Geometry and representation theory of symplectic singularities in the context of symplectic duality

by

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ABSTRACT

This thesis studies the geometry and representation theory of various symplectic resolutions of singularities from different perspectives. Specifically, following the ideas of Bellamy, Hilburn, Kamnitzer, Tingley, Webster, Weekes, and Yacobi, we establish a general approach to attack the Hikita-Nakajima conjecture and illustrate this approach in the example of ADHM spaces. We also study minimally supported representations of the quantizations of ADHM spaces and provide explicit formulas for their characters. Lastly, we describe the monodromy of eigenvalues of quantum multiplication operators for type A Nakajima quiver varieties by examining Bethe subalgebras in Yangians and linking their spectrum with Kirillov-Reshetikhin crystals.

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Chapter 1

Introduction

Symplectic singularities are an exciting new frontier of research in representation theory. During the last twenty years, people have actively studied them from various perspectives, both algebraic and geometric. Symplectic singularities are certain "nice" singular Poisson varieties $\mathfrak{M}_0$. Poisson algebras of functions $\mathbb{C}[\mathfrak{M}_0]$ can be quantized to associative algebras $\mathcal{A}$. Representation theory of these algebras is closely related to the geometry of resolutions $\mathfrak{M}$ of varieties $\mathfrak{M}_0$.

One particular example of a symplectic singularity is a nilpotent cone $\mathcal{N}$ of a semisimple Lie algebra $\mathfrak{g}$. In this example, algebras $\mathcal{A}$ are central quotients of the universal enveloping algebra $U(\mathfrak{g})$. We see that in this particular case, the representation theory of $\mathcal{A}$ boils down to the representation theory of $\mathfrak{g}$. So, the study of the representation theory of quantizations of symplectic singularities is the wide generalization of the "classical" representation theory of semisimple Lie algebras.

Another significance of symplectic resolutions of singularities lies in their origins within string theory. Namely, some symplectic resolutions are Higgs branches of certain $3D$ $\mathcal{N} = 2$ gauge theories. Moreover, Braverman, Finkelberg, and Nakajima have recently introduced a mathematical definition of Coulomb branches of these theories, and it turns out (see [119] and [4]) that these Coulomb branches are also particular examples of symplectic singularities. So, the theory of symplectic singularities is tightly connected with the modern physics. One might expect that every symplectic singularity should be a Higgs/Coulomb branch of some (maybe very complicated) theory.

Physicists predict that the Higgs and Coulomb branches of a theory should be related in some sophisticated and nontrivial way. This relation is called a *$3D$-mirror symmetry*. It turns out that even if we do not have a Higgs/Coulomb realization of our symplectic singularity, we still can guess the answer for the $3D$-dual variety in many examples. We can then study the relation between the dual varieties (in the setting of symplectic singularities, this is what people call the *symplectic duality*, it was developed independently of a $3D$-mirror symmetry by Braden, Licata, Proudfoot, and Webster).

There are several approaches to the $3D$-mirror symmetry/symplectic duality. One of them is via the enumerative geometry (proposed by Okounkov and collaborators). In this approach, solutions of quantum difference D-modules are compared for the resolutions $\mathfrak{M}$, $\mathfrak{M}^!$ of dual varieties $\mathfrak{M}_0$ and $\mathfrak{M}_0^!$ (this approach has its origins in string theory). Another view on symplectic duality was proposed in the paper [10], where the authors conjectured that the categories $\mathcal{O}$ of modules over quantizations of $\mathfrak{M}_0$, $\mathfrak{M}_0^!$ should be Koszul dual. So, there are two conceptually different ways to approach symplectic duality – one is via the geometry of the resolutions of $\mathfrak{M}_0, \mathfrak{M}_0^!$ (namely, by studying the equivariant quantum cohomology/K-theory of these varieties) and the second is via the representation theory of quantizations of these (Poisson, singular) varieties themselves.

One of the conjectures of symplectic duality that will be discussed in Chapter 2 of this thesis is the so-called Hikita-Nakajima conjecture proposed by Hikita in [46] and then generalized by Nakajima in an unpublished paper. It was also generalized further by Kamnitzer, McBreen, and Proudfoot in [57]. The most general version of this conjecture identifies certain specialization of the quantum D-module of $\mathfrak{M}$ with certain D-module (called D-module of graded traces) for the universal quantization of $\mathfrak{M}_0^!$. So, this conjecture provides a bridge between the geometry of $\mathfrak{M}$ and the representation theory of $\mathfrak{M}_0^!$. In this text, we will discuss the general approach that would lead us to the proof of (the classical version of) this conjecture in the case of ADHM spaces (these are certain varieties of great importance in physics; in physics literature, they are called moduli spaces of instantons on $\mathbb{P}^2$).

The other two main results of the thesis are also within the framework of the symplectic duality. Namely, in Chapter 3, we study the category $\mathcal{O}$ for quantizations of ADHM spaces. The main result of this chapter is the explicit formula for the characters of minimally supported modules in these categories $\mathcal{O}$. So, the objects appearing in this chapter are in the context of the algebraic approach to the symplectic duality. In Chapter 4, we study Bethe subalgebras in the Yangian of some simple Lie algebra $\mathfrak{g}$. We then restrict to type A and study the spectrum of these algebras in the tensor product of Kirillov-Reshetikhin modules. By the results of Maulik-Okounkov, geometrically this corresponds to studying eigenvalues of the operators of quantum multiplications for type A quiver varieties (a.k.a. solutions of Bethe ansatz equations). We observe that this set of eigenvalues has a combinatorial structure (called $\widehat{\mathfrak{g}}$-crystal) and use this to compute the monodromy of these solutions. So, this chapter studies objects arising on the enumerative side of the story.

1.1 Symplectic singularities and their quantizations

Let $\mathfrak{M}_0$ be an affine algebraic variety over $\mathbb{C}$. The following definition belongs to Beauville (see [2]).

Definition 1.1.1 *We say that* $\mathfrak{M}_0$ *is* singular symplectic *(or has symplectic singularities) if:*

(1) $\mathfrak{M}_0$ *is a normal Poisson variety,*

(2) there exists a smooth, dense open subset $U \subset \mathfrak{M}_0$ *on which the Poisson structure comes from the symplectic form that we denote by* ω*,*

(3) there exists a resolution of singularities (birational and projective morphism) $\mathfrak{M} \to \mathfrak{M}_0$ *such that the pullback of* ω *to* $\mathfrak{M}$ *has no poles.*

We say that $\mathfrak{M}_0$ is a *conical symplectic singularity* if in addition to (1) – (3) one has a $\mathbb{C}^\times$-action on $\mathfrak{M}_0$ which acts on ω with some positive weight and contracts $\mathfrak{M}_0$ to the unique fixed point. We will sometimes denote the contracting $\mathbb{C}^\times$ by $\mathbb{C}^\times_\hbar$.

Assume now that $\mathfrak{M}_0$ possesses a $\mathbb{C}^\times_\hbar$-equivariant symplectic resolution $\mathfrak{M} \to \mathfrak{M}_0$. We will call $\mathfrak{M} \to \mathfrak{M}_0$ a *conical symplectic resolution*. It is known (see [90, Lemmas 12, 22, Proposition 13], [38, Theorem 1.13]) that there exist canonical symplectic (resp. Poisson) deformations of $\mathfrak{M}$ (resp. $\mathfrak{M}_0$) over the base $\mathfrak{t} = \mathfrak{t}_{\mathfrak{M}_0} := H^2(\mathfrak{M}, \mathbb{C})$ that we denote by

$$\mathfrak{M}_\mathfrak{t} \to \mathfrak{t},\, \mathfrak{M}_{0,\mathfrak{t}} \to \mathfrak{t}.$$

Remark 1.1.2 One can show that the space $\mathfrak{t}$ does not depend on the choice of the resolution $\mathfrak{M}$ of $\mathfrak{M}_0$. This space can be defined even if $\mathfrak{M}_0$ does not have a symplectic resolution $\mathfrak{M}$. The deformation $\mathfrak{M}_\mathfrak{t}$ is the universal deformation of $\mathfrak{M}$, $\mathfrak{M}_{0,\mathfrak{t}}$ is a pullback of the universal deformation of $\mathfrak{M}_0$ along the morphism $\mathfrak{t} \to \mathfrak{t}/W$ for a certain group W acting on $\mathfrak{t}$ (called the Namikawa-Weyl group).

Let $\mathrm{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ be the group of Poisson automorphisms of $\mathfrak{M}_0$ commuting with the contracting $\mathbb{C}^\times_\hbar$. This is a finite-dimensional algebraic group (possibly disconnected). We denote by $S = S_{\mathfrak{M}_0} \subset \mathrm{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ a maximal torus and set $\mathfrak{s}_{\mathfrak{M}_0} := \mathrm{Lie}\, S_{\mathfrak{M}_0}$. One can show that the action of S on $\mathfrak{M}_0$ and the contracting action of $\mathbb{C}^\times_\hbar$ extend naturally to the action of $S \times \mathbb{C}^\times_\hbar$ on $\mathfrak{M}_{0,\mathfrak{t}}$ (the torus S acts fiberwise). It can also be shown that the $S \times \mathbb{C}^\times_\hbar$-action above lifts to the action on $\mathfrak{M}_\mathfrak{t}$.

1.1.1 Quantizations of $\mathfrak{M}$ and $\mathfrak{M}_0$

In this section, we assume that $\mathfrak{M}_0$ admits a symplectic resolution $\mathfrak{M}$. We also assume that $\mathbb{C}^\times_\hbar$ scales the symplectic form with weight 2. By a *graded formal quantization* of the structure sheaf $\mathcal{O}_\mathfrak{M}$ we mean (see for example [74, Section 2.2]):

- a sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{D}$ in Zariski topology on $\mathfrak{M}$ equipped with a $\mathbb{C}^\times_\hbar$-action by algebra automorphisms such that $t \cdot \hbar = t^2\hbar$,
- an isomorphism $\iota \colon \mathcal{D}/(\hbar) \xrightarrow{\sim} \mathcal{O}_\mathfrak{M}$ of sheaves

  such that:

- $\mathcal{D}$ is flat over $\mathbb{C}[[\hbar]]$,
- the $\hbar$-adic filtration on $\mathcal{D}$ is complete and separated,
- $[\mathcal{D}, \mathcal{D}] \subset \hbar\mathcal{D}$. This gives a Poisson bracket on $\mathcal{D}/(\hbar)$,
- ι is a graded Poisson isomorphism.

The graded formal quantizations of $\mathfrak{M}$ are parametrized by the vector space $\mathfrak{t} = H^2(\mathfrak{M}, \mathbb{C})$ via the so called *period map* Per from the isomorphism classes of quantizations to $\mathfrak{t}$ (see [7], [70, Section 2.3]). We will denote the formal quantization that corresponds to $\lambda \in \mathfrak{t}$ by $\mathcal{D}^{\mathrm{form}}_\lambda$. Recall that we have the universal (conical) deformation $\mathfrak{M}_\mathfrak{t} \to \mathfrak{t}$. We can talk about graded formal quantizations of $\mathfrak{M}_\mathfrak{t}/\mathfrak{t}$ that are now required to be sheaves of $\mathbb{C}^\times_\hbar$-graded $\mathbb{C}[\mathfrak{t}][[\hbar]]$-algebras, where the $\mathbb{C}^\times_\hbar$-action on $\mathfrak{t}$ is given by $t \cdot p = t^{-2}p$. It was shown in [7, Section

6.2] that $\mathcal{O}_{\mathfrak{M}_\mathfrak{t}}$ admits a canonical graded formal quantization to be denoted by $\mathcal{D}^{\mathrm{form}}_{\mathfrak{t}}$. It satisfies the following property: its specialization to $\lambda \in \mathfrak{t}$ coincides with $\mathcal{D}^{\mathrm{form}}_{\lambda}$. We set $\mathcal{A}^{\mathrm{form}}_{\lambda} := \Gamma(Y, \mathcal{D}^{\mathrm{form}}_{\lambda})$. By the Grauert-Riemenschneider theorem, $R\Gamma(\mathcal{O}_{\mathfrak{M}}) = \mathbb{C}[\mathfrak{M}_0]$. From here one deduces that the algebra $\mathcal{A}^{\mathrm{form}}_{\lambda}$ is a formal graded quantization of $\Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}) = \mathbb{C}[\mathfrak{M}_0]$. Similarly, the algebra $\mathcal{A}^{\mathrm{form}}_{\mathfrak{t}} = \Gamma(\mathfrak{M}_{0,\mathfrak{t}}, \mathcal{A}^{\mathrm{formal}}_{\mathfrak{t}})$ is a quantization of $\mathbb{C}[\mathfrak{M}_{0,\mathfrak{t}}]$ over $\mathbb{C}[\mathfrak{t}]$.

Recall that $\mathcal{D}^{\mathrm{form}}_{\lambda}, \mathcal{D}^{\mathrm{form}}_{\mathfrak{t}}, \mathcal{A}^{\mathrm{form}}_{\lambda}, \mathcal{A}^{\mathrm{form}}_{\mathfrak{t}}$ are $\mathbb{C}^\times_\hbar$-graded. Let $\mathcal{D}_{\hbar,\lambda}, \mathcal{D}_{\hbar,\mathfrak{t}}, \mathcal{A}_{\hbar,\lambda}, \mathcal{A}_{\hbar,\mathfrak{t}}$ be the corresponding $\mathbb{C}^\times_\hbar$-finite parts, they are modules over $\mathbb{C}[\hbar]$. In what follows we refer to them as *polynomial quantizations*. Finally, we consider filtered quantizations $\mathcal{D}_\lambda, \mathcal{D}_\mathfrak{t}, \mathcal{A}_\lambda, \mathcal{A}_\mathfrak{t}$ obtained as the specializations of the polynomial quantizations at $\hbar = 1$.

Remark 1.1.3 Recall that $\mathcal{D}_{\hbar,\lambda}, \mathcal{D}_\lambda, \mathcal{D}_{\hbar,\mathfrak{t}}, \mathcal{D}_\mathfrak{t}$ are sheaves on $\mathfrak{M}, \mathfrak{M}_{0,\mathfrak{t}}$ in *conical* topology.

Let us also mention that for a vector space $\mathfrak{a}$ mapping linearly to $\mathfrak{t}$, we can consider the corresponding base changes $\mathfrak{M}_{0,\mathfrak{a}}$, $\mathfrak{M}_\mathfrak{a}$ and the corresponding quantizations $\mathcal{D}^{\mathrm{form}}_{\mathfrak{a}}$, $\mathcal{A}^{\mathrm{form}}_{\mathfrak{a}}$, $\mathcal{D}_{\hbar,\mathfrak{a}}$, $\mathcal{A}_{\hbar,\mathfrak{a}}$.

1.1.2 Categories $\mathcal{O}$ and localizations

We assume that the set $\mathfrak{M}^S$ is finite. Let us fix a co-character $\nu\colon \mathbb{C}^\times \to S$. We say that a co-character ν is *regular* if the set $\mathfrak{M}^{\nu(\mathbb{C}^\times)}$ is finite (in other words, coincides with $\mathfrak{M}^S$). We obtain the decomposition of the vector space $\mathfrak{t}_{X,\mathbb{R}} := \mathrm{Hom}(\mathbb{C}^\times, S) \otimes_\mathbb{Z} \mathbb{R}$ into the union of *open chambers* separated by the walls corresponding to non-regular co-characters ν.

Categories $\mathcal{O}$

We fix a parameter $\lambda \in \mathfrak{t}$ and a regular co-character $\nu \in \mathrm{Hom}(\mathbb{C}^\times, S)$. Consider the corresponding filtered quantization $\mathcal{D}_\lambda$ of $\mathcal{O}_\mathfrak{M}$. Recall that $\mathcal{A}_\lambda = \Gamma(\mathfrak{M}, \mathcal{D}_\lambda)$. The action of $S \curvearrowright \mathcal{O}_\mathfrak{M}$ lifts canonically to the action $S \curvearrowright \mathcal{D}_\lambda$ and hence to the action $S \curvearrowright \mathcal{A}_\lambda$.

The co-character ν induces a grading $\mathcal{A}_\lambda = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{\lambda,i}$ on $\mathcal{A}_\lambda$. Let $\mathcal{A}^{>0}_\lambda := \bigoplus_{i>0} \mathcal{A}_{\lambda,i}$. It follows from [11, Proposition 3.11] that the action of ν on $\mathcal{A}_\lambda$ is Hamiltonian, so we have a comoment map $\mathbb{C} \to \mathcal{A}_{\lambda,0}$. Let $h \in \mathcal{A}_{\lambda,0}$ be the image of $1 \in \mathbb{C}$. Let $\mathcal{A}_\lambda$-mod be the category of finitely generated $\mathcal{A}_\lambda$-modules and let $\mathcal{O}_\nu(\mathcal{A}_\lambda) \subset \mathcal{A}_\lambda$-mod be the full subcategory, consisting of all modules where the action of $\mathcal{A}^{>0}_\lambda$ is locally nilpotent and the action of $\mathcal{A}_{\lambda,0}$ is locally finite.

Let $\mathrm{Coh}(\mathcal{D}_\lambda)$ be the category of all coherent $\mathcal{D}_\lambda$-modules and $\mathcal{O}_\nu(\mathcal{D}_\lambda) \subset \mathrm{Coh}(\mathcal{D}_\lambda)$ be the full subcategory of modules that come with a good filtration stable under h and are supported on the contracting locus $\mathfrak{M}_+$ of ν, i.e., $\mathfrak{M}_+ := \{x \in \mathfrak{M} \,|\, \lim_{t\to 0} \nu(t)\cdot x \ \text{ exists}\}$. The set $\mathrm{Irr}(\mathcal{O}_\nu(\mathcal{D}_\lambda))$ of irreducible objects can be naturally identified with $\mathfrak{M}^S$ (see [10, Proposition 5.17] and Section 1.3.4 below).

Localizations

We have the global section functor $\Gamma\colon \mathrm{Coh}(\mathcal{D}_\lambda) \to \mathcal{A}_\lambda$-mod. We denote by $\mathrm{Loc}\colon \mathcal{A}_\lambda\text{-mod} \to \mathrm{Coh}(\mathcal{D}_\lambda)$ the left adjoint functor given by $N \mapsto \mathcal{D}_\lambda \otimes_{\mathcal{A}_\lambda} N$ (see [11, Section 4.2]). For $\lambda \in \mathfrak{t}$, we say that abelian localization holds for $(\lambda, \mathfrak{M})$ when the functors Γ and Loc are quasi-inverse equivalences. Let $\eta \in H^2(\mathfrak{M}, \mathbb{C}) = \mathfrak{t}$ be the first Chern class of an ample line bundle on $\mathfrak{M}$. It is known (see [11, Corollary B.1]) that for a sufficiently large integer M (depending on $\lambda \in \mathfrak{t}$), the abelian localization holds for $\lambda + m\eta$ for all integers $m \geqslant M$. In particular, the irreducible objects of $\mathcal{O}_\nu(\mathcal{A}_{\lambda+m\eta})$ are parametrized by $\mathfrak{M}^S$.

1.2 Examples of symplectic singularities

1.2.1 Higgs and Coulomb branches

In this section, we briefly recall the main definitions of the Higgs and Coulomb branches of $3D$ $\mathcal{N} = 2$ gauge theories. Let G_{gauge} be a reductive group (over $\mathbb{C}$) and let $\mathbf{N}$ be a finite-dimensional representation of G_{gauge}. Let $\mu\colon T^*\mathbf{N} \to \mathfrak{g}^*_{\text{gauge}}$ be a moment map.

Higgs branches

We set

$$\mathfrak{M}_0 := T^*\mathbf{N}/\!\!/\!\!/G_{\text{gauge}} = \mu^{-1}(0)/\!\!/G_{\text{gauge}} = \operatorname{Spec}\mathbb{C}[(\mu^{-1}(0))^{G_{\text{gauge}}}].$$

We always assume that for generic enough stability parameter $\theta\colon G_{\text{gauge}} \to \mathbb{C}^\times$, the action $G_{\text{gauge}} \curvearrowright \mu^{-1}(0)^{\theta-\text{stable}}$ is free and the quotient $\mathfrak{M} := \mu^{-1}(0)^{\theta-\text{stable}}/G_{\text{gauge}}$ is the smooth (symplectic) resolution of $\mathfrak{M}_0$ to be called the (resolved) Higgs branch of the theory. Quantizations of $\mathfrak{M}_0$ can be constructed as quantum Hamiltonian reductions of (global) differential operators on $\mathbf{N}$.

Coulomb branches

Construction of the Coulomb branch is trickier and was recently proposed by Braverman, Finkelberg, and Nakajima in [14]. Let us recall the definition.

Set $\mathcal{T} := G_{\text{gauge}}((t)) \times^{G_{\text{gauge}}[[t]]} \mathbf{N}[[t]]$. We have the natural map $\mathcal{T} \to \mathbf{N}((t))$ given by $[g, n] \mapsto gn$. We define:

$$\mathcal{R} = \mathcal{R}(G_{\text{gauge}}, \mathbf{N}) = \mathbf{N}[[t]] \times_{\mathbf{N}((t))} \mathcal{T}.$$

We can consider the equivariant Borel-Moore homology $H_*^{G_{\text{gauge}}[[t]]}(\mathcal{R})$ of the space $\mathcal{R}$ (see [14, Section 2(ii)] for the definition and detailed discussion). This vector space is equipped with an algebra structure via the convolution $*$ (see [14, Section 3]). It follows from [14, Proposition 5.15] that the algebra $(H_*^{G_{\text{gauge}}[[t]]}(\mathcal{R}), *)$ is commutative. The Coulomb branch $\mathcal{M} = \mathcal{M}(G_{\text{gauge}}, \mathbf{N})$ is by the definition equal to the spectrum of this algebra:

$$\mathcal{M} := \operatorname{Spec} H_*^{G_{\text{gauge}}[[t]]}(\mathcal{R}).$$

Assume now that we are given an additional torus A acting on $\mathbf{N}$ and commuting with the action of G_{gauge} (we will call A a *flavor* torus). Set $\mathfrak{a} := \operatorname{Lie} A$. The *deformed* Coulomb branch over $\mathfrak{a}$ is defined as

$$\mathcal{M}_{\mathfrak{a}} := \operatorname{Spec} H_*^{G_{\text{gauge}}[[t]]\times A}(\mathcal{R}).$$

Remark 1.2.1 *Note that A naturally acts on the Higgs branch $\mathfrak{M}$, this is compatible with what symplectic duality predicts.*

Finally, *quantization* of $\mathcal{M}_{\mathfrak{a}}$ can be defined as follows. Consider the semidirect product $G_{\mathrm{gauge}}[[t]] \rtimes \mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $G_{\mathrm{gauge}}[[t]]$ by scaling t. This semidirect product acts naturally on $\mathcal{R}$ commuting with the action of A. Then,

$$\mathcal{A}_{\hbar,\mathfrak{a}}(\mathcal{M}) := H_*^{(G_{\mathrm{gauge}}[[t]]\rtimes\mathbb{C}^\times)\times A}(\mathcal{R}).$$

1.2.2 Case of quiver theories

An important class of pairs $(G_{\mathrm{gauge}}, \mathbf{N})$ to consider is the one coming from a choice of a quiver $I = (I_0, I_1)$ (I_0 are vertices and I_1 are edges) and a collection of nonnegative numbers $\mathbf{v} = (v_i)_{i\in I_0}$, $\mathbf{w} = (w_i)_{i\in I_0}$ ((v_i) is called the *dimension vector* and (w_i) is called the *framing*). For $i \in I_0$, let V_i, W_i be vector spaces of dimensions v_i, w_i. We define

$$G_{\mathrm{gauge}} = G_I := \prod_{i\in I_0} \mathrm{GL}(V_i),\ \mathbf{N} := \bigoplus_{(i\to j)\in I_1} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I_0} \mathrm{Hom}(W_i, V_i).$$

Varieties $\mathfrak{M}_0$, $\mathfrak{M}$, $\mathcal{M}$ corresponding to $(G_I, \mathbf{N})$ as above are called Higgs and Coulomb branches of the theory corresponding to I together with the choice of (v_i), (w_i). Note that the Higgs branch $\mathfrak{M}$ is nothing else but the *Nakajima quiver variety* corresponding to the quiver I (and the choice of (v_i), (w_i)), let us recall its definition in more detail.

We consider the cotangent space $\mathbf{M}_I(\mathbf{v},\mathbf{w}) = \mathbf{M}(\mathbf{v},\mathbf{w}) = \mathbf{M} := T^*\mathbf{N}$ that can be identified with

$$\bigoplus_{(i\to j)\in I_1} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{(i\to j)\in I_1} \mathrm{Hom}(V_j, V_i) \oplus \bigoplus_{i\in I_0} \mathrm{Hom}(W_i, V_i) \oplus \bigoplus_{i\in I_0} \mathrm{Hom}(V_i, W_i).$$

We can represent elements of $\mathbf{M}(\mathbf{v},\mathbf{w})$ as quadruples (X, Y, γ, δ), where

$$X \in \bigoplus_{(i\to j)\in I_1} \mathrm{Hom}(V_i, V_j),\ Y \in \bigoplus_{(i\to j)\in I_1} \mathrm{Hom}(V_j, V_i),$$

$$\gamma \in \bigoplus_{i\in I_0} \mathrm{Hom}(W_i, V_i),\ \delta \in \bigoplus_{i\in I_0} \mathrm{Hom}(V_i, W_i).$$

The space $\mathbf{M}(\mathbf{v},\mathbf{w}) = T^*\mathbf{N}$ carries a natural symplectic form. We set

$$G_{\mathbf{v}} := \prod_{i\in I_0} \mathrm{GL}(V_i),\ \mathfrak{g}_{\mathbf{v}} := \bigoplus_{i\in I_0} \mathfrak{gl}(V_i).$$

The group $G_{\mathbf{v}}$ acts naturally on the vector space $\mathbf{M}(\mathbf{v},\mathbf{w})$. This action is symplectic with the moment map:

$$\mu\colon \mathbf{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{g}_{\mathbf{v}},\ (X, Y, \gamma, \delta) \mapsto [X, Y] + \gamma\delta.$$

Definition 1.2.2 *A quadruple* $(X, Y, \gamma, \delta) \in \mathbf{M}(\mathbf{v},\mathbf{w})$ *is called* stable *if for every* X, Y*-invariant graded subspace* $S \subset V$ *such that* S *contains* $\mathrm{im}\,\gamma$ *we have* $S = V$. *We denote by* $\mathbf{M}(\mathbf{v},\mathbf{w})^{\mathrm{st}} \subset \mathbf{M}(\mathbf{v},\mathbf{w})$ *the (open) subset of stable quadruples.*

Definition 1.2.3 *The Nakajima quiver varieties* $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ *are defined as the following quotients:*

$$\mathfrak{M}(\mathbf{v},\mathbf{w}) := \mu^{-1}(0)^{\mathrm{st}}/G_{\mathbf{v}},\ \mathfrak{M}_0(\mathbf{v},\mathbf{w}) := \mu^{-1}(0)/\!\!/G_{\mathbf{v}}.$$

We have the natural (projective) morphism $\pi\colon \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$.

Assumption 1.2.4 *We assume that* π *is a resolution of singularities.*

Let $\mathfrak{z}_{\mathbf{v}} \subset \mathfrak{g}_{\mathbf{v}}$ be the center of $\mathfrak{g}_{\mathbf{v}}$. The varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ admit certain natural deformations over the space $\mathfrak{z}_{\mathbf{v}}$.

Definition 1.2.5 *The deformed quiver varieties* $\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}}$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}}$ *over* $\mathfrak{z}_{\mathbf{v}}$ *are defined as follows:*

$$\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}} := \mu^{-1}(\mathfrak{z}_{\mathbf{v}})^{\mathrm{st}}/G_{\mathbf{v}},\ \mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}} := \mu^{-1}(\mathfrak{z}_{\mathbf{v}})/\!\!/G_{\mathbf{v}}.$$

For $\mathbf{a} \in \mathfrak{z}_{\mathbf{v}}$, *we denote by* $\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathbf{a}}$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathbf{a}}$ *the fibers of these families over* $\mathbf{a}$.

1.3 Schematic fixed points and Cartan subquotients

1.3.1 Schematic fixed points

Given a variety Y with an action of some algebraic group G we can define the functor

$$Y^G\colon \mathbf{Schemes}_{\mathbb{C}} \to \mathbf{Sets}$$

from the category $\mathbf{Schemes}_{\mathbb{C}}$ of schemes over $\mathbb{C}$ to the category **Sets** of sets as follows (see [30], [24, Section 1.2]):

$$Y^G(S) := \mathrm{Maps}^G(S,Y),\ S \in \mathbf{Schemes}_{\mathbb{C}},$$

where the action of G on S is trivial and $\mathrm{Maps}^G(S,Y)$ is the set of G-equivariant morphisms from S to Y. It turns out that in some cases, functor Y^G is represented by a scheme that we call *schematic fixed points of* Y (for more details, see [30, Theorem 2.3]). Consider the case $Y = \operatorname{Spec} B$ for some $\mathbb{C}$-algebra B and $G = \mathbb{C}^\times$. Then the action $\mathbb{C}^\times \curvearrowright Y$ corresponds to the $\mathbb{Z}$-grading $B = \bigoplus_{i\in\mathbb{Z}} B_i$. Compare the following proposition with [24, Example 1.2.3].

Proposition 1.3.1 *If* $Y = \operatorname{Spec} B$ *is an affine variety and* $G = \mathbb{C}^\times$, *then* $Y^{\mathbb{C}^\times}$ *is represented by an affine scheme whose ring of functions can be described in two equivalent ways:*

$$\mathbb{C}[Y^{\mathbb{C}^\times}] = B_0/\sum_{i>0} B_{-i}B_i = B/(b_i \in B_i,\ i \neq 0). \tag{1.1}$$

Proof: It is enough to show that the functor $Y^{\mathbb{C}^\times}$ restricted to the category of affine schemes over $\mathbb{C}$ is represented by the affine scheme with the algebra of functions as in (1.1). Let $S = \operatorname{Spec} C$ be an affine scheme with trivial $\mathbb{C}^\times$-action. The set $\mathrm{Maps}^{\mathbb{C}^\times}(S,Y)$ can be identified with the set of graded homomorphisms $B \to C$, where the grading on C is the trivial one ($C = C_0$). Since $C = C_0$, we conclude that every such homomorphism $f\colon B \to C$ factors through $B/(b_i \in B_i,\, i \neq 0)$. Note now that every homomorphism $\overline{f}\colon B/(b_i \in B_i,\, i \neq 0) \to C$ induces the *graded* homomorphism $B \twoheadrightarrow B/(b_i \in B_i,\, i \neq 0) \to C$, so we must have $Y^{\mathbb{C}^\times} = \operatorname{Spec}(B/(b_i \in B_i,\, i \neq 0))$.

It remains to note that the natural morphism

$$B_0/\sum_{i>0} B_{-i}B_i \to B/(b_i \in B_i,\, i \neq 0),$$

given by

$$B_0/\sum_{i>0} B_{-i}B_i \ni [b] \mapsto [b] \in B/(b_i \in B_i,\, i \neq 0)$$

is an isomorphism. □

Remark 1.3.2 Proposition 1.3.1 can be easily generalized to the case when G is a torus of arbitrary rank (or an arbitrary reductive group).

The following proposition holds (see, for example, [54]).

Proposition 1.3.3 *If Y is a smooth algebraic variety over $\mathbb{C}$ and G is reductive, then Y^G is smooth.*

1.3.2 Cartan subquotients (B-algebras)

The following construction should be considered as a noncommutative version of taking schematic fixed points (compare with (1.1)). For any $\mathbb{Z}$-graded ring $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$, set:

$$\mathsf{C}(\mathcal{A}) := \mathcal{A}_0/\sum_{i>0} \mathcal{A}_{-i}\mathcal{A}_i.$$

Ring $\mathsf{C}(\mathcal{A})$ will be called the *Cartan subquotient* or a *B-algebra* of $\mathcal{A}$. See [73, Sections 4, 5] and [10, Section 5.1] for more details. We will apply this construction to the quantizations of $\mathbb{C}[\mathfrak{M}_0]$, $\mathbb{C}[\mathfrak{M}_{0,\mathfrak{t}}]$ with $\mathbb{Z}$-grading induced by a (generic) cocharacter $\nu\colon \mathbb{C}^\times \to S_{\mathfrak{M}_0}$. If $\mathcal{A}$ is one of these quantizations, then the corresponding Cartan subquotient will be denoted by $\mathsf{C}_\nu(\mathcal{A})$. We will also use the notation $\mathcal{A}^{\geqslant 0} := \bigoplus_{i \geqslant 0} \mathcal{A}_i$. Note that there exists the natural surjection $\mathcal{A}^{\geqslant 0} \twoheadrightarrow \mathsf{C}(\mathcal{A})$ induced by the surjection $\mathcal{A}^{\geqslant 0} \twoheadrightarrow \mathcal{A}_0$.

1.3.3 Sheaf version of Cartan subquotient

In this section, let $S = S_{\mathfrak{M}_0}$, let $\mathfrak{M}$ be a symplectic resolution of $\mathfrak{M}_0$, and assume that $\mathfrak{M}$ has finitely many S-fixed points. Following [73, Section 5.2] one can also talk about the Cartan subquotients of *sheaves* $\mathcal{D}$ on $\mathfrak{M}_{0,\mathfrak{a}}$ (see [73, Proposition 5.2]), the resulting object will be a *sheaf* of algebras on the S-fixed points of $\mathfrak{M}_{\mathfrak{a}}$ (recall that $\mathfrak{M}_{\mathfrak{a}}^S$ is smooth by Proposition 1.3.3 above).

Lemma 1.3.4 *We have a canonical isomorphism $\mathfrak{M}_{\mathfrak{a}}^S \simeq \mathfrak{M}^S \times \mathfrak{a}$ of varieties over $\mathfrak{a}$.*

Proof: Standard. □

Lemma 1.3.5 *There are canonical isomorphisms of sheaves of algebras*

$$\mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{a}}) \simeq \mathcal{O}_{\mathfrak{M}^S_{\mathfrak{a}}}[\hbar], \quad \mathsf{C}_\nu(\mathcal{D}_{\hbar,\lambda}) \simeq \mathcal{O}_{\mathfrak{M}^S}[\hbar], \quad \mathsf{C}_\nu(\mathcal{D}_\lambda) \simeq \mathcal{O}_{\mathfrak{M}^S}.$$

Proof: Since fixed S-points are isolated in $\mathfrak{M}$ by our assumption, the lemma follows from [10, Lemma 5.2]. □

Using Lemmas 1.3.4, 1.3.5 we obtain the natural homomorphism of $\mathbb{C}[\mathfrak{a},\hbar]$-algebras

$$\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{a}}) \to \Gamma(\mathfrak{M}^S_{\mathfrak{a}}, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{a}})) = \mathbb{C}[\mathfrak{M}^S] \otimes \mathbb{C}[\mathfrak{a},\hbar]$$

that at $\lambda \in \mathfrak{a}$ specializes to the homomorphism

$$\mathsf{C}_\nu(\mathcal{A}_{\hbar,\lambda}) \to \Gamma(\mathfrak{M}^S, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\lambda})) = \mathbb{C}[\mathfrak{M}^S] \otimes \mathbb{C}[\hbar]. \tag{1.2}$$

Remark 1.3.6 *It follows from [11, Proposition 5.3] (see also [73, Proposition 5.3]) that the composition (1.2) becomes $\mathbb{C}[\hbar]$-linear isomorphism for Zariski generic values of λ.*

1.3.4 Cartan subquotients and irreducible objects in categories $\mathcal{O}$

Let us now recall the relation between $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ and irreducible objects in the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. Pick λ such that the abelian localization holds and the morphism (1.2) specialized at $\hbar = 1$ is an isomorphism:

$$\varphi_\lambda \colon \mathsf{C}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \Gamma(\mathfrak{M}^S, \mathsf{C}_\nu(\mathcal{D}_\lambda)) = \bigoplus_{p \in \mathfrak{M}^S} \mathbb{C}.$$

We see that the algebra $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ is finite-dimensional semisimple and has $|\mathfrak{M}^S|$ irreducible (one dimensional) representations labeled by the fixed points of $\mathfrak{M}$. Let $\mathbb{C}_p$ be the irreducible representation of $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ corresponding to the point $p \in \mathfrak{M}^S$. We can then form the *standard* module (see [10, Section 5.2]):

$$\Delta_\lambda(p) = \Delta(p) = \mathcal{A}_\lambda \otimes_{\mathcal{A}_\lambda^{\geqslant 0}} \mathbb{C}_p \in \mathcal{O}_\nu(\mathcal{A}_\lambda),$$

where the action of $\mathcal{A}_\lambda^{\geqslant 0} \curvearrowright \mathbb{C}_p$ is induced by the natural surjection $\mathcal{A}_\lambda^{\geqslant 0} \twoheadrightarrow \mathsf{C}_\nu(\mathcal{A}_\lambda)$.

Remark 1.3.7 *Similary, one can define the* costandard *module $\nabla(p) = \nabla_\lambda(p)$ as follows (see [10, Section 5.2]): $\nabla(p) = (\mathbb{C}_p^* \otimes_{\mathcal{A}_\lambda^{\geqslant 0}} \mathcal{A})^\star$, here $\star$ denotes the restricted duality.*

By the resuls of [10], every $\Delta(p)$ has the *unique* irreducible quotient to be denoted by $L(p)$. Moreover, modules $\{L(p) \,|\, p \in \mathfrak{M}^S\}$ are pairwise non-isomorphic and form the complete list of simples in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. Recall that we have the equivalence of categories $\mathrm{Loc} \colon \mathcal{O}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{D}_\lambda)$. Applying Loc to $L(p)$, we obtain irreducible objects in $\mathcal{O}_\nu(\mathcal{D}_\lambda)$. In this way we obtain the labeling $\mathfrak{M}^S \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_\nu(\mathcal{D}_\lambda))$. This is exactly the labeling mentioned in Section 1.1.2.

Let us finally mention that every object $V \in \mathcal{O}_\nu(\mathcal{A}_\lambda)$ is naturally graded via the action by the commutator with h (recall that h is the image of 1 under the comoment map induced by

ν). It follows from the definitions that $\mathcal{A}_{\lambda,0}$ acts on each graded component. Now, the *highest weight* component of $L(p)$ is one-dimensional and the action of $\mathcal{A}_{\lambda,0}$ on the corresponding one-dimensional space factors through the quotient $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ and is precisely the character of $\mathsf{C}_\nu(\mathcal{A}_\lambda)$, corresponding to $\mathbb{C}_p$. In other words, we can read off the character $\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathbb{C}_p$ from the irreducible representation $L(p)$ as the "highest weight" of $L(p)$ considered as $\mathcal{A}_{\lambda,0}$-module.

Remark 1.3.8 *Note that highest weights of $\Delta(p)$, $\nabla(p)$, $L(p)$ are the same.*

From algebraic to sheaf Cartan subquotients

Recall that the algebra $\mathcal{A}_{\hbar,\mathfrak{a}}$ is $\mathbb{Z}_{\geqslant 0}$-graded (via the $\mathbb{C}^\times_\hbar$-action). We denote the i-th graded piece by $\mathcal{A}^i_{\hbar,\mathfrak{a}}$. Pick $\lambda \in \mathfrak{a}$ and $\hbar_0 \in \mathbb{C}$, we can consider the specialization $\mathcal{A}_{(\hbar_0,\lambda)}$ of $\mathcal{A}_{\hbar,\mathfrak{a}}$ to the point $(\hbar_0,\lambda) \in \mathbb{C} \oplus \mathfrak{a}$. We see that for $\hbar_0 \neq 0$ the action of $\mathbb{C}^\times_\hbar$ (corresponding to the $\mathbb{Z}_{\geqslant 0}$-grading above) identifies $\mathcal{A}_{(\hbar_0,\lambda)}$ and $\mathcal{A}_{(1,\hbar_0^{-1}\lambda)} = \mathcal{A}_{\hbar_0^{-1}\lambda}$. Consider the composition

$$\mathcal{A}_{\hbar,\mathfrak{a},0} \twoheadrightarrow \mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{a}}) \to \Gamma(\mathfrak{M}^S_{\mathfrak{a}}, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{a}})) = \mathbb{C}[\mathfrak{M}^S] \otimes \mathbb{C}[\mathfrak{a},\hbar]. \tag{1.3}$$

Pick $x \in \mathcal{A}^i_{\hbar,\mathfrak{a},0}$.

Proposition 1.3.9 *(a) For $\hbar_0 \neq 0$ and $\lambda \in \mathfrak{a}$ such that the abelian localization holds for $\lambda\hbar_0^{-1}$ and $\varphi_{\hbar_0^{-1}\lambda}$ is an isomorphism, the image of $x_{(\hbar_0,\lambda)} \in \mathcal{A}_{(\hbar_0,\lambda),0}$ under*

$$\mathcal{A}_{(\hbar_0,\lambda),0} \twoheadrightarrow \mathsf{C}_\nu(\mathcal{A}_{(\hbar_0,\lambda)}) \to \Gamma(\mathfrak{M}^S, \mathsf{C}_\nu(\mathcal{D}_{(\hbar_0,\lambda)})) = \mathbb{C}[\mathfrak{M}^S]$$

is equal to the collection of scalars by which $\hbar_0^i x$ acts on the highest components of the irreducible modules in $\mathcal{O}_\nu(\mathcal{A}_{\hbar_0^{-1}\lambda})$.

(b) For $\hbar_0 = 0$ the composition (1.3) identifies with the pull back homomorphism

$$\mathbb{C}[\mathfrak{M}_{0,\mathfrak{a}}]_0 \twoheadrightarrow \mathbb{C}[\mathfrak{M}^S_{0,\mathfrak{a}}] \to \mathbb{C}[\mathfrak{M}^S_{\mathfrak{a}}] = \mathbb{C}[\mathfrak{M}^S] \otimes \mathbb{C}[\mathfrak{a}].$$

(c) If the set of $\lambda \in \mathfrak{a}$ such that the abelian localization holds at λ is Zariski dense in $\mathfrak{a}$, then (a) determines the image of x uniquely. In particular, if the image of $\mathfrak{a}$ in $\mathfrak{t}$ contains the Chern class $\eta \in \mathfrak{t}_\mathbb{Z}$ of an ample line bundle $\mathcal{L} \in \operatorname{Pic}(\mathfrak{M})$, then (a) determines the image of x uniquely.

Proof: Since the homomorphism (1.3) is $\mathbb{C}^\times_\hbar$-equivariant, it is enough to prove (a) for $\hbar_0 = 1$. For $\hbar_0 = 1$ the claim follows from the definitions. Part (b) is clear. Part (c) follows from [11, Corollary B.1]. □

1.3.5 Families of costandard and "point" modules

The homomorphism (1.3) allows us to define the following $\mathcal{A}_{\hbar,\mathfrak{a}}$-module (compare with Remark 1.3.7 and [10, Section 5.2]):

$$\nabla_{\hbar,\mathfrak{a}}(p) := (\mathbb{C}[\mathfrak{a},\hbar]^*_p \otimes_{\mathcal{A}^{\geqslant 0}_{\hbar,\mathfrak{a}}} \mathcal{A}_{\hbar,\mathfrak{a}})^\star,$$

where $\mathbb{C}[\mathfrak{a},\hbar]_p$ is the $\mathbb{C}[\mathfrak{M}^S]\otimes\mathbb{C}[\mathfrak{a},\hbar]$-module corresponding to the evaluation at p homomorphism $\mathbb{C}[\mathfrak{M}^S]\otimes\mathbb{C}[\mathfrak{a},\hbar]\twoheadrightarrow\mathbb{C}[\mathfrak{a},\hbar]$ and $\star$ denotes the restricted duality. Clearly, the specialization of $\nabla_{\hbar,\mathfrak{a}}(p)$ to $(1,\lambda)\in\mathbb{C}\times\mathfrak{a}$ is nothing else but the dual Verma module $\nabla_\lambda(p)$. Proposition 1.3.9 describes the *highest weight* of $\nabla_{\hbar,\mathfrak{a}}(p)$. Note that the module $\nabla_{\hbar,\mathfrak{a}}(p)$ is in general *not* flat over $\mathbb{C}[\mathfrak{a},\hbar]$.

Let us now define another $\mathcal{A}_{\hbar,\mathfrak{a}}$-module $\Theta_{\hbar,\mathfrak{a}}(p)$ that *is* flat over $\mathbb{C}[\mathfrak{a},\hbar]$, has the same highest weight as $\nabla_{\hbar,\mathfrak{a}}(p)$ and is isomorphic to $\nabla_{\hbar,\mathfrak{a}}(p)$ generically.

We follow [10, Section 5.3]. Consider the formal neighbourhood $U_{\mathfrak{a}}(p)$ of $\{p\}\times\mathfrak{a}\subset\mathfrak{M}_{\mathfrak{a}}$. We can restrict $\mathcal{D}_{\hbar,\mathfrak{a}}$ to $U_{\mathfrak{a}}(p)$ and take the locally finite global sections that we denote by $D_{\hbar,\mathfrak{a}}(p)$. It follows from the formal Darboux lemma (compare with [10, Lemma 5.2]) that the Cartan subquotient of $D_{\hbar,\mathfrak{a}}(p)$ is isomorphic to $\mathbb{C}[\mathfrak{a},\hbar]$. Let

$$\Theta_{\hbar,\mathfrak{a}}(p):=D_{\hbar,\mathfrak{a}}(p)\otimes_{D_{\hbar,\mathfrak{a}}(p)^{\geqslant 0}}\mathbb{C}[\mathfrak{a},\hbar].$$

$\Theta_{\hbar,\mathfrak{a}}(p)$ is an $\mathcal{A}_{\hbar,\mathfrak{a}}$-module via the restriction map $\mathcal{A}_{\hbar,\mathfrak{a}}\to D_{\hbar,\mathfrak{a}}(p)$. It follows from [10, Proposition 5.20] that the S-character of $\Theta_{\hbar,\mathfrak{a}}(p)$ (considered as a free $\mathbb{C}[\mathfrak{a},\hbar]$-module) is equal to

$$\operatorname{ch}\Theta_{\hbar,\mathfrak{a}}(p)=\sum_{\mu\colon S\to\mathbb{C}^\times}\dim_{\mathbb{C}[\mathfrak{a},\hbar]}(\Theta_{\hbar,\mathfrak{a}}(p)_\mu)e^\mu=e^{\omega_p}\prod_{i=1}^{d}\frac{1}{1-e^{-\alpha_i}},$$

where $\alpha_1,\ldots,\alpha_d$ are ν-positive weights (with multiplicity) for the action of S on $T_p\mathfrak{M}$ and ω_p is the restriction of (1.3) on $\mathfrak{s}$ composed with the projection onto $\mathbb{C}[\mathfrak{a},\hbar]_p$.

It follows from [10, Lemma 5.21] that $\Theta_{\hbar,\mathfrak{a}}(p)$, $\nabla_{\hbar,\mathfrak{a}}(p)$ are isomorphic for *generic* values of $\hbar,\lambda$. For fixed $\hbar_0\in\mathbb{C}$, $\lambda\in\mathfrak{a}$, module $\Theta_{\hbar_0,\lambda}(p)$ will be called a *point* module so $\Theta_{\hbar,\mathfrak{a}}(p)$ is a *family* of point modules.

Warning 1.3.10 *Note that the $\mathcal{A}_{\hbar,\mathfrak{a}}$-modules $\Theta_{\hbar,\mathfrak{a}}(p)$, $\nabla_{\hbar,\mathfrak{a}}(p)$ may* not *be finitely generated in general.*

1.4 Symplectic duality

1.4.1 Structures associated to a conical singularity

We mostly follow [57] in this section. Assume that the Poisson bracket on $\mathfrak{M}_0$ has degree two. Let $\mathbb{C}[\mathfrak{M}_0]^2\subset\mathbb{C}[\mathfrak{M}]$ be the degree two component, this is a Lie algebra w.r.t. $\{\,,\,\}$. Assume that there exists a reductive group $\operatorname{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ consisting of Poisson automorphisms of $\mathfrak{M}_0$ commuting with the contracting $\mathbb{C}^\times_\hbar$-action and whose Lie algebra is $(\mathbb{C}[\mathfrak{M}_0]^2,\{\,,\,\})$.

Recall that $S_{\mathfrak{M}_0}\subset\operatorname{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ is a maximal torus and set $\mathfrak{s}^*_{\mathfrak{M}_0,\mathbb{Z}}:=\operatorname{Hom}(S_{\mathfrak{M}_0},\mathbb{C}^\times)$. The action of $S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar$ on $\mathfrak{M}_0$ induces the bigrading $\mathbb{C}[\mathfrak{M}_0]=\bigoplus_{\mu\in\mathfrak{s}^*_{\mathfrak{M}_0,\mathbb{Z}},\,k\in\mathbb{Z}_{\geqslant 0}}\mathbb{C}[\mathfrak{M}_0]^k_\mu$. Note that $\mathbb{C}[\mathfrak{M}_0]^2_0$ identifies with $\mathfrak{s}_{\mathfrak{M}_0}$.

Recall that $\mathfrak{M}_{\mathfrak{t}}$ is the deformation of $\mathfrak{M}$ over $\mathfrak{t}=H^2(\mathfrak{M},\mathbb{C})$. Let $\mathcal{A}_{\mathfrak{t}}$ be the canonical quantization of $\mathfrak{M}_{\mathfrak{t}}$ and let $\mathcal{A}^2_{\mathfrak{t}}$ be the degree two component. There is an exact sequence of Lie algebras (see [57, Section 2.2]):

$$0\to\mathfrak{t}\oplus\mathbb{C}\hbar\to\mathcal{A}^2_{\mathfrak{t}}\to\mathbb{C}[\mathfrak{M}_0]^2\to 0.$$

Passing to the zero weight space w.r.t. S we obtain the *quantization exact sequence*:

$$0 \to \mathfrak{t} \oplus \mathbb{C}\hbar \to \mathcal{A}^2_{\mathfrak{t},0} \to \mathfrak{s} \to 0. \tag{1.4}$$

Let us describe another exact sequence that one can associate with a symplectic singularity $\mathfrak{M}$. The odd cohomology of $\mathfrak{M}$ vanishes ([11, Proposition 2.5]) so we obtain a *cohomological exact sequence*:

$$0 \to \mathfrak{s} \oplus \mathbb{C}\hbar \to H^2_{S\times\mathbb{C}^\times_\hbar}(\mathfrak{M}, \mathbb{C}) \to \mathfrak{t} \to 0, \tag{1.5}$$

where the first map is the pull back homomorphism (we identify $\mathfrak{s} \oplus \mathbb{C}\hbar = H^2_{S\times\mathbb{C}^\times_\hbar}(\mathrm{pt}, \mathbb{C})$) and the second map is the restriction homomorphism (recall that $\mathfrak{t} = H^2(\mathfrak{M}, \mathbb{C})$).

1.4.2 Symplectic duality: basic properties

Often conical symplectic singularities come in dual pairs $(\mathfrak{M}_0, \mathfrak{M}^!_0)$. We refer the reader to [10] for details on symplectic duality. Let us recall some basic properties of it. One prediction of symplectic duality (formulated in [57, Section 5.1]) is that for the dual pairs, quantization and cohomological exact sequences (1.4), (1.5) identify with each other: there exists a natural isomorphism $H^2_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}, \mathbb{C}) \simeq \mathcal{A}^2_{\mathfrak{t}_{\mathfrak{M}^!_0},0}(\mathfrak{M}^!_0)$ that induces the identifications:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{t}_{\mathfrak{M}^!_0} \oplus \mathbb{C}\hbar & \longrightarrow & \mathcal{A}^2_{\mathfrak{t}_{\mathfrak{M}^!_0},0}(\mathfrak{M}^!_0) & \longrightarrow & \mathfrak{s}_{\mathfrak{M}^!_0} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathfrak{s}_{\mathfrak{M}_0} \oplus \mathbb{C}\hbar & \longrightarrow & H^2_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}, \mathbb{C}) & \longrightarrow & \mathfrak{t}_{\mathfrak{M}_0} & \longrightarrow & 0. \end{array}$$

In particular, we should have canonical identifications $\mathfrak{t}_{\mathfrak{M}_0} \simeq \mathfrak{s}_{\mathfrak{M}^!_0}$, $\mathfrak{s}_{\mathfrak{M}_0} \simeq \mathfrak{t}_{\mathfrak{M}^!_0}$, as was conjectured in [10].

Assume now that both $\mathfrak{M}_0$ and $\mathfrak{M}^!_0$ admit symplectic resolutions $\mathfrak{M}$ and $\mathfrak{M}^!$, respectively.

Remark 1.4.1 *It is expected that $\mathfrak{M}_0$ has a symplectic resolution iff $(\mathfrak{M}^!)^{S_{\mathfrak{M}^!_0}}$ consists of* one *point.*

The symplectic duality predicts the existence of a canonical identification of finite sets of torus fixed points:

$$\mathfrak{M}^{S_{\mathfrak{M}_0}} \xrightarrow{\sim} (\mathfrak{M}^!)^{S_{\mathfrak{M}^!_0}}, \ p \mapsto p^!.$$

1.5 Hikita-Nakajima conjecture

Hikita-Nakajima conjecture

Recall that $\mathfrak{M}_0$ is a symplectic singularity and let $\mathcal{M} = \mathfrak{M}^!_0$ be a dual variety. The main example for us is $\mathfrak{M}_0$ being a Nakajima quiver variety and $\mathcal{M}$ the corresponding Coulomb branch. We assume that $\mathfrak{M}_0$ is resolved by $\mathfrak{M} \to \mathfrak{M}_0$. Let $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ be the algebra of $S_{\mathfrak{M}_0}$-equivariant cohomology of $\mathfrak{M}$. This is an algebra over $H^*_{S_{\mathfrak{M}_0}}(\mathrm{pt}) = \mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$. The choice of $\mathfrak{M}$ (resolution of $\mathfrak{M}_0$) corresponds to a (generic) cocharacter $\nu_{\mathfrak{M}}\colon \mathbb{C}^\times \to S_{\mathcal{M}}$. We can consider the algebra of functions of schematic fixed points $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ that is an algebra over $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}] = \mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$.

Note also that the algebra $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ has a natural cohomological grading and the algebra $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ is graded via the contracting $\mathbb{C}^\times_\hbar$-action. The algebra $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ is finitely generated over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$. Now, the algebra $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ is finitely generated over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$ iff $\mathcal{M}^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)} = \mathcal{M}^{S_{\mathcal{M}}}$ (considered as a set) consists of one point. Since we want to identify the algebras above we must make the following assumption.

Assumption 1.5.1 *The set $\mathcal{M}^{S_{\mathcal{M}}}$ consists of one point.*

Remark 1.5.2 Let us recall that symplectic duality predicts that $\mathfrak{M}_0$ has a symplectic resolution iff $\mathcal{M}^{S_{\mathcal{M}}}$ consists of one point.

The following conjecture relates (equivariant) cohomology of $\mathfrak{M}$ with schematic fixed points of (the deformation of) $\mathcal{M}$ (part (i) goes back to Hikita [46], and part (ii) goes back to Nakajima). We will call (i) *Hikita conjecture* and (ii) will be called *Hikita-Nakajima conjecture*.

Conjecture 1.5.3 *(i) There is an isomorphism of $\mathbb{Z}$-graded algebras*

$$H^*(\mathfrak{M}, \mathbb{C}) \simeq \mathbb{C}[\mathcal{M}^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}].$$

(ii) There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$:

$$H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C}) \simeq \mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}].$$

We remark that (ii) implies (i) by specifying to the point $0 \in \mathfrak{s}_{\mathfrak{M}_0}$.

Remark 1.5.4 Let us note that if $\mathfrak{M}$, $\mathfrak{M}'$ are two symplectic resolutions of $\mathfrak{M}_0$, then the algebras $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$, $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}', \mathbb{C})$ are isomorphic. This follows from the fact that the universal deformations $\mathfrak{M}_{\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$, $\mathfrak{M}'_{\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$ are locally trivial in C^∞-topology (see [91, Section 1.2 and references therein]), so $\mathfrak{M}$, $\mathfrak{M}'$ are both diffeomorphic to a generic fiber of $\mathfrak{M}_{0,\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$. Similarly, one can see that for any generic cocharacter $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$ the schematic fixed points $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}}$ are the same and are isomorphic to the schematic fixed points $\mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}$. Indeed, note that $\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}}$ can be $S_{\mathcal{M}}$-equivariantly embedded in some vector space O with a linear action of $S_{\mathcal{M}}$. Then $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}}$, $\mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}$ are (schematic) intersections

$$\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}} = \mathcal{M}_{\mathfrak{t}_{\mathcal{M}}} \cap O^{\nu(\mathbb{C}^\times)},\ \mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}} = \mathcal{M}_{\mathfrak{t}_{\mathcal{M}}} \cap O^{S_{\mathcal{M}}}.$$

This reduces the claim to showing that $O^{\nu(\mathbb{C}^\times)} = O^{S_{\mathcal{M}}}$ for a generic $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$ that directly follows from Proposition 1.3.3 since O is smooth.

Warning 1.5.5 *Hikita-Nakajima conjecture is not true as stated for arbitrary pairs of symplectically dual varieties. A counterexample is a part of work in progress with K. Hoang and D. Matvieievskyi. It is still expected to be true as stated when $\mathfrak{M}$ is a Nakajima quiver variety (this is the setting in which Nakajima formulated it in [88, Section 1(viii)]).*

Equivariant Hikita-Nakajima conjecture

Let $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$ be a generic cocharacter corresponding to $\mathfrak{M}$. The following conjecture is due to Nakajima. It generalizes Conjecture 1.5.3 and will be called *equivariant Hikita-Nakajima conjecture*.

Conjecture 1.5.6 *There is an isomorphism of $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}, \hbar]$-algebras*

$$\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})) \simeq H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}).$$

We remark that setting $\hbar = 0$ in Conjecture 1.5.6, we obtain the classical Hikita-Nakajima conjecture.

Combinatorial corollary of (equivariant) Hikita-Nakajima conjecture

Let us mention one nontrivial combinatorial corollary of Conjecture 1.5.3. Pick a point $(s,\hbar_0) \in \mathfrak{s}_{\mathfrak{M}_0} \oplus \mathbb{C}$. We have an isomorphism of algebras $H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M})|_{(s,\hbar_0)} \simeq H^*(\mathfrak{M}^{(s,\hbar_0)})$. Note that the semisimple quotient of $H^*(\mathfrak{M}^{(s,\hbar_0)})$ is the algebra $H^0(\mathfrak{M}^{(s,\hbar_0)})$ that is isomorphic to the direct sum of copies of $\mathbb{C}$, the number of copies is equal to $|\mathrm{Comp}(\mathfrak{M}^{(s,\hbar_0)})|$, where $\mathrm{Comp}(\mathfrak{M}^{(s,\hbar_0)})$ is the set of connected components of $\mathfrak{M}^{(s,\hbar_0)}$. Conjecture 1.5.3 implies that the algebra $H^*(\mathfrak{M}^{(s,\hbar_0)})$ is isomorphic to the algebra $\mathsf{C}_\nu(\mathcal{A}_{(s,\hbar_0)}(\mathcal{M}))$, where $\mathcal{A}_{(s,\hbar_0)}(\mathcal{M})$ is the fiber over $(s,\hbar_0) \in \mathfrak{s}_{\mathfrak{M}_0} \oplus \mathbb{C}$ of the deformation $\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})$. For $\hbar_0 \neq 0$ the semisimple quotient of $\mathsf{C}_\nu(\mathcal{A}_{(s,\hbar_0)}(\mathcal{M}))$ is isomorphic to the direct sum of copies of $\mathbb{C}$ labeled by the set of irreducible objects in the category $\mathcal{O}_\nu(\mathcal{A}_{(s,\hbar_0)}(\mathcal{M}))$ (we use the bijection between simple objects in the category $\mathcal{O}$ and irreducible modules over the corresponding Cartan subquotient). For $\hbar_0 = 0$, the semisimple quotient above is the direct sum of copies of $\mathbb{C}$ labeled by the set $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{(s,\hbar_0)}(\mathbb{C})$ of $\mathbb{C}$-points of the scheme $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{(s,\hbar_0)}$.

So, Hikita-Nakajima conjecture *implies* the (natural) bijection between the sets of completely different nature:

$$\mathrm{Comp}(\mathfrak{M}^{(s,\hbar_0)}) \simeq \mathrm{Irr}\,\mathcal{O}_\nu(\mathcal{A}_{(s,\hbar_0)}(\mathcal{M})).$$

Remark 1.5.7 *When $\mathfrak{M} = \widetilde{S}(e)$ is the Slodowy variety for a nilpotent e in some semisimple Lie algebra $\mathfrak{g}$, the identification above provides an explicit relation between Lusztig's left cells for the Weyl group W of $\mathfrak{g}$ and the connected components of torus fixed points of Springer fibers. Note that the similar relation between the connected components of torus fixed points of affine Springer fibers and simple objects in categories $\mathcal{O}$ over certain vertex operator algebras was recently conjectured in [101] in the framework of the mirror symmetry for certain 4D-theories.*

Current state of the Hikita-Nakajima conjecture

The Hikita conjecture was proven for the case of Hilbert scheme of points on $\mathbb{A}^2$, type A Slodowy slices and hypertoric varieties in [46, Theorem 1.1, Theorem A.1, Theorem B.1]. In [104], Shlykov has proven the case of $Y = \mathbb{A}^2/\Gamma$ where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. In the paper [44, Theorem 1.0.5], Hatano proved that $H^*(\mathfrak{M}(n,r),\mathbb{C})$ and $\mathbb{C}[(\mathfrak{M}_0(n,r)^!)^{\mathbb{C}^\times}]$ are isomorphic as graded vector spaces (here $\mathfrak{M}(n,r)$ is the ADHM space, see Section 2.2.1 for the definition). In [56, Theorem 1.5], Kamnitzer, Tingley, Webster, Weekes, and Yacobi have proven Hikita conjecture for the ADE slices in the affine Grassmanian. They have also proven the equivariant version for type A quivers and some weaker form of this conjecture for DE quivers (see [56, Section 8.3], [58, Section 1.3 and Theorem 1.5], see also [55, Section 6.6]). In [57] Kamnitzer, McBreen and Proudfoot formulated a quantum version of the Hikita-Nakajima conjecture and proved it for the Springer resolution and for hypertoric varieties. The results of [104] were recently generalized to the equivariant (and even quantum) case in [19].

Chapter 2

Hikita-Nakajima conjecture for symplectic resolutions

2.1 General approach

In this section, following the ideas of Bellamy, Hilburn, Kamnitzer, Tingley, Webster, Weekes, and Yacobi, we describe the general approach that we propose to attack the Hikita-Nakajima conjecture as well as its relation to a certain statement proposed by Bullimore, Dimofte, Gaiotto, Hilburn, and Kim. We also briefly discuss a quantum version of this conjecture proposed by Kamnitzer, Proudfoot, and McBreen and mention how it fits into our approach.

2.1.1 Main conjecture

In this section, we are working with an arbitrary pair of symplectically dual varieties $\mathfrak{M}_0$, $\mathcal{M} = \mathfrak{M}_0^!$ (not necessary Higgs/Coulomb branches of a quiver theory), but we are assuming that both of them admit symplectic resolutions $\mathfrak{M}$, $\widetilde{\mathcal{M}}$.

Choice of a symplectic resolution $\mathfrak{M}$ corresponds to the choice of a chamber in $\mathfrak{t}_{\mathfrak{M}_0,\mathbb{R}} = H^2(\mathfrak{M},\mathbb{R})$. Using the identification $\mathfrak{t}_{\mathfrak{M}_0} \simeq \mathfrak{s}_{\mathcal{M}}$, chamber above gives us a generic cocharacter $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$. Our goal is to compare the $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0},\hbar]$-algebras:

$$H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}),\ \mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})). \tag{2.1}$$

Let us first of all note that these algebras should actually carry an additional structure. Namely, recall (see Section 1.4.2) that the symplectic duality predicts identification of vector spaces $\mathcal{A}^2_{\hbar,\mathfrak{t}_{\mathcal{M}},0} \simeq H^2_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C})$. The elements of these vector spaces should commute with each other. So, we see that our algebras are not only $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0},\hbar]$-algebras but also a modules over the bigger polynomial algebra $S^\bullet(\mathcal{A}^2_{\hbar,\mathfrak{t}_{\mathcal{M}},0}) = S^\bullet(H^2_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}))$.

Actually, it turns out that in many cases, we have even bigger polynomial algebras acting on the algebras (2.1). For example, in the case when $\mathfrak{M}$ is a Nakajima quiver variety corresponding to some quiver I, $\mathcal{M}$ is the corresponding Coulomb branch, and A is a flavor torus, then $\mathcal{A}_{\hbar,\mathfrak{a}} = H^{(G_I[[t]]\rtimes\mathbb{C}^\times_\hbar)\times A}(\mathcal{R})$ contains the polynomial subalgebra $B := H^*_{G_I\times A\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$ (sometimes called the "Cartan" subalgebra of $\mathcal{A}_{\hbar,\mathfrak{a}}$). The same algebra B acts naturally on

$H^*_{A\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}) = H^*_{G_I\times A\times\mathbb{C}^\times_\hbar}(\mu^{-1}(0)^{\mathrm{st}},\mathbb{C})$ via the restriction homomorphism (which is in this case surjective by the results of McGerty and Nevins [82]):

$$B = H^*_{G_I\times A}(\mu^{-1}(0),\mathbb{C}) \twoheadrightarrow H^*_{G_I\times A\times\mathbb{C}^\times}(\mu^{-1}(0)^{\mathrm{st}},\mathbb{C}) = H^*_{A\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}).$$

Remark 2.1.1 *Note that in the Coulomb branch case, we see that the algebra B is actually a subalgebra of $\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}$. I do not know if this should be a general phenomenon.*

So, let B be some commutative (polynomial) algebra that acts naturally on both of our algebras. Acting on 1, we obtain homomorphisms:

$$B \xrightarrow{a} H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}),\ B \xrightarrow{b} \mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})).$$

Recall that we have the *natural* identification

$$H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}^{S_{\mathfrak{M}_0}},\mathbb{C}) \simeq \Gamma(\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\widetilde{\mathcal{M}})))). \tag{2.2}$$

induced by Lemma 1.3.5, the bijection $\mathfrak{M}^{S_{\mathfrak{M}_0}} \simeq \widetilde{\mathcal{M}}^{T_{\mathcal{M}}}$ and the isomorphism $\mathfrak{s}_{\mathfrak{M}_0} \simeq \mathfrak{t}_{\mathcal{M}}$ (see Section 1.4.2). Note that both of the algebras in (2.2) are simply $\bigoplus_{p\in\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}}\mathbb{C}[\mathfrak{t}_{\mathcal{M}},\hbar]$.

The approach that we suggest to prove the equivariant Hikita-Nakajima conjecture is the following. One can first prove the following conjecture and then *deduce* the equivariant Hikita-Nakajima conjecture from it.

Conjecture 2.1.2 *The following diagram is commutative:*

$$\begin{array}{ccccc}
 & & H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C}) & \longrightarrow & H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}^{S_{\mathfrak{M}_0}},\mathbb{C}) \\
 & \overset{a}{\nearrow} & & & \downarrow \simeq \\
B & & & & \bigoplus_{p\in\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}}\mathbb{C}[\mathfrak{t}_{\mathcal{M}},\hbar] \\
 & \underset{b}{\searrow} & & & \uparrow \simeq \\
 & & \mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})) & \longrightarrow & \Gamma(\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\widetilde{\mathcal{M}}))))
\end{array} \tag{2.3}$$

In particular, there exists the isomorphism of B-algebras:

$$\mathrm{Im}(B \to H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C})) \simeq \mathrm{Im}(B \to \mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{h}_{\mathcal{M}}}(\mathcal{M})) \to \Gamma(\mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}, \mathsf{C}_\nu(\mathcal{D}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\widetilde{\mathcal{M}})))). \tag{2.4}$$

Warning 2.1.3 *Let us mention an important technical detail: for (2.3) to be commutative, we should* normalize *the map a by saying that $\hbar$ goes to $\frac{1}{2}\hbar$!*

Note that proving Conjecture 2.3 requires much less work than dealing with the original Hikita-Nakajima conjecture. Indeed to prove Conjecture 2.3, one have to compare B-highest weights of Verma modules over $\mathcal{A}_\lambda(\mathcal{M})$ with restrictions of B to S-fixed points on $\mathfrak{M}$ for *generic* values of λ. So, this conjecture is basically about comparing certain *numbers*.

Let me mention one possible way to think about Conjecture 2.1.2. We can consider $H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C})$, $\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(X)$ as $\mathbb{C}[\mathfrak{t}_{\mathcal{M}},\hbar]$-modules of rank $|\mathfrak{M}^{S_{\mathfrak{M}_0}}| = |\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}|$. The algebra

B acts on both of these modules by $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}, \hbar]$-linear endomorphisms. Let $\mathbb{K}$ be the algebraic closure of the field of fractions $\mathrm{Frac}(\mathbb{C}[\mathfrak{t}_{\mathcal{M}}, \hbar])$. Base changing to $\mathbb{K}$, our modules become $|\widetilde{\mathcal{M}}^{S_{\mathcal{M}}}|$-dimensional vector spaces over $\mathbb{K}$, and they decompose into the direct sum of one-dimensional modules over $B_{\mathbb{K}} := B \otimes_{\mathbb{C}[\mathfrak{t}_{\mathcal{M}},\hbar]} \mathbb{K}$ parametrized by certain characters of B (in other words, we diagonalize the action of $B_{\mathbb{K}}$ on these modules). Conjecture 2.1.2 gives a concrete recipe to find these characters and claims that they should be the same. In other words, the conjecture claims that the *solutions* of B-modules above should be the same.

Remark 2.1.4 *In [12, Definition 2.1], authors introduced a notion of a* localization algebra. *We observe that algebras $H^*_{S_{\mathfrak{M}_0} \times \mathbb{C}^\times_\hbar}(\mathfrak{M}, \mathbb{C})$, $\mathsf{C}_\nu(\mathcal{A}_{\hbar, \mathfrak{t}_{\mathcal{M}}}(X))$ have* natural *structures of localization algebras and our Conjecture 2.1.2 basically claims that these structures should be* compatible. *In this sense, Conjecture 2.1.2 should be considered in the framework of the duality of "localization algebras", see [10, Section 10.6]. Let us mention that in [10, Section 10.6], [12] authors study another interesting example of the localization algebra, namely the (universal) deformation $Z(\tilde{E})$ of the center of the* geometric *category $\mathcal{O}_\nu(\mathcal{D}_\lambda)$ for $\mathfrak{M}$. In [10, Conjecture 10.32] authors conjecture that for λ such that $\mathcal{O}_\nu(\mathcal{D}_\lambda)$ is* indecomposable, *localization algebras $Z(\tilde{E})$, $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ should be isomorphic. I* do not know *how to generalize this statement to arbitrary λ.*

Let us now explain the motivation for Conjecture 2.1.2 related to physics (this is interesting on its own since it includes the equivariant Hikita-Nakajima conjecture into the general context of $3D$ mirror symmetry). We restrict ourselves to the case, when $\mathfrak{M}$ is an actual Nakajima quiver variety and $\mathfrak{M}_0^! = \mathcal{M}$ is the corresponding Coulomb branch. Recall that A is a flavor torus. Pick an A-fixed point $p \in \mathfrak{M}^A$ In [16], authors conjecture that (roughly speaking) the quantization $\mathcal{A}_{\hbar,\mathfrak{a}}(\mathcal{M})$ should act on the equivariant (critical) cohomology $H^*_{A \times \mathbb{C}^\times_\hbar}(\mathrm{QMaps}_p(\mathbb{P}^1, \mathfrak{M}))$ of the space of based quasi-maps sending $\infty \in \mathbb{P}^1$ to p and the corresponding module should be the universal *point* module $\Theta_{\hbar,\mathfrak{a}}(p^!)$ over the quantized Coulomb branch (see Section 1.3.5). In general, it is not clear how to define the action of $\mathcal{A}_{\hbar,\mathfrak{a}}(\mathcal{M})$ on the space above (see [107] for the case of $\mathfrak{sl}_2$). On the other hand, the action of $B = H^*_{G_I \times A \times \mathbb{C}^\times}(\mathrm{pt})$ is easy to describe. Namely, consider the evaluation at 0 morphism $\mathrm{ev}_0\colon \mathrm{QMaps}_p(\mathbb{P}^1, \mathfrak{M}) \to \mu^{-1}(0)/G_I$, then the element τ should act via the multiplication by $\mathrm{ev}^*(\tau)$ (compare with [69]).

Now, the natural grading on the point module should correspond to the decomposition of the space $\mathrm{QMaps}_p(\mathbb{P}^1, \mathfrak{M})$ via the degree of the quasi-map, let us denote the degree d component by $\mathrm{QMaps}_p^d(\mathbb{P}^1, \mathfrak{M})$. So, the highest weight component is nothing else but the cohomology of $\mathrm{QMaps}_p^0(\mathbb{P}^1, \mathfrak{M}) := \{p\}$. We conclude that the action of $\tau \in B$ on the highest weight component $H^*_{A \times \mathbb{C}^\times_\hbar}(p) = \mathbb{C}[\mathfrak{a}, \hbar]$ is given by the multiplication by $\iota_p^* \tau$, where $\iota_p\colon \{p\} \hookrightarrow \mathfrak{M}$ is the obvious embedding. This is precisely what Conjecture 2.1.2 claims.

Remark 2.1.5 *As we see from the discussion above, Conjecture 2.1.2 is a shadow of the (conjectural) realization of the universal point module over the Coulomb branch via the space of based quasi-maps to the Higgs branch. This statement should, in particular, imply that characters of (universal) point modules (considered as functions on B) coincide with the $\hbar = q$ specializations of (normalized) vertex functions* with descendants *(introduced by Okounkov in [94, Section 7.2]) restricted to the corresponding fixed points (compare with [47,* *Remark 1.10]). It turns out that this statement is precisely the quantum analog of Conjecture 2.1.2 and can be used to prove the quantum Hikita conjecture (formulated in [57]). Namely, normalized vertex functions with descendants are* solutions *of the PSZ quantum D-module while characters of point modules are solutions of the D-module of graded traces. By identifying solutions, we obtain identifications of the corresponding D-modules. This is the joint work in progress with Hunter Dinkins, and Ivan Karpov.*

The natural question is: when Conjecture 2.1.2 implies the actual equivariant Hikita-Nakajima conjecture? Clearly, this is the case when morphisms a, b are surjective and $\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M}))$ is torsion-free over $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}, \hbar]$ (note that $H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M}, \mathbb{C})$ is always free over $H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$). The following claims hold.

(i) Morphisms a, b *are indeed surjective* if $\mathfrak{M}$ is a Nakajima quiver variety and $\mathcal{M}$ is the corresponding Coulomb branch (surjectivity of a follows from [82, Corollary 1.5], for the surjectivity of b see Proposition 2.1.7 below and references therein).

(ii) Module $\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M}))$ is known to be flat over $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}, \hbar]$ when $\mathcal{M}$ is a Coulomb branch of type ADE quiver corresponding to a framing (w_i) and dimension vector (v_i) such that $w_i \neq 0$ implies that ω_i is minuscule, $\mu = \sum_i w_i\omega_i - \sum_i v_i\alpha_i$ is dominant (follows from [56, Section 8.2]). This also holds when $\mathcal{M}$ is a Coulomb branch of the Jordan quiver (see Section 2.2.12 below).

Conjecture 2.1.6 *We expect that* $\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M}))$ *is flat over* $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}, \hbar]$ *for an arbitrary* quiver theory*.*

Note that to prove Conjecture 2.1.6, it is enough to show that

$$\dim \mathbb{C}[\mathcal{M}^{\nu(\mathbb{C}^\times)}] \leqslant |\mathcal{M}^{S_{\mathcal{M}}}|.$$

The proof of the following proposition was explained to me by Ben Webster and Alex Weekes and will appear in their joint work with Joel Kamnitzer and Oded Yacobi.

Proposition 2.1.7 *Morphism b is surjective for Coulomb branch corresponding to arbitrary quiver I.*

Proof: The claim follows from [118, Proposition 3.1] (see also [14, Remark 6.7]) using that the dressed minuscule monopole operators have a nonzero degree with respect to $S_{\mathcal{M}}$ so their images in $\mathbb{C}[(\mathrm{Spec}\, H_*^{A\times(G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{S_{\mathcal{M}}}]$ are zero. □

So, for Higgs/Coulomb branches of quiver theories, Conjectures 2.1.2, 2.1.6 indeed imply the equivariant Hikita-Nakajima conjecture.

Remark 2.1.8 Let us point out that even without assuming Conjecture 2.1.6, Conjecture 2.1.2 already implies that we have a surjective homomorphism of algebras $\mathsf{C}_\nu(\mathcal{A}_{\hbar,\mathfrak{t}_{\mathcal{M}}}(\mathcal{M})) \twoheadrightarrow H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M})$ that is an isomorphism generically. Conjecture 2.1.6 implies that this surjection is an isomorphism.

Let us mention that our approach towards the proof of the Hikita-Nakajima conjecture allows us to reduce the argument to comparing the highest weights of Verma modules over the quantized Coulomb branches with characters of tautological bundles on quiver varieties at torus fixed points (that can be done generically). It also allows us to describe the isomorphism explicitly on generators.

From the above, we already know that Conjecture 2.1.2 implies the Hikita-Nakajima conjecture when $\mathcal{M}$ is a Coulomb branch corresponding to a *finite* ADE quiver with (v_i), (w_i) as in (ii). Proof of Conjecture 2.1.2 in this particular case can be deduced from the results of [56], [58]; this is a joint work in progress with Pavel Shlykov. In the next Section, I will explain how this approach works for a quiver variety corresponding to the simplest *affine* quiver, namely, to the Jordan quiver (the corresponding quiver variety is the so-called ADHM space).

Overall, Conjecture 2.1.2 is a replacement of the Hikita-Nakajima conjecture that might be incorrect in general. The counterexample to the Hikita-Nakajima conjecture for $\mathfrak{M}$ being certain Slodowy variety as well as the proof of Conjecture 2.1.2 for *parabolic Slodowy varieties*[1] with finitely many torus fixed points is the joint work in progress with Do Kien Hoang and Dmytro Matvieievskyi. Our result with Hoang and Matvieievskyi, in particular, reproves the equivariant Hikita-Nakajima conjecture for type A quiver varieties (proved before by Alex Weekes in his thesis) using their realization as parabolic Slodowy varieties (see [77]).

Remark 2.1.9 *Note that, actually, the setting in which Nakajima formulated his conjecture in [88, Section 1(viii)] is precisely the setting in which we do believe the actual Hikita-Nakajima conjecture should hold. So, neither Hikita nor Nakajima made any incorrect conjectures.*

Let us finish this section by mentioning an application of the Conjecture 2.1.2 (recall that one application was already mentioned in Section 1.5). In [69], Liu realizes certain families of simple modules over shifted Yangians geometrically using critical cohomology of certain spaces resembling the moduli of quasimaps to Nakajima quiver varieties (see also [110]). For finite ADE quivers, it then follows from Conjecture 2.1.2 that the action that Liu constructs factors through the action of the corresponding quantized Coulomb branch (as it was predicted by [16]).

2.2 Hikita-Nakajima conjecture for the ADHM space

The results of this Section is the joint work with Pavel Shlykov. In this Section, we prove the Hikita-Nakajima conjecture for the ADHM spaces $\mathfrak{M}(n,r)$ using the approach suggested in the previous section. Actually, using the same approach, we also identify all the algebras that appear in the isomorphism with the center of the degenerate cyclotomic Hecke algebra (generalizing some results of Shan, Varagnolo, and Vasserot). Our approach allows us to describe all these isomorphisms explicitly on generators.

[1] Note that these varieties are neither Higgs nor Coulomb branches of quiver gauge theories outside of type A

2.2.1 Gieseker variety ($ADHM$ space)

Gieseker variety $\mathfrak{M}(n,r)$ depends on a pair $n,r \in \mathbb{Z}_{\geqslant 1}$ of positive integer numbers. It can be realized as a Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.2.10). Variety $\mathfrak{M}(n,r)$ also has a realization as the moduli space of torsion-free sheaves on $\mathbb{P}^2$ of rank r with the second Chern class being n and with a fixed trivialization at the line at infinity (see [85, Chapter 2] for details). The variety $\mathfrak{M}(n,r)$ is an important object that originally came from physics. The Gieseker variety $\mathfrak{M}(n,r)$ is a resolution of singularities of the variety $\mathfrak{M}_0(n,r) = \operatorname{Spec}\mathbb{C}[\mathfrak{M}(n,r)]$. The variety $\mathfrak{M}_0(n,r)$ has a realization as an (affine) Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.2.10).

So we are taking $\mathfrak{M} = \mathfrak{M}(n,r)$, $\mathfrak{M}_0 = \mathfrak{M}_0(n,r)$. Then the torus $S_{\mathfrak{M}_0}$ can be described as follows. We have a natural symplectic action of SL_r on $\mathfrak{M}(n,r)$ (via changing the trivialization at infinity) and also the action of $\mathbb{T} = \mathbb{C}^\times$ via the action on $\mathbb{P}^2$ that multiplies one coordinate by t and another by t^{-1} (so-called "hyperbolic action"). Let $T_r \subset \mathrm{SL}_r$ be a maximal torus. The torus $S_{\mathfrak{M}_0}$ is the image of $A := \mathbb{T} \times T_r$ in $\operatorname{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}(n,r))$ and $\mathfrak{s}_{\mathfrak{M}_0}$ naturally identifies with $\mathfrak{a} := \operatorname{Lie} A$. The space $\mathfrak{t}_{\mathfrak{M}_0(n,r)} = H^2(\mathfrak{M}(n,r),\mathbb{C})$ is known to be one-dimensional.

2.2.2 Symplectic dual to $\mathfrak{M}(n,r)$

Let us now give a very brief description of the variety $\mathcal{M}(n,r) = \mathfrak{M}_0(n,r)^!$ and its deformation $\mathcal{M}(n,r)_{\mathfrak{a}} \to \mathfrak{a}$.

One way to construct a dual to $\mathfrak{M}_0(n,r)$ is via the Coulomb branches introduced in [14]. In this approach, $\mathcal{M}(n,r)$ is equal to the spectrum of the algebra $H_*^{(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ of $(\mathrm{GL}_n)_{\mathcal{O}}$-equivariant Borel-Moore homology of the variety of triples $\mathcal{R}_{n,r}$ corresponding to the Jordan quiver (see Section 2.2.7 for details). The variety $\mathcal{M}(n,r)_{\mathfrak{a}}$ is then the spectrum of $A\times(\mathrm{GL}_n)_{\mathcal{O}}$ - equivariant homology of the same variety of triples $\mathcal{R}_{n,r}$ (see Section 2.2.7 for details). The Coulomb branch $\mathcal{M}(n,r)_{\mathfrak{a}}$ above can be realized (see [65], [13], [117]) as the spectrum of the center $Z(H_{n,r})$ of the rational Cherednik algebra $H_{n,r}$ corresponding to the group $\Gamma_n := S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ (see Section 2.2.7 for details). Thus we have

$$\mathfrak{M}_0(n,r)^!_{\mathfrak{a}} = \mathcal{M}(n,r)_{\mathfrak{a}} = \operatorname{Spec} Z(H_{n,r}).$$

The algebra $H_{n,r}$ has a natural $\mathbb{Z}$-grading (see (2.21)) that induces the action of $\mathbb{T} = \mathbb{C}^\times$ on $\operatorname{Spec} Z(H_{n,r}) = \mathcal{M}(n,r)_{\mathfrak{a}}$.

Remark 2.2.1 The action $\mathbb{T} \curvearrowright \mathcal{M}(n,r)_{\mathfrak{a}}$ in the Coulomb terms is described in [14, Section 3 (v)].

Remark 2.2.2 One can also construct $\mathfrak{M}_0(n,r)^!$ as the (affine) Nakajima quiver variety $\mathcal{X}_0(n,r)$ corresponding to the cyclic quiver with r vertices labeled by $\mathbb{Z}/r\mathbb{Z}$ and having n-dimensional vector spaces placed at these vertices and one-dimensional framing at the vertex corresponding to zero. The deformation $\mathcal{X}_0(n,r)_{\mathfrak{a}}$ can be constructed similarly (c.f. Definition 1.2.5 below). The identification $\mathcal{X}_0(n,r)_{\mathfrak{a}} \simeq \operatorname{Spec} Z(H_{n,r})$ is given by the Etingof-Ginzburg isomorphism (that goes back to the paper [36]). More detailed, in [95, Section 3.3] it is explained (following the proof of [36, Theorem 11.16]) how to construct isomorphisms between smooth fibers of the families $\mathcal{X}_0(n,r)_{\mathfrak{a}}$, $\operatorname{Spec} Z(H_{n,r})$ over $\mathfrak{a}$. One can show (I am grateful to Pavel Etingof and Ivan Losev for the explanations) that these isomorphisms extend to the desired isomorphism $\mathcal{X}_0(n,r)_{\mathfrak{a}} \simeq \operatorname{Spec} Z(H_{n,r})$. The idea is to first extend these isomorphisms to the smooth locus over fiber over *every* $c \in \mathfrak{a}$ and then use the normality of our varieties to extend in codimension two.

2.2.3 Cyclotomic Hecke algebra and its center

Let $Q_{n,r}$ be the algebra of functions on schematic $\mathbb{T}$-fixed points of $\operatorname{Spec} Z(H_{n,r})$ (that can also be considered as the algebra of functions on $\mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}$). Recall that by Proposition 1.3.1 we have

$$Q_{n,r} = Z(H_{n,r})_0 / \sum_{i>0} Z(H_{n,r})_{-i} Z(H_{n,r})_i = Z(H_{n,r})/(b \in Z(H_{n,r})_i,\ i \neq 0), \quad (2.5)$$

where the grading on $H_{n,r}$ is as in (2.21):

$$\deg x_j = 1,\ \deg y_j = -1,\ \deg \Gamma_n = \deg \mathbf{h} = 0.$$

Our goal is to identify the algebra $Q_{n,r}$ with the algebra $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$. It turns out that there is another algebra that is isomorphic to both of the algebras above. This algebra is the center $Z(R^r(n))^{JM}$ of the cyclotomic degenerate Hecke algebra $R^r(n)$ (see Section 2.2.48 for the definition). It was observed in [114] that algebras $Z(R^r(n))^{JM}$, $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$ are isomorphic. We give an independent (but certainly similar) proof of this fact.

2.2.4 Main idea of the proof

It turns out that there exists one "universal" approach that allows us to identify algebras

$$H^*_A(\mathfrak{M}(n,r),\mathbb{C}),\ Z(R^r(n))^{JM},\ Q_{n,r} \simeq \mathbb{C}[\mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}] \quad (2.6)$$

with each other simultaneously. The idea is simple: we embed all of the algebras above inside the algebra

$$E := \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}(r,n)} \mathbb{C}[\mathfrak{a}] = \mathbb{C}[\mathfrak{a}]^{\oplus |\mathcal{P}(r,n)|}$$

and show that their images coincide. This is precisely the classical analog of the approach suggested in Section 2.1. Here $\mathcal{P}(r,n)$ is the set of r-multipartitions of n (see Definition 2.2.42). In order to show that images are the same we consider natural generators of these algebras and show that their images in E are the same. In particular, we obtain explicit descriptions for isomorphisms between algebras in (2.6).

Remark 2.2.3 Let F be the function field of the parameter space $\mathfrak{a} = \mathbb{A}^r$. We will see that the embeddings above become isomorphisms after tensoring by F or, more precisely, after localizing at a certain finite set of elements of $\mathbf{k} := \mathbb{C}[\mathfrak{a}]$ (certain "walls"). Compare with Remark 2.2.45 below.

Remark 2.2.4 In [12, Definition 2.1] the authors introduce a notion of *localization algebra.* We observe that algebras that appear in (2.6) have natural structures of (strong, free) localization algebras and all of them are isomorphic (as algebras with this additional structure).

Embedding $Z(R^r(n))^{JM} \subset E$ and generators of $Z(R^r(n))^{JM}$

For every $\boldsymbol{\lambda} \in \mathcal{P}(r,n)$ one can consider the corresponding "universal" Specht modules over $R^r(n)$ that we denote by $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ (see Section 2.2.9 for details). Acting by the center $Z(R^r(n))^{JM}$ of $R^r(n)$ on these modules we obtain the desired embedding

$$\psi\colon Z(R^r(n))^{JM} \subset \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)} \operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})) = E.$$

It follows from [15, Theorem 1] that natural generators of $Z(R^r(n))^{JM}$ are classes of elements

$$e_k(z_1,\dots,z_n),\ k=1,\dots,n,$$

where $e_k \in \mathbb{C}[z_1,\dots,z_n]^{S_n}$ are elementary symmetric polynomials.

Embedding $H^*_A(\mathfrak{M}(n,r),\mathbb{C}) \subset E$ and generators of $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$

The set of A-fixed points $\mathfrak{M}(n,r)$ can be parametrized by $\mathcal{P}(r,n)$ (see Section 2.2.8 for details).

In particular, we have the natural embedding

$$\iota\colon \mathfrak{M}(n,r)^A \subset \mathfrak{M}(n,r)$$

that induces the desired embedding

$$\iota^*\colon H^*_A(\mathfrak{M}(n,r),\mathbb{C}) \subset H^*_A(\mathfrak{M}(n,r)^A,\mathbb{C}) = E.$$

It remains to note that the algebra $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$ is generated by the elements

$$c_k(\mathcal{V}),\ k=1,\dots,n,$$

where $c_k(\bullet)$ is the kth A-equivariant Chern class and $\mathcal{V}$ is the tautological n-dimensional vector bundle on $\mathfrak{M}(n,r)$. This, for example, follows from [82, Corollary 1.5].

Embedding $Q_{n,r} \subset E$ and generators of $Q_{n,r}$

Recall that $Q_{n,r}$ is a quotient of $Z(H_{n,r}) \subset H_{n,r}$. To every $\boldsymbol{\lambda} \in \mathcal{P}(r,n)$ one can associate the corresponding induced (graded) $H_{n,r}$-module $\Delta(\boldsymbol{\lambda})$ on which $Z(H_{n,r})$ will act by some character and this action factors through the action of $Q_{n,r}$ (see Section 2.2.10). This gives us the desired embedding

$$\phi\colon Q_{n,r} \subset \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)} \operatorname{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda})) = E.$$

Generators of $Q_{n,r}$ are classes of

$$e_k(u_1,\dots,u_n),\ k=1,\dots,n,$$

where $u_i \in H_{n,r}$ are Dunkl-Opdam elements (see Lemma 2.2.61).

Remark 2.2.5 The identification $Z(H_{n,r}) \simeq H_*^{A\times(\mathrm{GL}_n)_\mathcal{O}}(\mathcal{R}_{n,r})$ sends $e_k(u_1,\dots,u_n)$ to the function $m_k := c_k(V) * 1$, here $c_k(V) \in H^*_{A\times\mathrm{GL}_n}(\mathrm{pt})$ is the Chern class of the tautological $A\times\mathrm{GL}_n$-bundle $V=\mathbb{C}^n$ on pt and $*$ is the convolution product on $H_*^{A\times(\mathrm{GL}_n)_\mathcal{O}}(\mathcal{R}_{n,r})$ (see [117, Section 4] for details). This should be compared with the fact that generators on the dual side (i.e., generators of $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$) are the Chern classes $c_i(\mathcal{V})$ of the tautological bundle $\mathcal{V}$.

Remark 2.2.6 T*he embedding ϕ can be interpreted geometrically as a pullback homomorphism from schematic fixed points on $\mathcal{M}(n,r)_\mathfrak{a}$ to schematic fixed points on a resolution of $\mathcal{M}(n,r)_\mathfrak{a}$ (see Section 2.2.6 for the $r=1$ example and Section 2.1 for general conjectures in this direction).*

2.2.5 Main results and structure of the chapter

Identification of the parameters and the main Theorem

We have the parameter spaces

$$\mathbf{h} = \mathbb{C}[\kappa, c_1,\dots,c_{r-1}],\ \mathbf{k} = \mathbb{C}[\kappa, a_1,\dots,a_r]/(a_1+\ldots+a_r) = \mathbb{C}[\mathfrak{a}], \tag{2.7}$$

used in definitions of algebras (2.6) that we want to identify. We identify the parameters as follows ($i=1,\dots,r$, $k=1,\dots,r-1$):

$$a_i = p(\eta^{i-1}), \tag{2.8}$$

where $p(q) = \frac{1}{r}\sum_{l=1}^{r-1}\frac{c_l}{\eta^{-l}-1}q^l \in \mathbb{C}[q]$, $\eta = e^{\frac{2\pi\sqrt{-1}}{r}}$. We will denote by F the field of fractions of the algebras (2.7).

Theorem 2.2.7 *After the identification of parameters (2.8) the $\mathbb{Z}$-graded algebras*

$$H^*_A(\mathfrak{M}(n,r),\mathbb{C}),\ Z(R^r(n))^{JM},\ Q_{n,r},\ \mathbb{C}[\mathrm{Spec}(H_*^{A\times(\mathrm{GL}_n)_\mathcal{O}}(\mathcal{R}_{n,r}))^\mathbb{T}]$$

are isomorphic. The isomorphisms above identify generators as follows

$$c_k(\mathcal{V}) = [e_k(z_1,\dots,z_n)] = [e_k(u_1,\dots,u_n)] = [m_k], \tag{2.9}$$

where $\mathcal{V}$ is the tautological rank n vector bundle on $\mathfrak{M}(n,r)$, $m_k = c_k(V)*1$ and $u_i \in H_{n,r}$ are Dunkl-Opdam elements (Definition (2.2.58)).

Remark 2.2.8 Note that the isomorphisms above are automatically graded since they preserve the degree of the generators (2.9). Note that the isomorphism

$$H^*_A(\mathfrak{M}(n,r),\mathbb{C}) \simeq \mathbb{C}[\mathrm{Spec}(H_*^{A\times(\mathrm{GL}_n)_\mathcal{O}}(\mathcal{R}_{n,r}))^\mathbb{T}]$$

is already an isomorphism of $\mathbb{C}[\kappa,a_1,\dots,a_n]/(a_1+\ldots+a_n) = \mathbb{C}[\mathfrak{a}]$-algebras and there is no need to identify parameters.

Structure of this Section

In Section 2.2.6, we prove the Hikita-Nakajima conjecture for $r = 1$, i.e., for the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ using results of [112]. In Section 2.2.7, we consider the case of arbitrary r and describe symplectically dual variety to $\mathfrak{M}_0(n, r)$, along with its deformation using Coulomb branches and rational Cherednik algebras.

In Sections 2.2.8, 2.2.9 and 2.2.10, we provide the realization of the idea behind the proof of Hikita-Nakajima conjecture that we briefly introduced in Section 2.2.4. First, in Section 2.2.8 we describe the embedding $H^*_A(\mathfrak{M}(n, r), \mathbb{C}) \subset E$ and determine the image of the generators $c_i(\mathcal{V}) \in H^*_A(\mathfrak{M}(n, r), \mathbb{C})$ under this embedding. Then, in Section 2.2.9, we define the cyclotomic degenerate Hecke algebra $R^r(n)$ and review its representation theory. Using this, we describe the embedding $Z(R^r(n))^{JM} \subset E$ and determine its image. Next, in Section 2.2.10, we recall the representation theory of the rational Cherednik algebra $H_{n,r}$ and then describe the embedding $Q_{n,r} \subset E$ (using the representation theory of $H_{n,r}$). In Lemma 2.2.61, we describe generators of $Q_{n,r}$ and determine their images under the embedding $Q_{n,r} \subset E$. Finally, as a corollary of the results of Sections 2.2.8, 2.2.9, 2.2.10, we obtain Theorem 2.2.7 (see Theorem 2.2.64 in Section 2.2.11). Section 2.2.12 contains a (short) proof of the fact that $Q_{n,r} \simeq \mathbb{C}[\mathcal{M}(n, r)^{\mathbb{T}}_{\mathfrak{a}}]$ is flat over the space of parameters.

2.2.6 Hikita-Nakajima conjecture for Hilbert scheme

In this section, we provide an "elementary" proof of the Hikita-Nakajima conjecture for the Hilbert scheme of points on $\mathbb{A}^2$. This proof approach differs from the original method used by Hikita (see [46]) As a corollary, when we set the equivariant parameter to zero, we obtain a theorem that was originally proved by Hikita.

The idea is the following (and goes back to our Conjecture 2.1.2): we identify the algebra of schematic fixed points with the Rees algebra of the center $Z(\mathbb{C}S_n)$ of $\mathbb{C}S_n$ and then use the identification

$$\mathrm{Rees}(Z(\mathbb{C}S_n)) \simeq H^*_{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{A}^2)) \quad \text{(see [112])} \tag{2.10}$$

to obtain the Hikita-Nakajima conjecture for $\mathrm{Hilb}_n(\mathbb{A}^2)$. An alternative approach to the identification (2.10) appears in Sections 2.2.8, 2.2.9.

Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ and $S^n(\mathbb{A}^2)$ as Nakajima quiver varieties

Our main object of study in this section is $\mathrm{Hilb}_n(\mathbb{A}^2)$ the Hilbert scheme of n points on $\mathbb{A}^2$.

Definition 2.2.9 *The variety* $\mathrm{Hilb}_n(\mathbb{A}^2)$ *is the variety whose* $\mathbb{C}$*-points are ideals* $J \subset \mathbb{C}[x, y]$ *of codimension* n*. The (affine) variety* $S^n(\mathbb{A}^2)$ *is the categorical quotient* $(\mathbb{A}^2)^n/S_n$*.*

Recall that we have the Hilbert-Chow morphism:

$$\mathrm{Hilb}_n(\mathbb{C}^2) \to S^n(\mathbb{A}^2),\ J \mapsto \mathrm{Supp}(\mathbb{C}[x, y]/J).$$

This morphism is a symplectic resolution of singularities. Let us now recall the description of $\mathrm{Hilb}_n(\mathbb{C}^2)$ as a Nakajima quiver variety (corresponding to the Jordan quiver).

Definition 2.2.10 *We denote by $\mathfrak{M}(n,r)$, $\mathfrak{M}_0(n,r)$ the Nakajima quiver varieties corresponding to the quiver I consisting of one vertex and one loop with $\dim V = n$, $\dim W = r$.*

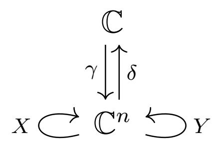
The following proposition holds by [85, Theorem 2.1 and Proposition 2.10].

Proposition 2.2.11 *There exist isomorphisms*

$$\mathfrak{M}(n,1) \xrightarrow{\sim} \mathrm{Hilb}_n(\mathbb{A}^2),\ \mathfrak{M}_0(n,1) \xrightarrow{\sim} S^n(\mathbb{A}^2) \tag{2.11}$$

compatible with natural morphisms $\mathfrak{M}(n,1) \to \mathfrak{M}_0(n,1)$, $\mathrm{Hilb}_n(\mathbb{A}^2) \to S^n(\mathbb{A}^2)$.

We conclude that the points of $\mathrm{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ can be represented as certain quadruples (X, Y, γ, δ) that can be considered as representations of the following quiver (here we identify $V = \mathbb{C}^n$, $W = \mathbb{C}$):



Calogero-Moser space, deformations of $\mathrm{Hilb}_n(\mathbb{A}^2)$ and torus actions

We see that varieties $\mathfrak{M}(n,1)_{\mathfrak{z}_n}$, $\mathfrak{M}_0(n,1)_{\mathfrak{z}_n}$ (see Definition 1.2.5) are one-parameter deformations of $\mathrm{Hilb}_n(\mathbb{A}^2)$ and $S^n(\mathbb{A}^2)$, where the base of the deformation is the center $\mathfrak{z}_n \subset \mathfrak{gl}(V)$ that can be identified with $\mathbb{A}^1$ via the map $\mathbb{A}^1 \ni \mathbf{a} \mapsto \mathbf{a} \cdot \mathrm{Id}_V \in \mathfrak{gl}(V)$.

Let us now discuss torus actions. Let $\mathbb{T}$, $\mathbb{C}^\times_\hbar$ be copies of $\mathbb{C}^\times$. We have an action of $\mathbb{T} \times \mathbb{C}^\times_\hbar$ on $\mathrm{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ that is induced by the action $\mathbb{T} \times \mathbb{C}^\times_\hbar \curvearrowright \mathbb{A}^2$ given by

$$(t, \hbar) \cdot (x, y) = (t\hbar^{-1}x, t^{-1}\hbar^{-1}y),\ t \in \mathbb{T},\ \hbar \in \mathbb{C}^\times_\hbar.$$

After identifications (2.11) the action of $\mathbb{T} \times \mathbb{C}^\times_\hbar$ can be described as follows: it is induced from the following action on $\mathfrak{M}(n,1)$:

$$(t, \hbar) \cdot (X, Y, \gamma, \delta) = (t\hbar^{-1}X, t^{-1}\hbar^{-1}Y, \hbar^{-1}\gamma, \hbar^{-1}\delta). \tag{2.12}$$

Remark 2.2.12 Note that $\mathbb{T}$ acts symplectically, while $\mathbb{C}^\times_\hbar$ scales the symplectic form with weight 2.

Formula (2.12) induces actions

$$\mathbb{T} \times \mathbb{C}^\times_\hbar \curvearrowright \mathfrak{M}(n,1)_{\mathfrak{z}_n},$$
$$\mathbb{T} \times \mathbb{C}^\times_\hbar \curvearrowright \mathfrak{M}_0(n,1)_{\mathfrak{z}_n}.$$

Consider $\mathbf{a} \in \mathbb{C}^\times$. Let us describe the fibers $\mathfrak{M}(n,1)_{\mathbf{a}}$, $\mathfrak{M}(n,1)_{\mathbf{a}}$ of the families $\mathfrak{M}(n,1)_{\mathfrak{z}_n}$, $\mathfrak{M}(n,1)_{\mathfrak{z}_n}$ over $\mathbf{a}$. First of all, note that the action of $\mathbb{C}_\hbar$ induces identifications

$$\mathfrak{M}(n,1)_{\mathbf{a}} \simeq \mathfrak{M}(n,1)_1,$$
$$\mathfrak{M}_0(n,1)_{\mathbf{a}} \simeq \mathfrak{M}_0(n,1)_1.$$

Definition 2.2.13 *Recall that V is a vector space of dimension n. We define Calogero-Moser variety $\mathcal{C}(n)$ as the following quotient:*

$$\mathcal{C}(n) := \{(X,Y) \in \operatorname{End}(V)^{\oplus 2} \mid \operatorname{rk}\Big([X,Y] - \operatorname{Id}_V\Big) = 1\}/\operatorname{GL}(V).$$

The following proposition will be useful for us (see, for example, [122, Section 1]):

Proposition 2.2.14 *Natural morphisms*

$$\mathfrak{M}(n,1)_1 \to \mathfrak{M}_0(n,1)_1 \to \mathcal{C}(n),$$

given by

$$[(X,Y,\gamma,\delta)] \mapsto [X,Y,\gamma,\delta] \mapsto [(X,Y)]$$

are isomorphisms.

Thus families $\mathfrak{M}(n,1)_{\mathfrak{z}_n}$, $\mathfrak{M}_0(n,1)_{\mathfrak{z}_n}$ are $\mathbb{C}^\times_\hbar$-equivariant deformations of $\operatorname{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ over $\mathbb{A}^1$. Over a non-zero parameter their fibers are isomorphic to the Calogero-Moser variety $\mathcal{C}(n)$.

Hikita-Nakajima conjecture for $\operatorname{Hilb}_n(\mathbb{A}^2)$

We denote by $H^*_{\mathbb{T}}(\operatorname{Hilb}_n(\mathbb{A}^2))$ the $\mathbb{T}$-equivariant cohomology of $\operatorname{Hilb}_n(\mathbb{A}^2)$. This is a $\mathbb{Z}$-graded algebra over $H^*_{\mathbb{T}}(\mathrm{pt})$ isomorphic to $\mathbb{C}[\operatorname{Lie}\mathbb{T}]$. We are now ready to state the Hikita-Nakajima conjecture for $\operatorname{Hilb}_n(\mathbb{A}^2)$.

Theorem 2.2.15 *We have an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathfrak{z}_n] \simeq \mathbb{C}[\operatorname{Lie}\mathbb{T}]$ (the identification induced by the isomorphism $\mathfrak{z}_n \simeq \mathbb{A}^1 \simeq \operatorname{Lie}\mathbb{T}$):*

$$\mathbb{C}\Big[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\Big] \simeq H^*_{\mathbb{T}}(\operatorname{Hilb}_n(\mathbb{A}^2)).$$

Our goal is to prove this theorem. We will do it by showing that both algebras in the theorem are isomorphic to the Rees algebra of the center $Z(\mathbb{C}S_n)$ of the group algebra of S_n.

Equivariant cohomology of $\operatorname{Hilb}_n(\mathbb{A}^2)$ and the center of $\mathbb{C}S_n$

Let $Z(\mathbb{C}S_n)$ in $\mathbb{C}S_n$ be the center of the group algebra $\mathbb{C}S_n$. Consider the grading on the vector space $\mathbb{C}S_n$ defined in the following way: pick a permutation $\sigma \in S_n$ and let $\ell(\sigma)$ be the number of cycles in the decomposition of σ as a product of disjoint cycles. We then define

$$\deg\sigma := 2(n - \ell(\sigma)).$$

The grading above induces the increasing $\mathbb{Z}_{\geqslant 0}$-filtration on Z_n:

$$\mathbb{C} = F_0Z_n = F_1Z_n \subset F_2Z_n = F_3Z_n \subset \ldots \subset Z_n = F_{2n-2}Z_n = F_{2n-1}Z_n = \ldots \tag{2.13}$$

We denote by $\operatorname{Rees}(Z(\mathbb{C}S_n))$ the Rees algebra corresponding to the filtration (2.13). Recall that the algebra $\operatorname{Rees}(Z(\mathbb{C}S_n))$ is defined as follows:

$$\operatorname{Rees}(Z(\mathbb{C}S_n)) := \bigoplus_{m \geqslant 0} \kappa^m F_{2m}Z(\mathbb{C}S_n) \subset Z(\mathbb{C}S_n)[\kappa],$$

where κ is a formal parameter of degree 2. We consider $\mathrm{Rees}(Z(\mathbb{C}S_n))$ as an algebra over $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[\kappa]$. The following result holds by [112] or Sections 2.2.8, 2.2.9 (see also [114, Theorem 4.7 and Corollary 4.8]).

Proposition 2.2.16 *There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathrm{Lie}\,\mathbb{T}] \simeq \mathbb{C}[\mathbb{A}^1]$ (the identification is induced by the isomorphism $\mathrm{Lie}\,\mathbb{T} \simeq \mathbb{A}^1$):*

$$H^*_{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{A}^2)) \simeq \mathrm{Rees}(Z(\mathbb{C}S_n)).$$

Remark 2.2.17 Proposition 2.2.16 can also be proved using the same argument as we use in the proof of Theorem 2.2.20 below.

Let us now recall the description of the center $Z(\mathbb{C}S_n)$. To every $k \in 1, \ldots, n$ we can associate the corresponding Jucys–Murphy element JM_k defined as follows:

$$JM_k := (1\,k) + (2\,k) + \ldots + (k-1\,k) \in \mathbb{C}S_n,$$

where $(i\,k) \in S_n$ is the transposition switching i, k.

Remark 2.2.18 Note that $JM_1 = 0$.

The following proposition is classical (see, for example, [84, Theorem 1.9]).

Proposition 2.2.19 *The center $Z(\mathbb{C}S_n)$ is generated (as a vector space) by the elements $f(JM_1, \ldots, JM_n)$, where f runs through the symmetric functions on n variables.*

Construction of the isomorphism between schematic fixed points of $\mathfrak{M}_0(n,1)_{\mathfrak{z}_n}$ and $\mathrm{Rees}(Z(\mathbb{C}S_n))$

We will prove the following theorem and, using Proposition 2.2.16, obtain Theorem 2.2.15 as a corollary.

Theorem 2.2.20 *There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$*

$$\mathrm{Rees}(Z(\mathbb{C}S_n)) \xrightarrow{\sim} \mathbb{C}\left[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\right] \tag{2.14}$$

that sends $f(JM_1, \ldots, JM_n) \in Z(\mathbb{C}Z_n)$ to the restriction of the function

$$\mathfrak{M}_0(n,1)_{\mathfrak{z}_n} \ni [(X, Y, \gamma, \delta)] \mapsto f(\alpha_1, \ldots, \alpha_n)$$

to $\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}$. Here f is a symmetric function on n variables and $\alpha_1, \ldots, \alpha_n$ are roots of the characteristic polynomial of $YX \in \mathrm{End}(V)$, the $\mathbb{Z}$-grading on the LHS of (2.14) is the natural grading on $\mathrm{Rees}(Z(\mathbb{C}S_n))$ and the $\mathbb{Z}$-grading on the RHS is the one induced by the action of $\mathbb{C}^\times_\hbar$.

The rest of the section is devoted to describing the idea of the proof of Theorem 2.2.20. We start with the following proposition, the proof of which is given in Section 2.2.12.

Proposition 2.2.21 *The algebra* $\mathbb{C}\Big[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\Big]$ *is flat (hence, free) over* $\mathfrak{z}_n = \mathbb{A}^1$. *In particular, we have an isomorphism of* $\mathbb{Z}$*-graded algebras*

$$\mathbb{C}\Big[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\Big] \simeq \operatorname{Rees}(\mathbb{C}[\mathfrak{M}_0(n,1)^{\mathbb{T}}_1]) = \operatorname{Rees}(\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]).$$

We conclude that to prove Theorem 2.2.20 it is enough to construct the isomorphism of filtered algebras $\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}] \simeq Z(\mathbb{C}S_n)$. In order to do so we need to describe the algebra $\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]$. Let us note that the variety $\mathcal{C}(n)$ is smooth, hence, the scheme $\mathcal{C}(n)^{\mathbb{T}}$ is also smooth (see Proposition 1.3.3) and, in particular, reduced.

The description of the set of fixed points $\mathcal{C}(n)^{\mathbb{T}}$ was given by Wilson in [122, Proposition 6.11], we recall it in Section 2.2.6. The set $\mathcal{C}(n)^{\mathbb{T}}$ is finite and can be parametrized by the set $\mathcal{P}(n)$ of partitions of n, we denote by $[(X^\lambda, Y^\lambda)]$ the fixed point corresponding to $\lambda \in \mathcal{P}(n)$ (see Definition 2.2.23). Since every finite reduced scheme over $\mathbb{C}$ is just the spectrum of the direct sum of copies of $\mathbb{C}$, we must have

$$\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}] = \bigoplus_{\lambda \in \mathcal{P}(n)} \mathbb{C}\chi_\lambda, \tag{2.15}$$

where $\chi_\lambda \in \mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]$ is the characteristic function of the $\mathbb{T}$-fixed point $[(X^\lambda, Y^\lambda)]$ corresponding to $\lambda \in \mathcal{P}(n)$. Recall now that we have the natural identification

$$Z(\mathbb{C}S_n) = \bigoplus_{\lambda \in \mathcal{P}(n)} \mathbb{C}\mathbf{e}_\lambda, \tag{2.16}$$

where $\mathbf{e}_\lambda \in Z(\mathbb{C}S_n)$ is the idempotent corresponding to the Specht module $S(\lambda)$ (in other words, for $\nu \in \mathcal{P}(n)$ the element $\mathbf{e}_\lambda \in Z(\mathbb{C}S_n)$ acts on $S(\nu)$ via $\delta_{\lambda\nu} \cdot \operatorname{Id}_{S(\nu)}$).

Composing (2.15) and (2.16), we obtain the isomorphism of algebras:

$$\Theta\colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}],\ \mathbf{e}_\lambda \mapsto \chi_\lambda.$$

To prove Theorem 2.2.20 it remains to show that the isomorphism Θ is the one that we need, i.e., it sends element $f(JM_1,\ldots,JM_n)$ to the restriction of the function $\mathcal{C}(n) \ni [(X,Y)] \mapsto f(\alpha_1,\ldots,\alpha_n)$ to $\mathcal{C}(n)^{\mathbb{T}}$. From this we would be able to conclude that the isomorphism Θ is filtration-preserving.

To prove that Θ sends element $f(JM_1,\ldots,JM_n)$ to the function $\mathcal{C}(n)^{\mathbb{T}} \ni [(X,Y)] \mapsto f(\alpha_1,\ldots,\alpha_n)$ we just need to show that for every symmetric function f on n variables we have

$$f(JM_1,\ldots,JM_n)|_{S(\lambda)} = f(\alpha_1,\ldots,\alpha_n)\operatorname{Id}_{S(\lambda)},$$

where $\alpha_1,\ldots,\alpha_n$ is the multiset of eigenvalues of $Y^\lambda X^\lambda$. Recall that $f(JM_1,\ldots,JM_n)$ acts on $S(\lambda)$ via the multiplication by $f(c_1,\ldots,c_n)$, where $c_1,\ldots,c_n$ is the multiset of contents of boxes of the Young diagram $\mathbb{Y}(\lambda)$ corresponding to λ (see (2.17)). It remains to check that the multiset of eigenvalues of $Y^\lambda X^\lambda$ is the same as the multiset of contents of boxes of $\mathbb{Y}(\lambda)$.

Description of $\mathcal{C}(n)^{\mathbb{T}}$ and eigenvalues of $Y^\lambda X^\lambda$

The parametrization of $\mathcal{C}(n)^{\mathbb{T}}$ by the elements of $\mathcal{P}(n)$ goes as follows (the description was obtained in [122], we follow [95, Section 5]). Pick $m \in \mathbb{Z}_{\geqslant 1}$ and $1 \leqslant k \leqslant m$.

Definition 2.2.22 *By D_m we will denote the $m \times m$ matrix with 1's on the first diagonal and 0's elsewhere, i.e., $D_m = \sum_{i=1}^{m-1} E_{i\,i+1}$. Now, let $Y(m,k)$ be the $m \times m$ matrix such that its only non-zero entries are on the -1st diagonal, and it satisfies the relation $[Y(m,k), D_m] + \mathrm{Id} = mE_{k\,k}$. In other words, the numbers below the diagonal are $1, 2, \ldots, k-1, -m+k, \ldots, -2, -1$:*

$$Y(m,k) = \begin{pmatrix} 0 & 0 & \cdots\cdots\cdots\cdots & 0 \\ 1 & 0 & \cdots\cdots\cdots\cdots & 0 \\ 0 & 2 & 0 \quad \cdots\cdots\cdots & 0 \\ \vdots & \ldots & \ddots \quad 0 \quad \cdots\cdots & 0 \\ \cdots\cdots & k-1 & 0 \quad \cdots & \\ \cdots\cdots\cdots & & -m+k \quad 0 \quad \cdots & \\ \cdots\cdots\cdots\cdots & & & \\ \cdots\cdots\cdots\cdots & & -1 & 0 \end{pmatrix}.$$

Pick $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n)$. Following [95, Section 4.1] we denote by

$$\mathbb{Y}(\lambda) := \{(i,j) \,|\, 1 \leqslant i \leqslant l,\, 1 \leqslant j \leqslant \lambda_i\} \tag{2.17}$$

the corresponding Young diagram. If $\square = (i,j) \in \mathbb{Y}(\lambda)$ is a cell let $c(\square) := j - i$ be the *content* of $\square$. By a *hook* associated to the cell (i,j) we call the set

$$\mathbb{H}_{(i,j)} := \{(i,j)\} \cup \{(i',j) \in \mathbb{Y}(\lambda) \,|\, i' > i\} \cup \{(i,j') \in \mathbb{Y}(\lambda) \,|\, j' > j\}.$$

Box (i,j) is called the *root* of the hook $\mathbb{H}_{(i,j)}$. By a *Frobenius hook* of $\mathbb{Y}(\lambda)$ we mean a hook of the form $\mathbb{H}_{(i,i)}$. The diagram $\mathbb{Y}(\lambda)$ is the disjoint union of its Frobenius hooks. Suppose that $(1,1), (2,2), \ldots, (s,s)$ are cells of $\mathbb{Y}(\lambda)$ with zero content. Let $\mathbb{H}_i$ be the Frobenius hook with root (i,i). Let k_i be the height of $\mathbb{H}_i$ and n_i be the size of $\mathbb{H}_i$. Now we are ready to describe the tuple $[(X^\lambda, Y^\lambda)] \in \mathbb{C}(n)^{\mathbb{T}}$ corresponding to λ.

Definition 2.2.23 *Tuple (X^λ, Y^λ) is defined as follows. We have X^λ to be a block diagonal matrix whose $n_i \times n_i$ diagonal blocks are given by D_{n_i}. The $n_i \times n_i$ diagonal blocks of Y^λ are given by the matrices $Y(n_i, k_i)$, and the off-diagonal blocks satisfy the following property: For $i \neq j$, $(Y^\lambda)_{ij}$ is the unique $n_i \times n_j$ matrix with non-zero entries on the diagonal $k_i - k_j - 1$, satisfying the following property:*

$$(Y^\lambda)_{ij} D_{n_j} - D_{n_i} (Y^\lambda)_{ij} = n_i E_{k_i\,k_j}.$$

Remark 2.2.24 If $\lambda \in \mathcal{P}(n)$ is a hook of height k, then we have $Y^\lambda = Y(n,k)$. Note that the diagonal matrix elements of $Y^\lambda X^\lambda = Y(n,k) D_n$ are precisely the contents of cells of the hook λ.

Proposition 2.2.25 *The eigenvalues of $Y^\lambda X^\lambda = Y^\lambda D_n$ are the same as the eigenvalues of blocks $(Y^\lambda)_{ii} D_{n_i} = Y(n_i, k_i) D_{n_i}$ as if off-diagonal blocks in $Y^\lambda X^\lambda$ were not present. So, the eigenvalues of $Y^\lambda X^\lambda$ are diagonal elements of $Y(n_i, k_i) D_{n_i}$ that are exactly the multiset of contents of boxes of λ.*

Proof: Follows from the proof of [122, Proposition 6.13]. □

Corollary 2.2.26 *The isomorphism $\Theta\colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)]^{\mathbb{T}}$ sends $f(JM_1, \ldots, JM_n)$ to the restriction of the function $[(X, Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ to $\mathbb{C}(n)^{\mathbb{T}}$, where f is a symmetric function on n variables.*

Proof of Theorem 2.2.20

We have already constructed (see Section 2.2.6) the isomorphism

$$\Theta\colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathbb{C}(n)^{\mathbb{T}}]$$

and have shown that this isomorphism sends generators $f(JM_1, \ldots, JM_n) \in Z(\mathbb{C}S_n)$ to functions $\mathcal{C}(n)^{\mathbb{T}} \ni (X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n)$ (see Corollary 2.2.26). To finish the proof of Theorem 2.2.20 it remains to show that the isomorphism Θ is filtered. Let us note that if f has degree k then $\deg f(JM_1, \ldots, JM_n) = 2k$ and the degree of the function $(X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n)$ is also equal to $2k$. Thus, in order to show that Θ is filtration-preserving it is enough to check that $F_{2k} Z(\mathbb{C}S_n)$ is generated (as a vector space) by

$$\Big\{ f(JM_1, \ldots, JM_n) \,|\, f \text{ is a homogeneous symmetric polynomial of degree } \leqslant k \Big\}.$$

This is a direct corollary of the following (classical) proposition.

Proposition 2.2.27 *The algebra $\operatorname{gr} Z(\mathbb{C}S_n)$ is generated by the elements*

$$\Big\{ \operatorname{gr} \Big(f(JM_1, \ldots, JM_n) \Big) \,|\, f \textit{ is a homogeneous symmetric polynomial} \Big\}.$$

Proof: The claim follows from the proof of [84, Theorem 1.9]. □

We finish this section with the following conjecture.

Conjecture 2.2.28 *The isomorphism*

$$\operatorname{gr} \Theta\colon \operatorname{gr} Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}]$$

coincides with the isomorphism constructed in [46, Section 2].

2.2.7 Description of the symplectic dual to the Gieseker variety

Recall that the Gieseker variety is the Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.2.10). It depends on the pair $n, r \in \mathbb{Z}_{\geqslant 1}$ and is denoted by $\mathfrak{M}(n, r)$. The corresponding affine Poisson variety is denoted by $\mathfrak{M}_0(n, r)$. In this section, we give two different ways to describe the symplectically dual variety $\mathfrak{M}_0(n, r)^!$ and its deformation $\mathfrak{M}_0(n, r)^!_{\mathfrak{a}}$.

In the paper [14], the candidate for symplectically dual variety to every Nakajima quiver variety was constructed.

Construction of Coulomb branch

Let us recall the construction in our case (when we start from the Jordan quiver with the dimension vector $n \in \mathbb{Z}_{\geqslant 1}$ and framing $r \in \mathbb{Z}_{\geqslant 1}$).

Recall the vector space $\mathbf{N} = \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,W)$ and the group $G_n = \mathrm{GL}(V)$ acting on $\mathbf{N}$ (see Section 1.2.2).

Definition 2.2.29 *We define $\mathrm{Gr}_{\mathrm{GL}(V)}$ as the moduli space of the data $(\mathcal{P}, \varphi)$, where:*

(a) $\mathcal{P}$ is a $\mathrm{GL}(V)$-bundle on $\mathbb{P}^1$;

(b) $\varphi\colon \mathcal{P}^{\mathrm{triv}}|_{\mathbb{P}^1\setminus\{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1\setminus\{0\}}$ is a trivialization of $\mathcal{P}$ restricted to $\mathbb{P}^1 \setminus \{0\}$.

We then consider the moduli space of triples $\mathcal{R}_{n,r}$ (corresponding to the Jordan quiver, dimension vector n and framing r) defined as follows.

Definition 2.2.30 *Let $\mathcal{R}_{n,r}$ be the moduli space of triples $\{(\mathcal{P}, \varphi, s)\}$, where $(\mathcal{P}, \varphi)$ is a point of $\mathrm{Gr}_{\mathrm{GL}(V)}$ and s is a section of the associated vector bundle $\mathcal{P}_{\mathbf{N}} = \mathcal{P} \times_{\mathrm{GL}(V)} \mathbf{N}$ such that it is sent to a regular section of a trivial bundle under φ.*

Set $\mathrm{GL}(V)_{\mathcal{O}} := \mathrm{GL}(V)[[z]]$. We can consider the equivariant Borel-Moore homology $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ (see [14, Section 2(ii)] for the definition and detailed discussion). This vector space is equipped with an algebra structure via convolution $*$ (see [14, Section 3]). It follows from [14, Proposition 5.15] that the algebra $(H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r}), *)$ is commutative.

Definition 2.2.31 *The Coulomb branch $\mathcal{M}(n,r)$ is defined as the spectrum of the algebra $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$:*

$$\mathcal{M}(n,r) := \mathrm{Spec}(H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})).$$

The variety $\mathcal{M}(n,r)$ is conjectured to be symplectically dual to $\mathfrak{M}(n,r)$. The deformation $\mathcal{M}(n,r)_{\mathfrak{a}} \to \mathfrak{a}$ of $\mathcal{M}(n,r)$ can be constructed as follows (compare with [14, Section 3(viii)]). Let $\mathbb{T}$ be the copy of $\mathbb{C}^\times$. Let $T_r \subset \mathrm{SL}(W)$ be a maximal torus. We have the natural action of $A = \mathbb{T} \times T_r$ on $\mathbf{N}$ given by

$$(t,g) \cdot (X, \gamma) = (tX, \gamma \circ g^{-1}),\ (t,g) \in \mathbb{T} \times T_r.$$

Warning 2.2.32 *Note that the action of $\mathbb{T}$ on $\mathbf{N}$ is not the scaling action (as in [14]). This action, for example, appears in [13, Section 5.5].*

We can identify $\mathrm{Lie}\,\mathbb{T} \simeq \mathbb{A}^1$, so $\mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[\kappa]$ for some variable κ. We can also identify $\mathfrak{t}_r := \mathrm{Lie}\,T_r$ with the subspace of $\mathbb{C}^r$ consisting of points with the sum of coordinates being zero, i.e., $\mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$.

Definition 2.2.33 *We define*

$$\mathcal{M}(n,r)_{\mathfrak{a}} := \mathrm{Spec}(H_*^{A \times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})).$$

Let us now give a more explicit description of the algebra $\mathbb{C}[\mathcal{M}(n,r)_{\mathfrak{a}}]$. We recall its realization as a spherical subalgebra of the rational Cherednik algebra $H_{n,r}$ corresponding to the group $\Gamma_n = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

Rational Cherednik algebra corresponding to $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$

We start by recalling some definitions and notations.

Consider the subgroup $\Gamma_n \subset \mathrm{GL}_n$ of monomial matrices with entries being rth roots of unity. Let $\eta \in \mathbb{C}^\times$ be a rth primitive root of unity. We set $\epsilon_j := \mathrm{diag}(1,\ldots,1,\eta,1,\ldots,1)$. Note that we have the natural embedding $S_n \subset \Gamma_n$. We obtain the identification $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n \xrightarrow{\sim} \Gamma_n$. Consider the standard representation $\Gamma_n \curvearrowright \mathfrak{h} := \mathbb{C}^n$ induced by the embedding $\Gamma_n \subset \mathrm{GL}_n$. Let $x_1,\ldots,x_n$ be the standard basis in $\mathfrak{h} = \mathbb{C}^n$ and denote by $y_1,\ldots,y_n \in \mathfrak{h}^*$ the dual basis.

Let $\hbar, \kappa, c_1, \ldots, c_{r-1}$ be formal parameters. We set

$$\mathbf{h} := \mathbb{C}[\kappa, c_1, \ldots, c_{r-1}], \quad c(q) := \sum_{i=1}^{r-1} c_i q^i,$$

here q is a formal variable. Following [117] and [95, Section 2.3] we define the rational Cherednik algebra $\mathcal{H}_{\Gamma_n}$ in the following way.

Definition 2.2.34 *Algebra $\mathcal{H}_{\Gamma_n} = \mathcal{H}_{n,r}$ is a quotient of the semi-direct product*

$$\Big(\mathbb{C}\Gamma \ltimes T^\bullet(\mathfrak{h} \oplus \mathfrak{h}^*)\Big) \otimes \mathbf{h}[\hbar]$$

subject to the relations:

$$[x_i, x_j] = [y_i, y_j] = 0, \tag{2.18}$$

$$[x_i, y_i] = -\hbar - \kappa \sum_{j \neq i} \sum_{p=0}^{r-1} (ij)\epsilon_i^p \epsilon_j^{-p} - c(\epsilon_i), \tag{2.19}$$

$$[x_i, y_j] = \kappa \sum_{p=0}^{r-1} \eta^p (ij)\epsilon_i^p \epsilon_j^{-p} \qquad (i \neq j). \tag{2.20}$$

We also set $H_{n,r} := \mathcal{H}_{n,r}/(\hbar)$.

Remark 2.2.35 Our parameters $(\hbar, \kappa, c_1, \ldots, c_{r-1})$ and the parameters $(\underline{t}, \underline{\kappa}, \underline{c}_1, \ldots, \underline{c}_{r-1})$ of [78] are related as follows: $\underline{t} = -\hbar$, $\underline{\kappa} = \kappa$, $\underline{c}_i = \frac{c_i}{(1-\eta^{-i})}$ (note that $x_i^{Martino} = y_i$, $y_i^{Martino} = x_i$).

Definition 2.2.36 *Set* $\mathbf{e} := \frac{1}{|\Gamma_n|}\sum_{g \in \Gamma_n} g$. *We denote by $\mathcal{H}_{n,r}^{\mathrm{sph}}$ the spherical subalgebra $\mathbf{e}\mathcal{H}_{n,r}\mathbf{e} \subset \mathcal{H}_{n,r}$, and by $H_{n,r}^{\mathrm{sph}}$ the subalgebra $\mathbf{e}H_{n,r}\mathbf{e} \subset H_{n,r}$.*

Recall that $Z(H_{n,r}) \subset H_{n,r}$ is the center. The following proposition holds by [36].

Proposition 2.2.37 *The composition $Z(H_{n,r}) \subset H_{n,r} \twoheadrightarrow \mathbf{e}H_{n,r}\mathbf{e}$ induces the identification $Z(H_{n,r}) \xrightarrow{\sim} \mathbf{e}H_{n,r}\mathbf{e}$.*

The algebra $H_{n,r}$ is $\mathbb{Z}$-graded as follows:

$$\deg x_j = 1,\ \deg y_j = -1,\ \deg \Gamma_n = \deg \mathbf{h} = 0. \tag{2.21}$$

Coulomb branch for Jordan quiver as the center of rational Cherednik algebra for $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$

Following [117] set

$$p(q) := \frac{1}{r}\sum_{l=1}^{r-1}\frac{c_l}{\eta^{-l}-1}q^l = \frac{1}{r}\sum_{l=1}^{r-1}\frac{c_l\eta^l}{1-\eta^l}q^l.$$

Then we can write

$$c(q) = r(p(\eta^{-1}q) - p(q)). \tag{2.22}$$

Proposition 2.2.38 *There exists the isomorphism of algebras*

$$H_{n,r}^{\mathrm{sph}} \simeq H_*^{A\times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$$

that identifies parameters in the following way $\kappa = \kappa$, $a_i = p(\eta^{i-1})$. The isomorphism above sends $e_i(u_1,\ldots,u_n)$ to m_i, where $m_i = c_i(V) * 1$.

The Proposition 2.2.38 was proven in [65, Theorem 1.1], see also [117, Theorem 4.1] and [13, Theorem 2.19].

Example 2.2.39 *Let us illustrate how (quantized version of) the isomorphism $H_{n,r}^{\mathrm{sph}} \simeq H_*^{A\times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ works for $n=1$. We have $A = \mathbb{C}^\times \times (\mathbb{C}^\times)^{r-1}$ and $\mathbf{N} = \mathbb{C} \oplus (\mathbb{C}^r)^*$. By [14, Section 4(iii)] the algebra $H_*^{A\times(\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}^\times)}(\mathcal{R}_{n,r})$ has the following description: it is generated over $\mathbb{C}[\kappa, a_1, \ldots, a_r]/(a_1+\ldots+a_r)$ by r_1, r_{-1}, b subject to relations:*

$$r_1r_{-1} = \prod_{i=1}^r (b-a_i),\ r_{-1}r_1 = \prod_{i=1}^r(b-a_i-\hbar),\ [r_1,b] = \hbar r_1,\ [r_{-1},b] = -\hbar r_{-1}.$$

The algebra $H_{1,r}$ is generated by $\mathbb{C}\mathbb{Z}/r\mathbb{Z}$ and x, y subject to the relations

$$[x,y] = -\hbar - c(\epsilon),\ \epsilon x = \eta x\epsilon,\ \epsilon y = \eta^{-1}y\epsilon.$$

Set $u := \frac{1}{r}(xy+\hbar) + p(\eta^{-1}\epsilon)$ (compare with Definition 2.2.58). The isomorphism

$$H_*^{A\times(\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}_\hbar^\times)}(\mathcal{R}_{1,r}) \xrightarrow{\sim} \mathcal{H}_{1,r}^{\mathrm{sph}}$$

is then given by

$$r_{-1} \mapsto \mathbf{e}x^r\mathbf{e},\ r_1 \mapsto \frac{1}{r^r}\mathbf{e}y^r\mathbf{e},\ b \mapsto \mathbf{e}u\mathbf{e},\ \kappa \mapsto \kappa,\ a_i \mapsto p(\eta^{i-1}) - \frac{(i-1)\hbar}{r}.$$

We conclude (using Proposition 2.2.37 and Proposition 2.2.38) that the deformation $\mathcal{M}(n,r)_{\mathfrak{a}}$ of the variety $\mathcal{M}(n,r)$ can be described as

$$\mathcal{M}(n,r)_{\mathfrak{a}} = \mathrm{Spec}(Z(H_{n,r})).$$

2.2.8 Equivariant cohomology $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$

We start by recalling the combinatorial parametrization of the $A = \mathbb{T}\times T_r$-fixed points of the Gieseker variety $\mathfrak{M}(n,r)$.

Consider the case $r=1$. The variety $\mathfrak{M}(n,1)$ coincides with the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ (see Proposition 2.2.11 above or [85, Section 2.2]). Recall that this Hilbert scheme parametrizes the codimension n ideals in $\mathbb{C}[x,y]$. The torus T_1 is zero-dimensional, thus

$$(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}\times T_1} = (\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}.$$

The fixed point set $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}$ is the set of monomial ideals $J\in \mathrm{Hilb}_n(\mathbb{A}^2)$. Such ideals are parametrized by the set $\mathcal{P}(n)$ of partitions of n as follows. Recall that (following notations of [95, Section 4.1]) we associate to $\lambda=(\lambda_1,\ldots,\lambda_l)\in\mathcal{P}(n)$ the Young diagram

$$\mathbb{Y}(\lambda)=\{(i,j)\,|\,1\leqslant i\leqslant l,\ 1\leqslant j\leqslant \lambda_i\}.$$

We fill $\mathbb{Y}(\lambda)$ with monomials by putting $x^{i-1}y^{j-1}$ into the box $(i,j)\in\mathbb{Y}(\lambda)$. The ideal J_λ that corresponds to λ is spanned by the monomials outside the diagram. We have the following classical result.

Proposition 2.2.40 *The fixed point set $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}$ is identified with the set $\mathcal{P}(n)$ via the map $\lambda\mapsto J_\lambda$ described above.*

Now we proceed to the case of arbitrary r. Let us introduce some notation. Denote by $w_1,\ldots,w_r\in W$ a basis of W consisting of eigenvectors of T_r.

The following lemma is classical (see, for example, [123, page 18]).

Lemma 2.2.41 *The variety $\mathfrak{M}(n,r)^{T_r}$ is $\mathbb{T}$-equivariantly isomorphic to the disjoint union*

$$\bigsqcup_{\sum_{l=1}^{r} n_l=n}\ \prod_{l=1}^{r}\mathfrak{M}(n_l,1)\ \simeq\ \bigsqcup_{\sum_{l=1}^{r} n_l=n}\ \prod_{l=1}^{r}\mathrm{Hilb}_{n_l}(\mathbb{A}^2).$$

Definition 2.2.42 *We say that an ordered r-tuple $\boldsymbol{\lambda}=(\lambda^0,\ldots,\lambda^{r-1})$ of partitions defines an r-multipartition of $n\in\mathbb{Z}_{\geqslant 0}$ if $\sum_{l=0}^{r-1}|\lambda^l|=n$. Let $\mathcal{P}(r,n)$ denote the set of r-multipartitions of n.*

Proposition 2.2.43 *The set of fixed points $\mathfrak{M}(n,r)^A$ is identified with the set $\mathcal{P}(r,n)$. A multipartition $\boldsymbol{\lambda}=(\lambda^0,\lambda^1,\ldots,\lambda^{r-1})$, corresponds to the following quiver data $V,W,\ X^{\boldsymbol{\lambda}},Y^{\boldsymbol{\lambda}}\in\mathrm{End}(V)$, $\gamma^{\boldsymbol{\lambda}}\in\mathrm{Hom}(W,V)$, $\delta^{\boldsymbol{\lambda}}\in\mathrm{Hom}(V,W)$:*

$$V:=\bigoplus_{l=0}^{r-1}\mathbb{C}[x,y]/J_{\lambda^l},\ \ W=\bigoplus_{l=0}^{r-1}\mathbb{C}w_l;$$

$$X^{\boldsymbol{\lambda}}:=\bigoplus_{i=0}^{r-1}L^l_x,\ \ Y^{\boldsymbol{\lambda}}:=\bigoplus_{l=0}^{r-1}L^l_y;$$

$$\gamma^{\boldsymbol{\lambda}}\text{ sends } w_i\in W \text{ to } [1] \text{ in } \mathbb{C}[x,y]/J_{\lambda^i},\ \ \delta^{\boldsymbol{\lambda}}=0,$$

by L^l_x, L^l_y we denote operators of multiplications by x,y on $\mathbb{C}[x,y]/J_{\lambda^l}$.

Proof: This follows from Lemma 2.2.41 and the description of $\mathbb{T}$-fixed points of Hilbert schemes above (see Proposition 2.2.40). □

We have the A-equivariant embedding

$$\iota\colon \mathfrak{M}(n,r)^A \subset \mathfrak{M}(n,r).$$

This embedding induces the homomorphism

$$\iota^*\colon H_A^*(\mathfrak{M}(n,r)) \to H_A^*(\mathfrak{M}(n,r)^A) = \mathbb{C}[\mathfrak{a}]^{\oplus|\mathcal{P}(r,n)|} = E.$$

Lemma 2.2.44 *The homomorphism ι^* is an embedding that becomes isomorphism after tensoring by $F = \mathbb{C}(\mathfrak{a})$.*

Proof: Follows from the Atiyah-Bott localization theorem (see [9]) together with the fact that $H_A^*(\mathfrak{M}(n,r),\mathbb{C})$ is a free $\mathbb{C}[\mathfrak{a}]$-module (see [86, Theorem 7.3.5]).

□

Remark 2.2.45 Actually, we do not need to tensor by the whole $\mathbb{C}(\mathfrak{a})$ but only have to localize at elements $h \in \mathbb{C}[\mathfrak{a}]$ corresponding to the cocharacters $\nu\colon \mathbb{C}^\times \to A$ such that $\mathfrak{M}(n,r)^{\nu(\mathbb{C}^\times)}$ is *infinite*. These elements h can be described explicitly.

Recall now that by [82] the algebra $H_A^*(\mathfrak{M}(n,r),\mathbb{C})$ is generated by the A-equivariant Chern classes $c_k(\mathcal{V})$, $k = 1,\ldots,n$, where $\mathcal{V}$ is the tautological rank n vector bundle on $\mathfrak{M}(n,r)$.

Lemma 2.2.46 *The image $\iota^*(c_k(\mathcal{V}))$ is equal to the collection*

$$(e_k(\kappa c_1^0 + a_1,\ldots,\kappa c^0_{|\lambda^0|} + a_1,\ldots,\kappa c_1^{r-1} + a_r,\ldots,\kappa c^{r-1}_{|\lambda^{r-1}|} + a_r))_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)},$$

where for $l = 0,1,\ldots,r-1$ $c_1^l, c_2^l,\ldots,c^l_{|\lambda^l|}$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Proof: The homomorphism ι^* sends $c_k(\mathcal{V})$ to

$$c_k(\mathcal{V}|_{\mathfrak{M}(n,r)^A}) = (c_k(\mathcal{V}|_{(X^{\boldsymbol{\lambda}},Y^{\boldsymbol{\lambda}},\gamma^{\boldsymbol{\lambda}},0)}))_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)}.$$

Note that $c_k(\mathcal{V}|_{(X^{\boldsymbol{\lambda}},Y^{\boldsymbol{\lambda}},\gamma^{\boldsymbol{\lambda}},0)})$ is nothing but $e_k(\alpha_1,\ldots,\alpha_n)$, where $\alpha_1,\ldots,\alpha_n$ is the multiset of $\mathfrak{a}$-weights of $\mathcal{V}|_{(X^{\boldsymbol{\lambda}},Y^{\boldsymbol{\lambda}},\gamma^{\boldsymbol{\lambda}},0)}$.

Recall that $\mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[\kappa]$, $\mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[a_1,\ldots,a_r]/(a_1+\ldots+a_r)$ and $\mathfrak{a} = (\mathrm{Lie}\,\mathbb{T})\oplus\mathfrak{t}_r$. We claim that the $\mathfrak{a}$-weight of $[x^iy^j] \in \mathbb{C}[x,y]/J_{\lambda^l}$ is equal to $\kappa(j-i) + a_l$ and this will conclude the proof. Indeed, taking $g = (t^\kappa, t^{a_1},\ldots,t^{a_r}) \in \mathbb{T}\times T_r$ we see that this element acts on $(X^{\boldsymbol{\lambda}},Y^{\boldsymbol{\lambda}},\gamma^{\boldsymbol{\lambda}},0)$ in the following way:

$$g\cdot X^{\boldsymbol{\lambda}} = \bigoplus_{l=0}^{r-1} t^\kappa L_x^l,\; g\cdot Y^{\boldsymbol{\lambda}} = \bigoplus_{l=0}^{r-1} t^{-\kappa} L_y^l,\; (g\cdot\gamma^{\boldsymbol{\lambda}})(w_l) = t^{-a_l}\gamma^{\boldsymbol{\lambda}}(w_l).$$

Consider the element $g' \in \prod_{l=0}^{r-1} \mathrm{GL}(\mathbb{C}[x,y]/J_{\lambda^l})$ that acts on $x^i y^j \in \mathbb{C}[x,y]/J_{\lambda^l}$ via the multiplication by $t^{a_l+\kappa j-\kappa i}$. Directly from the definitions, we see that

$$g \cdot (X^{\boldsymbol{\lambda}}, Y^{\boldsymbol{\lambda}}, \gamma^{\boldsymbol{\lambda}}, 0) = (g')^{-1} \cdot (X^{\boldsymbol{\lambda}}, Y^{\boldsymbol{\lambda}}, \gamma^{\boldsymbol{\lambda}}, 0).$$

The claim follows since the element $g \in \mathbb{T} \times T_r$ acts on the fiber V of $\mathcal{V}$ at $(X^{\boldsymbol{\lambda}}, Y^{\boldsymbol{\lambda}}, \gamma^{\boldsymbol{\lambda}}, 0)$ via g'. □

Combining the results of this section we obtain the following proposition.

Proposition 2.2.47 *The homomorphism*

$$\iota^* \colon H_A^*(\mathfrak{M}(n,r)) \hookrightarrow H_A^*(\mathfrak{M}(n,r)^A) = E$$

is injective and becomes an isomorphism after tensoring by $F = \mathbb{C}(\mathfrak{a})$. On generators $c_k(\mathcal{V})$, $k = 1, \ldots, n$ it is given by:

$$c_k(\mathcal{V}) \mapsto (e_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda^0|}^0 + a_1, \ldots, \kappa c_1^{r-1} + a_r, \ldots, \kappa c_{|\lambda^{r-1}|}^{r-1} + a_r))_{\boldsymbol{\lambda} \in \mathcal{P}(r,n)}.$$

2.2.9 The center $Z(R^r(n))^{JM}$ of the cyclotomic degenerate Hecke algebra

Recall the space $\mathbf{k} = \mathbb{C}[\kappa, a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$.

Definition 2.2.48 *The degenerate affine Hecke algebra $R(n) = R(S_n)$ is generated by $\mathbb{C}S_n$ and $\mathbf{k}[z_1, \ldots, z_n]$ subject to relations:*

$$s_i z_j = z_{s_i(j)} s_i + \kappa(\delta_{i+1,j} - \delta_{i,j}).$$

The cyclotomic degenerate Hecke algebra $R^r(n)$ is the quotient of $R(n)$ by the ideal generated by $\prod_{i=1}^r (z_1 - a_i)$.

Definition 2.2.49 *Let $Z(R^r(n))^{JM} \subset R^r(n)$ be the image of $\mathbf{k}[z_1, \ldots, z_n]^{S_n} \subset R(n)$ in $R^r(n)$.*

Remark 2.2.50 By [76, Theorem 6.5], the subalgebra $\mathbf{k}[z_1, \ldots, z_n]^{S_n} \subset R(n)$ is nothing else but the center of $R(n)$. It follows from [15, Theorem 1] that the algebra $Z(R^r(n))^{JM}$ is the center of $R^r(n)$.

To every $\boldsymbol{\lambda} \in \mathcal{P}(r,n)$ one can associate the "universal" Specht module $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ over $R^r(n)$ (compare with [15, Section 4]) in the following way.

Recall that $\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$. For $i = 0, 1, \ldots, r-1$ set $n_i := |\lambda^i|$. We slightly modify the algebras $R(n)$, $R^r(n)$ first.

Definition 2.2.51 *Let $R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n)$ be the algebra generated by $\mathbb{C}S_n$ and $\mathbb{C}[z_1, \ldots, z_n]$ over $\mathbb{C}[\kappa, a_1, \ldots, a_r]$ subject to the relations*

$$s_i z_j = z_{s_i(j)} s_i + \kappa(\delta_{i+1,j} - \delta_{i,j}).$$

We denote by $R^r_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n)$ the quotient of $R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n)$ by the ideal generated by $\prod_{i=1}^r (z_1 - a_i)$.

For every $i = 0, 1, \ldots, r-1$ consider the usual Specht module $S_{n_i} \curvearrowright S(\lambda^i)$, corresponding to λ^i. We can extend $S(\lambda^i) \otimes \mathbb{C}[\kappa, a_i]$ to the $R_{\mathbb{C}[\kappa,a_i]}(n_i)$-module by letting z_1 act via the multiplication by a_i. Note that we have the natural embedding

$$R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n_0, \ldots, n_{r-1}) := \\ = R_{\mathbb{C}[\kappa,a_1]}(n_0) \otimes_{\mathbb{C}[\kappa]} \ldots \otimes_{\mathbb{C}[\kappa]} R_{\mathbb{C}[\kappa,a_r]}(n_{r-1}) \subset R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n),$$

induced by the embedding $S_{n_0} \times \ldots \times S_{n_{r-1}} \subset S_n$. Then we define

$$\widetilde{S}_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(\boldsymbol{\lambda}) := R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n) \otimes_{R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n_0,\ldots,n_{r-1})} \\ \otimes \Big((S(\lambda^0) \otimes \mathbb{C}[\kappa, a_1]) \otimes_{\mathbb{C}[\kappa]} \ldots \otimes_{\mathbb{C}[\kappa]} (S(\lambda^{r-1}) \otimes \mathbb{C}[\kappa, a_r]) \Big)$$

that is an $R^r_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n)$-module. Modding out by $(a_1 + \ldots + a_r)$ we obtain the desired $R^r(n)$-module $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$.

Let us now compute the action of $Z(R^r(n))^{JM}$ on $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$. We start with the following lemma. Let μ be a partition of some $m \in \mathbb{Z}_{\geqslant 1}$ and consider the action $R_{\mathbb{C}[\kappa,a]}(m) \curvearrowright S(\mu) \otimes \mathbb{C}[\kappa, a]$. Recall that $S(\mu)$ has a basis labeled by the standard Young tableaux. such that JM_i act on such vectors with the content of $i's$ entry.

Lemma 2.2.52 *Let B be a standard Young tableau on μ and recall that $p_B \in S(\mu)$ is the corresponding vector. The element $z_i \in R_{\mathbb{C}[\kappa,a]}(m)$ acts on p_B via the multiplication by $\kappa\, \mathrm{ct}(B(i)) + a$.*

Proof: The i-th Jucys-Murphy element is given by

$$JM_i = \sum_{j<i} (ij) \in \mathbb{C}S_m.$$

Recall now that $s_i z_{i+1} = z_i s_i + \kappa$ so we have

$$z_{i+1} = s_i z_i s_i + s_i \kappa.$$

It follows that

$$z_j = s_{j-1} s_{j-2} \ldots s_1 z_1 s_1 \ldots s_{j-1} + \kappa JM_i$$

so the action of z_j on $S(\mu) \otimes \mathbb{C}[\kappa, a]$ coincides with the action of $\kappa JM_i + a$ and this concludes the proof. □

From Lemma 2.2.52 we obtain the following proposition (see also [15, Section 4]).

Proposition 2.2.53 *The class $[f(z_1, \ldots, z_n)] \in Z(R^r(n))^{JM}$ of the element f in $\mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(n)$ acts on the representation $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ via multiplication by*

$$f(\kappa c^0_1 + a_1, \ldots, \kappa c^0_{|\lambda^0|} + a_1, \ldots, \kappa c^{r-1}_1 + a_r, \ldots, \kappa c^{r-1}_{|\lambda^{r-1}|} + a_r),$$

where $c^l_1, c^l_2, \ldots, c^l_{|\lambda^l|}$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Since every element of $Z(R^r(n))^{JM}$ is central we can consider the homomorphism

$$\psi\colon Z(R^r(n))^{JM} \to \bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}))$$

induced by the action of $Z(R^r(n))^{JM}$ on representations $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$.

Lemma 2.2.54 *We have* $\operatorname{End}_{R^r(n)} \widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}) = \mathbf{k}$.

Proof: Let us first of all note that $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ is a free $\mathbf{k}$-module (see [7] and references therein). It follows that we have an embedding

$$\operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})) \subset \operatorname{End}_{R^r(n)\otimes_{\mathbf{k}}\overline{\operatorname{Frac}(\mathbf{k})}}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}) \otimes_{\mathbf{k}} \overline{\operatorname{Frac}(\mathbf{k})}) = \overline{\operatorname{Frac}(\mathbf{k})},$$

where $\overline{\operatorname{Frac}(\mathbf{k})}$ is the algebraic closure of the field of fractions $\operatorname{Frac}(\mathbf{k})$ and the last equality holds since $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}) \otimes_{\mathbf{k}} \overline{\operatorname{Frac}(\mathbf{k})}$ is simple over $R^r(n) \otimes_{\mathbf{k}} \overline{\operatorname{Frac}(\mathbf{k})}$ (see [15, Section 4]). Now it follows that $\operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})) = \mathbf{k}$. □

Proposition 2.2.55 *The homomorphism*

$$\psi\colon Z(R^r(n))^{JM} \to \bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)} \widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}) = E$$

becomes an isomorphism after tensoring by $F = \operatorname{Frac}(\mathbf{k})$. *This homomorphism is injective. It sends generators* $[e_k(z_1, \dots, z_n)]$ *to the collection*

$$(e_k(\kappa c^0_1 + a_1, \dots, \kappa c^0_{|\lambda^0|} + a_1, \dots, \kappa c^{r-1}_1 + a_r, \dots, \kappa c^{r-1}_{|\lambda^{r-1}|} + a_r))_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)},$$

where $c^l_1, c^l_2, \dots, c^l_{|\lambda^l|}$ *is the multiset of contents of boxes of the diagram* $\mathbb{Y}(\lambda^l)$.

Proof: The last part of the claim is Proposition 2.2.53. Recall now that by [15, Therorem 1] $Z(R^r(n))^{JM}$ is a free $\mathbf{k}$-module of rank $|\mathcal{P}(n,r)|$. So injectivity of ψ would follow if we show that ψ becomes isomorphism after tensoring by F. In order to do that it is enough to check that ψ becomes surjective after tensoring by F (using the equality of ranks of $Z(R^r(n))^{JM}$ and $\bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}))$ over $\mathbf{k}$). Surjectivity is a corollary of Proposition 2.2.53 and Proposition 2.2.47. □

2.2.10 Schematic fixed points of Spec $Z(H_{n,r})$

Standard representations of $H_{n,r}$, grading on them

We set $\zeta_{i,j} := \frac{1}{r}\sum_{p=0}^{r-1} \epsilon_i^p \epsilon_j^{-p}$ (projector to the invariants under $\epsilon_i\epsilon_j^{-1}$). The Jucys-Murphy elements generalize from S_n to Γ_n as

$$JM_{\Gamma_n,i} := \sum_{j<i} \zeta_{i,j}(ij) \in \mathbb{C}\Gamma_n. \tag{2.23}$$

Recall that $\mathcal{P}(r,n)$ is the set of r-multipartitions of n (Definition 2.2.42). Pick $\boldsymbol{\lambda} \in \mathcal{P}(r,n)$ and consider the corresponding r-tuple of Young diagrams

$$\mathbb{Y}(\boldsymbol{\lambda}) = (\mathbb{Y}(\lambda^0), \dots, \mathbb{Y}(\lambda^{r-1})). \tag{2.24}$$

Given a cell $b \in \mathbb{Y}(\boldsymbol{\lambda})$, define $\beta(b) = k$ if $b \in \mathbb{Y}(\boldsymbol{\lambda}^{k-1})$ and $\mathrm{ct}(b) = j - i$ if b is in the ith row and jth column of $\mathbb{Y}(\lambda^k)$. There is a bijection $\boldsymbol{\lambda} \mapsto S(\boldsymbol{\lambda})$ from the set of r-partitions of n to the set of irreducible Γ_n-modules such that $S(\boldsymbol{\lambda})$ has a basis p_B indexed by standard Young tableaux B on $\boldsymbol{\lambda}$, and p_B is determined up to scalars by the equations (see, for example, [42, Equation (2.16)]):

$$JM_{\Gamma_n,i} \cdot p_B = \mathrm{ct}(B(i)) p_B,\ \epsilon_i \cdot p_B = \eta^{\beta(B(i))} p_B. \tag{2.25}$$

We can consider $S(\boldsymbol{\lambda}) \otimes \mathbf{h}$ as a module over $(\mathbf{h} \otimes \mathbb{C}\Gamma_n) \ltimes S^\bullet \mathfrak{h}^*$ via the trivial action of the augmentation ideal of $S^\bullet \mathfrak{h}^*$. Let $\Delta(\boldsymbol{\lambda}) := \mathrm{Ind}^{H_{n,r}}_{(\mathbf{h}\otimes\mathbb{C}\Gamma)\ltimes S^\bullet\mathfrak{h}^*}(S(\boldsymbol{\lambda}) \otimes \mathbf{h})$ be the induced module (sometimes called the standard module corresponding to $\boldsymbol{\lambda}$). Recall that the algebra $H_{n,r}$ is graded via

$$\deg x_j = 1,\ \deg y_j = -1,\ \deg \Gamma_n = \deg \mathbf{h} = 0.$$

This grading induces a grading on $\Delta(\boldsymbol{\lambda})$. The following lemma describes all the *graded* endomorphisms of our modules $\Delta(\boldsymbol{\lambda})$.

Lemma 2.2.56 *We have* $\mathrm{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda})) = \mathbf{h}$. Proof*: Note that* $\Delta(\boldsymbol{\lambda})_0 = S(\boldsymbol{\lambda}) \otimes \mathbf{h}$ *and* $\Delta(\boldsymbol{\lambda})_0$ *generates* $\Delta(\boldsymbol{\lambda})$ *over* $H_{n,r}$*. Since we consider graded isomorphisms,* $\Delta(\boldsymbol{\lambda})_0$ *is mapped to itself. Now the claim follows from the equality* $\mathrm{End}_{\mathbf{h}\otimes\mathbb{C}\Gamma_n}(S(\boldsymbol{\lambda}) \otimes \mathbf{h}) = \mathbf{h}$. □

Recall the algebra

$$Q_{n,r} := Z(H_{n,r})_0 / \sum_{i>0} Z(H_{n,r})_{-i} Z(H_{n,r})_i = Z(H_{n,r})/(b \in Z(H_{n,r})_i,\ i \neq 0)$$

of functions on schematic fixed points $(\mathrm{Spec}\, Z(H_{n,r}))^{\mathbb{T}}$.

Note that the action of $Z(H_{n,r})_0$ on $\Delta(\boldsymbol{\lambda})_0$ factors through $Q_{n,r}$ so we obtain a homomorphism

$$\phi\colon Q_{n,r} \to \bigoplus_{\boldsymbol{\lambda}} \mathrm{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda})) = \bigoplus_{\boldsymbol{\lambda}} \mathbf{h} = E.$$

Remark 2.2.57 In [39] Gordon defines so-called baby Verma modules over certain quotients of $H_{n,r}$ (called restricted Cherednik algebras of Γ_n). Homomorphism ϕ can also be defined by replacing $\Delta(\boldsymbol{\lambda})$ by the (universal analogs) of baby Verma modules and acting by $Q_{n,r}$ on them (compare with [78]).

We have constructed the homomorphism ϕ, let us now describe generators of the algebra $Q_{n,r}$.

Definition 2.2.58 *Dunkl-Opdam operators* u_i*,* $i = 1, \dots, n$ *are the following elements of* $H_{n,r}$*:*

$$u_i := \frac{1}{r} y_i x_i - \kappa \sum_{j>i} (ij)\zeta_{i,j} + p(\epsilon_i) = \frac{1}{r} x_i y_i + \kappa JM_{\Gamma_n,i} + p(\eta^{-1}\epsilon_i), \tag{2.26}$$

where the last equality follows from the fact that

$$[x_i, y_i] = -\kappa \sum_{j\neq i} \sum_{p=0}^{r-1} (ij)\epsilon_i^p \epsilon_j^{-p} - c(\epsilon_i)$$

together with the equality $c(\epsilon_i) = rp(\eta^{-1}\epsilon_i) - rp(\epsilon_i)$.

Remark 2.2.59 Note that the element u_i identifies with the element $\frac{z_i}{r}$ defined in [78, Section 3.2].

Lemma 2.2.60 *The subalgebra* $\mathbf{h}[u_1, \ldots, u_n]^{S_n} \subset H_{n,r}$ *is central.*

Proof: This follows from [78, Theorem 3.4], the statement can also be deduced from the presentation of $H_{n,r}$ given in [117].

□

The following lemma describes generators of the algebra $Q_{n,r}$, see Proposition 2.1.7 for an alternative argument (that covers a much more general situation).

Lemma 2.2.61 *Classes of elements* $e_k(u_1, \ldots, u_n)$, $k = 1, \ldots, n$ *generate the algebra* $Q_{n,r}$.

Proof: The claim follows from the proof of [78, Theorem 5.5]. Let us repeat the argument. Recall that $Q_{n,r}$ is the quotient of the algebra $Z(H_{n,r})$. Consider the following filtration on $H_{n,r}$:

$$\deg x_i = \deg y_i = 1,\ \deg \kappa = \deg c_j = \deg \Gamma_n = 0.$$

We have

$$\operatorname{gr} H_{n,r} = \Big(\mathbb{C}\Gamma_n \ltimes S^\bullet(\mathfrak{h} \oplus \mathfrak{h}^*)\Big) \otimes \mathbf{h}.$$

We need to prove that classes of the elements $\sum_{i=1}^n u_i^{ra+c}$, $a, c \in \mathbb{Z}_{\geqslant 0}$ do generate the algebra $Q_{n,r}$. To see that it is enough to show that the elements $\operatorname{gr}\Big(\sum_{i=1}^n u_i^{ra+c}\Big) = \frac{1}{r^{ra+c}} \operatorname{gr}\Big(\sum_i (x_i y_i)^{ra+c}\Big)$ do generate $\operatorname{gr} Q_{n,r}$. Note that (by Theorem 3.3 from [36]) $\operatorname{gr} Q_{n,r}$ is the quotient of

$$\operatorname{gr} Z(H_{n,r}) = \mathbf{h}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n} = \\ = \mathbf{h}[x_1y_1, \ldots, x_ny_n, x_1^r, \ldots, x_n^r, y_1^r, \ldots, y_n^r]^{S_n}.$$

In particular, it is generated by $\{\sum_{i=1}^n (x_i^r)^a (y_i^r)^b (x_iy_i)^c,\ a, b, c \in \mathbb{Z}_{\geqslant 0}\}$ ([36, Lemma 11.17], [116, Lemma 1] see also Section 2.2.12 for more detailed discussion of the generators of $\operatorname{gr} Q_{n,r}$). It remains to note that the class of $\sum_{i=1}^n (x_i^r)^a (y_i^r)^b (x_iy_i)^c$ in $\operatorname{gr} Q_{n,r}$ is zero if $a \neq b$ and for $a = b$ we have $\sum_{i=1}^n (x_i^r)^a (y_i^r)^a (x_iy_i)^c = \sum_{i=1}^n (x_iy_i)^{ra+c}$. The claim follows. □

Lemma 2.2.62 *Let B be a Young tableau on $\boldsymbol{\lambda}$ and recall that $p_B \in S(\boldsymbol{\lambda})$ is the corresponding vector. The element u_i acts on p_B via the multiplication by*

$$\kappa \operatorname{ct}(B(i)) + p(\eta^{\beta(B(i))-1}).$$

Proof: Follows from the definition of u_i (see 2.26) together with (2.25). $\square$

Proposition 2.2.63 *The homomorphism $\phi\colon Q_{n,r} \to \bigoplus_{\boldsymbol{\lambda}} \mathrm{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda}))$ becomes isomorphism after tensoring by $F = \mathrm{Frac}(\mathbf{h})$. This homomorphism is injective. It sends generators $[e_k(u_1, \ldots, u_n)]$ to the collection*

$$(e_k(\kappa c^0_1 + p(1), \ldots, \kappa c^0_{|\lambda^0|} + p(1), \ldots, \kappa c^{r-1}_1 + p(\eta^{r-1}), \ldots, \kappa c^{r-1}_{|\lambda^{r-1}|} + p(\eta^{r-1})))_{\boldsymbol{\lambda} \in \mathcal{P}(r,n)},$$

where $c^l_1, c^l_2, \ldots, c^l_{|\lambda^l|}$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Proof: The proof repeats the argument in the proof of Proposition 2.2.55. The only difference is that we use Appendix 2.2.12 (flatness of $Q_{n,r}$ over $\mathbf{h}$) together with the fact that the fiber of $Q_{n,r}$ over a generic point is $\mathbb{C}^{\oplus|\mathcal{P}(r,n)|}$ (follows from [39]) instead of [15, Theorem 1] and Lemma 2.2.62 instead of Proposition 2.2.53. $\square$

2.2.11 Proof of Theorem 2.2.7

Proof of Theorem 2.2.7

Let us now recall the statement of Theorem 2.2.7.

Theorem 2.2.64 *After the identification of parameters (2.8) the graded algebras*

$$H^*_A(\mathfrak{M}(n,r)),\ Z(R^r(n))^{JM},\ Q_{n,r},\ \mathbb{C}[\mathrm{Spec}(H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r}))^{\mathbb{T}}]$$

are isomorphic. Isomorphisms above identify generators as follows:

$$c_k(\mathcal{V}) = [e_k(z_1, \ldots, z_n)] = [e_k(u_1, \ldots, u_n)] = [m_k]. \tag{2.27}$$

Proof: The claim follows from Propositions 2.2.47, 2.2.55, 2.2.63. The equality $e_k(u_1, \ldots, u_n) = m_k$ follows from [117, Section 4]. $\square$

Remark 2.2.65 Recall that $\mathfrak{M}_0(n,r)^\vee$ and its deformation have realization as certain quiver varieties (see Remark 2.2.2). It is a natural question to describe functions m_k in "quiver terms". For $r = 1$ these functions actually appear in Section 2.2.6 and are given by $(X, Y, \gamma, \delta) \mapsto e_k(\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are roots of the characteristic polynomial of $YX \in \mathrm{End}(V)$. Functions m_k can be similarly described for arbitrary r, see [95, Section 6] for details (for $r = l > 1$ there is a minor computational error in this paper that should be fixed, that's why we decided not to go into details here). Let us finally mention that the algebra generated by $\{m_k,\ k = 1, \ldots, n\}$ defines an integrable system on the Coulomb branch $\mathcal{M}(n,r)$. This Coulomb branch can be identified with a certain Cherkis bow variety, and this integrable system can be naturally described in these terms (see [89] for details, integrable systems are denoted by ϖ_C, Ψ in loc. cit.).

2.2.12 Flatness of schematic fixed points: approach of Hikita and Hatano

The goal of this appendix is to give a self-contained proof of the fact that the algebra $Q_{n,r}$ of functions on the schematic fixed points

$$(\operatorname{Spec} Z(H_{n,r}))^{\mathbb{T}} = \mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}$$

is a flat (hence, free) $\mathbf{h}$-module of rank $|\mathcal{P}(r,n)|$. Let us first note that by the graded Nakayama lemma together with the fact that $\dim_F(Q_{n,r} \otimes_{\mathbf{h}} F) = |\mathcal{P}(r,n)|$ (this follows from [39], see also [95, Section 5]) in order to prove this fact it is enough to show that

$$\dim_{\mathbb{C}} Q_{n,r}/(\kappa, c_1, \ldots, c_{r-1}) \leqslant |\mathcal{P}(r,n)|$$

i.e. that

$$\dim_{\mathbb{C}} \mathbb{C}[(\mathbb{A}^{2n}/\Gamma_n)^{\mathbb{T}}] \leqslant |\mathcal{P}(r,n)|. \tag{2.28}$$

The goal of this section is to prove the inequality (2.28). Our argument simply follows papers [46] (for $r=1$ case) and [44] (in general) but is much shorter since we do not need any explicit formulas for the multiplication rule of the elements of $\mathbb{C}[(\mathbb{A}^{2n}/\Gamma_n)^{\mathbb{T}}]$ and only need to estimate the dimension of this algebra from above since the estimate from below follows from the deformation argument (so we only need [46, Lemma 2.5] for $r=1$ case and [44, Lemma 2.1.4] for general r). We start from the case $r=1$ i.e. from the case when $\Gamma_n = S_n$.

Hilbert scheme case ($r = 1$)

Let us recall some notation (we follow [46]).

Definition 2.2.66 *An unordered sequence $\Lambda = (a_1,b_1)\ldots(a_l,b_l)$ with $(a_i,b_i) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ is called bipartite partition of $(a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ if $\sum_{i=1}^l a_i = a$, $\sum_{i=1}^l b_i = b$. We set $\ell(\Lambda) = l$, $|\Lambda| = (a,b)$.*

We have a natural surjection

$$\mathbb{C}[S^{n+1}(\mathbb{A}^2)] = \mathbb{C}[x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}]^{S_{n+1}} \twoheadrightarrow \\ \twoheadrightarrow \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n} = \mathbb{C}[S^n(\mathbb{A}^2)]$$

and denote by S the inverse limit (in the category of graded algebras)

$$S := \varprojlim \mathbb{C}[S^n(\mathbb{A}^2)].$$

We denote by $m_\Lambda \in S$ the symmetrization of the monomial $x_1^{a_1}y_1^{b_1}\ldots x_l^{a_l}y_l^{b_l}$. We have:

$$S = \mathbb{C}[m_{(a,b)} \,|\, (a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}].$$

For $(a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ we set $(a,b)\Lambda := (a,b)(a_1,b_1)\ldots(a_l,b_l)$. If $(a,b) = (a_i,b_i)$ for some $i \in \{1,2,\ldots,l\}$ we set

$$\Lambda \setminus (a,b) := (a_1,b_1)\ldots(a_{i-1},b_{i-1})(a_{i+1},b_{i+1})\ldots(a_l,b_l).$$

We set $\overline{S} := S/(m_{(a,b)},\ a \neq b)$ and denote by $\overline{m}_\Lambda$ the image of m_Λ in $\overline{S}$. Note that directly from the definitions for every $n \in \mathbb{Z}_{\geqslant 1}$ we have a surjective homomorphism $\overline{S} \twoheadrightarrow \mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}]$ which sends every $\overline{m}_\Lambda$ with $\ell(\Lambda) > n$ to zero. The following lemma is clear.

Lemma 2.2.67 *Let Λ be a bipartite partition. We have*

$$m_{(a,b)}m_\Lambda = km_{(a,b)\Lambda} + \sum_{(i,j)\in\Lambda} k_{(i,j)}m_{(a+i,b+j)\Lambda\setminus(i,j)}$$

for some $k, k_{(i,j)} \in \mathbb{Z}_{>0}$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we denote by $(\lambda, 0)$ the bipartite partition $(\lambda_1, 0), \ldots, (\lambda_l, 0)$.

This lemma is [46, Lemma 2.5].

Lemma 2.2.68 $\{\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}} \,|\, \lambda$ *is partition*$\}$ *spans* $\overline{S}$.

Proof: Let us first of all note that the functions $\overline{m}_{(a,a)(b,b)(c,c)\ldots}$ span $\overline{S}$. Indeed, to prove this, it is enough to show that every $\overline{m}_\Lambda$ can be obtained as a linear combination of $\overline{m}_{(a,a)(b,b)(c,c)\ldots}$. This can be proved by the induction on $\ell(\Lambda)$ using Lemma 2.2.67 together with the fact that $\overline{m}_{(a,b)} = 0$ for $a \neq b$.

It remains to show that every $\overline{m}_{(a,a)(b,b)(c,c)\ldots}$ can be expanded in terms of $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$. To see that it is enough to show that every $\overline{m}_{(a_1,b_1)\ldots(a_l,b_l)(0,1)^m}$ with $a_i \geqslant b_i$ can be obtained as a linear combination of $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$. We prove this by the induction on $d = \sum_{i=1}^l b_i$. For $d = 0$ the claim is clear.

For the induction step without losing the generality, we can assume that $b_1 > 0$. Using Lemma 2.2.67 we obtain:

$$m_{(a_1,b_1-1)}m_{(a_2,b_2)\ldots(a_l,b_l)(0,1)^{m+1}} = k_1 m_{(a_1,b_1)\ldots(a_l,b_l)(0,1)^m} + \\ + k_0 m_{(a_1,b_1-1)(a_2,b_2)\ldots(a_l,b_l)(0,1)^{m+1}} + \sum_{i=2}^{l} k_i m_{(a_2,b_2)\ldots(a_1+b_i,b_1+b_i-1)\ldots(a_l,b_l)(0,1)^{m+1}}$$

for some $k_0, k_1, \ldots, k_l \in \mathbb{Z}$ with $k_1 \neq 0$. Induction hypothesis together with the fact that $\overline{m}_{(a_1,b_1-1)} = 0$ finish the proof. □

Corollary 2.2.69 *The image of* $\{\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}, |\, \ell(\lambda)+|\lambda| \leqslant n\}$ *spans* $\mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}] = \mathbb{C}[(\mathbb{A}^{2n}/S_n)^{\mathbb{T}}]$. *In particular, we have*

$$\dim \mathbb{C}[(\mathbb{A}^{2n}/S_n)^{\mathbb{T}}] \leqslant |\mathcal{P}(n)|.$$

Proof: Clearly the elements $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$ with $\ell(\lambda) + |\lambda| > n$ lie in the kernel of $\overline{S} \twoheadrightarrow \mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}]$. Now the first claim follows from Lemma 2.2.68. It remains to note that we have a bijection

$$\{\lambda \,|\, \ell(\lambda) + |\lambda| \leqslant n\} \xrightarrow{\sim} \mathcal{P}(n)$$

that sends a partition $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots)$ to the partition $\hat{\lambda} \in \mathcal{P}(n)$ given by

$$\hat{\lambda} = 1^{n-\ell(\lambda)-|\lambda|}2^{\alpha_1}3^{\alpha_2}\ldots i^{\alpha_{i-1}}\ldots.$$

□

General case (r is arbitrary)

Let us now generalize the arguments of Section 2.2.12 to the arbitrary $r \in \mathbb{Z}_{\geqslant 1}$. We follow [44]. We start with some notation. Recall that $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$ is nothing else but

$$\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n\ltimes(\mathbb{Z}/r\mathbb{Z})^n} \simeq \Big(\mathbb{C}[x_1',\ldots,x_n',y_1',\ldots,y_n',z_1',\ldots,z_n']/(x_1'y_1'-(z_1')^r,\ldots,x_n'y_n'-(z_n')^r)\Big)^{S_n}$$

where the isomorphism is given by

$$x_i' \mapsto x_i^r,\ y_i' \mapsto y_i^r,\ z_i' \mapsto x_iy_i.$$

Remark 2.2.70 Geometrically isomorphism above corresponds to the identification $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n] \simeq S^n(\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z}))$.

Set

$$I_n := (x_1'y_1'-(z_1')^r,\ldots,x_n'y_n'-(z_n')^r) \subset \mathbb{C}[x_1',\ldots,x_n',y_1',\ldots,y_n',z_1',\ldots,z_n'],$$

and

$$S' := \varprojlim \mathbb{C}[x_1',\ldots,x_n',y_1',\ldots,y_n',z_1',\ldots,z_n']^{S_n},\ I := \varprojlim I_n,\ S := S'/I.$$

For every tripartition $\Lambda = (a_1,b_1,c_1)(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)$ let $m_\lambda' \in S'$ be the symmetrization of the monomial $(x_1')^{a_1}(y_1')^{b_1}(z_1')^{c_1}\ldots(x_l')^{a_l}(y_l')^{b_l}(z_l')^{c_l}$, we set $\ell(\Lambda) := l$. We denote by $m_\Lambda \in S$ the image of m_Λ'. The following lemma is clear.

Lemma 2.2.71 *The set $\{m_\Lambda' \,|\, \Lambda\text{-tripartition}\}$ spans S'. So*

$$\{m_\Lambda \,|\, \Lambda = (a_1,b_1,c_1)\ldots,\ c_i \leqslant r-1\}$$

spans S.

Lemma 2.2.72 *Let Λ be a tripartition and $(a,b,c) \in \mathbb{Z}^3_{\geqslant 0} \setminus \{(0,0,0)\}$. Then we have*

$$m_{(a,b,c)}'m_\Lambda' = ?\cdot m_{(a,b,c)\Lambda}' + \sum_{(i,j,k)\in\Lambda} ?\cdot m_{(a+i,b+j,c+k)\Lambda\setminus(i,j,k)}'.$$

with ? being some positive numbers. As a corollary we have

$$m_{(a,b,c)}m_\Lambda = ?\cdot m_{(a,b,c)\Lambda} + \sum_{(i,j,k)\in\Lambda,\, c+k\leqslant r-1} ?\cdot m_{(a+i,b+j,c+k)\Lambda\setminus(i,j,k)} + \sum_{(i,j,k)\in\Lambda,\, c+k\geqslant r} ?\cdot m_{(a+i+1,b+j+1,c+k-r)\Lambda\setminus(i,j,k)}.$$

with ? being some positive numbers.

For $\Lambda = (a_1,b_1,c_1)\dots(a_l,b_l,c_l)$ we set $\deg\Lambda := \sum_{i=1}^{l} a_i - \sum_{i=1}^{l} b_i$. Let $J \subset S$ be the ideal generated by $\{m_\Lambda \mid \deg\Lambda \neq 0\}$. We set $\overline{S} := S/J$. We denote by $\overline{m}_\Lambda \in \overline{S}$ the image of m_Λ.

For an r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ and a collection of nonnegative numbers $\mathbf{p} = (p_1, \dots, p_{r-1})$ we define tripartition to be denoted by the symbol

$$(\boldsymbol{\lambda},\mathbf{p}) := (\lambda^0_1,0,0)\dots(\lambda^0_{\ell(\lambda^0)},0,0),(0,0,1)^{p_1},(\lambda^1_1,0,1),\dots,(\lambda^1_{\ell(\lambda^1)},0,1),\dots \\ \dots,(0,0,r-1)^{p_{r-1}},(\lambda^{r-1}_1,0,r-1),\dots,(\lambda^{r-1}_{\ell(\lambda^{r-1})},0,r-1).$$

Recall that $\ell(\boldsymbol{\lambda}) = \sum_{i=0}^{r-1}\ell(\lambda^i)$, $|\boldsymbol{\lambda}| = \sum_{i=0}^{r-1}|\lambda^i|$. We set $|\mathbf{p}| := p_1 + \ldots + p_{r-1}$. The following lemma is [44, Lemma 2.1.4]. [2]

Lemma 2.2.73 *The set* $\{\overline{m}_{(\boldsymbol{\lambda},\mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}}\}$ *spans* $\overline{S}$.

Proof: We start from proving that elements $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)\dots}$ span $\overline{S}$. It is enough to show that every m_Λ can be presented as a linear combination of $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)\dots}$. We can assume that there exists $(a,b,c) \in \Lambda$ such that $a \neq b$ (otherwise, there is nothing to prove). From Lemma 2.2.72 it follows that

$$0 = \overline{m}_{(a,b,c)}\overline{m}_{\Lambda\setminus(a,b,c)} = k\overline{m}_\Lambda + \sum_{\Lambda',\,\ell(\Lambda')=\ell(\Lambda)-1} ?\cdot\overline{m}_{\Lambda'}$$

and the claim follows by the induction on the length of Λ.

It remains to show that every element $\overline{m}_{(a_1,a_1,c_1)\dots(a_l,a_l,c_l)}$ can be written as a linear combination of $\overline{m}_{(\boldsymbol{\lambda},\mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}}$. We prove a more general statement: that every element $\overline{m}_{(a_1,b_1,c_1)\dots(a_l,b_l,c_l)(0,1,0)^k}$ with $a_i \geqslant b_i$ can be written as a linear combination of $\overline{m}_{(\boldsymbol{\lambda},\mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}}$. We prove this claim by the induction on $b+l$, where $b := \sum_{i=1}^{l} b_i$. Let us, first of all, note that we can assume that $\sum_{i=1}^{l} a_i = b+k$. For $b=0$ we must have $b_i = 0$ for every i and then there is nothing to prove. Suppose now that $b > 0$. Without losing the generality we can assume that $b_1 > 0$. By Lemma 2.2.72 (also using that $a_1 \geqslant b_1 > b_1 - 1$) we have

$$\begin{aligned}0 = \overline{m}_{(a_1,b_1-1,c_1)}\overline{m}_{(a_2,b_2,c_2)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} &= \\ = ?\cdot\overline{m}_{(a_1,b_1-1,c_1)(a_2,b_2,c_2)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} &+ u\overline{m}_{(a_1,b_1,c_1)\dots(a_l,b_l,c_l)(0,1,0)^k} + \\ + \sum_{c_1+c_i\leqslant r-1} ?\cdot\overline{m}_{(a_2,b_2,c_2)\dots(a_i+a_1,b_i+b_1-1,c_i+c_1)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} &+ \\ \sum_{c_i+c_1\geqslant r} ?\cdot\overline{m}_{(a_2,b_2,c_2)\dots(a_i+a_1+1,b_i+b_1,c_i+c_1-r)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}}&\end{aligned}$$

with $u \in \mathbb{Z}_{>0}$. Now the claim follows from the induction hypothesis. □

Corollary 2.2.74 *The image of the set* $\{\overline{m}_{(\boldsymbol{\lambda},\mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}} \mid \ell(\boldsymbol{\lambda})+|\boldsymbol{\lambda}|+|\mathbf{p}| \leqslant n\}$ *spans* $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$. *In particular*

$$\dim\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n] \leqslant |\mathcal{P}(r,n)|.$$

[2] Hatano's statement had a minor typo, Lemma 2.2.73 is a corrected version of [44, Lemma 2.1.4].

Proof: Note that $\ell((\boldsymbol{\lambda}, \mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}) = \ell(\boldsymbol{\lambda}) + |\boldsymbol{\lambda}| + |\mathbf{p}|$ so the elements $\overline{m}_{(\boldsymbol{\lambda},\mathbf{p})(0,1,0)^{|\boldsymbol{\lambda}|}}$ with $\ell(\boldsymbol{\lambda}) + |\boldsymbol{\lambda}| + |\mathbf{p}| > n$ lie in the kernel of $\overline{S} \twoheadrightarrow \mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$. Now, the first claim follows from Lemma 2.2.73. It remains to note that we have a bijection

$$\{(\boldsymbol{\lambda}, \mathbf{p}) \,|\, \ell(\boldsymbol{\lambda}) + |\boldsymbol{\lambda}| + |\mathbf{p}| \leqslant n\} \xrightarrow{\sim} \mathcal{P}(r, n)$$

that sends $(\boldsymbol{\lambda}, \mathbf{p})$ with $\lambda^i = 1^{\alpha_1^i} 2^{\alpha_2^i} \ldots$ to the multipartition $\hat{\boldsymbol{\lambda}}$ given by:

$$\hat{\lambda}^0 = 1^{n-\ell(\boldsymbol{\lambda})-|\boldsymbol{\lambda}|-|\mathbf{p}|} 2^{\alpha_1^0} 3^{\alpha_2^0} \ldots, \text{ and } \hat{\lambda}^i = 1^{p_i} 2^{\alpha_1^i} 3^{\alpha_2^i} \ldots \text{ for } i = 1, \ldots, r-1.$$

The inverse map sends an r-partition $\boldsymbol{\mu} \in \mathcal{P}(r, n)$ with $\mu^i = 1^{\beta_1^i} 2^{\beta_2^i} \ldots$ to the pair $(\boldsymbol{\lambda}, \mathbf{p})$ given by:

$$\lambda^i = 1^{\beta_2^i} 2^{\beta_3^i} \ldots k^{\beta_{k+1}^i} \ldots, \; p_j = \beta_1^j \text{ for } i = 0, 1, \ldots, r-1, \; j = 1, \ldots, r-1.$$

□

Chapter 3

Reresentation theory of quantized ADHM spaces

3.1 Introduction

Results of this Chapter is the joint work with Pavel Etingof, Ivan Losev, and José Simental. In this Chapter, we continue the study of the representation theory of quantizations of Gieseker varieties started in [75]. The main focus is on the representations with minimal support. We will elaborate on what this means later in this section. We relate them to SL_n-equivariant D-modules on the nilpotent cone of $\mathfrak{sl}_n$ and to minimally supported representations of type A rational Cherednik algebras. Our main result is character formulas for minimally supported representations of quantized Gieseker moduli spaces. This chapter is self-consistent, and the notations slightly differ from those used in the previous chapter.

3.1.1 ADHM spaces

Pick two vector spaces V, W of dimensions $n, r \in \mathbb{Z}_{\geqslant 1}$ respectively. Consider the space $R := \mathfrak{gl}(V) \oplus \mathrm{Hom}(V, W)$ and a natural action of $\mathrm{GL}(V)$ on it, let $\mathfrak{gl}(V) \to TR$, $\xi \mapsto \xi_R$ be the infinitesimal action. We can form the cotangent bundle T^*R; this is a symplectic vector space. Identifying $\mathfrak{gl}(V)^*$ with $\mathfrak{gl}(V)$ and $\mathrm{Hom}(V, W)^*$ with $\mathrm{Hom}(W, V)$ by means of the trace form we identify T^*R with $\mathfrak{gl}(V)^{\oplus 2} \oplus \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V)$. The action of $\mathrm{GL}(V)$ on T^*R is symplectic so we get the moment map $\mu\colon T^*R \to \mathfrak{gl}(V)$. It can be described in two equivalent ways. First, we have $\mu(A, B, i, j) = [A, B] - ji$. Second, the dual map $\mu^*\colon \mathfrak{gl}(V) \to \mathbb{C}[T^*R]$ sends $\xi \in \mathfrak{gl}(V)$ to the vector field ξ_R considered as a polynomial function on T^*R.

We identify the character lattice of $\mathrm{GL}(V)$ with $\mathbb{Z}$ via the map

$$\mathbb{Z} \ni \theta \mapsto (\det{}^{\theta}\colon \mathrm{GL(V)} \to \mathbb{C}^{\times}).$$

Let us pick $\theta \in \mathbb{Z} \setminus \{0\}$ and consider the open set of θ-stable points $(T^*R)^{\theta-\mathrm{st}} \subset T^*R$. For $\theta > 0$ the subset of stable points consists of all quadruples (A, B, i, j) such that $\ker i$ does not contain nonzero A- and B-stable subspaces. For $\theta < 0$ the subset of stable points consists of all quadruples (A, B, i, j) such that V is the unique A- and B-stable subspace of V containing $\operatorname{im} j$. We can form the GIT Hamiltonian reduction $\mathfrak{M}^\theta(n,r) = T^*R /\!\!/\!\!/^\theta \operatorname{GL}(V) := \mu^{-1}(0)^{\theta-\mathrm{st}}/\operatorname{GL}(V)$. This is a smooth symplectic quasi-projective variety of dimension $2rn$ that is a resolution of singularities of the categorical Hamiltonian reduction $\mathfrak{M}(n,r) := \mu^{-1}(0)/\!\!/\operatorname{GL}(V)$ which is a Poisson variety. We note that $\mathfrak{M}^\theta(n,r)$ and $\mathfrak{M}^{-\theta}(n,r)$ are symplectomorphic via the isomorphism induced by $(A,B,i,j) \mapsto (B^*, -A^*, j^*, -i^*)$, so we always assume $\theta > 0$ unless otherwise explicitly stated. We also consider the varieties $\overline{\mathfrak{M}}(n,r)$, $\overline{\mathfrak{M}}^\theta(n,r)$ that are obtained similarly but with the space R replaced by $\overline{R} = \mathfrak{sl}(V) \oplus \operatorname{Hom}(V,W)$. We have natural isomorphisms $\mathfrak{M}^\theta(n,r) \simeq \mathbb{C}^2 \times \overline{\mathfrak{M}}^\theta(n,r)$, $\mathfrak{M}(n,r) \simeq \mathbb{C}^2 \times \overline{\mathfrak{M}}(n,r)$ and projective morphisms $\rho\colon \mathfrak{M}^\theta(n,r) \to \mathfrak{M}(n,r)$, $\overline{\rho}\colon \overline{\mathfrak{M}}^\theta(n,r) \to \overline{\mathfrak{M}}(n,r)$. The latter morphisms are resolutions of singularities.

We have an action of $\mathbb{C}^\times \times \operatorname{GL}(W)$ on R given by $(z,g)\cdot(A,i) = (zA, gi)$. This action naturally lifts to an action on T^*R and descends to $\mathfrak{M}^\theta(n,r)$ and $\mathfrak{M}(n,r)$. Let $T_0 \subset \operatorname{GL}(W)$ denote a maximal torus in $\operatorname{GL}(W)$. We set $T := \mathbb{C}^\times \times T_0$.

3.1.2 Quantizations of $\mathfrak{M}(n,r)$

We have a dilation action of $\mathbb{C}^\times$ on T^*R given by $t \cdot x = t^{-1}x$. It descends to both $\mathfrak{M}^\theta(n,r)$ and $\mathfrak{M}(n,r)$. The resulting grading on $\mathbb{C}[\mathfrak{M}(n,r)]$ is positive meaning that $\mathbb{C}[\mathfrak{M}(n,r)] = \bigoplus_{i\geqslant 0} \mathbb{C}[\mathfrak{M}(n,r)]_i$, where $\mathbb{C}[\mathfrak{M}(n,r)]_i$ is the i-th graded component. The Poisson bracket on $\mathbb{C}[\mathfrak{M}(n,r)]$ has degree -2 with respect to this grading. By a quantization of $\mathfrak{M}(n,r)$ we mean an associative unital algebra $\mathcal{A}$ together with an increasing filtration $\mathcal{A}_i \subset \mathcal{A}$, $i \in \mathbb{Z}$ such that $[\mathcal{A}_i, \mathcal{A}_j] \subset \mathcal{A}_{i+j-2}$ for any i, $j \in \mathbb{Z}$ and an isomorphism of graded Poisson algebras $\operatorname{gr}\mathcal{A} \simeq \mathbb{C}[\mathfrak{M}(n,r)]$.

Take $c \in \mathbb{C}$ and set

$$\mathcal{A}_c(n,r) := D(R)/\!\!/\!\!/_c \operatorname{GL}(V) := (D(R)/[D(R)\{\xi_R - c\operatorname{tr}\xi \mid \xi \in \mathfrak{gl}(V)\}])^{\operatorname{GL}(V)},$$

where $D(R)$ is the ring of global differential operators on R. The algebra $\mathcal{A}_c(n,r)$ has a filtration that is induced from the Bernstein filtration on $D(R)$, that is, the filtration, where both vector fields and functions on R have degree 1. There is a natural isomorphism $\mathbb{C}[\mathfrak{M}(n,r)] \xrightarrow{\sim} \operatorname{gr}\mathcal{A}_c(n,r)$. We analogously define quantizations $\overline{\mathcal{A}}_c(n,r)$ of $\overline{\mathfrak{M}}(n,r)$. Note that $\mathcal{A}_c(n,r) = D(\mathbb{C}) \otimes \overline{\mathcal{A}}_c(n,r)$.

3.1.3 Main results

The following theorem is proved in [75, Theorem 1.2].

Theorem 3.1.1 *The algebra $\overline{\mathcal{A}}_c(n,r)$ has a finite dimensional representation if and only if $c = \frac{m}{n}$ with $\gcd(m,n) = 1$ and c is not in the interval $(-r,0)$. If that is the case then there exists a unique simple finite dimensional $\overline{\mathcal{A}}_c(n,r)$-module to be denoted by $\overline{L}_{\frac{m}{n},r}$.*

We remark that proving the existence of a finite-dimensional representation is the most difficult part of the proof of Theorem 3.1.1 in [75]. We give an explicit construction of the representation $\overline{L}_{\frac{m}{n},r}$ which, in particular, simplifies the proof of Theorem 3.1.1. Moreover, our construction allows us to compute not only the dimension of $\overline{L}_{\frac{m}{n},r}$, but its $\mathbb{C}^\times \times \operatorname{GL}_r$-character as well. Let us elaborate on this. We have a natural action of the group $\mathbb{C}^\times \times \mathrm{GL}(W) = \mathbb{C}^\times \times \mathrm{GL}_r$ on the vector space $\overline{R}$. This action commutes with the action of $\mathrm{GL}(V)$ and thus induces an action of the group $\mathbb{C}^\times \times \mathrm{PGL}_r$ on the Gieseker variety $\overline{\mathfrak{M}}(n,r)$ as well as its resolution $\overline{\mathfrak{M}}^\theta(n,r)$. The action $\mathbb{C}^\times \times \mathrm{PGL}_r \curvearrowright \overline{\mathfrak{M}}(n,r)$ is Hamiltonian, it admits a quantum comoment map $\Upsilon\colon \mathbb{C}\oplus\mathfrak{sl}_r = \mathrm{Lie}(\mathbb{C}^\times \times \mathrm{PGL}_r) \to \overline{\mathcal{A}}_c(n,r)$ for any value of c. It is easy to see that the adjoint action of $\Upsilon(\mathbb{C}\oplus\mathfrak{sl}_r)$ on $\overline{\mathcal{A}}_c(n,r)$ is locally finite, and therefore it integrates to an action of $\mathbb{C}^\times \times \mathrm{PGL}_r$ on $\overline{\mathcal{A}}_c(n,r)$ by algebra automorphisms. Moreover, via the map Υ, every $\overline{\mathcal{A}}_c(n,r)$-module becomes a q-graded $\mathfrak{sl}_r$-representation. In particular, $\overline{L}_{\frac{m}{n},r}$ becomes a $\mathbb{C}^\times \times \mathrm{SL}_r$-representation. In the next section, we will show that the action of SL_r extends naturally to an action of GL_r and we will compute the $\mathbb{C}^\times \times \mathrm{GL}_r$-character of $\overline{L}_{\frac{m}{n},r}$.

Theorem 3.1.2 *Assume $m>0$ and $\gcd(m,n)=1$.*

1. *The $\mathbb{C}^\times \times \mathrm{GL}_r$-character of $\overline{L}_{\frac{m}{n},r}$ is*

$$\mathrm{ch}_{\mathbb{C}^\times\times\mathrm{GL}_r}(\overline{L}_{\frac{m}{n},r}) = \frac{1}{[n]_q}\sum_{\substack{\lambda\vdash m\\ r(\lambda)\leqslant\min(r;n)}} s_\lambda(q^{\frac{1-n}{2}},\dots,q^{\frac{n-1}{2}})[W_r(\lambda)^*],$$

where λ denotes a Young diagram with m boxes, $r(\lambda)$ is the number of rows of λ, s_λ is the Schur function corresponding to λ, $W_r(\lambda)$ is the irreducible GL_r-module corresponding to λ, the square brackets denote the class in $K_0(\mathrm{Rep}(\mathrm{GL}_r))$ and $[n]_q := (q^{\frac{n}{2}}-q^{-\frac{n}{2}})/(q^{\frac12}-q^{-\frac12})$.

2. *The dimension of $\overline{L}_{\frac{m}{n},r}$ equals $\frac{1}{n}\binom{nr+m-1}{m}$.*

Example 3.1.3 *Consider the case $n=1$, $r=2$, $\theta>0$. Then we have*

$$\overline{\mathfrak{M}}(1,2)\xrightarrow{\sim}\mathcal{N},\ (i,j)\mapsto(ij),\quad \overline{\mathfrak{M}}^\theta(1,2)\xrightarrow{\sim}T^*(\mathbb{P}^1),\ (i,j)\mapsto(ij,\operatorname{im} i),$$

where $\mathcal{N}\subset\mathfrak{sl}_2$ is the nilpotent cone. Note that the filtered quantizations of $\mathcal{N}$ are $\mathcal{U}(\mathfrak{sl}_2)_p := \mathcal{U}(\mathfrak{sl}_2)/(C-p(p+2))$, $p\in\mathbb{C}$, where $C:=2ef+2fe+h^2$ is the Casimir element and $\mathcal{U}(\mathfrak{sl}_2)$ is the universal enveloping algebra of $\mathfrak{sl}_2$. In our notations we have $\overline{\mathcal{A}}_c(1,2)=\mathcal{U}(\mathfrak{sl}_2)_c$. To see this, let us recall that

$$\overline{\mathcal{A}}_c(1,2) = (D(\mathbb{C}^2)/[D(\mathbb{C}^2)\{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-c\}])^{\mathrm{GL}_1},$$

where $x,y\in\mathbb{C}^{2*}$ are standard coordinate functions and the action of $\mathrm{GL}_1=\mathbb{C}^\times$ is given by $t\cdot x=tx$, $t\cdot y=ty$. Then the isomorphism $\mathcal{U}(\mathfrak{sl}_2)_c\xrightarrow{\sim}\overline{\mathcal{A}}_c(1,2)$ is induced by $e\mapsto -y\frac{\partial}{\partial x}$, $f\mapsto -x\frac{\partial}{\partial y}$, $h\mapsto y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$. This is nothing else but the infinitesimal action of $\mathfrak{sl}_2$ on $\mathbb{C}^2$ corresponding to the standard action $\mathrm{SL}_2\curvearrowright\mathbb{C}^2$. The module $\overline{L}_{m,2}$ is exactly $S^m(\mathbb{C}^2)$ with trivial action of $\mathbb{C}^\times$, the action of GL_2 is induced from the dual of the tautological action $\mathrm{GL}_2\curvearrowright\mathbb{C}^2$.

More generally, for $n=1$, $\theta>0$ one can identify $\overline{\mathcal{A}}_c(1,r)$ with a certain quotient of $\mathcal{U}(\mathfrak{sl}_r)$ and the $\mathbb{C}^\times\times\mathrm{GL}_r$-module $\overline{L}_{m,r}$ is nothing else but $S^m(\mathbb{C}^r)$ with trivial action of $\mathbb{C}^\times$ and the action of GL_r induced from the dual of the tautological action on $\mathbb{C}^r$.

One can generalize the theorem above to the case of irreducible representations with *minimal support*. Let us explain what we mean by this.

Fix a one parameter subgroup $\nu\colon \mathbb{C}^\times \to T$, it takes the form $(t^k, \nu_0(t))$, where $\nu_0\colon \mathbb{C}^\times \to T_0$. Assume this one parameter subgroup is generic, meaning that its fixed point locus in $\mathfrak{M}^\theta(n,r)$ coincides with that for T. We will assume that $k > 0$. To this subgroup one can assign the category $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ of certain $\mathcal{A}_c(n,r)$-modules. If $c \notin (-r, 0)$ or has denominator $> n$ (or is irrational), the irreducible objects in this category are labeled by the r-multipartitions of n. We can analogously define the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r))$. Recall that we have an isomorphism $\mathcal{A}_c(n,r) \simeq D(\mathbb{C}) \otimes \overline{\mathcal{A}}_c(n,r)$. It is clear that we have a label-preserving equivalence

$$\mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r)) \xrightarrow{\sim} \mathcal{O}_\nu(\mathcal{A}_c(n,r)), \quad M \mapsto \mathbb{C}[x] \otimes M,$$

where $\mathbb{C}[x]$ is the standard polynomial representation of the Weyl algebra $D(\mathbb{C})$, so the computation of the character of a module from $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ boils down to the same computation for the corresponding module in $\mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r))$. See Section 3.3.2 for references on categories $\mathcal{O}$.

In the special case when $\nu_0(t) = \mathrm{diag}(t^{d_1}, \ldots, t^{d_r})$ with $d_1 \gg d_2 \gg \ldots \gg d_r$, the third named author computed the GK dimensions of the irreducible modules in $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$, see [75, Section 6]. Assume that $c = \frac{m}{n}$ but m and n are no longer coprime. Let $d := \gcd(m,n)$. The minimal possible GK dimension of a module in $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ is then d and the simple modules with this GK dimension are precisely those labeled by r-multipartitions of the form $(\varnothing, \ldots, \varnothing, n_0\lambda)$. Here $n_0 := n/d$ and λ is a partition of d, by $n_0\lambda$ we mean the partition of n obtained from λ multiplying all parts by n_0. [1]

Using quantum Hamiltonian reduction, in Section 3.4 we define a certain simple module over the algebra $\overline{\mathcal{A}}_c(n,r)$ to be denoted $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$. We will see that it actually lies in the category $\mathcal{O}$ corresponding to any ν of the form $(t^k, \nu_0(t))$ for $k > 0$ and prove that $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is labeled by $(\varnothing, \ldots, \varnothing, n_0\lambda)$. Recall that $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is naturally a q-graded $\mathfrak{sl}_r$-module. We will show that the action of $\mathfrak{sl}_r$ on $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is integrable and induces a natural action $\mathrm{GL}_r \curvearrowright \overline{L}_{\frac{m}{n},r}(n_0\lambda)$. However, the q-grading on $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ does not necessarily integrate to a $\mathbb{C}^\times$-action.

Theorem 3.1.4 *The q-graded GL_r-character of $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is given by the following formula:*

$$\mathrm{ch}_{q,\mathrm{GL}_r}(\overline{L}_{\frac{m}{n},r}(n_0\lambda)) = \\ = (1-q^{-1}) \sum_{\substack{\mathrm{r}(\mu) \leqslant \min(n,r) \\ \mu,\beta \vdash m}} c^\beta_{\lambda,m_0} q^{-\frac{m-1}{2} + \frac{n}{m}\kappa(\beta)} \langle s_\beta \left[\frac{X}{1-q^{-1}}\right], s_\mu \rangle [W_r(\mu)^*],$$

where $\langle\, , \rangle$ is the Hall inner product on the algebra of symmetric functions Λ, we use plethystic notation, $\kappa(\beta)$ is the sum of contents of all boxes of the diagram β and the constants c^β_{λ,m_0} are defined as follows: $s_\lambda(x_1^{m_0}, x_2^{m_0}, \ldots) = \sum_\beta c^\beta_{\lambda,m_0} s_\beta(x_1, x_2, \ldots)$, where $m_0 := m/\gcd(m,n)$.

[1] Note that in [75] minimally supported modules are labeled by $(n_0\lambda, \varnothing, \ldots, \varnothing)$. The discrepancy here appears because there is a sign mistake in the proof of [75, Proposition 3.5] that leads to a reversal of the labeling. We fix the proof of [75, Proposition 3.5] in Proposition 3.3.4.

We remark that we have an isomorphism $\overline{\mathcal{A}}_c(n,r) \simeq \overline{\mathcal{A}}_{-c-r}(n,r)$, see for example [75, Lemma 3.1]. Thus, we will always assume $c \geqslant 0$ unless otherwise explicitly stated.

3.1.4 Organization of this Chapter

In Section 3.2, we define an associative algebra $H_c(n,r)$, which is isomorphic to the rational Cherednik algebra of $\mathfrak{sl}_n$ for $r = 1$. We study the representation theory of the algebra $H_c(n,r)$ and use it to prove Theorem 3.1.2. We also give a combinatorial interpretation to the dimension of $\overline{L}_{\frac{m}{n},r}$ in terms of parking functions, see Theorem 3.2.28. In Section 3.3, we use Theorem 3.1.2 to simplify the proof of the localization theorem for $\mathfrak{M}^\theta(n,r)$ given in [75]. Section 3.4 is devoted to the study of representations of $\mathcal{A}_c(n,r)$ with minimal support. We construct these representations explicitly as Hamiltonian reductions of certain D-modules on R. In Section 3.5, we prove Theorem 3.1.4.

3.2 Characters of finite-dimensional representations

In this section, we compute the character of the representation $\overline{L}_{\frac{m}{n},r}$ by means of a construction similar to that introduced in [22, Section 9]. This will lead us to study an algebra $H_c(n,r)$, explicitly defined by generators and relations, that is very similar to the rational Cherednik algebra that we obtain when setting $r = 1$. Our character computation will follow from our study of the representation theory of this algebra. For this reason, we first review the case of rational Cherednik algebras, which is well-known in the literature.

3.2.1 The case $r = 1$

In the $r = 1$ case, the algebra $\overline{\mathcal{A}}_c(n,1)$ is known to be a type A spherical rational Cherednik algebra, cf. [34], [71]. Let us define the *full* rational Cherednik algebra. The type A rational Cherednik algebra (of $\mathfrak{sl}_n$ type) is the algebra $H_c(n)$ that is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle \rtimes S_n$ by the relations

$$\textstyle\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n y_i = 0, \quad [x_i,x_j] = 0, \quad [y_i,y_j] = 0, \quad [x_i,y_j] = \frac{1}{n} - cs_{ij}, \tag{3.1}$$

where $s_{ij} \in S_n$ is the transposition $i \leftrightarrow j$ and, in the last equation, $i \neq j$. Let us remark that H_c contains a remarkable Euler element $\mathbf{h} := \frac{1}{2}\sum(x_iy_i + y_ix_i)$. This element satisfies $[\mathbf{h},x_i] = x_i$, $[\mathbf{h},y_i] = -y_i$ and $[\mathbf{h},w] = 0$ for $w \in S_n$. In particular, it gives a grading on H_c, and every finite-dimensional representation of H_c is graded by eigenvalues of $\mathbf{h}$.

We define a category $\mathcal{O}_c = \mathcal{O}(H_c)$ over H_c as the category of finitely generated modules over H_c on which elements x_i act locally nilpotently. Equivalently $\mathcal{O}_c$ is the category of finitely generated modules M over H_c such that $\mathbf{h}$ acts on M with finite dimensional generalized eigenspaces and real parts of the eigenvalues of $\mathbf{h}$ on M are bounded from *above*.

Remark 3.2.1 *Note that this definition is not the standard one (as for example in [6], [37]), where we ask y_i to act locally nilpotently or equivalently real parts of the eigenvalues of* $\mathbf{h}$ *on M to be bounded from* below.

If τ is a finite dimensional module over S_n, then we can extend it to a module $\widetilde{\tau}$ over $\mathbb{C}[x_1,\dots,x_n]\rtimes S_n$ by letting x_i act via 0 and define the *standard* H_c-module $M_c(\tau)$ as follows: $M_c(\tau) := H_c \otimes_{\mathbb{C}[x_1,\dots,x_n]\rtimes S_n} \widetilde{\tau}$. One can easily check that $M_c(\tau)\in\mathcal{O}_c$.

The finite-dimensional representations of the algebra $H_c(n)$ have been extensively studied from algebraic, combinatorial and geometric points of view, see [6], [40], [41], [113], for example.

Theorem 3.2.2 ([6], [41]) *The algebra $H_c(n)$ admits a finite-dimensional representation if and only if $c=\frac{m}{n}$ with $\gcd(m,n)=1$. If this is the case, there is a unique irreducible finite-dimensional representation that we will denote $\overline{F}_{\frac{m}{n}}$, and any other finite-dimensional representation is a direct sum of copies of $\overline{F}_{\frac{m}{n}}$. Moreover, if $m>0$, then the graded decomposition of $\overline{F}_{\frac{m}{n}}$ as an S_n-module is given by*

$$[\overline{F}_{\frac{m}{n}}] = \frac{1}{[m]_q}\bigoplus_{\lambda\vdash n} s_\lambda(q^{\frac{1-m}{2}},q^{\frac{3-m}{2}},\dots,q^{\frac{m-1}{2}})[V_\lambda], \tag{3.2}$$

where V_λ is the irreducible S_n-module labeled by the partition λ, s_λ is the corresponding Schur function and our normalization of quantum numbers is $[z]_q = \frac{q^{\frac{z}{2}}-q^{-\frac{z}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$.

Let us remark that the q-number $s_\lambda(q^{\frac{1-m}{2}},\dots,q^{\frac{m-1}{2}})$ can be explicitly computed via the following hook-length formula, see e.g. [96]:

$$s_\lambda(q^{\frac{1-m}{2}},\dots,q^{\frac{m-1}{2}}) = \prod_{(i,j)\in\lambda}\frac{[m+i-j]_q}{[h(i,j)]_q},$$

where $h(i,j)$ is the hook-length of the box $(i,j)\in\lambda$. In particular, $s_\lambda(q^{\frac{m-1}{2}},\dots,q^{\frac{1-m}{2}})=0$ if the partition λ has more than m rows.

Let us now elaborate on the connection between $H_c(n)$ and the algebra $\overline{\mathcal{A}}_c(n,1)$. Note that the algebra $H_c(n)$ contains the (trivial) idempotent $\mathbf{e} := \frac{1}{n!}\sum_{w\in S_n} w$ of S_n. So we can form the spherical subalgebra $H_c^{\mathrm{sph}}(n) := \mathbf{e}H_c(n)\mathbf{e}$. According to [34], [71], the algebras $H_c^{\mathrm{sph}}(n)$ and $\overline{\mathcal{A}}_c(n,1)$ are isomorphic. Thus, we have

$$\overline{L}_{\frac{m}{n},1} = \overline{F}_{\frac{m}{n}}^{S_n}$$

and the q-character of $\overline{L}_{\frac{m}{n},1}$ is given by

$$\frac{1}{[m]_q}s_{(n)}(q^{\frac{1-m}{2}},\dots,q^{\frac{m-1}{2}}) = \frac{1}{[m]_q}\begin{bmatrix} n+m-1 \\ n\end{bmatrix}_q.$$

3.2.2 The Calaque-Enriquez-Etingof construction

Our approach to the computation of the character of the module $\overline{L}_{\frac{m}{n},r}$ is based on a construction from [22, Section 9] that gives a construction of certain representations of type A rational Cherednik algebras via equivariant D-modules. Let us denote by $\chi\colon\mathfrak{gl}_n\to\mathbb{C}$ the character $\chi := \frac{m}{n}\,\mathrm{tr}$. Let M be a χ-twisted equivariant D-module on $\mathfrak{sl}_n$. Recall that this means that M is a D-module on $\mathfrak{sl}_n$ with a compatible GL_n-action, satisfying $\xi_R - \xi_M = \chi$ for every $\xi \in \mathfrak{gl}_n$. Note that, since constant matrices act trivially on $\mathfrak{sl}_n$, this implies that if $\xi = \mathrm{diag}(a, \dots, a)$, then $\xi_M = -am\,\mathrm{Id}_M$.

It follows that the invariant space $(M \otimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)])^{\mathrm{GL}_n}$ coincides with $(M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\mathrm{GL}_n}$. Moreover, since $\xi_R = 0$ for every constant matrix ξ and M is χ-equivariant, $\xi(M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)) = 0$ for every constant matrix ξ and it follows that

$$(M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\mathrm{GL}_n} = (M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\mathfrak{gl}_n} = (M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\mathfrak{sl}_n}.$$

In other words, $(M \otimes S^m \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\mathfrak{sl}_n}$ is a $\overline{\mathcal{A}}_{\frac{m}{n}}(n, r)$-module. Note that here we do not need to assume that n and m are coprime.

It will be convenient to study the larger space $(M \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)^{\otimes m})^{\mathfrak{sl}_n}$. To ease the notation, let us set $U := \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$. We also set $F_{n,m,r}(M) := (M \otimes U^{\otimes m})^{\mathfrak{sl}_n}$. A priori, $F_{n,m,r}$ is a functor from the category of χ-equivariant D-modules on $\mathfrak{sl}_n$ to the category of vector spaces, but we will put some extra structure on $F_{n,m,r}(M)$. First, for a matrix $P \in \mathfrak{gl}_n$, left multiplication by P defines a map that we will denote $P \colon U \to U$. Moreover, for $i = 1, \dots, m$, we denote $(P)_i \colon U^{\otimes m} \to U^{\otimes m}$ the map given by left multiplication by P on the i-th tensor factor.

We will also consider a pair of bases (ρ_j), (ρ^j) of $\mathfrak{sl}_n$ that are dual with respect to the trace form. We will need to make a distinction and consider $\rho_j \in \mathfrak{sl}_n$, $\rho^j \in \mathfrak{sl}_n^*$. In particular, ρ^j is a coordinate function on the space $\mathfrak{sl}_n$, while ρ_j can be thought of as a differentiation with respect to ρ^j. We will think of $\rho_j \in D(\mathfrak{sl}_n)$ as a degree 1 differential operator, and of $\rho^j \in D(\mathfrak{sl}_n)$ as a degree 0 differential operator. Clearly, $[\rho_i, \rho^j] = \delta_{ij}$.

Finally, for $\ell_1 \neq \ell_2$, let $\mathfrak{c}^{\ell_1,\ell_2} \colon U^{\otimes m} \to U^{\otimes m}$ denote the operator that acts as $\sum_{i,j=1}^r E_{ij} \otimes E_{ji}$ on the ℓ_1, ℓ_2-tensor factors of $U^{\otimes m}$. Here, $E_{ij} \in \mathrm{End}(\mathbb{C}^r)$ is given by $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$, and $Q \in \mathrm{End}(\mathbb{C}^r)$ acts on U by multiplication by Q^t on the right.

Proposition 3.2.3 *For $\ell = 1, \dots, m$, define the following operators on $F_{n,m,r}(M)$:*

$$X_\ell := \sum_j \rho^j \otimes (\rho_j)_\ell, \qquad Y_\ell := \frac{n}{m} \sum_j \rho_j \otimes (\rho^j)_\ell.$$

These operators satisfy the following relations:

$$\sum_{\ell=1}^m X_\ell = 0, \quad \sum_{\ell=1}^m Y_\ell = 0, \tag{3.3}$$

$$[X_{\ell_1}, X_{\ell_2}] = 0, \quad [Y_{\ell_1}, Y_{\ell_2}] = 0, \tag{3.4}$$

$$[X_{\ell_1}, Y_{\ell_2}] = \frac{1}{m} - \frac{n}{m} \mathfrak{c}^{\ell_1,\ell_2} s_{\ell_1,\ell_2}, \quad \ell_1 \neq \ell_2, \tag{3.5}$$

where s_{ℓ_1,ℓ_2} is the operator that permutes the ℓ_1, ℓ_2-tensor factors in $U^{\otimes m}$.

Proof: A direct computation. Relations (3.3) follow from $\mathfrak{sl}_n$-invariance. Relations (3.4) are obvious. Finally, for (3.5), we have

$$\begin{aligned} \tfrac{m}{n}[X_{\ell_1}, Y_{\ell_2}] &= \textstyle\sum_{i,j} [\rho^j \otimes (\rho_j)_{\ell_1}, \rho_i \otimes (\rho^i)_{\ell_2}] \\ &= \textstyle\sum_{i,j} [\rho^j, \rho_i] \otimes (\rho_j)_{\ell_1} (\rho^i)_{\ell_2} \\ &= \textstyle\sum_j [\rho^j, \rho_j] \otimes (\rho_j)_{\ell_1} (\rho^j)_{\ell_2} \\ &= -\textstyle\sum_j 1 \otimes (\rho_j)_{\ell_1} (\rho^j)_{\ell_2} \end{aligned}$$

and the result follows from the fact that $\sum_j (\rho_j)_{\ell_1}(\rho^j)_{\ell_2} = \mathfrak{c}^{\ell_1,\ell_2} s_{\ell_1,\ell_2} - \frac{1}{n} \colon U^{\otimes m} \to U^{\otimes m}$, which is straightforward. □

Note that on $(M \otimes U^{\otimes m})^{\mathfrak{sl}_n}$ we also have an action of $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$ by multiplying on the right by the transpose on the $U^{\otimes m}$ tensor factor, as well as an action of S_m by permuting the tensor factors on $U^{\otimes m}$. The action of $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$ commutes with the action of $X_1, \dots, X_m$, $Y_1, \dots, Y_m$, and S_m satisfies the obvious commutation relations with X's, Y's and $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$. This motivates the following definition.

Definition 3.2.4 *Let $m, r \in \mathbb{Z}_{>0}$ and $c \in \mathbb{C}$. We define the algebra $H_c(m,r)$ to be the quotient of the semi-direct product algebra*

$$(\mathbb{C}\langle x_1, \dots, x_m, y_1, \dots, y_m\rangle \otimes \mathrm{End}(\mathbb{C}^r)^{\otimes m}) \rtimes S_m$$

by the relations

$$\sum x_\ell = 0, \quad \sum y_\ell = 0, \tag{3.6}$$

$$[x_{\ell_1}, x_{\ell_2}] = 0, \quad [y_{\ell_1}, y_{\ell_2}] = 0, \tag{3.7}$$

$$[x_{\ell_1}, y_{\ell_2}] = \frac{1}{m} - c \sum_{i,j=1}^{r} (E_{ij})_{\ell_1} (E_{ji})_{\ell_2} s_{\ell_1,\ell_2}, \quad \ell_1 \neq \ell_2. \tag{3.8}$$

Example 3.2.5 *When $r = 1$, $H_c(m,1)$ is nothing but the rational Cherednik algebra $H_c(m)$ defined in Section 3.2.1.*

Then Proposition 3.2.3 can be reinterpreted as follows.

Proposition 3.2.6 *The correspondence sending M to $F_{n,m,r}(M)$ defines a functor*

$$F_{n,m,r} \colon D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi} \to H_{\frac{n}{m}}(m,r)\text{-mod}\,.$$

3.2.3 The Dunkl embedding for $H_c(m,r)$

Proposition 3.2.6 motivates the study of the algebra $H_c(m,r)$ and its representation theory. As it turns out, the algebras $H_c(m,r)$ and $H_c(m,1)$ are Morita equivalent. Before stating our result, let us establish some structural properties of the algebra $H_c(m,r)$. In particular, we will define a polynomial representation for $H_c(m,r)$.

Let $\mathfrak{h}$ denote the $(m-1)$-dimensional reflection representation of the symmetric group S_m. Recall that the algebra $H_c(m,1)$ acts on $\mathbb{C}[\mathfrak{h}]$, with $x_1, \dots, x_m$ acting by multiplication and $y_1, \dots, y_m$ acting by Dunkl operators. Let us denote these operators on $\mathbb{C}[\mathfrak{h}]$ by $\mathsf{x}_1, \dots, \mathsf{x}_m, \mathsf{y}_1, \dots, \mathsf{y}_m$.

Proposition 3.2.7 *The algebra $H_c(m,r)$ acts on the space $\mathbb{C}[\mathfrak{h}] \otimes (\mathbb{C}^r)^{\otimes m}$ as follows:*

- *$x_1, \dots, x_m, y_1, \dots, y_m$ act by $\mathsf{x}_1 \otimes 1, \dots, \mathsf{x}_m \otimes 1, \mathsf{y}_1 \otimes 1, \dots, \mathsf{y}_m \otimes 1$, respectively.*
- *$A_1 \otimes \cdots \otimes A_m \in \mathrm{End}(\mathbb{C}^r)^{\otimes m}$ acts by $1 \otimes A_1 \otimes \cdots \otimes A_m$.*

- *S_m acts diagonally.*

Proof: A direct computation, the main point here is that $\sum E_{ij} \otimes E_{ji}$ acts on $\mathbb{C}^r \otimes \mathbb{C}^r$ by switching the tensor factors. □

Let us now denote by $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$ the locus where the S_m-action is free, note that this is a principal open set. It is well-known that the algebra $H_c(m,1)$ admits an embedding $H_c(m,1) \hookrightarrow D(\mathfrak{h}^{\text{reg}}) \rtimes S_m$, by interpreting $x_1, \ldots, x_m$ as functions on $\mathfrak{h}$ and $y_1, \ldots, y_m$ as Dunkl operators. Thanks to Proposition 3.2.7, the algebra $H_c(m,r)$ admits a similar embedding to $(D(\mathfrak{h}^{\text{reg}}) \otimes \operatorname{End}(\mathbb{C}^r)^{\otimes m}) \rtimes S_m$. Since the action of $(D(\mathfrak{h}^{\text{reg}}) \otimes \operatorname{End}(\mathbb{C}^r)^{\otimes m}) \rtimes S_m$ on $\mathbb{C}[\mathfrak{h}] \otimes (\mathbb{C}^r)^{\otimes m}$ is faithful, this has the following consequence.

Proposition 3.2.8 (PBW property for $H_c(m,r)$) *Multiplication induces an isomorphism $S(\mathfrak{h}^*) \otimes (\operatorname{End}(\mathbb{C}^r)^{\otimes m} \rtimes S_m) \otimes S(\mathfrak{h}) \xrightarrow{\sim} H_c(m,r)$.*

Corollary 3.2.9 *Let $E := E_{11}^{\otimes m} \in \operatorname{End}(\mathbb{C}^r)^{\otimes m}$. Then E is an idempotent, $H_c(m,r)EH_c(m,r) = H_c(m,r)$ and $EH_c(m,r)E \cong H_c(m,1)$. In particular, $H_c(m,r)$ is Morita equivalent to the usual type A rational Cherednik algebra $H_c(m)$.*

Proof: The first two assertions are clear. For the last assertion, define a map

$$\mathbb{C}\langle x_1, \ldots, x_m, y_1, \ldots, y_m \rangle \rtimes S_m \to EH_c(m,r)E$$

by sending $x_\ell \mapsto Ex_\ell E = x_\ell E$, $y_\ell \mapsto Ey_\ell E = y_\ell E$ and $w \mapsto EwE = wE$ for $w \in S_m$. It is easy to check that this map factors through an algebra homomorphism $H_c(m,1) \to EH_c(m,r)E$. That it is an isomorphism follows from the PBW property. □

3.2.4 Representation theory of $H_c(m,r)$

Thanks to Corollary 3.2.9, the algebras $H_c(m,r)$ and $H_c(m,1)$ are Morita equivalent. In fact, we have a very concrete realization of this Morita equivalence, that generalizes Proposition 3.2.7.

Proposition 3.2.10 *Let $N \in H_c(m,1)\text{-mod}$. Denote the action of $x_1, \ldots, x_m, y_1, \ldots, y_m \in H_c(m,1)$ on N by $\mathsf{x}_1, \ldots \mathsf{x}_m$, $\mathsf{y}_1, \ldots \mathsf{y}_m$, respectively. Define $\Phi(N) := N \otimes (\mathbb{C}^r)^{\otimes m}$. Then $\Phi(N)$ becomes a $H_c(m,r)$-module by the same formulas as those in Proposition 3.2.7, and $\Phi \colon H_c(m,1)\text{-mod} \to H_c(m,r)\text{-mod}$ is an inverse to the functor $E \colon H_c(m,r)\text{-mod} \to H_c(m,1)\text{-mod}$.*

Proof: That the formulas do define an action of $H_c(m,r)$ on $\Phi(N)$ is a straightforward direct computation. Note that $E(N \otimes (\mathbb{C}^r)^{\otimes m}) = N$. So the functor $N \mapsto \Phi(N)$ is a right inverse to the Morita equivalence of Corollary 3.2.9 and the result follows. □

Thanks to Theorem 3.2.2 we can see the following.

Corollary 3.2.11 *The algebra $H_c(m,r)$ admits a finite-dimensional representation if and only if $c = \frac{n}{m}$ with $\gcd(m,n) = 1$. Moreover, the unique irreducible finite-dimensional representation is $\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^r)^{\otimes m}$ where, recall, $\overline{F}_{\frac{n}{m}}$ is the unique irreducible finite-dimensional representation of $H_{\frac{n}{m}}(m,1)$.*

Let us now see that we can get the unique irreducible finite-dimensional representation of $H_{\frac{n}{m}}(m, r)$ from a D-module on $\mathfrak{sl}_n$ via the functor $F_{n,m,r}$.

Proposition 3.2.12 *The equivalence in Corollary 3.2.9 intertwines the functors $F_{n,m,r}$ and $F_{n,m,1}$, that is, the following diagram commutes:*

$$\begin{array}{ccc} & D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi} & \\ {\scriptstyle F_{n,m,r}}\swarrow & & \searrow{\scriptstyle F_{n,m,1}} \\ H_{\frac{n}{m}}(m,r)\text{-mod} & \xrightarrow{M \mapsto EM} & H_{\frac{n}{m}}(m,1)\text{-mod} \end{array}$$

Proof: Since $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)E_{11} = \mathbb{C}^n$, it follows that $EF_{n,m,r}(M) = (M \otimes (\mathbb{C}^n)^{\otimes m})^{\mathfrak{sl}_n}$, with the correct formulas for the action of $x_1, \dots, x_m$, $y_1, \dots, y_m$. □

Now assume m and n are coprime and let O be the regular nilpotent orbit in $\mathfrak{sl}_n$. Consider the rank one local system on O that corresponds to the representation of the center $Z(\mathrm{SL}_n) \subset \mathrm{SL}_n$ given by $\mathrm{diag}(z, \dots, z) \to z^{-m}$, and let $M(O)$ be its minimal extension. This is an SL_n-equivariant D-module on $\mathfrak{sl}_n$. Extend the SL_n-action on $M(O)$ to a GL_n-action by requiring a matrix $\mathrm{diag}(a, \dots, a)$ to act by multiplication by a^{-m}. This makes $M(O)$ a χ-equivariant D-module on $\mathfrak{sl}_n$. The next result is now a consequence of Proposition 3.2.12 and [22, Section 9.12]

Corollary 3.2.13 *The module $F_{n,m,r}(M(O))$ is the unique irreducible finite-dimensional representation of $H_{\frac{n}{m}}(m, r)$. In particular, $F_{n,m,r}(M(O)) = \overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^r)^{\otimes m}$.*

Remark 3.2.14 *Let us compare the functor $F_{n,m,1}$ to that used in the work of Calaque-Enriquez-Etingof [22]. It is easy to see that, if M is a χ-equivariant D-module on $\mathfrak{sl}_n$ supported on the nilpotent cone $\mathcal{N}$, the action of $x_1, \dots, x_m \in H_{\frac{n}{m}}(m, r)$ on $F_{n,m,r}(M)$ is locally nilpotent and the action of the Euler element $\mathbf{h} = \frac{1}{2}\sum x_i y_i + y_i x_i$ is locally finite. In other words, the module $F_{n,m,r}(M)$ belongs to the category $\mathcal{O}$ of* highest *weight modules.*

In [22], the authors consider the functor $F^*_{n,m,1} := F_{n,m,1} \circ \mathcal{F}$, where $\mathcal{F}$ is the usual Fourier transform on D-modules. The reason is that, if M is supported on the nilpotent cone $\mathcal{N}$, then the action of $y_1, \dots, y_n$ on $F^*_{n,m,1}(M)$ is locally nilpotent, so $F^*_{n,m,1}(M)$ belongs to the category of lowest-*weight modules for $H_{\frac{n}{m}}(m, 1)$, which is more common in the Cherednik algebra literature, see Remark 3.2.1 above. Note, however, that since the D-module $M(O)$ considered in Corollary 3.2.13 is cuspidal, both $M(O)$ and $\mathcal{F}(M(O))$ are supported on the nilpotent cone, and the same reasoning as in [22, Section 9.12] implies that $F_{n,m,1}(M(O))$ is a finite-dimensional irreducible representation of $H_{\frac{n}{m}}(m, 1)$.*

3.2.5 Spherical subalgebra

Note that $H_c(m, r)$ contains the idempotent $\mathbf{e} := \frac{1}{m!}\sum_{w \in S_m} w$, so we have the spherical subalgebra $H_c^{\mathrm{sph}}(m, r) := \mathbf{e}H_c(m, r)\mathbf{e}$. As usual, we have a quotient functor $N \mapsto \mathbf{e}N = N^{S_m}$, $H_c(m, r)\text{-mod} \to H_c^{\mathrm{sph}}(m, r)\text{-mod}$ that is an equivalence provided $\mathbf{e}N \neq 0$ for every $N \in H_c(m, r)\text{-mod}$.

Proposition 3.2.15 *Assume that $c \notin (-1,0)$ or that $r \geqslant m$. Then the algebras $H_c(m,r)$ and $H_c^{sph}(m,r)$ are Morita equivalent. In particular, if $c = \frac{n}{m} > 0$ with $\gcd(m,n) = 1$, we have that $H_c^{sph}(m,r)$ admits a unique irreducible finite-dimensional representation, given by $\mathbf{e}(\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^r)^{\otimes m}) = (\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$.*

Proof: The case $c \notin (-1,0)$ follows from [8, Corollary 4.2], as follows. We need to show that $\mathbf{e}\widetilde{N} = \widetilde{N}^{S_m} \neq 0$ for every $\widetilde{N} \in H_c(m,r)$-mod. By Proposition 3.2.10, $\widetilde{N} = N \otimes (\mathbb{C}^r)^{\otimes m}$ for some $N \in H_c(m,1)$-mod. Now, by [8, Corollary 4.2], $N^{S_m} \neq 0$ provided $c \notin (-1,0)$. Then $0 \neq N^{S^m} \otimes S^m(\mathbb{C}^r) \subseteq \widetilde{N}^{S_m}$ and we are done.
If $r \geqslant m$, by Schur-Weyl duality we have that every irreducible representation of S_m appears with nonzero multiplicity in $(\mathbb{C}^r)^{\otimes m}$. So, using the notation of the previous paragraph, $\widetilde{N}^{S_m} = (N \otimes (\mathbb{C}^r)^{\otimes m})^{S_m} \neq 0$ for every nonzero $\widetilde{N} \in H_c(m,r)$-mod and the result now follows from Proposition 3.2.10. □

3.2.6 Functor $F^{\mathrm{sph}}_{n,m,r}$ vs. Hamiltonian reduction

Note that we have a functor $F^{\mathrm{sph}}_{n,m,r} \colon D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi} \to H^{\mathrm{sph}}_{\frac{n}{m}}(m,r)\text{-mod}$ which is defined by $F^{\mathrm{sph}}_{n,m,r} := \mathbf{e}F_{n,m,r}$. By the definition of the functor $F_{n,m,r}$, we have that $F^{\mathrm{sph}}_{n,m,r}(M) = (M \otimes S^m U)^{\mathfrak{sl}_n}$.

On the other hand, we have a Hamiltonian reduction functor

$$\mathbb{H} \colon D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi} \to \overline{\mathcal{A}}_{\frac{m}{n}}(n,r)\text{-mod},$$

given by taking the invariant space $\mathbb{H}(M) := (M \otimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)])^{\mathrm{GL}_n}$. Thanks to the discussion at the beginning of Section 3.2.2 we have that, as vector spaces, $\mathbb{H}(M) = F^{\mathrm{sph}}_{n,m,r}(M)$ for every $M \in D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi}$.

We claim that even more is true. Note that by Propositions 3.2.12 and 3.2.15, another formula for the functor $F^{\mathrm{sph}}_{n,m,r}$ is given by

$$F^{\mathrm{sph}}_{n,m,r}(M) = (F_{n,m,1}(M) \otimes (\mathbb{C}^r)^{\otimes m})^{S_m}$$

the space $F_{n,m,1}(M)$ has a grading coming from the Euler operator in $H_{\frac{n}{m}}(m,1)$, while the space $(\mathbb{C}^r)^{\otimes m}$ has a natural GL_r-action. Both the action of the Euler operator and the GL_r-action commute with the action of S_m, so we get commuting q-grading and GL_r-action on $F^{\mathrm{sph}}_{n,m,r}(M)$.

Remark 3.2.16 *It follows from [22, Section 8] that the action of the Euler element $\mathbf{h}$ on $F_{n,m,1}(M)$ extends to an action of $\mathfrak{sl}_2$ on $F_{n,m,1}(M)$ so if $F_{n,m,1}(M)$ is finite-dimensional, then our q-grading integrates to an action of $\mathbb{C}^\times$ on $F_{n,m,1}(M)$.*

Theorem 3.2.17 *Let $M \in D(\mathfrak{sl}_n)\text{-mod}^{\mathrm{GL}_n,\chi}$. Then, as q-graded $\mathfrak{sl}_r$-modules,*

$$\mathbb{H}(M) = (F_{n,m,1}(M) \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m}.$$

In particular, the $\mathfrak{sl}_r$-action on $\mathbb{H}(M)$ integrates to a SL_r-action that can be extended to a GL_r-action in a natural way. The q-grading integrates to a $\mathbb{C}^\times$-action provided the same is true for the $H_{\frac{n}{m}}(m,1)$-module $F_{n,m,1}(M)$.

Proof: Let us deal with the $\mathfrak{sl}_r$-action. First, we will consider the action of $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$. The action of $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$ on $\mathbb{H}(M) = (M \otimes S^m U)^{\mathfrak{sl}_n}$ is induced from the action on $U = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, so the embedding $\mathbb{H}(M) \hookrightarrow (M \otimes U^{\otimes m})^{\mathfrak{sl}_n} = F_{n,m,r}(M)$ is $\mathrm{End}(\mathbb{C}^r)^{\otimes r}$-equivariant. Now, the isomorphism $F_{n,m,r}(M) = F_{n,m,1}(M) \otimes (\mathbb{C}^r)^{\otimes m}$ is that of $H_{\frac{n}{m}}(m,r)$-modules, and is therefore both S_m and $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$-equivariant. Thus, as $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$-modules, we have that $\mathbb{H}(M)$ gets identified with the S_m-invariant part of $F_{n,m,1}(M) \otimes (\mathbb{C}^r)^{\otimes m}$.

Note, however, that the action of $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$ on $U^{\otimes m}$ is by multiplication by the transpose on the right, and thus it is not compatible with the action of $\mathfrak{sl}_r$. To fix this, one needs to first apply the automorphism $\xi \mapsto -\xi$ of $\mathfrak{sl}_r$. This induces the antipodal map at the level of the enveloping algebra $\mathcal{U}(\mathfrak{sl}_r)$ and thus we get $\mathbb{H}(M) = (F_{n,m,1} \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m}$ as $\mathfrak{sl}_r$-modules, as desired. The claim about the integrability of the $\mathfrak{sl}_r$-action follows easily. The action of SL_r on $(\mathbb{C}^{r*})^{\otimes m}$ naturally extends to an action of GL_r.

Note that the isomorphism $F_{n,m,r}(M) \cong F_{n,m,1}(M) \otimes (\mathbb{C}^r)^{\otimes m}$ is that of $H_{\frac{n}{m}}(m,r)$-modules and therefore preserves q-gradings. So it remains to show that the embedding $\mathbb{H}(M) \hookrightarrow F_{n,m,1}(M) \otimes (\mathbb{C}^r)^{\otimes m}$ that we produced in the first paragraph of this proof also preserves the q-gradings. The q-grading on $\mathbb{H}(M)$ is induced by the action of the operator $\sum_j \rho^j \rho_j$ where, recall, $\rho_j \in \mathfrak{sl}_n$, $\rho^j \in \mathfrak{sl}_n^*$ are a pair of dual bases. The coincidence of the actions now follows from the formulas in Proposition 3.2.3, see e.g. [22, Proposition 8.7]. Finally, as q-graded S_m-modules we clearly have $F_{n,m,1} \otimes (\mathbb{C}^r)^{\otimes m} = F_{n,m,1} \otimes (\mathbb{C}^{r*})^{\otimes m}$ and the claim follows. $\square$

Note that, since we are assuming that $c = \frac{m}{n} \geqslant 0$, we are always in one of the cases considered in Proposition 3.2.15 and therefore we have that $\mathbb{H}(M) \neq 0$ if and only if $F_{n,m,1}(M) \neq 0$.

Now let O be the regular nilpotent orbit in $\mathfrak{sl}_n$, and $M(O)$ the irreducible χ-equivariant D-module on $\mathfrak{sl}_n$ associated to O. Then $M(O) \otimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)]$ is an irreducible χ-equivariant $D(\overline{R})$-module and, since $\mathbb{H}$ is a quotient functor, $\mathbb{H}(M(O)) = (F_{\frac{n}{m}} \otimes (\mathbb{C}^r)^{\otimes m})^{S_m} \neq 0$ is an irreducible $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-module, which is finite-dimensional and therefore isomorphic to $\overline{L}_{\frac{m}{n},r}$. By Remark 3.2.16 the q-grading on $\overline{L}_{\frac{m}{n},r}$ integrates to a $\mathbb{C}^\times$-action. Then we obtain.

Corollary 3.2.18 *As $\mathbb{C}^\times \times \mathrm{GL}_r$-modules,*

$$\overline{L}_{\frac{m}{n},r} \cong (\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m}.$$

Let us explicitly compute the $\mathbb{C}^\times \times \mathrm{GL}_r$-character of $\overline{L}_{\frac{m}{n},r}$. From (3.2) we have that the $\mathbb{C}^\times$-character of $\overline{F}_{\frac{n}{m}}$ is

$$\frac{1}{[n]_q} \bigoplus_{\lambda \vdash m} s_\lambda(q^{\frac{1-n}{2}}, \ldots, q^{\frac{n-1}{2}}) V_\lambda.$$

Now, for a partition λ with at most r parts, denote by $W_r(\lambda)$ the irreducible GL_r-summand of $(\mathbb{C}^r)^{\otimes m}$ indexed by λ. Then, by Schur-Weyl duality, we can express the character of $\overline{L}_{\frac{m}{n},r}$ by

$$\frac{1}{[n]_q} \bigoplus_{\substack{\lambda \vdash m \\ r(\lambda) \leqslant r}} s_\lambda(q^{\frac{1-n}{2}}, \ldots, q^{\frac{n-1}{2}}) W_r(\lambda)^*,$$

where $r(\lambda)$ is the number of rows of λ. Even more is true. The Schur function $s_\lambda(q^{\frac{1-n}{2}}, \ldots, q^{\frac{n-1}{2}})$ is the graded dimension of the representation $W_n(\lambda)$ of $\mathfrak{gl}_n$ and therefore vanishes when

$r(\lambda) > n$. Then we obtain our character formula

$$\overline{L}_{\frac{m}{n},r} = \frac{1}{[n]_q} \bigoplus_{\substack{\lambda \vdash m \\ r(\lambda) \leqslant \min(n;r)}} s_\lambda(q^{\frac{1-n}{2}}, \dots, q^{\frac{n-1}{2}}) W_r(\lambda)^*. \tag{3.9}$$

Note that, if we ignore the $\frac{1}{[n]_q}$-factor in (3.9) we have the graded character of the $\mathrm{GL}_n \times \mathrm{GL}_r$-representation $S^m(\mathbb{C}^n \otimes \mathbb{C}^r)$, where the grading only affects the $\mathbb{C}^n$-part. It follows that the dimension of $\overline{L}_{\frac{m}{n},r}$ is $\frac{1}{n}\binom{nr+m-1}{m} = \frac{1}{n}\dim S^m(\mathbb{C}^n \otimes \mathbb{C}^r)$.

Corollary 3.2.19 *The $\mathbb{C}^\times$-character of $\overline{L}_{\frac{m}{n},r}$ is a Laurent polynomial in q that is symmetric under the change of variables $q \leftrightarrow q^{-1}$, i.e. $\mathrm{ch}_q(\overline{L}_{\frac{m}{n},r}) = \mathrm{ch}_{q^{-1}}(\overline{L}_{\frac{m}{n},r})$. Moreover, its degree is $(n-1)(m-1)/2$ and its leading coefficient is $\binom{r+m-1}{m}$.*

Proof: It is known that the action of the Euler element $\mathbf{h}$ on $\overline{F}_{\frac{n}{m}}$ extends to an action of $\mathfrak{sl}_2$ on $\overline{F}_{\frac{n}{m}}$ and, moreover, this action commutes with the action of S_m. Thus, the fact that $\mathrm{ch}_q(\overline{L}_{\frac{m}{n},r}) = \mathrm{ch}_{q^{-1}}(\overline{L}_{\frac{m}{n},r})$ follows immediately from Corollary 3.2.18. From (3.9) it follows that the degree of $\mathrm{ch}_q(\overline{L}_{\frac{m}{n},r})$ is independent of r, and in the case $r = 1$ it is easy to see that the degree is precisely $(m-1)(n-1)/2$. Finally, from (3.9) it is also easy to see that the leading coefficient is $\dim(S^m(\mathbb{C}^{r*})) = \binom{r+m-1}{m}$. $\square$

Example 3.2.20 *Consider now the example $m = 3, n = r = 2$. We have*

$$\overline{L}_{3/2,2} = W^*_{\tiny\yng(2,1)} + (q + q^{-1}) W^*_{\tiny\yng(3)}, \quad \dim \overline{L}_{3/2,2} = 10.$$

Remark 3.2.21 *Note that the roots of $\frac{1}{n}\binom{nr+m-1}{m} = \frac{1}{n}\binom{nr+m-1}{nr-1}$ considered as a polynomial of m (with fixed n) are precisely $-1, \dots, -nr+1$, i.e. we have $\dim \overline{L}_{\frac{m}{n},r} = 0$ for $-rn < m < 0$. This exactly corresponds to the fact that the algebra $\overline{\mathcal{A}}_c(n,r) = D(\mathfrak{sl}_n \times (\mathbb{C}^n)^{\oplus r}) /\!\!/\!\!/_{\frac{m}{n}} \mathrm{GL}_n$ has infinite homological dimension and does not have nontrivial finite dimensional representations in these cases, see [75, Theorems 1.1, 1.2]. We will use these observations in Section 3.3 to greatly simplify the most technical parts of the proof of Theorem 1.1 in loc.cit.*

Remark 3.2.22 *Let $\{E_{ii}\}$ be the standard basis of $\mathfrak{t}$, i.e. $\mathrm{diag}(a_1, \dots, a_r) = \sum_i a_i E_{ii}$. We can define the T-character of $\overline{L}_{\frac{n}{m},r}$ as follows:*

$$g(q, q_1, \dots, q_r) := \mathrm{Tr}_{\overline{L}_{\frac{n}{m},r}}(q^{\mathbf{h}} q_1^{E_{11}} \dots q_r^{E_{rr}}).$$

It is easy to deduce from the Cauchy identity that $g(q, q_1, \dots, q_r)$ is the coefficient in front of z^m of the following "generating function"[2]

$$D(z) = \frac{1}{[n]_q} \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant r}} \frac{1}{1 - zq^{\frac{n+1-2i}{2}} q_j^{-1}}. \tag{3.10}$$

[2]Note that if m and n are not coprime, the coefficient in front of z^m in $D(z)$ is not the character of a finite-dimensional representation and, in fact, does not need to be in $\mathbb{Z}[q, q^{-1}, q_j, q_j^{-1} \,|\, 1 \leqslant j \leqslant r]$. This is the reason why we write "generating function" inside quotation marks.

3.2.7 Semistandard parking functions and higher rank Catalan numbers

The goal of this section is to give a combinatorial interpretation of $\dim \overline{L}_{\frac{m}{n},r}$.

Definition 3.2.23 *We will call the number* $C_{\frac{m}{n},r} := \dim(\overline{L}_{\frac{m}{n},r}) = \frac{1}{n}\binom{nr+m-1}{m}$ *the rank r rational $\frac{m}{n}$-Catalan number.*

Let us, first, review the case $r=1$ which is well-known. Since m and n are coprime, the number $C_{\frac{m}{n},1} = \frac{1}{n}\binom{n+m-1}{m} = \frac{1}{n+m}\binom{n+m}{m}$ counts the number of $\frac{m}{n}$-*Dyck paths*, that is, paths in $\mathbb{Z}^2$ from $(0,0)$ to (n,m) that use only steps in the directions $(1,0)$ and $(0,1)$, and that always stay above the diagonal line $y = \frac{m}{n}x$. We will denote by $\mathcal{D}_{\frac{m}{n}}$ the set of $\frac{m}{n}$-Dyck paths.

Now let $D \in \mathcal{D}_{\frac{m}{n}}$. A *vertical run* of D is a maximal collection of consecutive vertical steps. Let $a_1, \ldots, a_\ell$ be the lengths of the vertical runs of D. Note that $a_1 + \ldots + a_\ell = m$. The following result will be very important for us.

Lemma 3.2.24 *As a representation of* S_m,

$$\overline{F}_{\frac{n}{m}} = \bigoplus_{D \in \mathcal{D}_{\frac{m}{n}}} \mathrm{Ind}_{S_{a_1}\times\cdots\times S_{a_\ell}}^{S_m} \mathrm{triv}\,.$$

Proof: Follows from [6, Proposition 1.7] and [1, Corollary 4]. □

A consequence of Lemma 3.2.24 is that, as GL_r-modules,

$$\begin{aligned}
\overline{L}_{\frac{m}{n},r} &= (\overline{F}_{\frac{n}{m}} \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m} \\
&= \mathrm{Hom}_{S_m}(\overline{F}_{\frac{n}{m}}, (\mathbb{C}^{r*})^{\otimes m}) \\
&= \bigoplus_{D\in\mathcal{D}_{\frac{m}{n}}} \mathrm{Hom}_{S_m}(\mathrm{Ind}_{S_{a_1}\times\cdots\times S_{a_\ell}}^{S_m} \mathrm{triv}, (\mathbb{C}^{r*})^{\otimes m}) \\
&= \bigoplus_{D\in\mathcal{D}_{\frac{m}{n}}} \mathrm{Hom}_{S_{a_1}\times\cdots\times S_{a_\ell}}(\mathrm{triv}, \mathrm{Res}_{S_{a_1}\times\cdots\times S_{a_\ell}}^{S_m}((\mathbb{C}^{r*})^{\otimes m})) \\
&= \bigoplus_{D\in\mathcal{D}_{\frac{m}{n}}} S^{a_1}(\mathbb{C}^{r*}) \otimes \cdots \otimes S^{a_\ell}(\mathbb{C}^{r*}).
\end{aligned} \tag{3.11}$$

Remark 3.2.25 *Note that* (3.11) *gives another expression for the* GL_r*-character of* $\overline{L}_{\frac{m}{n},r}$.

We will use equation (3.11) to give a combinatorial interpretation of $C_{\frac{m}{n},r}$. The key concept is the following.

Definition 3.2.26 *A rank r semistandard $\frac{m}{n}$-parking function consists of a pair (D,φ), where D is an $\frac{m}{n}$-Dyck path, $D \in \mathcal{D}_{\frac{m}{n}}$ and $\varphi\colon \{$vertical steps of $D\} \to \{1,\ldots,r\}$ is a function that is weakly increasing along each vertical run, reading from top-to-bottom. We will denote by $\mathcal{PF}^r_{\frac{m}{n}}$ the set of rank r semistandard $\frac{m}{n}$-parking functions.*

Remark 3.2.27 *Recall that an $\frac{m}{n}$-parking function consists of an $\frac{m}{n}$-Dyck path together with a bijection from its set of vertical steps to $\{1,\ldots,m\}$ that is strictly increasing along each vertical run. This explains the terminology in Definition 3.2.26.*

Theorem 3.2.28 *Assume m and n are coprime. Then* $|\mathcal{PF}^r_{\frac{m}{n}}| = C_{\frac{m}{n},r}$.

Proof: It is straightforward to see that the number of ways to label the vertical steps of a Dyck path $D \in \mathcal{D}_{\frac{m}{n}}$ to make a semistandard parking function of rank r is $\binom{r+a_1-1}{a_1} \times \cdots \times \binom{r+a_\ell-1}{a_\ell}$, where $a_1, \ldots, a_\ell$ are the lengths of the vertical runs of D. The result now follows from (3.11).
□

Example 3.2.29 *Let us consider the example $m = 3$, $n = r = 2$. There are $C_{\frac{3}{2},2} = 10$ $\frac{3}{2}$-semistandard parking functions of rank 2, given in Figure 3.1.*

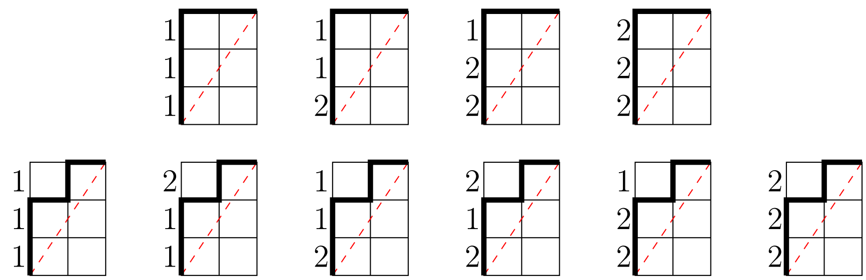


Figure 3.1: The $\frac{3}{2}$-semistandard parking functions of rank 2. Note that, if T_0 denotes a maximal torus of GL_2, then the T_0-character of $L_{\frac{3}{2},2}$ is given by $2q_1^{-3}+3q_1^{-2}q_2^{-1}+3q_1^{-1}q_2^{-2}+2q_2^{-3}$

Remark 3.2.30 *Recall that $T_0 \subseteq \mathrm{GL}_r$ denotes a maximal torus. It follows from (3.11) that the T_0-character of $\overline{L}_{\frac{m}{n},r}$ is given by*

$$\sum_{(D,\varphi)\in \mathcal{PF}^r_{\frac{m}{n}}} \prod_{i=1}^{r} q_i^{-|\varphi^{-1}(i)|},$$

see also Remark 3.2.22 above.

Remark 3.2.31 *It is an interesting question to find $|\mathcal{PF}^r_{\frac{m}{n}}|$ in the non-coprime case. It is possible that a Bizley-like formula exists for the generating function of $|\mathcal{PF}^r_{\frac{dm}{dn}}|$, but we will not pursue it here.*

3.2.8 q-analogues of $C_{\frac{m}{n},r}$

Let us compute $\mathrm{ch}_{\mathbb{C}^\times \times \mathrm{GL}_r} \overline{L}_{\frac{m}{n},r}$ in a couple of easy examples.

Example 3.2.32 *Let us consider the case $n = 3$, $m = 2$. Then*

$$\mathrm{ch}_{\mathbb{C}^\times \times \mathrm{GL}_r}(\overline{L}_{\frac{2}{3},r}) = (q + q^{-1})[S^2(\mathbb{C}^{r*})] + [\Lambda^2(\mathbb{C}^{r*})],$$

on the other hand, if $n = 2$, $m = 3$ we have

$$\mathrm{ch}_{\mathbb{C}^\times \times \mathrm{GL}_r}(\overline{L}_{\frac{3}{2},r}) = (q + q^{-1})[S^3(\mathbb{C}^{r*})] + [W_r(2,1)^*].$$

Note that the roots of $\mathrm{ch}_{\mathbb{C}^\times}(\overline{L}_{\frac{2}{3},r}) = \mathrm{Tr}_{\overline{L}_{\frac{m}{n},r}}(q^{\mathbf{h}}) = \frac{r}{2}((r+1)q + (r-1) + (r+1)q^{-1})$ are roots of unity if and only if $r = 1$. It follows, in particular, that $\mathrm{ch}_{\mathbb{C}^\times}(\overline{L}_{\frac{m}{n},r})$ does not admit an expression involving only products and quotients of q-numbers.

To remedy this, we propose an alternative evaluation of the trace that yields an expression that does factor. In fact, we have several of them, one for each divisor d of r. Let us adopt the notation of Remark 3.2.22, in particular, $g(q, q_1, \ldots, q_r) := \mathrm{Tr}_{\overline{L}_{\frac{m}{n},r}}(q^{\mathbf{h}} q_1^{E_{11}} \cdots q_r^{E_{rr}})$. Let d be a divisor of r, and set $k := r/d$. We consider the expression

$$\frac{[nr]_{\mathbf{q}}}{[n]_q} = [k]_{q^n}[d]_{\mathbf{q}},$$

where we set $\mathbf{q} := q^{\frac{1}{d}}$. Clearly, this is a Laurent polynomial in $\mathbf{q}$ with non-negative integer coefficients. Now let $N_d \in T_0 \subseteq \mathrm{SL}_r$ be

$$N_d = \mathrm{diag}(q_1, \ldots, q_r),$$

where $\{q_1, \ldots, q_r\} = \{\mathbf{q}^{\frac{dn(k+1-2\ell)+d+1-2s}{2}} \,|\, 1 \leqslant \ell \leqslant k,\ 1 \leqslant s \leqslant d\}$. Note that we have $\mathrm{Tr}(N_d) = [k]_{q^n}[d]_{\mathbf{q}}$. We define

$$\mathrm{ch}_q^d(\overline{L}_{\frac{m}{n},r}) := \mathrm{Tr}_{\overline{L}_{\frac{m}{n},r}}(q^{\mathbf{h}} N_d).$$

Equivalently, $\mathrm{ch}_q^d(\overline{L}_{\frac{m}{n},r}) = g(q, q_1, \ldots, q_r)$.

Proposition 3.2.33 *We have*

$$\mathrm{ch}_q^d(\overline{L}_{\frac{m}{n},r}) = \frac{[d]_{\mathbf{q}}}{[nd]_{\mathbf{q}}} \begin{bmatrix} nr+m-1 \\ m \end{bmatrix}_{\mathbf{q}}.$$

Proof: Note that, by (3.9), $\mathrm{ch}_q^d(\overline{L}_{\frac{m}{n},r})$ is

$$\mathrm{ch}_q^d(\overline{L}_{\frac{m}{n},r}) = \frac{1}{[n]_q} \mathrm{Tr}(S^m(q^{\rho_n} \otimes N_d^{-1})),$$

where $q^{\rho_n} = \mathrm{diag}(q^{\frac{-n+1}{2}}, q^{\frac{-n+3}{2}}, \ldots, q^{\frac{n-1}{2}})$. The matrix $q^{\rho_n} \otimes N_d^{-1}$ is diagonal:

$$q^{\rho_n} \otimes N_d^{-1} = \mathrm{diag}(q^{\frac{n+1-2i}{2}} (\mathbf{q}^{-1})^{\frac{dn(k+1-2\ell)+d+1-2s}{2}}),$$

where $1 \leqslant \ell \leqslant k$, $1 \leqslant s \leqslant d$ and $1 \leqslant i \leqslant n$. It is easy to see that, up to permuting the diagonal entries, this can be simplified as

$$q^{\rho_n} \otimes N_d^{-1} = \mathrm{diag}(\mathbf{q}^{\frac{-nr+1}{2}}, \mathbf{q}^{\frac{-nr+3}{2}}, \ldots, \mathbf{q}^{\frac{nr-1}{2}}) = \mathbf{q}^{\rho_{nr}}$$

and it is well-known, and easy to show, that

$$\mathrm{Tr}\, S^m(\mathbf{q}^{\rho_{nr}}) = \begin{bmatrix} nr+m-1 \\ m \end{bmatrix}_{\mathbf{q}},$$

the result now follows from the observation that $[n]_q = [nd]_{\mathbf{q}}/[d]_{\mathbf{q}}$. □

Thanks to Proposition 3.2.33 if d is a divisor of r the q-number

$$C^d_{\frac{m}{n},r}(q) := \frac{[d]_q}{[nd]_q}\begin{bmatrix} nr+m-1 \\ m \end{bmatrix}_q$$

is a Laurent polynomial in q with non-negative integer coefficients. Clearly, when $q = 1$ we recover the rank r Catalan number $C_{\frac{m}{n},r}$. When $r = 1 = d$, the q-number $C^1_{\frac{m}{n},1}(q)$ coincides with the usual q-Catalan number, that is the generating function for the area − dinv statistic on the set of $\frac{m}{n}$-Dyck paths. We do not know the combinatorial meaning of $C^d_{\frac{m}{n},r}(q)$ for $r > 1$.

Remark 3.2.34 *More generally, it would be interesting to find statistics on the set of $\frac{m}{n}$-semistandard parking functions of rank r whose generating function is $\mathrm{ch}_{\mathbb{C}^\times\times\mathrm{GL}_r}(\overline{L}_{\frac{m}{n},r}) \in \mathbb{Z}_{\geq 0}[q, q^{-1}, q_1, q_1^{-1}, \ldots, q_r, q_r^{-1}]$. The statistics corresponding to $q_1, \ldots, q_r$ are not hard, see Remark 3.2.30 above. One possible way to find a statistic corresponding to q is to introduce analogues of sweep maps for semistandard parking functions, see Section 6.1 in [1].*

3.3 Localization

3.3.1 Quantizations of $\mathfrak{M}^\theta(n, r)$ and localization

Let us now consider quantizations $\mathcal{A}^\theta_c(n, r)$ of $\mathfrak{M}^\theta(n, r)$ which are sheaves in conical topology of filtered algebras on $\mathfrak{M}^\theta(n, r)$ with $\mathrm{gr}\,\mathcal{A}^\theta_c(n, r) \simeq \mathcal{O}_{\mathfrak{M}^\theta(n,r)}$. These quantizations are defined via

$$\mathcal{A}^\theta_c(n, r) = D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V) := p_*(D_R/D_R\{\xi_R - c\,\mathrm{tr}(\xi) \,|\, \xi \in \mathfrak{gl}_n\}|_{\mu^{-1}(0)^{\theta\text{-st}}})^{\mathrm{GL}_n},$$

where D_R is the sheaf (in conical topology) of microlocal differential operators on T^*R and $p\colon \mu^{-1}(0)^{\theta\text{-st}} \twoheadrightarrow \mathfrak{M}^\theta(n, r)$ is the quotient morphism.

Remark 3.3.1 *Note that one can also define $\mathcal{A}^\theta_c(n, r)$ in the following way. We need to define its sections over principal open subsets $(T^*R)_f /\!\!/\!\!/^\theta \mathrm{GL}_n \subset \mathfrak{M}^\theta(n, r)$ where $f \in \mathbb{C}[T^*R]$ is a $\mathbb{C}^\times$-homogeneous θ-semiinvariant function of degree $\geqslant 1$. Let $\mathrm{R}_\hbar := \mathrm{R}_\hbar(D(R))$ be the Rees algebra of $D(R)$, so that $\mathrm{R}_\hbar/(\hbar) = \mathbb{C}[T^*R]$. For $k > 0$, let $S_k \subseteq \mathrm{R}_\hbar/(\hbar^k)$ be the preimage of the set $\{f^m \,|\, m \geqslant 0\}$ with respect to the natural morphism $\mathrm{R}_\hbar/(\hbar^k) \twoheadrightarrow \mathrm{R}_\hbar/(\hbar)$. It is easy to see that S_k is an Ore set in $\mathrm{R}_\hbar/(\hbar^k)$ and that we can form the inverse limit*

$$\widehat{\mathrm{R}_\hbar}[f^{-1}] := \varprojlim \left(\mathrm{R}_\hbar/(\hbar^k)\right)[S_k^{-1}].$$

We consider the subalgebra of locally finite vectors with respect to the natural $\mathbb{C}^\times$-action that we denote by $\widehat{\mathrm{R}_\hbar}[f^{-1}]_{\mathrm{l.f.}}$, and we define

$$D(R)[f^{-1}] := \widehat{\mathrm{R}_\hbar}[f^{-1}]_{\mathrm{l.f.}}/(\hbar - 1).$$

It is straightforward to see that $D(R)[f^{-1}]$ is a filtered quantization of $\mathbb{C}[T^*R][f^{-1}]$. Since f is θ-semiinvariant we still have an action of GL_n on $D(R)[f^{-1}]$ and a quantum comoment map $\mathfrak{gl}_n \to D(R)[f^{-1}]$. The sections of $\mathcal{A}^\theta_c(n, r)$ on $(T^*R)_f /\!\!/\!\!/^\theta \mathrm{GL}_n$ are then the quantum Hamiltonian reduction $D(R)[f^{-1}] /\!\!/\!\!/_c \mathrm{GL}_n$. All of this follows from the construction of the sheaf D_R, see [35, Section 1] for details.

We analogously define quantizations $\overline{\mathcal{A}}^{\theta}_c(n,r)$ of $\overline{\mathfrak{M}}^{\theta}(n,r)$. We write $\mathcal{A}^{\theta}_c(n,r)$-mod (resp. $\overline{\mathcal{A}}^{\theta}_c(n,r)$-mod) for the category of coherent $\mathcal{A}^{\theta}_c(n,r)$-modules (resp. $\overline{\mathcal{A}}^{\theta}_c(n,r)$-modules) and $\mathcal{A}_c(n,r)$-mod (resp. $\overline{\mathcal{A}}_c(n,r)$-mod) for the category of finitely generated $\mathcal{A}_c(n,r)$-modules (resp. $\overline{\mathcal{A}}_c(n,r)$-modules). We have the global sections functor $\Gamma^{\theta}_c\colon \mathcal{A}^{\theta}_c(n,r)\text{-mod} \to \mathcal{A}_c(n,r)\text{-mod}$ (resp. $\overline{\Gamma}^{\theta}_c\colon \overline{\mathcal{A}}^{\theta}_c(n,r)\text{-mod} \to \overline{\mathcal{A}}_c(n,r)\text{-mod}$). We say that the abelian localization holds for (θ,c) if the functor $\overline{\Gamma}^{\theta}_c$ (or equivalently Γ^{θ}_c) is an abelian equivalence.

In this section, using the fact that $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$ admits a finite-dimensional representation, we will simplify the proof of the following theorem (see [75, Theorem 1.1 (2)]).

Theorem 3.3.2 *For $\theta > 0$ (resp. $\theta < 0$), abelian localization holds for $c \in \mathbb{C}$ iff c is not of the form $\frac{s}{m}$, where $1 \leqslant m \leqslant n$ and $s < 0$ (resp. if c is not of the form $-r - \frac{s}{m}$, where $1 \leqslant m \leqslant n$ and $s < 0$).*

Recall that the proof of this theorem in [75, Section 5] consists of three steps (see the beginning of [75, Section 5]). In the first step the proof of Theorem 3.3.2 reduces to the case when $c = \frac{m}{n'} \geqslant 0$ with m, n' coprime, $n' \leqslant n$ and $\theta > 0$. In the second step the proof reduces to the case $n = n'$ and $c > 0$, $\theta > 0$. In the third step the claim reduces to the fact that the functor Γ^{θ}_c induces an equivalence between certain categories $\mathcal{O}_{\nu}(\mathcal{A}^{\theta}_c(n,r))$, $\mathcal{O}_{\nu}(\mathcal{A}_c(n,r))$ over $\mathcal{A}^{\theta}_c(n,r)$, $\mathcal{A}_c(n,r)$ (see Section 3.3.2 for the definitions of these categories). The last claim is proved via proving that the number of simples of the categories $\mathcal{O}_{\nu}(\mathcal{A}^{\theta}_c(n,r))$, $\mathcal{O}_{\nu}(\mathcal{A}_c(n,r))$ are equal. The last step is crucial and we will simplify its proof. So, from now on we assume that $c = \frac{m}{n} > 0$, $\gcd(m,n) = 1$ and $\theta > 0$.

Let us first of all define categories $\mathcal{O}$ and other notions and objects that we will use in the proof. We use the same notations as in [75].

3.3.2 Singular support and categories $\mathcal{O}$

Let $\nu\colon \mathbb{C}^{\times} \to T = \mathbb{C}^{\times} \times T_0$ be a cocharacter given by $t \mapsto (t, t^{d_1}, \ldots, t^{d_r})$ for $d_1 \gg \ldots \gg d_r$. We will denote by ν_0 the cocharacter of T given by $t \mapsto (1, t^{d_1}, \ldots, t^{d_r})$. The cocharacter ν (resp. ν_0) induces a grading $\mathcal{A}_c(n,r) = \bigoplus_i \mathcal{A}^{i,\nu}_c$ (resp. $\mathcal{A}_c(n,r) = \bigoplus_i \mathcal{A}^{i,\nu_0}_c$). We set $\mathcal{A}^{>0,\nu}_c := \bigoplus_{i>0} \mathcal{A}^{i,\nu}_c$. The action of ν on $\mathcal{A}_c(n,r)$ is Hamiltonian, let $h \in \mathcal{A}^{0,\nu}_c$ be the image of 1 under the comoment map. The grading $\mathcal{A}_c(n,r) = \bigoplus_i \mathcal{A}^{i,\nu}_c$ is inner and is given by h. Define the category $\mathcal{O}_{\nu}(\mathcal{A}_c(n,r))$ as the full subcategory of the category $\mathcal{A}_c(n,r)$-mod consisting of all modules where the action of $\mathcal{A}^{>0}_c(n,r)$ is locally nilpotent. Let us also define the category $\mathcal{O}_{\nu}(\mathcal{A}^{\theta}_c(n,r))$ as the full subcategory of $\mathcal{A}^{\theta}_c(n,r)$-mod consisting of modules that come with a good filtration stable under h and that are supported on the contracting locus $\mathfrak{M}^{\theta}_{+}(n,r)$ of ν in $\mathfrak{M}^{\theta}(n,r)$. Recall that this contracting locus is defined by

$$\mathfrak{M}^{\theta}_{+}(n,r) = \{x \in \mathfrak{M}^{\theta}(n,r) \mid \lim_{t\to 0} \nu(t)\cdot x \ \text{ exists}\}.$$

We set

$$\mathsf{C}_{\nu_0}(\mathcal{A}_c(n,r)) := \mathcal{A}^{0,\nu_0}_c(n,r) / \sum_{i>0} \mathcal{A}^{-i,\nu_0}_c(n,r)\mathcal{A}^{i,\nu_0}_c(n,r).$$

Remark 3.3.3 *Note that we have the natural isomorphisms*

$$\mathsf{C}_{\nu_0}(\mathcal{A}_c(n,r)) \xrightarrow{\sim} \mathcal{A}^{\geqslant 0,\nu_0}_c(n,r)/(\mathcal{A}^{\geqslant 0,\nu_0}_c(n,r) \cap \mathcal{A}_c(n,r)\mathcal{A}^{>0,\nu_0}_c(n,r)),$$

$$\mathsf{C}_{\nu_0}(\mathcal{A}_c(n,r)) \xrightarrow{\sim} \mathcal{A}_c^{\leqslant 0,\nu_0}(n,r)/(\mathcal{A}_c^{\leqslant 0,\nu_0}(n,r) \cap \mathcal{A}_c^{<0,\nu_0}(n,r)\mathcal{A}_c(n,r)).$$

The algebra $\mathsf{C}_{\nu_0}(\mathcal{A}_c(n,r))$ will be called the *Cartan subquotient* of $\mathcal{A}_c(n,r)$ with respect to ν_0. One can also define Cartan subquotients of sheaves $\mathcal{A}_c^\theta(n,r)$: it follows from [73, Proposition 5.2] that there exists a unique sheaf $\mathsf{C}_{\nu_0}(\mathcal{A}_c^\theta(n,r))$ in the conical topology on $\mathfrak{M}^\theta(n,r)^{\nu_0(\mathbb{C}^\times)}$ such that for any $\mathbb{C}^\times \times \nu_0(\mathbb{C}^\times)$-stable open subvariety $U \subset \mathfrak{M}^\theta(n,r)$ with $U^{\nu_0(\mathbb{C}^\times)} \neq \varnothing$ we have $\mathsf{C}_{\nu_0}(\mathcal{A}_c^\theta(n,r))(U^{\nu_0(\mathbb{C}^\times)}) = \mathsf{C}_{\nu_0}(\mathcal{A}_c^\theta(U))$.

Proposition 3.3.4 *For $\theta > 0$ we have*

1. $\mathfrak{M}^\theta(n,r)^{\nu_0(\mathbb{C}^\times)} = \bigsqcup_{n_1+\ldots+n_r=n} \prod_i \mathfrak{M}^\theta(n_i,1)$*, where the disjoint union runs over all ordered collections $(n_1,\ldots,n_r)$ of non-negative integers such that $n_1+\ldots+n_r = n$ (and we set $\mathfrak{M}^\theta(0,1) = \mathrm{pt}$).*

2. *For the connected component $Z \subset \mathfrak{M}^\theta(n,r)^{\nu_0(\mathbb{C}^\times)}$ which corresponds to the composition $(n_1,\ldots,n_r)$ of n we have $\mathsf{C}_{\nu_0}(\mathcal{A}_c^\theta(n,r))|_Z = \bigotimes_i \mathcal{A}_{c+r-i}^\theta(n_i,1)$, where we set $\mathcal{A}_c^\theta(0,1) = \mathbb{C}$ (a sheaf on $\mathfrak{M}^\theta(0,1) = \mathrm{pt}$).*

3. *For a Zariski generic $c \in \mathbb{C}$ we have $\mathsf{C}_{\nu_0}(\mathcal{A}_c(n,r)) = \bigoplus \bigotimes_i \mathcal{A}_{c+r-i}(n_i,1)$, where the direct sum is taken over all ordered collections $(n_1,\ldots,n_r)$ as in (1).*

Proof: Follows from [75, Proposition 3.5]. The proof should be changed as follows (see the footnote 1 above): note that the line bundle $\mathcal{O}(1)$ on $\mathfrak{M}^\theta(n_i,1)$ is the top exterior power of the bundle on $\mathfrak{M}^\theta(n_i,1)$ induced from the representation $\mathrm{GL}_{n_i} \curvearrowright \mathbb{C}^{n_i *}$, so we conclude that $c_1(Y_\mu) = \sum_{i=1}^r (2i-r-1)c_i$ (not $\sum_{i=1}^r (r+1-2i)c_i$), i.e. the period of $\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda^\theta)$ equals $\sum_{i=1}^r (\lambda + r + \frac{1}{2} - i)$, hence, $\mathsf{C}_{\nu_0}(\mathcal{A}_\lambda^\theta)|_Z \simeq \bigotimes_i \mathcal{A}_{c+r-i}^\theta(n_i,1)$. ☐

Let us now give an important property of the category $\mathcal{O}(\mathcal{A}_c^\theta(n,r))$.

Theorem 3.3.5 *The category $\mathcal{O}_\nu(\mathcal{A}_c^\theta(n,r))$ is a highest weight category, with simples indexed by the fixed points of ν and the order in the definition of a highest weight category is the contraction order on the fixed points.*

Proof: Since the action of ν on $\mathfrak{M}^\theta(n,r)$ has finitely many fixed points, this follows from [10, Proposition 5.17]. ☐

Starting from a module $M \in \mathcal{A}_c(n,r)$-mod (resp. $\mathcal{M} \in \mathcal{A}_c^\theta(n,r)$-mod) we can construct its *associated variety*, to be denoted $V(M) \subset \mathfrak{M}(n,r)$ (resp. $V(\mathcal{M}) \subset \mathfrak{M}^\theta(n,r)$), as follows. Consider any good filtration $F^\bullet M$ (resp. $F^\bullet\mathcal{M}$), then define $V(M)$ (resp. $V(\mathcal{M})$) as the support of $\operatorname{gr} F^\bullet M$ (resp. $\operatorname{gr} F^\bullet\mathcal{M}$) with the reduced scheme structure. It is straightforward, and well-known, that this does not depend on the choice of a good filtration. Moreover, $V(\mathcal{M})$ is simply the support of $\mathcal{M}$ considered as a sheaf on $\mathfrak{M}^\theta(n,r)$.

Proposition 3.3.6 *Let L be a simple module of the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r))$ and let $\mathcal{I} \subset \overline{\mathcal{A}}_c(n,r)$ be the annihilator of L. Then $2\dim(V(L)) = \dim(V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I}))$, where $V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I})$ is computed by considering $\overline{\mathcal{A}}_c(n,r)/\mathcal{I}$ as a left $\overline{\mathcal{A}}_c(n,r)$-module.*

Proof: Note that by [73, Section 4.4] every module of the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r))$ is holonomic in the sense of [72]. Now the claim follows from [72, Theorems 1.2, 1.3]. ☐

3.3.3 Properties of Γ_c^θ and categories $\mathcal{O}$

Let us now study the interaction between the global sections functor and the categories $\mathcal{O}$.

Proposition 3.3.7 *Assume that $\theta > 0$ and $c > -r$. Then the following holds.*

1. *The functor $\Gamma_c^\theta \colon \mathcal{O}_\nu(\mathcal{A}_c^\theta(n,r))) \twoheadrightarrow \mathcal{O}_\nu(\mathcal{A}_c(n,r)))$ is a quotient functor.*
2. *The canonical adjunction morphism $M \to \Gamma_c^\theta \circ \mathrm{Loc}(M)$ is an isomorphism for any $M \in \mathcal{O}_\nu(\mathcal{A}_c(n,r)))$.*

Proof: Part (1) follows from [81, Section 8] and [75, Proposition 5.1]. Part (2) follows from the fact that Loc is left adjoint to Γ_c^θ and general properties of quotient functors between finite categories. □

Let us now return to the proof of Theorem 3.3.2. Recall that we can assume that $c = \frac{m}{n} > 0$, $\gcd(m,n) = 1$, $\theta > 0$. It now follows from [75, Steps 2, 3 of Proposition 5.6] that to finish the proof of Theorem 3.3.2 it is enough to prove the following theorem.

Theorem 3.3.8 *Assume $\theta > 0$ and $c = \frac{m}{n} > 0$ with $\gcd(m,n) = 1$. Then $\overline{\Gamma}_c^\theta \colon \mathcal{O}_\nu(\overline{\mathcal{A}}_c^\theta(n,r)) \to \mathcal{O}_\nu(\overline{\mathcal{A}}_c(n,r))$ is an equivalence of categories.*

Thanks to Proposition 3.3.7, to prove Theorem 3.3.8 (and therefore also Theorem 3.3.2) it is enough to show that $\overline{\Gamma}_c^\theta(\mathcal{L}) \neq 0$ for every simple $\mathcal{L} \in \mathcal{O}_\nu(\overline{\mathcal{A}}_c^\theta(n,r))$. We consider two cases, according to the associated variety of simples.

3.3.4 Associated varieties of simples and proof of Theorem 3.3.8

First we show that in the situation of Theorem 3.3.8 if $\mathcal{L} \in \mathcal{O}(\overline{\mathcal{A}}_c^\theta(n,r))$ is such that $V(\mathcal{L}) \nsubseteq \overline{\rho}^{-1}(0)$, then $\overline{\Gamma}_c^\theta(\mathcal{L})$ is infinite dimensional so in particular nonzero.

Lemma 3.3.9 *Assume that $c = \frac{m}{n} > 0$ with m, n coprime. For any infinite dimensional simple module $L \in \mathcal{O}(\overline{\mathcal{A}}_c(n,r))$ we have $V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I}) = \overline{\mathfrak{M}}(n,r)$, where $\mathcal{I} \subset \overline{\mathcal{A}}_c(n,r)$ is the annihilator of L.*

Proof: Follows from the proof of Step 3 in the proof of Proposition 4.1 in [75]. Recall that $\mathcal{I}$ is a Harish-Chandra $\overline{\mathcal{A}}_c$-bimodule, so $V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I})$ is a union of symplectic leaves of $\overline{\mathfrak{M}}(n,r)$. Let $\mathcal{L} \subset V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I})$ be an open leaf which corresponds to a collection $\mu = (n_1, \ldots, n_k)$ and $x \in \mathcal{L} \cap V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I})$. The module L is infinite dimensional so $\dim(V(L)) > 0$, hence, by Proposition 3.3.6 we have $\dim V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I}) > 0$ so $n_i < n$ for any i. Consider the restriction $(\overline{\mathcal{A}}_c/\mathcal{I})_{\dagger,x}$. By our choice of x, this is a nonzero finite dimensional representation of $\overline{\mathcal{A}}_c(\mu) = \overline{\mathcal{A}}_c(n_1,r) \otimes \ldots \otimes \overline{\mathcal{A}}_c(n_k,r)$. It follows from Proposition **??** (which is independent of the intervening material) that we have $n_1 = \ldots = n_k = 0$ so $\mathcal{L}$ is the open symplectic leaf in $\overline{\mathfrak{M}}(n,r)$, hence, $V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I}) = \overline{\mathcal{L}} = \overline{\mathfrak{M}}(n,r)$. □

Remark 3.3.10 *We keep the notations of Lemma 3.3.9. Note that by [75, Theorem 1.3] $\mathcal{A}_c$ is a prime ring and $\mathcal{I}$ is primitive, hence, the equality $V(\overline{\mathcal{A}}_c(n,r)/\mathcal{I}) = \overline{\mathfrak{M}}(n,r)$ implies $\mathcal{I} = 0$.*

The variety $\overline{\mathfrak{M}}(n,r)$ is Poisson and we denote by $\overline{\mathfrak{M}}(n,r)^{\mathrm{reg}}$ its unique open symplectic leaf. The map $\overline{\rho} : \overline{\mathfrak{M}}^{\theta}(n,r) \to \overline{\mathfrak{M}}(n,r)$ is an isomorphism over $\overline{\mathfrak{M}}(n,r)^{\mathrm{reg}}$ and we denote by $\overline{\mathfrak{M}}^{\theta}(n,r)^{\mathrm{reg}}$ the preimage of $\overline{\mathfrak{M}}(n,r)^{\mathrm{reg}}$.

Lemma 3.3.11 *Assume that $c = \frac{m}{n} > 0$ with m, n coprime. Let $\mathcal{L} \in \mathcal{O}(\overline{\mathcal{A}}^{\theta}_c(n,r))$ be an irreducible sheaf such that $V(\mathcal{L}) \nsubseteq \overline{\rho}^{-1}(0)$, then $V(\mathcal{L}) \cap \overline{\mathfrak{M}}^{\theta}(n,r)^{\mathrm{reg}} \neq \varnothing$.*

Proof: By [11, Theorem A] there exists $N \gg 0$ such that the abelian localization holds for $(c + N\theta, \theta)$. Note that the categories $\mathcal{O}(\overline{\mathcal{A}}^{\theta}_c(n,r)), \mathcal{O}(\overline{\mathcal{A}}^{\theta}_{c+N\theta}(n,r))$ are equivalent via translation functors and these functors preserve associated varieties. Note also that $c + N\theta$ is obviously of the form $\frac{m'}{n}$ with m', n coprime. So, applying translation functors if needed, we can assume abelian localization holds. So $\mathcal{L} = \mathrm{Loc}(L)$ for some irreducible object $L \in \mathcal{O}(\overline{\mathcal{A}}_c(n,r))$ such that $V(L) \nsubseteq \{0\}$. It remains to show that $V(L) \cap \overline{\mathfrak{M}}(n,r)^{\mathrm{reg}} \neq \varnothing$. Let $\mathcal{I} \subset \overline{\mathcal{A}}$ be the annihilator of L. It follows from Lemma 3.3.9 that $V(\overline{\mathcal{A}}/\mathcal{I}) = \overline{\mathfrak{M}}(n,r)$. It now follows from [71, Theorem 1.1] that $V(L)$ has nonempty intersection with the open symplectic leaf of $\overline{\mathfrak{M}}(n,r)$, which is exactly $\overline{\mathfrak{M}}(n,r)^{\mathrm{reg}}$. □

Corollary 3.3.12 *Assume that $c = \frac{m}{n} > 0$ with m, n coprime. Let $\mathcal{L} \in \mathcal{O}(\overline{\mathcal{A}}^{\theta}_c(n,r))$ be an irreducible sheaf such that $V(\mathcal{L}) \nsubseteq \overline{\rho}^{-1}(0)$. Then $\overline{\Gamma}^{\theta}_c(\mathcal{L})$ is infinite dimensional.*

Proof: It follows from Lemma 3.3.11 that we have a point $x \in V(\mathcal{L}) \cap \overline{\mathfrak{M}}^{\theta}(n,r)^{\mathrm{reg}}$. Note now that the morphism $\overline{\rho}$ is an isomorphism over the open subvariety $\overline{\mathfrak{M}}^{\theta}(n,r)^{\mathrm{reg}} \subset \overline{\mathfrak{M}}^{\theta}(n,r)$, $\overline{\rho}\colon \overline{\mathfrak{M}}^{\theta}(n,r)^{\mathrm{reg}} \xrightarrow{\sim} \overline{\mathfrak{M}}(n,r)^{\mathrm{reg}}$. It follows that $\overline{\rho}(x) \in V(\overline{\Gamma}^{\theta}_c(\mathcal{L}))$, hence, $\overline{\Gamma}^{\theta}_c(\mathcal{L})$ is infinite dimensional. □

Let us now deal with the case $V(\mathcal{L}) \subseteq \overline{\rho}^{-1}(0)$.

Proposition 3.3.13 *The number of simple coherent $\overline{\mathcal{A}}^{\theta}_c(n,r)$-modules supported on $\overline{\rho}^{-1}(0)$ cannot be bigger then 1.*

Proof: This follows from Step 5 of the proof of Proposition 4.1 in [75]. To a module in the category $\mathcal{O}$ we can assign its characteristic cycle that is a formal linear combination of the irreducible components of the contracting locus $\overline{\mathfrak{M}}^{\theta}_{+}(n,r)$ of $\mathbb{C}^{\times}$ acting on $\overline{\mathfrak{M}}^{\theta}(n,r)$ via ν ($\overline{\mathfrak{M}}^{\theta}_{+}(n,r) := \{x \in \overline{\mathfrak{M}}^{\theta}(n,r) \mid \exists \lim_{t\to 0} \nu(t)\cdot x\}$). It follows from [10, Section 6] that this map is injective. Note also that the irreducible components of $\overline{\mathfrak{M}}^{\theta}_{+}(n,r)$ are lagrangian so have dimension $nr - 1 = \dim \overline{\rho}^{-1}(0)$ so it is enough to show that $\dim H^{2nr-2}(\overline{\rho}^{-1}(0), \mathbb{C}) = 1$. Recall now that $\overline{\rho}^{-1}(0) = \rho^{-1}(0)$ and the latter is homotopicaly equivalent to $\mathfrak{M}^{\theta}(n,r)$ so $H^{2nr-2}(\overline{\rho}^{-1}(0), \mathbb{C}) \simeq H^{2nr-2}(\mathfrak{M}^{\theta}(n,r), \mathbb{C})$. Now it follows from [123, Theorem 3.8] that $\dim H^{2nr-2}(\mathfrak{M}^{\theta}(n,r)) = 1$. □

If a simple coherent $\overline{\mathcal{A}}^{\theta}_c(n,r)$-module supported on $\overline{\rho}^{-1}(0)$ exists, we will denote it by $\mathcal{L}^{\mathrm{fin}}$.

We are now ready to prove Theorem 3.3.8.

Proof:[Proof of Theorem 3.3.8] From Proposition 3.3.7 it follows that it is enough to show that for any simple $\mathcal{L} \in \mathcal{O}(\overline{\mathcal{A}}^{\theta}_c(n,r))$ we have $\overline{\Gamma}^{\theta}_c(\mathcal{L}) \neq 0$. If $\mathcal{L} \not\simeq \mathcal{L}^{\mathrm{fin}}$, then the desired result follows from Corollary 3.3.12. It also follows from Corollary 3.3.12 that for any such $\mathcal{L}$ we have $\overline{\Gamma}_c^\theta(\mathcal{L}) \not\simeq \overline{L}_{\frac{m}{n},r}$. The functor $\overline{\Gamma}_c^\theta$ is a quotient functor and $\overline{\Gamma}_c^\theta(\mathcal{L}) \not\simeq \overline{L}_{\frac{m}{n},r}$ for any $\mathcal{L} \not\simeq \mathcal{L}^{\mathrm{fin}}$, hence, $\overline{\Gamma}_c^\theta(\mathcal{L}^{\mathrm{fin}}) \simeq \overline{L}_{\frac{m}{n},r} \neq 0$, where the last inequality follows from Remark 3.2.21. ☐

3.4 Representations with minimal support

3.4.1 Construction of minimally supported modules

Now we generalize results on finite-dimensional representations to representations with minimal support (see Section 3.3.2). We continue studying the algebra $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$, and set $d := \gcd(m,n)$. The difference now is that we do not assume $d = 1$. Let $m_0 := \frac{m}{d}$, $n_0 := \frac{n}{d}$. Let λ be a partition of d and consider the partition $n_0\lambda$ of n. We will denote by $M(O(n_0\lambda))$ the irreducible χ-equivariant D-module on $\mathfrak{sl}_n$ associated to the nilpotent orbit $O(n_0\lambda)$ where, recall, $\chi = \frac{m}{n}\operatorname{tr}$.

Lemma 3.4.1 *We have $F_{n,m,1}(M(O(n_0\lambda))) = S_{\frac{n}{m}}(m_0\lambda)$, where $S_{\frac{n}{m}}(m_0\lambda)$ is the irreducible highest-weight representation of $H_{\frac{n}{m}}(m,1)$ with highest weight $m_0\lambda$.*

Proof: Note that a similar result is proven in [22, Theorem 9.12] for $F^*_{n,m,1}(M(O(n_0\lambda)))$, where *highest* weight is replaced by *lowest* weight, see Remark 3.2.14. The same proof applies, *mutatis mutandis*, in our situation. More precisely, the functions constructed in [22, Lemma 9.13] are annihilated by $x_1, \dots, x_m$ and span a copy of the representation of S_m indexed by $m_0\lambda$, see [22, Lemma 9.15]. ☐

It follows that we have a q-graded GL_r-equivariant isomorphism

$$\overline{L}_{\frac{m}{n},r}(n_0\lambda) := (M(O(n_0\lambda)) \otimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)])^{\mathrm{GL}_n} \xrightarrow{\sim} (S_{\frac{n}{m}}(m_0\lambda) \otimes (\mathbb{C}^{r*})^{\otimes m})^{S_m}. \tag{3.12}$$

From here, we can read the character of $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$, we will do this explicitly in Section 3.5. Note that $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is an irreducible $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-module. We claim that it has minimal support. Recall from the introduction that, provided $\overline{L}_{\frac{m}{n},r}(n_0\lambda) \in \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r))$, this means that the GK dimension of $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is precisely $d-1$, where, as above, $d = \gcd(m,n)$. So we start by showing that $\overline{L}_{\frac{m}{n},r}(n_0\lambda) \in \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r))$.

Proposition 3.4.2 *Let $\nu\colon \mathbb{C}^\times \to T$ be a generic co-character given by $t \mapsto (t^k, t^{d_1}, \dots, t^{d_r})$ such that $k > 0$. Then $\overline{L}_{\frac{m}{n},r}(n_0\lambda) \in \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r))$.*

Proof: Recall that the action of ν on $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$ is Hamiltonian. Let $h \in \overline{\mathcal{A}}^{0,\nu}_{\frac{m}{n}}$ be the image of 1 under the comoment map. To check that $\overline{L}_{\frac{m}{n},r}(n_0\lambda) \in \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r))$ it is enough to show that generalized eigenvalues of h acting on $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ are bounded from above. This follows from the existence of the T-equivariant isomorphism (3.12), the fact that $(\mathbb{C}^{r*})^{\otimes m}$ is finite dimensional and the fact that $S_{\frac{n}{m}}(m_0\lambda)$ lies in the category $\mathcal{O}$ over the Cherednik algebra, see Lemma 3.4.1. ☐

Let us now show that the module $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ has minimal support. In order to do this, it is enough to compute its GK dimension.

Proposition 3.4.3 *The GK dimension of the $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-representation $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ is exactly $d-1$, where $d=\gcd(m,n)$. In particular, $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ has minimal support.*

Proof: Recall that we have $\overline{L}_{\frac{m}{n},r}(n_0\lambda) = (M(O(n_0\lambda))\otimes\mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)])^{\mathrm{GL}_n}$. The $D(\mathfrak{sl}_n\oplus\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r))$-module $M(O(n_0\lambda))\otimes\mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)]$ is holonomic, so it admits a good filtration that induces a good filtration on $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$, both as an $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-module and as a $H^{\mathrm{sph}}_{\frac{n}{m}}(m,r)$-module, where we take the Bernstein filtration on the ring of differential operators $D(\mathfrak{sl}_n\oplus\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r))$. Indeed, that this filtration is good for the $\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-module structure follows by definition, and that it is good for the $H^{\mathrm{sph}}_{\frac{n}{m}}(m,r)$-module structure follows from the formulas in Proposition 3.2.3. So it is enough to show that the GK dimension of the $H^{\mathrm{sph}}_{\frac{n}{m}}(m,r)$-module $(S_{\frac{n}{m}}(m_0\lambda)\otimes(\mathbb{C}^r)^{\otimes m})^{S_m}$ is exactly $d-1$. To do this, it is enough to show that the GK dimension of the $H_{\frac{n}{m}}(m,r)$-module $S_{\frac{n}{m}}(m_0\lambda)\otimes(\mathbb{C}^r)^{\otimes m}$ is $d-1$. But thanks to Proposition 3.2.10, this coincides with the GK dimension of the $H_{\frac{n}{m}}(m,1)$-module $S_{\frac{n}{m}}(m_0\lambda)$. This is precisely $d-1$ by [121, Theorem 1.6]. We are done. □

3.4.2 Coincidence of labels: $L_{\frac{m}{n},r}(n_0\lambda) = L(\varnothing,\ldots,\varnothing,n_0\lambda)$

By Proposition 3.4.3 the module $L_{\frac{m}{n},r}(n_0\lambda) := \mathbb{C}[x]\otimes(M(O(n_0\lambda))\otimes\mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)])^{\mathrm{GL}(V)}$ is minimally supported so it must have the form $L(\varnothing,\ldots,\varnothing,n_0\lambda')$ for some partition λ' of d. The goal of this section is to show that $\lambda'=\lambda$. We set $G:=\mathrm{GL}(V)$. Consider the $D(\overline{R})$-$\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-bimodule

$$\overline{\mathcal{Q}}_{\frac{m}{n}} := D(\overline{R})/(D(\overline{R})\{\xi_{\overline{R}} - \frac{m}{n}\operatorname{tr}\xi \mid \xi\in\mathfrak{gl}(V)\}).$$

Recall the character $\det\colon \mathrm{GL}(V)\to\mathbb{C}^\times$ and consider the corresponding space of $\mathrm{GL}(V)$-semiinvariants $\overline{\mathcal{Q}}^{\mathrm{GL}(V),\det}_{\frac{m}{n}}$ which is naturally a $\overline{\mathcal{A}}_{\frac{m}{n}+1}(n,r)$-$\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)$-bimodule. Recall that $\frac{m}{n}>0$, so by Theorem 3.3.2 the localization holds for $(\frac{m}{n},\det)$, $(\frac{m}{n}+1,\det)$. It now follows from [8, Proposition 5.2] and [11, Proposition 6.31] that the functor $\overline{\mathcal{Q}}^{\mathrm{GL}(V),\det}_{\frac{m}{n}}\otimes_{\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)}\bullet$ induces an equivalence

$$\overline{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}\colon \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r)) \xrightarrow{\sim} \mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}+1}(n,r))$$

which we will call a *translation equivalence*. Let us now prove the following lemma.

Lemma 3.4.4 *Under the equivalence $\overline{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}$ the module $\overline{L}_{\frac{m}{n},r}(n_0\lambda)$ maps to $\overline{L}_{\frac{m}{n}+1,r}(n_0\lambda)$.*

Proof: For $c\in\mathbb{C}$ we recall that $D(\overline{R})\text{-}\mathrm{mod}^{G,c}$ is the category of (G,c)-equivariant D-modules on $\overline{R}$. Note that we have an equivalence

$$\widetilde{\mathcal{T}}_{c\to c+1}\colon D(\overline{R})\text{-}\mathrm{mod}^{G,c} \xrightarrow{\sim} D(\overline{R})\text{-}\mathrm{mod}^{G,c+1}$$

given by tensoring with a one-dimensional $\mathrm{GL}(V)$-module $\mathbb{C}_{\det}$ on which $\mathrm{GL}(V)$ acts via det. Now for $c=\frac{m}{n}=\frac{m_0 d}{n_0 d}$ we set $\widetilde{L}_c := M(O(n_0\lambda))\otimes\mathrm{Hom}(V,W)$ and consider it as an object of the category $D(\overline{R})\text{-}\operatorname{mod}^{G,c}$. It follows from the definitions that $\widetilde{L}_{c+1} = \mathbb{C}_{\det} \otimes \widetilde{L}_c = \widetilde{\mathcal{T}}_{c\to c+1}(\widetilde{L}_c)$.

Recall that the categories $\mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}}(n,r))$, $\mathcal{O}_\nu(\overline{\mathcal{A}}_{\frac{m}{n}+1}(n,r))$ are highest weight. It follows from the definitions that the functor $\overline{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}$ sends $\Delta_{\frac{m}{n}}(p)$ to $\Delta_{\frac{m}{n}+1}(p)$ so it must be label preserving. It follows from [11, Proposition 6.27] that the functor $\mathcal{T}_{\frac{m}{n}\to\frac{m}{n}+1}$ preserves characteristic cycles, hence, is label preserving. We conclude that to prove lemma it is enough to show that the following diagram is commutative.

$$\begin{array}{ccc} D(\overline{R})\text{-}\operatorname{mod}^{G,\frac{m}{n}} & \xrightarrow{\overline{\pi}_{\frac{m}{n}}} & \overline{\mathcal{A}}_{\frac{m}{n}}(n,r)\text{-}\operatorname{mod} \\ \downarrow{\scriptstyle \widetilde{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}} & & \downarrow{\scriptstyle \overline{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}} \\ D(\overline{R})\text{-}\operatorname{mod}^{G,\frac{m}{n}+1} & \xrightarrow{\overline{\pi}_{\frac{m}{n}+1}} & \overline{\mathcal{A}}_{\frac{m}{n}+1}(n,r)\text{-}\operatorname{mod}, \end{array} \tag{3.13}$$

where we denote by $\overline{\pi}_{\frac{m}{n}}, \overline{\pi}_{\frac{m}{n}+1}$ the Hamiltonian reduction functors. Let us now define the sheaf versions of the functors $\overline{\pi}_{\frac{m}{n}}, \overline{\pi}_{\frac{m}{n}+1}, \overline{\mathcal{T}}_{c\to c+1}$. We denote

$$\overline{\pi}_c^\theta\colon D_{\overline{R}}\text{-}\operatorname{mod}^{G,c} \to \overline{\mathcal{A}}_c^\theta(n,r)\text{-}\operatorname{mod}, \quad \mathcal{F} \mapsto (\mathcal{F}|_{(T^*\overline{R})^{\theta-\mathrm{st}}})^{\mathrm{GL}(V)},$$

the functor $\overline{\pi}_{c+1}^\theta$ can be defined similarly. We denote by $\overline{\mathcal{T}}_{c\to c+1}^\theta$ the functor from $\overline{\mathcal{A}}_c^\theta(n,r)\text{-}\operatorname{mod}$ to $\overline{\mathcal{A}}_{c+1}^\theta(n,r)\text{-}\operatorname{mod}$ given by left tensoring with the sheaf of bimodules $(\overline{\mathcal{Q}}_c|_{(T^*\overline{R})^{\theta-\mathrm{st}}})^{\mathrm{GL}(V),\det}$. It follows from [11, Proposition 6.31] that the following diagram is commutative:

$$\begin{array}{ccc} \overline{\mathcal{A}}_c^\theta(n,r)\text{-}\operatorname{mod} & \xrightarrow{\overline{\Gamma}(\bullet)} & \overline{\mathcal{A}}_c(n,r)\text{-}\operatorname{mod} \\ \downarrow{\scriptstyle \overline{\mathcal{T}}_{c\to c+1}^\theta} & & \downarrow{\scriptstyle \overline{\mathcal{T}}_{c\to c+1}} \\ \overline{\mathcal{A}}_{c+1}^\theta(n,r)\text{-}\operatorname{mod} & \xrightarrow{\overline{\Gamma}(\bullet)} & \overline{\mathcal{A}}_{c+1}(n,r)\text{-}\operatorname{mod}, \end{array}$$

so to prove that (3.13) is commutative it remains to check the commutativity of the following diagram:

$$\begin{array}{ccc} D_{\overline{R}}\text{-}\operatorname{mod}^{G,\frac{m}{n}} & \xrightarrow{\overline{\pi}_{\frac{m}{n}}^\theta} & \overline{\mathcal{A}}_c^\theta(n,r)\text{-}\operatorname{mod} \\ \downarrow{\scriptstyle \widetilde{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}} & & \downarrow{\scriptstyle \overline{\mathcal{T}}_{\frac{m}{n}\to\frac{m}{n}+1}^\theta} \\ D_{\overline{R}}\text{-}\operatorname{mod}^{G,\frac{m}{n}+1} & \xrightarrow{\overline{\pi}_{\frac{m}{n}+1}^\theta} & \overline{\mathcal{A}}_{\frac{m}{n}+1}^\theta(n,r)\text{-}\operatorname{mod}. \end{array}$$

This was observed in [8, (5.1)]. Let us give a proof of this fact. We denote by $D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c}$ the category of (G,c)-equivariant sheaves of modules over $D_R|_{(T^*R)^{\theta-\mathrm{st}}}$. Recall that the functor π_c^θ can be obtained as the compostion of two functors: $\pi_c^\theta = \widetilde{\pi}_c^\theta \circ j^*$, where $j^*\colon D_R = \operatorname{mod}^{G,c} \to D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c}$ is given by $\mathcal{F} \mapsto \mathcal{F}|_{(T^*R)^{\theta-\mathrm{st}}}$ and the functor $\widetilde{\pi}_c^\theta\colon D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c} \to \mathcal{A}_c^\theta(n,r)$ sends a sheaf $\mathcal{P}$ to $\mathcal{P}^{\mathrm{GL}(V)}$.

The following diagram is commutative

$$\begin{array}{ccc} D(R)\text{-}\operatorname{mod}^{G,c} & \xrightarrow{j^*} & D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c} \\ \downarrow{\scriptstyle \widetilde{\mathcal{T}}_{c\to c+1}} & & \downarrow{\scriptstyle \widetilde{\mathcal{T}}_{c\to c+1}} \\ D(R)\text{-}\operatorname{mod}^{G,c+1} & \xrightarrow{j^*} & D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c+1} \end{array}$$

since for any $\mathcal{F} \in D_R\text{-}\operatorname{mod}^{G,c}$ we have

$$(\mathcal{Q}_c \otimes \mathcal{F})|_{(T^*R)^{\theta-\mathrm{st}}} = (\mathcal{Q}_c)_{(T^*R)^{\theta-\mathrm{st}}} \otimes \mathcal{F}|_{(T^*R)^{\theta-\mathrm{st}}}.$$

So we only need to check that the following diagram is commutative.

$$\begin{array}{ccc} D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c} & \xrightarrow{\widetilde{\pi}_c^\theta} & \mathcal{A}_c^\theta(n,r)\text{-}\operatorname{mod} \\ \downarrow \widetilde{\mathcal{T}}_{c\to c+1} & & \downarrow \mathcal{T}^\theta_{c\to c+1} \\ D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c+1} & \xrightarrow{\widetilde{\pi}_{c+1}^\theta} & \mathcal{A}_{c+1}^\theta(n,r)\text{-}\operatorname{mod}. \end{array}$$

Note that the (sheaf) Hamiltonian reduction functor

$$\widetilde{\pi}_c^\theta \colon D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c} \to \mathcal{A}_c^\theta(n,r)\text{-}\operatorname{mod},\ \mathcal{F} \mapsto \mathcal{F}^{\mathrm{GL}(V)}$$

is an equivalence since the action $\mathrm{GL}(V) \curvearrowright (T^*R)^{\theta-\mathrm{st}}$ is free. Let us denote by $p\colon (T^*R)^{\theta-\mathrm{st}} \to \mathfrak{M}(n,r)$ the natural morphism. The inverse functor $\widetilde{\pi}_c^{\theta,!}\colon \mathcal{A}_c^\theta(n,r)\text{-}\operatorname{mod} \to D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c}$ to $\widetilde{\pi}_c^\theta$ is given by

$$\mathcal{P} \mapsto \mathcal{Q}_c|_{(T^*R)^{\theta-\mathrm{st}}} \otimes_{p^{-1}(\mathcal{A}_c^\theta(n,r))} p^{-1}(\mathcal{P}).$$

It remains to prove that the following diagram is commutative

$$\begin{array}{ccc} D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c} & \xleftarrow{\widetilde{\pi}_c^{\theta,!}} & \mathcal{A}_c^\theta(n,r)\text{-}\operatorname{mod} \\ \downarrow \widetilde{\mathcal{T}}_{c\to c+1} & & \downarrow \mathcal{T}^\theta_{c\to c+1} \\ D_R|_{(T^*R)^{\theta-\mathrm{st}}}\text{-}\operatorname{mod}^{G,c+1} & \xrightarrow{\widetilde{\pi}_{c+1}^\theta} & \mathcal{A}_{c+1}^\theta(n,r)\text{-}\operatorname{mod} \end{array}$$

but this is clear from the definitions. □

It follows from Lemma 3.4.4 that we can assume that $\frac{m}{n} \in \mathbb{C}$ is Zariski generic, so we have an isomorphism $\mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r)) \simeq \bigoplus \bigotimes_i \mathcal{A}_{\frac{m}{n}+r-i}(n_i,1)$ (see Proposition 3.3.4), where the sum is taken over all ordered collections $(n_1,\ldots,n_r)$ of non-negative integers such that $n_1+\ldots+n_r = n$.

According to [75, Sections 6.3, 6.3] the module $L(\varnothing,\ldots,\varnothing,n_0\lambda')$ can be described as follows. Let $L^A(n_0\lambda')$ be the module in category $\mathcal{O}$ over $\mathcal{A}_{\frac{m}{n}}(n,1)$ corresponding to $n_0\lambda'$. We have the projection $\kappa\colon \mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r)) \twoheadrightarrow \mathcal{A}_{\frac{m}{n}}(n,1)$ which makes $L^A(n_0\lambda')$ a module over $\mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r))$. Then $L(\varnothing,\ldots,\varnothing,n_0\lambda')$ is the maximal quotient of $\Delta_{\nu_0}(L^A(n_0\lambda'))$ that does not intersect the highest weight space $L^A(n_0\lambda')$, where

$$\Delta_{\nu_0}(L^A(n_0\lambda')) := \mathcal{A}_{\frac{m}{n}}(n,r) \otimes_{\mathcal{A}^{\geqslant 0,\nu_0}_{\frac{m}{n}}(n,r)} L^A(n_0\lambda').$$

We conclude that $L(\varnothing,\ldots,\varnothing,n_0\lambda')^{\mathrm{hw}} \simeq L^A(n_0\lambda')$ as modules over $\mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r))$, here $L_{\frac{m}{n},r}(\varnothing,\ldots,\varnothing,n_0\lambda')^{\mathrm{hw}}$ is the highest weight component of $L_{\frac{m}{n},r}(n_0\lambda)$ w.r.t. ν_0. Note also that it follows from [26, Proposition 7.8] that we have an isomorphism between $\mathcal{A}_{\frac{n}{m}}(n,1)$-modules $L_{\frac{n}{m},1}(n_0\lambda')$ and $L^A(n_0\lambda')$. So to show that $\lambda' = \lambda$ it is enough to check that $L_{\frac{m}{n},r}(n_0\lambda)^{\mathrm{hw}} \simeq L_{\frac{m}{n},1}(n_0\lambda)$ as a module over $\mathcal{A}_{\frac{m}{n}}(n,1) \hookrightarrow \mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r))$.

Note that we have a natural isomorphism of vector spaces $L_{\frac{m}{n},r}(n_0\lambda)^{\mathrm{hw}} \simeq L_{\frac{m}{n},1}(n_0\lambda)$. To see this let us denote by $W^{\mathrm{lw}} \subset W$ the lowest weight component with respect to the $\mathbb{C}^\times$-action via ν_0. Set $R^{\mathrm{lw}} := \mathfrak{gl}(V) \oplus \mathrm{Hom}(V, W^{\mathrm{lw}}) \subset R$, $M := M(O(n_0\lambda))$. We have a tautological identification $\overline{L}_{\frac{m}{n},r}(n_0\lambda)^{\mathrm{hw}} = (M \otimes \mathbb{C}[\mathrm{Hom}(V, W^{\mathrm{lw}})])^{\mathrm{GL}(V)} = \overline{L}_{\frac{m}{n},1}(n_0\lambda)$. It remains to prove the following proposition.

Proposition 3.4.5 *The homomorphism $\kappa\colon \mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r)) \twoheadrightarrow \mathcal{A}_{\frac{m}{n}}(n,1)$ intertwines the actions*

$$\mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r)) \curvearrowright L_{\frac{m}{n},r}(n_0\lambda)^{hw} = L_{\frac{m}{n},1}(n_0\lambda) \curvearrowleft \mathcal{A}_{\frac{m}{n}}(n,1). \tag{3.14}$$

We will prove this Proposition in the end of this section. Recall that the cocharacter $\nu_0\colon \mathbb{C}^\times \to T$ is given by $t \mapsto (1, t^{d_1}, \ldots, t^{d_r})$ with $d_1 \gg \ldots \gg d_r$. Consider now the following cocharacter $\nu_0'\colon \mathbb{C}^\times \to T$, $t \mapsto (1, t^{d_1-d_r}, t^{d_2-d_r} \ldots, t^{d_{r-1}-d_r}, 1)$. Note that the $\mathbb{C}^\times$-actions on $\mathfrak{M}(n,r)$, $\mathfrak{M}^\theta(n,r)$ corresponding to ν_0, ν_0' coincide. So we have the identification $\mathsf{C}_{\nu_0'}(\mathcal{A}_{\frac{m}{n}}(n,r)) = \mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r))$. Note also that $R^{\mathrm{lw}} = R^{\nu_0'}$.

Let us now note that we have the natural isomorphism $\mathsf{C}_{\nu_0'}(D(R)) \simeq D(R^{\mathrm{lw}})$ induced by the embedding $D(R^{\mathrm{lw}}) \hookrightarrow D(R)$ which clearly intertwines the actions

$$\mathsf{C}_{\nu_0'}(D(R)) \curvearrowright (M \otimes \mathbb{C}[\mathrm{Hom}(V,W)])^{\mathrm{hw}} = M \otimes \mathbb{C}[\mathrm{Hom}(V, W^{\mathrm{lw}})] \curvearrowleft D(R^{\mathrm{lw}}). \tag{3.15}$$

Set $c := \frac{m}{n}$. Let us now understand the relation between the homomorphism

$$\kappa\colon \mathsf{C}_{\nu_0'}(D(R)/\!\!/\!\!/_{\frac{m}{n}} \mathrm{GL}(V)) \twoheadrightarrow D(R^{\mathrm{lw}})/\!\!/\!\!/_{\frac{m}{n}} \mathrm{GL}(V)$$

and the isomorphism $\mathsf{C}_{\nu_0'}(D(R)) \simeq D(R^{\mathrm{lw}})$ above. Note that both of them are induced by the corresponding isomorphisms of sheaves

$$\mathsf{C}_{\nu_0'}(\mathcal{A}^\theta_{\frac{m}{n}}(n,r))|_{\mathfrak{M}^\theta(n,1)} \overset{\kappa^\theta}{\simeq} \mathcal{A}^\theta_{\frac{m}{n}}(n,1),\ \mathsf{C}_{\nu_0'}(D_R) \simeq D_{R^{\mathrm{lw}}}.$$

We start with two general lemmas. Let $\mathcal{A}$ be an associative algebra equipped with a $\mathbb{Z}$-grading $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i$ and $J \subset \mathcal{A}$ is a $\mathbb{Z}$-graded two-sided ideal.

Lemma 3.4.6 *We have a natural isomorphism $\mathsf{C}(\mathcal{A})/[J^{\geqslant 0}] \xrightarrow{\sim} \mathsf{C}(\mathcal{A}/J)$, where C corresponds to taking the Cartan subquotient and $[J^{\geqslant 0}] \subset \mathsf{C}(\mathcal{A}) = \mathcal{A}^{\geqslant 0}/(\mathcal{A}^{\geqslant 0} \cap \mathcal{A}\mathcal{A}^{>0})$ is the image of $J^{\geqslant 0}$ under the natural morphism $J^{\geqslant 0} \to \mathsf{C}(\mathcal{A})$.*

Proof: Both of them can be naturally identified with $\mathcal{A}^{\geqslant 0}/((\mathcal{A}^{\geqslant 0} \cap \mathcal{A}\mathcal{A}^{>0}) + J^{\geqslant 0})$. □

Let $\mathcal{A}$ be an associative algebra as above and assume that the $\mathbb{Z}$-grading is induced by some element $h \in \mathcal{A}$, i.e. $\mathcal{A}^i = \{a \in \mathcal{A} \mid [h,a] = ia\}$. Let $\mathfrak{l}$ be a reductive Lie algebra. Assume that we are given a locally-finite completely reducible action $\mathfrak{l} \curvearrowright \mathcal{A}$ which commutes with the $\mathbb{Z}$-grading and is induced by a map of Lie algebras $\phi\colon \mathfrak{l} \to \mathrm{Der}(\mathcal{A})$. Assume also that we have a quantum comoment map $\upsilon\colon \mathcal{U}(\mathfrak{l}) \to \mathcal{A}$ for this action, i.e. a homomorphism of algebras υ such that $[\upsilon(x), -] = \phi(x)(-)$, $\forall x \in \mathfrak{l}$.

Remark 3.4.7 *Note that the image of υ lies in $\mathcal{A}^0$. Indeed, pick $x \in \mathfrak{l}$, we have to show that $\upsilon(x) \in \mathcal{A}^0$. Note that $h \in \mathcal{A}^0$ and $\phi(x)(\mathcal{A}^0) \subset \mathcal{A}^0$ so we must have $[h, \upsilon(x)] \in \mathcal{A}_0$. We can decompose $\upsilon(x) = \sum_{i\in\mathbb{Z}} a_i$ with $a_i \in \mathcal{A}_i$ and note that $[h,\upsilon(x)] = \sum_{i\in\mathbb{Z}} ia_i$ lies in $\mathcal{A}^0$ only if $a_i = 0$ for every $i \neq 0$. The claim follows.*

Fix a character $\lambda\colon \mathfrak{l} \to \mathbb{C}$. Let $P \subset \mathcal{A}$ be the left ideal generated by $\{\xi(x) - \lambda(x) \mid x \in \mathfrak{l}\}$. We define $\mathcal{A}/\!\!/\!\!/_\lambda \mathfrak{l} := (\mathcal{A}/P)^{\mathfrak{l}} = \mathcal{A}^{\mathfrak{l}}/P^{\mathfrak{l}}$. We analogously define $\mathsf{C}(\mathcal{A})/\!\!/\!\!/_\lambda \mathfrak{l}$ using the quantum comoment map $[\upsilon]\colon \mathcal{U}(\mathfrak{l}) \to \mathsf{C}(\mathcal{A})$ which is well defined by Remark 3.4.7. Note that $P^{\mathfrak{l}} \subset \mathcal{A}^{\mathfrak{l}}$ is a two-sided ideal.

Lemma 3.4.8 *For any character $\lambda\colon \mathfrak{l} \to \mathbb{C}$ we have a natural epimorphism*

$$\mathsf{C}(\mathcal{A}/\!\!/\!\!/_\lambda \mathfrak{l}) \twoheadrightarrow \mathsf{C}(\mathcal{A})/\!\!/\!\!/_\lambda \mathfrak{l}.$$

Proof: By Lemma 3.4.6 (applied to $\mathcal{A}^{\mathfrak{l}} \supset P^{\mathfrak{l}}$) we have $\mathsf{C}(\mathcal{A}/\!\!/\!\!/_\lambda \mathfrak{l}) = \mathsf{C}(\mathcal{A}^{\mathfrak{l}})/[(P^{\geqslant 0})^{\mathfrak{l}}]$. Note also that by the definitions $\mathsf{C}(\mathcal{A})/\!\!/\!\!/_\lambda \mathfrak{l} = \mathsf{C}(\mathcal{A})^{\mathfrak{l}}/[(P^{\geqslant 0})^{\mathfrak{l}}]$. Now the claim follows from the fact that $\mathrm{Id}\colon \mathcal{A} \to \mathcal{A}$ induces a surjective homomorphism $\mathsf{C}(\mathcal{A}^{\mathfrak{l}}) \twoheadrightarrow \mathsf{C}(\mathcal{A})^{\mathfrak{l}}$. $\square$

Using Lemma 3.4.8 and the fact that the open sets $((T^*R)_f /\!\!/\!\!/^\theta \mathrm{GL}(V))^{\nu_0'(\mathbb{C}^\times)}$, where $f \in \mathbb{C}[T^*R]^{\nu_0'(\mathbb{C}^\times)} \cap \mathbb{C}[T^*R]^{G,\theta}$ is homogeneous of positive degree, form a basis for the conical Zariski topology on $\mathfrak{M}^\theta(n,r)^{\nu_0'(\mathbb{C}^\times)}$, we obtain a homomorphism $\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,r)} \to \mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V)$ of sheaves on $T^*R^{\mathrm{lw}} /\!\!/\!\!/^\theta \mathrm{GL}(V)$.

Lemma 3.4.9 *The homomorphism $\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,1)} \to \mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V)$ is an isomorphism.*

Proof: Note that $\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,1)}$, $\mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V)$ are filtered and by [73, Proposition 5.2 (2)] we have

$$\mathrm{gr}\, \mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c G)|_{\mathfrak{M}^\theta(n,1)} = \mathcal{O}_{\mathfrak{M}^\theta(n,1)} = \mathrm{gr}\, \mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V).$$

The homomorphism $\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,1)} \to \mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V)$ preserves filtrations and the associated graded equals to $\mathrm{Id}\colon \mathcal{O}_{\mathfrak{M}^\theta(n,1)} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{M}^\theta(n,1)}$. The claim follows. $\square$

Lemma 3.4.9 gives us an explicit construction of an isomorphism

$$\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,1)} \xrightarrow{\sim} \mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V).$$

Note that the sheaf $\mathsf{C}_{\nu_0'}(D_R /\!\!/\!\!/^\theta_c \mathrm{GL}(V))|_{\mathfrak{M}^\theta(n,1)}$ is exactly $\mathsf{C}_{\nu_0'}(\mathcal{A}^\theta_c(n,r))$. We claim that the sheaves $\mathcal{A}^\theta_c(n,1)$, $\mathsf{C}_{\nu_0'}(D_R)/\!\!/\!\!/^\theta_c \mathrm{GL}(V)$ are canonically isomorphic and the isomorphism between them is induced by the isomorphism $D(R^{\mathrm{lw}}) \xrightarrow{\sim} \mathsf{C}_{\nu_0'}(D(R))$. To see that it is enough to prove the following lemma.

Lemma 3.4.10 *The isomorphism $D(R^{lw}) \xrightarrow{\sim} \mathsf{C}_{\nu_0'}(D(R))$ induces an isomorphism $I_{lw} \xrightarrow{\sim} [I^{\geqslant 0,\nu_0'}]$, where*

$$I = D(R)\{\xi_R - c\,\mathrm{tr}\,\xi \mid \xi \in \mathfrak{gl}(V)\},\ \text{and}\ I_{lw} = D(R^{lw})\{\xi_{R^{lw}} - c\,\mathrm{tr}\,\xi \mid \xi \in \mathfrak{gl}(V)\}.$$

Proof: It is enough to check that that the isomorphism $D(R^{\mathrm{lw}}) \xrightarrow{\sim} \mathsf{C}_{\nu_0'}(D(R))$ induces a surjective map $I_{\mathrm{lw}} \twoheadrightarrow [I^{\geqslant 0,\nu_0'}]$. The ideal I_{lw} is generated by the elements of the form $\xi_{R^{\mathrm{lw}}} - c\,\mathrm{tr}\,\xi$, $\xi \in \mathfrak{gl}(V)$. Note that $\xi_R - \xi_{R^{\mathrm{lw}}} \in D(R)^{>0,\nu_0'}$ because the action of $\nu_0'(\mathbb{C}^\times)$ contracts R to $R^{\mathrm{lw}} = R^{\nu_0'(\mathbb{C}^\times)}$. It then follows that, in the Cartan subquotient $\mathsf{C}_{\nu_0'}(D(R)) \simeq D(R)^{\geqslant 0,\nu_0'}/(D(R)^{\geqslant 0,\nu_0'} \cap D(R)D(R)^{>0,\nu_0'})$ we have $[\xi_{R^{\mathrm{lw}}} - c\,\mathrm{tr}\,\xi] = [\xi_R - c\,\mathrm{tr}\,\xi]$. The claim follows.

□

Proof:[Proof of Proposition 3.4.5] Combining Lemmas 3.4.8, 3.4.9, 3.4.10 and using the constructions therein we see that the isomorphism

$$\kappa^\theta\colon \mathsf{C}_{\nu_0'}(\mathcal{A}^\theta_{\frac{m}{n}}(n,r))|_{\mathfrak{M}^\theta(n,r)} \simeq \mathcal{A}^\theta_{\frac{m}{n}}(n,1)$$

is induced by the natural embedding $D(R^{\mathrm{lw}}) \hookrightarrow D(R)$. Now the desired statement about the intertwining property of $\kappa\colon \mathsf{C}_{\nu_0}(\mathcal{A}_{\frac{m}{n}}(n,r)) \twoheadrightarrow \mathcal{A}_{\frac{m}{n}}(n,1)$ follows from the fact that the isomorphism $\mathsf{C}_{\nu_0'}(D(R)) \simeq D(R^{\mathrm{lw}})$ intertwines the actions in (3.15). □

3.5 Character of $L_{\frac{m}{n},r}(n_0\lambda) = L_\nu(\varnothing,\ldots,\varnothing,n_0\lambda)$

Our construction of $L_{\frac{m}{n},r}(n_0\lambda)$ allows us to compute its character. Recall that we have set $m_0 := m/\gcd(m,n)$ and $n_0 := n/\gcd(m,n)$.

3.5.1 Characters of minimally supported modules over $H_{\frac{n}{m}}(m,1)$

Recall that $S_{\frac{n}{m}}(m_0\lambda)$ is the irreducible highest weight module over $H_{\frac{n}{m}}(m,1)$ with highest weight $m_0\lambda$. The character of $S_{\frac{n}{m}}(m_0\lambda)$ was computed in [26, Theorem 1.4]. Let us recall the answer. Let Λ be the ring of symmetric functions on infinitely many variables $z_1, z_2, \ldots$. For a partition β of m we define a constant c^β_{λ,m_0} by

$$s_\lambda(z_1^{m_0}, z_2^{m_0}, \ldots) = \sum_\beta c^\beta_{\lambda,m_0} s_\beta(z_1, z_2, \ldots),$$

where s_λ, $s_\beta \in \Lambda$ are the corresponding Schur polynomials.

Proposition 3.5.1 *The class $[S_{\frac{n}{m}}(m_0\lambda)] \in K_0(\mathcal{O}(H_{\frac{n}{m}}(m,1)))$ is given by the formula*

$$[S_{\frac{n}{m}}(m_0\lambda)] = \sum_{\beta\vdash m} c^\beta_{\lambda,m_0}[\Delta_{\frac{n}{m}}(\beta)],$$

where $\Delta_{\frac{n}{m}}(\beta)$ is the standard object with highest weight β in the category $\mathcal{O}(H_{\frac{n}{m}}(m,1))$.

Proof: The module $\Delta_{\frac{n}{m}}(\beta)$ is the graded dual of the co-standard module $\nabla_{\frac{n}{m}}(\beta) \in \mathcal{O}(H_{\frac{n}{m}}(m,1),\mathfrak{h})$ of modules with locally nilpotent action of $\mathfrak{h}$. In the category $\mathcal{O}$ for $H_{\frac{n}{m}}(m,1)$, the classes in K_0 of standard and co-standard modules coincide. Since taking the graded dual preserves the labels in category $\mathcal{O}$, the result now follows from [26, Theorem 1.4]. □

Recall that for a finite dimensional representation V of S_m its Frobenius character is

$$\mathrm{ch}_{S_m} V := \frac{1}{m!}\sum_{\sigma\in S_m} \mathrm{Tr}_V(\sigma) p_1^{k_1(\sigma)} \ldots p_l^{k_l(\sigma)} \in \Lambda, \tag{3.16}$$

here $p_i \in \Lambda$ are power sums, $k_i(\sigma)$ is the number of cycles of length i in σ, and Λ is the algebra of symmetric functions. For a partition β of m the Frobenius character of the irreducible representation V_β is given by the Schur polynomial $s_\beta \in \Lambda$. We will use plethystic notation, so that $f\left[\frac{X}{1-q}\right]$ denotes the image of $f \in \Lambda$ under the automorphism that sends power sums p_k to $p_k\left[\frac{X}{1-q}\right] = \frac{p_k}{1-q^k}$.

Lemma 3.5.2 *For a partition β of m we have*

$$\mathrm{ch}_{q,S_m}(\Delta_{\frac{n}{m}}(\beta)) = (1-q^{-1})q^{-\frac{m-1}{2}+\frac{n}{m}\kappa(\beta)} s_\beta\left[\frac{X}{1-q^{-1}}\right],$$

where $\kappa(\beta)$ is the sum of contents of all boxes of β.

Proof: It follows from [5] that the highest weight component of $\Delta_{\frac{n}{m}}(\beta)$ has weight $q^{\frac{n}{m}\kappa(\beta)-\frac{m-1}{2}}$. The module $\Delta_{\frac{n}{m}}(\beta)$ is isomorphic to $V_\beta \otimes \mathbb{C}[\mathfrak{h}]$ as $S_m \times \mathbb{C}^\times$-module and the $\mathbb{C}^\times$-action corresponds to the shifted standard negative grading $\mathbb{C}[\mathfrak{h}] = \bigoplus_{k\geqslant 0} S^k(\mathfrak{h}^*)$, $\deg(S^k(\mathfrak{h}^*)) = -k - \frac{m-1}{2} + \frac{n}{m}\kappa(\beta)$. Consider now a permutation $\sigma \in S_m$. It is clear that $\det_{\mathfrak{h}}(1-q^{-1}\sigma) = \frac{1}{1-q^{-1}}\prod_i (1-q^{-i})^{k_i(\sigma)}$. Note also that

$$\mathrm{Tr}_{V_\nu\otimes\mathbb{C}[\mathfrak{h}]}(\sigma q^h) = \frac{\mathrm{Tr}_{V_\nu}(\sigma)}{\det_{\mathfrak{h}}(1-q^{-1}\sigma)} = (1-q^{-1})\frac{\mathrm{Tr}_{V_\beta}(\sigma)}{\prod_i(1-q^{-i})^{k_i(\sigma)}}.$$

We conclude that

$$\mathrm{ch}_{q,S_m}(\Delta_{\frac{n}{m}}(\beta)) = \frac{1}{m!}\sum_{\sigma\in S_m}(1-q^{-1})q^{\frac{n}{m}\kappa(\beta)-\frac{m-1}{2}}\frac{\mathrm{Tr}_{V_\beta}(\sigma)\prod_i p_i^{k_i(\sigma)}}{\prod_i(1-q^{-i})^{k_i(\sigma)}} =$$

$$= (1-q^{-1})q^{\frac{n}{m}\kappa(\beta)-\frac{m-1}{2}} s_\beta\left[\frac{X}{1-q^{-1}}\right].$$

□

Corollary 3.5.3 *The q-graded S_m-character of $S_{\frac{n}{m}}(m_0\lambda)$ is given by*

$$\mathrm{ch}_{q,S_m}(S_{\frac{n}{m}}(m_0\lambda)) = (1-q^{-1})\sum_{\beta\vdash m} c^\beta_{\lambda,m_0} q^{-\frac{m-1}{2}+\frac{n}{m}\kappa(\beta)} s_\beta\left[\frac{X}{1-q^{-1}}\right].$$

Proof: Follows from Proposition 3.5.1 and Lemma 3.5.2. □

Computation of the character of $L_{\frac{m}{n},r}(n_0\lambda)$

Let us now finally compute the q-graded GL_r character of the module $L_{\frac{m}{n},r}(n_0\lambda)$. Recall that we have a q-graded GL_r-equivariant isomorphism

$$L_{\frac{m}{n},r}(n_0\lambda) \xrightarrow{\sim} (S_{\frac{n}{m}}(m_0\lambda)\otimes(\mathbb{C}^{r*})^{\otimes m})^{S_m}. \tag{3.17}$$

Proposition 3.5.4 *We have*

$$\mathrm{ch}_{q,\mathrm{GL}_r}(L_{\frac{m}{n},r}(n_0\lambda)) =$$

$$= (1-q^{-1}) \sum_{\substack{\mathrm{r}(\mu)\leqslant \min(n,r)\\ \mu,\beta\vdash m}} c^{\beta}_{\lambda,m_0} q^{-\frac{m-1}{2}+\frac{n}{m}\kappa(\beta)} \langle s_\beta \left[\frac{X}{1-q^{-1}}\right], s_\mu\rangle [W_r(\mu)^*],$$

where $\langle\, ,\, \rangle$ *is the Hall inner product on* Λ, *i.e. the inner product with respect to which* $\langle s_\alpha, s_\gamma\rangle = \delta_{\alpha\gamma}$ *for any two partitions* α, γ.

Proof: By Corollary 3.5.3 we have

$$\mathrm{ch}_{q,S_m}(S_{\frac{n}{m}}(m_0\lambda^t)) = (1-q^{-1}) \sum_{\beta,|\beta|=m} c^{\beta}_{\lambda,m_0} q^{-\frac{m-1}{2}+\frac{n}{m}\kappa(\beta)} s_\beta \left[\frac{X}{1-q^{-1}}\right].$$

By Schur-Weyl duality we have

$$\mathrm{ch}_{S_m\times \mathrm{GL}_r}((\mathbb{C}^{r*})^{\otimes m}) = \sum_{\substack{\mathrm{r}(\beta)\leqslant \min(n,r)\\ |\beta|=m}} s_\beta[W_r(\beta)^*].$$

So from (3.17) we obtain the desired equality. □

Chapter 4

Bethe subalgebras, Kirillov-Reshetikhin crystals and monodromy

4.1 Introduction

Results of this Chapter are joint with Leonid Rybnikov and Inna Mashanova-Golikova. Let $\mathfrak{g}$ be a complex simple finite dimensional Lie algebra and G be the adjoint Lie group with the Lie algebra $\mathfrak{g}$. To every $C \in G$ one can associate a commutative subalgebra $B(C)$ in the Yangian $Y(\mathfrak{g})$, which is responsible for the integrals of the (generalized) XXX Heisenberg magnet chain. Using the approach of [43], we construct a natural structure of affine crystals on spectra of $B(C)$ in Kirillov-Reshetikhin $Y(\mathfrak{g})$-modules in type A. We conjecture that such a construction exists for arbitrary $\mathfrak{g}$ and gives Kirillov-Reshetikhin crystals. Our main technical tool is the degeneration of Bethe subalgebras in the Yangian to commutative subalgebras $\mathcal{A}^{\mathrm{u}}_{\chi}$ in the universal enveloping of the current Lie algebra, $U(\mathfrak{g}[t])$, which depend on the parameter χ from the Lie algebra $\mathfrak{g}$ (and are of independent interest). We show that these subalgebras come from the Feigin-Frenkel center on the critical level as described by Feigin, Frenkel and Toledano Laredo in [29]. This allows to prove that our affine crystals in type A are indeed Kirillov-Reshetikhin by reducing to the crystal structure on the spectra of inhomogeneous Gaudin model which is already known ([43]). As an application, we obtain the description of the monodromy of eigenvalues of quantum multiplication operators for type A quiver varieties. This chapter is self-consistent and the notations slightly differ from the notations used in the previous chapters.

4.1.1 $\mathfrak{g}$-crystals and Gaudin subalgebras

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. Kashiwara $\mathfrak{g}$-crystals are combinatorial objects which model bases of representations of $\mathfrak{g}$. Namely, a $\mathfrak{g}$-crystal B is a set equipped with operators $\mathrm{e}_i, \mathrm{f}_i \colon B \to B \cup \{0\}$ attached to every simple root α_i of $\mathfrak{g}$, together with the weight function wt, satisfying certain conditions (see Section 4.9 for details). In particular, attached to each irreducible finite dimensional representation V_λ of $\mathfrak{g}$, we have a connected crystal B_λ. Given two $\mathfrak{g}$-crystals B, B', one can form their tensor product $B \otimes B'$. Crystals B_λ (more generally $B_{\lambda_1} \otimes \ldots \otimes B_{\lambda_k}$) can be constructed with the help of so-called inhomogeneous Gaudin subalgebras of $U(\mathfrak{g})^{\otimes k}$ (see [43] for details). Let us recall the latter and also recall the results of [43] that are important for us.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. To every $\chi \in \mathfrak{h}$ and distinct $z_1, \dots, z_k \in \mathbb{C}$ one can associate a commutative subalgebra $\mathcal{A}_\chi(\underline{z}) = \mathcal{A}_\chi(z_1, \dots, z_k) \subset U(\mathfrak{g})^{\otimes k}$ (so-called *inhomogeneous Gaudin subalgebra* of $U(\mathfrak{g})^{\otimes k}$, see, for example, [43, Section 9] and references therein). For $\chi \in \mathfrak{h}^{\mathrm{reg}}$, the algebra $\mathcal{A}_\chi(z_1, \dots, z_k)$ is a maximal commutative subalgebra of $U(\mathfrak{g})^{\otimes k}$ and $\mathfrak{h} \subset \mathcal{A}_\chi(z_1, \dots, z_k)$. These commutative subalgebras describe the quantum integrable spin chain called Gaudin magnet, see [25], [28], [29].

Let $\lambda_1, \dots, \lambda_k$ be dominant weights of $\mathfrak{g}$. We have the natural action $\mathcal{A}_\chi(z_1, \dots, z_k) \curvearrowright V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}$, where V_{λ_i} is the irreducible representation of $\mathfrak{g}$, corresponding to the dominant weight λ_i.

The following proposition holds by [43].

Proposition 4.1.1 *For every dominant $\lambda_1, \dots, \lambda_k$ and $\chi \in \mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}$, $z_1, \dots, z_k \in \mathbb{R}$, the action of $\mathcal{A}_\chi(\underline{z})$ on the tensor product $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}$ has a simple spectrum.*

Let us denote by $\mathcal{E}_\chi(\underline{\lambda})$ the set of eigenlines of $\mathcal{A}_\chi(\underline{z})$, acting on $\underline{V} := V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}$. Our goal for now is to describe the natural crystal structure on $\mathcal{E}_\chi(\underline{\lambda})$ (following [43]).

To every positive root α of $\mathfrak{g}$ we can associate the corresponding wall

$$H_\alpha := \{x \in \mathfrak{h}_{\mathbb{R}},\ \langle \alpha, x \rangle = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

Walls H_α separate $\mathfrak{h}_{\mathbb{R}}$ into (closed) chambers O_w parametrized by the elements w of the Weyl group W of $\mathfrak{g}$. We set $O_w^{\mathrm{reg}} := O_w \cap \mathfrak{h}^{\mathrm{reg}}$ (O_w^{reg} is the interior of O_w). Recall that $\chi \in \mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}$ and our goal is to describe the crystal structure on the set $\mathcal{E}_\chi(\underline{\lambda})$. For the simplicity of notation, we assume that $\chi \in O_1$. Recall that

$$O_1 = \{x \in \mathfrak{h}_{\mathbb{R}},\ \langle \alpha_i, x \rangle \geqslant 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

Recall that $\mathfrak{h} \subset \mathcal{A}_\chi(\underline{z})$, so every eigenline for $\mathcal{A}_\chi(\underline{z})$ must be an $\mathfrak{h}$-eigenline of some weight μ. Thus, for $L \in \mathcal{E}_\chi(\underline{\lambda})$, we can define $\mathrm{wt}(L) = \mu$ if L has $\mathfrak{h}$-weight μ.

Let us now define the crystal operators e_j, f_j for $\mathcal{E}_\chi(\underline{\lambda})$. Pick an element $\chi_0 \in H_{\alpha_i}$, lying in the interior of $H_{\alpha_i} \cap O_1$. Then the algebra $\mathcal{A}_{(\chi_0,\chi)} := \lim\limits_{\epsilon \to 0} \mathcal{A}_{\chi_0 + \epsilon\chi}(\underline{z})$ (see Section 4.2 for the discussion of limits of families of subalgebras) is generated by $\mathcal{A}_{\chi_0}(\underline{z})$ and the element $\Delta^k(h_{\alpha_i}) \in U(\mathfrak{g})^{\otimes k}$ (same argument as in the proof of [43, Lemma 10.9] works), where $\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes k}$ is the (iterated) comultiplication homomorphism and $h_{\alpha_i} \in \mathfrak{h}$ is the coroot, corresponding to α_i. We can decompose $\underline{V} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}$ as $\mathcal{A}_{\chi_0}(\underline{z})$-module in the direct sum of weight spaces $\underline{V} = \bigoplus_{\eta\colon \mathcal{A}_{\chi_0}(\underline{z}) \to \mathbb{C}} \underline{V}^\eta$. Since the action of $\mathcal{A}_{(\chi_0,\chi)}(\underline{z})$ on $\underline{V}$ has a simple spectrum (see [43, Section 11.1]), it follows that the action $h_{\alpha_i} \curvearrowright \underline{V}^\eta$ has a simple spectrum i.e. $\underline{V}^\eta = \bigoplus_{i \in \mathbb{Z}} \underline{V}_i^\eta$ with $\underline{V}_i^\eta$ being one-dimensional.

Recall also that the set $\{\underline{V}_i^\eta\}$ of eigenlines of $\mathcal{A}_{(\chi_0,\chi)}(\underline{z})$ on $\underline{V}$ canonically identifies with $\mathcal{E}_\chi(\underline{\lambda})$. We define

$$\mathrm{e}_i(\underline{V}_i^\eta) := \underline{V}_{i+1}^\eta,\ \mathrm{f}_i(\underline{V}_i^\eta) := \underline{V}_{i-1}^\eta.$$

This allows to define the $\mathfrak{g}$-crystal structure on the set $\mathcal{E}_\chi(\underline{\lambda})$ of eigenlines of $\mathcal{A}_\chi(\underline{z})$, acting on $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_k}$. Let us describe the crystal $\mathcal{E}_\chi(\underline{\lambda})$. We start from the case $k = 1$ i.e. $\underline{\lambda} = \lambda_1 = \lambda$. The following theorem is [43, Theorem 12.3].

Theorem 4.1.2 *For every dominant λ and regular $\chi \in O_1$, there is an isomorphism of $\mathfrak{g}$-crystals $\mathcal{E}_\chi(\lambda) \simeq B_\lambda$.*

One can generalize Theorem 4.1.2 to the case of arbitrary k-tuples of dominant weights $\lambda_1, \ldots, \lambda_k$. The following theorem follows from the results of [43].

Theorem 4.1.3 *For $z_1 < \ldots < z_k$ and regular $\chi \in O_1$, we have a canonical isomorphism of $\mathfrak{g}$-crystals*

$$\mathcal{E}_\chi(\underline{\lambda}) \simeq \mathcal{E}_\chi(\lambda_1) \otimes \ldots \otimes \mathcal{E}_\chi(\lambda_k)$$

so we obtain the isomorphism of $\mathfrak{g}$-crystals

$$\mathcal{E}_\chi(\underline{\lambda}) \simeq B_{\lambda_1} \otimes \ldots \otimes B_{\lambda_k}.$$

Remark 4.1.4 *This relation between Bethe ansatz in the Gaudin spin chain and crystal bases was first noticed by Varchenko in [111], for the case $\mathfrak{g} = \mathfrak{sl}_2$.*

4.1.2 Bethe subalgebras

Gaudin magnet is known to be a degenerate version of the XXX Heisenberg spin chain, whose integrals come from *Bethe subalgebras* in the Yangian $Y(\mathfrak{g})$, a certain Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g}[t])$ (see Section 4.7 for the definition of $Y(\mathfrak{g})$). Let G be the adjoint group with Lie algebra $\mathfrak{g}$. To every $C \in G$ one can associate the Bethe subalgebra $B(C) \subset Y(\mathfrak{g})$ that is a commutative subalgebra of the Yangian (see Section 4.8 for details).

Motivated by the classical results of Kirillov and Reshetikhin on the combinatorial description of asymptotics of solutions to Bethe ansatz equations for the XXX Heisenberg chain [63], we aim to formulate and prove a statement similar to Theorem 4.1.3 with Gaudin subalgebras being replaced by Bethe subalgebras in Yangian and $\mathfrak{g}$-crystals by Kirillov-Reshetikhin crystals (i.e. certain finite $\hat{\mathfrak{g}}$-crystals). For this, we first relate Bethe subalgebras and Gaudin subalgebras for arbitrary simple $\mathfrak{g}$ (see Theorem 4.8.12). We then restrict to type A and (using Theorem 4.8.12) prove the precise statement similar to Theorem 4.1.3 (see Theorem 4.1.6 below). It would be very interesting to generalize our results to the case of arbitrary simple Lie algebra $\mathfrak{g}$.

4.1.3 Gaudin subalgebras as limits of Bethe subalgebras

In this section we describe the precise relation between Bethe subalgebras in Yangians and (universal) inhomogeneous Gaudin subalgebras of $U(\mathfrak{g}[t])$.

Recall that $\mathcal{A}_\chi(\underline{z})$ is a subalgebra of $U(\mathfrak{g})^{\otimes k}$. It turns out that $\mathcal{A}_{-\chi}(\underline{z})$ can be realized as $\mathrm{ev}_{\underline{z}}(\mathcal{A}^{\mathrm{u}}_\chi)$, where $\mathrm{ev}_{\underline{z}} \colon U(\mathfrak{g}[t]) \to U(\mathfrak{g})^{\otimes k}$ is the evaluation homomorphism (at the points $z_1, \ldots, z_k$) and $\mathcal{A}^{\mathrm{u}}_\chi \subset U(\mathfrak{g}[t])$ is a certain subalgebra of $U(\mathfrak{g}[t])$, which we call the *inhomogeneous universal Gaudin subalgebra*. The algebra $\mathcal{A}^{\mathrm{u}}_\chi$ is defined as the image of the Feigin-Frenkel center $\mathcal{Z}$ in the quantum Hamiltonian reduction $U(\hat{\mathfrak{g}})_{-1/2} /\!\!/\!\!/_\chi t^{-1}\mathfrak{g}[[t^{-1}]]$ (see Section 4.3.3 for details).

It turns out that the algebra $\mathcal{A}^{\mathrm{u}}_\chi$ is the limit of Bethe subalgebras in the following sense. Consider the family $B(\exp(\epsilon\chi)) \subset Y(\mathfrak{g})$, $\epsilon \in \mathbb{C}^\times$. Recall that $Y(\mathfrak{g})$ is equipped with a filtration F_2 such that $\mathrm{gr}_2 Y(\mathfrak{g}) \simeq U(\mathfrak{g}[t])$ (see Section 4.7.3 for the discussion of filtrations on $Y(\mathfrak{g})$). We can pass to the Rees algebra $Y_\hbar(\mathfrak{g})$, corresponding to the filtration above ($Y_\hbar(\mathfrak{g})$ is nothing else but the well-known homogeneous version of the Yangian $Y(\mathfrak{g})$). Then, for every ϵ as above, we obtain the algebra $Y_\epsilon(\mathfrak{g}) := Y_\hbar(\mathfrak{g})/(\hbar - \epsilon)$. For $\epsilon \in \mathbb{C}^\times$, we have the natural isomorphism $Y(\mathfrak{g}) \xrightarrow{\sim} Y_\epsilon(\mathfrak{g})$ and denote by $B_\epsilon(\exp(\epsilon\chi)) \subset Y_\epsilon(\mathfrak{g})$ the image of $B(\exp(\epsilon\chi))$. The following Theorem gives a precise relation between Bethe subalgebras in $Y(\mathfrak{g})$ and universal inhomogeneous Gaudin subalgebras of $U(\mathfrak{g}[t])$.

Theorem 4.1.5 *We have*

$$\lim_{\epsilon \to 0} B_\epsilon(\exp(\epsilon\chi)) = \mathcal{A}^{\mathrm{u}}_\chi.$$

4.1.4 Kirillov-Reshetikhin crystals and Bethe subalgebras in type *A*

In this section we assume that $G = \mathrm{PGL}_n$ so $\mathfrak{g} = \mathfrak{sl}_n$ (we identify it with the quotient of $\mathfrak{gl}_n$ by the center). We denote by $\overline{\mathfrak{h}} \subset \mathfrak{sl}_n$ the (classes of) diagonal matrices. Recall that to every $C \in G$ one can associate the Bethe subalgebra $B(C) \subset Y(\mathfrak{g})$. Recall also that (since $\mathfrak{g}$ is of type A) we have the evaluation homomorphism $\mathbf{ev}_{\underline{z}} \colon Y(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes k}$ that depends on the collection of points $z_1, \ldots, z_k \in \mathbb{C}$.

Let $\lambda_1, \ldots, \lambda_k$ be a collection of dominant weights of $\mathfrak{g}$. The algebra $B(C)$ acts on $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_k}$ via the homomorphism $\mathbf{ev}_{\underline{z}}$, the corresponding $B(C)$-module will be denoted by $V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_k}(z_k)$. Assume that C is a regular element of the maximal compact subtorus $\overline{S} \subset \mathrm{PGL}_n$, consisting of (classes of) unitary diagonal matrices. By the results of [50] (see Section 4.14), for generic $z_1, \ldots, z_k$ in the appropriate shifts of $i\mathbb{R}$ and $\lambda_i = a_i \varpi_{b_i}$ being multiples of fundamental weights, the action of $B(C)$ on $V_{a_1\varpi_{b_1}}(z_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(z_k)$ has a simple spectrum. We denote by $\mathcal{E}_C(\underline{\lambda})$ the set of eigenlines of $B(C)$, acting on $V_{a_1\varpi_{b_1}}(z_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(z_k)$.

Consider the covering map

$$\exp \colon \overline{\mathfrak{h}}_{\mathbb{R}} \to \overline{S},\ \chi \mapsto \exp(2\pi i \chi).$$

As for the Gaudin case (see Section 4.1.1), $\overline{S}$ is separated by walls (that are root subtori of $\overline{S}$). Preimages of walls in $\overline{S}$ under the covering map exp induce the decomposition of $\overline{\mathfrak{h}}_{\mathbb{R}}$ into alcoves parametrized by the affine Weyl group, corresponding to $\mathfrak{g}$ (see Section 4.12 for details). Using the same approach as we described in Section 4.1.1 (for the Gaudin case) i.e. passing to limits $\varinjlim B(C)$ as $C \to C_0$ being generic element of the appropriate (affine) wall (see Section 4.13), we can endow $\mathcal{E}_C(\underline{\lambda})$ with a $\hat{\mathfrak{g}}$-crystal structure (see Section 4.16 for details).

The following theorem holds and should be considered as an analog of Theorem 4.1.3.

Theorem 4.1.6 *(a) For every dominant λ that is a multiple of a fundamental weight there is an isomorphism of $\hat{\mathfrak{g}}$-crystals $\mathcal{E}_C(\lambda) \simeq \mathbf{B}_\lambda$, where $\mathbf{B}_\lambda$ is the Kirillov-Reshetikhin crystal, corresponding to λ.*

(b) For a collection $\lambda_1, \ldots, \lambda_k$ of dominant weights such that every λ_i is a multiple of a fundamental weight and z_i are as above with $\operatorname{Im} z_1 \gg \operatorname{Im} z_2 \gg \ldots \gg \operatorname{Im} z_k$, we have a

canonical isomorphism

$$\mathcal{E}_C(\underline{\lambda}) \simeq \mathcal{E}_C(\lambda_1) \otimes \mathcal{E}_C(\lambda_2) \ldots \otimes \mathcal{E}_C(\lambda_k),$$

so we have the isomorphism of $\hat{\mathfrak{g}}$*-crystals*

$$\mathcal{E}_C(\underline{\lambda}) \simeq \mathbf{B}_{\lambda_1} \otimes \ldots \otimes \mathbf{B}_{\lambda_k}.$$

4.1.5 Main results and structure of the chapter

The chapter can be divided into two parts. The first part consists of Sections 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, where the universal inhomogeneous Gaudin subalgebras $\mathcal{A}_\chi^{\mathrm{u}} \subset U(\mathfrak{g}[t])$ are defined, studied and realized as limits of Bethe subalgebras, in this part $\mathfrak{g}$ is an arbitrary simple Lie algebra. Second part consists of Sections 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, 4.15, 4.16, 4.17. It illustrates one possible application of the results of the first part and is actually the main motivation for us. In this part we assume that $\mathfrak{g} = \mathfrak{sl}_n$. We discuss the action of Bethe subalgebras $B(C) \subset Y(\mathfrak{sl}_n)$ on the tensor products $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_k}$ of irreducible representations V_{λ_j} of $\mathfrak{sl}_n$. The action arises from the so-called evaluation homomorphism $\mathbf{ev}_{\underline{z}} \colon Y(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes k}$, which depends on $z_1, \ldots, z_k \in \mathbb{C}$. The action $B(C) \curvearrowright V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_k}(z_k)$ has a simple spectrum under certain conditions on $\underline{z}$, $\underline{\lambda}$ and C (for the proof of this statement we refer to [50]). We denote by $\mathcal{E}_C(\underline{\lambda})$ the set of eigenlines for the action $B(C) \curvearrowright V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_k}(z_k)$. Using the similar approach as in [43], we define on $\mathcal{E}_C(\underline{\lambda})$ the structure of the $\hat{\mathfrak{sl}}_n$-crystal and identify it with the tensor product of Kirillov-Reshetikhin crystals, corresponding to representations V_{λ_j} (see Theorem 4.17.4).

The chapter is organized as follows. In Section 4.2 we discuss various notions of limits of families of algebras (or more generally vector spaces) and recall the Rees construction. In Section 4.3 we recall various realizations of Gaudin subalgebras and their classical analogs, we then define the universal inhomogeneous Gaudin subalgebra $\mathcal{A}_\chi^{\mathrm{u}} \subset U(\mathfrak{g}[t])$ and its classical analogue $\overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])$. In Section 4.4 we consider the image of the algebra $\mathcal{A}_\chi^{\mathrm{u}}$ under the evaluation homomorphism $\mathrm{ev}_{z_1,\ldots,z_k} \colon U(\mathfrak{g}[t]) \to U(\mathfrak{g})^{\otimes k}$ and identify it with the so-called inhomogeneous Gaudin subalgebra of $U(\mathfrak{g})^{\otimes k}$ (see Proposition 4.4.10). In Section 4.5 we compute the Poincaré series of algebras $\mathcal{A}_\chi^{\mathrm{u}}$, $\overline{\mathcal{A}}_\chi^{\mathrm{u}}$ (see Propositions 4.5.7, 4.5.8). In Section 4.6 we realize $\mathcal{A}_\chi^{\mathrm{u}}$, $\overline{\mathcal{A}}_\chi^{\mathrm{u}}$ as centralizers of certain quadratic elements $\tilde{\Omega}_\chi \in \mathcal{A}_\chi^{\mathrm{u}}$, $\Omega_\chi \in \overline{\mathcal{A}}_\chi^{\mathrm{u}}$ (see Proposition 4.6.6), this is a generalization of the similar result for $\chi = 0$ (see [53, Theorem 5.1]). In Section 4.7 we recall the definition of the Yangian $Y(\mathfrak{g})$, corresponding to $\mathfrak{g}$ and discuss its RTT-realization. We also discuss various filtrations on the Yangian and recall the description of the associated graded and bigraded algebras. In the end of Section 4.7 we recall some facts from the representation theory of $Y(\mathfrak{g})$. In Section 4.8 we recall the definition of Bethe subalgebras $B(C) \subset Y(\mathfrak{g})$. The main result of this section is the realization of the universal inhomogeneous Gaudin subalgebra $\mathcal{A}_\chi^{\mathrm{u}}$ as an explicit limit of certain Bethe subalgebras (see Theorem 4.8.12). In Section 4.9 we recall the notion of $\mathfrak{sl}_n$, $\hat{\mathfrak{sl}}_n$-crystals and Kirillov-Reshetikhin crystals. The main result of this section is Proposition 4.9.14 (that is certainly well-known to experts). Section 4.10 recalls some properties of Yangians $Y(\mathfrak{sl}_n)$, $Y(\mathfrak{gl}_n)$, Bethe and Gaudin subalgebras, evaluation homomorphisms. In Section 4.11 we study generators of the universal inhomogeneous Gaudin subalgebras $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi \subset U(\mathfrak{gl}_n[t])$ (see Proposition 4.11.5). As a corollary, we obtain generators of the inhomogeneous Gaudin subalgebras $\tilde{\mathcal{A}}_\chi(z_1,\ldots,z_k) \subset U(\mathfrak{gl}_n)^{\otimes k}$ (see Corollary 4.11.7 and Proposition 4.11.8). In Section 4.12 we recall affine and extended affine Weyl groups of type A, alcoves and (affine) walls. In Section 4.13 we study limits of Bethe and Gaudin subalgebras to generic points of a wall. In Section 4.14 we formulate the results of [50] that give a criterion for the action of Bethe subalgebras on the tensor product $V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_k}(z_k)$ to have a simple spectrum. In Section 4.15 we study the image in $\mathrm{End}(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_k})$ of a certain two-parametric family. In Section 4.16 we define the structure of $\hat{\mathfrak{sl}}_n$-crystal on the set $\mathcal{E}_C(\underline{\lambda})$ of eigenlines of $B(C) \curvearrowright V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_k}(z_k)$. In Section 4.17 we prove that the crystal $\mathcal{E}_C(\underline{\lambda})$ is isomorphic to the tensor product $\mathbf{B}_{\lambda_1} \otimes \ldots \otimes \mathbf{B}_{\lambda_k}$ of Kirillov-Reshetikhin crystals. In Section 4.18, we compute the monodromy of the covering $\mathcal{E}_C(\underline{\lambda})$ and obtain the explicit description of the monodromy of eigenvalues of quantum multiplication operators for type A Nakajima quiver varieties.

4.2 Rees construction and various limits

4.2.1 Limits of families of subspaces of a fixed vector space

In this section we define limits of families of subspaces of a filtered vector space and discuss some properties of this construction that will be useful later. Let U be a vector space over $\mathbb{C}$ equipped with an increasing $\mathbb{Z}_{\geqslant 0}$-filtration $Q^\bullet U$ by finite dimensional vector subspaces. Let Z be either a formal disc $D = \operatorname{Spec}\mathbb{C}[[t]]$ (more generally $\operatorname{Spec} R$, where R is a discrete valuation ring) or an affine line $\mathbb{A}^1$ and let $\mathring{Z}$ be either $\mathring{D} = \operatorname{Spec}\mathbb{C}((t))$ or $\mathbb{A}^1 \setminus \{0\}$. By a point ϵ of Z we will mean a $\mathbf{k}$-point of Z, where $\mathbf{k} = \mathbb{C}$ for $Z = \mathbb{A}^1$ and $\mathbf{k}$ is either $\mathbb{C}$ or $\mathbb{C}((t))$ for $Z = D$ ($\mathbb{C}$ or $\operatorname{Frac} R$ for $Z = \operatorname{Spec} R$). An algebraic family of subspaces $H_\epsilon \subset U \otimes \mathbf{k}$, $\epsilon \in Z$ is a collection of compatible morphisms $\mathring{f}_i\colon \mathring{Z} \to \mathrm{Gr}(d(i), Q^i U)$, $i \in \mathbb{Z}_{\geqslant 0}$ i.e. a collection of morphisms $\mathring{f}_i$ as above such that for every $\epsilon \in Z$ and $i \leqslant j$ we have $Q^j H_\epsilon \cap (Q^i U \otimes \mathbf{k}) = Q^i H_\epsilon$, where $Q^j H_\epsilon = \mathring{f}_j(\epsilon)$, $Q^i H_\epsilon = \mathring{f}_i(\epsilon)$.

Remark 4.2.1 In other words, we are given an algebraic family of subspaces $H_\epsilon \subset U \otimes \mathbf{k}$, $\epsilon \neq 0$ such that $d(i) = \dim_{\mathbf{k}}(H_\epsilon \cap Q^i U)$ does not depend on ϵ.

Since $\mathrm{Gr}(d(i), Q^i U)$ is a proper variety, we can uniquely extend each $\mathring{f}_i$ to the map $f_i\colon Z \to \mathrm{Gr}(d(i), Q^i U)$ and define $\lim\limits_{\epsilon \to 0} H_\epsilon$ as $\bigcup_i f_i(0) \subset U$. We also set $\lim\limits_{\epsilon \to 0} Q^i H_\epsilon := f_i(0)$.

Lemma 4.2.2 *Pick $a \in \lim\limits_{\epsilon \to 0} Q^i H_\epsilon$. There exists an algebraic morphism $\tilde{a}\colon Z \to Q^i U$ such that $\tilde{a}(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ and $\tilde{a}(0) = a$.*

Proof: Recall that we have a morphism $f_i\colon Z \to \mathrm{Gr}(d(i), Q^i U)$, which sends $\epsilon \in Z$ to $Q^i H_\epsilon$ and sends 0 to $\lim\limits_{\epsilon \to 0} Q^i H_\epsilon$. Let $\mathcal{V}$ be the tautological vector bundle on $\mathrm{Gr}(d(i), Q^i U)$. Consider the pull-back $f_i^*\mathcal{V}$, it is a vector bundle on Z. Note that every vector bundle on Z is trivial. Note also that a can be considered as an element of the fiber of $f_i^*\mathcal{V}$ over a point $0 \in Z$. Vector bundle $f_i^*\mathcal{V}$ is trivial so there exists a section $\tilde{a}\colon Z \to f_i^*\mathcal{V}$ such that $\tilde{a}(0) = a$. Note that we have a natural embedding of $f_i^*\mathcal{V}$ into the trivial vector bundle $Z \times Q^i U \to Z$. In other words, one can consider a section $\tilde{a}(\epsilon)$ as a map $\tilde{a}\colon Z \to Q^iU$ such that $\tilde{a}(\epsilon) \in Q^iH_\epsilon$ for every $\epsilon \neq 0$ and $\tilde{a}(0) = a$. □

Lemma 4.2.3 *Let $\tilde{a}\colon Z \to Q^iU$ be an algebraic morphism such that $\tilde{a}(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ then $\tilde{a}(0) \in \lim\limits_{\epsilon\to 0} Q^iH_\epsilon$.*

Proof: Since $0 \in \lim\limits_{\epsilon\to 0} Q^iH_\epsilon$ we can assume that $\tilde{a}(0) \neq 0$. Let $e_1, \ldots, e_{d(i)-1}$ be any subset of $\lim\limits_{\epsilon\to 0} Q^iH_\epsilon$ such that $\{\tilde{a}(0), e_1, \ldots, e_{d(i)-1}\}$ are linearly independent. By Lemma 4.2.2, we can find morphisms $\tilde{e}_i\colon Z \to Q^iU$ such that $\tilde{e}_i(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ and $\tilde{e}_i(0) = e_i$. Note that there exists a Zariski open neighbourhood of zero $0 \in W \subset Z$ such that the morphism $\tilde{a} \wedge \tilde{e}_1 \wedge \ldots \wedge \tilde{e}_{d(i)-1}\colon W \to \Lambda^{d(i)}Q^iU$ maps to $\Lambda^{d(i)}Q^iU \setminus \{0\}$ (i.e. for $\epsilon \in W$ elements of the set $\{\tilde{a}(\epsilon), \tilde{e}_1(\epsilon), \ldots, \tilde{e}_{d(i)-1}(\epsilon)\}$ are linearly independent). Since $\dim Q^iH_\epsilon = d(i)$ and $\{\tilde{a}(\epsilon), \tilde{e}_1(\epsilon), \ldots, \tilde{e}_{d(i)-1}(\epsilon)\} \subset Q^iH_\epsilon$ it follows that the elements $\{\tilde{a}(\epsilon), \tilde{e}_1(\epsilon), \ldots, \tilde{e}_{d(i)-1}(\epsilon)\}$ form a basis of Q^iH_ϵ. Recall that we have a closed embedding $\mathrm{Gr}(d(i), Q^iU) \subset \mathbb{P}(\Lambda^{d(i)}Q^iU)$ and we can consider $Q^iH_\epsilon \in \mathrm{Gr}(d(i), Q^iU)$ as $[\tilde{a}(\epsilon) \wedge \tilde{e}_1(\epsilon) \wedge \ldots \wedge \tilde{e}_{d(i)-1}(\epsilon)] \in \mathbb{P}(\Lambda^{d(i)}Q^iU)$. Consider the algebraic morphism $[\tilde{a} \wedge \tilde{e}_1 \wedge \ldots \wedge \tilde{e}_{d(i)-1}]\colon W \to \mathbb{P}(\Lambda^{d(i)}Q^iU)$, its value at $\epsilon \neq 0$ is H_ϵ and the value at $\epsilon = 0$ is $[\tilde{a}(0) \wedge e_1 \wedge \ldots \wedge e_{d(i)-1}]$. We conclude that $\lim\limits_{\epsilon\to 0} Q^iH_\epsilon = [\tilde{a}(0) \wedge e_1 \wedge \ldots \wedge e_{d(i)-1}]$ so $\tilde{a}(0) \in \lim\limits_{\epsilon\to 0} Q^iH_\epsilon$. □

Corollary 4.2.4 *The limit $\lim\limits_{\epsilon\to 0} Q^iH_\epsilon$ can be described as follows: it consists of elements $a \in Q^iU$ such that there exists a morphism $\tilde{a}\colon Z \to Q^iU$ such that $\tilde{a}(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ and $\tilde{a}(0) = a$.*

Proof: Follows from Lemmas 4.2.2, 4.2.3. □

To every $\mathbf{k}$-vector subspace $W \subset U \otimes \mathbf{k}$ we can associate its dimension with respect to the filtration $Q^\bullet$:

$$\dim_Q W := \sum_{i\geqslant 0} \dim_{\mathbf{k}} \left(\frac{W \cap (Q^iU \otimes \mathbf{k})}{W \cap (Q^{i-1}U \otimes \mathbf{k})} \right) q^i \in \mathbb{Z}[q].$$

For two series $a(q) = \sum_{i\geqslant 0} a_i q^i$, $b(q) = \sum_{i\geqslant 0} b_i q^i \in \mathbb{Z}[q]$ we say that $a(q) \geqslant b(q)$ if $\sum_{i=0}^n a_i \geqslant \sum_{i=0}^n b_i$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Lemma 4.2.5 (1) *We have $\dim_Q H_\epsilon \leqslant \dim_Q(\lim\limits_{\epsilon\to 0} H_\epsilon)$.*

(2) If the filtration Q^iU was induced by some grading with respect to which H_ϵ are graded then we have $\dim_Q H_\epsilon = \dim_Q(\lim\limits_{\epsilon\to 0} H_\epsilon)$.

Proof: Follows from the definitions. □

Remark 4.2.6 Note that it is not true in general that $\dim_Q H_\epsilon = \dim_Q(\lim\limits_{\epsilon\to 0} H_\epsilon)$. Indeed, take, for example, $U = \mathbb{C}[x]$ with the filtration by the degree of the polynomial and let $H_\epsilon \subset \mathbb{C}[x]$ be the subalgebra generated by $\epsilon x^2 + x$. Then $\lim\limits_{\epsilon\to 0} H_\epsilon = \mathbb{C}[x]$, its dimension is strictly greater than the dimension of $H_1 = \mathbb{C}[x + x^2]$.

Lemma 4.2.7 (1) *If U has an algebra structure such that $Q^iU \cdot Q^jU \subset Q^{i+j}U$ and $H_\epsilon \subset U \otimes \mathbf{k}$ are subalgebras then $\lim_{\epsilon \to 0} H_\epsilon$ is a subalgebra of U. Moreover, if H_ϵ are commutative then $\lim_{\epsilon \to 0} H_\epsilon$ is commutative.*

(2) *If U is itself commutative and equipped with a Poisson bracket $\{\,,\,\}$ such that $H_\epsilon \subset U$ are Poisson commutative subalgebras then $\lim_{\epsilon \to 0} H_\epsilon$ is Poisson commutative.*

Proof: Let us prove part (1). Pick two elements $a_1 \in \lim_{\epsilon \to 0} Q^i H_\epsilon$, $a_2 \in \lim_{\epsilon \to 0} Q^j H_\epsilon$. By Lemma 4.2.2 we can find morphisms $\tilde{a}_1 \colon Z \to Q^iU$, $\tilde{a}_2 \colon Z \to Q^jU$ such that $\tilde{a}_i(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ and $\tilde{a}_i(0) = a_i$. Consider the morphism $\tilde{a}_1\tilde{a}_2 \colon Z \to Q^{i+j}U$ and note that $\tilde{a}_1(\epsilon)\tilde{a}_2(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ (use that $H_\epsilon \subset U \otimes \mathbf{k}$ is a subalgebra). It follows from Lemma 4.2.3 that $a_1a_2 \in \lim_{\epsilon \to 0} Q^{i+j}H_\epsilon$. Assume now that H_ϵ are commutative. The composition $[\tilde{a}_1, \tilde{a}_2] \colon Z \to Q^iU \times Q^jU \to U$ is clearly continuous. Note that $[\tilde{a}_1(\epsilon), \tilde{a}_2(\epsilon)] = 0$ for $\epsilon \neq 0$ so we must have $[\tilde{a}_1(0), \tilde{a}_2(0)] = 0$.

To prove part (2) consider two elements $a_1 \in \lim_{\epsilon \to 0} Q^i H_\epsilon$, $a_2 \in \lim_{\epsilon \to 0} Q^j H_\epsilon$. By Lemma 4.2.2 we can find $\tilde{a}_1 \colon Z \to Q^iU$, $\tilde{a}_2 \colon Z \to Q^jU$ such that $\tilde{a}_i(\epsilon) \in H_\epsilon$ for $\epsilon \neq 0$ and $\tilde{a}_i(0) = a_i$. The composition $Z \to Q^iU \times Q^jU \to U$ is clearly continuous. Note now that $\{\tilde{a}_1(\epsilon), \tilde{a}_2(\epsilon)\} = 0$ for $\epsilon \neq 0$ so we must have $\{\tilde{a}_1(0), \tilde{a}_2(0)\} = 0$. □

"Continuous" version of Corollary 4.2.4

In this section we formulate a lemma that should be considered as a "continuous" analog of Corollary 4.2.4 above. The results of this section will be used in Sections 4.13, 4.15 of the text.

Let W be a finite dimensional vector space over $\mathbb{C}$ and $d \in \mathbb{Z}_{\geqslant 1}$. Let $(P_n)_{n \in \mathbb{Z}_{\geqslant 1}}$ be a sequence of points of $\mathrm{Gr}(d, W)$ (considered as a smooth manifold) and assume that the limit $\lim_{n \to \infty} P_n$ exists and is equal to some vector space $P \in \mathrm{Gr}(d, W)$.

Lemma 4.2.8 *Vector space P can be described as follows: it consists of elements $a \in W$ such that there exists a sequence $a_n \in P_n$ with $\lim_{n \to \infty} a_n = a$.*

Proof: Let us show that if $a_n \in P_n$ and $\lim_{n \to \infty} a_n = a$ then $a \in P$. Consider the natural embedding $\mathrm{Gr}(d, W) \hookrightarrow \mathbb{P}(\Lambda^d W)$. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Lambda^d W)}(-1)$. It can be trivialized in some neighbourhood of P so we can lift P_n, P to some elements $\alpha_n, \alpha \in \Lambda^d W$ such that $\lim_{n \to \infty} \alpha_n = \alpha$. It follows that $\lim_{n \to \infty} \alpha_n \wedge a_n = \alpha \wedge a$. Note now that $\alpha_n \wedge a_n = 0$ so $\alpha \wedge a = 0$, hence, $a \in P$.

Let us show that if $a \in P$ then one can find a sequence $a_n \in P_n$ such that $\lim_{n \to \infty} a_n = a$. Consider liftings α_n, α as above. Note that $a \in P$ so there exists a functional $\xi \in (W^{\otimes(d-1)})^*$ such that $\partial_\xi(\alpha) = a$. Set $a_n := \partial_\xi(\alpha_n)$. It follows from the definitions that $a_n \in P_n$ and $\lim_{n \to \infty} a_n = a$. □

4.2.2 Rees construction and limits

In this section we recall the classical Rees construction and discuss its compatibility with taking limits. Let A be an algebra equipped with an increasing $\mathbb{Z}_{\geqslant 0}$-filtration by $\mathbb{C}$-vector spaces

$$0 = F^{-1}A \subset F^0 A \subset F^1 A \subset \dots$$

such that $F^iA \cdot F^jA \subset F^{i+j}A$ for $i, j \in \mathbb{Z}_{\geqslant 0}$ and $1 \in F^0A$. Consider the following $\mathbb{C}[\hbar]$ subalgebra of $A[\hbar]$

$$Rees(A) := \bigoplus_{i \geqslant 0} \hbar^i F^i A.$$

For $\epsilon \in \mathbb{C}$ we set $A_\epsilon := Rees(A)/(\hbar - \epsilon)$. More generally, for every $\mathbf{k}$-point $\epsilon \in \mathbb{A}^1$ we define A_ϵ as $A \otimes_{\mathbb{C}[\hbar]} \mathbf{k}$. We have a canonical isomorphism of $\mathbb{C}[\hbar^{\pm 1}]$-algebras:

$$Rees(A) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}] \xrightarrow{\sim} A[\hbar^{\pm 1}],\ (\hbar^i a) \otimes \hbar^l \mapsto \hbar^{l+i} a,$$

where $i \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}$ and $a \in F^iA$.

For $\epsilon \neq 0$ we obtain an isomorphism

$$A_\epsilon \xrightarrow{\sim} A_1 \xrightarrow{\sim} A,\ [\hbar^i a] \mapsto [\epsilon^i \hbar^i a] \mapsto \epsilon^i a.$$

Assume now that the algebra A is equipped with some filtration by $\mathbb{C}$-vector spaces

$$0 = L^{-1}A \subset L^0 A \subset L^1 A \subset \dots$$

such that $\dim L^jA < \infty$ for every $j \in \mathbb{Z}_{\geqslant 0}$.

We can define a filtration $L^\bullet$ on $Rees(A)$ in the following way:

$$L^j Rees(A) = \bigoplus_{i \geqslant 0} \hbar^i (F^i A \cap L^j A).$$

Note that $L^jRees(A)$ is a $\mathbb{C}[\hbar]$-submodule of $Rees(A)$ so the filtration $L^jRees(A)$ induces a filtration on every A_ϵ. Note also that $L^jRees(A)$ is a finitely generated $\mathbb{C}[\hbar]$ module: indeed, $L^jRees(A)$ is a submodule of the $\mathbb{C}[\hbar]$-module $\bigoplus_{i \geqslant 0} \hbar^i L^j A$ that is free over $\mathbb{C}[\hbar]$ with any basis of L^jA as the set of generators. Using that $\mathbb{C}[\hbar]$ is Noetherian, we conclude that $L^jRees(A)$ is finitely generated.

It follows that $\dim L^iA_\epsilon < \infty$ for every $i \in \mathbb{Z}_{\geqslant 0}$. Recall that for every $\mathbf{k}$-point ϵ of $\mathbb{A}^1$ and every vector subspace $B_\epsilon \subset A_\epsilon$, we can define the dimension $\dim_L B_\epsilon \in \mathbb{Z}[q]$ as

$$\dim_L B_\epsilon := \sum_{i \geqslant 0} \dim_{\mathbf{k}} \left(\frac{L^i A_\epsilon \cap B_\epsilon}{L^{i-1} A_\epsilon \cap B_\epsilon} \right) q^i \in \mathbb{Z}[q].$$

Recall that $L^iRees(A)$ is a finitely generated $\mathbb{C}[\hbar]$-module i.e. a coherent sheaf on $\mathbb{A}^1$. Note also that $L^iRees(A)$ is torsion free finitely generated, hence, is free. We denote by $(L^iRees(A))^\vee$ the $\mathbb{C}[\hbar]$-module $\operatorname{Hom}_{\mathbb{C}[\hbar]}(L^iRees(A), \mathbb{C}[\hbar])$. Consider the spectrum of the relative symmetric power $\operatorname{Spec}(S^\bullet_{\mathbb{C}[\hbar]}((L^iRees(A)))^\vee)$ and denote it by $L^iA_{\mathbb{A}^1}$. Note that we have a natural map $L^iA_{\mathbb{A}^1} \to \mathbb{A}^1$ that is a vector bundle with fiber over $\epsilon \in \mathbb{A}^1$ being equal to A_ϵ.

Recall that we have a vector bundle $L^iA_{\mathbb{A}^1} \to \mathbb{A}^1$ with fiber over $\epsilon \in \mathbb{A}^1$ being L^iA_ϵ. Consider the relative Grassmannian $\operatorname{Gr}_{\mathbb{A}^1}(d(i), L^iA_{\mathbb{A}^1}) \to \mathbb{A}^1$, its fiber over $\epsilon \in \mathbb{A}^1$ is $\operatorname{Gr}(d(i), L^iA_\epsilon)$.

Let Z be a scheme as in Section 4.2.1 and let us fix a morphism $Z \to \mathbb{A}^1$. We denote by $L^iA_Z \to Z$, $\mathrm{Gr}_Z(d(i), L^iA_Z) \to Z$ pull backs to Z of the corresponding $\mathbb{A}^1$-schemes.

Let us fix an algebraic family $B_\epsilon \subset A_\epsilon$ for $\epsilon \neq 0$ such that $d(i) = \dim(B_\epsilon \cap L^iA_\epsilon)$ does not depend on ϵ. In other words, consider a collection of compatible morphisms $\mathring{f}_i\colon \mathring{Z} \to \mathrm{Gr}_Z(d(i), L^iA_Z)$, then $L^iB_\epsilon = \mathring{f}(\epsilon)$ and $B_\epsilon = \cup_{i\geqslant 0} L^iB_\epsilon$. By the valuative criterion of properness (applied to $\mathrm{Gr}(d(i), L^iA_Z)$, $\mathring{Z}$ considered as schemes over Z) this section extends uniquely to the section $f\colon Z \to \mathrm{Gr}_Z(d(i), L^iA_Z)$. We define $\lim\limits_{\epsilon\to 0} L^iB_\epsilon$ as $f(0) \subset A_0$. We set $\lim\limits_{\epsilon\to 0} B_\epsilon := \bigcup_{i\geqslant 0} \lim\limits_{\epsilon\to 0} L^iB_\epsilon$.

Vector bundle $L^iA_{\mathbb{A}^1} \to \mathbb{A}^1$ is trivial so the following lemma can be proved in the same way as Lemmas 4.2.2, 4.2.3.

Lemma 4.2.9 *The limit $\lim\limits_{\epsilon\to 0} L^iB_\epsilon$ can be described as follows. Element $a \in L^iA_0$ lies in the limit $\lim\limits_{\epsilon\to 0} L^iB_\epsilon$ if and only if there exists a section $\tilde{a}\colon Z \to L^iA_Z$ of the vector bundle $L^iA_Z \to Z$ such that $\tilde{a}(\epsilon) \in L^iA_\epsilon$ for $\epsilon \neq 0$ and $\tilde{a}(0) = a$.*

Lemma 4.2.10 *We have $\dim_L B_1 \leqslant \dim_L(\lim\limits_{\epsilon\to 0} B_\epsilon)$.*

Proof: Follows from the definitions. □

4.3 Universal inhomogeneous Gaudin subalgebras: definitions

4.3.1 Three filtrations on $S^\bullet(\mathfrak{g}[t])$, $U(\mathfrak{g}[t])$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Let $(\,,\,)$ be the Killing form on $\mathfrak{g}$. We fix an orthonormal basis $\{x_a\}|_{a=1,\ldots,\dim\mathfrak{g}}$ of $\mathfrak{g}$. Set $\mathfrak{g}((t^{-1})) := \mathfrak{g} \otimes \mathbb{C}((t^{-1}))$. For an element $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we set $x[n] := x \otimes t^n \in \mathfrak{g}((t^{-1}))$. We define a Lie algebra structure on $\mathfrak{g}((t^{-1}))$ as follows: $[x[n], y[m]] := [x, y][n + m]$, $x, y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$. Set $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$, $t^{-1}\mathfrak{g}[[t^{-1}]] := \mathfrak{g} \otimes t^{-1}\mathbb{C}[[t^{-1}]]$. The natural decomposition $\mathfrak{g}((t^{-1})) = \mathfrak{g}[t] \oplus t^{-1}\mathfrak{g}((t^{-1}))$ is the decomposition of Lie subalgebras.

Let us discuss filtrations on $S^\bullet(\mathfrak{g}[t])$, $U(\mathfrak{g}[t])$ that we will use. We have the PBW-filtration on $U(\mathfrak{g}[t])$ defined by putting

$$\deg_{PBW} x[n] = 1.$$

Note that the associated graded $\mathrm{gr}_{PBW} U(\mathfrak{g}[t])$ is isomorphic to the $\mathbb{Z}_{\geqslant 0}$-graded algebra $S^\bullet(\mathfrak{g}[t]) = \bigoplus_{p\geqslant 0} S^p(\mathfrak{g}[t])$. Grading on $S^\bullet(\mathfrak{g}[t])$ induces the filtration that we will also call the PBW-filtration on $S^\bullet(\mathfrak{g}[t])$. We will denote by gr_{PBW} the associated graded with respect to the PBW-filtrations on $U(\mathfrak{g}[t])$, $S^\bullet(\mathfrak{g}[t])$. We also have filtrations F_1 on $U(\mathfrak{g}[t])$, $S^\bullet(\mathfrak{g}[t])$ defined by putting

$$\deg_1 x[n] = n + 1.$$

We will denote by gr_1 the associated graded with respect to these filtrations. Finally, we have filtrations F_2 on $U(\mathfrak{g}[t])$, $S^\bullet(\mathfrak{g}[t])$ defined by putting

$$\deg_2 x[n] = n.$$

We will denote by gr_2 the associated graded with respect to these filtrations.

Note that we have

$$\dim F_1^i S^\bullet(\mathfrak{g}[t]),\ \dim F_1^i U(\mathfrak{g}[t]) < \infty$$

for every $i \in \mathbb{Z}_{\geqslant 0}$. So, as in Section 4.2, for any vector subspace W of $S^\bullet(\mathfrak{g}[t])$ or of $U(\mathfrak{g}[t])$ we can define

$$\dim_{F_1} W = \sum_{i \geqslant 0} (\dim F_1^i W) q^i \in \mathbb{Z}[q], \tag{4.1}$$

where $F_1^\bullet W$ is the induced filtration on W.

4.3.2 Feigin-Frenkel center and its classical version

In this section we recall the Feigin-Frenkel center $\mathcal{Z}$ that is the center of a certain completion of the universal enveloping algebra of the affine Lie algebra $\hat{\mathfrak{g}}$ at the critical level. We also recall the description of the associated graded $\mathrm{gr}_{PBW} \mathcal{Z}$. The results of this sections follow from [3], [25], [28], [31].

Recall that

$$\hat{\mathfrak{g}} = \mathfrak{g}((t^{-1})) \oplus \mathbb{C}\mathsf{K},\ [x[n] + b\mathsf{K}, y[m] + c\mathsf{K}] = [x,y][n+m] + n(x,y)\delta_{n+m,0}\mathsf{K} \tag{4.2}$$

is the affine Kac-Moody algebra, the central extension of the loop Lie algebra $\mathfrak{g}((t^{-1}))$. Define the completion $\widetilde{U}(\hat{\mathfrak{g}})$ of $U(\hat{\mathfrak{g}})$ as the inverse limit of $U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(t^{-n}\mathfrak{g}[[t^{-1}]])$, $n > 1$ and set

$$\widetilde{U}(\hat{\mathfrak{g}})_{-1/2} := \widetilde{U}(\hat{\mathfrak{g}})/(\mathsf{K} + 1/2).$$

Algebra $\widetilde{U}(\hat{\mathfrak{g}})_{-1/2}$ is equipped with the PBW filtration. The associated graded $\mathrm{gr}_{PBW} \widetilde{U}(\hat{\mathfrak{g}})_{-1/2}$ is isomorphic to the completion:

$$\widetilde{S}^\bullet(\mathfrak{g}((t^{-1}))) := \varprojlim S^\bullet(\mathfrak{g}((t^{-1})))/S^\bullet(\mathfrak{g}((t^{-1})))(t^{-n}\mathfrak{g}[[t^{-1}]]).$$

Let $\mathcal{Z}$ be the center of $\widetilde{U}(\hat{\mathfrak{g}})_{-1/2}$. Note that $\mathcal{Z}$ contains the following quadratic elements:

$$S_1^{(r)} = \sum_{a,\, p+q=r} x_a[p]x_a[q] \in \mathcal{Z} \subset \widetilde{U}(\hat{\mathfrak{g}})_{-1/2},\ r \in \mathbb{Z}.$$

It is known that there are elements $S_1^{(r)}, \ldots, S_{\mathrm{rk}\,\mathfrak{g}}^{(r)} \in \mathcal{Z}$, $r \in \mathbb{Z}$, such that the image of $\mathcal{Z}$ in every $U(\hat{\mathfrak{g}})_{-1/2}/U(\hat{\mathfrak{g}})_{-1/2}(t^{-n}\mathfrak{g}[[t^{-1}]])$ is generated by the images of these elements.

There are no simple explicit formulas for the elements $S_i^{(r)}$ for $i > 1$, but one can describe $\mathrm{gr}_{PBW} S_i^{(r)} \in \widetilde{S}^\bullet(\mathfrak{g}((t^{-1})))$ explicitly. To do this, let us first describe the associated graded $Z := \mathrm{gr}_{PBW} \mathcal{Z}$ that is a Poisson-commutative subalgebra in the completion $\widetilde{S}^\bullet(\mathfrak{g}((t^{-1})))$.

To an element $x \in \mathfrak{g}$ we can associate the following infinite sum

$$x(z) := \sum_{r \in \mathbb{Z}} x[r] z^{-r}$$

that we consider as an element of $\prod_{r\in\mathbb{Z}}\widetilde{S}(\mathfrak{g}((t^{-1})))z^r$. We can uniquely extend the map $x\mapsto x(z)$ to the homomorphism of algebras

$$S^\bullet(\mathfrak{g})\to\prod_{r\in\mathbb{Z}}\widetilde{S}(\mathfrak{g}((t^{-1})))z^r,\ f\mapsto f(z).$$

We can then decompose $f(z)=\sum_{r\in\mathbb{Z}}f^{(r)}z^{-r}$.

Let $\{\Phi_i\in S^\bullet(\mathfrak{g})^{\mathfrak{g}}\}_{i=1,\dots,\operatorname{rk}\mathfrak{g}}$ be free homogeneous generators of the commutative algebra $S^\bullet(\mathfrak{g})^{\mathfrak{g}}$ (i.e. $S^\bullet(\mathfrak{g})^{\mathfrak{g}}=\mathbb{C}[\Phi_i\,|\,i=1,\dots,\operatorname{rk}\mathfrak{g}]$).

Proposition 4.3.1 *For $n\in\mathbb{Z}_{\geqslant 0}$ let π_n be the natural surjection from $\widetilde{S}^\bullet(\mathfrak{g}((t^{-1})))$ to $S^\bullet(\mathfrak{g}((t^{-1})))/S^\bullet(\mathfrak{g}((t^{-1})))(t^{-n}\mathfrak{g}[[t^{-1}]])$. The algebra $\pi_n(Z)$ is generated by the set*

$$\{\pi_n(\Phi_i^{(r)})\,|\,r\in\mathbb{Z},1\leqslant i\leqslant\operatorname{rk}\mathfrak{g},\ \Phi_i\in S^\bullet(\mathfrak{g})^{\mathfrak{g}}\}$$

so the subalgebra of Z generated by $\Phi_i^{(r)}$ is dense. We have $\operatorname{gr}_{PBW}S_i^{(r)}=\Phi_i^{(r)}$.

Recall that we have a Casimir element $\sum_a x_a^2\in S^2(\mathfrak{g})^{\mathfrak{g}}\subset S^\bullet(\mathfrak{g})^{\mathfrak{g}}$ that is nothing else but Φ_1. We conclude that the algebra Z contains the elements

$$\Phi_1^{(r)}=\sum_{a,\,p+q=r}x_a[p]x_a[q],\ r\in\mathbb{Z}.$$

4.3.3 Inhomogeneous universal Gaudin subalgebra $\mathcal{A}_\chi^{\mathrm{u}}$

Using the center $\mathcal{Z}$, one can define certain commutative subalgebras of the algebra $U(\mathfrak{g}[t])$. We regard the Lie algebra $\mathfrak{g}[t]$ as a "half" of the corresponding affine Kac-Moody algebra $\hat{\mathfrak{g}}$.

Pick $\chi\in\mathfrak{g}$. Note that χ defines a character of the Lie algebra $t^{-1}\mathfrak{g}[[t^{-1}]]=:\hat{\mathfrak{g}}_-$ that sends $x[n]$ to $(\chi,x)\delta_{-1,n}$. We will denote this character by the same letter χ and will sometimes denote the pairing (χ,x) by $\chi(x)$.

The image of the natural homomorphism from $\mathcal{Z}$ to the quantum Hamiltonian reduction

$$U(\hat{\mathfrak{g}})_{-1/2}/\!\!/\!\!/_\chi t^{-1}\mathfrak{g}[[t^{-1}]]:=\big(U(\hat{\mathfrak{g}})_{-1/2}/U(\hat{\mathfrak{g}})_{-1/2}\{u-\chi(u)\,|\,u\in t^{-1}\mathfrak{g}[[t^{-1}]]\}\big)^{t^{-1}\mathfrak{g}[[t^{-1}]]}\qquad(4.3)$$

is a commutative subalgebra there.

Using the decomposition $\hat{\mathfrak{g}}=\mathfrak{g}[t]\oplus t^{-1}\mathfrak{g}[[t^{-1}]]\oplus\mathbb{C}\mathsf{K}$ and the PBW decomposition, we obtain the embedding of the quantum Hamiltonian reduction algebra (4.3) into the algebra $U(\mathfrak{g}[t])$. The image of $\mathcal{Z}$ can be regarded as a commutative subalgebra $\mathcal{A}_\chi^{\mathrm{u}}\subset U(\mathfrak{g}[t])$, which we call the *inhomogeneous universal Gaudin subalgebra* of $U(\mathfrak{g}[t])$.

Lemma 4.3.2 *We have $\mathcal{A}_\chi^{\mathrm{u}}\subset U(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$.*

Proof: Pick $z\in\mathcal{Z}$ and $x\in\mathfrak{z}_{\mathfrak{g}}(\chi)$. Note that $[z,x]=0$ considered as elements of the completion $\widetilde{U}(\hat{\mathfrak{g}})_{-1/2}$. We claim that

$$[x,t^{-1}\mathfrak{g}[[t^{-1}]]]\subset\widetilde{U}(\hat{\mathfrak{g}})_{-1/2}\{u-\chi(u)\,|\,u\in t^{-1}\mathfrak{g}[[t^{-1}]]\}.\qquad(4.4)$$

Indeed, if we pick an element $y=y_1[-1]+y_2[-2]+\ldots\in t^{-1}\mathfrak{g}[[t^{-1}]]$ then we have $[x,y]=[x,y_1][-1]+[x,y_2][-2]+\ldots$. It remains to note that $(\chi,[x,y_1])=([\chi,x],y_1)=0$ so (4.4) indeed holds. It follows that the class of x in $U(\hat{\mathfrak{g}})_{-1/2}/U(\hat{\mathfrak{g}})_{-1/2}\{u-\chi(u)\,|\,u\in t^{-1}\mathfrak{g}[[t^{-1}]]\}$ defines an element of the quantum Hamiltonian reduction $U(\hat{\mathfrak{g}})_{-1/2}/\!\!/\!\!/_\chi t^{-1}\mathfrak{g}[[t^{-1}]]$. Now the claim follows from the equality $[z,x]=0$ above. □

4.3.4 Classical version of the inhomogeneous Gaudin subalgebra

Let us introduce the classical version $\overline{\mathcal{A}}_\chi^{\mathrm{u}}$ of the algebra $\mathcal{A}_\chi^{\mathrm{u}}$ that will be a Poisson commutative subalgebra of $S^\bullet(\mathfrak{g}[t])$. We relate $\mathcal{A}_\chi^{\mathrm{u}}$, $\overline{\mathcal{A}}_\chi^{\mathrm{u}}$ in the Proposition 4.6.5.

Recall the Poisson-commutative subalgebra $Z \subset \widetilde{S}^\bullet(\mathfrak{g}((t^{-1})))$. We can consider the Hamiltonian reduction

$$\left(S^\bullet(\mathfrak{g}((t^{-1})))/S^\bullet(\mathfrak{g}((t^{-1})))\{u-\chi(u)\,|\,u\in t^{-1}\mathfrak{g}[[t^{-1}]]\}\right)^{t^{-1}\mathfrak{g}[[t^{-1}]]}. \tag{4.5}$$

Again, using the decomposition $\mathfrak{g}((t^{-1})) = \mathfrak{g}[t] \oplus t^{-1}\mathfrak{g}[[t^{-1}]]$, we can embed the Hamiltonian reduction (4.5) in $S^\bullet(\mathfrak{g}[t])$. We denote by $\overline{\mathcal{A}}_\chi^{\mathrm{u}}$ the image of Z. In the same way as in Lemma 4.3.2 we see that $\overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}$.

One can describe explicitly the algebra $\overline{\mathcal{A}}_0^{\mathrm{u}}$, the classical Gaudin subalgebra. It is freely generated by all Fourier components of $\mathbb{C}[[t^{-1}]]$-valued functions $\Phi_l(x(t))$ on $t^{-1}\mathfrak{g}[[t^{-1}]] = \operatorname{Spec} S^\bullet(\mathfrak{g}[t])$ for all free homogeneous generators Φ_l of the algebra of adjoint invariants $S^\bullet(\mathfrak{g})^\mathfrak{g}$. Consider the derivation D of $S^\bullet(\mathfrak{g}[t])$ given by $D(x[n-1]) = nx[n]$. Recall that $\Phi_l, l = 1,\dots,\operatorname{rk}\mathfrak{g}$, are free generators of $S(\mathfrak{g}[0])^\mathfrak{g} \subset S^\bullet\mathfrak{g}[t]$.

Proposition 4.3.3 *([53, Proposition 4.6]) Classical universal Gaudin subalgebra $\overline{\mathcal{A}}_0^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])^\mathfrak{g}$ is the subalgebra freely generated by all $D^k\Phi_l, k \geqslant 0, l = 1,\dots,\operatorname{rk}\mathfrak{g}$.*

Recall that the element $\Phi_1^{(r)} = \sum_{a,\,p+q=r} x_a[p]x_a[q]$ lies in Z. We see that for $r < -2$, the image of $\Phi_1^{(r)}$ in $S^\bullet(\mathfrak{g}[t])$ is equal to zero. The image of $\Phi_1^{(-2)}$ is $\sum_a \chi(x_a)^2 = (\chi,\chi)$. The image of $\Phi_1^{(-1)}$ is $\sum_a \chi(x_a)x_a[0] = \chi[0]$. The image of $\Phi_1^{(0)}$ is

$$\sum_a x_a[0]^2 + \sum_a 2x_a[-1]x_a[1] = \sum_a x_a[0]^2 + \sum_a 2\chi(x_a)x_a[1] = \sum_a x_a[0]^2 + 2\chi[1].$$

The image of $\Phi_1^{(1)}$ is

$$\sum_a 2x_a[0]x_a[1] + \sum_a 2x_a[-1]x_a[2] = \sum_a 2x_a[0]x_a[1] + \sum_a 2\chi(x_a)x_a[2] =$$
$$= \sum_a 2x_a[0]x_a[1] + 2\chi[2].$$

We then set

$$\omega_\chi := \frac{1}{2}\sum_a x_a[0]^2 + \chi[1] \in \overline{\mathcal{A}}_\chi^{\mathrm{u}},$$
$$\Omega_\chi := \sum_a x_a[0]x_a[1] + \chi[2] \in \overline{\mathcal{A}}_\chi^{\mathrm{u}}.$$

4.4 Inhomogeneous Gaudin subalgebras and conformal blocks

Recall that we have constructed a commutative subalgebra $\mathcal{A}_\chi^{\mathrm{u}} \subset U(\mathfrak{g}[t])$. Note also that for every k-tuple of (distinct) points $z_1,\dots,z_k \in \mathbb{C}$ we can consider the evaluation homomorphism $\mathrm{ev}_{z_1,\dots,z_k}\colon U(\mathfrak{g}[t]) \to U(\mathfrak{g})^{\otimes k}$ and define the commutative algebra $\mathcal{A}_\chi(z_1,\dots,z_k) := \mathrm{ev}_{z_1,\dots,z_k}(\mathcal{A}_{-\chi}^{\mathrm{u}})$ that is a subalgebra of $U(\mathfrak{g})^{\otimes k}$.

Warning 4.4.1 *Note that $\mathcal{A}_\chi(z_1,\dots,z_k)$ is the image of $\mathcal{A}^{\mathrm{u}}_{-\chi}$, not of $\mathcal{A}^{\mathrm{u}}_{\chi}$.*

The goal of this section is to show that the algebra $\mathcal{A}_\chi(z_1,\dots,z_k)$ is the so-called inhomogeneous Gaudin subalgebra defined in [99], [29], see also [43, Section 9]. Let us start from the definition of the inhomogeneous Gaudin subalgebra.

Consider the affine Lie algebra $\hat{\mathfrak{g}}' := \mathfrak{g}((t)) \oplus \mathbb{C}\mathsf{K}$ and consider the quantum Hamiltonian reduction

$$U(\hat{\mathfrak{g}}')_{-1/2} /\!\!/\!\!/_0\, \mathfrak{g}[[t]] := (U(\hat{\mathfrak{g}}')_{-1/2}/U(\hat{\mathfrak{g}}')_{-1/2}\mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}. \tag{4.6}$$

Using the decomposition $\mathfrak{g}((t)) = \mathfrak{g}[[t]] \oplus t^{-1}\mathfrak{g}[t^{-1}]$, we can embed the Hamiltonian reduction (4.6) in the universal enveloping algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$. Let $\mathcal{A}' \subset U(t^{-1}\mathfrak{g}[t^{-1}])$ be the image of this embedding, $\mathcal{A}'$ is a commutative subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$.

Remark 4.4.2 It follows from [27], [31] that the center $\mathcal{Z}$ maps onto the Hamiltonian reduction (4.6) so there is no need to mention $\mathcal{Z}$ in the definition of the algebra $\mathcal{A}'$.

Note now that for nonzero z_i we can consider the evaluation homomorphism $U(t^{-1}\mathfrak{g}[t^{-1}]) \xrightarrow{\mathrm{ev}'_{z_1,\dots,z_k}} U(\mathfrak{g})^{\otimes k}$. We also have the "evaluation at the infinity" homomorphism $\mathrm{ev}'_\infty \colon U(t^{-1}\mathfrak{g}[t^{-1}]) \to S^\bullet(\mathfrak{g})$ that is induced by the homomorphism $t^{-1}\mathfrak{g}[t] \to \mathfrak{g}$ that extracts the coefficient in front of t^{-1}. We obtain the homomorphism

$$\mathrm{ev}'_{z_1,\dots,z_k,\infty} := \mathrm{ev}'_{z_1,\dots,z_k} \otimes \mathrm{ev}'_\infty \colon U(t^{-1}\mathfrak{g}[t^{-1}]) \to U(\mathfrak{g})^{\otimes k} \otimes S^\bullet(\mathfrak{g}).$$

Recall that we are given $\chi \in \mathfrak{g}$ that defines the evaluation at χ homomorphism $S^\bullet(\mathfrak{g}) \to \mathbb{C}$ (after the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ via the $\mathfrak{g}$-invariant bilinear form $(\, ,\,)$). We obtain the composite map to be denoted

$$\mathrm{ev}'_{z_1,\dots,z_k,\chi} \colon U(t^{-1}\mathfrak{g}[t^{-1}]) \to U(\mathfrak{g})^{\otimes k} \otimes S^\bullet(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes k}.$$

Let $\mathcal{A}'_\chi(z_1,\dots,z_k)$ be the image of $\mathcal{A}'$ under the homomorphism $\mathrm{ev}'_{w-z_1,\dots,w-z_k,\chi}(\mathcal{A}')$, $w \in \mathbb{C} \setminus \{z_1,\dots,z_k\}$.

Remark 4.4.3 Note that $\mathcal{A}'_\chi(z_1,\dots,z_k) = \mathrm{ev}_{z_1,\dots,z_k,-\chi}(\mathcal{A}')$.

Our goal is to show that

$$\mathcal{A}_\chi(z_1,\dots,z_k) = \mathcal{A}'_\chi(z_1,\dots,z_k). \tag{4.7}$$

Assume that we are given some $\mathfrak{g}$-modules $M_1,\dots,M_k$. Then the algebras

$$\mathcal{A}_\chi(z_1,\dots,z_k),\ \mathcal{A}'_\chi(z_1,\dots,z_k) \subset U(\mathfrak{g})^{\otimes k}$$

act naturally on the tensor product $M_1 \otimes \ldots \otimes M_k$ and their images in $\mathrm{End}(M_1 \otimes \ldots \otimes M_k)$ are commutative subalgebras that we denote by $\mathcal{A}_\chi(M_1,\dots,M_k)$, $\mathcal{A}'_\chi(M_1,\dots,M_k)$ respectively. To see that $\mathcal{A}_\chi(z_1,\dots,z_k) = \mathcal{A}'_\chi(z_1,\dots,z_k)$ it is enough to show that

$$\mathcal{A}_\chi(M_1,\dots,M_k) = \mathcal{A}'_\chi(M_1,\dots,M_k) \tag{4.8}$$

for every $M_1, \dots, M_k$. To prove (4.8) we will recall the general approach that is used to construct commutative subalgebras of such sort (approach via conformal blocks) and will identify both $\mathcal{A}_\chi(M_1, \dots, M_k)$ and $\mathcal{A}'_\chi(M_1, \dots, M_k)$ with a well-known commutative subalgebra of $\mathrm{End}(M_1 \otimes \ldots \otimes M_k)$ (see Proposition 4.4.10).

We start with some general recollections about conformal blocks (following [28], [29] and [27]). Recall that we fix distinct points $z_1, \dots, z_k \in \mathbb{P}^1$. In the neighbourhood of each point we have a local coordinate $t - z_i$. Set $\tilde{\mathfrak{g}}(z_i) := \mathfrak{g}((t-z_i))$ and let $\hat{\mathfrak{g}}(z_i) = \tilde{\mathfrak{g}}(z_i) \oplus \mathbb{C}\mathsf{K}_i$ be the one-dimensional central extension of $\tilde{\mathfrak{g}}(z_i)$ (see (4.2)). We set $\tilde{\mathfrak{g}}(\underline{z}) := \bigoplus_{i=1}^k \tilde{\mathfrak{g}}(z_i)$ and denote by $\hat{\mathfrak{g}}(\underline{z})$ the one-dimensional central extension of $\tilde{\mathfrak{g}}(\underline{z})$ that is $\left(\bigoplus_{i=1}^k \hat{\mathfrak{g}}(z_i)\right)/(\mathsf{K}_i - \mathsf{K}_j \mid i \neq j)$. Let $\mathsf{K} \in \hat{\mathfrak{g}}(\underline{z})$ be the central element (class of any K_i).

Let $M_1, \dots, M_k$ be any $\mathfrak{g}$-modules. We consider M_i as a module over $\mathfrak{g}[[t-z_i]] \oplus \mathbb{C}\mathsf{K}$, where $(t-z_i)\mathfrak{g}[[t-u_i]]$ acts trivially, K acts via the multiplication by $-1/2$ and $\mathfrak{g}$ acts via its given action on M_i. Consider also the induced modules $\mathbb{M}_{z_i} := \mathrm{Ind}_{\mathfrak{g}[[t-z_i]]\oplus\mathbb{C}\mathsf{K}}^{\hat{\mathfrak{g}}(z_i)} M_i$. For $z \in \mathbb{P}^1$ we set $\mathbb{V}_{0,z} := \mathrm{Ind}_{\mathfrak{g}[[t-z]]\oplus\mathbb{C}\mathsf{K}}^{\hat{\mathfrak{g}}(z)} \mathbb{C}$.

Remark 4.4.4 Note that $\mathbb{V}_{0,z}$ is nothing else than $\mathbb{M}_z$ for $M = \mathbb{C}$, the trivial $\mathfrak{g}$-module.

Consider the tensor product $\mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k}$. Let $\mathfrak{g}_{\underline{z}}$ be the Lie algebra of regular functions on $\mathbb{P}^1 \setminus \{z_1, \dots, z_k\}$. We have an embedding $\mathfrak{g}_{\underline{z}} \subset \hat{\mathfrak{g}}(\underline{z})$ and denote by $H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})$ the space of coinvariants

$$H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}) := (\mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k})/\mathfrak{g}_{\underline{z}}.$$

The following proposition is standard.

Proposition 4.4.5 *The embedding* $M_1 \otimes \ldots \otimes M_k \subset \mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k}$ *induces the isomorphism* $(M_1 \otimes \ldots \otimes M_k)/\mathfrak{g} \xrightarrow{\sim} H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})$.

Corollary 4.4.6 *Pick* $z \in \mathbb{P}^1 \setminus \{z_1, \dots, z_k\}$. *The embedding* $\mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k} \subset \mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k} \otimes \mathbb{V}_{0,z}$ *induces the isomorphism* $H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}) \xrightarrow{\sim} H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}, \mathbb{V}_{0,z})$.

We now define the commutative subalgebra

$$\mathcal{A}(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}) \subset \mathrm{End}((M_1 \otimes \ldots \otimes M_k)/\mathfrak{g})$$

as follows. We consider the natural action of $\mathcal{Z}^{\otimes k}$ on $H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}) \simeq (M_1 \otimes \ldots \otimes M_k)/\mathfrak{g}$ and denote by $\mathcal{A}(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})$ the image of $\mathcal{Z}^{\otimes k}$ in $\mathrm{End}((M_1 \otimes \ldots \otimes M_k)/\mathfrak{g})$.

Proposition 4.4.7 *Consider any nonempty subset* $\{i_1, \dots, i_l\} \subset \{1, 2, \dots, k\}$, $l \in \{1, 2, \dots, k\}$ *and consider the image of the natural homomorphism*

$$1 \otimes \ldots \otimes 1 \otimes \underset{i_1}{\mathcal{Z}} \otimes 1 \ldots \otimes 1 \otimes \underset{i_2}{\mathcal{Z}} \otimes 1 \ldots \otimes 1 \otimes \underset{i_l}{\mathcal{Z}} \otimes 1 \ldots \otimes 1 \to \mathrm{End}(H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})).$$

This image coincides with $\mathcal{A}(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})$. *In particular, the algebra* $\mathcal{A}(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k})$ *coincides with the image of the natural homomorphism*

$$\underbrace{1 \otimes 1 \otimes \ldots \otimes 1}_{k} \otimes \mathcal{Z} \to \mathrm{End}(H(\mathbb{M}_{z_1}, \dots, \mathbb{M}_{z_k}, \mathbb{V}_{0,z}))$$

and does not depend on the point $z \in \mathbb{P}^1 \setminus \{z_1, \dots, z_k\}$.

Proof: Set $\mathring{D} := \operatorname{Spec}\mathbb{C}((t))$, for $z \in \mathbb{P}^1$ we set $\mathring{D}_z := \operatorname{Spec}\mathbb{C}((t-z))$ i.e. $\mathring{D}_z$ is the punctured formal neighbourhood of $z \in \mathbb{P}^1$. Let X be a smooth curve or $\mathring{D}_z$. Let $\mathrm{Op}(X)$ be the moduli space of $G^\vee$-opers (see, for example, [33, Section 4.2] for the definition), here $G^\vee$ is the Langlands dual group to G. Recall now that by [27], [32] (see also [33]) we have the natural identification $\mathcal{Z} \simeq \mathcal{O}(\mathrm{Op}(\mathring{D}))$ which induces the identification $\mathcal{Z}^{\otimes k} \simeq \mathcal{O}(\mathrm{Op}(\mathring{D}_{z_1}) \times \ldots \times \mathrm{Op}(\mathring{D}_{z_k}))$. We have the homomorphism

$$\mathcal{O}(\mathrm{Op}(\mathring{D}_{z_1}) \times \ldots \times \mathrm{Op}(\mathring{D}_{z_k})) \to \mathcal{O}(\mathrm{Op}(\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\})) \tag{4.9}$$

induced by the natural (restriction) morphisms $\mathrm{Op}(\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\}) \to \mathrm{Op}(\mathring{D}_{z_i})$, $i = 1, \ldots, k$. It follows from the definitions (see the proof of [29, Theorem 5.7] for details) that the action of $\mathcal{Z}^{\otimes k}$ on $H(\mathbb{M}_{z_1}, \ldots, \mathbb{M}_{z_k})$ factors through (4.9). It remains to note that for every $i = 1, \ldots, k$, the homomorphism $\mathcal{O}(\mathrm{Op}(\mathring{D}_{z_i})) \to \mathcal{O}(\mathrm{Op}(\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\}))$ is surjective (since the corresponding morphism $\mathrm{Op}(\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\}) \to \mathrm{Op}(\mathring{D}_{z_i})$ is a closed embedding).

$\square$

Let $\mathbb{C}_\chi$ be the one-dimensional module over $t^{-1}\mathfrak{g}[[t^{-1}]] \oplus \mathbb{C}\mathsf{K}$, where $t^{-1}\mathfrak{g}[[t^{-1}]]$ acts via the character χ and K acts via $-1/2$. Consider the following module: $\mathbb{I}_{\chi,\infty} = \operatorname{Ind}^{\hat{\mathfrak{g}}}_{t^{-1}\mathfrak{g}[[t^{-1}]] \oplus \mathbb{C}\mathsf{K}} \mathbb{C}_\chi = \operatorname{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[[t^{-1}]] \oplus \mathbb{C}\mathsf{K}} I_\chi$, where $I_\chi = \operatorname{Ind}^{\mathfrak{g}[[t^{-1}]] \oplus \mathbb{C}\mathsf{K}}_{t^{-1}\mathfrak{g}[[t^{-1}]] \oplus \mathbb{C}\mathsf{K}} \mathbb{C}_\chi$.

Remark 4.4.8 Note that we can realize $\mathbb{I}_{\chi,\infty}$ as $\mathbb{M}_z$ for an appropriate choice of a $\mathfrak{g}$-module M and a point $z \in \mathbb{P}^1$. Indeed, taking $M = I_\chi$ and $z = \infty \in \mathbb{P}^1$, we see that $\mathbb{I}_{\chi,\infty} = \mathbb{M}_\infty$.

Lemma 4.4.9 *Assume that* $\{z_1, \ldots, z_k\} \subset \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. *The natural embedding* $M_1 \otimes \ldots \otimes M_k \subset \mathbb{M}_{z_1} \otimes \ldots \otimes \mathbb{M}_{z_k} \otimes \mathbb{I}_{\chi,\infty}$ *induces the isomorphism* $M_1 \otimes \ldots \otimes M_k \xrightarrow{\sim} H(\mathbb{M}_{z_1}, \ldots, \mathbb{M}_{z_k}, \mathbb{I}_{\chi,\infty})$.

Proof: Follows from Proposition 4.4.5 using that I_χ is isomorphic to $U(\mathfrak{g})$ as a $\mathfrak{g}$-module. $\square$

Proposition 4.4.10 *We have*

$$\mathcal{A}_\chi(z_1, \ldots, z_k) = \mathcal{A}(\mathbb{M}_{z_1}, \ldots, \mathbb{M}_{z_k}, \mathbb{I}_{-\chi,\infty}) = \mathcal{A}'_\chi(z_1, \ldots, z_k).$$

In particular,

$$\mathcal{A}_\chi(z_1, \ldots, z_k) = \mathcal{A}'_\chi(z_1, \ldots, z_k).$$

Proof: Recall that the algebra $\mathcal{A}^{\mathrm{u}}_\chi$ is the image of $\mathcal{Z}$ in the quantum Hamiltonian reduction (4.3) and the latter identifies naturally with $\operatorname{End}_{\hat{\mathfrak{g}}}(\mathbb{I}_{-\chi,\infty}) \subset \operatorname{End}_{\mathfrak{g}[t]}(\mathbb{I}_{-\chi,\infty}) = U(\mathfrak{g}[t])^{\mathrm{opp}}$. Pick $X \in U(\mathfrak{g}[t])$ and let us denote by $X_{z_i} \in U(\mathfrak{g}[[t-z_i]])$ the corresponding element of $U(\mathfrak{g}[[t-z_i]])$. Note that for every $v_i \in \mathbb{M}_{z_i}$, $v_\infty \in \mathbb{I}_{-\chi,\infty}$ the following equality holds in $H(\mathbb{M}_{z_1}, \ldots, \mathbb{M}_{z_k}, \mathbb{I}_{-\chi,\infty})$:

$$\Big[\Big(\bigotimes_{i=1}^{k} v_i\Big) \otimes X(v_\infty) + \sum_{i=1}^{k} v_1 \otimes \ldots \otimes v_{i-1} \otimes X_{z_i}(v_i) \otimes v_{i+1} \otimes \ldots \otimes v_k \otimes v_\infty\Big] = 0. \tag{4.10}$$

Now, taking $v_i \in M_i \subset \mathbb{M}_{z_i}$, we see that $X_{z_i}(v_i) = X(z_i)v_i$ so we conclude that

$$\sum_{i=1}^{k} v_1 \otimes \ldots \otimes v_{i-1} \otimes X_{z_i}(v_i) \otimes v_{i+1} \otimes \ldots \otimes v_k = \mathrm{ev}_{z_1,\ldots,z_k}(X)(v_1 \otimes \ldots \otimes v_k). \tag{4.11}$$

Equations (4.10), (4.11) imply that in $H(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty})$ we have

$$\Big[\mathrm{ev}_{z_1,\ldots,z_k}(X)(v_1\otimes\ldots\otimes v_k)\otimes v_\infty\Big] = -\Big[\Big(\bigotimes_{i=1}^{k} v_i\Big)\otimes X(v_\infty)\Big]$$

and the equality $\mathcal{A}_\chi(z_1,\ldots,z_k)=\mathcal{A}(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty})$ then follows.

Let us now prove the equality $\mathcal{A}'_\chi(z_1,\ldots,z_k)=\mathcal{A}(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty})$. Recall that by Proposition 4.4.7 the algebra $\mathcal{A}(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty})$ coincides with the image of $1\otimes\ldots\otimes1\otimes\mathcal{Z}$ in $\mathrm{End}(H(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty},\mathbb{V}_{0,z}))$. Directly from the definitions (c.f. [29, Sections 2.7, 2.8]) it then follows that the image of $\underbrace{1\otimes\ldots\otimes1}_{k+1}\otimes\mathcal{Z}$ in $\mathrm{End}(H(\mathbb{M}_{z_1},\ldots,\mathbb{M}_{z_k},\mathbb{I}_{-\chi,\infty},\mathbb{V}_{0,z}))$ is exactly $\mathcal{A}'_\chi(z_1,\ldots,z_k)$. $\square$

Remark 4.4.11 In type A the equality $\mathcal{A}_\chi(z_1,\ldots,z_k)=\mathcal{A}'_\chi(z_1,\ldots,z_k)$ can be deduced from the explicit description of the generating functions for the generators of the algebras $\mathcal{A}_\chi(z_1,\ldots,z_k)$, $\mathcal{A}'_\chi(z_1,\ldots,z_k)$ (see Section 4.11, Corollary 4.11.7 and [21, Theorem 3.1]).

4.5 The size of $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathcal{A}^{\mathrm{u}}_\chi$

The goal of this section is to compute dimensions of $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathcal{A}^{\mathrm{u}}_\chi$ with respect to the filtration F_1 (see Section 4.3.1). To do this, we first describe the associated graded subalgebras $\mathrm{gr}_{PBW}\,\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathrm{gr}_{PBW}\,\mathcal{A}^{\mathrm{u}}_\chi\subset S^\bullet(\mathfrak{g}[t])$. It turns out that both of them are isomorphic to the tensor product $\overline{\mathcal{A}}^{\mathrm{u}}_0\otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}}\overline{\mathcal{A}}_\chi$ (see Proposition 4.5.5), where $\overline{\mathcal{A}}_\chi\subset S^\bullet(\mathfrak{g})$ is the subalgebra of $S^\bullet(\mathfrak{g})$ generated by $\frac{\partial^k\Phi_l}{\partial^k\chi}$, $l=1,\ldots,\mathrm{rk}\,\mathfrak{g}$, $k\geqslant 0$.

Remark 4.5.1 Algebra $\overline{\mathcal{A}}_\chi$ is called the shift of argument subalgebra (or Mishchenko-Fomenko subalgebra). It can be considered as a classical version of the algebra $\mathcal{A}_\chi(z)$ defined in Section 4.4.

We then recall the dimensions (see (4.1)) of $\overline{\mathcal{A}}^{\mathrm{u}}_0$, $\overline{\mathcal{A}}_\chi$ and obtain the desired formula for the dimension of $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathcal{A}^{\mathrm{u}}_\chi$. Moreover, we conclude that $\dim_{F_1}\overline{\mathcal{A}}^{\mathrm{u}}_\chi=\dim_{F_1}\overline{\mathcal{A}}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}}(\chi))$, where $\overline{\mathcal{A}}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}}(\chi))$ is the universal (classical) Gaudin subalgebra of $S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi))$ (see [53, Remark in Section 4.7]). This equality will be important in the next section.

Set $\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}:=[\mathfrak{z}_{\mathfrak{g}}(\chi),\mathfrak{z}_{\mathfrak{g}}(\chi)]$. Let χ_1 be a regular Cartan element of $\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}$. Let $\overline{\mathcal{A}}_{(\chi,\chi_1)}$ be the subalgebra of $S^\bullet(\mathfrak{g})$ generated by $\overline{\mathcal{A}}_\chi$, $\overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi))$. According to [105], [108], the following holds.

Proposition 4.5.2 *(a) The subalgebra $\overline{\mathcal{A}}_{(\chi,\chi_1)}\subset S^\bullet(\mathfrak{g})$ is a free polynomial algebra with $\frac{1}{2}(\dim\mathfrak{g}+\mathrm{rk}\,\mathfrak{g})$ generators. The set of generators is the union of standard generators of $\overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi))$ and $\partial^k_\chi(\Phi_i)$ with $k=1,\ldots,d'_i$ for some $d'_i\leqslant d_i$.*

(b) The subalgebra $\overline{\mathcal{A}}_{(\chi,\chi_1)}\subset S^\bullet(\mathfrak{g})$ is maximal Poisson-commutative and is equal to the limit $\lim\limits_{\epsilon\to0}\overline{\mathcal{A}}_{\chi+\epsilon\chi_1}$, where the limit is taken with respect to the filtration on $S^\bullet(\mathfrak{g})$ by the degree.

Proposition 4.5.3 *For $\chi\in\mathfrak{g}$, the Poisson subalgebra $\overline{\mathcal{A}}_\chi\subset S^\bullet(\mathfrak{g})$ is maximal Poisson-commutative in $S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$.*

Proof: This follows from [43, Corollary 9.9]. We give a sketch of the argument. Let χ_1 be a regular element of $\mathfrak{z}_{\mathfrak{g}}(\chi)$. Recall the subalgebra $\overline{\mathcal{A}}_{(\chi,\chi_1)} \subset S^\bullet(\mathfrak{g})$ generated by $\overline{\mathcal{A}}_\chi$ and $\overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi)) \subset S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi))$.

Pick an element $x \in S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ that commutes with $\overline{\mathcal{A}}_\chi$. Since x is $\mathfrak{z}_{\mathfrak{g}}(\chi)$-invariant it follows that x commutes with $\overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi))$ so it commutes with $\overline{\mathcal{A}}_{(\chi,\chi_1)}$, hence, lies in $\overline{\mathcal{A}}_{(\chi,\chi_1)}$ (use part (b) of Proposition 4.5.2). Recall now that by the result of Knop from [64] we have

$$S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} \cdot S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi)) = S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} \otimes_{S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi))^{\mathfrak{z}_{\mathfrak{g}}(\chi)}} S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi))$$

so

$$\overline{\mathcal{A}}_{(\chi,\chi_1)} = \overline{\mathcal{A}}_\chi \cdot \overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi)) = \overline{\mathcal{A}}_\chi \otimes_{S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi))^{\mathfrak{z}_{\mathfrak{g}}(\chi)}} \overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi)).$$

It now follows from the fact that $\overline{\mathcal{A}}_{\chi,\underline{\chi_1}}$ is freely generated by the standard generators of $\overline{\mathcal{A}}_{\chi_1}(\mathfrak{z}_{\mathfrak{g}}(\chi))$ together with some $\partial_\chi^k \Phi_l \in \overline{\mathcal{A}}_\chi$ (see Proposition 4.5.2) that $S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} \cap \overline{\mathcal{A}}_{(\chi,\chi_1)} = \overline{\mathcal{A}}_\chi$. We conclude that $x \in S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} \cap \overline{\mathcal{A}}_{(\chi,\chi_1)} = \overline{\mathcal{A}}_\chi$ as desired. □

Proposition 4.5.4 *Consider the subalgebras $S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}}$, $S^\bullet(\mathfrak{g}) \subset S^\bullet(\mathfrak{g}[t])$. We have*

$$S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \cdot S^\bullet(\mathfrak{g}) = S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} S^\bullet(\mathfrak{g}) = Z_{S^\bullet(\mathfrak{g}[t])}(S^\bullet(\mathfrak{g})^{\mathfrak{g}}).$$

Proof: Follows from [53, Lemma 6.9]. □

Proposition 4.5.5 *The algebras $\mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi$, $\mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi \subset S^\bullet(\mathfrak{g}[t])$ are both equal to the subalgebra of $S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ generated by $\overline{\mathcal{A}}^{\mathrm{u}}_0 \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}}$ and $\overline{\mathcal{A}}_\chi \subset S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$. Moreover, this algebra coincides with the tensor product $\overline{\mathcal{A}}^{\mathrm{u}}_0 \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} \overline{\mathcal{A}}_\chi$ and is a maximal Poisson-commutative subalgebra of $S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$.*

Proof: Let us show that $S^\bullet(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi \cap \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi$. Note that for every $X = x_1 \dots x_l \in S^\bullet(\mathfrak{g})$ we can write

$$X^{(0)} = x_1[0] \dots x_l[0] + \text{other terms}.$$

The image of $X^{(0)}$ in the Hamiltonian reduction (4.5) inside $S^\bullet(\mathfrak{g}[t])$ is equal to

$$X + \text{lower degree terms}$$

so the image of $X^{(0)}$ in $\mathrm{gr}_{PBW}\, S^\bullet(\mathfrak{g}[t])$ is X. We conclude that $S^\bullet(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi$. In the same way we show that $S^\bullet(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi$.

Let us now show that the algebra generated by $\overline{\mathcal{A}}^{\mathrm{u}}_0$ and $\overline{\mathcal{A}}_\chi$ is contained in $\mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi \cap \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi$. To see that $\overline{\mathcal{A}}^{\mathrm{u}}_0$ lies in both $\mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi$, $\mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi$ recall (see Proposition 4.3.3) that $\overline{\mathcal{A}}^{\mathrm{u}}_0$ is generated by the elements $D^r\Phi$, $\Phi \in S^\bullet(\mathfrak{g})^{\mathfrak{g}}$, $r \in \mathbb{Z}_{\geqslant 0}$ and we already know that $S^\bullet(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{gr}_{PBW}\, \mathcal{A}^{\mathrm{u}}_\chi \cap \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}^{\mathrm{u}}_\chi$. Note now that if $X = x_1 \dots x_l \in S^\bullet(\mathfrak{g})$ then we can write

$$X^{(r)} = \sum_{j_1,\dots,j_l \geqslant 0,\, j_1+\dots+j_l=r} x_1[j_1] \dots x_l[j_l] + \text{other terms}.$$

The image of $X^{(r)}$ in $S^\bullet(\mathfrak{g}[t])$ is equal to

$$\frac{1}{r!} D^r X + \text{lower degree terms}.$$

We conclude that the image of $r!X$ in $\mathrm{gr}_{PBW}\, S^\bullet(\mathfrak{g}[t])$ is exactly $D^r X$. It follows that $\overline{\mathcal{A}}_0^{\mathrm{u}} \subset \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}}$. In the same way we see that $\overline{\mathcal{A}}_0^{\mathrm{u}} \subset \mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}}$.

To see that $\overline{\mathcal{A}}_\chi$ lies in $\mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}} \cap \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}}$ recall that $\overline{\mathcal{A}}_\chi$ is generated by $\partial_\chi^r \Phi$, $\Phi \in S^\bullet(\mathfrak{g})^{\mathfrak{g}}$. Note that if $X := x_1 \ldots x_k \in S^\bullet(\mathfrak{g})$ then we can write

$$X^{(-r)} = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant l} x_1[0] \ldots x_{i_1-1}[0] x_{i_1}[-1] x_{i_1+1}[0] \ldots x_{i_r-1}[0] x_{i_r}[-1] x_{i_r+1}[0] \ldots x_l[0] + \\ + \text{other terms.}$$

The image of $X^{(-r)}$ in $S^\bullet(\mathfrak{g}[t])$ is equal to

$$\frac{1}{r!} \partial_\chi^r X + \text{lower degree terms.}$$

We conclude that the class of the image of $r!X^{(-r)}$ in $\mathrm{gr}_{PBW}\, S^\bullet(\mathfrak{g}[t])$ is exactly $\partial_\chi^r X$. This observation finishes the proof of the fact that $\overline{\mathcal{A}}_\chi \subset \mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}}$. In the same way we show that $\overline{\mathcal{A}}_\chi \subset \mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}}$

Let us now describe the subalgebra of $S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ generated by $\overline{\mathcal{A}}_0^{\mathrm{u}}, \overline{\mathcal{A}}_\chi$. Note that $\overline{\mathcal{A}}_0^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}}$, $\overline{\mathcal{A}}_\chi \subset S^\bullet(\mathfrak{g})$ and by Proposition 4.5.4 we have

$$S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \cdot S^\bullet(\mathfrak{g}) = S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} S^\bullet(\mathfrak{g}) = Z_{S^\bullet(\mathfrak{g}[t])}(S^\bullet(\mathfrak{g})^{\mathfrak{g}}).$$

We conclude that

$$\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi = \overline{\mathcal{A}}_0^{\mathrm{u}} \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} \overline{\mathcal{A}}_\chi \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} = \\ = S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \cdot S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)} = Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(S^\bullet(\mathfrak{g})^{\mathfrak{g}}) \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}.$$

It then follows from [53, Corollary 4.10] and Proposition 4.5.3 that $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi$ is the maximal Poisson-commutative subalgebra of $S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \cdot S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$.

Note also that if $x \in S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ commutes with $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi$ then (since $S^\bullet(\mathfrak{g})^{\mathfrak{g}} \subset \overline{\mathcal{A}}_0^{\mathrm{u}} \cap \overline{\mathcal{A}}_\chi$) we must have $x \in \mathfrak{z}_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(S^\bullet(\mathfrak{g})^{\mathfrak{g}}) = S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \otimes_{S^\bullet(\mathfrak{g})^{\mathfrak{g}}} S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$. Recall that $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}} \cdot S^\bullet(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ is maximal Poisson-commutative so we conclude that $x \in \overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi$. We have shown that $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ is maximal Poisson-commutative.

Recall now that $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi$ is contained in both $\mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}}$, $\mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}}$ so from the maximality of $\overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi$ and commutativity of $\mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}}$, $\mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}}$ we conclude that

$$\mathrm{gr}_{PBW}\, \overline{\mathcal{A}}_\chi^{\mathrm{u}} = \overline{\mathcal{A}}_0^{\mathrm{u}} \cdot \overline{\mathcal{A}}_\chi = \mathrm{gr}_{PBW}\, \mathcal{A}_\chi^{\mathrm{u}}.$$

□

Corollary 4.5.6 *Subalgebras*

$$\mathcal{A}_\chi^{\mathrm{u}} \subset U(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}, \overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$$

are maximal (resp., maximal Poisson) commutative. In particular, $\overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}$ *is algebraically closed.*

Proof: Follows from Proposition 4.5.5 which claims that the associated graded $\mathrm{gr}_{PBW}\,\mathcal{A}^{\mathrm{u}}_\chi = \mathrm{gr}_{PBW}\,\overline{\mathcal{A}}^{\mathrm{u}}_\chi$ is maximal. $\square$

Let us now recall the dimensions of the algebras $\overline{\mathcal{A}}^{\mathrm{u}}_0$, $\overline{\mathcal{A}}_\chi$. Let d_i, $i=1,\ldots,\mathrm{rk}\,\mathfrak{g}$ be the degree of Φ_i and let p_i, $i=1,\ldots,\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}$ be the degrees of the generators of the algebra $S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}})^{\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}$.

Proposition 4.5.7 *We have*

$$\dim_{F_1}\overline{\mathcal{A}}^{\mathrm{u}}_0 = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=d_i}^{\infty}\frac{1}{1-q^l},\quad \dim_{PBW}\overline{\mathcal{A}}_\chi = \frac{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}\prod_{l=1}^{p_i-1}(1-q^l)}{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=1}^{d_i}(1-q^l)}.$$

Proof: The claim about the dimension of the algebra $\overline{\mathcal{A}}^{\mathrm{u}}_0$ follows from the fact that $\overline{\mathcal{A}}^{\mathrm{u}}_0$ is freely generated by $D^r\Phi_i$, $r\geqslant 0$ (see Proposition 4.3.3).

Let us now compute the dimension of $\overline{\mathcal{A}}_\chi$. Recall that by part (b) of Proposition 4.5.2 we have $\overline{\mathcal{A}}_{(\chi,\chi_1)} = \lim\limits_{\epsilon\to 0}\overline{\mathcal{A}}_{\chi+\epsilon\chi_1}$. Since $\chi+\epsilon\chi_1\in\mathfrak{g}$ is regular and $\overline{\mathcal{A}}_{\chi+\epsilon\chi_1}\subset S^\bullet(\mathfrak{g})$ are graded with respect to the grading by the degree (that induces the filtration that we use to define the limit) we conclude from Lemma 4.2.5 that

$$\dim_{PBW}\overline{\mathcal{A}}_{(\chi,\chi_1)} = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=1}^{d_i}\frac{1}{1-q^l}.$$

It now follows from part (a) of Proposition 4.5.2 that

$$\dim_{PBW}\overline{\mathcal{A}}_\chi\cdot\prod_{i=1}^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}\prod_{l=1}^{p_i-1}\frac{1}{1-q^l} = \dim_{PBW}\overline{\mathcal{A}}_{(\chi,\chi_1)} = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=1}^{d_i}\frac{1}{1-q^l}.$$

We conclude that

$$\dim_{PBW}\overline{\mathcal{A}}_\chi = \frac{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=1}^{d_i}\frac{1}{1-q^l}}{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}\prod_{l=1}^{p_i-1}\frac{1}{1-q^l}}.$$

$\square$

We are now ready to compute the dimension of the algebras $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathcal{A}^{\mathrm{u}}_\chi$.

Proposition 4.5.8 *We have*

$$\dim_{F_1}\overline{\mathcal{A}}^{\mathrm{u}}_\chi = \dim_{F_1}\mathcal{A}^{\mathrm{u}}_\chi = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}\prod_{l=p_i}^{\infty}\frac{1}{1-q^l}\cdot\prod_{l=1}^{\infty}\frac{1}{(1-q^l)^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)-\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}}.$$

Proof: Since

$$\mathrm{gr}_{PBW}\,\overline{\mathcal{A}}^{\mathrm{u}}_\chi = \mathrm{gr}_{PBW}\,\mathcal{A}^{\mathrm{u}}_\chi = \overline{\mathcal{A}}^{\mathrm{u}}_0\otimes_{S^\bullet(\mathfrak{g})^\mathfrak{g}}\overline{\mathcal{A}}_\chi$$

and

$$\dim_{F_1}\overline{\mathcal{A}}^{\mathrm{u}}_0 = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=d_i}^{\infty}\frac{1}{1-q^l},\ \dim_{PBW}\overline{\mathcal{A}}_\chi = \frac{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\prod_{l=1}^{d_i}\frac{1}{1-q^l}}{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{z}_\mathfrak{g}(\chi)^{\mathrm{der}}}\prod_{l=1}^{p_i-1}\frac{1}{1-q^l}},\ \dim_{PBW}S^\bullet(\mathfrak{g})^\mathfrak{g} = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}\frac{1}{1-q^{d_i}}$$

it follows that

$$\dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_{\chi} = \frac{\prod_{i=1}^{\operatorname{rk}\mathfrak{g}} \prod_{l=d_i+1}^{\infty} \frac{1}{1-q^l} \cdot \prod_{i=1}^{\operatorname{rk}\mathfrak{g}} \prod_{l=1}^{d_i} \frac{1}{1-q^l}}{\prod_{i=1}^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}} \prod_{l=1}^{p_i-1} \frac{1}{1-q^l}} =$$

$$= \frac{\prod_{l=1}^{\infty} \frac{1}{(1-q^l)^{\operatorname{rk}\mathfrak{g}}}}{\prod_{i=1}^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}} \prod_{l=1}^{p_i-1} \frac{1}{1-q^l}} = \prod_{i=1}^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}} \prod_{l=p_i}^{\infty} \frac{1}{1-q^l} \cdot \prod_{l=1}^{\infty} \frac{1}{(1-q^l)^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi) - \operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}}}.$$

□

Corollary 4.5.9 *We have* $\dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_{\chi} = \dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}}(\chi))$.

Proof: We have

$$\dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_{\chi} = \frac{\prod_{l=1}^{\infty} \frac{1}{(1-q^l)^{\operatorname{rk}\mathfrak{g}}}}{\prod_{i=1}^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}} \prod_{l=1}^{p_i-1} \frac{1}{1-q^l}} =$$

$$= \prod_{i=1}^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}} \prod_{l=p_i}^{\infty} \frac{1}{1-q^l} \cdot \prod_{l=1}^{\infty} \frac{1}{(1-q^l)^{\operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi) - \operatorname{rk}\mathfrak{z}_{\mathfrak{g}}(\chi)^{\mathrm{der}}}} = \dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}}(\chi)).$$

□

4.6 Universal inhomogeneous Gaudin subalgebras as centralizers

It follows from [53] that for $\chi = 0$ we have

$$\overline{\mathcal{A}}^{\mathrm{u}}_0 = Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{g}}}(\Omega_0),\ \mathcal{A}^{\mathrm{u}}_0 = Z_{U(\mathfrak{g}[t])^{\mathfrak{g}}}(\tilde{\Omega}_0).$$

The main goal of this section is to prove that the same equalities hold for $\overline{\mathcal{A}}^{\mathrm{u}}_{\chi}$, $\mathcal{A}^{\mathrm{u}}_{\chi}$ i.e. that

$$\overline{\mathcal{A}}^{\mathrm{u}}_{\chi} = Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\chi}),\ \mathcal{A}^{\mathrm{u}}_{\chi} = Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\tilde{\Omega}_{\chi}).$$

Lemma 4.6.1 *For every* $m > 0$ *we have* $Z_{S^\bullet(\mathfrak{g}[t])}(\chi[m]) = S^\bullet(\mathfrak{z}_{\mathfrak{g}}(\chi)[t])$.

Proof: Same proof as of [98, Lemma 4]. □

Proposition 4.6.2 *We have* $\overline{\mathcal{A}}^{\mathrm{u}}_{\chi} = Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\chi})$.

Proof: Consider the family $Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\kappa\chi})$, $\kappa \in \mathbb{C}^\times$. Note that

$$Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\kappa\chi}) \simeq Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\chi})$$

via the Poisson automorphism of $S^\bullet(\mathfrak{g}[t])$ given by the map $x[n] \mapsto \kappa^n x[n]$. It follows that we can consider the limit

$$\lim_{\kappa\to\infty} Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_{\mathfrak{g}}(\chi)}}(\Omega_{\kappa\chi}).$$

Pick an element

$$a_0 \in \lim_{\kappa\to\infty} Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_{\kappa\chi}).$$

Note that $a_0 \in F_1^i S^\bullet(\mathfrak{g}[t])$ for some $i \geqslant 0$. Recall that we have a map

$$f\colon \mathbb{A}^1 \to \mathrm{Gr}(d(i), F_1^i S^\bullet(\mathfrak{g}[t]))$$

that sends $\epsilon \in \mathbb{A}^1$ to $F_1^i S^\bullet(\mathfrak{g}[t])) \cap Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\epsilon\Omega_0 + \chi[2])$ and sends 0 to $\lim\limits_{\kappa\to\infty}(F_1^i S^\bullet(\mathfrak{g}[t]) \cap Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_{\kappa\chi}))$. It follows from Lemma 4.2.2 that there exists a morphism $a\colon \mathbb{A}^1 \to F_1^i S^\bullet(\mathfrak{g}[t]))$ such that $a(\epsilon) \in Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\epsilon\Omega_0 + \chi[2])$ for every $\epsilon \in \mathbb{A}^1 \setminus \{0\}$ and $a(0) = a_0$. Then we can write $a(\epsilon) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \dots$ with $a_i \in F_1^i S^\bullet(\mathfrak{g}[t])$.

We have $\{\epsilon\Omega_0 + \chi[2], a\} = 0$ so

$$\{\chi[2], a_0\} = 0,\ \{\Omega_0, a_0\} + \{\chi[2], a_1\} = 0.$$

From $\{\chi[2], a_0\} = 0$ we conclude by Lemma 4.6.1 that

$$a_0 \in S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t]). \tag{4.12}$$

Pick a basis of $\mathfrak{g}$, consisting of root vectors and consider the corresponding decomposition $\mathfrak{g} = \mathfrak{z}_\mathfrak{g}(\chi) \oplus \mathfrak{m}$. It induces the decomposition $S^\bullet(\mathfrak{g}[t]) = S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t]) \oplus S^\bullet(\mathfrak{g}[t])\mathfrak{m}[t]$. Let $\pi\colon S^\bullet(\mathfrak{g}[t]) \twoheadrightarrow S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t])$ be the projection. Note that

$$\{\chi[2], S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t])\} = 0,\ \ \{\chi[2], S^\bullet(\mathfrak{g}[t])\mathfrak{m}[t]\} \subset S^\bullet(\mathfrak{g}[t])\mathfrak{m}[t]$$

so it is clear that $\pi(\{\chi[2], a_1\}) = 0$. It is also clear (use (4.12)) that we have $\pi(\{\Omega_0, a_0\}) = \{\Omega_0(\mathfrak{z}_\mathfrak{g}(\chi)), a_0\}$, where $\Omega_0(\mathfrak{z}_\mathfrak{g}(\chi))$ is the element Ω_0 for $\mathfrak{z}_\mathfrak{g}(\chi)$.

We conclude that $\{\Omega_0(\mathfrak{z}_\mathfrak{g}(\chi)), a_0\} = 0$. We also know that $a_0 \in S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}$. It follows that

$$\lim_{\kappa\to\infty} Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_{\kappa\chi}) \subset Z_{S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi)[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_0(\mathfrak{z}_\mathfrak{g}(\chi))) = \overline{\mathcal{A}}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(\chi)),$$

where the last equality follows from [53, Proposition 4.9]. It remains to note that $\overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi)$, so

$$\dim_{F_1} \overline{\mathcal{A}}_\chi^{\mathrm{u}} \leqslant \dim_{F_1} Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi) \leqslant \dim_{F_1}\left(\lim_{\kappa\to\infty} Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_{\kappa\chi})\right) \leqslant \dim_{F_1} \overline{\mathcal{A}}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(\chi)). \tag{4.13}$$

Recall that $\dim_{F_1} \overline{\mathcal{A}}_\chi^{\mathrm{u}} = \dim_{F_1} \overline{\mathcal{A}}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(\chi))$ by Corollary 4.5.9 so the inequalities in (4.13) are actually equalities and we must have $\overline{\mathcal{A}}_\chi^{\mathrm{u}} = Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi)$ as desired.

□

Remark 4.6.3 It follows from the proof of Proposition 4.6.2 that we have

$$\lim_{\kappa\to\infty} \overline{\mathcal{A}}_{\kappa\chi}^{\mathrm{u}} = \overline{\mathcal{A}}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(\chi)).$$

Remark 4.6.4 Note that for regular χ we also have $\overline{\mathcal{A}}_\chi^{\mathrm{u}} = Z_{S^\bullet(\mathfrak{g}[t])}(\omega_\chi)$. Indeed, we have the filtration F_2 on $S^\bullet(\mathfrak{g}[t])$ with $\deg_2 x[m] = m$. Note that $\mathrm{gr}_2(\omega_\chi) = \mathrm{gr}_2(\chi[1]])$ so we conclude that $\mathrm{gr}_2(Z_{S^\bullet(\mathfrak{g}[t])}(\omega_\chi)) \subset Z_{S^\bullet(\mathfrak{g}[t])}(\chi[1]) = S^\bullet(\mathfrak{h}[t])$ (see Lemma 4.6.1). It follows that $\dim_{F_1} Z_{S^\bullet(\mathfrak{g}[t])}(\omega_\chi) \leqslant \dim_{F_1} S^\bullet(\mathfrak{h}[t])$. Recall also that $\overline{\mathcal{A}}_\chi^{\mathrm{u}} \subset Z_{S^\bullet(\mathfrak{g}[t])}(\omega_\chi)$ and $\dim_{F_1} \overline{\mathcal{A}}_\chi^{\mathrm{u}} = \dim_{F_1} S^\bullet(\mathfrak{h}[t])$. We conclude that $\overline{\mathcal{A}}_\chi^{\mathrm{u}} = Z_{S^\bullet(\mathfrak{g}[t])}(\omega_\chi)$.

So we have shown that the algebra $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$ coincides with the centralizer $Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi)$. Let us now quantize this statement i.e. prove that

$$\mathcal{A}^{\mathrm{u}}_\chi = Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi).$$

We first relate the algebras $\overline{\mathcal{A}}^{\mathrm{u}}_\chi$, $\mathcal{A}^{\mathrm{u}}_\chi$. Recall that the PBW-filtration on $U(\mathfrak{g}[t])$ and consider the Rees family $U_\epsilon(\mathfrak{g}[t])$, $\epsilon \in \mathbb{C}$ as in Section 4.2.2. Note that $U_\epsilon(\mathfrak{g}[t]) \simeq U(\mathfrak{g}[t])$ for $\epsilon \neq 0$ and $U_0(\mathfrak{g}[t]) = S^\bullet(\mathfrak{g}[t])$. For $\epsilon \neq 0$ we pick the natural isomorphisms

$$\varphi_\epsilon \colon U(\mathfrak{g}[t]) \xrightarrow{\sim} U_\epsilon(\mathfrak{g}[t])$$

that send an element a of degree r to $[\epsilon^{-r}\hbar^r a]$. Note that these isomorphisms are $\mathfrak{g}$-equivariant.

Recall that the algebra $U(\mathfrak{g}[t])$ is graded with the grading $\deg_2 x[n] = n$. Note that this grading induces the grading on $Rees(U(\mathfrak{g}[t]))$ ($\deg_2 \hbar x[n] = n$) such that $\mathbb{C}[\hbar] \subset U_\epsilon(\mathfrak{g}[t])$ lies in the zero graded component. Thus, we obtain the grading on every $U_\epsilon(\mathfrak{g}[t])$. We can consider the automorphism $d_\epsilon \colon U_\epsilon(\mathfrak{g}[t]) \xrightarrow{\sim} U_\epsilon(\mathfrak{g}[t])$ that sends a such that $\deg_2 a = i$ to $\epsilon^{-i}a$. The automorphism d_ϵ is $\mathfrak{g}$-equivariant. We obtain the composition

$$d_\epsilon \circ \varphi_\epsilon \colon U(\mathfrak{g}[t]) \xrightarrow{\sim} U_\epsilon(\mathfrak{g}[t]).$$

Set $\tilde{\varphi}_\epsilon := d_\epsilon \circ \varphi_\epsilon$. For $\epsilon \neq 0$ we can embed $\mathcal{A}^{\mathrm{u}}_\chi \subset U_\epsilon(\mathfrak{g}[t])$ via $\tilde{\varphi}_\epsilon$.

Proposition 4.6.5 *After the embedding $\tilde{\varphi}_\epsilon \colon \mathcal{A}^{\mathrm{u}}_\chi \subset U_\epsilon(\mathfrak{g}[t])$, we have $\lim\limits_{\epsilon\to 0} \mathcal{A}^{\mathrm{u}}_\chi = \overline{\mathcal{A}}^{\mathrm{u}}_\chi$.*

Proof: Let us check that $\Omega_\chi \in \lim\limits_{\epsilon\to 0} \mathcal{A}^{\mathrm{u}}_\chi$. Recall that $\tilde{\Omega}_\chi \in \mathcal{A}^{\mathrm{u}}_\chi$ (see Remark **??**). After the identification $\tilde{\varphi}_\epsilon$ this element becomes

$$\tilde{\Omega}_{\chi,\epsilon} = \sum_a \hbar^2 \epsilon^{-3} x_a x_a[1] + \epsilon^{-3}\hbar\chi[2]$$

and the limit of $\epsilon^3\tilde{\Omega}_{\chi,\epsilon}$ as $\epsilon \to 0$ is exactly Ω_χ.

We conclude (use Lemma 4.2.9) that $\lim\limits_{\epsilon\to 0} \mathcal{A}^{\mathrm{u}}_\chi \subset Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi) = \overline{\mathcal{A}}^{\mathrm{u}}_\chi$, where the last equality holds by Proposition 4.6.2. Since the dimension of $\lim\limits_{\epsilon\to 0} \mathcal{A}^{\mathrm{u}}_\chi$ is at least $\dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi$ (Lemma 4.2.10) and by Proposition 4.5.5 we have $\dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi = \dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_\chi$ we conclude that $\lim\limits_{\epsilon\to 0} \mathcal{A}^{\mathrm{u}}_\chi = \overline{\mathcal{A}}^{\mathrm{u}}_\chi$. □

Proposition 4.6.6 *We have $\mathcal{A}^{\mathrm{u}}_\chi = Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi)$.*

Proof: Since the element $\epsilon^3\tilde{\Omega}_{\chi,\epsilon} = \epsilon^3\tilde{\varphi}_\epsilon(\tilde{\Omega}_\chi)$ goes to Ω_χ as ϵ goes to zero we conclude that

$$\lim_{\epsilon\to 0} Z_{U_\epsilon(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_{\chi,\epsilon}) \subset Z_{S^\bullet(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\Omega_\chi) = \overline{\mathcal{A}}^{\mathrm{u}}_\chi.$$

Note now that since $\mathcal{A}^{\mathrm{u}}_\chi \subset Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi)$ and passing to limit may only increase the dimension then

$$\dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi \leqslant \dim_{F_1} Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi) \leqslant \dim_{F_1} \lim_{\epsilon\to 0} Z_{U_\epsilon(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_{\chi,\epsilon}) \leqslant \dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_\chi. \tag{4.14}$$

Since by Proposition 4.5.5 we have $\dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi = \dim_{F_1} \overline{\mathcal{A}}^{\mathrm{u}}_\chi$ we conclude that inequalities in (4.14) are equalities so $\mathcal{A}^{\mathrm{u}}_\chi = Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi)$. □

Remark 4.6.7 Note that for regular χ we also have $\mathcal{A}^{\mathrm{u}}_{\chi} = Z_{U(\mathfrak{g}[t])}(\tilde{\omega}_{\chi})$. The proof is the same as the one of Proposition 4.6.6, the only difference is that we use Remark 4.6.4 instead of Proposition 4.6.2.

4.7 Yangian

Recall that $\mathfrak{g}$ is a simple Lie algebra.

4.7.1 Definition of the Yangian

Definition 4.7.1 *Yangian $Y(\mathfrak{g})$ is the associative $\mathbb{C}$-algebra generated by*

$$\{x,\ J(x)\,|\,x \in \mathfrak{g}\}$$

subject to the following relations:

$$xy - yx = [x, y],\ J([x, y]) = [J(x), y],\ J(cx + dy) = cJ(x) + dJ(y),$$
$$[J(x), [J(y), z]] = [x, [J(y), J(z)]] = \sum_{a_1,a_2,a_3} ([x, x_{a_1}], [[y, x_{a_2}], [z, x_{a_3}]])\{x_{a_1}, x_{a_2}, x_{a_3}\},$$

$$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] =$$
$$= \sum_{a_1,a_2,a_3} (([x, x_{a_1}], [[y, x_{a_2}], [[z, w], x_{a_3}]]) + ([z, x_{a_1}], [[w, x_{a_2}], [[x, y]x_{a_3}]]))\{x_{a_1}, x_{a_2}, J(x_{a_3})\}$$

for all $x, y, z, w \in \mathfrak{g}$, $c, d \in \mathbb{C}$. Here $\{x_a\}|_{a=1,\ldots,\dim\mathfrak{g}}$ is an orthonormal basis and

$$\{x_1, x_2, x_3\} := \frac{1}{24} \sum_{\sigma \in S_3} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$$

for all $x_1, x_2, x_3 \in Y(\mathfrak{g})$.

By [23, Theorem 2] Yangian $Y(\mathfrak{g})$ is a Hopf algebra. It follows from [23, Theorem 3] that there is a unique formal series

$$\mathcal{R}(u) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k u^{-k} \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[[u^{-1}]]$$

satisfying

$$(\mathrm{id} \otimes \Delta)\mathcal{R}(u) = \mathcal{R}_{12}(u)\mathcal{R}_{13}(u),$$
$$\tau_{0,u}\Delta^{\mathrm{opp}}(x) = \mathcal{R}(u)^{-1}(\tau_{0,u}\Delta(x))\mathcal{R}(u) \text{ for all } x \in Y(\mathfrak{g}),$$

where $\tau_{0,u}$ is the automorphism of $Y(\mathfrak{g})[[u]]$ given by $x \mapsto x,\ J(x) \mapsto J(x) + ux$ for all $x \in \mathfrak{g}$.

We have

$$\mathcal{R}(u) = 1 + 2\omega' u^{-1} + \Big(\sum_a (J(x_a) \otimes x_a - x_a \otimes J(x_a)) + 2(\omega')^2\Big)u^{-2} + O(u^{-3}),$$

where $\omega' = \frac{1}{2}\sum_a x_a \otimes x_a$. See [23] and [120, Section 3] for details. We will call $\mathcal{R}(u)$ the universal R-matrix.

4.7.2 *RTT* realization of the Yangian

Let $\rho\colon Y(\mathfrak{g}) \to \mathrm{End}(V)$ be a finite dimensional representation of $Y(\mathfrak{g})$ that is not a direct sum of trivial representations w.r.t. $\mathfrak{g} \subset Y(\mathfrak{g})$. We fix a basis $e_1, \ldots, e_{\dim V}$ of V. Let $R(u-v) := (\rho \otimes \rho)(\mathcal{R}(u-v))$ be the image of the universal R-matrix in $\mathrm{End}(V)^{\otimes 2}$. Using this data, we define the RTT-realization $Y_V(\mathfrak{g})$ of $Y(\mathfrak{g})$ as follows.

Definition 4.7.2 *The Yangian $Y(\mathfrak{g})$ is a unital associative algebra generated by the elements $t_{ij}^{(r)}, 1 \leqslant i,j \leqslant \dim V; r \geqslant 1$ with the defining relations*

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \in \mathrm{End}(V)^{\otimes 2} \otimes Y(\mathfrak{g})[[u^{-1}, v^{-1}]],$$

$$\eta^2(T(u)) = T(u + \frac{1}{2}c_{\mathfrak{g}}),$$

where $\eta(T(u)) = T(u)^{-1}$ is the antipode map and $c_{\mathfrak{g}}$ is the value of the Casimir element $\sum_a x_a^2$ of $\mathfrak{g}$ on the adjoint representation. Here

$$T(u) = [t_{ij}(u)]_{i,j=1,\ldots,\dim V} \in \mathrm{End}\, V \otimes Y(\mathfrak{g}),$$

$$t_{ij}(u) = \delta_{ij} + \sum_{r \geqslant 1} t_{ij}^{(r)} u^{-r}$$

and $T_1(u)$ (resp. $T_2(u)$) is the image of $T(u)$ in the first (resp. second) copy of $\mathrm{End}\, V$.

Theorem 4.7.3 *([120, Theorem 6.2]) The assignment $T(u) \mapsto (\rho \otimes 1)\mathcal{R}(-u)$ extends to an isomorphism of algebras $\Phi\colon Y_V(\mathfrak{g}) \xrightarrow{\sim} Y(\mathfrak{g})$.*

Recall that we have a basis $e_1, \ldots, e_{\dim V}$ of V. We denote by $E_{ij}|_{1 \leqslant i,j \leqslant \dim V} \in \mathrm{End}(V)$ the corresponding matrix units. We define $\mathcal{F}_{ij} \in \mathrm{End}(V)$ by the following identity

$$\sum_{ij} E_{ij} \otimes \mathcal{F}_{ij} = -(\rho \otimes 1)\Big(\sum_a x_a \otimes x_a\Big).$$

Proposition 4.7.4 *([120, Corollary 6.3]) The map Φ^{-1} sends generators $\{\mathcal{F}_{ij},\, J(\mathcal{F}_{ij})\}$ of $Y(\mathfrak{g})$ to*

$$\mathcal{F}_{ij} \mapsto t_{ij}^{(1)},\; J(\mathcal{F}_{ij}) \mapsto t_{ij}^{(2)} - \frac{1}{2}\sum_{a=1}^{\dim V} t_{ia}^{(1)} t_{aj}^{(1)} + \sum_{p,l=1}^{\dim V} b_{pl}^{(ij)} t_{pl}^{(1)},$$

where b_{pl}^{ij} are certain scalars.

Remark 4.7.5 Note that for $\mathfrak{g} = \mathfrak{sl}_n$ and $V = \mathbb{C}^n$ (the standard representation of $\mathfrak{g}$), we have $\mathcal{F}_{ij} = -E_{ji}$.

4.7.3 Two filtrations on the Yangian

The first filtration F_1 on $Y(\mathfrak{g})$ is determined by putting $\deg_1 t_{ij}^{(r)} = r$. More precisely, the r-th filtered component $F_1^{(r)} Y(\mathfrak{g})$ is the linear span of all monomials $t_{i_1 j_1}^{(r_1)} \cdot \ldots \cdot t_{i_m j_m}^{(r_m)}$ with $r_1 + \ldots + r_m \leqslant r$. It follows from Proposition 4.7.4 that after the identification $Y(\mathfrak{g}) = Y_V(\mathfrak{g})$ we have $\deg_1 x = 1$, $\deg_1 J(x) = 2$ for every nonzero $x \in \mathfrak{g}$. By gr_1 we denote the operation of taking associated graded algebra with respect to F_1. For $x \in Y(\mathfrak{g})$ we denote by $\mathrm{gr}_1 x$ the class of x in in $F_1^{(i)} Y(\mathfrak{g}) / F_1^{(i-1)} Y(\mathfrak{g}) \subset \mathrm{gr}_2 Y(\mathfrak{g})$, where i is the minimal such that $x \in F_1^{(i)} Y(\mathfrak{g})$.

Let $G[[t^{-1}]]$ be the group of $\mathbb{C}[[t^{-1}]]$-points of G i.e. $g \in G[[t^{-1}]]$ is a morphism $g \colon \operatorname{Spec} \mathbb{C}[[t^{-1}]] \to G$. For $g \in G[[t^{-1}]]$, we denote by $ev_g \colon \mathbb{C}[G] \to \mathbb{C}[[t^{-1}]]$ the corresponding homomorphism of algebras. We have the evaluation at infinity homomorphism $G[[t^{-1}]] \to G$ and denote by $G_1[[t^{-1}]]$ its kernel. To any $f \in \mathbb{C}[G]$ one can assign the function

$$\tilde{f} \colon G_1[[t^{-1}]] \to \mathbb{C}[[t^{-1}]],\ \tilde{f} = \sum_{r \geqslant 0} f^{(r)} t^{-r}$$

given by

$$\tilde{f}(g) := ev_g(f),\ g \in G_1[[t^{-1}]].$$

The $\mathbb{C}^\times$-action on $G_1[[t^{-1}]]$ by dilations of the variable t determines a grading on $\mathbb{C}(G_1[[t^{-1}]])$ such that $\deg f^{(r)} = r$ for any $f \in \mathbb{C}[G]$.

Proposition 4.7.6 *([52, Proposition 2.24]) There is an isomorphism of graded algebras* $\mathrm{gr}_1 Y_V(\mathfrak{g}) \simeq \mathcal{O}(G_1[[t^{-1}]])$*, such that* $\mathrm{gr}_1 t_{ij}^{(r)} = \Delta_{ij}^{(r)}$*, where* $\Delta_{ij} \in \mathbb{C}[G]$ *are the matrix coefficients of the representation* V*.*

The second filtration F_2 on $Y(\mathfrak{g})$ is determined by putting $\deg t_{ij}^{(r)} = r - 1$ i.e. the r-th filtered component $F_2^{(r)} Y(\mathfrak{g})$ is the linear span of all monomials $t_{i_1 j_1}^{(r_1)} \cdot \ldots \cdot t_{i_m j_m}^{(r_m)}$ with $r_1 + \ldots + r_m \leqslant r + m$. It follows from Proposition 4.7.4 that after the identification $Y(\mathfrak{g}) = Y_V(\mathfrak{g})$ the filtration F_2 can by described as follows: we have $\deg_2 x = 0$, $\deg_2 J(x) = 1$ for every nonzero $x \in \mathfrak{g}$. By gr_2 we denote the operation of taking associated graded algebra with respect to F_2. For $x \in Y(\mathfrak{g})$ we denote by $\mathrm{gr}_2 x$ the class of x in $F_2^{(i)} Y(\mathfrak{g}) / F_2^{(i-1)} Y(\mathfrak{g}) \subset \mathrm{gr}_2 Y(\mathfrak{g})$, where i is the minimal such that $x \in F_2^{(i)} Y(\mathfrak{g})$.

Proposition 4.7.7 *([120, Theorem 6.5])* $\mathrm{gr}_2 Y(\mathfrak{g}) \simeq U(\mathfrak{g}[t])$*, where the grading is given by the* $\mathbb{C}^\times$*-action dilating* t*. Moreover, we have* $t^{r-1}\mathfrak{g} \subset \operatorname{span}\{t_{ij}^{(r)}\} / F_2^{(r-2)} Y(\mathfrak{g})$*.*

4.7.4 Filtrations on $\mathrm{gr}_1 Y(\mathfrak{g})$, $\mathrm{gr}_2 Y(\mathfrak{g})$ and the associated bigraded algebra

We follow [53, Sections 2.7, 2.13]. The filtration F_1 on $Y(\mathfrak{g})$ produces a filtration on $U(\mathfrak{g}[t]) = \mathrm{gr}_2 Y(\mathfrak{g})$ which we denote by the same letter F_1. The filtration F_2 on $Y(\mathfrak{g})$ produces the filtration on $\mathbb{C}[G[[t^{-1}]]_1] = \mathrm{gr}_1 Y(\mathfrak{g})$. The goal of this section is to describe these filtrations explicitly and also describe the corresponding associated bigraded algebra $\mathrm{bigr}\, Y(\mathfrak{g})$ (see (4.15) below).

Algebras with multiple filtrations

We begin with some general facts about algebras with multiple filtrations. For any algebra A endowed with two filtrations F_1, F_2 one can define the associated bigraded algebra of A as

$$\operatorname{bigr} A := \bigoplus_{i,j} (F_1^{(i)} A \cap F_2^{(j)} A)/((F_1^{(i-1)} A \cap F_2^{(j)} A) + (F_1^{(i)} A \cap F_2^{(j-1)} A)). \tag{4.15}$$

Algebra $\operatorname{bigr} A$ is naturally bigraded. For $x \in F_1^{(i)} A \cap F_2^{(j)} A$ let $\operatorname{bigr}^{(i,j)}(x) \in \operatorname{bigr} A$ be the class of x in $F_1^{(i)} A \cap F_2^{(j)} A/(F_1^{(i-1)} A \cap F_2^{(j)} A + F_1^{(i)} A \cap F_2^{(j-1)} A)$. Note that $\operatorname{bigr}^{(i,j)}(x)$ has bidegree (i,j).

Warning 4.7.8 *Note that* $\operatorname{bigr}^{(i,j)}(x)$ *may be zero.*

We have canonical identifications (see [53, Lemma 2.8])

$$\operatorname{gr}_2 \operatorname{gr}_1 A \simeq \operatorname{bigr} A \simeq \operatorname{gr}_1 \operatorname{gr}_2 A$$

so it makes sense to compare elements $\operatorname{gr}_{21} x := \operatorname{gr}_2 \operatorname{gr}_1 x$, $\operatorname{gr}_{12} x := \operatorname{gr}_1 \operatorname{gr}_2 x$. The following Lemma describes necessary and sufficient conditions on $x \in A$ for the equality $\operatorname{gr}_{21} x = \operatorname{gr}_{12} x$ to be true.

Lemma 4.7.9 (a) *Pick* $x \in A$, i, j *such that* $x \in F_1^{(i)} A \cap F_2^{(j)} A$.
The following are equivalent:
(i) *We have* $x \notin (F_1^{(i-1)} A \cap F_2^{(j)} A) + (F_1^{(i)} A \cap F_2^{(j-1)} A)$,
(ii) $\operatorname{gr}_{21} x$ *has bidegree* (i,j),
(iii) $\operatorname{gr}_{21} x = \operatorname{bigr}^{(i,j)} x$,
(iv) $\operatorname{gr}_{12} x$ *has bidegree* (i,j),
(v) $\operatorname{gr}_{12} x = \operatorname{bigr}^{(i,j)} x$,
(vi) $\operatorname{gr}_{21} x = \operatorname{bigr}^{(i,j)} x = \operatorname{gr}_{12} x$.
(b) *Pick* $x \in A$ *then* $\operatorname{gr}_{21} x = \operatorname{gr}_{12} x$ *if and only if there exist* i, j *such that* $x \in F_1^{(i)} A \cap F_2^{(j)} A$ *and one of six (equivalent) conditions* (i)-(vi) *holds.*

Proof: Let us prove part (a). Let us first of all show that

$$\operatorname{bigr}^{(i,j)} x = \operatorname{gr}_{21} x \text{ if and only if } x \notin (F_1^{(i)} A \cap F_2^{(j-1)} A) + F_1^{(i-1)} A. \tag{4.16}$$

Assume that $x \notin (F_1^{(i)} A \cap F_2^{(j-1)} A) + F_1^{(i-1)} A$. Note that by our assumptions we have $x \in F_1^{(i)} A \setminus F_1^{(i-1)} A$ so $\operatorname{gr}_1 x$ is equal to the class of x in $F_1^{(i)} A/F_1^{(i-1)} A$. We need to show that

$$\operatorname{gr}_1 x \in (F_1^{(i)} A \cap F_2^{(j)} A)/(F_1^{(i-1)} A \cap F_2^{(j)} A), \tag{4.17}$$

$$\operatorname{gr}_1 x \notin (F_1^{(i)} A \cap F_2^{(j-1)} A)/(F_1^{(i-1)} A \cap F_2^{(j-1)} A). \tag{4.18}$$

Equation (4.17) holds since $x \in F_1^{(i)} A \cap F_2^{(j)} A$, (4.18) holds since $x \notin (F_1^{(i)} A \cap F_2^{(j-1)} A) + F_1^{(i-1)} A$.

Assume that $\operatorname{bigr}^{(i,j)} x = \operatorname{gr}_{21} x$. It follows that (4.18) holds so $x \notin (F_1^{(i)} A \cap F_2^{(j-1)} A) + F_1^{(i-1)} A$ as desired.

Same observation shows that

$$\operatorname{bigr}^{(i,j)} x = \operatorname{gr}_{12} x \text{ if and only if } x \notin (F_1^{(i-1)}A \cap F_2^{(j)}A) + F_2^{(j-1)}A. \tag{4.19}$$

Note now that $x \in F_1^{(i)}A \cap F_2^{(j)}A$ implies that

$$x \notin (F_1^{(i-1)}A \cap F_2^{(j)}A) + (F_1^{(i)}A \cap F_2^{(j-1)}A) \text{ if and only if } x \notin (F_1^{(i)}A \cap F_2^{(j-1)}A) + F_1^{(i-1)}A$$

and

$$x \notin (F_1^{(i-1)}A \cap F_2^{(j)}A) + (F_1^{(i)}A \cap F_2^{(j-1)}A) \text{ if and only if } x \notin (F_1^{(i-1)}A \cap F_2^{(j)}A) + F_2^{(j-1)}A.$$

We conclude that (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). It is also clear that (ii) $\Leftrightarrow$ (iii), (iv) $\Leftrightarrow$ (v).

Let us now prove part (b). We only need to show that if $\operatorname{gr}_{21} x = \operatorname{gr}_{12} x$ has some bidegree (i,j) then $x \in F_1^{(i)}A \cap F_2^{(j)}A$. Indeed, since $\operatorname{gr}_1 x$ has degree i (w.r.t. the grading on $\operatorname{gr}_1 A$ induced by F_1) we must have $x \in F_1^{(i)}A$ and similarly since $\operatorname{gr}_2 x$ has degree j we must have $x \in F_2^{(j)}A$.

□

Warning 4.7.10 *Note that it is not true in general that* $\operatorname{gr}_{21} x = \operatorname{gr}_{12} x$ *for every* $x \in A$. *Take, for example,* $A = \mathbb{C}[a,b]$ *and define filtrations as follows:*

$$\deg_{F_1}(a) = 1,\ \deg_{F_1}(b) = 0,\ \deg_{F_2}(a) = 0,\ \deg_{F_2}(b) = 1.$$

Take $x = a + b$ *then*

$$\operatorname{gr}_{21}(x) = a \neq b = \operatorname{gr}_{12}(x).$$

Case of the Yangian

Let us now return to the case $A = Y(\mathfrak{g})$. Recall that $\operatorname{gr}_1 Y(\mathfrak{g}) = \mathbb{C}[G[[t^{-1}]]_1]$, $\operatorname{gr}_2 Y(\mathfrak{g}) = U(\mathfrak{g}[t])$. Let us describe the induced filtrations on $\mathbb{C}[G[[t^{-1}]]_1]$, $U(\mathfrak{g}[t])$ and the associated bigraded algebra $\operatorname{gr}_{21} Y(\mathfrak{g}) \simeq \operatorname{bigr} Y(\mathfrak{g}) \simeq \operatorname{gr}_{12} Y(\mathfrak{g})$.

Pick an identification $\exp\colon t^{-1}\mathfrak{g}[[t^{-1}]] \xrightarrow{\sim} G_1[[t^{-1}]]$ and identify $\mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]] \simeq S^\bullet(\mathfrak{g}[t])$ via the pairing given by

$$(x(t), y(t)) := \operatorname{Res}_{t=0}(x(t), y(t)),\ x(t) \in \mathfrak{g}[t],\ y(t) \in t^{-1}\mathfrak{g}[[t^{-1}]].$$

The following proposition holds by [53, proof of Proposition 2.12, Section 2.13].

Proposition 4.7.11 (a) *After an identification* $\operatorname{gr}_1 Y(\mathfrak{g}) \simeq \mathbb{C}[G_1[[t^{-1}]]] \simeq S^\bullet(\mathfrak{g}[t])$ *above the grading on* $S^\bullet(\mathfrak{g}[t])$ *is given by* $\deg_1 x[n-1] = n$ *and the filtration* F_2 *is given by* $\deg_2 x[n-1] = n$.

(b) *The grading on* $\operatorname{gr}_2 Y(\mathfrak{g}) \simeq U(\mathfrak{g}[t])$ *is given by* $\deg_2 x[n-1] = n-1$ *and the filtration* F_1 *is given by* $\deg_1 x[n-1] = n$.

(c) *We have a bigraded algebra isomorphism* $\operatorname{bigr} Y(\mathfrak{g}) \simeq S^\bullet(\mathfrak{g}[t])$, *where the bigrading on* $S^\bullet(\mathfrak{g}[t])$ *is given by* $\deg_1 x[n-1] = n$, $\deg_2 x[n-1] = n-1$.

4.7.5 Representation theory of $Y(\mathfrak{g})$

By [18, Section 12] to every dominant λ we can associate an irreducible finite dimensional representation of $Y(\mathfrak{g})$ to be denoted $V(\lambda, 0)$, $\rho\colon Y(\mathfrak{g}) \to \operatorname{End} V(\lambda, 0)$. As a $\mathfrak{g}$-module we have $V(\lambda, 0) = V_\lambda \oplus \bigoplus_{\mu<\lambda} V_\mu^{\oplus l_\mu}$. Moreover, if λ is minuscule then $V(\lambda, 0) = V_\lambda$ and $J(x)$ acts on $V(\lambda, 0)$ by zero for every $x \in \mathfrak{g}$.

Remark 4.7.12 If $\mathfrak{g}$ is of type A then $V(\lambda, 0) = V_\lambda$ for every dominant λ. The reason for this is the existence of so-called evaluation homomorphism $Y(\mathfrak{g}) \to U(\mathfrak{g})$ (which exists only in type A), see Section 4.10.2.

4.8 Bethe subalgebras and their degenerations

4.8.1 Bethe subalgebras in Yangian

Let $\rho_i\colon Y(\mathfrak{g}) \to \operatorname{End} V(\varpi_i, 0)$ be the i-th fundamental representation of $Y(\mathfrak{g})$. Let V be the direct sum of $V(\varpi_i, 0)$. Let $T^i(u)$ be the submatrix of $T(u)$-matrix, corresponding to the i-th fundamental representation $V(\varpi_i, 0)$. Let $\tilde{G}$ be the corresponding to $\mathfrak{g}$ connected simply-connected group and let G be the adjoint group with Lie algebra $\mathfrak{g}$. The action of $\mathfrak{g} \subset Y(\mathfrak{g})$ integrates to the action $\tilde{G} \curvearrowright V(\varpi_i, 0)$. We denote the corresponding map $\tilde{G} \to \operatorname{End} V(\varpi_i, 0)$ by ρ_i.

Definition 4.8.1 *Let $C \in \tilde{G}$. Bethe subalgebra $B(C) \subset Y(\mathfrak{g})$ is the subalgebra generated by all coefficients of the following series with the coefficients in $Y(\mathfrak{g})$*

$$\tau_i(u, C) := \operatorname{tr}_{V(\varpi_i,0)} \rho_i(C) T^i(u), 1 \leqslant i \leqslant n.$$

Remark 4.8.2 In fact, $B(C)$ depends only on the class of C in $G = \tilde{G}/Z(\tilde{G})$ so from now on we assume that $C \in G$.

It is easy to see that $B(C) \subset Y_V(\mathfrak{g})^{\mathfrak{z}_\mathfrak{g}(C)}$. Let $T \subset G$ be a maximal torus.

Proposition 4.8.3 *([92], [48], [52], [53])*

1. *Bethe subalgebra $B(C)$ is commutative for any $C \in G$.*

2. *$B(C)$ is a maximal commutative subalgebra of $Y(\mathfrak{g})$ for C in the regular part of T.*

3. *$B(C)$ is a maximal commutative subalgebra of $Y_V(\mathfrak{g})^{\mathfrak{z}_\mathfrak{g}(C)}$ for $C \in T$.*

4. *For $C \in T$ we have $\operatorname{gr}_2 B(C) = \mathcal{A}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(C))$ so, in particular, we have $\dim_{F_1} B(C) = \dim_{F_1} \mathcal{A}_0^{\mathrm{u}}(\mathfrak{z}_\mathfrak{g}(\chi))$.*

4.8.2 Limits of Bethe subalgebras

Recall now the filtration $F_2^\bullet$ on $Y(\mathfrak{g})$. We can consider the corresponding Rees algebra (see Section 4.2.2) $Rees(Y(\mathfrak{g})) = Y_\hbar(\mathfrak{g})$. For every $\epsilon \in \mathbb{C}$ we obtain the algebra $Y_\epsilon(\mathfrak{g}) := Y_\hbar(\mathfrak{g})/(\hbar - \epsilon)$. Recall also that for $\epsilon \neq 0$ we have the identification

$$Y_\epsilon(\mathfrak{g}) \xrightarrow{\sim} Y(\mathfrak{g}),\ [\hbar^i a] \mapsto \epsilon^i a. \tag{4.20}$$

We denote the inverse to (4.20) by ψ_ϵ. Pick now an element $\chi \in \mathfrak{h}$ and consider the element $C := \exp(\epsilon\chi) \in T$. We can consider the corresponding Bethe subalgebra $B(\exp(\epsilon\chi)) \subset Y(\mathfrak{g})$. Let us now denote by $B_\epsilon(C) = B_\epsilon(\exp(\epsilon\chi))$ the subalgebra $\psi_\epsilon(B(\exp(\epsilon\chi))) \subset Y_\epsilon(\mathfrak{g})$. We obtain the (formal) algebraic family $B_\epsilon(C)$ of subalgebras in $Y_\epsilon(\mathfrak{g})$.

Remark 4.8.4 Note that $B_\epsilon(\exp(\epsilon\chi))$ can be considered as an algebraic family over $\operatorname{Spec}\mathbb{C}[[\hbar]]$ or as a complex analytic family over $\mathbb{C}$.

Using the filtration $F_1^\bullet$, we can then define the limit $\lim\limits_{\epsilon\to 0} B_\epsilon(C)$ (see Section 4.2.2) that will be a commutative subalgebra of $Y_0(\mathfrak{g}) = \operatorname{gr}_2 Y(\mathfrak{g}) = U(\mathfrak{g}[t])$.

The goal of this section is to prove that $\lim\limits_{\epsilon\to 0} B_\epsilon(C) = \mathcal{A}_\chi^{\mathrm{u}}$. The strategy is the following. Recall that by Proposition 4.6.6 we have $\mathcal{A}_\chi^{\mathrm{u}} = Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi)$. We prove in Proposition 4.8.9 that $\tilde{\Omega}_\chi \in \lim\limits_{\epsilon\to 0} B_\epsilon(C)$, concluding that $\lim\limits_{\epsilon\to 0} B_\epsilon(C) \subset Z_{U(\mathfrak{g}[t])^{\mathfrak{z}_\mathfrak{g}(\chi)}}(\tilde{\Omega}_\chi) = \mathcal{A}_\chi^{\mathrm{u}}$. Then the comparison of the dimensions finishes the proof.

Let α_j, $j = 1,\ldots,\operatorname{rk}\mathfrak{g}$ be simple roots of $\mathfrak{g}$ and recall that ϖ_j, $j = 1,\ldots,\operatorname{rk}\mathfrak{g}$ are fundamental weights. We also denote by $h_j \in \mathfrak{h}$ the element, corresponding to the simple root $\alpha_j \in \mathfrak{h}^*$ via the identification $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ induced by the invariant scalar product $(\ ,\ )$. Similarly, $t_{\varpi_j} \in \mathfrak{h}$ is the element, corresponding to $\varpi_j \in \mathfrak{h}^*$. For every positive root $\alpha \in \Delta_+$ we denote by $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ elements of $\mathfrak{g}_{\pm\alpha}$ such that $(x_\alpha^+, x_\alpha^-) = 1$. We have

$$\mathcal{R}^{(2)} = \sum_{\alpha\in\Delta_+} (J(x_\alpha^\pm)\otimes x_\alpha^\mp - x_\alpha^\pm \otimes J(x_\alpha^\mp)) + \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} \frac{2}{(\alpha_j,\alpha_j)}(J(h_j)\otimes t_{\varpi_j} - h_j \otimes J(t_{\varpi_j})) + 2\omega'^2, \tag{4.21}$$

where

$$\omega' = \sum_{\alpha\in\Delta^+} \frac{1}{2} x_\alpha^\pm \otimes x_\alpha^\mp + \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} \frac{1}{(\alpha_j,\alpha_j)} h_j \otimes t_{\varpi_j}.$$

Proposition 4.8.5 *The element $\tilde{\omega}_\chi$ lies in the limit algebra $\lim\limits_{\epsilon\to 0} B_\epsilon(\exp(\epsilon\chi))$.*

Proof: Let V be a finite dimensional representation of $Y(\mathfrak{g})$ that is not a direct sum of trivial $\mathfrak{g}$ modules. Let $(\ ,\ )_V$ be the corresponding invariant form and let $c_V \in \mathbb{C}^\times$ be such that $(\ ,\ )_V = c_V(\ ,\ )$.

Pick an identification $\exp\colon t^{-1}\mathfrak{g}[[t^{-1}]] \xrightarrow{\sim} G_1[[t^{-1}]]$. Then (using Proposition 4.7.6) we obtain an isomorphism $\operatorname{gr}_1 Y(\mathfrak{g}) \simeq \mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]]$, which sends $\operatorname{gr}_1(\operatorname{tr}_V \rho(C))T$ to the function on $t^{-1}\mathfrak{g}[[t^{-1}]]$ given by $g(t^{-1}) \mapsto \operatorname{tr}_V \rho(C)\exp g(t^{-1})$, where $g(t^{-1}) = a_1 t^{-1} + a_2 t^{-2} + \ldots$, $a_i \in \mathfrak{g}$. We start from couple Lemmas.

Lemma 4.8.6 *We have* $\operatorname{tr}_V T^{(2)} \in F_2^{(0)}Y(\mathfrak{g})$ *and* $\operatorname{gr}_2(\operatorname{tr}_V T^{(2)}) = c_V\tilde{\omega}_0 + b$ *for some* $b \in \mathbb{C}$.

Proof:

Assume for the sake of contradiction that $\operatorname{tr}_V T^{(2)} \notin F_2^{(0)}Y(\mathfrak{g})$. It follows from the definitions that $\operatorname{tr}_V T^{(2)} \in F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})$. We claim that the condition (i) of Lemma 4.7.9 holds for $x = \operatorname{tr}_V T^{(2)}$ and $(i,j) = (2,1)$. Indeed, note that $F_1^{(1)}Y(\mathfrak{g}) \subset F_2^{(0)}Y(\mathfrak{g})$ so $(F_1^{(1)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})) + (F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(0)}Y(\mathfrak{g})) \subset F_2^{(0)}Y(\mathfrak{g})$ and $\operatorname{tr}_V T^{(2)} \notin F_2^{(0)}Y(\mathfrak{g})$ by our assumption. It follows from Lemma 4.7.9 that $\operatorname{gr}_{21}(\operatorname{tr}_V T^{(2)}) = \operatorname{bigr}^{(2,1)}(\operatorname{tr}_V T^{(2)})$ has bidegree $(2,1)$. Note also that $\operatorname{gr}_1(\operatorname{tr}_V T^{(2)})$ is the function

$$g \mapsto \frac{1}{2}(a_1, a_1)_V$$

that has degree $0 < 1$ w.r.t. the second filtration on $\mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]] \simeq S^\bullet(\mathfrak{g}[t])$ (use Proposition 4.7.11). Contradiction finishes the proof.

Let us now prove that $\operatorname{gr}_2(\operatorname{tr}_V T^{(2)}) = c_V\tilde{\omega}_0 + b$ for some $b \in \mathbb{C}$. Note that $\operatorname{tr}_V T^{(2)}$ is $\mathfrak{g}$-invariant and lies in $F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(0)}Y(\mathfrak{g})$ so its class in $\operatorname{gr}_2 Y(\mathfrak{g}) = U(\mathfrak{g}[t])$ must be a $\mathfrak{g}$-invariant element of $U(\mathfrak{g})$ that has degree at most two w.r.t. the PBW-filtration on $U(\mathfrak{g})$. It follows that $\operatorname{gr}_2(\operatorname{tr}_V T^{(2)}) = a\tilde{\omega}_0 + b$, $a, b \in \mathbb{C}$.

It remains to check that $a = c_V$. Let us first of all prove that

$$\operatorname{gr}_{21}(\operatorname{tr}_V T^{(2)}) = \operatorname{bigr}(\operatorname{tr}_V T^{(2)}) = \operatorname{gr}_{12}(\operatorname{tr}_V T^{(2)}). \tag{4.22}$$

To see that, we apply Lemma 4.7.9 (for $i = 2, j = 0$): we have $\operatorname{tr}_V T^{(2)} \in F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(0)}Y(\mathfrak{g})$, note also that $\operatorname{gr}_1 \operatorname{tr}_V T^{(2)}$ is $(g \mapsto \frac{1}{2}(a_1, a_1)_V)$ so (using Proposition 4.7.11) $\operatorname{gr}_{21} \operatorname{tr}_V T^{(2)} = c_V\omega_0$ has bidegree $(2,1)$. We conclude that (4.22) holds. It remains to note that

$$a\omega_0 = \operatorname{gr}_1(a\tilde{\omega}_0 + b) = \operatorname{gr}_{12} \operatorname{tr}_V T^{(2)} = \operatorname{gr}_{21} \operatorname{tr}_V T^{(2)} = \operatorname{gr}_2(x \mapsto \frac{1}{2}(a_1, a_1)_V) = c_V\omega_0$$

so $a = c_V$ as desired. □

Lemma 4.8.7 *Assume that* $\chi \neq 0$.

(*a*) *We have* $\operatorname{tr}_V \rho(\chi)T^{(2)} \in F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})$ *and* $\operatorname{tr}_V \rho(\chi)T^{(2)} \notin F_2^{(0)}Y(\mathfrak{g})$.

(*b*) *We have* $\operatorname{gr}_2\left(\frac{1}{c_V}\operatorname{tr}_V \rho(\chi)T^{(2)}\right) = \chi[1]$.

Proof: It follows from the definitions that $\operatorname{tr}_V \rho(\chi)T^{(2)} \in F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})$. Let us now check that $\operatorname{tr}_V \rho(\chi)T^{(2)} \notin F_2^{(0)}Y(\mathfrak{g})$. Assume for the sake of contradiction that $\operatorname{tr}_V \rho(\chi)T^{(2)} \in F_2^{(0)}Y(\mathfrak{g})$. It follows that the degree of $\operatorname{gr}_1(\operatorname{tr}_V \rho(\chi)T^{(3)})$ w.r.t. the second filtration on $\mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]] = S^\bullet(\mathfrak{g}[t])$ is at most zero. This contradicts to the fact that $\operatorname{gr}_1(\operatorname{tr}_V \rho(\chi)T^{(2)})$ is

$$x \mapsto \frac{1}{2}(a_1, a_1)_V + (\chi, a_2)_V,$$

which has degree one with respect to the second filtration (use that $\chi \neq 0$).

Let us now decompose $\chi = \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} \frac{2(\chi,\alpha_j)}{(\alpha_j,\alpha_j)} t_{\varpi_j}$ and consider the element

$$x := \sum_{j=1}^{\operatorname{rk}\mathfrak{g}} \frac{2(\chi,\alpha_j)}{(\alpha_j,\alpha_j)} J(t_{\varpi_j}) \in Y(\mathfrak{g}),$$

which lies in $F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})$. It is clear that $\mathrm{gr}_2(x) = \chi[1]$. Note now that $x \notin F_2^{(0)}$ (since $\mathrm{gr}_2(x) = \chi[1]$). Recall also that $\mathrm{tr}_V\, \rho(\chi)T^{(2)} \notin F_2^{(0)}$. We conclude from Lemma 4.7.9 (using $F_1^{(1)}Y(\mathfrak{g}) \subset F_2^{(0)}Y(\mathfrak{g})$) that

$$\mathrm{gr}_{21}(x) = \mathrm{bigr}^{(2,1)}(x) = \mathrm{gr}_{12}(x),$$
$$\mathrm{gr}_{21}(\mathrm{tr}_V\, \rho(\chi)T^{(2)}) = \mathrm{bigr}^{(2,1)}(\mathrm{tr}_V\, \rho(\chi)T^{(2)}) = \mathrm{gr}_{12}(\mathrm{tr}_V\, \rho(\chi)T^{(2)}).$$

Note also that

$$\mathrm{bigr}\left(\frac{1}{c_V}\mathrm{tr}_V\, \rho(\chi)T^{(2)}\right) = \mathrm{gr}_{21}\left(\frac{1}{c_V}\mathrm{tr}_V\, \rho(\chi)T^{(2)}\right) = \chi[1] = \mathrm{gr}_{12}(x) = \mathrm{bigr}(x)$$

so

$$\frac{1}{c_V}\mathrm{tr}_V\, \rho(\chi)T^{(2)} - x \in (F_1^{(1)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})) + (F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(0)}Y(\mathfrak{g})) \subset F_2^{(0)}Y(\mathfrak{g}),$$

hence,

$$\mathrm{gr}_2\left(\frac{1}{c_V}\mathrm{tr}_V\, \rho(\chi)T^{(2)}\right) = \mathrm{gr}_2(x) = \chi[1].$$

□

We are now ready to prove Proposition 4.8.5.

We start from the case $\chi = 0$. In this case we need to show that $\tilde{\omega}_0 \in \mathrm{gr}_2\, B(1)$. It follows from Lemma 4.8.6 that $\mathrm{gr}_2(\mathrm{tr}_V\, T^{(2)}) = c_V\tilde{\omega}_0 + b$ for some $b \in \mathbb{C}$. It follows that

$$\tilde{\omega}_0 = \frac{1}{c_V}\Big(\mathrm{gr}_2(\mathrm{tr}_V\, T^{(2)}) - b\Big) \in \mathrm{gr}_2\, B(1).$$

Let us now deal with arbitrary $\chi \neq 0$. Consider the image of the element

$$\frac{1}{c_V}\Big(\mathrm{tr}_V\, \rho_V(\exp(\epsilon\chi))T^{(2)} - b\Big)$$

in $B_\epsilon(C)$. Let $X(\chi)$ be the coefficient in front of $\epsilon^0 = 1$, which we consider as a function on χ with values in $Y_0(\mathfrak{g}) = U(\mathfrak{g}[t])$. It follows from Lemmas 4.8.6, 4.8.7 that we have

$$\psi_\epsilon\Big(\frac{1}{c_V}\Big(\mathrm{tr}_V\, \rho_V(\exp(\epsilon\chi))T^{(2)} - b\Big)\Big) = X(\chi) + O(\epsilon).$$

We claim that $X(\chi)$ is exactly $\tilde{\omega}_\chi$. It follows from the above that $X(0) = \tilde{\omega}_0$. Recall that terms in $X(\chi)$, depending on χ (i.e. $X(\chi) - X(0)$), appear only from

$$\psi_\epsilon(\mathrm{tr}_V\, \rho_V(\epsilon\chi + \ldots)T^{(2)}).$$

Recall now that $X(\chi)$ is the coefficient in front of 1 so the only term in $X(\chi)$, depending on χ, comes from $\mathrm{tr}_V\, \rho_V(\chi)T^{(2)}$ i.e. $X(\chi) - X(0) = \frac{1}{c_V}\mathrm{gr}_2(\mathrm{tr}_V\, \rho_V(\chi)T^{(2)})$. Recall that by Lemma 4.8.7 we have $\frac{1}{c_V}\mathrm{gr}_2(\mathrm{tr}_V\, \rho_V(\chi)T^{(2)}) = \chi[1]$. We conclude that

$$X(\chi) = X(0) + \chi[1] = \tilde{\omega}_\chi$$

so $\tilde{\omega}_\chi \in \lim_{\epsilon\to 0} B_\epsilon(\exp(\epsilon\chi))$. □

Remark 4.8.8 Note that for $\mathfrak{g}$, which has a minuscule representation (i.e. for $\mathfrak{g}$ not of type E_8, G_2) it is easy to see that the element $\tilde{\omega}_\chi$ lies in the limit, using the explicit formula (4.21) for $\mathcal{R}^{(2)}$.

We also need to find an element

$$\tilde{\Omega}_\chi = \sum_a x_a x_a[1] + \chi[2]$$

in the limit $\lim\limits_{\epsilon\to 0} B_\epsilon(C)$.

Proposition 4.8.9 *The element $\tilde{\Omega}_\chi$ lies in the limit algebra $\lim\limits_{\epsilon\to 0} B_\epsilon(\exp(\epsilon\chi))$.*

Proof: The proof is similar to the proof of Proposition 4.8.5. Recall that V is a finite dimensional representation of $Y(\mathfrak{g})$ that is not a direct sum of trivial $\mathfrak{g}$ modules, $(\ ,\ )_V$ is the corresponding invariant form and $c_V \in \mathbb{C}^\times$ is such that $(\ ,\ )_V = c_V(\ ,\ )$. We start from couple Lemmas (c.f. Lemmas 4.8.6, 4.8.7 above).

Lemma 4.8.10 (a) *We have* $\operatorname{tr}_V T^{(3)} \in F_2^{(1)} Y(\mathfrak{g})$.
(b) *We have* $\operatorname{gr}_2(\operatorname{tr}_V T^{(3)}) = c_V \tilde{\Omega}_0$.

Proof: Let us prove (a). Assume for the sake of contradiction that $\operatorname{tr}_V T^{(3)} \notin F_2^{(1)} Y(\mathfrak{g})$. It follows from the definitions that $\operatorname{tr}_V T^{(3)} \in F_1^{(3)} Y(\mathfrak{g}) \cap F_2^{(2)} Y(\mathfrak{g})$. We claim that the condition (i) of Lemma 4.7.9 holds for $x = \operatorname{tr}_V T^{(3)}$ and $(i,j) = (3,2)$. Indeed, note that $F_1^{(2)} Y(\mathfrak{g}) \subset F_2^{(1)} Y(\mathfrak{g})$ so $(F_1^{(2)} Y(\mathfrak{g}) \cap F_2^{(2)} Y(\mathfrak{g})) + (F_1^{(3)} Y(\mathfrak{g}) \cap F_2^{(1)} Y(\mathfrak{g})) \subset F_2^{(1)} Y(\mathfrak{g})$ and $\operatorname{tr}_V T^{(3)} \notin F_2^{(1)} Y(\mathfrak{g})$ by our assumption. It follows from Lemma 4.7.9 that

$$\operatorname{gr}_{21}(\operatorname{tr}_V T^{(3)}) = \operatorname{bigr}^{(3,2)}(\operatorname{tr}_V T^{(3)}) = \operatorname{gr}_{12}(\operatorname{tr}_V T^{(3)}).$$

Note now that $\operatorname{gr}_1(\operatorname{tr}_V T^{(3)})$ is the function

$$g \mapsto (a_1, a_2)_V + f(a_1),$$

where $f \in (S^3\mathfrak{g})^{\mathfrak{g}}$ is some $\mathfrak{g}$-invariant function of degree 3. We see that $\operatorname{gr}_1(\operatorname{tr}_V T^{(3)})$ has degree $1 < 2$ w.r.t. the second filtration on $\mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]]$. Contradiction finishes the proof.

Let us now prove (b). Let us note that $\operatorname{tr}_V T^{(3)} \in F_1^{(3)} Y(\mathfrak{g}) \cap F_2^{(1)} Y(\mathfrak{g})$ and

$$\operatorname{gr}_{21} \operatorname{tr}_V T^{(3)} = \operatorname{gr}_2(g \mapsto ((a_1, a_2)_V + f(a_1))) = c_V \Omega_0$$

has bidegree $(3,1)$. It follows from Lemma 4.7.9 that $c_V \Omega_0 = \operatorname{gr}_{21} \operatorname{tr}_V T^{(3)} = \operatorname{gr}_{12} \operatorname{tr}_V T^{(3)}$ so $\operatorname{gr}_1(\operatorname{gr}_2 \operatorname{tr}_V T^{(3)}) = c_V \Omega_0$. Note now that $\operatorname{tr}_V T^{(3)}$ is $\mathfrak{g}$-invariant so $\operatorname{gr}_2 \operatorname{tr}_V T^{(3)} \in U(\mathfrak{g}[t])^{\mathfrak{g}}$. Moreover, $\operatorname{gr}_2 \operatorname{tr}_V T^{(3)}$ is homogeneous of degree 1 w.r.t. the grading on $U(\mathfrak{g}[t])$. There exists the unique graded lift of $c_V \Omega_0$ to $U(\mathfrak{g}[t])^{\mathfrak{g}}$ so $\operatorname{gr}_2 \operatorname{tr}_V T^{(3)} = c_V \tilde{\Omega}_0$.

□

Lemma 4.8.11 *Assume that $\chi \neq 0$.*
(a) *We have* $\operatorname{tr}_V \rho(\chi) T^{(3)} \in F_1^{(3)} Y(\mathfrak{g}) \cap F_2^{(2)} Y(\mathfrak{g})$ *and* $\operatorname{tr}_V \rho(\chi) T^{(3)} \notin F_2^{(1)} Y(\mathfrak{g})$.
(b) *We have* $\operatorname{gr}_2 \left(\frac{1}{c_V} \operatorname{tr}_V \rho(\chi) T^{(3)} \right) = \chi[2]$.

Proof: It follows from the definitions that $\mathrm{tr}_V\,\rho(\chi)T^{(3)} \in F_1^{(3)}Y(\mathfrak{g}) \cap F_2^{(2)}Y(\mathfrak{g})$. Let us now check that $\mathrm{tr}_V\,\rho(\chi)T^{(3)} \notin F_2^{(1)}Y(\mathfrak{g})$. Assume for the sake of contradiction that $\mathrm{tr}_V\,\rho(\chi)T^{(3)} \in F_2^{(1)}Y(\mathfrak{g})$. It follows that the degree of $\mathrm{gr}_1(\mathrm{tr}_V\,\rho(\chi)T^{(3)})$ w.r.t. the second filtration on $\mathbb{C}[t^{-1}\mathfrak{g}[[t^{-1}]]]$ is at most one. This contradicts to the fact that $\mathrm{gr}_1(\mathrm{tr}_V\,\rho(\chi)T^{(3)})$ is

$$g \mapsto (\chi, a_3)_V + (a_1, a_2)_V + f(a_1)$$

that has degree two with respect to the second filtration (use Proposition 4.7.11).

Let us now decompose $\chi = \sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} \frac{2(\chi,\alpha_j)}{(\alpha_j,\alpha_j)} t_{\varpi_j}$ and consider the element

$$x := \sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} \frac{2(\chi,\alpha_j)}{(\alpha_j,\alpha_j)} [J(x_{\alpha_j}^+), J(x_{\alpha_j}^-)]$$

that clearly lies in $F_1^{(3)}Y(\mathfrak{g}) \cap F_2^{(2)}Y(\mathfrak{g})$ and $\mathrm{gr}_2(x) = \chi[2]$. Note now that $x \notin F_2^{(1)}Y(\mathfrak{g})$ (since $\mathrm{gr}_2(x) = \chi[2]$). Recall also that $\mathrm{tr}_V\,\rho(\chi)T^{(3)} \notin F_2^{(1)}$. We conclude from Lemma 4.7.9 (using $F_1^{(2)}Y(\mathfrak{g}) \subset F_2^{(1)}Y(\mathfrak{g})$) that

$$\mathrm{gr}_{21}(x) = \mathrm{bigr}^{(3,2)}(x) = \mathrm{gr}_{12}(x),$$

$$\mathrm{gr}_{21}(\mathrm{tr}_V\,\rho(\chi)T^{(3)}) = \mathrm{bigr}^{(3,2)}(\mathrm{tr}_V\,\rho(\chi)T^{(3)}) = \mathrm{gr}_{12}(\mathrm{tr}_V\,\rho(\chi)T^{(3)}).$$

Note now that

$$\mathrm{bigr}\left(\frac{1}{c_V}\,\mathrm{tr}_V\,\rho(\chi)T^{(3)}\right) = \mathrm{gr}_{21}\left(\frac{1}{c_V}\,\mathrm{tr}_V\,\rho(\chi)T^{(3)}\right) = \chi[2] = \mathrm{gr}_{12}(x) = \mathrm{bigr}(x)$$

so

$$\frac{1}{c_V}\,\mathrm{tr}_V\,\rho(\chi)T^{(3)} - x \in (F_1^{(2)}Y(\mathfrak{g}) \cap F_2^{(2)}Y(\mathfrak{g})) + (F_1^{(3)}Y(\mathfrak{g}) \cap F_2^{(1)}Y(\mathfrak{g})) \subset F_2^{(1)}Y(\mathfrak{g}),$$

hence,

$$\mathrm{gr}_2\left(\frac{1}{c_V}\,\mathrm{tr}_V\,\rho(\chi)T^{(3)}\right) = \mathrm{gr}_2(x) = \chi[2].$$

□

We are now ready to prove Proposition 4.8.9. We start from the case $\chi = 0$. Consider the element $\mathrm{tr}_V\,T^{(3)} \in B(1)$ and recall that by Lemma 4.8.10 (*b*) we have

$$\mathrm{gr}_2(\mathrm{tr}_V\,T^{(3)}) = c_V\tilde{\Omega}_0.$$

We conclude that

$$\tilde{\Omega}_0 = \frac{1}{c_V}\,\mathrm{gr}_2(\mathrm{tr}_V\,T^{(3)}) \in \mathrm{gr}_2\,B(1).$$

Let us now deal with arbitrary χ. Consider the image of the element

$$\frac{1}{c_V}\,\mathrm{tr}_V\,\rho_V(\exp(\epsilon\chi))T^{(3)}$$

in $B_\epsilon(C)$. Let $X(\chi)$ be the coefficient in front of ϵ^{-1}, which we consider as a function on χ with values in $Y_0(\mathfrak{g}) = U(\mathfrak{g})$. It follows from Lemmas 4.8.10, 4.8.11 that

$$\psi_\epsilon\Big(\frac{1}{c_V}\operatorname{tr}_V \rho_V(\exp(\epsilon\chi))T^{(3)}\Big) = X(\chi)\epsilon^{-1} + O(1).$$

We claim that $X(\chi)$ is exactly $\tilde{\Omega}_\chi$. It follows from the above that $X(0) = \tilde{\Omega}_0$. Recall that terms in $X(\chi)$, depending on χ (i.e. $X(\chi) - X(0)$) appear only from

$$\psi_\epsilon(\operatorname{tr}_V \rho_V(\epsilon\chi + \ldots)T^{(3)}).$$

Recall now that $X(\chi)$ is the coefficient in front of ϵ^{-1} so the only term in $X(\chi)$, depending on χ comes from $\operatorname{tr}_V \rho_V(\chi)T^{(3)}$ i.e. $X(\chi) - X(0) = \frac{1}{c_V}\operatorname{gr}_2(\operatorname{tr}_V \rho_V(\chi)T^{(3)})$. Recall that by Lemma 4.8.11 we have $\frac{1}{c_V}\operatorname{gr}_2(\operatorname{tr}_V \rho_V(\chi)T^{(3)}) = \chi[2]$. We conclude that

$$X(\chi) = X(0) + \chi[2] = \tilde{\Omega}_\chi$$

so $\tilde{\Omega}_\chi \in \lim\limits_{\epsilon\to 0} B_\epsilon(\exp(\epsilon\chi))$.

□

Theorem 4.8.12 *We have* $\lim\limits_{\epsilon\to 0} B_\epsilon(\exp(\epsilon\chi)) = \mathcal{A}^{\mathrm{u}}_\chi$.

Proof: We set $C = C(\epsilon) = \exp(\epsilon\chi)$. Recall that by Proposition 4.6.6 we have $\mathcal{A}^{\mathrm{u}}_\chi = Z_{U(\mathfrak{g}[t])}(\tilde{\Omega}_\chi)$. It follows from Proposition 4.8.9 that $\tilde{\Omega}_\chi \in \lim\limits_{\epsilon\to 0} B_\epsilon(C)$. We conclude that $\lim\limits_{\epsilon\to 0} B_\epsilon(C) \subset \mathcal{A}^{\mathrm{u}}_\chi$. It remains to note that by Lemma 4.2.10

$$\dim_{F_1} B(C) \leqslant \dim_{F_1} \lim_{\epsilon\to 0} B_\epsilon(C) \leqslant \dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi.$$

Using Corollary 4.5.9 and Proposition 4.8.3, we obtain $\dim_{F_1} B(C) = \dim_{F_1} \mathcal{A}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}}(\chi)) = \dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi$ and conclude that $\dim_{F_1} \lim\limits_{\epsilon\to 0} B_\epsilon(C) = \dim_{F_1} \mathcal{A}^{\mathrm{u}}_\chi$ so from $\lim\limits_{\epsilon\to 0} B_\epsilon(C) \subset \mathcal{A}^{\mathrm{u}}_\chi$ it follows that $\lim\limits_{\epsilon\to 0} B_\epsilon(C) = \mathcal{A}^{\mathrm{u}}_\chi$. □

Remark 4.8.13 Note that in type A Theorem 4.8.12 can be deduced from the results of Section 4.11.

4.9 Crystals: main properties and examples

From now on we assume that $\mathfrak{g} = \mathfrak{sl}_n$, $G = \mathrm{PGL}_n$. Slightly abusing notation, we denote by $\mathfrak{h}$ the subalgebra of diagonal matrices of $\mathfrak{gl}_n$ and by $\mathfrak{h}_0 \subset \mathfrak{h}$ we denote the subalgebra of traceless matrices. Let $T \subset \mathrm{GL}_n$ be the subgroup of diagonal matrices and let $T_0 \subset T$ be the subgroup of matrices with determinant one. We also denote by $\overline{T} \subset \mathrm{PGL}_n$ the subgroup of diagonal matrices.

In this section we introduce notions of $\mathfrak{sl}_n$, $\hat{\mathfrak{sl}}_n$ - crystals and discuss some of their properties.

4.9.1 Definitions of $\mathfrak{sl}_n, \hat{\mathfrak{sl}}_n$-crystals

Let us discuss $\mathfrak{sl}_n$-crystals. We denote by $P^\vee$ the weight lattice of $\mathfrak{sl}_n$.

Definition 4.9.1 *A $\mathfrak{sl}_n$-crystal is a finite set B together with maps:*

$$\mathrm{e}_i,\, \mathrm{f}_i : B \to B \cup \{0\},\ \mathrm{wt}\colon B \to P^\vee,\ i = 1, 2, \dots n-1,$$

such that for each $i = 1, 2, \dots, n-1$ we have:

(a) let $b \in B$. If $\mathrm{e}_i \cdot b \in B$ for some $i \in \{1, 2, \dots, n-1\}$, then $\mathrm{wt}(\mathrm{e}_i \cdot b) = \mathrm{wt}(b) + \alpha_i^\vee$, if $\mathrm{f}_i \cdot b \in B$ for some $i = \{1, 2, \dots, n-1\}$, then $\mathrm{wt}(\mathrm{f}_i \cdot b) = \mathrm{wt}(b) - \alpha_i^\vee$,

(b) for all $b, b' \in B$, $\mathrm{e}_i \cdot b = b'$ if and only if $b = \mathrm{f}_i \cdot b'$.

We use the identification $\{1, 2, \dots, n\} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$, $j \mapsto [j]$. For $[j] \in \mathbb{Z}/n\mathbb{Z}$ we denote by $\tau_{[j]} \in S_n$ the cyclic permutation which sends $[i]$ to $[i+j]$. We denote by $\tau_{[j]}^\vee$ the induced automorphism $\tau_{[j]}^\vee \colon P^\vee \xrightarrow{\sim} P^\vee$.

Let us now define $\hat{\mathfrak{sl}}_n$-crystals.

Definition 4.9.2 *A $\hat{\mathfrak{sl}}_n$-crystal is a finite set $\mathbf{B}$ together with maps*

$$\mathrm{e}_{[i]},\, \mathrm{f}_{[i]} \colon \mathbf{B} \to \mathbf{B} \cup \{0\},\ \mathrm{wt}\colon \mathbf{B} \to P^\vee,\ [i] \in \mathbb{Z}/n\mathbb{Z},$$

such that for every $[j] \in \mathbb{Z}/n\mathbb{Z}$ the operators

$$\mathrm{e}_i := \mathrm{e}_{[i-j]},\ \mathrm{f}_i := \mathrm{f}_{[i-j]},\ \mathbf{B} \xrightarrow{\mathrm{wt}} P^\vee \xrightarrow{\tau_{[j]}^\vee} P^\vee,\ i = 1, 2, \dots, n-1$$

define the structure of $\mathfrak{sl}_n$-crystal on $\mathbf{B}$ that will be denoted by $\mathbf{B}^{[j]}$.

Warning 4.9.3 *Note that our definition of $\hat{\mathfrak{sl}}_n$-crystal is not the standard one since we are assuming that the function* wt *maps to $P^\vee$ that is a weight lattice of $\mathfrak{sl}_n$ (not of the whole algebra $\hat{\mathfrak{sl}}_n$), we are also considering only finite crystals.*

4.9.2 Tensor products of crystals

Let us discuss tensor products of crystals. Given an $\hat{\mathfrak{sl}}_n$-crystal $\mathbf{B}$ and $[i] \in \mathbb{Z}/n\mathbb{Z}$ we can define maps $\varphi_{[i]}^+, \varphi_{[i]}^- \colon \mathbf{B} \to \mathbb{Z} \cup \{\infty\}$ as follows:

$$\varphi_{[i]}^+(\mathbf{b}) = \max\Big\{m \in \mathbb{N} \,|\, (\mathrm{e}_{[i]})^m(\mathbf{b}) \neq 0\Big\},\ \varphi_{[i]}^-(\mathbf{b}) = \max\Big\{m \in \mathbb{N} \,|\, (\mathrm{f}_{[i]})^m(\mathbf{b}) \neq 0\Big\}.$$

Given two integrable $\hat{\mathfrak{sl}}_n$-crystals $\mathbf{B}$, $\mathbf{B}'$ one can define an $\hat{\mathfrak{sl}}_n$-crystal structure on the set $\mathbf{B} \times \mathbf{B}'$ as follows:

$$\mathrm{wt}(\mathbf{b} \times \mathbf{b}') = \mathrm{wt}(\mathbf{b}) + \mathrm{wt}(\mathbf{b}')$$

$$\mathrm{e}_{[i]} \cdot (\mathbf{b} \times \mathbf{b}') = \begin{cases} (\mathrm{e}_{[i]} \cdot \mathbf{b}) \times \mathbf{b}', \text{ if } \varphi_{[i]}^+(\mathbf{b}) > \varphi_{[i]}^-(\mathbf{b}') \\ \mathbf{b} \times (\mathrm{e}_{[i]} \cdot \mathbf{b}'), \text{ otherwise}, \end{cases} \tag{4.23}$$

$$\mathrm{f}_{[i]} \cdot (\mathbf{b} \times \mathbf{b}') = \begin{cases} (\mathrm{f}_{[i]} \cdot \mathbf{b}) \times \mathbf{b}', \text{ if } \varphi_{[i]}^+(\mathbf{b}) \geqslant \varphi_{[i]}^-(\mathbf{b}') \\ \mathbf{b} \times (\mathrm{f}_{[i]} \cdot \mathbf{b}'), \text{ otherwise}. \end{cases} \tag{4.24}$$

This crystal will be denoted by $\mathbf{B} \otimes \mathbf{B}'$.

4.9.3 Examples of $\mathfrak{sl}_n$-crystals

To every dominant weight λ of $\mathfrak{sl}_n$ one can associate the crystal B_λ that should be considered as a "discrete model" of the representation V_λ (see [60], [61] for details). Explicit description of B_λ was given in [62], we recall the construction. Let $A(\lambda)$ be the Young diagram, corresponding to λ i.e. if we decompose $\lambda = \sum_{i=1}^{p} \varpi_{m_i}$ with $1 \leqslant m_1 \leqslant \ldots \leqslant m_p \leqslant n$ then $A(\lambda)$ consists of p columns of lengths $m_1, \ldots, m_p$. Let B_λ be the set of semi-standard tableaux of shape $A(\lambda)$. Operators f_i work as follows. Read the entries of the tableau from bottom to top left to right, ignoring all numbers except $i, i+1$. Replace $i+1$ by (and i by), turn i, corresponding to the rightmost unmached) by $i+1$. Similarly, operators e_i work as follows. Read the entries of the tableau from bottom to top left to right, ignoring all numbers except $i, i+1$. Replace $i+1$ by (and i by), turn $i+1$, corresponding to the leftmost unmached (by i.

Example 4.9.4 *For $\lambda = \omega_l$ the crystal B_{ω_l} can be described as follows. Note that the Young diagram $A(\omega_l)$ is a column of length l. It follows that the set B_{ω_l} is in bijection with l-tuples of positive numbers $(i_1, \ldots, i_l)$ such that $1 \leqslant i_1 < \ldots < i_l \leqslant n$. Action of e_i on $(i_1, \ldots, i_l)$ replaces (if possible) i by $i+1$ and sends $(i_1, \ldots, i_l)$ to zero otherwise. Action of f_i replaces (if possible) i by $i-1$ and sends $(i_1, \ldots, i_l)$ to zero otherwise. The weight function* wt *sends $(i_1, \ldots, i_l)$ to the linear function $\mathfrak{h}_0 \ni (x_1, \ldots, x_n) \mapsto x_{i_1} + \ldots + x_{i_l}$.*

Remark 4.9.5 Let us give another description of the crystal structure on B_λ. Consider the following embedding $B_\lambda \subset B_{\varpi_{m_1}} \otimes \ldots \otimes B_{\varpi_{m_p}}$. Let a be a semi-standard Young tableau of shape $A(\lambda)$. Let a_i be the i-th column with enumeration starting from the right. We send a to the element $a_1 \otimes a_2 \ldots \otimes a_p \in B_{\varpi_{m_1}} \otimes \ldots \otimes B_{\varpi_{m_p}}$. The crystal structure on B_λ is induced from the one on $B_{\varpi_{m_1}} \otimes \ldots \otimes B_{\varpi_{m_p}}$ (note that the crystal structure on B_{ϖ_l} is very simple, see Example 4.9.4 above).

Definition 4.9.6 *A $\mathfrak{sl}_n$-crystal is normal if it is isomorphic to the disjoint union of crystals B_λ.*

4.9.4 Kirillov-Reshetikhin crystals

Let us describe the so-called Kirillov-Reshetikhin crystals. They are $\hat{\mathfrak{sl}}_n$-crystals that will be most important for us (see Remark 4.9.7 for some motivation to restrict attention to these crystals). These $\hat{\mathfrak{sl}}_n$-crystals correspond to representations $V_{l\varpi_r}$ and will be denoted $\mathbf{B}_{l\varpi_r}$. The existence of such crystals is proven in [59] and the explicit construction is given in [102], [67] and [66], see also [93].

In this section we prove that if B_λ can be extended to an $\hat{\mathfrak{sl}}_n$-crystal $\mathbf{B}_\lambda$ such that $\mathbf{B}_\lambda^{[1]}$ is normal then $\lambda = l\varpi_r$ and such an extension is *unique*. One of the results of this chapter is a "geometric" construction of such an extension (see Section 4.16). So, the present chapter gives an alternative approach to the existence (and uniqueness) of $\mathbf{B}_{l\varpi_r}$.

Remark 4.9.7 The (arguably) most important class of $\hat{\mathfrak{sl}}_n$-crystals are the crystals that appear as crystal graphs of (finite dimensional) representations of $U_q(\hat{\mathfrak{sl}}_n)$. The following conjecture is due to Kashiwara (see [103, Conjecture 4.5]): every connected affine crystal graph is isomorphic to the tensor product of Kirillov-Reshetikhin crystals.

We first describe the $\hat{\mathfrak{sl}}_n$-crystals, corresponding to representations $\Lambda^l(\mathbb{C}^n) = V_{\varpi_l}$, $S^l(\mathbb{C}^n) = V_{l\varpi_1}$.

Warning 4.9.8 *Note that it is not true in general that $S^l(\Lambda^r\mathbb{C}^n) = V_{l\varpi_r}$. For example, for $n = 4$, $l = r = 2$, we have $S^2(\Lambda^2\mathbb{C}^4) = V_{2\varpi_2} \oplus \mathbb{C}$.*

Example 4.9.9 *The crystal $\mathbf{B}_{\varpi_l}$ can be defined as follows. Pick the standard basis $\{v_{[1]}, \dots, v_{[n]}\}$ of $\mathbb{C}^n$ and denote by $\{v^*_{[1]}, \dots, v^*_{[n]}\} \subset (\mathbb{C}^n)^*$ the dual basis. Consider the vector space $\Lambda^l\mathbb{C}^n$ and its projectivization $\mathbb{P}(\Lambda^l\mathbb{C}^n)$. For a vector $x \in \Lambda^l\mathbb{C}^n$ we denote by $[x]$ the corresponding element of $\mathbb{P}(\Lambda^l\mathbb{C}^n)$. We define*

$$\mathbf{B}_{\varpi_l} = \{[v_{[i_1]} \wedge v_{[i_2]} \dots \wedge v_{[i_l]}], \,|\, 1 \leqslant i_1 < i_2 < \dots < i_l \leqslant n\}.$$

The map $\mathrm{e}_{[i]}$ acts via $E_{[i],[i+1]}$, the map $\mathrm{f}_{[i]}$ acts via $E_{[i+1],[i]}$. The map wt sends $[v_{[i_1]} \wedge v_{[i_2]} \dots \wedge v_{[i_l]}]$ to the restriction of $v^*_{[i_1]} + \dots + v^*_{[i_l]}$ to $\mathfrak{h}_0 \subset \mathbb{C}^n$.

Example 4.9.10 *The crystal $\mathbf{B}_{l\varpi_1}$ can be described as follows. Consider the vector space $S^l\mathbb{C}^n$ and its projectivization $\mathbb{P}(S^l\mathbb{C}^n)$, for a vector $x \in S^l\mathbb{C}^n$ we denote by $[x]$ the corresponding element of $\mathbb{P}(S^l\mathbb{C}^n)$. We define*

$$\mathbf{B}_{l\varpi_1} = \{[v^{p_1}_{[i_1]} \dots v^{p_l}_{[i_k]}] \,|\, 1 \leqslant i_1 < i_2 < \dots < i_l \leqslant n,\, p_1 + \dots + p_l = l\}$$

The map $\mathrm{e}_{[i]}$ acts via $E_{[i],[i+1]}$, the map $\mathrm{f}_{[i]}$ acts via $E_{[i+1],[i]}$. The map wt sends $[v^{p_1}_{[i_1]} v^{p_2}_{[i_2]} \dots v^{p_l}_{[i_l]}]$ to the restriction of $p_1 v^*_{[i_1]} + \dots + p_l v^*_{[i_l]}$ to $\mathfrak{h}_0 \subset \mathbb{C}^n$.

Schützenberger involutions and the operator ϕ

We start from the following definition.

Definition 4.9.11 *[45] The Schützenberger involution $\xi_{B_\lambda} \colon B_\lambda \to B_\lambda$ is the unique map of sets which satisfies*

$$\mathrm{e}_i(\xi_{B_\lambda}(b)) = \xi_{B_\lambda}(\mathrm{e}_{n-i}(b)),$$
$$\mathrm{f}_i(\xi_{B_\lambda}(b)) = \xi_{B_\lambda}(\mathrm{f}_{n-i}(b)),$$
$$\mathrm{wt}(\xi_{B_\lambda}(b)) = w_0(\mathrm{wt}(b)).$$

Here $w_0 \in S_n$ is the longest element and $i \in \{1, \dots, n-1\}$.

There is an explicit combinatorial construction of ξ_{B_λ} (see, for example, [68]). If B is a disjoint union of B_λ then we denote by ξ_B the corresponding involution of B. Following [45, Section 2.2] let $\overline{B}$ be the crystal with underlining set $\{\overline{b} \,|\, b \in B\}$ and crystal structure

$$\mathrm{e}_i \cdot \overline{b} = \overline{\mathrm{f}_{n-i} \cdot b},\ \mathrm{f}_i \cdot b = \overline{\mathrm{e}_{n-i} \cdot b},\ \mathrm{wt}(\overline{b}) = w_0(\mathrm{wt}\, b).$$

It follows from the definitions that the involution ξ_B is the isomorphism of crystals $\xi_B \colon B \xrightarrow{\sim} \overline{B}$. We set $B^{(1)}_\lambda := (\overline{B}_\lambda)|_{\mathfrak{sl}_{n-1}}$. Let us describe the $\mathfrak{sl}_{n-1}$-crystal $B^{(1)}_\lambda$. It follows from the definitions that $B^{(1)}_\lambda = B_\lambda$ as a set i.e. $B^{(1)}_\lambda = \{b^{(1)} \,|\, b \in B_\lambda\}$ and the crystal structure is given by

$$\mathrm{e}_i \cdot b^{(1)} = (\mathrm{e}_{i+1} \cdot b)^{(1)},\ \mathrm{f}_i \cdot b^{(1)} = (\mathrm{f}_{i+1} \cdot b)^{(1)},\ \mathrm{wt}(b^{(1)}) = \tau_{[-1]}(\mathrm{wt}(b)).$$

Note that the composition $\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}$ induces the isomorphism

$$\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}} : B_\lambda^{(1)} \xrightarrow{\sim} (B_\lambda)|_{\mathfrak{sl}_{n-1}}$$

and such an isomorphism is *unique* (since $(B_\lambda)|_{\mathfrak{sl}_{n-1}}$ is isomorphic to the disjoint union of *distinct* irreducible $\mathfrak{sl}_{n-1}$-crystals). We conclude that the following lemma holds.

Lemma 4.9.12 *There exists the unique bijection* $\phi \colon B_\lambda \xrightarrow{\sim} B_\lambda$ *such that*
(1) $\mathrm{wt}(\phi(b)) = \tau_{[1]}(\mathrm{wt}(b))$
(2) $\phi(\mathrm{e}_i(b)) = \mathrm{e}_{i+1}(\phi(b))$ *and* $\phi(\mathrm{f}_i(b)) = \mathrm{f}_{i+1}(\phi(b))$ *for* $i = 1, \dots, n-2$.
The bijection ϕ *is given by* $\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}$, *where* $\xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}$ *is the Schützenberger involution of the* $\mathfrak{sl}_{n-1}$*-crystal* $(B_\lambda)|_{\mathfrak{sl}_{n-1}}$.

Description of ϕ as a promotion operator

Let us now give a combinatorial description of the operator ϕ of Lemma 4.9.12. Let us define the so-called Schützenberger's promotion operator $\mathbf{pr} \colon B_{l\varpi_r} \xrightarrow{\sim} B_{l\varpi_r}$. Recall that $A(l\varpi_r)$ is the Young diagram that corresponds to $l\varpi_r$. Starting from Young tableau $a \in B_{l\varpi_r}$ of shape $A(l\varpi_r)$, we denote by H the horizontal strip in $A(l\varpi_r)$ consisting of boxes of a with letter n. Tableau $\mathbf{pr}(a)$ is constructed as follows.

Step 1. We remove all the letters n in a.

Step 2. Slide the remaining subtableau to the southeast using the following procedure (Schützenberger's jeu-de-taquin). Entering the cells of H from left to right we shift the entry above down, or the entry to the left right, whichever is bigger (if it is same, take the vertical move) and continue this until there are no entries above and to the left of the empty square.

Step 3. We fill in the vacated cells with zeros.

Step 4. Add one to each entry. The resulting tableau is $\mathbf{pr}(a)$.

It is easy to see that the operator $\mathbf{pr}(a)$ satisfies conditions (1), (2) of Lemma 4.9.12 so we must have

$$\phi = \xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}} = \mathbf{pr}.$$

Example 4.9.13 *Let us consider the example* $l = r = 2$, $n = 4$. *We are dealing with the set* $B_{2\varpi_2}$, *consisting of Young tableaux of shape* $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$. *Let us describe the operator* $\mathbf{pr} = \phi$ *in this particular case. The set* $B_{2\varpi_2}$ *consists of tableaux:*

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 4&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 4&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 4&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 4&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&3\\\hline 4&4\\\hline\end{array}.$$

Then $\mathbf{pr}$ *acts on* $B_{2\varpi_2}$ *in the following way*

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 3&3\\\hline 4&4\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&2\\\hline 4&4\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&3\\\hline 4&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 2&3\\\hline 4&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 2&2\\\hline 4&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\mapsto\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}.$$

Description of $\mathbf{B}_{l\varpi_r}$

Pick a dominant $\lambda \in P^\vee$ and assume for a moment that there exists an $\hat{\mathfrak{sl}}_n$-crystal $\mathbf{B}_\lambda$ such that $\mathbf{B}_\lambda^{[0]} \simeq B_\lambda$ and $\mathbf{B}_\lambda^{[1]}$ is normal (we will see that this can happen only for $\lambda = l\varpi_r$ and the existence of such crystal will be proven in Section 4.16). The goal of this section is to prove that $\mathbf{B}_\lambda$ is unique and can be described explicitly.

Proposition 4.9.14 *Let $\mathbf{B}_\lambda$ be a $\hat{\mathfrak{sl}}_n$-crystal such that $\mathbf{B}^{[0]}$ is isomorphic to B_λ and $\mathbf{B}^{[1]}$ is normal. Then $\lambda = l\varpi_r$ and the operators $\mathrm{e}_{[0]}$, $\mathrm{f}_{[0]}$ are given by*

$$\mathrm{e}_{[0]} = (\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}})^{-1} \circ \mathrm{e}_1 \circ (\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}) = \mathbf{pr}^{-1} \circ \mathrm{e}_1 \circ \mathbf{pr}, \tag{4.25}$$

$$\mathrm{f}_{[0]} = (\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}})^{-1} \circ \mathrm{f}_1 \circ (\xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}) = \mathbf{pr}^{-1} \circ \mathrm{f}_1 \circ \mathbf{pr}. \tag{4.26}$$

Proof: Let us first of all note that since $\mathbf{B}^{[1]}$ is normal and has the same character as B_λ we must have $\mathbf{B}^{[1]} \simeq B_\lambda$. Pick isomorphisms of $\mathfrak{sl}_n$-crystals $\mathbf{B}^{[0]} \simeq B_\lambda$, $\mathbf{B}^{[1]} \simeq B_\lambda$ (note that these isomorphisms are unique). Composing one of them with the inverse of the other we obtain the bijection $\phi\colon B_\lambda \xrightarrow{\sim} B_\lambda$ that satisfies conditions (1), (2) of Lemma 4.9.12. It then follows from Lemma 4.9.12 that $\phi = \xi_{B_\lambda} \circ \xi_{(B_\lambda)|_{\mathfrak{sl}_{n-1}}}$. In particular, we see that the operator $f_{[0]}$ is equal to $\mathbf{pr}^{-1} \circ \mathrm{f}_1 \circ \mathbf{pr}$. Note that $\mathbf{pr}^n\colon B_\lambda \xrightarrow{\sim} B_\lambda$ should be the automorphism of the crystal B_λ so we must have $\mathbf{pr}^n = \mathrm{id}_{B_\lambda}$, hence, $\lambda = l\varpi_r$ (see [100] and [67, Section 6.2]). It now follows from the construction of $\mathbf{B}_{l\varpi_r}$ that $\mathbf{B} \simeq \mathbf{B}_{l\varpi_r}$ as $\hat{\mathfrak{sl}}_n$-crystals. ☐

Remark 4.9.15 *In Proposition 4.9.14 we prove that if $B_{l\varpi_r}$ can be extended to an $\hat{\mathfrak{sl}}_n$-crystal then the extension is* unique *and operators* $\mathrm{e}_{[0]}$, $\mathrm{f}_{[0]}$ must *be given by the formulas (4.25), (4.26) (compare with [102, Section 3]). It remains to check that the extension $\mathbf{B}_{l\varpi_r}$ indeed exists. This will be proven in Section 4.16. Let us recall that the existence of $\mathbf{B}_{l\varpi_r}$ is known and follows from [59]. Our approach gives an alternative construction of $\mathbf{B}_{l\varpi_r}$.*

4.10 Yangians in type A

4.10.1 Yangians of $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, Bethe and Gaudin subalgebras

Let us return to Yangians. We first of all describe the Yangian $Y(\mathfrak{gl}_n)$. It is generated by $t_{ab}^{(r)}$, $1 \leqslant a, b \leqslant n$, $r \geqslant 1$ subject to relations

$$[t_{ab}^{(r+1)}, t_{cd}^{(s)}] - [t_{ab}^{(r)}, t_{cd}^{(s+1)}] = t_{cb}^{(r)} t_{ad}^{(s)} - t_{cb}^{(s)} t_{ad}^{(r)},$$

where $t_{a,b}^{(0)} := \delta_{a,b}$. Yangian $Y(\mathfrak{sl}_n)$ is the subalgebra of $Y(\mathfrak{g}l_n)$, which is stable under all automorphisms of the form $T(u) \to f(u)T(u)$, where $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. It follows from [83, Theorem 1.8.2] that $Y(\mathfrak{gl}_n) = Y(\mathfrak{sl}_n) \otimes Z(Y(\mathfrak{gl}_n))$, where $Z(Y(\mathfrak{gl}_n))$ is the center of $Y(\mathfrak{gl}_n)$.

Let us also recall the description of Bethe subalgebras $\tilde{B}(C) \subset Y(\mathfrak{gl}_n)$. The symmetric group S_n acts on $Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes (\mathrm{End}(\mathbb{C}^n))^{\otimes n}$ by permuting the tensor factors. This action factors through the embedding $S_n \hookrightarrow (\mathrm{End}(\mathbb{C}^n))^{\otimes n}$, hence, the group algebra $\mathbb{C}[S_n]$ is a subalgebra of $Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes (\mathrm{End}(\mathbb{C}^n))^{\otimes n}$. Pick $m \in \{1, 2, \ldots, n\}$. Let S_m be the subgroup of S_n permuting the first m tensor factors. Denote by A_m the antisymmetrizer

$$A_m := \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma \sigma \in \mathbb{C}[S_m] \subset Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes (\mathrm{End}(\mathbb{C}^n))^{\otimes n}.$$

Note that for every $a \in \{1, \ldots, n\}$ there is an embedding

$$i_a \colon Y(\mathfrak{gl}_n) \otimes \mathrm{End}(\mathbb{C}^n) \hookrightarrow Y(\mathfrak{gl}_n) \otimes (\mathrm{End}(\mathbb{C}^n))^{\otimes n}$$

which is identity on $Y(\mathfrak{gl}_n)$ and embeds $\mathrm{End}(\mathbb{C}^n)$ as the a-th tensor factor in $(\mathrm{End}(\mathbb{C}^n))^{\otimes n}$. Suppose that $C \in \mathrm{GL}_n$. For any $a \in \{1, \ldots, n\}$ denote by C_a the element $i_a(1 \otimes C) \in Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes (\mathrm{End}(\mathbb{C}^n))^{\otimes n}$. For any $a \in \{1, 2, \ldots, n\}$ introduce the series with coefficients in $Y(\mathfrak{gl}_n)$ by

$$\tau_a(u, C) := \mathrm{tr}\, A_a C_1 \ldots C_a T_1(u) \ldots T_a(u - a + 1), \tag{4.27}$$

where we take the trace over all a copies of $\mathrm{End}(\mathbb{C}^n)$. The algebra $\tilde{B}(C)$ is the subalgebra of $Y(\mathfrak{gl}_n)$ generated by all coefficients of the series $\tau_a(u, C)$.

Remark 4.10.1 The fact that this definition of $\tilde{B}(C)$ coincides with the one given in Section 4.8 can be seen using that the irreducible representations of GL_n, corresponding to fundamental weights are wedge powers $\Lambda^l(\mathbb{C}^n)$.

For $C \in \mathrm{SL}_n$ let $B(C) \subset Y(\mathfrak{sl}_n)$ be the corresponding Bethe subalgebra. One can show (see [50]) that $\tilde{B}(C) = B(C) \otimes Z(Y(\mathfrak{gl}_n))$, where $Z(Y(\mathfrak{gl}_n))$ is the center of $Y(\mathfrak{gl}_n)$. It is easy to see that for $a \in \mathbb{C}^\times$ we have $\tilde{B}(aC) = \tilde{B}(C)$ and similarly if $a^n = 1$ then $B(aC) = B(C)$. It follows that actually $\tilde{B}(C)$, $B(C)$ depend on $C \in \mathrm{PGL}_n$.

Recall that $\mathfrak{h} \subset \mathfrak{gl}_n$ is the subalgebra of diagonal matrices. Let us finally recall that to $\chi \in \mathfrak{h}$ one can associate the universal inhomogeneous Gaudin subalgebra of $U(\mathfrak{gl}_n[t])$ to be denoted $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi \subset U(\mathfrak{gl}_n[t])$ (see Section 4.3.3). To the set of (distinct) points $u_1, \ldots, u_k \in \mathbb{C}$ one can associate the inhomogeneous Gaudin subalgebra of $U(\mathfrak{gl}_n)^{\otimes k}$ to be denoted by $\tilde{\mathcal{A}}_\chi(u_1, \ldots, u_k)$ (see Section 4.4). Recall that $\mathfrak{h}_0 \subset \mathfrak{h}$ is the subalgebra of traceless matrices. We also consider the quotient $\overline{\mathfrak{h}} := \mathfrak{h}/\mathbb{C}$. Note now that the embedding $\mathfrak{h}_0 \subset \mathfrak{h}$ induces the identification $\mathfrak{h}_0 \xrightarrow{\sim} \overline{\mathfrak{h}}$ so to every $\chi \in \overline{\mathfrak{h}}$ we can associate the corresponding subalgebras $\mathcal{A}^{\mathrm{u}}_\chi \subset U(\mathfrak{sl}_n[t])$, $\mathcal{A}_\chi(u_1, \ldots, u_k) \subset U(\mathfrak{sl}_n)^{\otimes k}$.

We will use the following lemma.

Lemma 4.10.2 *Pick $\chi \in \mathfrak{h}_0$, $s \in \mathbb{C}^\times$, $c \in \mathbb{C}$ and $z_1, \ldots, z_k \in \mathbb{C}$. Then we have the equality $\mathcal{A}_{s\chi}(z_1, \ldots, z_k) = \mathcal{A}_\chi(sz_1 + c, \ldots, sz_k + c)$. Same holds for the algebra $\tilde{\mathcal{A}}_\chi(z_1, \ldots, z_k)$.*

Proof: Follows from [99, Proposition 3] (see also [43, Lemma 9.2]). □

4.10.2 The evaluation homomorphisms for $Y(\mathfrak{gl}_n)$, $U(\mathfrak{gl}_n[t])$

Let $E = (E_{ab})_{a,b=1,\ldots,n} \in \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)$ be the matrix with coefficients in $U(\mathfrak{gl}_n)$, $E_{ab} \in \mathfrak{gl}_n$ is a matrix with 1 on (a, b)-entry and zeroes on other entries. Recall that for every $z \in \mathbb{C}$ we have the evaluation morphism

$$\mathbf{ev}_z \colon Y(\mathfrak{g}) \to U(\mathfrak{g}),\ t_{ab}^{(r)} \mapsto z^{r-1} E_{ab}.$$

Recall that we can identify $Y_0(\mathfrak{gl}_n) = \mathrm{gr}_2\, Y(\mathfrak{gl}_n)$ with $U(\mathfrak{gl}_n[t])$ by sending $\mathrm{gr}_2\, Y(\mathfrak{gl}_n) \ni [t_{ij}^{(r)}] \mapsto E_{ij}[r-1] \in U(\mathfrak{gl}_n[t])$.

Warning 4.10.3 *Note that this choice of the identification differs from the one in Proposition 4.7.4 by the automorphism of* $U(\mathfrak{gl}_n[t]) \xrightarrow{\sim} U(\mathfrak{gl}_n[t])$ *induced by the Cartan involution* $\mathfrak{gl}_n \xrightarrow{\sim} \mathfrak{gl}_n$ *given by* $x \mapsto -x^t$.

Recall the identification $Y_\epsilon(\mathfrak{gl}_n) \xrightarrow{\sim} Y(\mathfrak{gl}_n)$, $[\hbar^m a] \mapsto \epsilon^m a$. The composition $Y_\epsilon(\mathfrak{gl}_n) \xrightarrow{\sim} Y(\mathfrak{gl}_n) \xrightarrow{\mathbf{ev}_{z/\epsilon}} U(\mathfrak{gl}_n)$ is given by $[t_{ij}^{(r)} \hbar^{r-1}] \mapsto z^{r-1} E_{ij}$. We denote this composition by

$$\mathbf{ev}_{\epsilon;z} \colon Y_\epsilon(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n).$$

One can show that $\lim\limits_{\epsilon \to 0} \mathbf{ev}_{\epsilon;z} \colon U(\mathfrak{g}[t]) \to U(\mathfrak{g})$ is the standard evaluation at z homomorphism $\mathrm{ev}_z \colon U(\mathfrak{gl}_n[t]) \to U(\mathfrak{gl}_n)$ given by $x[l] \mapsto z^l x$ (see Lemma 4.10.5 and Corollary 4.10.6 below).

More generally, we can define the evaluation homomorphism

$$\mathbf{ev}_{(z_1,\ldots,z_k)} \colon Y(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes k},\ \mathbf{ev}_{(z_1,\ldots,z_k)} := (\mathbf{ev}_{z_1} \times \ldots \times \mathbf{ev}_{z_k}) \circ \Delta^k,$$

where $\Delta^k \colon Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)^{\otimes k}$ is the composition $(\Delta \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-1}) \circ \ldots \circ (\Delta \otimes 1) \circ \Delta$ and $\Delta \colon Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$ is the standard comultiplication on $Y(\mathfrak{gl}_n)$ given by $\Delta(T) = T \otimes T$, where $T(u) = \Big(t_{ij}(u)\Big) \in \mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{g})$ and $t_{ij}(u) = \delta_{ij} + \sum_{r \geqslant 1} t_{ij}^{(r)} u^{-r}$.

Similarly, we obtain the map $\mathbf{ev}_{(\epsilon;z_1,\ldots,z_k)} \colon Y_\epsilon(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes k}$ such that the limit $\lim\limits_{\epsilon \to 0} \mathbf{ev}_{(\epsilon;z_1,\ldots,z_k)}$ is the standard evaluation at $z_1, \ldots, z_k$ homomorphism $\mathrm{ev}_{z_1,\ldots,z_k} \colon U(\mathfrak{gl}_n[t]) \to U(\mathfrak{gl}_n)^{\otimes k}$ given by $E_{ij}[r] \mapsto \sum_{a=1}^k E_{ij}[r] z_a^r$ (see Lemma 4.10.5 and Corollary 4.10.6 below).

Remark 4.10.4 Explicitly we have

$$\mathbf{ev}_{z_1,\ldots,z_k}(T(u)) = \left(1 + \frac{E^{(1)}}{u - z_1}\right) \ldots \left(1 + \frac{E^{(k)}}{u - z_k}\right),$$

where $E^{(i)} = \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes E \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$ (E on the i-th place). It follows that

$$\mathbf{ev}_{z_1,\ldots,z_k}(t_{ij}^{(r)}) = \sum_{l_{a_1} + \ldots + l_{a_p} + p = r} E_{ij}^{(a_1)} z_{a_1}^{l_{a_1}} \ldots E_{ij}^{(a_p)} z_{a_p}^{l_p}. \tag{4.28}$$

Pick distinct $z_1, \ldots, z_k \in \mathbb{C}$ and $d_1, \ldots, d_k \in \mathbb{C}$.

Lemma 4.10.5 *Let $a(\epsilon) \in Y_\epsilon(\mathfrak{gl}_n)$, $\epsilon \in \mathbb{C}$ be an algebraic family and set $a := a(0)$. Then $\lim_{\epsilon\to 0} \mathbf{ev}_{(\epsilon; z_1+\epsilon d_1,\dots,z_k+\epsilon d_k)}(a(\epsilon))$ exists and is equal to $\mathrm{ev}_{\underline{z}}(a)$.*

Proof: Recall that $Y_\epsilon(\mathfrak{gl}_n)$ is equal to the quotient $Y_\hbar(\mathfrak{gl}_n)/(\hbar-\epsilon)$. Let $\tilde{a} \in Y_\hbar(\mathfrak{gl}_n)$ be the element that corresponds to the family $a(\epsilon)$ i.e. $[\tilde{a}]_\epsilon = a(\epsilon)$, where $[\tilde{a}]_\epsilon$ is the class of $\tilde{a}$ in $Y_\epsilon(\mathfrak{gl}_n)$. Note that $a = [\tilde{a}]_0$. Recall now that $Y_\hbar(\mathfrak{gl}_n)$ is a free $\mathbb{C}[\hbar]$-module. $Y(\mathfrak{gl}_n)$ is generated by different products of elements $t^{(k)}_{mn}$ so $Y_\hbar(\mathfrak{gl}_n)$ as a module over $\mathbb{C}[\hbar]$ is generated by the products of elements $t^{(k)}_{mn}\hbar^{k-1}$. We can then write $\tilde{a}$ as the linear combination of elements of the form $\hbar^{l+r_1+\dots+r_p-p} t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}$. Note that $[\hbar^{l+r_1+\dots+r_p-p} t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}]_\epsilon = \epsilon^l[\hbar^{r_1+\dots+r_p-p} t^{(r_1)}_{i_1j_1}\dots t^{(r_p)}_{i_pj_p}]_\epsilon$ and the map $\mathbf{ev}_{(\epsilon;u_1+\epsilon d_1,\dots,u_k+\epsilon d_k)}$ sends it to the element
$\epsilon^l \mathbf{ev}_{\epsilon,\underline{u}+\epsilon\underline{d}}([\hbar^{r_1-1}t^{(r_1)}_{i_1j_1}]_\epsilon)\dots\mathbf{ev}_{\epsilon,\underline{u}+\epsilon\underline{d}}([\hbar^{r_p-1}t^{(r_p)}_{i_pj_p}]_\epsilon)$ that goes to zero under the limit $\epsilon \to 0$ if $l > 0$. So it remains to show that for every $[\hbar^{r-1}t^{(r)}_{ij}]_\epsilon \in Y_\epsilon(\mathfrak{gl}_n)$ $\lim_{\epsilon\to 0}\mathbf{ev}_{\epsilon;\underline{u}+\epsilon\underline{d}}([\hbar^{r-1}t^{(r)}_{ij}]_\epsilon) = \mathrm{ev}_{\underline{u}}(E_{ij}[r-1])$.

Note that using (4.28) we have

$$\mathbf{ev}_{\epsilon;\underline{z}+\epsilon\underline{d}}([\hbar^{r-1}t^{(r)}_{ij}]_\epsilon) = \epsilon^{r-1}\mathrm{ev}_{\underline{z}/\epsilon+\underline{d}}(t^{(r)}_{ij}) = \sum_{a=1}^{k} E^{(a)}_{ij} z_a^{r-1} + O(\epsilon).$$

This observation finishes the proof of the lemma since $\mathrm{ev}_{\underline{z}}(E_{ij}[r-1]) = \sum_{a=1}^k E^{(a)}_{ij} z_a^{r-1}$. □

Corollary 4.10.6 *There exists a homomorphism of $\mathbb{C}[\hbar]$-algebras*

$$\mathbf{ev}_{(\hbar;z_1+\hbar d_1,\dots,z_k+\hbar d_k)}\colon Y_\hbar(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes k}[\hbar],$$

which fiber over $\hbar = \epsilon \in \mathbb{C}^\times$ is $\mathbf{ev}_{(\epsilon;z_1+\epsilon d_1,\dots,z_k+\epsilon d_k)}$ and the fiber over $\hbar = 0$ is $\mathrm{ev}_{\underline{z}}$.

Proof: Consider the natural embedding $Y_\hbar(\mathfrak{gl}_n) \subset Y_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}]$. Recal that we have the identification $Y_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm1}] \xrightarrow{\sim} Y(\mathfrak{gl}_n)[\hbar^{\pm1}]$ given by $(\hbar^i x)\otimes \hbar^l \mapsto \hbar^{l+i}x$. Consider the homomorphism $\mathbf{ev}_{\hbar^{-1}z_1+d_1,\dots,\hbar^{-1}z_k+d_k}\colon Y(\mathfrak{gl}_n)[\hbar^{\pm1}] \to U(\mathfrak{gl}_n)^{\otimes k}[\hbar^{\pm1}]$. Composing this homomorphism with the embedding $Y_\hbar(\mathfrak{gl}_n) \subset Y(\mathfrak{gl}_n)[\hbar^{\pm1}]$ we obtain the desired homomorphism (use Lemma 4.10.5 to see that this homomorphism satisfies the required properties)

$$\mathbf{ev}_{(\hbar;z_1+\hbar d_1,\dots,z_k+\hbar d_k)}\colon Y_\hbar(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes k}[\hbar].$$

□

4.11 Manin matrices and generators of $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi$, $\tilde{\mathcal{A}}_\chi(z_1,\dots,z_k)$

In this section following [21] and [106] we introduce certain generators of the algebras $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi$, $\tilde{\mathcal{A}}_\chi(z_1,\dots,z_k)$, $\tilde{B}(C)$, $\mathbf{ev}_{z_1,\dots,z_k}(\tilde{B}(C))$. We start from a general definition. Let R be an algebra over complex numbers. Pick $M \in \mathrm{End}(\mathbb{C}^n)\otimes R$ that we can consider as $n\times n$ matrix $M = (M_{ij})$ with $M_{ij} \in R$.

Definition 4.11.1 *We say that M is a Manin matrix if*

$$\forall p,l,r,s = 1,\dots,n \text{ we have } [M_{pl}, M_{rs}] = [M_{rl}, M_{ps}].$$

Given a matrix $M \in \mathrm{End}(\mathbb{C}^n)$ we define its column-determinant $\mathrm{cdet}\, M \in R$ by the following formula:

$$\mathrm{cdet}\, M := \sum_{\sigma \in S_n} (-1)^\sigma \cdot M_{\sigma(1)1} \dots M_{\sigma(n)n}.$$

For $i = 1, \dots, n$ we denote by $M_i \in \mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes R$ the image of M under the embedding $\mathrm{End}(\mathbb{C}^n) \otimes R \hookrightarrow \mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes R$ given by

$$\mathrm{End}(\mathbb{C}^n) \otimes R \ni f \otimes x \mapsto 1 \otimes \dots \otimes 1 \otimes \underset{i}{f} \otimes 1 \otimes \dots \otimes 1 \otimes x \in \mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes R.$$

Recall that $A_n \in \mathbb{C}[S_n] \subset \mathrm{End}(\mathbb{C}^n)^{\otimes n}$ is the antisymmetrizer normalized by the condition $A_n^2 = A_n$. Explicitly, we have $A_n = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma$.

Lemma 4.11.2 *Let $M \in \mathrm{End}(\mathbb{C}^n) \otimes R$ be a Manin matrix. Then*

$$\mathrm{tr}\, A_n M_1 \dots M_n = \mathrm{cdet}\, M.$$

Proof: We have

$$\mathrm{tr}\, A_n M_1 \dots M_n = \frac{1}{n!} \sum_{a,b \in S_n} (-1)^{ab} M_{a(1)b(1)} \dots M_{a(n)b(n)}.$$

It follows from [20] that for every $p \in S_n$ we have

$$\mathrm{cdet}(M) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) M_{\sigma(p(1))p(1)} \dots M_{\sigma(p(n))p(n)}.$$

We conclude that

$$n!\, \mathrm{cdet}\, M = \sum_{p,\sigma \in S_n} \mathrm{sgn}(\sigma) M_{\sigma(p(1))p(1)} \dots M_{\sigma(p(n))p(n)}.$$

Replacing $\sigma \circ p$ by a and σ by b we obtain that $\mathrm{tr}\, A_n M_1 \dots M_n = \mathrm{cdet}\, M$. □

Let us slightly modify generators of the algebra $\tilde{B}(C)$. Recall that the algebra $\tilde{B}(C)$ is generated by the elements

$$\tau_a(u, C) = \mathrm{tr}\, A_a C_1 \dots C_a T_1(u) \dots T_a(u - a + 1),\ a = 1, \dots, n.$$

Recall the antipode map $\eta\colon Y(\mathfrak{gl}_n) \xrightarrow{\sim} Y(\mathfrak{gl}_n)$ induced by the map $T(u) \mapsto T(u)^{-1}$. Then by [92, Section 1] the algebra $\eta(\tilde{B}(C)) = \tilde{B}(C^{-1})$ (see [51, Lemma 6.5]) is generated by the elements

$$\mathrm{tr}\, A_n T_1(u) \dots T_k(u - k + 1) C_{k+1} \dots C_n.$$

It follows that the the algebra $\tilde{B}(C)$ is generated by the elements

$$\mathrm{tr}\, A_n T_1(u) \dots T_a(u - a + 1) C_{a+1}^{-1} \dots C_n^{-1},\ a = 1, \dots, n.$$

These generators can be written in a generating function

$$\mathrm{tr}\, A_n (e^{-\partial_u} T_1(u) - C_1^{-1}) \dots (e^{-\partial_u} T_n(u) - C_n^{-1}).$$

Moreover, note that by [20, Proposition 4] matrix $e^{-\partial_u}T(u) - C^{-1}$ is Manin so we conclude from Lemma 4.11.2 that

$$\operatorname{tr} A_n(e^{-\partial_u}T_1(u) - C_1^{-1})\dots(e^{-\partial_u}T_n(u) - C_n^{-1}) = \operatorname{cdet}(e^{-\partial_u}T(u) - C^{-1}).$$

Recall the matrix $E \in \operatorname{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)$. Set $E(u) := \sum_{r\geqslant 0} E[r]u^{-r-1} \in \operatorname{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n[t])[[u^{-1}]]$.

Lemma 4.11.3 *We have*

$$[E_{ij}(u), E_{kl}(v)] = \frac{1}{u-v}\Big(E_{il}(v)\delta_{jk} - E_{il}(u)\delta_{jk} - E_{kj}(v)\delta_{li} + E_{kj}(u)\delta_{li}\Big)$$

i.e. $E(u)$ is a Lax matrix of $\mathfrak{gl}_n$-Gaudin type (see [20, Definition 5]).

Proof: Follows from the relation $[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{kj}\delta_{li},$. □

Corollary 4.11.4 *The matrix $\partial_u - E(u)$ is a Manin matrix.*

Proof: Follows from [20, Proposition 2], compare with [20, Equation (18)]. □

We are now ready to give an explicit description of the generators of the algebra $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi$. The following proposition is essentially due to Talalaev ([106]).

Proposition 4.11.5 *The subalgebra $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi \subset U(\mathfrak{gl}_n[t])$ is generated by the coefficients in front of $u^r\partial_u^l$, $l \in \mathbb{Z}_{\geqslant 0}$, $r \in \mathbb{Z}_{\leqslant 0}$ of $\operatorname{cdet}(E(u) - \partial_u + \chi)$.*

Proof: Recall that the generators of $\tilde{B}(C)$ are the coefficients of $\operatorname{cdet}(e^{-\partial_u}T(u) - C^{-1})$. Consider the family $\operatorname{cdet}\epsilon^{-1}(e^{-\epsilon\partial_u}T(u/\epsilon) - \exp(-\epsilon\chi))$ of elements of $\tilde{B}(\exp(\epsilon\chi))$. After the identification $\tilde{B}(\exp(\epsilon\chi)) \simeq \tilde{B}_\epsilon(\exp(\epsilon\chi))$ the element $\operatorname{cdet}\epsilon^{-1}(e^{-\epsilon\partial_u}T(u/\epsilon) - \exp(-\epsilon\chi))$ becomes

$$\operatorname{cdet}\epsilon^{-1}\Big(e^{-\epsilon\partial_u}\Big(1 + \epsilon\sum_{r\geqslant 1}[\hbar^{r-1}t_{ij}^{(r)}]u^{-r}\Big) - \exp(-\epsilon\chi)\Big) \in \tilde{B}_\epsilon(\exp(\epsilon\chi)).$$

We have

$$e^{-\epsilon\partial_u}\Big(1 + \epsilon\sum_{r\geqslant 1}[\hbar^{r-1}t_{ij}^{(r)}]u^{-r}\Big) - \exp(-\epsilon\chi) =$$
$$(1-\epsilon\partial_u)\Big(1 + \epsilon\sum_{r\geqslant 1}[\hbar^{r-1}t_{ij}^{(r)}]u^{-r}\Big) - 1 + \epsilon\chi + O(\epsilon^2) =$$
$$= \epsilon\Big(\sum_{r\geqslant 1}[\hbar^{r-1}t_{ij}^{(r)}]u^{-r} - \partial_u + \chi\Big) + O(\epsilon^2).$$

It follows that

$$\lim_{\epsilon\to 0}\operatorname{cdet}\epsilon^{-1}(e^{-\epsilon\partial_u}T(u/\epsilon) - \exp(-\epsilon\chi)) = \operatorname{cdet}(E(u) - \partial_u + \chi).$$

It remains to recall that by Theorem 4.8.12 we have $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi = \lim_{\epsilon\to 0}\tilde{B}_\epsilon(\exp(\epsilon\chi))$. The fact that the coefficients of $\operatorname{cdet}(E(u) - \partial_u + \chi)$ generate $\tilde{\mathcal{A}}^{\mathrm{u}}_\chi$ follow from the similar statement for the classical limit $\tilde{\overline{\mathcal{A}}}^{\mathrm{u}}_\chi \subset S^\bullet(\mathfrak{gl}_n[t])$ together with the fact that $\operatorname{gr}_{PBW}\tilde{\mathcal{A}}^{\mathrm{u}}_\chi = \operatorname{gr}_{PBW}\tilde{\overline{\mathcal{A}}}^{\mathrm{u}}_\chi$. □

Remark 4.11.6 For $\chi = 0$ the results of Proposition 4.11.5 should be compared with [21, Theorem 3.1].

Let us now pick distinct points $z_1, \dots, z_k \in \mathbb{C}$ and recall the corresponding standard Lax matrix for the Gaudin system

$$L_{\underline{z}}(u) := \sum_{i=1}^{k} \frac{E^{(i)}}{u - z_i}.$$

Corollary 4.11.7 *For every $\chi \in \mathfrak{gl}_n$ the algebra $\tilde{\mathcal{A}}_\chi(z_1, \dots, z_k)$ is generated by the coefficients in front of $u^r \partial_u^l$ of $\operatorname{cdet}(L(u) - \partial_u - \chi)$.*

Proof: Recall that $\mathcal{A}_\chi(z_1, \dots, z_k) = \mathrm{ev}_{z_1,\dots,z_k}(\mathcal{A}^{\mathrm{u}}_{-\chi})$. Now the claim follows from the fact that

$$\mathrm{ev}_{z_1,\dots,z_k}(E(u)) = \sum_{i=1}^{k} \sum_{r=0}^{\infty} E z_i^r u^{-r-1} = \sum_{i=1}^{k} \frac{E^{(i)}}{u - z_i} = L_{\underline{z}}(u).$$

□

Pick $\xi \in \mathfrak{h}$. Our goal for now is to introduce certain compatible families of generators of the algebras $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}\tilde{B}(\exp(\epsilon\chi))$, $\tilde{\mathcal{A}}_\chi(\underline{z})$.

Consider the following two-parametric family of differential operators, here $\epsilon, \varepsilon \in \mathbb{C}^\times$

$$\operatorname{cdet} \epsilon^{-1}(e^{-\epsilon\partial_u}(1 + L_{\underline{z}/\varepsilon+\underline{d}}(u/\epsilon)) - \exp(-\epsilon\chi)) = \sum_{k \geqslant 0} b_{k,\epsilon,\varepsilon}(u)\partial_u^k.$$

Since $e^{-\partial_u}u^r = u^{r-1}e^{-\partial_u}$ for every $r \in \mathbb{Z}$ it follows that $b_{k,\epsilon,\varepsilon}(u)$ are rational functions with poles at the points $\frac{\epsilon}{\varepsilon}z_i + \epsilon j + \epsilon d_i$, $i, j = 1, \dots, k$. Consider then the following elements

$$r_{k,z_i,l,\epsilon,\varepsilon} := \sum_{j=1}^{k} \mathrm{res}_{u=\frac{\epsilon}{\varepsilon}z_i+\epsilon j+\epsilon d_i}(u - z_i)^l b_{k,\epsilon,\varepsilon}(u)du \in \mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(\tilde{B}(\exp(\epsilon\chi))),\ l = 0, \dots, k.$$

We can also decompose

$$\operatorname{cdet}(L_{\underline{z}}(u) - \partial_u - \xi) = \sum_{k \geqslant 0} b^0_{k,\underline{z},\xi}(u)\partial_u^k$$

and consider the elements

$$r^0_{k,z_i,l,\xi} := \mathrm{res}_{u=z_i}(u - z_i)^l b^0_{k,\underline{z},\xi}(u)du,\ l = 0, \dots, k.$$

The following proposition is an immediate consequence of the results above.

Proposition 4.11.8 *The algebra $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(\tilde{B}(\exp(\epsilon\chi)))$ is generated by elements $r_{k,z_i,l,\epsilon,\varepsilon}$. The algebra $\tilde{\mathcal{A}}_\xi(\underline{z})$ is generated by the elements $r^0_{k,z_i,l,\xi}$*

Remark 4.11.9 Note that for every $c \in \mathbb{C}$ the elements $r_{k,z_i,l,\epsilon,\varepsilon}$, $r^0_{k,z_i,l,\xi}$ do not change under the simultaneous shift $(z_1, \dots, z_k) \mapsto (z_1 + c, \dots, z_n + c)$ so in particular

$$\mathbf{ev}_{((z_1/\varepsilon)+d_1,\dots,(z_k/\varepsilon)+d_k)}(\tilde{B}(\exp(\epsilon\chi))) = \mathbf{ev}_{((z_1+c)/\varepsilon+d_1,\dots,(z_k+c)/\varepsilon+d_k)}(\tilde{B}(\exp(\epsilon\chi))),$$

$$\tilde{\mathcal{A}}_\xi(z_1, \dots, z_n) = \tilde{\mathcal{A}}_\xi(z_1 + c, \dots, z_k + c).$$

4.12 Alcoves

Let $S \subset T$ be the compact torus i.e. $S := T \cap U(n)$, where $U(n) \subset \mathrm{GL}_n$ is the group of unitary matrices. We denote by $\overline{S}$ the quotient $S/U(1) \subset \mathrm{PGL}_n$, here $U(1) \subset U(n)$ is the center. Recall the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n$ consisting of diagonal matrices. We use the natural bijection $\{1, 2, \ldots, n\} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$, $j \mapsto [j]$ to parametrize coordinates of $\mathfrak{h}$ by elements of $\mathbb{Z}/n\mathbb{Z}$. We also recall the quotient $\overline{\mathfrak{h}} = \mathfrak{h}/\{\mathrm{diag}(a, \ldots, a),\, a \in \mathbb{C}\}$ and the natural identification $\mathfrak{h}_0 \xrightarrow{\sim} \overline{\mathfrak{h}}$ (here $\mathfrak{h}_0 \subset \mathfrak{h}$ is the subalgebra of traceless diagonal matrices). Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ be the real points. We have a covering:

$$\exp\colon \overline{\mathfrak{h}}_{\mathbb{R}} \to \overline{S},\ \chi \mapsto \exp(2\pi i \chi).$$

We denote by $\overline{S}^{\mathrm{reg}} \subset \overline{S}$ the subset of regular elements. Note that the preimage $\exp^{-1}(\overline{S} \backslash \overline{S}^{\mathrm{reg}})$ consists of the points $(a_{[1]}, \ldots, a_{[n]}) \in \overline{\mathfrak{h}}_{\mathbb{R}}$ such that $a_{[i]} - a_{[j]} \in \mathbb{Z}$ for some $[i] \neq [j]$. For $[i] \neq [j]$ and $k \in \mathbb{Z}$ we set

$$H^k_{[i],[j]} = \{(a_{[1]}, \ldots, a_{[n]}) \in \overline{\mathfrak{h}}_{\mathbb{R}} \,|\, a_{[i]} - a_{[j]} = k\}.$$

We will call the hyperplanes $H^k_{[i],[j]} \subset \mathfrak{h}_{\mathbb{R}}$ the *affine walls*, and the connected components of the complement to the union of all $H^k_{[i],[j]}$ the *alcoves*. Let Λ be the coweight lattice of $\mathfrak{gl}_n$. We have the natural identification $\Lambda = \mathbb{Z}^n$. Set $\overline{\Lambda} := \Lambda/\mathbb{Z}(1, 1, \ldots, 1)$. Consider the standard action $S_n \curvearrowright \Lambda$ via the permutation of the basis elements. Note that we have the left action of $S_n \ltimes \Lambda$ on $\mathfrak{h}$ given by

$$(\sigma; m_1, \ldots, m_n) \cdot (a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)} + m_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)} + m_{\sigma^{-1}(n)}).$$

Remark 4.12.1 Recall that we have $(\sigma_1, \lambda_1) \cdot (\sigma_2, \lambda_2) = (\sigma_1\sigma_2, \sigma_2^{-1}(\lambda_1) + \lambda_2)$ for $\sigma_1,\, \sigma_2 \in S_n,\, \lambda_1,\, \lambda_2 \in \Lambda$.

We define the *extended* affine Weyl group as $\widehat{W}^{\mathrm{ext}} := S_n \ltimes \overline{\Lambda}$. The natural action of $S_n \curvearrowright \overline{\mathfrak{h}}$ and the action of $\overline{\Lambda} \curvearrowright \overline{\mathfrak{h}}$ via translations induce the action $\widehat{W}^{\mathrm{ext}} \curvearrowright \overline{\mathfrak{h}}$.

The coroot lattice $\Lambda_r \subset \Lambda$ is generated by the vectors $E_{ii} - E_{i+1i+1} \in \mathfrak{h}$, $i = 1, \ldots, n-1$. Let $\overline{\Lambda}_r \subset \overline{\Lambda}$ be the image of Λ_r in $\overline{\Lambda}$. Note that $[\overline{\Lambda} : \overline{\Lambda}_r] = n$. The affine Weyl group $\widehat{W} := S_n \ltimes \overline{\Lambda}_r$ is an index n subgroup of the extended affine Weyl group $\widehat{W}^{\mathrm{ext}}$.

The $\widehat{W}^{\mathrm{ext}}$-action on $\mathfrak{h}_{\mathbb{R}}$ preserves the affine walls, so it acts on the set alcoves. It is easy to see that the alcoves are fundamental domains for the action $\widehat{W} \curvearrowright \overline{\mathfrak{h}}_{\mathbb{R}}$. By choosing the "base" alcove $Q = \{(a_{[1]}, \ldots, a_{[n]}) \in \overline{\mathfrak{h}}_{\mathbb{R}},\, |\, a_{[n]} + 1 \geqslant a_{[1]} \geqslant a_{[2]} \geqslant \ldots \geqslant a_{[n]}\}$, we obtain a bijection between the alcoves and the elements of $\widehat{W}$ and a surjective $n : 1$ map from $\widehat{W}^{\mathrm{ext}}$ to the set of alcoves. More precisely, to $\hat{w} = (\sigma, m_{[1]}, \ldots, m_{[n]}) \in \hat{W}^{\mathrm{ext}}$ we associate

$$\begin{aligned} Q_{\hat{w}} &:= \hat{w}Q = \\ &= \{(a_{[1]}, \ldots, a_{[n]}) \in \overline{\mathfrak{h}}_{\mathbb{R}} \,|\, a_{\sigma([n])} - m_{[n]} + 1 \geqslant a_{\sigma([1])} - m_{[1]} \geqslant \ldots \geqslant a_{\sigma([n])} - m_{[n]}\}. \end{aligned} \tag{4.29}$$

Warning 4.12.2 *Let us point out once again that the $\widehat{W}^{\mathrm{ext}}$-action on the set of alcoves is not free, so it may happen that $Q_{\hat{w}} = Q_{\hat{w}'}$ for distinct $\hat{w},\, \hat{w}' \in \widehat{W}^{\mathrm{ext}}$. This will be important in the proof of Proposition 4.16.6.*

The base alcove Q is separated by the walls

$$H^0_{[1],[2]}, \dots, H^0_{[n-1],[n]}, H^{-1}_{[n],[1]}$$

that we denote by $H_1, H_2, \dots, H_n$ respectively. We denote by $Q^{\mathrm{reg}}_{\hat{w}}$ the interior of $Q_{\hat{w}}$. The alcove $Q_{\hat{w}}$ is separated by the walls

$$H^{m_{[1]}-m_{[2]}}_{\sigma([1]),\sigma([2])}, \dots, H^{m_{[n-1]}-m_{[n]}}_{\sigma([n-1]),\sigma([n])}, H^{m_{[n]}-m_{[1]}-1}_{\sigma([n]),\sigma([1])}$$

that we enumerate by numbers $1, 2, \dots, n$ and denote by $H^{\hat{w}}_1, \dots, H^{\hat{w}}_n$. Note that $H^{\hat{w}}_i = \hat{w}(H_i)$.

4.13 Limits to the affine wall

In this section we describe the limits of different families of subalgebras depending on $\chi \in \overline{\mathfrak{h}}$ to a generic point of an (affine) wall $H^k_{[i],[j]}$.

We pick $\hat{w} \in \widehat{W}^{\mathrm{ext}}$ and consider the corresponding alcove $Q_{\hat{w}}$ (see (4.29)). Pick $\chi_0 \in Q_{\hat{w}}$ lying on some $H^k_{[i],[j]}$ but not on any other affine wall (such $\chi_0 \in H^k_{[i],[j]}$ will be called *subregular*). Set $C_0 := \exp(2\pi i \chi_0)$. Pick also $\chi \in Q^{\mathrm{reg}}_{\hat{w}}$ and consider the following family $C(\varepsilon) := C_0 \exp(2\pi i \varepsilon \chi) = \exp(2\pi i(\chi_0 + \varepsilon\chi))$, where $\varepsilon \in \mathbb{C}$ (one can also consider the formal version of this family). Set $B(C_0, \chi) := \lim\limits_{\varepsilon \to 0} B(C(\varepsilon))$. Recall that we have the identification $Y(\mathfrak{g}) \simeq Y_\epsilon(\mathfrak{g})$. We denote by $B_\epsilon(C_0, \chi)$ the image of $B(C_0, \chi)$ in $Y_\epsilon(\mathfrak{g})$. We also set $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} := \lim\limits_{\varepsilon \to 0} \mathcal{A}^{\mathrm{u}}_{\chi_0 + \varepsilon\chi}$.

Remark 4.13.1 The algebra $B(C_0, \chi)$ is the limit subalgebra for the family $\{B(C), C \in \overline{T}^{\mathrm{reg}}\}$. It follows from [51] that the space of all possible limit subalgebras is classified by the Deligne-Mumford space $\overline{M_{0,n+2}}$ of stable rational curves with $n + 2$ marked points.

We will use the following lemma in the proof of Proposition 4.13.3 below.

Lemma 4.13.2 *Pick $C \in \overline{T}^{\mathrm{reg}}$ then $\mathfrak{h}_0 \subset B(C)$ (we use the embedding $U(\mathfrak{g}) \subset Y(\mathfrak{g})$).*

Proof: Recall the filtration F_2 on $Y(\mathfrak{g})$ (see Section 4.7.3). It follows from [53] that $\mathrm{gr}_2 B(C) = U(\mathfrak{h}_0[t]) \subset U(\mathfrak{sl}_n[t]) = \mathrm{gr}_2 Y(\mathfrak{g})$ so $\mathfrak{h}_0 \subset \mathrm{gr}_2 B(C)$. Note that $\deg_{F_2} \mathfrak{h}_0 = 0$ and the only element of degree -1 in $Y(\mathfrak{g})$ is zero. The claim follows. □

The following proposition should be compared with [53, Theorem B].

Proposition 4.13.3 *We have*

$$B(C_0, \chi) = B(C_0) \otimes_{Z(U(\mathfrak{z}_{\mathfrak{g}}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_0)),\ \mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} = \mathcal{A}^{\mathrm{u}}_{\chi_0} \otimes_{Z(U(\mathfrak{z}_{\mathfrak{g}}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_0)).$$

Moreover, the subalgebra $B(C_0, \chi) \subset Y(\mathfrak{g})$ is generated by $B(C_0)$ and the element $h = h_{ij} = E_{ii} - E_{jj} \in U(\mathfrak{g}) \subset Y(\mathfrak{g})$, the subalgebra $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} \subset U(\mathfrak{g}[t])$ is generated by $\mathcal{A}^{\mathrm{u}}_{\chi_0}$ and the element $h_{ij} \in U(\mathfrak{g}) \subset U(\mathfrak{g}[t])$.

Proof: We have $B(C_0) \subset \lim_{\epsilon\to 0} B(C(\epsilon)) = B(C_0,\chi)$. We claim that h_{ij} lies in the limit. Indeed, this follows from the fact that $C = C(\epsilon)$ is regular for $\epsilon \neq 0$ so by Lemma 4.13.2 we have $\mathfrak{h}_0 \subset B(C(\epsilon))$. Note now that the derived subalgebra $\mathfrak{z}_\mathfrak{g}(\chi_0)^{\mathrm{der}}$ of $\mathfrak{z}_\mathfrak{g}(\chi_0)$ is equal to $\mathrm{Span}_\mathbb{C}(E_{ij}, h_{ij}, E_{ji}) \simeq \mathfrak{sl}_2$ so the algebra $\mathcal{A}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0))$ is equal to $\mathbb{C}[h_{ij}] \otimes Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))$ i.e. is (freely) generated by h_{ij} over the center $Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))$. We conclude that $B(C_0) \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0)) \subset B(C_0,\chi)$. Note that $Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0))) \subset B(C_0) \subset B(C_0,\chi)$ so $B(C_0,\chi)$ is contained in the centralizer $Z_{Y(\mathfrak{g})}(Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0))))$. It follows from [53, Lemma 6.14] that

$$Z_{Y(\mathfrak{g})}(Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))) = Y(\mathfrak{g})^{\mathfrak{z}_\mathfrak{g}(\chi_0)} \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} U(\mathfrak{z}_\mathfrak{g}(\chi_0)).$$

Algebra $B(C_0) \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0))$ is a maximal commutative subalgebra of $Y(\mathfrak{g})^{\mathfrak{z}_\mathfrak{g}(\chi_0)} \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} U(\mathfrak{z}_\mathfrak{g}(\chi_0))$ (compare with the proof of Proposition 4.5.3) so we must have the equality $B(C_0) \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0)) = B(C_0,\chi)$.

The proof for $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}$ is similar. □

Recall that $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} = \mathcal{A}^{\mathrm{u}}_{\chi_0} \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(\chi_0)))} \mathcal{A}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0))$. We set $\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)} := \overline{\mathcal{A}}^{\mathrm{u}}_{\chi_0} \otimes_{Z(S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi_0)))} \overline{\mathcal{A}}_\chi(\mathfrak{z}_\mathfrak{g}(\chi_0))$ that can be considered as a classical limit of $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}$.

Remark 4.13.4 In the same way as in the proof of Proposition 4.13.3 one can show that $\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)} = \lim_{\epsilon\to 0} \overline{\mathcal{A}}^{\mathrm{u}}_{\chi_0+\epsilon\chi}$ and $\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)}$ is generated by $\overline{\mathcal{A}}^{\mathrm{u}}_{\chi_0}$ and h.

Lemma 4.13.5 *We have*

$$\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)} = Z_{S^\bullet(\mathfrak{g}[t])}(\Omega_{\chi_0}, h),\ \mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} = Z_{U(\mathfrak{g}[t])}(\tilde{\Omega}_{\chi_0}, h).$$

Proof: The proof is similar to the proof of Proposition 4.6.2. Considering the family $Z_{S^\bullet(\mathfrak{g}[t])}(\Omega_{\kappa\chi_0}, h)$ and taking the limit $\kappa \to \infty$ we conclude (as in the proof of Proposition 4.6.2) that

$$\lim_{\kappa\to\infty} Z_{S^\bullet(\mathfrak{g}[t])}(\Omega_{\kappa\chi_0}, h) \subset Z_{S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi_0)[t])}(\Omega_{\mathfrak{z}_\mathfrak{g}(\chi_0)}, h).$$

It remains to show that $\overline{\mathcal{A}}^{\mathrm{u}}_0(\mathfrak{z}_{\mathfrak{g}(\chi_0)}) \cdot h = Z_{S^\bullet(\mathfrak{z}_\mathfrak{g}(\chi_0)[t])}(\Omega_{\mathfrak{z}_\mathfrak{g}(\chi_0)}, h)$. Note that $\mathfrak{z}_\mathfrak{g}(\chi_0)$ is a reductive Lie algebra with semisimple part being isomorphic to $\mathfrak{sl}_2$. Now the claim reduces to the $\mathfrak{sl}_2$-case when this is clear.

So we have shown that $\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)} = Z_{S^\bullet(\mathfrak{g}[t])}(\Omega_{\chi_0}, h)$. The proof of the equality $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)} = Z_{U(\mathfrak{g}[t])}(\tilde{\Omega}_{\chi_0}, h)$ follows from the equality $\overline{\mathcal{A}}^{\mathrm{u}}_{(\chi_0,\chi)} = Z_{S^\bullet(\mathfrak{g}[t])}(\Omega_{\chi_0}, h)$ in the same way as in the proof of Proposition 4.6.6. □

Lemma 4.13.6 *The two-parametric family $B_\epsilon(\exp(\epsilon(\chi_0+\varepsilon\chi)))$ depending on $(\epsilon,\varepsilon) \in (\mathbb{C}^\times)^2$ extends to the continuous family on the whole $\mathbb{C}^2$ by setting*

$$(\epsilon, 0) \mapsto B_\epsilon(\exp(\epsilon\chi_0), \chi),\ (0,\varepsilon) \mapsto \mathcal{A}^{\mathrm{u}}_{\chi_0+\varepsilon\chi},\ (0,0) \mapsto \mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}.$$

Proof: As in the proof of Proposition 4.8.9 we consider the family

$$X_\epsilon(\chi_0+\varepsilon\chi) := \frac{\epsilon}{c_V}\psi_\epsilon\Big(\operatorname{tr}_V \exp \rho_V(\epsilon(\chi_0+\varepsilon\chi))T^{(3)}\Big) \in B_\epsilon(\exp(\epsilon(\chi_0+\varepsilon\chi))),\ \epsilon \in \mathbb{C}^\times,\ \varepsilon \in \mathbb{C}.$$

It follows from the proof of Proposition 4.8.9 that the family $X_\epsilon(\chi_0+\varepsilon\chi)$ extends to $\epsilon=0$ via $X_0(\chi_0+\varepsilon\chi)=\tilde{\Omega}_{\chi_0+\varepsilon\chi}$. We conclude that every limit algebra to $(0,\varepsilon)$ contains $\tilde{\Omega}_{\chi_0+\varepsilon\chi}$ (here ε may be equal to zero). Note also that $h_{ij}\in B_\epsilon(\exp(\epsilon(\chi_0+\varepsilon\chi)))$ so every limit algebra contains h_{ij}. It follows that every limit algebra to $(0,\varepsilon)$ ($\varepsilon\in\mathbb{C}$) contains $\tilde{\Omega}_{\chi_0+\varepsilon\chi},h_{ij}$ so is contained in the centralizer $Z_{U(\mathfrak{g}[t])}(\tilde{\Omega}_{\chi_0+\varepsilon\chi},h_{ij})$ that is equal to $\mathcal{A}^{\mathrm{u}}_{\chi_0+\varepsilon\chi}$ for $\varepsilon\neq0$ (see Proposition 4.6.6) and to $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}$ for $\varepsilon=0$ (see Lemma 4.13.5). From the dimension estimates (same as in the proof of Theorem 4.8.12) we conclude that every limit algebra to $(0,\varepsilon)$ ($\varepsilon\in\mathbb{C}$) is $\mathcal{A}^{\mathrm{u}}_{\chi_0+\varepsilon\chi}$ for $\varepsilon\neq0$ and $\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}$ for $\varepsilon=0$. We also note that every limit algebra to $(\epsilon,0)$ ($\epsilon\neq0$) contains $B_\epsilon(\exp(\epsilon\chi_0))$, h_{ij}, hence, by Proposition 4.13.3 and the maximality of $B_\epsilon(\exp(\epsilon\chi_0,\chi))$ (see [51, Main Theorem of Section 1.2]) it must be equal to $B_\epsilon(\exp(\epsilon\chi_0,\chi))$.

To finish the proof we need to check that the family that we have constructed is indeed continuous. Let π be our two-parametric family and let $\pi_i\colon\mathbb{C}^2\to\mathrm{Gr}(d_i,Q_i)$ be the corresponding maps to Grassmannians. We need to check that if we have some convergent sequence $p_n\to p$ on $\mathbb{C}^2$ then the sequence of images $\pi_i(p_n)$ converges to $\pi_i(p)$. It is enough to show that every convergent subsequence of the sequence $\pi_i(p_n)$ converges to $\pi_i(p)$. Pick a convergent subsequence $\pi_i(p_{j_n^{(i)}})$ of the sequence $\pi_i(p_n)$ and assume that it converges to some P_i. Pick a subsequence $p_{j^{(i)}_{j_n^{(i+1)}}}$ of p_{j_n} such that $\pi_i(p_{j^{(i)}_{j_n^{(i+1)}}})$ converges to some P_{i+1}. Continuing in this way we obtain a sequence of vector spaces (use Lemma 4.2.8) $P_i\subset P_{i+1}\subset\ldots$ such that $P:=\bigcup_{j\geqslant i}P_i$ is a commutative algebra (use Lemma 4.2.8), its dimension can only jump. It now follows from the observations above together with Lemma 4.2.8 that the algebra P must be $\pi(p)$. We conclude that $P_i=\pi_i(p)$.

□

Remark 4.13.7 Note that there is an alternative proof of Lemma 4.13.6 that uses explicit generators described in Section 4.11 (compare with the proof of Proposition 4.11.5). Note also that we are not claiming that the family constructed in Lemma 4.13.6 is holomorphic (this should be true but we do not need this).

Pick distinct points $z_1,\ldots,z_k\in\mathbb{C}$. We set $\mathcal{A}_{(\chi_0,\chi)}(\underline{z}):=\lim\limits_{\epsilon\to0}\mathcal{A}_{\chi_0+\epsilon\chi}(\underline{z})\subset U(\mathfrak{g})^{\otimes k}$. Let us show that the algebra $\mathcal{A}_{(\chi_0,\chi)}(\underline{z})$ is equal to $\mathrm{ev}_{\underline{z}}(\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)})$ (see Proposition 4.13.9 below). We start from the following lemma.

Lemma 4.13.8 *We have $\mathcal{A}_{(\chi_0,\chi)}(\underline{z})=\mathcal{A}_{\chi_0}(\underline{z})\otimes_{Z(U(\mathfrak{z}_{\mathfrak{gl}_n}(\chi)))}\mathcal{A}_\chi(\mathfrak{z}_{\mathfrak{gl}_n}(\chi_0))$. The algebra $\mathcal{A}_{(\chi_0,\chi)}(\underline{z})$ is generated by $\mathcal{A}_{\chi_0}(\underline{z})$ and the element $\Delta^k(h_{ij})=h_{ij}^{(1)}+\ldots+h_{ij}^{(k)}$.*

Proof: Same proof as of Proposition 4.13.3. □

Proposition 4.13.9 *We have $\mathrm{ev}_{\underline{z}}(\mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)})=\mathcal{A}_{(-\chi_0,-\chi)}(\underline{z})$.*

Proof: Use Proposition 4.4.10 together with Proposition 4.13.3, Lemma 4.13.8 and the maximality of $\mathcal{A}_{(-\chi_0,-\chi)}(\underline{z})\subset U(\mathfrak{g})^{\otimes k}$ (follows from the fact that $\mathcal{A}_{-\chi_0}\subset U(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\chi_0)}$ is maximal that was proved in [43, Proposition 9.12]). □

4.14 Simplicity of spectra of $B(X)$ on a tensor product of KR modules

In this section we recall the main results of [50] that will be used later.

Pick $k \geqslant 1$ and let $V_{a_1\varpi_{b_1}}, \ldots, V_{a_k\varpi_{b_k}}$ be irreducible polynomial representations of $\mathfrak{gl}_n$, corresponding to rectangular Young diagrams. Let $\underline{z} = (z_1, \ldots, z_k)$ be a k-tuple of (distinct) complex numbers $z_i \in \mathbb{C}$, later we will put some restrictions on $\mathrm{Re}(z_j)$. We denote by $V_{a_j\varpi_j}(z_j)$ the corresponding irreducible representations of $Y(\mathfrak{gl}_n)$ (via $\mathbf{ev}_{z_j}$). The representations $V_{a_j\varpi_j}(z_j)$ are called Kirillov-Reshetikhin modules. Recall that to every $C \in G$, we can associate Bethe subalgebra $B(C) \subset Y(\mathfrak{sl}_n)$ (see Section 4.10.1) so we obtain the action $B(C) \curvearrowright V_1(z_1) \otimes \ldots \otimes V_k(z_k)$ (via $(\mathbf{ev}_{z_1} \otimes \ldots \otimes \mathbf{ev}_{z_k}) \circ \Delta^k$).

The following proposition was proved in [50].

Proposition 4.14.1 *For $C \in S$ and $\mathrm{Re}(z_j) = -b_j - n + a_j$ the algebra $B(C)$ acts on the tensor product $V_{a_1\varpi_{b_1}}(z_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(z_k)$ via normal operators.*

The following proposition holds by [50, Section 4.3].

Proposition 4.14.2 *Fix distinct $z_1, \ldots, z_k \in \mathbb{C}$ and $d_1, \ldots, d_k \in \mathbb{C}$. For almost every $s \in \mathbb{C}$ the action of $B(X)$ on $V_{a_1\varpi_{b_1}}(sz_1 + d_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(sz_k + d_k)$ has a cyclic vector for $X = C \in \overline{T}^{\mathrm{reg}}$ and $X = (C_0, \chi)$.*

Remark 4.14.3 Proposition 4.14.2 remains true for any $X \in \overline{M_{0,n+2}}$.

As a corollary of Proposition 4.14.1 and Proposition 4.14.2, we obtain the following proposition.

Proposition 4.14.4 *Fix distinct $z_1, \ldots, z_k \in i\mathbb{R}$ and let $d_j := -b_j - n + a_j$. Then the set of $s \in \mathbb{R}$ such that the action of $B(X)$ on $V_{a_1\varpi_{b_1}}(sz_1 + d_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(sz_k + d_k)$ does not have a simple spectrum for some $X = C \in \overline{S}^{\mathrm{reg}}$ or $X = (C_0, \chi)$ is finite.*

Remark 4.14.5 In the Proposition 4.14.4 one can replace X by any $X \in \overline{M_{0,n+2}}$.

From now on we fix distinct $z_1, \ldots, z_k \in i\mathbb{R}$. Using Proposition 4.14.4 we get

Corollary 4.14.6 *There exists a real number $N \in \mathbb{R}$ such that for every $s \in \mathbb{R}_{\geqslant N}$ and $X \in \overline{S}^{\mathrm{reg}}$ or $X = (C_0, \chi)$ the action of $B(X)$ on $V_{a_1\varpi_{b_1}}(sz_1 + d_1) \otimes \ldots \otimes V_{a_k\varpi_{b_k}}(sz_k + d_k)$ has a simple spectrum.*

4.15 Further properties of the two-parametric family $B_\epsilon(\exp(\epsilon(\chi_0 + \varepsilon\chi)))$

In this section we construct certain families relating images in $\mathrm{End}(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_k})$ of Bethe algebras and inhomogeneous Gaudin algebras. We consider T as a subset of $\mathfrak{h}$.

Lemma 4.15.1 *We have* $\mathbf{ev}_z(\tilde{B}(C)) = \tilde{\mathcal{A}}_{C^{-1}}$.

Proof: Follows from [92, Section 2], see also [50]. $\square$

Proposition 4.15.2 *For* $\chi \in \mathfrak{h}_0^{\mathrm{reg}}$ *we have*

$$\lim_{\varepsilon\to 0} B_\varepsilon(\exp(\chi)) = U(\mathfrak{h}_0[t]).$$

Moreover, for Weil generic $\chi \in \mathfrak{h}_0^{\mathrm{reg}}$ *(i.e. for* χ *in the complement to a certain countable union of Zariski closed subsets of* $\mathfrak{h}_0^{\mathrm{reg}}$*) the two-parametric family* $B_\varepsilon(\exp(\epsilon\chi))$ *depending on* $\epsilon, \varepsilon \in \mathbb{C}^\times$ *extends to the continuous two-parametric family on the blowup* $\mathrm{Bl}_{(0,0)}\,\mathbb{C}^2$ *as follows (here* $\epsilon \in \mathbb{C}$, $\varepsilon, c \in \mathbb{C}^\times$*):*

$$((\epsilon,0),[1:0]) \mapsto U(\mathfrak{h}_0[t]),\ ((0,\varepsilon),[0:1]) \mapsto B_\varepsilon(1)\cdot\mathcal{A}_\chi,$$
$$((0,0),[c:1]) \mapsto \mathcal{A}^{\mathrm{u}}_{c\chi},\ ((0,0),[0:1]) \mapsto \mathcal{A}^{\mathrm{u}}_0\cdot\mathcal{A}_\chi.$$

Proof: It follows from Lemma 4.13.2 that $\mathfrak{h}_0 \subset B(\exp(\epsilon\chi))$. Since $\deg_2 \mathfrak{h}_0 = 0$ we conclude that $\mathfrak{h}_0$ lies in every limit algebra of the family $B_\varepsilon(\exp(\epsilon\chi))$. Let us also note that according to [48] the algebra $B(\exp(\epsilon\chi))$ contains elements

$$\sigma_i(\epsilon) := 2J(t_{\varpi_i}) - \sum_{\alpha\in\Delta_+} \frac{e^\alpha(\exp(\epsilon\chi))+1}{e^\alpha(\exp(\epsilon\chi))-1}(\alpha,\alpha_i)x^+_\alpha x^-_\alpha.$$

We denote by $\sigma_{i,\varepsilon}(\epsilon)$ the corresponding elements of $B_\varepsilon(\exp(\epsilon\chi))$. We see that the the family $\sigma_{i,\varepsilon}(\epsilon)$ extends to $\mathrm{Bl}_{(0,0)}\,\mathbb{C}^2$ as follows:

$$((\epsilon,0),[1:0]) \mapsto 2t_{\varpi_i}[1],\ ((0,\varepsilon),[0:1]) \mapsto -\sum_{\alpha\in\Delta_+}\frac{(\alpha,\alpha_i)}{(\alpha,\chi)}x^+_\alpha x^-_\alpha,$$

$$((0,0),[c:1]) \mapsto 2t_{\varpi_i}[1] - \sum_{\alpha\in\Delta_+}\frac{(\alpha,\alpha_i)}{c(\alpha,\chi)}x^+_\alpha x^-_\alpha,$$

We conclude that every limit algebra at the point $((\epsilon,0),[1:0])$, $\epsilon\in\mathbb{C}$ contains $\mathfrak{h}_0[1]$ so lies in $Z_{U(\mathfrak{g}[t])}(\mathfrak{h}_0[1]) = U(\mathfrak{h}_0[t])$, hence, coincides with $U(\mathfrak{h}_0[t])$ since $\dim_{F_1} U(\mathfrak{h}_0[t]) = \dim_{F_1} B_\varepsilon(\exp(\epsilon\chi))$. Similarly, every limit algebra at the point $((0,\varepsilon),[0:1])$ clearly contains $B(1)$ and also contains the linear combinations of elements $\sum_{\alpha\in\Delta_+}\frac{(\alpha,\alpha_i)}{(\alpha,\chi)}x_\alpha x^-_\alpha$ i.e. the quadratic part of $\mathcal{A}_\chi$ (see [115]). Note also that the centralizer of $U(\mathfrak{g})^{\mathfrak{g}}$ in $Y(\mathfrak{g})$ is equal to $Y(\mathfrak{g})^{\mathfrak{g}}\cdot U(\mathfrak{g}) = Y(\mathfrak{g})^{\mathfrak{g}}\otimes_{U(\mathfrak{g})^{\mathfrak{g}}} U(\mathfrak{g})$ (see [53, Lemma 6.14]). Since for Weil generic $\chi\in\mathfrak{h}_0^{\mathrm{reg}}$ the algebra $\mathcal{A}_\chi$ is equal to the centralizer of its quadratic part (see [97]) we then conclude that for Weil generic χ every limit algebra at the point $((0,\varepsilon),[0:1])$ is contained in $B_\varepsilon(1)\cdot\mathcal{A}_\chi$, hence, is equal to $B_\varepsilon(1)\cdot\mathcal{A}_\chi$ since $\dim_{F_1} B_\varepsilon(\exp(\epsilon\chi)) = \dim_{F_1}(B_\varepsilon(1)\cdot\mathcal{A}_\chi)$ (compare with the proof of [53, Theorem 6.13]). Similarly, every limit algebra at the point $((0,0),[0:1])$ contains the elements $\sum_{\alpha\in\Delta_+}\frac{(\alpha,\alpha_i)}{(\alpha,\chi)}x_\alpha x^-_\alpha$ and also the element $\tilde{\omega}_0$ so coincides with $\mathcal{A}^{\mathrm{u}}_0\cdot\mathcal{A}_\chi$ (here we use the maximality of $\mathcal{A}^{\mathrm{u}}_0\cdot\mathcal{A}_\chi \subset U(\mathfrak{g}[t])$ which follows from Proposition 4.5.5). Finally, every limit algebra at the point $((0,0),[c:1])$ contains the element $\tilde{\omega}_{c\chi}$ therefore is contained in the centralizer $Z_{U(\mathfrak{g}[t])}(\tilde{\omega}_{c\chi})$ that is equal to $\mathcal{A}^{\mathrm{u}}_{c\chi}$ by Remark 4.6.7. The argument as in the end of the proof of Lemma 4.13.6 finishes the proof. $\square$

Remark 4.15.3 We expect that Proposition 4.15.2 is true for every $\chi \in \mathfrak{h}_0^{\mathrm{reg}}$. To prove this it is enough to show that the algebra $\mathcal{A}_\chi$ coincides with the centralizer of its quadratic part for every $\chi \in \mathfrak{h}_0^{\mathrm{reg}}$. Note also that we are not claiming that the family constructed in Proposition 4.15.2 is holomorphic (this should be true but we do not need this for our purposes).

Corollary 4.15.4 *For Weil generic $\chi \in i\mathfrak{h}_{0,\mathbb{R}}^{\mathrm{reg}}$ (i.e. for χ in the complement to a certain countable union of Zariski closed subsets in $i\mathfrak{h}_{0,\mathbb{R}}^{\mathrm{reg}}$) the two-parametric family $B_\varepsilon(\exp(\epsilon\chi))$ depending on $\epsilon, \varepsilon \in \mathbb{R}^\times$ extends to the two-parametric family on the blowup $\mathrm{Bl}_{(0,0)}\,\mathbb{R}^2$ as follows (here $\epsilon \in \mathbb{R}$, $\varepsilon, c \in \mathbb{R}^\times$):*

$$((\epsilon, 0), [1:0]) \mapsto U(\mathfrak{h}_0[t]),\ ((0,\varepsilon), [0:1]) \mapsto B_\varepsilon(1)\cdot\mathcal{A}_\chi,$$

$$((0,0),[c:1]) \mapsto \mathcal{A}^{\mathrm{u}}_{c\chi},\ ((0,0),[0:1]) \mapsto \mathcal{A}^{\mathrm{u}}_0\cdot\mathcal{A}_\chi.$$

Let us now introduce the notation. Recall that we have

$$\mathrm{Bl}_{(0,0)}\,\mathbb{R}^2 = \{((a,b),[c:d]) \in \mathbb{R}^2 \times \mathbb{RP}^1 \mid ad = bc\}.$$

We consider $\mathrm{Bl}_{(0,0)}\,\mathbb{R}^2$ as a topological space (with the standard induced topology on it). We denote by $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R}^\times))$ the following topological space:

$$\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R}^\times)) := \{((a,b),[c:d]) \in (\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R}^\times)) \times \mathbb{RP}^1 \mid ad = bc\}$$

Remark 4.15.5 Note that the topological space $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R}^\times))$ is not an algebraic variety (it is a constructible set).

We also consider the following subset of $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R}^\times))$:

$$\begin{aligned}\mathrm{Bl}_{(0,0)}(\mathbb{R}\times[0,1/N) \setminus (\{0\}\times(0,1/N)))) &:= \\ &= \{((a,b),[c:d]) \in (\mathbb{R}\times[0,1/N) \setminus (\{0\}\times(0,1/N))) \times \mathbb{RP}^1 \mid ad = bc\}. \quad (4.30)\end{aligned}$$

To simplify notation, we set $K := \mathbb{R}\times[0,1/N) \setminus (\{0\}\times(0,1/N))$ and will denote the space (4.30) by $\mathrm{Bl}_{(0,0)}\,K$.

We are now ready to prove the proposition that describes a partial compactification of the family $\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\chi)) \to \mathrm{End}(V_1\otimes\ldots\otimes V_k))$. Note that if representations $V_1,\ldots,V_k$ have no multiplicities for the action of $\mathfrak{h}_0$ then the proof of Proposition 4.15.6 below simplifies. Indeed, if this is the case then $\mathrm{Im}(U(\mathfrak{h}_0)^{\otimes k} \to \mathrm{End}(V_1\otimes\ldots\otimes V_k))$ is already the maximal commutative subalgebra of $\mathrm{End}(V_1\otimes\ldots\otimes V_k)$ so the statement of Proposition 4.15.6 can be deduced from Corollary 4.15.4.

Proposition 4.15.6 *For $z_1,\ldots,z_k \in i\mathbb{R}$ and $\chi \in i\mathfrak{h}_{0,\mathbb{R}}^{\mathrm{reg}}$ we have ($\varepsilon \in (0,1/N)$)*

$$\lim_{\varepsilon\to 0}\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\chi)) \to \mathrm{End}(V_1\otimes\ldots\otimes V_k)) = \mathrm{Im}(\mathcal{A}^{\otimes k}_{\exp(-\chi)} \to \mathrm{End}(V_1\otimes\ldots\otimes V_k)).$$

Moreover, for Weil generic $\chi \in i\mathfrak{h}_{0,\mathbb{R}}^{\mathrm{reg}}$ the two-parametric family

$$\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\epsilon\chi)) \to \mathrm{End}(V_1\otimes\ldots\otimes V_k))$$

depending on $\epsilon \in \mathbb{R}^\times$, $\varepsilon \in (0,1/N)$ extends to the (continuous) two-parametric family on $\mathrm{Bl}_{(0,0)} K$ as follows (here $c \in \mathbb{R}^\times$, $\epsilon \in (0,1/N)$):

$$((\epsilon,0),[1:0]) \mapsto \mathrm{Im}(\mathcal{A}^{\otimes k}_{\exp(-\epsilon\chi)} \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),$$

$$((0,0),[1:0]) \mapsto \mathrm{Im}(\mathcal{A}^{\otimes k}_{\chi} \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),$$
$$((0,0),[c:1]) \mapsto \mathrm{Im}(\mathcal{A}_{-c\chi}(\underline{z}) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),$$
$$((0,0),[0:1]) \mapsto \mathrm{Im}(\mathcal{A}_0(\underline{z}) \cdot \Delta^k(\mathcal{A}_\chi) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)).$$

Proof: Consider the topological space

$$\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\mathbb{R}^\times \times \{0\} \cup \{0\} \times \mathbb{R}^\times)) := \\ = \{((a,b),[c:d]) \in (\mathbb{R}^2 \setminus (\mathbb{R}^\times \times \{0\} \cup \{0\} \times \mathbb{R}^\times)) \times \mathbb{RP}^1 \,|\, ad = bc\}.$$

Let us first of all note that $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}} B(\exp(\epsilon\chi)) = \mathbf{ev}_{\varepsilon;\underline{z}+\varepsilon\underline{d}} B_\varepsilon(\exp(\epsilon\chi))$. Note also that by our assumptions on V_i the algebra $\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}} B(\exp(\epsilon\chi)) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k))$ is maximal commutative subalgebra of $\mathrm{End}(V_1 \otimes \ldots V_k)$ of dimension $\dim(V_1 \otimes \ldots \otimes V_k)$. Recall that the family $B_\varepsilon(\exp(\epsilon\chi))$ extends to $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2)$. We claim that the family of morphisms $\mathbf{ev}_{\varepsilon;\underline{z}+\varepsilon\underline{d}}$ also extends to $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2)$ by $\mathrm{ev}_{\underline{z}}$. Indeed, note that $\mathbf{ev}_{\varepsilon;\underline{z}+\varepsilon\underline{d}}$ does not depend on ϵ and $B_\varepsilon(\exp(\epsilon\chi)) \subset Y_\varepsilon(\mathfrak{g})$ so the claim easily follows from Lemma 4.10.5 (see Corollary 4.10.6). Note that the images of $\mathcal{A}_{-c\chi}(\underline{z})$, $(\mathcal{A}_0(\underline{z}) \cdot \Delta^k(\mathcal{A}_\chi))$ in $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$ are maximal commutative subalgebras (follows from [43, Corollary 11.9] using Lemma 4.10.2).

Taking the image in $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$ of the family considered in Corollary 4.15.4 and using that $\mathrm{Bl}_{(0,0)} \mathbb{R}^2$ is Noetherian (so the image of the whole family coincides with the image of a large enough filtration term of it) we conclude that the two-parametric family

$$\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}} B(\exp(\epsilon\chi)) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k))$$

depending on $(\epsilon,\varepsilon) \in (\mathbb{R}^\times)^2$ extends to the two-parametric family on $\mathrm{Bl}_{(0,0)}(\mathbb{R}^2 \setminus (\mathbb{R}^\times \times \{0\} \cup \{0\} \times \mathbb{R}^\times)) \setminus \{((0,0),[1:0])\}$ as follows (here $c \in \mathbb{R}^\times$, $\epsilon \in \mathbb{R}$):

$$((0,0),[c:1]) \mapsto \mathrm{Im}(\mathcal{A}_{-c\chi}(\underline{z}) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),$$

$$((0,0),[0:1]) \mapsto \mathrm{Im}((\mathcal{A}_0(\underline{z}) \cdot \Delta^k(\mathcal{A}_\chi)) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k).$$

It remains to show that this family extends to $\mathrm{Bl}_{(0,0)} K$ by setting

$$((\epsilon,0),[1:0]) \mapsto \mathrm{Im}(\mathcal{A}^{\otimes k}_{\exp(-\epsilon\chi)} \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),$$

$$((0,0),[1:0]) \mapsto \mathrm{Im}(\mathcal{A}^{\otimes k}_{\chi} \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)).$$

Note that the images of algebras $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(\exp(\epsilon\chi)))$, $\mathcal{A}^{\otimes k}_{\exp(-\epsilon\chi)}$, $\mathcal{A}^{\otimes k}_{\chi}$, $\mathcal{A}_{-c\chi}(\underline{z})$, $(\mathcal{A}_0(\underline{z}) \cdot \Delta^k(\mathcal{A}_\chi))$ in $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$ are maximal commutative so we can replace algebras above by the corresponding algebras for $\mathfrak{gl}_n$. Consider the following open subset $U \subset \mathrm{Bl}_{(0,0)} K$:

$$U := \{((\epsilon, c\epsilon),[1:c])\} \subset \mathrm{Bl}_{(0,0)} K.$$

Recall now the generators $r_{k,z_i,l,\epsilon,\varepsilon}$ of the algebra $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(\tilde{B}(\exp(\epsilon\chi)))$ (see Section 4.11). Recall also the generators $r^0_{k,z_i,l,\xi}$ of the algebra $\tilde{\mathcal{A}}_\xi(z_1,\dots,z_k)$, here $\chi \in \mathfrak{h}_0$. Let us first of all note that the family $r_{k,u_i,l,\epsilon,\varepsilon}$ extends continuously to the family on $U \setminus \{((\epsilon,0),[1:0])\}$ by setting

$$((0,c\epsilon),[1:c]) \mapsto r^0_{k,z_i,l,-c^{-1}\chi} \in \mathcal{A}_{-\chi}(c^{-1}z_1,\dots,c^{-1}z_k).$$

To see this, recall that $\varepsilon = c\epsilon$ and note that we have the equality (compare with the proof of Proposition 4.11.5):

$$\lim_{\epsilon\to 0} \epsilon^{-n}\,\mathrm{cdet}(e^{-\epsilon\partial_u}(1+L_{\underline{z}/\varepsilon+\underline{d}}(u/\epsilon)) - \exp(-\epsilon\chi)) = \mathrm{cdet}(L_{c^{-1}\underline{z}}(u) - \partial_u + \chi). \tag{4.31}$$

The equality (4.31) implies that $\lim_{\epsilon\to 0} b_{k,l,\epsilon,\varepsilon}(u) = b^0_{k,l,c^{-1}\underline{z},-\chi}(u)$, hence,

$$\lim_{\epsilon\to 0} r_{k,z_i,l,\epsilon,\varepsilon} = \lim_{\epsilon\to 0}\sum_{j=1}^k \mathrm{res}_{u=c^{-1}z_i+\epsilon j+\epsilon d_i}(u-z_i)^l b_{k,\epsilon,\varepsilon}(u)du =$$
$$= \mathrm{res}_{u=c^{-1}z_i}(u-z_i)^l b^0_{k,l,c^{-1}\underline{z},-\chi} = r^0_{k,c^{-1}z_i,l,-\chi}.$$

It remains to show that the families $r_{k,z_i,l,\epsilon,\varepsilon}$ extend to the points $((\epsilon,0),[1:0])$ and generate algebras $\tilde{\mathcal{A}}^{\otimes k}_{\exp(-\epsilon\chi)}$, $\tilde{\mathcal{A}}^{\otimes k}_{\chi}$. Indeed, recall that the elements $r_{k,z_i,l,\epsilon,\varepsilon}$ are sums of residues of the elements $(u-z_i)^l b_{k,\epsilon,\varepsilon}$ at the points $c^{-1}z_i+\epsilon j+\epsilon d_i$. Recall also that the generating function for $b_{k,\epsilon,\varepsilon}$ can be written in the following form:

$$\epsilon^{-n}\,\mathrm{tr}\,A_n(e^{-\epsilon\partial_u}(1+L_{\underline{z}/\varepsilon+\underline{d}}(u))_1 - \exp(-\epsilon\chi)_1)\dots(e^{-\epsilon\partial_u}(1+L_{\underline{z}/\varepsilon+\underline{d}}(u))_n - \exp(-\epsilon\chi)_n) =$$
$$= \epsilon^{-n}\sum_{a=0,\dots,n}(-1)^{n-a}\binom{n}{a}\mathrm{tr}\,A_n(1+\frac{\epsilon E_1^{(1)}}{u-c^{-1}z_1-\epsilon d_1})\dots(1+\frac{\epsilon E_a^{(1)}}{u-c^{-1}z_1-\epsilon(d_1+a-1)})\dots$$
$$\dots(1+\frac{\epsilon E_1^{(k)}}{u-c^{-1}z_k-\epsilon d_k})\dots(1+\frac{\epsilon E_a^{(k)}}{u-c^{-1}z_k-\epsilon(d_k+a-1)})(\exp(\epsilon\chi))_{a+1}\dots(\exp(\epsilon\chi))_n e^{-a\epsilon\partial_u}$$
$$= \sum_{k\geqslant 0} b_{k,\epsilon,\varepsilon}(u)\partial_u^k.$$

Since $c^{-1}\to\infty$ and ϵ is clearly bounded we see that at the point $((\epsilon,0),[1:0])$ the family $r_{k,z_i,l,\epsilon,\varepsilon}$ converges to the same element as the family $r^{(i)}_{1,z_i,l,\epsilon,\varepsilon} = 1\otimes\ldots\otimes 1\otimes \underset{i}{r_{1,z_i,l,\epsilon,\varepsilon}}\otimes 1\otimes\ldots\otimes 1$. So we can assume that $k=1$ and then we are dealing with the family $\mathbf{ev}_{z_i/\varepsilon+d_i}(\tilde{B}(\exp(\epsilon\chi)))$ that by Lemma 4.15.1 is exactly $\tilde{\mathcal{A}}_{\exp(-\epsilon\chi)}$. It remains to show that the limits as $c\to 0$ of the elements $r_{1,z_i,l,\epsilon,\varepsilon}$ generate $\tilde{\mathcal{A}}_{\exp(-\epsilon\chi)}$ for $\epsilon\neq 0$ and generate $\mathcal{A}_{-\chi}$ as $\epsilon\to 0$. Indeed, directly from the definitions the elements $r_{1,z_i,l,\epsilon,\varepsilon}$ do not depend on c, d_i (compare with Remark 4.11.9). Note also that as $\epsilon\to 0$ we have

$$\lim_{\epsilon\to 0}\mathrm{cdet}\,\epsilon^{-1}(e^{-\epsilon\partial_u}(1+L_0(u)) - \exp(\epsilon\chi)) = L_0(u) - \partial_u + \chi.$$

This observation finishes the proof.

□

Remark 4.15.7 We expect that for every $\chi \in \mathfrak{h}_0^{\mathrm{reg}}$ the family $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\epsilon\chi))$ extends to the blowup $\mathrm{Bl}_{(0,0)}\,\mathbb{C}^2$ by setting (here $\epsilon, \varepsilon, c \in \mathbb{C}^\times$)

$$((\epsilon,0),[1:0]) \mapsto \mathcal{A}^{\otimes k}_{\exp(-\epsilon\chi)},\ ((0,\varepsilon),[1:0]) \mapsto \mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(1))\cdot\mathcal{A}_\chi$$

$$((0,0),[c:1]) \mapsto \mathcal{A}_{-c\chi}(\underline{z}),\ ((0,0),[0:1]) \mapsto \mathcal{A}_0(\underline{z})\cdot\Delta^k(\mathcal{A}_\chi),\ ((0,0),[1:0]) \mapsto \mathcal{A}_\chi^{\otimes k}.$$

For example, it follows from our proof of Proposition 4.15.6 that for a fixed $\epsilon \in \mathbb{C}^\times$ we have $\lim\limits_{\varepsilon\to 0}\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(\exp(\epsilon\chi))) = \mathcal{A}^{\otimes k}_{\exp(-\epsilon\chi)}$. It is also easy to show that $\lim\limits_{\epsilon\to 0}\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\epsilon\chi)) \supset \mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(1))\cdot\mathcal{A}_\chi$ but the equality is not clear. We expect that $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(1))\cdot\mathcal{A}_\chi \subset U(\mathfrak{g})^{\otimes k}$ is the maximal commutative subalgebra and then the equality above will follow. Another problem with extending the family $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\epsilon\chi))$ to the blowup $\mathrm{Bl}_{(0,0)}\,\mathbb{C}^2$ is that we do not know Poincaré series of the algebras $\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}B(\exp(\epsilon\chi))$ so the method similar to the one used in the proof of Proposition 4.15.6 (see also the proof of [43, Proposition 10.17]) cannot be applied.

4.16 Crystal structure on $\mathcal{E}_C(\underline{\lambda})$

Let us now pick $k \geqslant 1$ and let $V_1,\ldots,V_k$ be irreducible polynomial representations of $\mathfrak{gl}_n$ with highest weights $\lambda_1,\ldots,\lambda_k$ respectively. Let us also fix distinct $z_1,\ldots,z_k \in i\mathbb{R}$. We are not assuming that V_j correspond to rectangular Young diagrams here. For our purposes it is enough to assume that Corollary 4.14.6 holds for tensor products of V_j. More precisely, we assume the following.

Assumption 4.16.1 *We assume that there exist $d_1,\ldots,d_k \in \mathbb{C}$ and $N \in \mathbb{R}$ such that for every $s \in \mathbb{R}_{\geqslant N}$, $X \in \overline{S}^{\mathrm{reg}}$ or $X = (C_0,\chi)$ and $1 \leqslant m_1 < \ldots < m_p \leqslant k$ the action of $B(X)$ on $V_{m_1}(sz_{m_1}+d_{m_1})\otimes\ldots\otimes V_{m_p}(sz_{m_p}+d_{m_p})$ has a simple spectrum.*

Remark 4.16.2 By the results of Section 4.14 this assumption holds automatically for V_j, corresponding to rectangular Young diagrams. We will see in Remark 4.17.2 that this assumption actually implies that V_j correspond to rectangular Young diagrams.

Recall that we have fixed z_i, d_i. For simplicity, let us denote the representation $V_1(sz_1+d_1)\otimes\ldots\otimes V_k(sz_k+d_k)$ of $Y(\mathfrak{g})$ by $\underline{V}(s)$, $s \in \mathbb{R}_{\geqslant N}$. For $X = C \in \overline{S}^{\mathrm{reg}}$ or $X = (C_0,\chi)$ let $\mathcal{E}_X(\underline{\lambda},s)$ be the set of eigenlines of $B(X)$ acting on $\underline{V}(s)$. Note that the set $\mathbb{R}_{\geqslant N}$ is contractible so simply-connected, hence, $\mathcal{E}_X(\underline{\lambda},s)$ are canonically identified for all $s \in \mathbb{R}_{\geqslant N}$ and we denote this set simply by $\mathcal{E}_X(\underline{\lambda})$.

Our goal for now is to define on $\mathcal{E}_C(\underline{\lambda})$ the structure of $\hat{\mathfrak{sl}}_n$-crystal (see Definition 4.9.2).

We pick $\hat{w} \in \widehat{W}^{\mathrm{ext}}$ and consider the corresponding alcove $Q_{\hat{w}} \supset Q^{\mathrm{reg}}_{\hat{w}}$. Taking $\chi \in Q^{\mathrm{reg}}_{\hat{w}}$ such that $C = \exp(2\pi i\chi)$ we define a crystal structure on $\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda})$ as follows. Pick $j = 1,\ldots,n$ and consider the corresponding wall $H^{\hat{w}}_j$ and pick a generic element χ_0 lying on the intersection of this wall with $Q_{\hat{w}}$. Set $C := \exp(2\pi i\chi)$, $C_0 := \exp(2\pi i\chi_0)$ and consider the algebra $B(C_0,\chi)$ acting on $\underline{V}(s)$. Note that we have the identification $\mathcal{E}_C(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{(C_0,\chi)}(\underline{\lambda})$. Recall that by Proposition 4.13.3 the algebra $B(C_0,\chi)$ is generated by $B(C_0)$ together with the element h, corresponding to the wall $H^{\hat{w}}_j$. We can then decompose $\underline{V}(s)$ as $B(C_0)$-module in the direct sum of weight spaces $\underline{V}(s) = \bigoplus_{\eta\colon B(C_0)\to\mathbb{C}} \underline{V}(s)^\eta$. The action of $B(C_0,\chi)$ on $\underline{V}(s)$ has a simple spectrum so the action $h \curvearrowright \underline{V}(s)^\eta$ has a simple spectrum i.e. $\underline{V}(s)^\eta = \bigoplus_{i\in\mathbb{Z}} \underline{V}(s)^\eta_i$ with $\underline{V}(s)^\eta_i$ being one-dimensional. Moreover, note that $\mathfrak{h}_0 \subset B(C_0,\chi)$ so $\mathfrak{h}_0$ acts on each $\underline{V}(s)^\eta_i$ via some weight $\mu \in \mathfrak{h}_0^*$. Since $\underline{V}(s)^\eta_i$ is one-dimensional it can be considered as an element of $\mathcal{E}_C(\underline{\lambda})$ and we can define

$$\mathrm{e}_{[j]}(\underline{V}(s)^\eta_i) := \underline{V}(s)^\eta_{i+1},\ \mathrm{f}_{[j]}(\underline{V}(s)^\eta_i) := \underline{V}(s)^\eta_{i-1},\ \mathrm{wt}(\underline{V}(s)^\eta_i) = \mu, \tag{4.32}$$

where $\mathrm{e}_{[j]}(\underline{V}(s)^\eta_i) := 0$ if $\underline{V}(s)^\eta_{i+1} = 0$, similarly, $\mathrm{f}_{[j]}(\underline{V}(s)^\eta_i) := 0$ if $\underline{V}(s)^\eta_{i-1} = 0$.

Let us now note that the definition of $\mathrm{e}_{[j]}, \mathrm{f}_{[j]}$ does not depend on the choice of χ_0 because the subset of $Q_{\widehat{w}}$, consisting of regular elements and subregular elements of $H_j^{\widehat{w}}$ is simply-connected (we also use the Main Theorem of [51], where the parametrizing space of limits of Bethe algebras is described). We will denote by $\mathcal{E}^{\hat{w}}_{\exp(2\pi i\chi)}(\underline{\lambda})$ the set $\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda})$ with a structure (4.32) defined as above.

Lemma 4.16.3 *Pick $\sigma \in S_n$ and $(m_1,\ldots,m_n)$, $(m_1',\ldots,m_n') \in \overline{\Lambda}$. Set $\hat{w} = (\sigma; m_1,\ldots,m_n)$, $\hat{w}' = (\sigma; m_1',\ldots,m_n')$. Pick also $\chi \in Q^{\mathrm{reg}}_{\hat{w}}$, $\chi' \in Q^{\mathrm{reg}}_{\hat{w}'}$. Then there is a bijection of sets $\mathcal{E}^{\hat{w}_1}_{\exp(2\pi i\chi_1)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\hat{w}_2}_{\exp(2\pi i\chi_2)}(\underline{\lambda})$ that commutes with $\mathrm{e}_{[j]}, f_{[j]}, \mathrm{wt}$.*

Proof: The map $\exp\colon \overline{\mathfrak{h}}_{\mathbb{R}} \to \overline{S}$ sends alcoves $Q_{\hat{w}}$, $Q_{\hat{w}'}$ to the same subset of $\overline{S}$. The image of $H_j^{\hat{w}}$ coincides with the image of $H_j^{\hat{w}'}$. The claim follows. □

Let $\mathcal{E}^{\hat{w},\mathrm{fin}}_{\exp(2\pi i\chi)}(\underline{\lambda})$ be the set $\mathcal{E}^{\hat{w}}_{\exp(2\pi i\chi)}(\underline{\lambda})$ together with operations $\mathrm{e}_{[j]}, \mathrm{f}_{[j]}$, $j = 1,\ldots,n-1$, wt (i.e. we forget $\mathrm{e}_{[n]}, \mathrm{f}_{[n]}$).

Proposition 4.16.4 *Pick $\sigma \in S_n$. There is a canonical isomorphism $\mathcal{E}^{\sigma,\mathrm{fin}}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi}(\underline{\lambda})$ compatible with $\mathrm{e}_j, \mathrm{f}_j, \mathrm{wt}$.*

Proof: Consider the family $B_\epsilon(\exp(2\pi i\epsilon\chi))$, $\epsilon \in (0,1/N)$. Note that by Theorem 4.8.12 the limit of this family to $\epsilon = 0$ is $\mathcal{A}^{\mathrm{u}}_{2\pi i\chi}$. It follows from Lemma 4.10.5 and Corollary 4.10.6 that the family of homomorphisms $\mathbf{ev}_{\epsilon;z_1+\epsilon d_1,\ldots,z_k+\epsilon d_k}$ ($\epsilon \in \mathbb{R}$) extends to $\epsilon = 0$ via $\mathrm{ev}_{\underline{z}}$. Taking the image of the family above in $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$, we obtain the family of commutative subalgebras of $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$. Every algebra of our family above acts on $V_1 \otimes \ldots \otimes V_k$ with a simple spectrum (for $\epsilon \in (0,1/N)$ this follows from the Assumption 4.16.1, for $\epsilon = 0$ this follows from Proposition 4.4.10 together with [28], see also [43, Secion 11]). Using Lemma 4.10.2, we obtain the identification of sets $\mathcal{E}^{\sigma}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi}(\underline{\lambda})$.

We now pick $[j] = [1],\ldots,[n-1]$ and our goal is to show that the operators $\mathrm{e}_{[i]}, \mathrm{f}_{[i]}, \mathrm{wt}$ are compatible with the identification above. To see that, we can consider the two-parametric family $B_\epsilon(\exp(\epsilon(\chi_0 + \varepsilon\chi)))$, $\epsilon \in (0,1/N)$, $\varepsilon \in \mathbb{R}^\times$ that by Lemma 4.13.6 extends to a family depending on $\epsilon \in [0,1/N)$, $\varepsilon \in \mathbb{R}$ as follows:

$$(\epsilon,0) \mapsto B_\epsilon(\exp(\epsilon\chi_0),\chi),\ (0,\varepsilon) \mapsto \mathcal{A}^{\mathrm{u}}_{\chi_0+\varepsilon\chi},\ (0,0) \mapsto \mathcal{A}^{\mathrm{u}}_{(\chi_0,\chi)}.$$

Recall that the family of homomorphisms $\mathbf{ev}_{\epsilon;z_1+\epsilon d_1,\ldots,z_k+\epsilon d_k}$ ($\epsilon \in (0,1/N)$) extends to $\epsilon = 0$ via $\mathrm{ev}_{\underline{z}}$ so we can consider the image of the family above (with $\epsilon \in [0,1/N)$, $\varepsilon \in \mathbb{R}$) in $\mathrm{End}(V_1 \otimes \ldots \otimes V_k)$. Every algebra in the image acts on $V_1 \otimes \ldots \otimes V_k$ with a simple spectrum: for $(\varepsilon,\epsilon) \in \mathbb{R} \times (0,1/N)$ this follows from Assumption 4.16.1, for $\epsilon = 0$ this follows from [43, Section 11] together with Propositions 4.4.10, 4.13.9. The claim follows since every two paths on $\mathbb{R} \times [0,1/N)$ from $(1,1)$ to $(0,0)$ are homotopic. □

Remark 4.16.5 Recall that for $x \in Y(\mathfrak{g})$ we have $\mathbf{ev}_{(\epsilon; z_1+\epsilon d_1, \ldots, z_r+\epsilon d_r)}(\psi_\epsilon(x)) = \mathrm{ev}_{z_1/\epsilon+d_1, \ldots, z_r/\epsilon+d_r}(x)$. We use this in the proof of Proposition 4.16.4.

For $[j] \in \mathbb{Z}/n\mathbb{Z}$ we denote by $\mathcal{E}^{\hat{w},[j]}_{\exp(2\pi i\chi)}(\underline{\lambda})$ the set $\mathcal{E}^{\hat{w}}_{\exp(2\pi i\chi)}(\underline{\lambda})$ together with operations $e_{[1]}, \ldots, e_{[j-1]}, e_{[j+1]}, \ldots, e_{[n]}$, $f_{[1]}, \ldots, f_{[j-1]}, f_{[j+1]}, \ldots, f_{[n]}$, wt (we forget the action of $\mathrm{e}_{[j]}, \mathrm{f}_{[j]}$). Recall that $\tau_{[j]} \in S_n$ is an element that sends $[i]$ to $[i+j]$.

Proposition 4.16.6 *For $\hat{w} = (\sigma; m_1, \ldots, m_n)$ we have an isomorphism $\mathcal{E}^{\hat{w},[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \simeq \mathcal{E}_{-\tau_{[j]}(\chi)}(\underline{\lambda})$, that identifies $\mathrm{e}_{[i]}$ with e_{i-j} and $\mathrm{f}_{[i]}$ with f_{i-j}.*

Proof: We have $\hat{w} = (\sigma; m_1, \ldots, m_n)$. Recall that $Q_{\hat{w}}$ consists of $(a_{[1]}, \ldots, a_{[n]})$ such that

$$a_{\sigma([n])} - m_{[n]} + 1 \geqslant a_{\sigma([1])} - m_{[1]} \geqslant \ldots \geqslant a_{\sigma([n])} - m_{[n]}.$$

Consider the element $\hat{w}' := \hat{w} \cdot (\tau_{[j]}, [0, \ldots, 0, \underbrace{1, \ldots, 1}_{j}]) = (\sigma \circ \tau_{[j]}, m'_1, \ldots, m'_n)$. Note that $Q_{\hat{w}} = Q_{\hat{w}'}$ and $H^{\hat{w}}_{[i]} = H^{\hat{w}'}_{[i-j]}$. We obtain the identification $\mathcal{E}^{\hat{w}}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\hat{w}'}_{\exp(2\pi i\chi)}(\underline{\lambda})$ such that the action of $\mathrm{e}_{[l]}, \mathrm{f}_{[l]}$ on $\mathcal{E}^{\hat{w}}_{\exp(2\pi i\chi)}(\underline{\lambda})$ corresponds to the action of $\mathrm{e}_{[l-j]}, \mathrm{f}_{[l-j]}$ on $\mathcal{E}^{\hat{w}'}_{\exp(2\pi i\chi)}(\underline{\lambda})$. So we obtain the identification $\mathcal{E}^{\hat{w},[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\hat{w}',\mathrm{fin}}_{\exp(2\pi i\chi)}$. It follows from Lemma 4.16.3 that we have an identification $\mathcal{E}^{\hat{w}',\mathrm{fin}}_{\exp(2\pi i\chi)} \xrightarrow{\sim} \mathcal{E}^{\sigma\circ\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\tau_{[j]}(\chi))}$. Note now that by Proposition 4.16.4 there is an isomorphism of crystals $\mathcal{E}^{\sigma\circ\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\tau_{[j]}(\chi))}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\tau_{[j]}(\chi)}(\underline{\lambda})$. After composing the identifications above we conclude that we have an isomorphism $\mathcal{E}^{\hat{w},[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\tau_{[j]}(\chi)}(\underline{\lambda})$ that sends $\mathrm{e}_{[i]} \mapsto \mathrm{e}_{[i-j]}$, $\mathrm{f}_{[i]} \mapsto \mathrm{f}_{[i-j]}$.

□

Corollary 4.16.7 *We have a canonical isomorphism $\mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \simeq \mathcal{E}_{-\tau_{[j]}(\chi)}(\underline{\lambda})$ that becomes an isomorphism of crystals if we identify $\mathrm{e}_{[i]}$ with e_{i-j} and $\mathrm{f}_{[i]}$ with f_{i-j}.*

4.17 Identification of $\mathcal{E}_C(\underline{\lambda})$ with tensor product of KR crystals

Pick $\chi \in Q_1$, $k \in \mathbb{Z}_{\geqslant 1}$. Recall that $\underline{\lambda} = \lambda_1, \ldots, \lambda_k$ is a collection of dominant weights (of $\mathfrak{gl}_n$), $z_1, \ldots, z_k \in i\mathbb{R}$ and the Assumption 4.16.1 holds.

Proposition 4.17.1 *Assume that $k = 1$. Then $\lambda = l\varpi_r$ and the crystal $\mathcal{E}_{\exp(2\pi i\chi)}(\lambda)$ coincides with the Kirillov-Reshetikhin crystal $\mathbf{B}_\lambda$, corresponding to the irreducible representation V_λ (see Section 4.9).*

Proof: Follows from Proposition 4.9.14, Corollary 4.16.7 and [43, Theorem 1.8]. □

Remark 4.17.2 Let us point out that Proposition 4.17.1, in particular, implies that if the Assumption 4.16.1 holds for $V_1, \ldots, V_k$ then each V_i is the irreducible representation, corresponding to a rectangular diagram.

Recall that in Section 4.9 the tensor product of $\hat{\mathfrak{sl}}_n$-crystals was defined. The goal of the rest of the Section is to construct the isomorphism of crystals $\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_1)\otimes \ldots \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_k)$ (see Proposition 4.17.3) and then to conclude (using Proposition 4.17.1) that we have an isomorphism $\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathbf{B}_{\lambda_1} \otimes \ldots \otimes \mathbf{B}_{\lambda_k}$ (see Theorem 4.17.4). The similar claims for the $\mathfrak{sl}_n$-crystals $\mathcal{E}_{\chi}(\underline{\lambda})$ follow from the results of [43].

Proposition 4.17.3 *For* $\operatorname{Im} z_1 \gg \operatorname{Im} z_2 \gg \ldots \gg \operatorname{Im} z_k$ *we have the canonical isomorphism of crystals*

$$\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_1) \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_2) \otimes \ldots \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_{k-1}) \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_k).$$

Proof: Let us construct an isomorphism of sets

$$\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_1) \otimes \ldots \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_k).$$

Consider the family

$$\operatorname{Im}(\mathbf{ev}_{(\underline{z}/\varepsilon+\underline{d})}(B(\exp(2\pi i\chi))) \to \operatorname{End}(V_1 \otimes \ldots \otimes V_k))$$

with $z_l \in i\mathbb{R}$, $\varepsilon \in (0, \frac{1}{N})$ (see Assumption 4.16.1) and note that by Proposition 4.15.6 the limit as $\varepsilon \to 0$ is $\operatorname{Im}(\mathcal{A}^{\otimes k}_{\exp(-2\pi i\chi)} \to \operatorname{End}(V_1 \otimes \ldots \otimes V_k))$ that is exactly $\operatorname{Im}(\mathbf{ev}_{z_1}(B(\exp(2\pi i\chi))) \otimes \ldots \otimes \mathbf{ev}_{z_k}(B(\exp(2\pi i\chi))) \to \operatorname{End}(V_1 \otimes \ldots V_k))$. Since the set of eigenlines of the action $\mathbf{ev}_{z_1}(B(\exp(2\pi i\chi))) \otimes \ldots \otimes \mathbf{ev}_{z_k}(B(\exp(2\pi i\chi)))$ is exactly $\mathcal{E}_{\exp(2\pi i\chi)}(\lambda_1) \otimes \ldots \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_k)$ we obtain the identification $\mathcal{E}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_1) \otimes \ldots \otimes \mathcal{E}_{\exp(2\pi i\chi)}(\lambda_k)$. We claim that this identification is an isomorphism of crystals. To check this, it is enough to show that for every $j = 1, \ldots, n$ the identification above induces the isomorphism of crystals $\mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\lambda_1) \otimes \ldots \otimes \mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\lambda_k)$. Consider the element $\hat{w}_j := (\tau_{[j]}; 0, \ldots, 0, \underbrace{1, \ldots, 1}_{j}) \in \hat{W}^{\mathrm{ext}}$ and let us also fix an element $\chi_j \in Q^{\mathrm{reg}}_{\tau_{[j]}}$. It follows from Lemma 4.16.3 and the proof of Proposition 4.16.6 that we have identifications $\mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\hat{w}_j,\mathrm{fin}}_{\exp(2\pi i\chi)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda})$, $\mathcal{E}^{[j]}_{\exp(2\pi i\chi)}(\lambda_l) \xrightarrow{\sim} \mathcal{E}^{\hat{w}_j,\mathrm{fin}}_{\exp(2\pi i\chi)}(\lambda_l) \xrightarrow{\sim} \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_l)$, $l = 1, \ldots, k$ that are compatible with the identifications above. Therefore it is enough to show that the bijection $\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_1) \times \ldots \times \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_k)$ is an isomorphism of crystals (with the tensor product crystal structure on the right-hand side). By Proposition 4.16.4 we have the isomorphism of crystals $\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_l) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\lambda_l)$ given by the monodromy along the path connecting $\epsilon = 0$ and $\epsilon = 1/2N$ in the family

$$\epsilon \mapsto \operatorname{Im}(\mathbf{ev}_{\underline{z}/\epsilon+\underline{d}}(B(\exp(2\pi i\epsilon\chi_j))) \to \operatorname{End}(V_1 \otimes \ldots \otimes V_k)),\ \epsilon \in (0, 1/N),$$

$$0 \mapsto \operatorname{Im}(\mathcal{A}_{-\chi_j}(2\pi i \cdot \underline{z}) \to \operatorname{End}(V_1 \otimes \ldots \otimes V_k))$$

so we just need to check that the bijection

$$\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\lambda_1) \times \ldots \times \mathcal{E}_{-\chi_j}(\lambda_k) \tag{4.33}$$

given by the composition of the product of monodromies above and the monodromy along the path connecting $\varepsilon = 0$ and $\varepsilon = 1$ of the family

$$\varepsilon \mapsto \mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}} B(\exp(2\pi i\chi_j)) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),\ \varepsilon \in \mathbb{R}^\times,$$

$$0 \mapsto \mathrm{Im}(\mathcal{A}^{\otimes k}_{\exp(-2\pi i\chi_j)} \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k))$$

is an isomorphism of crystals.

Consider the family $\mathrm{Im}(\mathbf{ev}_{\underline{z}/\varepsilon+\underline{d}}(B(\exp(2\pi i\epsilon\chi_j))) \to \mathrm{End}(V_1 \otimes \ldots \otimes V_k)),\ \epsilon, \varepsilon \in \mathbb{R}$. We can assume that χ_j is Weil generic so it follows from Proposition 4.15.6 that this family extends to the family parametrized by $\mathrm{Bl}_{(0,0)} K$. Our goal is to describe the path $p\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$ that induces the isomorphism (4.33) and then replace it by a homotopy equivalent path inside $\mathrm{Bl}_{(0,0)} K$.

We start from describing the path $p\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$. Note that $p(0) = ((1, 1/2N), 2N : 1])$, $p(1) = ((0,0), [1 : 0])$. Recall that the isomorphism (4.33) is the composition of two isomorphisms: one is the isomorphism

$$\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_1) \times \ldots \times \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_k)$$

and the second one is the isomorphism

$$\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_1) \times \ldots \times \mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\lambda_k) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\lambda_1) \times \ldots \times \mathcal{E}_{-\chi_j}(\lambda_k).$$

The first isomorphism is induced by the path $p_1\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$ given by $c \mapsto ((\frac{1}{2N}, \frac{1-c}{2N}), [1 : 1-c])$ and the second isomorphism is induced by the path $p_2\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$ given by $c \mapsto ((\frac{1-c}{2N}, 0), [1 : 0])$. So we get $p = p_1 * p_2$ i.e. p is obtained by gluing p_1, p_2.

Consider now the following path $q\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$ from $((0,0),[1:0])$ to $((1,1),[1:1])$. Path q will be the gluing of two different paths q_1, q_2, $q = q_1 * q_2$. Path $q_1\colon [0,1] \to \mathrm{Bl}_{(0,0)} K$ is given by $c \mapsto ((0,0),[1:c])$, path q_2 is given by $c \mapsto ((\frac{c}{2N}, \frac{c}{2N}), [1:1])$.

Consider the composition $p * q$. Note that $p * q$ is a cycle and it is easy to see that this cycle is homotopic to zero: indeed, note that we have a family of continuous maps $\gamma_t\colon \mathrm{Bl}_{(0,0)} K \to \mathrm{Bl}_{(0,0)} K$, $((a,b),[x:y]) \mapsto ((ta,tb),[x:y])$ that retracts $\mathrm{Bl}_{(0,0)} K$ on $\mathbb{RP}^1$. The cycles $\gamma_t(p*q)$ are homotopic and $\gamma_1(p*q) = p*q$. It remains to note that $\gamma_0(p*q)$ is homotopic to zero.

We conclude that the isomorphism induced by p is equal to the isomorphism $\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\lambda_1) \times \ldots \times \mathcal{E}_{-\chi_j}(\lambda_k)$ induced by q^{-1}. Note now that the isomorphism induced by $q^{-1} = q_2^{-1} * q_1^{-1}$ is the composition of the isomorphism $\mathcal{E}^{\tau_{[j]},\mathrm{fin}}_{\exp(2\pi i\chi_j)}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\underline{\lambda})$ induced by q_1^{-1} and the isomorphism $\mathcal{E}_{-\chi_j}(\underline{\lambda}) \xrightarrow{\sim} \mathcal{E}_{-\chi_j}(\lambda_1) \times \ldots \times \mathcal{E}_{-\chi_j}(\lambda_k)$ induced by q_2^{-1}. It follows from Proposition 4.16.4 that the first isomorphism is the isomorphism of crystals, it follows from [43] that the second isomorphism is the isomorphism of crystals, where the crystal structure on $\mathcal{E}_{-\chi_j}(\lambda_1) \times \ldots \times \mathcal{E}_{-\chi_j}(\lambda_k)$ is given by $\mathcal{E}_{-\chi_j}(\lambda_1) \otimes \ldots \otimes \mathcal{E}_{-\chi_j}(\lambda_k)$.

□

We are now ready to prove the main theorem of this section.

Theorem 4.17.4 *For $C \in \overline{S}^{\mathrm{reg}}$ and $\mathrm{Im}\, z_1 \gg \mathrm{Im}\, z_2 \gg \ldots \gg \mathrm{Im}\, z_k$, the crystal $\mathcal{E}_C(\underline{\lambda})$ coincides with the Kirillov-Reshetikhin crystal $\mathbf{B}_{\lambda_1} \otimes \mathbf{B}_{\lambda_2} \otimes \ldots \otimes \mathbf{B}_{\lambda_k}$, corresponding to the tensor product of irreducible representations $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_k}$.*

Proof: Follows from Propositions 4.17.1, 4.17.3.

□

4.18 Monodromy

4.18.1 Spaces $\overline{M_{0,n+2}}$, $\overline{S^{\text{reg}}}$

Space $\overline{M_{0,n+2}}$

Recall that $\overline{M_{0,n+2}}$ is the Deligne-Mumford space of stable rational curves with $n+2$ marked points. The points of $\overline{M_{0,n+2}}$ are isomorphism classes of curves of genus 0, with $n+2$ ordered marked points and possibly with nodes, such that each component has at least 3 distinguished points (either marked points or nodes).

The space $\overline{M_{0,n+2}}$ is a compactification of $M_{0,n+2}$ (configuration space of $n+2$ ordered pairwise distinct points $(z_1, \ldots, z_{n+2})$ of $\mathbb{P}^1$ modulo PGL_2). We have

$$M_{0,n+2} \simeq \{(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n \mid z_i \neq z_j\}/\mathbb{C}^\times = T^{\text{reg}}, \tag{4.34}$$

where $T^{\text{reg}} \subset \mathrm{PGL}_n$ is the subalgebra of diagonal matrices.

Space $\overline{M_{0,n+2}}$ has a natural stratification indexed by the combinatorial type of the curve with marked points. Namely, the strata are indexed by trees whose leaves correspond to the marked points $0, z_1, \ldots, z_n, \infty$ where the inner vertices represent the connected components and the edges correspond to the nodal points.

Space $\overline{S^{\text{reg}}}$

We will be interested in a certain real form of $\overline{M_{0,n+2}}$ that we will denote by $\overline{S^{\text{reg}}}$ since it is a compactification of S^{reg}. Namely, consider the (anti-holomorphic) involution on $\overline{M_{0,n+2}}$ sending $(C, z_0, z_1, \ldots, z_n, z_{n+1})$ to $(C, \overline{z}_{n+1}, \overline{z}_1, \ldots, \overline{z}_n, \overline{z}_0)$. Space $\overline{S^{\text{reg}}}$ are fixed points w.r.t. this involution. Note that after the identification (4.34) the involution on $M_{0,n+2}$ is given by $(z_1, \ldots, z_n) \mapsto (\overline{z}_1^{-1}, \ldots, \overline{z}_n^{-1})$ so its fixed points are precisely S^{reg}. The whole space $\overline{S^{\text{reg}}}$ is the moduli space of curves where all marked points are on the unit circle, and the points $0, \infty$ belong to the same component. We refer the reader to [49, Section 7.2.4] and [17] for details.

Stratification of $\overline{M_{0,n+2}}$ induces the stratification of $\overline{S^{\text{reg}}}$.

Extended affine cactus group

See [49, Section 10.3] for the details.

We recall that the affine cactus group AC_n as the group with generators s_{ij} for $1 \leqslant i \neq j \leqslant n$ and relations:

(1) $s_{ij}^2 = 1$

(2) $s_{ij}s_{kl} = s_{kl}s_{ij}$ if $[i,j] \cap [k,l] = \varnothing$

(3) $s_{ij}s_{kl} = s_{w_{ij}(l)w_{ij}(k)}s_{kl}$ if $[k,l] \subset [i,j]$, where w_{ij} is the element which reverses $[i,j]$ and leaves invariant the elements outside the interval.

Warning 4.18.1 *Note that we do* **not** *assume that $i < j$ in the definition of AC_n.*

We define an action of $Z/n\mathbb{Z}$ on AC_n by $r \cdot s_{ij} = s_{i+1\,j+1}$. The extended affine cactus group $\widetilde{AC}_n$ is the semi-direct product $(\mathbb{Z}/n\mathbb{Z}) \ltimes AC_n$.

Equivariant fundamental group of $\overline{S^{\mathrm{reg}}}$

Group S_n acts naturally on the space $\overline{S^{\mathrm{reg}}}$. We will be interested in the S_n-equivariant fundamental group of $\overline{S^{\mathrm{reg}}}$ which is defined as follows. Pick a basepoint $x \in \overline{S^{\mathrm{reg}}}$ which corresponds to a configuration of evenly spaced points of $U(1)$, namely $z_k = e^{\frac{2\pi i k}{n}}$.

$$\pi_1^{S_n}(\overline{S^{\mathrm{reg}}}) := \{(g, p) \,|\, g \in S_n,\ p \text{ is a homotopy class of paths from } x \text{ to } gx\}.$$

The multiplication in $\pi_1^{S_n}(\overline{S^{\mathrm{reg}}})$ is defined as follows:

$$(g_1, p_1) \cdot (g_2, p_2) = (g_1 g_2, p_1 * g_1(p_2)).$$

Abusing notations for $1 \leqslant i \neq j \leqslant n$ let $(w_{ij}, s_{ij}) \in \pi_1^{S_n}(\overline{S^{\mathrm{reg}}})$ be the element corresponding to the shortest path γ_{ij} connecting the base point x with $w_{ij}x$.

Proposition 4.18.2 ([17, Theorem 8.3] and [49, Theorem 11.12]) *There is an isomorphism $\widetilde{AC}_n \xrightarrow{\sim} \pi_1(\overline{S^{\mathrm{reg}}})$ that sends $s_{ij} \in \widetilde{AC}_n$ to $(w_{ij}, s_{ij}) \in \pi_1(\overline{S^{\mathrm{reg}}})$ and $[1] \in \mathbb{Z}/n\mathbb{Z}$ to the pair consisting of the cyclic permutation $(12\ldots n)$ and the shortest path connecting x with $(12\ldots n)x$.*

4.18.2 Description of the monodromy

The Deligne-Mumford space $\overline{M_{0,n+2}}$, being a compactification of T^{reg}, parametrizes all possible limits of the Bethe subalgebras $B(C)$ in the Yangian (see [51]). According to [50], any subalgebra $B(X)$ corresponding to any point $X \in \overline{M_{0,n+2}}$ acts on $\bigotimes V_{\lambda_i}(u_i)$ with a cyclic vector. Restricting to $\overline{S^{\mathrm{reg}}} \subset \overline{M_{0,n+2}}$, and requiring the evaluation parameters u_i to be as above, we guarantee that the action of the corresponding subalgebra on the corresponding module is Hermitian hence semisimple. So we can regard $\mathcal{E}_C(\underline{\lambda})$ as a S_n-equivariant covering of the space $\overline{S^{\mathrm{reg}}} \subset \overline{M_{0,n+2}}$. Our goal in this section is to describe the monodromy of this covering in terms of the above affine crystal structure (compare with [43]).

We claim that the monodromy of this covering can be expressed in terms of the partial Schützenberger involutions for the Kirillov-Reshetikhin crystals. Namely, for any proper subdiagram in the affine diagram $\widetilde{A_{n-1}}$ (i.e. for any proper subset in the set of affine simple roots), we can decompose our KR crystal into connected components with respect to the corresponding (finite-type) Levi subalgebra and apply the Schützenberger involution assigned to this Levi to each of the components.

Recall that if B is a normal $\mathfrak{g}$-crystal then the Schützenberger involution is the *unique* isomorphism $\xi_B \colon B \xrightarrow{\sim} B$ that preserves connected components of B and such that:

$$e_i(\xi_B(b)) = \xi_B(f_{\theta(i)}(b)),$$

$$f_i(\xi_B(b)) = \xi_B(e_{\theta(i)}(b)),$$

$$\mathrm{wt}(\xi_B(b)) = w_0\,\mathrm{wt}(b),$$

where $w_0 \in W$ is the longest element and θ is the involution of the set of simple roots induced by $-w_0$.

Recall that to $C \in S^{\mathrm{reg}}$ and $w \in \widehat{W}$ we can associate the Kirillov-Reshetikhin crystal $\mathcal{E}_C^w$. Abusing the notations, we denote by $\mathcal{E}$ the KR crystal corresponding to $C \in Q^{\mathrm{reg}}$, we can identify it with crystals corresponding to other alcoves using the action of $\widehat{W}$.

Theorem 4.18.3 *The generators s_{ij} of the monodromy group of $\overline{S^{\mathrm{reg}}}$ acts on $\mathcal{E}$ as a partial Schützenberger involution with respect to the corresponding subdiagram in the Dynkin diagram of $\widehat{A_{n-1}}$, see [50, Conjecture 7.1].*

Proof: Let us deal with the case of s_{1n}, the other cases are similar. We only need to check that the bijection induced by the monodromy along the path γ_{1n} is compatible with crystal structures and preserves connected components. The path γ_{1n} can be replaced by the following path γ. Let $\chi = \mathrm{diag}(u_1, \ldots, u_n)$ be an element of $\mathbb{R}^n$ such that $w_0(\chi) = -\chi$ (i.e., $u_j = -u_{n+1-j}$) and all u_i are distint. Consider the path $\gamma^+(t) = \exp(2\pi i t\chi) \in S^{\mathrm{reg}}$, $t \in (0, 1]$. We can extend it to 0 by setting $\gamma^-(0) := \lim_{t\to 0}\gamma(t) \in \overline{S^{\mathrm{reg}}}$. Set $\gamma^- \colon [0, 1] \to \overline{S^{\mathrm{reg}}}$, $\gamma^-(t) = w_0(\gamma^+(1-t))$. The path $\gamma \colon [0, 1] \to \overline{S^{\mathrm{reg}}}$ is the concatenation of the paths γ^+ and γ^-. Note now that $B(\gamma(\frac{1}{2}))$ is generated by $B(1)$ together with the image of $\mathcal{A}_\chi$. Let us now decompose our KR-module as the direct sum of weight spaces of $B(1)$. Each weight space will be a $\mathfrak{g}$-module without multiplicities since otherwise, the action of $\mathcal{A}_\chi$ on this component will not be cyclic (and so the action of $B(1) \cdot \mathcal{A}_\chi$ can not have a simple spectrum on the whole module). It then follows that our bijection presereves components. The same argument as in [43, Proposition 13.5] works to see the compatibility with crystal structure. □

4.18.3 Application: monodromy of the spectrum of $QH^*_{T\times\mathbb{C}^\times}(\mathfrak{M})$

Let $\mathfrak{M}$ be a disjoint union of type A quiver varieties with fixed framing vector (w_i) and arbitrary dimension vector (v_i) (w_i, v_i are collections of numbers, labeled by the vertices of the quiver). Let $QH^*_{T\times\mathbb{C}^\times_\hbar}(\mathfrak{M})$ be the algebra of *quantum cohomology* of $\mathfrak{M}$, one can think about this algebra as about the *family* of subalgebras in $\mathrm{End}(H^*_{T\times\mathbb{C}^\times_\hbar}(\mathfrak{M}))$ depending (in logarithmic coordinates) on an element of $H^2(\mathfrak{M}, \mathbb{C})$. Recall now that by the results Varagnolo [109] (following [87]), and Maulik-Okounkov ([79]) (see [80] for the comparison of the two approaches), $H^*_{T\times\mathbb{C}^\times_\hbar}(\mathfrak{M}) \simeq V_{\lambda_1}(z_1)\otimes\ldots\otimes V_{\lambda_k}(z_k)$ and the action of the Yangian on it has a *geometric realization* (here λ_j are fundamental weights that can be constructed from the framing (w_i)). After these identifications, the action of $QH^*_{T\times\mathbb{C}^\times_\hbar}(\mathfrak{M})$ on $V_{\lambda_1}(z_1)\otimes\ldots\otimes V_{\lambda_k}(z_k)$ identifies with the action of the family of Bethe subalgebras of the Yangian (our parameter C above naturally corresponds to an element of $H^2(\mathfrak{M}, \mathbb{C})$), see [79]. So, the following theorem is a corollary of our results.

Theorem 4.18.4 *The monodromy operators of eigenvalues of multiplication by elements of the quantum cohomology ring $QH^*_{T\times\mathbb{C}^\times_\hbar}(\mathfrak{M})$ are generated by partial Schützenberger involutions.*

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