical understanding of this compelling paradigm.

Chapter 4

An Empirical Analysis of Liquid Democracy's Epistemic Performance

Abstract

Liquid democracy promises to enhance collective decisions through a process deemed both legitimate (delegates are chosen endogenously by all) and accurate (experts tend to receive more delegations). Such assertions rely on both delegations improving the group's expertise post-delegation and no delegate amassing too much power. To investigate liquid democracy on binary issues for which there is a ground truth, Chapter 3 modeled delegation behavior stochastically and identified sufficient conditions such that liquid democracy performs better than direct democracy. Herein, we investigate whether these conditions are met empirically. Through six experiments with a total of $N = 168$ participants (and a pre-study involving $N = 101$ participants), we test the performance of liquid democracy by asking participants to either vote or delegate on *tasks* (group of questions from a unique theme). Regardless of their delegation choices, we collect participants' answers to all questions and compare the liquid vote with its counterfactual, the direct vote. We observe that higher-expertise

participants are statistically less likely to delegate than lower-expertise ones. Further, the average expertise of participants who delegate is lower than the expertise of those receiving delegations. These findings are aligned with Chapter 3's requirement and empirically suggest that delegation behaviors meet the conditions for positive theoretical guarantees.

4.1 Introduction

Liquid democracy is a voting paradigm that allows participants to either cast a vote directly or nominate a delegate to decide on their behalf. Delegations are transitive so that if A delegates to B, B delegates to C, and C votes herself, C effectively casts a vote on behalf of all three. The final decision is made through a weighted majority where a participant's weight equals the number of delegations she received; this is illustrated in Section 4.1.

Liquid democracy has been said to combine the best aspects of direct voting (where all participants cast a vote) and representative democracy (where participants elect representatives to vote on their behalf) [26]. Moreover, it is currently being proposed as an alternative to existing voting practices to elect per-issue bodies of experts (or congress-members) [229]. Evaluating such proposals is beyond the scope of this chapter; instead, we investigate the empirical performance of liquid democracy on closed questions, i.e., those with a correct answer. While such results cannot alone be used to advocate for or against liquid democracy, they would test a key assumption at the heart of this voting paradigm: local delegations will find experts in the electorate and lead to better decisions. We will focus on the *epistemic* setting, where participants decide on a binary issue for which there is a ground truth, and evaluate the epistemic dimension of decision-making investigating the performance of various rules in identifying the correct answer to given problems.[1]

[1] We will solely focus in this chapter on issues that have a unique correct answer. In Section 4.5, we will discuss the extent to which such insights extend beyond this to decisions that are not only fact-based but moral-based.

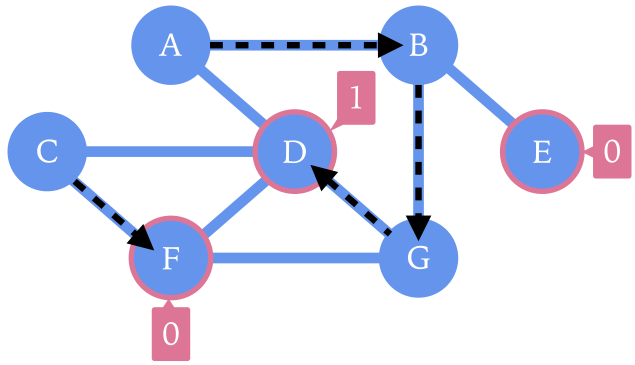


Figure 4.1: Liquid vote between propositions 0 and 1.
The figure represents the output of a liquid vote on two propositions (0 and 1) among $N = 7$ participants. The participants are connected through an underlying social structure illustrated by the blue lines. The dotted black arrows represent delegations: participant A delegates to participant B. The participants who delegate (participants A, B, C, and G) are called *delegators*. The participants who vote directly, that is, do not delegate and take part in the final vote (participants D, E, and F, circled in pink) are called *delegates*. In liquid democracy, votes are counted through a weighted majority where each participant $i \in [N]$ has weight w_i depending on their delegation behavior. Each delegate's weight equals the number of individuals they represent either directly or transitively (here $w_D = 4, w_E = 1$ and $w_F = 2$). Indeed, delegate D represents herself along with participants G, B and A. Delegate E, on the other hand, solely represents herself. The weight of the delegators in the final decision is equal to zero. Finally, the pink boxes display the policy for which each delegate votes. The decision is made among the delegates, each of their vote weighted by the number of delegations they received. Proposition 0 hence gathers $w_E + w_F = 3$ votes and proposition B $w_D = 4$ votes. In summary, this liquid assembly chose proposition 1.

Researchers on the epistemic dimension of collective decision-making have documented for over two centuries the power of *collective intelligence* that emerges when a group, through its collective agency, is wiser than any of its individual members. These results have theoretical underpinning as formalized mathematically by the Marquis de Condorcet in 1785 [64] (in what is known as the Condorcet Jury Theorem) and have been supported by considerable empirical evidence [94, 219], philosophical argument [144], and use in business applications, for instance, in prediction markets [10] and crowdsourcing [67, 232].

In the simple case of N participants facing a binary decision (think of the question "Will Joe Biden or Donald Trump win the American presidential election in 2024?"), a priori unknown, this phenomenon can be modeled as follows. We use the notation $[k] = \{1, \ldots, k\}$.

Each participant $i \in [N]$ has a *competence* or *expertise* level $p_i \in [0, 1]$ (we will use these terms interchangeably). This p_i represents the probability the participant votes correctly. We let 1 represent the correct answer and 0 the wrong answer, hence, their vote is a sample $X_i \sim \text{Ber}(p_i)$. In general, we will assume that these votes are mutually independent. Further, we will assume that the p_is are themselves drawn i.i.d. from some distribution $\mathcal{D}$. Note that the proportion of correct votes will approach $\mathbb{E}[\mathcal{D}]$ as N increases (simply from the Law of Large Numbers). Hence, if the average expertise of a group member $\mathbb{E}[\mathcal{D}]$ is strictly greater than 0.5, the probability that at least half of the participants are correct converges to 1 as N increases. In other words, for N large enough, even when no individual citizen is perfectly accurate, the group almost certainly converges to the correct answer.[2]

Of course, this result is flipped should the average expertise of a group member $\mathbb{E}[\mathcal{D}]$ be strictly smaller than 0.5. Empirically, groups are also known to fall into this sub-optimal regime, leading to the *confusion of the multitude* [156].[3]

The performance of this collective agency is therefore known to depend on the characteristics of the groups. In lay terms, for binary decision making, the premise of collective intelligence is that the average group member is better at voting than a random fair coin.

4.1.1 Contributions

Importantly, liquid democrats suggest displacing the necessary condition of collective intelligence from "knowing about an issue" to "knowing who knows about this issue." I might not know whether proposition 0 or proposition 1 is better suited to curb climate change, but I most likely know who knows more than me. Liquid democracy could leverage collec-

[2]Note that, for some distributions, this result still holds when comparing the group to the vote parametrized by the largest expertise drawn from of a fixed distribution (not changing with N) that has 1 in its support, as the probability that a sum of Bernouillis converges grows exponentially fast while the probability that the highest order statistics does grows as $1/N$ [153].

[3]The death of the philosopher Socrates can be taken as an example of collective confusion. Socrates was put on trial for "corrupting the youth" by politicians unhappy with Socrates' effort to teach students to have a critical spirit and sentenced to death by a majority vote (56%) of 501 Athenians [190].

tive intelligence to identify the knowledgeable agents and increase the likelihood of being collectively correct. In other words, it could use knowledge agents have *about each other* to foster quality decisions. Herein, we identify whether such a phenomenon (where participants identify more competent others through delegation) happens in practice.

Even if liquid democracy increases the expertise of the average group member endogenously through delegation, there are still ways for things to go wrong. It may potentially lead to excessive concentration of power where, in the extreme case, a unique delegate receives all the delegations. Beyond the philosophical concerns, the Condorcet Jury Theorem suggests that such a situation would be mathematically sub-optimal. Indeed, Kahng et al. [129] proved that, under a certain class of delegation behaviors, it is always possible to construct pathological network typologies such that a few agents amass too much power for liquid democracy to outperform a majority vote. Hence, besides testing whether participants identify more competent others, we will also comment on whether we observe concentration of power in our studies. This chapter constitutes, to the best of our knowledge, the first series of lab experiments on liquid democracy that can test the rules' promises in terms of epistemic performance.

4.1.2 Related Work

Liquid Democracy In chapter 3, the authors precisely studied this trade-off identifying sufficient conditions on the maximum number of delegations one may receive and the average increase in expertise post-delegation for liquid democracy to outperform direct democracy.

They further identify types of delegation behaviors that lead to liquid assemblies whose characteristics respect the trade-off mentioned above. They model delegation as dependent on participants' relative expertise. Concretely, they consider a function $q : [0, 1] \rightarrow [0, 1]$ that maps expertise to probability of delegation so that participant i with associated competence

p_i votes with probability $q(p_i)$. Next, if participant i delegates, she samples a peer j to delegate to with probability proportional to a value $\varphi(p_i, p_j)$ where $\varphi : [0,1]^2 \rightarrow [0,1]$ depends on both delegator i and potential delegate j's expertise and outputs the probability that this neighbor is chosen. The authors show that the following three classes of delegation behaviors are sufficient for liquid democracy to weakly outperform direct democracy:

- **Upward delegation:** participants delegate with a fixed probability p independent of their expertise but only delegate to more competent peers. In short: for all $i \in [n]$, $q(p_i) = p$ and for all $(i,j) \in [n]^2$, $\varphi(p_i, p_j) = \mathbb{I}[p_j > p_i]$.
- **Confidence-based delegation:** participants' propensity to delegate decreases with their expertise, and they choose someone randomly when they delegate. In short: $q(x)$ is a decreasing function and for all $(i,j) \in [n]^2$, $\varphi(p_i, p_j) = 1$.
- **General Continuous Delegation:** participants delegate with a fixed probability p independent of their expertise, but they put higher weight on more competent peers. In short: for all $i \in [n]$, $q(p_i) = p$ or $q(x)$ is a decreasing function and $\varphi(x, y)$ increases in its second coordinate.

The purpose of the present chapter is to investigate the validity of the delegation behaviors identified in chapter 3. Specifically, we will test whether participants delegate more often when they are less competent and when delegating, whether they tend to choose more competent agents.

While [129], Caragiannis and Micha [42], and Becker et al. [18] presented negative results for liquid democracy exhibiting pathological graphs with an intolerable amount of concentration of power and proving hardness results when trying to find the optimal delegation flows, Halpern et al. identify delegation behavior assuming connected social structure such that liquid democracy proves to be a better-performing voting system than direct democracy.

Note that liquid democracy has further been studied through many lenses other than this epistemic one. From a political economy perspective, Green-Armytage [105] studies whether utility-maximizing agents would rationally delegate; Bloembergen et al. [25], Escoffier et al. [81, 82] and Zhang and Grossi [243] analyze more sophisticated game-theoretic frames to motivate both participants and delegates' rationale in liquid democracy. Brill et al. [38], Colley and Grandi [53], Colley et al. [54], Zhang and Grossi [244] study different types of delegation formats, Kotsialou and Riley [139] consider incentive mechanisms, and Christoff and Grossi [48] investigate logically interdependent propositions connecting liquid democracy to a DeGroot model where participants "copy" their neighbor's signal.

Others have proposed various practical solutions to bypass empirical hurdles associated with liquid democracy: Brill and Talmon [37] proposed letting participants nominate multiple delegates in case some abstain and also suggest ways to let a central planner decide who would receive the delegation among the short-list. In a similar vein, Gölz et al. [101] let participants nominate k delegates and rely on a central planner to choose the delegation graph that would minimize concentration of power.

Empirically, some have looked into different aspects of liquid democracy through experiments in corporate [113] and political environments [137]. More recently, Campbell et al. [41] ran experiments to test a game-theoretic formulation of liquid democracy. Unlike our experiments, Campbell et al. [41] used online platforms to gather participants who did not know each other. They were assigned a probability of being correct and asked whether they would want to delegate to others, with experts (those with the highest probability of being correct) being publicly known. The delegations were randomly assigned to the pre-determined experts in one set-up, and through the random dot kinematogram task in another set-up. The group sizes considered are 5 people with one expert, 15 people with 3 experts and 125 people with 25 experts. While this study uncovers interesting connection between individuals' perceived expertise and delegation behavior, it cannot investigate how experts are (or are not) identified endogenously through interpersonal knowledge embedded in a social networks, since the participants do not know each other.

Finally, political philosophers have studied the normative properties of liquid democracy [26, 228] and have proposed it as an alternative to the current legislative processes [145, 229]. Such research often follows [142]'s minority view that representative democracy, if achieved through *cogent* selections, may be a better form of democracy than direct democracy.

Wisdom of Crowds in Practice This chapter provides an empirical analysis of liquid democracy's performance, also relying on the empirical literature focused on collective behaviors and the "psychology of crowds" [147].

In his controversial 1895 book, Gustave Le Bon, with the specter of the French Revolution in mind, defines different types of crowds and rationalizes their predictable irrationality through the concept of "popular mind." In his *Memoirs of Extraordinary Popular Delusions and the Madness of Crowds*, Charles Mackay references instances where groups' judgements resulted in disastrous outcomes [154]. Yet, examples dating back to the early twentieth century have exhibited a phenomenon called "Wisdom of Crowds" in which the quality of groups' judgements surpasses those of a few experts. For example, Francis Galton famously collected 787 predictions for the weight of an ox and observed that the "median of the guesses—1,207 pounds—was, remarkably, within 1% of the true weight" [94, 215].

Such experiments have been repeated over the years to assess knowledge [219], forecast stock prices [132], identify phishing websites [152, 176], forecast political or social events [39, 120, 131, 192], and predict sporting outcomes [117]. Predictions markets [10] have also promised to deliver more accurate forecasts on forums generating revenues for the prediction of highly uncertain events. Crowd-sourcing also was built on similar premises [67, 232].

The most comprehensive empirical investigation of the Wisdom of Crowds, to the best of our knowledge, is by Simoiu et al. [215]; they collected around 500,000 answers from almost 2,000 participants for about 1,000 questions spanning 50 different domains ordered in 5 categories (knowledge, tacit, popular culture, predictions, and spatial reasoning). They found that the crowd does better on individual questions and "considerably better than individuals when performance is computed on a full set of questions within a domain."

While these experiments use simple aggregation metrics (such as the mean or median of a sample), others were tested in an attempt to extract an even wiser substrate from the signals gathered from a group, such as in Prelec's *truth serum* [194].

Wisdom of Crowds has also been found to fail in certain setups, for example, situations "in which emotional, intuitive responses conflict with more rational, deliberative responses" [215]. For instance, Simmons et al. [214] found that participants' biases prevented them to make wise decisions in the sports betting context.

The notion of the Wisdom of Crowds is not, as it may first seem, at odds with the idea of expertise. On the contrary, researchers have identified that often, small crowds of identified *experts* perform better than the large less-informed crowds [39, 98, 118, 222]: there is a fine balance that must be struck between a crowd's size and its expertise. Liquid democracy promises to identify such a smaller crowd endogenously.

4.1.3 Experiment Goals

In what follows, we present a series of six experiments to test the efficacy of Liquid Democracy. The experiments were structured as follows. Participants were presented with several yes or no questions divided into themes. For the set of questions in each theme, which we call a "task," they could either choose to vote directly or delegate their vote (for all questions) to another participant. Even if they delegated on a given task, in a later phase, they were asked the same questions to see how they would have voted. This allows us to do a few things. First, this induces a matched-pair design where, for each task and experiment, we can compare the accuracy of voting under liquid and direct democracy. Second, we can use the answers to all questions to estimate participants' competencies.[4] From this, we can study how delegation behavior depends on expertise. We hypothesized that the behaviors match the sufficient theoretical conditions of chapter 3; that is, first, participants are less likely to delegate the more competent they are, and second, on average, more competent participants tend to receive more delegations. Finally, we investigate whether or not the estimated behaviors would lead to harmful concentration of power.

The rest of the chapter is organized as follows. In Sections 4.2 and 4.3, we describe the experimental design and the statistical methods used for inference. Next, in Section 4.4, we present our results: remarks about the observed delegation graphs, estimation of voter behaviors, and a comparison between the performance of liquid and direct democracy. Finally, Section 4.5 discusses the experiment's limitations and avenues for future work.

4.2 Experimental Design

In this section, we present the experiments set up and survey material, the survey flow and the experiments' demographics.

4.2.1 Experiments

We conducted $E = 6$ experiments between March 21st and November 27th, 2022.[5] In each experiment e, a group of participants performed $|\mathcal{T}_e|$ tasks ($|\mathcal{T}_e| = 4$, except for experiment $e = 6$ in which $|\mathcal{T}_e| = 12$).[6] Each task consisted of 8 questions on the corresponding theme.

[4] This was done taking into account question difficulty using the Item Response Theory framework [75, 141].

[5] Our protocols E-3766 and E-3948 were approved and exempted by the MIT Committee on the Use of Humans as Experimental Subjects.

[6] The final experiment was conducted over a longer period of time, allowing more tasks to be completed.

Liquid democracy's core tenet depends on the potential for beneficial delegation. It is therefore necessary to work with participants that have at least a passing familiarity with each other. To account for this, experiments were conducted in places such as classrooms and company workshops, where preexisting group structures guaranteed this. While significant preparation was needed to ensure correct experimental set-ups for these irregular environments, it did have the benefit of producing high-quality data with few missing entries and minimal drop-out.

A total of $N = 168$ individuals participated. Across all, participants hailed from over 30 countries; 33% were female, 1% were non-binary, 64% were male, and 2% preferred to self-describe. Each experiment e had a number of participants N_e ranging between 14 to 50. A description of the settings and group sizes are presented in Table 4.1.

Table 4.1: A description of experiment settings and sizes.

Group ID (e)	Setting	Group Size (N_e)
1	Company employees present at a workshop	14
2	Undergraduate students present in class	22
3	Research department meeting	19
4	Company employees present at a workshop	27
5	Participants at an academic conference	36
6	Participants at an academic conference	50

The described experiments were preceded by a pre-study on 6 different groups.[7] Lessons learned from the pre-study informed the current design, allowing for a more ergonomic survey design with more questions in the same amount of time, and filtering out ambiguous, wrong, and trivial questions. Further, the latter experiments coincide and add more power to the results initially observed in the pre-study.

[7] A description of the setup and results from the pre-study can be found in Appendix A.8

4.2.2 Material

Each participant in experiment e was faced with $|\mathcal{T}_e| \in \{4, 12\}$ tasks. To decide whether to vote or delegate on a task, voters were presented with prompts, described in Table 4.2. A task involved answering a series of 8 questions so that, in total, the experiments contained between 32 and 104 questions.[8] The questions were primarily taken and adapted from Simoiu et al. [215] which includes a curated list of epistemic questions (although several prediction questions pertained to events that had passed, so these were replaced with new ones).

To be consistent with the theoretical setup under study, we converted all categorical questions into binary questions. For example, for a question from Simoiu et al. [215] of the form "Where is this famous landmark from?" with four options (Italy, Tibet, Greece, or Brazil), we selected a possible answer (e.g., Brazil) to reformulate the question as: "Is this famous landmark from Brazil?" [9] A sample task and question can be found in Figure A.1.

4.2.3 Survey Flow

At the beginning of the survey, participants were asked to provide informed consent before inputting their name. Next, they were asked to complete all the $|\mathcal{T}_e|$ tasks (displayed in a random order).

In the first experimental stage, a participant could either answer a series of questions related to that theme or delegate the task to a peer. For instance, a task could read: "You will be shown images of architectural landmarks from around the world, and asked to select the country where the landmark is located," followed by "Do you want to vote yourself or delegate your vote to a trusted peer?" If they chose to vote themselves, they were taken to the 8 questions contained in the task. If they chose to delegate, they were asked to select

[8] All questions can be found in our Github repository: https://github.com/ManRev/liquiddemocracy.

[9] In more detail, we first randomly selected which questions would be correct (to not give a feel that most questions are incorrect) and then, for the incorrect ones, drew a wrong option at random.

Table 4.2: Prompts for Each Task

Task prompts presented to participants at the beginning of each task along with which experiments the task appeared in. After reading it, participants decided to delegate or perform the task themselves. If they delegated, they chose another participant to do the task on their behalf. If they did not delegate, they answered 8 questions related to the task (see all questions on our GitHub repository https://github.com/ManRev/liquiddemocracy).

ID	**Task Prompts**	**Corresponding Experiment(s)**
T_1	You will be shown images of architectural landmarks from around the world, and asked to select the country where the landmark is located.	$1, 2, 3, 4, 5, 6$
T_2	You will be provided with short audio files with theme songs from various movies, and asked to select the movie it was featured in.	$1, 2, 3, 4, 5, 6$
T_3	You will be given English idioms, and asked to identify their meaning. An idiom is a group of words that have a meaning not deducible from those of the individual words (e.g., rain cats and dogs, see the light).	$1, 2, 3, 4, 5, 6$
T_4	You will be given upcoming sport events (soccer and tennis games), and asked to predict the games' outcome?	1
T_5	You will be given the names of tennis players, and asked to predict which round they will make it to in the Tennis French Open (Roland Garros), taking place in May-June 2022?	$2, 3, 4$
T_6	You will be given the names of tennis players (women and men), and asked to predict which round they will make it to in the ongoing Wimbledon Tennis Tournament (The Championships, Wimbledon), taking place between June 27 and July 10, 2022.	5
T_7	You will be given upcoming European men soccer games and asked to predict the games' outcome.	6
T_8	You will be shown images of flags from around the world, and asked to identify their country of origin.	6
T_9	You will be shown 20 images of famous buildings from around the world, and asked to estimate the year in which the building was completed.	6
T_{10}	You will be shown images of constellations and asked to identify them.	6
T_{11}	You will be given headlines, and asked to identify the magazine that published the article, between The Economist and WIRED.	6
T_{12}	You will be given words, and asked to identify the correct synonym corresponding to each word.	6
T_{13}	You will be asked to listen to audio clips of classical compositions, and asked to identify the composer.	6
T_{14}	You will be given the names of American states, and be asked to predict whether the majority of Congress members elected in that state will be Republican or Democrat.	6
T_{15}	You will be given upcoming NBA games and asked to predict the games' outcome.	6

the name of their delegate and then immediately directed to the next task. Importantly, when deciding whether or not to delegate, participants did not see the questions, just the description.

After completing the tasks, a participant was taken to the second experimental stage to answer “additional questions.” These were, in fact, all the questions corresponding to tasks they had chosen to delegate in the first stage. This was done at the end of the experiment so as not to prime the participants on the exercise and interviews after the experiments revealed that participants could not guess why they were being asked these questions.

Finally, optional background questions were asked on the last page. Note that the order in which tasks, questions within each task, and the “True/False” options appeared were all randomized. The entire flow is summarized in Figure A.1.

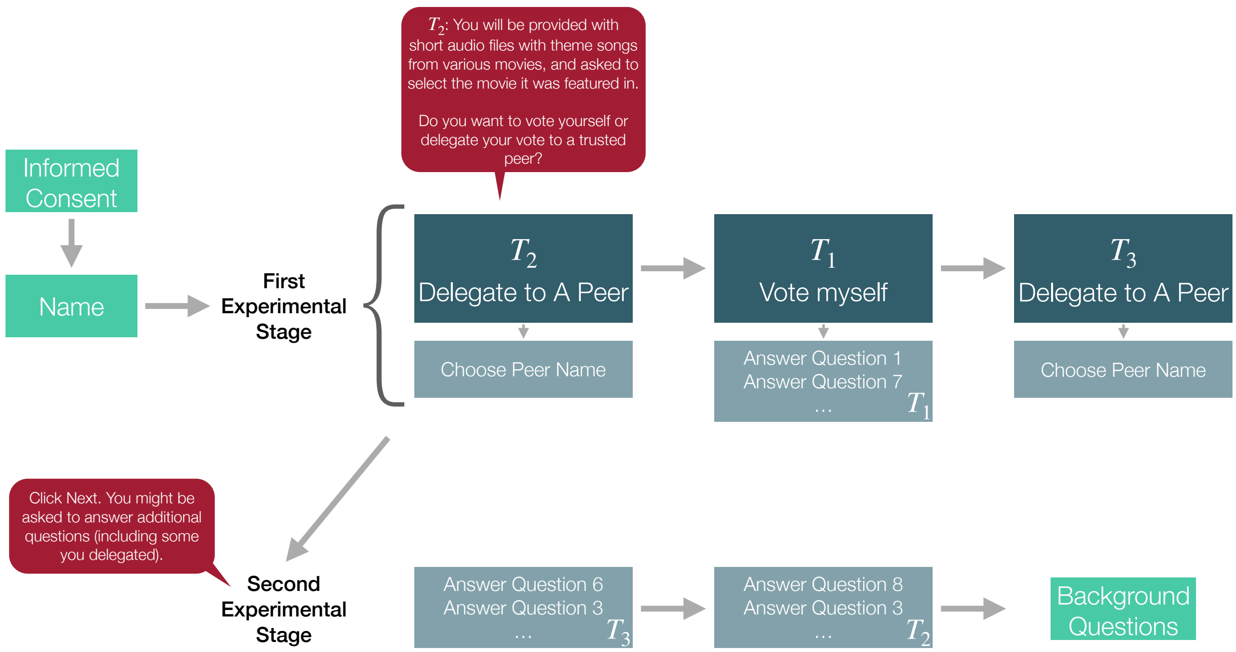


Figure 4.2: Survey Flow

Survey flow with only $|\mathcal{T}_e| = 3$ tasks. The green boxes represent the pre and post-survey steps (providing informed consent, name, and optional background questions). In the first stage, participants performed tasks, deciding to either delegate (providing a name) or vote (answering the 8 question). The upper red block exemplifies a task prompt (in which the options "delegate" and "vote" also appear in a random order). In the second stage, participants answer additional questions (those they delegated) and optional background questions.

4.3 Analysis Strategies

This section reviews the models used to capture voter behavior from chapter 3.

4.3.1 Notation

Let $[N]$ be the set of N participants and let $[E]$, the set of E experiments. Each experiment $e \in [E]$ has N_e participants so that $N = \sum_{e\in[E]} N_e$. We use $[N_e]$ to denote the subset of voters in experiment e. Let $\mathcal{T}$ be the set of tasks surveyed ($|\mathcal{T}| = 15$). For each task $t \in \mathcal{T}$

there is a set R_t of 8 corresponding questions. We let $R = \bigcup_t R_t$ be the set of all questions. For each participant i, we let $e(i) \in [E]$ be the experiment they participated in and for each question r, we let $t(r) \in \mathcal{T}$ be its corresponding task.

In the experiments, we collect:

(i) The direct vote to each question they answered $v_{i,r} \in \{0, 1\}$ where 1 means correct and 0 means incorrect.

(ii) The binary signal $\delta_{i,t}$ equal to 1 if i delegated on task t and 0 otherwise (note that $\delta_{i,t}$ is constant at the task level).

In general, these parameters should only be defined if a voter participated in an experiment where the corresponding task was present. However, we will abuse notation by ignoring this subtlety (writing as if all users participated in all tasks), but implicitly restricting to the voters that actually participated.

From this collected data, we can compute $w_{i,t}$, the weight of voter i on task t. This is their total weight after adding up all transitive delegations; it is set to 0 when they delegate.

In rare cases, a delegation could not be included for a couple of possible reasons. First, if a participant delegated to somebody who did not complete the survey. In this case, we would simply ignore the delegation (assuming they directly voted). Second, in an instance of a cycle (e.g., participant i delegated to participant j who delegated to participant i). These were also ignored (i.e., assumed that no voter on the cycle delegated). In many real-world implementations, such participants would be notified of the cycle and asked to choose a new delegate or vote directly.

4.3.2 Assessing Expertise

In order to evaluate how delegation behavior relates to expertise, we need to estimate participants' expertise. We denote by $\eta_{i,t}$ the estimated expertise of participant i in task t. A naive way to compute participants' expertise per task would be to average their number of correct answers given on all 8 questions of that task, $\eta_{i,t}^{\text{naive}} = \frac{\sum_{r \in R_t} v_{i,r}}{|R_t|}$. However, such a computation does not account for the varying difficulty of the questions.

Instead, we estimate $\eta_{i,t}$ using the Item Response Theory framework (IRT) [141], which provides a parametric model to estimate expertise and question difficulty from repeated measurements. To do this, we fit a one-parameter logistic model to estimate person i's latent ability to task t, $\eta_{i,t}$, as well as question r's latent difficulty, θ_r, where the probability that person i is correct to question r depends on the person's ability at task t and the question's difficulty.[10] Specifically, we assume a generative process of

$$\Pr[v_{i,r} = 1 | \eta_{i,t}, \theta_r] = \frac{1}{1 + \exp^{-(\eta_{i,t} - \theta_r)}}$$

and fit $\eta_{i,t}$ and θ_r to be consistent with the observed answers $v_{i,t}$. We fit these parameters in the canonical way using the Python package py-irt. See Natesan et al. [179] for more details.

While $\eta_{i,t}^{\text{naive}}$ takes on one of nine values (multiples of 1/8), $\eta_{i,t}$ is a continuous variable that can take on arbitrary values in $\mathbb{R}$. We normalize so that $\eta_{i,t} \in [0, 1]$, and assume this to be the expertise, the probability of being correct. The correlation between $\eta_{i,t}^{\text{naive}}$ and $\eta_{i,t}$ is above 94% (see Figure 4.3 for difference in distributions.)

4.3.3 Estimating the q function

Recall that $q(\eta)$ represents the probability that somebody of competence η chooses to delegate. We have observations $\delta_{i,t}$ encoding participant i's delegation choice for task t, and an estimate $\eta_{i,t}$ of i's expertise on task t. Intuitively, fitting a logistic model, regressing $\delta_{i,t}$

[10] We also fit a three-parameter logistic model estimating $\Pr[v_{i,r} = 1 | \eta_{i,t}, \theta_r, c_r, a_r] = c_r + \frac{1-c_r}{1+\exp^{-a_r(\eta_{i,t}-\theta_r)}}$, with c_r, the effect of guessing on question r and a_r, the degree to which question r' differentiate between participants. The resulting expertise levels are highly correlated and we stick with the one-parameter model as a result.

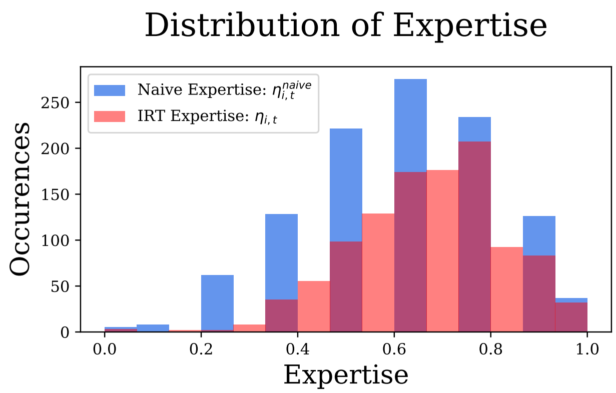


Figure 4.3: Distribution of Expertise using the naive and IRT frameworks

Distribution of expertise $\eta_{i,t}^{\text{naive}}$ and $\eta_{i,t}$ across all tasks and participants, computed with the naive and IRT frameworks respectively. The naive framework use the average correct answers per task per participant as expertise. The IRT framework involves parameters estimation to account for participants' expertise as well as questions' difficulty. The correlation between both frameworks is over 94%.

against $\eta_{i,t}$ estimates the probability that someone with expertise $\eta_{i,t}$ delegates. The following equation shows the relationship we wish to fit, where α_0 is the intercept and β^q is the effect size we measure:

$$\log\left(\frac{\Pr[\delta_{i,t}=1]}{1-\Pr[\delta_{i,t}=1]}\right) = \alpha_0 + \beta^q \eta_{i,t} + \varepsilon_i. \tag{4.1}$$

To account for potential correlation in the error term within participants' answers, when estimating the parameters in Equation (4.1), we cluster standard errors at the participant level. We test for the data normality in Appendix A.3.2.[11]

[11]For a more detailed analysis of what explains the trend we observe, we can add fixed effects to the model to account for intrinsic characteristics of the participants (such as confidence) or of the tasks (such as difficulty).

We can then fit a two-stage model that can be combined in the following generalized linear model with fixed effects:

$$\log\left(\frac{\Pr[\delta_{i,t}=1]}{1-\Pr[\delta_{i,t}=1]}\right) = \alpha_0 + \alpha_i + \alpha_t + \beta^q \eta_{i,t} + \varepsilon_i \tag{4.2}$$

where α_0 is the population average across participants and tasks, α_i is the fixed effect for person i and α_t is the fixed effect for task t, β^q is the model's estimate. See Table A.2 for results.

We repeat the procedure above on data sets filtered by task, this time having a distinct β_t for each task t to measure the task-specific estimates.

4.3.4 Estimating the φ function

Recall that in the theoretical model, a voter with competence η_1 delegates to another with competence η_2 with probability proportional to $\varphi(\eta_1, \eta_2)$. In comparison to q, estimating φ is more challenging. To make this more tractable, we first bucket the observed expertise levels into B clusters $c_1, \ldots, c_B$. We will assume that φ is constant accross inputs in the same bucket, and fit it based on bucket "centers", $\eta_1, \ldots \eta_B$, which are simply taken to be the mean values of the competences in each bucket, i.e., $\eta_\ell = \frac{\sum_{i,t:\eta_{i,t} \in c_\ell} \eta_{i,t}}{|\{(i,t)|\eta_{i,t} \in c_\ell\}|}$. This means we can estimate $\varphi(x, y)$ using the numbers of delegations from any expertise x' to expertise y' where x' and y' fall in the same bucket as x and y, respectively. Finally, we determine the Kendall tau rank correlation coefficient between $\varphi(x, y)$ and y with its associated p-value to test for monotonic relation between φ and its second coordinate.

4.3.4.1 Binning strategies

We discritize the segment $[0, 1]$ into B buckets. We do so using several methods (to ensure the robustness of our approach); we describe here the k-means clustering procedure and discuss the rest in Appendix A.6.4.

To bin using k-means, we optimize for B clusters, $c_1, \ldots, c_B$, that minimize

$$\sum_{k=1}^{B} \sum_{\eta_{i,t} \in c_k} \left(\eta_{i,t} - \frac{\sum_{\eta_{i,t} \in c_k} \eta_{i,t}}{|c_k|} \right)^2.$$

In words, we computer a partition of the $[0, 1]$ segment such that the total squared distance from elements to their cluster centers is minimized. We use the standard k-means clustering algorithm to find the clusters [114] and use the Kneedle algorithm [211] to optimize number of clusters B.

4.3.4.2 Estimation of φ for a given delegation graph

Next, we wish to fit a function φ that induces the relations we observe reasonably well. Note that because the number of participants in each bucket changes for different experiments/-tasks, it is difficult to fit a single function. Instead, we make this more tractable by first finding the most likely φ to have generated each experiment/task and then combining these to find an overall best fit.

When estimating, we only fit cluster centers, $\varphi(\eta_\ell, \eta_k)$ for each $\ell, k \in [B]$. Further, when doing the parametric fitting to combine these estimates, we will do it separately for each first coordinate input, so for conciseness, we write $\varphi^\ell(\eta) := \varphi(\eta_\ell, \eta)$.

Fix a delegation graph, corresponding to experiment e and task t. We are interested in reconstructing the most likely estimate of $\varphi^\ell_{e,t}(\eta_k)$ (we will drop e and t from the notation when they are clear from context). Let z^ℓ_k be the observed number of times that someone of type ℓ delegated to someone of type k. Let n_ℓ be the number of people of type ℓ, and let $\tilde{n}_\ell$ be the number of people of type ℓ that delegated.

Proposition 1. The maximum likelihood estimators for $\varphi^\ell_{e,t}(\eta_1), \ldots, \varphi^\ell_{e,t}(\eta_B)$ satisfy

$$\frac{z^\ell_k}{\tilde{n}_\ell} = \begin{cases} \frac{n_k \varphi^\ell(\eta_k)}{\sum_{j \in [b]} n_j \varphi^\ell(\eta_j) - \varphi^\ell(\eta_\ell)} & \text{if } k \neq \ell \\ \frac{(n_k - 1)\varphi^\ell(\eta_k)}{\sum_{j \in [b]} n_j \varphi^\ell(\eta_j) - \varphi^\ell(\eta_\ell)} & \text{otherwise.} \end{cases}.$$

Proof. Fix an experiment e and task t. These parameters correspond to a delegation graph with multiple delegations.

Fix an expertise type ℓ. We will investigate the delegation going out from participants of type ℓ. Recall that $\tilde{n}^\ell$ is the number of participants of type i that delegated and n_i is the total number of participants of type i in the graph. Let z_k^ℓ be the number of delegations given from participants of type ℓ to participants of type k. Let $\varphi^\ell(\eta_k)$ be the weight with which a participant of type ℓ selects a participant of type k (and it needs to be normalized by the sum of the weights to obtain a probability).

We can view the generation of the graph as $\tilde{n}^\ell$ trials of a B sided dice: each type $k \in \{1, \cdots, B\}$ is selected with probability $\frac{\hat{n_k}\varphi^\ell(\eta_k)}{\sum_{j=1}^B \hat{n_j}\varphi^\ell(\eta_j)}$, where $\hat{n_k} = \begin{cases} n_k & \text{if } k \neq \ell \\ n_\ell - 1 & \text{otherwise.} \end{cases}$ (This is because there are $n_\ell - 1$ participants of type ℓ available to receive the delegation from a participant of type ℓ.) The variable z_k^ℓ tracks the number of times that outcome k occurred in the trials and follows a multinomial distribution with $z_k^\ell \in \{1, \ldots, \tilde{n}^\ell\}$.

Let $\ell(\vec{\varphi})$ be the likelihood of the instance observed, with $\vec{\varphi} = \{\varphi^\ell(\eta_1), \ldots, \varphi^\ell(\eta_b)\}$, then

$$L(\vec{\varphi}) = \tilde{n}^\ell! \prod_{k=1}^{B} \left(\frac{\hat{n_k}\varphi^\ell(\eta_k)}{\sum_{j=1}^B \hat{n_j}\varphi^\ell(\eta_j)} \right)^{z_k^\ell} \frac{1}{z_k^\ell!}$$

and

$$\log(L(\vec{\varphi})) = \log(\tilde{n}^\ell!) + \sum_{k=1}^{B} z_k^\ell \log(\hat{n_k}\varphi^\ell(\eta_k)) - z_k^\ell \log(\sum_{j=1}^{B} \hat{n_j}\varphi^\ell(\eta_j)) - \log(z_k^\ell!).$$

To find the maximum likelihood estimator, we differentiate the log-likelihood above to get

$$\begin{aligned}\frac{\partial \log(L(\vec{\varphi}))}{\partial \varphi_m} &= \frac{z_m^\ell}{\varphi^\ell(\eta_m)} - \frac{\hat{n_k}}{\sum_{j=1}^B \hat{n_j}\varphi^\ell(\eta_j)} \sum_{j=1}^{B} z_j^\ell \\ &= \frac{z_m^\ell}{\varphi^\ell(\eta_m)} - \frac{\hat{n_k}}{\sum_{j=1}^B \hat{n_j}\varphi^\ell(\eta_j)} \tilde{n}^\ell \\ &= 0\end{aligned}$$

Re-arranging the equation above, we find that

$$\frac{z_k^\ell}{\tilde{n}_\ell} = \begin{cases} \frac{n_k \varphi^\ell(\eta_k)}{\sum_{j\in[b]} n_j \varphi^\ell(\eta_j) - \varphi^\ell(\eta_\ell)} & \text{if } k \neq \ell \\ \frac{(n_k - 1)\varphi^\ell(\eta_k)}{\sum_{j\in[b]} n_j \varphi^\ell(\eta_j) - \varphi^\ell(\eta_\ell)} & \text{otherwise.} \end{cases} \qquad \square$$

For consistency across the experiments, we also impose $\sum_{k=1}^B \varphi^\ell(\eta_k) = 1$. We hence have $B+1$ equations with B parameters, but note that $\sum_{k=1}^B \frac{z_k^\ell}{\tilde{n}_\ell} = 1$ so that the equations from section 4.3.4.2 are linearly dependent. Finally, it suffices to solve $\varphi^\ell(\eta_k) = \alpha \times \frac{z_k^\ell}{n^k}$ and normalize through $\sum_{k=1}^B \varphi^\ell(\eta_k) = 1$ to obtain our estimates for $\varphi^\ell(\eta_k)$.[12]

4.3.4.3 Testing for monotonic dependence of φ in its second coordinate

Finally, we test for potential monotonic dependence of $\varphi_{e,t}^\ell(\eta_k)$ as a function of η_k computing the Kendall tau rank correlation coefficient and its associated p-value. Let $R(\varphi_{e,t}^\ell(\eta_k))$ be the rank of $\varphi_{e,t}^\ell(\eta_k)$ amongst all the estimated weights and $R(\eta_k)$ be the rank of η_k amongst all the expertise, we then compute

$$\tau = \frac{\sum_{k\in[N]} \mathbb{I}[R(\varphi_{e,e,t,k}^\ell(\eta_k)) = R(\eta_k)] - \sum_{k\in[N]} \mathbb{I}[R(\varphi_{e,t}^\ell(\eta_k)) \neq R(\eta_k)]}{N - N_\varphi - N_\eta}.$$

[12]We show another approach of Proposition 1 in Appendix A.6.

where N_φ, N_η represent the number of ties in $R(\varphi^\ell_{e,t}(\eta_k))$ and $R(\eta_k)$ respectively. Next, we compare the z-score for the Kendall tau rank, $z = \frac{3\tau\sqrt{N(N-1)^2(N-2)}}{\sqrt{8(N(N-1)+5)}}$ that we compare to the values of a Gaussian distribution (or, more precisely, we let the function stats.kendalltau in python's library scipy handle all of this for us). We further check the Kendall tau rank correlation coefficient between $\varphi^\ell_{e,t}(\eta_k)$ and η_k for a fixed ℓ.[13]

4.3.5 Core Lemma Desiderata: Concentration of Power and Increase in Average Expertise Due to Delegation

We also measure the quantities of interest in chapter 3's core lemma, specifically, the maximum weight accumulated by any participant and the average increase in expertise post delegation. For every experiment e and task t, we note by $m_{e,t} = \max_{i\in[N_e]} w_{i,t}$ and by $\eta^L_{e,t} = \frac{\sum_{i\in[N_e]} w_{i,t}\eta_{i,t}}{N_e}$ (respectively $\eta^D_{e,t} = \frac{\sum_{i\in[N_e]} \eta_{i,t}}{N_e}$) the average expertise after delegation (respectively without delegation).

For each experiment, we display a blox plot of the different maximum weights $m_{e,t}$ per task. We further run a linear regression with fixed effects to measure the difference in mean expertise with and without delegation. Specifically, let ζ be the vector where all the $\eta^L_{e,t}$ and $\eta^D_{e,t}$ are stacked, and γ the vector indicating whether the j^{th} entry of ζ is with or without delegation ($\gamma_j = 1$ if ζ_j is the estimate for liquid democracy and 0 otherwise). Each entry j of ζ is then the average expertise under the different experimental conditions (either liquid or direct) for a given experiment and a given task, inducing a match paired design.

Then

$$\zeta_j = \alpha_0 + \alpha_{e(i)} + \alpha_t + \beta^{\text{lemma}}\gamma_j + \varepsilon_{t,e}, \tag{4.3}$$

estimates β^{lemma}, the difference between the mean expertise with delegation and the mean expertise without delegation, accounting for experiment and task fixed effects, and clusters

[13] We provide an analysis at the task level in Table A.2.

the standard error at the level of a pair (task, experiment) to account for the match paired design.[14]

4.3.6 Liquid Democracy versus Direct Democracy

Finally, we measure the results of liquid democracy versus direct democracy. Note that the result of chapter 3 for the comparison are asymptotic and not something we can replicate here. Instead, we investigate whether the liquid vote is more often right (that is, above 0.50) than the direct vote. We denote by $o^L_{e,t}$ (resp. $o^D_{e,t}$) the average number of times where liquid (resp. direct) democracy is correct for each task and experiment. That is: $o^L_{e,t} = \sum_r \mathbb{I}[\sum_{i\in[N_e]} v^{\ell}_{i,r} > N_e/2]/8$ and $o^D_{e,t} = \sum_r \mathbb{I}[\sum_{i\in[N_e]} v^{D}_{i,r} > N_e/2]/8$. We can run the same regression as Equation (4.3) with a response variable being the vector of $o^L_{e,t}$ and $o^D_{e,t}$ stacked accordingly. Let β^{LvD} be the resulting regression estimate.

4.4 Results

We briefly cover high-level statistics of the delegations and then dive into the analysis of voter behavior. After this, we examine the relative performance of liquid and direct democracy.

[14]Note that, for each experiment e and question r, we can also estimate the performance of direct democracy for the question and that group through $d_{e,r} = \frac{\sum_{i\in N_e} v^D_{i,r}}{N_e}$ (the proportion of correct answers across the group) and that of liquid democracy through $\ell_{e,r} = \frac{\sum_{i\in N_e} v^{\ell}_{i,r}}{N_e} = \frac{\sum_{i\in N_e} v^D_{i,r}\times w_{i,t}}{N_e}$ (the weighted proportion of correct answers across the group). There is, then, severe correlation within tasks: remember that the $\ell_{e,r}$ for each question r are computed with the same weights $w_{i,t}$). Let ρ be the vector that stacks all the outcome of direct and liquid democracy and γ be the vector that indicates whether the j^{th} entry of ρ is liquid or direct democracy. We can then specifying the following model,

$$\rho_j = \alpha_0 + \alpha_{e(i)} + \alpha_r + \beta^{\mathrm{outcome}}\gamma_j + \varepsilon_{r,e}, \tag{4.4}$$

with a fixed effect for the question r (as the questions are nested in the tasks t and adding a task fixed effect would be redundant) and clustering at the level of a pair (e, r) to account for the match paired design. If the expertise was computed using the naive method and the various experiments had the same number of participants, β^{outcome} and β^{lemma} would be estimating the exact same quantity. Given that the expertise computed through the IRT framework and that derived from the naive method are highly correlated, we observe that coefficients from both specifications are almost identical.

4.4.1 Delegation statistics and visuals

For each task t and experiment e, we show in Figures 4.4 and A.4 to A.7 examples of delegation graphs across different experiments and tasks, with all N_e participants in experiments e, represented by nodes labeled by their percentage of correct answers to that task $\eta_{i,t}^{\text{naive}}$. Figures 4.4, A.4 and A.6 display examples of relatively little concentration of power with delegations reaching participants with relatively high expertise, Figure A.5 displays some of the worst concentration of power observed across the 32 instances and Figure A.7 illustrates an example of relatively little concentration of power with delegations reaching participants with relatively low expertise.

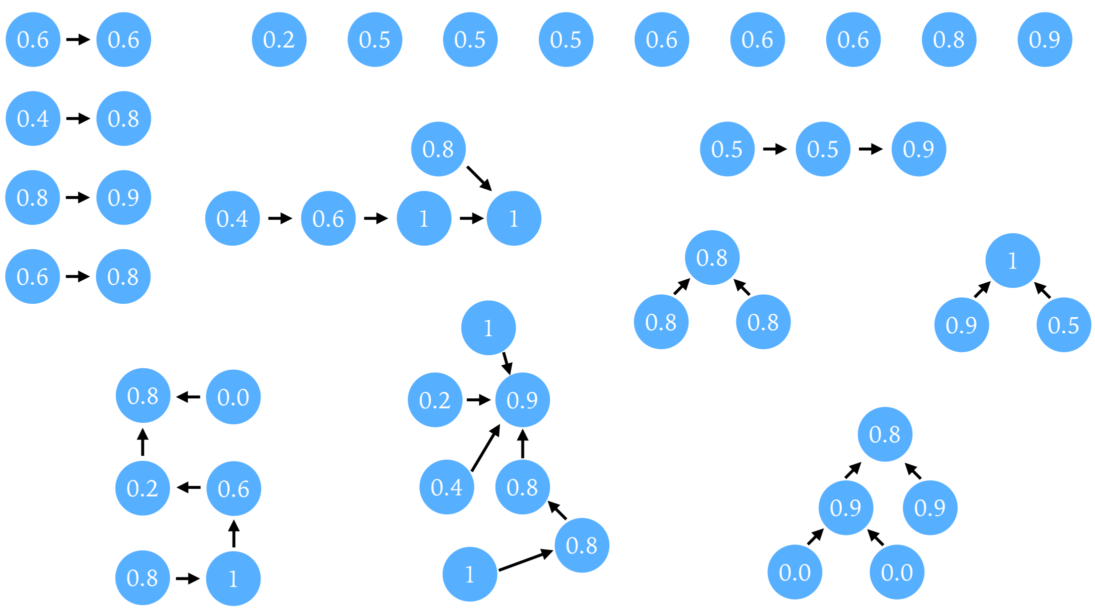


Figure 4.4: Delegation graphs for task T_7 from Experiment 6. Each node is a voter and the node's number represents the rounded proportion of correct answers given by the voter, $\frac{\sum_{r \in R_t} v_{i,r}}{8}$.

Over the course of the six experiments based on five tasks each, we observed only four delegation cycles, and all were only of size two (where a delegates to b, and b delegates back to a). These occurred in Experiment 4 with $N_4 = 27$, and in Experiment 6 with $N_6 = 50$.

Over the 1096 (participant/task) pairs, we observed a total of 505 delegations meaning participants delegated 47% of the time ($std = 0.49$ The rate varied across experiments from 32% ($std = 0.49$) in experiment 2 to 54% ($std = 0.50$) in experiment 5 and across tasks from 22% ($std = 0.50$) in task T_8 to 80% ($std = 0.40$) in task T_{15}. Among those who voted directly, 15% received only one delegation besides their own (hence had weight 2 in the decision), 6% received two delegations, and just about 5% received five or more delegations However, in one experiments, a participant concentrated more than half of the votes, for the prediction task in experiments 5.

While we might worry that delegation patterns vary across gender due to significant differences in confidence across gender [e.g., 78, 210], we actually find no significant differences in these experiments, neither in measured expertise in tasks nor in propensity to delegate. ANOVA tests for the propensity to delegate (resp. expertise) across gender shows no significant differences with $p = 0.464$ (resp. $p = 0.112$). Tukey tests for pairwise mean comparison further validate the absence of significant differences across the different genders (see Appendix A.3.1).

4.4.2 Estimating the q function

In this section, we show that the probability of delegating is decreasing with expertise, as a result of the regression analysis presented in Equation (4.1), confirming chapter 3's assumption on the probability to delegate.

We estimate the q function that models probability of delegating as a function of expertise, following the specifications in Equation (4.1) to assess the function overall and across each task respectively. We find $\beta^q = -2.24$, with standard error $s.e. = 0.42$, statistics $z = -7.12$

and p-value $p = 10^{-7}$. In turn, we estimated that

$$q(\eta_{i,t}) = \widehat{\Pr}[\delta_{i,t} = 1] = \frac{1}{1 + \exp^{-(-1.39 - 2.24 \times \eta_{i,t})}}$$

and that the probability of delegating decrease with expertise. Note that this characteristic is consistent with the confidence based model of chapter 3.

We relegate to Appendix A.5 further results and discussions: Table A.2 shows the results including the fixed-effects as specified in Equation (4.2), and Table A.3 displays the effects per task for task.

4.4.3 Estimating the φ function: Delegation Choice as a Function of Expertise

Herein, we report the results for the model specifications spelled out in Section 4.3.4 that measure potential connections between the delegation choices and expertise. In this section, we display the results based on k-means clustering, and provide further details on varying methods and bucket sizes in Appendix A.6, showing that the significant trends we observe are invariant across different bucket sizes.

4.4.3.1 k-means clustering results

We find an optimal number of clusters equal to 4 (that is the number of clusters at which the decay is within the sum of standard errors flattens as estimated by the kneedle algorithm). The resulting centroids are $0.43, 0.6, 0.75$ and 0.88, and the intervals span are, respectively, $c_1 = [0.00, 0.514], c_2 = [0.515, 0.674], c_3 = [0.677, 0.814]$ and $c_4 = [0.818, 1.00]$. There are, respectively, $16\%, 32\%, 35\%$ and 17% of the data points in each cluster.

4.4.3.2 Estimation of φ

Section 4.4.3.2 shows as a blue cross, for each expertise level η_ℓ, the value of $\varphi^{e,t}_{l,k}$ computed for every experiment, task and expertise level η_k. The pink points represent the average across all experiment and task for a given η_k, and the regression line corresponds to an ordinary least square regression on the mean values.

Table 4.3: Delegation Percentages by Bucket

Each row represents how often participants from a given bucket delegate to those in other buckets.

Bucket	c_1	c_2	c_3	c_4
c_1	15%	22%	27%	43%
c_2	18%	20%	36%	26%
c_3	18%	23%	33%	26%
c_4	12%	18%	34%	36%
Overall	16%	21%	32%	31%

Overall, participants from bucket c_1 receive 16% of the delegations, those from bucket c_2 receive 21%, those from bucket c_3 receive 32% and those from bucket c_4 receive 31%.

To test the significance of the trends observed in Section 4.4.3.2, we test whether the Kandall tau rank correlation coefficient between $\varphi_{e,t}(\eta_\ell, \eta_k) = \varphi^{\ell}_{e,t}(\eta_k)$ and η_k signals significant associations according to Section 4.3.4.3, both at the overall level, and when fixing ℓ, or η_ℓ, the first coordinate in $\varphi_{e,t}(\eta_\ell, \eta_k)$. Table 4.4 shows the resulting correlation coefficients and significance tests.

Table 4.4: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by Section 4.3.4.3.

	Overall	For fixed ℓ			
		c_1	c_2	c_3	c_4
Correlation	0.17****	0.29**	0.12*	0.11	0.28***
P-value	2×10^{-5}	2×10^{-2}	9×10^{-2}	1×10^{-1}	3×10^{-3}
Note:	*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001				

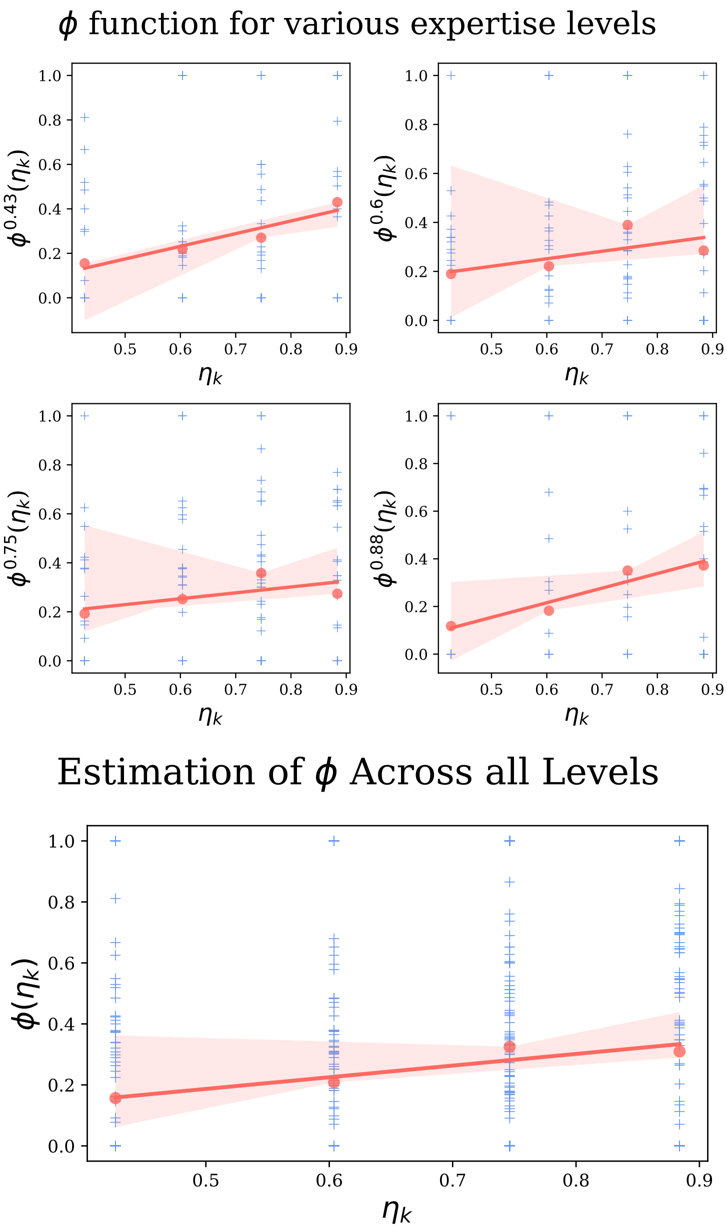


Figure 4.5: Estimates of φ

Each of four plots on the left represents the values of $\varphi_{l,k}^{e,t}$ for a fixed type ℓ. The blue crosses show the values computed for $\varphi_{l,k}^{e,t}$ at each possible η_k. The pink dots show the average across all $\varphi_{l,k}^{e,t}$ at a level η_k, and the pink line corresponds to a linear regression over the mean values. We observe increasing trends across the board, with slope (coefficient of determination) being $0.53(0.90), 0.28(0.46), 0.29(0.47)$ and $0.60(0.92)$ respectively. The plot on the right shows the values for $\varphi_{l,k}^{e,t}$ across all ℓ. The linear fits outputs a slope of 0.38 (coefficient of determination: 0.85).

Note that this characteristic is consistent with the general continuous model of chapter 3. In turn, the delegation trends we observe empirically seem to indicate that, with a complete graph and independent votes, participants behave in such a way that liquid democracy should satisfy positive gain and do no harm asymptotically, per chapter 3's theoretical results.

4.4.4 Core Lemma Desiderata: Concentration of Power and Increase in Average Expertise Due to Delegation

While it is challenging to estimate the relation between the maximum delegation weight $m_{e,t}$ per experiment and per task with little variability in the sample sizes N_e, we provide a box plot with the maximum weight reached across experiments for each task in Figure 4.6.[15]

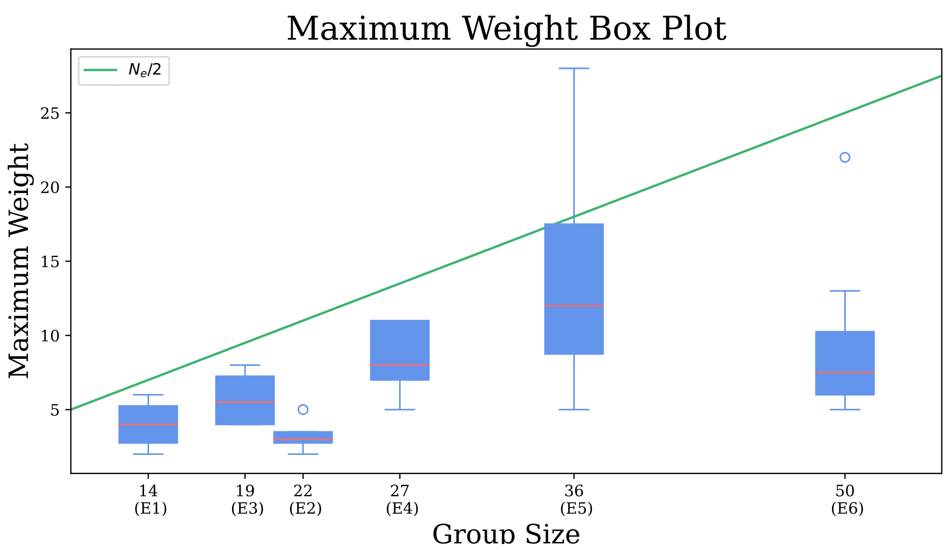


Figure 4.6: Maximum Weights

From left to right, a box plot of the maximum weight $m_{e,t}$ for each experiment e ordered in increasing N_e. The box plot represent the variations in $m_{e,t}$ for a fixed experiment.

Next, we estimate the average increase in expertise post delegation through the model

[15] Another interesting analysis, unrelated to the theoretical results in chapter 3 results, is to compute the minimal size of a majority coalition, that is the minimal number of participants that vote directly and received, in total, at least half of the total votes. This analysis can be found in Appendix A.7.

specification Equation (4.3). We find $\beta^{\text{lemma}} = 0.031$ with $s.e. = 0.006$, $t = 4.78$ and $p = 0.000004$. In other words, across all tasks and experiments, the mean average expertise post delegation is 3% higher than the mean average expertise without delegation.

Recall that chapter 3 found that delegation mechanisms that resulted in $m_{e,t} = o(N_e)$ and $\eta^{\ell}_{e,t} - \eta^{D}_{e,t} > \alpha$ for a fixed $\alpha > 0$ were likely to satisfy probabilistic do no harm and positive gain. While we cannot draw conclusions about the size of $m_{e,t}$, the latest fixed-effect linear model results indicate that the second condition of their core lemma is satisfied.

Finally, we report the results from the specification in Equation (4.4). We find $\beta^{\text{LvD}} = 0.0313$ with $s.e. = 0.025$, $t = 1.229$ and $p = 0.23$. In other words, liquid democracy's average proportion of correct answers is 3 points above that of direct democracy, but this increase is not significant. We show in Figure 4.7 the frequency at which liquid and direct democracies are correct for varying questions across each task.

Recall that we gathered one vote per question r, and that the results shown above for different tasks was averaged across all questions within that task. More work is needed to test the asymptotic results of chapter 3; the present analysis shows that, even if liquid democracy may outperform direct democracy, it is by a small margin.

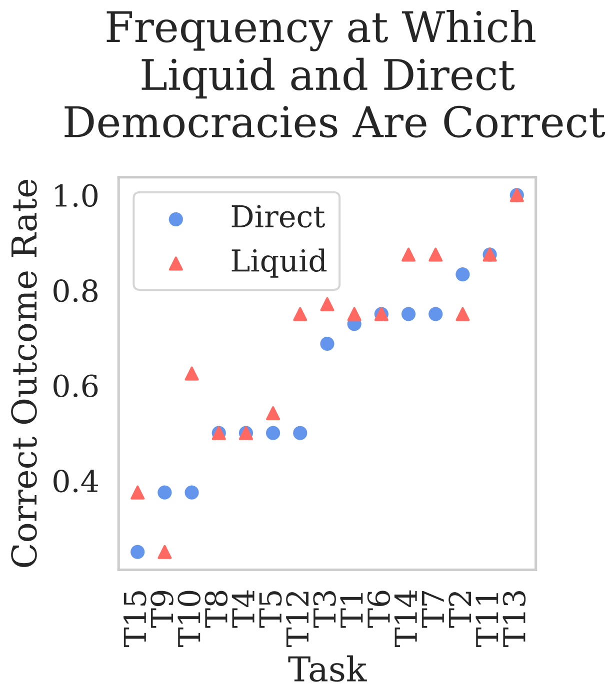


Figure 4.7: Frequency of Correctness for Liquid and Direct Democracies Averaged Per Task

A red triangle represents the proportion of correct liquid vote for a given task t, $\frac{\sum_{e,r} \mathbb{I}[\sum_{i\in[N_e]} v^D_{i,r} \times w_{i,t} > N_e/2]}{|R_t| \times \epsilon}$ where ϵ is the number of experiments that included task t. Similarly, a blue circle represents the proportion of correct direct vote for a given task t, $\frac{\sum_{e,r} \mathbb{I}[\sum_{i\in[N_e]} v^D_{i,r} > N_e/2]}{|R_t| \times \epsilon}$.

4.5 Discussion

This chapter was aimed at evaluating empirically the conditions identified in chapter 3 needed to guarantee the performance of liquid democracy with high probability asymptotically. While we find that delegation trends and aggregate delegation metrics are empirically aligned with chapter 3's theoretical work, this empirical analysis also has some shortcomings.

First, the analysis does not provide a framework to evaluate liquid democracy's asymptotic behavior. While we observe a marginal improvement using liquid democracy, it is a small improvement that varies greatly across tasks, hinting towards the idea that different tasks would benefit from liquid democracy to different extent. More research would be needed to test which tasks benefit from liquid democracy.

Second, while chapter 3 assume that anyone can delegate to anyone else, the groups with which we ran the experiment did not necessarily follow this structure: not all group members knew each other. More research is needed to understand how the graph topology impacts chapter 3's results.

Third, some delegations were problematic, either because they were given to someone who did not participate, or because they were cyclical. We removed such delegations under the assumptions that, due to their small number, such interventions would not impact our results. More research is needed, however, to understand whether problematic delegations are common and how to handle them.

In summary, this chapter focused on the epistemic performance of liquid democracy, where participants decide on a binary issue for which there is a ground truth, and tested the sufficient conditions on the delegation behaviors found by chapter 3. We find that participants are more likely to delegate when they are less competent and that delegation go more often to those with higher expertise, corroborating the most demanding of Halpern et al.'s conditions.

In addition to these comments, note that the current survey did not incentivize participants to make good or bad decisions. Future directions could include testing whether the epistemic performance of liquid democracy and the roles of expertise and confidence change in the presence of rewards for good answers, taken either directly or through transitive delegations.

Finally, note that the scope of this study is particularly narrow as it only considers binary questions with correct answers. In short, epistemic studies of voting relate primarily to the instrumental value of democracy. This informs efforts to deploy liquid democracy in prediction markets or to make corporate decisions with clear (but hard to achieve) goals.

However, deploying liquid democracy in political settings, for instance, would require further research and tests on the intrinsic value of delegations and how they relate to paradigms of representation of conflicting moral values that may not reduce to factual evidence. Further work at the intersection of political philosophy and social choice would be most needed to understand these other aspects of liquid democracy.[16]

[16]The two dimensions that are usually used to evaluate decision-making processes are the epistemic dimension (accuracy of the voting outcome.) and the procedural dimension (fairness of the procedure). Debates on the validity and comparison of the dimensions in institutional design are out of the scope of this chapter, but interested readers should refer to Chiara Destri's essay on the matter [68].

Chapter 5

A Descriptive Analysis of Liquid Democracy's Procedural Performance

Abstract

Delegation dynamics in large groups for contentious and potentially polarizing issues remain under-studied in the literature focusing on liquid democracy. This chapter presents the results of an experiment ran with an academic institution, in which 117 people participated, answering 11 questions about the institute's governance. While it has often been mentioned that cycles and concentration of power are practical threats to liquid democracy, we find that those are unlikely: there are only 3 two-cycles and no evidence of concentration of power. However, we find that a large portion of persons that are delegated to (about 19%) did not participate in the survey, posing serious practical concerns.

5.1 Introduction

Liquid democracy was defined as an (i) area-specific voting scheme (ii) based on transitive proxy voting (iii) with instant recall [230]. Practitioners have worried that it could lead to undue concentration of power [19] or to problematic cycles. Further, liquid democracy experiments where participants can delegate to different persons for different issues are rare, so that little is known about the delegation choices participants would make in varying contexts. As a first step towards improving our understanding of the empirical aspects of liquid democracy, we investigate herein characteristics of its procedure: we report descriptive metrics regarding delegation graphs, cycles, concentration of power, size of majority coalition, absenteeism and individual delegation behaviors.

5.1.1 Problem Statement

In this chapter, we are particularly interested in scenarios where groups have to decide on open and potentially divisive questions. How would group members delegate questions that are inherently subjective and that could impact the long-term governance of their organizations?

5.1.2 Contributions

This chapter provides, to the best of our knowledge, the first descriptive analysis of a lab experiment that can test procedural aspects of liquid democracy. The chapter reports descriptive metrics for high-level delegation trends and participants' behavior. We find rare examples of cycles and concentration of power but, instead, warn against the risk of delegations reaching people who abstain from participating in the decision-making process. While some started looking into the problems of delegations leading no-where [38] from a theoretical lens, it had not yet been explored empirically. Further, we observe frequent transitive delegations and issue-specific delegation choices, using liquid democracy in a way consistent with how it was envisoned in Valsangiacomo [228].

5.1.3 Related Work

To the best of our knowledge, there are only two liquid democracy experiments based on subjective questions: Hardt and Lopes [113] asked to decide on food menus to employees and Kling et al. [137] organically conducted an experiment in a political environment. Further, scholars and practitioners have developed a platform, *LiquidFeedback* ([21]), to support community governance based on liquid democracy (but they typically do not report on the case studies since those are conducted by their partners). For other pointers to experimental work related to liquid democracy, see Chapter 4.

5.2 Experimental Design

5.2.1 Recruitement

The survey was active between May, 11th 2022 and June, 1st 2022.[1] One hundred seventeen (117) persons responded, out of the 263 members listed on the IDSS' website. E-mails were sent to the community about the survey, and QR codes were distributed around the office spaces. The breakdown of participants per occupation can be found in Table 5.1.

[1]The protocol E-3948 was approved and exempted by the MIT Committee on the Use of Humans as Experimental Subjects.

	Faculty	Postdoc	Staff	Student	Total
IDSS Community	91	52	17	103	263
Survey Participants	24	15	14	64	117
Participation Rate	0.26	0.39	0.82	0.62	1
Composition of Survey Pool	0.20	0.12	0.11	0.54	1

Table 5.1: Participation per Occupation

5.2.2 Material

Eleven open questions spanning from operational problems to vision-setting dilemmas were asked – see Table 5.2.

Question ID	Prompt
Question 1	Which restaurants / caterers / vendors should IDSS use to provide IDSS lunches?
Question 2	What new practices could IDSS adopt to strengthen unusual connections across disciplines?
Question 3	What strategies would you recommend to help IDSS students navigate their search for advisors and projects?
Question 4	Should there be a requirement for *all* IDSS theses to include rigorous statistics?
Question 5	Should IDSS create its own academic journal?
Question 6	What are the the most significant factors in determining whether IDSS students enter a satisfying / rewarding / fulfilling career trajectory after graduate school?
Question 7	What are the key factors to consider in deciding between industry and academic careers?
Question 8	What issues do you think the MIT graduate student union should focus on?
Question 9	What should the topic(s) of the next IDSS faculty retreat be?
Question 10	Should IDSS have an undergraduate degree program?
Question 11	In previous years, IDSS launched large collective efforts to address challenges posed by Covid and systemic racism. What should the theme of IDSS's next large collective effort be?

Table 5.2: Prompts of the survey questions

5.2.3 Survey Flow

Participants were shown a series of questions in a random order, and could, for each question, either vote directly or delegate their decision to a trusted peer they could choose from a menu pre-filled with the community members listed on the website. Voting meant answering an open question by filling an unconstrained blank box.[2]

5.2.4 Participation

Out of the 117 participants, a little over a half were students, a fifth were faculty and about a tenth were either postdoctoral researchers or staff. The participation rate (number of people from the community that participated) was the highest among staff (82%), followed by students (62%), postdoctoral researchers (39%) and faculty (26%). See Table 5.1 for the details. Nineteen participants answered only some of the 11 questions, such that each question gathered, on average, 105 answers.

The participants included 16% Asian/Pacific Islanders, 2% Blacks or African-Americans (non-Hispanic), 42% Caucasians/Whites (non-Hispanic), 7% Latinos or Hispanics, 5% Middle Easterns, 3% self-described and 17% did not answer. Furthermore, there are 32% female, 46% male, 0.08% non-binary, 3% prefer not to say and 17% did not answer.

5.3 Analysis

In this section, we report descriptive delegation metrics observed over the different questions. We call a *delegator* someone who delegates and a *delegatee* someone who receives a delegation. *Gurus* are the participants that represent others in the final vote (hence did not delegate but received delegations). Last, *ghosts* are group members who were delegated to, but who

[2] The data was then anonimyzed.

did not participate in the survey. We denote by $[N]$ the set of participants, and by $[G]$ the set of ghosts.

5.3.1 Methods

For a fixed question q and a participant i, we denote by δ_i^q the binary variable that indicates whether person i delegated ($\delta_i^q = 1$) or not. Further, we denote by w_i^q the weight of person i in question q. That is, w_i^q is equal to the number of participants represented by person i for a given question. In turn, $w_i^q \in [0, 117]$, where it is 0 is person i delegated question q, 1 if person i did not delegate and was not delegated to for question q and strictly above 1 otherwise. Further, the edge $e_{i,j}^q$ takes value 1 if participant i delegated to participant j on question q.

We first show the directed delegation graphs (that is, the collection of edges $e_{i,j}^q$ corresponding to a specific question q) for a couple of questions. We next comment on the strongly connected components of the weighted directed graph that represents the delegation graphs for all questions combined (a component is strongly connected if every vertex is reachable from every other vertex).

Furthermore, we report the number of cycles observed. A cycle of size l occurs if a series of participants $\{i_1, \cdots, i_l\}$ are such that $e_{i_1,i_2}^q = 1, e_{i_2,i_3}^q = 1, \cdots, e_{i_l,i_1}^q = 1$. In other words, a group of l participants that delegated to one another constitute a delegation cycle, characterized by its size. Cycles are inherently problematic since no one is effectively representing the participants of a cycle.

We also report the amount of delegations accumulated by participants (with a particular interest for the maximum weight $m_q = \max_{i\in[117]} w_i^q$), and the size of the smallest majority coalition. Let fix a question q and order the weights $w_{(i)}^q$ in decreasing order. Let $n_q = \sum_{i=1}^{117} \mathbb{I}[w_{(i)}^q > 0]$ be the number of gurus in a graph. Let $\sum_{i=1}^{k} w_{(i)}^q$ be the weight accumulated by the k participants with the highest weights. Let m_q^* be the smallest k such that $\sum_{i=1}^{k} w_{(i)}^q > 0.5$; we call this *the smallest size of a potentially majority coalition* (SPMC).

In addition, we discuss the occurrences of ghost, that is, participants $g \in [G]$ such that $e_{i,g}^q = 1$ (or delegatees that did not participate in the survey).

Finally, we dig into delegation frequency across questions ($\sum_{i=1}^{117} \frac{\delta_i^q}{117}$) and participants ($\sum_{q=1}^{11} \frac{\delta_i^q}{11}$), and we check whether delegators tend to choose different delegatees across questions. We also compare the amount of delegations received to that given, where the number of delegations received by participant i is $in_i = \sum_{q=1}^{11} \sum_{j=1}^{117} e_{j,i}^q$, and the number of delegations given is $out_i = \sum_{q=1}^{11} \sum_{j=1}^{117} e_{i,j}^q$. Last, we compute participants' closeness centrality $c_i = \frac{n-1}{117-1} \frac{n-1}{\sum_{j=1}^{117} d(j,i)}$ where $n-1$ is the number of nodes reachable from i and $d(j,i)$ is the shortest-path distance between participants j and i. This weighted version accounts for the fact that the graph has more than one connected component (note that the edges' weight are ignored) [218].

5.3.2 Delegation Graphs

5.3.2.1 Directed Graphs Per Question

We show below delegation graphs for a couple of questions. A direct edge goes from a delegator to a delegatee. We show in the middle of each plot the number of voters who delegated to themselves and did not receive any delegations. The larger nodes represent voters that received delegations and are voting – the gurus. The transparent nodes received delegations but did not participate – the ghosts. The nodes are colored by position (faculty are blue, postdocs are green, staff are orange, students are red). On average, liquid democracy leads to 73 voters (std=10). On average, 56 (std=14) of these voters do not receive any delegation.

Most delegations involve only two participants (where the guru only represents themselves

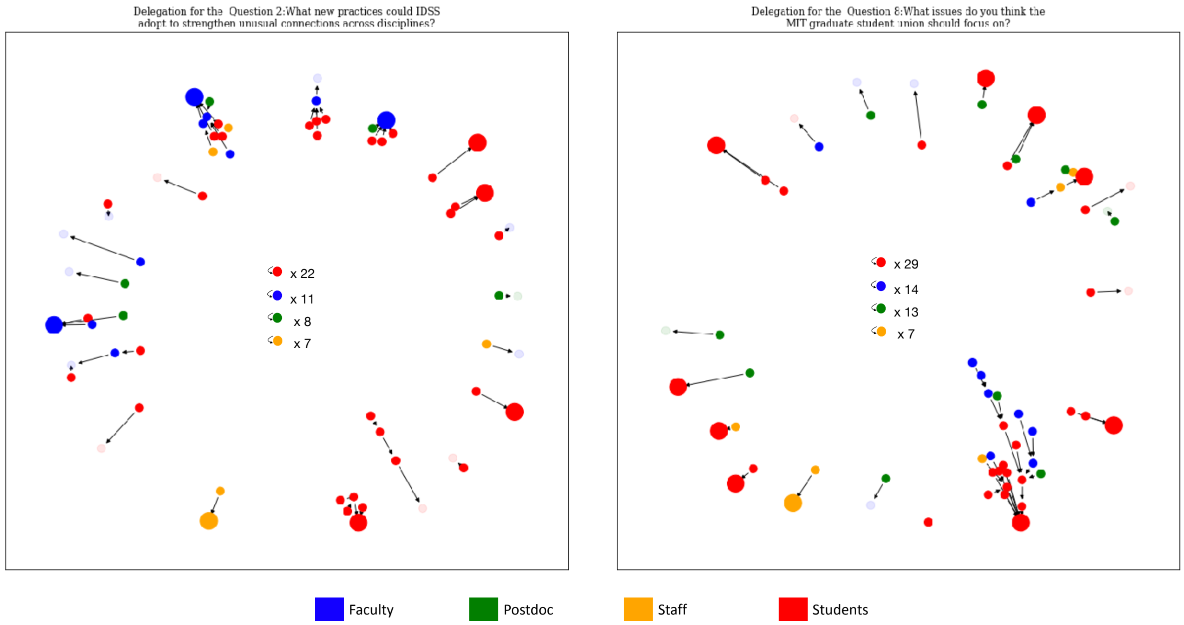


Figure 5.1: Delegation Graphs

Delegation graph for two questions. The color corresponds to the participants' occupation (faculty are blue, postdocs are green, staff are orange, students are red). The transparent nodes did not participate in the survey but received delegations. The large nodes are gurus (those who receive delegations and vote themselves). The nodes in the middle with a self-edge indicate voters who did not delegate and did not receive delegations.

and one delegator): we see 57 instances of such delegations out of 113. Next, there are respectively 15, 7 and 11 instances of delegations involving 3, 4 and 5 participants. In turn, 20% of the delegations involve more than 6 participants; 5% involve strictly more than 10 participants. There are 2 instances of delegation trees with 13 participants, 3 instances with 14 participants and the largest delegation tree involves 22 participants. We show in Figure 5.2 a histogram of the weights w_i^q carried by gurus across all the delegation graphs.

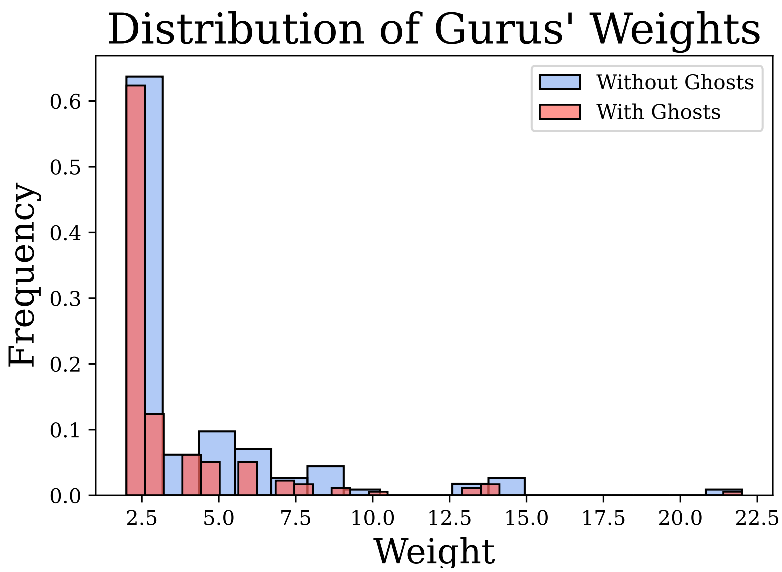


Figure 5.2: Delegation Weights w_i^q

Histogram of delegations across all questions. The blue histogram excludes the ghosts and the red one includes them.

5.3.2.2 Weighted Directed Graphs Across All Questions

Across all questions, 60 participants never received a single delegation across questions, and 11 among those never delegated either. Further, 16 participants received 1 direct delegation, 13 received 2, 4 received 3, 5 received 3, 3 received 5, and 16 received more than 6. One participant was delegated to 41 times across all questions.

We identify 96 strongly connected components in the weighted directed graph, 94 of which are of size 1. There is one component of size 2, and the last one is of size 21. This means that, accounting for all delegations across all questions, and picking a random participant in the strongly connected component, there always exists a directed path to any other participant for the component. However, the weights accumulated by those in the strongly connected components account only for 14% of the total delegation weights: the other participants do not seem likely to delegate *into* the component.

5.3.3 Cycles

Across the 11 questions, we found a total of 3 two-cycle. That is, it occurred three times that two participants delegated to each other, with i_1, i_2 and $e^q_{i_1,i_2} = 1, e^q_{i_2,i_1} = 1$.

5.3.4 Concentration of power and Majoritarian Coalitions

The maximum number of votes aggregated by one single person is 22. We show in Figure 5.3 a box plot of the maximum weights, m^q across the different questions. The median maximun weight across all questions is 8.

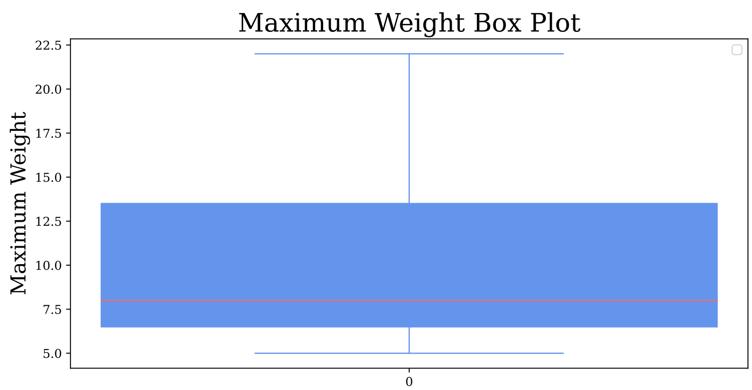


Figure 5.3: Box plot with the maximum number of delegations received across all 11 questions.

Next, we show in Figure 5.4 the fraction of votes gathered by the smallest coalition with the highest total weight for each question. Each color represents a question, and each point (x, y) represent the proportion of weights $y = \sum_{i=1}^{x} w^q_{(i)}$ that the x participants with the largest weight accumulated.

Further, we report in Figure 5.5 a box plot of m^*_q, the smallest numbers of gurus needed such that they control half of the votes. The median SPMC is 13. For question 9, the majority of the votes was controlled by just two participants. On the contrary, 30 participants were

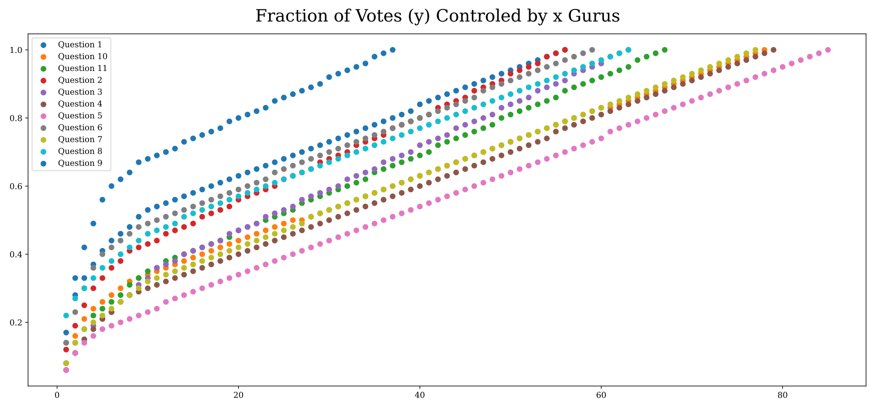


Figure 5.4: Fraction of votes gathered by the smallest coalition with the highest total weight per question.

needed to accumulate a majority of the votes in Question 5.

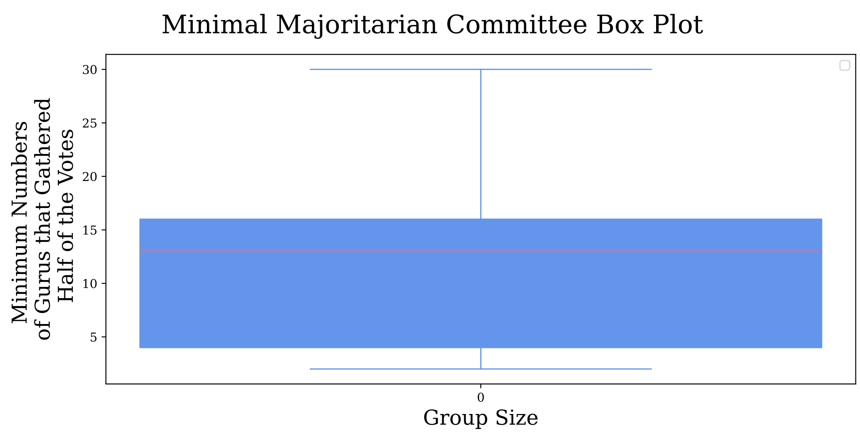


Figure 5.5: Box plot of the SPMC (m_q^*) across all questions.

5.3.5 Ghosts

This experiment showed little signs of cyclical delegations and concentration of power. However, we found that a lot of participants who received delegations did not participate – we call those persons *ghosts*. Overall, 39 persons that received a delegation did not participate, such that 80 delegations lead nowhere. On average, 6.8 persons per question received delegations without participating (std= 3.5). Based on our results, this is an important practical aspect of liquid democracy that requires further attention when participation is voluntary.

5.4 Delegation Behaviors

This section studies characteristics of delegation behaviors.

5.4.1 How often do people delegate?

Participants had up to 11 opportunities to delegate[3]. We measure the delegation rate as the frequency at which participants delegate (number of delegations divided by the number of questions answered).

Figure 5.6 shows a histogram for the $\sum_{q=1}^{11} \frac{\delta_i^q}{11}$.

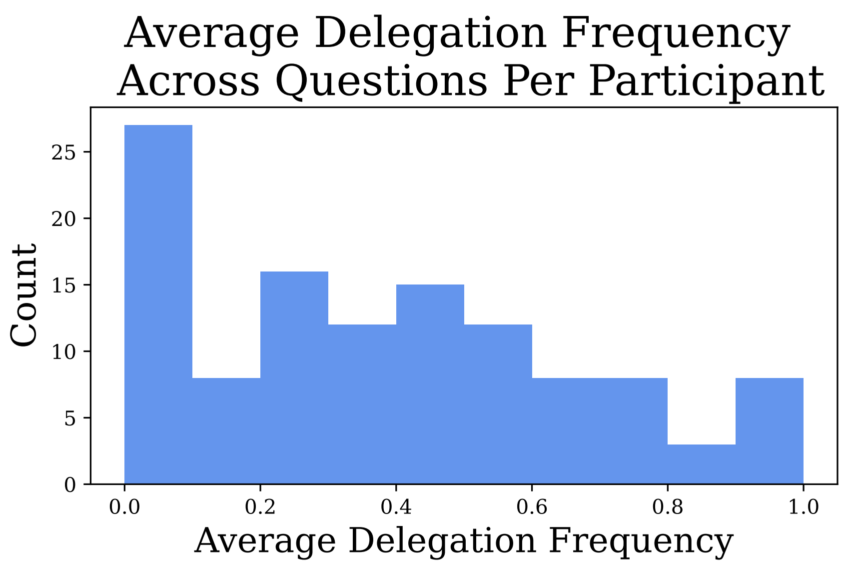


Figure 5.6: Proportion of people that delegate at certain delegation rates.

We see that many participants never delegated, and we observe that participants are less likely to delegate often than not.

[3]Most participants answered all 11 questions, but 19 participants answered only a subset of the 11 questions. Their delegation frequency is computed as a function of the number of questions they answered.

5.4.2 Are delegation frequencies different for different questions?

Delegation rates were significantly different across different questions (a chi-square test of independence results in $t = 74.9198$ with a p-value less than 10^{-5}.). Figure 5.7 shows delegation frequencies (with 95% confidence intervals) across the different questions, $\sum_{i=1}^{117} \frac{\delta_i^q}{117}$.

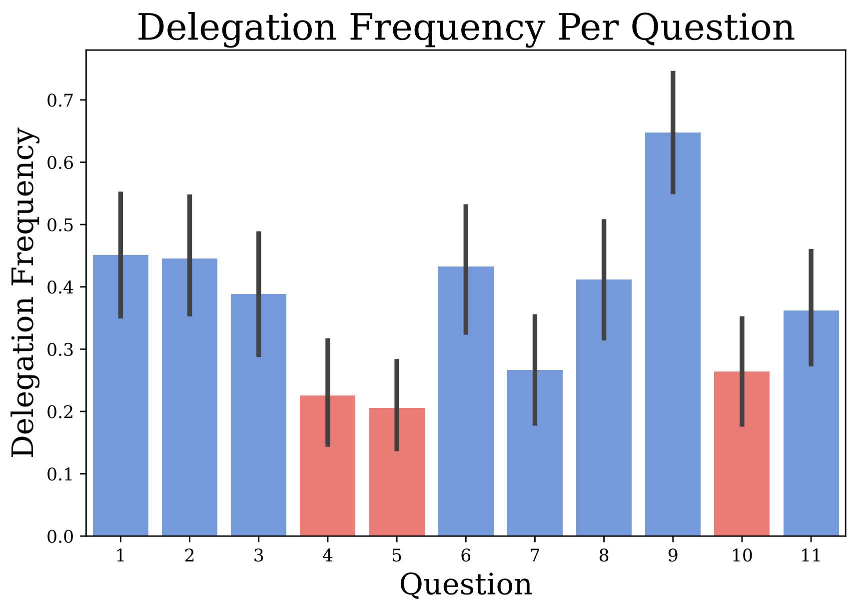


Figure 5.7: Delegation rates per question. The red bars represents yes/no questions.

Since Campbell et al. [41] reports that propensity to delegate play an important role in liquid democracy's performance in certain contexts, it may be interesting to further investigate whether there exists systematic heterogeneity in delegation behaviors for different kinds of problems.

5.4.3 Who do delegators delegate to?

Half of the participants that delegated always delegated to someone different. If we filter out participants who delegated only once, those who always delegate to different persons amount of 35% of the total. In Figure 5.8, we see in the square (x, y) the number of times that someone who delegated x times delegated to y different delegatee. The lower diagonal is necessarily empty and, interestingly, we see that the terms around the diagonal are typically the highest. This indicates that participants tend to choose different representatives for different questions, so that the delegation choices seem specific to the issue at stake.

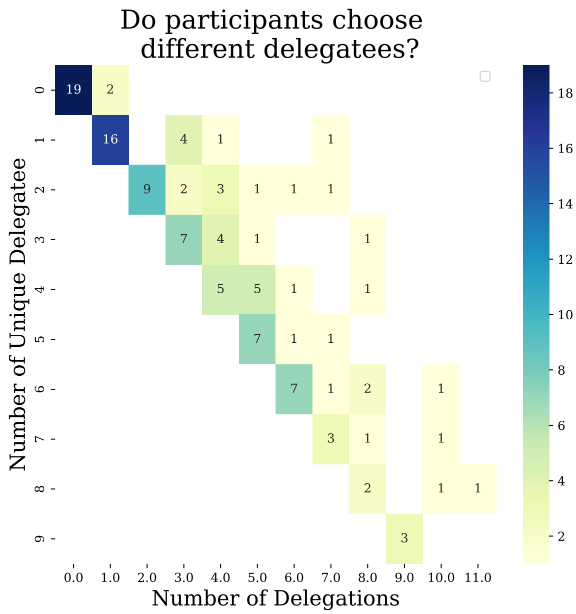


Figure 5.8: Heatmap of the choices of unique delegatees as a function of the number of delegations.

5.4.4 More on delegation behaviors

Last, we compare the number of delegations received in_i to that given out_i in Figure 5.9. We see that many of those who received many delegations tend to also delegate often. Further, performing a rank correlation coefficient test, we notice that those that received more delegations also tend to have a higher closeness centrality (Kendall Tau test gives $\tau = 0.55$, p-value $= 10^{-11}$, see Section 4.3.4.3). This indicates that we are unlikely to see many participants that receive many delegations and never delegates (in which case, no one would be reachable from these participants and the closeness centrality would be equal to

0). In turn, it seems that those who received delegations may act as connectors through the transitivty of delegations. If some participants gather more delegations due to their prominence in the community, it seems qualitatively that they are also somehow likely to delegate in turn to others community members. On the one hand, this may be a positive aspect of liquid democracy if the gurus identified through transitivity have some domain-knowledge regarding the issue at stake. On the other hand, this also raises questions in terms of the legitimacy of the gurus selected transitively.

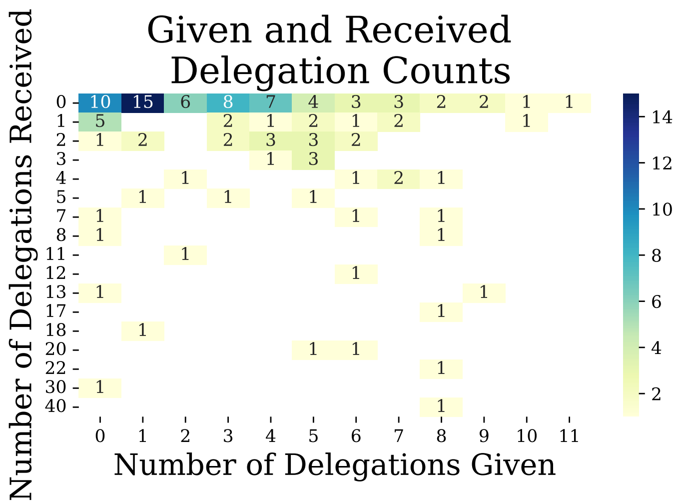


Figure 5.9: Heatmap of the choices of number of delegations received as a function of the number of delegations given.

5.5 Discussion

While many warned against liquid democracy leading to cycles and concentration of power, we find little signs that either are prevalent trends in these experiments. However, we observe that multiple members who received delegations did not participate, leading to an important practical hurtle: how should one, in real-world scenarios, handle delegations that are given to a guru that does not participate in decision-making?

Further, our experiments indicate that most participants prefer voting themselves than delegating their votes, and that delegation frequencies vary across questions. Liquid democracy was thought of as a per-issue voting scheme, where participants can delegate their power to issue-specific representatives and, interestingly, we observe that delegations tend to be given to distinct representatives across questions. Participant seem to appeal to the per-issue feature of liquid democracy.

While this chapter simply presents descriptive aspects of the experiment, it raises questions in terms of the behavioral and cognitive aspects of individual delegations (Why do participants delegate?), on the delegation trends for different community (How do received delegations travel transitively? How diverse and representative are the gurus in comparison to representatives selected to other processes?), and on the legitimacy of the selected assembly (How legitimate are representatives selected by a few that amass large weights through the transitivity of the delegations?).

Chapter 6

How to Open Democratic Representation to the Future?

Abstract

In recent years, various innovations aimed at counteracting perceived democratic decline and presentism have emerged. One primary concern is the issue of inadequate representation in parliaments, which has prompted the development of various proposals for reforming the selection mechanisms of parliamentarians. In this context, lottocracy (selection of representatives at random) and proxy democracy (selection models based on self-selection and flexible nominations that determine the relative influence of representatives) are candidates as selection rules to open democratic representation. Herein, I examine the normative and contextual trade-offs underpinning lottocracy and proxy democracy. While both systems outperform electoral alternatives on the dimensions under study, they induce tensions often overlooked. Nonetheless, clarifying the normative compromises is crucial to address the challenges facing democratic systems and inform the deployment of the future of representative democracy.

6.1 Introduction

Over the past decades, there have been trends of discontent with democracy and a perceived decline in trust in representative institutions. In 2022, Europeans had an average trust in national governments of 3.6 on a 10-point scale and, in the European Union, of 4.4 (against 4.7 and 4.6, respectively, in 2020) according to Ahrendt et al. [3]. Representative democracies are said to be afflicted by *presentism*, a blind spot for future-oriented policies and long-term risks [134, 155], blamed on short-term incentives of institutions with a tendency to misalign lawmaking with citizens' perspectives [122, 223]. The weakening of representative institutions' trust and performance further correlates with a rise of authoritarianism that threatens established representative democracies: as a second-order problem, these institutions' future itself is at risk. Reinforcing democratic institutions to better align with democratic values becomes a necessary investment in future generations' future. Optimistically, the concept of representative democracy remains popular — a survey found that a median of 78% of participants worldwide believe it to be a good way to govern [237]. In turn, proposals flourish to reform questioned representative institutions and increase their responsiveness. Some of these proposals involve enlarging the size of representative bodies, creating committees for the future within parliamentary chambers, and adopting different voting systems, such as ranked-choice voting, approval voting, or majority judgment (see respectively Allen et al. [4], Balinski and Laraki [15], Brams and Fishburn [34], Hernández [116], Koskimaa and Raunio [138].

6.1.1 Problem Statement

This chapter builds on the idea that engaging citizens in the political process could reduce blind-spots in risk management and limit the capture of decision-making processes through short-term incentives or special interests ("the public itself needs to be engaged [...] to ensure long-term public interests are protected" [p.129, 31]), and asks: how should ordinary citizens be engaged? In that vein, scholars have urged us to acknowledge representative democracies' oligarchic drifts and reconsider fundamental democratic principles underpinning current processes (see Van Reybrouck [231] and Guerrero [p.135, , 107]. Notably, Hélène Landemore coined open democracy, a paradigm founded on widespread participation in lawmaking, institutionalised deliberation, and accessible representation. [p.128-129, 145] She makes the "case for a new form of democratic representation in which elected officials are replaced with randomly selected ones", referred to as lottocracy [p.1061, 146]. Others echo that a lottocratic chamber could be tasked with [...] legislating for the long term." [p.337, 217]

This essay engages with selection mechanisms for representative democracy that attempt to broaden institutions' perspectives. While democracies historically tend to try out novel procedures that fit a particular normative ideal and evaluate other externalities after the fact, this essay benchmarks two selection procedures, lottocracy and proxy democracy, in an attempt to highlight the normative and contingent trade-offs. By understanding how different selection rules express democratic principles and respond to contexts, we shift from seeking an ultimate imperfect solution to debating how to prioritise competing objectives. In turn, citizens could make informed decisions about the values under which they institutionally live and shape the future of epistemically and procedurally responsible forms of representation that mitigate long-term risks and survive current turmoil.

6.1.2 Contributions

The contributions of this chapter are threefold. First, I consider the ecology of selection rules for representative assemblies (such as parliamentary chambers), introducing proxy democracy as a selection rule for representation in open democracies and comparing it to lottocracy. In proxy democracy, citizens can periodically choose to be in the legislature. Those who do not self-select flexibly nominate the self-selected citizen(s) they want to be represented by, and a legislator's vote is weighted by the number of nominations received. Second, it investigates how Landemore's accounts of democratic representation and legitimate representation are realised under lottocracy and proxy democracy, drawing on political and social choice theories to integrate these traditionally separate fields of study. While proxy democracy opens representative institutions reinforcing the current understanding of representative values, lottocracy cannot be fully justified in that context; this essay builds on recent political theory to characterise appropriate novel grasps on the concept of representation[57, 108, 145]. Third, it identifies a gap in the normative theory of lottocracy that raises a series of questions. Biased self-selection may impair lottocracy's promise to promote descriptive representation: should self-selection be handled by mandates, quotas, or ignored? In the first case, is there a moral duty to serve as a representative or a substantive argument that those in power should not seek it? In the second case, which fairness or equity standards should replace the equality principle? In the third case, why should equality be preferred over diversity?

In the following sections, I review the literature on democratic representation and selection models. I examine open representation through the lens of lottocracy and proxy democracy and compare these selection mechanisms' normative foundations.

6.1.3 Related Work

Political representation (through which certain individuals stand in for a group to perform specific functions on behalf of that group, [200]) has been the subject of much controversy. While some argued that democratic representation was a "defective substitute for direct democracy" in which constituents abdicated self-government, others believed that representation allowed the many to select the competent few through periodic elections[p.515, 164].[1] These theories of representation have primarily focused on electoral democracies characterised by exclusive competitions for limited seats. Once acclaimed as an ultimate form of democratic representation, elections are increasingly perceived as founded on elitist principles, decried for their oligarchic drift and remoteness, and criticised for experiencing high distrust and failing to focus on long-term risks.[2].

Looking back at John Stuart Mill's work, representation should provide all an equal opportunity to "take an actual part in the government by the personal discharge of some public function."[p.39, 171] For Urbinati and Warren, it also plays a crucial role in "unif[ying] and connect[ing] the plural forms of association within civil society, in part by projecting the horizons of citizens beyond their immediate attachments, and in part by provoking citizens to reflect on future perspectives and conflicts in the process of devising national politics."[p.391, 227] Representation further induces a relationship between the represented and the representatives, the nature of which has been extensively debated. The traditional view opposed the concept of delegate (mandated to fulfil the constituents' will) to that of trustee (trusted to exercise independent judgment), relying on the idea that constituents track and sanction the representatives' performance after the fact. However, Jane Mansbridge argued that sanction was peripheral to representation, proposing a selection model in which citizens screen candidates before they take office to choose self-motivated honest representatives with aligned preferences.[p.621, 166]

If political representation can be intrinsically democratic, instrumentally beneficial and understandable through a selection rationale, it is realised through selection processes with normative and empirical implications.[3] Investigating novel schemes, scholars have argued

[1] See also historical work by Rosanvallon [207].

[2] See the Triumph of Elections in Manin [chapt. 2, 161] On the withering of electoral democracies, see respectively Ahrendt et al. [3], Lessig [148], Thompson [223], Urbinati and Warren [227], Wike et al. [237]

[3] On normative considerations, see Landa and Pevnick [142]. See Lijphart [150] for empirical ones.

for increasing "degrees of openness of the sites of power to ordinary citizens" to promote a more accurate representation of the people and their interests.[p.134, 145] To reason around the justification of delegating power in open representative democracies, Landemore distinguishes between democratic representatives ("who [have] accessed the position of representative through a selection process characterized by inclusiveness and equality"), legitimate representatives ("who [have] been properly authorized to act as a representative") and good representatives ("that serves well the interests of the represented")[p.87, 145]. Further, she notes that "if the democratic principles of inclusiveness and equality are perfectly realised, then we should see a representative body that is statistically identical with the demos,"[p.89, 145] deriving descriptive representation from democratic representation. Descriptive representation (the idea that a legislature "should be an exact portrait, in miniature, of the people at large")[2] has a long instrumental history of enhancing democratic processes[p.628, 162].[4] It "speaks to the level at which those occupying positions of power reflect the population they represent" and aims to reflect the diversity of the constituents' experiences and perspectives, enhance long-term views and bring political power closer to the people.

Scholars have argued that contingent political risks associated with electoral designs, contextual political capture by special interests, and the complexity of the issues at stake were standing in the way of representative institutions delivering substantive outcomes to improve present and future citizens' life.[122] In contrast, cognitive diversity, inherent to descriptive assemblies, was reported to enhance the epistemic performance of a crowd when facing complex problems over deliberations [144] as "the range of arguments considered will be broader."[p.353, 217] Next, ordinary citizens in the right institutional design are said to be "more likely to feel accountable to future generations [than] to [...] electors (and in some cases to the donors who finance the elections)".[5]

[4]For a conceptual assessment of different views on representation, see Pitkin [188].

[5]Sintomer [p.353, 217] specifically writes this in the context of sortition chambers.

This essay builds on the assumption that engaging ordinary citizens will foster substantive outcomes (good representation) and engages with the procedural justification to (i) understand how lottocracy and proxy democracy respond to Landemore's interplay between democratic and descriptive representations and (ii) clarify how both selection rules understand legitimate representation.

6.1.3.1 Lottocracy

A lottocratic assembly is composed of congress members drawn randomly to participate in the political office that rotates over fixed periods, typically informed by appointed panels of experts. Lottocracy was famously used in Ancient Greece, re-introduced by Robert Dahl as mini-populous and suggested as a complementary form of representation.[6] They typically come with a side informational process through which the randomly selected citizens gather knowledge about the issues at stake. Recently, proposals to replace congresses with random chambers flourished and are contested.[7] Lottocracy is often defended for treating all more equally and being more inclusive, representative, and impartial than its electoral counterpart. Lottocratic assemblies have been composed worldwide to work on topics such as climate change, constitution drafting, same-sex marriage, etc.[p.257, 108]

6.1.3.2 Proxy Democracy

Proxy democracy is an alternative model in which citizens either self-select to be representatives or flexibly nominate self-selected citizen(s) through frequent nomination processes.[8]

[6] See respectively Barker [17], Dahl [63], Fung [91].

[7] On proposals, see for instance Callenbach et al. [40], Gastil and Wright [95], O'Leary [183]. On concerns, see Landa and Pevnick [143], Umbers [225].

[8] Proxy democracy generalises proxy voting [p.107, 174] and liquid democracy. Liquid democracy is an (i) area-specific (ii) transitive proxy voting with (iii) instant recall that has been used sporadically around the world (see Valsangiacomo [230]). I only focus on the potential of fractional transitive proxy voting as an alternative mechanism for parliamentary selection, all other things being equal. In particular, I do not consider instant recall in proxy democracy for its instability but rely on a rotative system such that nominations are held periodically. For an investigation of these concepts as representative processes, see

In turn, representatives have a weight equal to the number of citizens they represent, which scales their votes in congress. Nominations are fractional to allow the expression of plural preferences and choosing different representatives for different issues.[9] Each citizen would nominate a set of representatives, specifying the capacity in which each representative is chosen (a specialist or a generalist). While all representatives would participate in all votes, dedicated democratically selected per-issue committees would drive in-depth deliberations before voting.[10] Variations of this system have been used in political and corporate settings, but proxy democracy remains a fresh proposal with far fewer test cases than lottocratic alternatives.[p.71, 230]

Lottocracy and proxy democracy are committed to opening the set of potential representatives to virtually everyone , adding "to the mix of a new set of representatives, different from those we elect."[p.352, 217] While lottocracy works with pre-defined size and no direct intervention of the represented, proxy democracy theoretically admits unbounded parliament sizes and is realised through flexible nominations of those represented. Further, numerous lottocratic proposals suggest relying upon single-issue bodies connected through supra-chambers and trained independently. In proxy democracy, such single-issue deliberative pools are endogenously constituted and included in the broader institution to handle trans-issue consistency.

Valsangiacomo [229].

[9] The specific mechanics of fractional voting may vary, and quadratic voting may be better suited to prevent strategic behaviour while still allowing expressive nominations, as in Weyl [234].

[10] The approval-based multi-winner literature proposes ways to ensure a proportional representation of perspectives, see, Aziz et al. [e.g., 14]

6.2 Democratic, Descriptive and Legitimate Representation in Lottocracy and Proxy Democracy

In the remaining essay, I shall compare the two forms of selection models on similar grounds. First, I focus on Landemore's account of democratic representation and its interplay with descriptive representation.[p.81, 145] Second, I discuss how legitimate representation is mechanically derived in both models by those not included in the parliament.

6.2.1 On democratic and descriptive representation

Landemore suggests evaluating, in the non-electoral world "the democratic character of a representative assembly [. . .] in terms of the degree to which access to that assembly [. . .] is inclusive and equal (or fair)", in what is reminiscent of Robert A. Dahl's criteria for adequate participation and equality in the decisive stage in electoral democracy (see Landemore [p.81-82, 145] and Dahl [p.109, 61]). She further asserts that perfectly democratic representation leads to statistically descriptive representation.[144] Let us observe how the equation coupling inclusive participation and equality to descriptive representation plays out in lottocracy and proxy democracy.

6.2.1.1 Inclusive participation

In electoral democracy, all overaged citizens participate mainly through voting rights that vest peripheral and indirect access to power through episodical polls. In contrast, democratic representation requires that all could virtually participate substantively in policymaking through low entrance barriers and the assurance that they could reasonably have been included for any given term. Along these lines, lottocracy and proxy democracy virtually allow any citizen to become representative, effectively removing entrance barriers to congress.[11]

Next, lottocratic active participation is over one's lifetime (I may be selected to participate in policy-making throughout my life) and happens intermittently (when I am selected). In contrast, citizens in proxy democracy are actively included continuously in policymaking through their ability to self-select or nominate representatives per issue.[12] Unlike in lottocracy, citizens need not be willing to sit in congress; they can freely self-select (allowing direct inclusion) or exert their political power over influencing the legislature's composition (allowing indirect inclusion). It is unclear how many people would self-select to sit in the representative body in proxy democracy. While small congresses would include all continuously through direct and indirect participation, mechanisms would be necessary to mitigate massive assemblies — and filtering representatives could create new exclusion patterns.

Landemore notes that "if the number of seats and the frequency of rotation are insufficient for everyone to plausible expect to rule someday, then the comparative democratic advantage of lotteries over elections becomes quite thin."[p.91, 145] Unfortunately, the chance of being included in one's lifetime in modern examples is extremely small. In the Belgium case used by Landemore, 29 seats are filled randomly from a pool of $76,000$ citizens, and the rotation occurs every year and a half. The probability of being selected in a lifetime is less than 2.3%.[13] For a population of ten million and a congress of ten thousand members chosen

[11]Neither lotocracy without mandates nor proxy democracy proposes a framework to include those who do not engage in the political processes, or biased self-selection patterns based on, e.g., gender, which are, however, other crucial issues for open democracy.

[12]Unlike electoral democracy where entrance barriers to participating directly are high, proxy democracy allows every citizen to choose whether they want to participate directly (self-selecting) or indirectly (nominating) in policymaking.

[13]The probability of a citizen being chosen at least once in a lottocratic assembly is one minus the probability of never being chosen. Assuming that only four over five citizens are old enough to be selected and that the events of being selected for each term are independent, the probability of never being chosen is $(1 - 29/(0.8 * 76000))^m$, where m is the number of times one can be selected. We generously assume that a citizen can be chosen once every one year and a half over 70 years so that $m = 70/1.5$. This probability remains comparable if we take that a citizen may be selected only once in their lifetime and further shrinks if we include population dynamics. Landemore [p.91, 145] reports a probability of being chosen in one's lifetime of 67% but, to the best of my understanding, the assembly would need to be changed every ten days to reach this probability. Other sources [45] indicate that up to 174 citizens can be sorted through a combination of

yearly, the probability of any individual being selected once in a lifetime would not reach 7%. Guerrero finds that, even in a fully lottocratic American society where every political office at the local, state and national level is held randomly, the probability of being selected in one's lifetime in any of those is about 4% according to Guerrero [p.246, 108].[14] As such, only a small number of citizens would have the opportunity to participate in the policymaking process, even with frequent rotations and large parliaments in large states. This does not mean, however, that lottocracy is not inclusive. It does not favour active inclusion in the process but exemplifies passive inclusion of a broad range of perspectives: most individuals would have a high chance for their perspective and experience to matter at some point. Passive inclusiveness is not guaranteed in a majoritarian electoral framework, where some perspectives may never make it to a representative seat. It is also likely to be more prevalent in lottocracy than in proxy democracy (where all perspectives can be included, but some, being more weighted than others, could control voting outcomes).

In summary, lottocracy and proxy democracy virtually remove entrance barriers to the site of power. They differ in that active inclusiveness is intermittent through direct participation in the former and continuous through direct and indirect involvement in the latter. Yet, passive inclusion in lottocracy also allows citizens' perspectives to be represented and heard. Alternatively, self-selection and nominations in proxy democracy connect all to the site of power and allow citizens' multi-faceted interests to be represented, but self-selection and vote weighting may lead to some views struggling to be represented ans large congresses that would necessitate limiting mechanisms.

a permanent assembly with 24 members sorted every 18 months and three potential assemblies with 25 to 50 citizens called at most three times a year. Then, the probability is upper bounded by 18% in the most generous scenario.

[14]Selecting at random all elected officials would still induce imbalance in the stakes each individual has a chance to participate in.

6.2.1.2 Equal access, fair access or statistical representation

Another critical aspect of democratic representation is equal opportunity to share claims in exercising political power so that "the possibilities for political participation [are] equally distributed"[p.75, 103] among the citizenry.[47, 62] In electoral democracies, guardrails bias who can run for office, undermining political equality and preventing parliaments from being descriptively representative. Open democrats strive for equal access to substantive power among citizens, but this is not a sufficient condition to obtain diverse assemblies. Citizens' ability to choose whether to become representatives can prevent inclusiveness and equality from resulting in diversity: those who self-select may not be statistically representatives of all.[15]

In an idealised lottocracy, a parliament of size k in a citizenry of size n is constituted by randomly sampling citizens with probability k/n. In turn, all citizens have the same chance to sit in congress, control the agenda, deliberate, and vote. Also, groups constituting the citizenry have a proportional chance of being represented. This idealised view condones the citizens' right (given in current lottocratic implementations) to refuse the invitation to sit in parliament. Because active inclusion in lottocracy is understood as taking part directly in the policymaking process, it imposes a high participation cost that only some may tolerate. In sorted assemblies with low commitment, few citizens opt-in to serve in the short-lived sorted groups: "typically, only between 2 and 5% of citizens are willing to participate in the panel when contacted."Flanigan et al. [83]

Those who self-select "exhibit self-selection bias, i.e., they are not representative of the population, but rather skew toward certain groups with certain features,"[83] hurting a priori the chances for each group to be proportionally represented. Some argue one should simply limit the causes of abstention; others insist on limiting its effects, de-biasing it to

[15] This was also observed in Scheufele [212] and Landa and Pevnick [143].

"ensure that the assembly's eventual membership [is] representative of the population."[50] In turn, external checks such as quotas may enforce that the sorted assembly includes a certain number of people with specific characteristics. Practitioners prescribe first sampling a large pool of people and then using quotas to stratify the final assembly of size k, de-biasing those who accepted to participate in the larger pool through algorithmic procedures.[p.548, 85] Such stratified sampling is deemed necessary to "increase [sorted assemblies'] representativeness."[p.340, 217] While quotas may fail to account for "constituents' many-sided and cross-cutting interest" and be essentializing,"[p.30, 163] such "representative arrangements" are deemed valid "in the context of historical patterns of domination and subordination." They protect an ex-ante understanding of diversity (defined a priori) and could constitute modern guardrails to support democratic ideal. The explicit design of the quotas shall require meticulous attention to avoid being politicised (for instance, minimum thresholds over bi-partisan categories could promote mild guardrails).

Further, while equal chance to access power is unattainable in such scenarios, computer scientists have developed algorithms that enforce pre-defined quotas while treating participants fairly.[74] Some maximise the lowest probability of being selected; others sort the larger assembly with different probabilities that depend on citizens' attributes (such as age, gender, and education) to account for different likelihoods of opting in.[83, 85] Voters are not treated equally, as one's chance to be selected depends on the self-section pattern of the rest of the group, but these elegant approaches achieve procedures that guarantee descriptive representation while promoting fair access to power.

In all, attention must be devoted to the implications of self-selection in lottocracy in different contexts. Should participating be mandatory — if so, on which grounds? Should random sampling be procedurally sufficient to suffer the cost of potentially skewed representation? Or, could stratified sampling be the best option available to guarantee equitable representation — if so, what should be the fairness principles used instead of the equality principle, and how should such guardrails be normatively, empirically, and politically justified?

In contrast, proxy democracy intends to enforce equal opportunity to become a representative among those who self-select. Each citizen may deal with their voting power equally through nomination, provided that opening the set of representatives to virtually everyone will supply diverse choices. Valsangiacomo notes that, unlike in electoral democracy, proxies compete for political and legislative influence and not for seats, arguing this fundamental shift will "reduce the risk of strategic voting on the part of the voters, as well as the risk of anticipatory strategies on the part of the parties."[p.7, 229] Further, marginalised voices that struggle to gather the support needed to be heard in electoral setups would be included through self-selection in parliament, automatically taking a seat in deliberation phases.

In contrast, proxy democracy does not induce equality of influence in the decisions taken in parliament, as some representatives will carry more nominations than others. Closely related to that point, proxy democracy does not enforce a preconceived notion of diversity. Self-selection and flexible nominations are intended to couple equality of opportunities to become a representative with a diverse representation of interests. Diversity is understood as ex-post, resulting from popular nominations.[16] Philosophers have argued that there were reasons to believe that coupling self-selection with flexible nominations would lead to descriptive and "strongly" diverse parliaments.[26, 230] However, they do not provide ex-ante safeguards against popular nominations. In particular, proxy democracy may drift to nominations captured by coalition builders, charismatic leaders or special interests, as in electoral democracies, that could control enough voting weight to influence legislative outcomes.[17] While counter-popular guardrails could be deployed to prevent these cases, a context prone

[16] Proxy democracy is not per se incompatible with ex-ante diversity — external checks could randomly sample given features from a self-selected group.

[17] In the context of liquid democracy, an experiment documented extreme concentration of power, see Becker [19] and Guerrero [p.106, 108].

to political capture may find its way to game the system and reduce equality to access effective power.[102, 129]

In sum, self-selection creates tension between equality and diversity. Lottocracy may solve this tension by enforcing an ex-ante account of diversity through fair stratified sampling, arguing for a mandatory civic duty to serve when selected, or trading diversity for equal chances of being sorted. On the contrary, proxy democracy lets citizens who did not self-select balance out biases induced by self-selection. Proxy democrats admit an ex-post account of diversity revealed through the nominations and, in turn, not protected. Simultaneously, proxy voting does not guarantee equality in the representatives' chances to influence outcomes, and this may be explored to capture political processes. External checks to guardrail endogenous behaviours might be necessary (to promote diversity against self-selection or prevent concentration of power against nominations), and they pose a crucial challenge to open democracy, similar to that faced by electoral democrats two and a half centuries ago: when and why are guardrails (such as mandates, quotas, nominations cap) justifiable?

6.2.2 On legitimate representation

Those not directly included in parliaments need to authorise the representatives, consenting to their binding power. Authorisation constitutes a necessary condition for democratic legitimacy that pretends to accommodate individuals with an irreconcilable plurality of opinions to comply with a non-consensual decision and grounds what Landemore calls legitimate representation. Representatives in electoral democracy are authorised because they are chosen by a sufficiently large portion of the population and held accountable through period elections. However, those who vote for the election's winner authorise with greater intensity than those who do not, creating unbalanced authorisation theories in electoral democracies.

Authorisation in open democratic selection rules is deeply rooted in a procedural argument according to which citizens are all included and treated equally (to the extent possible), leading to an assembly whose diversity has instrumental credentials. Accordingly, citizens of open democracy are expected to authorise their parliament for its intrinsic and instrumental credentials. Intrinsically, citizens oscillate between "ruling and being ruled" by the sheer inclusiveness of the parliament to all.[p.117, 17] Instrumentally, the representative body is constituted by cognitive diversity that shall lead to better outcomes, either more sensitive to the plurality of opinion or epistemically dominant.[145]

The lottocratic assembly is selected by a voter-free process, primarily authorised via the procedural argument outlined above without a principal-agent relationship. Citizens do not exercise power when selecting a representative but consent to the power of a justly composed body and authorise it as a whole because it tends towards a statistical truth. Random assemblies are accompanied by knowledge-gathering and deliberative processes that may enhance lawmaking's outcome. This instrumental justification is sensitive to the assemblies' cognitive diversity.[145] Hence, mandatory participation or quotas may be necessary preconditions to outcome-oriented authorisation in lottocracy.

Beyond authorising a fair procedure, citizens in proxy democracy endorse "self-motivated agent[s] who can pursue their interests flexibly, adaptively and with internal commitment"[p.623, 162] and be political leaders during their term.[18] Unlike in electoral democracies, citizens who choose to be nominators have access to diverse and per-issue alternatives. Their weight in the decision is further effectively carried by their representative(s) so that authorisation is personalised through the indirect nature of inclusiveness. Further, proxy democracy purposes to rely on the concept of collective intelligence applied to the selection of topically competent peers to enhance lawmaking's outcome.[19] Proxies are expected to be authorised

[18] See Beerbohm [20] for a discussion about the compatibility between democracy and leadership.

[19] Our previous chapets found theoretically and empirically that, in well-connected and apolitical set-ups and in the context of liquid democracy, transitive nominations were reaching per-issue competent represen-

because they are perceived as "an especially competent set of individuals" selected through a democratic process.[142] This normative argument, however, shall carefully be confronted with the context in which proxy democracy may be deployed to look for external forces that may distort the nomination processes.

In lottocracy, those not directly included authorise a group that approaches a statistical truth. Lottocracy induces a discontinuity in the traditional theory of consent in representative democracy, basing authorisation on a voter-free procedural argument. Further, while non-random democratic processes may be theoretically proposed to achieve optimal epistemic performance[142], deliberative lottocracy is said to guarantee epistemically responsible assemblies better suited to resist risks of political capture. In contrast, proxy democracy reinforces how consent is understood in electoral theories: individual authorisation results from a free choice to nominate and is translated into a citizen's weight effectively represented in parliament. Relying on voters' collective intelligence, proxy democracy may fairly bring forth competent lawmakers, but nomination processes may be captured and risk being biased.

6.3 Discussion

Representation in democracy is due for an upgrade. The exact shape this update may take has yet to be made clear. In an open democracy, lawmakers could be selected through random draws of citizens or self-selected representatives weighted by popular votes. Normatively, both proposals promise to lead to more inclusive, egalitarian, and diverse representative bodies than current electoral systems. However, they lead to different readings of these principles. Proxy democracy lets citizens choose whether to directly or indirectly participate in the political craft and strengthens individualised authorisation of theoretically competent

tatives, see Chapters 3 and 4.

representatives. It reveals an ex-post diversity through endogenous nominations but may suffer from powerful forces capturing unbalanced influence. In contrast, lottocracy promotes rare active inclusion but broad passive inclusion of voters' various perspectives. Different handlings of biased self-selection appeal to mandatory participation or quotas safeguarding ex-ante diversity. Lottocratic authorisation relies on a radically new account of voter-free procedures. Since there may not exist an ultimate form of representation, choices about the future of democracies shall be driven by the principles we ought to prioritise and the contexts from which we start. This essay hopefully clarifies how lottocracy and proxy democracy respond to values and contingency.

Practically, handling large parliaments may become necessary to foster inclusive representation in both lottocracy and proxy democracy.[20] Operating with large congresses sounds preposterous — representation was invented to accommodate large population sizes and prevent chaotic debates. However, if deliberation is at its best in small assemblies, meaningful inclusiveness mechanically requires larger ones. To allow more citizens to be included in open democratic representation, specific attention shall be dedicated to rethinking lawmaking protocols so that they accommodate large groups and are compatible with non-political commitments citizens may have while serving as representatives. For instance, congresses could work by decoupling deliberation phases from voting phases in parliaments. Small, punctual, per-issue specialised committees geographically distributed would gather information, hear experts, deliberate, and draft laws before all representatives would cast a vote at the time of the decision.

Alternatively, to limit congress size while achieving a flexible understanding of inclusiveness and diversity and minimising the influence of charm in the nomination process, one could consider using mixed selection rules that incorporate elements of both lottocracy

[20] American founding fathers advocated for maintaining one representative for every thirty-six thousand citizens [112]. Similarly, proposals are made to enlarge the U.S. Congress, see Allen et al. [4].

and proxy democracy.[p.348, 217] For instance, citizens could be asked to announce their availability to serve in the parliament's next term or nominate a fellow citizen to represent them. Representatives would then be drawn randomly among self-selected candidates, with a probability of being chosen based on the number of nominations received. This approach intends to prevent excessive self-selection biases or incentives for charismatic capture while promoting authorisation through a procedure in which all citizens participate.

This chapter explores alternatives to mitigate first and second-order long-term risks associated with representative democracy from democratic, legitimate and descriptive perspectives. Further normative questions (regarding the rules' symbolic and substantive implications) and practical issues (about representatives' compensations, the trustworthiness of digital platforms used for sorting citizens or counting representative weights, etc.), let for future research, still stand in the way of a panoramic view of representative democracy's future.

Chapter 7

An Axiomatic View for Representative Democracy

Research is formalized curiosity. It is poking and prying with a purpose.

Zora Neale Hurston

Abstract

As the world's democratic institutions are challenged by dissatisfied citizens, political scientists and also computer scientists have proposed and analyzed various (innovative) methods to select representative bodies, a crucial task in every democracy. However, a unified framework to analyze and compare different selection mechanisms is missing, resulting in very few comparative works. To address this gap, we advocate employing concepts and tools from computational social choice in order to devise a model in which different selection mechanisms can be formalized. Such a model would allow for desirable representation axioms to be conceptualized and evaluated. We make the first step in this direction by proposing a unifying mathematical formulation of different selection mechanisms as well as various

social-choice-inspired axioms such as proportionality and monotonicity.

7.1 Introduction

It is often argued that representative democracy is in crisis (e.g., see Chapter 2 in the book by Landemore [145] and the references therein). In particular, the justification of *representative bodies* is called into question whenever they make decisions that appear to go against the interests of those they are supposed to represent. In line with this, a survey by the Pew Research Center [235] finds that, while there remains broad global support for representative democracy, there is also a strong sense that existing political systems need reform.

7.1.1 Problem Statement

In this chapter, we focus on the task of selecting a representative body, which is a crucial ingredient of all democratic institutions as argued by political scientists [142, 150, 165, 170, 172, 197]. There is no shortage of innovative proposals to change how representative bodies are selected around the world. For example, some propose to select representatives at random (a.k.a. *sortition*) [32], to elect them through transitive delegations (a.k.a. *liquid democracy*) [229], or to drastically increase the size of parliaments (see, e.g., https://thirty-thousand.org). Each proposed method has its benefits and drawbacks; however, we lack a systematic way to evaluate and compare them. Specifically, while there are numerous works in computer science and political science analyzing the strengths and weaknesses of specific methods, principled comparisons are rare.[1]

[1] The few existing exceptions mostly focus on epistemic aspects, the robustness of representation, and majority agreement [1, 6, 97, 104].

7.1.2 Contributions

We call for the development of a unified framework to formulate and compare innovative and traditional selection mechanisms on a more principled basis. Having formulated different mechanisms within the same framework then offers the possibility to formulate different desiderata in the same framework. While we believe that comparisons from different perspectives are possible and, in fact, urgently needed, we put forward an *axiomatic* view on selection mechanisms, drawing inspiration from the rich social choice literature on voting rules. In the main part of the chapter, we give a concrete example of how a systematic comparison from an axiomatic perspective of selection mechanisms could look like. Firstly, we present a simple yet rich mathematical framework to formulate different selection mechanisms. Secondly, we define various axioms capturing notions of *cogent representation*. We hope that these axioms can be used to quantitatively investigate inherent and poorly understood trade-offs at the heart of democratic innovations. Notably, our research program's focus is not on finding the "ideal" representation system. We rather envision building a navigator that maps selection mechanisms to axioms. We advocate for building a coherent picture of the advantages and disadvantages of competing selection proposals to gear public debates towards *what kind of trade-offs societies are facing*, instead of continuing to argue for competing selection mechanisms on disconnected grounds.

7.1.3 Related Work

In recent years, computer science and democratic innovations have become increasingly intertwined, with computer scientists tackling many algorithmic design and scalability problems arising in different representation schemes and analyzing such schemes axiomatically. In fact, only with advancements in information technology, the idea of more complex and interactive voting models is becoming more commonplace [36]. Miller [174] and Tullock [224], for instance, argued that richer political decision-making processes on a nationwide scale have recently become possible thanks to technology. Our envisioned research program again relies on the expertise of computer scientists. More specifically, many areas of the AAMAS community could contribute to our endeavour to mathematically formulate and analyze selection mechanisms and desiderata. For instance, (i) the *Social Choice and Cooperative Game Theory* community has expertise in the axiomatic analysis of voting rules, (ii) the *Coordination, Organisations, Institutions, and Norms* community can contribute a normative perspective, and (iii) the *Humans and AI / Human-Agent Interaction* area could help in the analysis of usability aspects.

In turn, making progress on representative selection mechanisms has also direct benefits for their applications in computer systems. For example, some blockchains select validators via a *nominated proof-of-stake* protocol, and the representativeness of the selection is essential for the security of the system [44]. Further afield, blockchain-enabled Decentralized Autonomous Organizations (DAOs) are at the forefront of testing innovative governance systems based on interactive procedures [7, 133, 242]. Good (e-)governance remains a vast open question [115].

7.2 Mathematical Framework

In this section, we outline a mathematical framework to model selection mechanisms. First, we define some preliminary notation. A matrix is stochastic if each row sums up to 1. For a natural number $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $e_n \in \mathbb{N}^{1\times n}$ denote the row vector containing all ones. For a vector $a \in \mathbb{R}^{1\times n}$, let $\|a\|_1$ denote the ℓ_1-norm of a, i.e., $\|a\|_1 = \sum_{i=1}^{n} |a_i|$.

7.2.1 Modeling Representation

We present a mathematical framework for the following task: A group $\mathbb{N} = [n]$ of n agents wants to select a subset of $\mathbb{N}$ to act as a representative body through a selection mechanism M. We additionally assume that the agents selected to be part of the representative body can have different voting weights, i.e., in a decision made by the representative body, some agents' votes have more weight than others. Formally, given $\mathbb{N}$, we want to select a *weight vector* $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$. For each $i \in \mathbb{N}$, if $\mathbf{w}_i > 0$, then i is selected as part of the representative body and has voting weight $\mathbf{w}_i$. The size of the induced representative body is given by $|\{i \in \mathbb{N} \mid \mathbf{w}_i > 0\}|$.

Representation Matrix. The relation of agents is captured by a *representation matrix* $\Gamma \in \mathbb{R}^{n \times n}$, where the entry Γ_{ij} describes how well agent j can represent agent i. Γ is a stochastic matrix that allows fractional entries to account for the fact that agent i may be best represented by a mixture of other agents. How well i feels represented by j may be based on complex interactions of multiple aspects such that the issues i cares about, the relative preferences of j and i on these and other issues, intrinsic characteristics of i and j, and the underlying social network capturing who knows who.[2] Hence, the matrix Γ captures the complex nature of potential representation between agents in a simple yet rich form.

Example 1. *Consider the following example:* $\mathbb{N} = \{A, B, C, D, E\}$ *such that* A *and* B *belong to some party, and* C, D *and* E *to another party. Imagine that* A *and* E *are extreme candidates seeking power in their respective parties. Moreover,* B *and* D *are completely partisan and would never want to be represented by someone outside their parties. In contrast,* C *is moderate in their beliefs and could be represented by other candidates with non-extreme*

[2] In line with this reasoning, political theorists have argued that quality of representation is multidimensional and depends on different factors such as similarities between the representative and the constituents (*descriptive representation* [162]), alignment of interests and values (*gyroscopic representation* [165]), or advancement of constituents' interests by the representative (*substantive representation* [188]).

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2/3 & 1/3 & 0 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 2/5 & 1/5 & 2/5 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Figure 7.1: Representation matrix Γ for the instance described in Example 1. Rows and columns are indexed with agents.

views. This situation could be represented by the representation matrix Γ given in Figure 7.1.

Interpreting the Representation Matrix. The representation matrix can be interpreted as giving rise to voting behavior. Specifically, assuming that each agent can split their vote in an arbitrary way, the matrix entries can be thought of as the ideal split of an agent's vote into fractional votes. In this chapter, we will focus on uninominal ballots, i.e., each agent can vote for exactly one other agent to be part of the representative body. Accordingly, we interpret the entry Γ_{ij} as the probability that agent i selects (i.e., votes for) agent j.[3]

Expected Vote Share. Using the probabilistic interpretation of Γ for uninominal ballots allows us to reason about the expected share of votes an agent receives. For this, let V_j be the random variable representing the vote share agent j receives under the representation matrix Γ. Let V be the vector of the n random variables $V_1, \dots, V_n$. Then, the expected vote share $\mathbb{E}[V_j]$ of agent j is the sum of the j-th column of Γ, i.e., $\mathbb{E}[V_j] = \sum_{i=1}^{n} \Gamma_{ij}$ and $\mathbb{E}[V] = \Gamma^T e_n$. In an idealistic setting, we would select all agents as members of the representative body and give each agent j a voting weight of $\mathbb{E}[V_j]$, i.e., $\mathbf{w}_j = \mathbb{E}[V_j]$ for all $j \in \mathbb{N}$. Accordingly, to evaluate the quality of different selection mechanisms, we will compare the (ideal) expected

[3]The translation of Γ to votes can be extended to other ballot formats such as approval ballots or ranked ballots.

vote share $\mathbb{E}[V]$ to the voting weights of agents returned by the mechanism, assuming that agents vote as described by Γ. For the representation matrix given in Figure 7.1, the vector of expected vote shares of the agents is $\mathbb{E}[V] = \left(\frac{5}{3}, \frac{2}{3}, \frac{16}{15}, \frac{1}{5}, \frac{7}{5}\right)$.

Properties of the Representation Matrix. We envision that the algebraic properties of a given representation matrix Γ could model salient societal characteristics of $\mathbb{N}$ relevant to an axiomatic analysis: polarized groups would be characterized by a block matrix Γ, the relative magnitude of Γ's trace would quantify the amount of power-seeking agents in the group, the rank of Γ would model how correlated agents are to each other, etc. In line with axiomatic results from social choice theory [9, 93], we expect to find impossibility theorems that can be circumvented by restricting the structure of Γ.

7.2.2 Selection Mechansisms

We want to analyze selection mechanisms M that, given the uninomial ballots from the agents, return a weight vector. As an additional part of the input, our mechanisms may take a pre-specified subset $\mathcal{C} \subseteq \mathbb{N}$ of m agents acting as *candidates* and an integer k describing the number of agents that can be selected to be part of the representative body. For a mechanism M and body size k, we define a function f^{M_k} that given Γ and $\mathcal{C}$ returns the candidates' expected voting weights under Γ and $\mathcal{C}$.[4] Formally, f^{M_k} is a function

$$f^{M_k} : \mathbb{R}^{n \times n} \times (2^{\mathbb{N}} \setminus \emptyset) \to \mathbb{R}^{n \times 1}$$

such that $\{i \in \mathbb{N} \mid f^{M_k}(\Gamma, \mathcal{C})_i > 0\}$ is a subset of C of size at most k. Here, $f^{M_k}(\Gamma, \mathcal{C})_i$ is the expected voting weight ($\mathbb{E}[\mathbf{w}_i]$) of candidate $i \in \mathcal{C}$ in a body of size k selected by mechanism M assuming that $i \in \mathbb{N}$ votes for $j \in \mathcal{C}$ with probability depending on Γ_{ij}.

[4] Some mechanisms need neither a candidate set $\mathcal{C}$ nor the size of the representative body k as input (in this case we drop k from f^{M_k}).

We now describe how the expected voting weights for different selection mechanisms can be computed, using the representation matrix given in Figure 7.1 as a running example.

7.2.2.1 Direct Democracy (D)

In direct democracy assemblies, all agents are elected in the represented body. Thus, $\mathcal{C} = \mathbb{N}$ and $f^D(\Gamma, \mathbb{N}) = e_n$ for all representation matrices Γ.

7.2.2.2 First-Past-The-Post (F)

First-past-the-post voting is widely used around the world but also widely criticized for, among other things, leaving voters feeling underrepresented [27]. In first-past-the-post, a voting weight of 1 is given to the candidate receiving the highest number of votes and 0 to all other candidates. Notably, in first-past-the-post elections the electorate is typically partitioned into different voting districts, each selecting its own representative. We focus on the single-district case; however, our model can be extended to parallel independent districts.

Continuing Example 1, let $\mathcal{C} = \{A, B, C, E\}$. Note that the function f^{F_1} alters the representation matrix to account for the set of candidates: agents can only vote for candidates, and we assume that candidates always select themselves. In the running example, the representation matrix projected on the set of candidates becomes

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

With probability 0.5, C receives 2 votes and A, B and E receive 1 vote (resulting in C having a voting weight of 1 in the representative body and all other agents having a voting weight of zero), and with probability 0.5, E receives 2 votes and A, B and C receive 1 vote. Consequently, we get $f^{F_1}(\Gamma, \mathcal{C}) = (0, 0, 1/2, 0, 1/2)^T$.

7.2.2.3 Proxy Voting (P)

In proxy voting, all agents are presented with a pre-defined pool of candidates, and each agent can delegate their voting power to one of the candidates. All candidates are *de facto* part of the representative body, and candidates have a voting power proportional to the number of votes delegated to them.[5] Proxy voting has been studied both within computer science [7, 51] and political sciences [174], with some works extending it to more flexible issue-based delegations [1].

The expected voting weight under proxy voting is the sum of expected delegations for the proxies (as in First-Past-The-Post, we adapt the representation matrix to account for the candidate set). Assuming again $\mathcal{C} = \{A, B, C, E\}$ in Example 1, since D's vote goes to C with probability 0.5 and to E with probability 0.5, we get $f^P(\Gamma, \mathcal{C}) = (1, 1, 3/2, 0, 3/2)^T$.

7.2.2.4 Liquid Democracy (L)

In liquid democracy, each agent can choose to be part of the representative body or delegate their vote to another agent. Delegations are transitive, i.e., if A delegates to B and B delegates to C, and C decides to be in the representative body, then C votes on behalf of themself, as well as A and B. The representative body consists of all agents who self-select, with their voting power being set to the number of votes (transitively) delegated to them

[5] Proxy voting is closely related to the widespread practice of *party-list elections* [195], where agents vote for parties and the seats in the representative body are distributed so that the number of seats of a party is proportional to the number of received votes. We focus on proxy voting as it allows for a cleaner mathematical formulation.

plus one. Liquid democracy has received considerable attention in the computer science community by studying it from a procedural and epistemic perspective [25, 38, 81, 102, 111, 130, 243], developing dedicated supporting software [21, 186], and examining possible extensions [38, 55, 87]. Liquid Democracy has also been scrutinized from a political science perspective [26, 174, 230].

To find the expected voting weights of the agents under transitive delegations, we leverage the representation matrix to exhaust all possible configurations of transitive delegations and compute their probability. For instance, one possible configuration in Example 1 is that every agent votes for themselves (which happens with probability $\frac{2}{45}$), resulting in all agents being part of the representative body and having voting weight 1.[6] Overall, the expected voting weights are as follows: $f^L(\Gamma, \mathbb{N}) = \left(\frac{89}{45}, \frac{22}{45}, \frac{14}{15}, \frac{1}{5}, \frac{7}{5}\right)^T$.

7.2.2.5 Sortition (S)

Sortition is a selection method which draws at random a subset of the population to act as the representative body [70, 83, 145, 178]. The method allows equal access to decision-making and does not require a voting phase. Agents who do not participate in the representative body despite being selected pose problems with the fairness guarantees offered by sortition. Computer scientists are investigating algorithmic ways to deal with this issue [83, 84]. The representative body to be found has a fixed size $k < n$ and is found uniformly at random from $\mathbb{N}$. All members of the representative body have an equal voting weight. Thus, the expected voting weight of each agent is $\frac{k}{n}$, i.e., $f^{S_k}(\Gamma, \mathbb{N}) = \frac{k}{n}e_n$.

Note that selection mechanisms differ with respect to different dimensions, in particular, (i) whether candidates are pre-selected ($m < n$) or anyone can be part of the representative body ($m = n$), (ii) whether the output representative body has a predefined size or not,

[6] Note that if there is a delegation cycle, the votes of agents in the cycle are lost. Accordingly, their voting weight is set to zero and the agents are effectively ignored.

and (iii) whether each agent has a direct link to some member of the representative body they support ($\|f^{M_k}(\Gamma, \mathcal{C})\|_1 = n$), or some agents are virtually represented (by someone they did not necessarily vote for) ($\|f^{M_k}(\Gamma, \mathcal{C})\|_1 < n$). We call these dimensions *open-closed*, *flexible-rigid*, and *direct-virtual*, respectively. The above-described selection mechanisms are all located on different positions of the induced 3-dimensional space because some allow more flexibility, or represent more agents by design. We want to understand the impact of these design choices on desirable axioms. In turn, the arising 3-dimensional space helps with comparing different mechanisms. We envision that mechanisms from a certain region of this 3-dimensional space perform particularly well (or not) with respect to some of our axioms.

7.3 Axioms

We focus on five axioms, each capturing different aspects of representation: proportionality, diversity, monotonicity, faithfulness, and effectiveness. We lean on both the field of (computational) social choice and political science for these axioms. Our described axioms are a first step toward clarifying what various selection mechanisms entail; this is not to pretend that these desiderata are the only ones that matter or that they are the "most desirable." For instance, one could want to study the selection mechanisms with respect to the quality of decisions made by the selected body or the accountability of the selected body. This, however, is outside the scope of our chapter.

7.3.1 $\varepsilon-$proportionality

Proportionality captures how "accurately" the expected voting weights of agents in the representative body reflect their expected vote share. Proportionality is particularly desirable to achieve descriptive representation [24, 33, 162, 229, 246], and relates to previous investigations in political science on proportionality metrics for different selection formulas [197].[7] We give a notion of ε-proportionality which insists for each candidate that their normalized expected vote share and normalized expected voting weight differ by at most ε. To define this, let

$$\mathit{diff}(\Gamma, \mathcal{C}, M_k) = \max_{j \in [n]} \left| \frac{\mathbb{E}[V_j]}{\|\mathbb{E}[V]\|_1} - \frac{f^{M_k}(\Gamma, \mathcal{C})_j}{\|f^{M_k}(\Gamma, \mathcal{C})\|_1} \right|.$$

Then, ε-proportionality requires that $\mathit{diff}(\Gamma, \mathcal{C}, M_k) \leq \varepsilon$. For selection mechanisms that depend on a closed set of candidates, define $\overline{\varepsilon^{M_k}} = \max_{\mathcal{C} \subset \mathbb{N}} \mathit{diff}(\Gamma, \mathcal{C}, M_k)$ and $\underline{\varepsilon^{M_k}} = \min_{\mathcal{C} \subset \mathbb{N}} \mathit{diff}(\Gamma, \mathcal{C}, M_k)$ as the maximum, respectively minimum, largest deviation of a candidate's expected voting weight from its expected vote share over all possible candidate sets.

To give an example for $\varepsilon-$proportionality, we again make use of the setting described in Example 1. In Table 7.1, we give the values of ε for which each of the selection mechanisms are ε-proportional on Example 1. We see that liquid democracy has the smallest value of ε in this case; whereas first-past-the-post does not manage to distribute the voting weight to the selected body as efficiently. An interesting takeaway from studying the selection mechanisms via this axiom is that we see that the proportionality of sortition is independent of the size of the body, including the case where the sortition is the size of the population ($k = n$) and direct democracy is recovered.

7.3.2 Diversity

Mostly relevant in the deliberation stage [46, 72, 145], we interpret diversity as requiring that all opinions should be present in the representative body. We formalize it as "if the expected vote share of a candidate is positive, then so should be their expected voting weight", i.e.,

[7] Note that proportionality can be defined as the descriptive representation of votes, attributes, or preferences [162]. Here, we only model the proportionality of expressed votes rather than any other characteristic of the electorate.

$\mathbb{E}[V_j] > 0$ implies $f^{M_k}(\Gamma, \mathcal{C})_j > 0$ for all $j \in \mathcal{C}$.

7.3.3 Monotonicity

This benchmark is standard in social choice theory [177]. Let Γ and Γ' be some representation matrices. If in the representation matrix Γ' the expected vote share of a candidate j is larger than in Γ and the expected vote share does not increase for any other candidates, then j's expected voting weight increases. That is, if Γ and Γ' be such that $\mathbb{E}[V_j'] > \mathbb{E}[V_j]$ and $\mathbb{E}[V_i'] \leq \mathbb{E}[V_i]$ for all $i \neq j$, then the inequality $f^{M_k}(\Gamma', \mathcal{C})_j \geq f^{M_k}(\Gamma, \mathcal{C})_j$ should hold.

7.3.4 Faithfulness

This axiom ensures that candidates are not hurt by having a higher expected vote share. The axiom requires that if a candidate has a higher vote share than some other candidate, then they also have a higher expected voting weight as computed by the mechanism M, i.e., $\mathbb{E}[V_i] \geq \mathbb{E}[V_j]$ implies $f^{M_k}(\Gamma, \mathcal{C})_i \geq f^{M_k}(\Gamma, \mathcal{C})_j$ for all $i, j \in \mathcal{C}$.

7.3.5 $\gamma-$effectiveness

Finally, *effectiveness* models potential deadlocks when no majoritarian coalition may come to an agreement. This benchmark measures the size of the smallest coalition needed to have majority support for some proposal. For a given mechanism M and candidate set $\mathcal{C}$, it is defined as the expected smallest number $\gamma_{\mathcal{C}}^{M_k}$ such that some coalition of $\gamma_{\mathcal{C}}^{M_k}$ representatives gather strictly more than half of the voting weight. For mechanisms that rely on a specified set of candidates, it would again be interesting to look at the worst and best-case scenarios for $\gamma_{\mathcal{C}}^{M_k}$ over all possible candidates set of fixed size.

	f^D	f^{F_1}	f^P	f^L	f^{S_k}
ε	0.16	[0.33, 0.86]	[0.13, 0.33]	0.06	0.16

Table 7.1: The minimum values of ε for $\varepsilon-$proportionality in Example 1 for each of the different selection mechanisms: direct democracy (D), first-past-the-post (F), proxy voting (P), liquid democracy (L), and sortition (S). The intervals denote the best and worst-case ε over all candidates sets. For sortition, the presented ϵ value holds for all sizes of the body $\mathbf{k} \in [\mathbf{n}]$.

7.4 Discussion

We have argued that there is a need for a more systematic comparison of different selection mechanisms within a unified framework to understand the trade-offs inherent to the selection mechanisms currently on the table to open democratic representation [145]. Taking a first step in this direction, we have presented a simple model that allows the formulation of many different selection mechanisms together with axioms derived from political science and social choice theory that can be used to compare and assess these mechanisms.

We do not see our model and axioms as final or exhaustive, and we believe that asking the right questions is already the first research challenge. Nevertheless, there are interesting open questions arising from our study: Which of these mechanisms always satisfy diversity, monotonicity, and faithfulness? Can we obtain meaningful bounds on the ϵ-proportionality or γ-effectiveness of the different mechanisms? While it seems unlikely that general bounds can be obtained, we hope that identifying characteristics of the representation matrix could correspond to a guarantee of certain proportionality and effectiveness values. Moreover, it would be interesting to obtain comparative statements in the sense that one mechanism is always guaranteed to be better than another (at least if the "right" candidate set is chosen). Lastly, one may also wonder about the influence of the number of candidates and the selected set of candidates on the axiomatic performance of our mechanisms. More generally, it would be interesting to identify general characteristics of the selection mechanisms that benefit axiomatic guarantees, potentially extending upon our discussed *open-closed*, *flexible-rigid*, and *direct-virtual* dimensions.

Chapter 8

Conclusion

In conclusion, this thesis has delved into various aspects of representative democracy, shedding light on different selection mechanisms, their epistemic performance, and procedural implications.

The thesis investigates the interplay between diversity and expertise in representation processes, started from the aggregation of votes in the epistemic framework in Chapter 2. Next, the exploration of liquid democracy, a novel selection model that combines elements of direct and representative democracy, has been a central theme explored in Chapters 3 to 5. Through theoretical analysis and empirical experiments, the potential benefits and challenges of liquid democracy have been examined. The concept of concentration of power, where certain individuals accumulate significant delegations, has been investigated, revealing insights into its impact on decision-making outcomes. By quantifying permissible levels of power concentration and proposing realistic delegation models, the thesis contributes to understanding the dynamics of liquid democracy and its viability as an emerging paradigm.

There remain several exciting avenues for future exploration.

- **Incorporating Network Structure:** An extension of the model to incorporate underlying social networks could provide a more realistic representation of delegation dynamics. Investigating how different network structures impact power concentration and decision outcomes would be valuable.

- **Dependent Voting:** Relaxing the assumption of independent voting could lead to more accurate models, especially when voters are locally dependent on a few neighbors. Investigating the impact of various degrees of voter interdependence is an interesting direction.

- **Reasons to Delegate:** Investigating the underlying reasons and context-dependencies driving participants' decisions to delegate or not would be valuable.

- **Strategic Behavior:** Considering strategic behaviors within the liquid democracy context, akin to game-theoretic analyses with or without incentives, would provide insights into how agents might manipulate delegation mechanisms for their advantage.

- **Transparency vs. Security:** Further examining the trade-off between transparency and security in liquid democracy systems is crucial, especially regarding the potential for voter coercion when delegation graphs are transparent.

- **Handling Problematic Delegations:** The experimental setup removed problematic delegations, such as those given to non-participants or resulting in cycles. Investigating the frequency and nature of such problematic delegations, as well as developing strategies to handle them effectively, would enhance the robustness of the empirical results.

- **Beyond the Epistemic Model:** Exploring scenarios where decisions are not binary and may involve subjective criteria would enhance the applicability of the findings to a broader range of real-world decision-making processes.

- **Diverse Decision Contexts:** Expanding the analysis to different decision contexts, such as political decisions and corporate governance, and studying how liquid democracy performs across diverse scenarios will enrich our understanding of its applicability.

- **Comparative Studies:** Conducting comparative studies that benchmark liquid democracy against other selection mechanisms, both theoretically and empirically, could offer a broader perspective on its advantages and drawbacks.

Another key aspect of this thesis has been to reflect on normative considerations for representative democracy and on democratic innovations to build a renewed case for representative democracy as democratic governance. Chapter 6 delved into the procedural aspects of selection mechanisms, shedding light on the potential enhancements that could be introduced to representative assemblies. It introduced an instantiation of proxy democracy as a selection rule for representation, comparing it to lottocracy and dissecting their respective implications. Proxy democracy, with its per-issue voting scheme and flexible representation, offers a new avenue for enhancing democratic representation. On the other hand, lottocracy raises novel questions about representation that challenge traditional democratic assumptions. The future trajectory of democratic representation holds significant implications. As we move forward, a myriad of normative and practical questions remain to be tackled.

- **Symbolic and Substantive Representation:** The discussion surrounding these selection mechanisms necessitates a broader exploration of their symbolic and substantive implications.

- **Concrete Implementation:** Practical considerations, ranging from the logistics of implementation to the reliability of digital platforms, further underscore the need for comprehensive research. The scale of parliaments presents a practical challenge that must be addressed to ensure inclusive representation within lottocracy and proxy democracy. While the idea of large congresses might seem counter intuitive, new law-making protocols could be devised to manage the increased scale of representation without sacrificing the benefits of deliberation. Additionally, exploring hybrid models that blend lottocracy and proxy democracy elements might strike a balance between inclusivity and preventing an undue concentration of power or biased self-selection.

Last, the absence of a unified framework for axiomatically analyzing and comparing representative processes has hindered comparative studies. Chapter 7 addresses this void by advocating the integration of computational social choice concepts and political philosophy concepts to establish a unifying axiomatic framework. We classified selection mechanisms based on whether they are *open-closed*, *flexible-rigid*, and *direct-virtual* and further discussed the following five axioms: proportionality, diversity, monotonicity, faithfulness, and effectiveness. While the taxonomy and axioms presented are a significant stride towards this goal, they are not intended to be final or exhaustive. Rather, they pave the way for posing what we think is an essential step that lies ahead.

The quest to shape the future of representative democracy requires us to confront these challenges head-on and navigate the intricate interplay of values, trade-offs, and real-world constraints. It is a pressing concern in today's democratic landscape. Refining our understanding and imaginary of representative democracy, we can contribute to shaping its future.

Appendix A

Supplemental Material for Chapter 4: An Empirical Analysis of Liquid Democracy's Epistemic Performance

A.1 Experiments

We report hereafter the average performance of liquid and direct democracy in each of the six experiments we ran. We observe positive deltas overall due to the use of liquid democracy, and we need specifications like in Equation (4.3) to test the magnitude and significance of this increase due to the correlation structure of the data.

Table A.1: Groups Characteristics

Qualitative groups description, sizes and performance under direct and liquid democracy across all questions.

Group ID	Group Description	Group Size	Direct Democracy	Liquid Democracy
1	Company Employees Present at a Workshop	14	0.614	0.625
2	Undergraduate Students Present in Class	22	0.675	0.703
3	Research Department Meeting	19	0.632	0.665
4	Company Employees Present at a Workshop	27	0.629	0.661
5	Participants at an Academic Conference	36	0.702	0.743
6	Participants at an Academic Conference	50	0.573	0.623

A.2 Material

This section contains examples of the delegating and the voting pages.

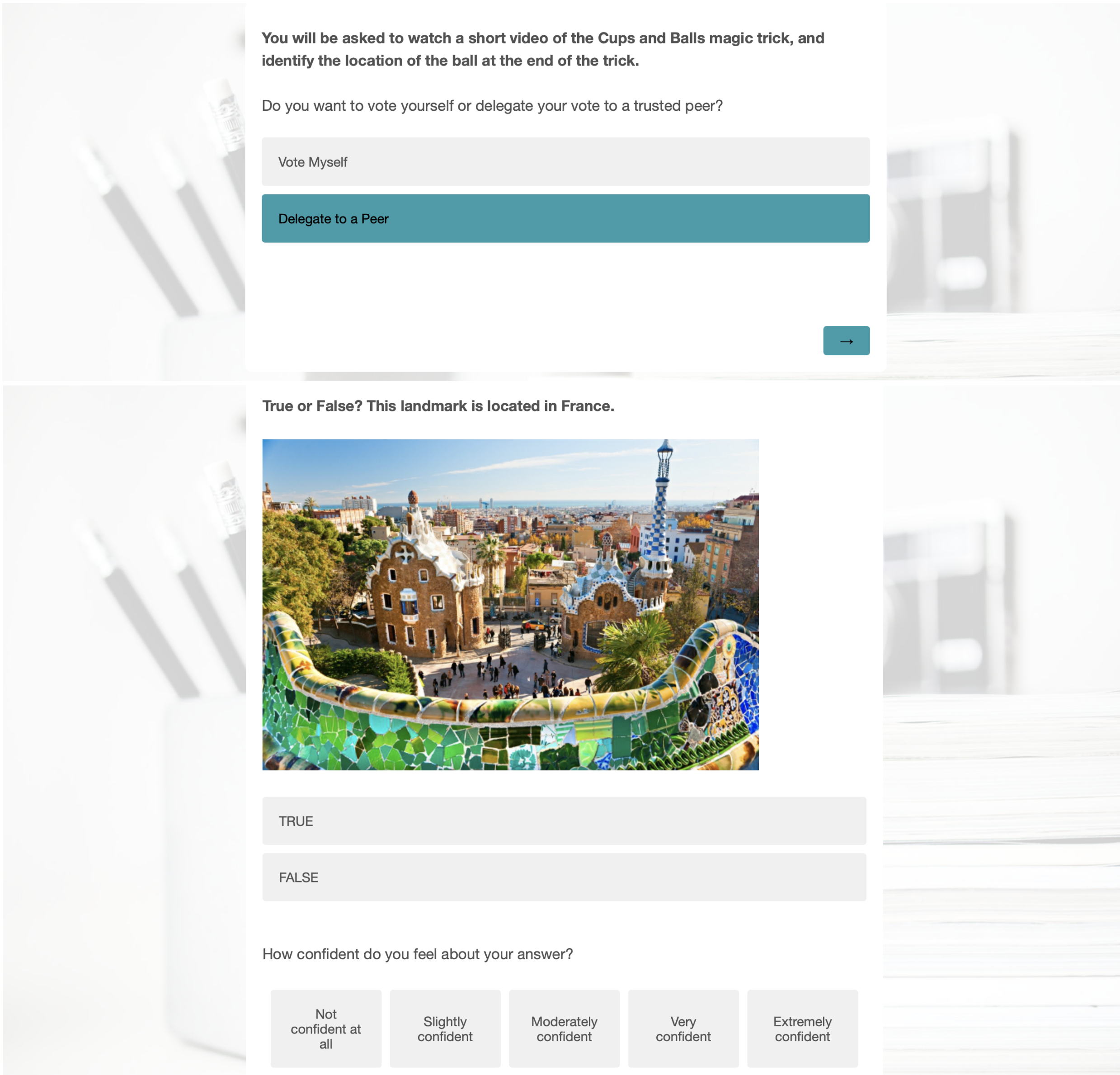


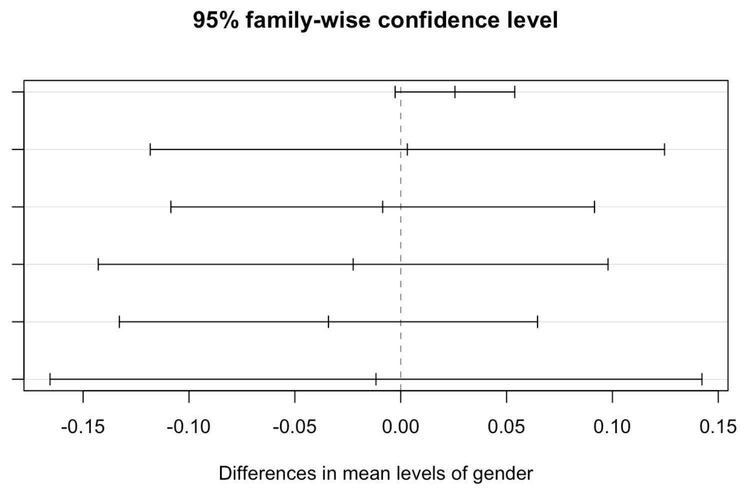
Figure A.1: Excerpts from the Liquid Democracy Survey.

Example of survey task when participants were asked to delegate at the category-level (top) and to answer at a specific questions (bottom).

A.3 Methods

A.3.1 Pairwise Tukey Tests

We show hereafter the results of the pairwise comparison of propensity of delegating and expertise for the following genders: non-binary, male, female, prefer to self-describe.



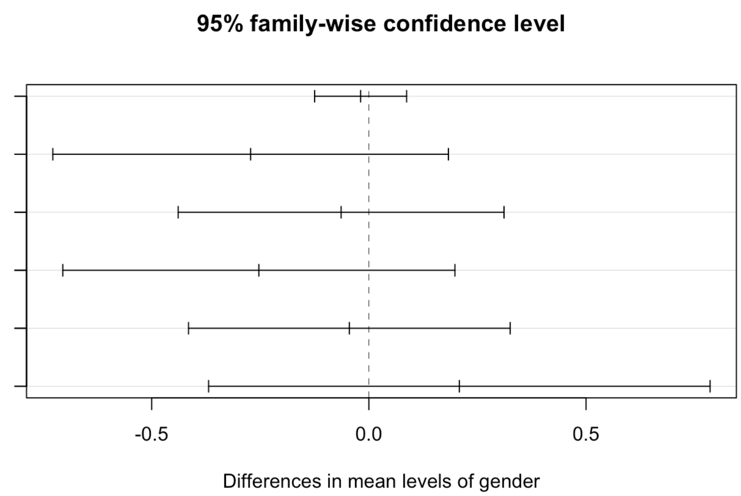


Figure A.2: Pairwise Tukey Tests

Pairwise Tukey test across different gender regarding expertise (top) and propensity to delegate (bottom). Pairwise test shows that expertise and propensity to delegate is indistinguishable across gender.

A.3.2 Normality Assumptions for Regressions

We check that the variables used in the different regressions are indeed normal, per the tests' assumptions.

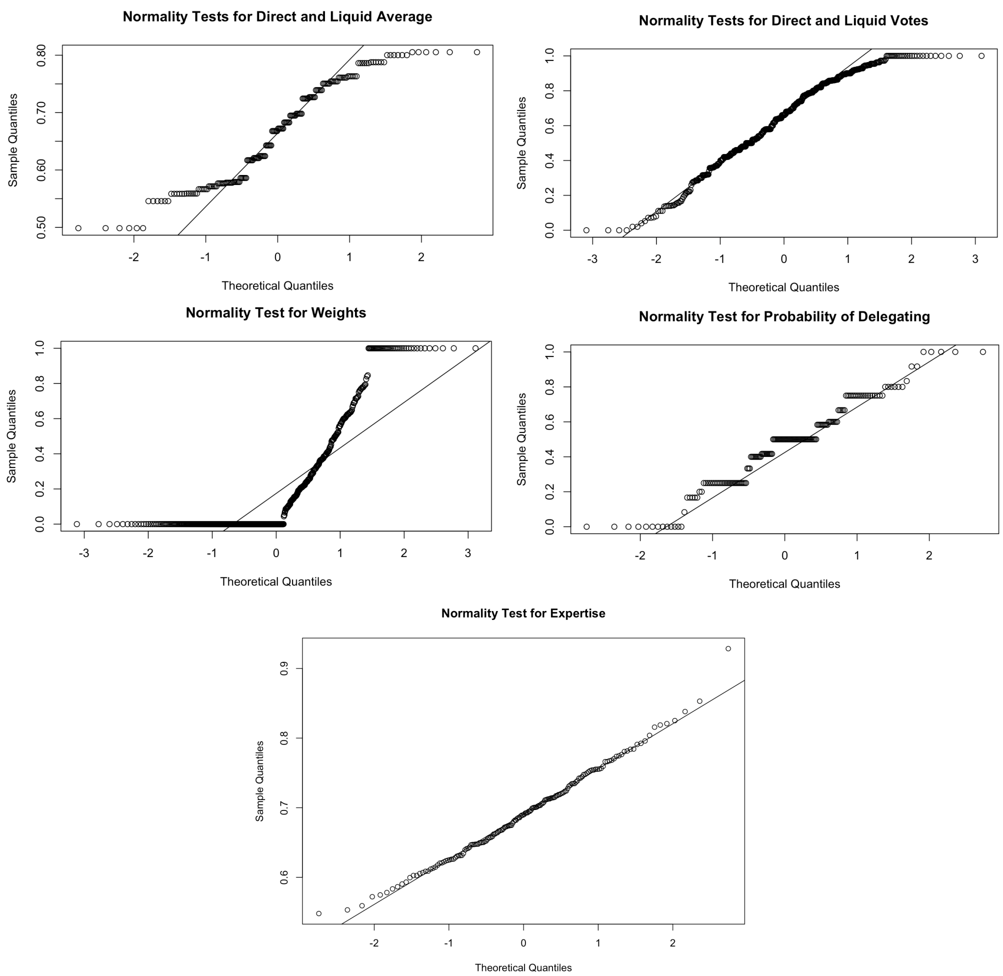


Figure A.3: Normality Tests

From top to bottom and left to right, normality tests for the average expertise in direct and liquid democracies, the average vote outcome in direct and liquid democracies, the estimated values of φ, the probability of delegating and the participants' expertise.

A.4 Examples of Delegation Graphs

A.4.1 Delegation Graph for Task T_3 and T_2

This page is for task T_3 and T_2 from Experiments 2 and 3 respectively. The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In this graph, we observe little concentration of power, and well-balanced delegation counts across relatively competent participants.

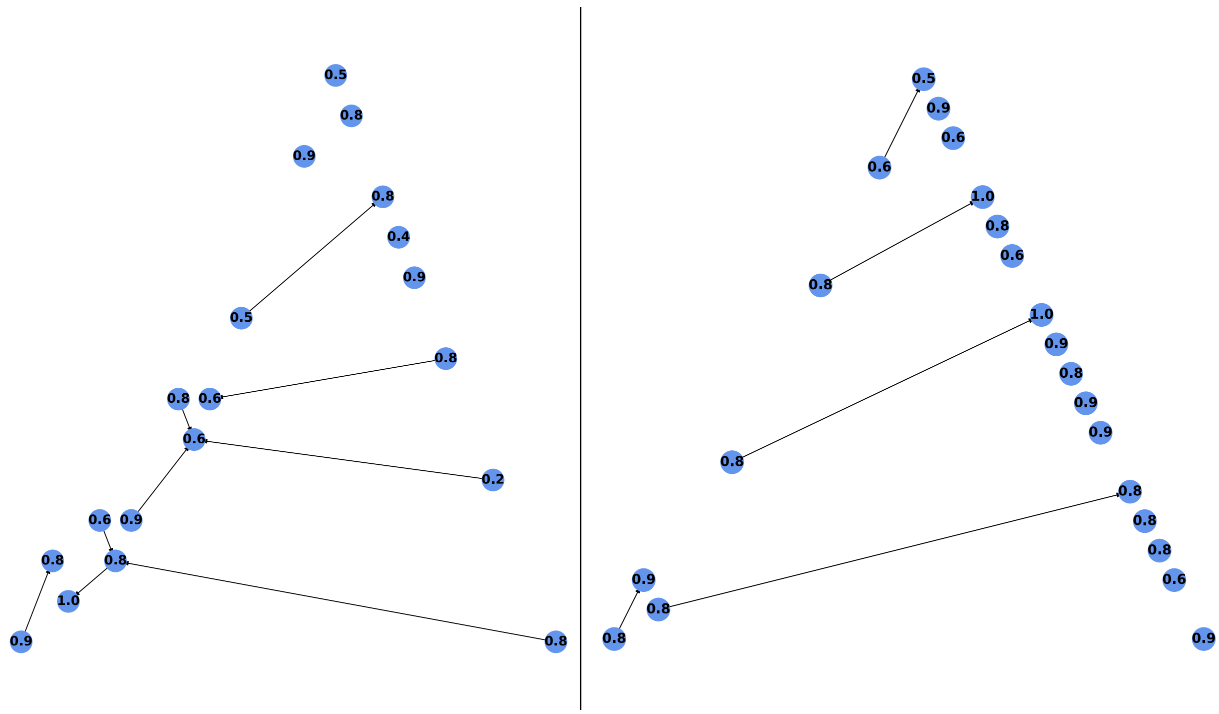


Figure A.4: Examples of Delegation Graphs

Delegation graphs for task T_3 from Experiment 2 (left) and task T_2 from Experiment 3 (right). The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In these graphs, we observe little concentration of power, a balanced set of participants post-delegation, and delegations towards relatively more expert participants.

A.4.2 Delegation Graph for Task $\boldsymbol{T_1}$ and $\boldsymbol{T_5}$

This page is for task T_1 and T_5 from Experiments 4 and 5 respectively. The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In these graphs, we observe rather extreme concentration of power.

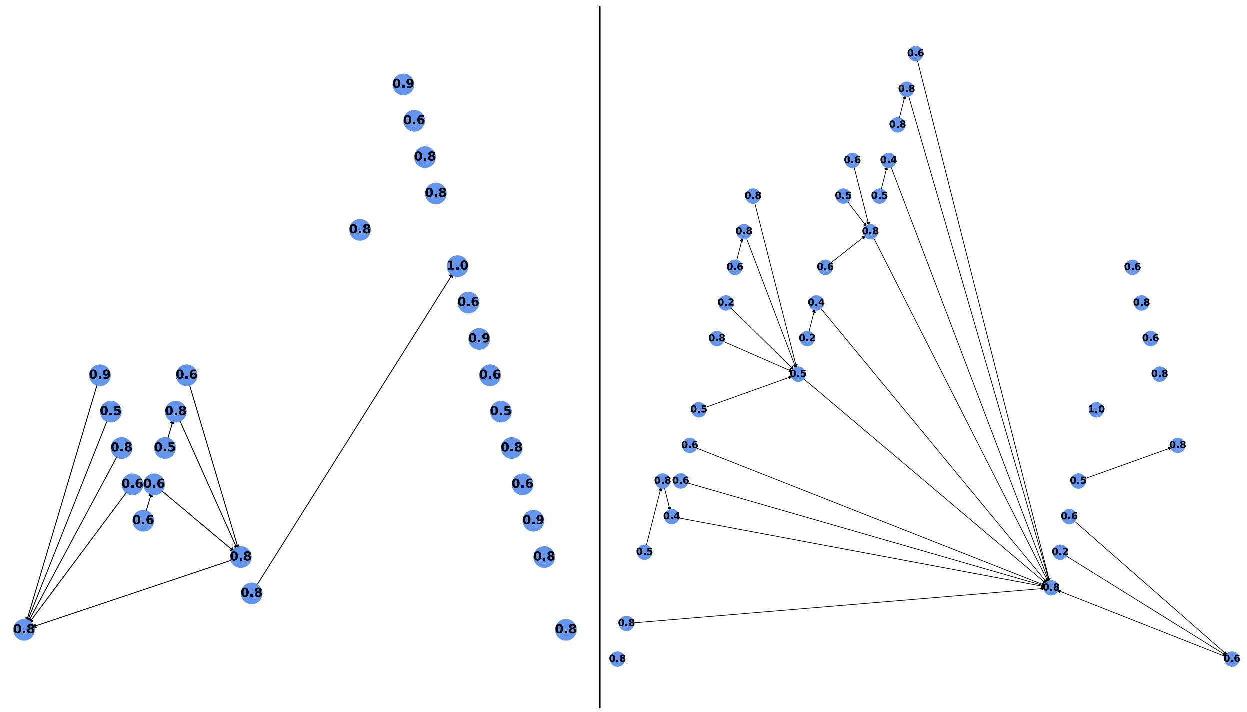


Figure A.5: Examples of Delegation Graphs

Delegation graphs for task T_1 from Experiment 4 (left) and task T_5 from Experiment 5 (right). The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In these graphs, we observe severe concentration of power in the hands of one participant. The right plot shows the worst concentration observed across the experiments where one participant received 28 transitive delegations (and 11 direct delegations) in s group of 36 participants.

A.4.3 Delegation Graph for Task T_7

This page is for task T_7 from Experiment 6. The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In this graph, we observe little concentration of power, and well-balanced delegation counts across relatively competent participants.

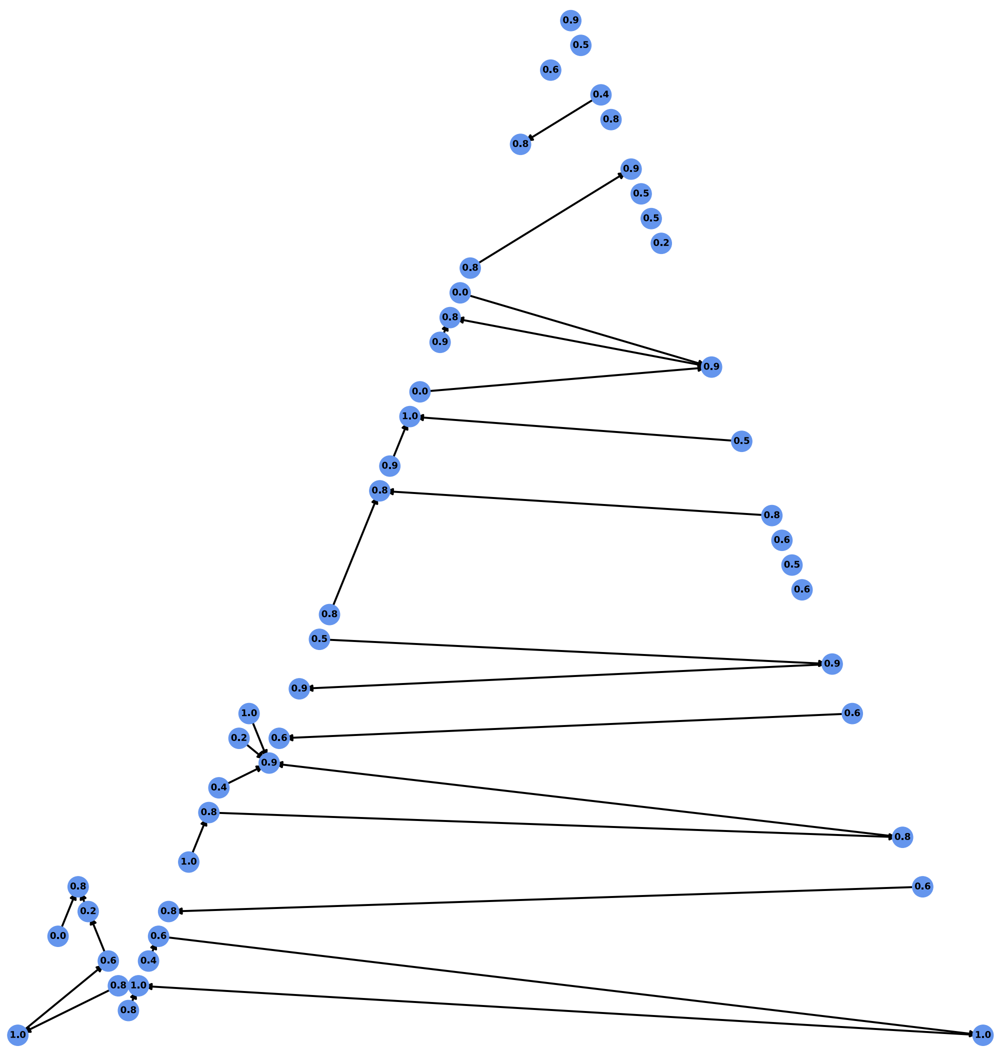


Figure A.6: Delegation Graph from experiment 6 and task 7.

A.4.4 Delegation Graph for Task T_{15}

This page is for task T_{15} from Experiment 6. The numbers represents the proportion of correct answers per participant per task, that is, $\eta_{i,t}^{\text{naive}}$. In this graph, we observe more concentration of power, and delegation across participants with relatively low expertise. We also note a delegation cycle in the upper right side.

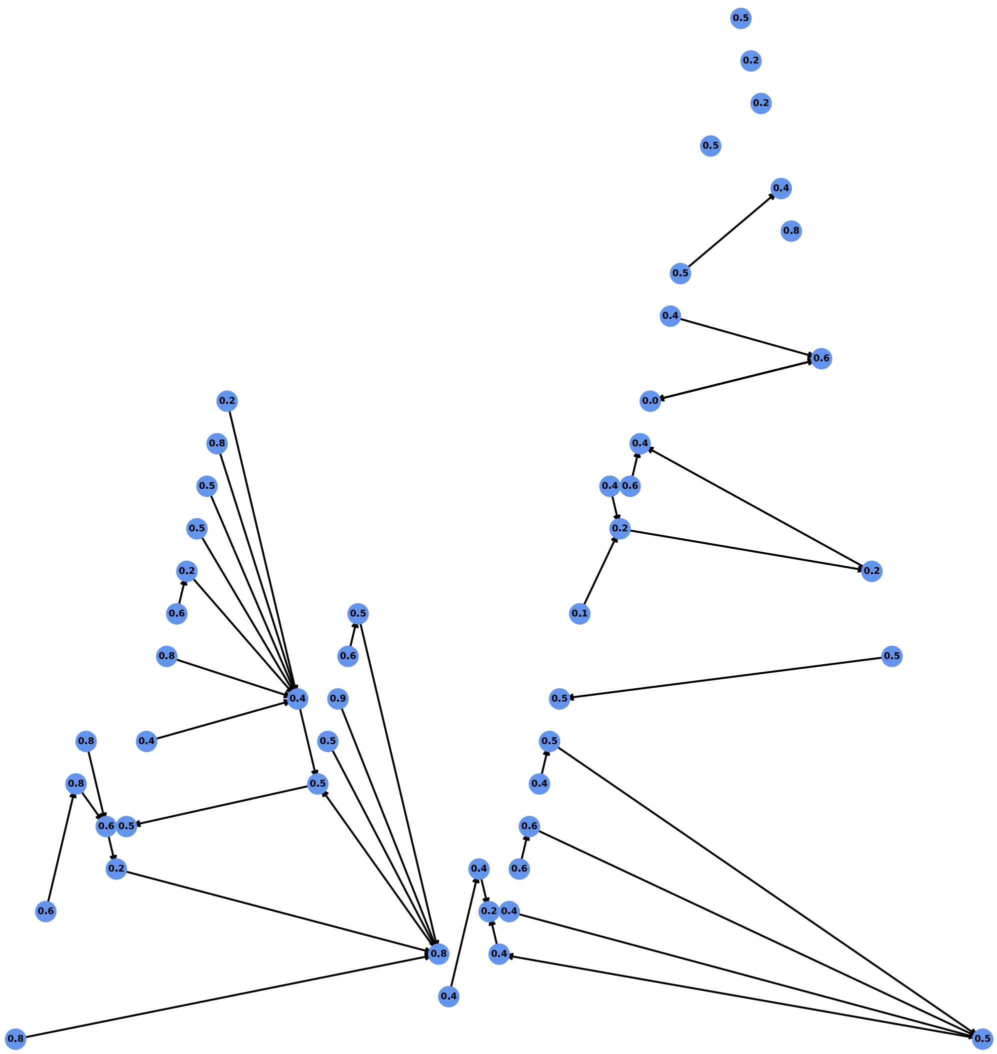


Figure A.7: Delegation Graph from experiment 6 and task 15.

A.5 Estimating the q function: Probability of Delegating as a Function of Expertise

A.5.1 Effects Sizes with Fixed Effects

Table A.2: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by the models in Equations (4.1) and (4.2).

	Overall Model	With Fixed Effects		
	Equation (4.1)	Equation (4.2)		
Effect Size β^q	-2.23^{****} (0.42)	-2.90^{****} (0.60)	-1.67^{****} (0.35)	-1.86^{****} (0.45)
Fixed Effects	NA	i	t	t, i
Clustered S.E.	i	i	i	i
Note:				****p<0.0001

Adding fixed effects, we account for participant-specific and/or task-specific characteristics that may affect the outcome variable. We see that within each participant, the probability of delegating decreases faster with expertise than in the model without fixed effect. This implies that participant-specific characteristics (such as confidence) is also at play in delegation decisions (and it seems that those more confident are not always more expert). We also see that within each task, the probability of delegating decreases more slowly with expertise than in the model without fixed effect. This implies that task-specific characteristics (such as difficulty) are at play in delegation decisions (and it seems that more difficult task are associated lower expertise).

A.5.2 Task-specific Effects

The results for the estimation of q per task are shown below.

Table A.3: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by the models in Equation (4.2) for the four main tasks conducted in each experiment. Recall that we have 6 times more data points for tasks $1, 2$ and 3 since they were performed across all experiments: sport tasks were different across different experiments while the others were constant.

	Overall Model	Tasks Models						
		(Tasks)						
		(T_1)	(T_2)	(T_3)	(T_4)	(T_5)	(T_6)	(T_7)
Effect Size β^q	-2.23^{****} (0.42)	-2.05 (1.58)	-1.96 (1.50)	-4.06^{***} (1.25)	2.33 (4.05)	0.21 (1.57)	-8.04^{*} (4.88)	-1.50 (0.23)
Clustered S.E.	i	NA	NA	NA	NA	NA	NA	NA
Note:								*p<0.1; **p<0.05; ***p<0.01

Table A.4: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence for questions specific to experiment 6.

The summary of the average and heterogeneous effects, estimated by the models in Equation (4.2) for the tasks specific to experiment 6. Note that we have many less data-points for these tasks since fewer subject took part into it overall.

	Tasks Models							
	(Tasks)							
	(T_8)	(T_9)	(T_{10})	(T_{11})	(T_{12})	(T_{13})	(T_{14})	(T_{15})
Effect Size β^q	-5.13 (5.09)	1.57 (2.15)	-3.87 (2.66)	-2.43 (2.38)	-2.50 (2.01)	-3.62^{*} (1.88)	3.70 (2.89)	-1.58 (1.57)
Clustered S.E.	NA	NA	NA	NA	NA	NA	NA	NA
Note:								*p<0.1; **p<0.05; ***p<0.01

A.5.3 q based on k-mean bucketing

We could also have estimated q from the buckets though this method would be less precise, relying on aggregate data. We show hereafter the propensity of delegating per bucket, as calculated using the k-means clustering algorithm, and the curve displays a strong decreasing tendency, corroborating Section 4.3.3.

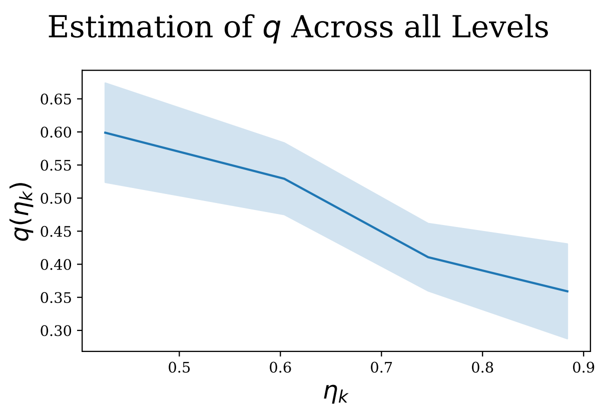


Figure A.8: Estimation of q using k-means clustering buckets

A.6 Estimating the φ function: Delegation Choice as a Function of Expertise

A.6.1 k-mean bucketing

We show the curve from which we learn the optimal number of buckets using k-means clustering, and the resulting bucketing.

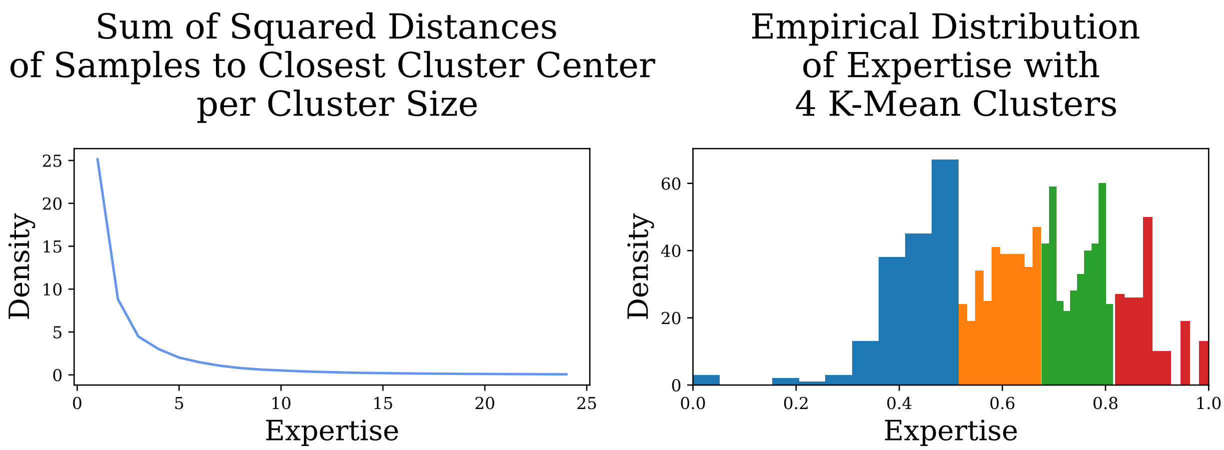


Figure A.9: Ouput of the k-means clustering procedure

A.6.2 Maximum Likelihood Estimation for the Multinomial Model

We next obtain a closed form for the maximum likelihood estimators that we invert herein. Let the matrix $M \in \mathbb{R}^{B\times B}$ be such that $M\vec{\varphi} = 0_B + e_1$ where 0_B is the vector of size B with only zeros and e_1 is the vector of size B with a 1 on the first coordinate and zeros otherwise. (Without loss of generality, we replace the first equation with $\sum_{k=1}^{B} \varphi^\ell(\eta_k) = 1$ and recover B equations with B parameters.) Re-writing Section 4.3.4.2, we get, if $m \neq i$:

$$\sum_{k=0,k\neq m,k\neq l}^{B} n_k\varphi^\ell(\eta_k)z_k^\ell + (n_\ell - 1)\varphi^\ell(\eta_\ell)z_\ell^\ell + n_m\varphi^\ell(\eta_m)(z_m^\ell - \tilde{n}_\ell) \tag{A.1}$$

and, if $m = i$:

$$\sum_{k=0,k\neq m}^{B} n_k\varphi^\ell(\eta_k)z_k^\ell + (n_m - 1)\varphi^\ell(\eta_m)(z_m^\ell - \tilde{n}_\ell) \tag{A.2}$$

and finally, matrix M can be written as follows:

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ n_1 z_1^\ell & n_2(z_2^\ell - \tilde{n}_\ell) & \cdots & (n_i - 1) z_i^\ell & n_{i+1} z_{i+1}^\ell & \cdots & n_B z_B^\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1)(z_i^\ell - \tilde{n}_\ell) & n_{i+1} z_{i+1}^\ell & \cdots & n_B z_B^\ell \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1) z_i^\ell & n_{i+1}(z_{i+1}^\ell - \tilde{n}_\ell) & \cdots & n_B z_B^\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1) z_i^\ell & n_{i+1} z_{i+1}^\ell & \cdots & n_B(z_B^\ell - \tilde{n}_\ell) \end{bmatrix}$$

so that

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ n_1 z_1^\ell & n_2(z_2^\ell - \tilde{n}_\ell) & \cdots & (n_i - 1) z_i^\ell & n_{i+1} z_{i+1}^\ell & \cdots & n_B z_B^\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1)(z_i^\ell - \tilde{n}_\ell) & n_{i+1} z_{i+1}^\ell & \cdots & n_B z_B^\ell \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1) z_i^\ell & n_{i+1}(z_{i+1}^\ell - \tilde{n}_\ell) & \cdots & n_B z_B^\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_1 z_1^\ell & n_2 z_2^\ell & \cdots & (n_i - 1) z_i^\ell & n_{i+1} z_{i+1}^\ell & \cdots & n_B(z_B^\ell - \tilde{n}_\ell) \end{bmatrix} \times \begin{bmatrix} \varphi^\ell(\eta_1) \\ \varphi^\ell(\eta_2) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \varphi^\ell(\eta_b) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

We finally compute $\vec{\varphi} = M^{-1}(0_B + e_1)$, so that we obtain the estimates for $\varphi^\ell(\eta_k)$ for all $k \in \{1, \ldots, B\}$ for the given task and experiment. We repeat the exercise for all possible $\ell \in \{1, \ldots, B\}$. We finally repeat the operations for each experiment and task. In turn, we collect, for each pair (experiment, task) a matrix $\Phi^{(e,t)}$ whose entry are the $\varphi_{e,t}^\ell(\eta_k)$ computed for that experiment as explained above.

Note that we could also reason from the observed proportion of delegations from type ℓ to type k, that should approach the expected proportion of delegations. The latter is equal to $\varphi^\ell(\eta_k)$ appropriately weighted by the weights carried by all potential recipients of the delegation. Per [111], we assume a complete graph: any participant of type m may receive the delegation with a probability equal to $\varphi^\ell(\eta_m)$. See Figure A.10 for an example. We recover the same equations as in Section 4.3.4.2.

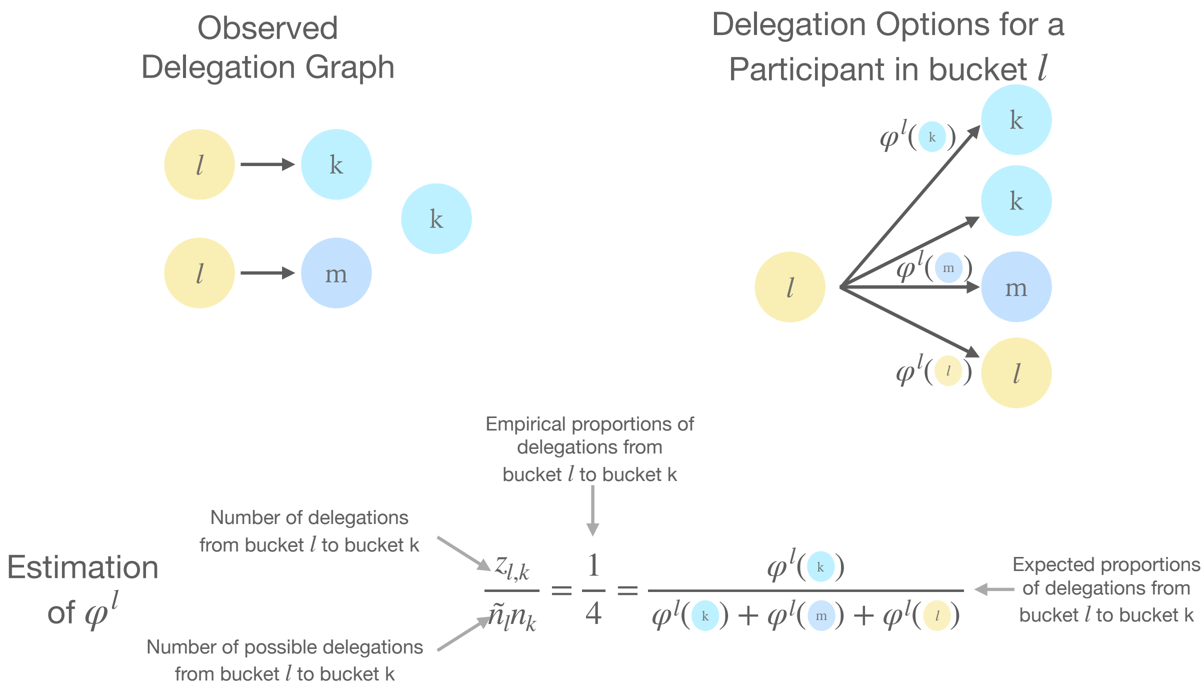


Figure A.10: Example on how to reconstruct φ^ℓ with 4 participants of three different types, ℓ, k and m.

The top left image shows the observed delegation graph among 4 participants. The top right image shows the probability weighting put on every neighbors from the perspective of a type ℓ's participant. The bottom image shows provides an example for the computation involved in Proposition 1.

A.6.3 Task-specific Effects with k-means Clustering

Lastly, we report in Table A.5 the the Kendall tau correlation coefficients for each task to understand whether the observed behavior is constant across tasks.

Table A.5: Correlation and p-values for Tasks

The Kendall tau rank correlation coefficient for each task. Note that the first three tasks, there are 96 data points per regression. For those, we observe clear trend that φ is increasing in its second coordinate across all three tasks. The following ones, however, rely on 16 data points. *p<0.1; **p<0.05; ***p<0.01; ****p<0.0001

Task	Correlation	p-value
T_1	0.3647***	0.0011
T_2	0.2621**	0.0079
T_3	0.3830****	< 0.0001
T_4	-0.1540	0.6233
T_5	-0.2055	0.1004
T_6	-0.1372	0.5666
T_7	0.4768*	0.0201
T_8	-0.1925	0.5392
T_9	-0.5231**	0.0127
T_{10}	-0.2490	0.2300
T_{11}	0.5427**	0.0108
T_{12}	0.0321	0.9115
T_{13}	0.5644**	0.0059
T_{14}	0.5768**	0.0051
T_{15}	-0.2689	0.1948

A.6.4 Robustenss of Bucketing

We experiment with varying bucketing methods, to ensure that the results are robust. We show in Figure A.11 the different bucketing methods we tried, and further report below the resulting φ. We described the first bucketing method in Section 4.3.4; we now discuss the remaining methods:

A.6.4.1 Equal cut:

With equal cut, we divide the $[0, 1]$ line in B buckets $c_\ell = [(\ell - 1)/B, \ell/B]$ for $\ell \in \{1, \ldots, B\}$ of equal size. We vary the number of buckets B from 3 to 10 to ensure robustness of the approach. When buckets are empty, we compute the weights on the existing types and re-normalize in the final stage.

A.6.4.2 Quantile cut:

We cut the $[0,1]$ line in B quantiles c_ℓ for $\ell \in [b]$ so that the number of expertise values in each bucket is the same and we take the mean expertise in the designated bucket to be the representative expertise, η_ℓ. That is, $\eta_\ell = \frac{\sum_{i,t} \eta_{i,t} \mathbb{I}[\eta_{i,t} \in c_\ell]}{\sum_{i,t} \mathbb{I}[\eta_{i,t} \in c_\ell]}$. In short, each expertise $\eta_{i,t}$ is assigned to a bucket η_ℓ such that $\ell \leq \eta_{i,t} < l + 1/b$. We vary the number of buckets B to ensure robustness of the approach (see Appendix A.6.4).

A.6.4.3 Gaussian Mixture Model:

We assume that the expertise level $\eta_{i,t}$ is drawn from B Gaussian distributions that we intend to reconstruct. To do so, we maximize the log-likelihood $\log \Pr(\vec{\eta}) = \sum_{i,t} \sum_k^b \log \left(\pi(k) \mathcal{N}(\eta_{i,t} | \eta_k, \sigma^2) \right)$, where $\vec{\eta}$ is the vector of $\eta_{i,t}$, $\pi(k)$ is the probability of being in Gaussian k, η_k, σ^2 are the mean and variance of the $k-$th Gaussian and $\mathcal{N}(x|\mu, s^2)$ denotes the probability density function of a Gaussian with mean μ and standard deviation s evaluated at x. This optimization cannot be solved in closed-form, and we use the Expectation-Maximization (EM) algorithm to estimate the Gaussian means η_k, as well as the marginal probabilities $\Pr(k|\eta_{i,k})$.[1] Last, we find the number of Gaussian B that maximizes the likelihood's cross-validation estimate. In turn, we obtain an assignment of expertise $\eta_{i,t}$ to B Guassian and B Gaussian's mean, that we denote η_ℓ for the $\ell-$th Gaussian. Each expertise $\eta_{i,t}$ is assigned to a Gaussian η_ℓ based on $\Pr(k|\eta_{i,k})$. We used the k-means clustering approach instead since the resulting cluster are very similar yet not normally distributed in either case, violating the underlying assumption of the Gaussiam Mixture estimation.

Next, we report the results for the estimation of φ for different buck sizes to test potential sensitivity of the results to the buck size. We run the same experiments as in Section 4.4.3 with $b \in \{3, 5, 7, 10.\}$

[1] See details here: https://people.csail.mit.edu/rameshvs/content/gmm-em.pdf.

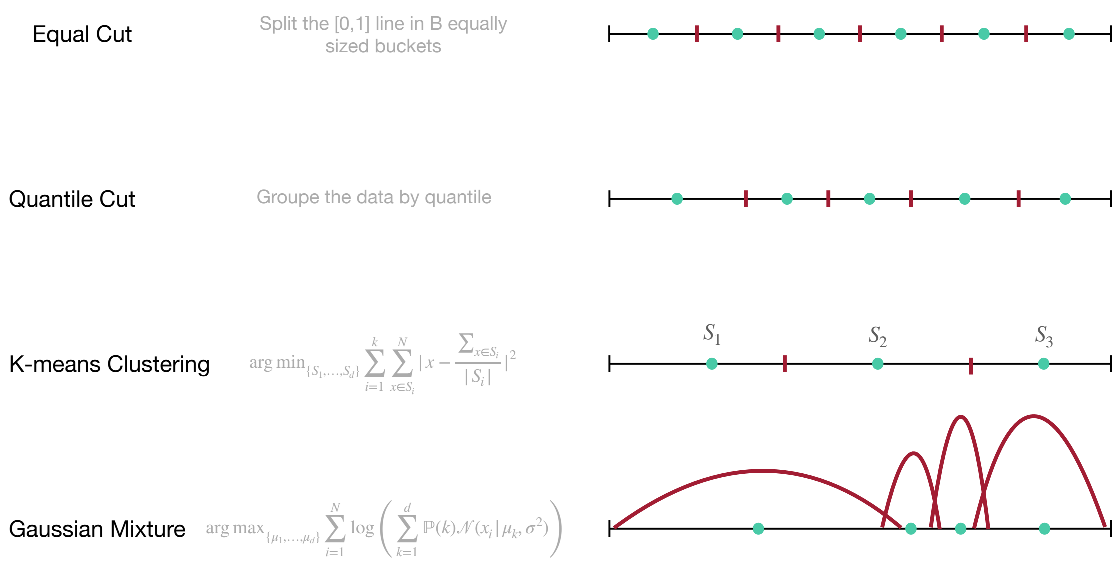


Figure A.11: Different Bucketing Methods Illustrated

A.6.4.4 Estimation with three buckets: $B = 3$

Table A.6: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

	Overall	For fixed ℓ		
		c_1	c_2	c_3
Correlation	0.44^{****}	0.60^{*88*}	0.43^{*888}	0.34^{****}
P-value	2×10^{-11}	2×10^{-3}	2×10^{-5}	7×10^{-4}
Note:	*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001			

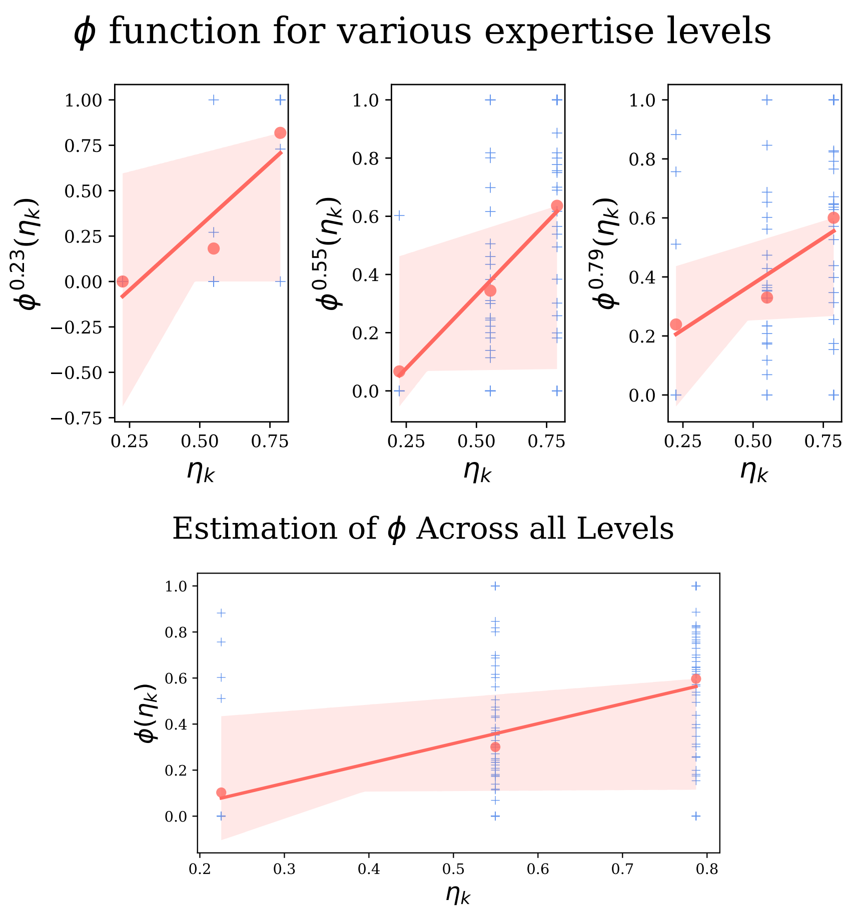


Figure A.12: Estimation of φ with $B = 3$

Table A.7: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by Section 4.3.4.3.

	Overall	For fixed ℓ				
		c_1	c_2	c_3	c_4	c_5
Correlation	0.13****	0.14**	0.06	0.15**	0.10	0.21***
P-value	6×10^{-5}	6×10^{-2}	3×10^{-1}	4×10^{-2}	10^{-1}	5×10^{-3}
Note:			*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001			

Table A.8: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by Section 4.3.4.3.

	Overall	For fixed ℓ						
		c_1	c_2	c_3	c_4	c_5	c_6	c_7
Correlation	0.23****	0.37	0.36**	0.05	0.18***	0.25****	0.16	0.45****
P-value	10^{-11}	3×10^{-1}	10^{-2}	6×10^{-1}	10^{-1}	2×10^{-4}	2×10^{-2}	3×10^{-6}

Note: *p<0.1; **p<0.05; ***p<0.01; ****p<0.0001

A.6.4.5 Estimation with five buckets: $B = 5$

A.6.4.6 Estimation with seven buckets: $B = 7$

A.6.4.7 Estimation with ten buckets: $B = 10$

A.6.4.8 Experimentation with quantile cut and $B = 7$

For simplicity, we only show the plots of the quantile split with $B = 7$.

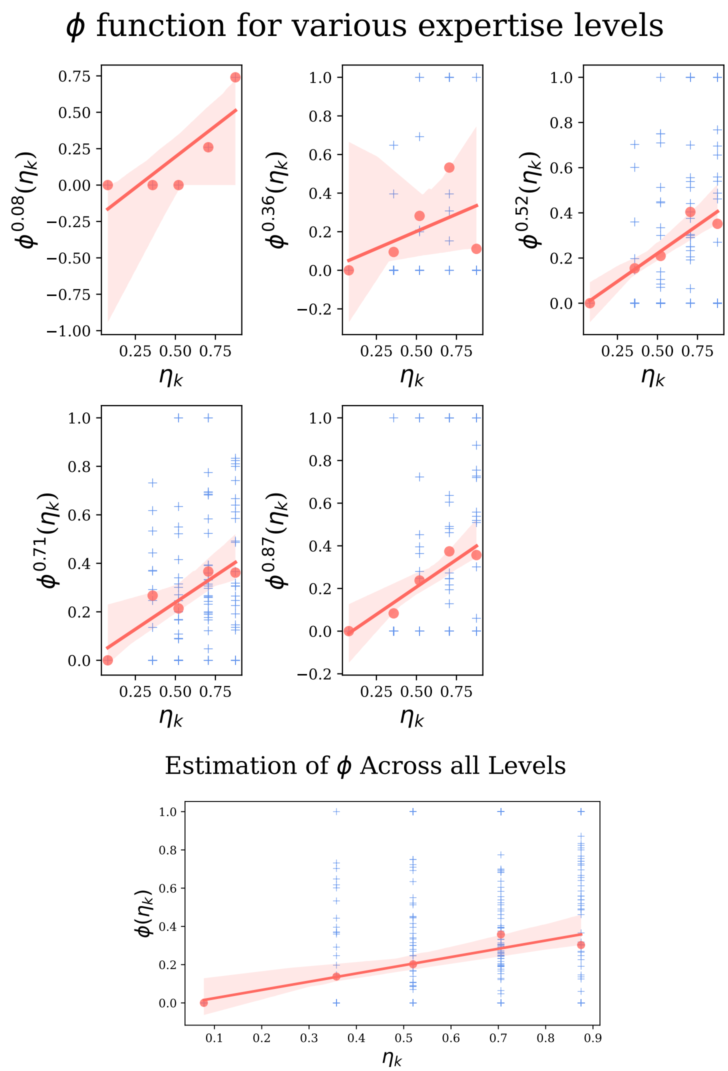


Figure A.13: Estimation of φ with $B = 5$

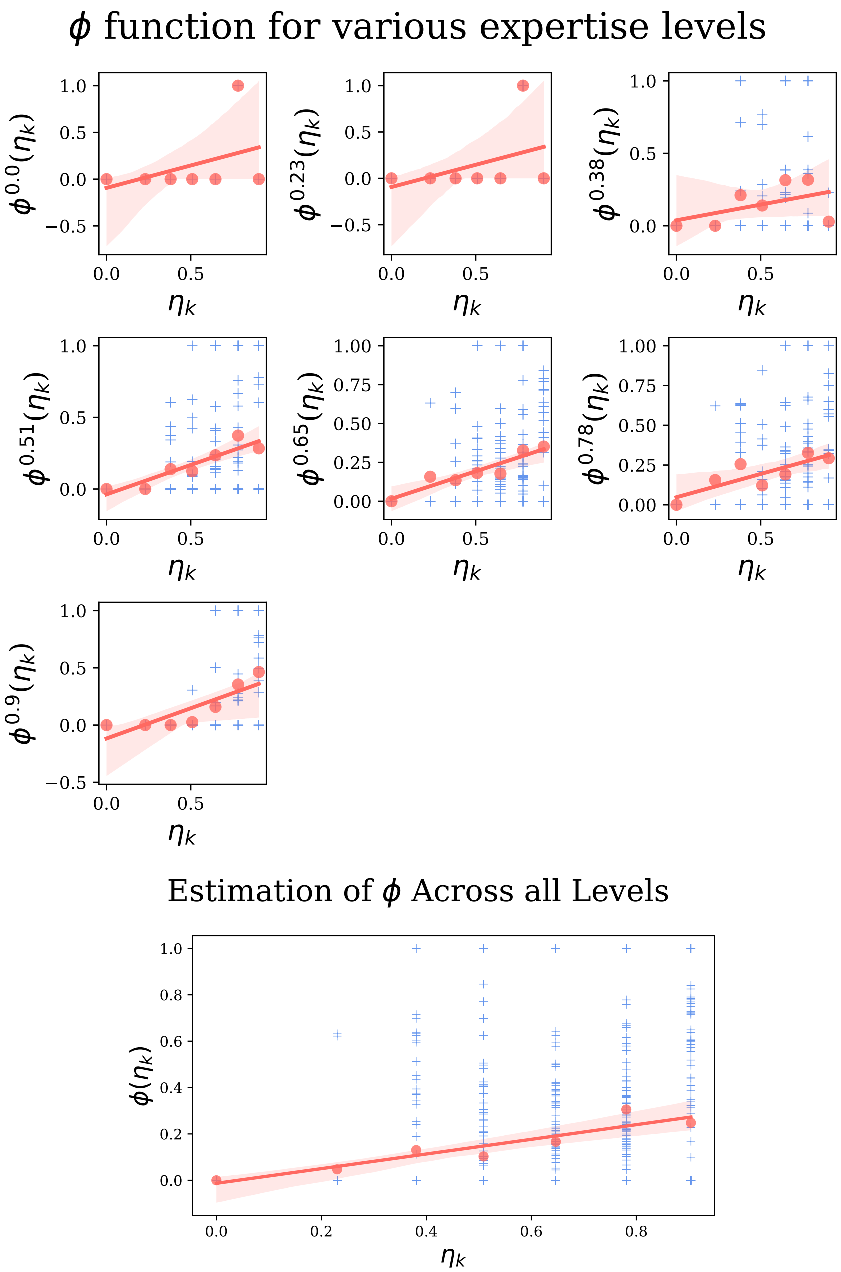


Figure A.14: Estimation of φ with $B = 7$

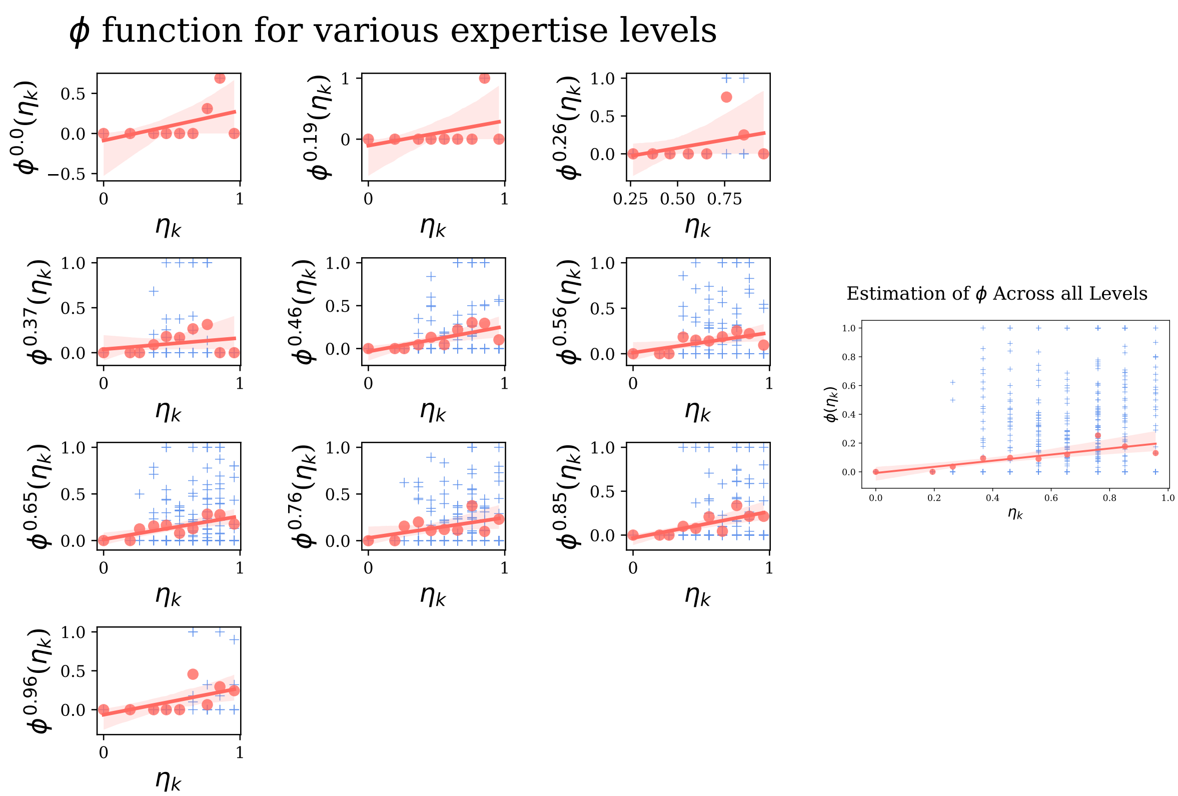


Figure A.15: Estimation of φ with $B = 10$

Table A.9: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by Section 4.3.4.3.

	Overall	For fixed ℓ									
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
Correlation	0.14****	0.47	0.35	0.35**	0.00	0.16**	0.03	0.15**	0.10*	0.21***	0.30**
P-value	1.2073×10^{-7}	0.1111	0.2453	0.0388	0.9763	0.0291	0.6754	0.0135	0.0957	0.0026	0.0418

Note: *p<0.1; **p<0.05; ***p<0.01; ****p<0.0001

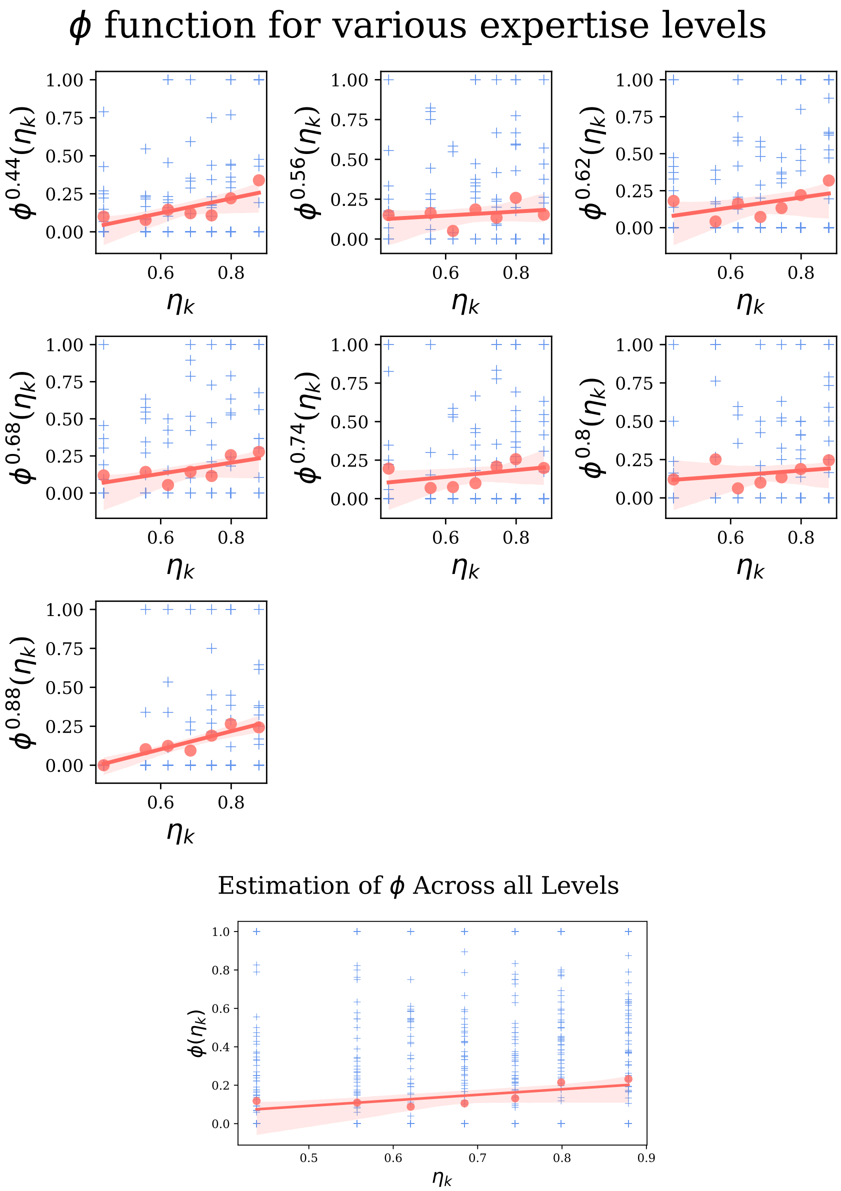


Figure A.16: Estimation of φ with $B = 7$

Table A.10: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by Section 4.3.4.3.

	Overall	For fixed ℓ						
		c_1	c_2	c_3	c_4	c_5	c_6	c_7
Correlation	0.10****	0.07	0.03	0.08	0.10*	0.11*	0.09	0.29***
P-value	3×10^{-5}	0.25	0.62	0.16	0.099	0.063	0.17	0.0003

Note: *p<0.1; **p<0.05; ***p<0.01; ****p<0.0001

A.7 Coalition Analysis

We investigate here the minimal number of participants that gathered half of the votes in total, so that their coalition would be a majority. We call a participant who votes directly a *guru*. Fix a graph (e,t) and order the weights $w_{(i,t)}$ in decreasing order. Let $n_{e,t} = \sum_{i=1}^{N_e} \mathbb{I}[w_{(i,t)} > 0]$ be the number of gurus in a graph. Let $\sum_{i=1}^{k} w_{(i,t)}$ be the weight accumulated by the k participants with the highest weights. Let $m^*_{e,t}$ be the smallest k such that $\sum_{i=1}^{k} w_{(i,t)} > 0.5$; we call this *the smallest size of a potentially majority coalition* (SPMC).

We first show in Figure A.17 per-experiment plots with the fraction of votes gathered by the smallest coalition with the highest total weight; that is, we plot $w_{(i,t)}$ as a function of i.. Each color represents a different task.

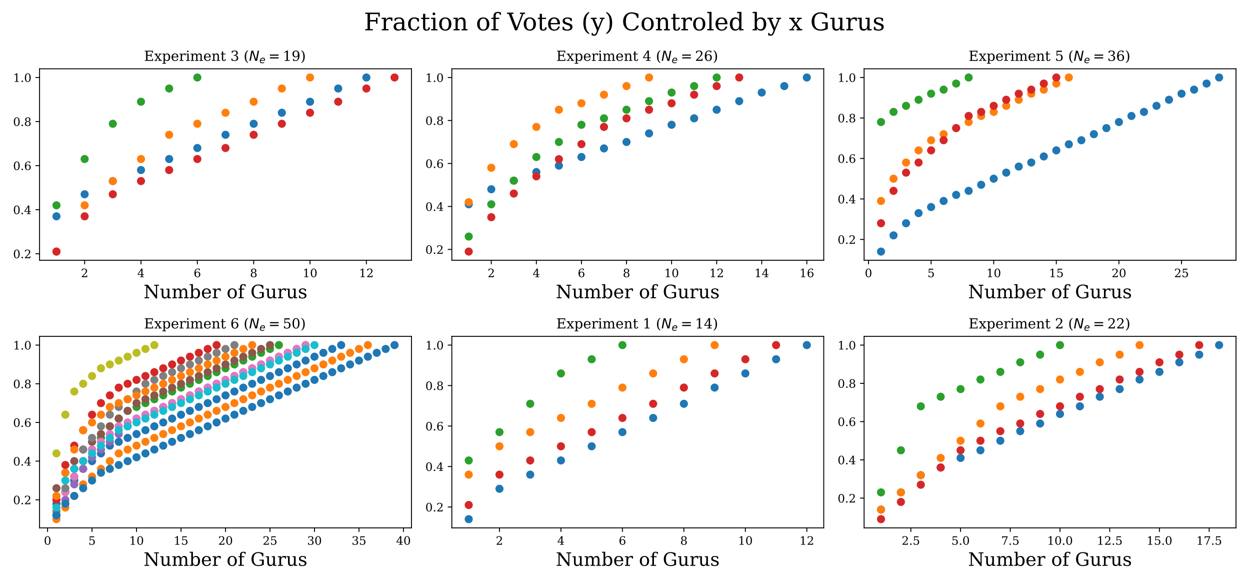


Figure A.17: Fraction of votes Gathered by Smallest Coalition that Maximizes Total Weight

Next, we show in Figure A.18 the number of gurus in a graph as a function of the minimal size of a majority coalition; that is, we plot $n_{e,t}$ as a function of $m^*_{e,t}$. We see rare occurrences of graphs with many gurus and small smallest majority coalition, indicating that votes tend to be spread among gurus.

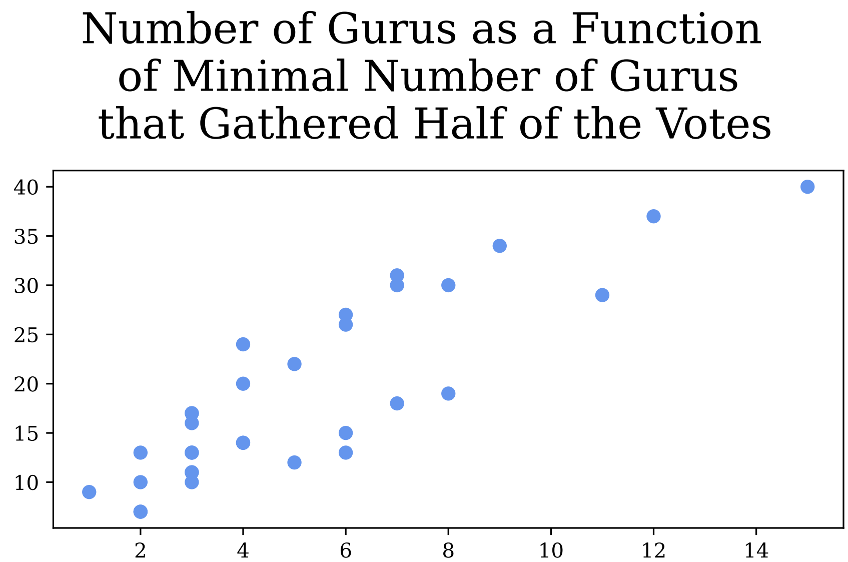


Figure A.18: Number of Gurus as a Function of the Size of the Smallest Potentially Majority Coalition (SPMC)

Last, we show the minimal size of the SPMC across experiments as a box plot in Figure A.19; that is, we plot $m^*_{e,t}$ as a function of N_e. We see multiple occurrences where a handful of gurus constitute the smallest majority coalition. This is a trend that requires further experimentation with larger groups to understand how it evolves asymptotically.

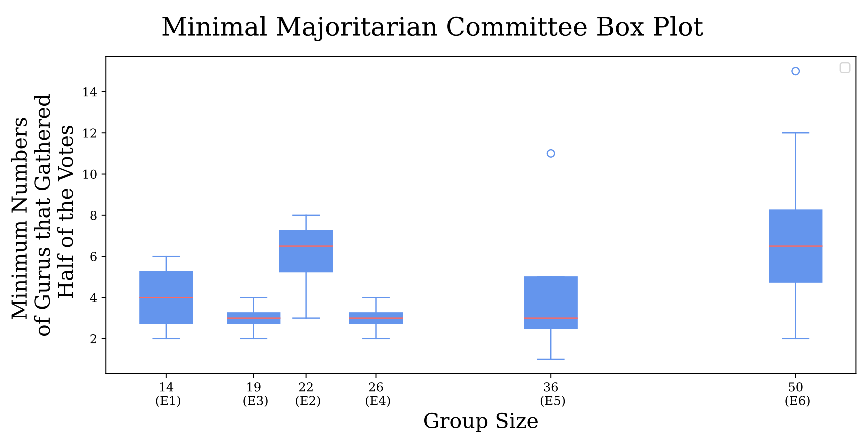


Figure A.19: Smallest Potentially Majority Coalition

A.8 Pre-Experiment

We provide in this section the results of the same analyses in the pre-study, that comprised 6 experiments and a similar design that was used to inform the survey flow and material of the main study.

A.8.1 Update in the Design for the Main Study

In the main study, the spatial reasoning task was dropped as participants were almost always correct. Ambiguous questions (for which multiple answers were true) taken from Simoiu et al. [215] were rephrased to enforce a clear epistemically answer. Questions that were mislabeled in Simoiu et al. [215] were removed. The survey flow was also changed so that more questions could be answered in the same amount of time, and all tasks were set to have the same number of questions $|R_t| = 8$.

A.8.2 Recruitment

The six groups with which the pre-study was ran are described below. There was a total of 102 participants that participated in the surveys between March 21st and April 5th, 2022. Of the participants across all experiments, 29% were native English speakers, 16% were female, 4% non-binary, and 80% were male.

A.8.3 Material

The tasks used in the pre-study are described below, Table A.12 detail the tasks and Table A.13 the questions within the tasks.

Table A.11: Groups Characteristics

Qualitative groups description, sizes and performance under direct and liquid democracy across all questions.

Group ID	Group Description	Group Size	Direct Democracy	Liquid Democracy
1	Research Group	11	0.682	0.713
2	Graduate and Undergraduate Class	12	0.726	0.730
3	Graduate Class	32	0.682	0.695
4	Sports Team	14	0.740	0.748
5	Financial Association	18	0.694	0.729
6	Group of employees, students and faculty	15	0.678	0.703

A.8.4 Assessing Expertise

We run the IRT framework and find a correlation of 97% between the naive expertise $\eta_{i,t}^{\text{naive}}$ and the expertise computed accounting for task difficulty $\eta_{i,t}$. Note that, while the expertise distribution is normal Appendix A.3.2, the expertise distribution in the pre-study is not, due to the spatial reasoning task for which almost all participants are correct.

We show the distribution of expertise computed with both the naive and IRT frameworks in both methods in Figure A.20.

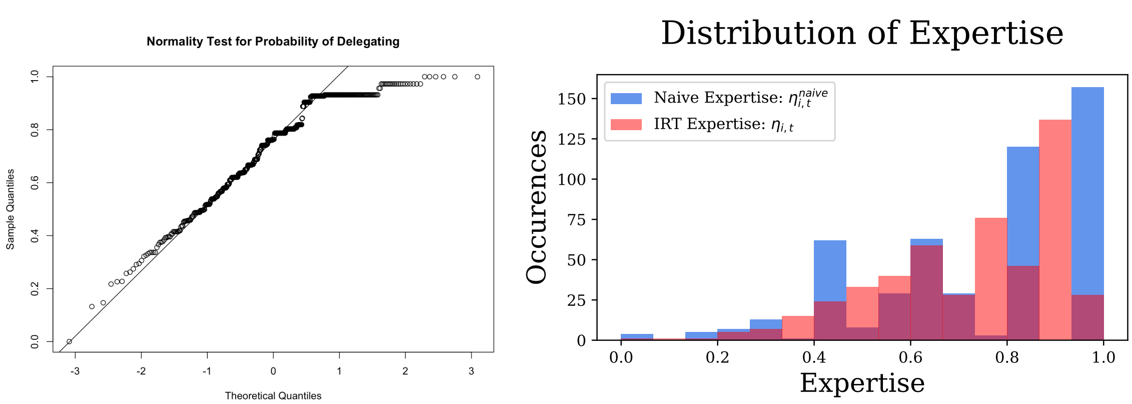


Figure A.20: Normality Test for expertise in pre-study (left) and Distribution of expertise per delegation behavior (right)

Table A.12: Prompts for Each Task

Prompts asked to participant for each task. After reading it, participants decided to delegate or perform the task themselves. If they delegated, they chose another participant to do the task on their behalf. If they did not delegate, they answered the questions related to the task (see all questions on Table A.13). The tasks Landmark and Movie had $|R_t| = 5$ questions, the task spatial had $|R_t| = 3$ questions and the prediction tasks for Sports had $|R_t| = 7$ questions. The harder the task (based on the success rate in the original work by [215], the more questions we asked).

ID	**Task Description**	**Included in Experiment(s)**
Landmark	You will be shown images of architectural landmarks from around the world, and asked to select the country where the landmark is located.	$1, 2, 3, 4, 5, 6$
Movie	You will be provided with short audio files with theme songs from various movies, and asked to select the movie it was featured in.	$1, 2, 3, 4, 5, 6$
English	You will be given English idioms, and asked to identify their meaning. An idiom is a group of words that have a meaning not deducible from those of the individual words (e.g., rain cats and dogs, see the light).	$1, 2, 3, 4, 5, 6$
Spatial	You will be asked to watch a short video of the Cups and Balls magic trick, and identify the location of the ball at the end of the trick.	$1, 2, 3, 4, 5, 6$
Sports	You will be given US college basketball teams, and asked to predict which round they will make it to in the NCAA Tournament, taking place in March 2022?	$1, 2$
Sports	You will be given upcoming soccer games, and asked to predict the games' outcome?	$3, 4, 5$
Sports	You will be given upcoming sport events (soccer and tennis games), and asked to predict the games' outcome?	6

Table A.13: Survey Material

Questions used in the liquid democracy survey. Note that different prediction questions were used for different experiments; this is simply because predicted outcomes were realized between the running of experiments. The questions in Knowledge, Popular Culture, and Spatial Reasoning relied on audio-visual documents that we can share upon request.

Task	Prompt	Answer
Landmark	This landmark is located in Italy.	False
	This landmark is located in Turkey.	True
	This landmark is located in Myanmar.	False
	This landmark is located in France.	False
	This landmark is located in Brazil.	False
Movie	This music was featured as a theme song in the movie The Hobbit.	False
	This music was featured as a theme song in the movie The Empire of Sun.	False
	This music was featured as a theme song in the movie Gravity.	True
	This music was featured as a theme song in the movie Goodfellas.	False
	This music was featured as a theme song in the movie The Pianist.	False
	This music was featured as a theme song in the movie A Passage through India.	False
	This music was featured as a theme song in the movie The Schindler's List.	True
English	“A man of straw" means “A very active person".	False
	“To drive home" means “To emphasize".	True
	“To smell a rat" means “To suspect foul dealings".	True
	“To end in smoke" means “To excite great applause".	False
	“To catch a tartar" means “To deal with a person who is more than one's match".	False
Prediction for Experiments 1-2	The US college basketball team West Virginia Mountaineers will make it to the Elite Eight in the 2022 NCAA Tournament.	False
	The US college basketball team Michigan State Spartans will make it to the First Round in the 2022 NCAA Tournament.	True
	The US college basketball team Syracuse Orange will win the 2022 NCAA Tournament.	False
	The US college basketball team Purdue Boilermakers will make it to the 2nd round in the 2022 NCAA Tournament.	True
	The US college basketball team Arizona Wildcats will make it to the Elite Eight in the 2022 NCAA Tournament.	False
Prediction for Experiments 3-5	Galatasaray SK will beat FC Barcelona during the Europa League game on March 17th.	False
	Olympic de Marseille and OGC Nice will tie during the French League game on March 20th.	False
	VFL Wolfsburg will beat Bayer 04 Leverkusen during the German League game on March 20th.	False
	Salernitana will lose against Juventus during the Italian League game on March 20th.	True
	FC Barcelona and Real Madrid CF will tie during the Spanish League game on March 20th.	False
Prediction for Experiment 6	Eintracht Frankfurt will beat FC Barcelona during the Europa League game on April 7th.	False
	Olympic de Lyon and West Ham United will tie during the Europa League game on April 7th.	True
	Brazil will lose to Spain during the Women's International Friendly game on April 7th.	False
	Neither Rafael Nadal nor Novak Djokovic will qualify for the ATP Masters 1000 Monte Carlo Final on April 17th.	NA
	Stefanos Tsitsipas will win the ATP Masters 1000 Monte Carlo Tournament on April 17th.	NA
Spatial Reasoning	The object is located in the middle cup at the end of the trick.	False
	The object is located in the middle cup at the end of the trick.	False
	The object is located in the right cup at the end of the trick.	True

A.8.5 Delegation Metrics

We collected 505 delegation data points, one per participant per task. Of those, 28% were delegations (like delegators A, B, C, G in Section 4.1) and 57% are direct participants that did not receive any delegation besides their own (like delegate E in Section 4.1). Among the delegates, 21% received only one delegation besides their own (hence had weight 2 in the decision, like delegate F in Section 4.1), 11% received two delegations besides their own and just about 1% received five or more delegations besides their own, showing little sign of concentration of power.

Next, we look at the delegation graphs across different tasks and experiments. Recall that delegations happened at the task-level so we represent delegation behaviors per task. For each task t and experiment e, we show in Figure A.21 the delegation graphs with all N_e participants in experiments e, represented by nodes labeled by their expertise $\eta_{i,t} = \sum_{i \in [N_e]} v^D_{i,e,t}/R_t$ that is the average number of correct answers given for that task.

The top left plot shows an example of a successful delegation chain to the right, where an expert from experiment 6 and task *landmarks*, with $\eta_{i,t} = 1$ was identified by six other participants either directly or transitively through a local expert j with $\eta_{j,t} = 0.8$. On the right, a smaller chain shows two participants delegating to a more competent expert, who in turn delegates to a non-expert.

Over the course of the six experiments based on five tasks each, we observed only two delegation cycles of size two (where A delegates to B, who delegates to A), both in Experiment 3 with $N_3 = 32$.

In the pre-surevy, only 27% of the tasks were delegated, which is much less than during the main study (delegation rate was 47% there).

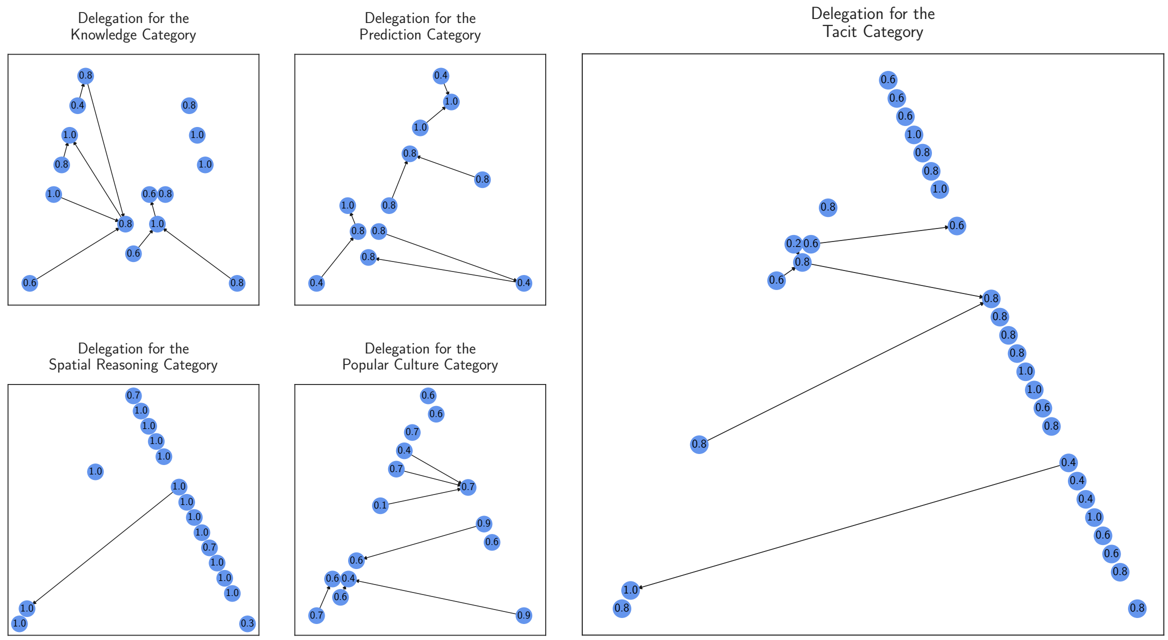


Figure A.21: Delegation Graphs for the Different Categories

Each graph represents the performance of a group on a given category. Recall that delegations happen at the task-level, so that one computes the average expertise across tasks $\eta_{i,t}$. Each node represent a participant i in the experiment e and the numbers within the node indicate their expertise $\eta_{i,t}$ on task t. An out-arrow from A to B indicates that A delegated to B. From left to right, top to bottom: Knowledge in Experiment 6, Prediction in Experiment 2, Tacit in Experiment 3, Spatial Reasoning in Experiment 5 and Popular Category in in Experiment 6. Knowledge stands for Landmarks, Prediction for Sport, Popular Culture for Movies and Tacit for English (these were the denominations used in the original work by Simoiu et al. [215].

A.8.6 Estimating the q function: Probability of Delegating as a Function of Expertise

Next, we estimate the q function that models probability of delegating as a function expertise, following the specifications in Equation (4.1) to assess the function overall and across each task respectively. The results, shown in Table A.14 for the overall effect and the effects per task.

We also report the q function estimated with the bucketing procedure described below,

Table A.14: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

The summary of the average and heterogeneous effects, estimated by the models in Equations (4.1) and (4.2) for the four main tasks conducted in each experiment. Recall that the prediction Sport tasks were different across different experiments while the others were constant. We indicate whether fixed effects were used (t stands for fixed effects at the task level, e at the experiment level and i at the individual level). We further indicate whether robust clustered standard errors were used to account for correlation within individuals i's answers.

	Overall Model	Tasks Models						
		(Tasks)						
		(T_1)	(T_3)	(T_2)	(T_4)	(T_5)	(T_6)	(T_7)
Effect Size β^q	-2.45^{****} (0.50)	-4.44^{**} (2.24)	0.02 (1.69)	-5.11^{***} (1.50)	0.48 (2.28)	-3.52 (2.19)	-3.10^{**} (1.34)	-3.11 (3.81)
Fixed Effects	NA	NA	NA	NA	NA	NA	NA	NA
Clustered S.E.	i	NA	NA	NA	NA	NA	NA	NA
Note:						*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001		

where we average the number of delegations per bucket and display, as in the main study, strong decreasing trends.

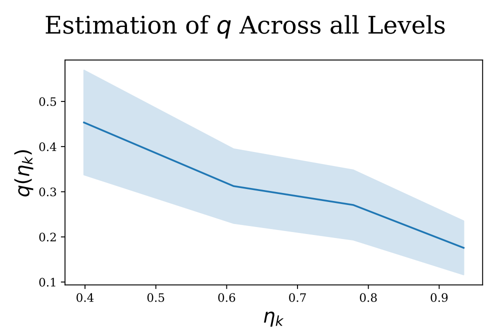


Figure A.22: Estimation of q using k-means clustering buckets

A.8.7 Estimating the φ function: Delegation Choice as a Function of Expertise

Herein, we estimate φ with $B = 4$ buckets estimated with the k-means procedure and show the estimation in Figure A.25.

A.8.7.1 k-means clustering

We find an optimal number of clusters equal to 4 (that is the number of clusters at which the decay in within the sum of standard errors flattens as estimated by the kneedle algorithm). The resulting centroids are $0.40, 0.61, 0.78$ and 0.94, and the intervals span are, respectively, $c_1 = [0.00, 0.50], c_2 = [0.51, 0.69], c_3 = [0.70, 0.84]$ and $c_4 = [0.89, 1.00]$. There are respectively $15\%, 26\%, 27\%$ and 33% of the data points in each cluster. We show below the resulting clustering and the loss curve.

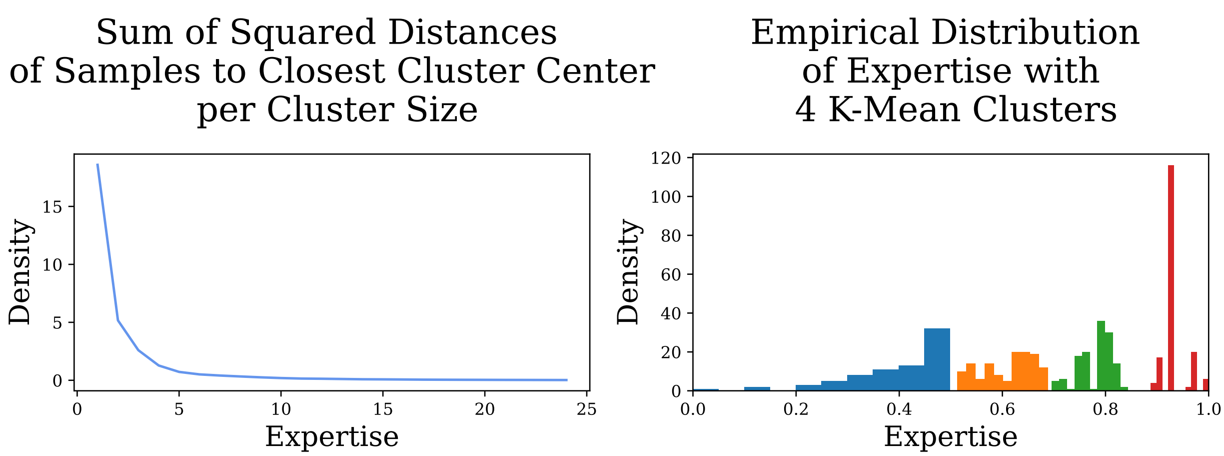


Figure A.23: Ouput of the k-means clustering procedure

A.8.7.2 Estimation of φ with k-means clustering bucketing

To test the significance of the trends observed in Figure A.25, we run the hypothesis testing specified by Section 4.3.4.3 and show the results in Table A.15. fixed effects at the task level,

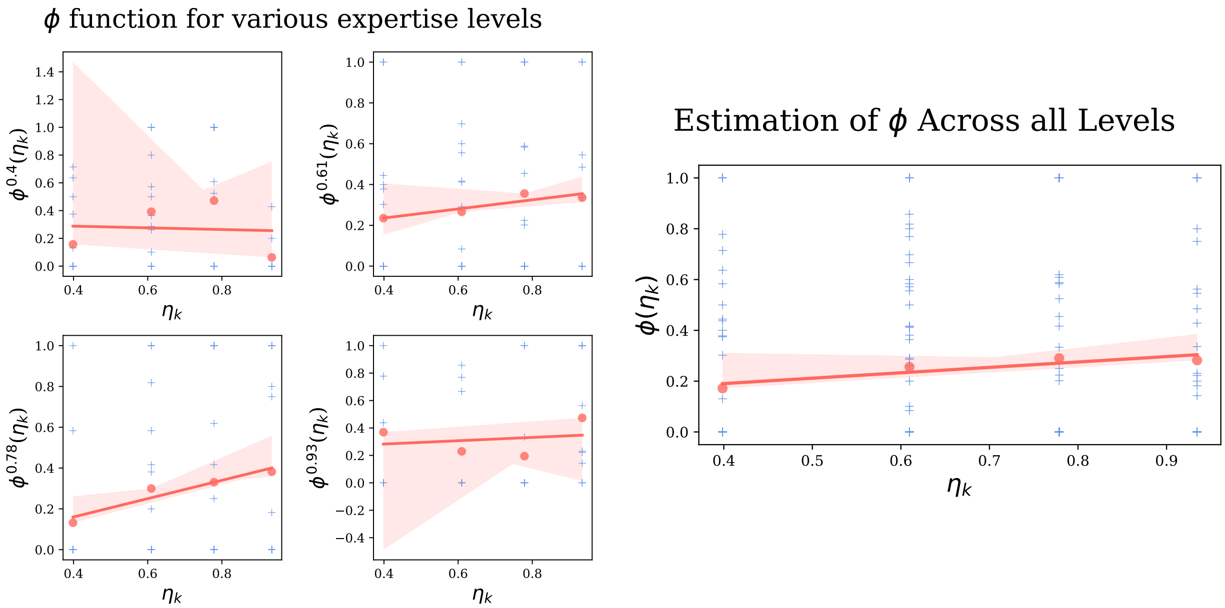


Figure A.24: Estimates of φ

Each of four plots on the left represents the values of $\varphi_{l,k}^{e,t}$ for a fixed type ℓ. The blue crosses show the values computed for $\varphi_{l,k}^{e,t}$ at each possible η_k. The pink dots show the average across all $\varphi_{l,k}^{e,t}$ at a level η_k, and the pink line corresponds to a linear regression over the mean values. We observe increasing trends across the board, with slope (coefficient of determination) being $-0.05(0.005), 0.18(0.81), 0.39(0.91)$ and $0.10(0.05)$ respectively. The plot on the right shows the values for $\varphi_{l,k}^{e,t}$ across all ℓ. The linear fits outputs a slope of 0.21 (coefficient of determination: 0.81).

e at the experiment level and η_ℓ at the bucket level)

Table A.15: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

	Overall	For fixed ℓ			
		c_1	c_2	c_3	c_4
Correlation	0.099*	−0.009	0.079	0.20*	0.19
P-value	8×10^{-1}	0.93	0.46	9×10^{-1}	0.11
Note:	*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001				

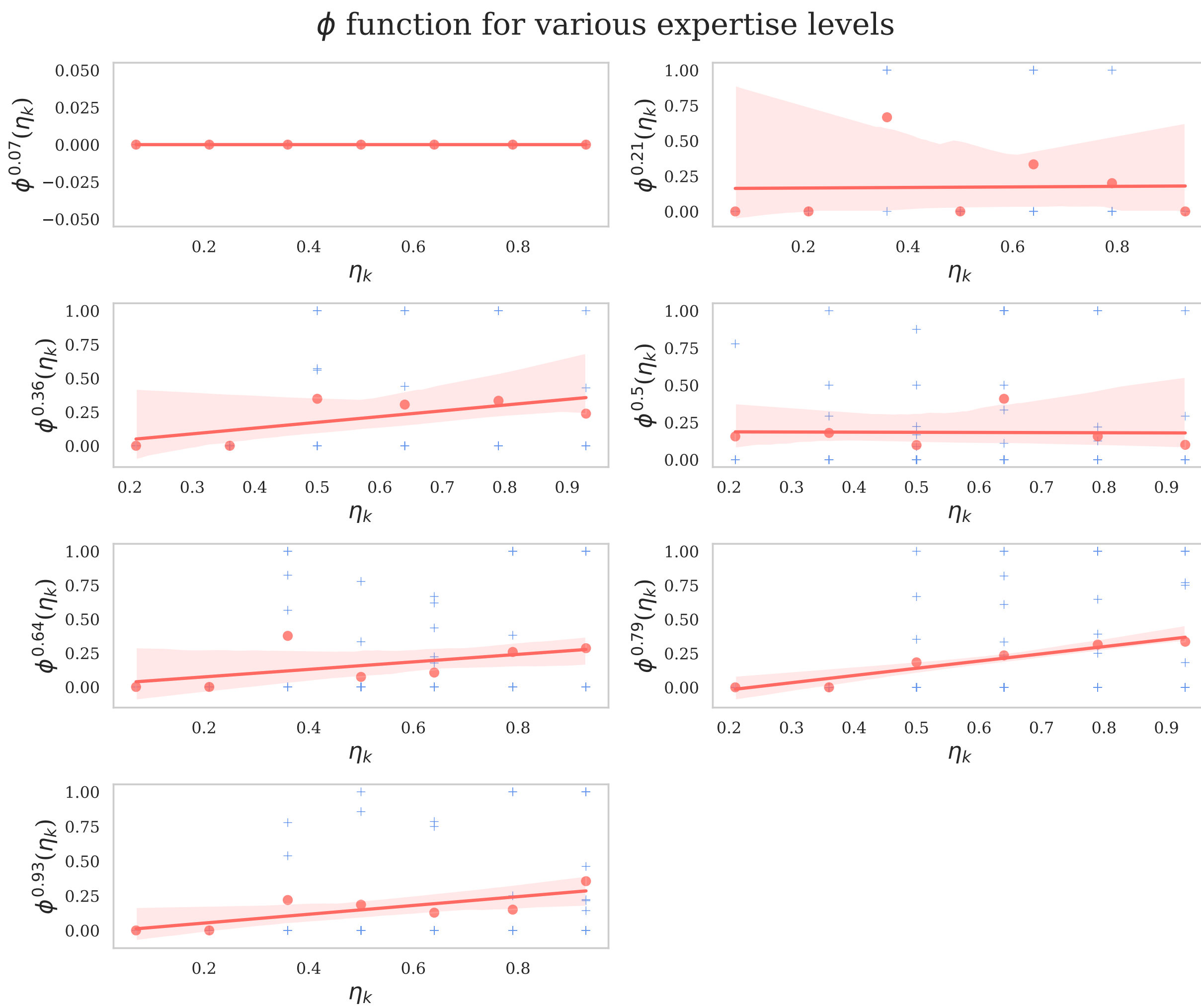


Figure A.25: Estimates of φ

Each plot represents the values of $\varphi^{e,t}_{l,k}$ for a fixed type ℓ. The blue crosses show the values computed for $\varphi^{e,t}_{l,k}$ at each possible η_k. The pink dots show the average across all $\varphi^{e,t}_{l,k}$ at a level η_k, and the pink line corresponds to a linear regression over the mean values. We observe increasing trends across the board, but the slopes seem rather small.

A.8.7.3 Estimation of φ with equal cut and $B = 7$

To test the significance of the trends observed in Figure A.25, we run the models specified by Section 4.3.4.3 and show the results in Table A.16.

We observe that φ is increasing in its second coordinate both at the aggregate level

Table A.16: Results on the Relation between Delegation Behaviors and Average Expertise or Confidence

	Overall	For fixed ℓ						
		c_1	c_2	c_3	c_4	c_5	c_6	c_7
Correlation	0.19****	0.15	0.20	0.05	0.07	0.35***	0.18*	0.26**
P-value	10^{-5}	0.41	0.18	0.70	0.49	0.0003	0.08	0.04
Note:					*p<0.1; **p<0.05; ***p<0.01; ****p<0.0001			

$\beta^{\varphi} = 0.25$ ($s.e. = 0.095$, $t = 2.69$, $p = 0.007$) and when fixing the first coordinate η_ℓ. Note that this characteristic is consistent with Halpern et al. [111]'s general continuous mechanism.

A.8.8 Core Lemma Desiderata: Concentration of Power and Increase in Average Expertise Due to Delegation

The maximum weights are displayed in Figure A.26. Next, we estimate the average in-

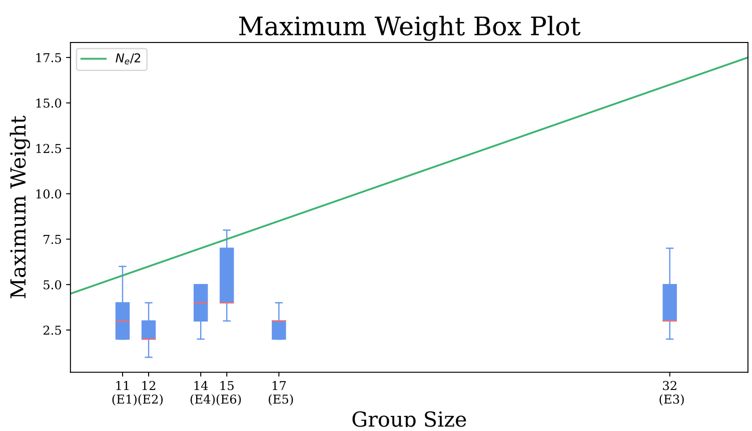


Figure A.26: Maximum Weights

From left to right, a box plot of the maximum weight $m_{e,t}$ for each experiment e ordered in increasing N_e. The box plot represent the variations in $m_{e,t}$ for a fixed e.

crease in expertise post delegation through the model specification Equation (4.3). We find $\beta^{\text{lemma}} = 0.025$ with $s.e. = 0.006$, $t = 4.13$ and $p = 0.00009$. In other words, across all tasks and experiments, the mean average expertise post delegation is 2.5% higher than the mean average expertise without delegation.

A.8.9 Liquid Democracy versus Direct Democracy

Finally, we report the results from the specification in Equation (4.4) with the response variable being the proportion of correct answers per task and experiment. We find $\beta^{\text{LvD}} = 0.00$ with $s.e. = 0.01$, $t = 1$ and $p = 0$. Liquid and direct democracies tended to agree in the pre-experiment.

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