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# Modeling entrainment volume due to surface-parallel vortex interactions with an air-water interface

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We consider the entrainment volume that results from the quasi two-dimensional interactions 8 of rising surface-parallel vorticity with an air-water interface. Based on systematic (three-9 dimensional) direct numerical simulations (DNS) of the canonical problem of a rectilinear 10 vortex pair impinging on and entraining air at the free surface, we develop a phenomenological 11 model to predict the resulting entrainment volume in terms of four key parameters. We 12 identify a new parameter, a circulation flux Froude number  $Fr_{\Xi}^2 = |\Gamma|W/a^2g$ , that predicts the dimensionless volume  $\forall$  of entrained air initiated by a coherent vortical structure of 13 14 circulation  $\Gamma$ , effective radius a, vertical rise velocity W with gravity g. For  $Fr_{\Xi}^2$  below some 15 critical value  $Fr_{\pm cr}^2$ , no air is entrained. For  $Fr_{\pm}^2 > Fr_{\pm cr}^2$ , the average initial entrainment 16  $\overline{\forall}_o$  scales linearly with  $(Fr_{\Xi}^2 - Fr_{\Xi cr}^2)$ . We also find that  $\overline{\forall}_o$  is linearly dependent on 17 circulation Weber number  $We_{\Gamma}$  for a range of vortex Bond number  $5 \leq Bo_{\Gamma} \leq 50$ , and 18 parabolically dependent on circulation Reynolds  $Re_{\Gamma}$  for  $Re_{\Gamma} \leq 2580$ . Outside of these 19 ranges, surface tension and viscosity have little effect on the initial entrainment volume. For 20 the canonical rectilinear vortex problem, the simple model predicts  $\overline{\forall}_o$  extremely well for individual coherent structures over broad ranges of  $Fr_{\Xi}^2$ ,  $We_{\Gamma}$ ,  $Bo_{\Gamma}$  and  $Re_{\Gamma}$ . We evaluate 21 22 the performance of this parameterization and phenomenological entrainment model for air 23 entrainment due to the complex periodic vortex shedding and quasi-steady wave breaking 24 behind a fully-submerged horizontal circular cylinder. For the range of parameters we 25 consider, the phenomenological model predicts the event-by-event dimensionless entrainment 26 volume measured in the DNS satisfactorily for this complex application. 27

28 Key words:

#### 29 **1. Introduction**

30 There are many possible mechanisms for entrainment observed at an air-water interface

- including four main processes: direct entrapment from droplet impacts and plunging breaking waves (e.g., Tran *et al.* 2013; Deane & Stokes 2002), plunging jets that entrain via the
- 32 waves (e.g., 11an et al. 2013, Deane & Stokes 2002), plunging jets that entrain via the
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# Abstract must not spill onto p.2

shear layer and/or jet surface disturbances (e.g., Biń 1993), surface aeration and surface 33 34 roller-bores (e.g., Leng & Chanson 2019; Ezure et al. 2011), and surface normal and tangential vortices interacting with the interface. Our interest is the latter mechanism of 35 air entrainment due to interaction of coherent vorticity with the free surface. Specifically, 36 recent numerical (Yu et al. 2019b; Masnadi et al. 2019) and experimental (André & Bardet 37 2017) investigations of strong free-surface turbulent flows identify tangential (i.e. surface-38 parallel) vortex structures interacting with an interface (and the associated near-surface vortex 39 instabilities/transformations) as an important mechanism for air entrainment. These three-40 dimensional vortex structures generally possess local quasi-two dimensional regions (such as 41 a segment of a vortex ring or the top of a horseshoe vortex structure) tangent to the free surface 42 interface and interact with it. Such originating vortex-interface interactions can be viewed as 43 a macroscopic entrainment event mechanism (Castro et al. 2016). Using this macroscopic 44 event framework, our interest is the development of a simple yet robust entrainment model to 45 predict the total volume of entrained air due to strong quasi two-dimensional, surface-parallel 46 vorticity interactions with the air-water interface. 47

The detailed interaction of horizontal (surface-parallel) vortices with a free surface in the 48 absence of entrainment is well documented (see Appendix A). Sarpkaya & Suthon (1991) 49 experimentally determined that surface-parallel vortex pairs rising towards and interacting 50 with an interface create a local depression (or a scar). Ohring & Lugt (1991) and Yu & 51 Tryggyason (1990) investigated an equivalent two-dimensional problem using numerical 52 simulations and found similar local surface deformations and ultimately identified gravity, 53 surface tension, and geometrical parameters as the main parameters influencing the interac-54 tion of the vortex pairs with the interface. While entrainment was not considered in these 55 and many subsequent studies, they do suggest the potential for entrainment to occur. Within 56 a macroscopic entrainment event framework, if such vortex interactions are strong enough to 57 overcome the stabilizing effects of gravity and surface tension, such interactions eventually 58 lead to entrainment. While the detailed interactions (eventually) involve well-documented 59 three-dimensional vortex instabilities (e.g., Sarpkaya & Suthon 1991; Dommermuth 1993; 60 André & Bardet 2017), our hypothesis is that a simple phenomenological model based 61 on the (macroscopic) parameters of the underlying quasi-two-dimensional surface-parallel 62 coherent vorticity may be able to predict the total entrainment volume (per unit length in the 63 dominant direction) without explicitly describing the details of the complex vortex breakup 64 65 and entrainment mechanisms.

Macroscopic entrainment modeling for turbulence-interface interactions initially intro-66 duced by Baldy (1993) assumes that the air entrainment is linearly related to the turbulence 67 dissipation (through similarity arguments). More recent versions of this model propose a 68 volumetric entrainment source proportional to turbulent dissipation and use experimentally 69 measured bubble distributions to populate entrained bubble models (Ma et al. 2011; Derakhti 70 & Kirby 2014). The volume source entrainment model of Castro et al. (2016) begins with 71 72 the model of a single vortex of given strength and distance from the interface and builds a probabilistic model to determine an entrainment rate and bubble distribution. All of these 73 models relate the rate of air entrainment to the underlying turbulent field (through local 74 dissipation and vorticity) with modest success. However, all of these models rely on a certain 75 amount of empiricism and assumptions that have yet to be confirmed by experiments. As yet, 76 it is not clear what key surface-parallel vorticity characteristics (and to what degree they need 77 to be resolved) are critical for predicting if, where, and how much entrainment will occur. 78 This understanding is critical for large-scale computational applications such as the complex 79 air-water flows in the near-wake behind surface ships where fully-resolved turbulent-interface 80 81 interactions is infeasible (e.g. Hendrickson & Yue 2019b; Castro et al. 2016; Drazen et al. 2010; Wyatt et al. 2008; Fu et al. 2006). 82

83 In this paper, we focus on developing a simple entrainment model based on our macroscopic entrainment event framework. We investigate entrainment initiated by coherent vortex 84 structures with a locally surface-parallel dominant direction. We perform three-dimensional 85 direct numerical simulations (DNS) of the incompressible, two-phase (air-water) Navier-86 Stokes equations on a three-dimensional Cartesian grid using a conservative volume-of-fluid 87 (cVOF) method (Weymouth & Yue 2010). To elucidate the basic underlying mechanism 88 89 and parameter dependencies, we first consider the canonical problem of air entrainment by horizontal rectilinear vortices of known circulation  $\Gamma$ , radius a, rising towards the 90 air-water interface with vertical velocity W. We perform systematic DNS over a broad 91 range of parameters ( $\Gamma$ , a, W, surface tension  $\sigma$ , gravity g, and water kinematic viscosity 92  $v_w = \mu_w / \rho_w$ ) and measure the entrainment volume  $\mathcal{V}_e$ . Using this extensive DNS dataset, 93 we identify the key dimensionless parameters relating the macroscopic vortex-interface 94 interaction and the measured entrainment. We find that the most important parameter 95 governing the initial dimensionless entrainment volume  $\overline{\forall}_o$  is the circulation flux Froude number  $Fr_{\Xi}^2 = \Xi/g$ , which measures the relative strength between the circulation flux 96 97  $\Xi = |\Gamma| W / a^2$  and gravity. The remaining key parameters we identified are the vortex 98 Reynolds number  $Re_{\Gamma}$ , Weber number  $We_{\Gamma}$  and relative depth  $z_c/a$  below the interface 99 at which these parameters are defined. We then use a representative subset of the DNS to 100 develop an explicit model based on this parameterization. This model predicts the initial 101 dimensionless entrainment volume extremely well when assessed across the entire scope of 102 parameters of the DNS cases. Finally, we assess the validity of the entrainment model for 103 a more complex problem of air entrainment due to periodic (plunging) quasi-steady wave 104 breaking in the wake of a fully-submerged horizontal circular cylinder in a uniform inflow. 105 106 For the range of parameters considered, our model predicts the event-by-event entrainment measured in the cylinder problem DNS satisfactorily for this complex problem. The modest 107 success of the model to generalize from a simple test problem to this more complex flow 108 demonstrates that we have identified the key dimensionless parameters that predict the onset, 109 general location and quantity of a (macroscopic) entrainment event for surface-parallel vortex 110 111 interactions. 112 The outline of this paper is as follows. Section 2 defines relevant quantities that describe the general interaction of a surface-parallel coherent structure with an interface. Section 3 details 113 the direct numerical simulations of the canonical problem of rectilinear vortex tube-interface 114

interactions. Section 4 identifies the set of parameters, including the circulation flux Froude

number, and summarizes the new phenomenological entrainment model. Section 5 assesses

117 the model for the periodic breaking behind a fully-submerged cylinder in a constant inflow.

118 Section 6 concludes the paper with a summary and discussion of our findings. The appendix

summarizes the salient features of air-entraining rectilinear vortex approaching a free surface

120 (comparing to non-entraining cases).

## 121 2. Air entrainment due to interaction of surface-parallel vorticity with an interface

#### 122

### 2.1. Definitions

123 Consider a general surface-parallel coherent vortex structure approaching an air-water

interface. Macroscopically, the coherent structure approaches the interface and interacts with it in a manner that ultimately results in the entrainment of a volume of air  $\mathcal{V}_e$ . The

coherent structure has volume  $V_{\Gamma}$ , circulation  $\Gamma$ , effective cylindrical radius *a* and a local

- 127 orientation that is surface parallel (see figure 1(a)). The bulk fluid (water) has density and
- kinematic viscosity  $\rho_w$  and  $\mu_w$ , and  $\sigma$  represents the surface tension coefficient of the air-
- 129 water interface. The coherent structure geometric centroid  $\mathbf{x_c} = (x_c, y_c, z_c)$  and vertical rise



Figure 1: Schematic of (a) a general, surface-parallel coherent vortex structure rising towards the air-water interface and entraining air and (b) a general set of coherent structures that satisfy the criteria  $\lambda < \lambda_{cr}$ .

velocity W strongly influence the subsequent surface interactions and deformations (e.g Yu
& Tryggvason 1990; Ohring & Lugt 1991; Lugt & Ohring 1992). Based on these given
parameters, we write the following parameterization:

133 
$$V_e = f(a, \Gamma, v_w = \mu_w / \rho_w, g, \sigma / \rho_w, W, V_{\Gamma}, z_c).$$
(2.1)

134

#### 135

149

#### 2.2. Coherent structure identification

To identify the geometric parameters of a coherent vortex structure as input to the model, we 136 construct a scalar value of the  $\lambda_2(\mathbf{x})$  eigenvalue (Jeong & Hussain 1995) as it provides a robust 137 (albeit imperfect) mechanism to spatially identify the location and orientation of a coherent 138 structure in an eigenplane across a vortex. We employ the Informed Component Labeling 139 method (ICL) (Hendrickson et al. 2020) to identify the connected regions that satisfy the 140 criterion  $\lambda_2 < \lambda_{cr}$ . ICL provides a method to identify the regions in the domain that satisfy 141 142  $\lambda_2 < \lambda_{cr}$  and for each region, determine its volume  $V_{\lambda 2}$ , centroid location  $\mathbf{x}_{\mathbf{c}}$ , extent in each Cartesian direction, and integrated quantities within the volume. With the coherent volumes 143 identified in this manner (see figure 1(b)), we define the geometric quantities of interest as the 144 vortex volume  $V_{\Gamma} \equiv V_{d2}$ , the length in the dominant direction  $L_d$  which is the largest extent 145 measured by ICL, the cross-sectional area normal to that dominant direction  $A_d = V_{\lambda 2}/L_d$ , 146 and an effective radius  $a_{\lambda 2} = \sqrt{A_d/\pi}$ . We compute circulation  $\Gamma$  and circulation flux  $\Xi$  of 147 the vorticity component in that dominant direction  $\omega_d$  using the local vertical velocity w by: 148

$$\Gamma = \frac{1}{L_d} \int \omega_d \, dV_{\lambda 2} \; ; \quad \Xi = \frac{1}{V_{\lambda 2}} \int |\omega_d| w \, dV_{\lambda 2} \; . \tag{2.2}$$

This method is geometrically simpler than the method of Masnadi *et al.* (2019), which uses the *Q*-criterion (Hunt *et al.* 1988) and constructs a spine of the coherent structure and estimates the vorticity and circulation on isoplanes perpendicular to the spine. Our method is robust and well suited for vortex structures with a clearly defined dominant direction such as those considered in this study. In the following section,  $\lambda_{cr} = 0$  as defined by Jeong & Hussain (1995). We note that the  $\lambda_2$  criterion suffers slightly when several vortices exist locally (Jeong & Hussain 1995) and it is common practice for more complex flows (e.g. §5)

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#### 3. Air entrainment due to surface-parallel vortex approaching an interface 159

160 To quantify the dependence of air entrainment on the coherent vortex properties and other physical parameters, we choose a simple canonical problem of a rectilinear surface-parallel 161 vortex tube rising towards the interface with an initially known rise velocity. This is a well-162 studied problem in the absence of entrainment (e.g., Sarpkaya 1996; Sarpkaya & Suthon 163 1991; Yu & Tryggvason 1990; Ohring & Lugt 1991; Dommermuth 1993), and the main 164 conclusions in this body of work relevant to our study are that gravity and the vortex 165 geometric parameters (depth, size, and strength) control the interaction of the vortex with 166 interface and that strong surface tension suppresses the surface curvature and formation of 167 secondary surface vorticity. 168

For this canonical problem, we perform high-resolution (three-dimensional) DNS of the 169 two-phase, incompressible Navier-Stokes equations with a fully-nonlinear interface using 170 171 the conservative Volume of Fluid method (cVOF) (Weymouth & Yue 2010). The threedimensional, two-phase incompressible Navier-Stokes equations in a single-fluid form with 172 variable density  $\rho$  and viscosity  $\mu$  are, 173

174 
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{\mathbf{g}}{Fr^2} + \frac{1}{Re}\frac{1}{\rho}\nabla \cdot \tau + \frac{1}{We}\kappa\delta_s\mathbf{n},$$
175 
$$\nabla \cdot \mathbf{u} = 0,$$
(3.1)

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0$$

where  $f(\mathbf{x}, t)$  represents the volume fraction implemented by cVOF that enables robust and 178 conservative transport of the gas-liquid interface. In (3.1), the volume fraction f provides 179 the density and viscosity via  $\rho(f) = f + (1 - f)\rho_a/\rho_w$ ,  $\mu(f) = f + (1 - f)\mu_a/\mu_w$ ; **u** 180 is the three-dimensional velocity field, p the total pressure field,  $\mathbf{g}$  the unit vector in the 181 direction of gravity,  $\tau$  the variable viscosity stress tensor,  $\delta_s$  the interfacial Dirac delta 182 function,  $\kappa$  the interface curvature, and **n** the normal vector to the interface. The equation is 183 184 non-dimensionalized by characteristic velocity scale U and length scale L and water density  $\rho_w$  and viscosity  $\mu_w$ . 185

The numerical implementation of (3.1) is a second-order (in space and time) finite-volume 186 scheme utilizing a staggered grid. Time integration is a two-stage Runge-Kutta method. We 187 employ a projection method to conserve mass and solve for the pressure. The resulting 188 variable coefficient Poisson equation is solved using the HYPRE library (Falgout et al. 189 190 2006). Finally, surface tension effects are implemented through the continuous surface force 191 (CSF) method (Brackbill et al. 1992) along with a height-function method for the interface curvature (Popinet 2009, 2018). Details of the formulation and validations of the numerical 192 method are available in Campbell et al. (2016); Yu et al. (2019a). 193

To create a known rise velocity for the vortex tube, we simulate a rectilinear surface-194 parallel vortex tube pair initially submerged below the air-water interface with initial radius 195  $a_0$ , spacing  $\ell$ , vorticity scaling  $\omega_c$ , circulation  $\Gamma_0 = \pi a_0^2 \omega_c$  and orientation angle  $\Theta$  with the x-196 axis (see figure 2). We non-dimensionalize the problem with  $L = a_0$  and  $U = \Gamma_0/a_0$  such that the simulation parameters in (3.1) are  $Fr^2 = \Gamma_0^2/ga_0^3$ ,  $Re = \Gamma_0/v_w$ , and  $We = \Gamma_0^2\rho_w/a_0\sigma$ . The density and viscosity ratios are respectively  $\rho_a/\rho_w = 0.001$  and  $\mu_a/\mu_w = 0.01$ . The 197 198 199 Cartesian domain size is  $(L_x, L_y, L_z) = (L_x, 15, 30)a_0$  with water depth  $H_w = 20a_0$ . The 200 domain width  $L_x$  varies depending on orientation angle. When  $\Theta = 0$ ,  $L_x = 30a_0$  and we 201



Figure 2: Schematic of a horizontal vortex pair rising towards the air-water interface with definitions for DNS

simulate only half of the vortex pair with a symmetry plane located at x = 0 and measure the entrainment volume based on the single vortex tube in the water. When  $\Theta > 0$ ,  $L_x = 60a_0$ and we simulate the pair of vortices and measure the entrainment volume based on the first vortex interaction with the surface. The remaining boundary conditions are symmetry planes in the *z* direction and periodic in the *y* direction.

We use (constant) grid spacing  $\Delta = 15a_0/128$ . Increased resolution of up to (constant) 207  $\Delta = 15a_0/480$  was required for large Re and small We to provide adequate resolution of 208 the laminar viscous free-surface boundary layer (Klettner & Eames 2012) and the relatively 209 small entrainment volume at small We. Based on these grid sizes, the cell Bond number 210  $Bo_{\Lambda} = g\Delta^2 \rho_w / \sigma$ , which governs entrainment volume (Yu *et al.* 2019*b*; Yu 2019), are fully 211 resolved. Grid sensitivity analyses determined that increasing the grid resolution beyond these 212 reported values resulted in a less than 1% relative change in entrained volume, which is the 213 objective of our simulations. The most grid sensitive aspect of these simulations is the surface 214 connectivity between cells that influences the average entrainment measurements, which we 215 address in §3.1. We note that if the full details of the small-scale capillary interactions are of 216 interest, conditions such as the cell Weber number  $We_{\Delta} \ll 1$  (Popinet 2018) may be relevant. 217 We initialize the DNS by assuming a Gaussian core vorticity distribution with  $\omega_v/\omega_c$  = 218  $e^{(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)/a_0^2}$ , where  $\mathbf{x}_0$  is the initial vortex core location. We solve for the one-dimensional 219 vector stream function  $\psi$  that satisfies  $\nabla^2 \psi = -\omega_{\rm v}$  using the same boundary conditions as 220 the simulation. We solve the resulting indeterminate problem up to  $\psi + const$  to determine 221 the initial velocity field  $\mathbf{u} = \nabla \times \psi$ . To construct an initial condition that reduces any initial 222 impulse on the free surface, we include the image vortices located in the air when constructing 223 the initial vorticity field. For cases when  $\Theta > 0$ , we construct the water vortex pair such that 224 the vortex closest to the surface is located at  $z_0$ . 225

We considered over a 100 DNS of rising vortex pairs covering a broad range of simulation 226 parameters: 90  $\leq$   $Fr^2 \leq$  2600, 150  $\leq$   $Re \leq$  2520, 90  $\leq$   $We \leq \infty$  and  $0 \leq \Theta \leq \pi/2$ 227 focusing mainly on entraining cases. Our interest is macroscopic quasi two-dimensional 228 229 processes, and we design the simulations to avoid three-dimensional instabilities using the following ideas. First, we set a shallow initial depth of the vortex pair  $z_0$  to suppress long-230 wavelength instabilities that arise from vortices translating/interacting over long distances 231 (Crow 1970; Sarpkaya & Suthon 1991). Second, we only consider vortices of within size 232 range  $0.1 \leq a_0/\ell \leq 0.3$  to limit short-wavelength instabilities (Widnall 1975; Sarpkaya 233 234 & Suthon 1991). Combined with the periodic boundary conditions in the y direction, we expect the initial vortex pair approach to the surface and initial interaction to be effectively 235



Figure 3: Visualization of two sample DNS simulation results. Blue isosurface f=0.5, purple  $\lambda_2 = -0.01$  for  $f \ge 0.5$ , with inset a representative vertical plane with  $\omega_y$  vorticity contours. (left)  $\ell/a_0=5$ ,  $Fr^2=790$ , Re = 628,  $We = \infty$  and  $\Theta = 0$  with vorticity contour details shown in figure 11a; (right)  $\ell/a_0=5$ ,  $Fr^2=494$ , Re = 628,  $We = \infty$  and  $\Theta = \pi/4$  with vorticity contour details shown in figure 13.

two dimensional. A detailed discussion of the vorticity and vortex core paths from these simulations appears in Appendix A, where we confirm the expected behavior of the vorticity in non-entraining cases and highlight the differences when entrainment occurs.

Generally, the surface-parallel vortex pair rises towards the interface causing the surface to deform as shown for two representative cases in figure 3. This deformation and the presence of viscosity induces a secondary vorticity of opposite sign in the water at the interface as well as a horizontal velocity component (Orlandi 1990; Ohring & Lugt 1991; Yu 2019). Depending on the parameters  $Fr^2$ , We and Re, entrainment occurs. As seen in figure 3, the initial entrainment event is essentially a two-dimensional process (as designed to provide the macroscopic modeling framework). As a result, the entrainment consists of almost uniform



Figure 4: Instantaneous non-dimensional total entrainment volume  $\forall = V_e/L_y\pi a_0^2$  as a function of Froude number  $Fr^2$  with Re=628,  $We = \infty$ , and  $\ell/a_0=5$ .  $\blacklozenge$  with error bars represent calculated average initial entrainment  $\overline{\forall}_o = \overline{V_e^o}/L_y\pi a_0^2$  with 95% confidence level.  $Fr^2=(--99;--148;---197;---395;---493;---592;---790).$ 

cylindrical tubes of air that become nonuniform and form smaller air cavities later in the entrainment process.

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#### 3.1. Quantification of the entrainment volume

We measure the total entrainment volume from the air volume fraction (1 - f) using ICL 249 as a two-level connected component algorithm, with the established optimum levels of 250  $\theta = (0.4, 0)$  (Hendrickson *et al.* 2020). Figure 4 shows a sample of the measured total 251 entrainment volume as a function of time for a series of  $Fr^2$ . As seen in this figure, the 252 total entrainment volume can fluctuate as grid resolution influences the connectivity between 253 cells in numerical simulations (Chan et al. 2021). In order to quantify the initial entrainment 254 volume  $\mathcal{V}_e^o$ , we first define the onset time ( $t_{\text{onset}}$ ) as the time where the entrained air volume exceeds and maintains a value above a threshold value of  $O(10^{-5})$ . We then sample the 255 256 total entrained volume over a set of increasingly longer time periods from 0.25 to 1.5 vortex 257 turnover times. The average initial entrainment volume  $\overline{\mathcal{V}_e^o}$  represents the sample with the 258 smallest standard deviation in that time period. Typically, the sample period is 1 or 1.5 vortex 259 turnover times. In all subsequent plots, the error bars are the 95% confidence level of this 260 261 average value.

#### 262

#### 3.2. General scaling observations

We performed DNS over a range of simulation parameters  $(Fr^2, Re, We)$  and vortex 263 parameters  $(a_0, \omega_c \text{ and } \Theta)$ . Prior to presenting the model, we first describe three key 264 observations (Yu 2019) using representative data in figure 5 to illustrate these trends. First, 265 we observe a strong dependence of  $\overline{\forall}_{o}$  on *Re* that then decreases for larger *Re*. For  $Fr^{2} = 790$ , 266 figure 5(a) shows this transition occurring at  $Re \approx 1500$  with the average initial entrainment 267 volume not increasing significantly for  $Re \gtrsim 2000$ . Second, we observe similar behavior 268 between  $\overline{\forall}_o$  and We where there is a strong dependence that then becomes less significant 269 at large We. Across the entire DNS dataset, we determined that  $\overline{\forall}_o$  is independent of We 270 for  $Bo = We/Fr^2 \gtrsim 50$ . For  $Bo \lesssim 50$ , we observe a linear relationship  $\overline{\forall}_o \propto We$  above a 271 critical value  $We_{cr}$  as shown in figure 5(b). This critical value corresponds to the minimum 272



Figure 5: Average initial dimensionless entrainment volume  $\overline{\forall}_o = \overline{\mathcal{V}_e^o} / L_y \pi a_0^2$  as a function of (a) *Re* and (b) *Bo*. In (b), <u>is</u>  $\overline{\forall}_o$  for  $We = \infty$ , inset linear axis.

273 Weber number to entrain air, which we determine to be  $Bo \sim 1-2$  as expected. Finally,

we note that in the absence of surface tension effects there is a similar critical  $Fr^2$  value above which entrainment occurs (cf. figure 4). This critical Froude number, which we find to be  $Fr^2 \sim 138$  for Re = 628, compares to the simulations of Yu & Tryggvason (1990) and Ohring & Lugt (1991) (after converting to equivalent length and velocity scales).

# 4. Parameterization and modeling of entrainment volume due to interaction of surface-parallel vorticity with an air-water interface

280

#### 4.1. Parameterization

After extensive analysis of the DNS data from the vortex pairs in §3, we identify the key parameters that govern the average initial dimensionless entrainment volume  $\overline{\forall}_o$ . The parameterization we obtain reads:

284 
$$\overline{\forall}_o \equiv \frac{\overline{V_e^o}}{V_{\Gamma}} = f\left(Fr_{\Xi}^2 = \frac{|\Gamma|}{a^2}\frac{W}{g}, We_{\Gamma} = \frac{\Gamma^2}{a(\sigma/\rho_w)}, Re_{\Gamma} = \frac{|\Gamma|}{v_w}, \frac{z_c}{a}\right).$$
(4.1)

The last three parameters are well established as being relevant to vortex interactions with 285 an interface (e.g., Yu & Tryggvason 1990; Ohring & Lugt 1991): the vortex Weber number 286  $We_{\Gamma}$  measures the effect of surface tension, the vortex Reynolds number  $Re_{\Gamma}$  viscous effects, 287 and  $z_c/a$  is the (centroid) depth relative to its effective radius a at which these parameters 288 are quantified. The key new parameter we identify is the circulation flux Froude number  $Fr_{\Xi}^2$  which measures the magnitude of the circulation flux density  $\Xi = |\Gamma| W/a^2$  relative to gravity g. The sign of the circulation  $\Gamma$  is not relevant for  $\Xi$  (or  $Re_{\Gamma}$ ) and we define  $Fr_{\Xi}^2 = \Xi/g$  for  $\Xi > 0$  and  $Fr_{\Xi}^2 = 0$  for  $\Xi < 0$  since we are only concerned with rising vorticity 289 290 291 292 W > 0. The circulation Bond number is defined as  $Bo_{\Gamma} = ga^2 \rho_w / \sigma$ . These model parameters 293 294 are calculated using the coherent structure identification in §2.2 and (2.2) with  $\lambda_{cr}=0$ . In particular,  $V_{\Gamma} = V_{\lambda 2}$  and  $a = a_{\lambda 2}$ . We propose a simple predictive model based on (4.1) by 295



Figure 6:  $\Xi/g$  as a function of  $z_c/a$  for (a) variable  $\Theta$  at fixed  $Fr^2$ , Re, and  $We = \infty$  and (b) a range of  $Fr^2$ , Re, We,  $\Gamma_0$  and  $a_0$ . In (a)  $\Theta=0$  (far right) to  $\Theta=\pi/2$  (far left).

assuming that the functional dependencies therein are separable, and write:

301

297 
$$\forall^{m} = \mathcal{F}\left(Fr_{\Xi}^{2}; \frac{z_{c}}{a}\right) \mathcal{W}\left(We_{\Gamma}; \frac{z_{c}}{a}\right) \mathcal{B}\left(Re_{\Gamma}; \frac{z_{c}}{a}\right).$$
(4.2)

Equation (4.2) with the explicit dependencies of the parameters on the depth  $z_c/a$  acknowledges that the parameter values change with the evolving vortex structure as it approaches the interface.

#### 4.2. Circulation Flux

Figure 6(a) shows  $\Xi/g$  as a function of relative distance to the static waterline  $z_c/a$  for a set 302 of cases with variable  $\Theta$ . For entraining cases, we show data for  $t < t_{onset}$ . For many oblique 303 rise angles,  $\Xi/g$  varies linearly with depth providing that  $z_c/a < -4$ . For  $z_c/a > -4$ , the 304 circulation flux depth dependence becomes more complex due to the interaction with the 305 interface. We note here that for  $\Theta = \pi/2$ , the actual vortex motion is effectively horizontal 306 and the implied rise towards the surface is due to the increase in a due to diffusion of the 307 coherent structure. We did not perform a study of horizontally translating vortices closer to 308 the surface than  $z_c/a_0 = -7.5$  so the behavior of  $\Xi/g$  for such physical cases is unclear. 309 Figure 6(b) shows the same for a range of simulation parameters Re,  $Fr^2$ , We,  $\Gamma_0$  and  $a_0$ . 310 Note in this figure that the boundary between entraining and non-entraining events is not well 311 defined as multiple parameters influence this behavior. The linear trend in  $\Xi/g$  persists for 312 significant depth  $z_c/a$  and  $\Xi/g$  for most cases. Based on figure 6(a), for simplicity, we choose 313 314  $z_c/a = -6$ , or effectively 3 coherent structure diameters below the interface, to develop the model. This choice is somewhat arbitrary and a model could conceivably be constructed with 315 316 variable  $z_c/a$  due to the linear trend observed in the data.

317 4.3. Development of the explicit entrainment volume model

#### To formally construct (4.2), we selected 60 representative DNS cases from §3 to develop respectively the functions $\mathcal{B}$ , $\mathcal{F}$ and $\mathcal{W}$ . We then estimate any relevant critical values from the resulting functions. The critical values presented in this section are estimates based on

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the DNS data and only apply to this model. When evaluated at  $z_c/a = -6$  these cases obtain the following parameter ranges for model development:  $240 \leq Re_{\Gamma} \leq 1640, 0.3 \leq Fr_{\Xi}^2 \leq 4$ , and  $40 \leq We_{\Gamma} \leq \infty$ .

Figure 7(a) shows average initial entrainment volume  $\overline{\forall}_o$  as a function of  $Re_{\Gamma}$  at  $z_c/a = -6$ for a set of simulations with variable Re,  $Fr^2$ , and  $We = \infty$ . Note that in this figure,  $\overline{\forall}_o$ is scaled by  $Fr_{\Xi}^2$ . We observe a similar trend for dependence on  $Re_{\Gamma}$  as in figure 5(a) in that  $\overline{\forall}_o \propto Re_{\Gamma}$ , and viscous effects become less important for greater  $Re_{\Gamma}$ . Based on these observations, we obtain a general fit for  $\mathcal{B}(Re_{\Gamma}; \frac{z_c}{a} = -6)$  given by:

$$\mathcal{B}(Re_{\Gamma}; \frac{z_{c}}{a} = -6) = \begin{cases} 0 & Re_{\Gamma} \leq 70 \\ c_{0}^{\nu} + c_{1}^{\nu}Re_{\Gamma} + c_{2}^{\nu}Re_{\Gamma}^{2} & 70 \leq Re_{\Gamma} \leq 2580 \\ \mathcal{B}_{max} & Re_{\Gamma} \geq 2580 \end{cases}$$
(4.3)

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Using the subset of the DNS data, we obtain  $c_0^{\nu} = -3.39 \cdot 10^{-3}$ ,  $c_1^{\nu} = 5.06 \cdot 10^{-5}$  and  $c_2^{\nu} = -9.79 \cdot 10^{-9}$ . The maximum value  $\mathcal{B}_{max}=0.062$ . The  $R^2$  value for this fit to the DNS is 0.97. We estimate the range  $70 < Re_{\Gamma} < 2580$  from the minimum root and maximum value of the function. Thus, for this  $z_c/a$ , we expect that entrainment should not occur below this  $Re_{\Gamma}$  range and should be almost independent of viscous effects above this range. Note that the quadratic fit (4.3) is quite robust and performs more satisfactorily than, say, a piece-wise linear fit with a transition at  $Re_{\Gamma} \approx 1000$ .

Figure 7(b) shows  $\overline{\forall}_o$  as a function of  $Fr_{\Xi}^2$  with  $We = \infty$ . Note that we have used (4.3) to factor out viscous effects in  $\overline{\forall}_o$ . The resulting dependence on  $Fr_{\Xi}^2$  is close to linear with a minimum critical value  $Fr_{\Xi cr}^2$  and no observed upper bound in the subset of DNS data considered. Thus, we propose as a model:

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$$\mathcal{F}(Fr_{\Xi}^2; \frac{z_c}{a} = -6) = \begin{cases} 0 & Fr_{\Xi}^2 \leq Fr_{\Xi cr}^2 \\ c_0^c + c_1^c Fr_{\Xi}^2 & \text{otherwise} \end{cases}$$
(4.4)

From the same subset of DNS data, we obtain  $c_0^c = -0.43$ ,  $c_1^c = 1.22$  with an  $R^2$  value of 0.95. The linear fit provides the critical value of  $Fr_{\Xi cr}^2 \approx 0.4$ . Similar to the bounds found in (4.3), the model predicts no entrainment below this  $Fr_{\Xi cr}^2$  for this  $z_c/a$ .

Finally, figure 7(c) shows  $\overline{\forall}_o$  as a function of  $We_{\Gamma}$  for a set of cases with  $Bo_{\Gamma} < 50$ , where we estimate surface tension is relevant (based on the entire DNS data set and illustrated in figure 5(b)); and we have used both (4.3) and (4.4) to factor out the effects of viscosity and circulation flux. We observe a linear dependence between  $\overline{\forall}_o$  and  $We_{\Gamma}$  and determine the following relationship:

$$\mathcal{W}(We_{\Gamma}; \frac{z}{a} = -6) = \begin{cases} 0 & Bo_{\Gamma} < 1\\ c_0^s + c_1^s We_{\Gamma} & 1 < Bo_{\Gamma} < 50\\ 1 & \text{otherwise} \end{cases}$$
(4.5)

where based on DNS, we have used  $Bo_{\Gamma} = 1$  as the lower limit for air entrainment. Fitting using the subset DNS obtains  $c_0^s = 1.47 \cdot 10^{-2}$  and  $c_1^s = 8.06 \cdot 10^{-5}$  with  $R^2 = 0.96$ .

Using the model (4.2) with (4.3), (4.4) and (4.5) to calculate the predicted  $\forall^m$ , figure 7(d) compares the model prediction against the DNS predictions for the 60 cases. The normalized root mean-square error we obtain is nRMSE=0.031. To verify that the model developed is sufficient and robust, we used the remaining (~40) cases of §3 for *a posteriori* model assessment. The overall model performance for DNS datasets now covering the parameter ranges:  $110 \leq Re_{\Gamma} \leq 1060, 0 \leq Fr_{\Xi}^2 \leq 13$ , and  $40 \leq We_{\Gamma} \leq \infty$ , remains excellent (see figure 7(d)).



Figure 7: (a-c) Model functions (4.3)-(4.5) with measured average initial entrainment  $\overline{\forall}_o$  for the properties at  $z_c/a = -6$ . (d) Comparison of predicted entrainment  $\forall^{model}$  with measured average initial entrainment  $\overline{\forall}_o$ . • model development cases; • *a posteriori* cases not used in the model development.

#### **5.** Performance of the entrainment volume model for a more general problem

In §4, we show that the entrainment volume parameterization and model perform quite well 361 over a broad range of the underlying physical parameters for the canonical case of a rectilinear 362 surface-parallel vortex approaching a free surface. As an illustration of the application of 363 this model, and to assess its general validity and robustness, we consider a more general and 364 complex problem of air entrainment in the wake of a fully-submerged horizontal cylinder in 365 a uniform flow. We choose this problem because, within an appreciable range of speeds and 366 submergence, quasi two-dimensional surface-parallel vortex structures behind the cylinder 367 interact with the interface leading to air entrainment. Briefly, for cylinders close to the 368 surface, the interaction of the underlying cylinder wake and the free surface creates two 369

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types of wake structures that produces quasi-steady breaking waves or spilling jet flows near 370 the first trough of the wave system behind the cylinder (e.g., Sheridan et al. 1995, 1997; 371 Reichl et al. 2005; Colagrossi et al. 2019). Although not explicitly addressed in these studies, 372 wave breaking and associated entrainment is implied. For deeper cylinders, we observed 373 both types of wake structures depending upon the cylinder Froude number (Hendrickson & 374 Yue 2019a). For large cylinder Froude numbers, we observed a turbulent bore-like region 375 376 riding the front face of the first wave behind the cylinder similar to that observed behind streamlined objects (Duncan 1981, 1983) and found no correlation between the entrainment 377 signal and the underlying cylinder wake. For small cylinder Froude numbers, we observed 378 periodic plunging breaking events at the first wave crest and determined that the measured 379 frequency of entrainment correlated with the cylinder-wake Strouhal number. It is within this 380 parameter range that we assess the predictions of the model. 381

We use a similar DNS described in §3 with a Boundary Data Immersion Method (BDIM) 382 (Maertens & Weymouth 2015) to model the no-slip body boundary condition on the fixed 383 cylinder surface. The method and numerical convergence results are described in detail 384 in (Hendrickson & Yue 2019a). Figure 8(a) is a schematic of the simulation setup and 385 parameters. For definiteness, we fix the cylinder center depth at 2d below a static water line 386 z = 0, where d is the cylinder diameter. We normalize all physical quantities by length, 387 velocity (and time) scales d, U, the uniform inflow velocity (and d/U) respectively. We 388 neglect surface tension and focus on the low cylinder Froude number  $(Fr_d = U/\sqrt{gd})$ 389 regime we identified that produces the periodic plunging breaking events. Figures 8(b)-8(c) 390 show an instantaneous visualization of a individual entrainment event for a representative 391 simulation at Reynolds number  $Re_d = Ud/v_w = 250$ . The sequence shows an individual 392 entrainment event, which occurs on the first wave behind the cylinder, in relation to a wide 393 view of the transverse vorticity field. In this sequence, the plunging breaker forms, plunges, 394 395 entrains air and resets. This event repeats with a frequency that correlates with the measured cylinder wake Strouhal number (Hendrickson & Yue 2019a). Figure 9(a) shows a sample 396 plot of the entrainment volume as a function of time. The periodicity of the entrainment 397 events correlates with the cylinder-wake Strouhal number for all cases used in this analysis. 398

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#### 5.1. Analysis of DNS data for the cylinder wake entrainment problem

We apply the analysis of §2.2 and §4 to the vortical air-entraining flow behind the fullysubmerged cylinder. This section provides an overview of the analysis process and steps taken to assess and reduce, where necessary, sensitivity of the estimates we obtain.

First, we measure the total entrainment volume of the *n*-the entrainment event as  $\mathcal{V}_e^{(n)} = \mathcal{V}_e(t_e^{(n)}) - \min(\mathcal{V}_e(t_e^{(n-1)} : t_e^{(n)}))$ , where  $t_e^{(n)}$  is the time of the *n*-th local peak in the 403 404 total entrainment volume (cf. figure 9(a)). We include only the initial (largest) peaks, 405 neglecting smaller secondary entrainment measurements close to the initial event in keeping 406 with the macroscopic framework. The average entrainment volume for  $N_e$  events is  $\overline{V_e}$  = 407  $(\Sigma_n \mathcal{V}_e^{(n)})/N_e$ . Figure 9(b) shows the average entrainment volume as a function of cylinder 408 Froude number, with the 95% confidence value bar based on the standard deviation. The 409 entrainment volume scales linearly with cylinder Froude number (as noted in Hendrickson 410 & Yue (2019a)). 411

Second, to identify the coherent structures in the flow field, we must choose a value of  $\lambda_{cr}$ to identify and quantify the coherent vortex structures, as discussed in §2.2. The value of  $\lambda_{cr}$ generally determines the size and extent of the identified vortex structure, with the threshold enstrophy scaled by  $|\omega|^2 \gtrsim -4\lambda_{cr}$  (Jeong & Hussain 1995). For the current problem, we find that the cylinder wake is not captured well for  $\lambda_{cr} \lesssim -15$ , while for  $\lambda_{cr} \gtrsim -5$  the resulting vortex structures are large and diffused. Thus we consider effect of variations in  $\lambda_{cr}$  centered



Figure 8: (a) A schematic for DNS of a flow past a fully-submerged cylinder; (b) center-plane instantaneous transverse vorticity contours  $\omega_y$  and air-water interface; and (c) close up visualization of individual entrainment event with blue isosurface f=0.5 and purple  $\lambda_2 = -10$  for  $f \ge 0.5$  with a representative vertical plane transverse vorticity (inset). Cylinder DNS parameters:  $Re_d=250$  and  $Fr_d^2 = 0.27$ .

around  $\lambda_{cr} = -10$  and assess the sensitivity in the model prediction. Overall, we find that varying  $\lambda_{cr}$  in the range  $-12 \le \lambda_{cr} \le -8$  resulted in a (general) change in average structure volume of 17-10% relative to  $\lambda_{cr} = -10$ . More details on the overall sensitivity of the model performance to the value of  $\lambda_{cr}$  appear in §5.2.

Third, unlike the laminar vortex pair dataset, it is necessary to estimate the local effective viscosity in the mixing region behind the cylinder containing the coherent vorticity (for



Figure 9: DNS of a fully submerged cylinder: (a) Total entrainment volume as a function of time for  $Fr_d^2$ =0.31 and  $Re_d$ =250; •  $\mathcal{V}_e(t_e^n)$ . (b) Average entrainment as a function of  $Fr_d^2$  for  $Re_d$ =250 with 95% confidence error bars.

obtaining  $Re_{\Gamma}$  in the model) as we expect the Reynolds number and turbulent viscosity to 424 increase along the streamwise direction (e.g., Pope 2000). To do this, we use a technique 425 from iLES (Aspden *et al.* 2008; Hendrickson *et al.* 2019) that calculates  $v_{\text{eff}} = \overline{\epsilon/\mathcal{D}}$  within a 426 control volume. Here,  $\epsilon$  is the (negative) time rate of change of the kinetic energy  $\rho_w \vec{u} \cdot \vec{u}/2$ 427 in the control volume, accounting for the appropriate fluxes in and out of the volume, and 428  $\mathcal{D} = V^{-1} \int \vec{u} \cdot \nabla^2 \vec{u} dV$  represents the work done by diffusion. To ensure that the control 429 volume is sufficiently large to contain the coherent structure of interest, we select the control 430 volume V as 2 < x/d < 3 and -2.5 < z/d < -1.5 to include the vertical extent of the 431 cylinder near the observed breaking events. Using this, we obtain  $860 < v_{eff}^{-1} < 870$  for the 432 four cases in figure 9(b). An extensive sensitivity analysis on the effect of the domain and 433 size of the V show that  $v_{\text{eff}}$  changes by less than 0.6% with a 5% change in V. 434

435 Finally, we identify which (if any) of the coherent vortex structures found in the cylinder wake correlates with an observed entrainment event. To do this, we develop an event-436 by-event analysis technique that is free from subjective bias. Between entrainment events 437  $t_e^{(n-1)} < t < t_e^{(n)}$ , we first bin the identified structures by their location behind the cylinder 438 and assess whether they meet the criteria developed in section \$4.3 — namely a positive 439 circulation flux and centroid locations  $-7 < z_c/a < -4$  below the static water line. We 440 observe that the coherent vortex structures shed from the lower portion of the cylinder 441 always meet the criteria when x/d > 1, and we use this subset of the coherent structures for 442 further analysis. 443

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#### 5.2. Model Predictions for Entrainment in Cylinder Wake

Using data analysis outlined in §5.1, we identify the coherent vortex structures and estimate the associated model parameters for each *n*-th entrainment event (across the four cylinder cases, the total number events is 37). Figure 10(a) shows the resulting  $Re_{\Gamma}^{(n)} = |\Gamma|^{(n)}/v_{\text{eff}}$ and  $z_c/a$  using  $\lambda_{cr} = -10$ . For all of the events, the relative vortex centroid depth below the static waterline has an average value of  $z_c/a = -6.4$ , thus enabling us to use (4.3)-(4.4)



Figure 10: (a) Whisker plot of  $(z_c/a)^{(n)}$  and  $Re_{\Gamma}^{(n)} = |\Gamma|^{(n)}/\nu_{eff}$  for all entrainment events (all cylinder cases). (b) Comparison of predicted entrainment  $\forall^m$  from (4.2)–(4.5) with measured entrainment  $\forall$ . In (b) for reference:  $Fr_d^2 = (\bullet 0.27; \bullet 0.29; \bullet 0.31; \bullet 0.33);$  $\bullet$  average value for each  $Fr_d^2$  case; and  $- - \forall = \forall^m$ .  $\lambda_{cr} = -10$ 

without modification. For reference, the average (effective) circulation Reynolds number is 1664 and all entrainment events fall within the range of (4.3) where Reynolds number is a factor in the entrainment volume prediction.

Figure 10(b) shows a comparison of the normalized observed event entrainment volume 453  $\forall = \mathcal{V}_e^{(n)} / \mathcal{V}_{\lambda 2}^{(n)}$  and the predicted entrainment volume using (4.2)–(4.5). The normalized RMSE for all of the events is nRMSE=0.175 (RMSE=0.0035). For information, we indicate 454 455 the  $Fr_d^2$  of each event and include an average estimate for each  $Fr_d^2$  case with  $\overline{Re_{\Gamma}}$  = 456  $N_e^{-1}\Sigma_n\Gamma^{(n)}/\nu_{\text{eff}}, \overline{Fr_{\Xi}^2} = N_e^{-1}\Sigma_n(Fr_{\Xi}^2)^{(n)}$  and  $\overline{\forall} = N_e^{-1}\Sigma_n \forall^{(n)}$ . The normalized RMSE is nRMSE=0.162 (RMSE=0.0023) using these average quantities. These results are for  $\lambda_{cr}$  = 457 458 -10. As previously noted, the model prediction is somewhat sensitive to this value of  $\lambda_{cr}$ . For 459  $-12 \leq \lambda_{cr} \leq -8$ , the nRMSE over all events varies between 0.18 and 0.21, not significantly 460 different from the  $\lambda_{cr} = -10$  baseline model prediction. Despite the differences between this 461 problem and the model development problem in §3 (such as the presence of weak free-surface 462 turbulence and a strong convective inflow velocity), the macroscopic framework satisfactorily 463 predicts the initial entrainment volume that correlates with the large vortex structures behind 464 the cylinder. This indicates that the key dimensionless parameters we identified, namely  $Fr_{\Xi}^2$ 465 and  $Re_{\Gamma}$ , and the functional dependence of entrainment on them, provide a useful model for 466 air entrainment by large-scale, surface-parallel coherent vortical structures. 467

#### 468 6. Summary

We perform direct numerical simulations (DNS), data analysis, and macroscopic entrainment event modeling of the initial entrainment volume due to the interaction of a rectilinear surface-parallel vortex rising towards an air-water interface. For the general case, we develop a coherent structure identification scheme to estimate and quantify the vortex geometric and kinematic quantities. These include the circulation  $\Gamma$  of the primary horizontal vorticity, equivalent radius *a*, and vertical rise velocity *W*.

As a canonical problem, we consider the air entrainment by a surface-parallel coherent 475 vortex structure rising toward the free surface as part of a rectilinear vortex pair. For this 476 problem we perform systematic three-dimensional DNS of the incompressible, two-phase 477 Navier-Stokes equations over broad ranges of physical parameters — Froude, Weber and 478 Reynolds numbers; as well as vortex radius a, vortex pair separation  $\ell$  and orientation angle 479  $\Theta$  (which affects the rise velocity W). Based on the extensive DNS data we generated for this 480 problem, we obtain the key parameterization for the air entrainment volume. We identify a 481 new parameter, the circulation flux Froude number  $Fr_{\Xi}^2 = \Xi/g$  which measures the relative importance of the circulation flux  $\Xi \equiv \Gamma W/a^2$  to gravity g, and controls the air volume 482 483 entrained. We show that it is the most important parameter governing the total volume of air 484 entrained. 485

We propose a simple, phenomenological model for the entrainment volume expressed as the product of three factors representing the separate effects of  $Fr_{\Xi}^2$ , the circulation Weber number  $We_{\Gamma} = \Gamma^2(a(\sigma/\rho_w))^{-1}$ , and the circulation Reynolds number  $Re_{\Gamma} = |\Gamma|/\nu_w$ , all defined/estimated at some distance  $z_c/a$  below the free surface. The model performs extremely well over a broad range of these parameters for the vortex pair DNS cases.

To assess the usefulness and robustness of the new model, we apply it to a more general 491 and complex problem of air entrainment by the horizontal vorticity shed from a submerged 492 circular cylinder in uniform flow. We focus on and perform DNS for the range of cylinder 493 Froude number characterized by large horizontal coherent vortical structures in the wake and 494 quasi-periodic wave breaking and air entrainment on the surface. Using a general vorticity 495 identification scheme to obtain the model parameters for the coherent horizontal vorticity 496 structures and kinematics, the phenomenological model performs satisfactorily compared to 497 DNS of the cylinder problem. 498

Our main interest in this work is predicting the total entrained volume in the context of 499 a macroscopic event framework rather than the detailed structures and mechanisms of the 500 vortex-interface interactions. In this context, we have focused on macroscopic air entrainment 501 502 events associated with coherent structures with primarily surface-parallel vorticity aligned in a dominant direction. Despite these restrictions, such scenarios are common, for example, 503 in air entrainment in large Froude number (isotropic) free-surface turbulence (see, e.g., Yu 504 et al. 2019a) and vertically immersed flat-plate boundary layers (Masnadi et al. 2019). Our 505 hypothesis that a phenomenological model based on macroscopic parameterization of the 506 vortical structure can capture the subsequent entrainment volume has guided and simplified 507 the current approach and modeling for the types of interactions studied here. This work 508 identifies the key parameters necessary for macroscopic entrainment modeling for large-509 scale computations where high-fidelity flow data is unavailable. An immediate application, 510 for example, is to obtain predictions using the present model of the entrainment volume in a 511 ship wake, based on resolved flow quantities obtained from large-scale uRANS calculations. 512 Such predictions, when compared to available measurements, can provide an assessment of 513 the validity and applicability of the present model. This work is now underway. 514

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# Appendix A. DNS of entrainment by a surface-parallel vortex pair near an air-water interface

The interaction of a horizontal vortex pair with a free surface in the absence of entrainment is well documented (Sarpkaya 1996; Rood 1994*b*,*a*, 1995; Sarpkaya & Suthon 1991; Ohring & Lugt 1991; Yu & Tryggvason 1990; Dommermuth 1993). This appendix verifies our simulations for non-entraining vortex behavior and highlights the differences of the behavior when entrainment occurs.

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#### A.1. Verification of non-entraining vortex behavior

Each column of figure 11 shows a sequence of instantaneous vorticity contours for different 530  $Fr^2$  and We and at increasing time instances during the evolution. The non-dimensional 531 entrainment volume  $\overline{\forall}_o = \mathcal{V}_e/(L_y \pi a_0^2)$  increases from left to right,  $\overline{\forall}_o = (0.0, 0.013, 0.097)$ . The bottom row represents the corresponding vortex motion of the two strongest vortices in 532 533 the water (as identified using the method in §2.2 with  $\lambda_{cr} = 0$ ). As the primary (positive) 534 vortex rises towards the interface, the surface deforms and secondary (negative) vorticity 535 536 forms in both fluids at the surface due to interface curvature and the presence of viscosity (Rood 1994b, a, 1995; Orlandi 1990). If the secondary vorticity is strong enough (columns 2 537 and 3), it couples with the primary vortex and this new pair (of unequal strengths) process 538 around each other via mutual induction (Ohring & Lugt 1991; Orlandi 1990). In column 1, 539 surface tension sufficiently suppresses the surface curvature such that no significant coupling 540 541 occurs and the primary vortex follows a path similar to the classical solution of the path of an inviscid point vortex near a wall (Lamb 1932). Our simulations showed that the primary 542 vortex path will always approach this classical solution when Froude number is reduced 543 or surface tension restricts the surface motion. This finding is consistent with the viscous 544 simulations of Ohring & Lugt (1991) and the inviscid simulations of Yu & Tryggvason 545 546 (1990).

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#### A.2. Vortex behavior for entrainment cases

Columns 2 and 3 of figure 11 include entrainment events. In both cases, the entrained air generally tracks with the secondary vortex as the coupled pair retreats deeper into the water bulk. For larger  $Fr^2$  (column 3), multiple secondary vortices appear in the water implying that the vortex procession involves three or more vortex structures, resulting in increased entrainment volume.

A survey of our simulations show that the coupling of the vortices and the procession of the 553 secondary vorticity into the bulk can correlate with entrainment. Figure 12 shows the path of 554 the primary vortex center for a range of  $Fr^2$  and We. First, we note that the gravity has a much 555 stronger influence on the primary vortex approach to the surface compared to surface tension, 556 as indicated by the point at which the vortex motion transitions from vertical to horizontal. 557 Second, the return of the primary vortex into the water bulk (below the classical theory line) 558 indicates stronger secondary surface vorticity and coupling. The steeper the return of the 559 primary vortex to the bulk, the stronger the coupling and the resulting entrainment. However, 560 as shown by Ohring & Lugt (1991) and confirmed in the vortex paths in figures 11 and 12, 561 this event is not a sufficient indicator for entrainment by itself. What our simulations do show 562 is that this coupling and procession is likely responsible for the transport of air to the bulk. 563

We summarize briefly the vortex behavior for the oblique cases using  $\Theta = \pi/4$  as an illustration in figure 13. The vortex pair rises obliquely to the interface consistent with the orientation angle and the surface deforms due to the influence of vortex 1 (the positive vortex closes to the interface at t = 0). This generates secondary negative vorticity at the interface. Vortex 1 turns away from inclination axis, as expected by the classical theory and shown in Lugt & Ohring (1994). As in the  $\Theta = 0$  case, if the secondary surface vorticity near vortex 1 is strong enough, they will couple and process downward into the bulk. After vortex 1 couples to its secondary surface vorticity, the contours representing vortex 2 become circular and it continues its approach to the surface following the original rise angle. In all cases considered, the surface deformation associated with vortex 2 is significantly less but can also cause entrainment. In our simulations we focused on entrainment events by vortex 1 only and observed entrainment for all angles  $\Theta < 75^{\circ}$  at the initial depth studied.



Figure 11: Contours of transverse vorticity  $\omega_y/\omega_c$  ( $\omega_y<0$ ) on the center-plane for  $Re = 628 \ \ell/a_0=5$ ; for  $(Fr^2, We) = (790, 99)$  (left); (197,  $\infty$ ) (center); and (790,  $\infty$ ) (right); for increasing time instances in the evolution (top 3 rows). Vorticity bounds are -1.5, 1.5. Black line near z = 0 is interface. Bottom row: Primary vortex path for  $\diamond \omega_y > 0$  and  $\diamond \omega_y < 0$  with — Lamb (1932) and - - z = 0.



Figure 12: Primary vortex path prior to entrainment onset (if entraining)  $t < t_{onset}$  for Re = 628 and  $\ell/a_0 = 5$ . Filled symbols are entraining cases. Lamb (1932); --- z = 0.



Figure 13: Contours of transverse vorticity  $\omega_y/\omega_c$  (---- $\omega_y<0$ ) on the center-plane for an oblique incidence angle  $\Theta = \pi/4$  with  $\ell/a_0=5$ ,  $Fr^2 = 494$ , Re = 628 and  $We = \infty$ . Time evolution is top-down, left-right. Red contours are negative vorticity. Vorticity bounds are -1.5, 1.5. Black line near z = 0 is interface.

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