CLASSICAL SOLUTIONS IN BAG THEORY

by

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ABSTRACT

A class of solutions to the bag theory equations is studied. They are shown not to be eigenstates. The stability of the bag in such states is investigated under two different radial perturbations. The need for a more general solution is indicated. A scheme for general solution is given. This scheme is shown to require a linear spectrum of eigenvalues.

Thesis Supervisor: Alan Chodos

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I. INTRODUCTION

The bag theory has been proposed[1] in which a strongly interacting particle is a finite region of space comprised of fields confined in a Lorentz-invariant way. Based on the single parameter of the theory, the constant energy per unit volume, B, a relativistic quark model for baryons has been given.

The low-lying baryon states were explored in this quark model[2] by considering solutions to the bag equations

$$i\not\partial\Psi = 0, \text{ inside the bag, } S_1 \ , \quad (1.1)$$

$$in_\mu\gamma^\mu\Psi = \Psi, \quad (1.2)$$

$$\left(n_\mu\frac{\partial}{\partial x_\mu}\right)\overline{\Psi}\Psi = 2B, \text{ on the surface } S, \quad (1.3)$$

for the case in which the confining region (the "bag") has a fixed radius R_o. Eq.(1.1) is just the Dirac equation[4] for a massless field Ψ and

$$n_\mu = \frac{-(m_\lambda\dot{X}^\lambda)\eta_\mu + m_\mu}{\surd(1-(m_\lambda\dot{X}^\lambda)^2)} \quad (1.4)$$

is the inward normal to the space-time surface swept out by S_1. m is the interior spatial normal, η is the unit time-like vector, and $\dot{X}^\lambda$ are the time derivatives of the surface coordinates X^λ.

While incorporating the desirable features of the quark, parton, and dual models[3], the bag is distinct from other extended[5] models because the geometrical variables in bag theory are defined (by the equations of motion) in terms of the field variables rather than being separate dynamical variables. Quark field confinement in the bag provides a simple explanation of the fact that free quarks have not been observed.[6]

The boundary conditions (1.2) and (1.3) were derived by requiring energy-momentum conservation on the Dirac field Ψ, which permeates all space, in the limit in which it is confined to the bag.[1] In determining the baryon states, the linear boundary condition (1.2) was used to generate the eigenvalue condition

$$j_o(\omega_{nk}) = -kj(\omega_{nk}) \qquad (1.5)$$

for the eigenfrequencies ω_{nk}, where the j_i are the usual spherical Bessel functions. The quadratic condition (1.3) restricts Ψ to angular momentum $j = 1/2$.

Furthermore, to insure time-independence in (1.3), only excitations of a single mode were considered for this fixed-radius case. Thus,

$$\Psi = \psi_{n\kappa m}(\vec{x},t) = N_{nk}\begin{pmatrix} i\, j_0(\omega_{n\kappa} r/R_0)\, U_m \\ -j_1(\omega_{n\kappa} r/R_0)\, \vec{\sigma}\cdot\hat{r}\, U_m \end{pmatrix} e^{-i\omega_{n\kappa} t/R_0}, \quad \kappa=-1$$

$$= \psi_{n\kappa m}(\vec{x},t) = N_{nk}\begin{pmatrix} i\, j_1(\omega_{n\kappa} r/R_0)\, \vec{\sigma}\cdot\hat{r}\, U_m \\ j_0(\omega_{n\kappa} r/R_0)\, U_m \end{pmatrix} e^{-i\omega_{n\kappa} t/R_0}, \quad \kappa=1$$

where U_m is a Pauli spinor and N_{nk} is a suitable normalization constant. However, time-independence is also possible with the choice

$$\Psi = a_{n'k'm} N_{n'k'm} \psi_{n'k'm} + a_{nk-m} N_{nk-m} \psi_{nk-m} \tag{1.6}$$

where $m = \pm 1/2$. This follows since the time-dependent cross terms $\exp(\pm i(\omega_n - \omega_{n'})t)$ of $\bar{\Psi}\Psi$ involve factors of products of mutually orthogonal Pauli spinors. Thus, they vanish.

Eq. (1.6) apparently represents the most general time dependence for a solution. For example, the bag equations for a charged scalar bag with spherical symmetry have been solved with the result

$$\phi(r,t) = \frac{1}{r}\left(g(t+r) - g(t-r)\right)$$

$$\frac{dg(u)}{du} = \sqrt{\frac{BR_0^2}{4}}\, e^{i\psi(u)}$$

$$\psi(u+2R_0) = \psi(u) + 2\pi n \quad , \quad n = \text{integer}$$

One may consider analogously the ansatz

$$\psi_\alpha = \begin{pmatrix} f(r)\, e^{iF(r,t)}\, u_m \\ \vec{\sigma}\cdot\hat{r}\, g(r)\, e^{iF(r,t)}\, u_m \end{pmatrix}$$

as an alternative to Eq. (1.6). It may be shown (See Appendix A) however that

$$F = At + B(r)$$

so that

$$e^{iF(r,t)} = H(r)\,e^{iAt}, \qquad A = \text{constant.}$$

Hence, the form (1.7) reduces to the single-frequency case.

In the calculations which follow, it is shown that the function (1.6) is not generally an eigenstate of energy. If the bag theory is indeed a correct one for the description of natural phenomena, such a special state may be of particular interest. For example, whereas the eigenstates are expected to be stable, (1.6) may under perturbation exhibit instability. This possibility is investigated here by introducing a small spherically symmetric radial motion. Later, angular disturbance is introduced. Finally, a scheme is suggested for a general description of the perturbed bag.

II. CALCULATIONS

Using Eq. (1.6)

$$\overline{\Psi}\Psi = N_{n'}^2 a_{n'}^* a_{n'} \overline{\Psi}_{n'} \Psi_{n'} + N_n^2 a_n^* a_n \overline{\Psi}_n \Psi_n$$

where the indices

$$K \equiv K' \equiv 1, \qquad m \equiv -m' \equiv -1/2$$

have been suppressed. The quadratic boundary condition (1.3) becomes for a fixed radius R_o

$$n_\mu \frac{\partial \overline{\Psi}\Psi}{\partial x_\mu} = \left[\vec{\nabla}(R_o - \vec{r})\right] \cdot \nabla(\overline{\Psi}\Psi) = -\hat{r}\cdot\nabla(\overline{\Psi}\Psi)$$

$$= -\frac{\partial}{\partial r}(\overline{\Psi}\Psi)$$

$$= -\frac{\partial}{\partial r}\Big\{ a_{n'}^* a_{n'} N_{n'}^2 \left(j_1^2\left(\frac{\omega_{n'} r}{R_o}\right) + j_0^2\left(\frac{\omega_{n'} r}{R_o}\right)\right) +$$

$$a_n^* a_n N_n^2 \left(j_1^2\left(\frac{\omega_n r}{R_o}\right) + j_0^2\left(\frac{\omega_n r}{R_o}\right)\right) \Big\} \tag{2.1}$$

$$= \Big\{ a_{n'}^* a_{n'} N_{n'}^2 \, j_0^2(\omega_{n'}) \left(\frac{4}{R_o}\right)(1 + \omega_{n'}) +$$

$$a_n^* a_n N_n^2 \, j_0^2(\omega_n) \left(\frac{4}{R_o}\right)(1 + \omega_n) \Big\} = -2B$$

Quantization[2] is now possible by defining a state $|0\rangle$, such that

$$a_i |0\rangle = |0\rangle, \qquad K \equiv K_i, \qquad i > 0$$

and

$$a^{+}_{-i}\,|0\rangle = |0\rangle\ , \qquad K \equiv -K_i$$

with all anticommutators equal to zero except

$$\{a_i, a_i^{+}\} = 1 = \{a^{+}_{-i}, a_{-i}\}$$

Then if

$$N_i^2 \equiv \frac{\omega_i}{2(\sin^2\omega_i)(4\pi)R_0^3(1+\omega_i)}$$

the following integral will have integer eigenvalues[6]:

$$\int \psi^{+}\psi\, d^3\vec{r} = \int \sum_{\{n',n\}} a_i^{*}a_i N_i^2\left[j_1^2(\omega_i r/R_0) + j_0^2(\omega_i r/R_0)\right] d^3\vec{r}$$

$$= \Big\{4\pi \int r^2 dr\, \Sigma\Big[\frac{\sin^2(\omega_i r/R_0)}{(\omega_i r/R_0)^4} + \frac{\cos^2(\omega_i r/R_0)}{(\omega_i r/R_0)^2}$$

$$-2\,\frac{\sin(\omega_i r/R_0)\cos(\omega_i r/R_0)}{(\omega_i r/R_0)^3} + \frac{\sin^2(\omega_i r/R_0)}{(\omega_i r/R_0)^2}\Big]\cdot$$

$$\left(a_i^{*}a_i N_i^2\right)\Big\}$$

$$= \Big\{4\pi\,\Sigma\, a_i^{*}a_i N_i^2 \int r^2 dr\left[1 - \frac{d\left(\frac{\sin^2(\omega_i r/R_0)}{(\omega_i r/R_0)}\right)}{d(\omega_i r/R_0)}\right]\cdot$$

$$\left(\frac{R_0^2}{(\omega_i r)^2}\right)\Big\}$$

$$= 4\pi\,\Sigma\, a_i^{*}a_i N_i^2 \left(\frac{R_0}{\omega_i}\right)^3\left[\omega_i - \frac{\sin^2\omega_i}{\omega_i}\right]$$

$$= a^{*}_{n'}a_{n'} + a^{*}_{n}a_{n}$$

In operator notation,

$$\int \Psi^{+} \Psi \, d^{3}\vec{r} = \sum \left(a_i^{+} a_i - a_{-i}^{+} a_{-i} \right) \left(\delta_{in'} + \delta_{in} \right)$$

and Eq. (2.1) may be written

$$4\pi B R_0^{4} = \sum \omega_i \left(a_i^{+} a_i + a_{-i}^{+} a_{-i} \right) \left(\delta_{in'} + \delta_{in} \right) \equiv \Omega \tag{2.2}$$

where zero-point fluctuations have not been written and use has been made of the identity $\omega_{n,1} = -\omega_{-n,-1}$ (from (1.5)).

The virial theorem,

$$E = 4B\langle V \rangle = \frac{16\pi}{3} B R_0^{3}$$

relates the bag energy to the bag size. Eliminating R_o using (2.2)

$$E = \frac{4}{3} B^{1/4} (4\pi)^{1/4} \Omega^{3/4}$$

For $n \neq n'$ we see that (1.6) is not an energy eigenstate:

$$E|\Phi\rangle = E\left(|n\rangle + |n'\rangle \right) = \frac{4}{3} (4\pi B)^{\frac{1}{4}} \left(\omega_n^{3/4} |n\rangle + \omega_{n'}^{3/4} |n'\rangle \right)$$

Such a state may not exhibit the stability expected for an eigenstate ψ_{nkm}. Indeed, its radius will have a value between those of states n and n' as may be seen from (2.2). We will therefore study its behavior by introducing a small, spherically symmetric radial motion.

Let

$$r = R_0 + R_1 , \qquad R_1 \ll R_0$$

$$\psi = \psi_0 + \psi_1 , \qquad |\psi_1| \ll |\psi_0|$$

With Ψ_1 of the same order as R_1, we expand Ψ about R_o. Ψ_o will be of the form (1.6) above.

$$\psi \approx \psi_0(R_0, \Omega) + \psi_0'(R_0, \Omega) R_1 + \psi_1(R_0, \Omega)$$

The normal (1.4) becomes

$$n_\mu = \frac{\partial}{\partial x^\mu}\left(R_0 + R_1 - r \right)$$

where

$$R_1 \equiv A e^{-i\lambda t} + A^* e^{i\lambda^* t} , \quad \lambda \equiv \lambda_1 + i\lambda_2$$

To zeroth order, the Dirac equation is satisfied by Ψ_o,

$$\not{\partial}\psi = \not{\partial}\psi_0 + \not{\partial}\psi_1 = 0$$

so that

$$\not{\partial}\psi_1 = 0$$

To zeroth order, the equation (1.5) follows from

$$i n_\mu \gamma^\mu \psi = \psi$$

where

$$n_\mu = (\dot{R}_1 , -\hat{r})$$

The first order linear condition is

$$\left(I + i\hat{r}\cdot\vec{\gamma}\right)\left(\psi_1(R_0,\Omega) + R_1\psi'(R_0,\Omega)\right)$$
$$= i\dot{R}_1\gamma^0\psi_0(R_0,\Omega)$$

I is just the unit matrix. With $n\to\beta$, and $n'\to\alpha$, (1.6) is

$$\psi_0 = \alpha_0\begin{pmatrix} i\,j_1(\omega_\alpha r)\,\vec{\sigma}\cdot\hat{r}\,U_\alpha \\ j_0(\omega_\alpha r)\,U_\alpha \end{pmatrix} e^{-i\omega_\alpha t} \tag{2.3}$$

$$+ \beta_0\begin{pmatrix} i\,j_1(\omega_\beta r)\,\sigma\cdot\hat{r}\,U_\beta \\ j_0(\omega_\beta r)\,U_\beta \end{pmatrix} e^{-i\omega_\beta t}$$

$$U_\alpha^{+} U_\beta = 0$$

The linear condition suggests the following form for Ψ_1

$$\psi_1 = \alpha_1\begin{pmatrix} i\,j_1(\omega_{\alpha_1} r)\,\vec{\sigma}\cdot\hat{r}\,U_\alpha \\ j_0(\omega_{\alpha_1} r)\,U_\alpha \end{pmatrix} e^{-i\omega_{\alpha_1} t} + \tilde{\alpha}_1\begin{pmatrix} i\,j_1(\omega_{\tilde{\alpha}_1} r)\,\vec{\sigma}\cdot\hat{r}\,U_\alpha \\ j_0(\omega_{\tilde{\alpha}_1} r)\,U_\alpha \end{pmatrix} e^{-i\omega_{\tilde{\alpha}_1} t}$$

$$+ \beta_1\begin{pmatrix} i\,j_1(\omega_{\beta_1} r)\,\vec{\sigma}\cdot\hat{r}\,U_\beta \\ j_0(\omega_{\beta_1} r)\,U_\beta \end{pmatrix} e^{-i\omega_{\beta_1} t} + \tilde{\beta}_1\begin{pmatrix} i\,j_1(\omega_{\tilde{\beta}_1} r)\,\vec{\sigma}\cdot\hat{r}\,U_\beta \\ j_0(\omega_{\tilde{\beta}} r)\,U_\beta \end{pmatrix} e^{-i\omega_{\tilde{\beta}_1} t}$$

The coefficients outside the parentheses are determined from the linear condition.

By the linear condition, equations similar to the following

$$\alpha_1\left(i\, j_1(w_{\alpha_1}R_0) + i\, j_0(w_{\alpha_1}R_0)\right) =$$
$$-\alpha_0 A\, w_\alpha\, j_0(w_\alpha R_0)\left(i\left(1+\frac{2}{w_\alpha R_0}\right) + i\right)$$
$$-\lambda A\, \alpha_0\, j_0(w_\alpha R_0)\, i$$

give

$$\alpha_1 = \frac{-A\alpha_0\, j_0(w_\alpha R_0)\left[w_\alpha\left(2+\frac{2}{w_\alpha R_0}\right)+\lambda\right]}{j_1(w_{\alpha_1}R_0) + j_0(w_{\alpha_1}R_0)}$$

$$\tilde{\alpha}_1 = \frac{-A\alpha_0\, j_0(w_\alpha R_0)\left[w_\alpha\left(2+\frac{2}{w_\alpha R_0}\right)-\lambda^*\right]}{j_1(w_{\tilde{\alpha}_1}R_0) + j_0(w_{\tilde{\alpha}_1}R_0)}$$

$$\beta_1 = \frac{-A\beta_0\, j_0(w_\beta R_0)\left[w_\beta\left(2+\frac{2}{w_\beta R_0}\right)+\lambda\right]}{j_1(w_\beta R_0) + j_0(w_\beta R_0)}$$

$$\tilde{\beta}_1 = \frac{-A\beta_0\, j_0(w_\beta R_0)\left[w_\beta\left(2+\frac{2}{w_\beta R_0}\right)-\lambda^*\right]}{j_1(w_{\beta_1}R_0) + j_0(w_{\tilde{\beta}_1}R_0)} \quad .$$

Here,

$$w_{\alpha_1} \equiv w_\alpha + \lambda \qquad w_{\tilde{\alpha}_1} \equiv w_\alpha - \lambda^*$$

$$w_{\beta_1} \equiv w_\beta + \lambda \qquad w_{\tilde{\beta}_1} \equiv w_\beta - \lambda^*$$

The quadratic condition

$$R_1(\bar{\Psi}_0\Psi_0)'' = -(\bar{\Psi}_1\Psi_0 + \bar{\Psi}_0\Psi_1)'$$
$$= -2\,\mathrm{Re}\left[(\bar{\Psi}_0\Psi_1)'\right], \quad r = R_0$$

is then

$$-A(\alpha_0^*\alpha_0\bar{\Psi}_\alpha\Psi_\alpha + \beta_0^*\beta_0\bar{\Psi}_\beta\Psi_\beta)''$$
$$\begin{aligned} &= \alpha_0\tilde{\alpha}_1^*\,T(\alpha,\tilde{\alpha}_1) + \beta_0\tilde{\beta}_1^*\,T(\beta,\tilde{\beta}_1) \\ &+ \alpha_0^*\alpha_1\,T(\alpha,\alpha_1) + \beta_0^*\beta_1\,T(\beta,\beta_1) \end{aligned} \tag{2.4a}$$

and the conjugate condition

$$-A^*(\alpha_0^*\alpha_0\bar{\Psi}_\alpha\Psi_\alpha + \beta_0^*\beta_0\bar{\Psi}_\beta\Psi_\beta)''$$
$$\begin{aligned} &= \alpha_0^*\tilde{\alpha}_1\,T(\alpha,\tilde{\alpha}_1) + \beta_0^*\tilde{\beta}_1\,T(\beta,\tilde{\beta}_1) \\ &+ \alpha_0\alpha_1^*\,T(\alpha,\alpha_1) + \beta_0\beta_1^*\,T(\beta,\beta_1) \end{aligned} \tag{2.4b}$$

$$T(a,b) \equiv j_0(\omega_a R_0)\left[\left(\omega_a + \omega_b + \frac{4}{R_0}\right)j_1(\omega_b R_0) - (\omega_a + \omega_b)j_0(\omega_b R_0)\right]$$

Eqs. (2.4) are an eigenvalue condition on λ (and λ^*). One condition results if $\lambda = \pm\lambda^*$. Solution is clearly difficult. An improved structure, (e.g., which more clearly suggests instability) may result if we impart an angular momentum to the surface. We therefore define

$$R_1 \equiv \left(A e^{-i\lambda t} + A^* e^{i\lambda^* t}\right) Y_{10} \sqrt{\tfrac{4\pi}{3}}$$
$$= \left(A e^{-i\lambda t} + A^* e^{i\lambda^* t}\right) \cos\theta$$

and rewrite (2.3) with a slight change in notation

$$\psi_0 = \alpha_0 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_1(\omega_{\alpha 0} r)\, u\uparrow \\ j_0(\omega_{\alpha 0} r)\, u\uparrow \end{pmatrix} e^{-i\omega_{\alpha 0} t} + \beta_0 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_1(\omega_{\beta 0} r)\, u\downarrow \\ j_0(\omega_{\beta} r)\, u\downarrow \end{pmatrix} e^{-i\omega_{\beta 0} t}$$

Compatible with the linear condition and the Dirac equation, we try

$$\begin{aligned}
\psi_1 = {} & \alpha_1 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_2(\omega_\alpha r)\, \mathcal{Y}_{mjl} \\ j_1(\omega_\alpha r)\, \mathcal{Y}_{mjl} \end{pmatrix} e^{-i\omega_\alpha t} + \beta_1 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_2(\omega_\beta r)\, \mathcal{Y}_{mjl} \\ j_1(\omega_\beta r)\, \mathcal{Y}_{mjl} \end{pmatrix} e^{-i\omega_\beta t} \\
& + \tilde{\alpha}_1 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_2(\omega_{\tilde{\alpha}} r)\, \mathcal{Y}_{mjl} \\ j_1(\omega_{\tilde{\alpha}} r)\, \mathcal{Y}_{mjl} \end{pmatrix} e^{-i\omega_{\tilde{\alpha}} t} + \tilde{\beta}_1 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_2(\omega_{\tilde{\beta}} r)\, \mathcal{Y}_{mjl} \\ j_1(\omega_{\tilde{\beta}} r)\, \mathcal{Y}_{mjl} \end{pmatrix} e^{-i\omega_{\tilde{\beta}} t} \\
& + \alpha_2 \begin{pmatrix} i\, j_0(\omega_\alpha r)\, u\uparrow \\ -j_1(\omega_\alpha r)\, \vec{\sigma}\cdot\hat{r}\, u\uparrow \end{pmatrix} e^{-i\omega_\alpha t} + \beta_2 \begin{pmatrix} i\, j_0(\omega_\beta r)\, u\downarrow \\ -j_1(\omega_\beta r)\, \vec{\sigma}\cdot\hat{r}\, u\downarrow \end{pmatrix} e^{-i\omega_\beta t} \\
& + \tilde{\alpha}_2 \begin{pmatrix} i\, j_0(\omega_{\tilde{\alpha}} r)\, u\uparrow \\ -j_1(\omega_{\tilde{\alpha}} r)\, \vec{\sigma}\cdot\hat{r}\, u\uparrow \end{pmatrix} e^{-i\omega_{\tilde{\alpha}} t} + \tilde{\beta}_2 \begin{pmatrix} i\, j_0(\omega_{\tilde{\beta}} r)\, u\downarrow \\ -j_1(\omega_{\tilde{\beta}} r)\, \vec{\sigma}\cdot\hat{r}\, u\downarrow \end{pmatrix} e^{-i\omega_{\tilde{\beta}} t}
\end{aligned}$$

where the usual Clebsch-Gordon decomposition is

$$\mathcal{Y}_{m j \ell} = \sum_{m_s} \langle \ell\, m_\ell\, \tfrac{1}{2}\, m_s \mid \ell\, \tfrac{1}{2}\, j\, m \rangle\, Y_{\ell m}\, U_{m_s}$$

$$j = 3/2$$

With the normal

$$n_\mu = \left(\dot{R}_1 ,\; -\hat{r} + (A e^{-i\lambda t} + A^* e^{i\lambda^* t})\left(\frac{\hat{k}}{r} - \frac{z\hat{r}}{r^2}\right)\right)$$

the linear condition gives

$$\tilde{\alpha}_1 = \sqrt{8\pi}\, \alpha_0 A^* \left[-j_1'(\omega_{\alpha_0} R_0) - j_0'(\omega_{\alpha_0} R_0) + \lambda^* j_0'(\omega_{\alpha_0} R_0) - \frac{1}{R_0} j_1'(\omega_{\alpha_0} R_0)\right] \cdot \left(j_2(\omega_{\tilde{\alpha}} R_0) + j_1(\omega_{\tilde{\alpha}} R_0)\right)^{-1} ,$$

$$\alpha_1 = \sqrt{8\pi}\, \alpha_0 A \left[-j_1'(\omega_{\alpha_0} R_0) - j_0'(\omega_{\alpha_0} R_0) - \lambda j_0'(\omega_{\alpha_0} R_0) - \frac{1}{R_0} j_1'(\omega_{\alpha_0} R_0)\right] \cdot \left(j_2(\omega_{\alpha} R_0) + j_1(\omega_{\alpha} R_0)\right)^{-1} ,$$

$$\tilde{\alpha}_2 = \alpha_0 A^* \left[j_1'(\omega_{\alpha_0} R_0) + j_0'(\omega_{\alpha_0} R_0) - \lambda^* j_0'(\omega_{\alpha_0} R_0) + j_1'(\omega_{\alpha_0} R_0) R_0^{-1}\right] \cdot \left(j_1(\omega_{\tilde{\alpha}} R_0) - j_0(\omega_{\tilde{\alpha}} R_0)\right)^{-1} ,$$

$$\alpha_2 = \alpha_0 A \left[j_1'(\omega_{\alpha_0} R_0) + j_0'(\omega_{\alpha_0} R_0) + \lambda j_0'(\omega_{\alpha_0} R_0) + j_1'(\omega_{\alpha_0} R_0) R_0^{-1}\right] \cdot \left(j_1(\omega_{\alpha} R_0) - j_0(\omega_{\alpha} R_0)\right)^{-1} ,$$

and

$$\tilde{\beta}_1 = \sqrt{8\pi}\,\left(j_2(\omega_{\tilde{\beta}}R_0) + j_1(\omega_{\tilde{\beta}}R_0)\right)^{-1}\beta_0 A^*\left[j_1' + j_0' - \lambda^* j_0' - j_1' R_0^{-1}\right]\Big|_{\omega_{\beta 0}R_0}$$

$$\beta_1 = \sqrt{8\pi}\,\left(j_2(\omega_\beta R_0) + j_1(\omega_\beta R_0)\right)^{-1}\beta_0 A\left[j_1' + j_0' + \lambda j_0' - j_1' R_0^{-1}\right]\Big|_{\omega_{\beta 0}R_0}$$

$$\tilde{\beta}_2 = \left(j_1(\omega_{\tilde{\beta}}R_0) - j_0(\omega_{\tilde{\beta}}R_0)\right)^{-1}\beta_0 A^*\left[j_1' + j_0' - \lambda^* j_0' - 2\,j_1' R_0^{-1}\right]\Big|_{\omega_{\beta 0}R_0}$$

$$\beta_2 = \left(j_1(\omega_\beta R_0) - j_0(\omega_\beta R_0)\right)^{-1}\beta_0 A\left[j_1' + j_0' + \lambda j_0' - 2\,j_1' R_0^{-1}\right]\Big|_{\omega_{\beta 0}R_0}$$

We now proceed to the quadratic boundary condition. We shall see that it suggests the need for a more general calculation. The first order condition

$$-2\,\mathrm{Re}\left[(\bar{\Psi}_0\Psi_1)'\right] = (\bar{\Psi}_0\Psi_0)''R_1$$

becomes

$$H + \tilde{H} + \tilde{H}^* + H^* = (\bar{\Psi}_0\Psi_0)''\left(Ae^{-i\lambda_1 t} + A^*e^{i\lambda_1 t}\right)\cos\theta$$

where

$$H \equiv e^{-i\lambda_1 t}\left[T_1(\alpha 0,\alpha)\sqrt{\tfrac{2}{3}}\,Y_{10}\,\alpha_0^*\alpha_1 + e^{i\Delta t}\,T_1(\alpha 0,\beta)\sqrt{\tfrac{1}{3}}\,Y_{1-1}\,\alpha_0^*\beta_1\right]$$

$$+\,e^{-i\lambda_1 t}\left[T_1(\beta 0,\alpha)\sqrt{\tfrac{1}{3}}\,Y_{11}\,\beta_0^*\alpha_1\,e^{-i\Delta t} - T_1(\beta 0,\beta)\sqrt{\tfrac{2}{3}}\,Y_{10}\,\beta_0^*\beta_1\right]$$

$$+$$

$$e^{i\lambda_1 t}\Big\{\alpha_0^*\alpha_2(\cos\theta)\,T_2(\alpha o,\alpha) + \beta_0^*\alpha_2(\sin\theta)e^{i\varphi}\,T_2(\beta o,\alpha)\,e^{-i\Delta t}$$
$$+\alpha_0^*\beta_2(\sin\theta)e^{-i\varphi}\,T_2(\alpha o,\beta)\,e^{i\Delta t} - \beta_0^*\beta_2(\cos\theta)\,T_2(\beta o,\beta)\Big\},$$

$\tilde{H}$ corresponds to the transformation

$$\alpha \to \tilde{\alpha}\,,\quad \beta \to \tilde{\beta}\,,\quad \lambda \to -\lambda^* \quad ,$$

and

$$T_1(a,b) \equiv j_0(\omega_a R_0)\Big[j_2(\omega_b R_0)\Big(\frac{5}{R_0}+\omega_a\Big) + j_1(\omega_b R_0)\Big(\frac{2}{R_0}-\omega_b-\omega_a\Big) - \omega_b j_0(\omega_b R_0)\Big]$$

$$T_2(a,b) \equiv j_0(\omega_a R_0)\Big[j_1(\omega_b R_0)\Big(\omega_a+\omega_b-\frac{2}{R_0}\Big) + j_0(\omega_b R_0)\Big(\omega_a+\omega_b+\frac{2}{R_0}\Big)\Big].$$

Eigenvalue conditions on λ result from setting the linearly independent terms equal to zero. Though difficult to interpret, their structures suggest that they are not all equivalent. For example, note the presence of the $\cos\theta$, (Y_{10}), factor multiplying $(\bar{\Psi}_o\Psi_o)''$ but not $Y_{1\pm1}$ factors. More than one condition on λ would place strong restrictions on or overspecify it.

The asymmetry associated with the Y_{lm}'s results from the choice of R_1. We expect a corresponding asymmetry with respect to any Y_{lm} to which R_1 is made proportional. A partial remedy to this situation is to make R_1 a linear combination of all Y_{lm}, or more generally, the complete set of the $Y_{\ell m}$'s . The exponentials in $i\Delta t$ also generate new conditions. A more general temporal behavior in R_1 may tend to compensate for such terms. To investigate these considerations a more general calculation has been attempted. The scheme of the rather lengthy derivation is outlined here.

The choice for R_1 is

$$R_1 = \sum_{\ell m} Y_{\ell m}(\theta,\varphi) \int_{-\infty}^{\infty} g_{\ell m}(\lambda)\, e^{-i\lambda t}\, d\lambda$$

where for real R_1

$$g_{\ell m}(\lambda) = g^{*}_{\ell m}(-\lambda)$$

The g's must be determined as well as λ which may now be taken as real. The first order linear condition is then

$$(I + i\hat{r}\cdot\vec{\gamma})\,\psi_1 = -(I + i\hat{r}\cdot\vec{\gamma})\,\psi_0' R_1 + i\Big[\dot{R}_1\gamma^0 + \sum_{\ell m} \vec{\gamma}\cdot\nabla Y_{\ell m} \int_{-\infty}^{\infty} g_{\ell m}(\lambda)\, e^{-i\lambda t}\, d\lambda\Big]\psi_0$$

As before,

$$\psi_0 = \alpha_0 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_1(w_{\alpha 0} r)\, U\!\uparrow \\ j_0(w_{\alpha 0} r)\, U\!\uparrow \end{pmatrix} e^{-iw_{\alpha 0} t} + \beta_0 \begin{pmatrix} i\vec{\sigma}\cdot\hat{r}\, j_1(w_{\beta 0} r)\, U\!\downarrow \\ j_0(w_{\beta 0} r)\, U\!\downarrow \end{pmatrix} e^{-iw_{\beta 0} t}$$

The most general choice for Ψ_1 is

$$\Psi_1 = \sum_{jm} \int_{-\infty}^{\infty} \left[A_{jm}\, e^{-i\lambda t - i\omega_{\alpha 0} t} + B_{jm}\, e^{-i\lambda t - i\omega_{\beta 0} t} \right] d\lambda$$

where

$$A_{jm} \equiv \alpha_{jm}^{-}(\lambda)\, \Psi_{jm,-1}((\omega_{\alpha 0}+\lambda)r, \Omega, t) + \alpha_{jm}^{+}(\lambda)\, \Psi_{jm1}((\omega_{\alpha 0}+\lambda)r, \Omega, t)$$

$$B_{jm} \equiv \beta_{jm}^{-}(\lambda)\, \Psi_{jm,-1}((\omega_{\beta 0}+\lambda)r, \Omega, t) + \beta_{jm}^{+}(\lambda)\, \Psi_{jm1}((\omega_{\beta 0}+\lambda)r, \Omega, t)$$

The Ψ_{jmk} are solutions to the massless free-particle Dirac equation of a given angular momentum j, z-component m, and parity k. The α and β coeffients of Ψ_1 are determined by setting terms linearly independent in time and angles equal to zero in the linear boundary condition. The resulting equations are solved simultaneously to yield equations of the form

$$a_{jm}(\lambda) = \sum_{n=n_1}^{n_2} \sum_{\ell=\ell_1}^{\ell_2} f_{\ell n}(\lambda, j, m)\, g_{\ell n}(\lambda)$$

The quadratic condition

$$-2\, \mathrm{Re}\left[(\bar{\Psi}_0 \Psi_1)' \right] = (\bar{\Psi}_0 \Psi_0)'' R_1$$

becomes an equation of the form

$$\sum_{a} \sum_{jm} \int_{-\infty}^{\infty} a_{jm}(\lambda)\left[h_{1jm}(\lambda) + h_{2jm}(\lambda)e^{-i\Delta t} + h_{3jm}(\lambda)e^{i\Delta t}\right] \cdot e^{i\lambda t}\, d\lambda = 0$$

Time independence on the left-hand side can be achieved by changing variables. Then, setting the integrand equal to zero yields the condition

$$\sum_{a\,jm} \left[a_{jm}(\lambda)h_{1jm}(\lambda) + a_{jm}(\lambda+\Delta)h_{2jm}(\lambda+\Delta) + a_{jm}(\lambda-\Delta)\cdot h_{3jm}(\lambda-\Delta)\right] = 0$$

The change of variables is possible only if

$$\lambda = \lambda_0 + p\Delta$$

exists for some λ_0 and any integer p. Then defining

$$g_{\ell n}(\lambda \pm \Delta) \equiv g_{\ell n m}(\lambda) \quad , \quad m = \pm 1$$

$$q_{\ell n i}(\lambda \pm \Delta) \equiv q_{\ell n i m}(\lambda) \quad , \quad m = \pm 1$$

$$q_{\ell n i o}(\lambda) \equiv q_{\ell n i}(\lambda) \quad , \quad g_{\ell n o}(\lambda) \equiv g_{\ell n}(\lambda) ,$$

where the q's are assumed known, the quadratic condition becomes

$$\sum_{\ell}\sum_{n}\sum_{m} g_{\ell nm}(\lambda)\, q_{\ell nim}(\lambda) = 0$$

Labeling the indices l , n , and m by k,

$$\sum_{k} g_{k}(\lambda)\, q_{ki}(\lambda) = 0 .$$

This is an infinite set of coupled equations for the g's. A solution exists if the determinant

$$\mathrm{Det}\left[q_{ki}(\lambda)\right] = 0$$

This requirement when met gives the eigenvalues of λ . We may finally note that to account for the static bag, zero is expected to be an eigenvalue. If so then λ_o must be zero.

CONCLUSION

We have shown that solutions exist to the bag equations which are not eigenstates. The stability of the bag in such states was studied by introducing radial perturbations. The resulting equations were complicated and also suggested the need for a general solution to the problem. A scheme for such a solution was outlined. Although complete solution will be difficult, this scheme does suggest that λ must have an infinite linear spectrum.

APPENDIX

It is shown here that

$$\psi \equiv \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} f(r)\, e^{iF(r,t)}\, u \\ \vec{\sigma}\cdot\hat{r}\; g(r)\, e^{iF(r,t)}\, u \end{pmatrix} \tag{A.1}$$

must reduce to a function with single-frequency behavior if it is to satisfy the Dirac equation.

In terms of spinor components, the Dirac equation is

$$\dot{\chi}_2 = -\vec{\sigma}\cdot\vec{\nabla}\chi_1 \tag{A.2}$$

$$\dot{\chi}_1 = -\vec{\sigma}\cdot\vec{\nabla}\chi_2 \tag{A.3}$$

Evaluating (A.2) ,

$$\vec{\sigma}\cdot\hat{r}\; g(r)\, \frac{\partial}{\partial t} e^{iF(r,t)} = -\vec{\sigma}\cdot\vec{\nabla} f(r)\, e^{iF(r,t)}$$
$$= -\vec{\sigma}\cdot\hat{r}\, \frac{\partial}{\partial r}(f\, e^{iF})$$

$$\vec{\sigma}\cdot\hat{r}\; e^{iF}\, i g \dot{F} = -\vec{\sigma}\cdot\hat{r}\; e^{iF}\left(f'(r) + i f F'(r,t)\right)$$

$$i g \dot{F} = -(f' + i f F') \quad , \quad \left(\text{“}\,'\,\text{”} \equiv \frac{\partial}{\partial r} \right). \tag{A.4}$$

(A.3) gives

$$f\, i\, e^{iF} \dot{F} = -(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\hat{r})\, g(r)\, e^{iF(r,t)}$$

when (A.1) is substituted. Using the algebraic properties of the Pauli matrices,

$$(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\hat{r}) = \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)$$

when applied to a function of r alone. Thus,

$$f i e^{iF}\dot{F} = -\left(\frac{2}{r} + \frac{\partial}{\partial r}\right) g(r)\, e^{iF}$$

$$f i \dot{F} = -\frac{2}{r} g - i g F' - g' \qquad \text{(A.5)}$$

Adding or subtracting (A.4) from (A.5) gives

$$f = 0 = g\,, \quad \text{if} \quad f = \pm g$$

We then time differentiate and add and subtract (A.4) and (A.5)

$$0 = \ddot{F}(f+g) + \dot{F}'(f+g)$$

$$0 = \ddot{F}(f-g) - \dot{F}'(f-g)$$

If $f \neq \pm g$, then

$$0 = \ddot{F} - \dot{F}'$$

$$0 = \ddot{F} + \dot{F}'$$

Together these imply

$$\ddot{F} = 0 \quad , \quad \dot{F}' = 0$$

Their solution is

$$F = At + B(r) , \qquad A = \text{constant}$$

$$e^{iF} = e^{iAt} e^{iB(r)} = H(r) e^{iAt}$$

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