Sparse Fourier restriction for the cone

by

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ABSTRACT

In Fourier restriction theory, weighted inequalities allow us to probe the shape of level sets. In this thesis, we describe a new weighted Fourier extension estimate for the cone and its connection with the Mizohata–Takeuchi conjecture. The main result Theorem 3.1 builds on techniques from geometry originally explored by Tom Wolff in this context. The proof uses circular maximal function estimates first proved by Wolff and later generalized by Pramanik–Yang–Zahl in their work on restricted projections as a black box.

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Fourier restriction and decoupling

1.1 Introduction

This thesis makes a study of Fourier extension for the cone segment

$$\mathbb{C}one^2 = \{(\xi, |\xi|) \in \mathbb{R}^2 \times \mathbb{R} : 1 < |\xi| < 2\} \subset \mathbb{R}^3,$$

but it makes sense to begin by introducing the Fourier extension of arbitrary submanifolds of $\mathbb{R}^n$.

Given a smooth function $\psi \colon B^k(0,1) \to \mathbb{R}^{n-k}$, consider the graphical k-dimensional submanifold

$$\mathcal{M}_\psi = \{(\xi, \psi(\xi)) : \xi \in B^k(0,1)\} \subset \mathbb{R}^n.$$

We define the Fourier extension operator of $\mathcal{M}_\psi$ by

$$E_{\mathcal{M}_\psi} f(x', x'') = \int_{B^k(0,1)} f(\xi) e^{2\pi i (x' \cdot \xi + x'' \cdot \psi(\xi))} \, d\xi, \quad (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

Slightly more generally, given a submanifold $\mathcal{M} \subset \mathbb{R}^n$ (not necessarily a graphical submanifold) with a smooth surface measure σ, we define the Fourier extension of $(\mathcal{M}, \sigma)$ by

$$E_{\mathcal{M}} f(x) = \int_{\mathcal{M}} f(\xi) e^{2\pi i x \cdot \xi} \, d\sigma(\xi), \quad x \in \mathbb{R}^n.$$

In other words, $E_{\mathcal{M}} f$ is just the inverse Fourier transform of the tempered distribution $f\sigma$. In this thesis, we will mainly consider graphical submanifolds, and these two different definitions of the Fourier extension will not have any meaningful difference for our purposes. We will frequently drop the subscript entirely and write Ef for $E_{\mathcal{M}_\psi} f$ or $E_{\mathcal{M}} f$, as appropriate.

A few model manifolds of interest in restriction theory include the $(n-1)$-sphere

$$S^{n-1} = \{\xi \in \mathbb{R}^n : \sum_{j=1}^{n} \xi_j^2 = 1\},$$

the unit paraboloid

$$P^{n-1} = \{\xi \in B^n(0,1) : \xi_n = \sum_{j=1}^{n-1} \xi_j^2\},$$

the moment curve

$$\Gamma_n = \{(\xi, \xi^2, \dots, \xi^n) : \xi \in [0,1]\},$$

as well as the cone

$$\mathbb{C}one^{n-1} = \{(\xi, |\xi|) \in \mathbb{R}^{n-1} \times \mathbb{R} : 1 < |\xi| < 2\}.$$

It is an observation due to Stein [1] that the curvature of $\mathcal{M}$ influences the mapping properties of $E_{\mathcal{M}} f$, for instance, whether Ef is in $L^q(\mathbb{R}^n)$ for some $q < \infty$.

As a non-example, if $\mathcal{M}$ is an open disk in $\mathbb{R}^k \times \{0\}$, then $Ef(x', x'') = Ef(x', 0)$ is constant in the x'' variables, and in particular, $\|Ef\|_{L^q(\mathbb{R}^n)} = \infty$ for all $q < \infty$. In the presence of curvature, the situation is much different.

Proposition 1.1. *If $\mathcal{M} = S^{n-1}$, then the estimate*

$$|E1(x)| \lesssim \frac{1}{(1+|x|)^{\frac{n-1}{2}}}, \quad x \in \mathbb{R}^n$$

holds. In particular, $E1 \in L^q(\mathbb{R}^n)$ if and only if $q > \frac{2n}{n-1}$.

Proof. The function $E1(x)$ is radial, so it suffices to prove $|E1(\lambda e_n)| \lesssim |\lambda|^{\frac{1-n}{2}}$ for $\lambda \gg 1$. By definition,

$$E1(\lambda e_n) = \int_{S^{n-1}} e(\lambda \xi_n)\, d\sigma(\xi).$$

The contribution to the integral from $|\lambda \xi_n| \gg 1$ is negligible, so the value of the integral is approximately $\sigma(\{\xi \in S^{n-1} : |\lambda \xi_n| \lesssim 1\}) \sim (\lambda^{-\frac{1}{2}})^{n-1}$. □

Stein conjectured that if f is a function on the $(n-1)$-sphere with L^∞ norm bounded by 1, then Ef should obey the same L^q bounds as $E1$.

Conjecture 1.1 (Restriction conjecture)**.** *If f is a function on S^{n-1} that is bounded by 1, then*

$$\|Ef\|_{L^q(\mathbb{R}^n)} \lesssim 1 \quad \textit{if and only if } q > \frac{2n}{n-1}.$$

The conjecture is known in $\mathbb{R}^2$, and is open in all dimensions $n \geq 3$. Generally speaking, the *restriction problem* refers to this or the various equivalent forms of this problem for the sphere. The reason for the terminology is that the Fourier extension operator of a manifold $\mathcal{M}$ is the formal adjoint of the operation of taking a function on $\mathbb{R}^n$, computing its Fourier transform, and restricting the resulting function to the manifold $\mathcal{M}$.

Proposition 1.2 (Adjoint of extension is restriction of the Fourier transform)**.** *The L^2 adjoint of the Fourier extension $E_{\mathcal{M}}$ is the Fourier restriction operator $R_{\mathcal{M}}$ defined by*

$$R_{\mathcal{M}} g(\xi) = \widehat{g}|_{\mathcal{M}}(\xi).$$

Proof. We set up the integral $\int_{\mathbb{R}^n} Ef(x)\overline{g(x)}\,dx$ and compute:

$$\begin{aligned}\int_{\mathbb{R}^n} Ef(x)\overline{g(x)}\,dx &= \int_{\mathbb{R}^n}\int_{\mathcal{M}} f(\xi)e(x\cdot\xi)\,d\sigma(\xi)\,\overline{g(x)}\,dx\\ &= \int_{\mathcal{M}} f(\xi)\overline{\int_{\mathbb{R}^n} g(x)e(-x\cdot\xi)\,dx\,d\sigma(\xi)}\\ &= \int_{\mathcal{M}} f(\xi)\overline{R_{\mathcal{M}}g(\xi)}\,d\sigma(\xi).\end{aligned}$$

□

Therefore, by duality, a Fourier extension estimate of the form

$$E_{\mathcal{M}}\colon L^p(\mathcal{M}) \to L^q(\mathbb{R}^n)$$

is equivalent to a restriction estimate of the form

$$R_{\mathcal{M}}\colon L^{q'}(\mathbb{R}^n) \to L^{p'}(\mathcal{M}),$$

where p', q' are the Hölder conjugate exponents of p, q, respectively.

The next Example for hypersurfaces is useful for determining necessary conditions on p, q so that $E\colon L^p(\mathcal{M}) \to L^q(\mathbb{R}^n)$. The presentation comes from Demeter's book [2].

Example 1.1 (Knapp example for a hypersurface)**.** *Consider $\mathcal{M}_\psi$ which is a graphical hypersurface for some $\psi\colon B^{n-1}(0,1) \to \mathbb{R}$, and let $N > 1$ be a large number, f be a nonnegative smooth function supported in $B^{n-1}(0, \frac{1}{N})$. Then changing variables $\xi = \frac{\eta}{N}$,*

$$Ef(x', x_n) = \int_{B^{n-1}(0,\frac{1}{N})} f(\xi)e(x'\cdot\xi + x_n\psi(\xi))\,d\xi = \frac{1}{N^{n-1}}\int_{B^{n-1}(0,1)} f(\frac{\eta}{N})e(\frac{x'}{N}\cdot\eta + x_n\psi(\frac{\eta}{N}))\,d\eta.$$

Expanding the function $\psi(\frac{\eta}{N})$ around $\eta = 0$, we have

$$\psi(\frac{\eta}{N}) = \psi(0) + \frac{\nabla\psi(0)}{N} + \frac{1}{N^2}O(|\eta|^2).$$

Plugging this in, we see

$$Ef(x', x_n) = \frac{1}{N^{n-1}}\int_{B^{n-1}(0,1)} f(\frac{\eta}{N})e(\eta\cdot(\frac{x' + x_n\nabla\psi(0)}{N}) + \frac{x_n}{N^2}O(|\eta|^2))\,d\eta.$$

In particular, when (x', x_n) belong to the set

$$T = \{(x', x_n) \in \mathbb{R}^n : |x' + x_n\nabla\psi(0)| \ll N, |x_n| \ll N^2\},$$

the phase $\eta\cdot(\frac{x'+\nabla\psi(0)}{N}) + \frac{x_n}{N^2}O(|\eta|^2)$ will be close to zero, and $|Ef(x', x_n)| \geq \frac{1}{2}|Ef(0,0)| \sim \int f$. The set T is a tube of length $\sim N^2$ and direction $(-\nabla\psi(0), 1) \in \mathbb{R}^n$ containing the origin.

As a consequence, for this f, we see

$$(\int_{\mathbb{R}^n} |Ef|^q)^{1/q} \gtrsim |T|^{1/q}(\int f) = N^{\frac{n+1}{q}}N^{1-n}$$

and $(\int |f|^p)^{1/p} \sim N^{\frac{1-n}{p}}$. Comparison of these two quantities shows that E can only map $L^p(\mathcal{M}_\psi) \to L^q(\mathbb{R}^n)$ provided

$$\frac{n+1}{q} \leq \frac{n-1}{p'}.$$

In the context of Fourier extension for S^{n-1}, when $p = \infty$, we see that the condition $q > \frac{2n}{n-1}$ (which guarantees $E1 \in L^q(\mathbb{R}^n)$) is simply stronger than the necessary condition $\frac{n+1}{q} \le \frac{n-1}{p'}$ provided by the Knapp example.

1.2 Local estimates and Tomas–Stein

By means of local-to-global reductions, local estimates of the form $L^p(\mathcal{M}) \to L^q(B_R)$ for $R > 1$ can be bootstrapped to global $L^p(\mathcal{M}) \to L^q(\mathbb{R}^n)$ estimates, possibly with slightly worse q. See Tao's article [3] for a particularly clean example.

If $\mathcal{M}$ is a curved submanifold of $\mathbb{R}^n$, then the distribution of $|Ef|$ is spread out over space, so we fix a large parameter $R > 1$ and localize to a large ball $B_R \subset \mathbb{R}^n$. We can ask for $L^p(\mathcal{M}) \to L^q(B_R)$ type estimates. For which $p, q \ge 1$ do we have

$$\|Ef\|_{L^q(B_R)} \le C(n,p,q,R)\|f\|_{L^p(\mathcal{M})}?$$

The dependence of the constant C on R is ideally mild, such as $O_\epsilon(R^\epsilon)$ for every $\epsilon > 0$, or even $O(\log R)$, but all estimates of this type are still interesting.

An even weaker estimate we can look for is the local weak-type estimate:

$$\lambda^q|\{x \in B_R : |Ef(x)| > \lambda\}| \le C(n,p,q,R)\|f\|^q_{L^p(\mathcal{M})}, \quad \text{for all } \lambda > 0.$$

Any "strong-type" $L^p \to L^q$ estimate, either local or global, implies a corresponding weak-type estimate, and in many cases, a local weak-type estimate implies a corresponding local strong-type estimate with an R^ϵ-loss via dyadic pigeonholing.

Proposition 1.3. *Suppose $\|f\|_{L^p(\mathcal{M})} = 1$ and the local weak-type estimate*

$$\lambda^q|\{x \in B_R : |Ef(x)| > \lambda\}| \le 1, \quad \textit{for all } \lambda > 0$$

holds. Then the local strong-type estimate

$$\int_{B_R} |Ef|^q \lesssim \log R$$

holds.

Proof. First note that $\|Ef\|_{L^\infty} \le \|f\|_{L^1} \lesssim \|f\|_{L^p} = 1$. Let $N > 1$ and for each dyadic $R^{-N} < \lambda \le 1$, let

$$U(\lambda) = \{x \in B_R : |Ef(x)| \sim \lambda\}.$$

Then

$$\int_{B_R} |Ef|^q \le \sum_{R^{-N} < \lambda \lesssim 1} \int_{U(\lambda)} |Ef|^q + |B_R|R^{-Nq}.$$

If N is large enough, the term $|B_R|R^{-Nq} \le 1$, so it is negligible. There are $O(\log R)$-many dyadic values of λ in the sum, so by the pigeonhole principle, there is some particular λ such that

$$\int_{B_R} |Ef|^q \lesssim (\log R)\lambda^q |U(\lambda)|.$$

By our assumption, this is bounded by $\log R$, and the claim is proved. □

Because of the duality between Fourier extension and restriction, it is also possible to prove estimates using the method of TT^*. Perhaps the most famous example of this in practice is the Tomas–Stein restriction theorem.

Theorem 1.1 (Tomas–Stein, non-endpoint version)**.** *For every $\epsilon > 0$, the estimate*

$$\|E_{S^{n-1}} f\|_{L^{\frac{2(n+1)}{n-1}+\epsilon}(\mathbb{R}^n)} \le C_\epsilon \|f\|_{L^2(S^{n-1})}$$

holds.

Proof. Let σ be the surface measure for the sphere S^{n-1}. We set $T = E_{S^{n-1}}$ and note that by Proposition 1.2, $TT^* g = g * K$, where

$$K(x) = \widehat{\sigma}(x), \quad x \in \mathbb{R}^n.$$

(Strictly speaking, K is the inverse Fourier transform of σ, but by symmetry of the sphere, this is the same as the Fourier transform.) Therefore, by the method of TT^*, the estimate

$$\|E_{S^{n-1}} f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathcal{M})}$$

is equivalent to

$$\|g * \widehat{\sigma}\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}.$$

To prove this estimate, we let $p = \frac{2(n+1)}{n-1}$, and we let $\{\phi_0\} \cup \{\phi_k\}_{k=1}^\infty$ be a smooth partition of unity subordinate to a decomposition of $\mathbb{R}^n$ into

- U_0, the ball of radius ~ 1,
- U_k, the annulus of frequencies $|x| \sim 2^k$,

and we use the triangle inequality to estimate

$$\|g * \widehat{\sigma}\|_{L^p(\mathbb{R}^n)} \le \|g * (\widehat{\sigma} \cdot \phi_0)\|_{L^p(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \|g * (\widehat{\sigma} \cdot \phi_k)\|_{L^p(\mathbb{R}^n)}.$$

For the first piece, we have Young's inequality to say

$$\|g * (\widehat{\sigma} \cdot \phi_0)\|_{L^p(\mathbb{R}^n)} \le \|\widehat{\sigma} \cdot \phi_0\|_{L^r(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p'}(\mathbb{R}^n)},$$

where $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{p'}$.

For the second term, we show that

$$\|g * (\widehat{\sigma} \cdot \phi_k)\|_{L^p(\mathbb{R}^n)} \lesssim_\epsilon 2^{-\epsilon k} \|g\|_{L^{p'-\epsilon}(\mathbb{R}^n)} \tag{1.1}$$

in a few steps. First we note that

$$\|g * (\widehat{\sigma} \cdot \phi_k)\|_{L^\infty(\mathbb{R}^n)} \le \|\widehat{\sigma} \cdot \phi_k\|_{L^\infty(\mathbb{R}^n)} \cdot \|g\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-k(\frac{n-1}{2})} \|g\|_{L^1(\mathbb{R}^n)}$$

by the stationary phase estimate Proposition 1.1. Next we analyze the L^2-norm of $g*(\widehat{\sigma}\cdot\phi_k)$. By Plancherel,

$$\|g*(\widehat{\sigma}\cdot\phi_k)\|_{L^2(\mathbb{R}^n)} = \|\widehat{g}\cdot(\sigma*\widehat{\phi}_k)\|_{L^2(\mathbb{R}^n)}.$$

We claim that in fact $\|\sigma*\widehat{\phi}_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^k$, so

$$\|\widehat{g}\cdot(\sigma*\widehat{\phi}_k)\|_{L^2(\mathbb{R}^n)} \lesssim 2^k\|\widehat{g}\|_{L^2(\mathbb{R}^n)} = 2^k\|g\|_{L^2(\mathbb{R}^n)}.$$

To see the claim, we use the heuristic that $\widehat{\phi}_k$ is essentially constant at scale 2^{-k} and hence the integral $\int_{|x-y|\gg 2^{-k}} \widehat{\phi}_k(x-y)\,d\sigma(y)$ will have a lot of cancellation. The main contribution will come from $\int_{|x-y|\lesssim 2^{-k}} \widehat{\phi}_k(x-y)\,d\sigma(y)$. On this region, we can just use a base times height estimate since $|\widehat{\phi}_k| \lesssim 2^{kn}$:

$$\int_{|x-y|\lesssim 2^{-k}} |\widehat{\phi}_k(x-y)|\,d\sigma(y) \le \sigma(B(x,c2^{-k}))2^{kn} \sim (2^{-k})^{n-1}\cdot 2^{kn} = 2^k.$$

Interpolating the bound $\|g*(\widehat{\sigma}\cdot\phi_k)\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k(\frac{n-1}{2})}\|g\|_{L^1(\mathbb{R}^n)}$ with $\|g*(\widehat{\sigma}\cdot\phi_k)\|_{L^2(\mathbb{R}^n)} \lesssim 2^k\|g\|_{L^2(\mathbb{R}^n)}$ gives Equation (1.1), and finishes the proof. □

1.3 Decoupling and the shape of level sets

Besides the weak and strong $L^p \to L^q$ estimates, another important class of estimates which motivated the body of work in this thesis are the *decoupling estimates* introduced by Wolff in [4]. To motivate this kind of estimate, we first discuss the *wave packet decomposition* of Ef, which is a tool introduced by Bourgain in [5] to study the restriction problem in 3 and higher dimensions.

The wave packet decomposition is based on the following heuristic of Fourier analysis.

Heuristic 1.1 (Locally constant property)**.** *Suppose $C \subset \mathbb{R}^n$ is a bounded convex set containing* 0 *and* supp $\widehat{f} \subset C$. *If*

$$C^* = \{x \in \mathbb{R}^n : |x\cdot\xi| \le 1\}$$

is the dual body of C, and $x \in \mathbb{R}^n$, then

$$f(x) \approx \frac{1}{|C^*|}\int_{C^*+x} f$$

In words, $f(x)$ is approximately its average over the dual body C^* centered at x.

"Proof." Let ϕ be a Schwartz function equal to 1 on C and rapidly decaying away from C. Then $f = f*\check{\phi}$, and by the Schwartz decay of $\check{\phi}$, for any $N > 1$,

$$|\check{\phi}(x)| \lesssim_N |C|\sum_{j=1}^{\infty} 2^{-jN} 1_{2^jC^*}(x),$$

the dominant contribution to the integral $f(x) = \int f(y)\check{\phi}(x-y)\,dy$ comes from the $j=0$ term in the sum appearing above. Lastly, we note $|C| \sim \frac{1}{|C^*|}$. □

Suppose $\mathcal{M} = S^{n-1}$, or another hypersurface of curvature ~ 1, and we are studying $E_{\mathcal{M}}f$ on a large ball $B_R \subset \mathbb{R}^n$. We let ψ be a nonnegative Schwartz function with compact support in $B^n(0, R^{-1})$ which is equal to 1 on B_R, so

$$E_{\mathcal{M}}f = E_{\mathcal{M}}f \cdot \psi,$$

and $E_{\mathcal{M}}f \cdot \psi$ has Fourier support in $\mathcal{M}^{(R^{-1})}$, the R^{-1}-neighborhood of $\mathcal{M}$. The thin neighborhood of $\mathcal{M}$ is not convex, so Heuristic 1.1 does not apply to $E_{\mathcal{M}}f$. We let Θ be a partition of $\mathcal{M}^{(R^{-1})}$ into almost rectangular boxes θ of dimensions

$$\theta : R^{-1} \times \underbrace{R^{-1/2} \times \cdots \times R^{-1/2}}_{n-1 \text{ factors}}$$

and let $\{\phi_\theta\}_{\theta \in \Theta}$ be a partition of unity subordinate to Θ. Let $f_\theta = f\phi_\theta$. By linearity of the Fourier transform,

$$Ef = \sum_{\theta \in \Theta} Ef_\theta.$$

Each piece Ef_θ is supported in a convex set, so in particular it is approximately constant on translates of the dual body θ^*, which is a tube of dimensions $R \times R^{1/2} \times \cdots \times R^{1/2}$ with direction determined by θ. Each Ef_θ, or a spatially localized piece $Ef_\theta \cdot 1_T$, where T is a translate of θ^*, is referred to as a *wave packet.* We note that the wave packet decomposition is essentially a superposition of Knapp examples—cf. Example 1.1.

The wave packets Ef_θ enjoy an almost orthogonality relation. For any such decomposition of $\mathcal{M}$ into finitely overlapping sets θ, we have the orthogonality relation

$$\int |Ef|^2 \lesssim \sum_\theta \int |Ef_\theta|^2$$

by Plancherel. Decoupling is a version of this orthogonality relation in L^p when Θ is a decomposition of the thin neighborhood of $\mathcal{M}$ into almost-rectangular boxes.

Theorem 1.2 (Bourgain–Demeter ℓ^2-decoupling for the paraboloid [6])**.** *Consider a decomposition Θ of P^{n-1} into $R^{-1/2}$-caps $\theta \in \Theta$. Then for $2 \le p \le \frac{2(n+1)}{n-1}$, and every $\epsilon > 0$, we have*

$$\|Ef\|^2_{L^p(\mathbb{R}^n)} \lesssim_\epsilon R^\epsilon \sum_\theta \|Ef_\theta\|^2_{L^p(\mathbb{R}^n)}.$$

There is also a local version of decoupling which is equivalent to the global version in Theorem 1.2.

Theorem 1.3 (Local ℓ^2-decoupling for the paraboloid)**.** *Suppose* supp $\widehat{f} \subset P^{n-1}(N^{-2})$*, the N^{-2}-neighborhood of the truncated paraboloid in $\mathbb{R}^n$, and $\Theta = \{\theta\}$ is a partition of $P^{n-1}(N^{-2})$ into boxes θ of dimensions $N^{-1} \times \cdots \times N^{-1} \times N^{-2}$. Then for each $2 \le p \le \frac{2(n+1)}{n-1}$, and every $\epsilon > 0$, we have*

$$\|f\|^2_{L^p(B_{N^2})} \lesssim_\epsilon N^\epsilon \sum_\theta \|f_\theta\|^2_{L^p(w_{B_{N^2}})},$$

where $w_{B_{N^2}}$ is a nonnegative weight that is ~ 1 on B_{N^2} and decays quickly outside of B_{N^2}.

We will sketch a proof of a corollary of the local decoupling Theorem 1.3 in $\mathbb{R}^2$ which contains all of the key ideas that go into the proof of the full theorem. We follow the presentation in Guth's paper [7]. It underlines the role that the *shape* of level sets plays in Fourier analysis, a topic we take up in Chapter 3.

Corollary 1.1. *Suppose* $\operatorname{supp} \widehat{f} \subset P^1(N^{-2})$*, the* N^{-2} *neighborhood of the truncated parabola, and* $f = \sum_\theta f_\theta$*. where the* θ *form a tiling of* $P^1(N^{-2})$ *by almost-rectangular boxes* θ *of dimensions* $N^{-1} \times N^{-2}$*. Moreover, suppose each* $\|f_\theta\|_{L^\infty(\mathbb{R}^2)} \leq 1$*. For every* $\epsilon > 0$*,*

$$|U_{\frac{N}{10}}(f) \cap B_{N^2}| \leq C_\epsilon N^{1+\epsilon}. \tag{1.2}$$

The estimate of Equation (1.2) follows from the decoupling Theorem 1.3 taking $p = 6$, since

$$(\frac{N}{10})^6 |U_{\frac{N}{10}}(f) \cap B_{N^2}| \leq \int_{B_{N^2}} |f|^6 \lessapprox (\sum_\theta \|f_\theta\|^2_{L^6(w_{B_{N^2}})})^3 \lesssim (\#\theta)^3 |B_{N^2}| = N^7.$$

Moreover, the estimate of Equation (1.2) is sharp as the following example shows.

Example 1.2 (Exponential sum)**.** *Let*

$$f(x) = \sum_{n=1}^N e(x_1 \frac{n}{N} + x_2 \frac{n^2}{N^2}),$$

so $\operatorname{supp} \widehat{f} = \{(\frac{n}{N}, \frac{n^2}{N^2}) : n = 1, \dots, N\} \subset P^1$*. For each* $(x_1, x_2) = (mN, 0)$*,* $m = 1, \dots, N$*, we have* $f(mN, 0) = N$*. For* x *in a small neighborhood of each lattice point* $(mN, 0)$*, we have* $|f(x)| > \frac{N}{10}$*. Therefore, since the points* $(mN, 0)$*,* $m = 1, \dots, N$ *are* N*-separated,*

$$|U_{\frac{N}{10}}(f) \cap [0, N^2]^2| \sim N.$$

Since $|f(x)| \leq \sum_\theta |f_\theta(x)| \leq N$, the level set

$$U_{\frac{N}{10}}(f) = \{x \in \mathbb{R}^2 : |f(x)| > \frac{N}{10}\}$$

represents where $|f(x)|$ is close to maximal. To prove estimates for its size, we will use properties of the f_θ together with induction on scales to argue about the shape of the level set.

Induction on scales. If $N = O(1)$, then the estimate we want to prove is trivial, so suppose by induction that for every $\tilde{N} < \frac{1}{2}N$, we have proved that Corollary 1.1 holds with "$\tilde{N}$" in place of "N".

The first thing we want to use our induction hypothesis for is to establish a bound for

$$|U_{\frac{N_\tau}{10}}(f_\tau) \cap B_{N^2}|.$$

By parabolic rescaling, studying f_τ on B_{N^2} is equivalent to studying a certain function h on a box Q of dimensions $N^{3/2} \times N$, whose Fourier support is contained in $P^1(N^{-1})$, and such that

$$h = \sum_{\tilde{\theta}} h_{\tilde{\theta}},$$

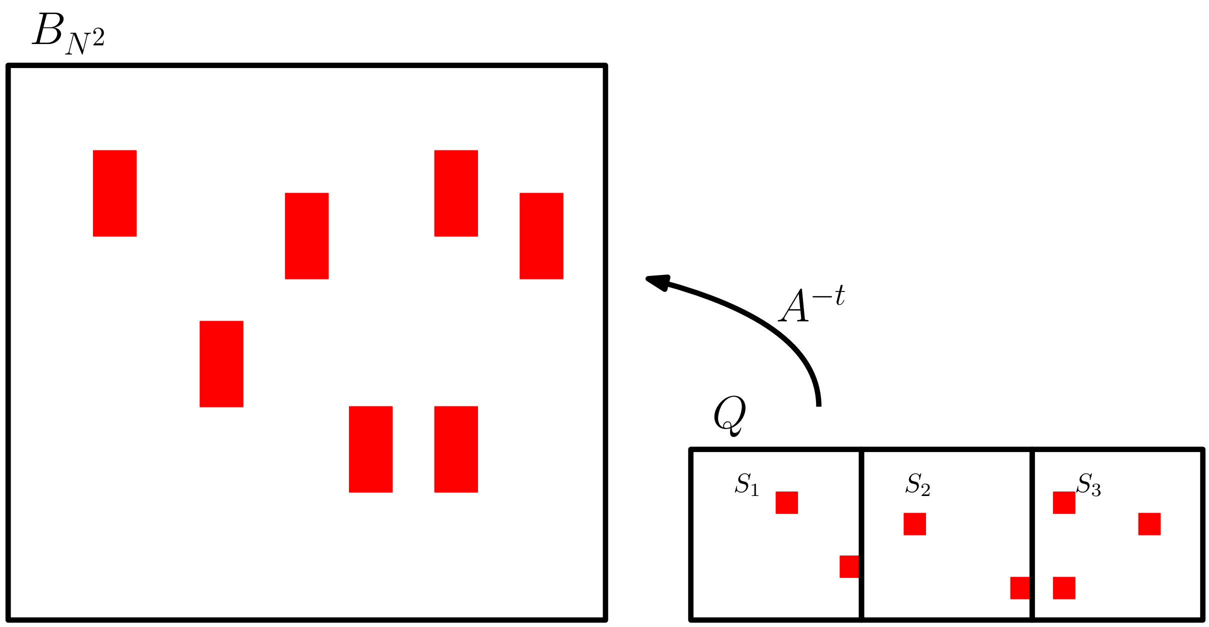

Figure 1.1: Parabolic rescaling in the proof of decoupling

where each $\|h_{\tilde\theta}\|_{L^\infty(\mathbb{R}^2)} \le 1$, each $\tilde\theta$ is a box of dimensions $N^{-1/2} \times N^{-1}$, and the $\tilde\theta$ cover the N^{-1}-neighborhood of the parabola.

To see this unfold, suppose for simplicity that $\tau = [0, N^{-1/2}] \times [0, N^{-1}]$, and consider the linear transformation

$$\xi = A\eta = (N^{-1/2}\eta_1, N^{-1}\eta_2).$$

The map A sends $[0,1]^2$ to the box τ, and the preimage of each $\theta \subset \tau$ is a box $\tilde\theta$ of dimensions $N^{-1/2} \times N^{-1}$, all of which together cover the N^{-1}-neighborhood of P^1. Consider the function

$$\widehat{h}(\eta) = |A|\widehat{f_\tau}(A\eta),$$

where $|A| = N^{-3/2}$ is the determinant of A. By definition and Fourier inversion,

$$h(x) = f_\tau(A^{-t}x) = \sum_{\theta\subset\tau} f_\theta(A^{-t}x) = \sum_{\tilde\theta} h_{\tilde\theta}(x).$$

where A^{-t} is the inverse transpose of A. (In this case $A^{-t} = A^{-1}$, but generally A will be an affine transformation, and the actual formulas involved are slightly more complex.) Now, if $A^{-t}x \in B_{N^2}$, then $x \in A(B_{N^2})$, which is a box Q of dimensions $N^{3/2} \times N$. By the locally constant property of f_τ, the level set $U_{\frac{N_\tau}{10}}(f_\tau) \cap B_{N^2}$ is a union of $N^{1/2} \times N$-tiles contained in B_{N^2}. These tiles correspond to 1×1-squares in Q, which together comprise the level set $U_{\frac{N_\tau}{10}}(h) \cap Q$. We may divide Q into $N^{1/2}$-many $N \times N$-boxes S, each of which contains some of the unit squares of $U_{\frac{N_\tau}{20}}(h) \cap Q$—see Figure 1.1. Since each square S is an $N \times N$-square, by our induction hypothesis,

$$|U_{\frac{N_\tau}{10}}(h) \cap S| \le C_\epsilon N_\tau^{1+\epsilon} = C_\epsilon N^{\frac{1}{2}+\frac{\epsilon}{2}}.$$

Therefore,

$$|U_{\frac{N_\tau}{10}}(h) \cap Q| \le C_\epsilon N^{1+\frac{\epsilon}{2}}. \tag{1.3}$$

Translating this back to a statement about f_τ, Equation (1.3) says there are at most $C_\epsilon N^{1+\frac{\epsilon}{2}}$-many *"heavy"* $N^{1/2} \times N$-tiles in $U_{\frac{N_\tau}{10}}(f_\tau) \cap B_{N^2}$. We can divide B_{N^2} into N^2-many N-balls B_N. Say such a ball B_N is *rich* if

$$\#\{\tau : U_{\frac{N_\tau}{20}}(f_\tau) \cap B_N \neq \emptyset\} > \frac{1}{1000} N^{1/2}.$$

If $x \in U_{\frac{N}{10}}(f) \cap B_N$, then B_N is rich because

$$\begin{aligned} \frac{N}{10} < |f(x)| &\leq \sum_{\tau : U_{\frac{N_\tau}{20}}(f_\tau) \cap B_N = \emptyset} |f_\tau(x)| + \sum_{\tau : U_{\frac{N_\tau}{20}}(f_\tau) \cap B_N \neq \emptyset} |f_\tau(x)| \\ &\leq \frac{N}{20} + N^{1/2} \cdot \#\{\tau : U_{\frac{N_\tau}{20}}(f_\tau) \cap B_N \neq \emptyset\}. \end{aligned}$$

Rearranging shows that B_N is rich. How many rich balls are there? Induction on scales in the form of Equation (1.3) tells us that for each τ, there are at most $N^{1+\epsilon/2}$-many heavy wave-packet tiles of f_τ in B_{N^2}, but at this point we haven't uncovered whether those heavy tiles are spread out, or whether they are packed close together.

This is a question about the shape of the level set $U_{\frac{N_\tau}{20}}(f_\tau)$. To address it, we will use local L^2-orthogonality and transversality. Within a fixed B_N, the wave packets of f_τ are parallel and essentially disjoint so if $H_\tau(B_N)$ is the number of wave packets of f_τ in B_N such that $|f_\tau| \geq \frac{N_\tau}{20}$ on B_N, we have

$$N_\tau^2 \cdot H_\tau(B_N) \cdot |\text{tile}| \lesssim \int_{B_N} |f_\tau|^2. \tag{1.4}$$

By local L^2-orthogonality of the f_θ, we have

$$\int_{B_N} |f_\tau|^2 \lesssim \sum_{\theta \subset \tau} \int_{B_N} |f_\theta|^2 \leq \#(\theta \subset \tau)|B_N| \tag{1.5}$$

Strictly speaking, there should be a weight in the second integral in (1.5) which is ~ 1 on B_N and decaying quickly away from B_N, but it is safe to ignore. Combining inequalities (1.4) and (1.5) yields $H_\tau(B_N) \lesssim 1$.

We have learned geometric information about the level set $U_{\frac{N_\tau}{20}}(f_\tau)$: for each τ, there are at most about 1 heavy wave-packet tiles in a ball B_N. Let $\mu(B_N)$ be the number of τ such that $U_{\frac{N_\tau}{20}}(f_\tau) \cap B_N \neq \emptyset$. Recall that B_N is rich if $\mu(B_N) > \frac{1}{1000} N^{1/2} \sim \#\tau$. Thus,

$$\#\tau \cdot \#\text{rich } B_N \lesssim \sum_{B_N \subset B_{N^2}} \mu(B_N),$$

and each τ contributes at most ~ 1 heavy wave-packet tiles to B_N. When we combine this geometric information about the shape of the level sets of the f_τ we got from Equation (1.5) with what we learned from induction on scales in Equation (1.3)—that there are only at most $N^{1+\epsilon/2}$-many heavy tiles from f_τ in B_{N^2}—we get

$$\sum_{B_N \subset B_{N^2}} \mu(B_N) \lesssim (\#\tau) \cdot N^{1+\epsilon/2}.$$

Therefore,

$$\#\text{rich } B_N \lesssim N^{1+\epsilon/2}.$$

Finally, if a ball B_N is not rich, $U_{\frac{N}{10}}(f) \cap B_N = \emptyset$, so

$$|U_{\frac{N}{10}}(f)| = \sum_{\substack{B_N \subset B_{N^2} \\ B_N \text{ rich}}} |U_{\frac{N}{10}}(f) \cap B_N|.$$

We will be done if we can show that $|U_{\frac{N}{10}}(f) \cap B_N| \lesssim N^{\epsilon/2}$ for each rich ball B_N. This will be a consequence of the multilinear Kakeya inequality, which says that tubes pointing in different directions overlap in a set of significantly smaller size than the union of the tubes (see Guth's article [8] for a clean statement and proof).

Since each rich ball contains many $N^{1/2} \times N$ heavy wave packets of the various f_τ, at most one for each of the different τ, the level set $U_{\frac{N}{10}}(f) \cap B_N$ is contained in $O(1)$ balls of radius $N^{1/2}$, by the multilinear Kakeya inequality. Combined with the fact that there are only $O(N^{1+\epsilon/2})$ many rich N-balls, we see that $U_{\frac{N}{10}}(f) \cap B_{N^2}$ is contained in $O(N^{1+\epsilon/2})$-many $N^{1/2}$-balls.

We study f on each of the $N^{1+\epsilon/2}$-many $N^{1/2}$-balls, decomposing $f = \sum_\sigma f_\sigma$ for some tiles σ of dimensions $N^{-1/4} \times N^{-1/2}$ covering the $N^{-1/2}$-neighborhood of the parabola. Analogous considerations using the multilinear Kakeya inequality show that $U_{\frac{N}{10}}(f) \cap B_{N^2}$ is contained in $O(N^{1+\epsilon/2})$-many $N^{1/4}$-balls. Continuing this reasoning until the diameter of the balls is $N^{1/2^k} < N^{\epsilon/4}$ (say), we see that the level set $U_{\frac{N}{10}}(f) \cap B_{N^2}$ is contained in $C_\epsilon N^{1+\epsilon}$-many unit balls, which finishes the sketch of the proof.

1.4 Refined decoupling

Bourgain–Demeter's ℓ^2-decoupling theorem is a remarkable inequality with numerous applications to PDE, geometry, combinatorics, and Fourier analysis. Decoupling is sharp, as the exponential sum Example 1.2 shows, but there are also situations where decoupling does not provide the best possible estimate. For instance, if the supports of the various f_θ pieces were disjoint, we have the much stronger (and trivial) inequality $\|f\|_p^p = \sum_\theta \|f_\theta\|_p^p$. We will sketch another less trivial scenario where decoupling does not provide the best estimate in the next Example.

Example 1.3. *Let $f = \sum_T a_T f_T$ be a sum of $R^{1/2} \times R$ wave packets in the plane, one for each direction, where each $|a_T| \sim 1$, and* supp $\widehat{f}$ *is contained in the R^{-1}-neighborhood of the parabola. We think of each f_T as the indicator function 1_T of the tube T. The region of near-maximal overlap of the tubes T is contained in $O(1)$-many $R^{1/2}$ balls, by the multilinear Kakeya inequality, and it turns out that this overlap region dominates the integral. We choose the coefficients (randomly if necessary) so that on each such ball, we have square-root cancellation, i.e. $|f| \sim R^{1/4}$. Therefore, if X is the union of these $R^{1/2}$-balls,*

$$\int |f|^6 \sim \int_X R^{6/4} \sim R^{6/4} \cdot R.$$

Decoupling says that $\int |f|^6$ *is bounded by* $R^\epsilon(\sum_T \|f_T\|_{L^6}^2)^{6/2} \sim R^{\epsilon+1}\sum_T \|f_T\|_{L^6}^6$, *but*

$$R^{\epsilon+1}\sum_T \|f_T\|_{L^6}^6 \sim R^{\epsilon+1}\cdot R^{1/2}\cdot R^{3/2} \approx R^3.$$

In Example 1.3, the supports of the various wave packets f_θ are localized and mostly disjoint outside a union of a few $R^{1/2}$-balls. In recent years, some geometric problems such as Falconer's distance set problem [9]–[11], and restricted projection problems [12] have found applications of *refined* decoupling theorems, which improve the statement of decoupling when we have more information about the wave packets of Ef. Intuitively speaking, refined decoupling theorems prove a gain over what a sharp non-refined decoupling theorem says in a scenario where the wave packet supports are "sparse."

In the next Theorem, we will state the version of refined decoupling for the cone we will use to compare with our results in Chapter 3. This Theorem appears in Terence Harris' paper on the length of sets under restricted projections to lines [13], where he adapts it from Gan–Guo–Guth–Harris–Maldague–Wang's study of restricted projections to planes in $\mathbb{R}^3$ [12].

Theorem 1.4 (Refined decoupling for cones, stated in [13])**.** *Let I be a compact interval, and let $\gamma\colon I \to S^2$ be a C^2 unit -speed curve with $\det(\gamma,\gamma',\gamma'')$ nonvanishing on I. If $c > 0$ is sufficiently small (depending only on γ), then for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that the following holds for all $0 < \delta < \delta_0$, and any $R \geq 1$. Let Θ_R be a maximal $cR^{-1/2}$-separated subset of I, and for each $\theta \in \Theta_R$, let*

$$\tau(\theta) = \{\lambda_1\gamma(\theta) + \lambda_2\gamma'(\theta) + \lambda_3(\gamma\times\gamma')(\theta) : \frac{1}{2} \leq \lambda_1 \leq 1, |\lambda_2| \leq R^{-1/2}, |\lambda_3| \leq R^{-1}\}.$$

For each $\tau = \tau(\theta)$, let $\mathbb{T}_\tau$ be a finitely overlapping cover of $\mathbb{R}^3$ by translates of

$$\{\lambda_1\gamma(\theta) + \lambda_2\gamma'(\theta) + \lambda_3(\gamma\times\gamma')(\theta) : |\lambda_1| \leq R^\delta, |\lambda_2| \leq R^{1/2+\delta}, |\lambda_3| \leq R^{1+\delta}\}.$$

Suppose $2 \leq p \leq 6$, and

$$\mathbb{W} \subset \bigcup_{\theta\in\Theta_R} \mathbb{T}_{\tau(\theta)},$$

and

$$\sum_{T\in\mathbb{W}} f_T$$

is such that $\|f_T\|_p$ is constant over $T \in \mathbb{W}$ up to a factor of 2, with $\operatorname{supp}\widehat{f_T} \subset \tau(T)$. If

$$\|f_T\|_{L^\infty(B(0,R)\setminus T)} \leq AR^{-10000}\|f_T\|_p,$$

and X is a disjoint union of balls in $B(0,R)$ of radius 1, such that each ball $Q \subset X$ intersects at most M planks $2T$ with $T \in \mathbb{W}$, then

$$\|\sum_{T\in\mathbb{W}} f_T\|_{L^p(X)} \lesssim_{A,\gamma,c,\epsilon,\delta} R^\epsilon\left(\frac{M}{|\mathbb{W}|}\right)^{\frac{1}{2}-\frac{1}{p}}\left(\sum_{T\in\mathbb{W}}\|f_T\|_p^2\right)^{1/2}.$$

The example we will focus on later is the case where $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$, in which case the implicit "cone" in Theorem 1.4 is the usual lightcone segment

$$\mathbb{C}one^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3^2 = x_1^2 + x_2^2, 1/2 < x_1^2 + x_2^2 < 1\}.$$

1.5 Main results and the Mizohata–Takeuchi conjecture

To motivate the main result of this thesis, Theorem 3.1, we introduce the open Mizohata–Takeuchi conjecture of restriction theory. Local and global $L^p(\mathcal{M}) \to L^q$ estimates give us a measure of the size of level sets $\{|E_{\mathcal{M}}f| > \lambda\}$. To test the shape of level sets, we can consider weighted inequalities of the following form:

$$\int |E_{\mathcal{M}}f|^2 w \leq C\|f\|^2_{L^2(\mathcal{M})}. \tag{1.6}$$

If $\mathcal{M}$ is a smooth compact hypersurface in $\mathbb{R}^n$, $n \geq 2$, then the following conjecture is an open problem in Fourier analysis. If T is a tube in $\mathbb{R}^n$, then $\nu(T)$ is one of the two unit vectors parallel to the central axis of T. If a vector v is orthogonal to $T_x\mathcal{M}$ for some point $x \in \mathcal{M}$, we say $v \perp \mathcal{M}$.

Conjecture 1.2 (Mizohata–Takeuchi)**.** *For every $\epsilon > 0$, there is a constant C_ϵ such that the following holds for every $R > 1$. If $\mathcal{M} \subset \mathbb{R}^n$ is a smooth compact hypersurface, $n \geq 2$, let*

$$\mathbb{T}(\mathcal{M}) = \{T \subset \mathbb{R}^n : \nu(T) \perp \mathcal{M},\ T \textit{ is a } 1 \times \cdots \times 1 \times R\textit{-tube}\}$$

be the collection of $1 \times \cdots \times 1 \times R$-tubes whose direction is orthogonal to a tangent space of $\mathcal{M}$. If $X \subset B^n(0, R)$ is a disjoint union of unit balls, let

$$\mathbf{T}_{\mathcal{M}}(X) = \sup\{|X \cap T| : T \in \mathbb{T}(\mathcal{M})\}$$

be the "maximal tube occupancy of X" by tubes orthogonal to the hypersurface $\mathcal{M}$. Then the following estimate holds:

$$\int_X |E_{\mathcal{M}}f|^2 \leq C_\epsilon R^\epsilon\, \mathbf{T}_{\mathcal{M}}(X)\|f\|^2_{L^2(\mathcal{M})}. \tag{1.7}$$

It's important to underline that Conjecture 1.2 is made without any assumptions on the curvature of $\mathcal{M}$, unlike the restriction Conjecture 1.1. To give some sense for the conjecture when $\mathcal{M}$ is a sphere in $\mathbb{R}^n$, (1.7) holds if X is the 1-neighborhood of an affine subspace of $\mathbb{R}^n$. See the article [14] and the references therein for more discussion on Mizohata–Takeuchi-type estimates for spheres and the with the Bochner–Riesz problem on spherical Fourier summation. Section 3.3 of this thesis outlines the state of the art on MT for spheres due to Carbery–Iliopoulou–Wang [15] based on refined decoupling estimates (see Theorem 1.4), and makes an analogy between our main Theorem 3.1 for $\mathbb{C}one^2$ and Carbery–Iliopoulou–Wang's result, Theorem 3.11, for the circle S^1.

Part of what makes Conjecture 1.2 difficult to prove is that certain methods of restriction theory based on wave packet analysis cannot break through a quantitative barrier, due to an *enemy scenario* of Guth [16]. In Chapter 3, we will prove the following Theorem, which appeared in the authors work [17]. This result makes progress past this barrier on a special case of the Mizohata–Takeuchi-type problem for $\mathbb{C}one^2$ when the set X is 1-dimensional. It implies inequality (3.5) for $\mathbb{C}one^2$ and the 1-dimensional sets X with a power $R^{1/4}$-loss.

Theorem 1.5 (Ortiz, 2023)**.** *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose $X \subset B_R$ is a disjoint union of unit balls that satisfies the 1-dimensional Frostman condition*

$$|X \cap B(x,r)| \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(X)$ be the quantity

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then the estimate

$$\int_X |E_{\mathbb{C}one^2} f|^2 \le C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/2} \|f\|^2_{L^2(\mathbb{C}one^2)}$$

holds.

See Definition 2.8 in Chapter 2 for the definition of $1 \times R^{1/2} \times R$-lightplanks. Theorem 1.5 builds on an approach originally due to Mattila [18] to study the decay of Fourier averages of measures in $\mathbb{R}^n$ with finite α-energy:

$$I_\alpha(\mu) = \iint \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha}, \quad 0 < \alpha < n.$$

Briefly, by duality, an estimate of the form (1.6) is related to a L^2 average of $\widehat{1}_X$ over $\mathcal{M}$. See Section 3.2 for a discussion of the connection between weighted Fourier extension estimates of type (1.6) and the problem of determining the optimal rate of decay of Fourier averages in detail. Mattila's idea was to use the decay rate from of $\check{\sigma}_{S^{n-1}}$ Proposition 1.1 to prove power-law decay for $\int_{RS^{n-1}} |\widehat{1}_X|^2 \, d\sigma$.

In Theorem 1.5, we follow the duality approach. When σ is the smooth surface measure for $\mathbb{C}one^2$, $|\check{\sigma}(x)| \le C/(1+|x|)^{1/2}$, following the same numerology as that of the arclength measure on the circle in $\mathbb{R}^2$, but it also obeys another principle of decay we can use:

$$|\check{\sigma}(x)| \le C(\epsilon, N) \frac{1}{(1+|x|)^{1/2-\epsilon}} \frac{1}{(1+\operatorname{dist}(x,\Gamma_0))^N}, \quad x \in \mathbb{R}^3 \tag{1.8}$$

where $\Gamma_0 = \{(a,r) \in \mathbb{R}^2 \times \mathbb{R} : ||a| - r| = 0\}$ is the lightcone in $\mathbb{R}^3$ with vertex 0. By Fourier transform properties,

$$\int_{\mathbb{C}one^2} |\widehat{1}_X|^2 \, d\sigma = \iint_{X \times X} \check{\sigma}(x-y) \, dx \, dy.$$

By (1.8), we can pretend that the only $(x,y) \in X^2$ which contribute to this integral are those satisfying $d(x-y, \Gamma_0) = 0$:

$$0 = d(x-y, \Gamma_0) = ||x'-y'| - |x_3 - y_3|| \equiv \Delta(x,y).$$

The equation $\Delta(x,y) = 0$ has a separate interpretation when we regard x and y as circles in the plane with center-radius pairs given by $(x', x_3), (y', y_3) \in \mathbb{R}^2 \times \mathbb{R}_+$. The condition $\Delta(x,y) = 0$ implies that x and y—thought of as circles in the plane—are internally tangent. Tom Wolff studied a related problem for $E_{\mathbb{C}one^2} f$ in [4], and he had a creative idea of how to use this association between points and circles. In Chapter 2 we discuss this correspondence between points and circles, and explain Wolff's idea to use circular maximal function estimates (see Theorem 2.1) to produce information about the level sets of $E_{\mathbb{C}one^2} f$ in Section 2.3.

Chapter 2

Points, circles, and cones

2.1 Circular maximal function estimates and examples

In [19], Wolff made a study of the following circular maximal function.

Definition 2.1 (Wolff circular maximal function)**.** *For $\delta > 0$, $a \in \mathbb{R}^2$ and $1 < r < 2$, let*

$$C_{\delta,a,r} = \{z \in \mathbb{R}^2 : ||z-a| - r| < \delta\}$$

be the δ-thick annulus of center a and central radius r. For a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, define the Wolff circular maximal function

$$W_\delta f(r) = \sup_{a\in\mathbb{R}^2} \frac{1}{|C_{\delta,a,r}|} \int_{C_{\delta,a,r}} |f|, \quad 1 < r < 2.$$

By definition, $\|W_\delta f\|_{L^\infty([1,2])} \leq \|f\|_{L^\infty(\mathbb{R}^2)}$, and the general question we are interested in regarding W_δ is the range of exponents $p < \infty$ such that $\|W_\delta f\|_{L^p([1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$.

Example 2.1 (Clamshell, version 1)**.** *Let $\Omega = [0,\delta] \times [0,\sqrt{\delta}]$ be a $\delta \times \sqrt{\delta}$-rectangle in the plane.* Claim: *for each $a = (a_1, a_2) \in [1,2] \times [0,\sqrt{\delta}]$,*

$$\Omega \subset C_{100\delta,a,a_1}.$$

Given the claim, $W_\delta 1_\Omega(r) \gtrsim \frac{|\Omega|}{\delta}$ for each $r \in [1,2]$. Consequently, if $\|W_\delta f\|_{L^p([1,2])} \lesssim \|f\|_{L^p}$ holds for all f, we must have

$$\delta^{p/2} = (\frac{|\Omega|}{\delta})^p \lesssim \int_1^2 W_\delta 1_\Omega(r)^p\, dr \lesssim |\Omega| = \delta^{3/2}.$$

Since $\delta > 0$ can be arbitrarily small, we must have $p \geq 3$.

Proof of the claim. $\Omega \subset C_{100\delta,a,r}$ *if and only if for all $(x,y) \in \Omega$,*

$$|(x-a_1)^2 + (y-a_2)^2 - r^2| < 100\delta.$$

Expanding the left-hand side, we have

$$x^2 + y^2 - 2xa_1 - 2ya_2 + a_1^2 + a_2^2 - r^2.$$

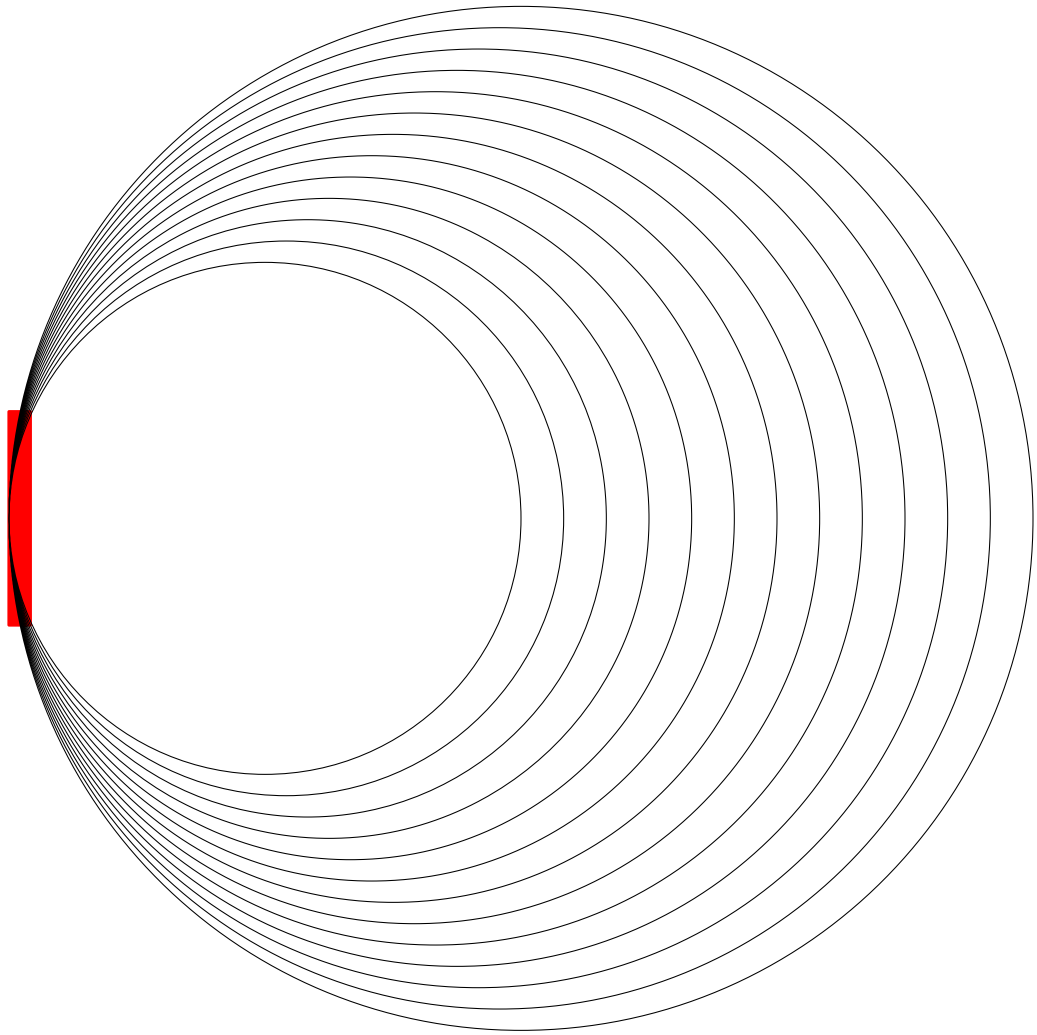

Figure 2.1: A clamshell of circles all tangent to a common $\delta \times \sqrt{\delta}$-rectangle

Plugging in $(x, y) \in \Omega$, *so* $x = O(\delta)$ *and* $y = O(\sqrt{\delta})$ *we have*

$$O(\delta^2) + O(\delta) + O(\delta)a_1 + O(\sqrt{\delta})a_2 + a_1^2 + a_2^2 - r^2.$$

In particular, taking $a_1 \in [1, 2]$, $a_2 = O(\sqrt{\delta})$, *and* $r = a_1$ *we can make the left-hand side smaller than say* 50δ.

Wolff's circular maximal function estimate has an important dual formulation. To discover it, we set up the integral

$$\int_1^2 W_\delta f(r) w(r)\, dr$$

for $w \in L^{p'}([1, 2])$ with $\|w\|_{L^{p'}([1,2])} = 1$. For each r, let $a(r) \in \mathbb{R}^2$ be a (measurable) choice of center so that $W_\delta f(r) = \frac{1}{|C_{\delta,a(r),r}|} \int_{C_{\delta,a(r),r}} |f|$. Plugging this in and changing the order of integration, we get

$$\int_{\mathbb{R}^2} f(z) \int_1^2 w(r) \frac{C_{\delta,a(r),r}(z)}{|C_{\delta,a(r),r}|}\, dr\, dz.$$

Let $g[w,a](z) = \int_1^2 w(r)\frac{C_{\delta,a(r),r}(z)}{|C_{\delta,a(r),r}|}\,dr$. By Hölder's inequality,

$$|\int f(z)g(z)\,dz| \le \|f\|_{L^p(\mathbb{R}^2)}\|g\|_{L^{p'}(\mathbb{R}^2)}.$$

Therefore, the dual form of Wolff's circular maximal estimate takes the form

$$\|g\|_{L^{p'}(\mathbb{R}^2)} \lesssim \|w\|_{L^{p'}([1,2])}.$$

To explore this form, we revisit the clamshell example.

Example 2.2 (Clamshell, version 2). *Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Choose a 10δ-net $\{r_i\} \subset [1,2]$, set $w(r) = 1_{\bigcup_i [r_i, r_i+\delta]}(r)$, and let*

$$a(r) = \sum_i r_i 1_{[r_i,r_i+\delta]}(r) e_1.$$

By definition,

$$g[w,a](z) \sim \frac{1}{\delta}\sum_i \int_{[r_i,r_i+\delta]} C_{\delta,r_ie_1,r}(z)\,dr \sim \sum_i C_{\delta,r_ie_1,r_i}(z).$$

Each of the annuli C_{δ,r_ie_1,r_i} contains the rectangle $\Omega = [0,\delta/100]\times[0,\sqrt{\delta}/100]$, and there are $\sim \delta^{-1}$ annuli, so if $\|g\|_{L^{p'}(\mathbb{R}^2)} \lesssim \|w\|_{L^{p'}([1,2])}$ holds for all w,

$$\delta^{-p'}|\Omega| \lesssim \int g(z)^{p'}\,dz \lesssim 1.$$

Rearranging with $|\Omega| = \delta^{3/2}$, the inequality says

$$\delta^{-p'} \lesssim \delta^{-3/2}.$$

Since this must hold for all $\delta > 0$, it implies $p' \le 3/2$.

In [19], Wolff proved that his circular maximal function is bounded for all $p \ge 3$, with a $\delta^{-\epsilon}$-loss.

Theorem 2.1 (Wolff, [19]). *For all $\delta > 0$, and $\epsilon > 0$, the estimate*

$$\|W_\delta f\|_{L^3([1,2])} \lesssim_\epsilon \delta^{-\epsilon}\|f\|_{L^3(\mathbb{R}^2)}$$

holds.

Example 2.3 (δ-separated radii). *Let $\{r_i\}_{i=1}^N \subset [1,2]$ be 10δ-separated set of radii (not necessarily maximal). For real numbers b_i, set*

$$w(r) = \sum_i b_i 1_{[r_i,r_i+\delta]}(r),$$

and for each i*, let* $a_i \in \mathbb{R}^2$ *be an arbitrary point. Set*

$$a(r) = \sum_i a_i 1_{[r_i, r_i+\delta]}(r).$$

By definition,

$$g[w,a](z) \approx \sum_i b_i C_{\delta,a_i,r_i}(z),$$

hence the dual form of Wolff's circular maximal estimate implies

$$\int (\sum_{i=1}^N b_i C_{\delta,a_i,r_i}(z))^{3/2}\, dz \lesssim \delta^{-\epsilon}\delta \sum_{i=1}^N |b_i|^{3/2}.$$

As a special case, setting each $b_i = 1$ *we get*

$$\int (\sum_{i=1}^N C_{\delta,a_i,r_i}(z))^{3/2}\, dz \lesssim \delta^{-\epsilon}\delta N.$$

The two forms of Wolff's circular maximal estimate are equivalent and they imply one another. We present the proof of this fact in the following Proposition.

Proposition 2.1. *Let* $A > 0$*. Then for every* $f(z)$*,*

$$\|W_\delta f\|_{L^p([1,2],dr)} \leq A\|f\|_{L^p(\mathbb{R}^2,dz)} \tag{2.1}$$

if and only if for every $w(r)$ *and* $a(r)$*,*

$$\|g[w,a]\|_{L^{p'}(\mathbb{R}^2,dz)} \leq A\|w\|_{L^{p'}([1,2],dr)}. \tag{2.2}$$

Proof. Suppose that (2.1) holds, and let $g = g[w,a]$. By duality, for an appropriate $f \in L^p(\mathbb{R}^2, dz)$ with $\|f\|_p = 1$,

$$\begin{aligned}
\|g\|_{L^{p'}(\mathbb{R}^2,dz)} &= \int_{\mathbb{R}^2} g(z) f(z)\, dz \\
&= \int_{\mathbb{R}^2} \left(\int_1^2 w(r) \frac{C_{\delta,a(r),r}}{|C_{\delta,a(r),r}|}(z)\, dr \right) f(z)\, dz \\
&= \int_1^2 w(r) \left(\frac{1}{|C_{\delta,a(r),r}|} \int_{C_{\delta,a(r),r}} f(x)\, dz \right) dr \\
&\leq \int_1^2 w(r) W_\delta f(r)\, dr \\
&\leq \|w\|_{L^{p'}([1,2],dr)} \|W_\delta f\|_{L^p([1,2],dr)} \\
&\leq A\|w\|_{L^{p'}([1,2],dr)}.
\end{aligned}$$

Likewise, if (2.2) holds, then by linearizing the maximal function, given $f \in L^p(\mathbb{R}^2, dz)$, for an appropriate $a(r)$ we have

$$M_\delta f(r) = \frac{1}{|C_{\delta,a(r),r}|} \int_{C_{\delta,a(r),r}} |f(z)|\, dz.$$

By duality, for an appropriate $w \in L^{p'}([1,2], dr)$ with $\|w\|_{p'} = 1$,

$$\begin{aligned}
\|W_\delta f\|_{L^p([1,2],dr)} &= \int_1^2 W_\delta f(r) w(r)\, dr \\
&= \int_1^2 \left(\frac{1}{|C_{\delta,a(r),r}|} \int_{C_{\delta,a(r),r}} |f(z)|\, dz \right) w(r)\, dr \\
&= \int_{\mathbb{R}^2} |f(z)| \left(\int_1^2 w(r) \frac{C_{\delta,a(r),r}}{|C_{\delta,a(r),r}|}(z)\, dr \right) dz \\
&\le \|f\|_{L^p(\mathbb{R}^2,dz)} \|g\|_{L^{p'}(\mathbb{R}^2,dz)} \\
&\le A\|f\|_{L^p(\mathbb{R}^2,dz)}.
\end{aligned}$$

□

Wolff's circular maximal function estimate implies that the dimension of certain Kakeya sets of circles have maximal dimension.

Definition 2.2. *A compact subset $E \subset \mathbb{R}^2$ is a* Besicovitch–Rado–Kinney set of circles *(or just a* BRK set of circles*) if for each $r \in [1,2]$, E contains a circle of radius r.*

As Wolff notes in his survey on then-recent work on the Kakeya problem [20], since every circle is a 1-dimensional curve, and a BRK set of circles E contains a circle for each $r \in [1,2]$, a 1-dimensional set of parameters, deciding whether $\dim E = 2$ can be seen as a borderline question. The following argument is based on Lemma 1.6 of [20].

Proposition 2.2. *If $\|W_\delta f\|_{L^p([1,2])} \lesssim \delta^{-\alpha}\|f\|_{L^p(\mathbb{R}^2)}$ holds for some $p < \infty$, then the dimension of every BRK set of circles is at least $2 - \alpha p$. In particular, if α can be taken arbitrarily small, the dimension of every BRK set of circles is 2.*

Proof. Let E be a BRK set of circles, and for each $r \in [1,2]$, let $C_{a(r),r} \subset E$ be a circle with center $a(r)$ and radius r. Let $s = 2 - \alpha p - \epsilon$. We have to bound $\mathcal{H}^s(E)$ from below, so let $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ be a cover of E by disks of radii $r_j \le 1$.

Let $J_k = \{j : r_j \sim 2^{-k}\}$, $\nu_k = |J_k|$, and $E_k = E \cap \bigcup_{j \in J_k} B_j$. Let $\widetilde{B}_j = B(x_j, 2r_j)$ and $\widetilde{E}_k = \bigcup_{j \in J_k} \widetilde{B}_j$. Since $E = \bigcup_k E_k$, for each $r \in [1,2]$, the pigeonhole principle implies that $\mathcal{H}^1(C_{a(r),r} \cap E_k) \gtrsim k^{-2}$ for some $k = k(r)$. For each $r \in [1,2]$, the value of $k(r)$ can be different, but by the pigeonhole principle again, there is a value of k and a subset $S \subset [1,2]$ such that $|S| \gtrsim k^{-2}$, and for each $r \in S$, $k(r) = k$.

For this value of k we found, we note that $\widetilde{E}_k$ contains a disk of radius 2^{-k} centered at every point of E_k, from which it follows that if $r \in S$, then $|C_{2^{-k},a(r),r} \cap \widetilde{E}_k| \gtrsim k^{-2}|C_{2^{-k},a(r),r}|$. With $f = 1_{\widetilde{E}_k}$, we therefore have

$$|\{r \in [1,2] : W_{2^{-k}} f(r) \gtrsim k^{-2}\}| \gtrsim k^{-2}.$$

On the other hand, the maximal estimate we are assuming implies

$$k^{-2p}|\{r \in [1,2] : W_{2^{-k}} f(r) \gtrsim k^{-2}\}| \lesssim 2^{\alpha p k}|\widetilde{E}_k|.$$

By the union bound, $|\widetilde{E}_k| \lesssim 2^{-2k}\nu_k$, so combining these estimates gives a lower bound for ν_k:

$$\nu_k \gtrsim 2^{(2-\alpha p)k} k^{-2(1+p)} = 2^{\epsilon k} 2^{sk} k^{-2(1+p)}.$$

Therefore,

$$\sum_j r_j^s \gtrsim 2^{-sk}\nu_k \gtrsim 2^{\epsilon k} k^{-2(1+p)} \gtrsim 1.$$

□

An inspection of the proof shows that if $p < \infty$ and $\|W_\delta f\|_{L^p([1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$ with no loss which tends to infinity as δ tends to zero, then in fact the Lebesgue outer measure of a BRK set must be positive. However, due to the existence of measure zero BRK sets [21], [22], some growing loss is necessary.

Another circular maximal function which is a natural counterpart to Wolff's was studied by Jean Bourgain [23].

Definition 2.3. *For a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $\delta > 0$, define the* Bourgain circular maximal function

$$B_\delta f(a) = \sup_{1<r<2} \frac{1}{|C_{\delta,a,r}|} \int_{C_{\delta,a,r}} |f|, \quad a \in \mathbb{R}^2.$$

Bourgain's circular maximal function is a natural relative of Wolff's circular maximal function, where we consider for a fixed center the radius which gives the maximal circular average. Like Wolff's circular maximal function, $\|B_\delta f\|_{L^\infty(\mathbb{R}^2)} \leq \|f\|_{L^\infty(\mathbb{R}^2)}$, so we are interested in the range of $p < \infty$ such that $\|B_\delta f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$.

Example 2.4 (Lightplank, version 1)**.** *Let $\Omega = [0,\delta] \times [0,\sqrt{\delta}]$. As we saw in Example 2.1, for each $a \in [1,2] \times [0,\sqrt{\delta}]$, we know that*

$$\Omega \subset C_{100\delta,a,a_1},$$

so $B_\delta 1_\Omega(a) \gtrsim \frac{|\Omega|}{\delta}$ for each $a \in [1,2] \times [0,\sqrt{\delta}]$. Therefore, if $\|B_\delta f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$ holds for all f, we must have

$$\delta^{1/2}\left(\frac{|\Omega|}{\delta}\right)^p \lesssim \int_{[1,2]\times[0,\sqrt{\delta}]} B_\delta 1_\Omega(a)^p \, da \lesssim |\Omega| = \delta^{3/2}.$$

Rearranging this inequality and using $|\Omega| = \delta^{3/2}$, we have

$$\delta^p \lesssim \delta^2.$$

Letting $\delta \to 0$ gives $p \geq 2$.

In [23], Bourgain proved that his maximal function is bounded on $L^2(\mathbb{R}^2)$ with a $\delta^{-\epsilon}$-loss.

Theorem 2.2 (Bourgain, [23])**.** *For all $\delta > 0$, and $\epsilon > 0$, the estimate*

$$\|B_\delta f\|_{L^2(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}.$$

When we count parameters as we did for BRK sets, if E is a compact set in the plane which contains a circle (a 1-dimensional set) centered at every point in $[0,1]^2$, a 2-dimensional set of parameters, we might expect it to be possible for there to exist such Kakeya sets of circles with positive measure. In fact, every such Kakeya set must have positive measure, a fact due independently to Bourgain [23] and Marstrand [24].

Despite the fact that every such Kakeya set of circles has positive measure, some growing loss in Bourgain's maximal function estimate is unavoidable, as examples show [25].

Like Wolff's circular maximal estimate, it is possible to express Bourgain's maximal estimate in a dual form. As before, we set up the integral

$$\int B_\delta f(a) w(a)\, da$$

and linearize the maximal function. For each center a, choose $r(a)$ which maximizes the average of f and change the order of integration:

$$\int B_\delta f(a) w(a)\, da = \int_{\mathbb{R}^2} f(z) \int_{\mathbb{R}^2} w(a) \frac{C_{\delta,a,r(a)}(z)}{|C_{\delta,a,r(a)}|}\, da\, dz.$$

Set $g[w,r](z) = \int_{\mathbb{R}^2} w(a) \frac{C_{\delta,a,r(a)}(z)}{|C_{\delta,a,r(a)}|}\, da$. Then Theorem 2.2 implies the following estimate holds for g:

$$\|g[w,r]\|_{L^2(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-\epsilon} \|w\|_{L^2(\mathbb{R}^2)}.$$

As a special case of this estimate, we revisit the lightplank example.

Example 2.5 (Lightplank, version 2)**.** *Let* $\{a_i\}_{i=1}^N$ *be a* 10δ*-net in the rectangle* $[1,2]\times[0,\sqrt{\delta}]$*, and let* $B_i = B(a_i,\delta)$*. Define*

$$r(a) = \sum_i |a_i| 1_{B_i}(a),$$

and set $w(a) = \sum_i 1_{B_i}(a)$*.*

By definition,

$$g[w,r](z) = \frac{1}{\delta} \sum_i \int_{B_i} C_{\delta,a,|a_i|}(z)\, da \sim \delta \sum_i C_{\delta,a_i,|a_i|}(z).$$

Each of the N*-many* δ*-annuli* $C_{\delta,a_i,|a_i|}$ *contains the rectangle* $\Omega = [0,\delta/100] \times [0,\sqrt{\delta}/100]$*, so*

$$N^2\delta^2|\Omega| \lesssim \int_\Omega g(z)^2\, dz \lesssim_\epsilon \delta^{-\epsilon}\|w\|^2_{L^2(\mathbb{R}^2)} \sim \delta^{-\epsilon} N \delta^2.$$

Since $N \sim \delta^{-3/2}$*, the left-hand side evaluates to* $\delta^{1/2}$*, while the right-hand side is* $\delta^{-\epsilon}\delta^{1/2}$*, so Bourgain's maximal estimate is sharp.*

Example 2.6 (δ-separated centers)**.** *Let* $\{a_i\}_{i=1}^N$ *be a* 10δ*-separated subset of* $[1,2]\times[0,1]$ *(not necessarily maximal) and let* $B_i = B(a_i,\delta)$*. Set* $w(a) = \sum_i 1_{B_i}(a)$*, and for each* i*, let* $1 < r_i < 2$ *be arbitrary. Define*

$$r(a) = \sum_i r_i 1_{B_i}(a).$$

As before, we calculate the multiplicity function $g[w,r](z) \approx \delta \sum_i C_{\delta,a_i,r_i}(z)$. *By Bourgain's maximal function estimate, we have*

$$\int (\sum_{i=1}^{N} C_{\delta,a_i,r_i}(z))^2\, dz \lesssim \delta^{-2}\delta^{-\epsilon}\|w\|^2_{L^2(\mathbb{R}^2)} \sim \delta^{-\epsilon}N.$$

2.2 The tangency rectangle point of view

At this point, we have described Wolff's and Bourgain's circular maximal functions, their dual formulations, as well as examples that illustrate their sharpness. The original proofs of Theorem 2.1 and Theorem 2.2 are very different. However, as pointed out by Schlag in his expository paper on continuum incidence problems in harmonic analysis [26], there is a unified approach to proving both estimates based on counting the number of high-multiplicity *tangency rectangles* of a configuration of circles. In this subsection, we will illustrate this approach by exploring how it applies to Wolff's circular maximal estimate in Theorem 2.1.

Given $X = \{x_i\}_{i=1}^N$, a set of N circles with δ-separated radii in the interval $[1,2]$, we consider the multiplicity function

$$g^X_\delta(z) = \sum_{i=1}^{N} C_{\delta,x_i}(z).$$

The function $g^X_\delta(z)$ counts the number of thin annuli from X which contain the point z. Our first Proposition explains how we can equivalently regard Wolff's circular maximal function estimate as a statement regarding the level sets of g^X_δ for arbitrary X.

Proposition 2.3 (Schlag, [26] Lemma 3)**.** *The following are equivalent:*

(i) Let $\epsilon > 0$ be arbitrary. Then for $\delta > 0$ sufficiently small depending on ϵ, the restricted weak-type estimate

$$\lambda^3|\{r \in [1,2] : W_\delta 1_E(r) > \lambda\}| \le A_\epsilon \delta^{-\epsilon}|E| \tag{2.3}$$

holds for all $E \subset [0,1]^2$ and $0 < \lambda < 1$.

(ii) Given a family of circles X with δ-separated radii, the following holds. For any $\epsilon > 0$ and $\delta > 0$ sufficiently small depending on ϵ, there exists $X' \subset X$ such that $|X'| > \frac{1}{2}|X|$ and for each $x \in X'$ and $0 < \eta < 1$,

$$|\{z \in C_{\delta,x} : g^{X'}_\delta(z) > \delta^{-\epsilon}\eta^{-2}\}| \le \eta|C_{\delta,x'}|. \tag{2.4}$$

(iii) Same statement as in (ii), but with $|X'| > c_\epsilon|X|$ for some small constant c_ϵ.

Proof. Clearly (ii) implies (iii), so we start with showing that (iii) implies (i). Let $0 < \lambda < 1$ be fixed.

Let $F = \{r \in [1,2] : W_\delta 1_E(r) > \lambda\}$, and let $\{r_j\}_{j=1}^M$ be a 10δ-net inside F. Because $W_\delta 1_E$ is approximately constant on scale δ, $|F| \sim M\delta$. For each $j = 1, \ldots, M$, let C_{a_j, r_j} be a circle such that

$$\frac{|E \cap C_{\delta,a_j,r_j}|}{|C_{\delta,a_j,r_j}|} = W_\delta 1_E(r_j) > \lambda. \tag{2.5}$$

Let $X = \{x_j = (a_j, r_j)\}_{j=1}^M$ be the set of circles so obtained. Let $\epsilon > 0$, and pass to a subset $X' \subset X$ with $|X'| > c_\epsilon |X|$ satisfying the assumption of statement (iii) for $0 < \eta < 1$ to be determined. On the one hand,

$$\int_{E : g_\delta^{X'} < \delta^{-\epsilon}\eta^{-2}} g_\delta^{X'}(z)\, dz \leq \delta^{-\epsilon}\eta^{-2}|E|. \tag{2.6}$$

On the other hand,

$$\int_{E : g_\delta^{X'} < \delta^{-\epsilon}\eta^{-2}} g_\delta^{X'}(z)\, dz = \int_E g_\delta^{X'}(z)\, dz - \int_{E : g_\delta^{X'} > \delta^{-\epsilon}\eta^{-2}} g_\delta^{X'}(z)\, dz.$$

By Equation (2.5), the first term on the right-hand side satisfies

$$\int_E g_\delta^{X'}(z)\, dz \geq |X'|\lambda\delta.$$

For the second term on the right-hand side, we throw out E and apply Equation (2.4) to get an upper bound for the integral:

$$\begin{aligned} \int_{E : g_\delta^{X'} > \delta^{-\epsilon}\eta^{-2}} g_\delta^{X'}(z)\, dz &\leq \sum_{x \in X'} |\{z \in C_{\delta,x} : g_\delta^{X'}(z) > \delta^{-\epsilon}\eta^{-2}\}| \\ &\leq |X'|\eta\delta. \end{aligned}$$

Now we see that by choosing $\eta = \lambda/2$ and combining these bounds with Equation (2.6) and the assumption $c_\epsilon|X| < |X'|$, we have

$$c_\epsilon |X|\lambda\delta \lesssim |X'|\frac{\lambda}{2}\delta \leq \int_{E : g_\delta^{X'} < \delta^{-\epsilon}(\frac{\lambda}{2})^{-2}} g_\delta^{X'}(z)\, dz \leq 4\delta^{-\epsilon}\lambda^{-2}|E|.$$

Plugging in $|X|\delta = M\delta \sim |F|$ and rearranging the inequality shows

$$\lambda^3 |F| \leq A_\epsilon \delta^{-\epsilon}|E|,$$

as desired.

To show that (i) implies (ii), we rely on Example 2.3. We will also be careful with the value of "ϵ." As a particular corollary to the dual statement of the strong form of Equation (2.3), for any collection $X = \{x_i = (a_i, r_i)\}_{i=1}^N$ of N circles with δ-separated radii, and for any $\bar{\epsilon} > 0$, we have

$$\int g_\delta^X(z)^{3/2}\, dz \lesssim \delta^{-\bar{\epsilon}}\delta N. \tag{2.7}$$

Suppose for the sake of contradiction that for at least half the circles $x \in X$, for some $\eta = \eta(x)$, we have

$$|\{z \in C_{\delta,x} : g_\delta^X(z) > \delta^{-\epsilon}\eta^{-2}\}| > \eta |C_{\delta,x}|. \tag{2.8}$$

By dyadically pigeonholing, we may assume that η is fixed for all $x \in X'$ with $|X'| \gtrsim |\log\delta|^{-1}N \geq \delta^\epsilon N$, the last inequality holding as long as δ is sufficiently small. Then set

$$E = \{z : g_\delta^{X'}(z) > \delta^{-\epsilon}\eta^{-2}\}.$$

For each r_i a radius corresponding to a circle in X', we have $W_\delta 1_E(r) > \eta$ for all $r \in [r_i, r_i+\delta]$ by Equation (2.8). Let $F = \{r \in [1,2] : W_\delta 1_E(r) > \eta\}$, and note that by our assumption (i),

$$\eta^3(\delta^\epsilon N\delta) \lesssim \eta^3|F| \lesssim \delta^{-\bar{\epsilon}}|E|. \tag{2.9}$$

Applying the dual estimate Equation (2.7) and the definition of E,

$$(\delta^{-\epsilon}\eta^{-2})^{3/2}|E| \leq \int_E g_\delta^{X'}(z)^{3/2}\,dz \lesssim \delta^{-\bar{\epsilon}}\delta N. \tag{2.10}$$

Simplifying Equations (2.9) and (2.10), we have

$$\eta^3\delta^{\epsilon+\bar{\epsilon}+1}N \lesssim |E|$$
$$|E| \lesssim \eta^3\delta^{\frac{3}{2}\epsilon-\bar{\epsilon}+1}N.$$

Choosing $\bar{\epsilon} = \frac{\epsilon}{10}$ gives a contradiction if δ is sufficiently small depending on ϵ. □

By (ii) of Proposition 2.3, in order to prove a restricted-weak type version of Wolff's circular maximal function estimate Theorem 2.1, it suffices to gain some control on the high multiplicity regions within the thin annuli $C_{\delta,x}$ for x in an arbitrary collection X of δ-separated circles. Set $\mu = \delta^{-\epsilon}\eta^{-2}$, fix $x \in X$, and set

$$U_\mu(x) = \{z \in C_{\delta,x} : g_\delta^X(z) > \mu\}.$$

Suppose that at the typical point $z \in U_\mu(x)$, there are μ circles $x_1, \ldots, x_\mu \in X$ such that

$$z \in C_{\delta,x_1} \cap \cdots \cap C_{\delta,x_\mu},$$

and for the sake of discussion all μ of the δ-thin annuli overlap "tangentially" at z (the scenario where the circles overlap transversally is actually easier to deal with). In particular, all μ of the thin annuli contain a common $\delta \times \sqrt{\delta}$-rectangle $\Omega \subset C_{\delta,x}$. We refer to such a rectangle Ω as a *tangency rectangle.* In the special case where the circles overlap tangentially at a typical point of $U_\mu(x)$, we see what is at stake is obtaining a favorable bound for the number of high multiplicity tangency rectangles.

In his paper "Local smoothing type estimates on L^p for large p," [4] Wolff revisits circle tangencies, and remarks that the proof of the L^3 bound for the circular maximal function W_δ follows by means of "fairly standard arguments" from a nontrivial bound on the number of high multiplicity tangency rectangles. The lengthy details of the "fairly standard arguments" are presented elegantly in Schlag's exposition [26], but our focus here is the bound on the high multiplicity tangency rectangles. We recall the notation Wolff used in [4] to state the next Proposition.

Given two finite sets of circles X_1, X_2 which are ~ 1-separated (in the metric of Euclidean distance between center-radius pairs), we say a $\delta \times \sqrt{\delta}$-rectangle Ω in the plane is of type $(\geq \mu, \geq \nu)$ if at least μ circles of X_1 and at least ν circles of X_2 have the property that their 10δ-neighborhoods contain the rectangle Ω. The following Proposition is the case of $\mu = \nu$ and $|X_1| = |X_2|$ of Lemma 1.4 in Wolff's paper [4]:

Proposition 2.4 (Wolff, [4] Lemma 1.4)**.** *Let X_1, X_2 each be sets of N-many circles such that* $\mathrm{dist}(X_1, X_2) \geq 1/2$*, and with δ-separated radii in the interval* $[1, 2]$*. For each $\epsilon > 0$, there is a constant C_ϵ such that for any collection $\mathcal{R}$ of pairwise incomparable $\delta \times \sqrt{\delta}$ rectangles in the plane, the cardinality of the rectangles of type $(\geq \mu, \geq \mu)$ in $\mathcal{R}$ is at most*

$$C_\epsilon N^\epsilon (\frac{N}{\mu})^{3/2}. \tag{2.11}$$

"Pairwise incomparable" will be defined precisely in Definition 2.5, but intuitively, two rectangles are pairwise incomparable if they do not have a significant overlap. Let us sketch how the bound on high multiplicity tangency rectangles translates into the bound for $U_\mu(x)$.

Proposition 2.5. *Let X be a collection of N many circles with δ-separated radii in the interval* $[1, 2]$*. Then there exists $X' \subset X$, $|X'| > \frac{1}{2}|X|$ such that for each $0 < \eta < 1$ and each $x \in X'$, the bound*

$$|\{z \in C_{\delta,x} : g_\delta^X(z) > \delta^{-\epsilon}\eta^{-2}\}| \leq \eta\delta$$

holds.

Heuristic proof using (2.11)*.* Set $\mu = \eta^{-2}$, and let us say that that typical circle $x \in X$ contains about A many tangency rectangles of multiplicity μ. In particular, the number of *pairs* of nearly tangent circles is about $NA\mu$. Each pair of nearly tangent circles corresponds to a tangency rectangle of multiplicity μ, so letting $\mathcal{R}_\mu$ denote the tangency rectangles of multiplicity about μ, and using Equation (2.11), we have

$$NA\mu \lesssim \mu^2 \# \mathcal{R}_\mu \lesssim \mu^2 (\frac{N}{\mu})^{3/2} = \mu^{1/2} N^{3/2}.$$

Rearranging, it gives

$$A \lesssim \mu^{-1/2} N^{1/2} = \eta N^{1/2}.$$

Therefore, the typical $x \in X$ satisfies

$$|\{z \in C_{\delta,x} : g_\delta^X(z) > \eta^{-2}\}| \lesssim A\delta^{3/2} \lesssim \eta N^{1/2} \delta^{3/2} \lesssim \eta\delta,$$

where the last inequality used that the circles of X have δ-separated radii, so $N \leq \delta^{-1}$. $\square$

In the remark following Lemma 1.17 of [4], Wolff asks whether $(\frac{N}{\mu})^{3/2}$ is the sharp bound for the number of tangency rectangles of type $(\geq \mu, \geq \mu)$, or whether it could be replaced by $(\frac{N}{\mu})^{4/3}$, which would be essentially best possible for δ-separated sets of circles X_1, X_2 with $\mathrm{dist}(X_1, X_2) \sim 1$.

2.3 Point-circle duality

In "Local smoothing type estimates on L^p for large p," [4] Wolff had a creative idea to study the Fourier extension of $\mathbb{C}one^2$, whose definition we recall here:

$$\mathbb{C}one^2 = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : 1 < |(\xi_1, \xi_2)| < 2, \xi_3 = \sqrt{\xi_1^2 + \xi_2^2}\},$$

$$E_{\mathbb{C}one^2} f(x', x_3) = \int_{1<|\xi'|<2} f(\xi') e(x' \cdot \xi' + x_3|\xi'|)\, d\xi', \quad (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}.$$

Wolff's goal was partly to understand the large level sets of $E_{\mathbb{C}one^2} f$.

To apply facts about circles to the Fourier extension of the cone successfully, Wolff needed to partly describe a dictionary of lemmas that relates the geometry of circle tangencies in the plane to the geometry of points in the upper half space. Part of our goal in this section is to establish the facts from this dictionary we need for our own problems facing $E_{\mathbb{C}one^2} f$ in Chapter 3. Another goal of this presentation is to clarify and expand on some of the same facts Wolff used in [4].

To motivate the facts from this dictionary we will need, we will consider the following problem about points in $\mathbb{R}^3$ that we will seek an answer for in the proof of Theorem 3.4 using point-circle duality. First a definition, and then the problem. For $x = (a, r), x' = (a', r')$ in $\mathbb{R}^2 \times \mathbb{R}$, we define

$$\Delta(x, x') = ||a - a'| - |r - r'||.$$

Up to a multiplicative constant, $\Delta(x, x')$ is the same as the distance from x to the lightcone with vertex x', and vice-versa. Now the problem.

Problem 2.1. *Given a finite set of δ-separated points $X \subset \mathbb{R}^3$, estimate the cardinality of the set of "nearly lightlike" pairs*

$$\mathcal{L}_{<\delta}(X) = \{(x, x') \in X \times X : \Delta(x, x') < \delta\}.$$

Problem 2.1 can be turned into an equivalent problem of counting pairs of nearly internally tangent δ-separated circles using point-circle duality. Depending on the assumptions we make about X, we can give nontrivial estimates for $|\mathcal{L}_{<\delta}(X)|$ using (dual) circular maximal function estimates (such as those in Examples 2.3 or 2.6) which apply to X as a black box. We will come back to Problem 2.1 in Section 2.4 after developing the facts of point-circle duality we need.

Point-circle duality. We consider circles in the plane with radii in $[1 - \alpha_0, 1 + \alpha_0]$ for a small but absolute constant α_0, so we do not have to consider very small or very large circles. We parametrize circles by their center-radius pairs. Given a point $(a, r) \in \mathbb{R}^2 \times [1 - \alpha_0, 1 + \alpha_0]$, we will equivalently regard it as the circle

$$C_{a,r} = \{z \in \mathbb{R}^2 : ||z - a| - r| = 0\}.$$

Likewise, the δ-neighborhood of the point (a, r) is identified with the δ-thin annulus

$$C_{\delta,a,r} = \{z \in \mathbb{R}^2 : ||z - a| - r| < \delta\}.$$

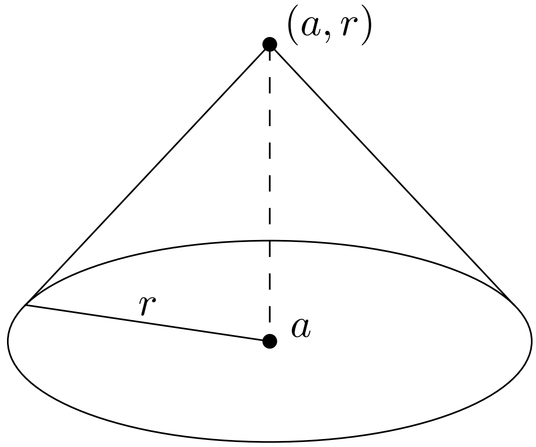

Figure 2.2: Point-circle duality

From now until the end of Section 2.3, let $\epsilon > 0$ be fixed. We will assume that $\delta < \delta_0(\epsilon)$ is small enough so that $\delta_0^\epsilon < 10^{-3}$ to ensure that approximations such as $\cos\theta \sim 1 - \theta^2/2$ hold up to constant factors if $|\theta| \leq \delta^\epsilon$. We identify points $x = (a, r)$ with their corresponding circles $C_{a,r}$. All the circles we consider will be parametrized by points in $Q = B(e_3, \alpha_0)$ for a small but absolute constant $\alpha_0 > 0$ unless mentioned otherwise.

Remark 2.1 (Notation)**.** *In this section, we will make use of the following notation. For $\epsilon > 0$ fixed:*

- $A \ll B$*: there is a constant $c > 0$ so $A \lesssim \delta^{c\sqrt{\epsilon}} B$.*
- $A \lessapprox B$*: there is a constant $C > 0$ so that $A \lesssim \delta^{-C\epsilon} B$. Note that with this definition, as long as δ is sufficiently small depending on ϵ, $A \ll B \lessapprox C$ implies $A \ll C$.*
- $A \approx B$*: $A \lessapprox B$ and $B \lessapprox A$ (with possibly different implied constants).*
- $\Gamma_0 = \{(a, r) \in \mathbb{R}^2 \times \mathbb{R} : ||a| - |r|| = 0\}$ *is the lightcone with vertex* 0.
- $\Gamma_y = \Gamma_0 + y$ *is the lightcone with vertex* y.
- $Q = B(e_3, \alpha_0)$*, the ball of radius α_0 about e_3, which we take as the parameter space of circles.*
- *When we are using point-circle duality, we will not necessarily distinguish between circles and the points of Q that parametrize them. For instance, we may say things like "given a set of circles in Q," or "take circles arranged within a lightray," where in both cases we mean precisely consider points of $\mathbb{R}^3$ with these properties, but we think of them as circles.*
- *If E is a set, we will sometimes use $E(x)$ to denote $1_E(x)$, the indicator function of E.*
- *If $x = (a, r), x' = (a', r') \in Q$, then $\Delta(x, x') = ||a - a'| - |r - r'||$ is (up to an absolute constant) the distance from x to $\Gamma_{x'}$, and vice-versa.*

- *If $(E_1, o_1), (E_2, o_2)$ are sets with designated "centers" $o_1 \in E_1$, $o_2 \in E_2$, and $A^{-1}E_1 \subset E_2 \subset AE_1$ for some $A \approx 1$, where the notation AE_1 denotes the dilation of E_1 by A about its center o_1, then we may write $E_1 \asymp E_2$.*

Given $X \subset Q$ a set of N circles, recall the multiplicity functions

$$g_{\lambda\delta}(a) = \sum_{x \in X} C_{\lambda\delta,x}(a), \quad a \in \mathbb{R}^2, \quad \delta, \lambda > 0.$$

We want to understand the shape of the large level sets of $g_{\lambda\delta}$ for some small fixed δ, and $\lambda \lessapprox 1$. As we saw in Section 2.2, we can partition these level sets (which are unions of thin annuli) into curvilinear rectangles of width $\lambda\delta$, and variable length $0 < \tau \lesssim 1$. It turns out that the range $\sqrt{\delta} \leq \tau \ll 1$ is the most important for us to understand.

Definition 2.4 (δ, τ-rectangle, core circle, center)**.** *For $\delta^{1/2} \leq \tau \ll 1$, a* δ, τ-rectangle *is the δ-neighborhood of an arc of length τ on some circle of radius $r \in [1 - \alpha_0, 1 + \alpha_0]$. We will sometimes refer to the implicit circle in this definition as the* core circle *of Ω, and we may write $\Omega = \Omega^{(v)}$ if v is the core circle of Ω. The midpoint of the core arc of Ω will be referred to as the* center *of Ω.*

Definition 2.5 (Comparable)**.** *We say two δ, τ-rectangles Ω_1, Ω_2 are* A-comparable *if each is contained in the A-dilation of the other about their centers. If Ω_1, Ω_2 are A-comparable for some $A \approx 1$, then we say they are simply* comparable*. If Ω_1, Ω_2 are not A-comparable, we say they are* A-incomparable. *A collection $\mathcal{R}$ of δ, τ-rectangles is* pairwise A-incomparable *if no two members of $\mathcal{R}$ are A-comparable.*

If $\tau \lesssim \delta^{1/2}$, then a δ, τ-rectangle is approximately a rectangle in the usual sense, while if τ is much larger than $\delta^{1/2}$, a δ, τ-rectangle is a curvilinear rectangle.

Definition 2.6 (Tangency)**.** *We say a δ, τ-rectangle Ω is* λ-tangent *to the circle x if $\Omega \subset C_{\lambda\delta,x}$. We let $\mathbf{D}_{\lambda\delta}(\Omega) = \{x \in Q : \Omega \subset C_{\lambda\delta,x}\}$ be the collection of λ-tangent circles to Ω in Q.*

Note that in the definition of tangency, we restrict the "dual set" $\mathbf{D}_{\lambda\delta}(\Omega)$ to be contained in Q. We record the following useful and easily proved facts about taking tangency.

Proposition 2.6. *For every $\delta > 0$, the following hold.*

(i) (Monotonicity) If $\Omega' \subset \Omega$, then $\mathbf{D}_\delta(\Omega) \subset \mathbf{D}_\delta(\Omega')$.

(ii) (Intersection) If $\Omega = \bigcup_k \Omega_k$, then $\mathbf{D}_\delta(\Omega) = \bigcap_k \mathbf{D}_\delta(\Omega_k)$

(iii) For every δ, τ-rectangle $\Omega = \Omega^{(v)}$, we have $v \in \mathbf{D}_\delta(\Omega^{(v)})$ and $B(v, \delta) \cap Q \subset \mathbf{D}_{10\delta}(\Omega)$. In particular, $\mathbf{D}_\delta(\Omega^{(v)}) \neq \emptyset$ for every $v \in Q$.

Later on we will refine property (iii) of Proposition 2.6 substantially. If $\Omega = \Omega^{(v)}$ is a δ, τ-rectangle, then the core circle v is 1-tangent to Ω. Besides v, there are other nearby circles $w \in Q$ which are ≈ 1-tangent to Ω. The set of all such w takes the shape of an "essentially unique" $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-*lightplank* centered on v when regarded as a subset of $\mathbb{R}^3$. To describe the shape of $\mathbf{D}_{\lambda\delta}(\Omega)$ more precisely, we introduce the following definition.

Definition 2.7. *A* lightlike basis *for* $\mathbb{R}^3$ *is an orthonormal basis* $\mathcal{E} = e_m, e_l, e_s$ *of* $\mathbb{R}^3$ *such that for some* $\theta \in [-\pi/2, \pi/2)$*, with respect to the standard basis of* $\mathbb{R}^3$*,*

$$e_m = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, e_l = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 1 \end{pmatrix}, e_s = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}.$$

A lightlike coordinate system $(x_m, x_l, x_s), o$ *is the Cartesian coordinate system with respect to a lightlike basis with the point* $o \in \mathbb{R}^3$ *as the designated origin, i.e., for any* x *in* $\mathbb{R}^3$*,* $x = o + x_m e_m + x_l e_l + x_s e_s$*. If we want to emphasize the angle* θ*, we will write* $x_m(\theta), x_l(\theta), x_s(\theta), e_m(\theta), e_l(\theta), e_s(\theta)$*.*

Note that with this definition, a lightlike basis is ordered and right-handed, and in particular, it is completely determined by the first basis vector.

Definition 2.8 (Lightplank, comparable, essentially unique)**.** *Let* $\delta^{1/2} \le \tau \ll 1$*. A rectangular parallelepiped* $P \subset \mathbb{R}^3$ *is a* $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank *if the edge lengths of* P *are* $A_1\delta < A_2\delta\tau^{-1} < A_3\delta\tau^{-2}$ *for numbers* $A_1, A_2, A_3 \approx 1$*, and if for some lightlike basis* e_m, e_l, e_s *of* $\mathbb{R}^3$*, the edges of* P *are parallel to the vectors* e_m, e_l, e_s*, in the order "intermediate, long, short." We will adopt a similar notation as with* δ, τ*-rectangles where we write* $P = P^{(v)}$ *if* $v \in \mathbb{R}^3$ *is the center of* P*.*

Two lightplanks P, P' *are* comparable *if they are both contained in the* A*-dilation of the other for some* $A \approx 1$*.*

We say a lightplank P *satisfying a property* π *is* essentially unique *if any other lightplank* P' *which also satisfies* π *is comparable to* P*.*

The following calculation of $\mathbf{D}_{C\delta}(\Omega)$ when Ω is a $\delta, \sqrt{\delta}$-rectangle will be the basis for the calculation of $\mathbf{D}_{\lambda\delta}(\Omega)$ when Ω is a δ, τ-rectangle, and $\lambda \approx 1$, $\tau \gg \sqrt{\delta}$.

Proposition 2.7 (Dual of $\delta, \sqrt{\delta}$-rectangle)**.** *Let* $o = e_3$ *and* $\Omega^{(o)} = [1-\delta, 1+\delta] \times [-\sqrt{\delta}/2, \sqrt{\delta}/2]$*. Let*

$$e_m = e_2, \quad e_l = \frac{-e_1 + e_3}{\sqrt{2}}, \quad e_s = \frac{e_1 + e_3}{\sqrt{2}}$$

be a lightlike basis for $\mathbb{R}^3$*, and let* $(x_m, x_l, x_s), o$ *be the associated lightlike coordinate system. Recall that* $Q = B(e_3, \alpha_0)$*. If* C *is a sufficiently large absolute constant, then the following hold.*

(i) If $x \in P^{(o)} \cap Q$*, where*

$$P^{(o)} = \{(x_m, x_l, x_s) : |x_m| \le \sqrt{\delta}, |x_l| \le 1, |x_s| \le \delta\}$$

then $x \in \mathbf{D}_{C\delta}(\Omega^{(o)})$*.*

(ii) If $x \in \mathbf{D}_{10\delta}(\Omega^{(o)})$*, then* $x \in Q \cap CP^{(o)}$*, where* CP *is the dilation of* P *by a factor of* C *about its center* o*.*

Remark 2.2. *Proposition 2.7 shows in what precise sense the* $\delta, \sqrt{\delta}$*-rectangle* $\Omega^{(o)}$ *is dual to "the"* $1 \times \sqrt{\delta} \times \delta$*-lightplank* $P^{(o)}$*: any other lightplank* P' *which satisfies (i) and (ii) would be comparable to* $P^{(o)}$*. Another way to say it is that there is an essentially unique* $\approx \delta \times \sqrt{\delta} \times 1$*-lightplank* P *satisfying (i) and (ii).*

Proof of Proposition 2.7. First we prove (i). By definition of P and the lightlike coordinate system $(x_m, x_l, x_s), o$, it is clear that $P \subset Q$. Let $x \in P$, and let (x_1, x_2, x_3) be the usual Euclidean coordinates of x with respect to the standard basis e_1, e_2, e_3. To verify $x \in \mathbf{D}_{C\delta}(\Omega)$, let $(a_1, a_2) \in \Omega$ be arbitrary, and consider the quantity

$$I = |(x_1 - a_1)^2 + (x_2 - a_2)^2 - x_3^2|. \tag{2.12}$$

We need to show that $I \leq C\delta$. Let $x' = (x_1, x_2)$, and $a_1 = 1 + h$ with $|h| \leq \delta$. Expanding Equation (2.12) and applying the triangle inequality,

$$\begin{aligned} I &= |(x_1 - 1)^2 + x_2^2 - x_3^2 + h^2 - 2(x_1 - 1)h - 2x_2 a_2| \\ &\leq \big(||x' - e_1| - x_3| \cdot ||x' - e_1| + x_3|\big) + \delta^2 + 2\delta|x_1 - 1| + 2\sqrt{\delta}|x_2|. \end{aligned}$$

Now we use that $x \in P$ to make some estimates of this last expression. First, $||x' - e_1| - x_3| = \Delta(x, e_1) \leq C\delta$, because $\Delta(x, e_1)$ is nearly the distance from x to the lightcone with vertex e_1. The quantity $||x' - e_1| + x_3| < 10$ because $x \in Q$. Similarly, $|x_1 - 1| \leq 2$. Note $|x_2| = |x_m| \leq \sqrt{\delta}$, so $2\sqrt{\delta}|x_2| \leq 2\delta$. Altogether this shows that $I \leq C\delta$, as desired.

Now we prove (ii). Let $\ell = \{e_1 + t(e_3 - e_1) : t \in \mathbb{R}\}$ be a lightray intersecting e_1. Consider the infinite $2\delta \times \sqrt{\delta}$ rectangular prism R we get by sliding Ω along ℓ. (Concretely, R is the Minkowski sum $e_1 + (\ell - e_1) + (\Omega - e_1)$.) If $x \in Q$, but x is not in CR, the dilation of R about its central axis ℓ by a factor of C, and C is sufficiently large, then $\Omega \not\subset C_{10\delta,x}$. This shows that if $x \in \mathbf{D}_{10\delta}(\Omega)$, then $x \in Q \cap CP$ for an appropriate $C > 1$. □

Remark 2.3 (Our favorite position, coordinates)**.** *The coordinate system and position of $\Omega^{(o)}$ in Proposition 2.7 are so convenient for computations that we will say a δ, τ-rectangle $\Omega^{(o)}$ is* in our favorite position *if $o = e_3$, and the center of $\Omega^{(o)}$ as defined in Definition 2.8 is e_1.*

If $P^{(o)}$ is a $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, we say $P^{(o)}$ is in our favorite position *if $o = e_3$ is the center of P, and if the intermediate axis of P is parallel to $e_m := e_2$, the long axis is parallel to $e_l := \frac{-e_1+e_3}{\sqrt{2}}$, and the short axis is parallel to $e_s := \frac{e_1+e_3}{\sqrt{2}}$.*

We let $(x_m, x_l, x_s), o$ be our favorite lightlike coordinate system.

By changing coordinates, we can study any $\delta, \sqrt{\delta}$-rectangle by first changing coordinates so that the transformed rectangle is in our favorite position, applying Proposition 2.7, and then transforming back to the original coordinates. Alternatively, we have the following "coordinate-invariant" description of an essentially unique dual lightplank to a $\delta, \sqrt{\delta}$-rectangle in the plane.

Construction 2.1 (Dual of $\delta, \sqrt{\delta}$-rectangle)**.** *Proposition 2.7 describes how to construct an essentially unique $\delta \times \sqrt{\delta} \times 1$-lightplank dual to a $\delta, \sqrt{\delta}$-rectangle in our favorite position. Now we consider more arbitrary $\delta, \sqrt{\delta}$-rectangles.*

Let $\Omega^{(v)}$ be an arbitrary $\delta, \sqrt{\delta}$-rectangle in the plane with core circle $v \in Q = B(e_3, \alpha_0)$ and center c_Ω. Let e_m be a unit vector in $\mathbb{R}^2$ parallel to the long edge of $\Omega^{(v)}$, forming an angle in $[-\pi/2, \pi/2)$ with the standard basis vector e_1. Consider the lightlike basis e_m, e_l, e_s determined by e_m and the lightrays $\ell_+ = c_\Omega + \mathbb{R}e_l, \ell_- = c_\Omega + \mathbb{R}e_s$ passing through c_Ω.

Define infinite $\delta \times \sqrt{\delta}$ rectangular prisms R_+, R_- by

$$R_+ = \Omega^{(v)} - c_\Omega + \ell_+, \quad R_- = \Omega^{(v)} - c_\Omega + \ell_-.$$

Since $v \in Q$, v lies in exactly one of the sets

$$Q \cap R_+, \quad Q \cap R_-$$

By relabeling R_+, R_- if necessary, we may assume $v \in Q \cap R_+$. Then $P^{(v)} = R_+ \cap 2Q$ is an essentially unique $\approx \delta \times \sqrt{\delta} \times 1$-lightplank $P^{(v)}$ satisfying $P^{(v)} \asymp \mathbf{D}_{10\delta}(\Omega^{(v)})$.

It will be convenient for future computations to describe how different lightlike coordinates are related when o is the same.

Proposition 2.8 (Change of coordinates, fixed o)**.** *If $\theta \in [-\pi/2, \pi/2)$ and $o \in Q$ is fixed, we have the following relationship between the lightlike coordinates $(x_m(\theta), x_l(\theta), x_s(\theta)), o$ and $(x_m(0), x_l(0), x_s(0)), o$:*

$$\begin{pmatrix} \cos\theta & \frac{-\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} \\ \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} & \frac{1-\cos\theta}{2} \\ \frac{-\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} & \frac{1+\cos\theta}{2} \end{pmatrix} \begin{pmatrix} x_m(\theta) \\ x_l(\theta) \\ x_s(\theta) \end{pmatrix} = \begin{pmatrix} x_m(0) \\ x_l(0) \\ x_s(0) \end{pmatrix}. \tag{2.13}$$

Proof. Since the lightlike bases $e_m(\theta), e_l(\theta), e_s(\theta)$ and $e_m(0), e_l(0), e_s(0)$ are orthonormal, the proof is just the calculation of the 9 inner products $\langle e_m(\theta), e_m(0)\rangle$, $\langle e_l(\theta), e_m(0)\rangle, \ldots,$ etc. □

The next Proposition is the first of our dictionary results which does not appear in [4].

Proposition 2.9 (Dual of δ, τ-rectangle)**.** *Let $\delta^{1/2} < \tau \ll 1$, and let $\Omega^{(o)}$ be a δ, τ-rectangle with core circle $o = e_3$ in our favorite position. In our favorite lightlike coordinate system $(x_m, x_l, x_s), o$, let*

$$P^{(o)} = \{(x_m, x_l, x_s) : |x_m| \le \delta\tau^{-1}, |x_l| \le \delta\tau^{-2}, |x_s| \le \delta\}.$$

Recall that $Q = B(e_3, \alpha_0)$. If C is a sufficiently large absolute constant, then the following hold.

(i) If $x \in P^{(o)} \cap Q$, then $x \in \mathbf{D}_{C\delta}(\Omega^{(o)})$.

(ii) If $x \in \mathbf{D}_{10\delta}(\Omega^{(o)})$, then $x \in Q \cap CP^{(o)}$, where CP is the dilation of P by a factor of C about its center.

Proof snapshot. See Figure 2.3. Assume Ω is in our favorite position and take intersections of Construction 2.1 $\mathbf{D}_{C\delta}(\Omega_k)$ for some sub-$\delta, \sqrt{\delta}$-rectangles $\{\Omega_k\}_k$ which cover Ω but not 2Ω. □

Proof. First we prove (i). Let $x \in P^{(o)}$ and consider the $\delta, \sqrt{\delta}$-rectangles

$$\Omega_k^{(o)} = \{z \in \mathbb{R}^2 : ||z| - 1| < \delta, |z - e^{ik\sqrt{\delta}}| < \frac{1}{2}\sqrt{\delta}\}, \quad k \in \mathbb{Z}, |k| \le N.$$

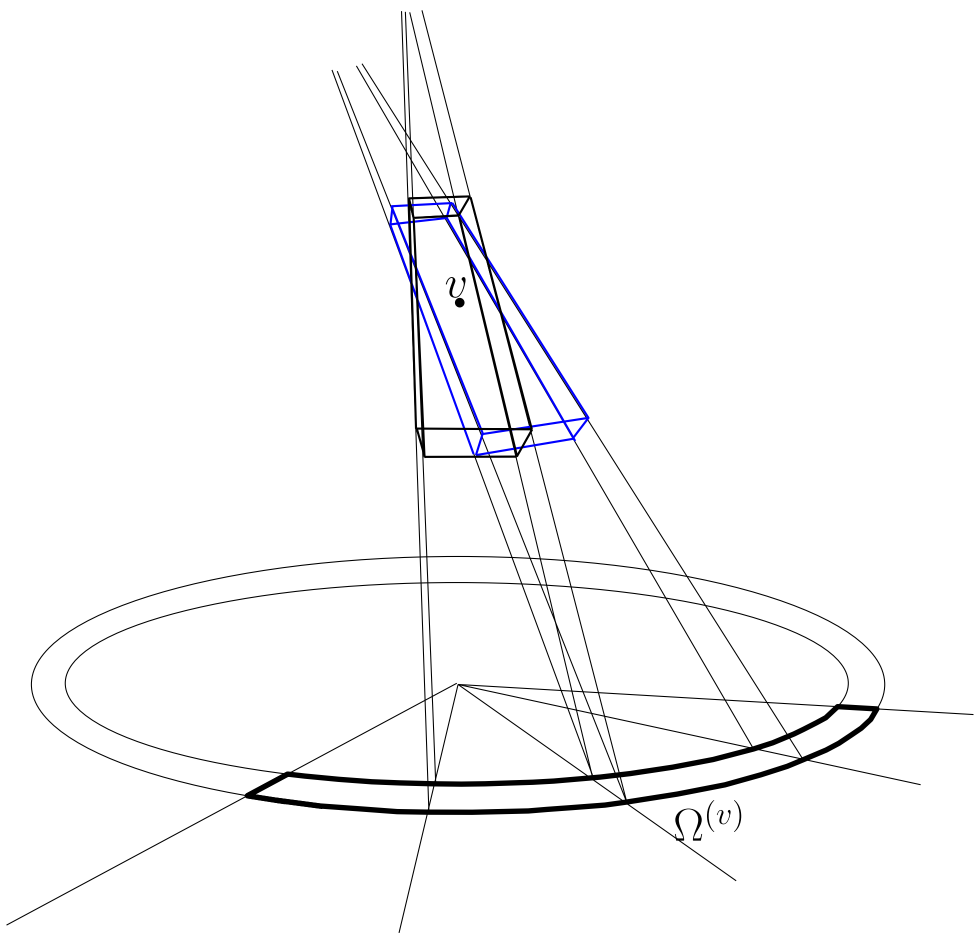

Figure 2.3: A snapshot of "the" dual $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank to a δ, τ-rectangle $\Omega^{(v)}$

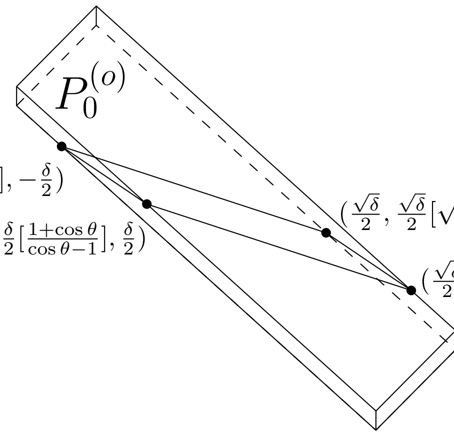

Figure 2.4: The boundary vertices of two intersecting lightplanks

We choose N so that $\tau < N\sqrt{\delta} < 2\tau$, and let $\overline{\Omega}^{(o)} = \bigcup_{|k|\leq N} \Omega_k^{(o)}$. It is clear from the definition that $\Omega^{(o)} \subset \overline{\Omega}^{(o)}$so by Proposition 2.6, it suffices to prove that

$$x \in \mathbf{D}_{C\delta}(\Omega_k^{(o)}) \quad \text{for each } |k| \leq N. \tag{2.14}$$

By Construction 2.1, let $P_k^{(o)}$ be an essentially unique $\approx \delta \times \sqrt{\delta} \times 1$-lightplank $P_k^{(o)}$ such that

$$P_k^{(o)} \approx \mathbf{D}_{C\delta}(\Omega_k^{(o)}).$$

It thus suffices to prove

$$x \in P_0^{(o)} \cap P_k^{(o)} \cap P_{-k}^{(o)} \quad \text{for each } 1 \leq k \leq N.$$

We claim that for each $1 \leq k \leq N$, in our favorite lightlike coordinate system $(x_m(0), x_l(0), x_s(0)), o$,

$$P_0^{(o)} \cap P_k^{(o)} \cap P_{-k}^{(o)} \approx \{|x_m| \leq k^{-1}\sqrt{\delta}, |x_l| \leq k^{-2}, |x_s| \leq \delta\},$$

in the sense that both sets are contained in constant dilations of each other in the (x_m, x_l) directions.

To prove this, let $\theta \in \{\pm k\sqrt{\delta}\}$ and consider the lightlike coordinate system $(x_m(\theta), x_l(\theta), x_s(\theta)), o$. The lightplank $P_{\frac{\theta}{\sqrt{\delta}}}^{(o)}$ is contained in the δ-neighborhood of the affine plane

$$\Pi_{\frac{\theta}{\sqrt{\delta}}} = o + \{x \in \mathbb{R}^3 : \langle x, e_s(\theta)\rangle = 0\}.$$

To determine how $P_{\frac{\theta}{\sqrt{\delta}}}^{(o)}$ intersects $P_0^{(o)}$, we just need to determine the $(x_m(0), x_l(0), x_s(0)), o$-coordinates of the points $x \in \Pi_{\frac{\theta}{\sqrt{\delta}}} \cap \partial P_0^{(o)}$ (see Figure 2.4). The calculation to determine the coordinates shown in Figure 2.4 is not difficult, but it is lengthy, so we just describe the method we use to obtain the coordinates here. The four marked points in Figure 2.4 are the points where the plane $\Pi_{\frac{\theta}{\sqrt{\delta}}}$ intersects $P_0^{(o)}$ on its faces $x_m(0) = \pm\frac{\sqrt{\delta}}{2}$ and $x_s(0) = \pm\frac{\delta}{2}$. We use Equation (2.13) to solve for $x_m(\theta), x_l(\theta)$ (we know $x_s(\theta) = 0$ by definition of the plane $\Pi_{\frac{\theta}{\sqrt{\delta}}}$)

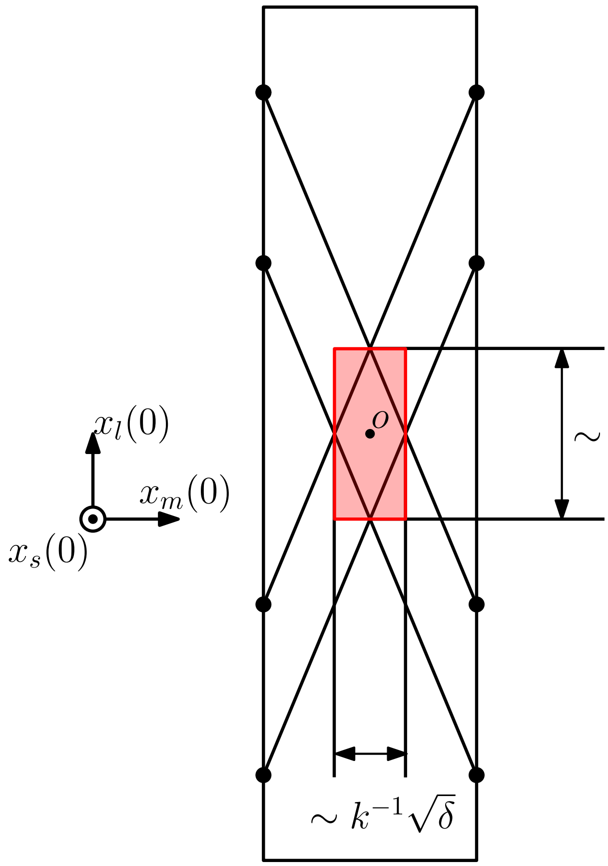

Figure 2.5: The dimensions of the intersection $P_0^{(o)} \cap P_k^{(o)} \cap P_{-k}^{(o)}$

in terms of $x_m(0), x_s(0)$, and then use Equation (2.13) again with $x_m(0) = \pm\frac{\sqrt{\delta}}{2}, x_s(0) = \pm\frac{\delta}{2}$ to determine the value of $x_l(0)$ for each of these four points.

By computing these points for each $\theta \in \{\pm k\sqrt{\delta}\}$, we can find the region of overlap $P_0^{(o)} \cap P_k^{(o)} \cap P_{-k}^{(o)}$ in the $(x_m(0), x_l(0), x_s(0))$, o-coordinates—see Figure 2.5. Using $k \leq N$ and $N\sqrt{\delta} < 2\tau$, this shows that

$$
\begin{aligned}
P_0^{(o)} \cap P_k^{(o)} \cap P_{-k}^{(o)} &\asymp \{|x_m| \leq k^{-1}\sqrt{\delta}, |x_l| \leq k^{-2}, |x_s| \leq \delta\} \\
&\subset \{|x_m| \leq \frac{1}{2}\delta\tau^{-1}, |x_l| \leq \frac{1}{4}\delta\tau^{-2}, |x_s| \leq \delta\} \approx P^{(o)}.
\end{aligned}
$$

Analogous considerations using $N\sqrt{\delta} > \tau$ show

$$P^{(o)} \asymp P_0^{(o)} \cap P_N^{(o)} \cap P_{-N}^{(o)}$$

This finishes the proof of (i) that $P^{(o)} \cap Q \subset \mathbf{D}_{C\delta}(\Omega^{(o)})$ provided C is sufficiently large.

With the work we used to prove (i), the proof of statement (ii) will be comparatively much simpler. We choose N again so that $\tau < N\sqrt{\delta} < 2\tau$, and let $\Omega_k^{(o)}$ be $\delta, \sqrt{\delta}$-rectangles as in the proof of (i). By the intersection property of Proposition 2.6, suppose

$$x \in \mathbf{D}_{10\delta}(\Omega_k^{(o)}) = \bigcap_{|k| \leq N} \mathbf{D}_{10\delta}(\Omega_k^{(o)}). \tag{2.15}$$

By Construction 2.1 and the work we did in the proof of (i), in our favorite coordinate system, the intersection in Equation (2.15) is comparable to the lightplank

$$\{|x_m| \leq N^{-1}\sqrt{\delta}, |x_l| \leq N^{-2}, |x_s| \leq \delta\}.$$

Using $N \sim \frac{\tau}{\sqrt{\delta}}$ finishes the proof. □

We can study any particular δ, τ-rectangle $\Omega^{(v)}$ by transforming coordinates so that the rectangle under consideration is in our favorite position. Then the dual lightplank to $\Omega^{(v)}$ is the $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank obtained by applying Proposition 2.9 and then undoing the coordinate transformation. Alternatively, we record the following "coordinate-invariant" description of the dual lightplank to a δ, τ-rectangle analogous to Construction 2.1.

Construction 2.2 (Dual to δ, τ-rectangle, $\sqrt{\delta} \leq \tau \ll 1$)**.** *Given an arbitrary δ, τ-rectangle $\Omega^{(v)}$ with $v \in Q$, take the intersection of essentially unique $\delta \times \sqrt{\delta} \times 1$-lightplanks dual to sub-$\delta, \sqrt{\delta}$-rectangles contained in Ω. The result is an essentially unique $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank with long edge parallel to the lightray connecting v and the center of $\Omega^{(v)}$.*

Remark 2.4. *Construction 2.2 is "continuous" in the sense that if $\Omega^{(v)}$ and $\Omega^{(w)}$ are comparable δ, τ-rectangles for some $v, w \in Q$, then $\mathbf{D}_{10\delta}(\Omega^{(v)}), \mathbf{D}_{10\delta}(\Omega^{(w)})$ are comparable $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplanks. A precise version of this remark appears later in Proposition 2.11.*

Like the sets $\mathbf{D}_\delta(\Omega) \subset Q$ for $\Omega \subset \mathbb{R}^2$, there is an appropriate "dual" for subsets $E \subset Q = B(e_3, \alpha_0)$.

Definition 2.9. *If $E \subset Q$, and $\delta > 0$, $\lambda \lessapprox 1$, let*

$$\mathbf{D}^*_{\lambda\delta}(E) = \{z \in \mathbb{R}^2 : E \subset \Gamma_{\delta,z}\}$$

where $\Gamma_{\delta,z}$ is the δ-neighborhood of Γ_z.

Like $\mathbf{D}_\delta$, the map $\mathbf{D}^*_\delta$ obeys a few simple properties. Based on Proposition 2.9, we can refine the third point substantially with essentially no comment. Point (iv) on continuity follows from (iii).

Proposition 2.10. *For every $\delta > 0$, the following hold.*

(i) (Monotonicity) If $E' \subset E$, then $\mathbf{D}^*_\delta(E) \subset \mathbf{D}^*_\delta(E')$.

(ii) (Intersection) If $E = \bigcup_k E_k$, then $\mathbf{D}^*_\delta(E) = \bigcap_k \mathbf{D}^*_\delta(E_k)$.

(iii) For every $\sqrt{\delta} \leq \tau \ll 1$, and every $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-rectangle P, $\mathbf{D}^*_{10\delta}(P) \neq \emptyset$.
If $P = P^{(o)}$ is a $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$ lightplank in our favorite position, and $\Omega^{(o)}$ is a δ, τ-rectangle in our favorite position, then

$$\mathbf{D}^*_{10\delta}(P^{(o)}) \asymp \Omega^{(o)}.$$

(iv) (Continuity) For $v, w \in Q$, if $P^{(v)} \asymp P^{(w)}$ then $\mathbf{D}^*_{10\delta}(P^{(v)}) \asymp \mathbf{D}^*_{10\delta}(P^{(w)})$.

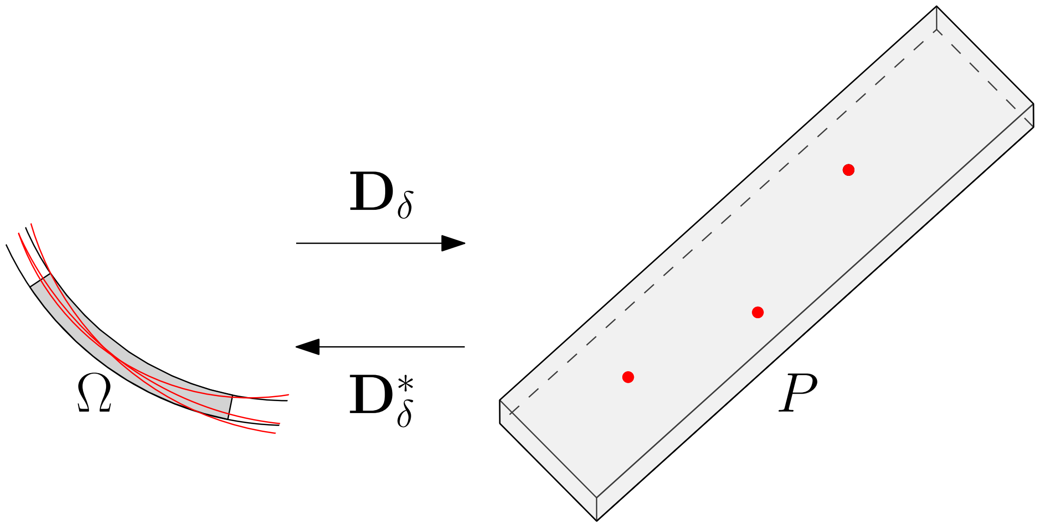

Figure 2.6: Rectangle-lightplank duality

To summarize the results of this Section, we record the following Theorem and accompanying Figure 2.6.

Theorem 2.3 (Rectangle-lightplank duality)**.** *If $\Omega^{(v)}$ is a δ, τ-rectangle with $v \in Q$, then for appropriate $C \approx 1$,*

$$\mathbf{D}^*_{C\delta}(\mathbf{D}_{10\delta}(\Omega^{(v)})) \asymp \Omega^{(v)}.$$

Likewise, if $v \in Q$ and $P^{(v)}$ is a $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, then

$$\mathbf{D}_{C\delta}(\mathbf{D}^*_{10\delta}(P^{(v)})) \asymp P^{(v)}.$$

Until now, we have considered one rectangle $\Omega^{(v)}$ and its corresponding dual lightplank $P^{(v)} \asymp \mathbf{D}_{10\delta}(\Omega^{(v)})$. In the next Section we will prove that if Ω_1, Ω_2 are two distinct rectangles with dual lightplanks P_1, P_2, then Ω_1 and Ω_2 are comparable if and only if P_1, P_2 are comparable, refining what we said in Remark 2.4 about the continuity of the dual constructions $\mathbf{D}$(rectangle), $\mathbf{D}^*$(lightplank).

2.4 Geometry of comparability

In this Section, we will supply an answer to Problem 2.1 in the case where our set of circles $X \subset Q$ satisfies the following 1-dimensional Frostman non-concentration condition:

$$|X \cap B_r|_\delta \leq \frac{r}{\delta}, \quad \text{for all } r\text{-balls } B_r \subset \mathbb{R}^3 \text{ and } r \geq \delta. \tag{2.16}$$

Since our circles are always assumed to be δ-separated, the δ-covering number $|X \cap B_r|_\delta$ is the same as the cardinality $|X \cap B_r|$. If X satisfies 2.16, we will say X is 1-*dimensional*. The main reason for the assumption (2.16) is that we plan to use the following dual circular maximal function estimate due to Pramanik, Yang, and Zahl [27] as a black box.

Theorem 2.4 (Pramanik–Yang–Zahl [27], 2022)**.** *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds. Suppose $X \subset Q$ is a set of δ-separated circles obeying the 1-dimensional Frostman non-concentration condition (2.16). Then the following estimate holds:*

$$\int_{\mathbb{R}^2} \Big(\sum_{x \in X} C_{\delta,x}(z) \Big)^{3/2} dz \leq C_\epsilon \delta^{-\epsilon} \delta |X|. \tag{2.17}$$

The estimate of Theorem 2.4 is a strengthening of Wolff's dual circular maximal function estimate which we recorded in Example 2.3. There, the family of circles X is assumed to have δ-separated radii, so in particular X is 1-dimensional.

First we will sketch how we plan to use Theorem 2.4 to address Problem 2.1. It will take some justification (see Proposition 2.17), but the integral appearing on the left-hand side of (2.17) controls the cardinality of any maximal pairwise incomparable collection $\mathcal{R}$ of δ, τ-rectangles Ω contained in $\bigcup_{x \in X} C_{\delta,x}$ satisfying

$$|X \cap \mathbf{D}_{10\delta}(\Omega)| \approx \mu, \quad \text{for all } \Omega \in \mathcal{R} \tag{2.18}$$

to the effect

$$\int_{\bigcup \mathcal{R}} \Big(\sum_{x \in X} C_{10\delta,x}(z) \Big)^{3/2} dz \geq \mu^{3/2} \int_{\bigcup \mathcal{R}} \sum_{\Omega \in \mathcal{R}} \Omega(z)\, dz = \mu^{3/2} \delta\tau |\mathcal{R}|. \tag{2.19}$$

If $v, w \in X$ is a pair of points that is D-separated, and nearly lightlike-separated, meaning $\Delta(v,w) < \delta$, then v, w belong to a common $\approx \delta \times \sqrt{D\delta} \times D$-lightplank P, so if

$$\mathcal{L}_{D,<\delta} = \{(v,w) \in X \times X : d(v,w) \sim D, \Delta(v,w) < \delta\},$$

then we have

$$|\mathcal{L}_{D,<\delta}(X)| \leq \sum_{\substack{P:\ \text{incomparable} \\ \delta \times \sqrt{D\delta} \times D \ \text{lightplanks}}} |X \cap P|^2.$$

By dyadic pigeonholing, we may assume a fraction ≈ 1 of pairs satisfy $d(v,w) \approx \delta\tau^{-2}$ for some $\sqrt{\delta} \leq \tau \ll 1$ depending on D. Fixing a maximal pairwise incomparable collection $\mathcal{P}$ of $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$ lightplanks P with $|X \cap P| \approx \mu$ and using rectangle-lightplank duality, the set

$$\mathcal{R}(\mathcal{P}) \equiv \{\mathbf{D}^*_{10\delta}(P) : P \in \mathcal{P}\}$$

of dual $\approx \delta, \tau$-rectangles $\Omega = \mathbf{D}^*_{10\delta}(P)$ is a maximal pairwise incomparable collection of $\approx \delta, \tau$-rectangles satisfying (2.18). Therefore, applying rectangle-lightplank duality and Equation (2.18),

$$\begin{aligned} |\mathcal{L}_{\delta\tau^{-2},<\delta}(X)| &\leq \sum_{P \in \mathcal{P}} |X \cap P|^2 \\ &\leq \sup\{|X \cap P| : P \in \mathcal{P}\}^{1/2} \sum_{P \in \mathcal{P}} |X \cap P|^{3/2} \\ &\approx \sup\{|X \cap P| : P \in \mathcal{P}\}^{1/2} \cdot \mu^{3/2} |\mathcal{R}(\mathcal{P})|. \end{aligned}$$

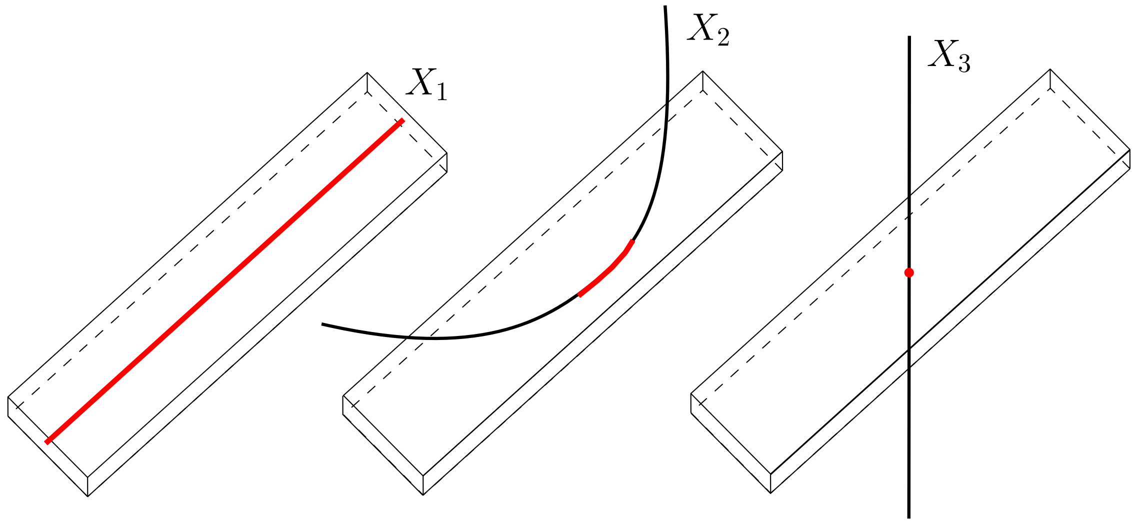

Figure 2.7: Three sets X_1, X_2, X_3 with $\mathbf{P}(X_1) \gg \mathbf{P}(X_2) \gg \mathbf{P}(X_3)$

Now, combining Equations (2.17) and (2.19),

$$\begin{aligned} |\mathcal{L}_{\delta\tau^{-2},<\delta}(X)| &\lessapprox \sup\{|X \cap P| : P \in \mathcal{P}\}^{1/2} \cdot (\delta\tau)^{-1} \cdot \delta|X| \\ &= \mathbf{P}(X)^{1/2}\tau^{-1}|X|, \end{aligned}$$

where we introduce the shorthand notation $\mathbf{P}(X) = \sup\{|X \cap P| : P \in \mathcal{P}\}$. Since $\tau \geq \sqrt{\delta}$, and there are $\approx$ 1-many dyadic values of τ,

$$|\mathcal{L}_{<\delta}(X)| \lessapprox \mathbf{P}(X)^{1/2}\delta^{-1/2}|X|. \tag{2.20}$$

Inequality (2.20) is our answer to Problem 2.1 when X is 1-dimensional (see Theorem 2.5 for the precise statement). The quantity $\mathbf{P}(X)^{1/2}$ depends on the shape of X. For example, if X manages to avoid lightplanks, so $\mathbf{P}(X) = 1$, our bound for the size of $|\mathcal{L}(X)|$ is a significant gain over the trivial bound of $|X|^2$. See Figure 2.7 for three different 1-dimensional configurations X with different values of $\mathbf{P}(X)$. In Chapter 3, we will use the bound of Equation (2.20) to prove new $L^2(w)$ estimates for $E_{\mathbb{C}one^2} f$ when the weight w is 1-dimensional.

The rest of this Section, and the accompanying Appendix A will be devoted to making the proof sketch of inequality (2.20) we've just presented rigorous. To keep the exposition in this section somewhat clean, we will prove a number of geometric lemmas regarding circles satisfying $\Delta(x, x') < \delta$ within Appendix A.

Our first minor goal is to count the number of A-incomparable δ, τ-rectangles contained within a slightly larger rectangle in Proposition 2.12. To do so, we first give a rigorous proof of Remark 2.4, which says that the tangency map $\Omega \mapsto \mathbf{D}_{10\delta}(\Omega)$ is "continuous" when Ω is a δ, τ-rectangle.

Proposition 2.11 (Continuity of $\mathbf{D}_{10\delta}$). *Suppose $\Omega^{(v)} \subset \overline{\Omega}^{(w)}$, where $\Omega^{(v)}$ is a δ, τ-rectangle, and $\overline{\Omega}^{(w)}$ is an $A\delta, A\tau$-rectangle. Let $P^{(v)} = \mathbf{D}_{10\delta}(\Omega^{(v)})$ and $\overline{P}^{(w)} = \mathbf{D}_{10A\delta}(\overline{\Omega}^{(w)}$. Then for an absolute constant $C > 1$, $P^{(v)} \subset CA^C\overline{P}^{(w)}$.*

Proof. Let $\mathcal{E} = e_m, e_l, e_s$ and $\overline{\mathcal{E}} = \bar{e}_m, \bar{e}_l, \bar{e}_s$ be the lightlike bases associated with the lightplanks $P^{(v)}$ and $\overline{P}^{(w)}$, respectively. Let $\theta = \angle e_m, \bar{e}_m$. Because $\Omega^{(v)} \subset C_{A\delta,w}$, we have $w \in \mathbf{D}_{A\delta}(\Omega^{(v)})$ by Definition 2.6. By Proposition 2.9, $\mathbf{D}_{A\delta}(\Omega^{(v)}) \subset CA^C P^{(v)}$ for an appropriate large constant C.

Hence, by Proposition 2.8, we have

$$\begin{array}{rcccccc} |\langle v-w, \bar{e}_s\rangle| & \leq & |\langle v-w, e_s\rangle| & + & O(\theta)|\langle v-w, e_m\rangle| & + & O(\theta^2)|\langle v-w, e_l\rangle| \\ & \lesssim & A^C\delta & + & O(\theta)A^C\delta\tau^{-1} & + & O(\theta^2)A^C\delta\tau^{-2}. \end{array}$$

By Proposition A.2, $|\theta| \lesssim A^C\tau$, so $|\langle v-w, \bar{e}_s\rangle| \lesssim A^C\delta$. Analogous considerations using Proposition 2.8 and $|\theta| \lesssim A^C\tau$ show $|\langle v-w, \bar{e}_m\rangle| \lesssim A^C\delta\tau^{-1}$ and $|\langle v-w, \bar{e}_l\rangle| \lesssim A^C\delta\tau^{-2}$. Since $\overline{P}^{(w)}$ is an $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, we find $v \in CA^C\overline{P}$. Now it suffices to prove that for any $x \in P^{(v)}$, the inequalities

$$|\langle x-v, \bar{e}_s\rangle| \lesssim A^C\delta$$
$$|\langle x-v, \bar{e}_m\rangle| \lesssim A^C\delta\tau^{-1}$$
$$|\langle x-v, \bar{e}_l\rangle| \lesssim A^C\delta\tau^{-2}$$

all hold. We provide the details to estimate $|\langle x-v, \bar{e}_s\rangle|$ since the proofs of the remaining inequalities are entirely analogous. By Proposition 2.8 again, we have

$$|\langle x-v, \bar{e}_s\rangle| \lesssim O(1)|\langle x-v, e_s\rangle| + O(\theta)|\langle x-v, e_m\rangle| + O(\theta^2)|\langle x-v, e_l\rangle|. \tag{2.21}$$

Since $x \in P^{(v)}$, a $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, we have

$$|\langle x-v, e_s\rangle| \lesssim \delta$$
$$|\langle x-v, e_m\rangle| \lesssim \delta\tau^{-1}$$
$$|\langle x-v, e_l\rangle| \lesssim \delta\tau^{-2}.$$

Substituting these bounds into (2.21) with $|\theta| \lesssim A^C\tau$, we obtain

$$|\langle x-v, \bar{e}_s\rangle| \lesssim A^C\delta.$$

Finally, we use Proposition 2.8 and $|\theta| \lesssim A^C\tau$ again to bound $|\langle x-v, \bar{e}_m\rangle| \lesssim A^C\delta\tau^{-1}$ and $|\langle x-v, \bar{e}_l\rangle| \lesssim A^C\delta\tau^{-2}$, and this finishes the proof. □

The next proposition is a minor refinement of Lemma 1.2 in [4]. The refinement comes in the form of being more explicit about the shape of the constant, and the only important point is it is at most $(A_0A)^C$ for an absolute constant $C > 1$ (rather than an intolerable exponential bound, e.g. e^{A_0A}).

Proposition 2.12 (Packing incomparable rectangles)**.** *For any $A_0 \geq 1$, the number of pairwise A-incomparable δ, τ-rectangles contained in an $A_0A\delta, A_0A\tau$-rectangle is $\lesssim (A_0A)^C$.*

Proof. Let $\overline{\Omega}^{(o)}$ be an $A_0A\delta, A_0A\tau$-rectangle, and let $\{\Omega^{(v_i)}\}_{i=1}^M$ be a maximal pairwise A-incomparable collection of δ, τ-rectangles contained in $\overline{\Omega}^{(o)}$. Let $\overline{P}^{(o)} \asymp \mathbf{D}_{10A_0A\delta}(\overline{\Omega}^{(o)})$ be the essentially unique $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank dual to $\overline{\Omega}^{(o)}$.

Let $\overline{\mathcal{E}} = \bar{e}_m, \bar{e}_l, \bar{e}_s$ be the lightlike basis associated to the lightplank $\overline{P}^{(o)}$. Since $\Omega^{(v_i)} \subset \overline{\Omega}^{(o)}$ for each i, by Proposition 2.11, we have $v_i \in (A_0A)^C\overline{P}^{(o)}$, so each of the following inequalities holds for every $i, j \in \{1, \ldots, M\}$:

- $|\langle v_i - v_j, \bar{e}_s\rangle| \lesssim (A_0A)^C\delta$
- $|\langle v_i - v_j, \bar{e}_m\rangle| \lesssim (A_0A)^C\delta\tau^{-1}$
- $|\langle v_i - v_j, \bar{e}_l\rangle| \lesssim (A_0A)^C\delta\tau^{-2}$.

As the circles $v_1, \ldots, v_M$ contained in $(A_0A)^C\overline{P}^{(o)}$ are A-incomparable, for each $i \neq j$, at least one of the following inequalities must hold:

- $|\langle v_i - v_j, \bar{e}_s\rangle| \gtrsim A^C\delta$
- $|\langle v_i - v_j, \bar{e}_m\rangle| \gtrsim A^C\delta\tau^{-1}$
- $|\langle v_i - v_j, \bar{e}_l\rangle| \gtrsim A^C\delta\tau^{-2}$.

Therefore, $M \lesssim (A_0A)^C$, and the claim is proved. $\square$

2.5 Application of the maximal function estimate as a black box

For $\delta > 0$ fixed, and each $\lambda \approx 1$, define a multiplicity function

$$g_{\lambda\delta}(z) = \sum_{x\in X} C_{\lambda\delta,x}(z), \qquad z \in \mathbb{R}^2.$$

If $v \in Q$ and $\Omega^{(v)}$ is a δ, τ-rectangle, recall that v is by Definition 2.6 1-tangent to Ω, and that by Proposition 2.9 if $\lambda \approx 1$, the collection $\mathbf{D}_{\lambda\delta}(\Omega^{(v)})$ is comparable to an essentially unique $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank $P^{(v)}$.

Proposition 2.13. *There is an absolute constant $C \geq 1$ such that the following holds. Let $\mathcal{R}$ be an A-incomparable collection of δ, τ-rectangles contained in $\bigcup_{x\in X} C_{\delta,x}$. For each $x \in X$, and each $\lambda \geq A$, let*

$$\mathcal{R}_{\lambda\delta}(x) = \{\Omega \in \mathcal{R} : x \in \mathbf{D}_{\lambda\delta}(\Omega)\}$$

be the rectangles in $\mathcal{R}$ which are λ-tangent to x (see Definition 2.6). Then for every $z \in \mathbb{R}^2$, we have

$$\sum_{\Omega\in\mathcal{R}_{\lambda\delta}(x)} \Omega(z) \lesssim \lambda^C C_{\lambda\delta,x}(z).$$

Proof. The δ, τ-rectangles $\Omega \in \mathcal{R}$ satisfying

$$x \in \Omega \subset C_{\lambda\delta,x}$$

are all contained in a $\lambda\delta, C\tau$-rectangle. The $\Omega \in \mathcal{R}$ are pairwise A-incomparable, so by Proposition 2.12 on packing rectangles, the number of such Ω is at most λ^C for some absolute constant C. $\square$

Definition 2.10 (Multiplicity of a rectangle)**.** *For $\lambda \approx 1$, and a δ, τ-rectangle Ω, let*

$$\mu_{\lambda\delta}(\Omega) = |X \cap \mathbf{D}_{\lambda\delta}(\Omega)|$$

be the number of points in X that are λ-tangent to Ω. We refer to $\mu_{\lambda\delta}(\Omega)$ as the multiplicity *of Ω (or the X*-multiplicity *if we want to emphasize the set X of circles).*

We recall the notation that if $E \subset \mathbb{R}^2$ is a set, we may use $E(z)$ as shorthand for the indicator function $1_E(z)$. Recall the following notation we used in Proposition 2.13: if $\mathcal{R}$ is any set of δ, τ-rectangles, then

$$\mathcal{R}_{\lambda\delta}(x) = \{\Omega \in \mathcal{R} : x \in \mathbf{D}_{10\delta}(\Omega)\}$$

is the set of rectangles of $\mathcal{R}$ that are λ-tangent to x. If $a(x, \Omega)$ is any quantity that depends on $x \in X$ and $\Omega \in \mathcal{R}$, then we have the following double-counting/Fubini relationship:

$$\sum_{\Omega \in \mathcal{R}} \sum_{x \in X \cap \mathbf{D}_{\lambda\delta}(\Omega)} a(x, \Omega) = \sum_{x \in X} \sum_{\Omega \in \mathcal{R}_{\lambda\delta}(x)} a(x, \Omega).$$

Proposition 2.14. *If $\mathcal{R}$ is a pairwise A-incomparable collection of δ, τ-rectangles contained in $\bigcup_{x \in X} C_{\delta,x}$ and $\lambda \geq A$, then*

$$\sum_{\Omega \in \mathcal{R}} \mu_{\lambda\delta}(\Omega)\Omega(z) \lesssim \lambda^C g_{\lambda\delta}(z) = \lambda^C \sum_{x \in X} C_{\lambda\delta,x}(z), \quad z \in \mathbb{R}^2.$$

Proof. By Definition 2.10 of $\mu_{\lambda\delta}(\Omega)$ and double-counting,

$$\begin{aligned}\sum_{\Omega \in \mathcal{R}} \mu_{\lambda\delta}(\Omega)\Omega(z) &= \sum_{\Omega \in \mathcal{R}} \Big(\sum_{x \in X \cap \mathbf{D}_{\lambda\delta}(\Omega)} C_{\lambda\delta,x}(z) \Big) \Omega(z) \\ &= \sum_{x \in X} C_{\lambda\delta,x}(z) \sum_{\Omega \in \mathcal{R}_{\lambda\delta}(x)} \Omega(z).\end{aligned}$$

By Proposition 2.13, for each $\lambda \geq A$, the inner sum is bounded by $\lambda^C C_{\lambda\delta,x}(z)$. This finishes the proof by the definition of $g_{\lambda\delta}(z)$. $\square$

For $\delta^{1/2} \leq \tau \ll 1$, let

$$T(\tau) = \sup\{\mu_{10\delta}(\Omega) : \Omega \subset \mathbb{R}^2 \text{ is a } \delta, \tau\text{-rectangle}\}$$

be the maximal number of circles of X 10-tangent to a δ, τ-rectangle.

Proposition 2.15 (Maximal tangency number and lightplank occupancy)**.** *For $\delta^{1/2} \leq \tau \ll 1$, we have*

$$T(\tau) \approx \sup\{|X \cap P| : P \textit{ is a } \delta \times \delta\tau^{-1} \times \delta\tau^{-2}\textit{-lightplank}\}.$$

Proof. For this proof, let $\mathbf{P}(\tau)$ denote the maximal $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank occupancy of X:

$$\mathbf{P}(\tau) = \sup\{|X \cap P| : P \text{ is a } \delta \times \delta\tau^{-1} \times \delta\tau^{-2}\text{-lightplank}\}.$$

Let Ω be a δ, τ-rectangle such that $\mu_{10\delta}(\Omega) = T(\tau)$, and consider that $\mathbf{D}_{10\delta}(\Omega)$ can be covered by $\approx$ 1-many pairwise comparable $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplanks. By the pigeonhole principle, there exists a $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank P_0 such that

$$|X \cap P_0| \gtrapprox |X \cap \mathbf{D}_{10\delta}(\Omega)| = \mu_{10\delta}(\Omega) = T(\tau).$$

This shows that $\mathbf{P}(\tau) \gtrapprox T(\tau)$. Conversely, let P be a $\delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank such that $|X \cap P| = \mathbf{P}(\tau)$. There are $\approx$ 1-many pairwise comparable δ, τ-rectangles Ω which cover $\mathbf{D}^*_{10\delta}(P)$. By the pigeonhole principle, there exists a δ, τ-rectangle Ω_0 such that

$$|X \cap \mathbf{D}_{10\delta}(\Omega_0)| \gtrapprox |X \cap P| = \mathbf{P}(\tau).$$

This finishes the proof that $T(\tau) \approx \mathbf{P}(\tau)$. □

Proposition 2.16. *If Ω is a δ, τ-rectangle, then for each $\lambda \approx 1$,*

$$\mu_{\lambda\delta}(\Omega) \lessapprox T(\tau).$$

Proof. This would follow immediately from the definition of $\mu_{\lambda\delta}(\Omega)$ (with "$\le$" in place of "$\lessapprox$") if $T(\tau)$ were defined with $\mathbf{D}_{\lambda\delta}(\Omega)$ instead of $\mathbf{D}_{10\delta}(\Omega)$. By Proposition 2.9, $\mathbf{D}_{\lambda\delta}(\Omega)$ is comparable to a $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, which can be covered by $\approx$ 1-many translates $\{\mathbf{D}_{10\delta}(\Omega) + v_i\}_i$, at least one of which must satisfy by the pigeonhole principle,

$$|X \cap (\mathbf{D}_{10\delta}(\Omega) + v_i)| \gtrapprox \mu_{\lambda\delta}(\Omega).$$

Therefore, $T(\tau) \gtrapprox \mu_{\lambda\delta}(\Omega)$. □

For a dyadic number $1 \le M \le T(\tau)$, a number $\lambda \approx 1$, and a collection $\mathcal{R}$ of δ, τ-rectangles, let

$$\mathcal{R}_{M,\lambda} = \{\Omega \in \mathcal{R} : \mu_{\lambda\delta}(\Omega) \sim M\}.$$

We are now ready to rigorously justify the proof sketch we presented at the beginning of this Section.

Proposition 2.17 (Counting tangency rectangles)**.** *Suppose $X \subset Q$ is a set of δ-separated circles obeying the 1-dimensional Frostman non-concentration condition*

$$|X \cap B_r|_\delta \le \frac{r}{\delta}, \quad \textit{for all } r\textit{-balls } B_r \subset \mathbb{R}^3 \textit{ and } r \ge \delta.$$

If $\mathcal{R}$ is any pairwise A-incomparable collection of δ, τ-rectangles contained in $\bigcup_{x \in X} C_{\delta,x}$, then for each $M \in [1, T(\tau)]$ and $A \le \lambda \lessapprox 1$,

$$M^{3/2} |\mathcal{R}_{M,\lambda}| \lessapprox \tau^{-1} |X|.$$

Proof. By Proposition 2.14, if $\lambda \ge A$, we have

$$\lambda^C g_{\lambda\delta}(z) \gtrapprox \sum_{\Omega \in \mathcal{R}} \mu_{\lambda\delta}(\Omega)\Omega(z), \quad z \in \mathbb{R}^2.$$

We organize the sum on the right-hand side by the dyadic level sets of $\mu_{\lambda\delta}(\Omega)$, keeping in mind Proposition 2.16:

$$\lambda^C g_{\lambda\delta}(y) \gtrsim \sum_{\substack{1<M<T(\tau)\\ M \text{ dyadic}}} M \sum_{\Omega\in\mathcal{R}_{M,\lambda}} \Omega(y).$$

By Theorem 2.4, and the embedding $\ell^1 \hookrightarrow \ell^{3/2}$,

$$\delta|X| \gtrapprox \lambda^C \int_{\mathbb{R}^2} g_{\lambda\delta}(z)^{3/2}\, dz \gtrapprox M^{3/2}|\mathcal{R}_{M,\lambda}| \cdot |\Omega|.$$

Dividing by $|\Omega| \sim \delta\tau$ finishes the proof. □

2.5.1 Nearly lightlike pairs

For dyadic numbers $\delta < \Delta \leq D < 1$, define the collection

$$\mathcal{L}_{D,\Delta}(X) = \{(v,w) \in X \times X : d(v,w) \sim D, \Delta(v,w) \sim \Delta\}.$$

We will refer to a pair $(v,w) \in \mathcal{L}_{D,\Delta}(X)$ as *nearly lightlike* when $\Delta \lessapprox \delta$. Let $\tau_D = \delta^{1/2}D^{-1/2}$, and recall

$$T(\tau_D) = \sup\{|X \cap \mathbf{D}_{10\delta}(\Omega)| : \Omega \subset \mathbb{R}^2 \text{ is a } \delta, \tau_D\text{-rectangle}\}$$

is the maximal number of circles of X that are 10-tangent to any δ, τ_D-rectangle. The reason for the choice $\tau_D = \delta^{1/2}D^{-1/2}$ is that if $d(v,w) \approx D$, then both v and w belong to an $\approx \delta \times \sqrt{\delta D} \times D$-lightplank, and the circles v, w are $\gtrapprox \delta, \tau_D$-tangent, in the sense of the following Definition 2.11.

Definition 2.11. *We say two circles C_v, C_w are $\gtrapprox \delta, \tau$-tangent if there are ≈ 1-comparable δ, τ-rectangles $\Omega^{(v)} \subset C_{\delta,v}$, $\Omega^{(w)} \subset C_{\delta,w}$.*

Proposition 2.18. *If $(v,w) \in \mathcal{L}_{D,\Delta}(X)$ for $D \gg \delta$ and $\Delta \lessapprox \delta$, then C_v, C_w are $\gtrapprox \delta, \tau_D$-tangent.*

Proof. Suppose $D \gg \delta$, $\Delta \lessapprox \delta$, and $(v,w) \in \mathcal{L}_{D,\Delta}(X)$. We will find a $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank P such that both $v, w \in P$. By duality, for appropriate $\lambda \approx 1$,

$$\mathbf{D}^*_{\lambda\delta}(P) \cap C_{\delta,v} \text{ and } \mathbf{D}^*_{\lambda\delta}(P) \cap C_{\delta,w}$$

are ≈ 1-comparable $\approx \delta, \tau_D$-rectangles contained in $C_{\delta,v}$ and $C_{\delta,w}$, respectively, so this will finish the proof.

Recall Γ_v is the lightcone with vertex v. Let $w_0 \in \Gamma_v$ be the nearest point to w. By definition, $v - w_0$ is a lightlike vector, and since $\delta \ll D$, we have $|v - w_0| \sim |v - w| \sim D$ and $|w_0 - w| \sim \Delta(v,w) \lessapprox \delta$. Choose a unit vector e_l parallel to $v - w_0$, another unit vector e_m orthogonal to $(v - w_0)'$ (the projection of $v - w_0$ to $\mathbb{R}^2$), and a third unit vector e_s so that e_m, e_l, e_s is a lightlike basis for $\mathbb{R}^3$. It suffices to check

(i) $|\langle v - w, e_l\rangle| \lessapprox \delta\tau_D^{-2}$,

(ii) $|\langle v - w, e_m\rangle| \lessapprox \delta\tau_D^{-1}$, and

(iii) $|\langle v-w, e_s\rangle| \lessapprox \delta$.

Since $\delta\tau_D^{-2} = D \sim |\langle v-w, e_l\rangle|$, and $\Delta(v,w) = |w-w_0| \sim |\langle v-w, e_s\rangle| \lessapprox \delta$, only point (ii) needs elaboration. But by elementary geometry considerations, this is a simple consequence of the assumption $d(v,w) \sim D$ and $\Delta(v,w) \lessapprox \delta$. □

Proposition 2.19 (Covering by lightplanks)**.** *There is an absolute constant $C > 1$ so that the following holds. If $(v,w) \in \mathcal{L}_{D,\Delta}(X)$ and $\mathcal{R}$ is a maximal pairwise A-incomparable collection of δ, τ_D-rectangles contained in $\bigcup_{x\in X} C_{\delta,x}$, then there exists $\Omega_0 \in \mathcal{R}$ so that $v, w \in \mathbf{D}_{CA^C\delta}(\Omega_0)$.*

Proof. By Proposition 2.18, there are ≈ 1-comparable δ, τ_D-rectangles $\Omega^{(v)}, \Omega^{(w)}$ in $C_{\delta,v}, C_{\delta,w}$ respectively. By maximality of $\mathcal{R}$ with respect to A-incomparability, there is some $\Omega_0 \in \mathcal{R}$ such that $\Omega^{(v)}$ (say) is comparable to Ω_0, hence $v \in \mathbf{D}_{CA^C}(\Omega_0)$ and since $\Omega^{(v)}$ and $\Omega^{(w)}$ are comparable, by almost-transitivity (Proposition A.4), Ω_0 and $\Omega^{(w)}$ are CA^C-comparable. Hence $w \in \mathbf{D}_{CA^C\delta}(\Omega_0)$ for a large enough absolute constant C, and the claim is proved. □

Theorem 2.5 (Nearly lightlike pairs for a 1-dimensional configuration of circles)**.** *Suppose $X \subset Q$ is a set of δ-separated circles obeying the 1-dimensional Frostman non-concentration condition*

$$|X \cap B_r|_\delta \leq \frac{r}{\delta}, \quad \textit{for all } r\textit{-balls } B_r \subset \mathbb{R}^3 \textit{ and } r \geq \delta.$$

If $\Delta \lessapprox \delta$ and $D \gg \delta$, then $|\mathcal{L}_{D,\Delta}(X)| \lessapprox T(\tau_D)^{1/2}(\delta^{-1}D)^{1/2}|X|$, where $\tau_D = \delta^{1/2}D^{-1/2}$.

Proof. Let $A \approx 1$ be a parameter (take $A = \delta^{-\epsilon}$ for definiteness), and fix an arbitrary maximal pairwise A-incomparable collection $\mathcal{R}$ of δ, τ_D-rectangles contained in $\bigcup_{x\in X} C_{\delta,x}$.

By Proposition 2.19, for a given $(v,w) \in \mathcal{L}_{D,\Delta}(X)$, we can find a rectangle $\Omega \in \mathcal{R}$ such that $v, w \in \mathbf{D}_{\lambda\delta}(\Omega)$ for some $\lambda = A^{O(1)}$, and we can write

$$\mathcal{L}_{D,\Delta}(X) \subset \bigcup_{\Omega\in\mathcal{R}} \{(v,w) \in X \times X : v, w \in \mathbf{D}_{\lambda\delta}(\Omega)\}.$$

By the union bound,

$$|\mathcal{L}_{D,\Delta}(X)| \leq \sum_{\Omega\in\mathcal{R}} |X \cap \mathbf{D}_{\lambda\delta}(\Omega)|^2 = \sum_{\Omega\in\mathcal{R}} \mu_{\lambda\delta}(\Omega)^2. \tag{2.22}$$

Recall that by Proposition 2.16, $\mu_{\lambda\delta}(\Omega) \lessapprox T(\tau_D)$. We organize the last sum on the right-hand side of (2.22) by the dyadic value of $\mu_{\lambda\delta}(\Omega)$, up to $T(\tau_D)$. Letting $\mathcal{R}_{M,\lambda} = \{\Omega \in \mathcal{R} : \mu_{\lambda\delta}(\Omega) \sim M\}$, we estimate (2.22) by

$$\sum_{\substack{1<M<T(\tau_D)\\ M \text{ dyadic}}} M^2|\mathcal{R}_{M,\lambda}| \leq T(\tau_D)^{1/2} \sum_{1<M<T(\tau_D)} M^{3/2}|\mathcal{R}_{M,\lambda}|.$$

By Proposition 2.17, for each M, $M^{3/2}|\mathcal{R}_{M,\lambda}| \lessapprox \tau_D^{-1}|X| = (\delta^{-1}D)^{\frac{1}{2}}|X|$. As $T(\tau_D) \leq |X \cap Q| \lessapprox \delta^{-1}$ for any 1-dimensional set of δ-separated circles in $Q = B(e_3, \alpha_0)$, there are ≈ 1-many values of M in the sum, so we have shown $|\mathcal{L}_{D,\Delta}(X)| \lessapprox T(\tau_D)^{1/2}(\delta^{-1}D)^{1/2}|X|$. This finishes the proof. □

Chapter 3

Sparse Fourier restriction estimates for the cone

In this chapter, we will prove the following sparse Fourier restriction theorem for the cone.

Theorem 3.1 (O., 2023)**.** *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose $X \subset B_R$ is a disjoint union of unit balls that satisfies the 1-dimensional Frostman condition*

$$|X \cap B(x,r)| \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(X)$ be the quantity

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then the estimate

$$\int_X |E_{\mathbb{C}one^2} f|^2 \leq C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/2} \|f\|^2_{L^2(\mathbb{C}one^2)}$$

holds.

Theorem 3.1 appeared in [17]. Along the way, we will describe sparse Fourier restriction estimates in some generality, as well as discuss the connection with Fourier averages over submanifolds in $\mathbb{R}^n$ and some of the background on Fourier average estimates. We will also give some comparison of the relative strength of Theorem 3.1 compared with what refined decoupling for the cone, Theorem 1.4, says.

3.1 Sparse restriction estimates

In Chapter 1, we described Fourier extension and restriction estimates. In this chapter, we take a more refined look at the Fourier extension operator and ask about weighted estimates of the form

$$\left(\int_{\mathbb{R}^n} |E_{\mathcal{M}} f|^q w\right)^{\frac{1}{q}} \leq C \|f\|_{L^2(\mathcal{M})} \tag{3.1}$$

for $w\colon \mathbb{R}^n \to [0,\infty)$ a nonnegative weight function. This type of inequality will be referred to as a "sparse Fourier restriction estimate" or a "weighted Fourier extension estimate."

Definition 3.1. *The number $C_q(\mathcal{M}, w)$ is the smallest constant such that (3.1) holds for every f.*

We will quickly specify the weights under consideration. We will assume that $\mathcal{M}$ is a compact submanifold of $\mathbb{R}^n$ contained in the unit ball with a smooth surface measure σ. By the uncertainty principle, $E_{\mathcal{M}}f$ is approximately constant on scale 1, so we will focus on weights w which equal 1 on a disjoint union of unit balls in $\mathbb{R}^n$. If $w = 1_X$, the indicator function for a set X, we will write $C_q(\mathcal{M}, X)$ instead of $C_q(\mathcal{M}, 1_X)$.

Our primary focus will be on $q = 1$ and $q = 2$, but other values of q are also interesting. By Hölder's inequality, we have the relationship

$$C_1(\mathcal{M}, w) \leq (\int_{\mathbb{R}^n} w)^{\frac{1}{q'}} C_q(\mathcal{M}, w), \quad q \geq 1.$$

The role of $q = 1$ is somewhat special because of its close connection with Fourier averages, a topic we will pick up in Section 3.2. For now, we point out that there is a converse which will allow us to prove estimates for $C_q(\mathcal{M}, X)$, $q > 1$, provided we have an estimate for $C_1(\mathcal{M}, w1_X)$ for every measurable function $w\colon \mathbb{R}^n \to [0, 1]$.

Theorem 3.2. *Fix X a disjoint union of N unit balls, and let $1 < q < \infty$. Let $q' = \frac{q}{q-1}$ be the Hölder conjugate exponent of q. Suppose that for some $A > 1$ and for every measurable function $w\colon \mathbb{R}^n \to [0, 1]$, we know*

$$\int_X |Ef| w \leq A(\int_X w)^{\frac{1}{q'}} \|f\|_{L^2(\mathcal{M})}.$$

Then for every f with $\|f\|_{L^2(\mathcal{M})} = 1$,

$$(\int_X |Ef|^q)^{\frac{1}{q}} \lesssim A(\log N)^{\frac{1}{q}}.$$

In other words, if $C_1(\mathcal{M}, w1_X) \leq A(\int w1_X)^{1/q'}$ for every nonnegative $w \leq 1$, then $C_q(\mathcal{M}, X) \lesssim A(\log N)^{1/q}$.

Proof. Let $\|f\|_{L^2(\mathcal{M})} = 1$, and let $c > 0$ be very small to be determined. Note $|Ef| \leq \|f\|_{L^1(\mathcal{M})} \lesssim \|f\|_{L^2(\mathcal{M})} = 1$, so we can write

$$\int_X |Ef|^q = \int_{X \cap \{|Ef| < c\}} |Ef|^q + \sum_{c < \lambda \lesssim 1} \int_{X \cap \{|Ef| \sim \lambda\}} |Ef|^q$$

where the sum is over dyadic values of λ. For the first term we can bound it by

$$\int_{X \cap \{|Ef| < c\}} |Ef|^q \leq c^q N < 1,$$

if $c < N^{-1/q}$. If this term dominates, we are done, so we assume the second term dominates. We write out the second term as

$$\sum_{c < \lambda \lesssim 1} \int_{X \cap \{|Ef| \sim \lambda\}} |Ef|^q \sim \sum_{c < \lambda < 1} \lambda^{q-1} \int_X |Ef| 1_{\{|Ef| \sim \lambda\}}.$$

By assumption, since $0 \le 1_{\{|Ef|\sim\lambda\}} \le 1$ and $\|f\|_{L^2} = 1$, this is bounded by

$$A \sum_{c<\lambda\lesssim 1} \lambda^{q-1} |X \cap \{|Ef| \sim \lambda\}|^{\frac{q-1}{q}}.$$

By Hölder's inequality, this is bounded by

$$A(\log N)^{\frac{1}{q}} \Big(\sum_{c<\lambda\lesssim 1} \lambda^q |X \cap \{|Ef| \sim \lambda\}|\Big)^{\frac{q-1}{q}} \lesssim A(\log N)^{\frac{1}{q}} \Big(\int_X |Ef|^q\Big)^{\frac{q-1}{q}}.$$

Finally, we cancel $(\int_X |Ef|^q)^{\frac{q-1}{q}}$ from both sides to obtain

$$\Big(\int_X |Ef|^q\Big)^{\frac{1}{q}} \lesssim A(\log N)^{\frac{1}{q}}.$$

□

With Theorem 3.1 in mind, by Theorem 3.2, our main goal will be to prove the following theorem:

Theorem 3.3. *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose $X \subset B_R$ is a disjoint union of unit balls that satisfies the 1-dimensional Frostman condition*

$$|X \cap B(x, r)| \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(X)$ be the quantity

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then for any measurable function $w\colon \mathbb{R}^3 \to [0, 1]$, the estimate

$$\int_X |Ef|w \le C_\epsilon R^\epsilon\, \mathbf{P}(X)^{1/4} \Big(\int_X w\Big)^{1/2} \|f\|_{L^2(d\sigma)}$$

holds.

Theorem 3.3 will follow as a corollary of the following Fourier average estimate.

Theorem 3.4 (O., [17])**.** *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose ν is a measure that agrees with the Lebesgue measure on a disjoint union of unit balls $X \subset B_R$ and satisfies the 1-dimensional Frostman condition*

$$\nu(B(x, r)) \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(\nu)$ be the quantity

$$\mathbf{P}(\nu) = \sup\{\nu(P) : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then the estimate

$$\int |\widehat{\nu}|^2\, d\sigma \le C_\epsilon R^\epsilon\, \mathbf{P}(\nu)^{1/2} \|\nu\|$$

holds, where $\|\nu\| = |X|$ is the total mass of ν.

Before giving the proof that Theorem 3.4 implies Theorem 3.3 (and hence Theorem 3.1 by Theorem 3.2), we discuss the connection between the L^1 sparse restriction problem and Fourier averages.

3.2 Fourier averages

Suppose we are interested in determining $C_1(\mathcal{M}, w)$ for a weight w. The example to keep in mind is $w = 1_X$, for X a disjoint union of unit balls. The following Proposition relates $C_1(\mathcal{M}, w)$ to $L^2(\mathcal{M})$-averages of the Fourier transform $\widehat{hw}$ for h a measurable function with $|h| \leq 1$.

Proposition 3.1. *Given a manifold with smooth surface measure $(\mathcal{M}, \sigma)$ and a weight w, we have*

$$C_1(\mathcal{M}, w) \leq \sup_{\|h\|_{L^\infty}=1} \|\widehat{hw}\|_{L^2(\mathcal{M},\sigma)}.$$

Proof. Given $f \in L^2(\mathcal{M}, \sigma)$, by duality, for an appropriate $|h| \leq 1$, we have

$$\int |Ef| w = \int Ef \cdot hw.$$

By Parseval and Cauchy–Schwarz, this is

$$\int_{\mathcal{M}} f \cdot \overline{\widehat{hw}}\, d\sigma \leq \|\widehat{hw}\|_{L^2(\mathcal{M},\sigma)} \|f\|_{L^2(\mathcal{M},\sigma)}.$$

The claim then follows by the definition of $C_1(\mathcal{M}, w)$. □

Given a nonnegative finite measure ν in $\mathbb{R}^n$, we define its Fourier transform by

$$\widehat{\nu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi}\, d\nu(x).$$

The Fourier transform of a finite measure need not decay as $|\xi| \to \infty$. Perhaps the most extreme example of this is the Dirac mass δ_0, whose Fourier transform is identically 1.

Another set of non-examples includes measures supported in affine subspaces of $\mathbb{R}^n$:

Example 3.1. *Let $1 \leq k < n$, $\phi(x_1, \dots, x_k)$ be a non-negative Schwartz function, and consider the measure*

$$\nu(dx_1 \cdots dx_n) = \phi(x_1, \dots, x_k)\, dx_1 \cdots dx_k \otimes \delta_0(dx_{k+1} \cdots dx_n).$$

The measure ν is supported in $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. The Fourier transform of ν is

$$\widehat{\nu}(\xi_1, \dots, \xi_n) = \widehat{\phi}(\xi_1, \dots, \xi_k),$$

which is independent of its last $n - k$ coordinates. In particular, $\widehat{\nu}$ doesn't decay in the "vertical directions."

As a special case, consider $n = 2$, $k = 1$, so

$$\widehat{\nu}(\xi_1, \xi_2) = \widehat{\phi}(\xi_1),$$

and consider the average value of $|\widehat{\nu}|^2$ on a large circle:

$$\int_0^{2\pi} |\widehat{\nu}(Re^{i\theta})|^2\, d\theta = 2 \int_{-\pi}^{0} |\widehat{\phi}(R\cos\theta)|^2\, d\theta = 2 \int_{-\pi/2}^{\pi/2} |\widehat{\phi}(R\sin\theta)|^2\, d\theta.$$

For $\delta < |\theta| < \pi/2$, $R|\sin\theta| > R\delta$, so by the Schwartz decay of $\widehat{\phi}$,

$$2\int_{-\pi/2}^{\pi/2} |\widehat{\phi}(R\sin\theta)|^2 \, d\theta \lesssim_N \delta + (R\delta)^{-N}.$$

Taking $\delta = R^{-1+\epsilon}$, and choosing $N > 1/\epsilon$, we get

$$\int_0^{2\pi} |\widehat{\nu}(Re^{i\theta})|^2 \, d\theta \lesssim_\epsilon R^{\epsilon-1}.$$

This upper bound is nearly tight for this example because $\int_{|\theta|<R^{-1}} |\widehat{\phi}(R\sin\theta)|^2 \, d\theta \gtrsim R^{-1}$.

Remark 3.1. *When we say "measure," we shall always mean a positive Borel measure. We will take care to note when measures we consider are signed or complex.*

The calculation of Example 3.1 shows that if the measure ν is not *so singular,* then in spite of the lack of decay of $\widehat{\nu}$, the averages of $\widehat{\nu}$ over dilations of a curved set may still decay as the dilation parameter tends to infinity. This is analogous to the Tomas–Stein Theorem 1.1 which shows that as long as $\mathcal{M}$ has nonvanishing Gaussian curvature, there are nontrivial restriction theorems $E_{\mathcal{M}} \colon L^p(\mathcal{M}) \to L^q(\mathbb{R}^n)$.

To make the notion of a measure not being *so singular* precise, we introduce the α-dimensional energy of a measure.

Definition 3.2 (Energy of a measure)**.** *Let ν be a Borel measure in $\mathbb{R}^n$. For $0 < \alpha < n$, let*

$$I_\alpha(\nu) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{d\nu(x)d\nu(y)}{|x-y|^\alpha}.$$

By Fourier transform properties, the α-dimensional energy can be expressed in terms of $\widehat{\nu}$:

Proposition 3.2. *There is an absolute constant $c_{\alpha,n}$ so that the α-dimensional energy of ν can be expressed in terms of the Fourier transform $\widehat{\nu}$:*

$$I_\alpha(\nu) = c_{\alpha,n} \int_{\mathbb{R}^n} \frac{|\widehat{\nu}(\xi)|^2}{|\xi|^{n-\alpha}} \, d\xi.$$

See Theorem 3.10 of Mattila's book *Fourier Analysis and Hausdorff Dimension* [28] for the proof by Parseval's relation, and the fact that if $k(x) = |x|^{-\alpha}$, then $\widehat{k}(\xi) = c|\xi|^{-(n-\alpha)}$. In polar coordinates, the Fourier expression for the energy $I_\alpha(\nu)$ becomes

$$\int_0^\infty \Big(\int_{S^{n-1}} |\widehat{\nu}(Re)|^2 \, d\sigma(e) \Big) \, R^{\alpha-1} \, dR.$$

If $I_\alpha(\nu) < \infty$, then it's not unreasonable to expect that $\int_{RS^{n-1}} |\widehat{\nu}|^2 \, d\sigma$ goes to zero like a polynomial of R.

3.2.1 Fourier averages over circles and spheres

We recall here some of the history on Fourier averages over circles and spheres.

In 1987, Mattila [18] proposed the problem of identifying the optimal decay rate of spherical Fourier averages for compactly supported measures with finite α-energy.

Definition 3.3 (Optimal spherical Fourier average decay rate). *For $n \geq 1$ and $0 < \alpha < n$, $\beta_n(\alpha)$ is the supremum of the numbers $\beta > 0$ such that there exists C with*

$$\int_{S^{n-1}} |\widehat{\nu}(Re)|^2 \, d\sigma(e) \leq C R^{-\beta} I_\alpha(\nu), \quad \text{for all } R > 1,$$

for all measures ν supported in the unit ball of $\mathbb{R}^n$.

In 1987, Mattila [18] established the sharp exponent $\beta_n(\alpha) = \alpha$ when $0 < \alpha \leq \frac{n-1}{2}$. For the lower bound $\beta_n(\alpha) \geq \alpha$, Mattila used the pointwise bound $|\widehat{\sigma}(x)| \lesssim |x|^{-(n-1)/2}$, where σ is the surface measure of the sphere (see Proposition 1.1), by rewriting the Fourier average as a double integral of the form

$$\int_{S^{n-1}} |\widehat{\nu}|^2 \, d\sigma = \iint \widehat{\sigma}(x-y) \, d\nu(x) \, d\nu(y).$$

Mattila also made some partial progress when $\alpha > \frac{n-1}{2}$ using properties of Bessel functions, and furthermore showed that when $n = 2$, $\beta_2(\alpha) = \frac{1}{2}$ for $\frac{1}{2} < \alpha \leq 1$. Mattila's examples to demonstrate $\beta_2(\alpha) \leq \frac{1}{2}$ in the range $\frac{1}{2} < \alpha \leq 1$ were based on modifications of Example 3.1 where the planar piece of ν in that example is replaced with a Cantor set.

In 1993, Sjölin [29] made some more partial progress on lower bounds using a more efficient Fourier analysis when $\alpha > \frac{n-1}{2}$ and described new Knapp-type examples to illustrate upper bounds of $\beta_n(\alpha)$.

In 1994, using techniques from restriction theory, Bourgain [30] made more partial progress, showing $\beta_2(\alpha) \geq \frac{\alpha}{2} - \frac{1}{6} - \epsilon$. Bourgain also made some further improved lower bounds on $\beta_3(\alpha)$ using a partial improvement on the restriction problem in $\mathbb{R}^3$ beyond the Tomas–Stein exponent (see Theorem 1.1).

In 1999, building on Bourgain's approach based on restriction theory, Wolff [31] proved the best possible lower bound $\beta_2(\alpha) \geq \frac{\alpha}{2}$ when $1 < \alpha < 2$, closing the question on $\beta_2(\alpha)$ for the last unknown range $1 < \alpha < 2$ in $\mathbb{R}^2$. We will describe examples exhibiting the sharpness of Wolff's result following the statement of Theorem 3.5.

Theorem 3.5 (Wolff, 1999 [31]). *Fix $\alpha \in (0, 2)$. Then for any $\epsilon > 0$, there is a constant C_ϵ such that the following is true. Let ν be a positive measure in $\mathbb{R}^2$ supported in the unit disk and with α-dimensional energy $I_\alpha(\nu) = 1$. Then for any $R \geq 1$,*

$$\int_0^{2\pi} |\widehat{\nu}(Re^{i\theta})|^2 \, d\theta \leq C_\epsilon R^\epsilon R^{-\alpha/2}.$$

By combining Wolff's estimate with the other sharp values of $\beta_2(\alpha)$, we have the following Theorem.

Theorem 3.6. *The following sharp decay rates are known for circular Fourier averages of measures with finite α-dimensional energy:*

$$\beta_2(\alpha) = \begin{cases} \alpha, & \alpha \in (0, 1/2], \quad \text{(Mattila, 1987 [18])} \\ 1/2, & \alpha \in [1/2, 1], \quad \text{(Mattila —)} \\ \alpha/2, & \alpha \in (1, 2), \quad \text{(Wolff, 1999 [31])}. \end{cases}$$

In [31], Wolff gave examples of α-dimensional measures for each regime of α illustrating the sharpness of $\beta_2(\alpha)$. The example in the range $\frac{1}{2} \le \alpha \le 1$ is essentially the same as Sjölin's, so we will focus on describing the examples Wolff gave for the ranges $0 < \alpha < 1/2$ and $1 < \alpha < 2$. In order to describe his examples, first we recall another lemma which relates spherical decay rates of Fourier transforms of measures and $L^1(\mu)$-Fourier extension estimates.

Lemma 3.1 (Wolff, [31] Lemma 3.1 (b))**.** *If $\beta < \beta_2(\alpha)$, then there is a constant C such that the following are true.*

(i) Let f be a function on the circle $|\xi| = R$ with $L^2(\sigma)$-norm 1; and let G be the inverse Fourier transform of the measure $f(R\xi)\, d\sigma(\xi)$. Then

$$\int |G| \, d\mu \le 4CR^{-\frac{\beta}{2}} \sqrt{I_\alpha(\mu)}$$

for all measures μ supported in the unit ball.

(ii) Let $f \colon \mathbb{R}^2 \to \mathbb{C}$ be a function supported in the annulus $A_R = \{\xi \in \mathbb{R}^2 : R - 1 \le |\xi| \le R + 1\}$ with L^2-norm 1, and let G be the inverse Fourier transform of f. Then

$$\int |G| \, d\mu \le 4CR^{-\frac{\beta}{2} + \frac{1}{2}} \sqrt{I_\alpha(\mu)}$$

for all measures supported in the unit ball.

Proof. The proofs are very similar, and the proof of (i) is in Wolff's paper, so we only give the proof of (ii). By assumption, for any measure μ supported in the unit ball,

$$\begin{aligned} \int_{A_R} |\widehat{\mu}||f| &\le \Big(\int_{R-1}^{R+1} \int_{S^1} |\widehat{\mu}(re)|^2 \, d\sigma(e) \, r \, dr\Big)^{\frac{1}{2}} \|f\|_{L^2(A_R)} \\ &\le CR^{-\frac{\beta}{2} + \frac{1}{2}} \sqrt{I_\alpha(\mu)}. \end{aligned}$$

It then follows that

$$\int |\widehat{\mu}||f| \le 4CR^{-\frac{\beta}{2} + \frac{1}{2}} \sqrt{I_\alpha(|\mu|)} \tag{3.2}$$

for measures that are not necessarily positive by expressing μ as a linear combination of four positive measures in the natural way. Set $d\nu = (\bar{G}/|G|) \, d\mu$. Thus, by Parseval's theorem,

$$\begin{aligned} \int |G| \, d\mu &= \int G \, d\nu \\ &= \int_{A_R} \widehat{\nu} \bar{f} \\ &\le 4CR^{-\frac{\beta}{2} + \frac{1}{2}} \sqrt{I_\alpha(\mu)}, \end{aligned}$$

by (3.2). □

Example 3.2 ($\beta_2(\alpha) \leq \alpha$ for $0 < \alpha < \frac{1}{2}$). *This construction is based on periodicity. Let* $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ *be a nonnegative smooth compactly supported function whose Fourier transform is nonzero on the closed unit disk, let* e_1, e_2 *be the standard basis of* $\mathbb{R}^2$, *and let* $\phi_R(x) = R^2\phi(Rx)$. *Let* $J \sim R^\alpha$ *and define*

$$a(x) = \frac{1}{2J+1}\sum_{|j|\leq J}\phi_R(x - \frac{j}{J+\frac{1}{2}}e_2)$$

and $d\mu = a(x)\,dx$. *Since* $\alpha < 1$, *we have* $I_\alpha(\mu)$ *is bounded independently of* R. *By the explicit form of the Dirichlet kernel (see Equation (3.3) in Example 3.3), it follows* $|\widehat{\mu}(\xi)| \gtrsim 1$ *when* $|\xi| \leq R$ *and* ξ_1 *is within a small fixed distance of a point of* $(2J+1)\pi\mathbb{Z}$. *Thus,*

$$\int_{S^1} |\widehat{\mu}(Re)|^2\,d\sigma(e) \gtrsim R^{-\alpha},$$

so $\beta_2(\alpha) \leq \alpha$, *which matches the lower bound* $\beta_2(\alpha) \geq \alpha$ *due to Mattila [18].*

Example 3.3 ($\beta_2(\alpha) \leq \alpha/2$ for $1 < \alpha < 2$). *This example is also based on periodicity. Let* ψ *be a* C_c^∞ *function supported in* $B(0, 1/10)$ *whose inverse Fourier transform* ϕ *is nonzero on the unit ball. Fix* R, *let* e_1, e_2 *be the standard basis for* $\mathbb{R}^2$, *and consider*

$$f_J(\xi) = \sum_{|j|\leq J}\psi(\xi - Re_2 - \frac{j}{J+\frac{1}{2}}R^{\frac{1}{2}}e_1)$$

where J *is a large positive integer* $\leq R^{\frac{1}{2}}$. *It is then clear that* f_J *is supported in* A_R *and that* $\|f_J\|_2 \lesssim J^{\frac{1}{2}}$. *Let* G_J *be the inverse Fourier transform of* f_J; *so*

$$|G_J(x)| = \left|\phi(x)\frac{\sin(R^{\frac{1}{2}}x_1)}{\sin(\frac{R^{\frac{1}{2}}x_1}{2J+1})}\right|. \tag{3.3}$$

If c *is a small constant, then* $|G_J(x)| \gtrsim J$ *whenever* x *belongs to the set*

$$S = \{x : \operatorname{dist}(x, \frac{(2J+1)\pi}{R^{\frac{1}{2}}}\mathbb{Z}) \leq cR^{-\frac{1}{2}}\}.$$

See Figure 3.1 We now choose $J \sim R^{1-(\alpha/2)}$, *which is possible by our assumption on* α. *The intersection of* S *with the unit ball* $B(0,1)$ *consists of roughly* $R^{(\alpha-1)/2}$ *equally spaced vertical strips of width* $2cR^{-\frac{1}{2}}$. *Let* μ *be the restriction of Lebesgue measure to* $S \cap B(0,1)$, *normalized so that* μ *is a probability measure. In other words,*

$$\mu = R^{1-(\alpha/2)}m|_{S\cap B(0,1)}.$$

We now verify two facts about the measure μ:

(a) $\mu(B(a,r)) \lesssim r^\alpha$.

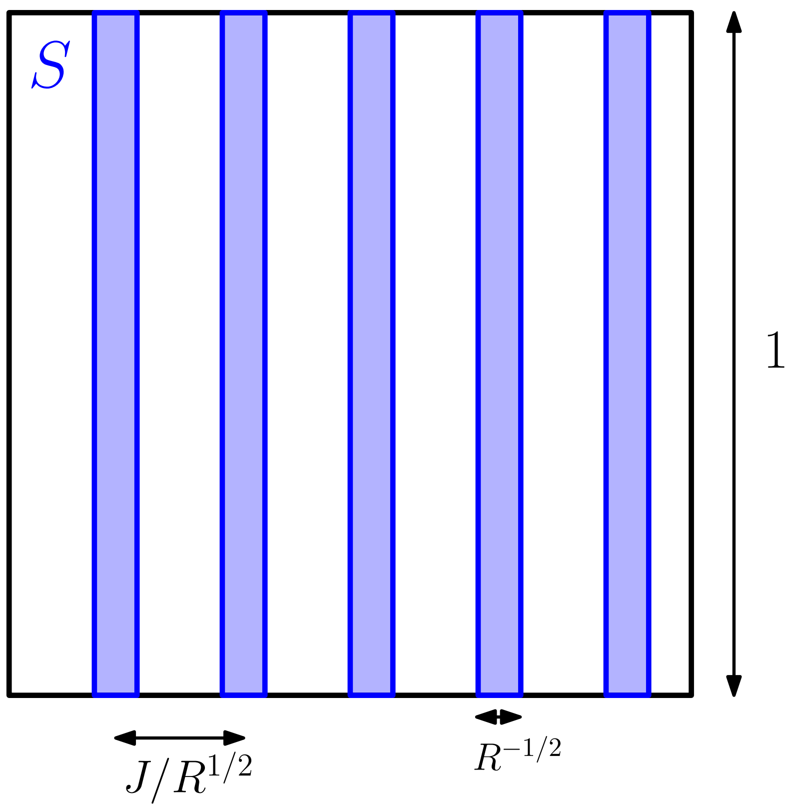

Figure 3.1: The set S of Wolff's example illustrating $\beta_2(\alpha) \leq \alpha/2$ for $1 < \alpha < 2$

Proof. The support of μ consists of $\sim R^{\frac{\alpha-1}{2}}$ vertical strips of width $\sim R^{-\frac{1}{2}}$ and spacing $\sim R^{\frac{1-\alpha}{2}}$ in the unit ball. Thus for $1 \leq L \lesssim R^{\frac{\alpha-1}{2}}$, it suffices to consider $r \sim LR^{\frac{1-\alpha}{2}}$. By definition,

$$\mu(B(a,r)) \sim R^{1-(\alpha/2)} L \cdot rR^{\frac{1}{2}} = LR^{\frac{1-\alpha}{2}} r \sim r^2 \lesssim r^\alpha,$$

the last inequality holding since $r \lesssim 1$ and $\alpha < 2$. □

(b) $I_\alpha(\mu) \lesssim \log R$.

Proof. Since $\mu(B(a,r)) \lesssim r$, we have

$$I_\alpha(\mu) = \iint_{|x-y|\leq 1/R} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} + \sum_{\substack{r \text{ dyadic} \\ R^{-1}<r<1}} r^{-\alpha} \iint_{|x-y|\sim r} d\mu(x)d\mu(y).$$

For the integral over $|x-y| \leq 1/R$, note that on this set, $d\mu(x)d\mu(y) \ll R^{2-\alpha} dx\, dy$, so

$$\begin{aligned} \iint_{|x-y|\leq 1/R} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} &\lesssim R^{2-\alpha} \int_{B(0,1)} \int_{|x-y|\leq 1/R} |x-y|^{-\alpha}\, dy\, dx \\ &= R^{2-\alpha} \int_0^{1/R} r^{-\alpha+1}\, dr \sim 1. \end{aligned}$$

For $R^{-1} < r < 1$, the set

$$\{y \in \mathbb{R}^2 : |x-y| \sim r\}$$

can be covered by $\sim$ 1-many r-balls, so

$$\iint_{|x-y|\sim r} d\mu(x)d\mu(y) = \int \mu(\{y : |x-y| \sim r\})\, d\mu(y) \lesssim \int r^\alpha \, d\mu(y) = r^\alpha.$$

Lastly, there are about $\log R$-many different dyadic values of r, so $I_\alpha(\mu) \lesssim \log R$. □

By part (ii) of Lemma A.1 and $I_\alpha(\mu) \lesssim \log R$, *it follows that*

$$\int |G_J|\, d\mu \lessapprox R^{-\frac{\beta}{2}+\frac{1}{2}} \|f_J\|_2$$

for any $\beta < \beta_2(\alpha)$. *We conclude that*

$$R^{1-(\alpha/2)} \lesssim \int |G_J|\, d\mu \lessapprox R^{-\frac{\beta}{2}+\frac{1}{2}} \|f_J\|_2 \lesssim R^{-\frac{\beta}{2}+\frac{1}{2}+(1/2)(1-(\alpha/2))}.$$

Rearranging, we see $\beta \leq \alpha/2$, *as claimed.*

3.2.2 Fourier averages over cones

In 2004, Erdoğan [32] gave new proofs of Mattila's and Wolff's sharp lower bounds for $\beta_2(\alpha)$, and proved new results regarding Fourier averages over cones. We recall that

$$\mathbb{C}one^2 = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : |\xi| = \tau, \tau \in [1, 2]\}.$$

The basic question we are interested in is if ν is a measure with finite α-energy and σ is the surface measure for the cone segment, how quickly do the Fourier averages $\int_{R\mathbb{C}one^2} |\widehat{\nu}|^2 \, d\sigma$ decay as $R \to \infty$?

Definition 3.4 (Optimal conical Fourier average decay rate)**.** *For* $0 < \alpha < 3$, $\gamma_3(\alpha)$ *is the supremum of the numbers* $\gamma > 0$ *such that there exists* C *with*

$$\int_{\mathbb{C}one^2} |\widehat{\nu}(Re)|^2 \, d\sigma(e) \leq CR^{-\gamma} I_\alpha(\nu), \quad \textit{for all } R > 1,$$

for all measures ν *supported in the unit ball of* $\mathbb{R}^3$.

In 2004, Erdoğan [32] used a Whitney decomposition and techniques from bilinear restriction theory involving wave packets to prove $\gamma_3(\alpha) = \alpha/2$ for $\alpha \in (1, 2)$. For α not within this range, Erdoğan also proved the sharp $\gamma_3(\alpha) = \max(\min(\alpha, 1/2), \alpha - 1)$. We focus here on the case $\alpha = 1$ of Erdoğan's work, which we state separately as its own Theorem.

Theorem 3.7 (Erdoğan [32], Theorem 4, case $\alpha = 1$)**.** *For any* $\epsilon > 0$, *there is a constant* C_ϵ *such that the following is true. Let* ν *be a measure in* $\mathbb{R}^3$ *supported in the unit ball and with* 1*-dimensional energy* $I_1(\nu) = 1$. *Then for any* $R \geq 1$,

$$\int_{\mathbb{C}one^2} |\widehat{\nu}(Re)|^2 \, d\sigma(e) \leq C_\epsilon R^\epsilon R^{-1/2}.$$

For the sake of comparison with our main theorem on Fourier averages—Theorem 3.4—we record the following Corollary of Theorem 3.7.

Corollary 3.1. *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose ν is a measure that agrees with the Lebesgue measure on a disjoint union of unit balls $X \subset B(0, R)$ and satisfies the 1-dimensional Frostman condition*

$$\nu(B(x, r)) \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Then the estimate

$$\int |\widehat{\nu}|^2 \, d\sigma \leq C_\epsilon R^\epsilon R^{3/2}$$

holds.

Proof using $\gamma_3(1) = 1/2$. To make use of Theorem 3.7, we have to push the measure ν forward to $B(0, 1)$ by the map

$$Tx = R^{-1}x,$$

as well as normalize the 1-energy of our measure. Let $\mu = T\nu$ be the pushforward of ν under T. By the definition of μ and our assumption on ν,

$$I_1(\mu) = \iint \frac{d\nu(x)d\nu(y)}{|R^{-1}x - R^{-1}y|} \lesssim R \int \sum_{j=1}^{O(\log R)} \frac{\nu(B(x, 2^j))}{2^j} \, d\nu(x) \lesssim (\log R) R^2.$$

Hence $\mu_0 = R^{-1}\mu$ satisfies $I_1(\mu_0) \approx 1$, so by Theorem 3.7,

$$\int |\widehat{\mu}_0(R\xi)|^2 \, d\sigma(\xi) \lessapprox R^{-1/2}.$$

On the other hand, $\widehat{\mu}_0(R\xi) = R^{-1}\widehat{\nu}(\xi)$, so substituting and rearranging, we obtain

$$\int |\widehat{\nu}|^2 \, d\sigma \lessapprox R^{3/2}.$$

□

In 2023, I improved Corollary 3.1 by proving Theorem 3.4 in [17], which we recall and prove here:

Theorem 3.8 (O., [17])**.** *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose ν is a measure that agrees with the Lebesgue measure on a disjoint union of unit balls $X \subset B_R$ and satisfies the 1-dimensional Frostman condition*

$$\nu(B(x, r)) \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(\nu)$ be the quantity

$$\mathbf{P}(\nu) = \sup\{\nu(P) : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then the estimate

$$\int |\widehat{\nu}|^2 \, d\sigma \leq C_\epsilon R^\epsilon \, \mathbf{P}(\nu)^{1/2} \|\nu\|$$

holds, where $\|\nu\| = |X|$ is the total mass of ν.

Before diving into the proof, we say a few words about the technique. The strategy of proof initially follows the same lines as Mattila [18]: using Fourier transform properties, we can write

$$\int |\widehat{\nu}|^2 \, d\sigma = \iint \check{\sigma}(x-y)\, d\nu(x)\, d\nu(y) \leq \iint |\check{\sigma}(x-y)|\, d\nu(x)\, d\nu(y). \tag{3.4}$$

If σ is a smooth surface measure supported in $\mathbb{C}one^2$, then $|\check{\sigma}(x)| \leq \frac{C}{(1+|x|)^{1/2}}$, following the same numerology as that for the Fourier transform of the arclength measure on the circle in the plane. However, owing to its lack of curvature, $\check{\sigma}$ also obeys another principle of decay which we record in the following Lemma.

Lemma 3.2 (Pointwise bound of $|\check{\sigma}(x)|$)**.** *Let σ be a smooth surface measure supported in $\mathbb{C}one^2$. For any $\epsilon > 0$ and any $N > 1$, there is a constant $C(\epsilon, N)$ so that*

$$|\check{\sigma}(x)| \leq C(\epsilon, N) \frac{1}{(1+|x|)^{\frac{1}{2}-\epsilon}} \frac{1}{(1+d(x,\Gamma_0))^N}$$

holds for all $x \in \mathbb{R}^3$, where Γ_0 is the lightcone $\{(a,r) \in \mathbb{R}^2 \times \mathbb{R} : ||a| - r| = 0\}$ with vertex 0.

Lemma 3.2 is almost surely not new, but I could not find its precise statement in the literature. To keep the exposition here focused, we present the proof of Lemma 3.2 in Appendix B.

Plugging in the bound of $|\check{\sigma}(x-y)|$ given to us by Lemma 3.2, we see that the pairs $(x, y) \in \operatorname{supp} \nu \times \operatorname{supp} \nu$ which contribute most to the integral on the right-hand side of Equation (3.4) are those such that $x - y$ is close to the lightcone Γ_0. We are thus left with an instance of Problem 2.1 where the configuration X is 1-dimensional. Now we can invoke our answer to Problem 2.1 in the case X is 1-dimensional which we proved in Theorem 2.5 via Pramanik–Yang–Zahl's circular maximal function theorem and point-circle duality to finish.

Now we present the proof of Theorem 3.4, our main result on conical Fourier averages of 1-dimensional sets.

Proof of Theorem 3.4. By duality and Fourier transform properties,

$$\|\widehat{\nu}\|^2_{L^2(d\sigma)} = \int \widehat{\nu}\overline{\widehat{\nu}}\, d\sigma = \iint_{B_R \times B_R} \check{\sigma}(x-y)\, d\nu(x) d\nu(y).$$

We will estimate this integral by dividing the domain of integration into regions where we have good control on the integrand. For instance,

$$\begin{aligned}\iint \check{\sigma}(x-y)\, d\nu(x) d\nu(y) &= \iint_{|x-y| \leq R^{10\epsilon}} \check{\sigma}(x-y)\, d\nu(x) d\nu(y) + \iint_{|x-y| > R^{10\epsilon}} \check{\sigma}(x-y)\, d\nu(x) d\nu(y) \\ &=: I + II.\end{aligned}$$

Because $|\check{\sigma}| \lesssim 1$ everywhere, we can estimate $|I| \lesssim R^{10\epsilon}|X|$. To estimate $|II|$, for any $x \in B_R$, we have a corresponding point $\tilde{x} \in B_1$ defined by $\tilde{x} = R^{-1}x$. For each pair $x, y \in X$, we consider the numbers

$$d(\tilde{x}, \tilde{y}) = R^{-1}|x-y| \quad \text{and} \quad \Delta(\tilde{x}, \tilde{y}) = R^{-1}||x' - y'| - |x_3 - y_3||.$$

These are simply the scaled down values of $d(x,y)$ and $\Delta(x,y)$. We write things this way to use the results of Section 2.5 which are phrased at scales ≤ 1. For any dyadic number $\delta^{1-10\epsilon} < D < 1$, we let

$$\mathcal{L}_{D,\lessapprox\delta} = \{(x,y) \in X \times X : d(\tilde{x},\tilde{y}) \sim D, \Delta(\tilde{x},\tilde{y}) \leq \delta^{1-\epsilon}\}$$
$$\mathcal{L}_{D,\gg\delta} = \{(x,y) \in X \times X : d(\tilde{x},\tilde{y}) \sim D, \Delta(\tilde{x},\tilde{y}) > \delta^{1-\epsilon}\}.$$

We organize the integral II by writing

$$II \leq \sum_{\delta^{1-10\epsilon}<D<1} \iint_{\mathcal{L}_{D,\lessapprox\delta}} \check{\sigma}(x-y)\, d\nu(x)d\nu(y) + \sum_{\delta^{1-10\epsilon}<D<1} \iint_{\mathcal{L}_{D,\gg\delta}} \check{\sigma}(x-y)\, d\nu(x)d\nu(y).$$

We claim that the second sum in this decomposition of II is $O(R^{-95})$, so it is negligible. Postponing the proof of this for a moment, we only have to show that the first sum is bounded by $\mathbf{P}(\nu)^{1/2}|X|$.

For $D > \delta^{1-10\epsilon}, \Delta \leq \delta^{1-\epsilon}$, and $(x,y) \in \mathcal{L}_{D,\Delta}$ we use the Fourier transform estimate of Lemma 3.2,

$$|\check{\sigma}(x-y)| \lessapprox \frac{1}{(RD)^{1/2}},$$

together with the estimate of Theorem 2.5 for $|\mathcal{L}_{D,\Delta}|$ when $\Delta \lessapprox \delta$:

$$|\mathcal{L}_{D,\lessapprox\delta}| \lessapprox T(\tau_D)^{1/2}(RD)^{1/2}|X|.$$

By Proposition 2.15 on the maximal tangency number $T(\tau_D)$ and our definition of $\mathbf{P}(\nu)$ and $\delta = R^{-1}$,

$$|\mathcal{L}_{D,\lessapprox\delta}| \lessapprox \mathbf{P}(\nu)^{1/2}(RD)^{1/2}|X|.$$

Putting these estimates together gives

$$\begin{aligned}\sum_{\delta^{1-10\epsilon}<D<1} \iint_{\mathcal{L}_{D,\lessapprox\delta}} |\check{\sigma}(x-y)|\, d\nu(x)d\nu(y) &\lessapprox \sum_{\delta^{1-10\epsilon}<D<1} |\mathcal{L}_{D,\lessapprox\delta}| \cdot \frac{1}{(RD)^{1/2}} \\ &\lessapprox \mathbf{P}(\nu)^{1/2}|X| \sum_{\delta^{1-10\epsilon}<D<1} 1 \approx \mathbf{P}(\nu)^{1/2}|X|.\end{aligned}$$

Now we estimate the contribution from the second sum in the decomposition of II. We write the contribution as

$$\sum_{\delta^{1-10\epsilon}<D<1} \sum_{\delta^{1-\epsilon}<\Delta<D} \iint_{\substack{d(\tilde{x},\tilde{y})\sim D \\ \Delta(\tilde{x},\tilde{y})\sim\Delta}} |\check{\sigma}(x-y)|\, d\nu(x)d\nu(y).$$

By the estimate of Lemma 3.2, for $(x,y) \in \mathcal{L}_{D,\Delta}$ with $\Delta > \delta^{1-\epsilon}$, we have

$$|\check{\sigma}(x-y)| \lesssim_\epsilon \frac{1}{(RD)^{1/2}} \cdot \frac{1}{(R\Delta)^{100\epsilon^{-1}}} \leq \frac{1}{R^{100}}.$$

Since $|\mathcal{L}_{D,\Delta}| \leq R^2$ for any D, Δ, and since there are $O(\log R)$-many summands, we have a total contribution of no more than (say) $C_\epsilon R^{-100+2+\epsilon}$. This finishes the estimate of $|II|$, and the proof of Theorem 3.4. □

We are ready to give the proof of the L^1 sparse restriction Theorem 3.3, whose statement we recall here.

Theorem 3.9. *For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose $X \subset B_R$ is a disjoint union of unit balls that satisfies the 1-dimensional Frostman condition*

$$|X \cap B(x,r)| \lesssim r, \qquad x \in \mathbb{R}^3, r > 1.$$

Let $\mathbf{P}(X)$ be the quantity

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \text{ is a lightplank of dimensions } 1 \times R^{1/2} \times R\}.$$

Then for any measurable function $w\colon \mathbb{R}^3 \to [0,1]$, the estimate

$$\int_X |E_{\mathbb{C}one^2} f| w \le C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/4} (\int_X w)^{1/2} \|f\|_{L^2(d\sigma)}$$

holds.

Proof. For each dyadic $\eta \le 1$, let X_η be the union of unit balls $Q \subset X$ satisfying $\int_Q w \sim \eta$. The sets X_η are pairwise disjoint, and we can write

$$\int_X |Ef| w = \sum_\eta \int_{X_\eta} |Ef| w.$$

Moreover,

$$\int_X w = \sum_\eta \int_{X_\eta} w = \sum_\eta \sum_{Q \subset X_\eta} \int_Q w \sim \sum_\eta \eta |X_\eta|.$$

Set $d\nu_\eta = w 1_{X_\eta} dx$. By duality, for an appropriate $|g| \le 1$,

$$\int |Ef| \, d\nu_\eta = \int Ef \cdot g \, d\nu_\eta.$$

By Fourier transform properties and Cauchy–Schwarz, this is bounded by

$$(\int_{\mathbb{C}one^2} |\widehat{g\nu_\eta}|^2 \, d\sigma)^{1/2} \|f\|_{L^2(d\sigma)}.$$

Next, we use Fourier transform properties, $|g| \le 1$, and the triangle inequality to bound this by

$$(\iint_{X_\eta \times X_\eta} |\check{\sigma}(x-y)| w(x) w(y) \, dx \, dy)^{1/2} \|f\|_{L^2(d\sigma)}.$$

By dyadic pigeonholing and Lemma 3.2, up to a rapidly decaying tail, for some $1 < \rho < R$ this is bounded by

$$(\frac{1}{\rho^{1/2}} \sum_{\substack{Q_1, Q_2 \subset X_\eta \\ d(Q_1,Q_2) \sim \rho \\ \Delta(Q_1,Q_2) < R^\epsilon}} \iint_{Q_1 \times Q_2} w(x) w(y) \, dx dy)^{1/2} \|f\|_{L^2(d\sigma)}.$$

By the definition of X_η, $\iint_{Q_1\times Q_2} w(x)w(y)\,dxdy \sim \eta^2$, so this is equal to

$$\Big(\frac{\eta^2}{\rho^{1/2}}\#\big\{(Q_1,Q_2)\subset X_\eta\times X_\eta : d(Q_1,Q_2)\sim\rho, \Delta(Q_1,Q_2)<R^\epsilon\big\}\Big)^{1/2}\|f\|_{L^2(d\sigma)}$$

Since X_η is a disjoint union of unit balls satisfying $|X_\eta\cap B_r|\le|X\cap B_r|\lesssim r$ for all r-balls and $r\ge 1$, we use Pramanik–Yang–Zahl's maximal estimate in the form of Theorem 2.5 to bound this by

$$\big(\frac{\eta^2}{\rho^{1/2}}\cdot\rho^{1/2}\mathbf{P}(X_\eta)^{1/2}|X_\eta|\big)^{1/2}\|f\|_{L^2(d\sigma)}.$$

Simplifying and using $\mathbf{P}(X_\eta)\le\mathbf{P}(X)$, we arrive at

$$\int_{X_\eta}|Ef|w \lessapprox \eta|X_\eta|^{1/2}\mathbf{P}(X)^{1/4}\|f\|_{L^2(d\sigma)}.$$

Summing over $\eta > R^{-500}$ gives

$$\int_X|Ef|w\lessapprox\Big(\sum_{\substack{\eta>R^{-500}\\ \eta\text{ dyadic}}}\eta|X_\eta|^{1/2}\Big)\mathbf{P}(X)^{1/4}\|f\|_{L^2(d\sigma)}+O(R^{-100})\|f\|_{L^2(d\sigma)}.$$

There are ≈ 1 summands, so using Cauchy–Schwarz and neglecting the error term gives

$$\int_X|Ef|w\lessapprox\Big(\sum_{\substack{\eta>R^{-500}\\ \eta\text{ dyadic}}}\eta^2|X_\eta|\Big)^{1/2}\mathbf{P}(X)^{1/4}\|f\|_{L^2(d\sigma)}.$$

Finally, $\eta\le 1$, and since $\sum_\eta \eta|X_\eta|\sim\int_X w$, we have arrived at the desired bound

$$\int_X|Ef|w\lessapprox\Big(\int_X w\Big)^{1/2}\mathbf{P}(X)^{1/4}\|f\|_{L^2(d\sigma)}.$$

□

Lastly, we describe examples that establish the sharpness of Theorem 3.1.

Theorem 3.10. *For each $R>1$, and each $T\in[1,R]$, there is a nonzero function $f\in L^2(\mathbb{C}one^2)$ and disjoint union of unit balls X satisfying the 1-dimensional Frostman condition*

$$|X\cap B(x,r)|\lesssim r,\qquad x\in\mathbb{R}^3, r>1,$$

such that $\mathbf{P}(X)\sim T$, and

$$\int_X|E_{\mathbb{C}one^2}f|^2\gtrapprox T^{1/2}\|f\|_{L^2(d\sigma)}.$$

Proof. We give one example with a small variation.

1. Let $f_0(\xi) = 1_\theta(\xi)$, where $\theta = [1, 1+c] \times [0, T^{-1/2}] \subset \{1 < |\xi| < 2\}$, be the Knapp example of the given dimensions. Let f be an appropriate modulation of f_0 so that $|Ef| \gtrsim |\theta| 1_P = T^{-1/2} 1_P$, where P is a lightplank in B_R of dimensions $1 \times T^{1/2} \times T$.

   Let X be any $1 \times 1 \times T$-tube contained in the lightplank P. By construction, X is 1-dimensional and $\mathbf{P}(\nu) \sim T$. Also, $\|f\|^2_{L^2(d\sigma)} = |\theta| = T^{-1/2}$, and

$$\int_X |Ef|^2 \gtrsim T^{-1}|X \cap P| \sim 1 \sim T^{1/2}\|f\|^2_{L^2(d\sigma)},$$

   as desired.

2. As a small variation on the last example, we can also normalize $|X| = R$, with $\mathbf{P}(X) \sim T$. Let f and P be the same as in the first example, let X_P be any $1 \times 1 \times T$ tube contained in the lightplank P, and let X_V be a $1 \times 1 \times (R-T)$-tube with direction e_3. Let $X = X_P \cup X_V$. By construction, $|X| = R$, X is 1-dimensional, and $\mathbf{P}(X) \sim T$ as any lightplank intersects X_V in a ball of radius $O(1)$. By the same calculation of the first example, $\int_X |Ef|^2 \gtrsim T^{1/2}\|f\|^2_{L^2(d\sigma)}$.

□

3.3 The Mizohata–Takeuchi conjecture and refined decoupling

We recall the *Mizohata–Takeuchi Conjecture* 3.1 we introduced in Chapter 1. If T is a tube in $\mathbb{R}^n$, then $\nu(T)$ is one of the two unit vectors parallel to the central axis of T. If S is a smooth hypersurface in $\mathbb{R}^n$ and a vector v is orthogonal to T_xS for some point $x \in S$, we say $v \perp S$.

Conjecture 3.1 (Mizohata–Takeuchi)**.** *For every $\epsilon > 0$, there is a constant C_ϵ such that the following holds for every $R > 1$. If $S \subset \mathbb{R}^n$ is a smooth compact hypersurface, $n \geq 2$, let*

$$\mathbb{T}(S) = \{T \subset \mathbb{R}^n : \nu(T) \perp S,\ T \textit{ is a } 1 \times \cdots \times 1 \times R\textit{-tube}\}$$

be the collection of $1 \times \cdots \times 1 \times R$-tubes whose direction is orthogonal to a tangent space of S. If $X \subset B^n(0, R)$ is a disjoint union of unit balls, let

$$\mathbf{T}_S(X) = \sup\{|X \cap T| : T \in \mathbb{T}(S)\}$$

be the "maximal tube occupancy of X" by tubes orthogonal to the hypersurface S. Then the following estimate holds:

$$\int_X |E_S f|^2 \leq C_\epsilon R^\epsilon\, \mathbf{T}_S(X) \|f\|^2_{L^2(S)}. \tag{3.5}$$

The recent work from 2023 of Carbery–Iliopoulou–Wang [15] is the state of the art on the problem in all dimensions when S is the sphere, or a C^2 hypersurface of bounded curvature.

The discussion and references in [15] provide more background on Conjecture 3.1 in the case of the sphere and its connection with the Bochner–Riesz problem. Currently, the best known bound when S is the unit sphere in $\mathbb{R}^n$ (or a similar C^2 surface with Gaussian curvature ~ 1) follows as a consequence of the following Theorem from [15].

Theorem 3.11 (Carbery–Iliopoulou–Wang)**.** *Let S be the unit sphere in $\mathbb{R}^n$. For each $\epsilon > 0$, there is a constant C_ϵ so the following holds for each $R > 1$. Suppose $X \subset B_R$ is an arbitrary disjoint union of unit balls, and let $\mathbf{P}(X)$ be the quantity*

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \textit{ is a } R^{1/2} \times \cdots \times R^{1/2} \times R\textit{-tube (with any orientation)}\}.$$

Then the estimate

$$\int_X |E_S f|^2 \leq C_\epsilon R^\epsilon \, \mathbf{P}(X)^{\frac{2}{n+1}} \|f\|^2_{L^2(S)}$$

holds.

The sets P that are $R^{1/2} \times \cdots \times R^{1/2} \times R$-tubes with directions orthogonal to S have the same shape as the supports of wave packets in the wave packet decomposition of $E_S f$. As a consequence of Theorem 3.11, Carbery–Iliopoulou–Wang deduce (3.5) holds with an $R^{\frac{n-1}{n+1}}$-loss. Key to the authors' point is that the $R^{\frac{n-1}{n+1}}$-loss they get is unavoidable *given* the methods they use. The tools the authors used were wave packet analysis of $E_S f$, in particular local L^2-orthogonality, the locally constant property (see Heuristic 1.1), and refined decoupling (a version of Theorem 1.4 for spheres). Because of an *enemy scenario* due to Guth [16], these methods alone cannot break the $R^{\frac{n-1}{n+1}}$-barrier.

Besides spheres, it is also interesting to consider what we can say about the MT conjecture for other model manifolds, such as $\mathbb{C}one^2$. We note that in Theorem 3.11, the shapes P appearing in the definition of $\mathbf{P}(X)$ have the same shape as wave packets of $E_S f$. In this sense, our work on 1-dimensional sets in Theorem 3.1 with its definition of $\mathbf{P}(X)$ in terms of $1 \times R^{1/2} \times R$-lightplanks is a direct analogy of Theorem 3.11. By covering a $1 \times R^{1/2} \times R$-lightplank with $R^{1/2}$-many $1 \times 1 \times R$ lightrays, Theorem 3.1 implies (3.5) for $\mathbb{C}one^2$ with a power $R^{1/4}$-loss. It is interesting to note that the $R^{1/4}$-loss we have for the 1-dimensional sets X is better than the $R^{1/3}$-loss we get for general sets X using the decoupling approach of Carbery–Iliopoulou–Wang.

The following argument was shown to me by Ciprian Demeter regarding $\mathbb{C}one^2$ and general sets X. The method of proof using refined decoupling is a direct analog of the proof of Theorem 3.11.

Theorem 3.12 (Demeter, 2023)**.** *For each $\epsilon > 0$ there is a constant C_ϵ so the following holds for every $R > 1$. Let $X \subset B_R$ be an arbitrary disjoint union of unit cubes. Then*

$$\int_X |E_{\mathbb{C}one^2} f|^2 \leq C_\epsilon R^\epsilon \, \mathbf{P}(X)^{2/3} \|f\|_2^2.$$

Proof. (Ciprian Demeter, private communication) Consider the wave packet decomposition of Ef into $1 \times R^{1/2} \times R$-lightplanks:

$$Ef = \sum_{P \in \mathcal{P}} f_P,$$

where each f_P has Fourier support in a dual lightplank $\theta(P)$ contained in the R^{-1}-neighborhood of $\mathbb{C}one^2$. By dyadically pigeonholing and also ignoring Schwartz tails, we may assume $|f_P| \approx 1_P$ for each $P \in \mathcal{P}$. So,

$$\|Ef\|_{L^2(B_R)}^2 \sim |\mathcal{P}|R^{3/2} \text{ and } \|f\|_{L^2}^2 \sim \frac{\|Ef\|_{L^2(B_R)}^2}{R} \sim |\mathcal{P}|R^{1/2}. \tag{3.6}$$

For each $r \leq R^{1/2}$, consider the set $X_r \subset X$ of unit cubes of X which intersect between $r/2$ and r of the lightplanks $2P$, where $P \in \mathcal{P}$. By pigeonholing, we may assume that $X = X_r$ for some r. By Hölder's inequality and the refined decoupling Theorem 1.4, we have

$$\|\sum_{P\in\mathcal{P}} f_P\|_{L^6(X)} \lessapprox r^{\frac{1}{3}}(\sum_{P\in\mathcal{P}} \|f_P\|_{L^6(\mathbb{R}^3)}^6)^{1/6} \approx r^{1/3}(|\mathcal{P}|R^{3/2})^{1/6}. \tag{3.7}$$

On the other hand, by Hölder's inequality,

$$\int_{X_r} |Ef|^2 \leq |X_r|^{2/3}(\int_{X_r} |Ef|^6)^{1/3}.$$

Combining with inequality (3.7), we have

$$\int_{X_r} |Ef|^2 \lessapprox r^{2/3}|X_r|^{2/3}(|\mathcal{P}|R^{3/2})^{1/3} \approx (r|X|)^{2/3}(|\mathcal{P}|R^{3/2})^{1/3} \tag{3.8}$$

We say a unit ball $q \subset X$ and a lightplank $P \in \mathcal{P}$ are *incident* if $q \cap 2P \neq \emptyset$. By double-counting incidences, and using the definition of $\mathbf{P}(X)$, we have $r|X_r| \lesssim \mathbf{P}(X)|\mathcal{P}|$. Plugging this into (3.8) and invoking (3.6) yields

$$\begin{aligned}\int_X |Ef|^2 &\lessapprox \mathbf{P}(X)^{2/3}\,|\mathcal{P}|^{2/3}(|\mathcal{P}|R^{3/2})^{1/3} \\ &= \mathbf{P}(X)^{2/3}\,|\mathcal{P}|R^{1/2} \\ &\sim \mathbf{P}(X)^{2/3}\|f\|_{L^2}^2.\end{aligned}$$

□

Theorem 3.12 is sharp apart from $C_\epsilon R^\epsilon$ factors for $\mathbb{C}one^2$ and general sets X, as the example of X equal to a $1 \times R^{1/2} \times R$-lightplank, and Ef equal to the Knapp example shows. (In particular, the exponent "2/3" on $\mathbf{P}(X)^{2/3}$ cannot be improved without further assumptions on the set X.)

We note that since the refined decoupling Theorem 1.4 for the cone and that for the parabola (see Theorem 4.2 of [10]) have the same numerology, the exact same proof we used in Theorem 3.12 can be used to prove the $n = 2$ case of Theorem 3.11, and indeed this is how Carbery–Iliopoulou–Wang prove Theorem 3.11. By covering a $1 \times R^{1/2} \times R$-lightplank with $R^{1/2}$-many $1 \times 1 \times R$ lightrays, Theorem 3.12 implies MT for $\mathbb{C}one^2$ with a power $R^{1/3}$-loss. It would be very interesting to know whether there is a similar enemy scenario [16] for $\mathbb{C}one^2$ which tells us whether the power loss $R^{1/3}$ on Theorem 3.12 is impossible to improve on with decoupling axioms alone.

Appendix A

Lemmas of rectangle-lightplank duality

In this Appendix we record a few facts about the geometry of overlapping circles and their corresponding dual lightplanks for use in Chapter 2. The first Lemma describes the region of intersection of two δ-annuli: generally two δ-annuli intersect in two δ, τ-rectangles for some τ depending on the distance and degree of internal tangency.

We recall that $Q = B(e_3, \alpha_0)$ for a small absolute constant ($\alpha_0 = 1/100$ will do).

Lemma A.1 (Lemma 3.1 (a), [20])**.** *Assume that v, w are two circles in Q. Let $d = d(v, w)$, and $\Delta = \Delta(v, w)$. Then*

$$|C_{\delta,v} \cap C_{\delta,w}| \lesssim \delta \cdot \frac{\delta}{\sqrt{(d+\delta)(\Delta+\delta)}}.$$

Corollary A.1. *If $\Omega \subset C_{\delta,v} \cap C_{\delta,w}$ is a δ, τ-rectangle, then*

$$\tau \lesssim \frac{\delta}{\sqrt{(d(v,w)+\delta)(\Delta(v,w)+\delta)}}$$

Proof. Let $d = d(v, w)$, $\Delta = \Delta(v, w)$. By Lemma A.1,

$$\delta\tau \sim |\Omega| \leq |C_{\delta,v} \cap C_{\delta,w}| \lesssim \delta \cdot \frac{\delta}{\sqrt{(d+\delta)(\Delta+\delta)}}.$$

Canceling δ from both sides of the inequality gives the desired result. □

The next Proposition calculates the angle of intersection of two circles in terms of their distance and degree of internal tangency.

Proposition A.1. *Suppose C_1, C_2 are two circles in Q which intersect at a point $a \in \mathbb{R}^2$. Let u_1, u_2 be the positively oriented unit tangent vectors to C_1, C_2, respectively at the point a. Then $\angle u_1, u_2 \sim \sqrt{d(C_1, C_2)\Delta(C_1, C_2)}$*

Proof. Without loss of generality, suppose $C_1 = (0, 0, r)$ and $C_2 = (b, 0, s)$ with $1 - \alpha_0 \leq s \leq r \leq 1 + \alpha_0$ and $b > 0$. With these choices, we have $d(C_1, C_2) = b + (r - s)$ and $\Delta(C_1, C_2) = |b - (r - s)|$. Consider the triangle T in the plane whose vertices are a, $(0, 0)$

and $(b, 0)$. By elementary geometry, the angle ϕ at the vertex a of T is the same as $\angle u_1, u_2$. By the law of cosines,

$$b^2 = r^2 + s^2 - 2rs\cos\phi.$$

Adding and subtracting $2rs$ and completing the square yields

$$b^2 = (r - s)^2 + 2rs(1 - \cos\phi).$$

Note that by definition, $d(C_1, C_2)\Delta(C_1, C_2) = |b^2 - (r - s)^2|$. Suppose $b > r - s$ (with a similar conclusion in case $b \leq r - s$), so rearranging, we have

$$\frac{d(C_1, C_2)\Delta(C_1, C_2)}{rs} = 2(1 - \cos\phi).$$

Using the approximations $r, s \sim 1$ and $\cos\phi \sim 1 - \frac{\phi^2}{2}$, we obtain

$$d(C_1, C_2)\Delta(C_1, C_2) \sim \phi^2.$$

Taking square roots yields the claim. □

Proposition A.2. *Suppose $\Omega^{(v)} \subset \overline{\Omega}^{(w)}$, where $\Omega^{(v)}$ is a δ, τ-rectangle, and $\overline{\Omega}^{(w)}$ is an $A\delta, A\tau$-rectangle. Let $a, \bar{a}$ be the center points of $\Omega^{(v)}$ and $\overline{\Omega}^{(w)}$, respectively, and let $e_m, \bar{e}_m$ be the positively oriented unit tangent vectors to v, w at the points $a, \bar{a}$, respectively. Then $\angle e_m, \bar{e}_m \lesssim A^2\tau$.*

Proof. Let $d = d(v, w)$, $\Delta = \Delta(v, w)$, and let $\gamma, \overline{\gamma}$ denote the core arcs of $\Omega, \overline{\Omega}$. We make the same simplifying technical assumption that there exists a point $x \in \gamma \cap \overline{\gamma}$ to make use of Proposition A.1. To remove this assumption, we note by replacing v with a concentric circle of a slightly smaller or larger radius, we can arrange for $\gamma \cap \overline{\gamma} \neq \emptyset$, while keeping $\angle e_m, \bar{e}_m$ the same and $\Omega^{(v)} \subset \overline{\Omega}^{(w)}$. By Proposition A.1, the angle between v and w at x is $\sim \sqrt{d\Delta}$.

By assumption $\Omega^{(v)} \subset \overline{\Omega}^{(w)}$,

$$|C_{A\delta,v} \cap C_{A\delta,w}| \geq |\Omega| \sim \delta\tau. \tag{A.1}$$

On the other hand, by Lemma A.1,

$$|C_{A\delta,v} \cap C_{A\delta,w}| \lesssim \frac{(A\delta)^2}{\sqrt{(d + A\delta)(\Delta + A\delta)}}. \tag{A.2}$$

Combining inequalities (A.1) and (A.2) and using $\delta^{1/2} \leq \tau$ gives

$$\sqrt{d\Delta} \lesssim (\delta\tau)^{-1}(A\delta)^2 \leq A^2\tau.$$

Finally, because $\operatorname{dist}(a, x) + \operatorname{dist}(x, \bar{a}) \lesssim A\tau$, by comparing angles at a and $\bar{a}$, we conclude $\angle e_m, \bar{e}_m = O(A^2)\tau + O(A)\tau = O(A^2)\tau$. □

The following Proposition is similar to Lemma 1.2 of [4]. In that context, the assumption that Ω is contained in the intersection of two thin annuli is replaced with the assumption that the intersection of the annuli is nonempty, and the conclusion of Lemma 1.2 gives an estimate of the size of τ such that $C_{\delta,v} \cap C_{A\delta,w}$ contains a δ, τ-rectangle in terms of $d(v, w)$ and $\Delta(v, w)$.

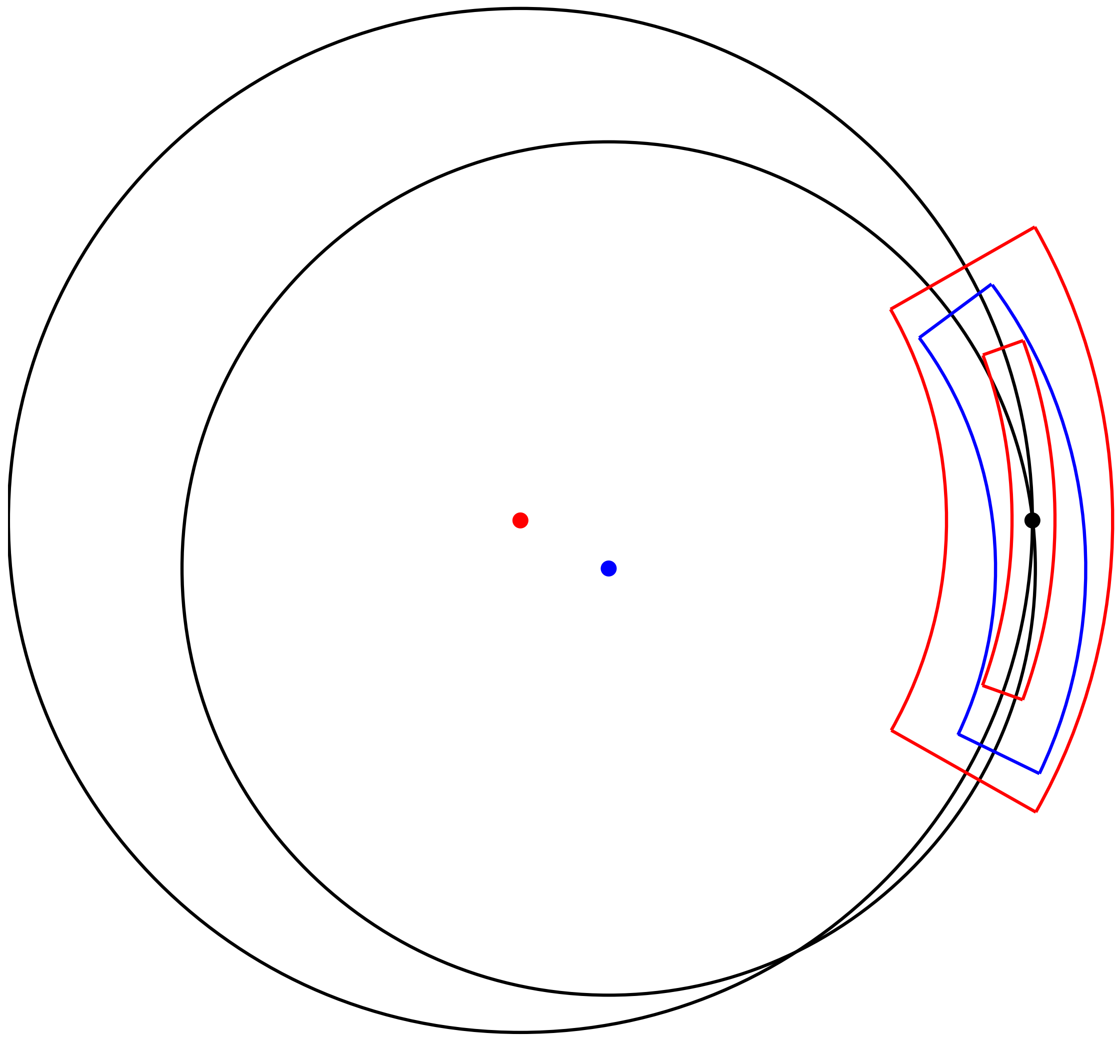

Figure A.1: Engulfing rectangles

Proposition A.3 (Engulfing)**.** *Let* $1 < A, B < \delta^{-C\epsilon}$ *and* $\Omega^{(v)}$ *be a* δ, τ*-rectangle contained in* $C_{\delta,v} \cap C_{A\delta,w}$ *for* $v, w \in Q$*. Suppose* $\overline{\Omega}^{(w)}$ *is an* $A\delta, B\tau$*-rectangle contained in* $C_{A\delta,w}$ *which contains* Ω*. Then there exists* $A_1 \approx 1$ *such that* $\overline{\Omega} \subset C_{A_1 AB^2\delta,v} \cap C_{A\delta,w}$*.*

Proof. Let $d = d(v, w)$, $\Delta = \Delta(v, w)$, and let $\gamma, \overline{\gamma}$ denote the core arcs of $\Omega, \overline{\Omega}$. We make the simplifying technical assumption that there exists a point $x \in \gamma \cap \overline{\gamma} \subset \Omega$. To remove this assumption, we note by replacing v with a concentric circle of a slightly smaller or larger radius, we can arrange for $\gamma \cap \overline{\gamma} \neq \emptyset$, while keeping $\Omega^{(v)} \subset \overline{\Omega}^{(w)}$.

By translating, scaling by a factor ~ 1, and rotating our coordinate system if necessary, assume that $w = (0, 0, 1)$, $v = (a_1, a_2, s) =: (a, s)$, and that $x = 1$ is on the positive real axis (see Figure A.1). In particular, we may assume that $\Omega^{(v)}$ is in our favorite position (see Remark 2.3), and its center is e_1.

With this choice of coordinate system, it suffices to show (using complex notation) that for $|r - 1| < A\delta$ and $|\theta| \lesssim B\tau$, we have

$$|re^{i\theta} - a| = s + O(AB^2)\delta,$$

since in our chosen coordinate system, $\overline{\Omega} \subset \{re^{i\theta} : |r-1| < A\delta, |\theta| \lessapprox B\tau\}$.

So assume $|r-1| < A\delta$ and $|\theta| \lessapprox B\tau$. It follows by the triangle inequality we can replace $re^{i\theta}$ with $e^{i\theta}$ at the cost of $A\delta$. Next, because we assume $1 \in \gamma \cap \overline{\gamma}$, we have $s = |1-a|$, so we can substitute $|1-a|$ for s, and we are left with estimating

$$||e^{i\theta} - a| - |1-a||.$$

Because our circles lie in $Q = B(e_3, \alpha_0)$, we have the estimate $|e^{i\theta} - a| + |1-a| \sim 1$. Therefore multiplying by this expression, we only have to show

$$|e^{i\theta} - a|^2 - |1-a|^2 = O(AB^2)\delta.$$

The upshot is we can use the trigonometric identity

$$\begin{aligned}|e^{i\theta} - a|^2 - |1-a|^2 &= 2\,\mathrm{Re}(a)(1-\cos\theta) - 2\,\mathrm{Im}(a)\sin\theta \\ &= O(\mathrm{Re}(a))\theta^2 + O(\mathrm{Im}(a))|\theta| \\ &\lessapprox O(\mathrm{Re}(a))B^2\tau^2 + O(\mathrm{Im}(a))B\tau,\end{aligned}$$

and it suffices to estimate the components $a_1 = \mathrm{Re}(a)$ and $a_2 = \mathrm{Im}(a)$. We can use rectangle-lightplank duality to estimate both components simultaneously. We note $\Omega \subset C_{A\delta,w}$, so we have $w \in \mathbf{D}_{A\delta}(\Omega)$, which is contained in an $\approx \delta \times \delta\tau^{-1} \times \delta\tau^{-2}$-lightplank, by Proposition 2.9. By projecting this lightplank down to the plane $\mathbb{R}^2 \times \{0\}$, we see that $|\mathrm{Re}(a)| \lessapprox \delta\tau^{-2}$, and $|\mathrm{Im}(a)| \lessapprox \delta\tau^{-1}$.

Collecting the estimates we have made so far, we have shown for arbitrary $|r-1| < A\delta$ and $|\theta| \lessapprox B\tau$,

$$||re^{i\theta} - a| - s| \lessapprox A\delta + \delta\tau^{-2} \cdot B^2\tau^2 + A\delta\tau^{-1} \cdot B\tau \lessapprox AB^2\delta.$$

This finishes the proof. □

Now we are ready to prove that being comparable is almost a transitive relation. For the purpose of stating it succinctly, if Ω, Ω' are A-comparable, we write $\Omega \asymp_A \Omega'$.

Proposition A.4 (Being comparable is almost transitive)**.** *There is an absolute constant $C > 1$ such that if $\Omega_1 \asymp_A \Omega_2$ and $\Omega_2 \asymp_A \Omega_3$, then $\Omega_1 \asymp_{CA^C} \Omega_3$.*

Proof. Suppose the core circles of Ω_i are $v_i \in Q$, $i = 1, 2, 3$. Consider the radial projection $\pi\colon \mathbb{R}^2 \setminus \{v_2'\} \to C_{v_2}$ onto C_{v_2}. By assumption, $\pi(\Omega_1 \cup \Omega_3)$ is contained in an arc of length $\sim A\tau$ containing the core arc of Ω_2. Therefore, by Proposition A.3, $\Omega_1 \cup \Omega_3 \subset C_{A^{O(1)}\delta, v_2}$, which finishes the proof. □

Appendix B

Fourier transform of surface carried measure for the cone

Here we record Lemma 3.2 and its proof.

Lemma B.1 (Pointwise bound of $|\check{\sigma}(x)|$)**.** *Let σ be a smooth surface measure supported in $\mathbb{C}one^2$. For any $\epsilon > 0$ and any $N > 1$, there is a constant $C(\epsilon, N)$ so that*

$$|\check{\sigma}(x)| \leq C(\epsilon, N)\frac{1}{(1+|x|)^{\frac{1}{2}-\epsilon}}\frac{1}{(1+d(x,\Gamma_0))^N}$$

holds for all $x \in \mathbb{R}^3$, where Γ_0 is the lightcone $\{(a,r) \in \mathbb{R}^2 \times \mathbb{R} : ||a| - r| = 0\}$ with vertex 0.

Proof. We will prove this by combining two estimates for $|\check{\sigma}(x)|$:

(i) $|\check{\sigma}(x)| \lessapprox (1+|x|)^{-\frac{1}{2}+\epsilon}$

(ii) For every N, $|\check{\sigma}(x)| \lesssim_N (1+d(x,\Gamma_0))^{-N}$.

The conclusion follows by taking an appropriate geometric average of these two estimates. We may assume that $|x| \geq C$ for an appropriately large constant since $|\check{\sigma}(x)| \lesssim 1$ for $|x| \lesssim 1$.

We will start with (i). Suppose $|x| \sim r \gg 1$; our aim is to show $|\check{\sigma}(x)| \lessapprox r^{-\frac{1}{2}+\epsilon}$. We divide $\mathbb{C}one^2$ into $\sim r^{\frac{1}{2}-\epsilon}$-many strips θ of angular width $r^{-\frac{1}{2}+\epsilon}$ and let $\{\eta_\theta\}$ be a smooth partition of unity subordinate to $\{\theta\}$. Then with $\sigma_\theta = \sigma\eta_\theta$,

$$\check{\sigma}(x) = \sum_\theta \check{\sigma}_\theta(x).$$

For each θ, we let θ^* be the lightplank containing the origin of dimensions $1 \times r^{\frac{1}{2}-\epsilon} \times r^{1-2\epsilon}$ dual to the $r^{-1+2\epsilon}$-neighborhood of θ. By the Schwartz decay of $\check{\sigma}_\theta(x)$, we have

$$|\check{\sigma}_\theta(x)| \lesssim_N |\theta| \sum_{j=0}^{\infty} 2^{-jN} 1_{2^j\theta^*}(x).$$

Since we assume $|x| \sim r \gg r^{1-2\epsilon}$, and the directions of θ^* are $r^{-\frac{1}{2}+\epsilon}$-separated, x lies in at most $\lessapprox 1$ of the θ^*. Therefore,

$$|\check{\sigma}_\theta(x)| \lessapprox |\theta| \sim r^{-\frac{1}{2}+\epsilon}.$$

Now we prove (ii), but instead of using wave packets, we give a proof based on stationary phase considerations. Let $x = (x', x_3)$ with $|x| \gg 1$. Suppose that x is spacelike and lies in the upper half-space, so $|x'| > x_3 > 0$. The case of $|x'| < x_3$ is similar. For an appropriate smooth and compactly supported function $a(\xi)$ in $\{1 < |\xi| < 2\}$, we can write

$$\check{\sigma}(x) = Ea(x) = \int_{1<|\xi|<2} a(\xi) e^{i(x'\cdot\xi + x_3|\xi|)}\, d\xi.$$

Here E is the Fourier extension operator for the cone.

Let w be the nearest point on the cone Γ_0 to x. By elementary geometry, $x - w$ is orthogonal to the lightcone at w, and from this, we can compute the coordinates of w in terms of x:

$$w = w(x) = \left(\frac{|x'| + x_3}{2} \frac{x'}{|x'|}, \frac{|x'| + x_3}{2} \right).$$

Note from this formula for w that

$$d(x, \Gamma_0) = |x - w| \le |x'| - x_3 \lesssim |x - w| = d(x, \Gamma_0),$$

so $d(x, \Gamma_0) \sim ||x'| - x_3|$. Write

$$\begin{aligned}
Ea(x) = Ea(w + (x - w)) &= \int a(\xi) e^{i(w'\cdot\xi + w_3|\xi|)} e^{i[(x'-w')\cdot\xi + (x_3 - w_3)|\xi|]}\, d\xi \\
&= \int a(\xi) e^{i(w'\cdot\xi + w_3|\xi|)} e^{i\frac{|x'|-x_3}{2}(\frac{x'}{|x'|}\cdot\xi - |\xi|)} \\
&= \int a(\xi) e^{i\phi_1(\xi)} e^{i\lambda\phi_2(\xi)}\, d\xi,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(\xi) &= w' \cdot \xi + w_3|\xi| \\
\phi_2(\xi) &= \frac{x'}{|x'|} \cdot \xi - |\xi| \\
\lambda = \lambda(x) &= \frac{|x'| - x_3}{2}.
\end{aligned}$$

Let $\Sigma_1 = \{1 < |\xi| < 2 : \nabla\phi_1(\xi) = 0\}$, and similarly denote $\Sigma_2 = \{1 < |\xi| < 2 : \nabla\phi_2(x) = 0\}$. Since w is the nearest point to x lying in Γ_0, the critical points of $\phi_1(\xi)$ in $\{1 < |\xi| < 2\}$ are precisely the line segment

$$\Sigma_1 = \{1 < |\xi| < 2 : \frac{\xi}{|\xi|} = -\frac{x'}{|x'|}\}.$$

Likewise, Σ_2 is the line segment

$$\Sigma_2 = \{1 < |\xi| < 2 : \frac{\xi}{|\xi|} = \frac{x'}{|x'|}\}.$$

Consider the open sets

$$U_1 = \{1 < |\xi| < 2 : \angle(\xi, -x') > 0.1\}, \qquad U_2 = \{1 < |\xi| < 2 : \angle(\xi, -x') < 0.2\},$$

and a smooth partition of unity η_1, η_2 subordinate to U_1, U_2. Then $Ea = E(a\eta_1) + E(a\eta_2)$. Since the phase $x' \cdot \xi + x_3|\xi|$ has no critical points in U_1, $|E(a\eta_1)(x)| \lesssim_N (1 + |x|)^{-N}$ via integration by parts. So we only have to show that $|E(a\eta_2)(x)| \lesssim_N d(x, \Gamma_0)^{-N}$.

Since we only work with $a\eta_2$ from now on, to reduce clutter, we let a denote $a\eta_2$, so $\operatorname{dist}(\operatorname{supp} a, \Sigma_2) \gtrsim 1$. Lastly, we note that the phase ϕ_2 satisfies

$$\sup\{|\partial^\alpha \phi_2(\xi)| : 1 < |\xi| < 2, |\alpha| \le N\} \le C_N.$$

Consider the following vector field and its transpose

$$L = \frac{1}{i\lambda} \mathbf{v} \cdot \nabla, \quad L^t f = -\frac{1}{i\lambda} \nabla \cdot (f\mathbf{v})$$

where $\mathbf{v}(\xi) = \nabla\phi_2(\xi)/|\nabla\phi_2(\xi)|$, which is well defined and smooth throughout $\operatorname{supp} a$. By definition, $Le^{i\lambda\phi_2} = e^{i\lambda\phi_2}$, and consequently integrating by parts one time,

$$\begin{aligned} Ea(x) &= \int L^t(ae^{i\phi_1})e^{i\lambda\phi_2} \\ &= -\frac{1}{i\lambda} \int \nabla \cdot (ae^{i\phi_1}\mathbf{v})e^{i\lambda\phi_2}. \end{aligned}$$

Using the vector calculus identity

$$\nabla \cdot (fg\mathbf{v}) = f\nabla g \cdot \mathbf{v} + g\nabla f \cdot \mathbf{v} + fg\nabla \cdot \mathbf{v},$$

applied with $f = a$ and $g = e^{i\phi_1}$, we get

$$Ea(x) = -\frac{1}{i\lambda}\left(\int a(ie^{i\phi_1}\nabla\phi_1 \cdot \mathbf{v})e^{i\lambda\phi_2} + \int e^{i\phi_1} \underbrace{(\nabla a \cdot \nabla \mathbf{v} + a\nabla \cdot \mathbf{v})}_{\equiv a'} e^{i\lambda\phi_2} \right).$$

Note that

$$\nabla\phi_1(\xi) = w' + w_3\frac{\xi}{|\xi|} = (\frac{|x'| + x_3}{2})(\frac{x'}{|x'|} + \frac{\xi}{|\xi|})$$

and

$$\mathbf{v} = \frac{1}{|\nabla\phi_2|}(\frac{x'}{|x'|} - \frac{\xi}{|\xi|}).$$

Therefore, $\nabla\phi_1 \cdot \mathbf{v} = 0$, so $Ea(x)$ simplifies to

$$Ea(x) = -\frac{1}{i\lambda} \int a' e^{i\phi_1} e^{i\lambda\phi_2}.$$

Since a' is a smooth amplitude with all the same essential properties as those of $a(= a\eta_2)$, we are ready to run the same integration by parts argument N times to get

$$|Ea(x)| \lesssim_N \frac{1}{\lambda^N} = \frac{1}{(|x'| - x_3)^N}.$$

Since $||x'| - x_3| \sim d(x, \Gamma_0)$, we have proved

$$|Ea(x)| \lesssim_N \frac{1}{d(x, \Gamma_0)^N}.$$

Together with the proof of (i), this finishes the proof. □

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