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Subregion independence in gravity

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Abstract: In gravity, spacelike separated regions can be dependent on each other due to the constraint equations. In this paper, we give a natural definition of subsystem independence and gravitational dressing of perturbations in classical gravity. We find that extremal surfaces, non-perturbative lumps of matter, and generic trapped surfaces are structures that enable dressing and subregion independence. This leads to a simple intuitive picture for why extremal surfaces tend to separate independent subsystems. The underlying reason is that localized perturbations on one side of an extremal surface contribute negatively to the mass on the other side, making the gravitational constraints behave as if there exist both negative and positive charges. Our results support the consistency of islands in massless gravity, shed light on the Python's lunch, and provide hints on the nature of the split property in perturbatively quantized general relativity. We also prove a theorem bounding the area of certain surfaces in spherically symmetric asymptotically de Sitter spacetimes from above and below in terms of the horizon areas of de Sitter and Nariai. This theorem implies that it is impossible to deform a single static patch without also deforming the opposite patch, provided we assume spherical symmetry and an energy condition.

Keywords: AdS-CFT Correspondence, Black Holes, Differential and Algebraic Geometry, Gauge-Gravity Correspondence

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1 Introduction

When are spacelike separated regions in a quantum theory of gravity independent? This is a central question that holography forces us to confront. On the one hand, the constraint equations of GR tells us that some gravitational subregions depend on each other, since generically one cannot specify initial data on two regions independently. On the other hand, the theory of entanglement wedges and subregion-subregion duality [1–13] in AdS/CFT demonstrates that independent regions do exist — at least perturbatively in the $G_N \to 0$ limit. Thus, some spatially separated regions are mutually dependent, while others are not.

For perturbative canonical quantization of GR around flat space, any compact region in the bulk in some sense depends on a region surrounding spatial infinity, since every compactly supported operator must be dressed to infinity when coupled to gravity [14], due to diffeomorphism invariance. No excitation can be turned on in the center of the spacetime without disturbing the gravitational field at infinity. An even stronger notion of dependence on infinity has been shown for perturbations of pure AdS: to first order in perturbation theory, two Wheeler-deWitt (WdW) functionals that agree on observables in a time-band near the boundary must be equal [15]. It might be tempting to think that perturbatively quantized GR itself is holographic, in the sense that perturbative data near the conformal boundary can be used to access perturbative data in the deep bulk, without relying on non-perturbative corrections in G_N^{-1}.[1] But what about other backgrounds? While gravity contains no truly local operators, once the background has sufficiently rich structure, there exist quasilocal operators [22–28] that are dressed features other than conformal infinity. For such backgrounds, one can construct operators that commute with operators near the boundary to all orders in perturbation theory [26, 29, 30]. This shows that we must be cautious about extrapolating lessons about subregion dependence learned from working perturbatively around featureless spacetimes like Minkowski or AdS. At the classical level, these satisfy rigidity theorems [31–34] that strongly constrain the nature of perturbations around these spacetimes.

In this paper, we will show that the question of subregion independence is surprisingly interesting and illuminating already at the classical level. We will demonstrate that the nature of the background strongly influences how and when regions become independent. This will lead to a simple physical picture where classical extremal surfaces play an important role in ensuring subregion independence.

To approach the precise question we will ask in this paper, consider two spatial boundary subregions L, R on the conformal boundary of an asymptotically AdS (AAdS) spacetime. Assume that these are spacelike separated by a small but finite gap — see figure 1. By the so-called split property of local QFT [35–39], we then have a notion of independence for the CFT states on L, R. Namely, for any two independent choices of states ω_L and ω_R on these regions, there exists a pure state $|\psi\rangle$ on the full Hilbert space that agrees with ω_L, ω_R on L, R:[2]

$$\langle\psi| \mathcal{O}_L \mathcal{O}_R |\psi\rangle = \langle\mathcal{O}_L\rangle_{\omega_L} \langle\mathcal{O}_R\rangle_{\omega_R}, \tag{1.1}$$

where $\mathcal{O}_L, \mathcal{O}_R$ are operators in the algebra of (the domain of dependence of) L, R, respectively. Assuming we are in a holographic CFT, ω_L, ω_R encodes data about the entanglement wedges of L, R. Now, let Σ_L, Σ_R be Cauchy slices of the entanglement wedges of L, R, contained in a full Cauchy slice Σ shown in figure 1. In the strictly classical limit of AdS/CFT, we expect that Σ_L, Σ_R inherit some notion of independence from the QFT split property. A natural expectation would be the following: for any two small perturbations of initial data in Σ_L, Σ_R, there exists some complete set of GR initial data on all of Σ that agrees with our perturbed data on Σ_L, Σ_R. Furthermore, this complete initial data is perturbatively close to the initial data we started with. In more pedestrian language: we can wiggle the initial data in the two subregions independently, without making drastic non-perturbatively large changes to the full spacetime.[3] However, there is no one-liner argument in GR showing that this is true, since the constraint equations are elliptic — they have no lightcones. Generically,

[1] See [16–19] for the development of the idea that a version of holography is implied by the fact that the Hamiltonian is boundary term, and also for discussion of the importance of non-perturbative corrections. See also [20, 21] for recent work on these ideas.

[2] Strictly speaking, the split property guarantees a state on $R \cup L$ with this property. But we expect that this state can be purified on the full Hilbert space.

[3] In AdS/CFT language, we do not have to leave the classical limit of our code subspace where the entanglement wedges were defined.

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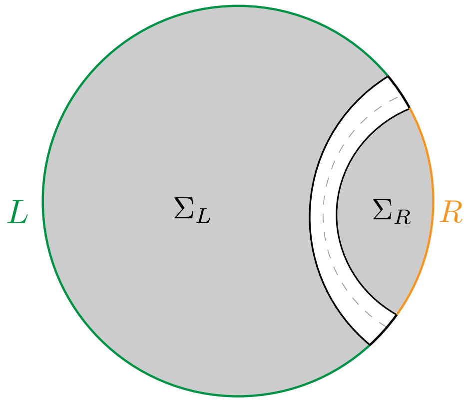

Figure 1. Illustration of an AdS Cauchy slice Σ containing Cauchy slices Σ_L, Σ_R for the entanglement wedges of the boundary regions L, R. Taking the classical limit of AdS/CFT, we expect that we can make perturbations of the classical initial data in Σ_L and Σ_R independently.

modifications of matter (or pure gravity degrees of freedom) in some compact region A source gravitational fields spacelike to A, so how do we know that there are no perturbations of initial data in Σ_L that requires us to change fields in Σ_R? We might be tempted to mumble words about dressing observables to R and L, but this is not quite satisfactory. First, we should be able to ask this question at the level of solutions of the classical constraint equations of GR. Second, and more importantly, this leaves no room for the extremal surface playing any kind of special role — this explanation would lead us to conclude that any two disjoint regions $\Sigma_1, \Sigma_2 \subset \Sigma$ connected to the asymptotic boundary and separated by a gap are independent. It is not clear that this is true.

To illustrate the problem more clearly, let us for a brief moment consider a different example where we model gravity as electromagnetism coupled to matter with only positive charge density, living on a fixed background geometry. In this case, Gauss' law is a stand-in for the Hamiltonian constraint of GR, the electric field E^i for the gravitational field, and the charge density ρ for the energy density. Consider now this theory on a fixed AdS-Schwarzschild background, and let Σ_L, Σ_R be the left and right sides of the bifurcation surface on a canonical $t = 0$ slice. We let Σ_L, Σ_R terminate slightly to the left and right of the bifurcation surface, so that there is a small open region C between these slices that contains the bifurcation surface.[4] See figure 2. A perturbation of the trivial $E^i = \rho = 0$ initial data on Σ_L is the following: add a spherically symmetric shell of charged matter and choose that the electric field is unchanged near left infinity. This is compatible with Gauss' law on Σ_L. However, now Gauss' law requires the new electric field lines sourced by the shell to travel towards the right. Because our theory contains no negative charges, there is nothing we can do in C to absorb the field lines before they reach Σ_R. They will travel all the way to right infinity. See figure 2. The initial data on Σ_L and Σ_R cannot be chosen independently.

[4]This case has a qualitative difference from the previous example. Σ_L and Σ_R are slightly smaller than Cauchy slices of the entanglement wedges of the left and right boundaries, R, L. This is because ω_L, ω_R now cannot be chosen completely independently. If their entanglement entropies disagree, there is no pure state $|\psi\rangle$ on the joint Hilbert space that reduces to ω_R, ω_L. So we leave a small number of degrees of freedom flexible, to avoid this obstruction.

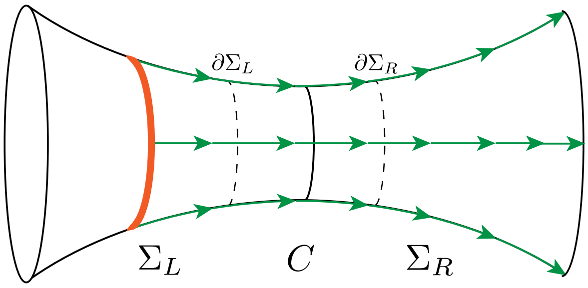

Figure 2. Initial data for electromagnetism coupled to matter on a fixed AdS-Schwarzschild background, on a canonical $t = 0$ slice. The regions Σ_L, Σ_R are not independent when electromagnetism is coupled to charge of a fixed sign. If we pick initial data on Σ_L corresponding to adding a charged shell of matter, and with the $E^i = 0$ at left infinity, the Gauss law imposes that $E^i \neq 0$ at right infinity.

Now let us turn to gravity proper. If the Hamiltonian constraint behaved like Gauss' law coupled to matter with only positive charge densities, this would not bode well for the independence of initial data perturbations in Σ_R and Σ_L in gravity. However, except in the special case of perturbations around pure AdS or Minkowski, Gauss' law coupled to purely positive charge densities is a poor analogy for the Hamiltonian constraint. Crucially, the total gravitational mass is not just an integral of the local energy density weighted by something positive. As we will explain, extremal surfaces play a very practical role in ensuring that complementary entanglement wedges are independent (after you cut out a small splitting region). Specifically, we are going to show that if you live in Σ_R, positive energy density added to the other side of the bifurcation surface, from your perspective, acts like a negative energy density! Furthermore, this effect is present for general extremal surfaces. This means that positive energy densities can be used to terminate gravitational fields sourced by positive energy densities on the opposite sides of an extremal surface.[5] For compact extremal surfaces, this effect cannot be understood by studying the Hamiltonian constraints perturbatively around AdS or flat space, or by relying on analogies to Gauss' law. It relies on the non-linear nature of gravity, which causes structures of the background to significantly affect when subregions allow independent perturbations of initial data. We argue that this effect relieves a tension that was argued to exist between islands and massless gravity in [40] (see also [29, 30] for arguments that islands and massless gravity are consistent). It also ensures the independence of Σ_L and Σ_R in the example considered above.

Next, we will prove or argue, depending on the setup, that trapped regions of spacetime always have a kind of indeterminate energetic behavior. From the perspective of the mass in any asymptotic region, there are always some modes in a trapped region that, when turned on, increase the mass, and others that reduce it.[6] Together, these enable perturbations of spatial compact support. This gives subregions of a trapped region a larger degree of

[5]Strictly speaking, we are able to show this in all dimensions only with spherical symmetry, while in four spacetime dimensions we are able to remove this assumption.

[6]This has of course long been understood for stationary black holes, where the Killing vector that defines the asymptotic energy flips signature in the bulk, but we will make it clear more generally that trapped regions always have this behavior.

independence from nearby regions, and we will use it to argue that the islands found in the evaporating black holes of [41, 42] easily can host a large number of localized excitations that never affect the black hole exterior. Along the way we will also show the following fact: an object's contribution to the ADM mass approaches zero as the object localizes on an extremal surface (see sections 4.2, 4.3, for precise statements). So around an extremal surface, we can add arbitrarily high energy objects (as measured locally) with no or arbitrarily small change in the ADM mass.[7]

In the following, we first give a precise definition of subregion independence in GR in section 2 and argue why it is a useful definition. We explore our definition in a series of illuminating examples in section 3. We find that regions separated by extremal surfaces "shields" of matter tend to classically independent. Sometimes trapped surfaces can play a similar role. Of course, we do not expect that this independence generally persists (or can be defined) in full non-perturbative quantum gravity, but it does appear plausible that this independence could be inherited by a perturbative quantization of GR around a fixed background.

Then, in section 4, we conduct a more general analysis. For spacetimes with spherical symmetry, we prove that the behaviors we found in the examples are general: gravity truly behaves as if it has negative and positive energies when extremal or trapped surfaces are present. In four-dimensional spacetime, we are also able to remove the spherical symmetry assumption, giving strong arguments at the physics level of rigor. We do expect that the physical effects we identify to survive in other dimensions, although we do not know what are the right tools to show this in spacetimes without symmetries.

Then, guided by findings when studying independence of subregions of de Sitter in section 3, in section 5 we prove a new theorem upper and lower bounded the area of a family of surfaces that include certain extremal and trapped surfaces in asymptotically dS spacetimes. A corollary of the theorem states that under spherical symmetry and given an energy condition, adding matter in one static patch necessitates adding matter in the opposite patch. Furthermore, adding matter always reduces the area of a certain extremal/trapped surface. This result holds at the full non-linear level.

Finally, in section 6 we discuss implications of our findings for semiclassical gravity, quantum extremal surfaces, islands in massless gravity, localized observables, and the Python's lunch.

2 Subregion independence

2.1 The constraint equations

To understand subsystem independence, we need to deal with the gravitational constraint equations. Let Σ be a manifold, possibly with boundary, and let

$$S \equiv (h_{ab}, K_{ab}, \Phi) \tag{2.1}$$

[7]This plays nicely with the ideas of [43], which propose that Wilson lines threading a wormhole really, in the bulk UV, are split into factorizing operators by heavy charges residing near the bifurcation surface. This ensures factorization. The fact that general classical extremal surfaces can host heavy objects without large cost in energy supports the viability of this idea in a large class of backgrounds.

be a regular initial dataset for the Einstein equations. Here h_{ab} is a Riemannian metric on Σ, K_{ab} a symmetric tensor, and Φ a label that collectively denotes initial data for all of the matter fields. For example, when the matter is a scalar field, Φ consists of the value of the scalar and its time-derivative on Σ. For any choice of S, assuming the matter is reasonably well behaved, there is a unique (up to diffeomorphism) globally hyperbolic spacetime (M, g) corresponding to the maximal evolution of (Σ, S) [44–47]. The pair (Σ, S) represents a moment of time in (M, g), with Σ being an embedded spatial slice, and with h_{ab}, K_{ab} being the induced metric and extrinsic curvature of Σ, respectively. Note that we never consider initial data such that Σ has a kink or ends in a singularity. If we extract an initial data slice (Σ, S) from an existing spacetime (N, g), unless otherwise specified, we always assume it is globally (AdS-)hyperbolic, with Σ being a Cauchy slice.[8]

Due to diffeomorphism invariance, initial data is not freely specifiable. Instead, it has to satisfy a set of constraint equations. In the case of Einstein gravity minimally coupled to matter, the constraint equations read (in units of $8\pi G_N = 1$)

$$\mathcal{R} - K_{ab}K^{ab} + K^2 - 2\Lambda = 2\mathcal{E}, \tag{2.2}$$

$$\mathcal{D}^b K_{ba} - \mathcal{D}_a K = \mathcal{J}_a, \tag{2.3}$$

where $\mathcal{R}$ is the Ricci scalar of h_{ab}, $\mathcal{D}_a$ the h_{ab}-compatible connection on Σ, $K = K^a{}_a$, and Λ the cosmological constant. $\mathcal{E}$ and $\mathcal{J}_a$ are the matter energy and momentum densities, respectively. From the spacetime perspective, if n^a is the future unit normal to Σ, P_{ab} the projector onto the tangent space of Σ, and T_{ab} the matter stress tensor, then $\mathcal{E} = T_{ab}n^a n^b$ and $\mathcal{J}_a = T_{cb}n^b P^c{}_a$. Equations (2.2) and (2.3) are known as the Hamiltonian and momentum constraints, respectively. Unless otherwise stated, we will assume the weak energy condition (WEC), which says that the local energy density is positive:

$$\mathcal{E} \geq 0. \tag{2.4}$$

As will become clear soon, we are in an unusual situation where we are assuming the WEC to make our lives harder, not easier. The positivity of local energy densities is a significant obstruction to achieving independence of subregions in gravity, so if two regions are independent despite the WEC, then we expect them to be independent also when the WEC is violated.[9]

As alluded to in the introduction, the central property of the constraint equations for us is that they are not hyperbolic equations. There are no lightcones. A change in the matter fields with support on some region $A \subset \Sigma$ might demand a change in the gravitational or gauge fields in a different spacelike separated region $B \subset \Sigma$. This type of behavior is of course already present in the Maxwell equations. For example, at $t = 0$ in Minkowski spacetime, it is impossible to add an electron without also adding an electric field which either reaches infinity, or alternatively, also adding a positron that the field can terminate on. From the QED perspective, this is because there are no gauge-invariant operators that create a single

[8] For AAdS spacetimes, which are technically not globally hyperbolic, we assume the AdS notion of global hyperbolicity [5].

[9] Modulo other complications that might arise if WEC breaking comes from considering quantum effects.

electron at a point x. Instead one needs to dress the operator to restore gauge invariance, for example by adding a Wilson line that reaches out to infinity:

$$e^{i\int_x^\infty A}\psi(x), \tag{2.5}$$

where A, ψ are the gauge and Dirac fields, respectively. This adds precisely the required electric field to satisfy the constraints. However, there is no need study the quantum theory to see the need for dressing — the effect is already there in the classical Gauss' law. This suggests that a careful understanding of the constraints, even at the classical level, can give hints about what gauge invariant operators, and thus what algebras of observables, can exist. This is an additional motivation for this study. We will see the Einstein constraint equations yield behavior significantly richer than Gauss' law.

2.2 Subregion independence and dressing

To define subsystem independence, we need to look at perturbations to the constraints. We will use $\delta S \equiv (\delta h_{ab}, \delta K_{ab}, \delta\Phi)$ to denote a regular perturbative solution of the constraint equations around some initial dataset S (Σ is often left implicit). The small perturbation parameter is the amplitude of the perturbation. Unless otherwise specified, we assume δS is a formal series solution to all orders in perturbation theory in this amplitude, and not just a linearized solution. We always choose the lowest order terms in the perturbation so that the leading order correction to the stress tensor and the metric is at the same order. Note that some perturbations do not correspond to a physical change of the spacetime, but instead to a diffeomorphism, either changing coordinates within Σ, or moving us to a different slice nearby Σ in the same spacetime. We allow these.

Next, consider a subregion $A \subset \Sigma$. We will use the notation $\delta S|_A$ to refer to a perturbative constraint solution restricted to A, and we will refer to a perturbative constraint solution δS specified on all of Σ as an *extension* of $\delta S|_A$ if it restricts to $\delta S|_A$ on A. If an extension exists, it will generally not be unique. If we specify a perturbation of the constraint on two disjoint subregions A, B, we take $\delta S|_A \cup \delta S|_B$ to mean the obvious concatenated perturbation on $A \cup B$. Finally, since we have in mind working in some particular theory, when we talk about perturbations, we only ever talk about perturbations that respect the boundary conditions imposed on the boundaries of Σ by our theory — either at finite boundaries or at spatial infinity.

We are now ready to define when A and B are independent:

Definition 1 (Independence)**.** *Let (Σ, S) be an initial dataset, and let $A, B \subset \Sigma$ be two disjoint subregions. We say that A and B are independent if for any two perturbations $\delta S|_A$ and $\delta S|_B$, there exists a perturbative extension δS of $\delta S|_A \cup \delta S|_B$ to all of Σ.*

Definition 1 simply says that it is possible to wiggle the initial data in A, B completely independently, provided these wiggles are perturbatively small.[10] If A and B are not

[10] Rather than perturbative deformations, we could alternatively work with finite but arbitrarily small deformations with respect a C^k or Sobolev norm defined by the background metric, as is common in mathematical studies of the constraints. See for example [46, 47] for a discussion of such norms. We expect that the physical effects we discuss in this paper are robust to taking this approach instead.

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independent, we say that they are dependent. If A and B are independent, it is natural to say that classical gravity satisfies the split property across $C \equiv \Sigma - A - B$. This means that for any (perturbative) choice of the classical state on A and B, there exist a classical state on all of Σ that agrees with the chosen states on A and B. Note that it is important that we work within a fixed theory, which comes with a set of allowed matter fields and boundary conditions. We might think that we can use gluing constructions supported by delta functions shocks to make more or less anything independent, but this is not so. Gluings will generically produce distributional stress tensors that do not satisfy any energy conditions. Even if they do, there is no guarantee that regularizations of these shocks can be achieved by matter in our theory. So we might as well stick with genuinely regular initial data.

Next, note that since pure diffeomorphisms are allowed, perturbing the boundaries of the regions A, B count as perturbing the initial data in A and B. To see this, consider the data S and assume that the regions A, B are fixed in some gauge. Assume now that a diffeomorphism $\psi : \Sigma \to \Sigma$ maps the boundaries of A, B to new locations. An initial dataset in which the boundaries have moved, but with the spacetime and the slice Σ left fixed, is $(\Sigma, \psi[A], \psi[B], S)$. By diffeomorphism invariance, this is equivalent to $(\Sigma, A, B, \psi_*[S])$, where ψ_* is the pull-back.[11] This logic can easily be generalized to a diffeomorphism perturbing A, B out of the slice. Thus, the regions A and B are not completely fixed — they are fuzzy at the perturbative scale. Alternatively said, the perturbative edges modes [48] of A and B can be activated when testing for independence. Before we apply a physical perturbation in A that changes the spacetime, we are allowed to deform the boundaries of A by a perturbative amount first — or the other way around.

Definition 1 can be applied to classical non-gravitational theories as well. For theories without constraints, such as a scalar field ϕ, all regions A, B separated by an open set C are independent. We can always match the initial data $\phi, \dot{\phi}$ in A, B smoothly across C, since they are completely unconstrained there. However, the *splitting region* C was important. Even in free scalar theory, two regions with intersecting closures are dependent. Next, even in the presence of a constraint like Gauss' law of electromagnetism, on Minkowski space all spacelike separated subregions are completely independent, provided we couple to charged matter with both signs, as explained in the introduction. If we add a positive charge density in A, we just need to make sure to compensate with a negative charge density in C to screen any new multipole moments incurred in B.[12]

In gravity things are more interesting. Because of the positivity of local energy densities, it is generally harder to screen perturbations, and the question of independence is non-trivial. As an example, consider Σ to be a canonical $t = 0$ slice of Minkowski or AdS. Let $A \subset \Sigma$ be a compact region and $B \subset \Sigma$ an infinite annulus containing spatial infinity. By the positive mass

[11] Note that if ψ is a "large diffeomorphism" (not large in the perturbative sense, but in the sense that it has non-trivial action on the boundary of spacetime), then we have both changed the full physical state, in addition to moving A and B.

[12] In vacuum electromagnetism on Minkowski, we also have independence of subregions [49]. However, on non-trivial manifolds, the question of independence is more interesting. For example, as pointed out in [50], vacuum electromagnetism on a spatial torus will not have subregion independence. For example, on T^2, this is because the value of the electric charge on two homologous non-contractible S^1's must match in different regions.

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theorem (PMT) [32–34, 51, 52], any perturbation in A which is not a pure diffeomorphism must increase the spacetime's mass. Since the ADM Hamiltonian is a boundary term localized to B, this implies that the gravitational data in B must be perturbed as well. Thus A and B are dependent. Pure AdS and Minkowski support no perturbations of compact support, except for pure gauge.[13] From the perspective of perturbative quantum gravity around flat space and AdS, this can be seen as the classical avatar of the statement that operators must be dressed to infinity [14], for example with a gravitational Wilson line (see for example [53–56]). Effectively, there is no negative energy density to terminate the Wilson line on.

Going away from pure AdS or Minkowski, since matter has positive energy density, it naively seems like we are always forced to dress a perturbation to asymptotic infinity, in turn causing every compact subregion to depend on an annulus around spatial infinity. However, this conclusion is much too quick. When certain geometric features are present, such as extremal and trapped surfaces or lumps of matter, localized dressings become possible.

To streamline the discussion going forward, it is useful to give a precise notion of classical dressing of a perturbation (as opposed to an observable):

Definition 2 (Dressing)**.** *Let (S, Σ) be an initial dataset. Let A be a subregion and $\delta S|_A$ a perturbation of the constraints on A. If δS is an extension of $\delta S|_A$, we say that δS is a dressing of $\delta S|_A$ to the region supp(δS).*

We should emphasize that given some $\delta S|_A$, the choice of gravitational dressing is highly non-unique. The independence of A and B implies that for every perturbation of A, there exists some choice of dressing that does not intersect B. Furthermore, after dressing any perturbation on A to C, we still have the ability to dress any perturbation in B to C as well. The statement of independence is not that a certain perturbation in A could not be dressed to B, if so desired.

Next, let us address two questions that naturally arise from our definition. What about regions in spacetime that are not conveniently described as lying on the same slice? Why only perturbative deformations?

First, in globally (AdS-)hyperbolic spacetimes,[14] it is straightforward to extend our notion of independence to two causal diamonds D_A and D_B. First, if D_A and D_B overlap or are timelike separated, they should clearly not be considered independent, so assume that they are spacelike separated. Let Σ be any (AdS-)Cauchy slice Σ, and define

$$\begin{aligned} A_\Sigma &= \Sigma \cap (J^+[D_A] \cup J^-[D_A]), \\ B_\Sigma &= \Sigma \cap (J^+[D_B] \cup J^-[D_B]), \end{aligned} \tag{2.6}$$

where $J^+[X]$ ($J^-[X]$) is the usual causal future (past) of a given set X. We say that D_A and D_B are independent if there exists a Cauchy slice Σ such that B_Σ and A_Σ are independent, in the sense already defined. See figure 3. This is natural, since by causality and global hyperbolicity, the gravitational initial data on A_Σ, B_Σ contains all information about D_A, D_B.

[13] It is crucial to not just work to linear order in perturbations, since compact perturbations of Minkowski exist at linear order [49].

[14] We can also apply this definition within a causal diamond of an AdS spacetime without reflecting boundary conditions.

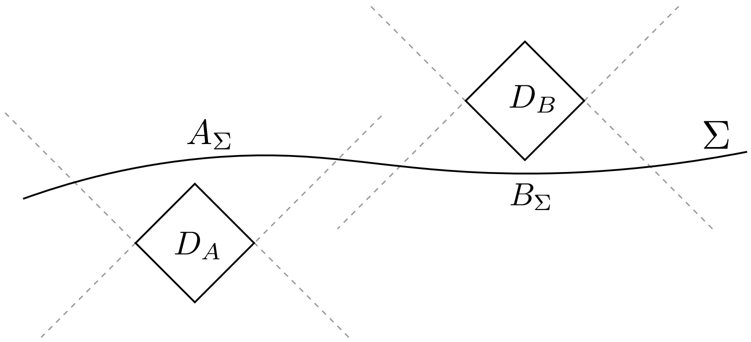

Figure 3. D_A and D_B are independent if there exist a slice Σ such that A_Σ and B_Σ are independent.

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Thus, any perturbation in D_A and D_B can, through the evolution equations, be translated into a corresponding perturbation on A_Σ and B_Σ.

Second, the reason that we only consider perturbative deformations is that if we allow large deformations of the initial data, there is no sense in which the regions are fixed. If we could modify the data in A, B arbitrarily much, we could deform it to any data that is compatible with the topologies of A, B, or use a diffeomorphism to move the boundaries of A, B arbitrarily much. But then we would in the end only be asking the following question: does there exist interpolating initial data between every possible choice of initial data on two regions with the given topologies of A and B. But why do we even care to keep topologies fixed at this point? Thus, going beyond the perturbative level, we quickly lose all connection to our original question. This problem is reflective of the deeper underlying fact that, on the global phase space of GR, there is probably no useful diffeomorphism invariant notion of a subregion. However, when we restrict to perturbative deformations, we let the regions and the states become fuzzy, but only at the scale of the perturbative parameter. In language used in AdS/CFT contexts, asking about independence of two spacetime regions seems to be a natural question only in a code subspace built of perturbations around a single background (and, at the quantum level, in the small$-G_N$ limit).

It is worth making a few more comments on the non-perturbative case. However, this is a detour, and the reader can freely skip the next section without any loss of coherence in the rest of the paper.

Gluing constructions and independence. Consider making small deformations of initial data in $A, B \subset \Sigma$, but not requiring that the change outside $A \cup B$ is small — i.e. we allow jumping to a point in the GR phase space that is not near our original point. Can we then find an initial dataset interpolating between A and B, and which still has the requisite number of conformal boundaries? While this question has little relevance to understanding perturbative quantization around a background, it is nevertheless interesting to see what happens in this case. In vacuum gravity, it can be partially answered, thanks to the gluing result of [57]. This result says that in vacuum gravity, if Σ is a smooth initial dataset and $\Omega_1, \Omega_2 \subset \Sigma$ are two open sets whose domains of dependence have no Killing vectors, then we can cut out geodesic balls in both Ω_1, Ω_2 and glue in a handle $[0, 1] \times S^{d-1}$ to connect Ω_1, Ω_2

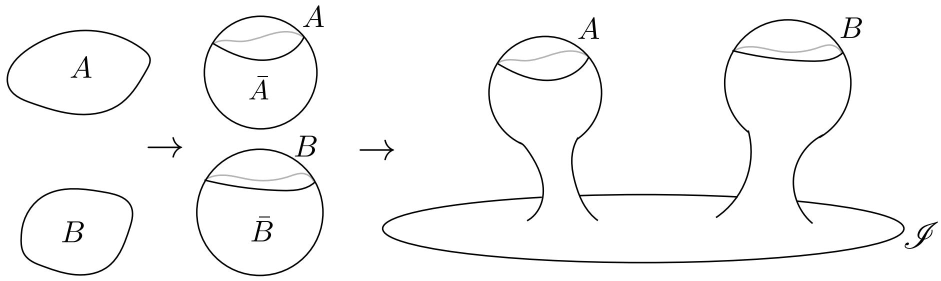

Figure 4. Using the gluing construction of [57] to embed two compact datasets in a complete dataset with a pre-specified conformal infinity $\mathscr{I}$.

with a wormhole. Furthermore, this can be achieved so that the final initial data is smooth and unchanged outside $\Omega_1 \cup \Omega_2$. Using this, if $A, B \subset \Sigma$ are compact and sufficiently generic, then we can embed them in a common initial dataset through the following procedure: adjoin A and B to some compact initial datasets $\bar{A}$ and $\bar{B}$ such that $\bar{A} \cup A$ and $\bar{B} \cup B$ are complete compact initial datasets, with $\bar{A}, \bar{B}$ both having at least one neighbourhood whose domain of dependence has no Killing vectors. Then take another sufficiently generic initial dataset with the required number of conformal boundaries, and use the gluing result to attach two wormholes to it — one connecting to $\bar{A}$ and one to $\bar{B}$. See figure 4. One can see that a similar procedures works if A and/or B merely have compact boundaries, so that A and B might contain complete connected components of conformal infinity. However, if ∂A and ∂B themselves are anchored to conformal boundaries, while we again can find a common initial dataset containing A and B using gluings, the complete set of initial data might have too many conformal boundaries, since the naive type of procedure described above will not connect any of the subregions of the conformal boundary that might be present in A and B.

3 How background structures enable independence

In this and the following section, we will show how extremal surfaces, matter, and generic trapped surfaces can be used to dress perturbations, avoiding dressing to asymptotic regions. They are background structures that enable subregion independence. We will first study a set of illustrative examples in this section, where we restrict to particular backgrounds with spherical symmetry. Then, in section 4 we will give a more general discussion, relaxing many of the simplifications made in our examples.

3.1 Setup

Our example will be Einstein gravity in $d+1$ spacetime dimensions coupled to a massless scalar, with action

$$I = \int \mathrm{d}^{d+1}x\sqrt{-g}\left[R + \frac{d(d-1)}{L^2} - \nabla_a\phi\nabla^a\phi\right]. \tag{3.1}$$

For positive cosmological constant, L is imaginary, and we set $L = iL_{\mathrm{dS}}$.

Consider now a spherically symmetric spacelike initial data slice Σ. Because of spherical symmetry, we can cover Σ with coordinate patches U_i, such that on each patch, we have

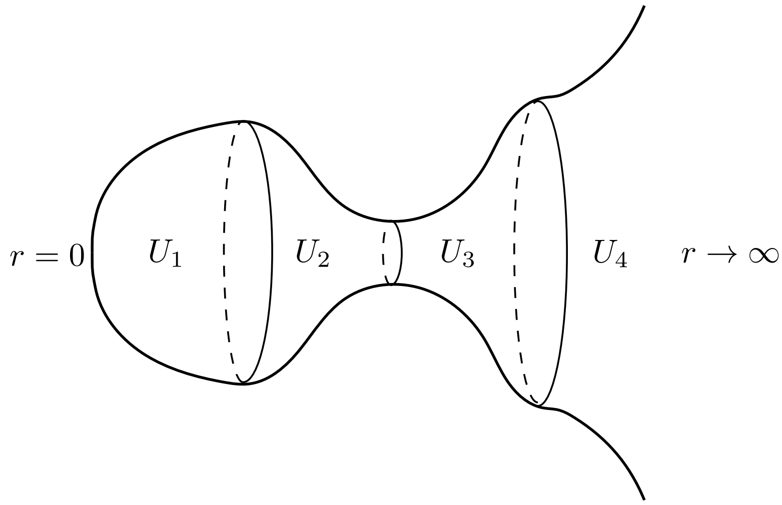

Figure 5. An example of a $d = 2$ spacelike slice Σ covered by four coordinate patches of the type (3.2), separated by every type of stationary surface. At these surfaces, $B = \infty$.

local coordinates

$$h_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu = B(r)\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{3.2}$$

where $\mathrm{d}\Omega^2$ is the round metric S^{d-1}. These coordinates break down at some given radius $\hat{r}$ if and only if $B(\hat{r}) = \infty$. $B(\hat{r}) = 0$ can be shown to correspond to curvature singularity for h_{ab}, which we do not consider. By computing the mean curvature of a sphere of radius r within Σ, we can see that $B = \infty$ precisely corresponds to the sphere having vanishing mean curvature [58], meaning that the sphere is locally minimal, maximal, or saddle-like within Σ. See figure 5.

Next, assume furthermore the mean extrinsic curvature in spacetime vanishes: $K_a{}^a = 0$. For nonpositive cosmological constant, this usually means that Σ is a maximal volume slice, while for a positive cosmological constant, it can also mean that Σ has minimal volume. The extrinsic curvature is conveniently parametrized by a single function $\mathcal{K}(r)$ as[15]

$$K_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\mu = \mathcal{K}(r)\left[B(r)\mathrm{d}r^2 - \frac{r^2}{B(r)(d-1)}\mathrm{d}\Omega^2\right]. \tag{3.3}$$

$\mathcal{K}(r)$ is simply the rr-component in an orthonormal basis, and so must be everywhere finite, otherwise Σ would have a singular embedding in spacetime. Furthermore, on a slice Σ whose embedding in spacetime does not have any kinks, $\mathcal{K}(r)$ must be continuous across coordinate patches.

It turns out that our analysis will greatly simplify if we introduce the function $\omega(r)$ as

$$B(r) = \frac{1}{1 + \frac{r^2}{L^2} - \frac{\omega(r)}{r^{d-2}}}. \tag{3.4}$$

[15]There is only one function's worth of degrees of freedom in K_{ab}, thanks to spherical symmetry and $K^a{}_a = 0$.

In terms of this variable, a stationary surface is characterized by

$$0 = 1 + \frac{\hat{r}^2}{L^2} - \frac{\omega(\hat{r})}{\hat{r}^{d-2}}. \tag{3.5}$$

When we patch together two coordinate systems at some $r = \hat{r}$, ω must match in the two patches, since $\omega(\hat{r})$ is uniquely fixed by $\hat{r}$. Next, we have that in a patch containing conformal infinity, where we can take $r \to \infty$, that [58][16]

$$M = \frac{(d-1)\mathrm{Vol}[S^{d-1}]}{16\pi G_N}\omega(\infty), \tag{3.6}$$

where M is the ADM mass, or the AdS analogue. We refer to ω as the Hawking mass.[17]

Computing the constraints in our setup, we find the equations

$$(d-1)\frac{\omega'(r)}{r^{d-1}} = \left(1 + \frac{r^2}{L^2} - \frac{\omega}{r^{d-2}}\right)\phi'(r)^2 + \dot{\phi}(r)^2 + \frac{d}{d-1}\mathcal{K}(r)^2, \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^d\mathcal{K}(r)\right] = r^d\phi'(r)\dot{\phi}(r), \tag{3.8}$$

where the first equation is the Hamiltonian constraint, and the second the momentum constraint. Since each coordinate patch has its own set of functions $\{\omega, \mathcal{K}, \phi, \dot{\phi}\}$, we will sometimes write ω_{U_i} to indicate the ω-function on the i-th patch, and similarily for other quantities. To obtain the solution on all of Σ, we solve these equations on each patch, matching $\omega, \mathcal{K}$ across each junction. In a spacetime with two asymptotic regions, there are two integration constants, corresponding to ω and $\mathcal{K}$ at a single value of r. These are the purely gravitational degrees of freedom in spherical symmetry. Without matter, where every solution is (AdS-)Schwarzschild by the Birkhoff-Jebsen theorem [66–69], these numbers determine M and $t_L + t_R$, where t_L, t_R are the times on the left and right conformal boundaries at which Σ is anchored. With only one conformal boundary, or in a spacetime without any conformal boundary, $r = 0$ is present on at least one coordinate patch. The spacelike nature of Σ ($B > 0$) then requires that $\omega(r = 0) = 0$, while smoothness of the embedding of Σ in spacetime requires $\mathcal{K}(r = 0) = 0$.

Now come the important observations. Since we are on a spacelike slice, we must have $B(r) > 0$, meaning that the prefactor of $\phi'(r)^2$ term in (3.7) is positive, and thus also the full right hand side. We conclude that

$$\omega'(r) \geq 0. \tag{3.9}$$

ω is monotonically non-decreasing as we move towards increasing r. This conclusion does not rely on our particular theory. It is true as long as we have the WEC (2.4). It is the monotonicity of ω that makes excitations in gravity more difficult to screen. However, even with one conformal boundary, the direction of increasing r does not always point towards the

[16]This relies on $K^a{}_a = 0$, or that $K_a{}^a \to 0$ sufficiently fast at infinity.

[17]There are two versions of the Hawking mass: the Riemannian (also known as the Geroch-Hawking mass) [33, 59–61] and the Lorentzian versions [62–65], with the former not being directly sensitive to K_{ab}. ω is the Riemannian version, suitably generalized to $d \neq 3$ and $\Lambda \neq 0$, in the spacial case of spherical symmetry.

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conformal boundary. This is at the heart of all our subsequent results. It means that adding matter in a patch of spacetime were r is decreasing towards the boundary can decrease the ADM mass, despite the fact that the matter has positive local energy density.

We are now going to use eqs. (3.7) and (3.8) to study subregion independence and dressing of gravitational perturbations. This requires us to study the linearized versions of (3.7) and (3.8) around some background geometry. We will restrict to spherical perturbations preserving $K_a^a = 0$ in our explicit analysis, and give arguments that our conclusions are unchanged under general perturbations. Now, the scalar matter corresponds to completely free data; thus we can organize our perturbative expansion as follows:

$$\begin{aligned}
\phi &= \phi_0(r) + \kappa\delta\phi(r), \\
\dot{\phi} &= \dot{\phi}_0(r) + \kappa\delta\dot{\phi}(r), \\
\omega &= \omega_0(r) + \sum_{i=1}^{\infty}\kappa^i\delta_i\omega(r), \\
\mathcal{K} &= \mathcal{K}_0(r) + \sum_{i=1}^{\infty}\kappa^i\delta_i\mathcal{K}(r),
\end{aligned} \tag{3.10}$$

where $\delta\phi, \delta\dot{\phi}$ is free data, and with $\delta_i\omega, \delta_i\mathcal{K}$ determined by the constraints. κ is the small perturbative parameter controlling our expansion.[18] We could also adjust $\delta\phi, \delta\dot{\phi}$ more finely at higher orders, but we will not need this.

Note that the fully non-perturbative solutions of (3.7), (3.8) can be written down exactly as nested integrals over the sources $\phi(r), \dot{\phi}(r)$ (see for example [58]), but we will not need these solutions.

3.2 Subregion independence in asymptotically flat and AdS spacetimes

We have already shown how a compact region in the center of AdS or Minkowski depends on an annulus around infinity. From the perspective of spherically symmetric perturbations, this follows from integrating (3.7) from $r = 0$ to $r = \infty$ and using $\omega(r = 0) = 0$, giving that

$$\omega(\infty) = \omega(\infty) - \omega(0) = \int_0^{\infty} \mathrm{d}r(\text{positive}) > 0. \tag{3.11}$$

Thus, let us now consider more interesting geometries. For notational convenience we take $d = 3$ and a vanishing cosmological constant — the analysis is virtually identical for AdS and/or $d \neq 3$. We will treat the $\Lambda > 0$ case separately.

Schwarzschild. Let us consider the simplest non-trivial background: Schwarzschild with mass M. Let Σ be a canonical $t = 0$ slice, which has $K_{ab} = 0$. Let B be an infinite annulus containing right infinity, and let A be a finite annulus in the right region, restricted to $r \in [r_{A,\text{in}}, r_{A,\text{out}}]$, with $r_{A,\text{in}} > 2M$. See figure 6. The perturbed leading order constraints read[19]

$$\begin{aligned}
2\delta_2\omega'(r) &= r^2 f(r)\delta\phi(r)^2 + r^2\delta\dot{\phi}^2, \\
\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3\delta_2\mathcal{K}(r)\right) &= r^3\delta\phi'(r)\delta\dot{\phi}(r),
\end{aligned} \tag{3.12}$$

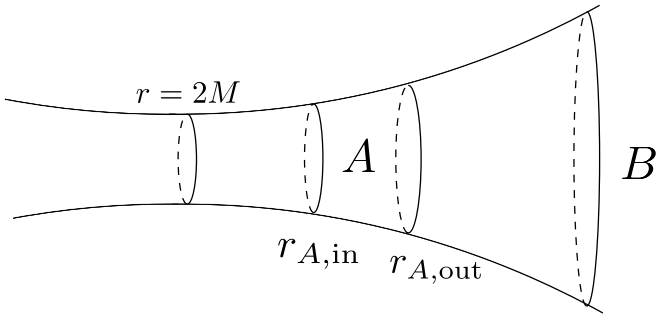

Figure 6. A bounded annular region A on a $t = 0$ slice in Schwarzschild, with one spatial dimension suppressed. A and B are dependent regions.

with $f(r) = 1 - 2M/r$. The free data in A are the functions $\delta\phi(r), \delta\dot{\phi}(r)$ for $r \in [r_{A,\text{in}}, r_{A,\text{out}}]$, and two numbers, corresponding to $\delta_2\omega, \delta_2\mathcal{K}$ a single value of the radius in A. These two numbers are the only spherically symmetric gravitational degrees of freedom. To test for independence, we are free to consider a perturbation with $\delta_2\omega(r_{A,\text{in}}) = \delta_2\mathcal{K}(r_{A,\text{in}}) = 0$, and add some matter in A. With the choice $\delta_2\omega(r_{A,\text{in}}) = \delta_2\mathcal{K}(r_{A,\text{in}}) = 0$, we are choosing $\delta S|_A$ such that the new gravitational field sourced by $\delta\phi$ is forced to travel to the right. Then, integrating (3.12) from $r_{A,\text{in}}$ to $r = \infty$ gives that $\delta_2\omega(\infty) > 0$ on the right, and so A and B are dependent. The background has no matter that we can remove outside A to screen the perturbation. One could wonder if a non-symmetric perturbation outside A could be used to screen the perturbation, but in this case, we can appeal to a rigorous theorem, the so-called Riemannian Penrose inequality [70–72],[20] to show that no perturbation whatsoever can save us.[21] This theorem says that if we have an asymptotically flat initial dataset with ${K_a}^a = 0$, with one asymptotic end with mass M, and with an inner boundary σ that is an outermost minimal surface, then

$$M \geq \sqrt{\frac{\text{Area}[\sigma]}{16}}, \tag{3.13}$$

with equality if and only if we are identically Schwarzschild. Our chosen perturbation in A was compatible with keeping the geometry at the minimal surface fixed, and so if a non-symmetric dressing that did not reach B existed, it would violate the rigidity statement of the Riemannian Penrose inequality. So graviton degrees of freedom cannot make A and B independent.

We thus see that any bounded region A lying strictly in the right exterior is dependent on right infinity. Is A dependent on left infinity? To test this, again pick a perturbation where we add some matter in A. However, we now use our gravitational degree of freedom to

[18]Morally, we could think of κ as $\sqrt{G_N}$, although in the purely classical theory, there is no preferred universal scale. If we were to sum the series, κ should be adjusted according to the scales of the background solution.

[19]Since the leading order perturbation of δT_{ab} is $\mathcal{O}(\kappa^2)$, we do not consider $\delta_1\omega \neq 0$.

[20]In the AdS case, this inequality is still conjectural, and only partial proofs exist [58, 73–76].

[21]We do assume that $K^a_a = 0$ is preserved under these perturbations, but is not a severe restriction, since small perturbations should not break the existence of a slice with maximal volume. So even if we perturbed away from $K^a_a = 0$, there should be a nearby slice in spacetime where $K^a_a = 0$, so we can always take our perturbation to be a combination of the perturbation that changes spacetime, together with a diffeomorphism that moves us to this slice.

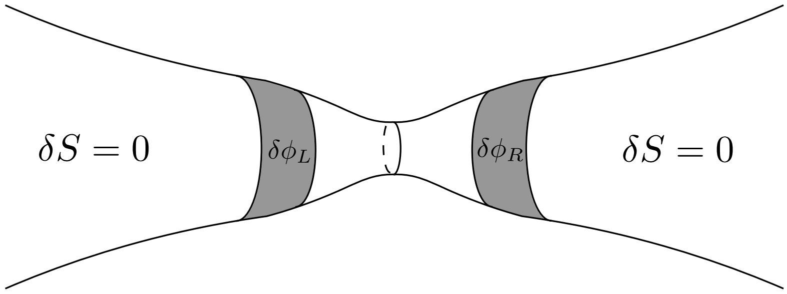

Figure 7. A perturbation $\delta\phi_R$ on the right side dressed to a perturbation on the left side, leaving the initial data both asymptotic regions unchanged. This causes the area of the minimal surface to decrease.

set $\delta_2\omega(r_{A,\text{out}}) = \delta_2\mathcal{K}(r_{A,\text{out}}) = 0$ instead, forcing the new gravitational field lines sourced by $\delta\phi$ to travel to the left. This seems dangerous: will we now be forced to dress to left infinity? The answer is no! We can screen the perturbation from left infinity by **adding** more matter, provided we add it to the left of the bifurcation surface. Integrating the constraints, and matching $\delta_2\omega, \delta_2\mathcal{K}$ across the coordinate systems, we find that the functions on the left side are[22]

$$2\delta_2\omega_L(r) = \int_{2M}^{r} \mathrm{d}\rho\rho^2 \left[f(\rho)(\delta\phi_L')^2 + (\delta\dot{\phi}_L)^2\right] + \int_{r_{A,\text{out}}}^{2M} \mathrm{d}\rho\rho^2 \left[f(\rho)(\delta\phi_R')^2 + (\delta\dot{\phi}_R)^2\right], \quad (3.14)$$

$$\delta_2\mathcal{K}_L(r) = \int_{2M}^{r} \mathrm{d}\rho\rho^3 \delta\phi_L'(r)\delta\dot{\phi}_L + \int_{r_{A,\text{out}}}^{2M} \mathrm{d}\rho\rho^3 \delta\phi_R'(r)\delta\dot{\phi}_R. \quad (3.15)$$

The crucial thing to note in these equations is that the last term in (3.14) is negative, caused by that fact that r increases in opposite directions on each side of the bifurcation surface. Locally positive energy densities on one-side of an extremal surface contribute negatively to the Hawking (and thus ADM) mass on the other side. Thus, we see that we can always pick $\delta\phi_L, \delta\dot{\phi}_L$ such that $\delta_2\omega_L(r) = 0, \delta_2\mathcal{K}_L(r) = 0$ for all $r > 2M + \epsilon$ for any $\epsilon > 0$ (assuming ϵ does not scale with κ to a positive power). So a positive energy density shell can be dressed to another positive energy density shell, provided they are separated by an extremal surface. See figure 7. As advertised earlier, now gravity behaves more like electromagnetism. Positive energy densities seen from the other side of an extremal surface behaves as a negative energy density.

What about perturbations in A that are not spherically symmetric, but where we keep the solution in A fixed near the rightmost boundary, so that we are forcing the gravitational field to change towards the left. Can these perturbations be screened from left infinity? It is physically quite clear that they can, although we will give an argument rather a proof. First, using a technique known as inverse mean curvature flow, which we explain in section 4.3, it can be shown for $d = 3$ that non-spherical additions of matter on the right will again contribute

[22]The reader might be worried that $\delta_2\omega \neq 0$ at $r = 2M$, since the rr-metric perturbation there reads, in our gauge, $\delta_2 h_{rr} = r\delta\omega(r - 2M)^{-2}$, which diverges at $r = 2M$. However, this is simply a gauge artifact that can be fixed with the addition of a term $\nabla_{(a}\xi_{b)}$. It is simply reflecting the fact the location of the minimal surface, and thus coordinate breakdown location of our gauge, has moved by a perturbative amount.

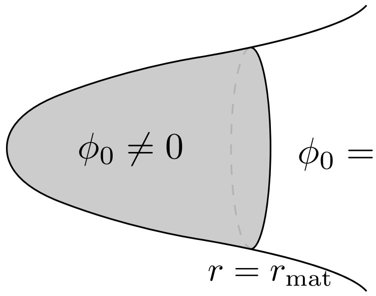

Figure 8. A one-sided spacetime with a lump of matter around $r = 0$. We argue that a ball A with radius r_A is independent from an annulus B around infinity if and only if $r_A < r_{\rm mat}$.

negatively to the mass on the left. So again we can simply add matter on the left to bring this back up, keeping left the mass unchanged. Of course, now we might also have to cancel the momentum and angular momentum, but this is much easier since these are quantities without a preferred sign. Let us assume we added some amount of angular momentum in A that is backreacting leftwards. We might worry that the required matter or gravitons we need to add to cancel this angular momentum forces us to overshoot the mass, causing a left mass increase. But this can always be avoided. Rather than cancelling the angular momentum by adding matter on the left, we can add it on the right side, between A and the minimal surface, so that we bring the angular momentum closer to zero, while at the same time decreasing the mass. Thus, as long as A is slightly separated from the bifurcation surface, so that we have the flexibility of perturbing on both sides of it, A ought to be independent of B.

What we see from this discussion is that gravity appears to have a classical version of the split property across the Schwarzschild bifurcation surface. Let A and B be the regions $r \geq 2M + \epsilon$ on the left and right, respectively, with $\epsilon > 0$ not scaling with κ. By the discussion above, we see that A, B are likely independent of each other. However, it is crucial that ϵ is non-zero, and in fact non-perturbatively large, so that we can always fit enough matter without leaving perturbation theory. Thanks to the fact that the small band between A, B contains an extremal surface, we have access to perturbations with all combinations of signs of charges within this band. We will see later that this property is a feature of general extremal surfaces.

A spacetime with a lump of matter. Consider now a one-sided spherically symmetric geometry at a moment of time-symmetry, $\mathcal{K}_0 = \dot{\phi}_0 = 0$. Assume that it has no stationary surfaces, so that one coordinate patch covers everything. Next, we take there to be a lump of matter $\phi_0(r)$ with compact support on $r \in [0, r_{\rm mat}]$. See figure 8. We demand that $\phi_0'(r)$ is non-zero at least in $r \in [r_{\rm mat} - \epsilon, r_{\rm mat})$ for some $\epsilon > 0$.

Let A be a ball with radius r_A, and take B to be an annulus around infinity. Consider turning on any scalar field profiles $\delta\phi(r), \delta\dot{\phi}(r)$ in A. The perturbative constraints are now non-trivial at first order in κ, and read

$$\begin{aligned} \delta_1\omega'(r) &= r^2 \left[1 - \frac{\omega_0(r)}{r}\right] \phi_0'(r)\delta\phi'(r) - \frac{1}{2} r \delta_1\omega(r)\phi_0'(r)^2, \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 \delta_1 \mathcal{K}\right) &= r^3 \phi_0'(r)\delta\dot{\phi}(r). \end{aligned} \tag{3.16}$$

Can we then extend the perturbation to $r > r_A$ such that no backreaction leaks to infinity? If $r_A < r_{\text{mat}}$, then the answer is yes. The first order constraint solutions are

$$\delta_1\omega(r) = e^{-\frac{1}{2}\int_0^r \mathrm{d}z z\phi_0'(z)^2} \int_0^r \mathrm{d}\rho\rho^2 e^{\frac{1}{2}\int_0^\rho \mathrm{d}z z\phi_0'(z)^2}\left[1 - \frac{\omega_0(\rho)}{\rho}\right]\phi_0'(\rho)\delta\phi'(\rho) \tag{3.17}$$

$$\delta_1\mathcal{K}(r) = \frac{1}{r^3}\int_0^r \mathrm{d}\rho\rho^3\phi_0'(\rho)\delta\dot{\phi}(\rho) \tag{3.18}$$

We now see that neither integral has fixed sign, so as long as $r_A < r_{\text{mat}}$,[23] we can always pick $\delta\phi, \delta\dot{\phi}$ on $(r_A, r_{\text{mat}}]$ so that there is some $\hat{r} \in (r_A, r_{\text{mat}}]$ such that

$$\delta_1\omega(r \geq \hat{r}) = 0, \quad \delta_1\mathcal{K}(r \geq \hat{r}) = 0. \tag{3.19}$$

Physically, what has happened is that we have removed some matter from the background shell in order to avoid having the backreaction of our newly added matter leak out to infinity. Since we are considering infinitesimal perturbations, the background reservoir is effectively infinitely large, so we can keep doing this to any order in perturbation theory. All spherically symmetric perturbations in A can be dressed to the background lump of matter. Similarly, if we alter the mass in B, we can always add or remove some of the matter in the shell lying in C to compensate, keeping A fixed. Assuming we can dress non-symmetric perturbations to the lump as well, we thus find that A, B now are independent. It is physically reasonable that it should also be possible to dress these perturbations to B. If our non-spherical perturbation in, say, A, adds some momentum or angular momentum, we can just add more moving or rotating matter between A and B to cancel these charges, since they do not have a preferred sign. The only real worry is that this matter will increase the energy at the same time, but since we are working perturbatively, we have an infinite background reservoir of matter, so we can always remove a bit of matter from the lump to keep the energy fixed. So the background lump of matter restores independence of the ball A from infinity, and perturbations in A need not be dressed to infinity.

If we instead took $r_A > r_{\text{mat}}$, independence breaks down. This is because the spacetime looks like Schwarzschild for $r > r_{\text{mat}}$, and if we pick a perturbation on A that changes the region $(r_{\text{mat}}, r_A]$, but keeps everything fixed for $r \in [0, r_{\text{mat}}]$, then we have specified a perturbation where no matter from the background is allowed to be removed, and we have no choice but to let the backreaction leak to infinity. This time there is no minimal surface to dress to. Non-symmetric perturbations in C cannot save us, since these would constitute all-order compactly supported perturbations of pure Schwarzschild in one exterior, and these perturbations could be used contradict the rigidity-part of the Riemannian Penrose inequality.

This example illustrated dressing of perturbations to background matter. It was important that this background matter was not just perturbatively small, since then the argument would break down — there would exist perturbations that could not be dressed to matter, since the added matter then exceed what we might have the ability to remove. At the level of the full perturbative series, the radius of convergence of κ thus has to be less than infinity.

Schwarzschild in pure gravity. We already discussed subsystem independence in Schwarzschild, but we crucially relied on matter. What happens in vacuum gravity? Consider again

[23] And $r_A - r_{\text{mat}} = \mathcal{O}(\kappa^0)$.

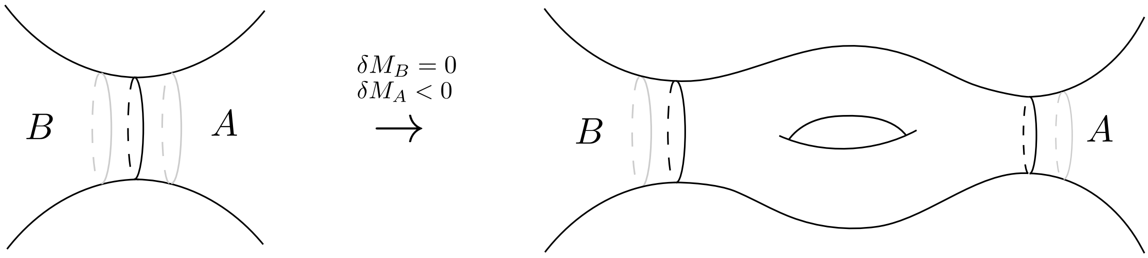

Figure 9. Using pure gravity degrees of freedom to change the mass of the BTZ region on the A-side without changing it on the B-side. The right geometry looks identical to BTZ in A and B.

a canonical $t = 0$ slice of (AdS-)Schwarzschild, and let A and B be the regions $r \geq 2M + \epsilon$ on the right and left, respectively. Consider now picking a perturbation $\delta S|_A$ that corresponds to simply deforming A to Schwarzschild with a lower mass. Pure gravity has no local spherically symmetric degrees of freedom, so we can clearly not hide this perturbation from B within spherical symmetry. However, it appears likely that the perturbation can be hidden from B if we break spherical symmetry, so that we unlock the graviton degrees of freedom. Let us see how this happens in a similar situation: the BTZ black hole. Unlike higher dimensions, we do not have local degrees of freedom. However, vacuum gravity still possesses topological degrees of freedom. Once we leverage these, it is well known (see for example [77, 78]) that we can reduce the BTZ mass in A without changing B by filling in a higher-genus surface between A and B — see figure 9. Of course, in this case, the change is not perturbative, since we changed the topology, so pure 3d gravity at best has a non-perturbative notion of independence. Nevertheless, the example is brought up to make it less surprising that pure gravity degrees of freedom might do the job for $d \geq 3$. In fact, there are mathematical results in GR supporting this possibility. As long as we are allowed to change the geometry at the bifurcation surface, we are not constrained by the Riemannian Penrose inequality, and Corvino and Schoen [79, 80] have shown that being identically Schwarzschild in a neighbourhood of infinity is not a rigid feature.[24] Furthermore, by Theorem 1.1 in [81], Schwarzschild is known to support first order pure gravity perturbations of compact support. Thus, it is a possibility that any perturbation on A can be shielded from B by utilizing pure gravity degrees of freedom in C.

3.3 Subregion independence in de Sitter

Let us now treat a case with positive cosmological constant. Let the background be a minimal slice of dS_4, which has $K_{ab} = 0$, and which can be covered by two coordinate patches U_L, U_R, each having a range $r \in [0, L_{\mathrm{dS}}]$. The metric on each patch is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{r^2}{L_{\mathrm{dS}}^2}} + r^2 \mathrm{d}\Omega^2, \tag{3.20}$$

i.e. $\omega_0(r) = 0$. Together these two patches make up a round sphere of radius L_{dS}. The spherically symmetric constraints are non-trivial at $\mathcal{O}(\kappa^2)$, and given by (3.12), except now

[24] Specifically, they proved that for any asymptotically flat vacuum solution (h_{ab}, K_{ab}) of the constraints on $\mathbb{R}^3$, and for any choice of compact subset $\Omega \subset \mathbb{R}^3$, there exists new vacuum initial data (h'_{ab}, K'_{ab}) that looks identical to Kerr near infinity, and which agrees with (h_{ab}, K_{ab}) on Ω.

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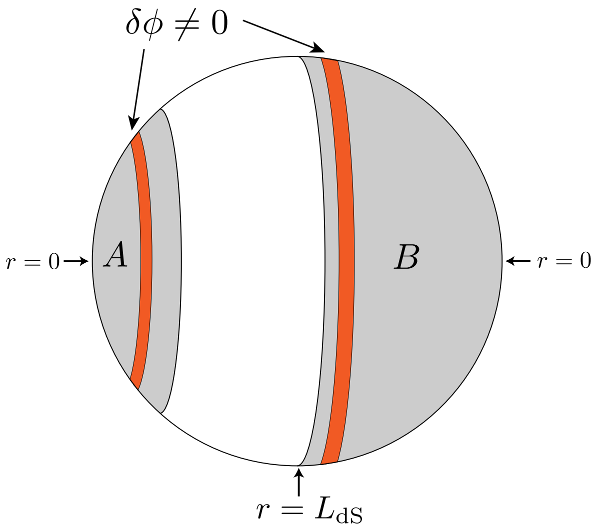

Figure 10. A perturbation of a minimal slice of de Sitter, which is just a round sphere of radius L_{dS}. The two shells are dressed to each other, so the solution looks like pure de Sitter near the poles of A and B. At the level of spherical symmetry, adding a shell to the one static patch requires the addition of one to the other. Thus it is possible that A and B are dependent in the left scenario. On the right, we can screen perturbations in A from B by adding matter around the maximal surface.

$f(r) = 1 - r^2/L_{\mathrm{dS}}^2$. At $r = 0$, we as usual need that $\delta_2\omega(0) = 0$. Integrating (3.12) and using that $\delta_2\omega$ is continuous at $r = L_{\mathrm{dS}}$, all perturbations are constrained to satisfy

$$\int_0^{L_{\mathrm{dS}}} \mathrm{d}\rho\rho^2 \left[f(\rho)(\delta\phi_L')^2 + (\delta\dot{\phi}_L)^2\right] = \int_0^{L_{\mathrm{dS}}} \mathrm{d}\rho\rho^2 \left[f(\rho)(\delta\phi_R')^2 + (\delta\dot{\phi}_R)^2\right] \tag{3.21}$$

This is just a special case of the first law of thermodynamics for positive cosmological constant [82] applied to spherical perturbations of pure dS. Since U_L and U_R are Cauchy slices for the left and right static patches, we see that within the domain of spherical symmetry, it is impossible to perturb one static patch without also perturbing the other. See figure 10. It is easy to show the same result for all $d \geq 2$. Thus, it seems plausible in dS_{d+1} for $d \geq 2$, any subregion of a given static patch depends on the full opposite static patch. However, we have no proof that we cannot break spherical symmetry on the right and leverage pure vacuum gravity degrees of freedom to shield this perturbation from the left. As we discuss in section 5, there very likely exist at least some non-symmetric non-trivial initial data that is strictly localized to one static patch. Either way, A is probably independent of any B that is a strict subset of the opposite static patch, provided B is such that C contains a neighbourhood of the cosmological horizon. In this case, we can satisfy (3.21) by adding matter in a small neighbourhood around the cosmological horizon, as illustrated in figure 10.

In recent work, a von Neumann algebra of diffeomorphism invariant operators was constructed for a static patch of de Sitter that contains an observer [83]. This algebra consisted of operators dressed to the observer. It was found that in order to have a consistent description also describing the opposite static patch, it was necessary to include an observer in the second patch as well. While our setup is different, it is worth pointing out that an observer ought to backreact on the geometry, and the Hamiltonian constraint forbids backreaction in a single static patch (in spherical symmetry). If the observer were to travel on a worldline, this perturbation preserves spherical symmetry, and so some backreaction must thus be added in the other patch. Adding a second observer solves this. In our setup, it would however be appropriate to treat an observer as a feature of the background, rather than a perturbation, so that dressed observables act as perturbations with respect to the observer. To treat this case, and to get a better sense of when we can deform a single static patch, we prove a stronger theorem in section 5, showing that for full spherically symmetric non-linear backreaction and $d \geq 2$, it is not possible to change a single static patch. We do this by showing that the area of the cosmological horizon must be reduced by any of these deformations. However, before we do this, we first study gravitational independence at the more general level.

4 Dressing across extremal and trapped surfaces

Above we studied examples of spherically symmetric backgrounds, and we argued that at moments of time-symmetry, we can always dress perturbations across minimal or maximal surfaces, at least when the theory has matter. The assumption of time-symmetry and spherical symmetry is however restrictive. Furthermore, it would be more useful to characterize the surfaces that we are able to dress across in terms of their spacetime properties, rather than their properties on a particular slice. We are now going to argue that we can dress across extremal surfaces and generic trapped surfaces. We start with general spherically symmetric spacetimes and perturbations in sections 4.1, 4.2. Then in section 4.3, we remove symmetry assumptions in the case of four spacetime dimensions.

4.1 Spherical symmetry

In spherical symmetry, the general constraint equations on a ${K_a}^a = 0$ slice Σ read

$$(d-1)\frac{\omega'(r)}{r^{d-1}} = 2\mathcal{E} + \frac{d}{d-1}\mathcal{K}(r)^2, \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^d\mathcal{K}\right) = r^d\mathcal{J}_r. \tag{4.2}$$

By virtue of $\omega'(r) \geq 0$, as we cross any minimal or maximal surface σ, we always have a flip in the direction in which ω grows as it encounters matter. So if we add some matter on one side of σ, we can add some matter on the other side to compensate, keeping ω unchanged outside a neighbourhood of σ. The assumption of time-symmetry in our examples was irrelevant to this central point. However, at first sight, minimal and maximal surfaces in a $K_a^a = 0$ slice seem like a very restricted set of surfaces. It turns out this is not true. A simple computation shows that stationary surfaces on a ${K_a}^a = 0$ slice are always either strictly (anti)trapped or extremal (see for example eq. (A.55) of [58]). In a moment we are going to show a partial converse in spherical symmetry: maximal or minimal surfaces on ${K_a}^a = 0$ slices include all (spherical) extremal surfaces and also all generic trapped surfaces, provided we do not require Σ to satisfy ${K_a}^a = 0$ globally.[25] This will be enough for us.

Consider a general spherically symmetric spacetime (N, g_{ab}) and let σ be any sphere. Then there always exists a one-parameter family Σ_η of spherically symmetric ${K_a}^a = 0$ slices

[25]Allowing saddle-type surfaces, all trapped surfaces are included.

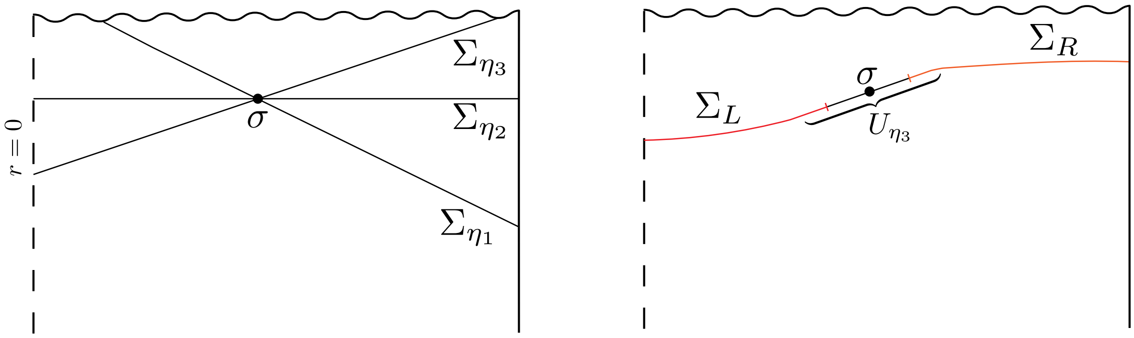

Figure 11. (Left) Three $K^a{}_a = 0$ slices fired off σ, with $\eta_3 > \eta_2 > \eta_1$. Only Σ_{η_2} is smooth and complete. (Right) Deforming Σ_{η_3} to a smooth complete slice, preserving a neighbourhood U_{η_3} around Σ that has $K^a{}_a = 0$.

locally defined in a neighbourhood around σ. To see this, note that the equation that determines Σ_η is a second order ODE. This ODE is obtained by extremizing the induced volume of Σ with respect to a single embedding coordinate, say $t(r)$. One integration constant is fixed by demanding that Σ_η passes through σ. The second integration constant is just the boost angle η at which Σ_η is "fired off" σ. This is completely analogous to how we can fire a one-parameter family of radial spacelike geodesics off some given point. Once η is fixed we can integrate the ODE determining the location of Σ_η to try to extend it to a full slice, but generically one does not get a full smooth Cauchy slice. See the left panel of figure 11. In a one-sided spacetime, for example, this could happen because we hit $r = 0$ with the wrong boost angle, so that Σ_η has a kink, corresponding to $\mathcal{K} \to \infty$. Or we could fall into a singularity. However, crucially, we will not need a full $K_a{}^a = 0$ slice. We can just terminate the integration at some finite value and then arbitrarily continue the slice in a smooth way, now giving up the requirement that the mean curvature vanishes. This way we produce a one-parameter family of Cauchy slices Σ_η, each containing σ and each having a neighbourhood $U_\eta \subset \Sigma_\eta$ that has $K^a{}_a = 0$. See the right panel of figure 11 for one such slice.

Consider now σ being a surface such that there exists an η so that σ is minimal or maximal on Σ_η. Let Σ_L and Σ_R be the parts of Σ_η lying to the left and right sides of σ, respectively, except for a small neighbourhood around σ that is included in neither — see figure 11. Thus, Σ_L and Σ_R are separated by an open neighbourhood containing σ that has $K^a{}_a = 0$. We now see that any spherical perturbation in Σ_L can be screened from Σ_R, and vice versa, by exactly the mechanism we described earlier.[26]

Note that $\omega(r)$ is defined everywhere on the slice by (3.4), but ω generally has no monotonicity properties away from U_η. This does not matter to the argument, however. Thus, what we now to show is that for any extremal or generic trapped surface, there is some η on which σ is maximal or minimal on Σ_η.

Let us now first assume that σ is extremal. That means that the area of σ is stationary under all variations, so for every choice of η, σ is either maximal, minimal, or a saddle in Σ_η.

[26] Strictly speaking, we should also consider perturbations that satisfy $\delta K_a{}^a \neq 0$ at $\partial\Sigma_R, \partial\Sigma_L$, so we should technically have consider the spherically symmetric Einstein equations for general $K^a_a = 0$. This leads to no complications. This should be clear from the following Lorentzian analysis.

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Being a saddle is clearly a fine-tuned case, and it can always be avoided. To see this, note that when the boost angle η approaches $+\infty$ or $-\infty$, U_η gets closer and closer to one of the two the null congruences fired off σ. By the focusing theorem, the area of these congruences are shrinking both to the future and past, so once $|\eta|$ is large enough, σ will be a locally maximal surface on U_η. In the fine tuned case where the congruences fired from σ are stationary, we are just locally in a standard spherical black hole, in which case we can just pick one of the other slices, where we know that σ is minimal or maximal.

Next, assume that σ is a trapped or antitrapped surface, meaning that the two future null expansions have the same sign, with both strictly nonzero. Fixing an arbitrary zero-point of the boost angle η, let n^a be the future timelike unit normal to $\Sigma_{\eta=0}$, and let r^a be a unit normal to σ that is tangent to $\Sigma_{\eta=0}$. Two independent future directed null normals to σ can be taken to be

$$k_\pm^a = \frac{1}{\sqrt{2}}(n^a \pm r^a), \tag{4.3}$$

where k_+^a conventionally defines "outwards". As shown in the appendix of [58], with this normalization of the null normals, the mean curvature $H_0[\sigma]$ of σ within $\Sigma_{\eta=0}$ reads

$$H_0[\sigma] = D_a r^a = \frac{1}{\sqrt{2}}(\theta_+ - \theta_-), \tag{4.4}$$

where the null expansions are given by $\theta_\pm = (g^{ab} + 2k_+^{(a}k_-^{b)})\nabla_a k_b$. Consider now boosting our slice to Σ_η. Since n^a, r^a form an orthonormal basis of the normal bundle of σ, they transform in the canonical way under a Lorentz boost. It is easily seen that the null normals canonically normalized with respect to the boosted n^a, r^a are

$$k_{\pm,\eta}^a = e^{\pm\eta} k_\pm^a, \tag{4.5}$$

so the mean curvature of σ within Σ_η is

$$H_\eta[\sigma] = \frac{1}{\sqrt{2}}(e^\eta \theta_+ - e^{-\eta}\theta_-), \tag{4.6}$$

where θ_+, θ_- still are the expansions at $\eta = 0$. Because θ_+, θ_- are nonzero and have the same sign, we can always take $\eta = \frac{1}{2}\log(\theta_-/\theta_+)$, giving $H_\eta[\sigma] = 0$. Hence, locally there is always a $K^a{}_a = 0$ slice on which σ is stationary. Assuming we do not have the fine tuned situation where this surface is a saddle, we see that σ posses a slice one which the left and right sides are independent — at least with respect to spherical perturbations.

So in total, if D_A and D_B are two spacelike separated causal diamonds whose edges are spacelike to a common extremal or generic trapped surface, then we expect that these diamonds are independent, since they are contained in the domains of dependence of the regions Σ_R and Σ_L that we expect to be independent — see figure 12.[27]

[27] Assuming that the edges are not parametrically close to null separated, so that the splitting region is parametrically small.

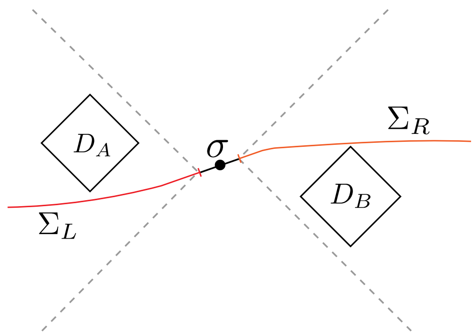

Figure 12. Two causal diamonds separated by a trapped or extremal surface σ.

4.2 A Lorentzian analysis

It is illustrative to reach the same conclusion using a spacetime approach. Let us now do this, at the same type generalizing to also allow planar or hyperbolic symmetry, in addition to spherical. This will also let us show that arbitrarily large energy densities near an extremal surface make arbitrarily small contributions to the ADM mass.

Let (N, g_{ab}) be a $(d+1)$-dimensional spacetime with one of these three types of symmetries. This means that N can be foliated by a two-parameter family of codimension-2 surfaces σ that have the intrinsic metric of a sphere, a plane, or the hyperbolic plane (or quotients thereof). We can pick a double null gauge

$$\mathrm{d}s^2 = -2e^{-f(x^+,x^-)}\mathrm{d}x^+\mathrm{d}x^- + r(x^+,r^-)^2\mathrm{d}\Sigma_k^2, \tag{4.7}$$

where $\mathrm{d}\Sigma_k^2$ is the metric of the unit sphere, plane, and hyperbolic plane for $k = -1, 0, 1$, respectively (or quotients thereof). For the leaves of our symmetric foliation, given by constant x^+, x^-, we now define the following function[28]

$$\mu(x^+, x^-) = kr^{d-2} + \frac{r^d}{L^2} - \frac{2\theta_+\theta_-}{k_+ \cdot k_-(d-1)^2}, \tag{4.8}$$

where k_+^a, k_-^a are any two independent future-directed null normals to the surface, and θ_+, θ_- the corresponding null expansions. The quantity on the r.h.s. is covariantly defined, since the area radius r can be viewed as a coordinate independent scalar on spacetime.[29] μ is a spacetime analog of ω, and it has some special properties. First, in AAdS or AF spacetimes, it can be shown to reduce to the mass at spatial infinity:[30]

$$M = \frac{(d-1)\mathrm{Vol}[\Sigma_k]}{16\pi G_N}\mu|_{r=\infty}. \tag{4.9}$$

[28] μ is a generalization of the Lorentzian Hawking mass [62–65] to $d \neq 3$, in the special case of spatial symmetries. Without symmetries, the proper definition of the Hawking mass is unknown for $d \neq 3$.

[29] This is not strictly true in planar symmetry. In this case, there is an ambiguity in an overall scaling of r, reflective of the fact that there is no canonical conformal frame for the boundary of AdS, when the boundary is conformal to Minkowski. In this case, there is also a scaling ambiguity in the mass. But this ambiguity can be fixed in some particular spacetime.

[30] Provided matter fields fall off fast enough. The important thing is that μ is a spacetime function with certain monotonicities that sometimes act as an obstruction to independence.

Second, using the Einstein equations, it was shown in [76] to satisfy

$$\partial_{\pm}\mu = \frac{2e^f r^d}{(d-1)^2}(T_{+-}\theta_{\pm} - \theta_{\mp}T_{\pm\pm}). \tag{4.10}$$

Let now X^a be any spacelike vector pointing outwards ($X_a k_+^a \geq 0$), and assume the dominant energy condition (DEC), which says that

$$T_{ab}U^aV^b \geq 0 \quad \forall \text{ timelike } U^a, V^a, \tag{4.11}$$

and which implies the WEC. The DEC implies that $T_{\pm\pm} \geq 0, T_{+-} \geq 0$. Thus, in a "normal" region, where $\theta_+ \geq 0, \theta_- \leq 0$, μ is monotonically non-decreasing in any outwards spacelike direction:

$$X^a\nabla_a\mu \geq 0 \qquad \text{when} \quad \theta_+ \geq 0, \quad \theta_- \leq 0. \tag{4.12}$$

Next, in an "anti-normal" region, we have monotonicity in the inwards direction instead:

$$X^a\nabla_a\mu \leq 0 \qquad \text{when} \quad \theta_+ \leq 0, \quad \theta_- \geq 0. \tag{4.13}$$

This is the spacetime analog of monotonicity of ω, and it constitutes an obstruction to subregion independence (if we break the DEC, this obstruction only becomes weaker). However, we see that if σ is an extremal surface that separates a normal and an anti-normal region, then the insertion of matter causes μ to increase in opposite directions on opposite sides of σ, and so we regain independence between the opposite sides. If we instead find ourselves in an (anti)trapped region of spacetime, μ is no longer monotonic, and by the appropriate choice of turning on either T_{++} or T_{--}, we can push μ up or down as we are moving in any fixed spacelike direction.

We also see from (4.9) and (4.10) that the contribution to the asymptotic mass is given by a integral of $T_{++}\theta_-, T_{--}\theta_+, T_{+-}\theta_-, T_{+-}\theta_+$ weighted by positive factors that are bounded in a neighbourhood of an extremal surface. Thus, arbitrarily large energy densities make arbitrarily small contributions to the ADM mass, provided they are localized to an extremal surface. Similarly, a marginally trapped surface (say, $\theta_+ = 0$) can support modes with very large T_{--} at low cost to the ADM mass, provided we do not make T_{+-} large as well.

4.3 No symmetries

We now want to understand what happens if our spacetime does not have any symmetries. Furthermore, even if our spacetime has symmetries, we might want to consider perturbations or surfaces that break the symmetries of the spacetime. A prototypical example of the latter occurs in AdS/CFT, where we might consider two regions A and B separated by a region containing a boundary anchored extremal surface, as illustrated in figure 13.

To deal with the case of no symmetry, we need an appropriate generalization of ω. To the author's knowledge, the appropriate generalization is known only in four spacetime dimensions, and so we will only treat this case. However, based on expectations from AdS/CFT and the fact that the spherically symmetric considerations work for all $d \geq 2$, we expect a similar story holds in other dimensions, but we do not know what is the right tool to use.

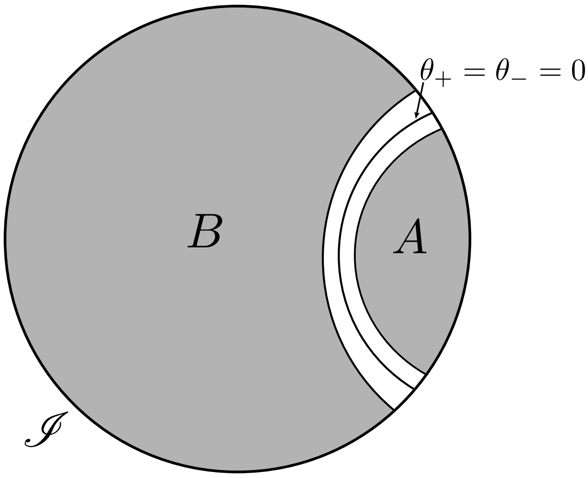

Figure 13. A timeslice Σ of an asymptotically AdS spacetime containing two subregions A and B that are separated by a set containing an HRT surface anchored to the conformal boundary $\mathscr{I}$.

Let us thus assume four spacetime dimensions. If (Σ, S) is some initial dataset with a spacelike two-dimensional surface σ, we define

$$\omega[\sigma] = \frac{1}{16\pi}\sqrt{\frac{\text{Area}[\sigma]}{4\pi}}\int_\sigma \left[2\mathfrak{R} - H[\sigma]^2 + \frac{4}{L^2}\right], \tag{4.14}$$

where the integral is taken in the induced volume form on σ, and where $\mathfrak{R}$ is the Ricci scalar for the induced metric on σ. Consider now a one-parameter family of surfaces σ_τ, where increasing τ corresponds to flowing the surfaces along the vector field

$$v^a = \frac{1}{H[\sigma_\tau]} r^a_\tau, \tag{4.15}$$

with r^a_τ being a unit normal to Σ in σ. Thus, our one-parameter family of surfaces correspond to a flow where the velocity of the flow at $p \in \sigma_\tau$ is set by the inverse of the mean curvature of σ_τ at p. This flow is known as inverse mean curvature (IMC) flow,[31] and it can be shown that on a slice with $K^a{}_a = 0$, assuming the WEC, we have [59]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\text{Area}[\sigma_\tau] \geq 0, \tag{4.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\omega[\sigma_\tau] \geq 0. \tag{4.17}$$

This is the generalization of $\omega'(r) \geq 0$. Minimal and maximal area surfaces are special locations where the flow terminates. See [86] for a review of these facts for compact surfaces, and for the proof that (4.16) and (4.17) remains true also for boundary anchored surfaces in asymptotically AdS_4 spacetimes (ω is finite even when σ_τ is boundary anchored [86]).

Consider a surface that is a minimal or maximal surface on a $K^a{}_a = 0$ slice. Then, as illustrated in figure 14, IMC flow goes in opposite directions, so again we have a mechanism

[31] In four spacetime dimensions, a Lorentzian version of IMC flow also exists [65, 84, 85], but is less understood. It generalizes the monotonicities (4.12) and (4.13) to cases with no symmetries.

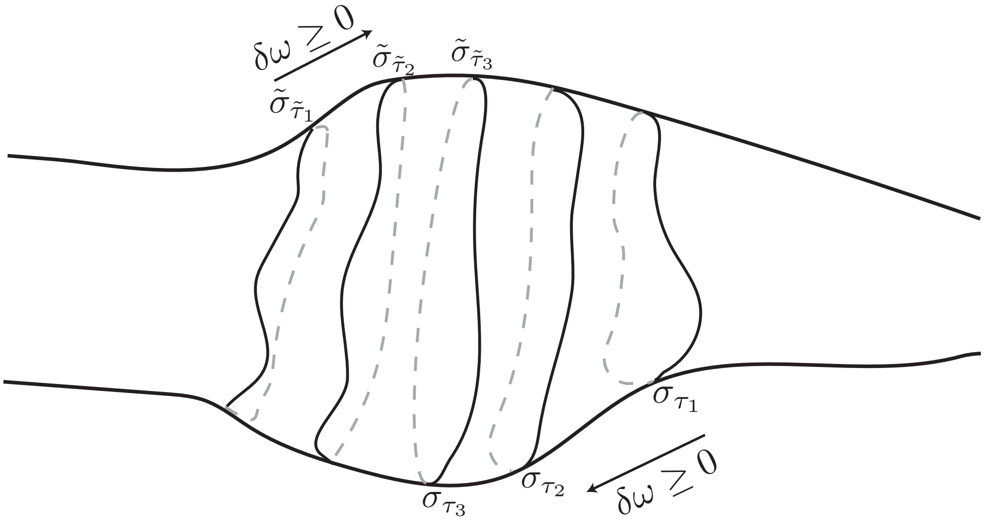

Figure 14. Two IMC flows σ_τ and $\tilde{\sigma}_{\tilde{\tau}}$ accumulating at a maximal surface $\tilde{\sigma}_{\tilde{\tau}_3} = \sigma_{\tau_3}$.

to bypass the monotonicity of $\omega[\sigma_\tau]$. Positive energy density can now likely screen positive energy density. Of course, to show this rigorously is challenging,[32] but it would not be surprising if it is true. It would fit perfectly with what we know about AdS/CFT, and it is plausible that the monotonicity of ω is the sole obstruction to subregion independence in gravity, since charges other than the mass tend to not have a preferred sign. In fact, in various gluing results, there is typically only a finite-dimensional space of obstructions to gluing, corresponding to a set of charges that must match or have a certain relationship [79, 87]. In four spacetime dimensions, this is a set of 10 charges, and the mass is the only one with a preferred sign.[33] Thus, we conjecture the following:

Conjecture. *Let (Σ, S) be a smooth d-dimensional initial dataset in Einstein gravity minimally coupled to matter, with $K_a{}^a = 0$, and $d \geq 3$. Let A, B be two closed disjoint subregions such that $\Sigma - A - B$ contains a locally minimal or maximal surface that is homologous to ∂A and ∂B. For any sufficiently small deformations $\delta S|_A$ and $\delta S|_B$, there exists a small extension δS of $\delta S|_A \cup \delta S|_B$.*

We are deliberately vague about what we mean with small here. We might either consider formal perturbative solutions, or perhaps better, we could try to require that with an appropriate choice of C^k or Sobolev norm defined by the background metric, a small extension always exists once the norm of $\delta S|_A \cup \delta S|_B$ is sufficiently small. Note that the above might also be true for $d = 2$ if matter is included, but for $d = 2$ vacuum gravity, we saw that we were forced to change the topology of C to match perturbations made in A, B, so there was no sense in which the perturbation was small.

[32]Note that IMC flow can have singularities. In this case, a weak version of the flow can be defined. In this case, when a surface becomes singular, it makes a jump to a surface with larger area and Hawking mass. However, since we are trying to screen perturbative excitations by adding matter near the minimal/maximal surface at which we start the flow, should be able to screen the perturbation before we reach a jump, since the jump-time is non-perturbative, so it will not get radically affected by our perturbations.

[33]The other charges corresponds to linear momentum, angular momentum, and center of mass.

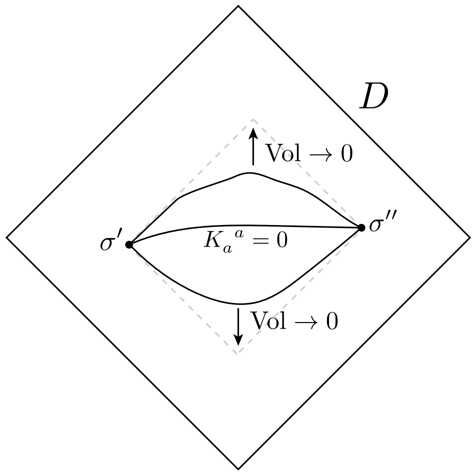

Figure 15. A small domain of dependence D. For any two spacelike separated surfaces σ', σ'', we expect there to be a maximal volume slice Σ with $\partial\Sigma = \sigma' \cup \sigma''$.

Next, we again really want to talk about surfaces that are trapped or extremal, rather that maximal or minimal on a $K_a{}^a = 0$ slice. By the same logic as in the spherically symmetric discussion, we thus want that every extremal and generic trapped surface σ is minimal or maximal on some slice Σ that satisfies $K_a{}^a = 0$ locally in a neighbourhood of σ. While we will not attempt a proof, this appears very likely to be true. Consider a region of spacetime D corresponding to a domain of dependence containing σ. Let us shrink D enough so that its closure is compact, so that the null generators of its boundary do not terminate on singularities or future infinity. Then for any two spacelike separated surfaces σ', σ'' in D, there should exist a maximal volume hypersurface Σ bounded by σ', σ''. See figure 15. This makes sense, since if Σ is deformed towards the future or past null congruences fired from σ', σ'', its volume goes to zero. Provided we pick D small enough so that it does not contain a portion of a future/past infinity where volume of space could be forever expanding, like future infinity of de Sitter, we should not be able to make the volume of Σ arbitrary large. Thus, there ought to be a maximal volume slice bounded by σ', σ''. This surface is necessarily a $K_a{}^a = 0$ slice, and so in D we ought to have a $K_a{}^a = 0$ slice for every choice of two spacelike separated surfaces σ', σ''. Allowing these surfaces to vary, this likely provides more than enough freedom for finding such a slice that contains σ, again suggesting that the setup described by figure 12 is true without spherical symmetry as well.

5 de Sitter rigidity and area bounds

In our discussion of pure AdS and Minkowski, we saw that these spacetimes were very rigid. No deformation in the interior of the spacetime can be made without it altering the geometry at infinity. Can some form of rigidity statement be made for dS? A natural question to ask is the following: is it possible to deform the initial data in one static patch without altering the other? At the level of spherical symmetry, we saw that the answer was no at first order in perturbation theory. We will now prove a non-linear version of this statement. However, as we discuss below, once we break spherical symmetry, rigidity appears to no longer hold, unlike the case of Minkowski and AdS. This likely gives rise to large class of spacetimes that look identical to dS in a single static patch.

We prove the following[34]

Theorem 1. *Let* (Σ, S) *be a regular spherically symmetric initial dataset for the Einstein equations with positive cosmological constant,* $K_a{}^a = 0$*, and satisfying the WEC. If* Σ *has the topology of a hemisphere (i.e. a ball) of dimension* $d \geq 2$*, then every sphere* σ *in* Σ *has an area radius* r *satisfying*

$$r \leq L_{\mathrm{dS}}. \tag{5.1}$$

Equality is achieved if and only if $K_{ab} = 0$*, with* h_{ab} *the metric of a round sphere. Next, if* σ_* *is a locally maximal sphere of radius* r_**, then*

$$r_* \geq \sqrt{\frac{d-2}{d}} L_{\mathrm{dS}} = r_{\mathrm{Nariai}}. \tag{5.2}$$

Before the proof, let us make a few comments. First, at leading order in perturbation theory around pure dS, the upper bound (5.1) is just a special case of the first law [88]. Next, for $d = 3$, the upper bound (5.1), which is equivalent to $A(\sigma) \leq 4\pi L_{\mathrm{dS}}^2$, was proven without spherical symmetry in [89], given certain other assumptions (see also [90]). Second, extremal surfaces are always stationary on any slice, so for an extremal surface X in a spherical asymptotically dS spacetime with a simply connected Cauchy slice, we expect the area bounds (5.1), (5.2) to apply to X, provided X is a cosmological horizon type surface — i.e. it is maximal rather than minimal/a saddle on Σ. Third, we are not aware if a bound like (5.2), which lower bounds the area of cosmological horizon-type surfaces in terms of the event horizon of the Nariai black hole, has been shown before (beyond the dS-Schwarzschild family). Finally, the result shows that a spacetime with the following properties cannot exist when the WEC holds: (1) σ is a sphere that splits a Cauchy slice in two, (2) the wedge D_R of points right-spacelike to σ looks identically like a static patch of de Sitter, and (3) the wedge of D_L points left-spacelike to σ has a Cauchy slice Σ with $K^a{}_a = 0$. See figure 16. Now to the proof:

Proof. By the fact that Σ is spacelike, we have

$$1 - \frac{r^2}{L_{\mathrm{dS}}^2} - \frac{\omega(r)}{r^{d-2}} \geq 0, \tag{5.3}$$

as discussed in section 3.1. Thus, if we can show that $\omega(r) \geq 0$ on every patch of Σ, then we get that $r \leq L_{\mathrm{dS}}$. Let us now show this. The Hamiltonian constraint together with $K_a{}^a = 0$ implies that $\omega'(r) \geq 0$. Furthermore, by assumption, Σ contains the point $r = 0$, where we must have $\omega(0) = 0$. Hence, integrating (4.1) outwards, we find that $\omega \geq 0$ on the patch containing $r = 0$. If one patch covers Σ, we are done. Thus, assume that Σ is covered by multiple patches. Then the $r = 0$ patch is separated from the next patch by either a maximal surface or a saddle — see figure 17. If it is a saddle, it does not alter the monotonicity properties of ω as we move from $r = 0$ towards $\partial\Sigma$, so ω remains positive on the next patch.

[34]The author is grateful to Matt Headrick for proposing that Nariai might set a lower area bound, and for extensive discussions that lead directly to this result.

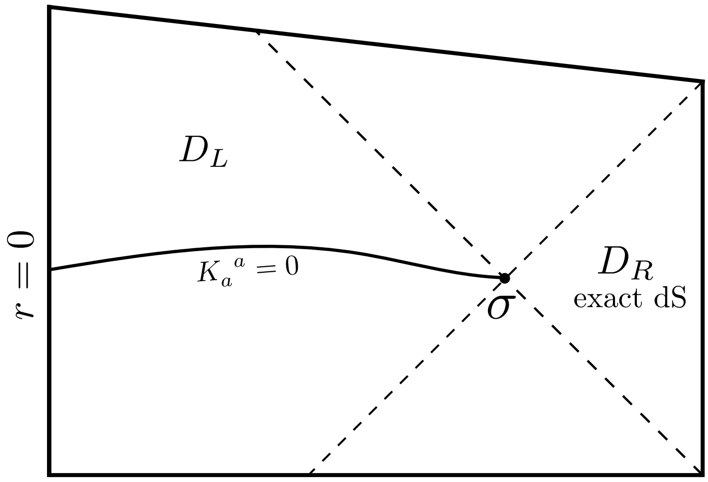

Figure 16. A hypothetical spherically symmetric spacetime that is ruled out by Theorem 1 when the WEC holds.

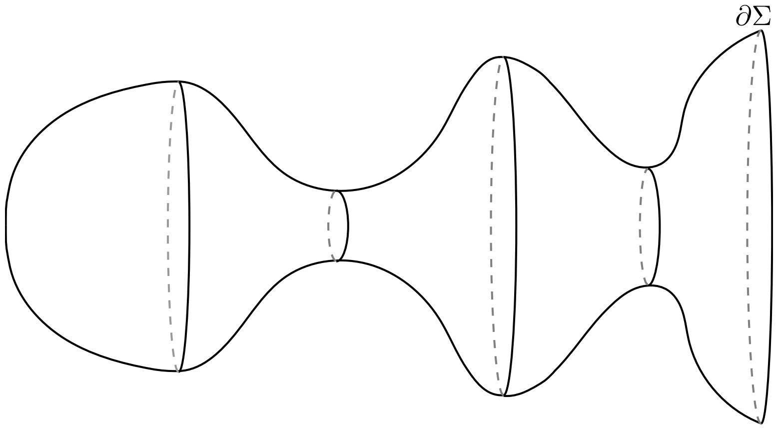

Figure 17. A spherically symmetric spatial manifold with a smooth $r = 0$ and a locally maximal boundary. When $K_a{}^a = 0$, the WEC implies that all spheres satisfy $r \leq L_{\text{dS}}$.

Thus, assume we have a maximal surface instead. Consider first integrating ω from $r = 0$ to the first maximal surface, with radius r_{max}. We already showed that $\omega \geq 0$ there, so we find that $r_{\text{max}} \leq L_{\text{dS}}$. Transitioning to the next patch, r and ω must now be decreasing as we move towards $\partial\Sigma$. Either we hit $\partial\Sigma$ and we are done since then $r|_{\partial\Sigma} < r_{\text{max}} \leq L_{\text{dS}}$, or we hit a minimal surface with radius $r = r_{\text{min}} < r_{\text{max}} \leq L_{\text{dS}}$ (again, we can hit a saddle, but nothing interesting happens at these). By stationarity we get

$$\frac{\omega(r_{\text{min}})}{r_{\text{min}}^{d-1}} = 1 - \frac{r_{\text{min}}^2}{L_{\text{dS}}^2} > 1 - \frac{r_{\text{max}}^2}{L_{\text{dS}}^2} \geq 0. \tag{5.4}$$

Repeating this exact argument as we move past any number of minimal, maximal, or saddle surfaces, we find that $\omega \geq 0$ everywhere, showing that $r \leq L_{\text{dS}}$. Next, we only find a sphere with $r = L_{\text{dS}}$ if Σ a subset of exact de Sitter. This is seen by the fact that chain of potential equalities above is broken if we encounter matter anywhere. This proves rigidity.

To prove the lower bound, note that approaching a stationary surface implies that $1/B(r)$ is approaching 0 from a positive value. The surface being maximal with radius $r = r_*$ implies that we are approaching r_* from a smaller value of r. Thus, $\frac{\mathrm{d}}{\mathrm{d}r} B^{-1}|_{r=r_*} \leq 0$, giving that

$$
\begin{aligned}
0 &\geq r_* \frac{\mathrm{d}}{\mathrm{d}r}\left[1 - \frac{r^2}{L_{\mathrm{dS}}^2} - \frac{\omega(r)}{r^{d-2}}\right]\bigg|_{r=r_*} \\
&= -\frac{2r_*^2}{L_{\mathrm{dS}}^2} - \frac{\omega'(r_*)}{r_*^{d-3}} + (d-2)\frac{\omega(r_*)}{r_*^{d-2}} \\
&= -\frac{dr_*^2}{L_{\mathrm{dS}}^2} + d - 2 - \frac{\omega'(r_*)}{r_*^{d-3}}.
\end{aligned}
\tag{5.5}
$$

Using that $\omega' \geq 0$, this then gives the lower bound on r_*. □

Let us finally discuss the case without spherical symmetry, which turns out to be interesting. It would be tempting to conjecture the following: if Σ is a $K_a{}^a = 0$ slice where $\partial\Sigma$ is an extremal surface, then $A[\partial\Sigma] \leq A[\partial\Sigma_{\mathrm{dS}}]$, with equality only in the case of pure dS (assuming the WEC). While the area inequality might hold, the rigidity part, i.e. equality only in pure dS, is overwhelmingly likely to be false for $d \geq 3$. The rigidity part is closely related to a former conjecture by Min-Oo [91], who conjectured the following: assume h_{ab} is a metric on a hemisphere Σ with (1) a Ricci scalar lower bound $\mathcal{R} \geq d(d-1)L_{\mathrm{dS}}^{-2}$,[35] and (2) with the intrinsic and extrinsic geometry of $\partial\Sigma$ matching that of the boundary of a round hemisphere of radius L_{dS}. Then Σ must be the round hemisphere.[36] While according to [92] this conjecture was proven in [93] for $d = 2$, after being open for 16 years, Min-Oo's conjecture was disproven in [92] for all $d \geq 3$. They showed that there are Riemannian metrics distinct from the canonical round sphere metric that nevertheless looks identical to it in a neighbourhood around $\partial\Sigma$, for any $d \geq 3$. Several properties of these solutions are known. They can be constructed to have arbitrarily large volume [94],[37] they always have a minimal surface [96], and they can have a wide range of topologies [94, 97]. Furthermore, for h_{ab} sufficiently close to pure dS, the metric must disagree with the round metric somewhere in the band [98]

$$
\sqrt{\frac{d-1}{d+3}}L_{\mathrm{dS}} < r < L_{\mathrm{dS}},
\tag{5.6}
$$

so these deformations cannot localize in a tiny cap. If any of these metrics can be realized as a solution of the constraints, either with some choice of $K_{ab} \neq 0$ and/or with some choice of matter, then the rigidity part of the upper bound in Theorem 1 cannot be true without spherical symmetry (when $d > 2$). Thus, in four spacetime dimensions and higher, when we break symmetries there might exist a large flexibility in changing the initial data in only one static patch. The behavior of these solutions are reminiscent of another example we have already discussed: black holes in GR with a negative cosmological constant and three spacetime dimensions. In this case there is an infinite number of one-sided black holes that

[35] This is the lower bound one would get from the constraints, assuming the WEC and $K_a{}^a = 0$.

[36] With hemisphere, we mean that $\partial\Sigma$ is totally geodesic in Σ — it is really a half sphere, not just a cap.

[37] None of the solutions with volume greater than that of the static patch lie perturbatively close to the static patch, since the results of [95] imply that WEC-respecting metrics C^2-close to the static patch must have smaller volume.

look identical to the BTZ black hole in the exterior(s). However, they all break spherical symmetry in the interior, and none are close to a spherically symmetric metric. Like black holes, the de Sitter metrics discussed above also have minimal surfaces, so it seems likely that they will be black holes as well. It would be interesting if these could be used to give a semiclassical counting interpretation of the area of the cosmological horizon using the Euclidean path integral approach of [99, 100].

6 Discussion

In this work, we have given a simple natural definition of subregion independence in classical gravity. We argued that extremal surfaces, generic trapped surfaces, and background distributions of matter are structures that enable subregion independence, i.e. independent initial data perturbations. For extremal surfaces X, we saw that excitations added on the opposite side of X contribute with opposite sign to any given asymptotic mass. This enables positive energy densities to terminate the gravitational fields sourced by positive energy densities on the other side of X, providing a simple physical picture for why an extremal surface is a good location to separate independent subregions.

We now discuss some further implications of our perspective.

6.1 The semiclassical case and quantum extremal surfaces

Consider studying semiclassical gravity, where we couple the expectation value of the stress tensor of quantum matter to the classical Einstein equation. In this case, the constraint equations are unchanged, so our analysis is still relevant, albeit incomplete. The matter source is now more exotic, since classical energy conditions are violated. However, in our case, the classical energy conditions were an obstruction to independence, so we expect that this particular effect pushes subregions to a greater tendency for independence. Nevertheless, while quantum fields typically do not have local energy conditions, they often have global ones, so we do not expect that the problem is trivial — the Hawking mass in a normal region still ought to have certain monotonicity properties over sufficiently large distances.

Next, while our analysis does not directly say anything about general QESes [7], our results are informative in many particular cases. We might for example find that QESes themselves are classically (anti)trapped, so that we still avoid monotonicity of the Hawking mass, boding well for the perturbative independence of the complement wedges of the QES. Next, if a QES is a Planckian distance away from a classical extremal or (anti)trapped surface, we can tell a similar story, since we always want to draw some splitting region around the QES, which then contain a classically trapped or extremal surface. However, we leave a more careful study of QESs to the future.

6.2 Islands in massless gravity

The new QESs discovered after the Page time [41, 42], giving rise to the island phenomenon, are at the heart of recent breakthroughs on the information paradox [41, 42, 99, 101]. Islands are regions of spacetime that furnish entanglement wedges for the Hawking radiation after the Page time. They have the special property that they are compact, not reaching any asymptotic boundaries. Since the island is supposed to be encoded in the radiation, acting with operators in the island should correspond to acting with operators on the radiation.

In [40] it was argued that this behavior is inconsistent with having massless gravity. An important part of their argument was the following claim: any excitation localized to the island must have compact support, and by the Heisenberg uncertainty principle it should have finite energy, thus altering the ADM energy. However, as should be clear by now, non-linearities of the Einstein equations make this is false at the classical level. While turning on excitations adds positive energy density locally, this does not imply that the ADM energy is changed. It is true for perturbations around Minkowski space or AdS, but these examples turn out to not be good analogies for the general case. As discussed in section 4.1, in trapped regions of spacetime, ingoing and outgoing modes can be added so as to affect the ADM mass with either sign. In the case of the evaporating black holes studied in [41, 42], the island is in a classically trapped region of spacetime, so there is no obstruction to implementing large classes of localized perturbations of the constraints with no influence on the geometry outside the island.

Another part of the argument in [40] was the closely related claim that diffeomorphism invariant operators that probe local physics in the island must be dressed to the boundary, and so they must have non-zero commutators with operators in the complement entanglement wedge, leading to a contradiction with subregion-subregion duality. However, at the level of perturbative quantization in $\sqrt{G_N}$ around some background, there does exist localized operators that are not dressed to the boundary [22–28]. Operators like this were constructed in the CFT by [29, 30], and they argued this relieves the tension between islands and massless gravity. We agree, and will add a few additional remarks.[38]

It was suggested in [29, 30] that one might need to dress operators to the Hawking radiation. While perhaps possible, this can be avoided for the evaporating black holes of [41, 42]. To begin with, let Φ_0 be some scalar field that is part of the background we are considering, and let ϕ be a dynamical quantum field.[39] Then

$$O(\alpha) = \int \mathrm{d}^{d+1}x\sqrt{-g}\,\delta\left(\Phi_0(x) - \alpha\right)\phi(x) \tag{6.1}$$

is a one-paramater family of diffeomorphism invariant observables. In the quantum case, we should smear this over a window of α-values to get a proper operator, but as long as the gradient of Φ_0 is not extremely small (i.e. scaling with G_N to a positive power), this gives an approximately localized operator on appropriate backgrounds. The existence of single-integral observables like (6.1) was acknowledged in [40]. However, they argued that single-integral observables cannot probe large parts of the island. This is based on the fact that once we reach the Page time, the black hole interior volume (on some nice slice) has grown approximately linearly for a time of order $t \sim \mathcal{O}(G_N^{-1})$, so unless the black hole has been constantly fed matter, there are large regions in the black hole where there is no matter, or where the matter is extremely dilute. So if we try to get the delta function in $O(\alpha)$ to "click" somewhere in the dilute region, once we smear α a tiny bit, the operator strongly delocalizes. There are however other better operators that do not have this problem. First,

[38] See also [102, 103], which finds the appearance of islands in braneworlds with massless gravity.
[39] Φ_0 could also be quantum field with a non-zero VEV.

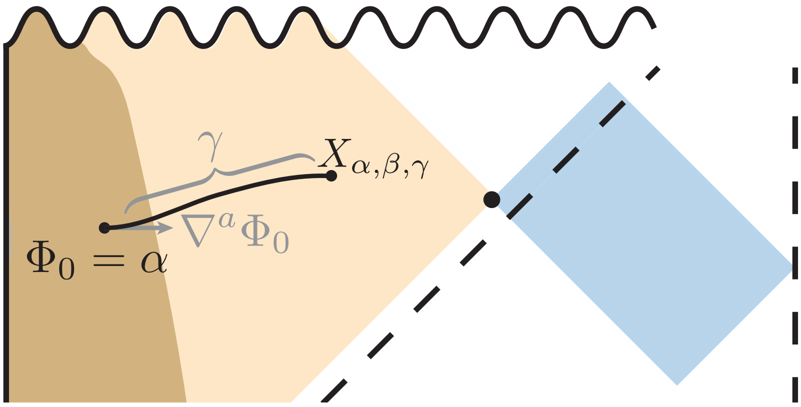

Figure 18. A diffeomorphism invariant operator localized to the island, dressed to matter that formed the black hole, and which probes the "barren" part of the island without becoming highly delocalized as $G_N \to 0$.

note that the matter that collapsed to form the black hole never dilutes, and it is present in the island.[40] So for some windows of α, we could get operators localized inside the matter distribution that formed the original black hole. But we can do better. Let us now consider forming the following three-parameter family of diffeomorphism invariant observables[41]

$$O(\alpha,\beta,\gamma) = \int \mathrm{d}^{d+1}x\sqrt{-g}\delta\left(\Phi_0(x)-\alpha\right)\delta\left(\nabla_a\Phi_0(x)\nabla^a\Phi_0(x)-\beta\right)\phi(X_{\alpha,\beta,\gamma}(x)) \tag{6.2}$$

where the $X_{\alpha,\beta,\gamma}(x)$ is the point obtained when firing a geodesic from x along the direction of $t^a = \nabla^a\Phi_0$ and following it for a proper distance/time γ. See figure 18. Since we get to fix the sign of β, we also get to fix the signature of t^a, so we know whether we are using a spacelike or timelike geodesic — i.e. the observable is well defined. To get an observable with the potential to be promoted to a proper operator when quantizing, we could smear over a small window of α,β,γ. If we pick α-window appropriately, then the x-integral can localize within the concentrated matter that formed the black hole. Next, by increasing γ up to values of order $\mathcal{O}(1/G_N)$, we can reach into the "barren" matterless region of the island.[42] We could also reach this region by dressing to matter that falls into the black hole around the Page time, if such matter exist.

All in all, perturbation theory in massless gravity appears to be consistent with compact entanglement wedges. Of course, once we worry about exponential corrections in G_N, things

[40]Alternatively, if we instead evaporate a past-eternal black hole in AdS, the same goes through for the shell of matter that falls into the black hole due to the sudden coupling of the CFT to the reservoir where we dump the Hawking radiation.

[41]Here g_{ab}, ∇_a are the full metric and connections, so these should be expanded in a perturbative series. Similar for $X_{\alpha,\beta,\gamma}$.

[42]Note that setting $\gamma \sim \mathcal{O}(1/G_N)$ raises questions as to whether the operator is well controlled in perturbation theory. However, the same issue would arise for any boundary-dressed operator probing sufficiently deep into the interior around the Page time. Thus, the issue of the consistency of islands in massless gravity does not appear to have any bearing on this subtlety. At the very least, this argument against the existence of islands would be an equally good argument against the existence of the interior.

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are much more subtle.[43] But once we consider exponential corrections, we should worry about what right we have to talk about concepts like spacetime regions or entanglement wedges, and it is hard to draw strong conclusions either way. But we see no clear sign of inconsistencies between islands and massless gravity at the perturbative level.

6.3 Gravitational splitting and local algebras

In this paper, we have argued that classical gravity has a sort of perturbative split property, provided the splitting region is sufficiently generic — i.e. it contains an extremal surface, a non-perturbative amount of matter, or a generic (anti)trapped surface. It would be very interesting to study the perturbative WdW equation to see whether this remains true in the quantum case. Specifically, working around a sufficiently generic background, can we choose the WdW wavefunctional on generic separated spacelike subregions separated by a finite gap independently? Ref. [15] showed that this is not true around pure AdS, but there are many hints that this story changes on other backgrounds. If we indeed can choose the WdW wavefunctional independently on different generic regions (up to the local constraints on these regions), this suggest that non-trivial perturbative algebras of localized observables exist for generic compact regions. This would be good news, given the recent interesting developments on algebras and their entropies in gravity [83, 105–117].

Note that we have not directly discussed the relation between independence and localized observables, which is strictly speaking more directly tied to algebras that the states themselves. We now turn to this.

6.4 From independence to localized observables

Our notion of independence deals with the structure of the phase space $\mathcal{P}$ of GR in a sufficiently small neighbourhood $U \subset \mathcal{P}$ around a point $\Psi \in \mathcal{P}$, which corresponds to our background. It would be interesting to understand whether the independence of $A, B \subset \Sigma$ means that there exist observables $f_{AC}, f_{BC} : U \mapsto \mathbb{R}$ that are sensitive only to the dynamical fields in the domains of dependence of $A \cup C$ and $B \cup C$, respectively. For example, if B is a collar around spatial infinity, f_{AC} would correspond to a (perturbatively) diffeomorphism invariant observable that is not dressed to the boundary, and it would Poisson-commute with observables localized to B [26]. It would be natural if independence implied that such functions exist. If they do, they are candidates for localized operators upon quantization. Let us outline a non-rigorous argument that localized observables exist. We do not claim that this completely settles the issue — there might be devils in the details.

Assume that there exist some gauge fixing procedure, so that the coordinate values of $\partial A, \partial B$ are fixed, and so that every point $\delta\Psi \in U$ has a unique value of $\delta S|_A$, $\delta S|_B$ and $\delta S|_C$. These three collections of classical field values on A, B, C are generally dependent on each other. However, if A and B are independent, then $\delta S|_A$ and $\delta S|_B$ can be specified independently. So they provide valid coordinates on U, and we can write a coordinate representation

$$\delta\Psi = \begin{pmatrix} \delta S|_A \\ \delta S|_B \\ \widehat{\delta S}|_C \end{pmatrix}, \tag{6.3}$$

[43] See [104] for an illuminating discussion of the challenges of defining non-perturbatively diffeomorphism invariant observables.

where $\widehat{\delta S}_C$ is a collection of coordinates parametrizing the remaining freedom in C independent of the degrees of freedom in A and B. Clearly, by the constraints, some of the degrees of freedom in C are dependent on the degrees of freedom in $A \cup B$, so $\widehat{\delta S}|_C$ must be a strict subset of $\delta S|_C$.

Now, since $\delta S|_A, \delta S|_B$ can be used as coordinates, the phase space coordinate functions in the A– and B-slots of (6.3) then seem like candidates for f_{AC}, f_{BC}. It might be tempting to say that these functions are sensitive to the dynamical fields in just A or just B, since these functions always return information about the dynamical fields in A or B. However, this is not true. A choice of $\delta S|_C$, influences the possible values of $\delta S|_A$, and thus the possible output of the coordinate functions of the A-slot. Disregarding diffeomorphism invariance for a second, an analogy in a theory with a dynamical field ϕ and two points $x_A \in A, x_C \in C$ would be a function like $f_{AC} = \phi(x_A)\theta(\phi(x_C))$, where θ is the heaviside step function.

Note that if A and B were dependent, this construction would not work. The possible values of $\delta S|_A$ would depend on the state on B, so any function of $\delta\Psi$ that returned $\delta S|_A$ could not be sensitive only to the fields in A, since the possible value of the fields there depends on the fields in B.

6.5 The Python's lunch

Consider an asymptotically AdS spacetime, and let R be a spatial subregion of the conformal boundary — or perhaps a complete connected component. Next, assume that R has three extremal surfaces homologous to it: X_{min}, X_{out} and X_{bulge}, as shown in figure 19 in the case where R is a complete boundary component. X_{min} is the HRT surface, which defines the entanglement wedge. X_{out} is an extremal surface forming a candidate HRT surface, but it has area greater than X_{min}. X_{bulge} lies in between the two, has greater area than either, and it is not minimal on any spatial slice. These three surfaces together form a structure known as a "Python's lunch" — see figure 19. It was conjectured in [118] that the computational complexity required to reconstruct operators in the Python's lunch from the CFT scales as

$$\propto \exp\left(\frac{\text{Area}[X_{\text{bulge}}] - \text{Area}[X_{\text{out}}]}{8G_N}\right). \tag{6.4}$$

In the semiclassical case, we replace HRT with QES and Area$/4G_N$ with the generalized entropy. Thus, the reconstruction of operators in the lunch from operators near the boundary is non-perturbative in G_N, and should not be achievable when just accessing the perturbative bulk description (including gravitons). Further evidence for this was found in [119], where they showed that classical boundary sources, together with repeated forwards and background time evolution (i.e. time-folds), cannot be used to expose the lunch region and make its data causally accessible from the boundary.

The findings in this paper play nicely with the Python's conjecture. The bulge provides a surface that can be dressed across, giving it a functional role. It enables perturbations with spatial compact support around the bulge, never disturbing the complement wedges to the lunch, including the exterior of X_{out}, which is the part of the entanglement wedge that is believed to be simple to reconstruct. Furthermore, at least in spherical symmetry, bulge surfaces are supported by matter, which provides additional background structures to anchor diffeomorphism invariant with respect observables to. So it is quite reasonable that the

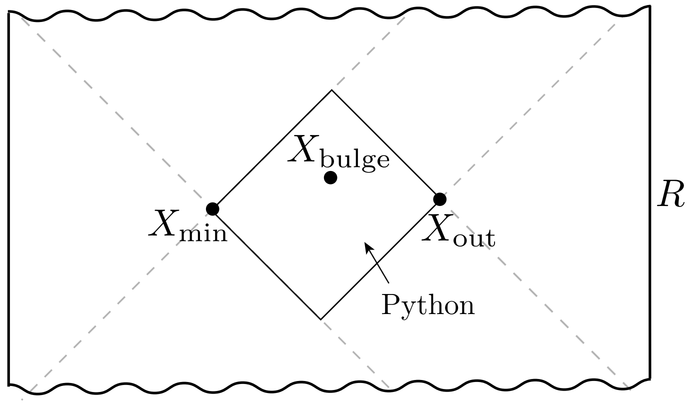

Figure 19. A spacetime with a Python's lunch for a complete boundary component R. The wedge to the left and right of $X_{\min}$ are the entanglement wedges of the left and the right CFT, respectively.

perturbatively quantized theory supports a non-trivial localized algebra of observables inside the Python that commutes with observables in the simple wedge, and thus observables near the boundary, to all orders in perturbation theory in G_N (see [120] for a recent discussion of this algebra).

6.6 Entanglement wedges of gravitating regions

In [121, 122] they proposed how to define entanglement wedges of general gravitational subregions A. More precisely, they defined a min-entanglement wedge $e_{\min}(A)$ and a max-entanglement wedge $e_{\max}(A)$. $e_{\max}(A)$ is proposed to be the region where the semiclassical description can be fully reconstructed from operators associated to A, while $e_{\min}(A)$ is the complement of the largest wedge about which nothing can be learned from operators associated to A. Consider now two regions A, B such that $A \subsetneq e_{\min}(B)^c$ and $B \subsetneq e_{\min}(A)^c$, where c denotes the complement, and where $X \subsetneq Y$ is used to indicate $\partial X \cap \partial Y = \varnothing$. Let's call this EW-independence of A and B. From the above proposal, we should then have that semiclassical operators in A and B can be implemented completely independently. It is then natural to conjecture that

$$\text{EW-independence of } A \text{ and } B \Rightarrow \text{classical independence of } A \text{ and } B, \tag{6.5}$$

where classical independence is the notion studied in this paper. Note however that the operators associated to A that can reconstruct the relevant wedges are part of some yet-to-be-understood quantum mechanical system associated to A, rather than the semiclassical degrees of freedom of A. As a consequence, we do not expect the converse of (6.5) to hold. Even if A and B are classically independent regions, the fundamental degrees of freedom associated to A and B that are capable of entanglement wedge reconstruction might not be independent. It is interesting to consider a case of a region A and a region $B \subsetneq e_{\max}(A) - A$, with A and B classically independent. In this case A can presumably still reconstruct B with access to potentially non-semiclassical quantum effects. Could this gap signify that it is non-perturbatively difficult to reconstruct B from A?

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