ESSAYS IN ECONOMIC THEORY

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Abstract

Essay One develops a dynamic programming approach to the theory of optimal social insurance with variable retirement. Moral hazard problem is caused by the inability on the part of government to distinguish between those unable to work and those who choose not to work. Discontinuity is inherent in the optimization problem with this type of moral hazard problem. With "Piecewise Regularity, a new concept introduced to deal with the discontinuity problem, the optimal structure of wage and benefit is studied. In addition, a comparative analysis concerning the effect upon the planned retirement date of an increase in the government subsidy to the retirement insurance fund is presented.

In Essay Two similar questions are considered about unemployment insurance, using the same method. The moral hazard problem, however, becomes more complicated since, in this case, workers make multiple participation decisions and are allowed to re-enter the labor market. The analysis shows that net wage should be nondecreasing overtime during each spell of employment and that the unemployment benefit should be nonincreasing during each spell of unemployment.

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Essay One

Moral Hazard and Optimal Retirement Insurance

I. Introduction

In employment related social insurances, such as unemployment insurance or retirement insurance, it is frequently impossible or prohibitively costly for government to distinguish between those who happen to be out of work, being unfavorably affected by nature, and those who choose not to work. In these circumstances, government usually lessens the cost of this moral hazard problem by making its insurance payoffs depend upon the employment histories of the insured. The purpose of this essay is to analyse the optimal features of retirement insurance when government can make insurance benefits depend upon work histories.

In each period, workers are assumed to face uncertainty about their abilities to work. Disability is permanent and, therefore, even those capable are not allowed to re-enter the labor market once they retire. There is no saving, and the only variable controlled by a worker is the date of planned retirement. On the assumption that workers are identical, we consider three questions about the optimal features of retirement insurance : how consumption should be made to vary with age when working, how the initial retirement benefit should depend upon the age of retirement, and what is the effect upon the optimal date of planned retirement of an increase in the government subsidy to the retirement insurance fund.

The first two questions were studied by Diamond and Mirrlees in [1] and [2] under the assumption that moral hazard problem is always effective. The optimal features of retirement insurance were characterized by them as follows : optimal consumption and the initial retirement benefit should increase with age when working such that, in each period, workers are made just indifferent to continued work.

The primary purpose of the present essay is to show that this optimal feature of consumption over time is robust. Under very general assumptions, whether the moral hazard problem is actually effective or not, optimal consumption should be made nondecreasing with age. The actual effectiveness of moral hazard problem only makes this monotonic feature of optimal consumption strict. However, the optimal feature of the initial retirement benefit will be studied under the same assumptions as those of Diamond and Mirrlees' model.

The second purpose of this essay is to present an analysis of the effect upon the optimal date of planned retirement of an increase in the government subsidy to the retirement insurance fund. Under the same assumptions as those of Diamond and Mirrlees' model, we will show that an increase in the government subsidy to the retirement insurance fund tends to induce earlier retirement.

II. The Model

Let instantaneous utilities be specified by the following utility functions,

$u_1(c)$ utility of consumption c when working

$u_2(c)$ utility when able to work but not working

$u_3(c)$ utility when unable to work.

We assume that, at the start of period t, workers (not yet disabled) face permanent disability with probability p_t, and that workers are not allowed to re-enter the labor market once they retire. Any able worker has a marginal product equal to one.

Remaining lifetime expected utilities are specified as follows,

$U_1(x_t;t)$ remaining lifetime expected utility at period t of a worker who works in that period, which is the maximum expected utility attainable (by government social welfare maximization) over the remaining n-t+1 periods with initial resource x_t (including period t output)

$U_2(b_t,\ldots,b_n;t)$ remaining lifetime expected utility at period t of a worker who, though capable, does not work in that period (and therefore in all remaining periods) and consumes $b_t,\ldots,b_n$ over the remaining n-t+1 periods

$U_3(b_t,\ldots,b_n;t)$ remaining lifetime expected utility at period t of a worker who is disable at the start of that period and consumes $b_t,\ldots,b_n$ over the remaining n-t+1 periods.

$U_3(b_t,\ldots,b_n;t)$ is simply expressed as

$$U_3(b_t,\ldots,b_n;t) = u_3(b_t) + \ldots + u_3(b_n). \tag{1}$$

$U_2(b_t,\ldots,b_n;t)$ is recursively defined as

$$\begin{aligned} U_2(b_t,\ldots,b_n;t) \\ &= u_2(b_t) + (1 - p_{t+1})\, U_2(b_{t+1},\ldots,b_n;t+1) \\ &\quad + p_{t+1}\, U_3(b_{t+1},\ldots,b_n;t+1). \end{aligned} \tag{2}$$

For u_2 and u_3 concave and differentiable, U_2 and U_3 are concave and differentiable.

$U_1(x_t;t)$, on the other hand, depends upon whether a worker should continue to work in period t+1 if able, as part of social welfare maximization. Let $v_A(x_t;t)$ denote the remaining lifetime expected utility at period t of a worker who works in both period t and t+1. It is the maximum expected utility of the following constrained maximization problem,

$$v_A(x_t;t) =$$

$$\max \quad u_1(c_t) + (1 - p_{t+1})\, U_1(x_{t+1};t+1) + p_{t+1}\, U_3(b_{t+1},\ldots,b_n;t+1) \tag{3}$$

subject to

$$\text{(i)} \quad c_t + (1 - p_{t+1})(x_{t+1} - 1) + p_{t+1}(b_{t+1} + \ldots + b_n) \leqq x_t \tag{4}$$

$$\text{(ii)} \quad U_1(x_{t+1};t+1) \geqq U_2(b_{t+1},\ldots,b_n;t+1). \tag{5}$$

The maximum is taken over c_t, x_{t+1}, and $b_{t+1}, \ldots, b_n$. The first constraint is the resource constraint and the second is the moral hazard constraint. Similarly, let $v_B(x_t;t)$ denote the remaining lifetime expected utility at period t of a worker who retires in period t+1. It is the maximum expected utility of the following constrained maximization problem,

$$v_B(x_t;t) =$$

$$\max \quad u_1(c_t) + (1 - p_{t+1})\, U_2(b_{t+1},\ldots,b_n;t+1) + p_{t+1}\, U_3(b_{t+1},\ldots,b_n;t+1) \tag{6}$$

subject to $\quad c_t + b_{t+1} + \ldots + b_n \leqq x_t.$ (7)

The maximum is taken over c_t and $b_{t+1}, \ldots, b_n$. $U_1(x_t;t)$ is then defined as the maximum of $v_A(x_t;t)$ and $v_B(x_t;t)$.

$$U_1(x_t;t) = \max \left\{ v_A(x_t;t),\ v_B(x_t;t) \right\}. \tag{8}$$

For $t = n$, $U_1(x_n;n)$ is equal to $u_1(x_n)$.

When the government's initial resource is Y and when we assume that a worker should work at least one period, the n-period retirement insurance model is given by

$$\text{maximize} \quad (1 - p_1)\, U_1(x_1;1) + p_1\, U_3(b_1,\ldots,b_n;1) \tag{9}$$

subject to

$$\text{(i)} \quad (1 - p_1)(x_1 - 1) + p_1(b_1 + \ldots + b_n) \leqq Y \tag{10}$$

$$\text{(ii)} \quad U_1(x_1;1) \geqq U_2(b_1,\ldots,b_n;1). \tag{11}$$

Before we are engaged in the analysis of the recursive system defined above, we should here note the equivalence between the recursive model and the non-recursive model formulated by Diamond and Mirrlees.

First, an optimal retirement insurance scheme for the recursive model, with the date of planned retirement R+1, is specified by a series of optimal solutions,

$$(c_t^*,\ x_{t+1}^*,\ b_{t+1}^*(t+1), \ldots, b_n^*(t+1)),$$

to the constrained maximization problems defined by (3), (4), and (5) for t = 1,...,R-1, and an optimal solution,

$$(c_R^*, b_{R+1}^*(R+1), \ldots, b_n^*(R+1)),$$

to the constrained maximization problem defined by (6) and (7) for t = R. The numbers in the parentheses refer to the date of (voluntary or involuntary) retirement.

	x_1^*	$b_1^*(1)$, $b_2^*(1)$	....		$b_n^*(1)$
c_1^*	x_2^*	$b_2^*(2)$	....		$b_n^*(2)$
....					
c_{R-1}^*	x_R^*	....	$b_R^*(R)$, $b_{R+1}^*(R)$	...	$b_n^*(R)$
c_R^*		....	$b_{R+1}^*(R+1)$	...	$b_n^*(R+1)$

The first row is an optimal solution to the constrained maximization problem defined by (9), (10), and (11). The pair of these optimal solutions for any consecutive two periods are interrelated by the condition that the initial resource x_t in the constrained maximization problem of period t is given at the level x_t^* which is a component of the optimal vector to the constrained maximization problem of period t-1.

Now let's consider the non-recursive system. Let V^{s+1} denote the lifetime expected utility of a worker who plans to retire in period s+1.

$$V^{s+1} =$$

$$\sum_{t=0}^{s} q_{t+1} \left\{ \sum_{r=1}^{t} u_1(c_r) + U_3(b_{t+1}(t+1),\ldots,b_n(t+1);t+1) \right\}$$

$$+ (1 - \sum_{t=0}^{s} q_{t+1}) \left\{ \sum_{r=1}^{s} u_1(c_r) + U_2(b_{s+1}(s+1),\ldots,b_n(s+1);s+1) \right\}$$

where

$$q_{t+1} = (1 - p_1)(1 - p_2) \ldots (1 - p_t)\, p_{t+1} .$$

An optimal retirement insurance scheme for the non-recursive system, with the date of planned retirement R+1, is then maximizing V^{R+1} subject to the resource constraint

$$\sum_{t=0}^{R} q_{t+1} \left(\sum_{r=1}^{t} c_r - t + \sum_{r=t+1}^{n} b_r(t+1) \right)$$

$$+ (1 - \sum_{t=0}^{R} q_{t+1}) \left(\sum_{r=1}^{R} c_r - R + \sum_{r=R+1}^{n} b_r(R+1) \right)$$

$$\leqq Y$$

and the moral hazard constraints

$$V^{R+1} \geqq V^{s+1} \qquad \text{for } s = 0, \ldots, R-1.$$

We start from a given optimal retirement insurance scheme for the non-recursive system

$$\begin{array}{llll}
 & b_1^{**}(1),\ b_2^{**}(1) & \ldots\ldots\ldots\ldots\ldots & b_n^{**}(1) \\
c_1^{**} & b_2^{**}(2) & \ldots\ldots\ldots\ldots\ldots & b_n^{**}(2) \\
\multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
c_{R-1}^{**} & \ldots\ldots\ldots\ldots & b_R^{**}(R),\ b_{R+1}^{**}(R) & \ldots\ b_n^{**}(R) \\
c_R^{**} & \ldots\ldots\ldots\ldots & b_{R+1}^{**}(R+1) & \ldots\ b_n^{**}(R+1)
\end{array}$$

From this optimal insurance scheme, we recursively define auxiliary variables x_1^{**} ,..., x_R^{**} as follows.

$$x_R^{**} = c_R^{**} + \sum_{r=R+1}^{n} b_r^{**}(R+1)$$

$$x_t^{**} = c_t^{**} + (1 - p_{t+1})(x_{t+1}^{**} - 1) + p_{t+1} \sum_{r=t+1}^{n} b_r^{**}(t+1).$$

We prove that the given optimal insurance scheme for the non-recursive system, together with the auxiliary variables, constitutes a feasible insurance scheme for the recursive system. Using the equality

$$\sum_{t=0}^{s} q_{t+1} \sum_{r=1}^{t} (c_r^{**} - 1) = \sum_{t=1}^{s} (c_t^{**} - 1) \sum_{r=t}^{s} q_{r+1} ,$$

we can transform the resource constraint into

$$p_1 \sum_{r=1}^{n} b_r^{**}(1)$$

$$+ \sum_{t=1}^{R} (1 - \sum_{r=0}^{t-1} q_{r+1})(c_t^{**} - 1 + p_{t+1} \sum_{r=t+1}^{n} b_r^{**}(t+1))$$

$$+ (1 - \sum_{r=0}^{R} q_{r+1}) \sum_{r=R+1}^{n} b_r^{**}(R+1)$$

$$= p_1 \sum_{r=1}^{n} b_r^{**}(1) + (1 - p_1)(x_1^{**} - 1)$$

$$\leqq Y.$$

The equality was derived from the definition of x_t^{**}'s and the relationship

$$1 - \sum_{r=0}^{t-1} q_{r+1} = (1 - p_1)(1 - p_2) \ldots (1 - p_t) .$$

The given optimal insurance scheme is maximizing the lifetime expected utility of a worker who plans to retire in period R+1 subject to the resource constraint and the R moral hazard constraints. The variables in the first row of this optimal insurance scheme $(b_1^{**}(1), \ldots, b_n^{**}(1))$ appear on the left-hand side of the moral hazard constraint

$$V^{R+1} \geqq V^1$$

in the form

$$p_1 \; U_3(b_1^{**}(1),\ldots,b_n^{**}(1);1)$$

and on the right-hand side in the form

$$p_1 \; U_3(b_1^{**}(1),\ldots,b_n^{**}(1);1) + (1 - p_1) \; U_2(b_1^{**}(1),\ldots,b_n^{**}(1);1).$$

However, in the remaining R-1 moral hazard constraints

$$V^{R+1} \geqq V^{s+1} \qquad \text{for} \quad s = 1 ,\ldots, R-1$$

they appear on both sides of each constraint exactly in the same term. Therefore, they are cancelled out. The remaining R rows of the optimal insurance matrix, on the other hand, appear only on the left-hand side of the first moral hazard constraint, and on both sides of each of the remaining R-1 moral hazard constraints. This implies that the variables in these remaining R rows must be maximizing the lifetime expected utility V^{R+1} subject to the resource constraint

$$\sum_{t=1}^{R} (1 - \sum_{r=0}^{t-1} q_{r+1})(c_t - 1 + p_{t+1} \sum_{r=t+1}^{n} b_r(t+1))$$

$$+ (1 - \sum_{r=0}^{R} q_{r+1}) \sum_{r=R+1}^{n} b_r(R+1) \leqq (1 - p_1)(x_1^{**} - 1)$$

and the R-1 moral hazard constraints

$$V^{R+1} \geqq V^{s+1} \qquad \text{for} \quad s = 1, \ldots, R-1.$$

This maximum value of the lifetime expected utility is equal to

$$(1 - p_1)\, U_1(x_1^{**};1) + p_1\, U_3(b_1^{**}(1),\ldots,b_n^{**}(1);1)$$

of (9) in its economic sense, although we have not proved the equivalence between the mathematical expressions of these two terms yet.

Now the variables $(c_1^{**},\ b_2^{**}(2), \ldots,\ b_n^{**}(2))$ in the second row of the optimal insurance matrix appear only in the moral hazard constraint

$$V^{R+1} \geqq V^2.$$

The remaining R-1 rows of this matrix appear only on the left-hand side of the above moral hazard constraint and on both sides of each of the remaining R-2 moral hazard constraints. We can therefore repeat the same argument. In this way, we can show that the given optimal retirement insurance scheme for the non-recursive system, together with the auxiliary variables, constitutes a feasible insurance scheme for the recursive system. The equivalence of the mathematical expressions which we have left unsettled is automatically established once we reach the end of this repetition. Since an optimal retirement insurance scheme for the recursive system is always feasible for the non-recursive system, we have thus established the equivalence of these two systems.

III. Remaining Lifetime Expected Utility

The optimal structure of retirement insurance depends upon the nature of remaining lifetime expected utility $U_1(x_t;t)$. In this section, we inquire of properties of that function. The following concept has the primary importance.

Definition (Piecewise Regularity) A real-valued function $f(x)$, defined on an interval $(-d,\infty)$ for a nonnegative number d, is called piecewise regular if $f(x)$ has the following properties,

(i) Monotonicity : $f(x)$ is strictly increasing

(ii) Continuity : $f(x)$ is continuous

(iii) Left- and Right-Hand Derivatives : the right-hand derivative, $f^+(x)$, is at least as great as the left-hand derivative, $f^-(x)$

(iv) End-Point Properties :

$$\lim_{x \to -d} f(x) = -\infty \qquad \lim_{x \to -d} f^+(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \infty .$$

A piecewise regular function which is not differentiable is not concave by property (iii). A differentiable piecewise regular function is called regular. A regular function is not necessarily concave. In this section, we assume regularity and concavity for instantaneous utility functions, and, by induction, derive piecewise regularity for $U_1(x_t;t)$.

Assumption R : u_1, u_2, and u_3 are regular and concave.

We also derive important relationships among instantaneous utilities and remaining lifetime expected utilities in equilibrium. For convenience, we assume $0 < p_t < 1$ for any t. In the following lemmas, we use the notations

$u_1'(c_t) = du_1(c_t)/dc_t$

$U_1^-(x_t;t)$ and $U_1^+(x_t;t)$ are the left- and right-hand derivatives of $U_1(x_t;t)$ with respect to x_t. If $U_1(x_t;t)$ is differentiable at x_t, $U_1'(x_t;t) = dU_1(x_t;t)/dx_t$.

$U_{ij}(b_{t+1},\ldots,b_n;t+1)$ is the partial derivative of $U_i(b_{t+1},\ldots,b_n;t+1)$ with respect to b_j ($i = 2$ and 3, and $j = t+1, \ldots, n$).

Lemma 1 Let $U_1(x_{t+1};t+1)$ be a piecewise regular function. Under Assumption R, $U_1(x_t;t)$ can be expressed in one of the following three forms (labelled Ai, Aii, and B).

(A) When, for a given x_t, a worker should work both in period t and t+1,

$$U_1(x_t;t) = v_A(x_t;t)$$

$$= u_1(c_t^*) + (1 - p_{t+1})\, U_1(x_{t+1}^*;t+1)$$

$$+ p_{t+1}\, U_3(b_{t+1}^*,\ldots,b_n^*;t+1) \qquad (12)$$

where

(i) if the moral hazard constraint (5) is binding,

$$U_1'(x_{t+1}^*;t+1) < u_1'(c_t^*) < U_{3j}(b_{t+1}^*,\ldots,b_n^*;t+1)$$

$$\text{for} \quad j = t+1, \ldots, n \qquad (13)$$

and (c_t^*, x_{t+1}^*, b_{t+1}^*, ..., b_n^*) are obtained by solving the following n-t+2 equations

$$\frac{u_1'}{U_1'} = \frac{p_{t+1}\, U_{3j} + (1 - p_{t+1})\, U_{2j}}{p_{t+1}\, U_1' + (1 - p_{t+1})\, U_{2j}}$$

$$\text{for} \quad j = t+1, \ldots, n \qquad (14)$$

$$U_1(x_{t+1}^*;t+1) = U_2(b_{t+1}^*,\ldots,b_n^*;t+1) \qquad (15)$$

$$c_t^* + (1 - p_{t+1})(x_{t+1}^* - 1) + p_{t+1}(b_{t+1}^* + \ldots + b_n^*)$$

$$= x_t. \qquad (16)$$

(Note that, in general, it is not the case that U_{3j} is constant over j. Note also that this statement implies that $U_1(x_{t+1};t+1)$ is differentiable at x^*_{t+1}).

(ii) if the moral hazard constraint is not binding, $(c^*_t, x^*_{t+1}, b^*_{t+1}, \ldots, b^*_n)$ are obtained by solving the following n-t+1 equations together with the resource constraint (16).

$$U_1'(x^*_{t+1};t+1) = u_1'(c^*_t) = U_{3j}(b^*_{t+1},\ldots,b^*_n;t+1) \quad \text{for } j = t+1, \ldots, n \qquad (17)$$

(B) When, for a given x_t, a worker should retire in period t+1,

$$U_1(x_t;t) = v_B(x_t;t)$$
$$= u_1(c^*_t) + (1 - p_{t+1})\, U_2(b^*_{t+1},\ldots,b^*_n;t+1)$$
$$+ p_{t+1}\, U_3(b^*_{t+1},\ldots,b^*_n;t+1) \qquad (18)$$

where $(c^*_t, b^*_{t+1}, \ldots, b^*_n)$ are derived by solving the following n-t+1 equations

$$u_1'(c^*_t) = (1 - p_{t+1})\, U_{2j}(b^*_{t+1},\ldots,b^*_n;t+1)$$
$$+ p_{t+1}\, U_{3j}(b^*_{t+1},\ldots,b^*_n;t+1) \quad \text{for } j = t+1, \ldots, n \qquad (19)$$

$$c_t^* + b_{t+1}^* + \dots + b_n^* = x_t. \tag{20}$$

Proof :

$U_1(x_t;t)$ is defined as the expected utility of an optimal solution, $(c_t^*, x_{t+1}^*, b_{t+1}^*, \dots, b_n^*)$, to the constrained maximization problem defined by (3), (4), and (5), or that of an optimal solution, $(c_t^*, b_{t+1}^*, \dots, b_n^*)$, to the constrained maximization problem defined by (6) and (7). The only complication in the proof of the lemma is the differentiability of $U_1(x_{t+1};t+1)$ in case A. We first derive the equilibrium relationships among c_t^* and $b_{t+1}^*, \dots, b_n^*$ for case A (both (i) and (ii)), keeping x_{t+1} fixed at an optimal level. We can then apply the Kuhn-Tucker theorem. The first-order conditions are given by

$$u_1' = r \tag{21}$$

$$U_{3j} = r + (s/p_{t+1})\, U_{2j} \qquad \text{for } j = t+1, \dots, n \tag{22}$$

where r and s are the Lagrange multipliers corresponding to the resource constraint (4) and the moral hazard constraint (5) respectively.

We shall now consider the effect upon the total expected utility (3) of an increase in x_{t+1}, keeping the resource constraint and the moral hazard constraint. When the moral hazard constraint is binding at $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*)$, the deviations in $(c_t, b_{t+1}, \ldots, b_n)$ compensating an increase in x_{t+1} must satisfy the equations

$$dc_t + (1 - p_{t+1})\, dx_{t+1} + p_{t+1}(db_{t+1} + \ldots + db_n) = 0 \qquad (23)$$

$$U_1^+\, dx_{t+1} = U_{2t+1}\, db_{t+1} + \ldots + U_{2n}\, db_n \qquad (24)$$

The effect upon the total expected utility is given by

$$dV = u_1'\, dc_t + (1 - p_{t+1})\, U_1^+\, dx_{t+1}$$
$$+ p_{t+1}(U_{3t+1}\, db_{t+1} + \ldots + U_{3n}\, db_n) \qquad (25)$$

Multiplying the left-hand side of (23) by r, using (21) and (22), and subtracting the resulting expression from (25),

$$dV = (1 - p_{t+1})(U_1^+ - r)\, dx_{t+1} + p_{t+1} \sum_{j=t+1}^{n} (U_{3j} - r)\, db_j$$

$$= (1 - p_{t+1})(U_1^+ - r)\, dx_{t+1} + s \sum_{j=t+1}^{n} U_{2j}\, db_j \qquad (26)$$

By (24) and (26), dV is then expressed as

$$dV = (1 - p_{t+1})(U_1^+ - r)\, dx_{t+1} + s\, U_1^+\, dx_{t+1}$$
$$= ((1 - p_{t+1}) + s)\, U_1^+\, dx_{t+1} - (1 - p_{t+1})\, r\, dx_{t+1} \qquad (27)$$

This must be nonpositive and, therefore,

$$U_1^+ \leqq \frac{1 - p_{t+1}}{(1 - p_{t+1}) + s} \; r \tag{28}$$

since dx_{t+1} is positive.

When x_{t+1} is reduced from x^*_{t+1} by $dx_{t+1} < 0$, the corresponding effect upon the total expected utility (3) is obtained by simply replacing U_1^+ in (27) by U_1^- .

$$dV = ((1 - p_{t+1}) + s) \; U_1^- \, dx_{t+1} - (1 - p_{t+1}) \; r \; dx_{t+1}.$$

Since dx_{t+1} is negative, $dV \leqq 0$ implies

$$U_1^- \geqq \frac{1 - p_{t+1}}{(1 - p_{t+1}) + s} \; r \tag{29}$$

Combining (28) with (29) and the piecewise regularity of $U_1(x_{t+1};t+1)$, $U_1(x_{t+1};t+1)$ becomes differentiable at x^*_{t+1} and

$$U_1' = \frac{1 - p_{t+1}}{(1 - p_{t+1}) + s} \; r. \tag{30}$$

This equation holds whether s is positive or zero. When s is positive, we obtain, from (21) , (22), and (30), the inequalities

$$U_1' < u_1' < U_{3j}' . \qquad \text{for} \quad j = t+1 , \ldots , n$$

Solving (30) for s,

$$s = \frac{(1 - p_{t+1})(r - U_1')}{U_1'} . \tag{31}$$

Substituting r and s into (22),

$$U_{3j} = u_1' + \frac{1 - p_{t+1}}{p_{t+1}} \quad \frac{(u_1' - U_1')\, U_{2j}}{U_1'}$$

or

$$\frac{u_1'}{U_1'} = \frac{p_{t+1}\, U_{3j} + (1 - p_{t+1})\, U_{2j}}{p_{t+1}\, U_1' + (1 - p_{t+1})\, U_{2j}} .$$

When the moral hazard constraint is not binding at $(c_t^*, x_{t+1}^*, b_{t+1}^* , \ldots, b_n^*)$, (21), (22), and (30) imply

$$U_1' = u_1' = U_{3j} .$$

This completes the analysis of case A. There is no complication in analysing case B since $U_1(x_{t+1};t+1)$ does not enter the expression for $U_1(x_t;t)$ in (18).

In order to apply Lemma 1 for t = 1 ,..., n-1, we derive the piecewise regularity of $U_1(x_t;t)$ by induction.

$U_1(x_n;n) = u_1(x_n)$ is piecewise regular by the regularity of u_1. A function $f(x) = \max\left\{g(x), h(x)\right\}$ is piecewise regular if g(x) and h(x) are both piecewise regular. In (8), $v_B(x_t;t)$ is a regular function since it is the optimum value of the ordinary constrained maximization problem, (6) and (7). It remains to show the piecewise regularity of $v_A(x_t;t)$.

Lemma 2 If $U_1(x_{t+1};t+1)$ is a piecewise regular function, the optimum expected utility, $v_A(x_t;t)$, of the constrained maximization problem defined by (3), (4), and (5) is also a piecewise regular function of x_t. In addition,

$$v_A^-(x_t;t) = \min\left\{u_1'(c_t^*) \,\middle|\, (c_t^*, x_{t+1}^*, b_{t+1}^*, \dots, b_n^*) \in E^A(x_t;t)\right\}$$

$$v_A^+(x_t;t) = \max\left\{u_1'(c_t^*) \,\middle|\, (c_t^*, x_{t+1}^*, b_{t+1}^*, \dots, b_n^*) \in E^A(x_t;t)\right\} \tag{32}$$

where $E^A(x_t;t)$ is the set of optimal solutions to the constrained maximization problem. $v_A^-(x_t;t)$ and $v_A^+(x_t;t)$ denote the left- and right-hand derivatives of $v_A(x_t;t)$.

Remark : The relevance of the conditions in (32) is explained as follows. If we can derive the piecewise regularity of $U_1(x_t;t)$, for $t = 1, \dots, n$, by induction, we obtain, by Lemma 1, a series of optimal conditions (13) or (17) or (19) for optimal retirement insurance. But the pair of these optimal conditions of any consecutive two periods cannot be linked together, unless $u_1'(c_t^*)$ in (13), (17), and (19) and $U_1'(x_t;t)$ are interrelated to each other. (32) provides us with this interrelation. It can be interpreted as a kind of the envelope theorem.

Proof :

Before we derive the four conditions of the piecewise regularity, we must show that there always exists an optimal solution to the constrained maximization problem.

Non-emptiness of $E^A(x_t;t)$: Let us suppose $(-d,\infty)$, $d > 0$, to be the domain of $U_1(x_{t+1};t+1)$. Let $F(x_t;t)$ be the feasible set of the constrained maximization problem. $F(x_t;t)$ is the set of vectors $(c_t, x_{t+1}, b_{t+1}, \dots, b_n)$ which satisfy the resource constraint, the moral hazard constraint, and the boundary constraints

$$c_t \geqq 0, \quad x_{t+1} \geqq -d, \quad b_j \geqq 0 \qquad (j = t+1, \dots, n)$$

$F(x_t;t)$ is not empty since we can construct a feasible solution $(\bar{c}_t, \bar{x}_{t+1}, \bar{b}_{t+1}, \dots, \bar{b}_n)$ as follows. Let $\bar{x}_{t+1}$ be an arbitrary number in the interval $(-d, x_t + (1 - p_{t+1}))$. We then choose $(\bar{b}_{t+1}, \dots, \bar{b}_n)$ sufficiently small so that the moral hazard constraint and the inequality

$$(1 - p_{t+1})\, \bar{x}_{t+1} + p_{t+1}(\bar{b}_{t+1} + \dots + \bar{b}_n) < x_t + (1 - p_{t+1})$$

are both satisfied. Finally we set $\bar{c}_t$ as

$$\bar{c}_t = x_t - (1 - p_{t+1})(\bar{x}_{t+1} - 1) - p_{t+1}(\bar{b}_{t+1} + \dots + \bar{b}_n).$$

Let $\bar{U}$ be the expected utility of $(\bar{c}_t, \bar{x}_{t+1}, \bar{b}_{t+1}, \dots, \bar{b}_n)$ and define $K(x_t;t)$ as the set of feasible solutions

$(c_t, x_{t+1}, b_{t+1}, \ldots, b_n)$ such that

$$u_1(c_t) + (1 - p_{t+1}) U_1(x_{t+1}; t+1) + p_{t+1} U_3(b_{t+1}, \ldots, b_n; t+1)$$
$$\geqq \bar{U} .$$

$K(x_t; t)$ is a compact set and does not contain any boundary point. Therefore, $u_1(c_t)$, $U_1(x_{t+1}; t+1)$, and $U_3(b_{t+1}, \ldots, b_n; t+1)$ are all continuous in $K(x_t; t)$, and there must be an optimal solution.

We shall now derive the four conditions of the piecewise regularity.

Monotonicity of $v_A(x_t; t)$: This is obvious from the nature of the constrained maximization problem.

Continuity : Since $v_A(x_t; t)$ is the maximum of a continuous function over a well-behaved constraint set, it is clearly continuous.

Left- and Right-Hand Derivatives : Let $\bar{u}$ and $\bar{\bar{u}}$ be defined as

$$\bar{u} = \min \left\{ u_1'(c_t^*) \,\middle|\, (c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t; t) \right\}$$
$$\bar{\bar{u}} = \max \left\{ u_1'(c_t^*) \,\middle|\, (c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t; t) \right\}$$

We first prove that

$$\bar{\bar{u}} = \lim_{k \to \infty} \inf_{0 < h < 1/k} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h}$$

$$= \lim_{k \to \infty} \sup_{0 < h < 1/k} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h}$$

The following properties are derived from the definition of the optimum expected utility $v_A(x_t;t)$.

1° For any x_t, if $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t;t)$, then $(c_t^* + h, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*)$ is feasible for $x_t + h$. Therefore, when $|h|$ is sufficiently small,

$$v_A(x_t + h;t) - v_A(x_t;t) \geqq u_1'(c_t^*)\, h + o(h) \qquad (33)$$

where $o(h)$ is a term such that $o(h)/h \to 0$ as $h \to 0$. In addition, dividing both sides of (33) by h,

$$\frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \geqq u_1'(c_t^*) + o(h)/h \qquad (h > 0)$$

(34)

and

$$\frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \leqq u_1'(c_t^*) + o(h)/h \qquad (h < 0)$$

(35)

2^o Let $E^A(x_t + h_k;t)$ be the set of optimal solutions to the constrained maximization problem corresponding to $x_t + h_k$, and $(c_t^{(k)}, x_{t+1}^{(k)}, b_{t+1}^{(k)}, \ldots, b_n^{(k)}) \varepsilon E^A(x_t + h_k;t)$. Then the sequence of vectors

$$(c_t^{(k)}, x_{t+1}^{(k)}, b_{t+1}^{(k)}, \ldots, b_n^{(k)}) \qquad (h_k \to 0)$$

contains a convergent subsequence whose limit vector $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*)$ is contained in $E^A(x_t;t)$.

In 2^o, the existence of a convergent subsequence is shown by the compactness of $K(x_t;t)$, which we constructed in the proof of the non-emptiness of $E^A(x_t;t)$, and the monotonicity of $v_A(x_t + h_k;t)$ with respect to h_k. It is easily shown that the limit vector of a sequence of optimal vectors corresponding to $x_t + h_k$ is optimal for x_t.

From Property 1^o, we can immediately derive the inequality

$$\bar{\bar{u}} \leqq \lim_{k \to \infty} \inf_{0<h<1/k} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \tag{36}$$

We shall now prove the inequality

$$\bar{\bar{u}} \geqq \lim_{k \to \infty} \sup_{0<h<1/k} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \tag{37}$$

Let us suppose the contrary. Then there must exist a sequence of positive numbers $\{h_k\} \to 0$ such that

$$\frac{v_A(x_t + h_k;t) - v_A(x_t;t)}{h_k} \geqq \bar{\bar{u}} + a \tag{38}$$

for a positive constant a. We can choose $\{h_k\}$ such that, for each k, there exists an optimal vector corresponding to $x_t + h_k$, $(c_t^{(k)}, x_{t+1}^{(k)}, b_{t+1}^{(k)}, \ldots, b_n^{(k)})$, which satisfies the inequality

$$u_1'(c_t^{(k)}) \geqq \bar{\bar{u}} + a. \tag{39}$$

If we can choose such a sequence, then, by Property 2^o, we can construct a convergent subsequence of these optimal vectors whose limit $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*)$ is contained in $E^A(x_t;t)$. But this contradicts the definition of $\bar{\bar{u}}$ since

$$u_1'(c_t^*) \geqq \bar{\bar{u}} + a.$$

It remains to show that we can actually choose a sequence $\{h_k\} \to 0$ which has the required properties. Let h_k be a positive number for which (38) is satisfied but (39) is not. That is

$$u_1'(c_t^*) < \bar{\bar{u}} + a$$

for any $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t + h_k;t)$

Then, by Property 1°,

$$\frac{v_A(x_t + h;t) - v_A(x_t + h_k;t)}{h - h_k} \leqq u_1'(c_t^*) + o(h-h_k)/(h-h_k)$$

$$< \bar{\bar{u}} + a + o(h-h_k)/(h-h_k)$$

$$\text{for } h < h_k \qquad (40)$$

Let us consider the graph of $v_A(x_t + h;t)$ in fig.1.

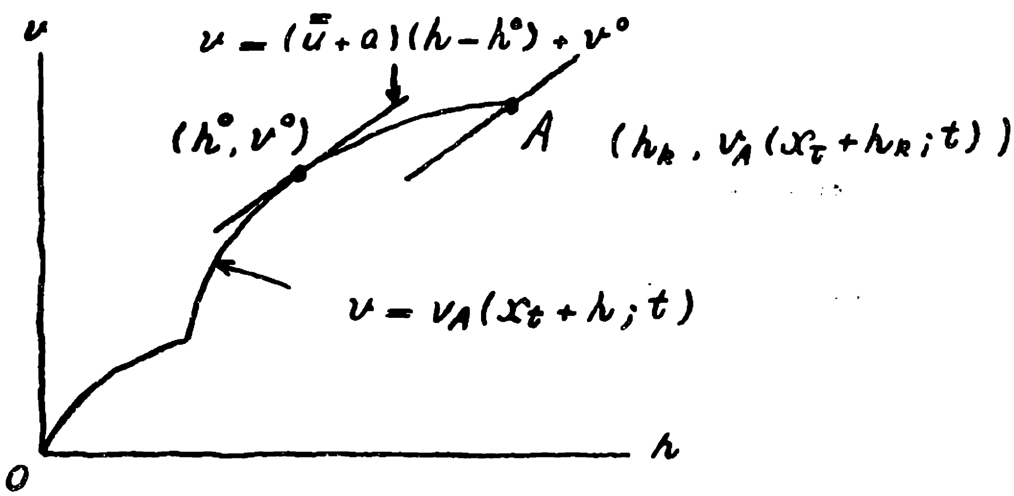


fig. 1.

We take $(x_t, v_A(x_t;t))$ as the origin, and draw the curve $v = v_A(x_t + h;t)$ in h - v coordinates. Inequality (40) then says that, if we draw a straight line passing through point A with a slope equal to $\bar{\bar{u}} + a$, there is a point on the curve $v = v_A(x_t + h;t)$ between A and the origin which lies above that straight line. Since the curve $v = v_A(x_t + h;t)$ is continuous, we can choose point (h°, v°) on the curve such that the straight line passing through (h^o, v^o)

$$v = (\bar{\bar{u}} + a)(h - h^o) + v^o$$

supports the curve at (h^o, v^o) over the interval $(0, h_k)$. In other words,

$$v_A(x_t + h;t) \leqq (\bar{\bar{u}} + a)(h - h^o) + v^o \tag{41}$$

for any h such that $0 \leqq h \leqq h_k$. Dividing both sides of (41) by $h - h^o < 0$, we obtain

$$\frac{v_A(x_t + h;t) - v_A(x_t + h^o;t)}{h - h^o} \geqq \bar{\bar{u}} + a. \tag{42}$$

On the other hand, by Property 1^o,

$$\frac{v_A(x_t + h;t) - v_A(x_t + h^o;t)}{h - h^o} \leqq u_1'(c_t^*) + o(h-h^o)/(h-h^o) \tag{43}$$

for any $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t + h^o;t)$. Combining (42) with (43), we obtain the desired inequality

$$u_1'(c_t^*) \geqq \bar{\bar{u}} + a$$

for $(c_t^*, x_{t+1}^*, b_{t+1}^*, \ldots, b_n^*) \in E^A(x_t + h^o;t)$. Therefore, if we redefine h_k by h^o, h_k satisfies the required condition.

We have shown the existence of the right-hand derivative,

$v_A^+(x_t;t)$, and the equality

$$v_A^+(x_t;t) = \bar{\bar{u}}.$$

The existence of the left-hand derivative, $v_A^-(x_t;t)$, can similarly be shown. First, Property 1^o implies

$$\bar{u} \geqq \lim_{k \to \infty} \sup_{-1/k < h < 0} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \qquad (44)$$

The inequality

$$\bar{u} \leqq \lim_{k \to \infty} \inf_{-1/k < h < 0} \frac{v_A(x_t + h;t) - v_A(x_t;t)}{h} \qquad (45)$$

must also be satisfied. Otherwise we can construct a convergent sequence of optimal vectors whose limit violates the definition of $\bar{u}$. (44) and (45) then imply the existence of the left-hand derivative, which is equal to $\bar{u}$.

The third requirement of the piecewise regularity that the right-hand derivative is not less than the left-hand derivative then follows from the definition of $\bar{u}$ and $\bar{\bar{u}}$.

End-Point Properties : This condition is also obvious from the nature of the constrained maximization problem. When $U_1(x_{t+1};t+1)$ is defined upon $(-d, \infty)$, the domain of $U_1(x_t;t)$ is derived from the resource constraint by substituting $c_t = b_{t+1} = \ldots = b_n = 0$ and $x_{t+1} = -d$. The lower end of the domain of $U_1(x_t;t)$ is $x_t = -(1 - p_{t+1})(d + 1)$.

Q. E. D.

IV. The Structure of Optimal Consumption

In this section, we consider the implications of Lemma 1 and 2 of the previous section for the structure of optimal consumption, $(c_1^*, c_2^*, \ldots, c_{R-1}^*, c_R^*)$, over time. In the next section, we study the same problem under the assumption that the moral hazard problem actually exists. The relationship between the initial retirement benefit and the age of retirement will be discussed in Section VI. In the final section, we will analyse the effect upon the optimal date of planned retirement of an increase in the government subsidy to the retirement insurance fund, Y. We will also consider the optimal relationship between x_{t+1}^* and the sum of $b_{t+1}^*(t+1), \ldots, b_n^*(t+1)$.

Theorem 1 If instantaneous utility functions are regular and concave, under optimal retirement insurance,

(i) consumption should be made nondecreasing with age when working, and, in addition,

(ii) workers are just indifferent to continued work while consumption is increasing, and consumption is constant during the periods when workers prefer to continue work if able.

Proof :

By Lemma 1, $(c^*_{t-1}, x^*_t, b^*_t(t), \ldots, b^*_n(t))$ and $(c^*_t, x^*_{t+1}, b^*_{t+1}(t+1), \ldots, b^*_n(t+1))$ must satisfy either (13) or (17).

$$U_1'(x^*_t; t) \leqq u_1'(c^*_{t-1}) \tag{46}$$

$$U_1'(x^*_{t+1}; t+1) \leqq u_1'(c^*_t). \tag{47}$$

In this pair of inequalities, $U_1'(x^*_t;t)$ in (46) and $u_1'(c^*_t)$ in (47) are interrelated by the conditions

$$U_1(x^*_t; t) = \max\left\{v_A(x^*_t; t),\ v_B(x^*_t; t)\right\}$$

and

$$v_A^-(x^*_t; t) \leqq u_1'(c^*_t) \leqq v_A^+(x^*_t; t). \tag{48}$$

The second inequality is obtained from (32) of Lemma 2.

When a worker is working $(t < R)$, $v_A(x^*_t;t)$ is at least as great as $v_B(x^*_t;t)$. If $v_A(x^*_t;t)$ is strictly greater than $v_B(x^*_t;t)$,

$$U_1^-(x^*_t; t) = v_A^-(x^*_t; t) \qquad \text{and} \qquad U_1^+(x^*_t; t) = v_A^+(x^*_t; t).$$

But $U_1(x_t;t)$ is differentiable at x^*_t. Therefore

$$U_1'(x^*_t; t) = v_A'(x^*_t; t) = u_1'(c^*_t).$$

If $v_A(x_t^*;t)$ and $v_B(x_t^*;t)$ are equal,

$$U_1^-(x_t^*;t) = \min \left\{ v_A^-(x_t^*;t),\ v_B'(x_t^*;t) \right\}$$

$$U_1^+(x_t^*;t) = \max \left\{ v_A^+(x_t^*;t),\ v_B'(x_t^*;t) \right\}$$

since $v_B(x_t;t)$ is always differentiable. Therefore, the differentiability of $U_1(x_t;t)$ at x_t^* again implies

$$U_1'(x_t^*;t) = v_A'(x_t^*;t) = v_B'(x_t^*;t) = u_1'(c_t^*). \qquad (50)$$

Now substituting $U_1'(x_t^*;t) = u_1'(c_t^*)$ into (46), we obtain the desired inequality

$$u_1'(c_t^*) \leqq u_1'(c_{t-1}^*).$$

That is $c_t^* \geqq c_{t-1}^*$.

If consumption is increasing with age (i.e., c_t^* is greater than c_{t-1}^*), only (13) is relevant instead of (46). This is the case when the moral hazard constraint is binding, that is

$$U_1(x_t^*;t) = U_2(b_t^*(t),\ldots,b_n^*(t);t).$$

A worker is just indifferent to continued work.

Similarly, if a worker prefers to continue work when able, only (17) is relevant instead of (46). In this case, (49) or (50) implies that $u_1'(c_t^*) = u_1'(c_{t-1}^*)$.

V. Moral Hazard Condition

We have shown in Theorem 1 that workers are just indifferent to continued work while consumption is increasing. In their studies of social insurance with variable retirement, Diamond and Mirrlees characterized the optimal structure of consumption that consumption should increase with age when working if the moral hazard problem is actually effective. They then proposed a sufficient condition for the moral hazard problem to be effective, which they called moral hazard condition.

Definition (Moral Hazard Condition) u_1, u_2, and u_3 are said to satisfy moral hazard condition when

$$u_1'(c_1) = u_3'(c_2) \quad \text{implies} \quad u_1(c_1) < u_2(c_2) \tag{51}$$

In our terminology, moral hazard problem exists when the moral hazard constraint (5) is binding. If the moral hazard constraint is binding, the corresponding Lagrange multiplier s is positive and, therefore, the optimal solution must satisfy (13). From the analysis in the previous section, this implies that $c^*_{t-1} < c^*_t$. The only difference between this statement and that of Theorem 1 is that indifference to continued work does not necessarily mean the existence of moral hazard problem.

In this section, we prove that consumption should actually be made to increase with age when working if the moral hazard problem is effective.

Assumption M : Instantaneous utility functions satisfy the moral hazard condition (51).

The moral hazard condition (51) is a special case of the following condition for t = n.

$$U_1^+(x_t;t) = U_{3j}(b,\dots,b;t) \qquad (j = t, \dots, n)$$

$$\text{imply} \qquad U_1(x_t;t) < U_2(b,\dots,b;t). \tag{52}$$

For convenience, we state as a lemma an immediate implication of this condition for an optimal solution to the constrained maximization problem defined by (3), (4), and (5).

Lemma 3 When $U_1(x_{t+1};t+1)$, $U_2(b_{t+1},\dots,b_n;t+1)$ and $U_3(b_{t+1},\dots,b_n;t+1)$ satisfy condition (52), the moral hazard constraint (5) is binding and, therefore, an optimal solution must satisfy (13).

It remains to derive moral hazard condition (52) for remaining lifetime expected utility functions by induction from the moral hazard condition of instantaneous utility functions.

For this purpose, we need an additional assumption.

Assumption U : $u_2'(c) \leqq u_3'(c)$ for any c .

This assumption is satisfied, for instance, if the disutility of disability can be expressed in monetary term as

$$u_3(c) = u_2(c - l)$$

where l is the monetary cost of disability.

Now we prove the following lemma.

Lemma 4 Let $U_1(x_{t+1};t+1)$, $U_2(b_{t+1},\ldots,b_n;t+1)$, and $U_3(b_{t+1},\ldots,b_n;t+1)$ satisfy the moral hazard condition (52), and u_2 and u_3 Assumption U. Then $U_1(x_t;t)$, $U_2(b_t,\ldots,b_n;t)$, and $U_3(b_t,\ldots,b_n;t)$ also satisfy the moral hazard condition.

Proof :

Let us suppose x_t and b to satisfy

$$U_1^+(x_t;t) = U_{3j}(b,\ldots,b;t) \qquad (j = t ,\ldots, n) \tag{53}$$

From (32) of Lemma 2, there exists an optimal solution $(c_t^*, x_{t+1}^*, b_{t+1}^* ,\ldots, b_n^*)$ which satisfies the equality

$$U_1^+(x_t;t) = u_1'(c_t^*). \tag{54}$$

or, if $v_A(x_t;t) < v_B(x_t;t)$, there exists an optimal solution

$(c_t^*, b_{t+1}^* , \ldots, b_n^*)$ which satisfies (54).

In the first case,

$$U_1(x_t;t) = u_1(c_t^*) + (1 - p_{t+1}) U_1(x_{t+1}^*;t+1) + p_{t+1} U_3(b_{t+1}^*,\ldots,b_n^*;t+1) \qquad (55)$$

where, by Lemma 3,

$$U_1'(x_{t+1}^*;t+1) < u_1'(c_t^*) < U_{3j}(b_{t+1}^*,\ldots,b_n^*;t+1) \quad \text{for} \quad j = t+1 ,\ldots, n \qquad (56)$$

Then from (53) and (54)

$$u_1'(c_t^*) = U_{3t}(b,\ldots,b;t) = u_3'(b), \qquad (57)$$

and from (53), (54), and (56)

$$U_{3j}(b_{t+1}^*,\ldots,b_n^*;t+1) > U_{3j}(b,\ldots,b;t) \quad \text{for} \quad j = t+1 ,\ldots, n \qquad (58)$$

By Assumption M, (57) implies

$$u_1(c_t^*) < u_2(b). \qquad (59)$$

On the other hand, (58) implies

$$b_j^* < b \qquad \text{for} \quad j = t+1 ,\ldots, n.$$

Therefore, combining these results together with (55), we obtain

$$U_2(b,\ldots,b;t) = u_2(b) + (1 - p_{t+1})\, U_2(b,\ldots,b;t+1) + p_{t+1}\, U_3(b,\ldots,b;t+1)$$

$$> u_1(c_t^*) + (1 - p_{t+1})\, U_2(b_{t+1}^*,\ldots,b_n^*;t+1) + p_{t+1}\, U_3(b_{t+1}^*,\ldots,b_n^*;t+1)$$

$$= u_1(c_t^*) + (1 - p_{t+1})\, U_1(x_{t+1}^*;t+1) + p_{t+1}\, U_3(b_{t+1}^*,\ldots,b_n^*;t+1)$$

$$= U_1(x_t;t).$$

In the second case,

$$U_1(x_t;t) = u_1(c_t^*) + (1 - p_{t+1})\, U_2(b_{t+1}^*,\ldots,b_n^*;t+1) + p_{t+1}\, U_3(b_{t+1}^*,\ldots,b_n^*;t+1) \qquad (60)$$

where

$$u_1'(c_t^*) = (1 - p_{t+1})\, U_{2j}(b_{t+1}^*,\ldots,b_n^*;t+1) + p_{t+1}\, U_{3j}(b_{t+1}^*,\ldots,b_n^*;t+1)$$

$$\text{for} \quad j = t+1, \ldots, n \qquad (61)$$

Since $U_{2j}(b_{t+1}^*,\ldots,b_n^*;t+1)$ is a weighted average of $u_2'(b_j^*)$

and $u_3'(b_j^*)$, (61) implies, by Assumption U,

$$u_1'(c_t^*) \leqq U_{3j}(b_{t+1}^*, \ldots, b_n^*; t+1) = u_3'(b_j^*). \tag{62}$$

Combining (53), (54), and (62), we obtain the inequalities

$$b_j^* \leqq b \qquad \text{for} \quad j = t+1, \ldots, n.$$

Therefore, from (60) and (59),

$$\begin{aligned} U_1(x_t; t) &= u_1(c_t^*) + (1 - p_{t+1})\, U_2(b_{t+1}^*, \ldots, b_n^*; t+1) \\ &\qquad + p_{t+1}\, U_3(b_{t+1}^*, \ldots, b_n^*; t+1) \\ &< u_2(b) + (1 - p_{t+1})\, U_2(b, \ldots, b; t+1) \\ &\qquad + p_{t+1}\, U_3(b, \ldots, b; t+1) \\ &= U_2(b, \ldots, b; t). \end{aligned}$$

Q. E. D.

Combining Lemma 3 and 4, we have shown

Theorem 2 If instantaneous utility functions satisfy Assumption R, M, and U, then consumption should be made to increase with age when working under optimal retirement insurance.

VI. The Structure of Optimal Retirement Benefit

In the rest of this essay, we analyse the special case where the difference between instantaneous utilities when a worker is unable to work and when capable but not working is constant and independent of consumption.

Assumption U' : $u_2'(c) = u_3'(c)$ for any c .

In this case, the retirement benefit for a worker who retires in period t, $b_j^*(t)$, is constant over j but the level $b_j^*(t) = b^*(t)$ depends upon the age of retirement t. The question is how the retirement benefit should depend upon the age of retirement. In this section, we consider this question under Assumption R, M, and U'.

For convenience, let us introduce two notations for constant terms

$$\bar{z}(t) = (n-t+1)\, u_2(b) - U_2(b,\ldots,b;t) \tag{63}$$

$$\underline{z}(t) = U_2(b,\ldots,b;t) - (n-t+1)\, u_3(b) \tag{64}$$

$(n-t+1)\, u_2(b)$ and $(n-t+1)\, u_3(b)$ are the maximum and minimum remaining lifetime utilities of a worker who retires in period t and receives retirement benefit b over the remaining (n-t+1) periods. $\bar{z}(t)$ is the difference between this maximum utility and the remaining lifetime expected utility of a worker who is capable but retires in period t. Similarly, $\underline{z}(t)$ is the difference between the remaining lifetime expected utility and the minimum lifetime utility. $\bar{z}(t)$ and $\underline{z}(t)$ are both independent of retirement benefit b. $\bar{z}(t)$ and $\underline{z}(t)$ have the following relationship

$$\bar{z}(t) = \bar{z}(t+1) + p_{t+1}\, \underline{z}(t+1). \tag{65}$$

This is derived from the definition of $U_2(b,\ldots,b;t)$ as follows

$$\begin{aligned}
\bar{z}(t) &= (n-t+1)\, u_2(b) - U_2(b,\ldots,b;t) \\
&= (n-t+1)\, u_2(b) - \Big\{ u_2(b) + (1 - p_{t+1})\, U_2(b,\ldots,b;t+1) \\
&\qquad + p_{t+1}\, U_3(b,\ldots,b;t+1) \Big\} \\
&= (n-t)\, u_2(b) - U_2(b,\ldots,b;t+1) + p_{t+1}\, \underline{z}(t+1) \\
&= \bar{z}(t+1) + p_{t+1}\, \underline{z}(t+1).
\end{aligned}$$

Since the moral hazard constraint is binding for $t = 1, \ldots, R$, a worker is just indifferent to continued work.

$$\begin{aligned}
U_1(x_t^*;t) &= U_2(b^*(t),\ldots,b^*(t);t) \\
&= (n-t+1)\, u_2(b^*(t)) - \bar{z}(t).
\end{aligned} \tag{66}$$

For $t = 1, \ldots, R-1$, $U_1(x_t^*;t)$ is written as

$$U_1(x_t^*;t) = u_1(c_t^*) + (1 - p_{t+1})\, U_1(x_{t+1}^*;t+1)$$
$$+ p_{t+1}\, U_3(b^*(t+1),\ldots,b^*(t+1);t+1)$$
$$= u_1(c_t^*) + (1 - p_{t+1}) \left\{ (n-t)\, u_2(b^*(t+1)) - \bar{z}(t+1) \right\}$$
$$+ p_{t+1} \left\{ (n-t)\, u_2(b^*(t+1)) - \bar{z}(t+1) - \underline{z}(t+1) \right\}$$

The last equality was derived from the definition

$$U_3(b,\ldots,b;t+1) = (n-t)\, u_3(b),$$

together with (63) and (64). Therefore, using (65),

$$U_1(x_t^*;t) = u_1(c_t^*) + (n-t)\, u_2(b^*(t+1)) - \bar{z}(t). \qquad (67)$$

Similarly, $U_1(x_t^*;t)$ is written for $t = R$ as

$$U_1(x_R^*;R) = u_1(c_R^*) + (1 - p_{R+1})\, U_2(b^*(R+1),\ldots,b^*(R+1);R+1)$$
$$+ p_{R+1}\, U_3(b^*(R+1),\ldots,b^*(R+1);R+1)$$
$$= u_1(c_R^*) + (n-R)\, u_2(b^*(R+1)) - \bar{z}(R)$$

which is expressed as a special case of (67).

If we compare (66) with (67), it will immediately be recognized that

$$u_1(c_t^*) - u_2(b^*(t)) = (n-t) \left\{ u_2(b^*(t)) - u_2(b^*(t+1)) \right\} \qquad (68)$$

This is a direct implication of the equality between the remaining lifetime expected utilities of a worker when he retires in period t and when he continues work. That is, if a worker is just indifferent to continued work but the instantaneous utility is less when working than when he retires, then he must expect to receive a higher retirement benefit by postponing his retirement.

We shall now prove

Lemma 5 Under Assumption R, M, and U', an optimal solution $(c_1^*, \ldots, c_R^*, x_1^*, \ldots, x_R^*, b^*(1), \ldots, b^*(R+1))$ satisfy

$$U_1(x_t^*;t) > (n-t+1)\, u_1(c_t^*) - \bar{z}(t) \tag{69}$$

$$U_2(b^*(t),\ldots,b^*(t);t) = (n-t+1)\, u_2(b^*(t)) - \bar{z}(t) \tag{70}$$

Remark : The equality between $U_1(x_t^*;t)$ and $U_2(b^*(t),\ldots,b^*(t);t)$, together with (69) and (70), then implies

$$u_1(c_t^*) < u_2(b^*(t)). \tag{71}$$

Proof :

(70) is the definition of $\bar{z}(t)$ itself, and we need only to prove (69). For t = R, $(c_R^*, b^*(R+1))$ must satisfy (19).

$$u_1'(c_R^*) = (1 - p_{R+1})\, U_{2j}(b^*(R+1),\ldots,b^*(R+1);R+1)$$

$$+ p_{R+1}\, U_{3j}(b^*(R+1),\ldots,b^*(R+1);R+1)$$

$$= u_3'(b^*(R+1)).$$

By the moral hazard condition, this implies

$$u_1(c_R^*) < u_2(b^*(R+1)).$$

Then (69) follows from (67) for t = R.

Let us assume that (69) holds for t+1. Then (71) also holds for t+1. Therefore, from (67), (71), and the relationship between c_t^* and c_{t+1}^*, we obtain

$$U_1(x_t^*;t) = u_1(c_t^*) + (n-t)\, u_2(b^*(t+1)) - \bar{z}(t)$$

$$> u_1(c_t^*) + (n-t)\, u_1(c_{t+1}^*) - \bar{z}(t)$$

$$> (n-t+1)\, u_1(c_t^*) - \bar{z}(t).$$

Thus we have proved

Theorem 3 Under Assumption R, M, and U', the retirement benefit should increase with the age of retirement.

VII. Optimal Date of Planned Retirement and Government Subsidy

In this section, we analyse the effect upon the optimal date of planned retirement of an increase in the government subsidy to the retirement insurance fund, Y. We consider this question in the special case which we discussed in the previous section.

For convenience, we change the order of maximization in the definition of $U_1(x_t;t)$. Let $x_{t+1}(y)$ and $b_{t+1}(y)$ be the unique solution to the pair of equations

$$U_1(x_{t+1};t+1) = (n-t)\, u_2(b) - \bar{z}(t+1) \tag{72}$$

$$(1 - p_{t+1})(x_{t+1} - 1) + p_{t+1}\,(n-t)\, b = y. \tag{73}$$

We must note that we changed the meaning of the subscript of $b_{t+1}(y)$. From now on, the subscript of $b_{t+1}(y)$ refers to the date of retirement by which the level of the retirement benefit is determined. Previously we used the subscript of b_{t+1} to refer to the date of consumption. The remaining subscripts do not change their meaning.

(72) and (73) are the moral hazard constraint and the resource constraint, respectively, with $y = x_t - c_t$ being fixed. Since

$$(1 - p_{t+1})\, U_1(x_{t+1};t+1) + p_{t+1}\, U_3(b,\ldots,b;t+1)$$

$$= U_1(x_{t+1};t+1) - p_{t+1}\, \underline{z}(t+1) \qquad (74)$$

when the moral hazard constraint is binding, the remaining lifetime expected utility of a worker who works both in period t and t+1 can be written as

$$v_A(x_t^*;t) = u_1(c_t^*) + U_1(x_{t+1}^*;t+1) - p_{t+1}\, \underline{z}(t+1)$$

$$= u_1(c_t^*) + U_1\left[x_{t+1}(x_t^* - c_t^*);t+1\right] - p_{t+1}\, \underline{z}(t+1)$$

$$= \max_c \; u_1(c) + U_1\left[x_{t+1}(x_t^* - c);t+1\right]$$

$$- p_{t+1}\, \underline{z}(t+1). \qquad (75)$$

Similarly, from (63), (64), and (65),

$$(1 - p_{t+1})\, U_2(b,\ldots,b;t+1) + p_{t+1}\, U_3(b,\ldots,b;t+1)$$

$$= U_2(b,\ldots,b;t+1) - p_{t+1}\, \underline{z}(t+1)$$

$$= (n-t)\, u_2(b) - \bar{z}(t). \qquad (76)$$

Therefore the remaining lifetime expected utility of a worker who retires in period t+1 can be written as

$$v_B(x_t^*;t) = u_1(c_t^*) + (n-t)\, u_2(b^*(t+1)) - \bar{z}(t)$$

$$= u_1(c_t^*) + (n-t)\, u_2\left(\frac{x_t^* - c_t^*}{n-t}\right) - \bar{z}(t)$$

$$= \max_{c} \; u_1(c) + (n-t)\, u_2\left(\frac{x_t^* - c}{n-t}\right) - \bar{z}(t) \qquad (77)$$

Let us define

$$V(y;t+1) = \max \left\{ U_1\left[x_{t+1}(y);t+1\right],\; (n-t)\, u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \right\} \qquad (78)$$

Then, changing the order of maximization and using (65), $U_1(x_t^*;t)$ can be written as

$$U_1(x_t^*;t) = \max \left\{ v_A(x_t^*;t),\; v_B(x_t^*;t) \right\}$$

$$= \max_{c} \; u_1(c) + V(x_t^* - c;t+1) - p_{t+1}\, \underline{z}(t+1)$$

Rewriting $U_1(x_t^*;t)$ in this way, we can now analyse the effect upon the optimal date of planned retirement of an increase in the government subsidy to the retirement insurance fund. When the optimal date of planned retirement is R+1, the corresponding optimal insurance scheme $(c_1^*, \ldots, c_R^*, x_1^*, \ldots, x_R^*, b^*(1), \ldots, b^*(R+1))$ must satisfy

$$V(x_t^* - c_t^*;t+1)$$

$$= \max \left\{ U_1\left[x_{t+1}(x_t^* - c_t^*);t+1\right],\; (n-t)\, u_2\left(\frac{x_t^* - c_t^*}{n-t}\right) - \bar{z}(t+1) \right\}$$

$$= U_1\left[x_{t+1}(x_t^* - c_t^*);t+1\right]$$

$$= U_1(x^*_{t+1};t+1) \qquad \text{for} \quad t = 1, \ldots, R-1 \tag{79}$$

and

$$V(x^*_R - c^*_R;R+1)$$

$$= \max\left\{U_1\left[x_{R+1}(x^*_R - c^*_R);R+1\right],\ (n-R)\ u_2\left(\frac{x^*_R - c^*_R}{n-R}\right) - \bar{z}(R+1)\right\}$$

$$= (n-R)\ u_2\left(\frac{x^*_R - c^*_R}{n-R}\right) - \bar{z}(R+1)$$

$$= (n-R)\ u_2(b^*(R+1)) - \bar{z}(R+1) \tag{80}$$

It should be noted that an optimal insurance scheme for a given date of optimal planned retirement is unique. This is because function $x_{t+1}(y)$ uniquely determines the value of x^*_{t+1} for the given value of $x^*_t - c^*_t$ and c^*_t must satisfy the inequality

$$U_1^-(x^*_t;t) \leqq u_1'(c^*_t) \leqq U_1^+(x^*_t;t).$$

Since $U_1(x_t;t)$ is differentiable at x^*_t, c^*_t must be unique for the given value of x^*_t. The initial value of x^*_t for $t = 1$ is determined by $x^*_1 = x_1(Y)$ where Y is the government subsidy to the retirement insurance fund.

The importance of (79) and (80) is that work-retirement decision depends upon the relative values of

$$U_1\left[x_{t+1}(x^*_t - c^*_t);t+1\right] \qquad \text{and} \qquad (n-t)\ u_2\left(\frac{x^*_t - c^*_t}{n-t}\right) - \bar{z}(t+1)$$

Our first two propositions concern the critical value $x_t^* - c_t^*$ of this work-retirement decision.

Lemma 6

$$U_1[x_{t+1}(y);t+1] \geqq (n-t)\, u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \qquad (81)$$

if and only if

$$U_1(1+y;t+1) \geqq (n-t)\, u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \qquad (82)$$

Proof :

We defined $x_{t+1}(y)$ and $b_{t+1}(y)$ as the unique solution to the pair of equations (72) and (73). $(1+y,\ y/(n-t))$ is also feasible for the resource constraint (73), and, therefore, $x_{t+1}(y) \leqq 1+y$ if and only if $b_{t+1}(y) \geqq y/(n-t)$. Comparing (81) with (72), (81) holds if and only if $b_{t+1}(y)$ is at least as great as $y/(n-t)$. Comparing (82) with (72), (82) is satisfied if and only if $x_{t+1}(y)$ is not greater than $1+y$. Therefore (81) is equivalent to (82).

Lemma 7 There exists at most one value of y which satisfies the equality

$$U_1(1+y;t+1) = (n-t)\, u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \qquad (83)$$

In addition, if $\bar{y}_{t+1}$ is the value which satisfies this equality, then

$$U_1(1+y;t+1) > (n-t)\ u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \qquad \text{for} \quad y < \bar{y}_{t+1}$$

and

$$U_1(1+y;t+1) < (n-t)\ u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \qquad \text{for} \quad y > \bar{y}_{t+1}$$

Proof :

This is a direct implication of the moral hazard condition of remaining lifetime expected utilities. Under Assumption U', the moral hazard condition is simplified to

$$U_1^+(x_{t+1};t+1) = u_2'(b)$$

implies

$$U_1(x_{t+1};t+1) < (n-t)\ u_2(b) - \bar{z}(t+1). \tag{84}$$

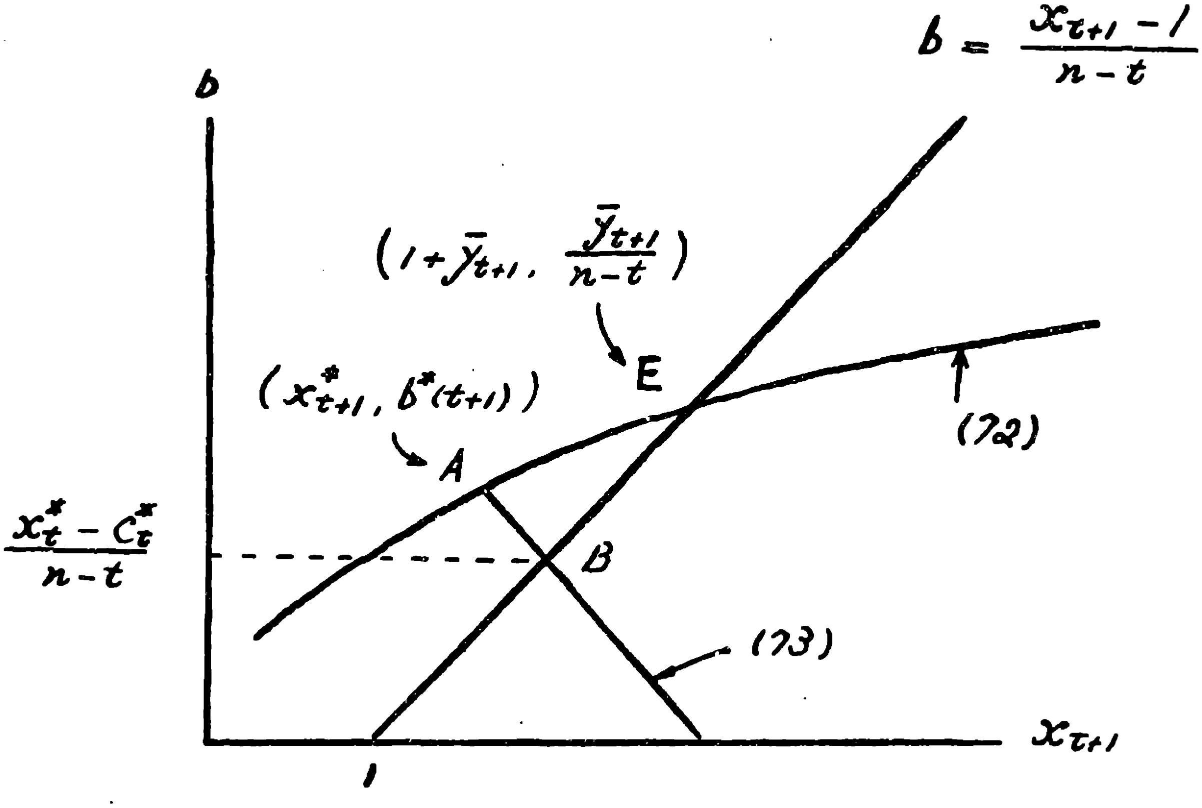
Since u_2 is concave, this is equivalent to

$$U_1(x_{t+1};t+1) = (n-t)\ u_2(b) - \bar{z}(t+1)$$

implies

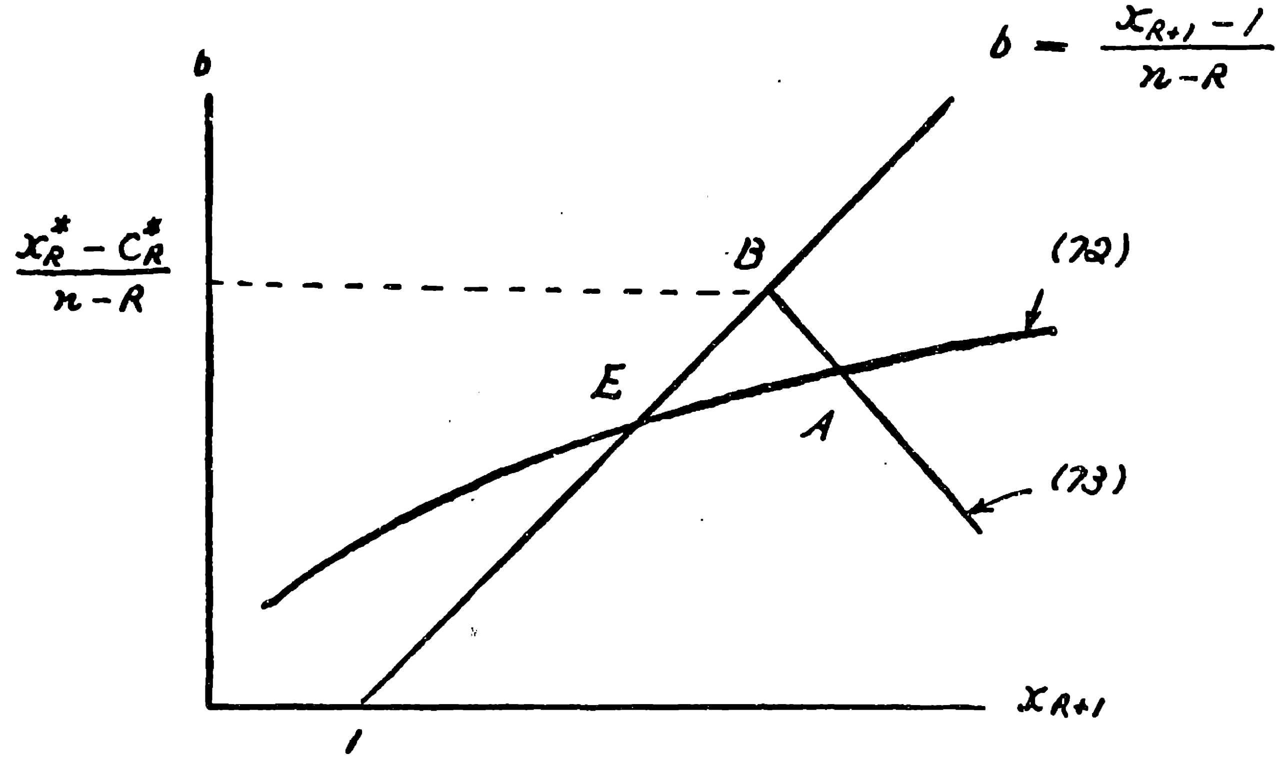
$$U_1^+(x_{t+1};t+1) < u_2'(b). \tag{85}$$

The first derivative of the expression on the right-hand side of (83) with respect to y is $u_2'(y/(n-t))$. Therefore, this



$b = \frac{x_{t+1} - 1}{n - t}$
b
$(1 + \bar{y}_{t+1}, \frac{\bar{y}_{t+1}}{n-t})$
E
$(x^*_{t+1}, b^*(t+1))$
A
(72)
$\frac{x^*_t - C^*_t}{n-t}$
B
(73)
x_{t+1}
1

(fig. 2)



$b = \frac{x_{R+1} - 1}{n - R}$
b
$\frac{x^*_R - C^*_R}{n-R}$
B
(72)
E
A
(73)
x_{R+1}
1

(fig. 3)

equivalent form of the moral hazard condition says that the curve $u = U_1(1+y;t+1)$ intersects the curve

$$u = (n-t)\ u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1) \tag{86}$$

at most once and always cuts from above.

$\bar{y}_{t+1}$ is the critical value of $x_t^* - c_t^*$ which determines whether a worker should work or retire in period t+1. From Lemma 6 and 7, a worker should work if $x_t^* - c_t^* \leqq \bar{y}_{t+1}$ and retire if $x_t^* - c_t^* > \bar{y}_{t+1}$.

Let us consider this problem in a diagram. In fig. 2, the determination of x_{t+1}^* and $b^*(t+1)$ are portrayed. The straight line (73) is the resource constraint line, and the curve (72) is the moral hazard constraint curve. Above this curve, the remaining lifetime expected utility of a worker is higher when he retires in period t+1. The straight line $b = (x_{t+1} - 1)/(n-t)$ is the locus of points where transfer incomes for those who retire in period t+1, (n-t) b, and for those who continue work, $x_{t+1} - 1$, are equal. On this straight line, there is no transfer of income from those who continue work to those who retire, and any worker is unconditionally subsidized the amount equal to the initial resource at the start of this period.

The critical value $\bar{y}_{t+1}$ is given at point E where the moral hazard constraint curve intersects the equal transfer income line. It is the value of initial resource which, if given unconditionally to any worker, leaves workers just indifferent to continued work. B is the point of intersection of the resource constraint line and the equal transfer income line. The vertical coordinate of this point therefore gives a (n-t) th of the initial resource, $x_t^* - c_t^*$. Lemma 7 says that point B lies below the moral hazard constraint curve if and only if B is below point E on the equal transfer income line.

Point A is the point of intersection of the resource constraint line and the moral hazard constraint curve. Work-retirement decision depends upon the position of A relative to that of B, since their vertical coordinates indicate the remaining lifetime expected utilities for those who continue work and those who retire. Lemma 6 says that point A lies above point B on the resource constraint line, if and only if B is below the moral hazard constraint curve.

Combining Lemma 6 and 7, work-retirement decision depends upon the position of the resource constraint line relative to point E. The situation for period t+1 ($t \leqq R-1$) is portrayed in fig.2, and that for period R+1 is portrayed in fig.3.

If, therefore, the initial resource at the start of period t+1 ($t \leqq R-1$), $x_t^* - c_t^*$, is significantly raised as a result of an increase in the government subsidy to the retirement insurance fund, it is likely to induce workers to retire before period R+1. More formally, let us define R+1 and R'+1 as the optimal dates of planned retirement before and after the increase in the government subsidy. The corresponding optimal insurance schemes are denoted by

$$(c_1^* ,\dots, c_R^*, x_1^* ,\dots, x_R^*, b^*(1) ,\dots, b^*(R+1))$$

$$(c_1^{**} ,\dots, c_{R'}^{**}, x_1^{**} ,\dots, x_{R'}^{**}, b^{**}(1) ,\dots, b^{**}(R'+1))$$

If

$$x_t^* - c_t^* < x_t^{**} - c_t^{**} \qquad \text{for} \quad t \leqq R' \tag{87}$$

then R' must not be greater than R. This is so because, if $R' \geqq R$, the analysis of the diagrams shows that workers certainly retire in period R+1 even after the increase in the government subsidy. Since $\bar{y}_{t+1}$ may not exist for $t \leqq R-1$ and the increase in the government subsidy may not be large enough, (87) does not necessarily imply an earlier retirement. But at least the direction of the impact upon the optimal date of planned retirement is unambiguous. In the rest of this essay, we prove the proposition in (87).

When the government subsidy is increased from Y to Y', the analysis of the diagram shows that

$$x_1^* = x_1(Y) < x_1(Y') = x_1^{**}. \tag{88}$$

Similarly,

$$x_t^* - c_t^* < x_t^{**} - c_t^{**}$$

implies

$$x_{t+1}^* = x_{t+1}(x_t^* - c_t^*) < x_{t+1}(x_t^{**} - c_t^{**}) = x_{t+1}^{**} \tag{89}$$

$$b^*(t+1) = b_{t+1}(x_t^* - c_t^*) < b_{t+1}(x_t^{**} - c_t^{**}) = b^{**}(t+1) \tag{90}$$

Therefore, it remains to show that

$$x_t^* < x_t^{**} \qquad \text{implies} \qquad x_t^* - c_t^* < x_t^{**} - c_t^{**} \tag{91}$$

Let $c_t(x_t)$ be defined by

$$\begin{aligned} u_1(c_t(x_t)) + V(x_t - c_t(x_t); t+1) - \bar{z}(t+1) \\ = \max_c \; u_1(c) + V(x_t - c; t+1) - \bar{z}(t+1) \\ = U_1(x_t; t). \end{aligned} \tag{92}$$

When there exist multiple solutions, we choose the optimal solution that maximizes $u_1'(c_t)$ among them. With this notation, c_t^* and c_t^{**} are written as

$$c_t^* = c_t(x_t^*) \qquad \text{and} \qquad c_t^{**} = c_t(x_t^{**}).$$

We shall prove

Lemma 8 Let us assume that $u_1(c)$ is regular and concave, and that $V(y;t+1)$ is a piecewise regular function of y which is differentiable except for at most finite number of points and, when differentiable, has negative second derivative (i.e., concave on each piece). Then

(i) $x_t - c_t(x_t)$ is increasing in x_t, and

(ii) $U_1(x_t;t)$ is also differentiable except for at most finite number of points and, when differentiable, has negative second derivative.

Proof :

First, we have to find the optimality condition for the maximization problem defined by (92). Since a deviation from the optimal solution $c_t(x_t)$ in either direction would reduce or leave unchanged the value of the objective function,

$$u_1'(c_t(x_t)) - V^-(x_t - c_t(x_t);t+1) \leqq 0$$

$$- u_1'(c_t(x_t)) + V^+(x_t - c_t(x_t);t+1) \leqq 0$$

As before, V^- and V^+ denote the left- and right-hand derivatives. Since V is piecewise regular, these two inequalities imply

$$u_1'(c_t(x_t)) = V'(x_t - c_t(x_t); t+1). \tag{93}$$

where V' denotes the first derivative of V with respect to y.

Now we consider the behavior of $c_t(x_t)$ as we increase x_t continuously from x_t^* to x_t^{**}. When $c_t(x_t)$ is the unique solution to the maximization problem for $x_t = x_t^o$, we can analyse the behavior of $c_t(x_t)$ in some small neighborhood of x_t^o by simply differentiating the optimality condition. This is possible since V is differentiable except for at most finite numher of points and, therefore, always differentiable in a small neighborhood of a differentiable point. Totally differentiating the optimality condition (93),

$$\frac{d\ c_t(x_t^o)}{dx_t} = \frac{V''}{u_1'' + V''} \tag{94}$$

where V" is the second derivative of V. Therefore, $c_t(x_t)$ and $x_t - c_t(x_t)$ are both increasing in x_t at x_t^o. In this case,

$$\frac{d\ U_1(x_t^o; t)}{dx_t} = \frac{u_1''\ V''}{u_1'' + V''}$$

and $U_1(x_t; t)$ has negative second derivative at x_t^o.

When $c_t(x_t^o)$ is not a unique solution corresponding to

x_t^o, we cannot analyse the behavior of $c_t(x_t)$ in a small neighborhood of x_t^o in this way. However, we have already known that

$$U_1^-(x_t^o;t) \leqq u_1'(c_t(x_t^o)) \leqq U_1^+(x_t^o;t).$$

This indicates that $u_1'(c_t(x_t))$ is very close to $U_1^-(x_t^o;t)$ for x_t slightly below x_t^o and very close to $U_1^+(x_t^o;t)$ for x_t slightly above x_t^o. That is, if $U_1^-(x_t^o;t) < U_1^+(x_t^o;t)$, then $u_1'(c_t(x_t))$ jumps at x_t^o as x_t increases. Since this means that $c_t(x_t)$ drops at x_t^o, $x_t - c_t(x_t)$ must jump at x_t^o correspondingly.

Summing up, $x_t - c_t(x_t)$ is always increasing in x_t. It remains to show that $U_1(x_t;t)$ is differentiable except for at most finite number of points. $U_1(x_t;t)$ is not differentiable at point x_t^o when $c_t(x_t)$ drops at x_t^o as x_t increases. Since $u_1'(c_t(x_t))$ then jumps at x_t^o, $V(x_t - c_t(x_t);t+1)$ must jump onto a new piece. Otherwise, the equality (93) cannot be kept, since $V'(x_t - c_t(x_t);t+1)$ drops if $V(x_t - c_t(x_t);t+1)$ remains on the same old piece. Therefore, the number of points where $U_1(x_t;t)$ is not differentiable cannot exceed the number of pieces of $V(y;t+1)$. This completes the proof.

We shall now derive piecewise regularity for $V(y;t+1)$ by induction and complete the arguments of this section.

Lemma 9 Let $U_1(x_{t+1};t+1)$ be a piecewise regular function which is differentiable except for at most finite number of points and, when differentiable, has negative second derivative. Then $V(y;t+1)$, which was defined by (78), is also a piecewise regular function of y which is differentiable except for at most finite number of points and, when differentiable, has negative second derivative.

Proof :

Since $u_2(b)$ is regular and concave and the curve

$$u = (n-t)\, u_2\left(\frac{y}{n-t}\right) - \bar{z}(t+1)$$

intersects the curve $u = U_1[x_{t+1}(y);t+1]$ at most once by the moral hazard condition, we need only to show that $U_1[x_{t+1}(y);t+1]$ is a piecewise regular function of y which is differentiable except for at most finite number of points and, when differentiable, has negative second derivative. Let us consider the case when $U_1(x_{t+1};t+1)$ in (72) is differentiable at $x_{t+1}(y)$. In this case, $x_{t+1}(y)$ and $b_{t+1}(y)$ are twice continuously differentiable by the Implicit Function Theorem. Totally differentiating (72) and (73)

$$\begin{bmatrix} U'[x_{t+1}(y);t+1] & -(n-t)\, u'(b_{t+1}(y)) \\ 1 - p_{t+1} & p_{t+1}\,(n-t) \end{bmatrix} \begin{bmatrix} dx_{t+1}(y)/dy \\ db_{t+1}(y)/dy \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore

$$\frac{d\ x_{t+1}(y)}{dy} = \frac{u_2'}{p_{t+1}\ U_1' + (1 - p_{t+1})\ u_2'} \tag{95}$$

$$\frac{d\ b_{t+1}(y)}{dy} = \frac{U_1'}{p_{t+1}\ U_1' + (1 - p_{t+1})\ u_2'} \tag{96}$$

(95) corresponds to the expression on the right-hand side of (14). From (95)

$$\frac{d\ U_1\left[x_{t+1}(y);t+1\right]}{dy} = \frac{U_1'\ u_2'}{p_{t+1}\ U_1' + (1 - p_{t+1})\ u_2'} \tag{97}$$

and

$$\frac{d^2\ U_1\left[x_{t+1}(y);t+1\right]}{dy^2}$$

$$= \frac{\left\{p_{t+1}U_1' + (1-p_{t+1})u_2'\right\}\left\{U_1''u_2'(dx_{t+1}/dy) + U_1'u_2''(db_{t+1}/dy)\right\}}{\left\{p_{t+1}U_1' + (1-p_{t+1})u_2'\right\}^2}$$

$$- \frac{U_1'u_2'\left\{p_{t+1}U_1''(dx_{t+1}/dy) + (1-p_{t+1})u_2''(db_{t+1}/dy)\right\}}{\left\{p_{t+1}U_1' + (1-p_{t+1})u_2'\right\}^2}$$

$$= \frac{(1-p_{t+1})U_1''(u_2')^2(dx_{t+1}/dy) + p_{t+1}(U_1')^2u_2''(db_{t+1}/dy)}{\left\{p_{t+1}U_1' + (1-p_{t+1})u_2'\right\}^2}$$

$$= (1-p_{t+1})U_1''(dx_{t+1}/dy)^3 + p_{t+1}u_2''(db_{t+1}/dy)^3.$$

We have thus shown that $U_1[x_{t+1}(y);t+1]$ has negative second derivative whenever it is differentiable.

$U_1[x_{t+1}(y);t+1]$ is not differentiable with respect to y only when $U_1(x_{t+1};t+1)$ is not differentiable at $x_{t+1}(y)$. Therefore $U_1[x_{t+1}(y);t+1]$ is differentiable except for at most finite number of points.

It remains to derive piecewise regularity for $U_1[x_{t+1}(y);t+1]$ as a function of y. We need only to prove the property about the left- and right-hand derivatives. The remaining three conditions are not difficult to prove. Since $x_{t+1}(y)$ and $b_{t+1}(y)$ are increasing functions of y, the left- and right-hand derivatives of $U_1[x_{t+1}(y);t+1]$ with respect to y

$$(d/dy)^- \, U_1[x_{t+1}(y);t+1] \qquad \text{and} \qquad (d/dy)^+ \, U_1[x_{t+1}(y);t+1]$$

are given by (97) as

$$(d/dy)^- \, U_1[x_{t+1}(y);t+1] = \frac{U_1^- \, u_2'}{p_{t+1}\, U_1^- + (1 - p_{t+1})\, u_2'}$$

$$(d/dy)^+ \, U_1[x_{t+1}(y);t+1] = \frac{U_1^+ \, u_2'}{p_{t+1}\, U_1^+ + (1 - p_{t+1})\, u_2'}$$

Since the expression

$$\frac{A\,B}{p_{t+1}\,A + (1 - p_{t+1})\,B}$$

as a function of A is increasing, the left-hand derivative of $U_1[x_{t+1}(y);t+1]$ with respect to y is less than the right-hand derivative when $U_1[x_{t+1}(y);t+1]$ is not differentiable.

Concluding the arguments of this section, we state the results we obtained as a theorem.

Theorem 4 Under Assumption R, M', and U', an increase in the government subsidy to the retirement insurance fund tends to induce earlier retirement and uniformly raises the retirement benefits.

References

1. P. A. Diamond and J. A. Mirrlees, "A Model of Social Insurance with Variable Retirement," Journal of Public Economics, 10 (1978), 295-336.

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Essay Two

Payroll-Tax Financed Unemployment Insurance with Human Capital

I. Introduction

The purpose of this essay is to analyse the optimal structure of payroll-tax financed unemployment insurance where government cannot distinguish between those who actually participate in the labor market but fail to get a job and those who choose not to work. Payroll-tax financed unemployment insurance is different from general-revenue financed unemployment insurance, such as the unemployment insurance model considered by Shavell and Weiss, in the sense that government resource constraint depends upon choices on the part of workers and not only the unemployment benefits but also net wages are determined by the government.

The dependence of the government resource constraint upon workers' choices apparently makes the government's problem very complicated, especially when workers are allowed to choose in each period whether to participate in the labor market. However, if participation decisions have no ensuing effect upon prospects and marginal products, moral hazard problems of different periods are not interwoven and we can reduce the apparently complicated unemployment insurance model into a simple recursive system. To emphasize this point, we present a payroll-tax financed unemployment insurance model where prospects and marginal products depend upon work history but are independent of participation history.

We show that, under optimal unemployment insurance, consumption should be made nondecreasing over time when working. In addition, when marginal utility of those not working is independent of whether they actually participate in the labor market or merely choose not to work, the unemployment benefit should be made nonincreasing over time while out of work.

II. The Model

At the start of each period, a worker, whether he worked in the previous period or not, makes a decision whether to participate in the labor market. If he chooses to take part in the labor market (this represents his search effort), his employment is determined by a lottery. The lottery for a worker depends upon his work history. That is, the probability that a worker fails to get a job in the current period, p_h, depends upon his work history up to the previous period which is indicated by subscript h. When employed, a worker has a marginal product equal to m_h which also depends upon his work history. Work experience thus has two human capital effects, upon prospects and upon marginal products.

The government is assumed to be unable to distinguish between those unemployed who actually fail to get a job and those who merely choose not to work. However, the government can use experience rating. That is, a worker whose work history is h receives net wage c_h when working and the unemployment benefit b_h when not working.

Since we are interested in the comparison of unemployment benefits and net wages of successive two periods, we need notations to indicate positions and branching out in the tree of work history. Let e_t denote the eligibility for unemployment benefit in the t th period, e_t being equal to 0 when not working and 1 when working. Work history index h is an n dimensional vector consisting of a number of e_t's and a number of D's, the latter being a dummy parameter that fills the positions in the vector corresponding to the periods for which h does not report a history. For instance, if h reports a history for the first t-1 periods, it is expressed as

$$h = (e_1, e_2, \ldots, e_{t-1}, D, D, \ldots, D) \qquad (1)$$

Let H be the totality of work history indices. We define correspondences e(h) and $\bar{e}(h)$ from H into H such that, if h is expressed as (1),

$$e(h) = (e_1, e_2, \ldots, e_{t-1}, 1, D, \ldots, D)$$

$$\bar{e}(h) = (e_1, e_2, \ldots, e_{t-1}, 0, D, \ldots, D). \qquad (3)$$

That is, e(t) is the work history index which is obtained by replacing the t th element of h, D, with 1. If h reports a complete history up to the n th period, we define that $e(h) = h$ and $\bar{e}(h) = h$.

The work history index does not explicitly indicate the periods for which it reports a history although it contains that information. We cannot recognize, for instance, the period to which the probability of unemployment p_h refers without explicitly writing h as (1). This is not a problem for our mathematical formulation, but it is convenient, for exposition, to have some term to indicate the t th period when h reports a history up to the t-1 st period. We use the 'current' period for this purpose.

Let instantaneous utilities be specified by the following utility functions.

$u_1(c)$ utility of consumption c when working

$u_2(c)$ utility when not participating in the labor market

$u_3(c)$ utility when participating in the labor market but unable to get a job

Remaining lifetime expected utility of a worker whose work history is h is denoted by $U(x_h,h)$, which is the maximum expected utility attainable (by government social welfare maximization) over the remaining periods with initial resource x_h. If h reports a history up to the t-1 st period, $U(x_h,h)$ is the lifetime expected utility at the start of the t th (current) period over the remaining n-t+1 periods.

When h reports a history up to the n th period, we set $U(x_h,h) = 0$. When h reports a history up to the n-1 st period, $U(x_h,h)$ is the optimum expected utility of the following one-period constrained maximization problem.

$$U(x_h, h) = \max \quad (1 - z_h) \left\{ (1 - p_h)\, u_1(c_h) + p_h\, u_3(b_h) \right\} + z_h\, u_2(b_h) \tag{4}$$

subject to

$$\text{(i)} \quad (1 - z_h) \left\{ (1 - p_h)\, u_1(c_h) + p_h\, u_3(b_h) \right\} + z_h\, u_2(b_h) = \max \left\{ (1 - p_h)\, u_1(c_h) + p_h\, u_3(b_h),\ u_2(b_h) \right\} \tag{5}$$

$$\text{(ii)} \quad (1 - z_h) \left\{ (1 - p_h)(c_h - m_h) + p_h\, b_h \right\} + z_h\, b_h \leqq x_h. \tag{6}$$

The first constraint is the moral hazard constraint and determines the value of participation variable z_h. The expression of this constraint indicates the sequence of moves between individual and nature. In case of retirement insurance, a worker decides whether to work or not after he knows his health. But, in the present case, a worker must make a decision whether to participate in the labor market without being ensured beforehand to get a job.

The second constraint is the resource constraint. When unemployment insurance is financed by payroll-tax, the government resource constraint depends upon workers' participation decisions. This is the main difference between payroll-tax financed unemployment insurance and general-revenue financed unemployment insurance such as the one considered by Shavell and Weiss in [1] .

An apparent difficulty in the payroll-tax financed unemployment insurance model involving multiple participation variables is that moral hazard constraints of different periods are interwoven and, therefore, the multi-period model becomes a very complicated maximization problem. The difficulty arises partly from the possibility that those workers among whom the government cannot differentiate in the current period (although they choose differently) may respond differently to the same insurance policy of the later periods, and partly from the possibility that the participation decisions of the current period affect the feasibility of an insurance policy of the later periods differently. The first possibility occurs when prospects depend upon participation history, and the second possibility occurs when marginal products depend upon participation history.

We look for a sufficient condition for the interwoven multi-period moral hazard problems to be decomposed into a series of simple moral hazard problems, so that the multi-period model of payroll-tax financed unemployment insurance is reduced to a recursive system of simple insurance problems. A sufficient condition which is general enough for the present purpose is given as follows : the participation decisions made for the current period by those workers among whom the government cannot differentiate have no ensuing effects upon their prospects and marginal products of the later periods which would induce different responses to the same insurance policy for the later periods or which would affect the feasibility of an insurance policy of the later periods differently. In other words, prospects and marginal products do not depend upon participation history although they do depend on work history.

We consider two kinds of activities, productive labor and search effort. Workers are classified into three types according to the combinations of these two activities : search-work, search-no-work, and no-search-no-work. The government cannot differentiate between those who search and are unemployed and those who do not search. If productive labor has human capital effects whereas search effort has no human capital effect, then those who participate in the labor market of the current period but cannot get a job and those who choose not to work will have the same prospects and marginal products in the later periods and respond similarly to the same insurance policy of the remaining periods. They respond similarly even though their responses are different from that of those who work in the current period.

Let us consider the role of the search decision on the possibility of decomposing in the two-period model. In order to keep the continuity of our arguments and for notational convenience, however, we define this two-period example as the government's problem of determining wages and benefits for the n-1 st and n th periods when the work history up to the n-2 nd period is given by h and when the initial resource at the start of the n-1 st period is x_h. We ignore the moral hazard problems of the preceding periods for a while. The government's problem is to determine the values of the n-1 st period wage and benefit, c_h and b_h, and the n th period wages and benefits, $c_{e(h)}$, $b_{e(h)}$, $c_{\bar{e}(h)}$, and $b_{\bar{e}(h)}$, so as to maximize the expected utility

$$U = (1 - z_h)\left\{(1 - p_h)\left[u_1(c_h) + U_{e(h)}\right] + p_h\left[u_3(b_h) + U_{\bar{e}(h)}\right]\right\} + z_h\left\{u_2(b_h) + U_{\bar{e}(h)}\right\} \tag{7}$$

where $U_{e(h)}$ denotes the expected utility of the n th period when working in the n-1 st period, i.e.,

$$U_{e(h)} = (1 - z_{e(h)})\left\{(1 - p_{e(h)})\, u_1(c_{e(h)}) + p_{e(h)}\, u_3(b_{e(h)})\right\} + z_{e(h)}\, u_2(b_{e(h)}) \tag{8}$$

and similarly

$$U_{\bar{e}(h)} = (1 - z_{\bar{e}(h)})\left\{(1 - p_{\bar{e}(h)})\, u_1(c_{\bar{e}(h)}) + p_{\bar{e}(h)}\, u_3(b_{\bar{e}(h)})\right\}$$

$$+ z_{\bar{e}(h)} u_2(b_{\bar{e}(h)}) \tag{9}$$

z_h, $z_{e(h)}$, and $z_{\bar{e}(h)}$ are the participation variables which should be chosen by a worker. Here we have already incorporated the condition that those who participate in the labor market of the n-1 st period but cannot get a job and those who choose not to work respond similarly (that is, choose $z_{\bar{e}(h)}$) to the same insurance policy for the n th period, $c_{\bar{e}(h)}$ and $b_{\bar{e}(h)}$. This is the optimum strategy from the worker's point of view since he simply tries to maximize the overall expected utility.

The nature of this worker's concern leads to the first group of constraints to the government's maximization problem, that is the moral hazard constraints. Participation variables z_h, $z_{e(h)}$, and $z_{\bar{e}(h)}$ must satisfy the following three conditions

$$U_{e(h)} = \max \left\{ (1 - p_{e(h)}) u_1(c_{e(h)}) + p_{e(h)} u_3(b_{e(h)}), \quad u_2(b_{e(h)}) \right\} \tag{10}$$

$$U_{\bar{e}(h)} = \max \left\{ (1 - p_{\bar{e}(h)}) u_1(c_{\bar{e}(h)}) + p_{\bar{e}(h)} u_3(b_{\bar{e}(h)}), \quad u_2(b_{\bar{e}(h)}) \right\} \tag{11}$$

and

$$U = \max \left\{ (1 - p_h)\left[u_1(c_h) + U_{e(h)}\right] + p_h\left[u_3(b_h) + U_{\bar{e}(h)}\right], \; u_2(b_h) + U_{\bar{e}(h)} \right\} \tag{12}$$

In this specification, it is implied that $z_{e(h)}$ and $z_{\bar{e}(h)}$ can be chosen independently of the value of z_h although the latter depends upon the values of $z_{e(h)}$ and $z_{\bar{e}(h)}$. This is an implication of the nature of the worker's concern.

Next, the government's maximization problem is also subject to the resource constraint

$$(1 - z_h)\left\{ (1 - p_h)(c_h + x_{e(h)} - m_h) + p_h(b_h + x_{\bar{e}(h)}) \right\} + z_h\,(b_h + x_{\bar{e}(h)}) \leqq x_h \tag{13}$$

where $x_{e(h)}$ denotes the initial resource at the start of the n th period when working in the n-1 st period, i.e.,

$$x_{e(h)} = (1 - z_{e(h)})\left\{ (1 - p_{e(h)})(c_{e(h)} - m_{e(h)}) + p_{e(h)}\, b_{e(h)} \right\} + z_{e(h)}\, b_{e(h)} \tag{14}$$

and similarly

$$x_{\bar{e}(h)} = (1 - z_{\bar{e}(h)})\left\{ (1 - p_{\bar{e}(h)})(c_{\bar{e}(h)} - m_{\bar{e}(h)}) + p_{\bar{e}(h)}\, b_{\bar{e}(h)} \right\} + z_{\bar{e}(h)}\, b_{\bar{e}(h)} \tag{15}$$

The government's problem is therefore to choose (c_h, b_h $c_{e(h)}$, $b_{e(h)}$, $c_{\bar{e}(h)}$, $b_{\bar{e}(h)}$) so as to maximize the overall expected utility (7) subject to the moral hazard constraints (10), (11), and (12) and the resource constraint (13).

Now we define the recursive system for this two-period model. Let $U(x_{e(h)}, e(h))$ and $U(x_{\bar{e}(h)}, \bar{e}(h))$ be the maximum expected utilities attainable by government social welfare maximization at the start of the n th period with initial resources $x_{e(h)}$ and $x_{\bar{e}(h)}$ respectively. They are the optimum expected utilities of the problem defined by (4), (5), and (6) when h and x_h are replaced with e(h) and $x_{e(h)}$, and $\bar{e}(h)$ and $x_{\bar{e}(h)}$, respectively. In other words, $U(x_{e(h)}, e(h))$ is obtained by maximizing $U_{e(h)}$ of (8) subject to (10) and (14). Similarly, $U(x_{\bar{e}(h)}, \bar{e}(h))$ is obtained by maximizing $U_{\bar{e}(h)}$ of (9) subject to (11) and (15).

The government's problem in the recursive form is then to choose c_h, b_h, $x_{e(h)}$, and $x_{\bar{e}(h)}$ so as to maximize

$$U = (1 - z_h)\Big\{(1 - p_h)\big[u_1(c_h) + U(x_{e(h)}, e(h))\big] + p_h\big[u_3(b_h) + U(x_{\bar{e}(h)}, \bar{e}(h))\big]\Big\} + z_h\Big\{u_2(b_h) + U(x_{\bar{e}(h)}, \bar{e}(h))\Big\} \qquad (16)$$

subject to the moral hazard constraint (12) with $U_{e(h)}$ and $U_{\bar{e}(h)}$ there being replaced with $U(x_{e(h)}, e(h))$ and $U(x_{\bar{e}(h)}, \bar{e}(h))$ and the resource constraint (13).

We shall now prove that the recursive system is equivalent to the original two-period model. Let

$$(c^*_h,\ b^*_h,\ c^*_{e(h)},\ b^*_{e(h)},\ c^*_{\bar{e}(h)},\ b^*_{\bar{e}(h)})$$

be a given optimum insurance policy for the original two-period model. With this optimum insurance policy, we define the n th period expected utilities $U^*_{e(h)}$ and $U^*_{\bar{e}(h)}$ by (8) and (9). Similarly, we define the initial resources at the start of the n th period, $x^*_{e(h)}$ and $x^*_{\bar{e}(h)}$, by (14) and (15) with this optimum policy. We need to show that

$$U^*_{e(h)} = U(x^*_{e(h)}, e(h)) \quad \text{and} \quad U^*_{\bar{e}(h)} = U(x^*_{\bar{e}(h)}, \bar{e}(h)) \tag{17}$$

That is, the n th period wage and benefit of an insurance policy which is optimal from the point of view at the start of the n-1 st period is also optimal from the point of view at the start of the n th period, whether a worker works in the n-1 st period or not.

The value of the participation variable for the current period, z_h, is determined by comparing the expected utility

$$(1 - p_h)\left\{u_1(c^*_h) + U^*_{e(h)}\right\} + p_h\left\{u_3(b^*_h) + U^*_{\bar{e}(h)}\right\} \tag{18}$$

with

$$u_2(b^*_h) + U^*_{\bar{e}(h)} \ . \tag{19}$$

On the other hand, c^*_h, b^*_h, $x^*_{e(h)}$, and $x^*_{\bar{e}(h)}$ satisfy the resource constraint

$$(1 - z_h)\left\{(1 - p_h)(c_h^* + x_{e(h)}^* - m_h) + p_h(b_h^* + x_{e(h)}^*)\right\}$$
$$+ z_h\ (b_h^* + x_{e(h)}^*) \leqq x_h. \qquad (20)$$

Therefore, if (17) holds, $(c_h^*, b_h^*, x_{e(h)}^*, x_{e(h)}^*)$ is feasible for the recursive system and the given optimum policy cannot be preferred to the insurance policy which is obtained from an optimum solution to the recursive system. Since the insurance policy obtained from an optimum policy to the recursive system is always feasible for the original two-period model, (17) implies the equivalence of these two formulations.

Let us first suppose

$$U_{e(h)}^* < U(x_{e(h)}^*, e(h)).$$

In this case, we can increase the overall expected utility (7) by reallocating resources to attain $U(x_{e(h)}^*, e(h))$ for the n th period when working in the n-1 st period. This does not violate the moral hazard constraint of the n-1 st period, since the expected utility (18) must originally be at least as great as (19). Otherwise, $c_{e(h)}^*$ and $b_{e(h)}^*$, and therefore $U_{e(h)}^*$, would have been indeterminate. This reallocation also leaves the resource constraint (20) intact. Therefore, the given insurance policy cannot become optimum from the point of view at the start of the n-1 st period.

Next, let us suppose

$U^*_{e(h)} < U(x^*_{e(h)}, \bar{e}(h))$.

There are two cases. When the expected utility (18) is originally at least as great as (19), we may not be able to reallocate resources to attain $U(x^*_{e(h)}, \bar{e}(h))$ for the n th period because it may violate the moral hazard constraint of the n-1 st period. In this case, however, we can spare some resources from $x^*_{e(h)}$ without reducing the expected utility when not working in the n-1 st period from the original level $U^*_{e(h)}$ and reallocate them to increase the expected utility when working in the n-1 st period. This is then reduced to the first case.

When the expected utility (18) is originally less than (19), we can simply reallocate resources to attain $U(x^*_{e(h)}, \bar{e}(h))$ without violating the moral hazard constraint of the n-1 st period. Therefore, in each case, we can show that the given insurance policy cannot become optimal from the point of view at the start of the n-1 st period.

We have shown that the two-period model of payroll-tax financed unemployment insurance can be reduced to a recursive system. We now leave the two-period world and return to the n-period model. However, the arguments for the two-period case can be applied to the n-period model without any essential change.

We start from a given optimum insurance policy to the non-recursive system which is optimal from the point of view at the start of the first period. For the first period, $h = (D, D, \ldots, D)$. Let us denote this given insurance policy by

$$(c^*_h, b^*_h) \qquad \ldots \quad \text{1st period}$$

$$(c^*_{e(h)}, b^*_{e(h)})\ (c^*_{\bar{e}(h)}, b^*_{\bar{e}(h)}) \qquad \ldots \quad \text{2nd period}$$

$$\left.\begin{array}{l} (c^*_{e(e(h))}, b^*_{e(e(h))})\ (c^*_{\bar{e}(e(h))}, b^*_{\bar{e}(e(h))}) \\ (c^*_{e(\bar{e}(h))}, b^*_{e(\bar{e}(h))})\ (c^*_{\bar{e}(\bar{e}(h))}, b^*_{\bar{e}(\bar{e}(h))}) \end{array}\right\} \quad \ldots \quad \text{3rd period}$$

....

....

We define $U^*_{e(h)}$ and $x^*_{e(h)}$ as the expected utility and the initial resource at the start of the second period corresponding to the subprogram of this insurance policy

$$(c^*_{e(h)}, b^*_{e(h)}) \qquad \ldots \quad \text{2nd period}$$

$$(c^*_{e(e(h))}, b^*_{e(e(h))})\ (c^*_{\bar{e}(e(h))}, b^*_{e(e(h))}) \qquad \ldots \quad \text{3rd period}$$

....

Similarly, $U^*_{\bar{e}(h)}$ and $x^*_{\bar{e}(h)}$ are defined as the expected utility and the initial resource at the start of the second period corresponding to the subprogram

$(c^*_{e(h)}, b^*_{e(h)})$ ... 2nd period

$(c^*_{e(\bar{e}(h))}, b^*_{e(\bar{e}(h))})$ $(c^*_{\bar{e}(\bar{e}(h))}, b^*_{\bar{e}(\bar{e}(h))})$ ... 3rd period

....

The value of the participation variable for the first period, z_h, is determined by comparing (18) with (19), and c^*_h, b^*_h, $x^*_{e(h)}$, and $x^*_{\bar{e}(h)}$ satisfy the resource constraint (20). Therefore, we can use the same arguments as before. (17) must hold, where $U(x^*_{e(h)}, e(h))$ and $U(x^*_{\bar{e}(h)}, \bar{e}(h))$ are defined as the maximum expected utilities attainable by government social welfare maximization over the remaining n-1 periods with initial resources $x^*_{e(h)}$ and $x^*_{\bar{e}(h)}$ respectively. (17) shows that the two subprograms are optimal for the remaining n-1 periods, with initial resources $x^*_{e(h)}$ and $x^*_{\bar{e}(h)}$ respectively, from the point of view at the start of the second period.

Since the given subprograms are optimal from the point of view at the start of the second period, we can repeat the same arguments again. The first period variables and the first period moral hazard constraint do not enter the arguments for the remaining periods any more. Using the same arguments repeatedly, therefore, we can decompose the original n-period non-recursive model into a recursive system.

Since we have shown that the multi-period model of payroll-tax financed unemployment insurance can be reduced to a recursive system, we shall now give the formal recursive definition of $U(x_h,h)$. Let $v_A(x_h,h)$ be the remaining lifetime expected utility at the start of the current period of a worker whose work history is h when he should participate in the current labor market. It is the maximum expected utility of the following constrained maximization problem.

$$v_A(x_h,h) = \max \quad (1 - p_h)\left\{u_1(c_h) + U(x_{e(h)},e(h))\right\} + p_h\left\{u_3(b_h) + U(x_{\bar{e}(h)},\bar{e}(h))\right\} \qquad (21)$$

subject to

$$\text{(i)} \quad (1 - p_h)(c_h + x_{e(h)} - m_h) + p_h(b_h + x_{\bar{e}(h)}) \leqq x_h \qquad (22)$$

$$\text{(ii)} \quad (1 - p_h)\left\{u_1(c_h) + U(x_{e(h)},e(h))\right\} + p_h\left\{u_3(b_h) + U(x_{\bar{e}(h)},\bar{e}(h))\right\} \geqq u_2(b_h) + U(x_{\bar{e}(h)},\bar{e}(h)) \qquad (23)$$

The maximum is taken over c_h, b_h, $x_{e(h)}$, and $x_{\bar{e}(h)}$.

Similarly, let $v_B(x_h,h)$ be the remaining lifetime expected utility at the start of the current period of a worker whose work history is h when he should not participate in the current labor market. It is the optimum value of the following constrained maximization problem.

$$v_B(x_h,h) = \max \left\{ u_2(b_h) + U(x_{\bar{e}(h)},\bar{e}(h)) \;\middle|\; b_h + x_{\bar{e}(h)} \leqq x_h \right\} \tag{24}$$

$U(x_h,h)$ is then defined as the maximum of $v_A(x_h,h)$ and $v_B(x_h,h)$.

$$U(x_h,h) = \max \left\{ v_A(x_h,h),\; v_B(x_h,h) \right\} \tag{25}$$

When the government's initial resource is Y and when the equilibrium unemployment rate is P, the government's overall problem is to maximize the social welfare

$$(1 - P)\, U(x_{e(h_D)},e(h_D)) + P\, U(x_{\bar{e}(h_D)},\bar{e}(h_D)) \tag{26}$$

subject to

$$(1 - P)\, x_{e(h_D)} + P\, x_{\bar{e}(h_D)} \leqq Y \tag{27}$$

where $h_D = (D, D, \ldots, D)$.

III. Remaining Lifetime Expected Utility

In this section, we analyse the nature of remaining lifetime expected utility $U(x_h, h)$ and derive important relationships among instantaneous utilities and lifetime expected utilities. The properties with which we will endow the remaining lifetime expected utility are summarized by the following four conditions.

Definition (Piecewise Regularity). A real valued function $f(x)$, defined on an interval $(-d, \infty)$ for a nonnegative number d, is called piecewise regular if $f(x)$ has the following properties,

(i) Monotonicity : $f(x)$ is strictly increasing

(ii) Continuity : $f(x)$ is continuous

(iii) Left- and Right-Hand Derivatives : the right-hand derivative, $f^+(x)$, is at least as great as the left-hand derivative, $f^-(x)$.

(iv) End-Point Properties :

$$\lim_{x \to -d} f(x) = -\infty \qquad \lim_{x \to -d} f^+(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \infty$$

A piecewise regular function which is twice continuously differentiable is called regular. In this section, we assume regularity and concavity for instantaneous utility functions, and derive, by induction, the piecewise regularity of $U(x_h,h)$

Assumption R u_1, u_2, and u_3 are regular and concave.

In the following lemmas, we use the notations

$u_j'(c) = du_j(c)/dc$ (j = 1, 2, and 3)

$U^-(x_h,h)$ and $U^+(x_h,h)$ are the left- and right-hand derivatives of $U(x_h,h)$ with respect to x_h. When $U(x_h,h)$ is differentiable at x_h, we denote $U'(x_h,h) = dU(x_h,h)/dx_h$.

Lemma 1 Let $0 < p_h < 1$ and let us assume that $U(x_{e(h)},e(h))$ and $U(x_{\bar{e}(h)},\bar{e}(h))$ are piecewise regular functions. Under Assumption R, $U(x_h,h)$ can be expressed in one of the following three forms.

(A) When, for a given x_h, a worker should participate in the labor market of the current period,

$$U(x_h,h) = v_A(x_h,h)$$
$$= (1 - p_h)\left\{u_1(c_h^*) + U(x^*_{e(h)},e(h))\right\}$$
$$+ p_h\left\{u_3(b_h^*) + U(x^*_{\bar{e}(h)},\bar{e}(h))\right\} \quad (28)$$

where

(i) if the moral hazard constraint (23) is binding, c_h^*, b_h^*, $x^*_{e(h)}$, and $x^*_{\bar{e}(h)}$ are obtained by solving the two constraints and the following equilibrium conditions

$$u_1'(c_h^*) = U'(x^*_{e(h)}, e(h)) \tag{29}$$

$$\frac{U'(x^*_{\bar{e}(h)}, \bar{e}(h))}{u_1'(c_h^*)} = \frac{u_2'(b_h^*)}{u_1'(c_h^*) + u_2'(b_h^*) - u_3'(b_h^*)} \tag{30}$$

and they satisfy the inequalities

$$u_3'(b_h^*) > u_1'(c_h^*) = U'(x^*_{e(h)}, e(h)) < U'(x^*_{\bar{e}(h)}, \bar{e}(h)) \tag{31}$$

$$u_1'(c_h^*) + u_2'(b_h^*) - u_3'(b_h^*) > 0. \tag{32}$$

(ii) if the moral hazard constraint is not binding, c_h^*, b_h^*, $x^*_{e(h)}$, and $x^*_{\bar{e}(h)}$ are obtained by solving the resource constraint and the following equilibrium conditions

$$u_1'(c_h^*) = u_3'(b_h^*) = U'(x^*_{e(h)}, e(h)) = U'(x^*_{\bar{e}(h)}, \bar{e}(h)). \tag{33}$$

(B) When, for a given x_h, a worker should not participate in the market of of the current period,

$$U(x_h, h) = v_B(x_h, h)$$

$$= u_2(b_h^*) + U(x^*_{\bar{e}(h)}, \bar{e}(h)) \tag{34}$$

where b_h^* and $x_{e(h)}^*$ satisfy the equilibrium condition

$$u_2'(b_h^*) = U'(x_{e(h)}^*, \bar{e}(h)). \tag{35}$$

Proof :

We first consider case A. In this case, $U(x_h, h)$ is equal to the expected utility, $v_A(x_h, h)$, of an optimal solution $(c_h^*, b_h^*, x_{e(h)}^*, x_{\bar{e}(h)}^*)$ to the constrained maximization problem defined by (21), (22), and (23).

Any deviations in c_h and b_h from these optimum values which keep the resource constraint change the total expected utility by

$$dV = (1 - p_h)(u_1' - u_3')\, dc_h.$$

Therefore, if $u_1' - u_3'$ is positive, an increase in c_h must violate the moral hazard constraint. That is

$$(1 - p_h)(u_1' - u_3') + \frac{1 - p_h}{p_h}\, u_2' < 0.$$

But this is impossible when $u_1' - u_3' > 0$. This implies

$$u_1' - u_3' \leqq 0. \tag{36}$$

When (36) holds with strict inequality, any decrease in c_h must also violate the moral hazard constraint. Therefore

$$p_h(u_1' - u_3') + u_2' > 0. \tag{37}$$

Since (37) also holds when $u_1' - u_3' = 0$, (36) and (37) must always be satisfied for case A.

Since any deviations in c_h and $x_{e(h)}$ from the optimum values which keep the resource constraint should not increase the expected utility,

$$u_1' - U^-(x^*_{e(h)}, e(h)) \leqq 0 \tag{38}$$

and

$$- u_1' + U^+(x^*_{e(h)}, e(h)) \leqq 0. \tag{39}$$

These inequalities, together with the piecewise regularity of $U(x_{e(h)}, e(h))$, imply the differentiability of $U(x_{e(h)}, e(h))$ at $x^*_{e(h)}$ and

$$u_1' = U'(x^*_{e(h)}, e(h)). \tag{40}$$

We shall now consider the effect upon the total expected utility of an increase in $x_{\bar{e}(h)}$ and the corresponding deviations in c_h and b_h to keep the resource constraint and the moral hazard constraint. The deviations in c_h and b_h must satisfy the equations

$$\begin{bmatrix} (1 - p_h)u_1' & -(u_2' - p_h u_3') \\ 1 - p_h & p_h \end{bmatrix} \begin{bmatrix} dc_h/dx_{\bar{e}(h)} \\ db_h/dx_{\bar{e}(h)} \end{bmatrix} = \begin{bmatrix} (1 - p_h)U^+(x^*_{\bar{e}(h)}, \bar{e}(h)) \\ - p_h \end{bmatrix}$$

That is

$$\frac{dc_h}{dx_{\bar{e}(h)}} = \frac{p_h}{1 - p_h} \frac{(1 - p_h)\, U^+(x^*_{\bar{e}(h)}, \bar{e}(h)) - (u_2' - p_h u_3')}{p_h u_1' + u_2' - p_h u_3'}$$

$$\frac{db_h}{dx_{\bar{e}(h)}} = - \frac{(1 - p_h)\, U^+(x^*_{\bar{e}(h)}, \bar{e}(h)) + p_h u_1'}{p_h u_1' + u_2' - p_h u_3'}$$

Therefore

$$(1 - p_h)\, u_1'\, dc_h + p_h\, u_3'\, db_h$$

$$= \frac{p_h \left\{ (1 - p_h)(u_1' - u_3')\, U^+(x^*_{\bar{e}(h)}, \bar{e}(h)) - u_1' u_2' \right\}}{p_h u_1' + u_2' - p_h u_3'}\, dx_{\bar{e}(h)}$$

and the effect upon the total expected utility dV can be expressed as

$$dV = (1 - p_h)\, u_1'\, dc_h + p_h\, u_3'\, db_h + p_h\, U^+(x^*_{\bar{e}(h)}, \bar{e}(h))\, dx_{\bar{e}(h)}$$

$$= \frac{p_h \left\{ (u_1' + u_2' - u_3')\, U^+(x^*_{\bar{e}(h)}, \bar{e}(h)) - u_1' u_2' \right\}}{p_h u_1' + u_2' - p_h u_3'}\, dx_{\bar{e}(h)} \qquad (41)$$

Since dV must be nonpositive and the denominator is positive by (37), (41) implies

$$\frac{u_1' + u_2' - u_3'}{u_1'\, u_2'} \leqq \frac{1}{U^+(x^*_{\bar{e}(h)}, \bar{e}(h))} \tag{42}$$

For $dx_{\bar{e}(h)} < 0$, the corresponding effect upon the total expected utility is obtained by simply replacing $U^+(x^*_{\bar{e}(h)}, \bar{e}(h))$ in (41) with $U^-(x^*_{\bar{e}(h)}, \bar{e}(h))$.

$$dV = \frac{p_h \left\{ (u_1' + u_2' - u_3')\, U^-(x^*_{\bar{e}(h)}, \bar{e}(h)) - u_1' u_2' \right\}}{p_h u_1' + u_2' - p_h u_3'}\, dx_{\bar{e}(h)}$$

Since $dx_{\bar{e}(h)}$ is negative, $dV \leqq 0$ implies

$$\frac{u_1' + u_2' - u_3'}{u_1'\, u_2'} \geqq \frac{1}{U^-(x^*_{\bar{e}(h)}, \bar{e}(h))} \tag{43}$$

Combining (42) with (43), together with the piecewise regularity of $U(x_{\bar{e}(h)}, \bar{e}(h))$, $U(x_{\bar{e}(h)}, \bar{e}(h))$ becomes differentiable. Therefore,

$$u_1' + u_2' - u_3' > 0$$

and

$$U'(x^*_{\bar{e}(h)}, \bar{e}(h)) = \frac{u_1'\, u_2'}{u_1' + u_2' - u_3'} \tag{44}$$

When $u_1' - u_3' = 0$, (44) implies

$$u_1' = u_3' = U'(x^*_{e(h)}, e(h)) = U'(x^*_{\bar{e}(h)}, \bar{e}(h)). \tag{45}$$

In this case, any infinitesimal deviations in c_h, b_h, $x_{e(h)}$, and $x_{\bar{e}(h)}$ from the optimum values which keep the resource constraint have no effect upon the total expected utility and the moral hazard constraint is not binding. Therefore, when the moral hazard constraint is effective, $u_1' - u_3'$ must be strictly negative.

This completes the analysis of case A. In case B, $U(x_h,h)$ is equal to the expected utility, $v_B(x_h,h)$, of an optimal solution $(b_h^*, x_{\bar{e}(h)}^*)$ to the constrained maximization problem (24). Therefore, the desired equilibrium condition follows from the inequalities exactly similar to (38) and (39).

Q. E. D.

For Lemma 1 to be applicable for any h, we must derive the piecewise regularity of $U(x_h,h)$ by induction.

For work history index h which reports a history up to the n-1 st period, $U(x_h,h)$ is the maximum of $v_A(x_h,h)$ and $v_B(x_h,h)$ where

$$v_A(x_h,h) = \max \quad (1 - p_h)\, u_1(c_h) + p_h\, u_3(b_h)$$

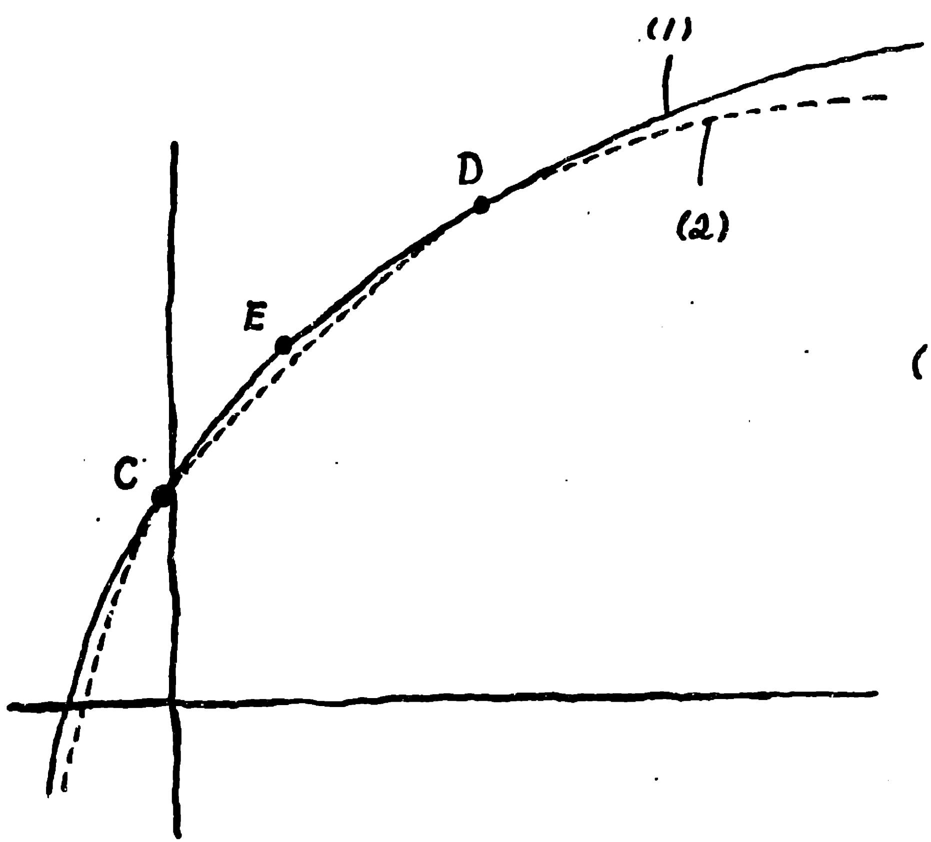
subject to

$$\text{(i)} \quad (1 - p_h)(c_h - m_h) + p_h\, b_h \leqq x_h$$

$$\text{(ii)} \quad (1 - p_h)\, u_1(c_h) + p_h\, u_3(b_h) \geqq u_2(b_h)$$

and $v_B(x_h,h) = u_2(x_h)$. The graph of $v_A(x_h,h)$ is depicted in fig. 1. The curve (1) is the graph of the expected utility of first-best insurance policy. The curve (2) depicts the graph of the expected utility of the pair which satisfy the moral hazard constraint with equality (besides the resource constraint). The graph of $v_A(x_h,h)$ is obtained by imposing upon curve (2) the segments of curve (1) which satisfy the moral hazard constraint. When the first-best pair (c_h, b_h) corresponding to point E satisfy the moral hazard constraint, the whole segment CD of curve (1) has the corresponding first-best pair which also satisfy the moral hazard constraint. Therefore, $v_A(x_h,h)$ is piecewise regular.

Since the maximum of two piecewise regular functions is also piecewise regular, $U(x_h,h)$ is a piecewise regular function.



(fig. 1)

It remains to show the piecewise regularity of $U(x_h,h)$ when h reports a history for the first t-1 periods where t is less than n. Since $U(x_h,h)$ is the maximum of $v_A(x_h,h)$ and $v_B(x_h,h)$, we must derive piecewise regularity for both $v_A(x_h,h)$ and $v_B(x_h,h)$.

Lemma 2 If $U(x_{e(h)},e(h))$ and $U(x_{\bar{e}(h)},\bar{e}(h))$ are both piecewise regular, the optimum expected utility, $v_A(x_h,h)$, of the constrained maximization problem defined by (21), (22), and (23) is also a piecewise regular function. In addition,

$$v_A^-(x_h,h) = \min \left\{ \frac{u_1' \, u_2'}{p_h u_1' + u_2' - p_h u_3'} \;\middle|\; (c_h^*, b_h^*, x_{e(h)}^*, x_{\bar{e}(h)}^*) \in E^A(x_h,h) \right\}$$

$$v_A^+(x_h,h) = \max \left\{ \frac{u_1' \, u_2'}{p_h u_1' + u_2' - p_h u_3'} \;\middle|\; (c_h^*, b_h^*, x_{e(h)}^*, x_{\bar{e}(h)}^*) \in E^A(x_h,h) \right\} \tag{46}$$

where $E^A(x_h,h)$ is the set of optimal solutions to the constrained maximization problem. $v_A^-(x_h,h)$ and $v_A^+(x_h,h)$ are, as before, the left- and right-hand derivatives of $v_A(x_h,h)$.

(46) is a kind of the envelope theorem for the constrained maximization problem which has multiple solutions. The lemma can be proved by the same argument as that we applied to prove the Lemma 2 of the first essay. Property 1° in that argument must be replaced with the inequality

$$v_A(x_h + \delta x_h, h) - v_A(x_h, h)$$

$$\geqq \frac{u_1' u_2'}{p_h u_1' + u_2' - p_h u_3'} \delta x_h + o(\delta x_h) \qquad (47)$$

where $o(\delta x_h)$ is a term such that $o(\delta x_h)/\delta x_h \to 0$ as $\delta x_h \to 0$. (47) can be derived from the Taylor's expansion of the moral hazard constraint. When the initial resource is changed from x_h to $x_h + \delta x_h$, by the Taylor's expansion of the moral hazard constraint, the partial responses in c_h and b_h which keep the constraint must satisfy

$$0 = (1 - p_h)\left\{u_1(c_h^* + \delta c_h) - u_1(c_h^*)\right\} + p_h\left\{u_3(b_h^* + \delta b_h) - u_3(b_h^*)\right\}$$
$$- \left\{u_2(b_h^* + \delta b_h) - u_2(b_h^*)\right\}$$
$$= (1 - p_h)\, u_1' \,\delta c_h + p_h\, u_3' \,\delta b_h - u_2' \,\delta b_h + o(\delta c_h) + o(\delta b_h)$$

From the resource constraint,

$$(1 - p_h)\,\delta c_h + p_h\,\delta b_h = \delta x_h.$$

Combining these two equations, we can easily show that

$$\delta c_h = \frac{1}{1 - p_h} \frac{u_2' - p_h u_3'}{p_h u_1' + u_2' - p_h u_3'} \delta x_h + o(\delta x_h)$$

$$\delta b_h = \frac{u_1'}{p_h u_1' + u_2' - p_h u_3'} \delta x_h + o(\delta x_h).$$

Since the effect upon the expected utility of these partial responses must not exceed the difference

$$v_A(x_h + \delta x_h, h) - v_A(x_h, h),$$

(47) must be satisfied.

Lemma 3 If $U(x_{e(h)}, e(h))$ and $U(x_{\bar{e}(h)}, \bar{e}(h))$ are both piecewise regular, the optimum expected utility, $v_B(x_h, h)$, of the constrained maximization problem (24) is also a piecewise regular function. In addition,

$$\begin{aligned} v_B^-(x_h, h) &= \min \left\{ u_2'(b_h^*) \mid (b_h^*, x_{e(h)}^*) \in E^B(x_h, h) \right\} \\ v_B^+(x_h, h) &= \max \left\{ u_2'(b_h^*) \mid (b_h^*, x_{e(h)}^*) \in E^B(x_h, h) \right\} \end{aligned} \qquad (48)$$

where $E^B(x_h, h)$ is the set of optimal solutions to the constrained maximization problem. $v_B^-(x_h, h)$ and $v_B^+(x_h, h)$ are the left- and right-hand derivatives of $v_B(x_h, h)$.

The proof of this lemma is similar to that of Lemma 2.

IV. Optimal Structure of Unemployment Insurance

In this section, we stucy the implications of Lemma 1, 2, and 3 for the structure of optimal unemployment insurance over time.

Theorem 1 If instantaneous utility functions are regular and concave, under optimal unemployment insurance,

(i) net wage should be made nondecreasing over time when working, and, in addition,

(ii) workers are just indifferent to continued participation in the labor market while net wage is increasing and net wage is constant during the periods when workers prefer to continue participation.

Proof :

When the pair of remaining lifetime expected utilities, $U(x_h, h)$ and $U(x_{e(h)}, e(h))$, for any consecutive two periods are interrelated by the condition that $x_{e(h)}$ is equal to $x^*_{e(h)}$ which is a component of an optimal vector to the constrained maximization problem defined by (21), (22), and (23), $U(x_{e(h)}, e(h))$ is differentiable at $x^*_{e(h)}$ according to Lemma 1. Since $U(x_{e(h)}, e(h))$ is the maximum of $v_A(x_{e(h)}, e(h))$ and

$v_B(x_{e(h)}, e(h))$, the differentiability of $U(x_{e(h)}, e(h))$ at $x^*_{e(h)}$ implies

$$U'(x^*_{e(h)}, e(h)) = v'_A(x^*_{e(h)}, e(h)) \qquad \text{if} \quad v_A(x^*_{e(h)}, e(h)) \geqq v_B(x^*_{e(h)}, e(h))$$

and

$$U'(x^*_{e(h)}, e(h)) = v'_B(x^*_{e(h)}, e(h)) \qquad \text{if} \quad v_A(x^*_{e(h)}, e(h)) < v_B(x^*_{e(h)}, e(h))$$

When a worker should participate in the labor market in both periods, Lemma 1 and 2, therefore, imply the following equilibrium relationship

$$u'_1(c^*_h) = U'(x^*_{e(h)}, e(h))$$

$$= \frac{u'_1(c^*_{e(h)})\ u'_2(b^*_{e(h)})}{p_{e(h)}u'_1(c^*_{e(h)}) + u'_2(b^*_{e(h)}) - p_{e(h)}u'_3(b^*_{e(h)})} \qquad (49)$$

The first proposition of the theorem immediately follows from (49) since $u'_1(c^*_{e(h)}) - u'_3(b^*_{e(h)})$ is nonpositive in equilibrium. The second proposition follows from the condition that the moral hazard constraint is binding when $u'_1(c^*_{e(h)})$ is strictly less than $u'_3(b^*_{e(h)})$.

The counterpart of Theorem 1 with regard to the relationship between b_h^* and $b_{e(h)}^*$ does not generally hold. From the differentiability of $U(x_{\bar{e}(h)}, \bar{e}(h))$ at $x_{\bar{e}(h)}^*$ where $x_{\bar{e}(h)}^*$ is a component of an optimal vector to the constrained maximization problem defined by (21), (22), and (23),

$$U'(x_{\bar{e}(h)}^*, \bar{e}(h)) = v_A'(x_{\bar{e}(h)}^*, \bar{e}(h))$$

when $v_A(x_{\bar{e}(h)}^*, \bar{e}(h))$ is not less than $v_B(x_{\bar{e}(h)}^*, \bar{e}(h))$. Therefore, when a worker should participate in the labor market in both periods, Lemma 1 and 2 imply

$$\frac{u_1'(c_h^*)\ u_2'(b_h^*)}{u_1'(c_h^*) + u_2'(b_h^*) - u_3'(b_h^*)} = U'(x_{\bar{e}(h)}^*, \bar{e}(h))$$

$$= \frac{u_1'(c_{\bar{e}(h)}^*)\ u_2'(b_{\bar{e}(h)}^*)}{p_{\bar{e}(h)} u_1'(c_{\bar{e}(h)}^*) + u_2'(b_{\bar{e}(h)}^*) - p_{\bar{e}(h)} u_3'(b_{\bar{e}(h)}^*)} \qquad (50)$$

We cannot compare $u_3'(b_h^*)$ with $u_3'(b_{\bar{e}(h)}^*)$ from (50) alone. However, if we can assume that $u_2'(c) = u_3'(c)$ for any c, then (50) is reduced to

$$u_3'(b_h^*) = \frac{u_1'(c_{\bar{e}(h)}^*)}{p_{\bar{e}(h)} u_1'(c_{\bar{e}(h)}^*) + (1 - p_{\bar{e}(h)}) u_3'(b_{\bar{e}(h)}^*)}\ u_3'(b_{\bar{e}(h)}^*)$$

Therefore we obtain

Theorem 2 If instantaneous utility functions are regular and concave, and if $u_2'(c) = u_3'(c)$ for any c, then, under optimal unemployment insurance,

(i) the unemployment benefit should be made nonincreasing over time when out of work, and, in addition,

(ii) workers are just indifferent to continued participation in the labor market while the unemployment benefit is decreasing and it is constant during the periods when workers prefer to continue participation.

The first proposition is also true when workers prefer not to participate in the labor market in either period.

It seems to me that the necessity of the additional qualification in Theorem 2 is another instance of a general property of optimal structure of the employment-related insurance. In the previous essay, we had to assume the same qualification to derive the optimal feature of the retirement benefit with respect to the age of retirement, although the optimal structure of consumption over time could be deduced without such qualification. In their model of optimal unemployment insurance, Shavell and Weiss also assumed that instantaneous utility functions are separable with regard to consumption and search effort. The indeterminacy in the optimal structure of the insurance benefit over time is probably caused by the inability on the part of the government to distinguish between those who happen to be out of work, being unfavorably affected by nature, and those who choose not to work. When the marginal utility of the unemployed is independent of whether they actually participate in the labor market, the inability of the government to monitor does not cause any trouble for it to determine the distribution of a given income for the unemployed between current unemployment benefit and future consumption. This is because any distribution of income y between b_h and $x_{\bar{e}(h)}$ which maximizes the sum of utilities

$$u_3(b_h) + U(x_{\bar{e}(h)}, \bar{e}(h))$$

always maximizes

$$u_2(b_h) + U(x_{\bar{e}(h)}, \bar{e}(h)).$$

The moral hazard constraint of the current period is ineffective in the determination of the distribution of income y between the current benefit b_h and the future consumption $x_{\bar{e}(h)}$. It is effective only for the determination of the distribution of income between those working and those unemployed.

Reference

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Biographical Note

Yasuo Usami was born July 9, 1945 in Japan. He entered Keio University in 1965 and graduated in 1969. He received M.A. from Keio University in 1971 and, from 1971 to 1976, was a research assistant and then an assistant professor of the Department of Economics of Keio University. He entered M.I.T. in 1973.