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OPTIMAL OPEN-LOOP CONTROL OF INDUSTRIAL ROBOTS

by

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ABSTRACT

The purpose of this project is to investigate a class of difficult optimization problems: The minimization of the energy consumption for the determination of the trajectories of displacement for industrial robots and the determination of their near minimum time open-loop control.

It is a difficult problem because the equations of motion of robots are highly nonlinear and the problem is probably non convex. In addition, there are equality and inequality constraints on both control and state variables. Furthermore, the number of state variables is large (at least three for each degree of freedom).

The issues to be included here are:
1_The dynamics of the robot.

The model includes the Coriolis and centrifugal forces, the limitations on the capability of the motors, their maximum peak current, their maximum average heat power.

2_The optimization technique.

The preliminary work has been performed using OHNO's differential dynamic programming algorithm and successful results have been achieved. The algorithm is the only one of which we are aware that includes equality and inequality constraints on both control and state variables and for which a proof of convergence to optimality is available.

3_The qualitative behavior.

Preliminary results indicate that the energy consumption can often be reduced by more than 30%, and sometimes more than 80% on the minimum energy trajectory compared to currently used trajectories such as straight lines in cartesian or joint coordinates. The time of displacement has also a significant effect on the energy consumption.

Thesis Advisors : S.B. Gershwin
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I_INTRODUCTION

In recent years, the problem of factory productivity has become very important. A solution is to modernize manufacturing facilities by means of automation and in particular by the implementation of robots. A close examination of the basic structures and controls of robots reveals some limitations which lead to unnatural specifications and inefficient task operations.

Among many factors, the reduction of a robot cycle time for a given repetitive task can contribute to an increase of productivity. The reduction of the energy consumption can also lead to a decrease of the operating cost. More specifically, the rate at which a production line can produce goods is the inverse of the longest cycle time of any station in the line. The long standing "line-balancing" problem is that of minimizing the time of each task, and then allocating the task to a set of robots so that their cycle time are as nearly equal as possible. Once the tasks are allocated, all stations except those with the longest operations will have some idle time. The trajectories can then be recalculated to equalize the operation times and to minimize the energy consumption.

For this thesis, a robot problem has always a fixed initial point and a fixed terminal point. By fixing the terminal time, one could think of minimizing the the deviation from a planned path or minimizing the energy of displacement. A real optimum control problem would be an optimum path planning followed by optimum path tracking.

1) Literature Survey

Iemenschot and Whitney (4) have proposed a method to find optimal trajectories for mechanical arms, by minimizing the integral of the kinetic energy between the initial and final time. But their work is based on the assumption that the trajectory can be expanded in a series of simple functions of time. Furthermore, the effect of gravity has been neglected. In addition, no constraint conditions are available for the state or the control variable. Kahn and Roth (5), Takegaki and Arimoto (6) have attempted to minimize the time or a quadratic performance index, but because of the difficulty due to highly nonlinear equations, either the Coriolis and centrifugal terms or constraint conditions have been omitted. According to Luh (3), there have been no results on trajectory optimization which included all the above conditions.

2) Purpose

The purpose of this thesis is to apply Ohno's differential dynamic programming algorithm (1) to solve the optimum path planning of mechanical manipulators, by means of minimizing the energy consumption, during a given time of displacement, between fixed initial and final points. His algorithm is well suited for this case, because it solves discrete time optimal control problems with equality and inequality constraints on both control and state variables. In addition, a proof of convergence to optimality is available. The minimum time open-loop control can then, in principle, be found by computing the minimum time for which a trajectory exists.

3) Brief Discussion of Chapters

In the next Chapter, a optimal control problem with general constraints is described. A model of the robot is presented in Chapter III. The state transition function (Chapter IV), the cost function (Chapter V) and the constraints (Chapter VI and VII) are shown to fit the form described in Chapter II. Special attention is given to the cost function, to emphasize the different possible models of electrical and mechanical systems. Dynamic programming and Ohno's algorithm are described in Chapter VIII and IX. The generation of the initial trajectory is proposed in Chapter X and important remarks are discussed in Chapter XI. Numerical results appear in Chapter XII and the last Chapter concerns the summary and the conclusion.

II_CONTROL PROBLEM WITH GENERAL CONSTRAINTS

The dynamic control of an industrial robot involves the determination of the inputs (voltages, currents, pressures , etc...) for the actuators, so that a set of desired values for the positions and the velocities is achieved. The robot is driven electrically, hydraulically or pneumatically with measurements for feedback control generally provided by encoders and tachimeters connected to the motors shafts.

The control problem solved in this thesis is the computation of the trajectory and the corresponding forces, which minimize the energy provided by the source, for a given time between two fixed points. If we suppose that the problem is convex, that is, if the algorithm converges effectively to the minimum energy trajectory (not a local minimum), then the minimum time problem can be solved by finding the minimum time for which a minimum energy trajectory still exists.

Consider a general discrete time problem where x_n is a state a-vector and u_n is a control b-vector. We define the state transition function $f_n(x_n,u_n)$ where

[1] $x_{n+1} = f_n(x_n,u_n)$, $n = 1,\ldots,N-1$

$\dim(x_n) = a$
$\dim(u_n) = b$

The cost function is

[2] $$V = \sum_{n=0}^{N-1} L_n(x_n,u_n) + L_N(x_N)$$

and the constraints are

[3] $g_n(x_n,u_n) \leqslant 0$,
$h_n(x_n,u_n) = 0$, $n = 0,\ldots,N-1$
$\dim(g_n) = m_n$
$\dim(h_n) = l_n$

The problem is to determine an optimal control u_n^* ($n = 0,\ldots,N-1$) that minimizes V subject to [1] and [3].

III_DESCRIPTION OF THE MANIPULATOR

Consider a P-degree of freedom manipulator driven by P d.c. electric motors through appropriate reduction systems. The energy is provided by an electric source, and the power is transmitted to the motors and regulated by amplifiers commanded by the controller. This electric source can be regenerative or non-regenerative. A source is called regenerative if it can store absorbed power in a form that can be later returned to the system. In a non-regenerative source, the power is dissipated. Obviously, the type of source has an important effect on the minimization problem, and is directly related to the cost function. Both types of source will be considered.

To describe the motion of the manipulator, let q(t) be a P-dimensional vector representing the displacement of the joints. The equations of motion may be developed in a number of ways, using the direct application of the classical mechanics (Newton-Euler, Lagrange). In any case, the equations can be written as follows:

$$[4] \quad I(q)\,\ddot{q} = \dot{q}^{T}\,Q(q)\,\dot{q} + D(q)\,\dot{q} + G(q) + A\,u$$

where

$\dot{q}$ and $\ddot{q}$	are P-dimensionnal vectors representing the velocities and accelerations of the joints,
$I(q)$	is a P×P inertia matrix,
$Q(q)$	is a P×P×P tensor, refering to the Coriolis and centrifugal terms,
$D(q)$	is a P×P viscous friction matrix,
$G(q)$	is a P vector defining the gravitational forces
A	is a P×P diagonal scaling matrix,
u	is a P vector of control input forces.

Let $T = t_f - t_o$, the time of displacement between the initial state $x(t_o)$ and the final state $x(t_f)$. Because Ohno's algorithm solves discrete time control problems, it is necessary to discretize the time during the motion of the robot.

[5] $t_{n+1} = t_n + \Delta t, \quad n = 0,\dots,N-1$

with

$t_o = 0$ and

$t_f = T = N\,\Delta t$

If we consider the first two terms in the Taylor series for the position and the velocity, we obtain

[6] $$q(t + \Delta t) = q(t) + \Delta t\, \dot{q}(t) \quad \text{and}$$
$$\dot{q}(t + \Delta t) = \dot{q}(t) + \Delta t\, \ddot{q}(t)$$

We define, for n = 0,...,N,

[7] $$q_n = q(t)\Big|_{t = n\,\Delta t}$$
$$\dot{q}_n = \dot{q}(t)\Big|_{t = n\,\Delta t}$$

and for n = 0,...,N-1

[8] $$i_n = i(t)\Big|_{t = n\,\Delta t}$$

where i(t) is a P-dimensional vector representing the current is the motors. It is to be noted that for a d.c. electric motor, the torque (or the force, if the motor is translational) is proportional to the current in the armature.

This minimization problem is done for robots working, for example, in an assembly line. Some have to work without interuption and it is necessary to limit the average heat power in the electric circuit of the motors. For the p^{th} motor, the average heat power, when the motion is completed, can be written

$$[9] \quad P_{ave\,p} = \frac{1}{T} \int_0^T R_p \, i_p^2(t) \, dt$$

where R_p is the resistance of the armature of motor p.

After discretisation we obtain

$$P_{ave\,p} = \frac{1}{N} \sum_{n=0}^{N} R_p \, i_{pn}^2$$

If we define the variable w_{pn} by

$$[10] \quad w_{pn} = \sum_{j=0}^{n} R_p \, i_{pj}^2 ,$$

then the average heat power for each motor p is

$$[11] \quad P_{ave\,p} = \frac{1}{N} \quad w_{pN}$$

From the equations [7], [8] and [10], we define the state variable x_n and the control variable u_n by

$$[12] \quad x_n = \begin{pmatrix} q_n \\ \dot{q}_n \\ w_n \end{pmatrix} , \qquad n = 0,\ldots,N$$

[13] $u_n = i_n$, $n = 0,\ldots,N-1$

We have

$\dim(x_n) = 3P$
$\dim(u_n) = P$

IV_STATE TRANSITION FUNCTION

Denote

[14] $d = \Delta t = T/N$

From equation [6], [9] and [10] we deduce the state transition function f_n by

[15] $$f_n(x_n,u_n) = \begin{pmatrix} q_{n+1} \\ \dot{q}_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} q_n + d\,\dot{q}_n \\ \dot{q}_n + d\,\ddot{q}_n(q_n,\dot{q}_n,i_n) \\ w_n + (\mathrm{diag}\ R)\ i_n^2 \end{pmatrix}$$

where (diag R) is the PxP diagonal matrix with the p^{th} diagonal element R_p and $\ddot{q}_n(q_n,\dot{q}_n,i_n)$ is the vector acceleration given by the dynamic equation [4].

V_COST FUNCTION

There are many ways to derive the cost function which is here the energy consumed by the robot during the time T. An easy one is to look only at the behavior of the motors which transmit all the power to the robot. We know that all the electric energy provided by the source is transformed into heat and work. (Assuming that the field circuits of the motors are independent of the armatures, the field circuits have no effects on the minimization problem.)

We can consider that the heat is only dissipated in the armature resistance and can be written as

$$[16] \quad H = \sum_{p=1}^{P} \int_{0}^{T} R_p \, i_p^2(t) \quad dt$$

where $i_p(t)$ is the current in the armature of resistance R_p of motor p. After discretization, we obtain

$$[17] \quad H = d \sum_{n=0}^{N} u_n^T \, (\mathrm{diag}\ R) \, u_n$$

If we consider now the work of the torques or forces of the motors, we can write

$$[18] \quad W = \sum_{p=1}^{P} \int_0^T Pm_p(t)\, dt$$

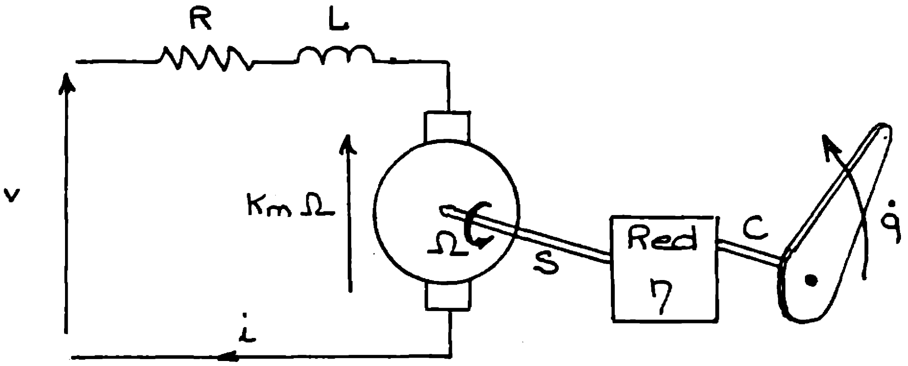
where $Pm_p(t)$ is the instantaneous mechanical power provided by motor p. The total instantaneous mechanical power can be written

$$[19] \quad Pm = S\, \Omega$$

where S is a P-vector representing the torques or forces at the shafts of the motors and Ω their velocities.

We need now to find a mathematical model of the motors and their reduction systems in order to express the cost as a function of the state and control variables.

Figure 1: model for the motor and the reduction system.



R is the resistance of the armature,
L is the inductance of the armature,
Km is the torque constant of the motor,
v is the input voltage,
i is the current in the induction circuit,
Red is the value of the reduction,
η is the efficiency of the reduction system,
Ω is the velocity of the shaft of the motor,
$\dot{q}$ is the velocity of the link of the robot,
S is the torque (force) on the shaft,
C is the torque (force) on the link.

We have

$$[20] \quad \Omega = q/Red$$

$$S = Km\ i$$

and

$$C = T\ Red\ \eta \qquad \text{if } R\ i^2 + S\Omega \geqslant 0,$$

$$C = T\ Red/\eta \qquad \text{if } R\ i^2 + S\Omega < 0.$$

For one motor p, we can write

$$[21] \quad Pm\ dt = \left(Km_p/Red_p\right) i_p(t)\ \dot{q}_p(t)\ dt$$

After discretisation of equation [18], we obtain

$$[22] \quad W = d \sum_{p=1}^{P} \sum_{n=0}^{N} \left(Km_p/Red_p\right) i_{pn}\ \dot{q}_{pn}$$

Note that d is a constant, and has no effect on on the minimum of the cost function. For simplification, it will be substituted from the cost function.

At this point, it is necessary to consider several different cases, each of which gives rise to a different cost function. The electric system can be regenerative or not, and the mathematical model may or may not include the friction effect.

Since we are more interested in the behavior of the robot for high speed motion, it is a reasonable assumption to neglect the effect of dry friction compared to the effect of viscous friction. The resistant force due to dry friction only depends on the direction of motion, while the resistant force due to viscous friction depends on the speed of displacement.

We now state the cost function for various combinations of the above conditions.

1) Regenerative source and no friction

The assumption of no friction stands for
_ no dry friction,
_ no viscous friction,
_ efficiency of the reduction system equal to 1,
but includes electrical friction (i.e. resistance of the motors)

In this case, the mechanical energy consumed or provided by the motors is independent of the trajectory (if the velocities at the extreme points are equal to zero, then the mechanical energy is the difference of potential between those two points). The loss is the heat dissipated in the resistances of the motors. The cost function is derived from equation [17], and the instantaneous cost can be written

$$L_n = u_n^T \text{ (diag } R) \, u_n \tag{23}$$

2) Regenerative source and friction

Because there is some energy dissipated in the mechanical devices, we need to consider the electrical and mechanical energies consumed by the global system.

Using equation [17] and [22], the instantaneous cost function can then be written

[24] $$L_n = u_n^T \, (\text{diag } R) \, u_n + u_n^T \, (\text{diag } Km/Red) \, \dot{q}_n$$

3) Non-regenerative source

In the case of a non-regenerative electric system with or without the friction effect , we cannot recuperate the mechanical energy when a motor becomes a generator. The loss is the energy consumed by the motors. (We don't benefit when the energy is coming back to the voltage source). So for each motor,and using equation [20], we define the variable δ_{pn} by

[25] $\delta_{pn} = 1,$ when $R_p \, i_{pn}^2 + (Km_p/Red_p) \, i_{pn} \, \dot{q}_{pn} \geqslant 0,$
(i.e. when the actuator is a motor)
$\delta_{pn} = 0,$ when $R_p \, i_{pn}^2 + (Km_p/Red_p) \, i_{pn} \, \dot{q}_{pn} < 0.$
(i.e. when the actuator is a generator)

and the instantaneous cost function by

[26] $$L_n = u_n^T \, (\text{diag } \delta_n R) \, u_n + u_n^T \, (\text{diag } \delta_n \, Km/Red) \, \dot{q}_n$$

VI_EQUALITY CONSTRAINTS

For the purpose of this thesis, the only equality constraint is that the final position must be equal to its expected value and that the velocity at this position is equal to zero. We have

[27] $q_N - qf = 0$ and
$\dot{q}_N = 0.$

The inequality constraints are defined from $n = 0,\dots,N-1$. But using the state transition function we can express the final constraints at step N-1. Furthermore, the algorithm will converge more rapidly if we define the constraints as function of the control and the state variables. Using the fact that $\dot{q}_{N-1}$ is function of $\ddot{q}_{N-2}$, we define the position constraints at step N-2. We obtain

[28] $q_N = q_{N-2} + 2\ d\ \dot{q}_{N-2} + d^2\ \ddot{q}_{N-2}(x_{N-2}, u_{N-2})$

[29] $\dot{q}_N = \dot{q}_{N-1} + d\ \ddot{q}_{N-1}(x_{N-1}, u_{N-1})$

It is to be noted that it is possible to add other constraints. For example, one could think of defining a configuration of the robot at a certain step s, which would allow the robot to arrive at the final position from a given direction. It would cross this point without stopping. This constraint can be written

$$[30] \quad q_{s-2} + 2\,d\,\dot{q}_{s-2} + d^2\,\ddot{q}_{s-2}(x_{s-2}, u_{s-2}) - q_{rs} = 0$$

where q_{rs} is the required position of the robot at step s.

VII_INEQUALITY CONSTRAINTS

The robot we are dealing with is driven by P d.c. electric motors which are limited in power. From Figure 1, we can write the equation of motion of an electric d.c. motor. Compared to the inertias of the links, the inductance of the armature in negligible. Thus

[31] $v = R\, i + Km\, \Omega$

where $Km\, \Omega$ is the back-EMF of the motor.

From equation [16], we can deduce

[32] $v_n = (\text{diag } R)\, u_n + (\text{diag } Km/Red)\, \dot{q}_n$

where v_n is a P-vector representing the voltages at the inputs of the motors. These voltages are provided by amplifiers which are limited in voltage and current. From these remarks, we can deduce three inequality constraints defined for every step n, which are

[33] $(\text{diag } R)\, u_n + (\text{diag } Km/Red)\, \dot{q}_n - V_{max} \leqslant 0$
$-(\text{diag } R)\, u_n - (\text{diag } Km/Red)\, \dot{q}_n - V_{max} \leqslant 0$

[34] $(\text{diag } R)\ u_n^2 - P_{peak} \leqslant 0.$

where V_{max} is a P-vector representing the maximum voltages at the output of the amplifiers, and P_{peak} is a P-vector representing the maximum peak power admissible in the armature of the motors.

The inequality constraint concerning the maximum average heat power has been introduced in section III, and using equation [15], we can write the constraint for the final step at step N-1 as follows

[35] $w_{N-1} + (\text{diag } R)\ u_{N-1}^2 - N\ P_{ave} \leqslant 0.$

where P_{ave} is a P-vector representing the maximum average heat powers during the motion of the robot. This inequality constraint is defined at step N-1.

Other inequality constraints are given by the limits of displacement of the links. (We will not consider obstacles avoidance in this paper.) As an example on the limits of displacement, we can define a minimum and a maximum value for each joint displacement, and we have

[36] $q_{min} \leqslant q_n \leqslant q_{max}$ for $n = 0,\ldots,N$

so

$$[37] \quad \begin{aligned} & q_n + 2\,d\,\dot{q}_n + d^2\,\ddot{q}_n(x_n,u_n) - q_{max} \leqslant 0 \\ & q_{min} - q_n - 2\,d\,\dot{q}_n - d^2\,q_n(x_n,u_n) \leqslant 0. \end{aligned}$$

These constraints must be defined at all steps n.

In Chapters IV, V, VI and VII, we have determined all the necessary functions for solving by dynamic programming the minimum energy or minimum time control of a general P degree of freedom manipulator using d.c. electric motors. Let us recall all these functions:

The state transition function is given by equations [4] and [15].

The equality constraints are given by equations [28],[29] and [30]. Equation [28] is defined at step N-2, equation [29] at step N-1 and equation [30] at step s-2.

The inequality constraints are given by equations [33], [34], [35] and [37]. Equations [33], [34], [35] are defined at all steps n and equation [37] at step N-1.

The cost function is given by equations [23], or [24], or [25] and [26].

VIII_DYNAMIC PROGRAMMING

Conventional dynamic programming is described in this section. Define $V_n(X_n)$ the optimal cost of the trajectory from step n to the final step N as

$$[38] \quad V_n(x_n) = \min\left\{ \sum_{i=n}^{N-1} L_i(x_i,u_i) + L_N(x_N) \text{ subject to} \right.$$

$$\left.\begin{array}{ll} g_i(x_i,u_i) \leqslant 0, & \\ h_i(x_i,u_i) = 0, & i = n,\dots,N-1 \end{array}\right\}$$

$$\text{for } n = 0,\dots,N-1$$

Then the principle of optimality leads to

$$[39] \quad V_n(x_n) = \min\left\{ L_n(x_n,u_n) + V_{n+1}\{x_{n+1}\} \text{ subject to} \right.$$

$$\left.\begin{array}{l} g_n(x_n,u_n) \leqslant 0, \\ h_n(x_n,u_n) = 0 \end{array}\right\}$$

$$\text{for } n = 0,\dots,N-1$$

and by [2], $V_N(x_N)$ is given by

$$[40] \quad V_N(x_N) = L_N(x_N).$$

As is well known, the optimal control { u_n^* } can be obtained by starting from [40] and solving [39] recursively for n = N-1,...,0. Since this approach is computationally impossible when the number of state variables is greater than 4, we will follow the differential dynamic programming method invented by Jacobson and Mayne (2).

Define the Lagrangian function F_n (n = 0,...N-1) for one variable minimization problem [37] as

$$[41] \quad F_n(x_n,u_n,\lambda_n,\mu_n) = L_n(x_n,u_n) + V_{n+1}\{f_n(x_n,u_n)\} + \lambda_n^T g_n(x_n,u_n) + \mu_n^T h_n(x_n,u_n)$$

where λ_n and μ_n are the Lagrangian multipliers. Note that the minimum value of $V_n(x_n^*)$ is attained at $u_n = u_n^*$. If V_{n+1} is twice continuously differentiable, then the following Kuhn-Tucker conditions hold as second-order necessary conditions that u_n^* be an optimal solution of [39]. There exist λ_n^* and μ_n^* such that

[42] $F_{un}^{*} = 0,$

$(\mathrm{diag}\ \lambda_{n}^{*})\ g_{n}^{*} = 0,$

$h_{n}^{*} = 0,$

$g_{n}^{*} \leqslant 0,$

$\lambda_{n}^{*} \geqslant 0,$

and such that for every vector z satisfying

[43] $g_{iun}^{*}\ z = 0$ for all i $\{i;\ g_{in}^{*} = 0\}$

and

$h_{jun}^{*}\ z = 0$ for all j,

$z^{T}\ F_{uun}\ z \geqslant 0.$

IX_DIFFERENTIAL DYNAMIC PROGRAMMING ALGORITHM

Put for n = 0,...N-1

$$y_n = (u_n^T, \lambda_n^T, \mu_n^T)^T$$

and define

[44] $T_n(x_n, y_n) = (F_{un}, g_n^T \ (\mathrm{diag}\,\lambda_n), h_n^T)^T$

With this notation, condition [40] can be rewritten as $T_n(x_n^*, y_n^*) = 0$. Therefore, if y_n^* is an isolated solution of $T_n(x_n^*, y_n) = 0$, Then y_n^* can be obtained by solving $T_n(x_n, y_n) = 0$ in its appropriate neighborhood. If the Jacobian matrix of T_n with respect to y_n is nonsingular at y_n^* , then y_n^* is an isolated solution of $T_n(x_n^*, y_n) = 0$.

Denote by J_n the Jacobian matrix of T_n with respect to y_n, and by K_n the Jacobian matrix of T_n with respect to x_n.

Let $\{ y_n^o, n = 0,\dots,N-1 \}$ be given, and let $\{ x_n^o, n = 0,\dots,N \}$ be the trajectory corresponding to $\{ u_n^o \}$. Then a conceptual algorithm is as follows.

Calculate y_n^{k+1} using Newton's method for $n = N-1,\ldots,0$. Newton's method is described as

$$[45] \quad y_n^{k+1} = y_n^k - J_n^{-1}(x_n^k, y_n^k)\; T_n(x_n^k, y_n^k)$$

It is to be noted that T_n, J_n, K_n for $n = 0,\ldots,N-2$ include unknown values $V_{xn+1}(x_{n+1}^k)$ and $V_{xxn+1}(x_{n+1}^k)$. Consequently it is essential to obtain their approximate values which guarantee that $\{ y_n^* \}$ is a point of attraction. We know that for the optimal control $\{ u_n^* \}$, the cost function is equal to the Lagrangian function. It is then possible to determine $V_{xn+1}(x_{n+1})$ and $V_{xxn+1}(x_{n+1})$ by an approximation of the Lagrangian function. With a second order approximation, we obtain for $\Delta y_n^*(x_n)$

$$[46] \quad \Delta y_n^*(x_n) = - J_n^{-1}(x_n, y_n^*(x_n))\; K_n(x_n, y_n^*(x_n))$$

Denote by $\tilde{.}$ the approximate value of . .
So calculate for $n = N-1,\ldots,1$, $\tilde{y}_n^{k+1}$ by

$$[47] \quad \tilde{y}_n^{k+1} = y_n^k - \tilde{J}_n^{-1}(x_n, u_n)\; \tilde{T}_n(x_n, u_n)$$

and consequently

$\tilde{V}_{xn+1}(x_{n+1}^k)$ and $\tilde{V}_{xxn+1}(x_{n+1}^k)$ using equation [46].

Compute the new control variable and the new state variable for n = 1,...,N-1 by

[48] $$x_n^{k+1} = f_{n-1}(x_{n-1}^{k+1}, u_{n-1}^{k+1}).$$
$$y_n^{k+1} = \tilde{y}_n^{k+1} - [J_n^{-1} K_n(x_n, y_n)](x_n^{k+1} - x_n^k)$$

and $$x_N^{k+1} = f_{N-1}(x_{N-1}^{k+1}, u_{N-1}^{k+1})$$

Choose ε , and if

[49] $$\max_n(\| y_n^{k+1} - y_n^k \|) < \varepsilon$$

then stop, otherwise set k = k+1 and use this trajectory as initial guess for a new iteration.

For more details, refer to Ohno's algorithm (1).

X_INITIAL TRAJECTORY

We could guess any trajectory starting from the initial point, but it is better to choose a trajectory as near as possible to the optimal one. This would require fewer iterations for optimality. It would be interesting to choose a trajectory currently used for industrial robots so that we are able to compare the energy or the time reduction. In the space domain, straight lines in joint or cartesian coordinates are the most often used (Luh (3)). Along these paths, there is no specific method used to determine the positions and the velocities. For our case, let us choose a trajectory such that the velocities are zero at the initial and final points. A cubic function of the time for each parameter is well adapted here, because it has one minimum and one maximum which can correspond to the extreme points. This would represent the straight line in the joint coordinates. For the straight line in cartesian coordinates, it is sufficient to calculate X(n), Y(n), Z(n) in the cartesian frame and use the inverse kinematics to find the joint coordinates.

The cubic function for each parameter can be written as follows:

[50] $$q_n = A n^3 + B n^2 + C n + D.$$

The initial and final conditions are

[51] $$\begin{aligned} q_0 &= D \\ q_N &= A N^3 + B N^2 + C N + D \\ 0 &= C \\ 0 &= 3 A N + 2 B N + C \end{aligned}$$

So, we find for $n = 0,\ldots,N$

[52] $$q = 2 (q_0 - q_f)(n/N)^3 + 3 (q_f - q_0)(n/N)^2 + q$$

The corresponding control $\{ u_n \}$ will be calculated by the dynamic equation [4].

Ohno's algorithm will converge to optimality, only if the initial control corresponding to the initial trajectory is in the neighborhood of the optimal control u_n^* . For this reason, defining initial trajectories only by straight lines in joint and cartesian coordinates is not sufficient. Those trajectories can be far from optimal. Consequently, we must be able to use previously calculated trajectories as initial ones for related problems. As an example, we can use the optimal trajectory corresponding to a given time of displacement, to solve the optimal control for the problem with a shorter or longer time. This can be the basis for an algorithm to find the minimum time trajectory.

XI_REMARKS

This DDP algorithm is the only one of which we are aware that includes equality and inequality constraints on both control and state variables and whose convergence has been proved. This proof of convergence (1) holds when the functions L_n,f_n,g_n and h_n are twice differentiable and their second derivatives are locally Lipschitz continuous. In the cases of a non-regenerative source and some friction effects in the reduction systems, those derivatives are not continuous. Some further work is required to approximate those functions by continuous functions. Those cases should not be forgotten since they are more appropriate models for many real problems.

XII_EXAMPLE

Let us take now an an example of this optimization problem applied to an industrial robot. The example treated is based on the assumption of a regenerative source and no friction. In order to simplify the equations of motion and especially the number of state variables and inequality constraints, the effect of the end-effector (wrist) will be neglected. The robot chosen for this example will have spherically defined coordinates, such as the UNIMATE 2000 or the BENDIX AA-160 robots. A model of the robot and its parameters are presented in Figure 2.

For these robots, without considering the wrist, three parameters are sufficient to define the configuration. That is, we represent a three degree of freedom robot. The parameters are

R = the length between the center of rotation and the effective end,
θ = the deviation angle of the arm from the vertical axis,
ϕ = the deviation angle of the arm from a fixed horizontal axis.

The control variables will be the currents in the three motors.

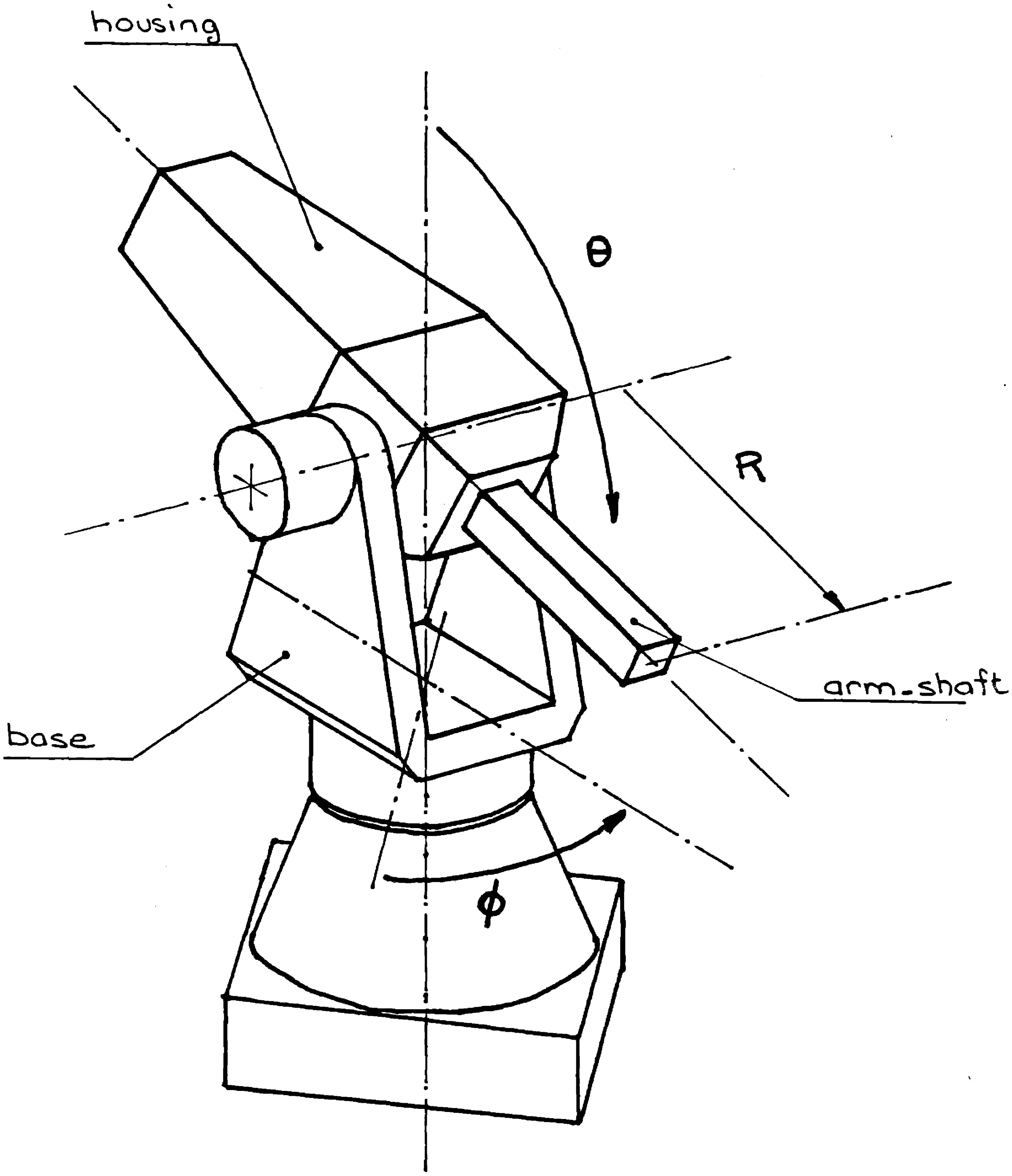
I_R = current in the translationnal motor,
I_θ = current in the theta rotation motor,
I_ϕ = current in the phi rotation motor.

From equations [11] and [12], we define

$$[53] \quad x_n = \begin{pmatrix} R_n \\ \theta_n \\ \phi_n \\ \dot{R}_n \\ \dot{\theta}_n \\ \dot{\phi}_n \\ w_{Rn} \\ w_{\theta n} \\ w_{\phi n} \end{pmatrix} \qquad \text{and} \qquad u_n = \begin{pmatrix} I_R \\ I_\theta \\ I_\phi \end{pmatrix}$$

1)Model of robot

Figure 2: Model of the robot


housing
θ
R
arm-shaft
base
φ

Hypothesis :

The shaft is a uniformaly distributed mass

L = length

m_s = mass

$J_{s\theta}$ = inertia for rotation theta

$J_{s\phi}$ = inertia for rotation phi

Housing

r_h = algebraic distance between the center of gravity and the center of rotation

m_h = mass

$J_{h\theta}$ = inertia for rotation theta

$J_{h\phi}$ = inertia for rotation phi

J_r = inertia of shaft and housing for rotation around their long axis

Base

J_b = inertia for rotation phi

The workpiece is a point mass m and we write for simplification

$$m_t = m + m_s$$

2)Equations of motion

The equations of motion are derived from Lagrange's equation in appendix A. They are expressed as follows

$$[54]\quad \begin{aligned} \ddot{R} = {} & [C_R\, I_R + (m_t\, R - m_s\, L/2)\, (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)]/m_t \\ & - g\, \cos\theta \end{aligned}$$

$$\begin{aligned} \ddot{\theta} = {} & [\, C_\theta\, I_\theta + (m_t\, R - m_s\, L/2 + m_h\, r_h)\, g\, \sin\theta \\ & - 2\, (m_t\, R - m_s\, L/2)\, \dot{R}\, \dot{\theta} \\ & + 1/2\, (m_t\, R^2 - m_s\, L\, R + m_s\, L^2/3 + m_h\, r_h^2 + J_{h\phi} \\ & \quad - J_r)\, \dot{\phi}^2 \sin 2\theta\,] \\ & /(m_t\, R^2 - m_s\, L\, R + m_s\, L^2/3 + m_h\, r_h^2 + J_{h\theta}) \end{aligned}$$

$$\begin{aligned} \ddot{\phi} = {} & [\, C_\phi\, I_\phi - 2\, (m_t\, R^{?} - m_s\, L/2)\, \dot{R}\, \dot{\phi}\, \sin^2\theta \\ & - (m_t\, R^2 - m_s\, L\, R + m_s\, L^2/3 + m_h\, r_h^2 + J_{h\phi} - J_r) \\ & \quad \dot{\phi}\, \dot{\theta}\, \sin 2\theta\,] \\ & /\, [\, J_b + J_r\, \cos^2\theta + (m_t\, R^2 - m_s\, L\, R + m_s\, L^2/3 \\ & \quad + m_h\, r_h^2 + J_{h\phi}\,)\, \sin^2\theta\,] \end{aligned}$$

with

$$C_p = Km_p\, \eta_p\, /Red_p \qquad \text{if } R_p\, I_p^2 + (Km_p/Red_p)\, I_p\, \dot{q}_p \geq 0$$

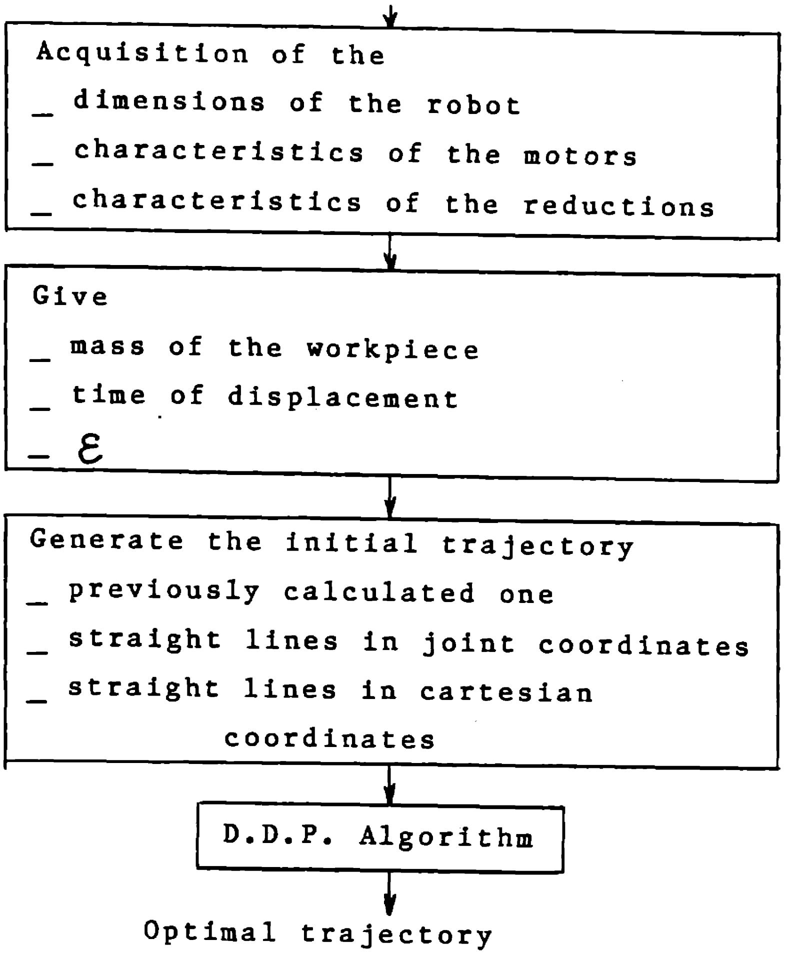
$$C_p = Km_p/(Red_p\, \eta_p\,) \qquad \text{if } R_p\, I_p^2 + (Km_p/Red_p)\, I_p\, \dot{q}_p < 0$$

In order to apply Ohno's algorithm, we need to find the derivatives of all the equations f_n, L_n, g_n, h_n with respect of all the state and control variables. They are listed in appendix B.

3) Programs

FORTRAN programs have been written. As expected, we have not been able to get succesful results with friction or non-regenerative conditions. But on the other hand, very interesting and unanticipated results have been achieved with the regenerative case without friction. A general flowchart of the program appears in Figure 3.

Figure 3: General flowchart of the main program


Acquisition of the
_ dimensions of the robot
_ characteristics of the motors
_ characteristics of the reductions
Give
_ mass of the workpiece
_ time of displacement
_ ε
Generate the initial trajectory
_ previously calculated one
_ straight lines in joint coordinates
_ straight lines in cartesian coordinates
D.D.P. Algorithm
Optimal trajectory

More details, using the names of the subroutines and the files, are shown in Figure 4.

Figure 4: detailed flowchart of the main program

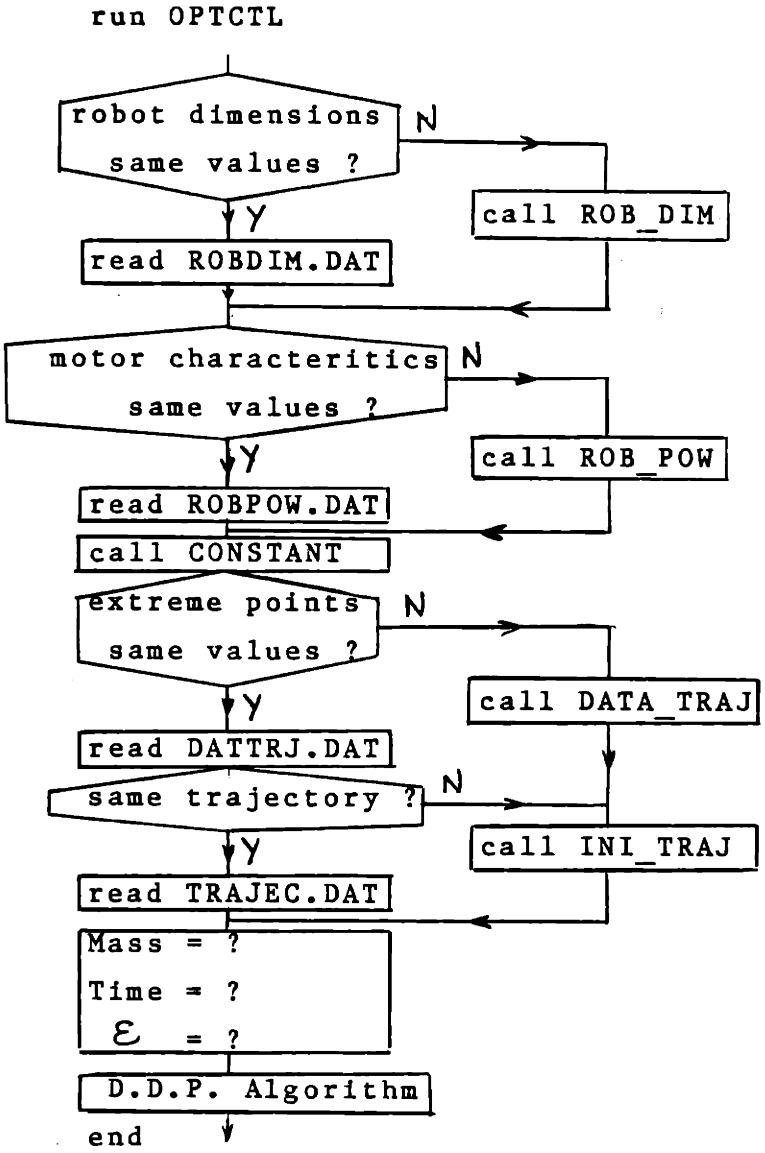
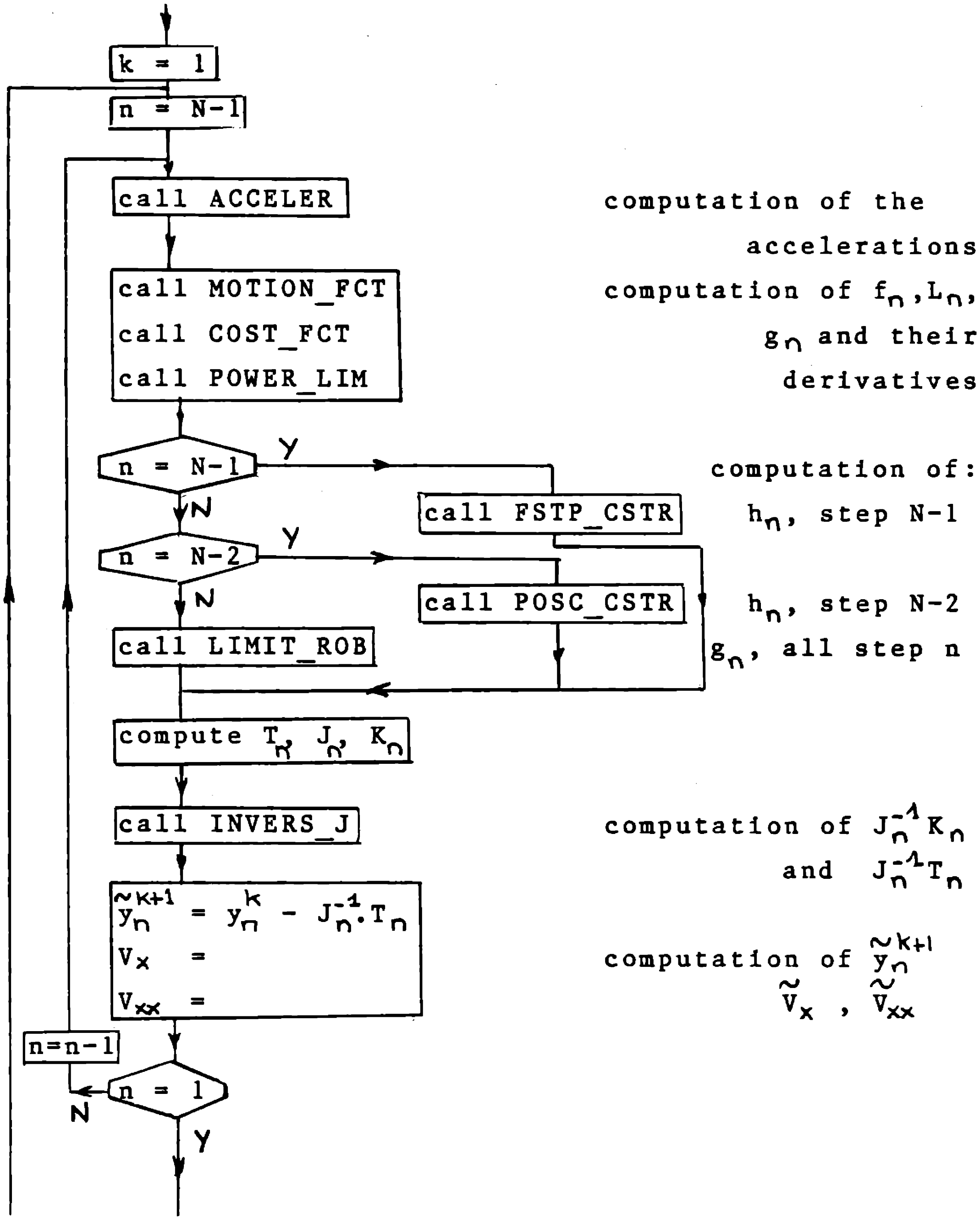

run OPTCTL
robot dimensions same values ?
N
call ROB_DIM
Y
read ROBDIM.DAT
motor characteritics same values ?
N
call ROB_POW
Y
read ROBPOW.DAT
call CONSTANT
extreme points same values ?
N
call DATA_TRAJ
Y
read DATTRJ.DAT
same trajectory ?
N
call INI_TRAJ
Y
read TRAJEC.DAT
Mass = ?
Time = ?
ε = ?
D.D.P. Algorithm
end

Figure 5:Description of the D.D.P. Algorithm


k = 1
n = N-1
call ACCELER
call MOTION_FCT
call COST_FCT
call POWER_LIM
n = N-1
Y
N
call FSTP_CSTR
n = N-2
Y
N
call POSC_CSTR
call LIMIT_ROB
compute T_n, J_n, K_n
call INVERS_J
$\tilde{y}_n^{k+1} = y_n^k - J_n^{-1} \cdot T_n$
$V_x =$
$V_{xx} =$
n=n-1
n = 1
N
Y

computation of the accelerations

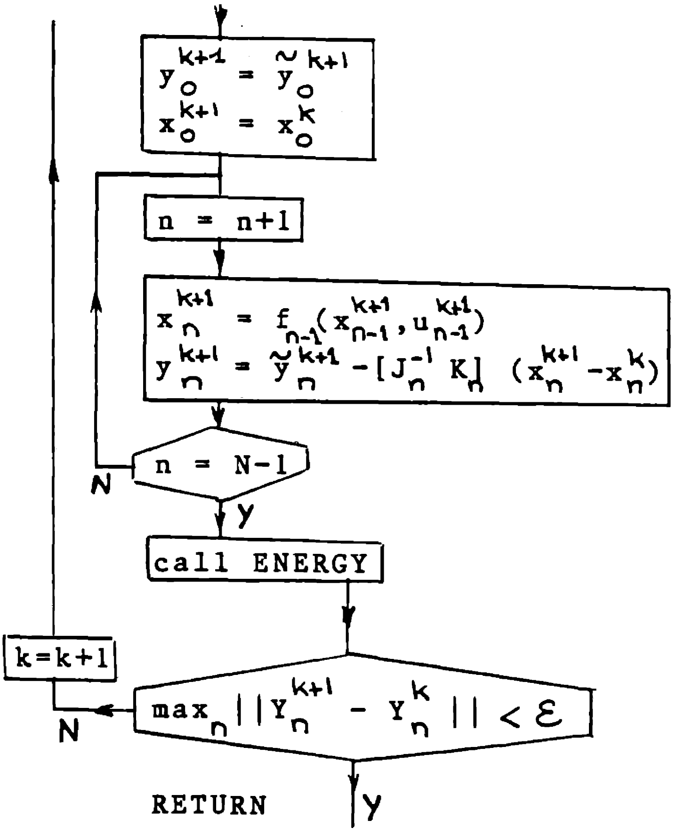
computation of f_n, L_n, g_n and their derivatives

computation of:
h_n, step N-1

h_n, step N-2
g_n, all step n

computation of $J_n^{-1} K_n$ and $J_n^{-1} T_n$

computation of $\tilde{y}_n^{k+1}$
$\tilde{V}_x$, $\tilde{V}_{xx}$


$y_0^{k+1} = \tilde{y}_0^{k+1}$
$x_0^{k+1} = x_0^k$
n = n+1
$x_n^{k+1} = f_{n-1}(x_{n-1}^{k+1}, u_{n-1}^{k+1})$
$y_n^{k+1} = \tilde{y}_n^{k+1} - [J_n^{-1} K_n] (x_n^{k+1} - x_n^k)$
n = N-1
N
Y
call ENERGY
k=k+1
$\max_n \| Y_n^{k+1} - Y_n^k \| < \varepsilon$
N
RETURN
Y

computation of x_n^{k+1}
computation of y_n^{k+1}

computation of the energy, iteration k

In order to display all the results, three other programs have been written :

The program GRAF shows the graphs of CURRENT vs VELOCITY for all the motors, specifying the feasible region (i.e. all the possible values of current and velocity satisfying the inequality constraint conditions).

The program POW shows the graphs of POWER vs TIME for all the motors.

The program MOV shows the motion of the robot along a calculated trajectory, under different views.

XIII_RESULTS

As described in the flowchart in Figure 3, the program can solve the optimal control for any robot defined in spherical coordinates for any choice of the parameters for its dimensions and for the characteristics of its motors and reductions. All the following examples have been treated with the same robot. The parameters are listed in Table 1 for the dimensions and in Table 2 for the motors and reductions. The following results describe general observations on minimum energy and near minimum time control trajectories.

Table 1 : Dimensions of the robot

lenght in meters
mass in kilos
angle in degrees
inertias in kg.m**2

DIMENSIONS OF THE ROBOT

SHAFT	ls= 1.000	ms= 3.00
HOUSING	rh= -0.100	mh= 8.00
	Iht= 0.150000	Ihp= 0.150000
	Ir= 0.050000	
BASE	Ib= 0.400000	

LIMITS

RADIUS	Rmin= 0.160	Rmax= 0.910
ANGLE	Tmin= -135.0	Tmax= 250.0

Table 2 : Characterisics of the motors and reduction systems

M : 1: MOTOR TRANSLATION
2: MOTOR THETA
3: MOTOR PHI

voltage in Volts
resistance in Ohms
constant Km in Newton.meter/amp.
power in Watts
Red(1) in meter/rev

M\I	1	2	3	4	5	6	7
1	100.0	1.0	80.0	200.0	0.80	0.04	1.00
2	100.0	1.0	80.0	200.0	1.70	0.15	1.00
3	100.0	1.0	80.0	200.0	1.70	0.15	1.00
	Vmax	R	Ppeak	Pave	Km	Red	ef

REGENERATIVE ELECTRIC SYSTEM

1) General observations

a) Control

For a P degree of freedom robot with a regenerative source and no friction, we have seen in Chapter V, that the cost function is defined by equation [23], but can be written

$$[55] \quad Et = \sum_{p=1}^{P} R_p \left[\int_0^T I_p^2 \, dt \right]$$

When a motor is applying a torque, a current is set in the armature, and heat is dissipated in the resistance. It therefore consumes energy. From equation [50], we can say that the energy is the sum of the energy of each motor. One way to minimize this energy is to reduce the current in each motor. We must not forget that the robot has to meet all the constraint conditions and in particular the final position constraint. We have observed that the minimum energy control to go from one point to an other in a given time is to apply some torque just at the beginning and at the latest moment. An intuitive explanation of this behavior is that it should reach some speed at the earliest time, then move without consuming any energy, then stop by the opposite torque. This behavior is achieved for many examples, in particular, for the rotation of the robot around its vertical axis. The graphs $R.I^2$ vs TIME from Figure 6 show that each motor is only excited at the beginning and at the end of the trajectory. For this trajectory, note that the energy reduction compared to a straight line in joint coordinates (see Chapter X) is 76 %. Other examples demonstrate similar behavior.

Figure 6 a

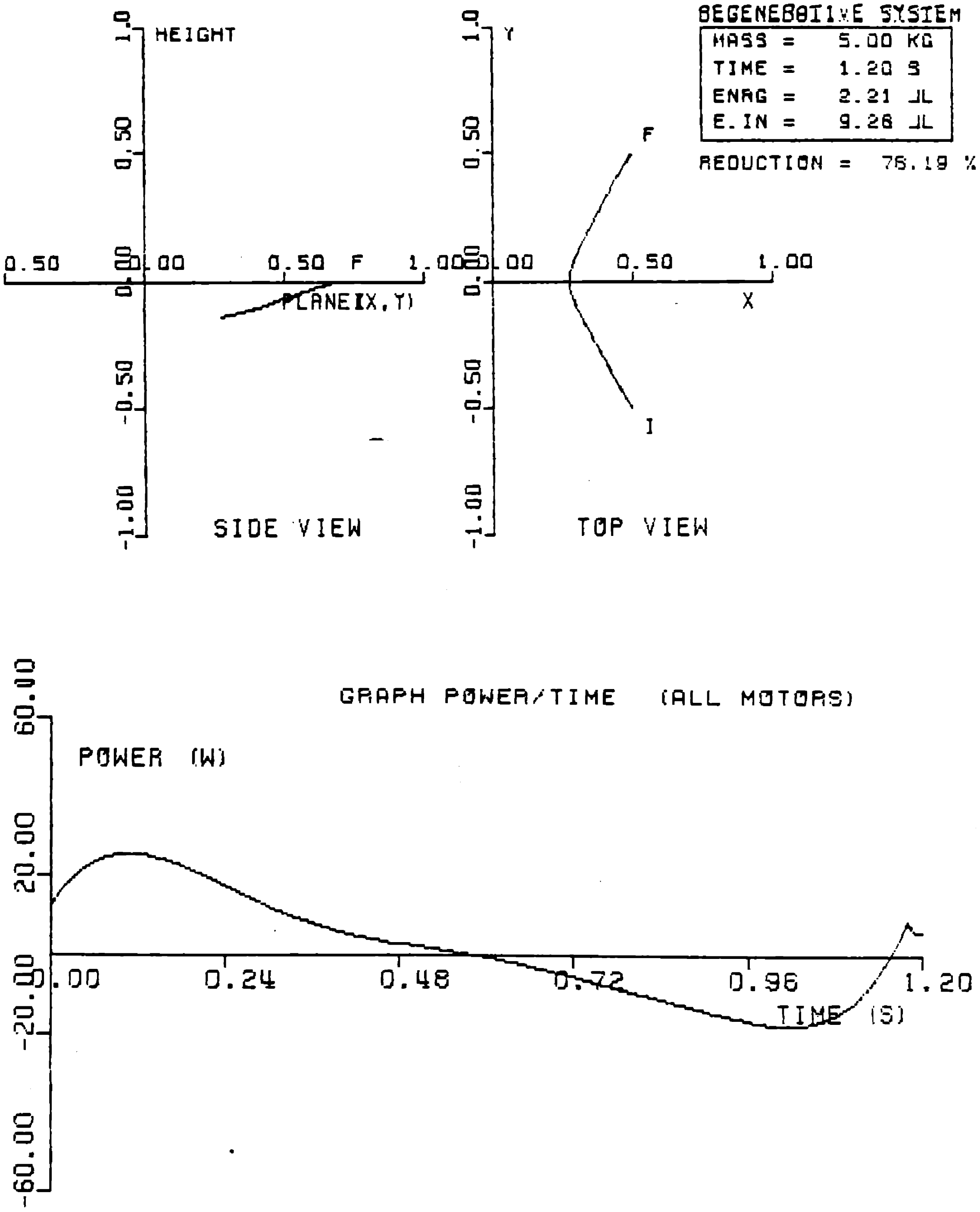
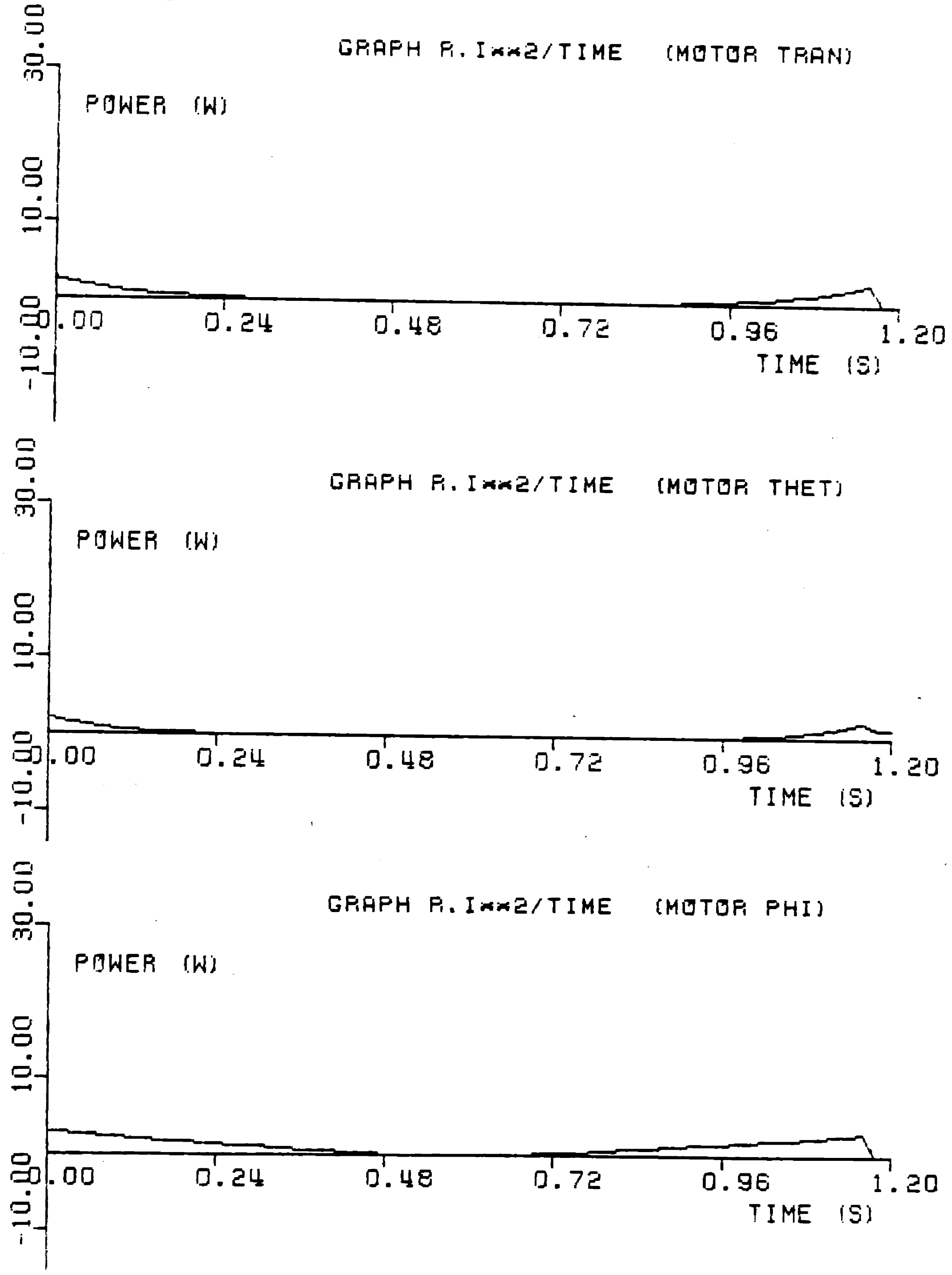

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.20 S
ENRG = 2.21 JL
E.IN = 9.28 JL
REDUCTION = 76.19 %
HEIGHT
1.0
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
F
1.00
PLANE(X,Y)
SIDE VIEW
Y
F
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
0.00
-20.00
-60.00
0.24
0.48
0.72
0.96
1.20
TIME (S)

Figure 6 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
30.00
10.00
-10.00
0.00
0.24
0.48
0.72
0.96
1.20
TIME (S)
GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
30.00
10.00
-10.00
0.00
0.24
0.48
0.72
0.96
1.20
TIME (S)
GRAPH R.I**2/TIME (MOTOR PHI)
POWER (W)
30.00
10.00
-10.00
0.00
0.24
0.48
0.72
0.96
1.20
TIME (S)

b) Choice of the trajectory

If we consider the example of reaching the antipodal point, we note that several possibilities for the initial trajectory exist. It is possible to reach the final position by the top (from $\theta = 90^\circ$ to $\theta = -90^\circ$), by the bottom (from $\theta = 90^\circ$ to 270°), or by the side (from $\phi = 0^\circ$ to 180°). Figure 7 shows the optimum trajectory for the rotation by the top. Note that the energy consumption is 4.94 Joules. For the rotation by the bottom (Figure 8) the energy consumption is 4.28 Joules, and for the rotation by the side (Figure 9), the energy consumption is 0.95 Joules. In the previous section we mentioned that it is better to reach some speed as early as possible. It is therefore not surprising that the trajectory by the top demands more energy. The trajectory by the bottom is not so good, because the centrifugal force reacts in the same direction as the gravity. For this specific robot, even if you need to activate the motor for the rotation the base, we find that it is worthwhile to move by the side.

Figure 7 : Rotation by the top

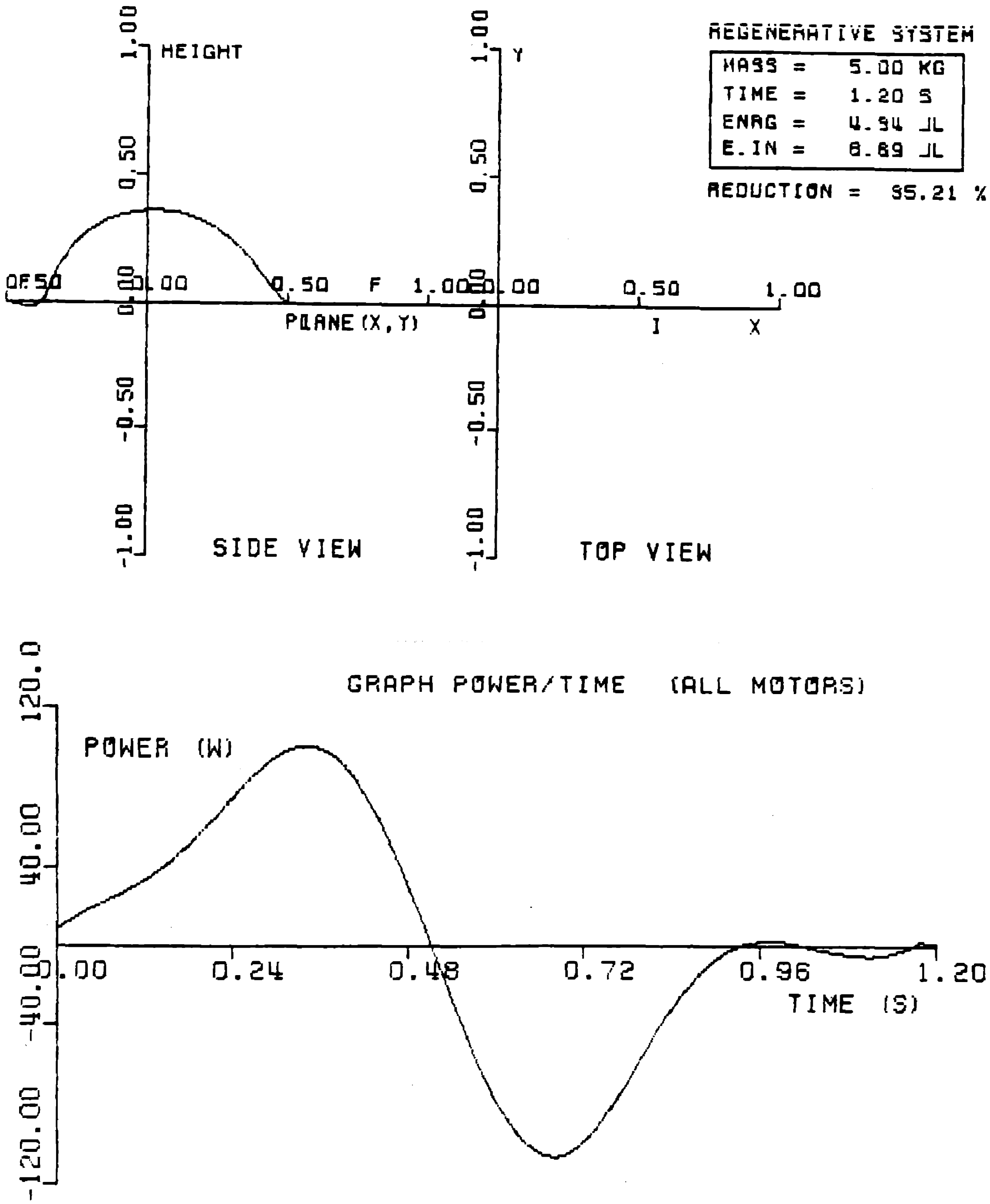


Figure 8 : Rotation by the bottom

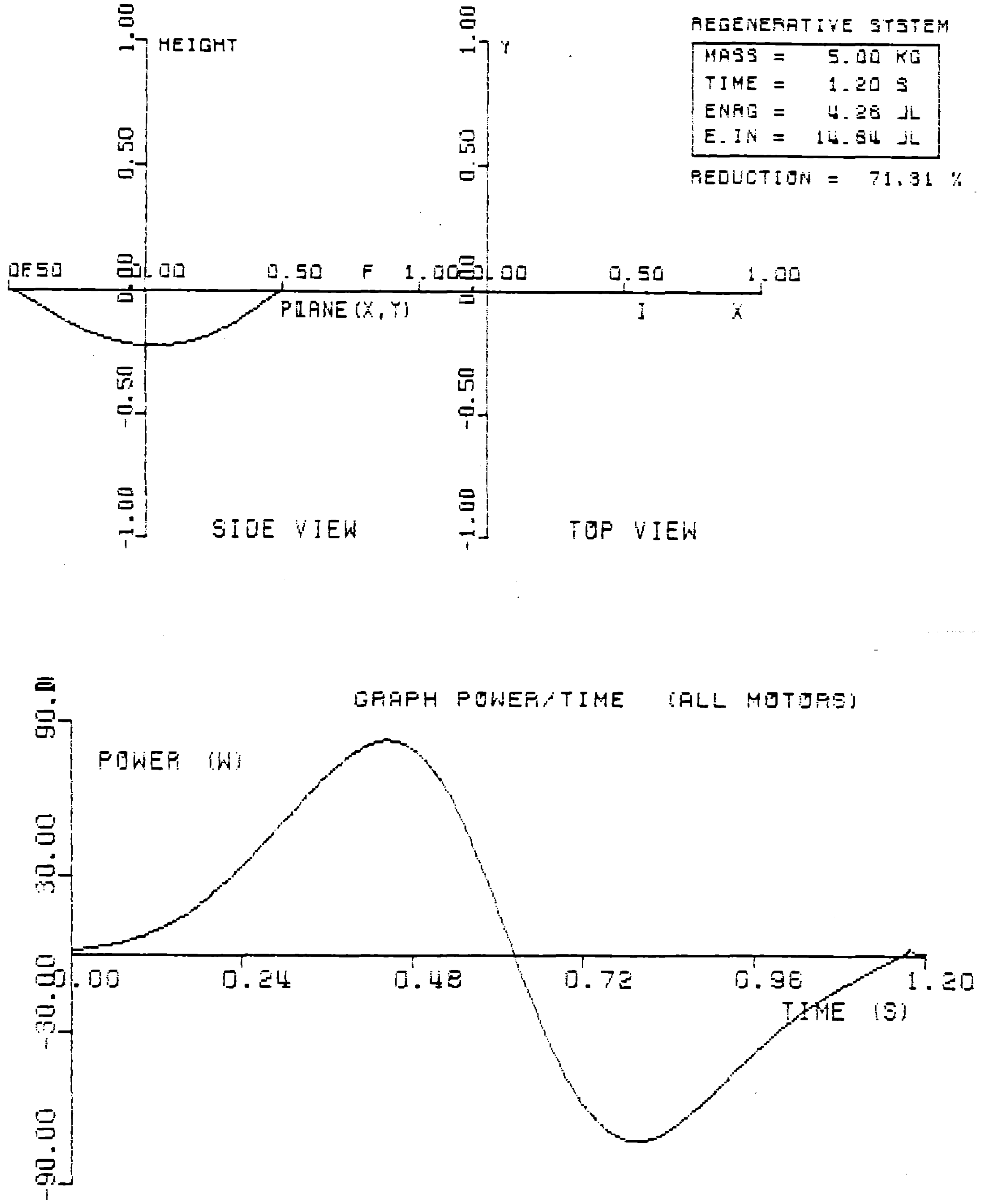
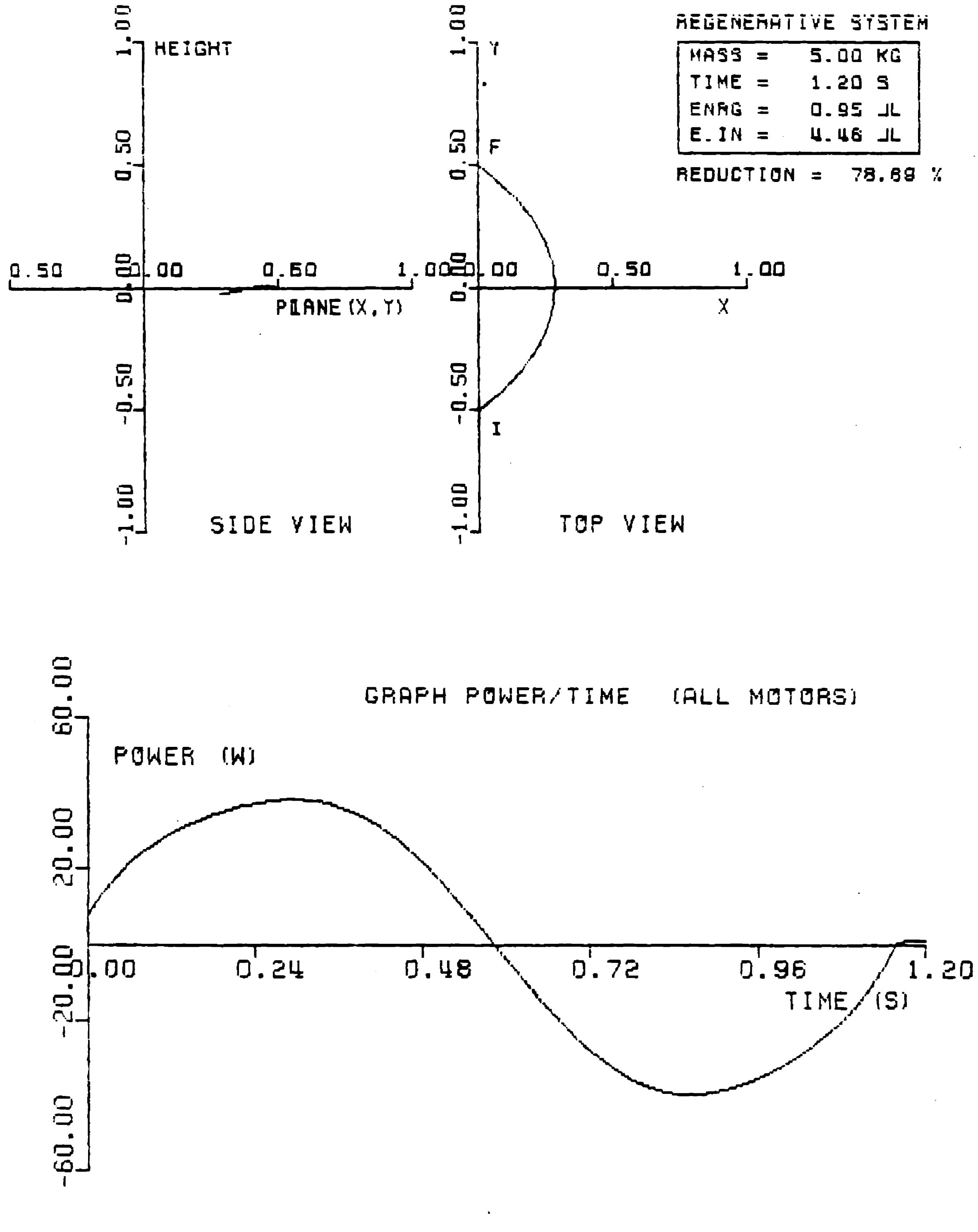


Figure 9 : Rotation by the side



c) Transfer of energy

There is an interesting observation if equation [55] is written as follows

$$[56] \quad Et = \int_{0}^{T} [\sum_{p=1}^{P} R_p \, I_p^2] . \, dt$$

With this notation, we can say that the energy is the time integral of the total instantaneous power consumed by the motors. Hence, the total instantaneous power could be reduced by exciting the motors which would need less current to accomplish the task. The current in the motors depends on the dynamics of the system, but in particular on the reduction systems. In other words, this is a transfer of energy from some motors to others. To give an example, suppose that the robot is holding a piece and has to be at the same location a few moments later (see Figure 10). If it stays at rest, it needs constant torque and force. For the minimum energy trajectory, the power needed to apply the torque has been transfered to the translational motor which uses it to pull the arm towards a position of less potential energy. This observation also comes from the example of the extension of the arm in the horizontal plane (X,Y) (Figure 11). In this case, the θ rotation motor is commanded so that the arm can be extended only by its own weight. Then it must bring the robot to its final position.

Figure 10 a

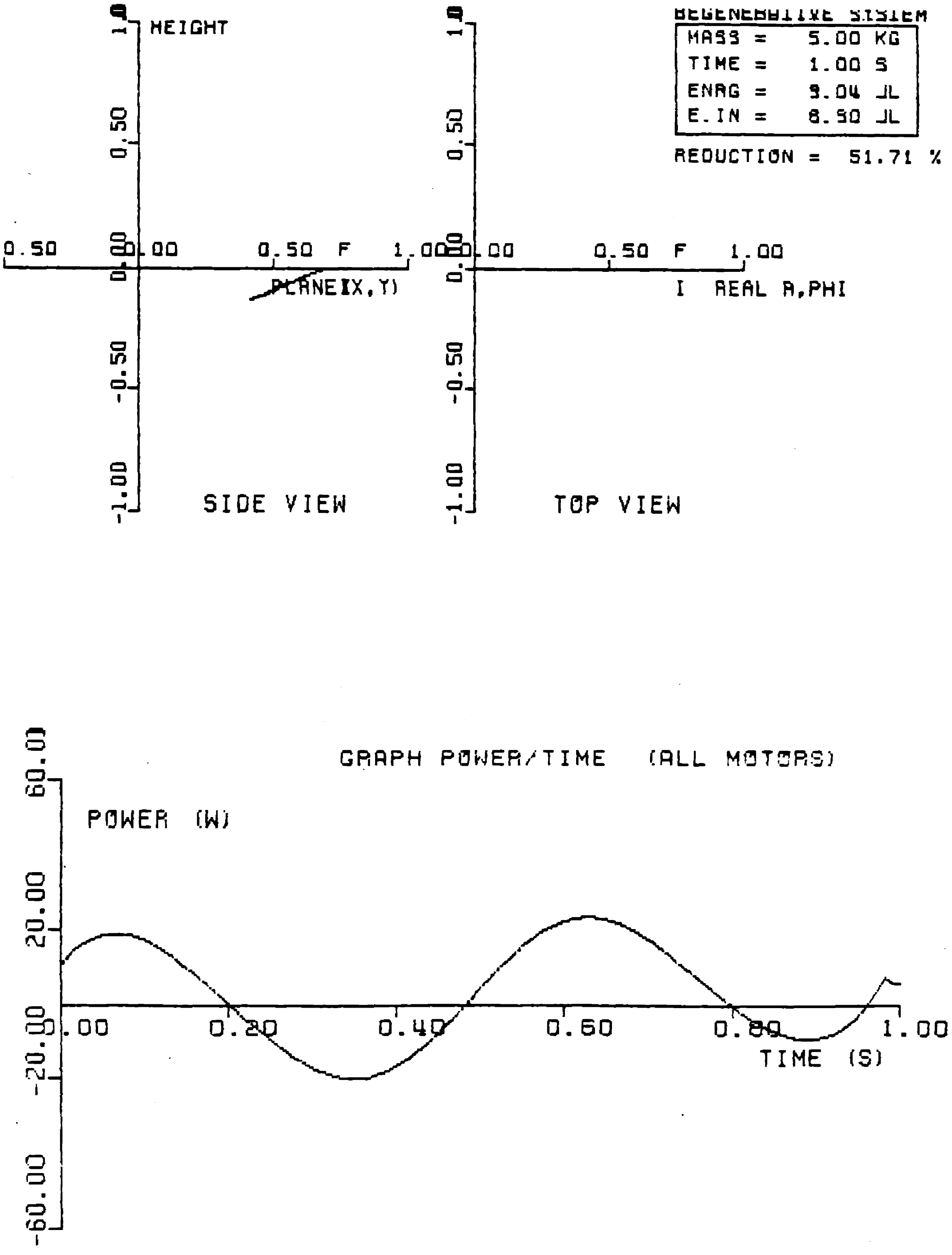
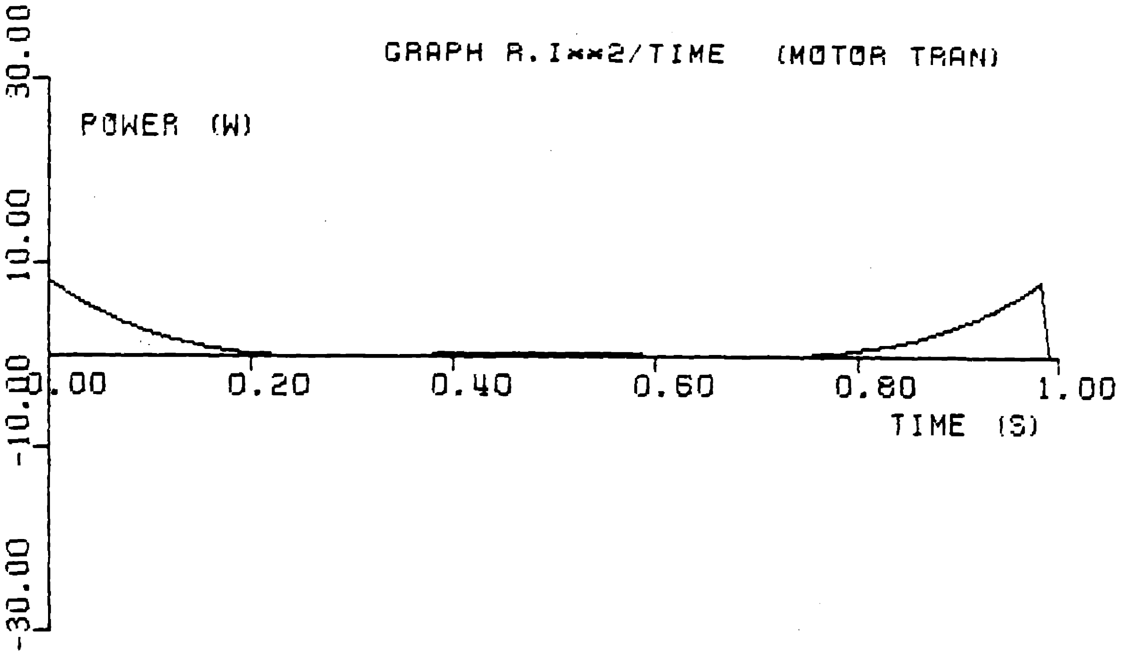

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.00 S
ENRG = 9.04 JL
E.IN = 8.50 JL
REDUCTION = 51.71 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
F
1.00
PLANE(X,Y)
SIDE VIEW
I
REAL R,PHI
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

Figure 10 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

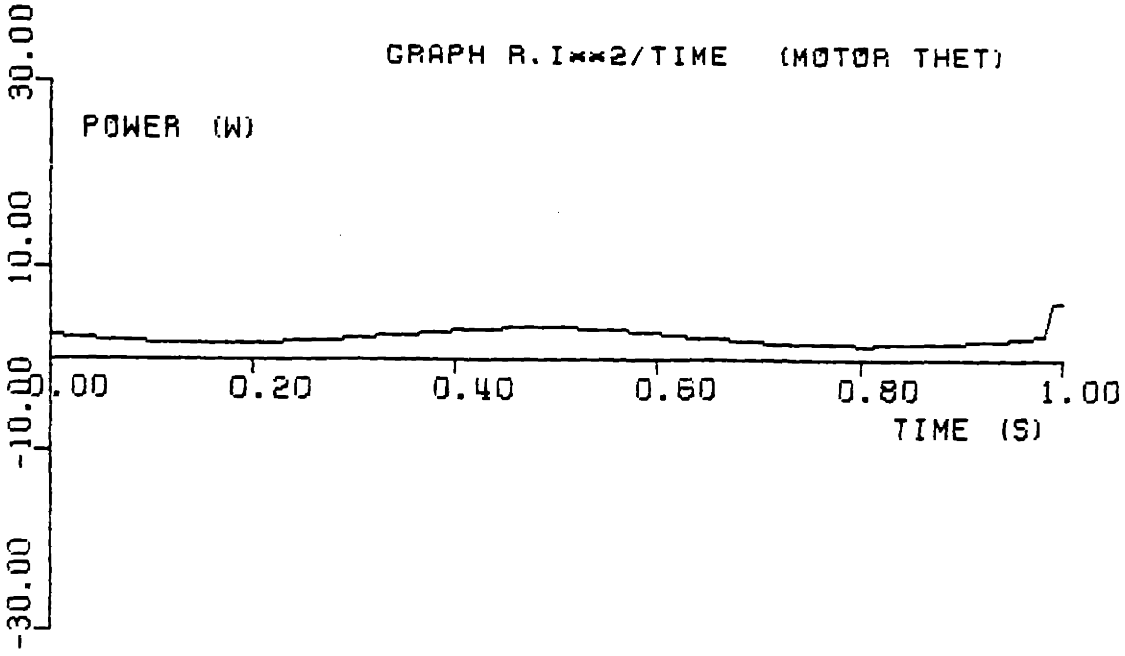

GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

Figure 11 a

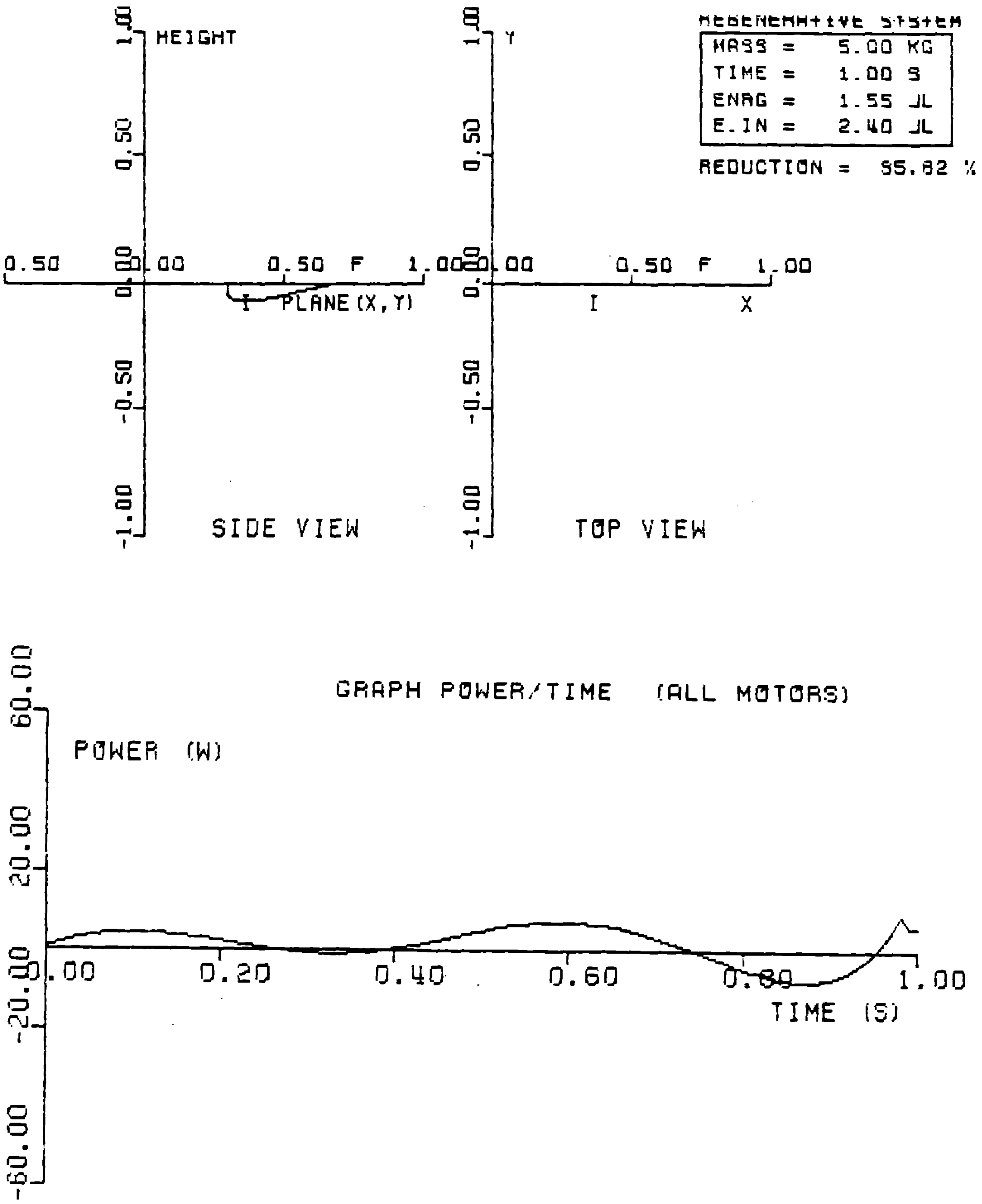
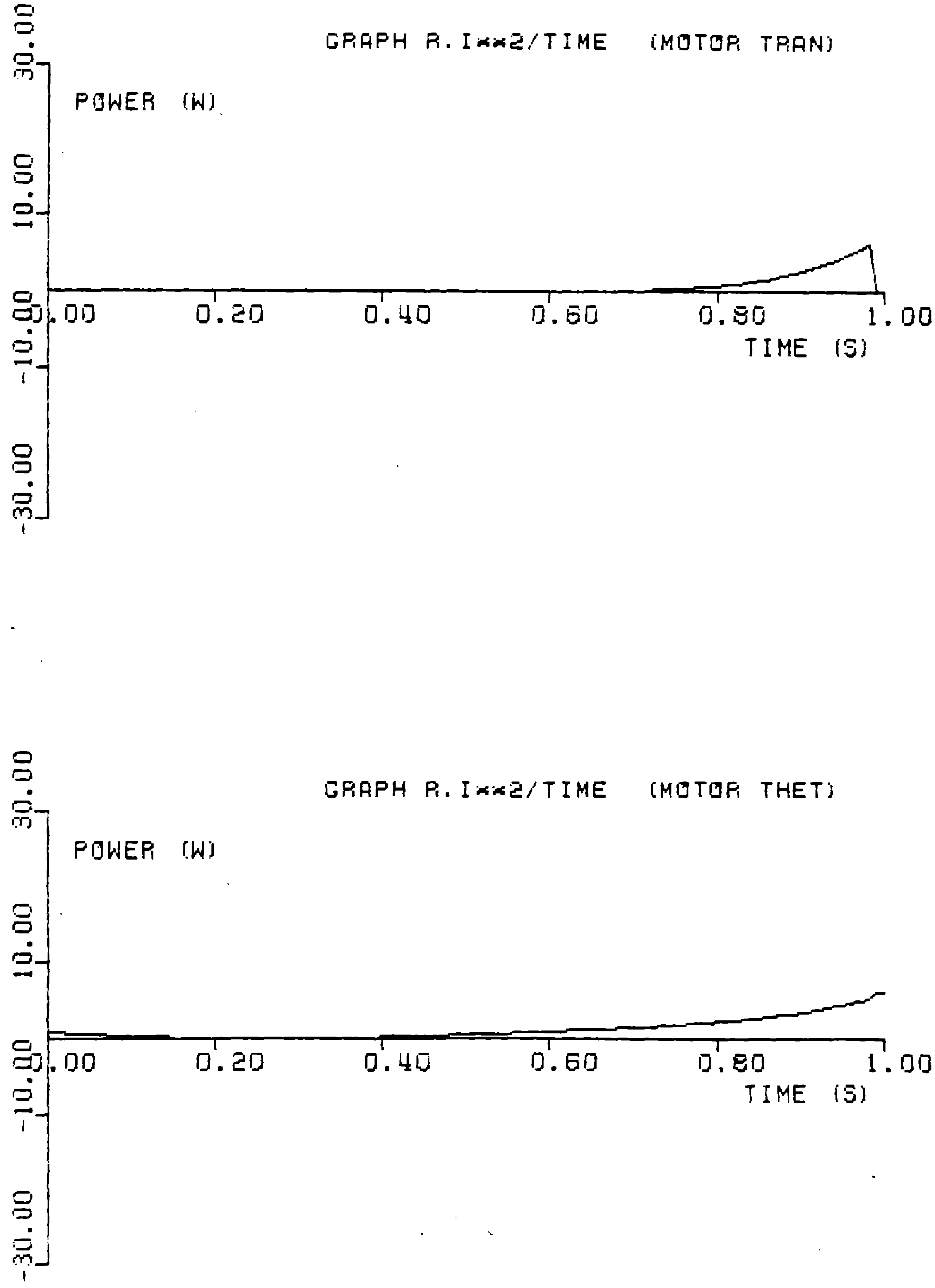

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.00 S
ENRG = 1.55 JL
E.IN = 2.40 JL
REDUCTION = 55.82 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
F
1.00
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
0.00
-20.00
-60.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

Figure 11 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)
GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

d) resonance of the robot

It is important to know how the energy consumption varies as a function of the final time. In Section 1)a) we observed that in a minimum energy trajectory, the torques are applied only at the beginning and at the end, this behavior is even more accentuated when the time of displacement is long. It is a reasonable guess to say that the robot tends to move towards a trajectory where it does not consume energy. If the dimensions of the robots and the inequality constraints permit it, trajectories of zero energy exist, and are defined by the resonances of the system. Since the robot does not consume any energy on these trajectories, the electric components have no effects. The equations of motion at the resonance are found by solving the dynamic equation [4] when the input forces are equal to zero, that is

$$[52] \quad I(q).\ddot{q} = \dot{q}^T.Q(q).\dot{q} + D(q).\dot{q} + G(q)$$

Note that the equilibrium is a solution of this equation and is defined by q_e such that

$G(q_e) = 0$ (at the equilibrium $\dot{q}_e = 0$ and $\ddot{q}_e = 0$).

The equilibrium position is achieved when no forces are needed to maintain the robot at rest. Necessarily, the arm must be in the horizonal plane, and the radius is determined by the dimensions of the housing, the arm and the workpiece. For a long time of displacement, it could have been possible that the robot moved to this position, rested, and moved finally to the final position. But in all the studied cases, this behavior never happened. It oscillates around its equilibrium position.

In order to visualize this phenomenon, let us take the two previous examples but with a longer time. T = 2 seconds has been chosen for the examples shown in Figures 12 and 13. Note that the energy reduction is 67 % for Figure 12 and 88 % for Figure 13

Figure 12 a

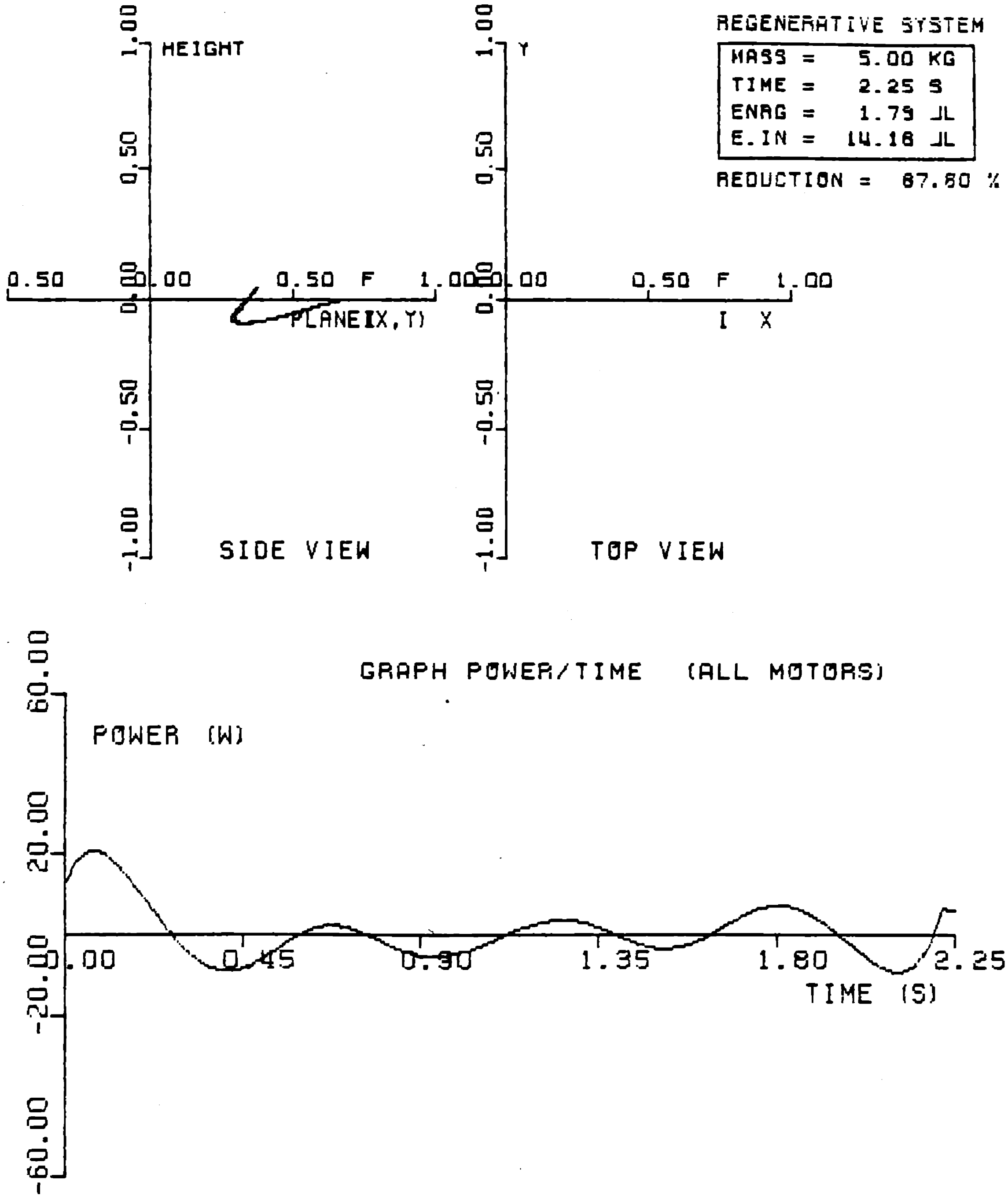
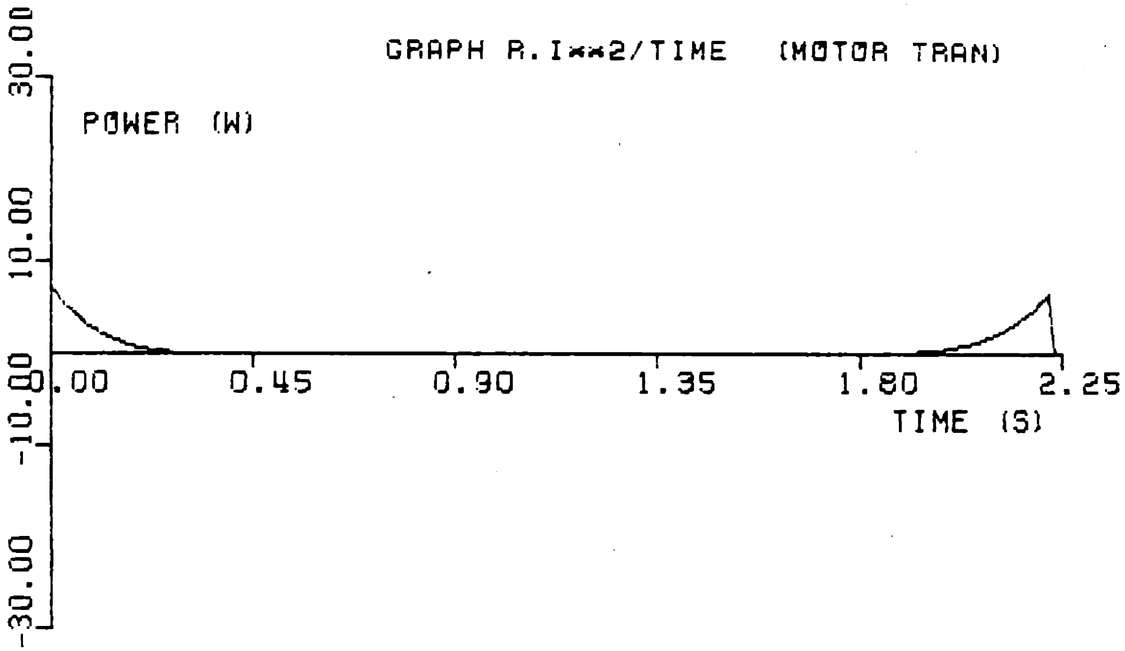

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 2.25 S
ENRG = 1.79 JL
E.IN = 14.18 JL
REDUCTION = 87.80 %
HEIGHT
Y
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
F
1.00
PLANE(X,Y)
I
X
SIDE VIEW
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

Figure 12 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

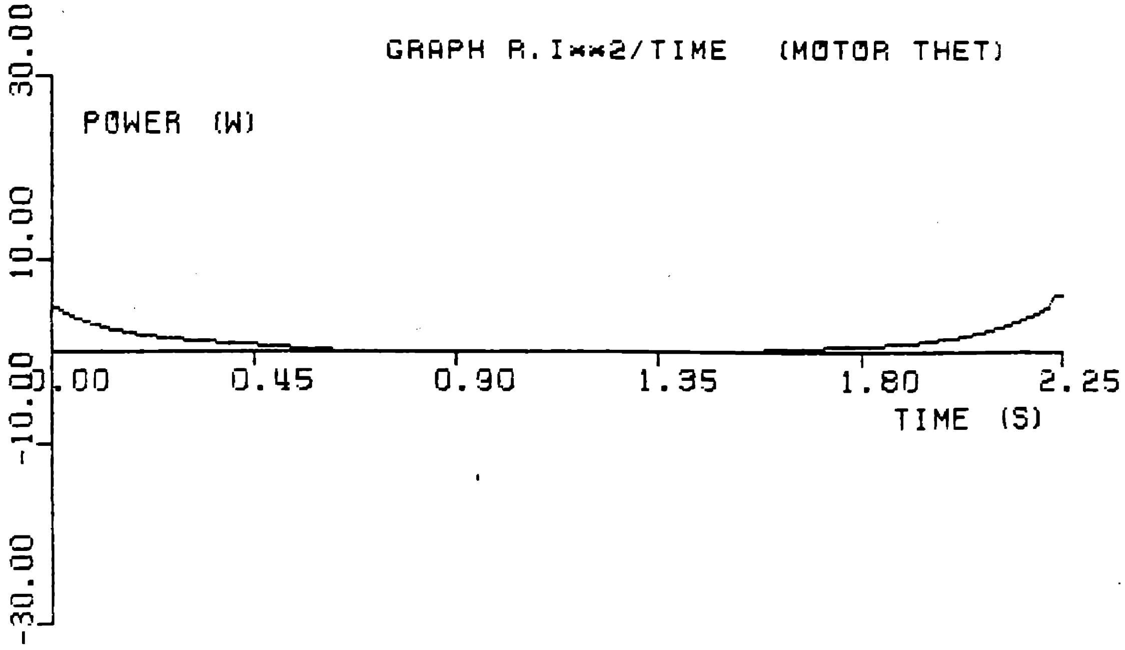

GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

Figure 13 a

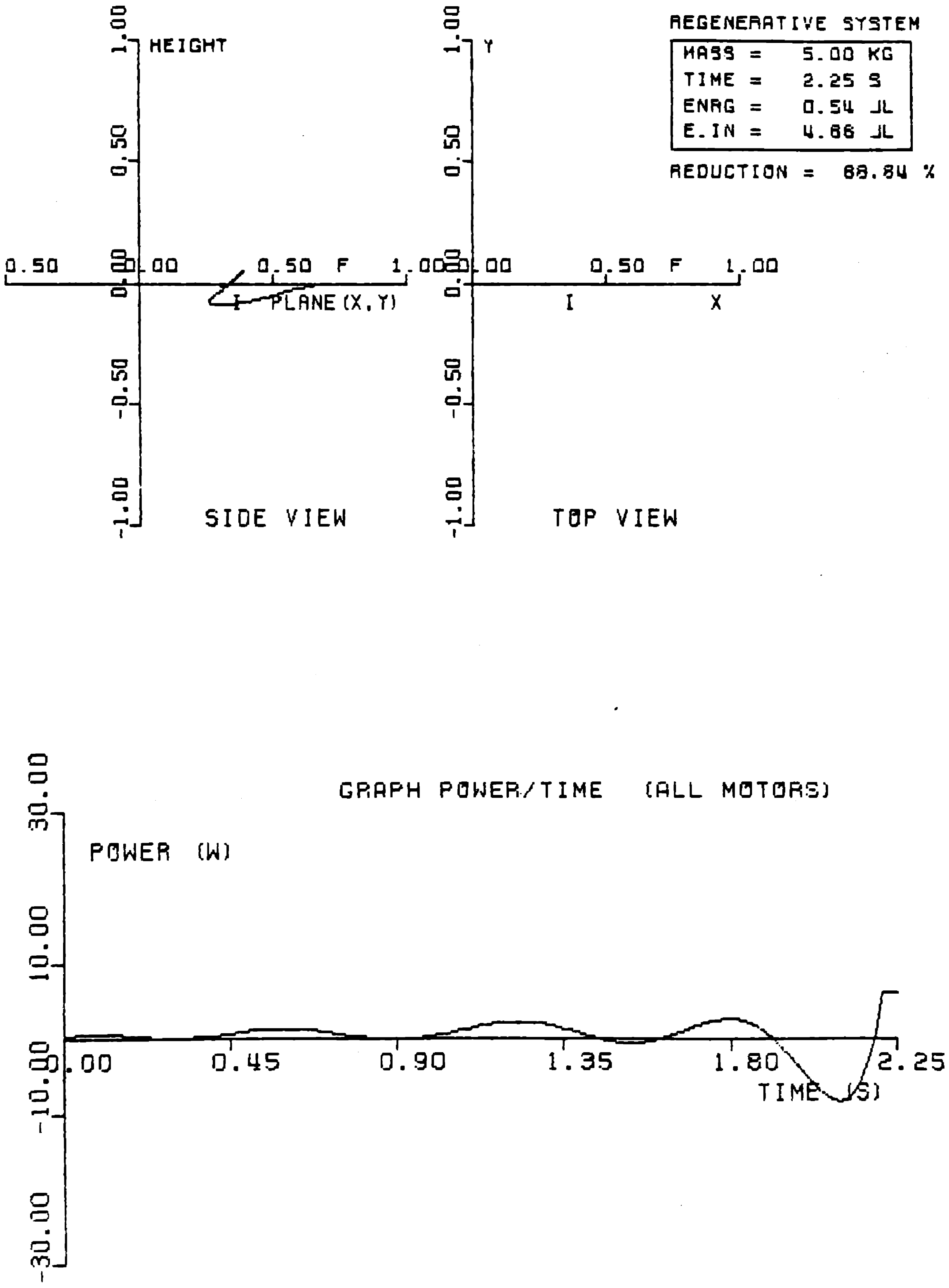
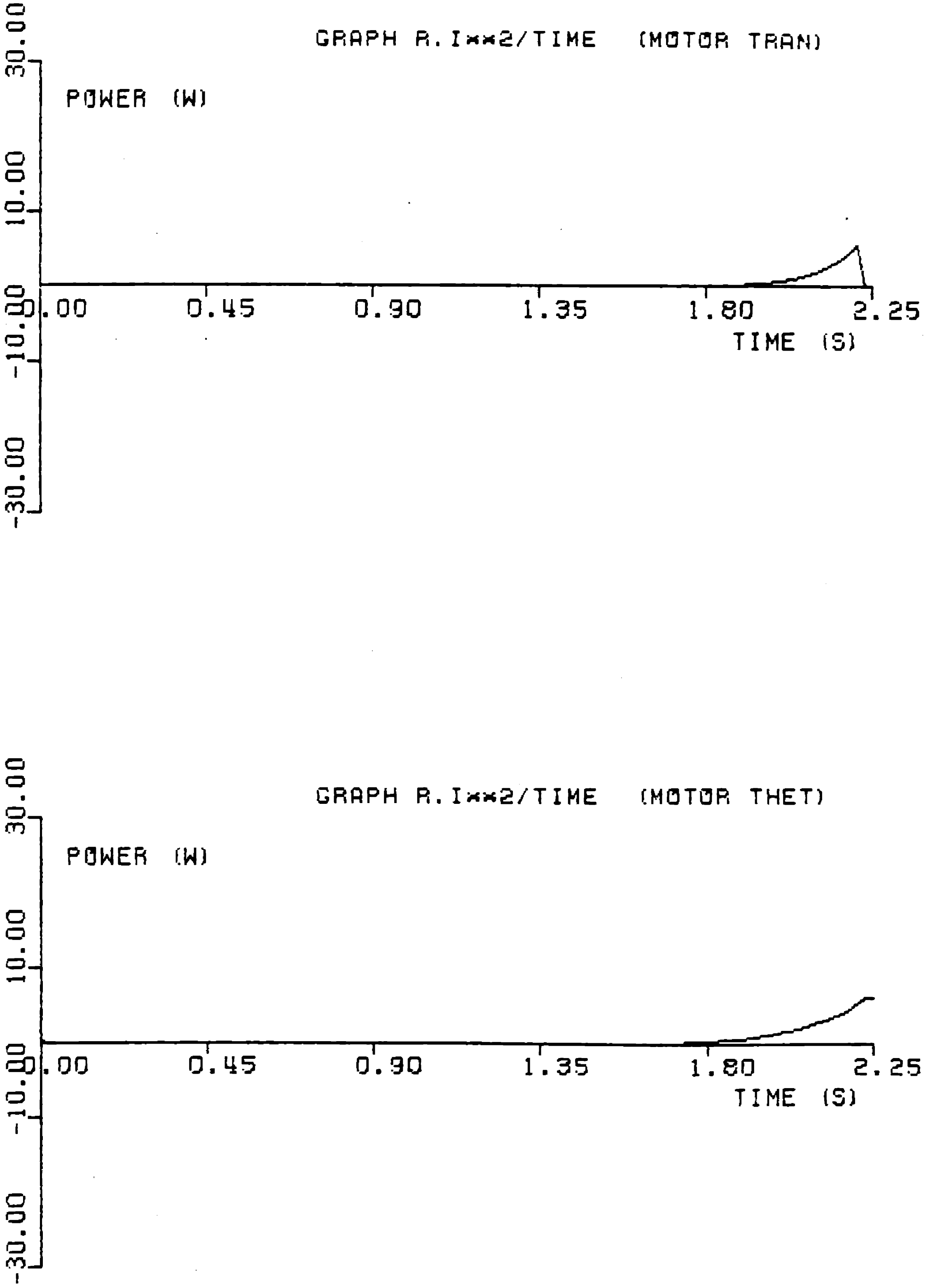

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
F
1.00
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 2.25 S
ENRG = 0.54 JL
E.IN = 4.86 JL
REDUCTION = 88.84 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

Figure 13 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)
GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
30.00
10.00
-10.00
-30.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

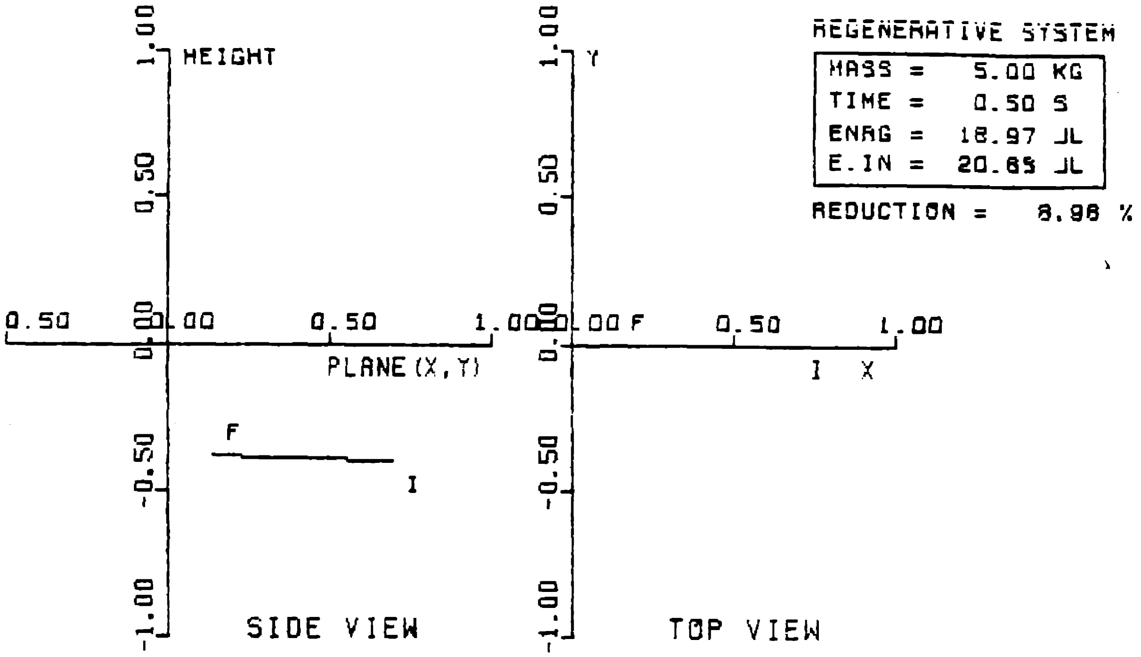
2) energy consumption vs time of displacement

In this section, we show the effect of the total time of displacement on the energy consumption. Two experiments are studied and illustrated by Figures 14 and 15.

Those figures show very interesting results for the choice of the final time. For each task, there are some local minimum energy trajectories. One appears when the robot has not enough time to oscillate. Another one appears at the time where one oscillation at the resonance is possible. There should be another minimum, at the time when the robot can complete two oscillations, but the present algorithm has not been able to converge for such long time. Between these times, the robot has no time to complete the oscillation, and it has to maintain the piece. Hence it consumes more energy.

The results given by the graphs ENERGY vs FINAL TIME (Figure 16 and Figure 17) suggest that if the operation time of the robot is longer that the time of the first minimum, it is better to accomplish the task in less time, and wait for the end of the operation by using a brake.

Figure 14 A


REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.50 S
ENRG = 18.97 JL
E.IN = 20.85 JL
REDUCTION = 8.98 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00 F
0.50
1.00
I
X
TOP VIEW

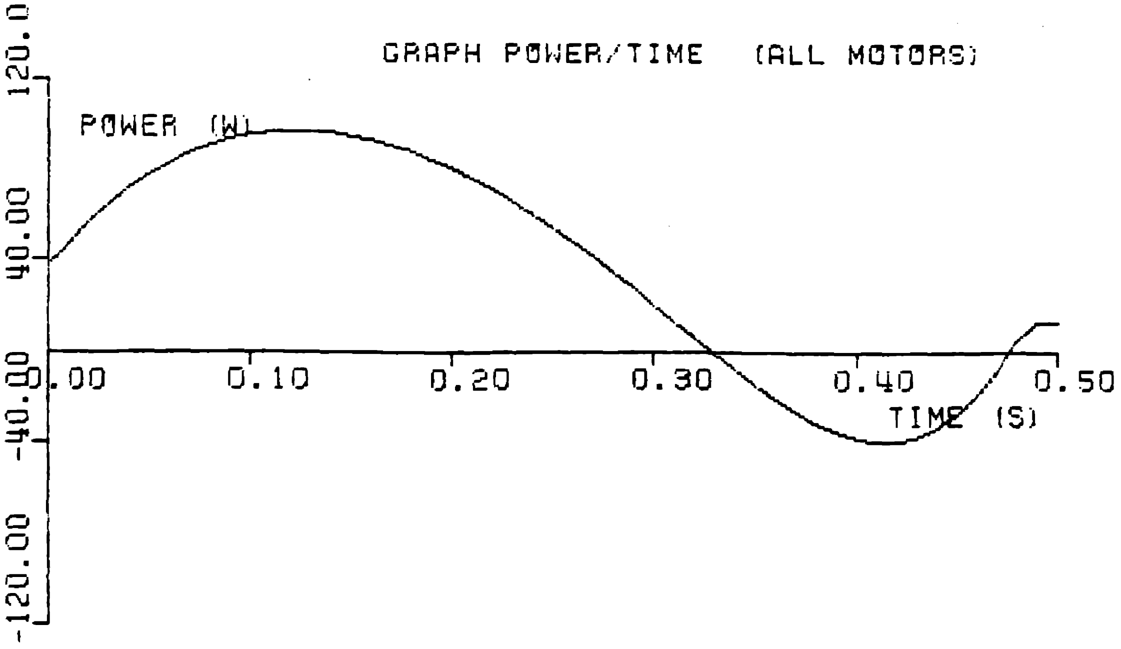
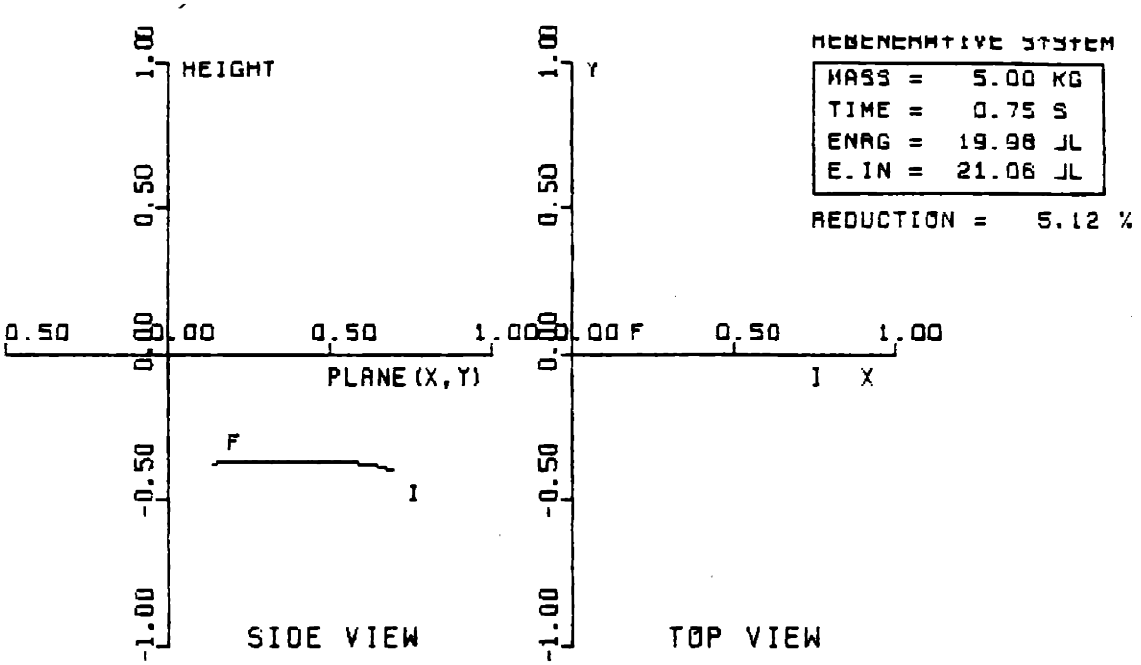

GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
-40.00
-120.00
0.00
0.10
0.20
0.30
0.40
0.50
TIME (S)

Figure 14 B


REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.75 S
ENRG = 19.98 JL
E.IN = 21.06 JL
REDUCTION = 5.12 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE(X,Y)
F
I
SIDE VIEW
Y
0.00 F
0.50
1.00
I X
TOP VIEW

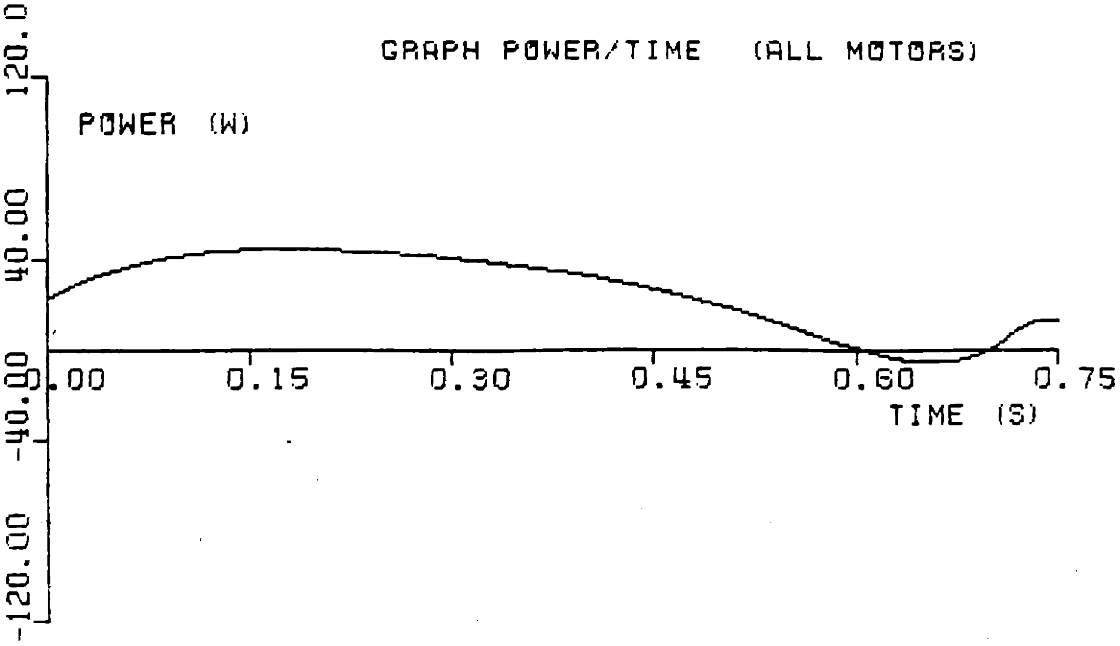

GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
-40.00
-120.00
0.00
0.15
0.30
0.45
0.60
0.75
TIME (S)

Figure 14 C

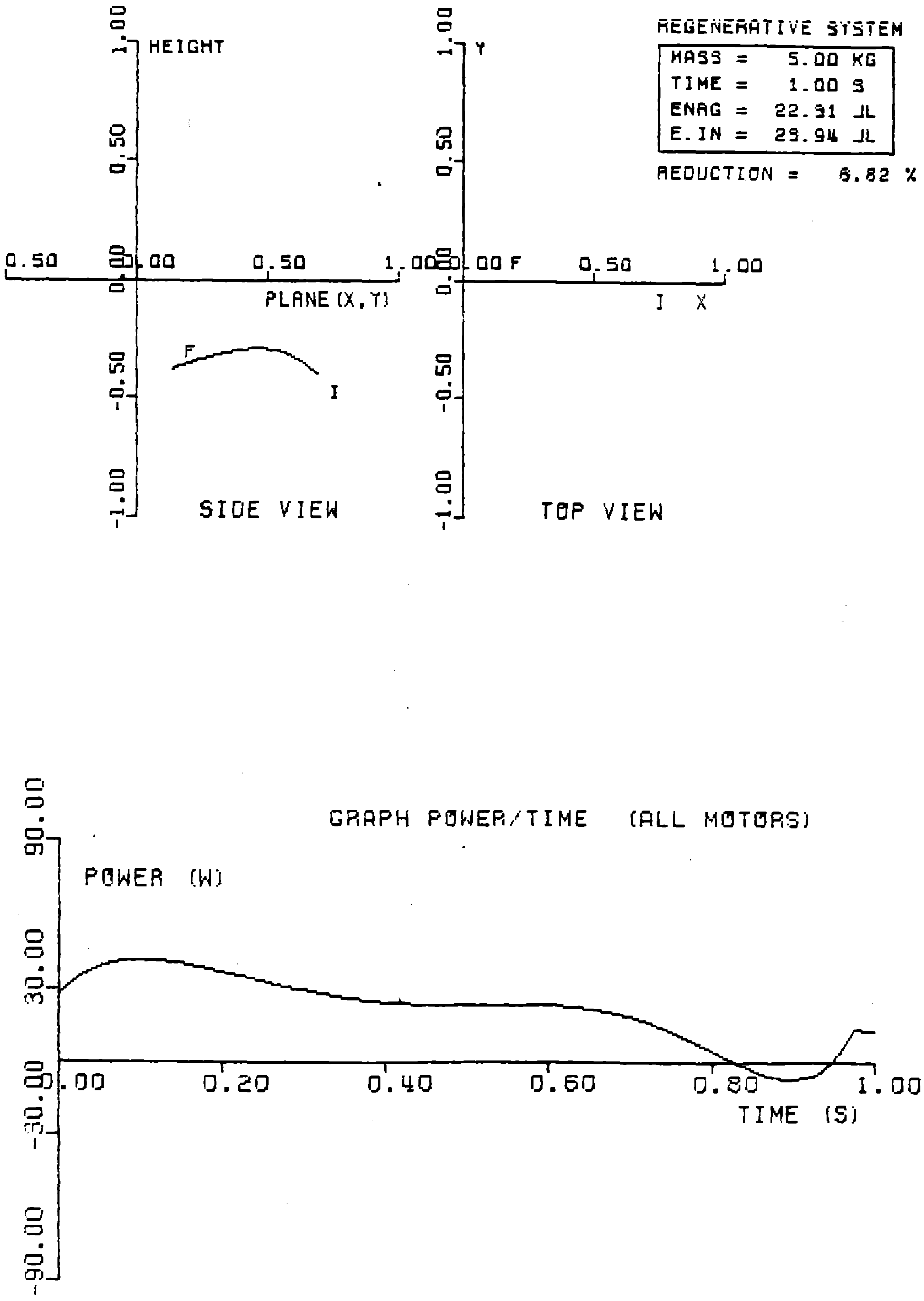

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.00 S
ENRG = 22.31 JL
E.IN = 23.94 JL
REDUCTION = 6.82 %
HEIGHT
PLANE (X,Y)
SIDE VIEW
Y
X
TOP VIEW
F
I
1.00
0.50
0.00
-0.50
-1.00
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
90.00
30.00
0.00
-30.00
-90.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

Figure 14 D

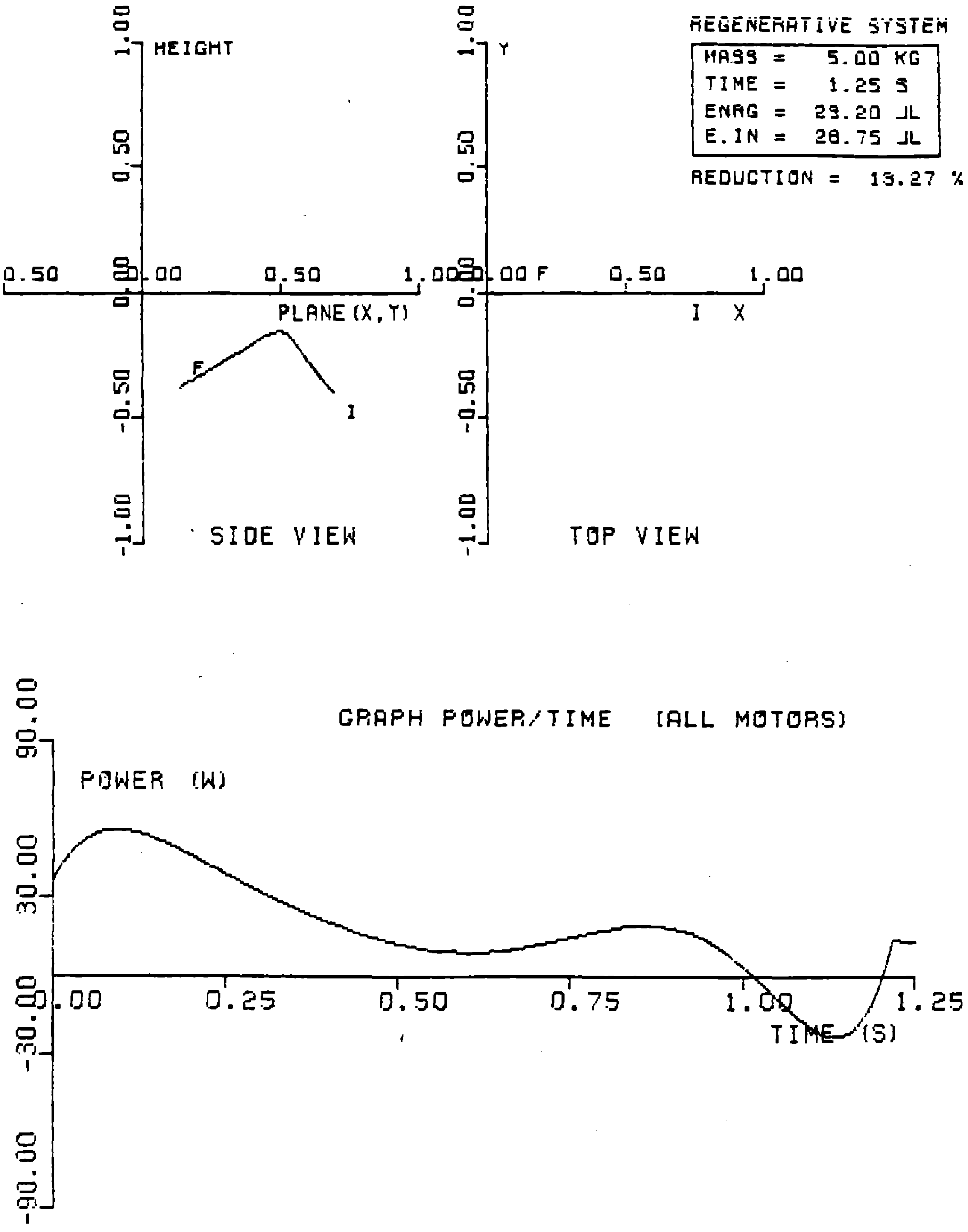
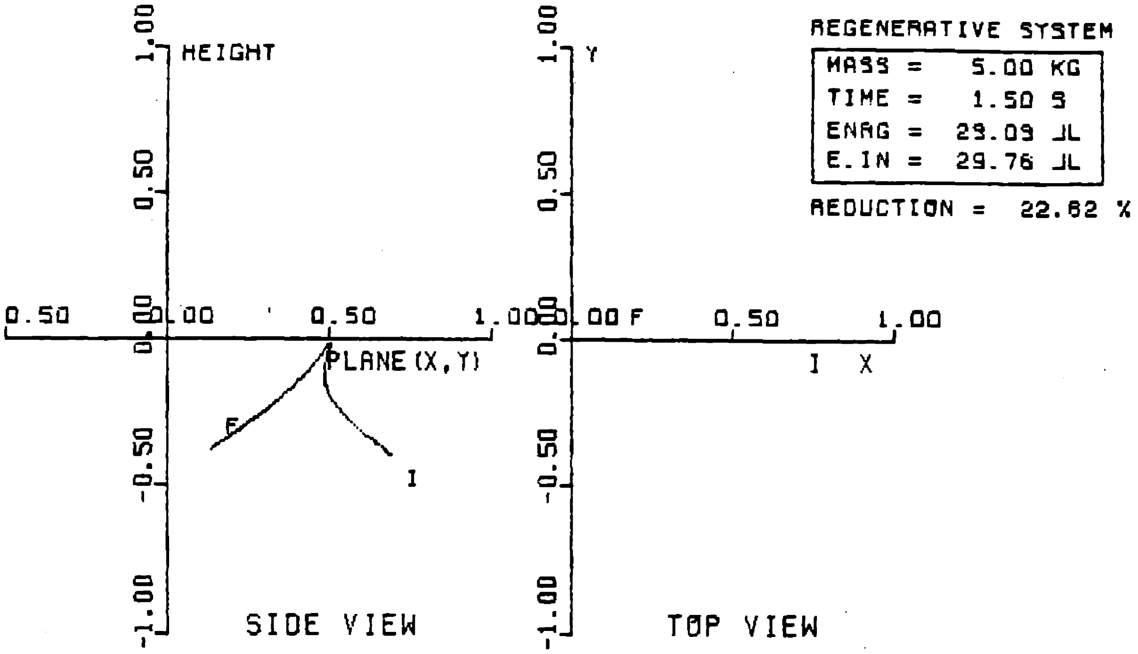

1.00
HEIGHT
0.50
0.50
0.00
0.00
0.50
1.00
PLANE (X,Y)
F
I
-0.50
-1.00
SIDE VIEW
1.00
Y
0.50
0.00
0.00 F
0.50
1.00
I X
-0.50
-1.00
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.25 S
ENRG = 29.20 JL
E.IN = 28.75 JL
REDUCTION = 13.27 %
GRAPH POWER/TIME (ALL MOTORS)
90.00
POWER (W)
30.00
-30.00
-90.00
0.00
0.25
0.50
0.75
1.00
1.25
TIME (S)

Figure 14 E


HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00 F
0.50
1.00
I X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.50 S
ENRG = 23.09 JL
E.IN = 29.76 JL
REDUCTION = 22.62 %

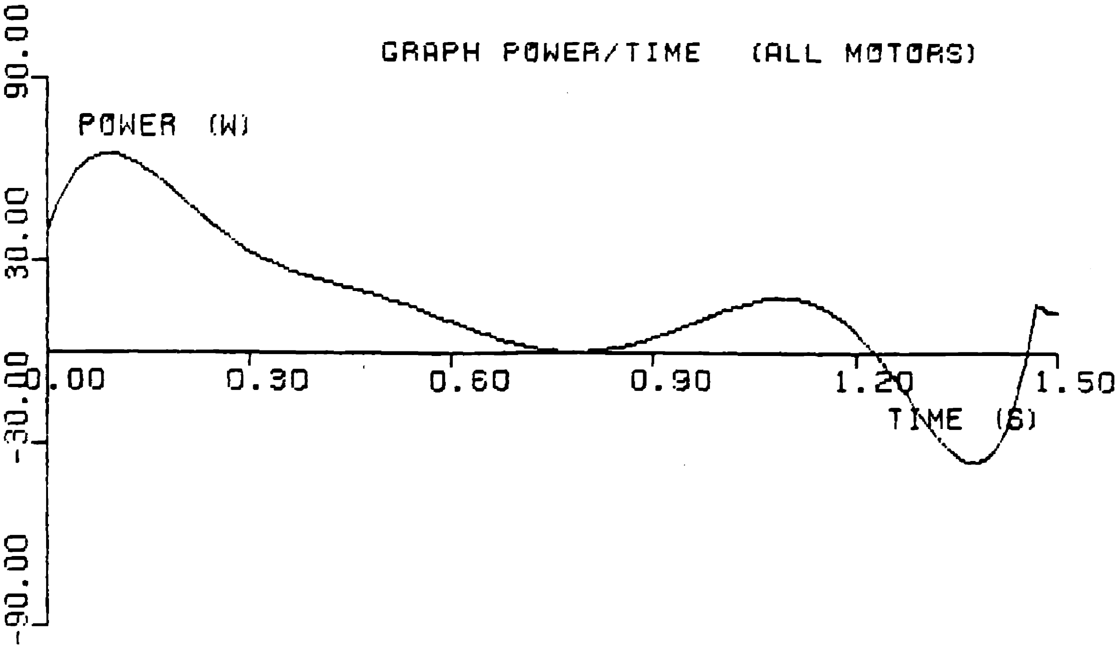

GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
90.00
30.00
-30.00
-90.00
0.00
0.30
0.60
0.90
1.20
1.50
TIME (S)

Figure 14 F

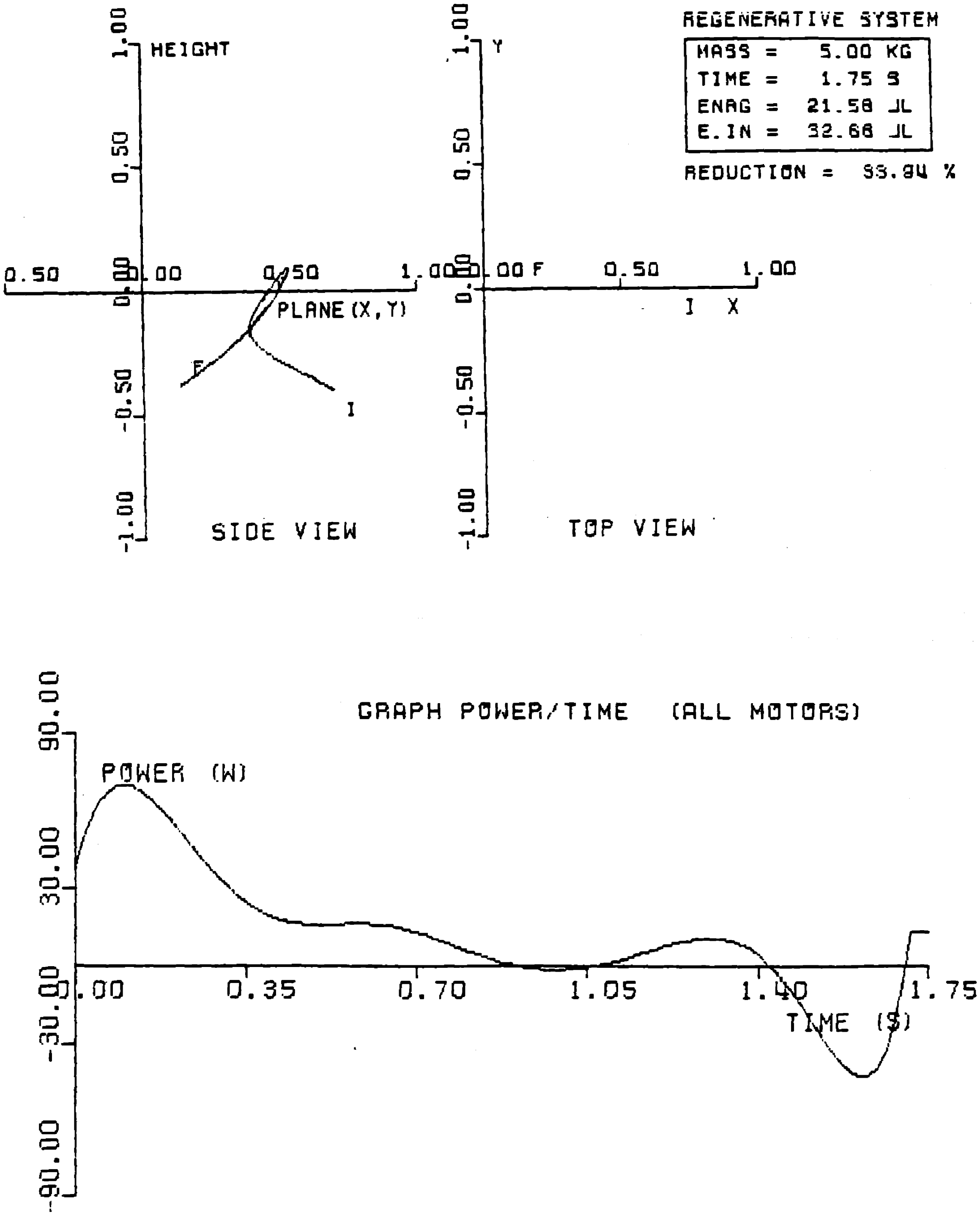

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.75 S
ENRG = 21.58 JL
E.IN = 32.66 JL
REDUCTION = 33.94 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00 F
0.50
1.00
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
90.00
30.00
-30.00
-90.00
0.00
0.35
0.70
1.05
1.40
1.75
TIME (S)

Figure 15 A

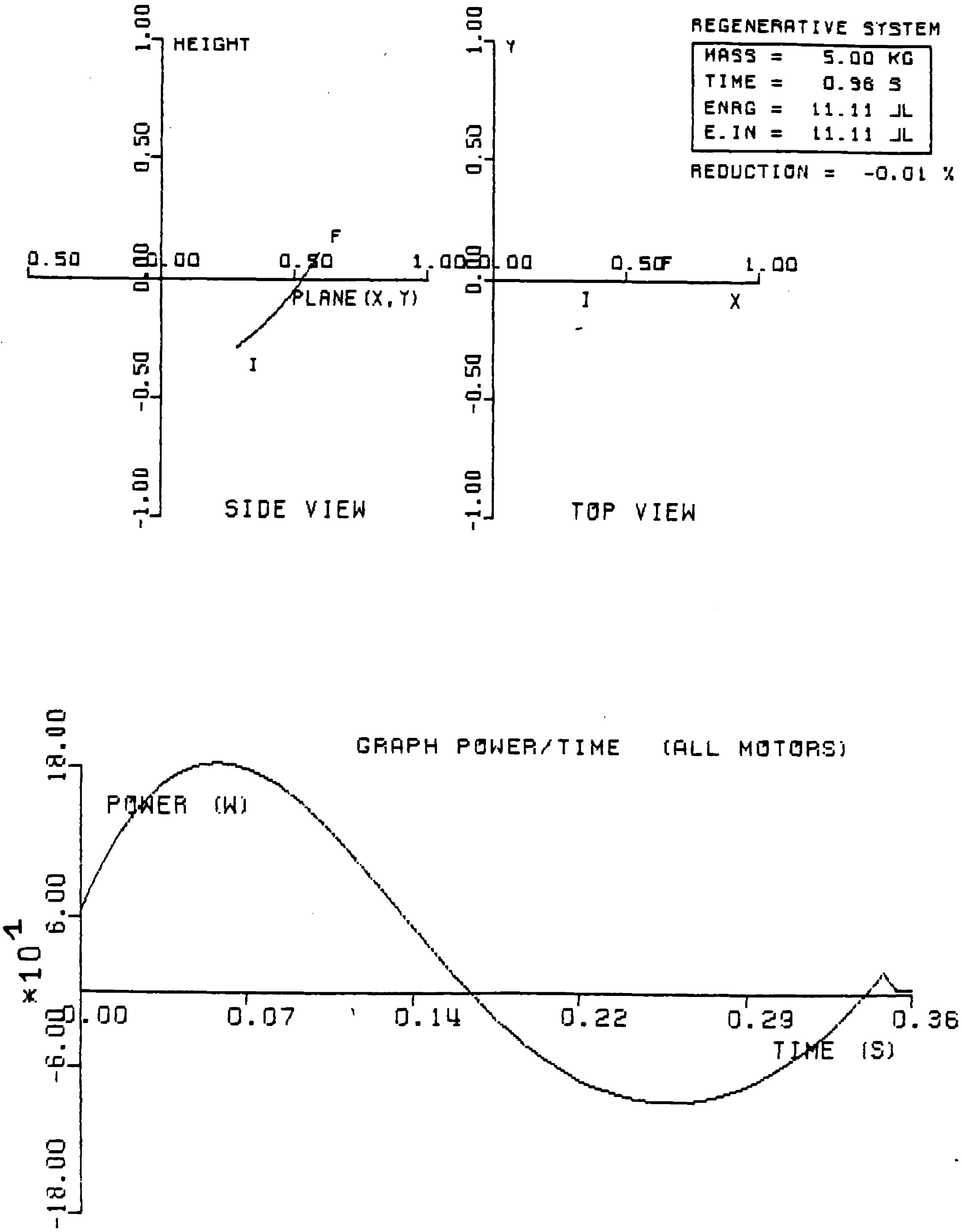

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
PLANE (X, Y)
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.36 S
ENRG = 11.11 JL
E.IN = 11.11 JL
REDUCTION = -0.01 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
*10^1
18.00
6.00
-6.00
-18.00
0.00
0.07
0.14
0.22
0.29
0.36
TIME (S)

Figure 15 B

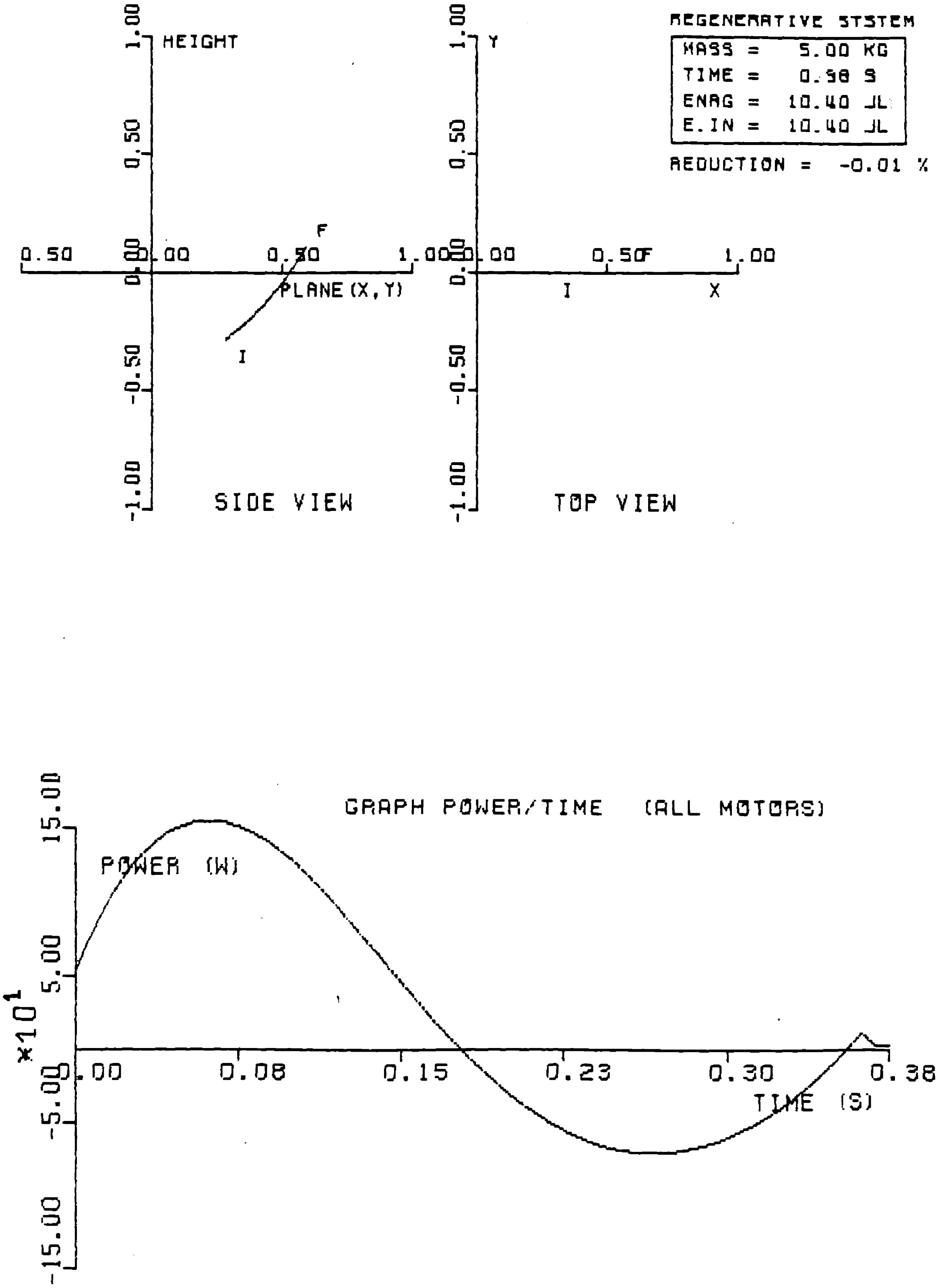
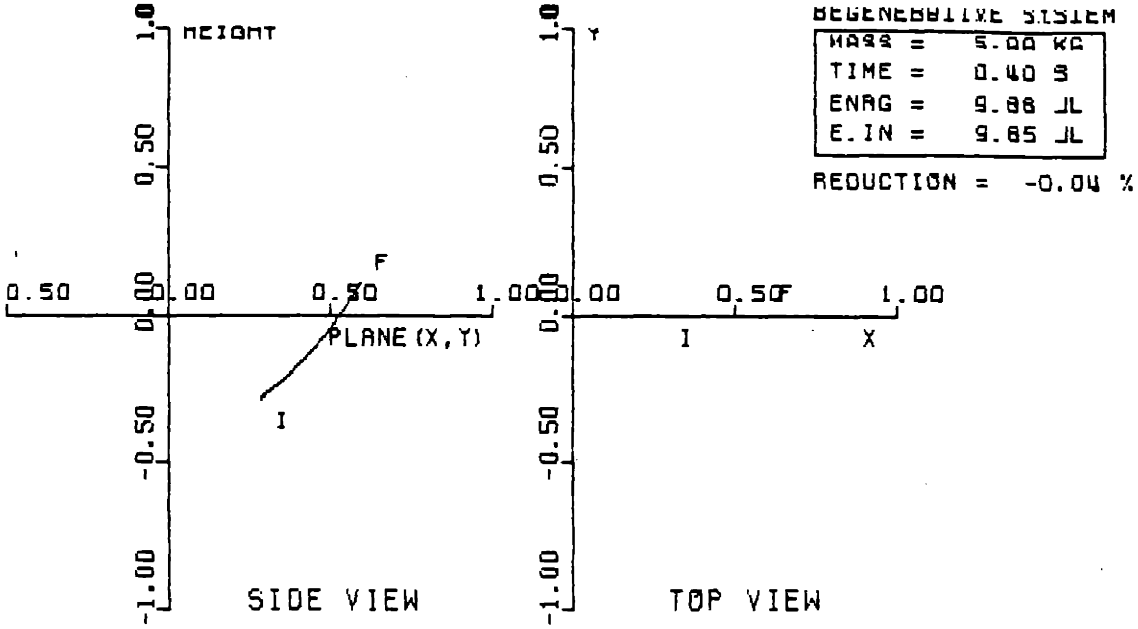

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.38 S
ENRG = 10.40 JL
E.IN = 10.40 JL
REDUCTION = -0.01 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
15.00
5.00
-5.00
-15.00
*10^1
0.00
0.08
0.15
0.23
0.30
0.38
TIME (S)

Figure 15 C


REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.40 S
ENRG = 9.88 JL
E.IN = 9.85 JL
REDUCTION = -0.04 %
HEIGHT
1.0
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
0.00
0.50
F
1.00
I
X
TOP VIEW

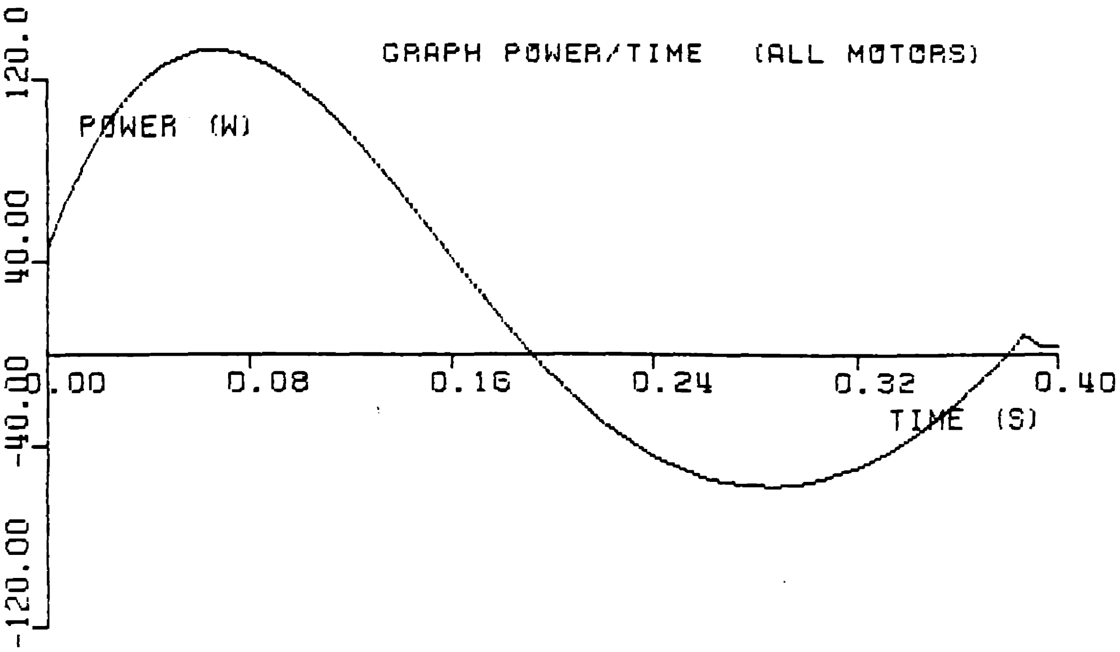

GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
0.00
-40.00
-120.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME (S)

Figure 15 D

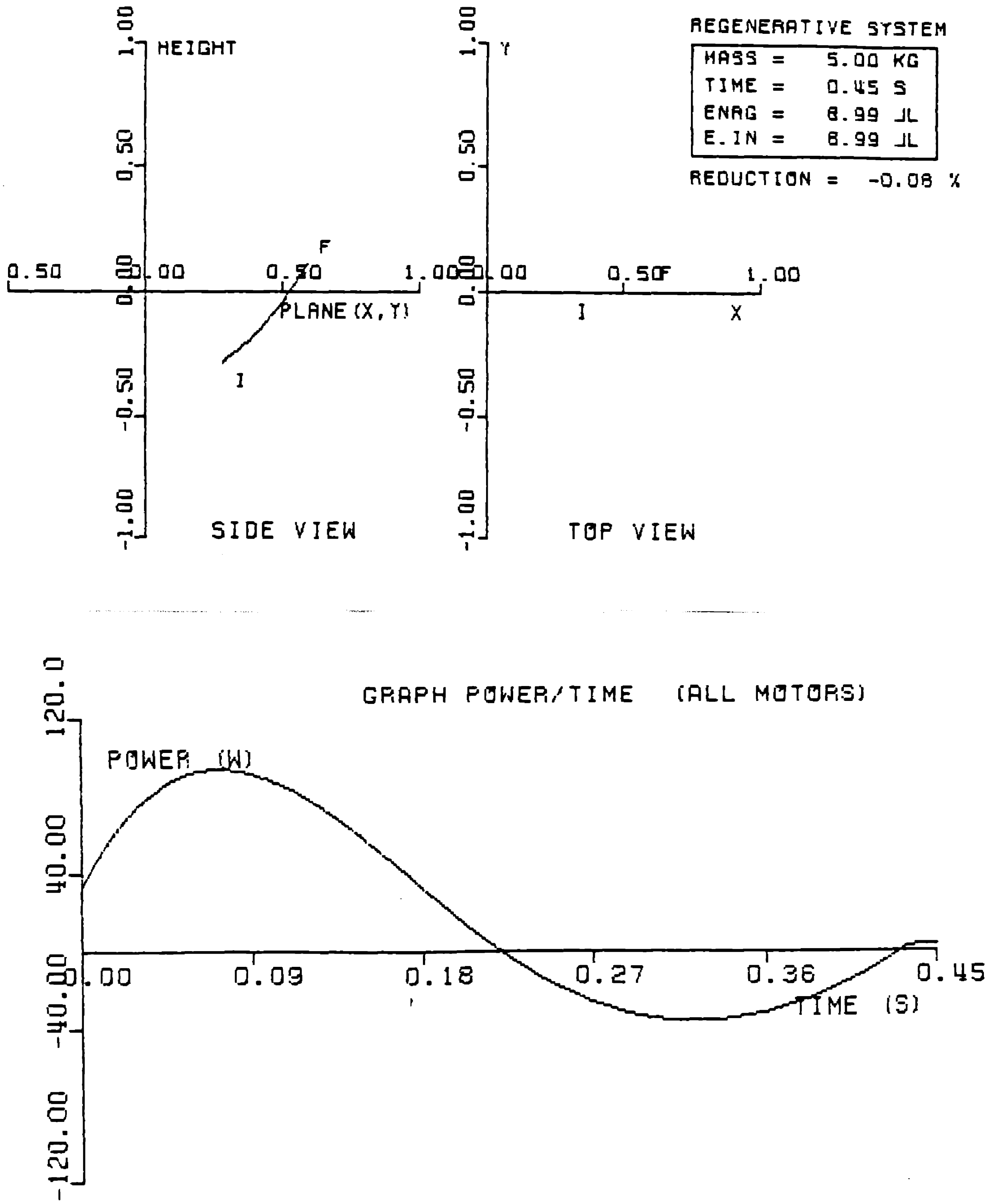

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
X
0.50F
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.45 S
ENRG = 8.99 JL
E.IN = 8.99 JL
REDUCTION = -0.08 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
-40.00
-120.00
0.00
0.09
0.18
0.27
0.36
0.45
TIME (S)

Figure 15 E

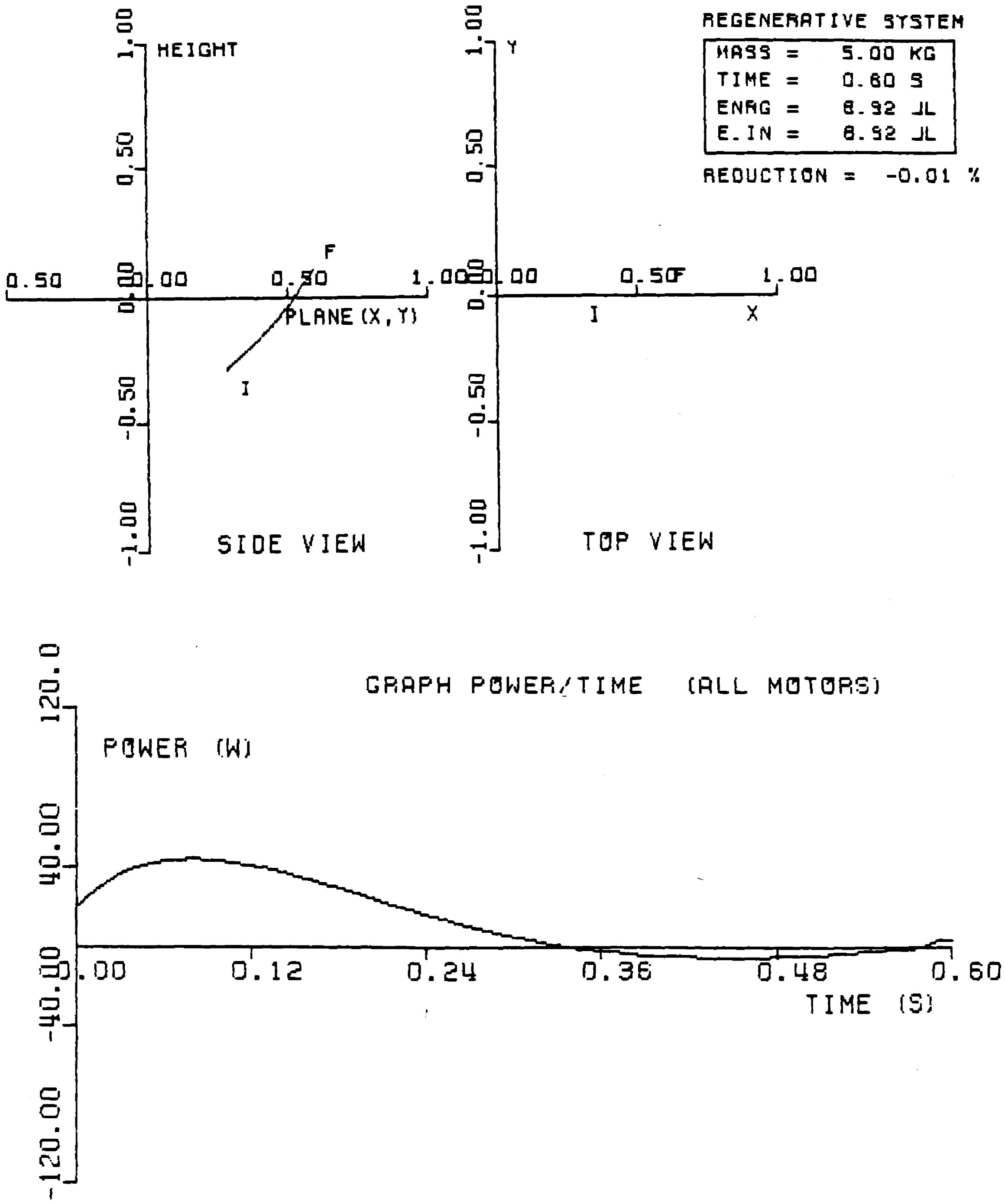

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.60 S
ENRG = 8.92 JL
E.IN = 8.92 JL
REDUCTION = -0.01 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
0.00
-40.00
-120.00
0.00
0.12
0.24
0.36
0.48
0.60
TIME (S)

Figure 15 F

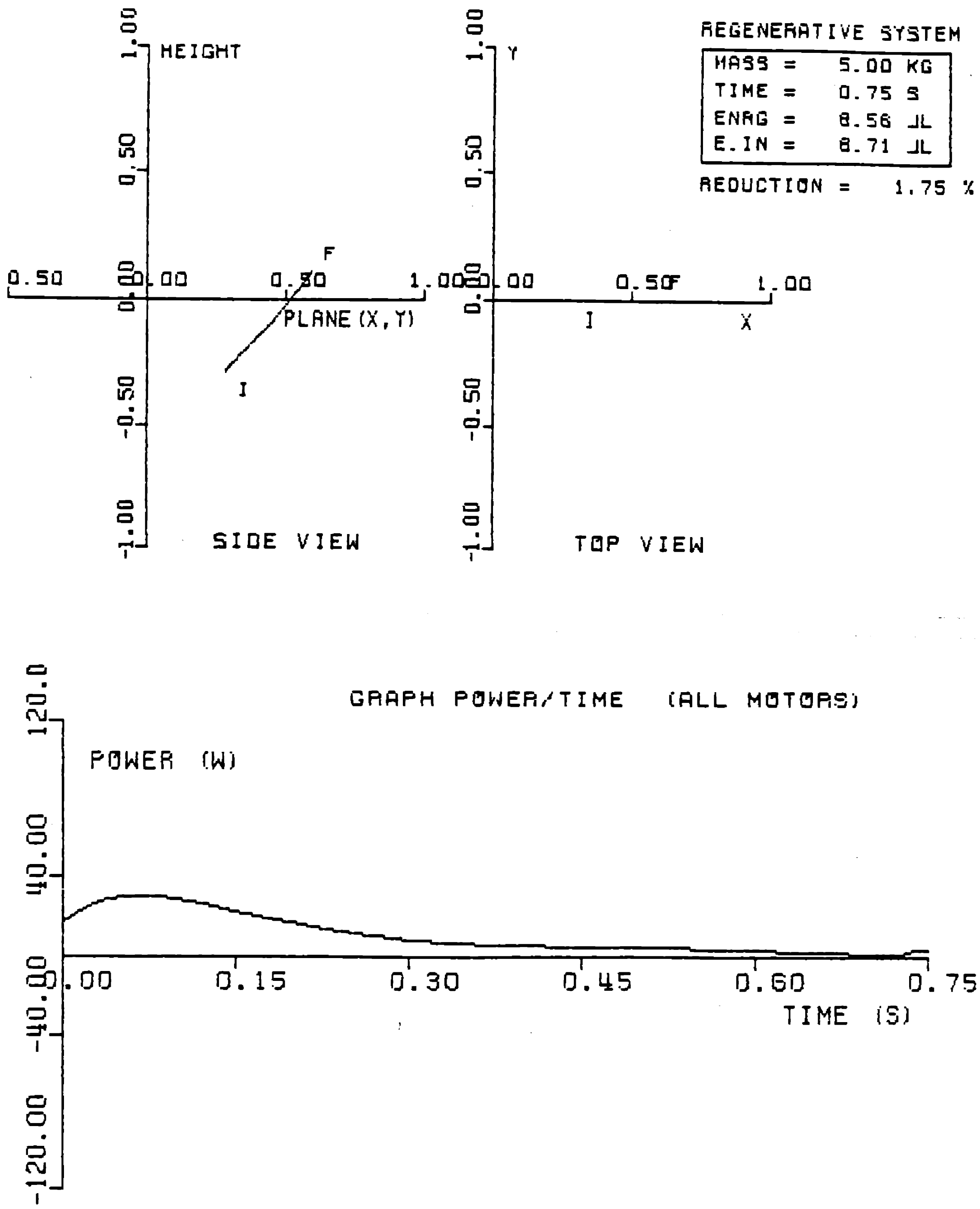

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.75 S
ENRG = 8.56 JL
E.IN = 8.71 JL
REDUCTION = 1.75 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
1.00
F
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
120.0
40.00
-40.00
-120.00
0.00
0.15
0.30
0.45
0.60
0.75
TIME (S)

Figure 15 G

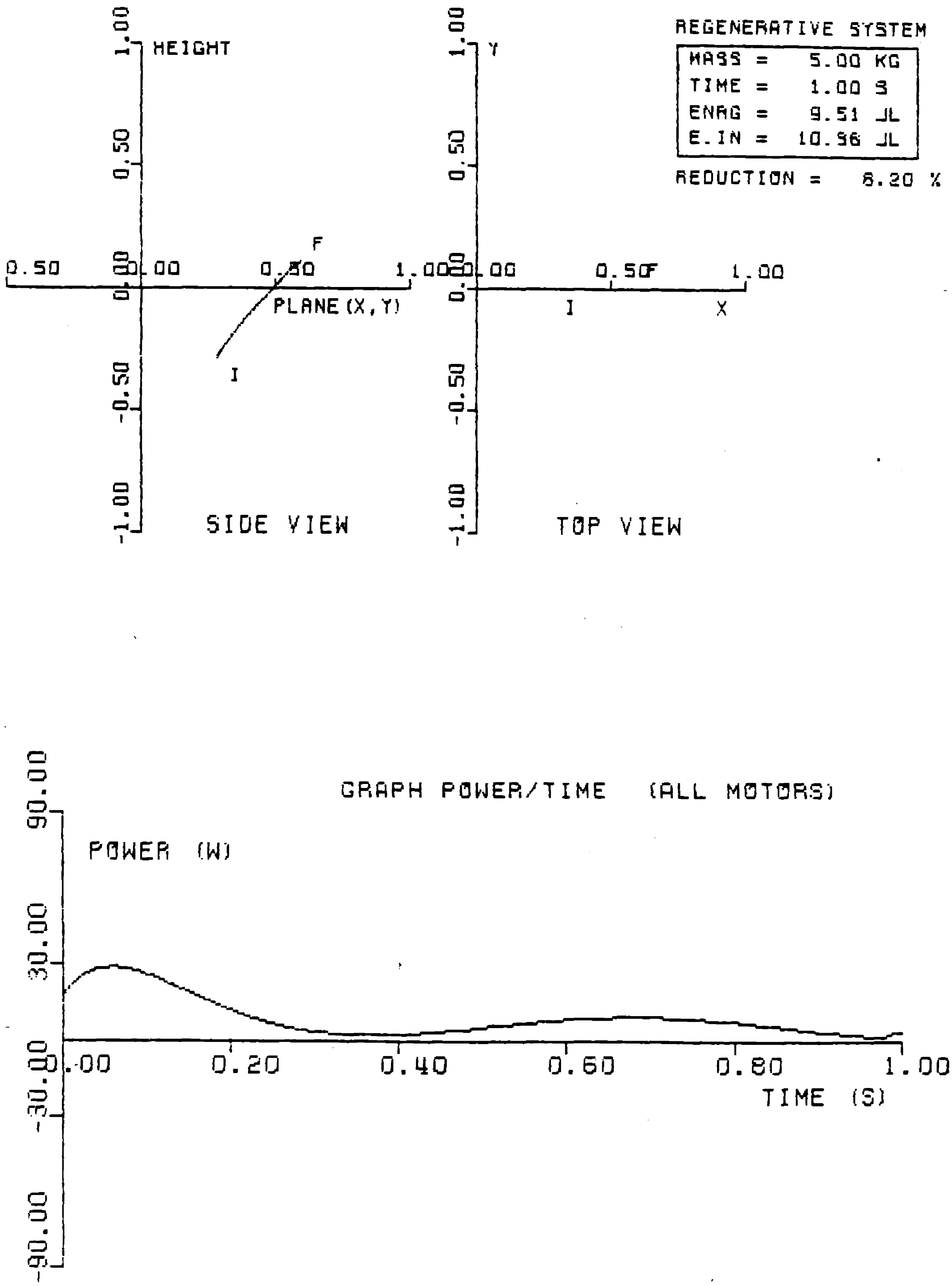

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
0.00
0.50
1.00
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.00 S
ENRG = 9.51 JL
E.IN = 10.36 JL
REDUCTION = 8.20 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
90.00
30.00
-30.00
-90.00
0.00
0.20
0.40
0.60
0.80
1.00
TIME (S)

Figure 15 H

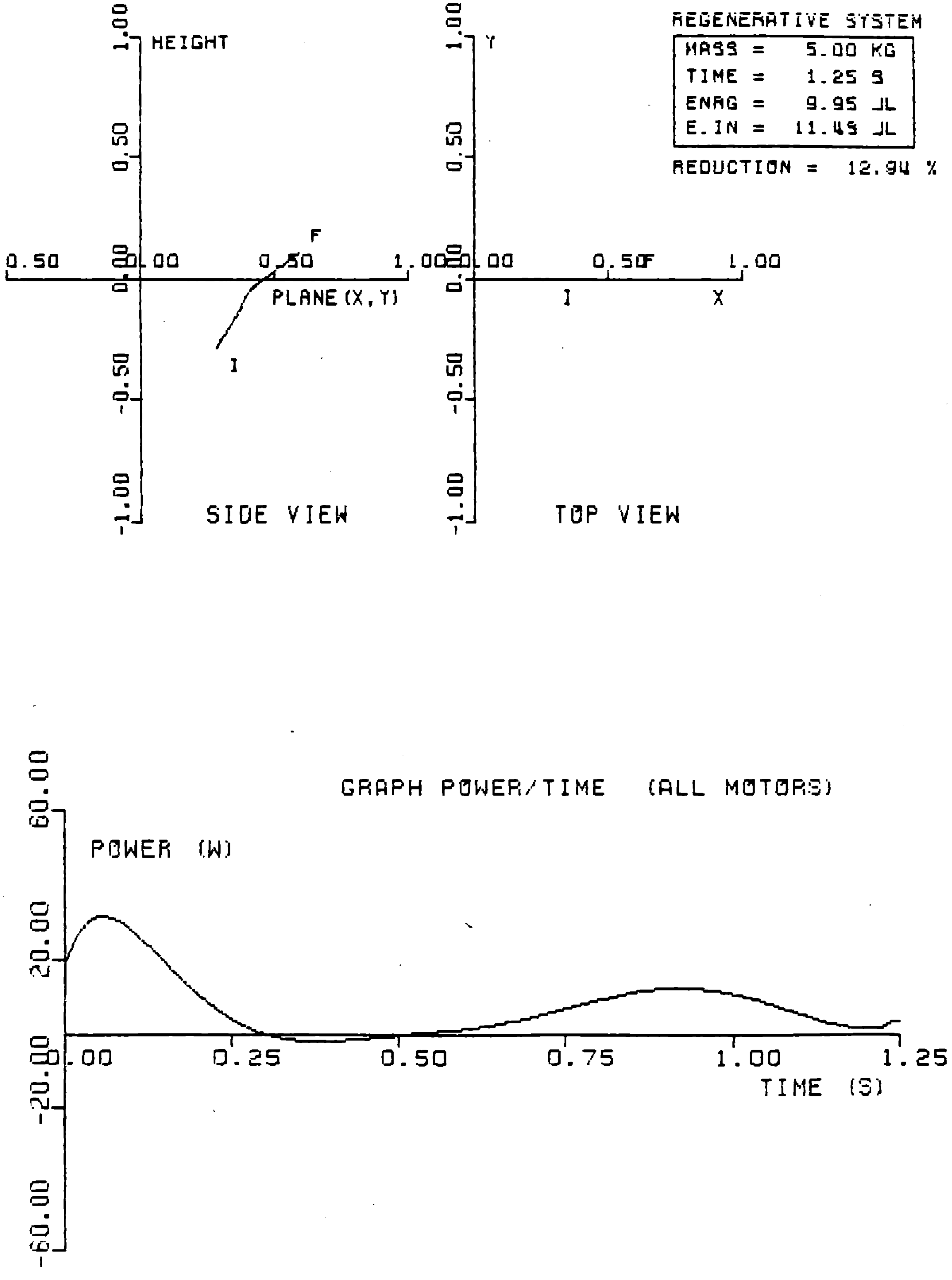

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.25 S
ENRG = 9.95 JL
E.IN = 11.49 JL
REDUCTION = 12.94 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
PLANE (X,Y)
F
I
SIDE VIEW
Y
0.00
0.50
1.00
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.25
0.50
0.75
1.00
1.25
TIME (S)

Figure 15 I

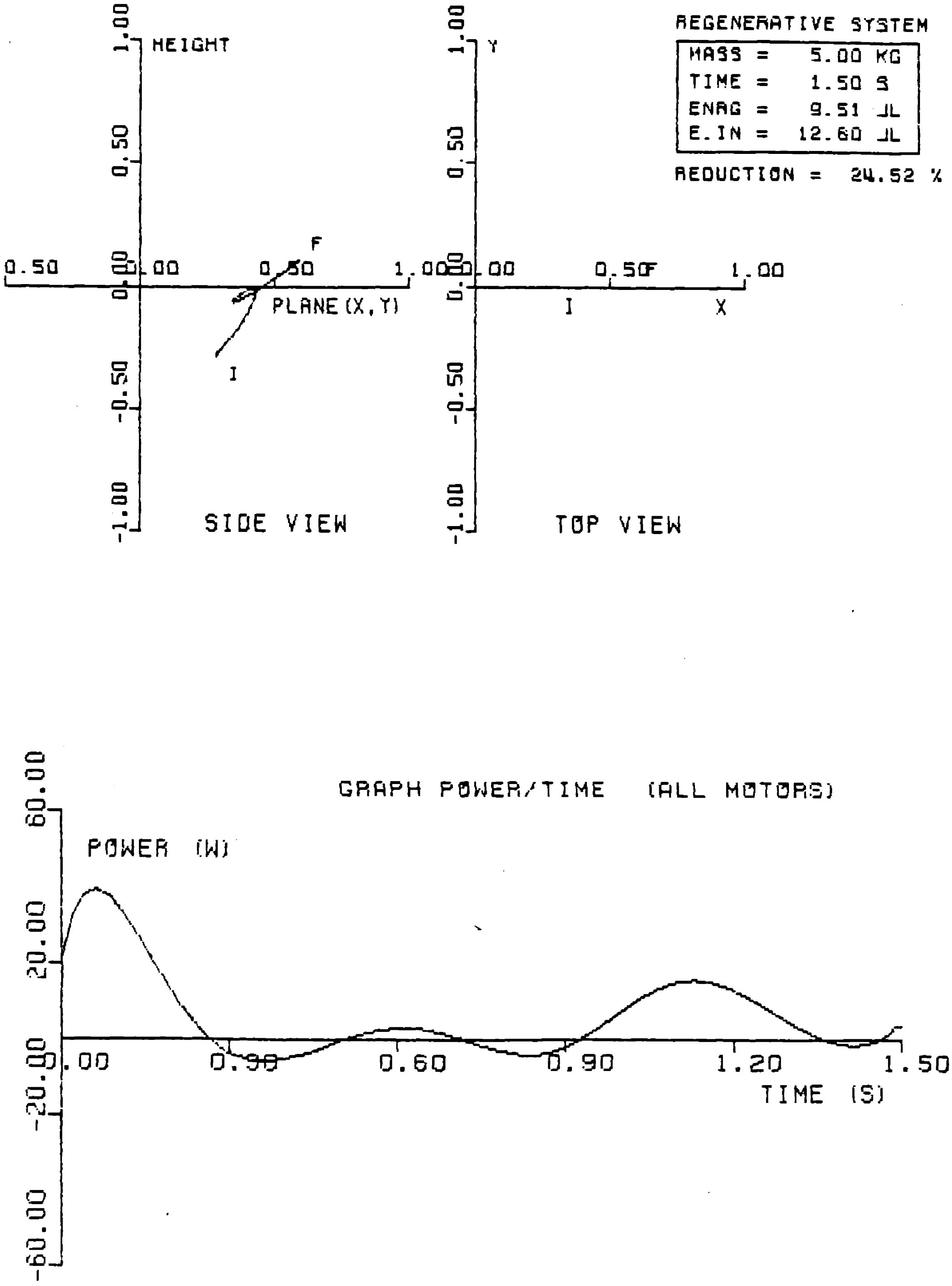

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.50 S
ENRG = 9.51 JL
E.IN = 12.60 JL
REDUCTION = 24.52 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.30
0.60
0.90
1.20
1.50
TIME (S)

Figure 15 J

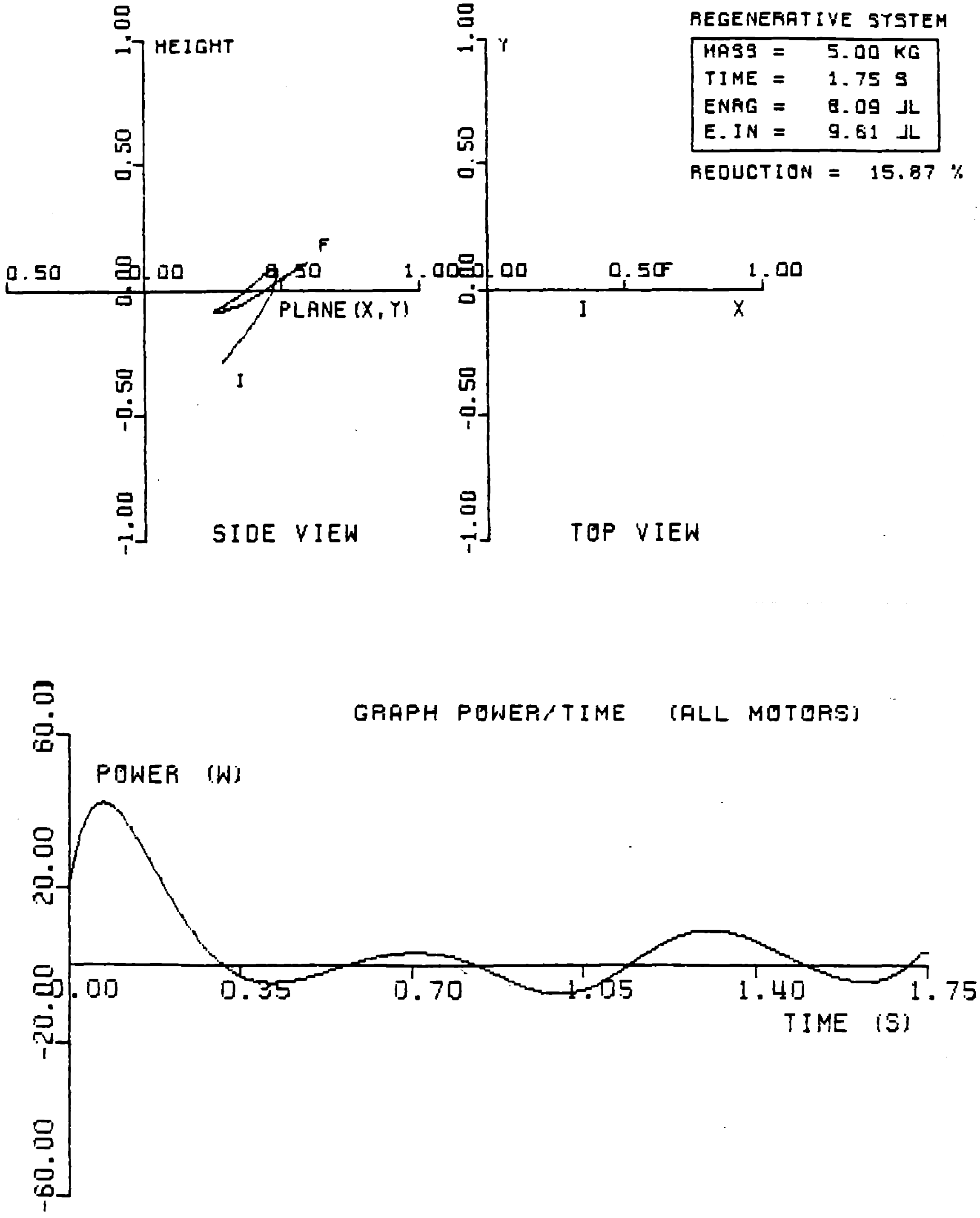

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 1.75 S
ENRG = 8.09 JL
E.IN = 9.61 JL
REDUCTION = 15.87 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.35
0.70
1.05
1.40
1.75
TIME (S)

Figure 15 K

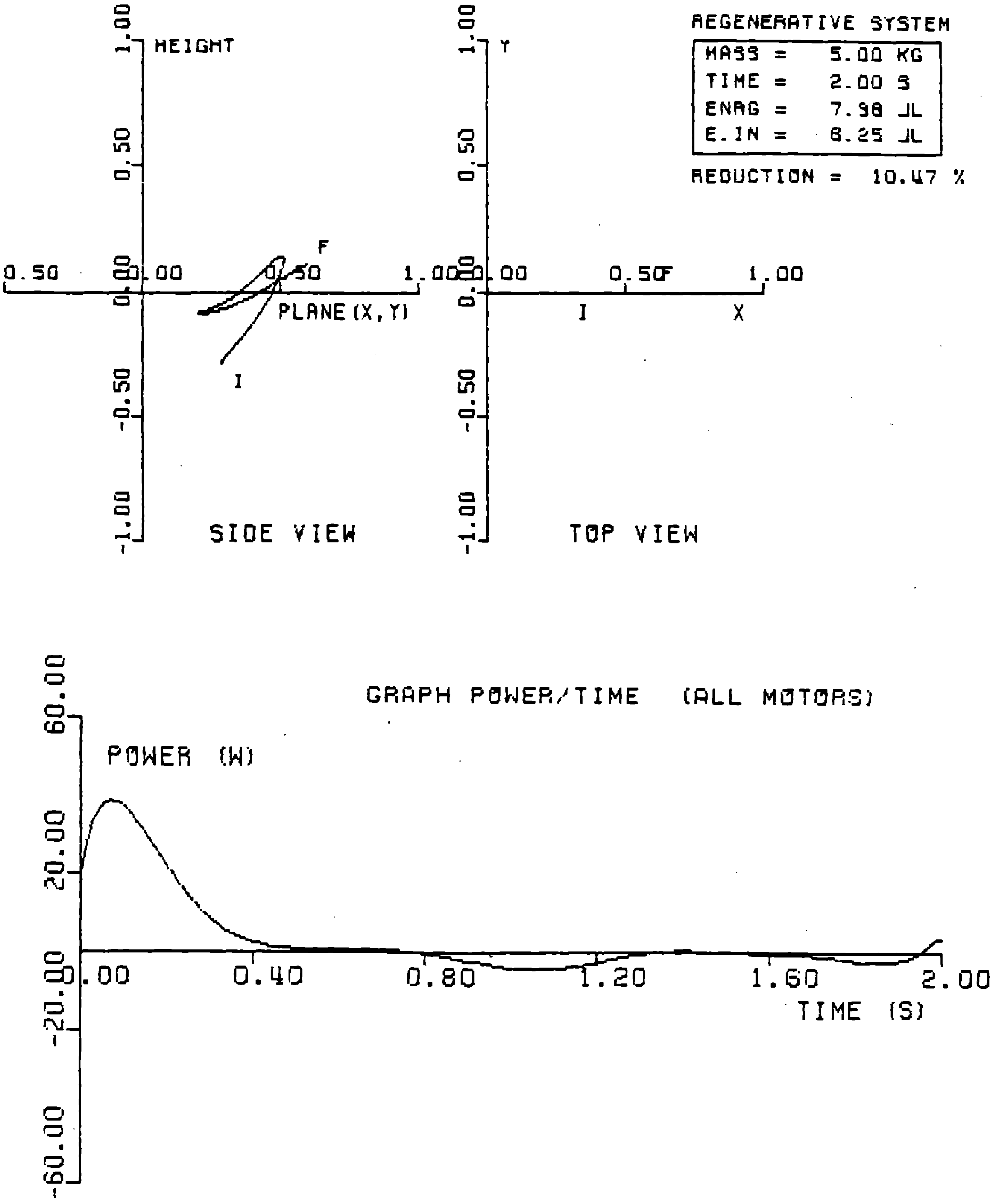

REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 2.00 S
ENRG = 7.38 JL
E.IN = 8.25 JL
REDUCTION = 10.47 %
HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
1.00
F
I
X
TOP VIEW
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
0.00
-20.00
-60.00
0.00
0.40
0.80
1.20
1.60
2.00
TIME (S)

Figure 15 L

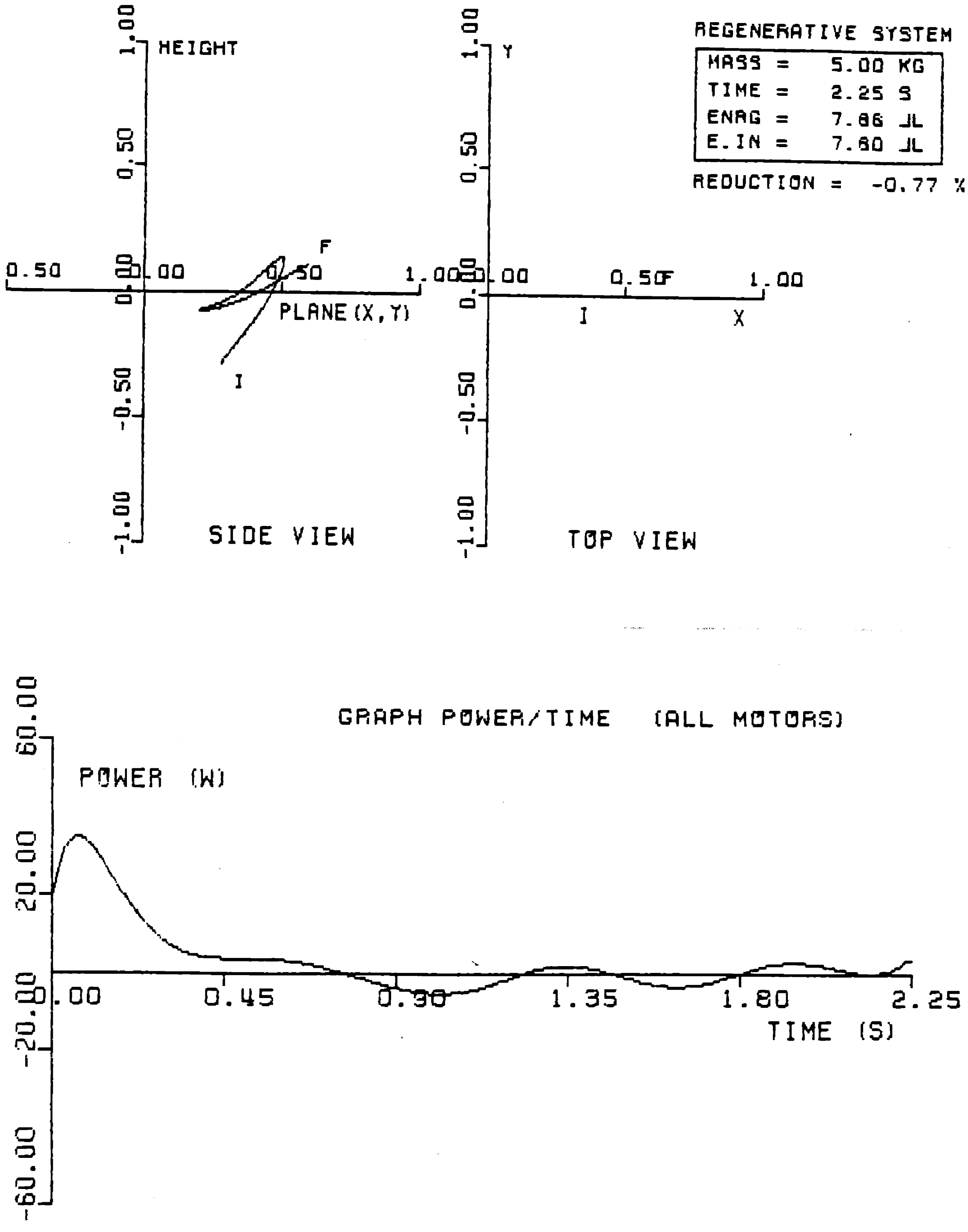

HEIGHT
1.00
0.50
0.00
-0.50
-1.00
0.50
0.00
0.50
1.00
F
I
PLANE (X,Y)
SIDE VIEW
Y
1.00
0.50
0.00
-0.50
-1.00
0.00
0.50
F
1.00
I
X
TOP VIEW
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 2.25 S
ENRG = 7.86 JL
E.IN = 7.80 JL
REDUCTION = -0.77 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
60.00
20.00
-20.00
-60.00
0.00
0.45
0.90
1.35
1.80
2.25
TIME (S)

Figure 16

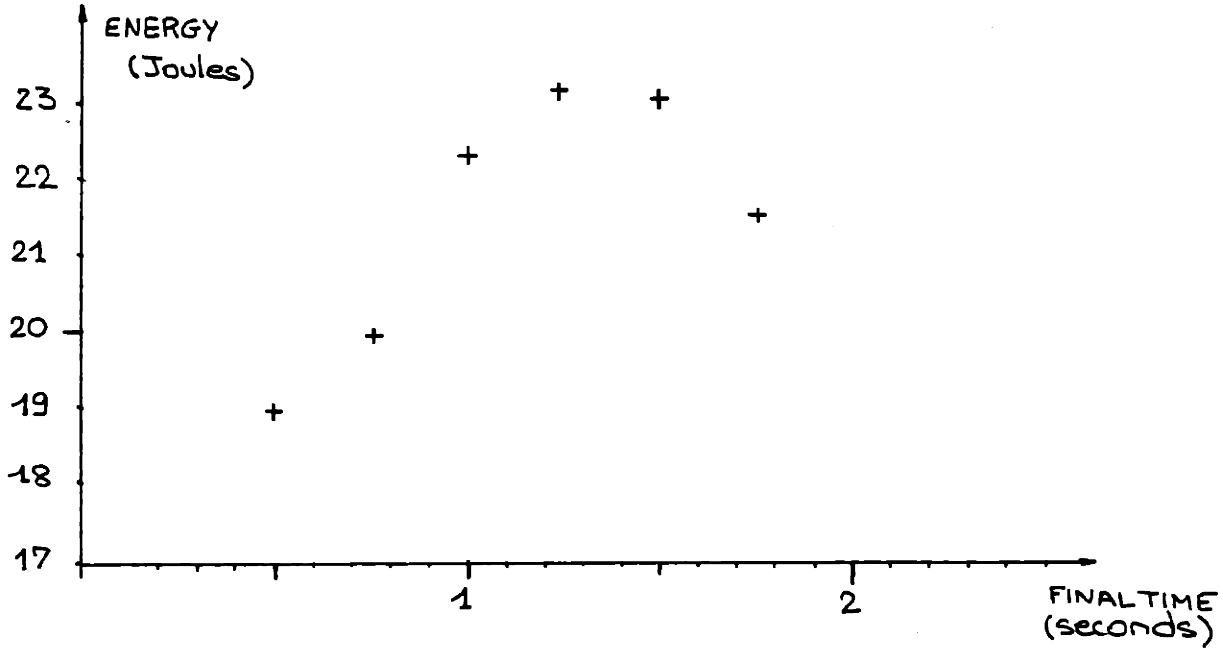
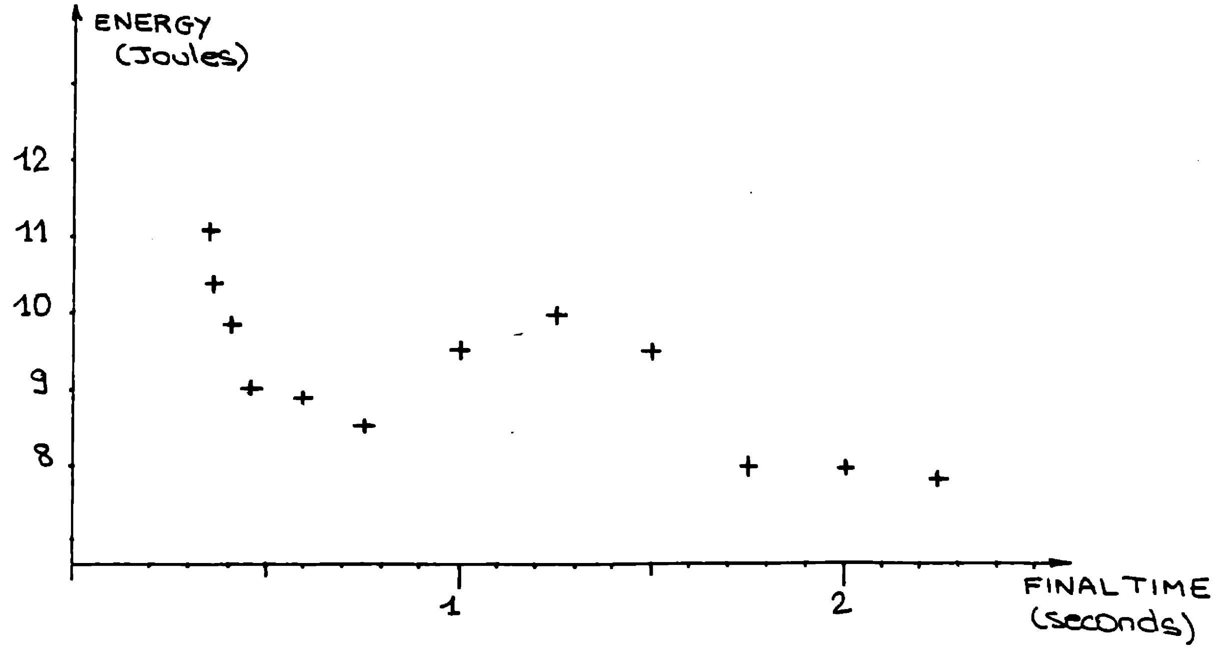


Figure 17



3) Minimum time

Minimum time trajectories have been difficult to obtain. Nevertheless, one successful result is presented in Figure 18, and show that the behavior is quite similar to those of Section 1). The forces are applied at the beginning and at the end of the motion, but they are limited by the maximum peak power constraints. A very interesting result is shown in comparison with Figures 16 and 17: The first local minimum energy trajectory appears at a time close to the minimum time trajectory.

Figure 18 a

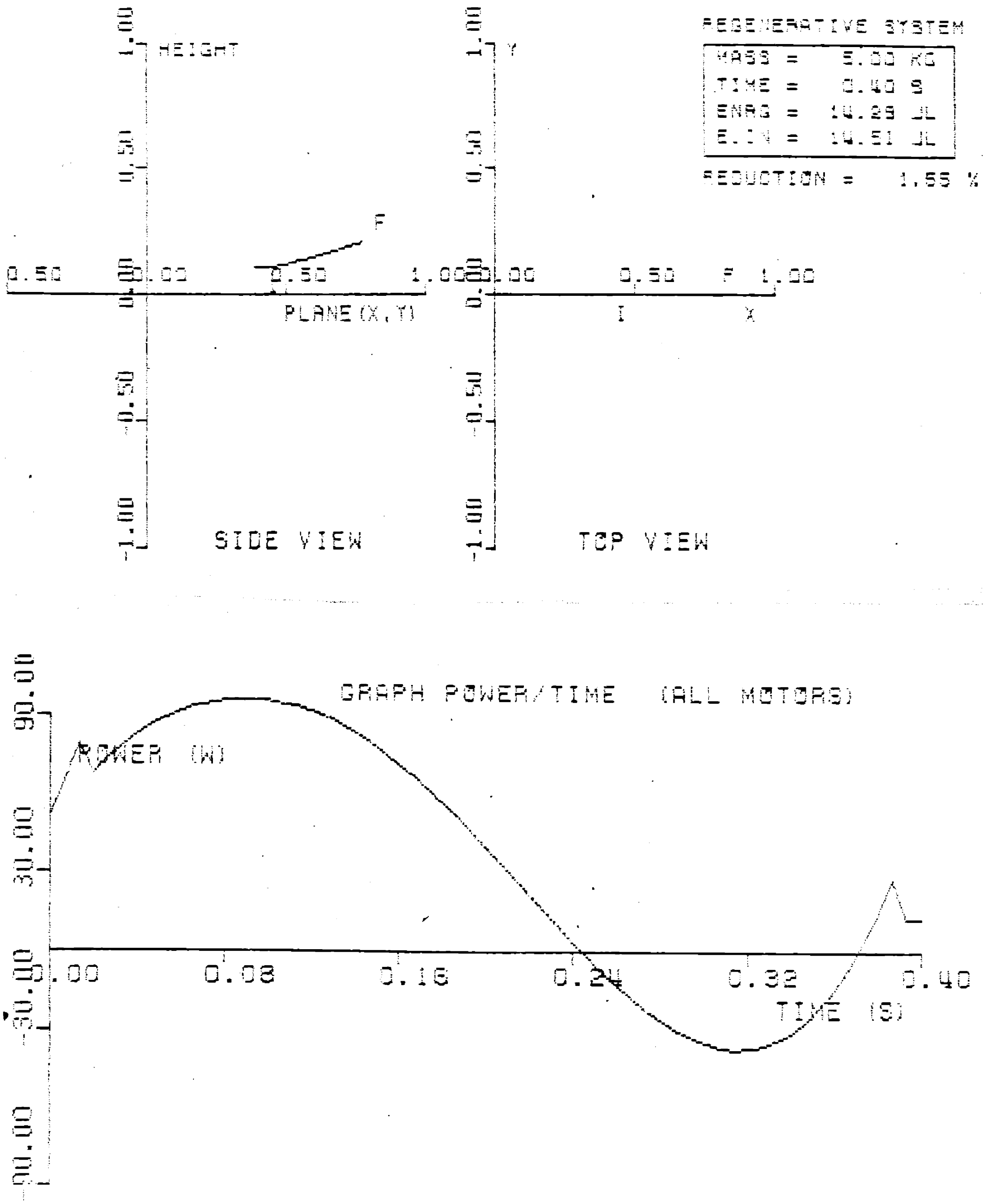
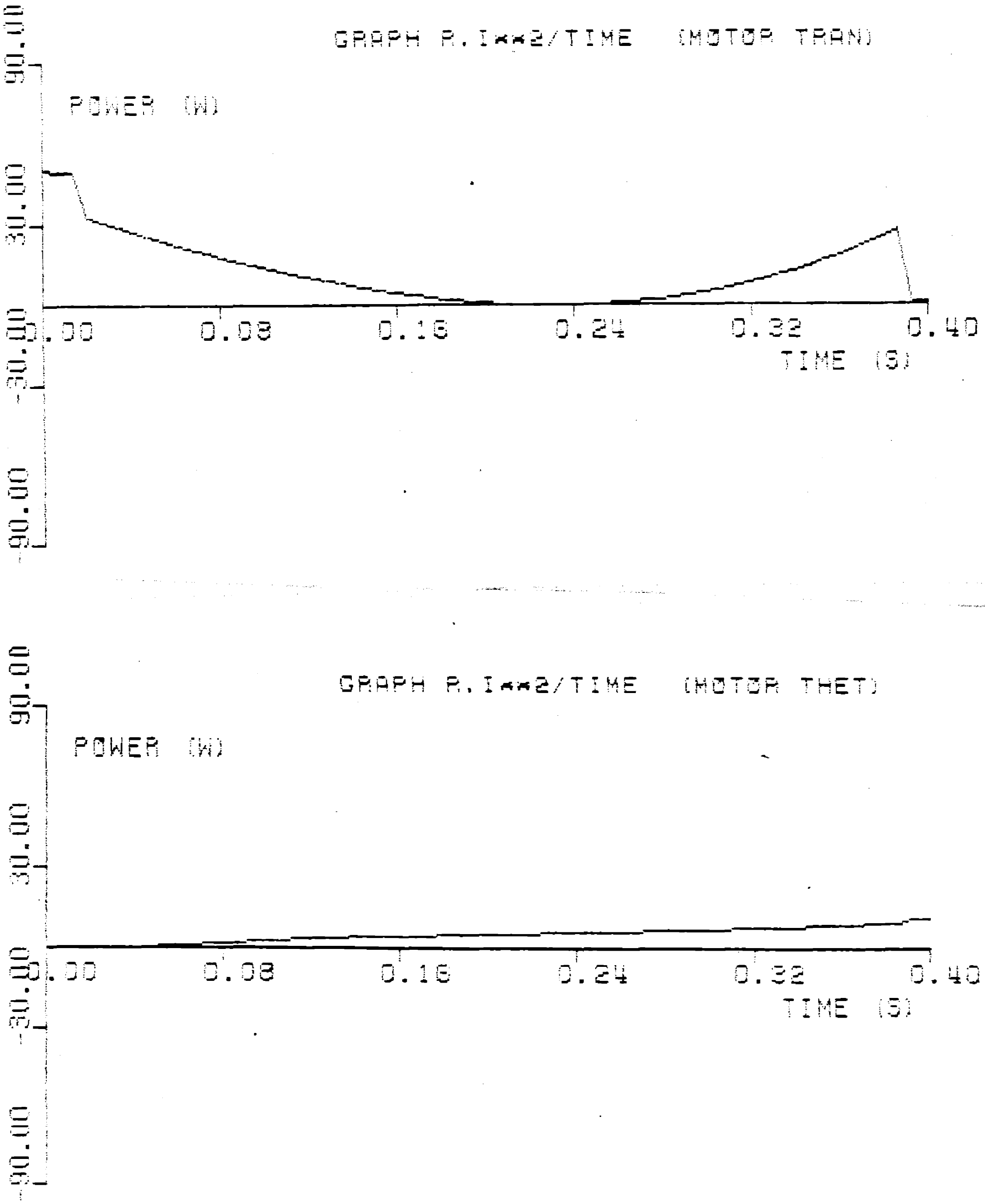

HEIGHT
PLANE(X,Y)
SIDE VIEW
Y
X
TOP VIEW
F
I
1.00
0.50
0.00
-0.50
-1.00
REGENERATIVE SYSTEM
MASS = 5.00 KG
TIME = 0.40 S
ENRG = 14.29 JL
E.IN = 14.51 JL
REDUCTION = 1.55 %
GRAPH POWER/TIME (ALL MOTORS)
POWER (W)
TIME (S)
90.00
30.00
-30.00
-90.00
0.00
0.08
0.16
0.24
0.32
0.40

Figure 18 b


GRAPH R.I**2/TIME (MOTOR TRAN)
POWER (W)
90.00
30.00
-30.00
-90.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME (S)
GRAPH R.I**2/TIME (MOTOR THET)
POWER (W)
90.00
30.00
-30.00
-90.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME (S)

XIV_SUMMARY AND CONCLUSION

1) Summary

A method for solving the optimal open-loop control of industrial robots has been presented. The determination of the trajectories of minimum energy for a given time of displacement, and the minimum time trajectories are difficult problems because the equations of motion are highly nonlinear and there are some equality and inequality constraints on both control and state variales. In addition, the number of state variables is large.

Several numerical examples have been studied and succesful results have been achieved. The work has been performed with Ohno's D.D.P. algorithm. This algorithm includes equality and inequality constraints on both control and state variables.

In Chapters III, IV, V, VI, VII, we have presented the equations for a general robot, including to the state transition function, the cost function and the constraint conditions for the minimization problem. We have emphasize the importance of the cost function for different types of model. The results have been achieved only with one type, but some further work must be done to transform the model to meet all the convergence conditions of the algorithm. In Chapter X, we have shown the necessity to use different initial trajectories. One fundamental reason is that the initial control must be in the neighborhood of convergence of the optimal control, and the time of computation is smaller if there are fewer iterations.

Important observations have to be noted for a robot with a regenerative source and no friction :

_ It is suitable to retract the arm in order to reduce the moments of inertia.

_ The optimal control tends to give forces only at the beginning and at the end of the trajectory.

_ It is beneficial to use the gravitational forces whenever possible to get some kinetic energy.

_ The results show that local minimum energy trajectories exist as a function of the time of displacement.

_ These trajectories make use of the mechanical resonance of the robot.

_ The minimum time trajectory is not far for the first local minimum trajectory.

2) Conclusion

More results can be achieved with the present method. For example, the determination of the optimum dimensions of robot assigned to a given task. This method can also lead to the determination of the motors and their reduction system. The time of computation is long, further work is required to make this procedure more efficient. The cases of friction and non-regenerative source have to be studied. The determination of the minimum time trajectories also need more attention. It will be possible to extend the method for the avoidance of obstacles by using the inequality constraint.

XIV_REFERENCES

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APPENDIX A

Determination of the equations of motion of the three link robot.

Hypothesis : (see Figure 2)

_ The two axes of rotation and the axis of translation converge.

_ The moments of inertia of each part are defined with respect of their center of gravity, except for the base, for which it is with respect of its axis of rotation.

_ The workpiece is a point mass attached to the end of the shaft.

_ The shaft is a uniformaly distributed mass along its length L. Its moment of inertia for rotation θ is

$$J_{s\theta} = m_s L^2/12$$

The equations of motion will be derived from the Lagrange's equation. Denote by K, the kinetic energy of the global system, and by P its potential energy. Lagrange's equation can be written

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_p}\right) - \frac{\partial K}{\partial q_p} = F_p - \frac{\partial P}{\partial q_p}$$

$$\text{with } q = \begin{pmatrix} R \\ \theta \\ \phi \end{pmatrix} \qquad \text{and } F = \begin{pmatrix} F_R \\ T_\theta \\ T_\phi \end{pmatrix}$$

Kinetic Energy

workpiece : $Kw = 1/2\ m\ (\dot{R}^2 + R^2\ \dot{\theta}^2 + R^2\ \dot{\phi}^2\ \sin^2\theta)$

shaft : $Ks = 1/2\ m_s\ (\dot{R}^2 + (R-L/2)^2\ \dot{\theta}^2 + (R-L/2)^2\ \dot{\phi}^2\ \sin^2\theta)$
$+ 1/2\ m_s\ L^2/12\ \dot{\theta}^2$
$+ 1/2\ m_s\ (L^2/12)\ \dot{\phi}^2\ \sin^2\theta$

housing : $Kh = 1/2\ m_h\ (r_h^2\ \dot{\theta}^2 + r_h^2\ \dot{\phi}^2\ \sin^2\theta)$
$+ 1/2\ J_{h\theta}\ \dot{\theta}^2$
$+ 1/2\ (J_{h\phi}\ \sin^2\theta + J_r\ \cos^2\theta)\ \dot{\phi}^2$

base : $Kb = 1/2\ J_b\ \dot{\phi}^2$

If we write $m_t = m_s + m$, then the total kinetic energy can be expressed as

$$
\begin{aligned}
K = {} & 1/2\ m_t\ \dot{R}^2 \\
& + 1/2\ (m_t\ R^2 - m_s\ L\ R + m_s\ L^2/12 + m_h\ r_h^2 + J_{h\theta})\ \dot{\theta}^2 \\
& + 1/2\ [(m_t\ R^2 - m_s\ L\ R + m_s\ L^2/12 + m_h\ r_h^2 + J_{h\phi})\ \sin^2\theta \\
& \qquad + J_r\ \cos^2\theta + J_b]\ \dot{\phi}^2
\end{aligned}
$$

Potential Energy

$$P = (m\ R + m_s\ (R - L/2) + m_h\ r_h)\ g\ \cos\theta$$

Equations of motion

a) $q_p = R$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{R}}\right) = m_t \ddot{R} \qquad \frac{\partial K}{\partial R} = (m_t R - m_s \tfrac{L}{2})(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$$\frac{\partial P}{\partial R} = m_t g \cos\theta$$

b) $q_p = \theta$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = (m_t R^2 + m_s L(\tfrac{L}{3} - R) + m_h r_h^2 + J_{h\theta})\, \ddot{\theta} + 2(m_t R - m_s \tfrac{L}{2})\, \dot{R}\, \dot{\theta}$$

$$\frac{\partial K}{\partial \theta} = (m_t R^2 + m_s L(\tfrac{L}{3} - R) + m_h r_h^2 + J_{h\phi} - J_r) \sin\theta \cos\theta\, \dot{\phi}^2$$

$$\frac{\partial P}{\partial \theta} = -(m_t R - m_s \tfrac{L}{2} + m_h r_h)\, g \sin\theta$$

c) $q_p = \phi$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}}\right) = \Big[\left(m_t R^2 + m_s L\left(\frac{L}{3} - R\right) + m_h r_h^2 + J_{h\phi}\right)\sin^2\theta$$
$$+ J_r \cos^2\theta + J_b\Big]\ddot{\phi} + 2\left(m_t R - m_s \frac{L}{2}\right)\dot{R}\,\dot{\phi}\,\sin^2\theta$$
$$+ \left(m_t R^2 + m_s L\left(\frac{L}{3} - R\right) + m_h r_h^2 + J_{h\phi} - J_r\right)\sin 2\theta\,\dot{\theta}\,\dot{\phi}$$

$$\frac{\partial K}{\partial \phi} = 0 \quad , \quad \frac{\partial P}{\partial \phi} = 0$$

we can deduce

$$\ddot{R} = [C_R\ I_R + (m_t\ R - m_s\ L/2)\ (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 0)]/m_t$$
$$- g \cos\theta$$

$$\ddot{\theta} = [C_\theta\ I_\theta + (m_t\ R - m_s\ L/2 + m_h\ r_h)\ g \sin\theta$$
$$- 2\ (m_t\ R - m_s\ L/2)\ \dot{R}\ \dot{\theta}$$
$$+ 1/2\ (m_t\ R^2 - m_s\ L\ R + m_s\ L^2/3 + m_h\ r_h^2 + J_{h\phi}$$
$$- J_r)\ \dot{\phi}^2 \sin 2\theta\ ]$$
$$/(m_t\ R^2 - m_s\ L\ R + m_s\ L^2/3 + m_h\ r_h^2 + J_{h\theta})$$

$$\ddot{\phi} = [C_\phi\ I_\phi - 2\ (m_t\ R^2 - m_s\ L/2)\ \dot{R}\ \dot{\phi}\ \sin^2\theta$$
$$- (m_t\ R^2 - m_s\ L\ R + m_s\ L^2/3 + m_h\ r_h^2 + J_{h\phi} - J_r)$$
$$\dot{\theta}\ \dot{\phi}\ \sin 2\theta\ ]$$
$$/\ [\ J_b + J_r \cos^2\theta + (m_t\ R^2 - m_s\ L\ R + m_s\ L^2/3$$
$$+ m_h\ r_h^2 + J_{h\phi}\ )\ \sin^2\theta\ ]$$

Notations

$g = 9.81$ m/s

constants CST(i) :

CST(1) = m_t
CST(2) = 1/CST(1)
CST(3) = $- m_s L/2$
CST(4) = $m_s L^2/3 + m_h r_h^2$
CST(5) = $m_s L^2/3 + m_h r_h^2 + J_{h\phi}$
CST(6) = J_r
CST(7) = $m_s L^2/3 + m_h r_h^2 + J_{h\phi} - J_r$
CST(8) = J_b
CST(9) = $m_s L^2/3 + m_h r_h^2 + J_{h\theta}$
CST(10) = $m_h r_h g$

constants CX(i) :

CX(1) = Km_R/Red_R
CX(2) = Km_θ/Red_θ
CX(3) = Km_ϕ/Red_ϕ

global inertias IRT(i) :

$$\begin{aligned}
\mathrm{IRT}(1) &= \mathrm{CST}(1)\, R^2 + 2\, \mathrm{CST}(3)\, R + \mathrm{CST}(9) \\
\mathrm{IRT}(2) &= \mathrm{CST}(8) + \mathrm{CST}(6) \cos^2\theta \\
&\quad + (\mathrm{IRT}(1) - \mathrm{CST}(9) + \mathrm{CST}(5)) \sin^2\theta \\
\mathrm{IRT}(3) &= \mathrm{IRT}(1) - \mathrm{CST}(9) + \mathrm{CST}(7) \\
\mathrm{IRT}(4) &= \mathrm{CST}(1)\, R + \mathrm{CST}(3)
\end{aligned}$$

We can deduce the accelerations

$$\begin{aligned}
\ddot{R} &= \mathrm{CST}(2)\, (FR + \mathrm{IRT}(4)\, (\dot{\theta}^2 + (\dot{\phi}^2 \sin^2\theta)\ )) - g \cos\theta \\
\ddot{\theta} &= (T\theta + \mathrm{IRT}(3)/2\ \dot{\phi}^2 \sin 2\theta - 2\, \mathrm{IRT}(4)\, \dot{R}\, \dot{\theta} \\
&\quad + (g\, \mathrm{IRT}(4) + \mathrm{CST}(10)) \sin\theta)/\mathrm{IRT}(1) \\
\ddot{\phi} &= (T\phi - \mathrm{IRT}(3)\, \dot{\theta}\, \dot{\phi} \sin 2\theta \\
&\quad - 2\, \mathrm{IRT}(4)\, \dot{R}\, \dot{\phi} \sin^2\theta\ )/\mathrm{IRT}(2)
\end{aligned}$$

APPENDIX B

Determination of the derivatives of the state transition function, the cost function, the inequality constraint function and the equality constraint function for the three link robot.

Notation : $L_{un} = \left(\frac{\partial L_n}{\partial u_{in}} \right)$, $L_{uxn} = \left(\frac{\partial^2 L_n}{\partial u_{in} \partial x_{jn}} \right)$

$$f_{un} = \left(\frac{\partial f_{in}}{\partial u_{jn}} \right)$$

$$z^T f_{uun} = \sum_{i=1}^{m} z_i f_{iuun} .$$

STATE TRANSITION FUNCTION f

$f(1) = R + \Delta t \dot{R}$

$f(2) = \theta + \Delta t \dot{\theta}$

$f(3) = \phi + \Delta t \dot{\phi}$

$f(4) = \dot{R} + \Delta t \ddot{R}$

$f(5) = \dot{\theta} + \Delta t \ddot{\theta}$

$f(6) = \dot{\phi} + \Delta t \ddot{\phi}$

$f(7) = w_R + R_R I_R^2$

$f(8) = w_\theta + R_\theta I_\theta^2$

$f(9) = w_\phi + R_\phi I_\phi^2$

$fu(4,1) = \Delta t\ CI(1)\ CST(2)$

$fu(5,2) = \Delta t\ CI(2)/IRT(1)$

$fu(6,3) = \Delta t\ CI(3)/IRT(2)$

$fu(7,1) = 2\ R_R\ I_R$

$fu(8,2) = 2\ R_\theta\ I_\theta$

$fu(9,3) = 2\ R_\phi\ I_\phi$

$fuu(7,1,1) = 2\ R_R$

$fuu(8,2,2) = 2\ R_\theta$

$fuu(9,3,3) = 2\ R_\phi$

$fx(1,1) = 1$

$fx(2,2) = 1$

$fx(3,3) = 1$

$fx(4,4) = 1$

$fx(1,4) = \Delta t$

$fx(2,5) = \Delta t$

$fx(3,6) = \Delta t$

$fx(4,1) = \Delta t\ (\dot{\theta}^2 + (\dot{\phi}\ \sin\theta)^2\ )$

$fx(4,2) = \Delta t\ (CST(2)\ IRT(4)\ \dot{\phi}^2\ \sin 2\theta + g\ \sin\theta)$

$fx(4,5) = 2\ \Delta t\ CST(2)\ IRT(4)\ \dot{\theta}$

$fx(4,6) = 2\ \Delta t\ CST(2)\ IRT(4)\ \dot{\phi}\ \sin^2\theta$

$$fx(5,1) = \Delta t\ (IRT(4)\ \dot{\phi}^2 \sin 2\theta - 2\ IRT(4)\ \ddot{\theta} \\ + CST(1)\ (g\ \sin\theta - 2\ \dot{R}\ \dot{\theta}))/IRT(1)$$

$$fx(5,2) = \Delta t\ (IRT(3)\ \dot{\phi}^2\ \cos 2\theta + (g\ IRT(4) \\ + CST(10))\ \cos\theta)/IRT(1)$$

$fx(5,4) = -\ 2\ \Delta t\ IRT(4)\ \dot{\theta}/IRT(1)$

$fx(5,5) = 1 - \Delta t\ 2\ IRT(4)\ \dot{R}/IRT(1)$

$fx(5,6) = \Delta t\ IRT(3)\ \dot{\phi}\ \sin 2\theta/IRT(1)$

$$\begin{aligned} \mathrm{fx}(6,1) = {} & -2\,\Delta t\,(\mathrm{IRT}(4)\,\dot{\theta}\,\dot{\phi}\,\sin 2\theta \\ & + (\mathrm{CST}(1)\,\dot{R}\,\dot{\phi} + \mathrm{IRT}(4)\,\ddot{\phi})\,\sin^2\theta)/\mathrm{IRT}(2) \end{aligned}$$

$$\begin{aligned} \mathrm{fx}(6,2) = {} & -\Delta t\,(2\,\mathrm{IRT}(3)\,\dot{\theta}\,\dot{\phi}\,\cos 2\theta \\ & + (\mathrm{IRT}(3)\,\ddot{\phi} + 2\,\mathrm{IRT}(4)\,\dot{R}\,\dot{\phi})\,\sin 2\theta)/\mathrm{IRT}(2) \end{aligned}$$

$$\mathrm{fx}(6,4) = -2\,\Delta t\,\mathrm{IRT}(4)\,\dot{\phi}\,\sin^2\theta/\mathrm{IRT}(2)$$

$$\mathrm{fx}(6,5) = -\Delta t\,\mathrm{IRT}(3)\,\dot{\phi}\,\sin 2\theta/\mathrm{IRT}(2)$$

$$\begin{aligned} \mathrm{fx}(6,6) = {} & 1 - \Delta t\,(\mathrm{IRT}(3)\,\dot{\theta}\,\sin 2\theta + 2\,\mathrm{IRT}(4)\,\dot{R}\,\sin^2\theta) \\ & /\mathrm{IRT}(2) \end{aligned}$$

$$\mathrm{fx}(7,7) = 1$$

$$\mathrm{fx}(8,8) = 1$$

$$\mathrm{fx}(9,9) = 1$$

$$\mathrm{fxx}(4,1,2) = \Delta t\,\dot{\phi}^2\,\sin 20$$

$$\mathrm{fxx}(4,1,5) = 2\,\Delta t\,\dot{\theta}$$

$$\mathrm{fxx}(4,1,6) = 2\,\Delta t\,\dot{\phi}\,\sin^2\theta$$

$$\mathrm{fxx}(4,2,2) = \Delta t\,(2\,\mathrm{CST}(2)\,\mathrm{IRT}(4)\,\dot{\phi}^2\,\cos 2\theta + g\,\cos\theta)$$

$$\mathrm{fxx}(4,2,6) = 2\,\Delta t\,\mathrm{CST}(2)\,\mathrm{IRT}(4)\,\dot{\phi}\,\sin 2\theta$$

$$\mathrm{fxx}(4,5,5) = 2\,\Delta t\,\mathrm{CST}(2)\,\mathrm{IRT}(4)$$

$$\begin{aligned} \mathrm{fxx}(5,1,1) = {} & (\Delta t\,\mathrm{CST}(1)\,(\dot{\phi}^2\,\sin 2\theta - 2\,\ddot{\theta}) \\ & - 4\,\mathrm{IRT}(4)\,\mathrm{fx}(5,1))/\mathrm{IRT}(1) \end{aligned}$$

$$\begin{aligned} \mathrm{fxx}(5,1,2) = {} & (\Delta t\,(2\,\mathrm{IRT}(4)\,\dot{\phi}^2\,\cos 2\theta + g\,\mathrm{CST}(1)\,\cos\theta) \\ & - 2\,\mathrm{IRT}(4)\,\mathrm{fx}(5,2))/\mathrm{IRT}(1) \end{aligned}$$

$$\begin{aligned} \mathrm{fxx}(5,1,4) = {} & -2\,(\mathrm{CST}(1)\,\Delta t\,\dot{\theta} + \mathrm{IRT}(4)\,\mathrm{fx}(5,4)) \\ & /\mathrm{IRT}(1) \end{aligned}$$

$$\mathrm{fxx}(5,1,5) = -2\,\Delta t\,\mathrm{CST}(1)\,\dot{R}/\mathrm{IRT}(1)$$

$$\begin{aligned} \mathrm{fxx}(5,1,6) = {} & 2\,(\Delta t\,\mathrm{IRT}(4)\,\dot{\phi}\,\sin 2\theta - \mathrm{IRT}(4)\,\mathrm{fx}(5,6)) \\ & /\mathrm{IRT}(1) \end{aligned}$$

$$\begin{aligned} \mathrm{fxx}(5,2,2) = {} & -\Delta t\,(2\,\mathrm{IRT}(3)\,\sin 2\theta\,\dot{\phi}^2 \\ & + (g\,\mathrm{IRT}(4) + \mathrm{CST}(10))\,\sin\theta)/\mathrm{IRT}(1) \end{aligned}$$

$$\mathrm{fxx}(5,2,6) = 2\,\Delta t\ \mathrm{IRT}(3)\ \dot{\phi}\ \cos 2\theta/\mathrm{IRT}(1)$$

$$\mathrm{fxx}(5,4,5) = -\,2\,\Delta t\ \mathrm{IRT}(4)/\mathrm{IRT}(1)$$

$$\mathrm{fxx}(5,6,6) = \Delta t\ \mathrm{IRT}(3)\ \sin 2\theta/\mathrm{IRT}(1)$$

$$\begin{aligned}\mathrm{fxx}(6,1,1) = {} & -\,2\,(\Delta t\ \mathrm{CST}(1)\ (\dot{\theta}\ \dot{\phi}\ \sin 2\theta + \ddot{\phi}\ \sin^2\theta)\\ & + 2\ \mathrm{IRT}(4)\ \mathrm{fx}(6,1)\ \sin^2\theta)/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,1,2) = {} & (\,-\,2\,\Delta t\ (2\ \mathrm{IRT}(4)\ \dot{\theta}\ \dot{\phi}\ \cos 2\theta\\ & + (\mathrm{CST}(1)\ \dot{R}\ \dot{\phi} + \mathrm{IRT}(4)\ \ddot{\phi})\ \sin 2\theta)\\ & - 2\ \mathrm{IRT}(4)\ \sin^2\theta\ \mathrm{fx}(6,2)\\ & - \mathrm{IRT}(3)\ \sin 2\theta\ \mathrm{fx}(6,1))/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,1,4) = {} & (\,-\,2\,(\Delta t\ \mathrm{CST}(1)\ \dot{\phi}\\ & \qquad + \mathrm{IRT}(4)\ \mathrm{fx}(6,4))\ \sin^2\theta)/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,1,5) = {} & (\,-\,2\,(\Delta t\ \dot{\phi}\ \sin 2\theta\\ & \qquad + \mathrm{fx}(6,5)\ \sin^2\theta)\ \mathrm{IRT}(4))/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,1,6) = {} & -\,2\,(\Delta t\ \mathrm{IRT}(4)\ \dot{\theta}\ \sin 2\theta\\ & + (\mathrm{CST}(1)\ \Delta t\ \dot{R}\\ & \qquad + \mathrm{IRT}(4)\ (\mathrm{fx}(6,6) - 1))\ \sin^2\theta)\\ & \qquad\qquad /\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,2,2) = {} & (2\,\Delta t\ (2\ \mathrm{IRT}(3)\ \dot{\theta}\ \dot{\phi}\ \sin 2\theta\\ & - (2\ \mathrm{IRT}(4)\ \dot{R}\ \dot{\phi} + \mathrm{IRT}(3)\ \ddot{\phi})\ \cos 2\theta)\\ & - 2\ \mathrm{IRT}(3)\ \mathrm{fx}(6,2)\ \sin 2\theta)/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,2,4) = {} & -\,(2\,\Delta t\ \mathrm{IRT}(4)\ \dot{\phi}\\ & \qquad + \mathrm{IRT}(3)\ \mathrm{fx}(6,4))\ \sin 2\theta/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,2,5) = {} & -\,(2\,\Delta t\ \mathrm{IRT}(3)\ \dot{\phi}\ \cos 2\theta\\ & \qquad + \mathrm{IRT}(3)\ \mathrm{fx}(6,5)\ \sin 2\theta)/\mathrm{IRT}(2)\end{aligned}$$

$$\begin{aligned}\mathrm{fxx}(6,2,6) = {} & -\,(2\,\Delta t\ (\mathrm{IRT}(3)\ \dot{\theta}\ \cos 2\theta + \mathrm{IRT}(4)\ \dot{R}\ \sin 2\theta)\\ & + \mathrm{IRT}(3)\ (\mathrm{fx}(6,6) - 1)\ \sin 2\theta)/\mathrm{IRT}(2)\end{aligned}$$

$$\mathrm{fxx}(6,4,6) = -\,2\,\Delta t\ \mathrm{IRT}(4)\ \sin^2\theta/\mathrm{IRT}(2)$$

$$\mathrm{fxx}(6,5,6) = -\,\Delta t\ \mathrm{IRT}(3)\ \sin 2\theta/\mathrm{IRT}(2)$$

$\mathrm{fux}(5,2,1) = -2\,\Delta t\ \mathrm{CI}(2)\ \mathrm{IRT}(4)/\mathrm{IRT}(1)$

$\mathrm{fux}(6,3,1) = -2\,\Delta t\ \mathrm{CI}(3)\ \mathrm{IRT}(4)\ \sin^2\theta/\mathrm{IRT}(2)$

$\mathrm{fux}(6,3,2) = -\,\Delta t / \mathrm{CI}(3)\ \mathrm{IRT}(3)\ \sin 2\theta/\mathrm{IRT}(2)$

and for all i,j,k

$\mathrm{fxx}(i,j,k) = \mathrm{fxx}(i,k,j)$

COST FUNCTION L

If the electric system is regenerative and we make the assumption that there is no friction, then the cost function is R I

$Lu(1) = 2\ R_R\ I_R$

$Lu(2) = 2\ R_\theta\ I_\theta$

$Lu(3) = 2\ R_\phi\ I_\phi$

$Luu(1,1) = 2\ R_R$

$Luu(2,2) = 2\ R_\theta$

$Luu(3,3) = 2\ R_\phi$

If the electric system is non regenerative then the cost function is $\sum_{p=1}^{3} R_p\ I_p^2 + CX(p)\ \dot{q}_p\ I_p$

```
If $R_R\ IR + CX(1)\ IR\ \dot{R} \geqslant 0$, then
        $Lu(1) = 2\ R_R\ I_R + \dot{R}_R CX(1)$
        $Lux(1,4) = CX(1)$
        $Lx(4) = I\ \ CX(1)$
        $Luu(1,1) = 2\ R_R$
        else
        $Lu(1) = 0$
        $Lux(1,4) = 0$
        $Luu(1,1) = 0$
        $Lx(4) = 0$
```

```
If $R_\theta I\theta + CX(2) I_\theta \dot{\theta} \geqslant 0$, then
        $Lu(2) = 2 R_\theta I_\theta + \dot{\theta} CX(2)$
        $Lux(2,5) = CX(2)$
        $Lx(5) = I_\theta CX(2)$
        $Luu(2,2) = 2 R_\theta$
        else
        $Lu(2) = 0$
        $Lux(2,5) = 0$
        $Luu(2,2) = 0$
        $Lx(5) = 0$

If $R_\phi I_\phi + CX(3) I_\phi \dot{\phi} \geqslant 0$, then
        $Lu(3) = 2 R_\phi I_\phi + \dot{\phi} CX(3)$
        $Lux(3,6) = CX(3)$
        $Lx(6) = I_\phi CX(3)$
        $Luu(3,3) = 2 R_\phi$
        else
        $Lu(3) = 0$
        $Lux(3,6) = 0$
        $Lx(6) = 0$
        $Luu(3,3) = 0$
```

INEQUALITY CONSTRAINTS g

_ maximum voltage V_{max} (all steps)

$g(1) = R_R \ IR + CX(1) \ \dot{R} - V_{maxR}$

$g(2) = R_\theta \ I\theta + CX(2) \ \dot{\theta} - V_{max\theta}$

$g(3) = R_\phi \ I\phi + CX(3) \ \dot{\phi} - V_{max\phi}$

$g(4) = - g(1) - 2 \ V_{maxR}$

$g(5) = - g(2) - 2 \ V_{max\theta}$

$g(6) = - g(3) - 2 \ V_{max\phi}$

$gu(1,1) = R_R$

$gu(2,2) = R_\theta$

$gu(3,3) = R_\phi$

$gu(4,1) = - R_R$

$gu(5,2) = - R_\theta$

$gu(6,3) = - R_\phi$

$gx(1,4) = CX(1)$

$gx(2,5) = CX(2)$

$gx(3,6) = CX(3)$

$gx(4,4) = - CX(1)$

$gx(5,5) = - CX(2)$

$gx(6,6) = - CX(3)$

_ maximum peak power P (all steps)

g(7) = R_R IR - P_{peakR}
g(8) = R_θ Iθ - $P_{peak\theta}$
g(9) = R_ϕ I_ϕ - $P_{peak\phi}$
gu(7,1) = 2 R_R I_R
gu(8,2) = 2 R_θ I_θ
gu(9,3) = 2 R_ϕ I_ϕ
guu(7,1,1) = 2 R_R
guu(8,2,2) = 2 R_θ
guu(9,3,3) = 2 R_ϕ

_ maximum average power P (step N-1)

g(10) = f(7) - N P_{aveR}
g(11) = f(8) - N $P_{ave\theta}$
g(12) = f(9) - N $P_{ave\phi}$
gx(10,7) = 1
gx(11,8) = 1
gx(12,9) = 1
gu(10,1) = fu(7,1)
gu(11,2) = fu(8,2)
gu(12,3) = fu(9,3)
guu(10,1,1) = fuu(7,1,1)
guu(11,2,2) = fuu(8,2,2)
guu(12,3,3) = fuu(9,3,3)

_ limits of the robot (all steps except N-1)

for $i = 1,\dots,9$, $j = 1,\dots,9$

$g(10) = R + 2\,\Delta t\,\dot{R} + \Delta t^2\,\ddot{R} - R_{max}$
$g(11) = \theta + 2\,\Delta t\,\dot{\theta} + \Delta t^2\,\ddot{\theta} - \theta_{max}$
$g(12) = R_{min} - R_{max} - g(10)$
$g(13) = \theta_{min} - \theta_{max} - g(11)$
$gu(10,1) = \Delta t\ fu(4,1)$
$gu(11,2) = \Delta t\ fu(5,2)$
$gu(12,1) = -\,gu(10,1)$
$gu(13,2) = -\,gu(11,2)$
$gx(10,i) = t\ fx(4,i)$
$gx(11,i) = t\ fx(5,i)$
$gx(10,1) = gx(10,1) + 1$
$gx(11,2) = gx(11,2) + 1$
$gx(10,4) = gx(10,4) + \Delta t$
$gx(11,5) = gx(11,5) + \Delta t$
$gx(12,i) = -\,gx(10,i)$
$gx(13,i) = -\,gx(11,i)$
$gux(10,1,i) = \Delta t\ fux(4,1,i)$
$gux(11,2,i) = \Delta t\ fux(5,2,i)$
$gux(12,1,i) = -\,gux(10,1,i)$
$gux(13,2,i) = -\,gux(11,2,i)$
$gxx(10,i,j) = \Delta t\ fxx(4,i,j)$
$gxx(11,i,j) = \Delta t\ fxx(5,i,j)$
$gxx(12,i,j) = -\,gxx(10,i,j)$
$gxx(13,i,j) = -\,gxx(11,i,j)$

EQUALITY CONSTRAINTS h

_ position constraint (step N-2)

for $i = 1,\dots,6$, $j = 1,\dots,6$

$h(1) = R + 2\,\Delta t\,\dot{R} + \Delta t^2\,\ddot{R} - R_f$

$h(2) = \theta + 2\,\Delta t\,\dot{\theta} + \Delta t^2\,\ddot{\theta} - \theta_f$

$h(3) = \phi + 2\,\Delta t\,\dot{\phi} + \Delta t^2\,\ddot{\phi} - \phi_f$

$hu(1,1) = \Delta t\; fu(4,1)$

$hu(2,2) = \Delta t\; fu(5,2)$

$hu(3,3) = \Delta t\; fu(6,3)$

$hx(1,i) = \Delta t\; fx(4,i)$

$hx(2,i) = \Delta t\; fx(5,i)$

$hx(3,i) = \Delta t\; fx(6,i)$

$hx(1,1) = hx(1,1) + 1$

$hx(2,2) = hx(2,2) + 1$

$hx(3,3) = hx(3,3) + 1$

$hx(1,4) = hx(1,4) + \Delta t$

$hx(2,5) = hx(2,5) + \Delta t$

$hx(3,6) = hx(3,6) + \Delta t$

$hux(1,1,i) = \Delta t\; fux(4,1,i)$

$hux(2,2,i) = \Delta t\; fux(5,2,i)$

$hux(3,3,i) = \Delta t\; fux(6,3,i)$

$hxx(1,i,j) = \Delta t\; fxx(4,i,j)$

$hxx(2,i,j) = \Delta t\; fxx(5,i,j)$

$hxx(3,i,j) = \Delta t\; fxx(6,i,j)$

_ velocity equal zero (step N-1)

```
for i = 1,...,6
    j = 1,...,6

h(1) = f(4)
h(2) = f(5)
h(3) = f(6)
hu(1,1) = fu(4,1)
hu(2,2) = fu(5,2)
hu(3,3) = fu(6,3)
hx(1,i) = fx(4,i)
hx(2,i) = fx(5,i)
hx(3,i) = fx(6,i)
hxx(1,i,j) = fxx(4,i,j)
hxx(2,i,j) = fxx(5,i,j)
hxx(3,i,j) = fxx(6,i,j)
hux(1,1,i) = fux(4,1,i)
hux(2,2,i) = fux(5,2,i)
hux(3,3,i) = fux(6,3,i)
```

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