A DECISION SUPPORT MODEL FOR THE INVESTMENT PLANNING OF
THE RECONSTRUCTION AND REHABILITATION
OF MATURE WATER DISTRIBUTION SYSTEMS

by

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ABSTRACT

The problem of decaying American infrastructure has lately been given considerable emphasis, due to its importance to societal needs and to the large amount of capital expenditures needed to bring the concerned systems to an adequate level of serviceability. In particular, the deterioration of aging water distribution systems in many service areas calls for better maintenance decision-making related to the future repair, replacement or rehabilitation of the equipment. An evaluation of these alternatives would make it possible to optimally allocate budgeted resources, which are based on the estimation of future needs for different maintenance and capital improvement measures.

With an estimated total investment of the order of 50 billion dollars over the next ten to twenty years required for the maintenance, rehabilitation and rebuilding of water mains, a thorough analysis of this category of decision problems is bound to cause considerable savings for water utilities.

The purpose of this thesis is to build a decision support model for the planning of capital improvement programs in a mature water distribution system. Such a model would be part of a comprehensive maintenance management system, which would also determine the required operation and maintenance standards, as well as the needs for future maintenance and capital improvement measures.

The two major components of the decision support model are: a) the economic analysis of alternative measures such as rehabilitation, replacement or expansion at the level of each pipe or bundle of pipes of similar characteristics, and b) the scheduling of capital improvement expenditures over time for the whole distribution network.

The first component requires the prediction of future repair and operational costs due respectively to the problems of breakage and loss of carrying capacity. Predictive models for pipe failure are examined, and two approaches are suggested: a regression-based model for small diameter pipes with relatively higher frequency of breakage, and a probabilistic model based on the Cox regression for large diameter pipes with low frequency, high outcome breakage characteristics.
Based on such predictive models, optimal measures can be derived at the level of each pipe or bundle of similar pipes, taking into account the dynamic evolution of maintenance and operational costs. Given the streams of these cash flows, optimal replacement or rehabilitation times can be evaluated.

Based on the economic analysis, the planning of future capital expenditures can be performed. A general scheduling optimization model with a network substructure is first derived, where the expansion alternative is considered at the network level. In real situations, the expansion alternative is considered at the pipe level, as the future flows are exogenously obtained from a flow model based on changing needs in the vicinity of the pipe. Within this context, the scheduling problem is approached as a capital budgeting problem. Two possible structures are suggested for scheduling bundles of pipes of similar characteristics.

a) A linear programming model, particularly relevant for large utilities. The network is divided into a number of large bundles. The decision variables are the fractions of each bundle to be implemented in different years. A case example illustrates this model.

b) A mixed integer multi-period programming model for the case of small bundles implemented as projects. This model is quite suitable for small utilities, but can be also applied to large utilities, provided that the network is decomposed into small bundles. A case study illustrates this model.

In both case examples, it is shown that the existence of a resource transfer mechanism from one period to the next one, enhances the scheduling strategy.

The outputs of different multi-period models have to be sequenced into yearly plans. A branch-and-bound Lagrangian relaxation procedure is suggested as a general solution approach to the sequencing problem.

The problem of real-time response to breaks in major water mains is analyzed within the framework of Crisis Decision Analysis, as the predicted performance of the pipe is updated through the Cox regression model.

The aforementioned scheduling models are integrated into a global planning process which is adaptive with regard to the external planning parameters such as demands and system performance.

Thesis Supervisor: Professor David H. Marks
Title: Professor of Civil Engineering
Preliminary Note

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Dr. Robert Clark, whose research and comments provided significant insights to this work deserves special thanks. We are also grateful to Mr. Jeff Adams and other EPA officials for their pertinent inputs related to the New Haven data set which analysis supported an important part of this work.
ACKNOWLEDGEMENTS

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I was fortunate to work with Professor David Marks, whose exceptional conceptual capabilities, sense of humor, and friendly style added a special dimension to my excellent experience at M.I.T.

Professor William Dumouchel and Stefanos Andreou are thanked for their input related to the analysis of the New Haven data set. Special thanks to Connie Choquet who typed the major part of this thesis never losing patience or her smile. Jackie Winton is also thanked for her contribution.

Finally, I am grateful to my friends, Vanessa, Nathalie and Michael and to my family for their support throughout the shaping of this work. To them this thesis is dedicated.
# Table of Contents

<table>
<thead>
<tr>
<th>Chapter/Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title Page</td>
<td>i</td>
</tr>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Preliminary Note</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgement</td>
<td>v</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>List of Figures</td>
<td>x</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xii</td>
</tr>
<tr>
<td>Chapter 1: Maintenance Policies and Issues in Urban Water Distribution Systems</td>
<td>1</td>
</tr>
<tr>
<td>I. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>II. Problems in Water Distribution Systems</td>
<td>4</td>
</tr>
<tr>
<td>III. Maintenance Practices and Their Impacts on the System</td>
<td>7</td>
</tr>
<tr>
<td>1. What is Maintenance?</td>
<td>7</td>
</tr>
<tr>
<td>2. Present Situation in U.S. Distribution Systems</td>
<td>11</td>
</tr>
<tr>
<td>IV. Past and Present Approaches to Maintenance Policy</td>
<td>15</td>
</tr>
<tr>
<td>1. Past Maintenance Policy</td>
<td>15</td>
</tr>
<tr>
<td>2. Current Maintenance Thrusts</td>
<td>16</td>
</tr>
<tr>
<td>Chapter 2: Development of Maintenance Management Systems</td>
<td>21</td>
</tr>
<tr>
<td>I. Understanding the System Status</td>
<td>21</td>
</tr>
<tr>
<td>II. Scope of this Work</td>
<td>24</td>
</tr>
<tr>
<td>III. Specific Issues Addressed in the Work</td>
<td>28</td>
</tr>
<tr>
<td>IV. Issues Related to the Development of Predictive Models for Future Pipe Performance</td>
<td>30</td>
</tr>
</tbody>
</table>
Chapter 3: Economic Analysis of Alternative Maintenance Measures at the Single Pipe Level

A. Economic Analysis of the Rehabilitation of Water Mains
   I. Conditions for Rehabilitation
   II. Evaluation of the Additional Operating Costs in Aging Water Mains

B. Economic Analysis of the Replacement Alternative
   I. Optimal Replacement Time for Pipes Failing and Experiencing Head Losses
   II. Optimal Replacement Time for a Pipe Experiencing only Failure Events
      1. Predictive Model Used

C. Analysis of the Rehabilitation Followed by Replacement Alternative

D. Selection of the Optimal Alternative at the Pipe Level

Chapter 4: The Prediction of Pipe Failure

I. Causes of Water Main Failure

II. Predicting Water Main Failures
   1. Shamir and Howard's Models
   2. Event Estimating Equations (from Clark et al. (1982))
   3. A Probabilistic Model for the Prediction of Pipe Failure
      A. Description of the Model
      B. The Proportional Hazards Model

III. Evaluation of the Expected Number of Breaks in a Given Year

Chapter 5: A Planning Model for the Scheduling of Maintenance and Expansion of Mature Water Distribution Systems

I. A Decision Support System Perspective of Distribution Planning
II. Mature Water Distribution System Planning: The Incremental Approach 107

III. The Scheduling Models for Mature Water Distribution Systems 110
   A. The Multi-Period Approach 111
      1. The Choice of the Time Horizon and Its Temporal Disaggregation 111
      2. The Planning Process 111
   B. The Scheduling Models Under Deterministic Peak Demand 116
      I. The Network Models 118
      II. A Capital Budgeting Structure of the Scheduling Model 134
         a) The Integer Programming Model 135
      III. A Linear Programming Structure of the Capital Budgeting Model 142
      IV. The Incorporation of Uncertainties in The Planning Process 146
   V. Case Examples 152
      1. Application of the Linear Programming Model for Capital Budgeting 152
      2. Application of the Integer Programming Model for Capital Budgeting 162

Chapter 6: The Short-term Sequencing Model for Capital Improvement Measures 167

I. Purpose of the Short-term Sequencing Model 167
II. The Sequencing Model 168
   A. Problem Background 168
   B. Problem Formulation 170
   C. Solution Procedure Using a Brand-and-Bound Lagrangian Relaxation Approach 172
III. An Inference into the Financing Issues of the
Upgrading Program Associated with the Mainte-
nance Management System  

Chapter 7.: Real time Response to Breaks in Major Water Mains  

I. The Hidden Costs of Breaks in Major Water Mains 177  

II. The Importance of a Probabilistic Model of 
Failure Rate of Major Water Mains 178  

III. Analytical Evaluation of Replacement and 
Repair Alternatives 179  

IV. Replacement Versus Repair Decision-Making: 
The Crisis Decision Analysis Framework 181  

Chapter 8: Summary and Conclusions 186  

Recommendations for Further Research 193  

Appendix A: - Foundations of the Cox Regression Method and 
Its Application to Pipe Failure 195  

Appendix B: - An Application of the Cox Regression Model 207  

Appendix A II: Linear Programming Case 223  

Appendix A III: Integer Programming Model Case 244  

References 252
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Problems in water distribution systems at a given point of time: their respective relationship and their impacts</td>
<td>6</td>
</tr>
<tr>
<td>1.2</td>
<td>Maintenance programs in aging systems: still a supply Related Approach</td>
<td>19</td>
</tr>
<tr>
<td>1.3</td>
<td>Influence diagram for a maintenance management system</td>
<td>20</td>
</tr>
<tr>
<td>2.1</td>
<td>Impact of the evaluation of the system status/performance on planning policy and practices: A feedback scheme</td>
<td>25</td>
</tr>
<tr>
<td>2.2</td>
<td>Flow of tasks in a maintenance Management System</td>
<td>31</td>
</tr>
<tr>
<td>3.1</td>
<td>Trend curves for head loss tests</td>
<td>51</td>
</tr>
<tr>
<td>3.2</td>
<td>Decision tree at the single pipe level</td>
<td>63</td>
</tr>
<tr>
<td>3.3</td>
<td>Project management approach to pipe maintenance decision-making</td>
<td>66</td>
</tr>
<tr>
<td>4.1</td>
<td>Break history for the data set</td>
<td>80</td>
</tr>
<tr>
<td>4.2</td>
<td>Predicted actual breaks for combined data set</td>
<td>85</td>
</tr>
<tr>
<td>5.1</td>
<td>Interactive Use of the scheduling model</td>
<td>106</td>
</tr>
<tr>
<td>5.2</td>
<td>The two sides of the maintenance/expansion decision support system</td>
<td>108</td>
</tr>
<tr>
<td>5.3</td>
<td>Schematic diagram of a real network</td>
<td>109</td>
</tr>
<tr>
<td>5.4</td>
<td>The Planning Process</td>
<td>114</td>
</tr>
<tr>
<td>5.5</td>
<td>Iterative Decomposition of the set of pipes into bundles</td>
<td>141</td>
</tr>
<tr>
<td>5.6</td>
<td>Major software tools for maintenance planning</td>
<td>144</td>
</tr>
<tr>
<td>5.7</td>
<td>Performance and economic resiliency indices</td>
<td>151</td>
</tr>
<tr>
<td>Fig. No.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>7.1</td>
<td>Sample of key parameters (crisis decision analysis)</td>
<td>183</td>
</tr>
<tr>
<td>8.1</td>
<td>From the <em>New York Times</em>, February 24, 1984</td>
<td>187</td>
</tr>
<tr>
<td>AII.1</td>
<td>Data distribution by age in 1983</td>
<td>214</td>
</tr>
<tr>
<td>AII-2</td>
<td>Data distribution by number of breaks</td>
<td>215</td>
</tr>
<tr>
<td>AII-3</td>
<td>Data distribution by diameter</td>
<td>216</td>
</tr>
<tr>
<td>AII-4</td>
<td>Data distribution by length</td>
<td>217</td>
</tr>
<tr>
<td>AII-5</td>
<td>Data distribution by pressure</td>
<td>218</td>
</tr>
<tr>
<td>AII-6</td>
<td>Basic survival function for different strata</td>
<td>220</td>
</tr>
<tr>
<td>AII-7</td>
<td>Log minus log survival function for different strata</td>
<td>220</td>
</tr>
<tr>
<td>AII-8</td>
<td>Goodness-of-fit test for the stratified case</td>
<td>221</td>
</tr>
<tr>
<td>AII-9</td>
<td>Goodness-of-fit test for the unstratified case</td>
<td>222</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table No.</th>
<th>Table Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Estimated Water Leakage in 12 US Cities</td>
<td>9</td>
</tr>
<tr>
<td>1.2</td>
<td>Unaccounted-for Water-1978 (Boston)</td>
<td>10</td>
</tr>
<tr>
<td>1.3</td>
<td>Comparison of Water Main Failures in 14 Cities</td>
<td>12</td>
</tr>
<tr>
<td>1.4</td>
<td>Importance of Different Criteria for Replacement/Rehabilitation Decision-making, Under Different Types of Maintenance Policies</td>
<td>14</td>
</tr>
<tr>
<td>1.5</td>
<td>Average Costs, Benefits, and Net Benefits of Leak Detection and Repair for Three Surveys - Dollars per Leak</td>
<td>18</td>
</tr>
<tr>
<td>1.6</td>
<td>Boston Water System Capital Improvement Program</td>
<td>29</td>
</tr>
<tr>
<td>3.1</td>
<td>Unit Cost for Rehabilitation of Water Mains</td>
<td>45</td>
</tr>
<tr>
<td>3.2</td>
<td>Typical Values and Ranges for Parameter</td>
<td>57</td>
</tr>
<tr>
<td>4.1</td>
<td>Categorization of External Factors Affecting Main Breaks</td>
<td>71</td>
</tr>
<tr>
<td>4.2</td>
<td>Partial Correlations</td>
<td>81</td>
</tr>
<tr>
<td>4.3</td>
<td>Partial Correlations</td>
<td>83</td>
</tr>
<tr>
<td>4.4</td>
<td>Categorization of Pipes According to Reliability Standards</td>
<td>102</td>
</tr>
<tr>
<td>5.1</td>
<td>Criteria for Bundle Selection</td>
<td>138</td>
</tr>
<tr>
<td>5.1a</td>
<td>Bundles and Characteristics</td>
<td>154</td>
</tr>
<tr>
<td>5.2</td>
<td>Budget Projections (Example 1)</td>
<td>155</td>
</tr>
<tr>
<td>5.3</td>
<td>Summary of the Solution of the Linear Programming Model</td>
<td>156</td>
</tr>
<tr>
<td>5.4</td>
<td>Budget Projections (Example 2)</td>
<td>158</td>
</tr>
<tr>
<td>5.5</td>
<td>Summary of the Solution of the Linear Programming Model (Example 2)</td>
<td>160</td>
</tr>
<tr>
<td>5.6</td>
<td>Summary of the Solution of the Linear Programming Model (Example 2 with resource transfer)</td>
<td>161</td>
</tr>
<tr>
<td>5.7</td>
<td>Bundles and Characteristics (Integer Programming Example)</td>
<td>164</td>
</tr>
<tr>
<td>Table No.</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>5.8</td>
<td>Budget Projections for the Planning Horizon</td>
<td>165</td>
</tr>
<tr>
<td>5.9</td>
<td>Comparative Results of Integer Programming Models with and Without Transfer of Resources Between Periods</td>
<td>166</td>
</tr>
<tr>
<td>AII-1</td>
<td>Descriptive Statistics for Covariates in the Data Set</td>
<td>213</td>
</tr>
<tr>
<td>AII-2</td>
<td>Estimates of the $\beta$- Values</td>
<td>213</td>
</tr>
<tr>
<td>AII-3</td>
<td>Significance of Results</td>
<td>213</td>
</tr>
</tbody>
</table>
CHAPTER 1: Maintenance Policies and Issues in Urban Water Distribution Systems

I. Introduction

Meeting water requirements in the coming decades is a task of high complexity involving balancing the demands for water and the investments necessary to meet them at a time of tight municipal budgets. With other infrastructure work in serious need as well, strategies such as regionalization for future water supply will be necessary to capture the important economies of scale in water "production" and treatment and large-scale distribution to aggregated demand centers.

However, future water supply planning is not limited to the regional capacity expansion. Other problems related to the existing local urban systems have to be dealt with as well. There has been significant public concern over the past year for a problem that has long been known to the water utility professionals. Carrying the banner of "America in Ruin" and focusing on old, poorly maintained, and possibly dangerous infrastructure, considerable media attention has been given to the need to rectify the deterioration/maintenance problem within the institutional and financial structures now available.

In particular, as was reported in a survey of previous water supply projects (Clark et al. (1982)), the distribution facilities in water supply systems account for the largest cost item in future maintenance budgets. The aging distribution systems in a number of service areas raise a whole category of maintenance decision-making problems.
These maintenance decision problems are further complicated by the necessary ongoing capacity expansion of such urban systems in accordance with the regional facility expansion and regional population shifts.

Planning of future water supply systems can be pictured as a two-level analysis:

1) A "macro" analysis at the level of aggregate demand centers of the size of urban communities: Such a task is concerned with the scheduling and sizing of the supply alternatives (at the overall regional level) and the associated inter-urban distribution facilities.

2) A "micro" analysis at the local distribution system level to handle the category of problems related to the expansion and maintenance of urban distribution systems.

The present work is concerned with the second or micro analysis of future water supply planning. It is aimed at providing the means for better maintenance decision-making related to water distribution networks. In particular, the emphasis is on mature and deteriorating water distribution networks.

Such a deterioration of the water distribution systems in many service areas is translated into a high proportion of unaccounted-for water due to the leakages in different components of the system. Such leakage not only indicates the loss of a valuable resource but raises concern for safe drinking water caused by possible contamination through cracked pipes. As much as 30% of the water captured and treated is presently lost in a number of urban distribution systems. These
systems are usually aging, with a large part of their pipes made of cast iron and a continuous program of comprehensive leak detection and repair is required. But, which pipes to replace, and when, especially given budget constraints on activities? In spite of incomplete information about the status of the system, water utility managers are still faced with the decision problem related to the best allocation of their maintenance budget both to counter present problems and anticipate future ones.

Many estimates put the capital needs to rehabilitate urban water distribution systems in the United States at very high figures. One such estimate (Choate and Walter (1981)) provided the $75-110 billion (in 1972 dollars) range over the next twenty years. However, traditionally, investments in infrastructure maintenance have been small and given little attention.

While many underfunded municipalities are still deferring needed maintenance and replacement measures, some cities have started rehabilitation programs. However, most of these programs are still quite modest.
II. Problems in Water Distribution Systems

Aging water distribution systems have raised considerable public concern about the safety and quality of drinking water in many localities throughout the U.S.. While the public authorities became aware of the seriousness of the "infrastructure problem" in general, water distribution emerged as an area where urgent upgrading was required to bring the systems to an acceptable level of serviceability and reliability. As water utilities discovered that they had to fight against such problems as scale, rust, internal and external corrosion, tuberculation and inadequate sizes, let alone leaks and breaks in the systems, the idea of preventive maintenance was making its way. In Figure 1.1, the major problems in water distribution systems are represented as well as their impacts. Rust and external corrosion lead to leaks and breaks. The combined effect of these two problems on water quality is obvious. Leaks and breaks affect water supply availability, as well as the maintenance costs. Internal corrosion can lead to head losses and the latter problem could affect both water supply availability and operational costs. Also internal corrosion affects water quality. Until very recently, many municipalities have been unwilling or unable to finance rehabilitation of deteriorating pipelines and have deferred needed maintenance and replacement until catastrophe threatens or the magnitude of the problem justifies the required expenses.

Many separate efforts were started. Leak detection and repair programs were phased in some states, as a first attempt to launch
preventive maintenance. Rehabilitation programs in some cities are now in progress and Boston, for one, has made inroads during the first five years of its 25-year rehabilitation program. While these joined efforts are providing some preventive maintenance, they still were initiated by the realization that the systems were falling below acceptable quality standard levels. After a long history of supply related* and insufficient maintenance, budgets were designed to bring the systems again to an acceptable quality standard, with "due consideration" to long-term maintenance costs. However, while this approach to maintenance is necessary in the short run, there is definitely a need for a more demand-responsive approach to maintenance, which would generate a maintenance management system providing the "right" levels of maintenance on a dynamic basis, by timing the necessary measures to implement taking into account the evolution of maintenance requirements and the external parameters such as consumer demand, water quality, etc. For example, based solely on maintenance economics, it might not be efficient to replace a given pipe at a certain time. If this pipe was undersized within the structure of the network, it might still be globally efficient to replace it with a pipe of larger capacity, instead of adding incremental capacity at high marginal cost. Identically, if internal corrosion and rust are affecting water quality, then one might consider replacing the pipe earlier, instead of cleaning (and lining) the pipe immediately and replacing it in a few years based on the maintenance economics criterion.

It follows that maintenance management is a complex integrated problem, which has to be decomposed into its major components in order

* This term will be explained in a later section.
FIGURE 1.1 Problems in water distribution systems at a given point of time: their respective relationship and their impacts.
to build an efficient maintenance management system. The skeleton of such a system will be described in section III. In the next section, the central issues in maintenance planning are recognized, after the different maintenance practices have been reviewed.

III. Maintenance Practices and Their Impacts on the System

III.-1 What is maintenance?

It seems necessary to suggest a definition of "maintenance" and the different types of action that it encompasses. In its general context, maintenance includes two major categories of measures.

1. Routine type of maintenance, related to the inspection of different components of the system and the repair of detected leaks and breaks occurring in the network.

The common level of routine maintenance depends on the standards set by the water utility. Setting up such standards is in itself part of the maintenance management system. Routine maintenance may result from a careful monitoring of flows in the system and regular pitometer inspections, which would be the case under the scenario of a "well-maintained" system, i.e. with high standards of maintenance. The other extreme of routine maintenance would be the case of minimal repair taking place due to complaints of customer for insufficient pressure or fire protection regulations, or simply the occurrence of a major leak or a break. Obviously, the type of routine maintenance practice affects directly the evolution of the state of the system over time, as the existence of unrepaired leaks leads to major breaks. The "accumulation" of these leaks explains the high proportion of
unaccounted-for water due to the leakage problem. This figure varied between 10 and 17 percent in 11 of 12 U.S. cities in 1978 (see table I-1), with about one-fourth of it due to leaks in mains, the remaining part being mostly due to service pipe leaks (table I-2).

2. **Upgrading type of maintenance**, related to the improvement of the deteriorated system by such measures as:

   a) The rehabilitation of pipes, by cleaning and "mortar lining" the interior walls of the pipe, if internal corrosion has taken place. Also the construction of a structurally independent lining (using plastic) might be envisaged as a means to fight against external corrosion. As was explained previously, internal corrosion affects the performance of the system due to lower Hazen-Williams coefficients and higher head losses. Though internal corrosion might affect the failure rate of the pipes, it is believed (Corless (1982)) that this effect is significantly smaller than external corrosion in causing the failure mechanism. Rehabilitation is an alternative to increased energy costs due to pumping and lower water quality due to internal corrosion.

   b) The replacement of the section of pipe as an alternative to continually increasing failure rates and repair costs, but also in order to deal with other problems, as new pipes have none of the undesired features mentioned in figure 1.1.
Estimated Water Leakage in 12 US Cities - 1978

<table>
<thead>
<tr>
<th>Location</th>
<th>Percent of Water Lost Through Leaks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston, Mass.</td>
<td>17</td>
</tr>
<tr>
<td>Cleveland, Ohio</td>
<td>15</td>
</tr>
<tr>
<td>St. Louis, Mo.</td>
<td>15</td>
</tr>
<tr>
<td>Pittsburgh, Pa.</td>
<td>14</td>
</tr>
<tr>
<td>Tulsa, Okla</td>
<td>14</td>
</tr>
<tr>
<td>Philadelphia, Pa</td>
<td>12</td>
</tr>
<tr>
<td>Hartford, Conn.</td>
<td>11</td>
</tr>
<tr>
<td>Kansas City, Mo.</td>
<td>11</td>
</tr>
<tr>
<td>Cincinnati, Ohio</td>
<td>11</td>
</tr>
<tr>
<td>Buffalo, N.Y.</td>
<td>10</td>
</tr>
<tr>
<td>Baltimore, Md.</td>
<td>10</td>
</tr>
<tr>
<td>Portland, Ore.</td>
<td>8</td>
</tr>
</tbody>
</table>

**TABLE 1-1**

(from Sullivan (1982))
### Unaccounted-for Water-1978 (Boston)

<table>
<thead>
<tr>
<th>Cause</th>
<th>Amount</th>
<th>Percent of Unaccounted for Water</th>
<th>Percent of Total Water Purchased</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ML/d</td>
<td>mgd</td>
<td></td>
</tr>
<tr>
<td>Undermetering</td>
<td>117.7</td>
<td>31.1</td>
<td>46.4</td>
</tr>
<tr>
<td>Leaks and breaks</td>
<td>94.6</td>
<td>25.0</td>
<td>37.3</td>
</tr>
<tr>
<td>Blowoffs and flushings</td>
<td>4.5</td>
<td>1.2</td>
<td>1.8</td>
</tr>
<tr>
<td>Firefighting</td>
<td>7.1</td>
<td>1.9</td>
<td>2.8</td>
</tr>
<tr>
<td>Unmetered public usage</td>
<td>15.8</td>
<td>4.2</td>
<td>6.3</td>
</tr>
<tr>
<td>Other</td>
<td>4.5</td>
<td>1.2</td>
<td>1.8</td>
</tr>
</tbody>
</table>

**TABLE 1-2**

(from Sullivan (1982))

**Units:**

- ML/d = million liters per day
- mg/d = million gallons per day
III.2 Present Situation in U.S. Distribution Systems

The number of breaks recorded in different systems has been quite variable (Table I.3). Also, routine maintenance has varied between different systems in the U.S. Those systems with good maintenance practices suffer less today from the need to the second category of maintenance, i.e., rehabilitation, replacement, or at least leak detection and repair. In systems with good "quality standards", the accumulation of repair events and the occasional replacement of bad sections of pipes have prevented the system from deteriorating below minimal acceptable levels of reliability. Using past maintenance history (if recorded) in these systems would provide significant insight for future failure rates, or repair costs. The regression of past maintenance data using different statistical techniques would yield predictive models for future pipe failure, or at least of future repair costs under the maintenance standards used, as past data is related to repair events.

In poorly maintained systems (usually in underfunded municipalities), repair events are quite scarce and the urgency of all types of maintenance including leak detection and repair, replacement and rehabilitation is felt. Also, the problem of reliability is more serious in these systems due to the deteriorating system and existing leaks. While in well maintained systems, analytical models to predict future repair costs can provide means of comparison with upgrading alternatives, predictive models in poorly maintained systems with rare repair events can at best provide insights on the probability of
## Comparison of Water Main Failures in 14 Cities

<table>
<thead>
<tr>
<th>Location</th>
<th>Reporting Period</th>
<th>Length of System</th>
<th>Number of Breaks*</th>
<th>Main Breaks Per Year Per 1609 km (1000 mi) of Main</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston, Mass.</td>
<td>1969-1978</td>
<td>1,737 km</td>
<td>1080 mi</td>
<td>39</td>
</tr>
<tr>
<td>Los Angeles, Calif.</td>
<td>1973-1974</td>
<td>10,941 km</td>
<td>6800 mi</td>
<td>290</td>
</tr>
<tr>
<td>Chicago, Ill.</td>
<td>1973</td>
<td>6,674 km</td>
<td>4148 mi</td>
<td>223</td>
</tr>
<tr>
<td>New York, N.Y.</td>
<td>1976</td>
<td>10,152 km</td>
<td>6310 mi</td>
<td>476</td>
</tr>
<tr>
<td>St. Louis, Mo.</td>
<td>1973</td>
<td>2,209 km</td>
<td>1373 mi</td>
<td>106</td>
</tr>
<tr>
<td>Indianapolis, Ind.</td>
<td>1969-1978</td>
<td>3,234 km</td>
<td>2010 mi</td>
<td>167</td>
</tr>
<tr>
<td>San Francisco, Calif.</td>
<td>1973</td>
<td>1892 km</td>
<td>1176 mi</td>
<td>125</td>
</tr>
<tr>
<td>Louisville, Ky.</td>
<td>1964-1976</td>
<td>3,924 km</td>
<td>2439 mi</td>
<td>300</td>
</tr>
<tr>
<td>Denver, Colo.</td>
<td>1973</td>
<td>2,884 km</td>
<td>1793 mi</td>
<td>280</td>
</tr>
<tr>
<td>Troy, N.Y.</td>
<td>1969-1978</td>
<td>241 km</td>
<td>150 mi</td>
<td>25</td>
</tr>
<tr>
<td>Milwaukee, Wis.</td>
<td>1973</td>
<td>2,806 km</td>
<td>1800 mi</td>
<td>421</td>
</tr>
<tr>
<td>New Orleans, La.</td>
<td>1969-1978</td>
<td>2,444 km</td>
<td>1519 mi</td>
<td>1033</td>
</tr>
<tr>
<td>Houston, Texas</td>
<td>1973</td>
<td>6,432 km</td>
<td>3998 mi</td>
<td>5144</td>
</tr>
</tbody>
</table>

### TABLE 1-3

(from Sullivan (1982))

*a break is recorded following a repair event.*
major failures in the future. The decision to replace or rehabilitate, versus routine maintenance is therefore more the outcome of a rational economic analysis in the first instance, while it is much closer to a multi-criteria problem in the second case.

It follows from this reality that past maintenance policy affects our understanding of the present problem and the means to deal with it optimally. The argument that the upgrading of well-maintained systems can be based on the economic analysis of alternative measures is reinforced by the fact that other measures of performance of the system (or lack of performance) such as water quality, loss of pressure and reliability, are usually less significant in these systems, due to better maintenance. Thus, the importance of different criteria for new capital outlay or rehabilitation decision-making changes with the type of maintenance policy/practices, as shown in Table I-4.

The definition of a comprehensive maintenance management system must therefore take into account previous maintenance practices. Understanding the strengths and limitations of different methodologies that can be used to derive a global maintenance strategy is a preliminary step to any analysis. This task starts by recognizing that different water distribution systems would require different approaches to the problem of short-run and long-run maintenance. However, more general guidelines can be laid out to provide the common structure of all maintenance problems. These will be presented in the next section, within a generalized maintenance management system. Using these guidelines as a starting point, the identification of the "type" of distribution system and its
<table>
<thead>
<tr>
<th>Criteria</th>
<th>Proactive Maintenance Practices</th>
<th>Reactive Maintenance Practices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Intensive</td>
<td>Fair</td>
</tr>
<tr>
<td>Economic analysis</td>
<td>I.F.</td>
<td>P.F.</td>
</tr>
<tr>
<td>Loss of Pressure</td>
<td>P.F.</td>
<td>P.F.</td>
</tr>
<tr>
<td>Water Quality</td>
<td>P.F.</td>
<td>P.F.</td>
</tr>
<tr>
<td>Reliability</td>
<td>M.F.</td>
<td>P.F.</td>
</tr>
</tbody>
</table>

**TABLE 1-4**

Importance of Different Criteria for Replacement/Rehabilitation Decision-making, Under Different Types of Maintenance Policies

I.F. = Important Factor

P.F. = Partial Factor

M.F. = Marginal Factor
associated maintenance history can help tailor a comprehensive maintenance management system specific to that distribution network. The examination of past failure rates, routine maintenance practices and upgrading measures allows the required categorization of the system.

IV. Past and Present Approaches to Maintenance Policy.

IV-1. Past Maintenance Policy

Maintenance policy has been largely based on a supply oriented approach. Early trends were set on the "threshold of intervention", i.e. what constitutes the basis for action in the form of a repair event or a replacement of a pipe. Based on past maintenance costs, a sought level of maintenance effort would be chosen and an associated budget developed. In predicting future maintenance requirements, the primary focus is on the ability to supply maintenance services, by estimating the dollar resources needed to meet the usual quality standards. The factors that initiated and developed the maintenance requirement in the first place, are totally neglected in such an approach. Although this approach is open to criticism, it is understandable in light of the difficulty of understanding these factors and their impacts, let alone the organizational and administrative realities surrounding maintenance program development.

Under past maintenance policy, many systems had therefore insufficient maintenance budgets, which explains the existence of undetected leaks and the total deterioration of parts of such systems. However, according to a case study by Moyer, Male et al (1983), the early detection of leaks in water mains leads to a net benefit of about
2,000 dollars per leak. Therefore unrepaired leaks represent foregone opportunities to save global costs that would ultimately be carried on to the consumer, in both public and investor-owned utilities. The costs and benefits per leak in different components of the water conveyance system are reported in Table I-5, and show that the net benefit is highest in the case of main leaks.

IV-2. **Current Maintenance Thrusts**

The awareness of the "infrastructure" problem has led to actions such as leak detection and repair programs (sometimes initiated at the State level for small communities), as well as capital improvement programs including rehabilitation and replacement (Figure 1.2). However, if such programs are required to bring the systems to an acceptable level of serviceability and reliability, they still need to be coordinated taking into account the real maintenance requirements. In other words, the approach remains supply-oriented (Table I-6), but the standards are higher because of the current concern about maintenance. What is still needed is a maintenance management system, based on a demand-responsive approach, using predictions of future maintenance requirements, and striking a better balance between these future needs and seasonal resource constraints.

The different modules of such a maintenance management system are reported in the influence diagram of Figure 1.3. The dotted arrows represent the demand-responsive approach. Before designing a maintenance management system, some decisions have to be made about "standards" such as water quality, reliability and maintenance policy. Given these
standards, the maintenance management system can be implemented to generate the required maintenance strategies.
### Average Costs, Benefits, and Net Benefits of Leak Detection and Repair for Three Surveys - Dollars per Leak

<table>
<thead>
<tr>
<th>Type of Leak</th>
<th>All Surveys</th>
<th>Survey 1</th>
<th>Survey 2</th>
<th>Survey 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrant</td>
<td>332</td>
<td>227</td>
<td>-105</td>
<td>363</td>
</tr>
<tr>
<td>Service Customer</td>
<td>274</td>
<td>1081</td>
<td>807</td>
<td>287</td>
</tr>
<tr>
<td>Service-W/WW</td>
<td>836</td>
<td>1202</td>
<td>366</td>
<td>880</td>
</tr>
<tr>
<td>Main</td>
<td>839</td>
<td>2781</td>
<td>1942</td>
<td>822</td>
</tr>
<tr>
<td>No leak found</td>
<td>645</td>
<td>0</td>
<td>-645</td>
<td>970</td>
</tr>
<tr>
<td>Other</td>
<td>267</td>
<td>0</td>
<td>-267</td>
<td>232</td>
</tr>
<tr>
<td>All types</td>
<td>480</td>
<td>806</td>
<td>326</td>
<td>541</td>
</tr>
</tbody>
</table>

**TABLE 7-5**

*(from Meyer, Male, Moore and Hock (1983))*
FIGURE 1.2: Maintenance Programs in Aging Systems:

Still a Supply Related Approach

(Presence of dotted arrows shows the need for an Integrated System)

TABLE 1-6: Boston Water System Capital Improvement Program

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount Planned - x$1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1981</td>
<td>3,292</td>
</tr>
<tr>
<td>1982</td>
<td>5,248</td>
</tr>
<tr>
<td>1983</td>
<td>6,291</td>
</tr>
<tr>
<td>1984</td>
<td>6,920</td>
</tr>
<tr>
<td>1985</td>
<td>7,612</td>
</tr>
<tr>
<td>1986</td>
<td>8,375</td>
</tr>
<tr>
<td>1987</td>
<td>9,212</td>
</tr>
<tr>
<td>1988</td>
<td>10,133</td>
</tr>
<tr>
<td>1989</td>
<td>11,146</td>
</tr>
<tr>
<td>Total</td>
<td>68,229</td>
</tr>
</tbody>
</table>

(from Sullivan (1982))
Figure 1.3: Influence Diagram for a Maintenance Management System
CHAPTER 2: Development of Maintenance Management Systems

I. Understanding the System Status

The design of a maintenance management system applicable to a water distribution network starts from the understanding of the status of the system. The status of the system encompasses a wide variety of performance measures, some of which are operational, such as carrying capacity (loss of pressure) and the leakage through the system. Other performance measures are related to the economic cost of repairing water main leaks and breaks, as well as the impact of system deterioration on water quality and the reliability of water supply threatened by high break rates in some systems. As a former top government official recently declared, "Every other day a water main break occurs on 57th Street in New York, reminding us of the seriousness of the infrastructure problem in the nation."

In order to maintain the required serviceability levels in mature infrastructure, important capital layouts will be needed for the maintenance, rehabilitation and reconstruction of deteriorated facilities.

It is therefore critical that an understanding of the system status, i.e., its deterioration level, as well as its measures of performance, be reached in order to be able to derive an adequate maintenance management system. In Chapter I, the major factors influencing the design of a maintenance management system were represented. Each of these factors has to be evaluated to make the planning exercise for future maintenance activities most efficient.

It is necessary to emphasize, at this point, the fact that the understanding of the aforementioned performance measures and their
evolution has to be system specific under the present data availability. This reality is of critical importance to the design of a maintenance management system, which relies heavily on the assessment of maintenance requirements.

The reason why the evaluation of the performance measures of the system has to remain system specific is that the factors that led to these levels of performance had different impacts, depending on some "structural" system specific context. The elements of this context are usually:

1. The maintenance history in the system, including the levels of intervention in routine maintenance, but also the improvement in the quality of the components of the system, undertaken at different periods in the past.

2. The design characteristics of the system, including both the initial design philosophy, but also the design of the capacity increments throughout the history. The design of a system influences its present status, in such issues as redundancy, but also the operational characteristics originally tailored in the system design such as the pressure/capacity trade-offs, as well as the evolution of these operational characteristics through time. For example, a delay in capacity increments in the past might have been accommodated by a change in the operational characteristics (or rules) in the system. The impacts of these "out-of-design standards" effects, as well as these design standards
themselves on the performance measure have yet to be evaluated. Other non-structural or "natural" system specific factors include such items as water hardness and geological conditions.

However, this does not exclude the possibility of common denominators between a number of systems which have been operated and maintained according to a similar set of rules. A number of insights can be gained from pooling systems into different categories depending on their maintenance/design and development/operation practices. Such a pooling would produce and number of major results:

1. Within each pool, it allows the evaluation of how other system specific factors affect the measure of performance. The effect of water hardness, for instance, on the loss of carrying capacity can be analyzed by considering the performance of different systems (with varying degrees of water hardness) in the pool.

2. The comparison between the different central tendencies of each pool allows one to make a judgment on the impact of maintenance history, design and operation of these systems, and upon the evolution of their status.

3. The evaluation of non-system specific factors and their impacts on the deterioration of the system and other measures of performance.

An important result can be obtained from undertaking these steps. Previous design criteria could be reconsidered, if it was
proven that under such criteria, maintenance events were more frequent. Such design criteria can vary from the equipment design (capacity, thickness of the wall, etc.) to the actual layout (depth under the ground level, methods of pipe support and joint sealing, etc.). The initial design and implementation of a system, its expansion over time and its maintenance have to be considered as a continuum of planning problems. This realization broadens the scope of the maintenance problem and justifies the fact that maintenance cannot be separated from the other major components of urban water distribution system planning.

The analysis of the status and performance of a system and the factors which contributed to the evolution of these parameters to their current levels allows one to suggest changes in future water distribution system planning policies and practices. One major component of these proposed changes would be a comprehensive maintenance management system, continuously interfaced with the expansion and operational sides of water supply planning. This work attempts to analyze the maintenance management alternatives at the network level. It assumes therefore that the other sides of water distribution system planning are given. Ideally, the major parameters of the global planning policy have been set at their optimal level, due to the feedback mechanism diagrammed in Figure 2.1.

II. Scope of this Work

The foundations of this work are based on the philosophy that a demand-responsive approach, as explained in Chapter 1, is required
FIGURE 2.1

Impact of the Evaluation of the System Status/Performance on Planning Policy and Practices: A Feedback Scheme
to derive an "efficient" maintenance management system. In order to assess the different future maintenance requirements in every single section of the network, prediction models for the future evolution of the different measures of performance discussed above are needed. Predictive models for future failure rate of pipes have been studied by some authors, who derived regression-based models. A probabilistic model to predict future failure rate of pipes based on a statistical methodology called Cox's regression is presented in this work. Its advantages and applications are discussed in Chapter 4, dedicated to the comprehensive study of predictive models for pipe failure rate. Its use in the analysis of the failure rates of water distribution systems provides better insights for the prediction of the economic costs associated with different maintenance alternatives, but also for the reliability analysis, as the serviceability of some major water mains is essential to the provision of safe drinking water to a whole district.

Besides failures leading to repair events, the loss of carrying capacity with age is another problem which is rectified by additional operating costs which are discussed in Chapter 3. Still according to the demand-responsive approach, the ability to project future additional operating costs required to rectify the problem of carrying capacity loss, is contingent on a predictive model for the future evolution of the Hazen-William coefficient. Such a model is not available, but is within the domain of researchable projects, provided that the required data is collected. In the next section, the type of data required to build these models is analyzed.
Most of the recent publications have focused on the economic costs associated with the rectification of the first type of problem, i.e., the repair of leaks and breaks in the system. Obviously, these works attempted to analyze the different alternatives allowing to rectify "the infrastructure problem," by allowing the system to "function" at lowest (present value) maintenance costs. However, the ability to make the system function still leaves a leeway in the magnitude of the operating costs associated with the status of the system. The variability of operating costs is mainly attributed to the loss of carrying capacity. If such a problem is negligible, which is certainly the case in some systems again depending on system specific factors such as water hardness and layout, the analysis of the "physical" maintenance costs is bound to provide a fairly accurate approximation of the optimal alternative.

It follows that the relevant alternative to evaluate at the single pipe level, as well as the network level depend on the context of the system itself. Some cities with low break frequency are mainly focusing on the rehabilitation of their old pipes which require cleaning and lining. Other cities have higher break and maintenance event frequency and are now concerned about the timing of the replacement alternative, as an ultimate solution to the observed trend. The replacement versus repair issues are the central ones in this work as the replacement alternative is the best way to deal with the degradation of different performance measures. However, the evaluation of the rehabilitation alternative, and the alternative of a rehabilitation
followed at a future date by a replacement alternative is theoretically necessary for each single pipe. Obviously, the trend of repair costs and additional operations costs allows one to decide on the range of alternatives to consider for the associated pipe.

III. Specific Issues Addressed in this Work

This work is concerned with the derivation of a scheduling program for maintenance alternatives in a mature (aging and slowly expanding) water distribution network, and the development of a comprehensive framework for a maintenance management system.

In Chapter 3, the analysis of different measures at the single pipe level is performed. The optimal replacement time and optimal rehabilitation time associated with both alternatives are evaluated, to allow their comparison.

In Chapter 4, the causal factors for the failure mechanism (leaks and breaks) are recognized. Predictive models for future failure rate are presented. In particular a probabilistic model for the prediction of future major breaks based on a statistical approach known as Cox's regression is suggested.

In Chapter 5, a long-term planning model for the scheduling of maintenance and expansion measures at an integrated network level is presented. It uses as inputs the results of the techniques presented in Chapter 3, which are applied, in the context of the predictive models of Chapter 4. The scheduling model is formulated as a mixed-integer programming problem. A case example illustrates the model.
In Chapter 6, a short term sequencing model using the outputs of the long-term planning model is presented. It is also a mixed-integer programming model, and a suggested solution based on a Lagrangian relaxation approach is derived.

In Chapter 7, a method to cope with major breaks occurring in major water mains (in which interruption of serviceability has a direct impact on the reliability of drinking water for a whole district), is presented. It is based on a technique called Crisis Decision Analysis and uses the probabilistic model obtained in Chapter 4, using the Cox regression approach.

In Chapter 8, the conclusions of this work are summarized and recommendations for further research to accomplish the goals of a comprehensive well-documented maintenance management system are suggested.

The sequence of steps described above within the different chapters of this thesis are represented in Figure 2.2. The use of predictive models for future pipe degradation in performance (failure rate, loss of carrying capacity), allows us to perform a first level economic analysis of alternative measures that can be undertaken on each single pipe, including rehabilitation and replacement. Depending on the optimal alternative, an optimal replacement or rehabilitation time is evaluated for each pipe and the pipes are then sorted according to the values of these optimal replacement or rehabilitation times. As is pictured in Figure 2.2, a two-level hierarchical analysis is suggested. In the first level economic analysis mentioned above, some of the characteristics of a given pipe are evaluated using the data network concept:
that is, by "borrowing" data from other pipes subject to the same physical and external conditions. Clearly, the data network concept introduces more errors in the data, but it is quite convenient in order to reduce the needs for gathering data, especially as data collection often requires considerable effort.

The outcome of the first level economic analysis is an optimal replacement or rehabilitation time, depending on the more economically efficient alternative. This leads to the sorting of these pipes according to their replacement or rehabilitation times, and the identification of "critical" pipes, or pipes nearing their optimal replacement or rehabilitation times. The collection of further information related to this set of "critical" pipes would then allow one to update their records in the data base and perform a second level economic analysis, leading to a better estimation of the optimal replacement or rehabilitation time.

The outputs of the second level economic analysis can be used to define priorities and replacement or rehabilitation constraints in the scheduling of maintenance (or possibly expansion) measures at the network level. The scheduling model is also subjected to reliability and financial constraints as represented in the lower part of Figure 2.2.

IV. Issues Related to the Development of Predictive Models for Future Pipe Performance

In the general context of a demand-responsive approach, it is quite clear that accurate predictions of future performance (failure rate, loss of carrying capacity) is found to significantly improve
FIGURE 2.2

Flow of Tasks in a Maintenance Management System
the quality of maintenance decision-making. Well-calibrated models for the prediction of the major performance indicators are therefore needed to perform any analytical exercise, leading to the recommendation of the optimal alternative.

In the case of pipe failure, it has been shown by Moyer, Male and Moore (1983) that a water main leak detection and repair leads to positive savings. We therefore start from the premises that all leaks and breaks should be repaired, and that capital improvement alternatives (replacement and rehabilitation) are evaluated under this assumption. It is therefore necessary to predict the occurrence of future failures (leading to repair events) in every single pipe. This leads to the examination of two first data "format" issues, the definition of a pipe and that of a repair event.

1. **Definition of a "pipe"

The selection of a unit of length of watermain as a pipe is clearly quite subjective. However, when it comes to decide on the replacement versus repair of a pipe, with a capital cost of about $30,000 per 1000 feet, the definition of the adequate order of magnitude to represent a pipe is an important task. Once the network has been decomposed into a number of separate pipes of links, each link may constitute more than one entity or be simply a part of an entity. The choice of what section or assembly of sections of watermain represents an "entity" should be based on the spatial variability of the physical characteristics which affect the state of the pipe. Obviously, for a 4,000 foot pipe, a number of replacement possibilities can be considered, depending on
the number of separate entities within this pipe. However, the repair
history previously recorded already constrains the data management
options, as the exact location of the different repair events is
usually not available. In such cases, the part of watermain is pre-
declared, due to the previous data collection practices.

2. **Definition of a repair event**

Once an "entity" has been defined, the future repair events associ-
ated with it are to be predicted. Past maintenance data, as mentioned
previously, is incomplete and inaccurate. Leaks and breaks are not
distinguished. The reason why such a distinction would be useful is
tied to the reliability analysis, which is mostly influenced by the
occurrence of breaks in the major water mains. The semantic problem of
defining a leak and a break has been mentioned in the recent literature.
As was suggested by O'Day (1982), it is reasonable to distinguish between
leaks and breaks based on the discharge rate. Beyond a certain threshold
of discharge rate (such as 250 gallons per minute), leaks are considered
as breaks.

However, solving the semantic problem does not solve the intrinsic
problem due to the type of data at hand. The available data is usually
in terms of repair events, and does not make the distinction between
leaks and breaks. Besides lacking accuracy, the high level of unaccounted-
for water in many systems suggests the existence of unrepaired leaks (that
should be repaired, according to the findings by Moyer, Male and Moore
(1983)). The completeness of the data is therefore highly dependent on
the previous maintenance practices. The prediction of future repair
events, assuming a more adequate level of maintenance requires a number of data adjustment and collection tasks.

As was mentioned at the beginning of this chapter, the best understanding of all the system specific and non-system specific factors and their impacts on the failure mechanisms can be achieved by undertaking the series of steps underlined above, that is, by doing a cross-system analysis of past system performance. However this task will require a large amount of effort that is beyond the feasibility domain in the near future. However, it is still necessary to predict at the best achievable level of accuracy, the future system performance. Based on the past maintenance data collected from one single system a predictive model can be derived where the major causal factors are non-system specific, as the system specific factors are by definition constant. Such a model can generate the required projected performance with enough accuracy provided that the quality of the data at hand is acceptable. Most of the models derived by previous authors, such as Shamir and Howard (1979) and Clark et al. (1983) were based on data collected from a given system. Also, in our discussion of predictive models in Chapter 4, most of these models are related to a given system. Due to the subjectivity of some of the system specific factors (such as level of intervention in maintenance, design and implementation criteria), the task of analyzing a cross-system data base is very complex. Obviously, if these subjective factors are standardized, or if their direct underpinnings are rectified, the task of accounting for the other system specific factors such as water hardness, general geological conditions, etc., is quite simplified.
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:
It is interesting to note that the analysis of maintenance data from one given system normally requires adjustments which will make it usable in a cross-system analysis. In order to build a predictive model, the level of previous maintenance practices directly impacts maintenance records. Ideally, it is desirable to have access to all previous failure events and their classification into leaks and breaks, for the purposes of the reliability and economic analysis. The total number of events allows one to predict direct maintenance costs, while the number of major breaks allows one to perform a risk analysis of the system.

Unfortunately, previous repair events are not representative of the comprehensive set of failure events which occurred in a given pipe. Depending on such factors as whether the system is privately owned or publicly owned, the past availability of adequate maintenance budgets is an indicator of the level of intervention for maintenance measures, and therefore the relevance of the past data for prediction purposes.

However, even in systems with previous insufficient maintenance effort, it is still possible to build a predictive model, provided that a leak detection program is implemented. Such a program would provide a cumulative number of "unrepaired leaks", which added to the previously repaired leaks, would make a global analysis of the data much more insightful, provided that pipes of different ages and characteristics are available in the network.

Therefore, two major sources of data can help undertake the data analysis for a given system, irrespective of the previous maintenance policy.
1. The previous maintenance history in terms of repair events.

2. A leak detection program in a significantly representative sample of pipes in the network. The design of the data collection task can be enhanced by using the data network concept and would decrease the global costs of the leak detection program.

With regard to the predictive model for the loss of carrying capacity, the monitoring of the present operational levels (head losses) and their comparison to the originally set levels in the design plans, provide the necessary information for the derivation of the model parameters. Using an adequately designed sample of data, it is possible to predict the evolution of the Hazen-Williams coefficient for a given pipe, by means of a multiple regression model.

These predictive models make it possible to perform the economic analysis of alternative measures for each single pipe, which is the topic of Chapter 3. In addition to that, a probabilistic model for the prediction of major breaks presented in Chapter 4, allows one to make inferences in the reliability analysis, and the case of capital improvement decisions, including reliability as a major measure of performance. Using the Crisis Decision Analysis approach, maintenance decision-making would account for reliability issues, as well as economic costs of different alternatives.
 CHAPTER 3: Economic Analysis of Alternative Maintenance Measures at the Single Pipe Level

A. Economic Analysis of the Rehabilitation of Water Mains

I. Conditions for Rehabilitation

The review of the problems experienced by a water main throughout its service life allows the identification of the conditions under which the rehabilitation alternative is relevant. Clearly, there are two major sources of additional economic costs associated with the maintenance and operation of aging water mains, in order to keep them at the required level of serviceability:

1. The repair costs for leaks and breaks occurring in the main.
2. The operational costs associated with the additional energy required for pumping purposes to correct for the head losses caused by the deterioration of the inside wall of the pipe. Also additional capital costs on new pumping equipment could be necessary once the head loss requirement can no longer be met by the specified capacity of the pumping facility.

The projection of the future streams of these costs for a given pipe is bound to provide the necessary insights for capital improvement decisions, such as rehabilitation, replacement or expansion. The detailed analysis of these decisions is undertaken in this chapter devoted to the evaluation of alternative measures to implement on a given pipe. It is useful to mention, at this point, that the trade-offs between the savings in future repair costs, the savings in future operational costs, as well as the additional incremental costs
due to increasing needs in the vicinity of the water main are the major inputs to this decision problem.

In the specific case of the rehabilitation alternative, two major problems are handled: internal corrosion affecting water quality and loss of head affecting energy costs for pumping purposes. The energy costs item can be evaluated, given certain projections which will be further discussed. This leads to an economic analysis of the rehabilitation alternative, and more importantly, an evaluation of the optimal rehabilitation time, based on the assumption that the rehabilitation alternative is adopted. It is useful to note that the consideration of the rehabilitation alternative in the first place is contingent on the order of magnitude of the additional operating costs relative to the other maintenance costs (mainly repair costs), incurred in aging water mains. For example, in water mains where repair costs are minor (low break frequency), it is rational to consider principally the rehabilitation alternative as the replacement alternative is more costly and would only add savings in repair costs, over the savings obtained through the rehabilitation of the pipe. The examination of the relevant factors to the economic analysis, i.e. failure rate, loss of carrying capacity, increasing needs in the vicinity, is a necessary first step for the evaluation of the optimal alternative at the level of a single pipe. The loss of carrying capacity in particular seems to be "system specific" as well as pipe specific, in the same way failure rate is both dependent on the system and the pipe under study. The "system specificity" of the loss of carrying capacity is represented in
figure 3.1, representing the changes in the Hazen-William coefficient with the age of the pipe in different distribution systems. The head losses vary in the opposite direction to the Hazen-William coefficient. Figure 3.1 was presented in the paper by McBean et al. (1983). These results were based on the work by Hudson (1966) who analyzed the evolution of seven water distribution systems. While the trends shown in figure 3.1 represent the "system-specific" drop of the Hazen-Williams coefficient with time, it is important to note that such system specific effect is composed of an environmental factor and a "human" factor. The environmental factor is related to the water type and its mineral composition. The human factor is related to the treatment method as well as the operation of treatment plants. In other words, water corrosiveness can be controlled by undertaking the "adequate" treatment method, given the type of water treated. This fact was verified by Larson and Sollo (1967), who analyzed the possible "control" measures needed to minimize the effects of corrosion.

The central performance measure related to operational costs is the loss of carrying capacity. It is due to many factors, corrosion being its major source, due to its impact or velocity and the Hazen-Williams coefficient C. The quality of the lining of the interior of the pipe therefore affects the evolution of the local performance in terms of carrying capacity, as the timely effect of corrosion varies with the material type and the interior lining of the pipe. Another major source of loss of carrying capacity is "operational" and is related to improper filtration of water or the operation of treatment plants beyond their design capacity. As described by Hudson (1966), the previous schemes
will result in deposits such as aluminum hydroxide on the interior walls of the pipes, regardless of the type of lining. A thin layer of deposit may develop ripples which cause eddying and turbulence. As the thickness of this layer of deposit increases, a reduction in diameter becomes a factor in the loss of carrying capacity, as shown in the Hazen-Williams formula. Also, organic growths may become attached to the interior walls in systems carrying new surface water, resulting in significant loss of carrying capacity. The evolution of carrying capacity is initially affected by the physical layout and installation of the pipe, as poor alignment and a large number of fittings and bends reduce capacity quite substantially.

If corrosive water does affect considerably the evolution of carrying capacity, it is interesting to delineate the properties of a "corrosive water". Chemically corrosive water for example is high in total dissolved solids, low in pH, high in hardness, dissolved oxygen and carbon dioxide. Morris (1967) mentions the major approaches to controlling the corrosiveness of such waters, based on pH adjustment, the formation of a protective film, and chlorination. It is important to note that water corrosiveness is not simply an environmental factor, but that water treatment methods designed to control the water quality often increases its corrosive properties. As reported by Hudson (1966), waters that must receive major treatment to be made potable "invariably increase the loss of carrying capacity in the system". Larson and Sollo (1967) studied the different corrosion rates in water of diverse chemical contents and suggested methods of "chemical control" to reduce the problem of loss of carrying capacity.
The problem of loss of carrying capacity is therefore a complex one. Its system specific component, represented in the trend curves of figure 3.1 is a result of the chemical balance of the conveyed water which is partly environmental, but also affected by the treatment methods required, and the operation of the treatment plants. Other pipe specific effects (non system of location specific) are related to the type of lining, the materials used and the "local" operation of the pipe.

As the minimization of both system specific and pipe specific effects within a cost-effectiveness framework is often neglected, it is important to note that good planning practice, concerned with the most efficient allocation of resources, should be initiated after such quasi-exogenous parameters have been set to their "optimal" levels. The analysis of both the failure rates and the loss of carrying capacity should firstly provide insights for a better policy of water quality control, pipe material selection, and system operation including treatment plants. The correction of such inherent inefficiencies within the possibility domain is then followed by a prediction of future system performance, based on past performance but also understanding of the mechanisms affecting such performance measures as loss of carrying capacity, breakage, water quality.

In this context, the derivation of predictive models for both the loss of carrying capacity and the failure rates allows the identification and the scheduling of the capital improvement measures in the system. While predictive models for failure rates have been researched and are discussed in detail in this work (Chapter 4) the
derivation of a predictive model for the evolution of the head loss in a pipe is still required. One major source of data is the present survey of such head losses in the system using pitometer measurements. When the evolution of such head losses has been monitored, it is quite possible to derive the aforementioned predictive model. In the absence of such a model, system specific approximations such as in figure 3.1 can help make decisions related to capital improvement measures. In any case, it is important to note that data analysis and inference has to be conducted individually for each system, because of the difficulties involved in separately analyzing system specific and pipe specific effects.

II. Evaluation of the Additional Operating Costs in Aging Water Mains

The existence of additional operating costs increasing with the age of the pipe is easily understood by examining the Hazen-Williams formula, which is recognized as the fundamental equation describing the flow in each pipe. This equation relates the flow Q to a series of parameters as

$$Q = KC_{HW} D^{2.63} \left( \frac{\Delta H}{L} \right)^N$$

(1)

where $C_{HW}$ is the Hazen-Williams coefficient, D is internal diameter, $\Delta H$ is the head loss due to friction, L is length, N is the exponent of the hydraulic slope, usually taken as 0.54, and K is a constant dependent on the choice of units. Q is generally expressed in gallons per minute, D in inches, $\Delta L$ and H in feet.
Given the trends shown in figure 3.1, a drop in the value of $C_{HW}$ would lead to an increase in $\Delta H$, i.e. a higher head loss over the length $L$ of the pipe, at a given level of the flow $Q$. It is important to mention another factor affecting $\Delta H$, which is the internal diameter $D$. As stated by McBeen et al. (1983), a reduction of the diameter may occur as a result of the formation of various deposits. Such a reduction could reach 50% of the diameter, due to the deposition of magnesium hydroxide. However, this is a fairly extreme case and in the following discussion, it is assumed that a rehabilitation (cleaning and lining) affects only the Hazen-Williams coefficient. In other words, although cleaning the pipe would increase the internal diameter $D$, this latter effect can be neglected, relative to the drop in $C_{HW}$ with age, in a generic sense. In specific cases where the reduction in the internal diameter was significant, the impact of the rehabilitation has to be examined, taking into account both changes in $C_{HW}$ and the internal diameter.

The additional operational costs incurred by the drop in $C_{HW}$ were identified by Walski (1982), who suggested a static rule to decide on whether to rehabilitate a pipe or not at a given point of time. The approach to the economic analysis adopted in this work is a dynamic one, consistent with the demand-responsive principle based on the assessment of maintenance requirements, as the "objective" is to define the optimal rehabilitation time based on projections of future behavior of the pipe.

The major additional energy costs are evaluated using the head loss formula, as given in the next paragraph.
FIGURE 3.1: Trend Curves for Head Loss Tests
(from McBean et al. (1983))
Energy costs in pumping requirements

Given a flow Q in the pipe, the head loss in feet per foot is given by the Hazen-William equation, adapted to the units below.

\[ h(Q) = \frac{\Delta H}{L} = \frac{10.43}{D^{4.87}} \times \left( \frac{Q}{C_{\text{HW}}} \right)^{1.85} \]  \hspace{1cm} (2)

where Q = flow in gallons per minute

\( C_{\text{HW}} \) = Hazen-Williams coefficient

D = diameter in inches.

Let us assume that the frequency distribution of Q is \( \psi(Q) \). Such a distribution is easily obtained by "probing" the flow Q at different times and in different periods.

The cost of energy to outweigh this head loss, assuming a constant price of energy P (other pricing structures could necessitate the modification of this expression), is under the choice of units below

\[ C_e = 0.0164 \int_{Q_{\text{min}}}^{Q_{\text{max}}} Q \ h(Q) \times \psi(Q) \times P \ dQ \]  \hspace{1cm} (3)

where \( Q_{\text{min}} \) = lower bound of Q in gallons per minute

\( Q_{\text{max}} \) = upper bound of Q in gallons per minute

\( C_e \) = cost of energy in dollars per foot of pipe per year

P = price of energy in cents per kilowatt hour

Q is expressed in gallons per minute

h(Q) = head loss in feet per foot
Evaluation of the additional energy costs - an example

Under the assumption that the flow rate Q is proportional to the square of the diameter of the pipe, the head loss is proportional to \( D^{-1.2} \), as obtained from the Hazen-William formula. The energy cost (3) is therefore proportional to \( D^{0.8} \), which is almost the same result found for the capital costs for pipelines. A function of the form \( C = a_0 D^a \), where C is the capital cost per foot of water main and D is the diameter of the main, was fitted to data reported by Walski and Pelliccia (1982) for a range of diameters varying from 6 to 24 inches. The best fit was obtained for \( a_0 = 3.75 \) and \( a_1 = 0.93 \). The trade-offs between different capital improvement measures are therefore quite similar for different sizes of diameters.

To illustrate the order of magnitude of the additional energy costs incurred by the problem of loss of carrying capacity, let us consider an example related to a 24-inch diameter pipe in which the Hazen-Williams coefficient is \( C=140 \). The pipe operates within the range 0-20 mgd. The head loss reported by Hudson (1966) for such a pipe averages 13.7 feet for 5,000 feet of pipe, or 2.75 feet per 1,000 feet of pipe.

The energy cost per year for this pipe (1,000 feet) is almost equal to \$3,120 as computed from (3), assuming a price per kilowatt-hour of 10 cents. Let us assume that C drops to 0.75C, which in eastern cities can occur for pipes ranging between 20 and 40 years of age. The head loss per foot rises to 1.7 h(Q), according to (2). The "additional" energy costs are at that point of time, almost equal to: \( 0.7 \times 3,120 = 2,184 \) per year and would continue to increase with time as shown in figure 3.1. As C drops to 0.5C, the head loss becomes 3.6 h(Q) which represents "additional" energy costs of the order of \$8,100 per year.
As the rehabilitation costs for 1,000 feet of a 24-inch pipe are equal to $23,100 (table 3.1), it is easy to see the rationale behind the implementation of a rehabilitation program in aging systems.

The elementary change in head loss for a change dC in the Hazen-Williams coefficient C (at constant Q) is

$$\frac{\partial h}{\partial C} = -1.85 \frac{h}{C}$$

This result can be formulated as follows:

$$\frac{\partial h/\partial C}{h/C} = -1.85 \quad (4)$$

Identically, at constant Q, as $C_e$ is proportional to $h$, the following result is obtained:

$$\frac{\partial C_e/\partial C}{C_e/C} = -1.85 \quad (5)$$

Thus, for every 1 percent decrease in C, $C_e$ increases by 1.85 percent, which gives an idea of the impact of the deterioration of the interior walls of the pipe on the energy costs required to maintain the level of serviceability in the vicinity of the pipe.
Using predicted future values of $C$, based on a predictive model taking into account both system specific and pipe specific effects, the stream of additional energy costs in the future can be derived. This allows the evaluation of the rehabilitation alternative at any point of time. Assuming that cleaning and lining a pipe restores the Hazen-William coefficient to a value $\bar{C}$, which is close to the value associated to the pipe if it were new, the cost of energy is obtained using the "modified" head loss:

$$\bar{h}(Q) = \frac{10.43}{D^{4.87}} \times \left(\frac{Q}{\bar{C}}\right)^{1.85}$$

The expression for the "modified" cost of energy is

$$\bar{C}_e = 0.0164 \int_{Q_{\text{min}}}^{Q_{\text{max}}} Q \bar{h}(Q) \psi(Q) \times P \, dQ.$$ 

The economic viability of the rehabilitation of the pipe in a given year $t$, can be "statically" examined by comparing the ensuing energy costs under both scenarios. If the energy costs are assumed to become constant in real terms at the value $\bar{C}_e(t)$ in the following years, then the present value of future energy costs is that of a perpetuity and is equal to $\frac{\bar{C}_e(t)}{r}$, where $r$ is the real discount rate. It is also possible to predict the future energy costs after the rehabilitation alternative is implemented, by evaluating the evolution of such costs
for a newly relined pipe predicting future trends in the pipe. For simplicity, let us assume that the first scenario prevails (no significant performance deterioration after the rehabilitation alternative). Under such conditions, the comparison of the rehabilitation alternative to the "do nothing" alternative at time $t$ is obtained by comparing $\frac{C_e(t)}{e^r}$, to the present value of the future stream of energy costs under "do nothing", assuming that no additional pumping capacity is required in either case. (REH represents the rehabilitation costs for the pipe.) However, the comparison of these two alternatives does not exhaust the whole range of alternatives obtained from allowing the timing of the rehabilitation to vary. This dynamic view of the decision problem is taken in the determination of an optimal rehabilitation time, as derived further.

Table 3.1 hereafter reports the unit cost for rehabilitation of water mains. This table was given by Walski (1982). The other category of additional costs due to the loss of carrying capacity is related to the capital cost for pumping capacity.

As can be seen from Table 3.1, the rehabilitation alternative seems to be more economically attractive for larger pipes, other things being equal. As seen earlier, the additional energy costs from head losses are proportional to $D^{0.8}$, where $D$ is the diameter of the pipe, while rehabilitation costs per foot seem to be quite flat relative to the diameter. Furthermore, as repair events are more infrequent in large diameter pipes, the rehabilitation alternative will often be selected for this category of pipes as the optimal capital improvement measure, especially as these pipes are often oversized (underutilized). Therefore, rehabilitation seems quite relevant for large pipes. In the case of smaller diameter pipes with higher breakage rates, replacement
<table>
<thead>
<tr>
<th>Diameter, D, in inches (millimeters) (1)</th>
<th>Unit cost, c. (1981), in dollars per foot (dollars per meter) (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>25.00</td>
</tr>
<tr>
<td>(100)</td>
<td>(82.00)</td>
</tr>
<tr>
<td>6</td>
<td>25.00</td>
</tr>
<tr>
<td>(150)</td>
<td>(82.00)</td>
</tr>
<tr>
<td>8</td>
<td>25.00</td>
</tr>
<tr>
<td>(200)</td>
<td>(82.00)</td>
</tr>
<tr>
<td>12</td>
<td>25.00</td>
</tr>
<tr>
<td>(300)</td>
<td>(82.00)</td>
</tr>
<tr>
<td>16</td>
<td>23.10</td>
</tr>
<tr>
<td>(400)</td>
<td>(75.80)</td>
</tr>
<tr>
<td>20</td>
<td>23.10</td>
</tr>
<tr>
<td>(500)</td>
<td>(75.80)</td>
</tr>
<tr>
<td>24</td>
<td>23.10</td>
</tr>
<tr>
<td>(600)</td>
<td>(75.80)</td>
</tr>
<tr>
<td>30</td>
<td>32.20</td>
</tr>
<tr>
<td>(750)</td>
<td>(105.64)</td>
</tr>
<tr>
<td>36</td>
<td>34.50</td>
</tr>
<tr>
<td>(900)</td>
<td>(113.20)</td>
</tr>
<tr>
<td>48</td>
<td>42.20</td>
</tr>
<tr>
<td>(1,200)</td>
<td>(138.45)</td>
</tr>
</tbody>
</table>

(from Walski (1982))
costs tend to be quite close to rehabilitation costs (table 3.2) making the analysis more likely to favor replacement over rehabilitation.

Costs of Pumping Capacity

As carrying capacity decreases with age, the head provided by the pumps becomes insufficient to maintain the required level of serviceability. Therefore, given a certain projection of the evolution of head losses, it is possible to predict at a given point of time, both the exact time and the incremental pumping capacity required and evaluate its present cost.

Optimal Rehabilitation Time

The information provided by the projections of future energy costs in pumping requirements and costs of pumping capacity to be installed in the future allows one to derive the optimal rehabilitation time for a pipe. If a pipe is rehabilitated at time $t_1$, then the total present costs incurred are the sum of the additional energy costs due to head losses and the capital costs associated with the increased pumping capacity, if such was required before $t_1$.

The problem can therefore be stated as:

$$\min_{t_1} \left\{ \frac{t_1}{t} \frac{\Delta C_e(t)}{(1+r)^t} + \frac{CPC(t_2)}{(1+r)^{t_2}} + \frac{REH(t_1)}{(1+r)^{t_1}} \right\} (1 + r)^{t_1}$$

where

$t_p$ = present date

$\Delta C_e(t) = $ additional energy costs incurred in year $t = C_e(t) - \bar{C}_e(t)$

$CPC(t_2) = $ capital pumping capacity cost incurred in year $t_2$, where the value of $t_2$ is obtained in a separate analysis. The magnitude of $CPC(t_2)$ depends on the value of $t_1$. For $t_1$ small enough, no capital pumping capacity is required.
\[ \text{REH}(t_1) = \text{rehabilitation cost at time } t_1 \]
\[ r = \text{annual discount rate (in real terms)} \]

The output of this minimization problem is an estimate of the optimal rehabilitation time for the specific pipe considered. The evaluation of the additional energy costs involved, as well as the timing and value of the capital costs needed for the incremental pumping capacity depends on the projection of future head losses in the pipe. This projection can be generated through a predictive model for the future variation with time of the Hazen-Williams coefficient. In figure 3.1, the "system specific" variation of the Hazen-Williams coefficient was represented. A predictive model to generate estimates of future values of \( C_{\text{HW}} \) in a given pipe, based on its specific parameters is therefore required to allow for a more accurate analysis of the optimal rehabilitation time. As the system specific trend of variation of \( C_{\text{HW}} \) incorporates the central tendency of the behavior of different pipes of the system, a more elaborate predictive model is required, including both system specific parameters such as water hardness, materials specification and pipe specific parameters which affect the Hazen-Williams coefficient.

**Evaluation of the rehabilitation alternative**

As was shown above, it is possible to optimize the rehabilitation time, assuming that the rehabilitation alternative is the one retained. The evaluation of the net present value of the total costs associated with the rehabilitation alternative, under the optimal conditions can be undertaken by adding to the present value of the additional
operational costs (represented in the objective function of the previous paragraph), the present value of future repair costs due to leaks and breaks occurring in the section of pipe as these latter costs are not affected by the rehabilitation. The value of these repair costs can be estimated using the predictive model for future failure rate applied to the pipe.

B. Economic Analysis of the Replacement Alternative

The category of maintenance measures considered herein is related to repair versus replacement of a pipe. It is useful to note that, under the assumption that only repair costs for leaks and breaks occurring in the system are truly "material", the timing of the replacement measure is more readily derivable, due to the existence of predictive models for pipe failure rate.

In the most general content, it can be assumed that new pipes do not experience breaks and do not require additional operational costs due to the head losses observed in aging pipes. The formulation of the optimal replacement time will be first derived in this general case. An illustration of the evaluation of the optimal replacement time follows in the special case where additional operational costs are neglected, i.e. head losses are not significant. The model derived by Shamir and Howard (1979) to predict failure rates is then used.
I. Optimal Replacement Time for Pipes Failing and Experiencing Head Losses

Assuming that an old pipe experiences both failures and head losses, the costs associated with these defects in year $t$ are denoted by $(C_m(t) + C_0(t))$, where:

- $C_m(t)$ = repair costs in year $t$
- $C_0(t)$ = additional operational costs in year $t$

It is easy to evaluate the first item as:

$C_m(t) = C_b N(t)$, where:

- $C_b$ = single repair cost
- $N(t)$ = expected repair events in year $t$

The evaluation of $C_0(t)$, as given in the previous section related to the rehabilitation of water mains can be undertaken provided that a model predicting the head losses which cause this increase in operations costs is available.

The optimal replacement time $t^*_r$ is thus found by minimizing the net present value of future maintenance, operational and replacement expenditures. In other words, $t^*_r$ is the solution of the minimization problem of

$$
\sum_{t=t_p}^{t_r} \frac{C_m(t) + C_0(t)}{(1+r)^{t-t_p}} + \frac{C_r}{(1+r)^{t_r-t_p}}
$$

where:

- $r$: real discount rate (inflation adjusted)
- $t_p$: present date
$C_r$ = replacement cost of pipe

More generally, the continuous case is formulated as the minimization of:

$$\int_{t_p}^{t_r} \left[ C_m(t) + C_0(t) \right] e^{-r(t-t_p)} dt + C_r e^{-r(t_r-t_p)}$$

This leads to the estimation of $t_r^*$ in the general case of additional repair and operational costs. The evaluation of the net present value of future repair, operational and replacement costs can then be compared to the present value of future costs incurred under the rehabilitation alternative.

II. Optimal Replacement Time of a Pipe Experiencing Only Failure Events

1. Predictive model used

For the purposes of this section, the predicted number of breaks per year in a link $l$ derived by Shamir and Howard (1979), will be assumed. The form that was found by the latter authors is:

$$N_l(t) = N_l(t_0) e^{A(t-t_0)}: \text{where the exponential growth rate with time is in agreement with Clark's results.}$$

Notations:

$t$ = time in years

$l$ = index of link

$t_0$ = base year for the analysis (the year the pipe was installed for instance)
\[ N(t) = \text{number of breaks per 1000-ft length of pipe in year } t \]
\[ A = \text{growth rate coefficient (per year); depends on the} \]
\[ \text{causal factors previously seen.} \]

Table 3.2 (from Shamir and Howard) gives the typical values and ranges for the parameters.

**Table 3.2**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Typical Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual growth rate of breaks-( A )</td>
<td>0.05</td>
<td>0.01-0.15</td>
</tr>
<tr>
<td>Discount rate-( R )</td>
<td>0.10</td>
<td>0.05-0.15</td>
</tr>
<tr>
<td>Number of breaks per 1000 ft in year ( t_o ) (1961 in the study) - ( N(t_o) )</td>
<td>0.10</td>
<td>0.01-0.20</td>
</tr>
<tr>
<td>Cost of repairing a break (1977 $) - ( C_b )</td>
<td>1000</td>
<td>500-2000</td>
</tr>
<tr>
<td>Cost of replacing 1000 ft of pipe (1977 $) - ( C_r )</td>
<td>50,000</td>
<td>10,000-150,000</td>
</tr>
</tbody>
</table>
The repair costs in year $t$ can be expressed as:

$$C_{ni}(t) = C_b N(t),$$

where $N(t)$ is the expected number of breaks in year $t$ if the pipe is not replaced in that year.

The optimal replacement time $t_r^*$ is thus found by minimizing the net present costs of future maintenance and replacement expenditures. In other words, $t_r^*$ is the solution of the minimization problem of:

$$t_r^* = \min_{t \geq t_p} \left( C_m(t) + \frac{C_r}{(1+R)^{t-t_p}} \right)$$

where:

- $R =$ real discount rate (inflation adjusted)
- $t_p =$ present date

More generally, the continuous case can be formulated as the minimization of

$$\min_{t \geq t_p} \left( \int_{t_p}^{t} C_m(t) e^{-R(t-t_p)} dt + C_r e^{-R(t_r-t_p)} \right) .$$

The solution of this latter problem can be found by differentiating the above function and setting the differential to zero.

This yields the optimal value:

$$t_r^* = t_o + \frac{1}{A} \ln \left( \frac{RC_r}{N(t_o)C_b} \right)$$

where $A$ and $N(t_o)$ are the coefficients in the expression of $N(t)$. 
Thus, it is possible to find the optimal replacement time for each section of pipe given its predicted future failure rate. It is also interesting to see how sensitive the optimal replacement time is to different factors appearing in the aforementioned formula.

**Sensitivity analysis**

The sensitivity of $t_r$ to variations in the parameters $A, N(t_o), C_b, C_r$ and $R$ can be studied by evaluating the partial derivatives.

\[
\frac{\partial t_r}{\partial A} = - \frac{1}{A^2} \ln \left( \frac{RC_r}{N(t_o)C_b} \right)
\]

\[
\frac{\partial t_r}{\partial R} = \frac{1}{AR}
\]

\[
\frac{\partial t_r}{\partial N(t_o)} = -\frac{1}{AN(t_o)}
\]

\[
\frac{\partial t_r}{\partial C_b} = -\frac{1}{AC_b}
\]

\[
\frac{\partial t_r}{\partial C_r} = \frac{1}{AC_r}
\]

Using typical values from Shamir and Howard (1979), Clark (1982) computed the above values as:

\[
\frac{\partial t_r}{\partial A} = -248.13
\]
which indicates that $t_r$ will decrease by 1 year for an increase of 0.004 in $A$, thus underlining the importance of a goal

\[
\frac{\Delta t_r}{\Delta R} = 160.3, \text{ i.e., } t_r \text{ will increase by 1 year for an increase of 0.6 percent in the real discount rate.}
\]

\[
\frac{\Delta t_r}{\Delta N(t_o)} = -11.13, \text{ i.e., } t_r \text{ will decrease by 1 year for an increase of 0.09 in } N(t_o).
\]

\[
\frac{\Delta t_r}{\Delta C_b} = -0.0081, \text{ i.e., } t_r \text{ will decrease by 1 year for an increase of $123$ in } C_b.
\]

\[
\frac{\Delta t_r}{\Delta C_r} = 0.0001, \text{ i.e., } t_r \text{ will increase by 1 year for an increase of $12,000$ in } C_r.
\]
C. Analysis of the "Rehabilitation Followed by Replacement" Alternative

In this case, it is assumed that a rehabilitation occurs at time $t_1$, followed by a replacement at time $t_2$.

The problem of identifying the optimal values for $t_1$ and $t_2$ can therefore be formulated as the minimization of:

$$
\sum_{t=t_p}^{t_1} \frac{C_m(t) + C_0(t)}{(1+R)^{t-t_p}} + \frac{C_{REH}}{(1+R)^{t_1-t_p}} + \sum_{t=t_1+1}^{t_2} \frac{C_m(t)}{(1+R)^{t-t_p}} + \frac{C_{REP}}{(1+R)^{t_2-t_p}}
$$

under the constraint that

$$
t_1 < t_2
$$

where

- $C_m(t)$ = repair costs incurred in year $t$
- $C_0(t)$ = additional operational costs in year $t$
- $C_{REH}$ = rehabilitation costs for the pipe
- $C_{REP}$ = replacement costs for the pipe.

D. Selection of the Optimal Alternative at the Pipe Level

The selection of the optimal alternative at the single pipe level is schematized in the decision tree of figure 3.2. Under the rehabilitation alternative, it is possible to minimize the net present value of future operational costs by timing optimally the rehabilitation alternative. The outcome of the decision in present value terms is obtained by adding the net present value of operational and rehabilitation costs to that of future repair costs due to leaks and
breaks in the pipe.

Similarly, the outcome of the replacement alternative and the rehabilitation followed by a replacement alternative can be evaluated and the optimal alternative is then identified.

In practice, the alternatives considered over a twenty-year time span would probably be either a rehabilitation or a replacement, as the costs associated with both alternatives are of the same order of magnitude. The "project management" approach to such a decision problem is reported by Corless (1982), where the replacement decision is selected automatically, if the pipe is undersized. "Structural lining" is selected if external corrosion is taking place and normal cleaning and lining is selected if bacterial tuberculation takes place, due to internal corrosion.

In the aforementioned project management approach, a pipe is considered as undersized if it cannot meet the projected demands in its vicinity. This is determined via a computer model that generates the effects of adding supply or demand to the system on the flow rates in the large mains. The output of the model allows the identification of the undersized sections and the required capacity increment.

The decision over the type of remedial action required, as depicted in figure 3.3, relies heavily on the qualitative criteria such as tuberculation, external corrosion (suspected to cause main breaks). Upgrading of high velocity pipes is suggested based on an engineering minimal allowable C value. If the Hazen-Williams coefficient falls below 80, the pipe is a candidate for upgrading.

The prediction of the future performance of the system is not
FIGURE 3.2: Decision Tree at the Single Pipe Level.
available in this approach, and the estimation of future economic costs associated with different alternatives is therefore an impossible task. Only the minimum allowable $C_{HW}$ value results from an appropriate "break-down analysis". Even undersizing is determined separately from the additional operational costs and repair costs that are incurred in the system.

The shortcomings of the approach represented in figure 3.3 can be summarized as follows:

1. Budgets are taken as exogenous parameters, whereas a demand-responsive approach is required to undertake the planning of future maintenance expenditures.

2. The engineering criteria mentioned above are static. "Rehabilitate if $C_{HW} < 80$" is one such static rule, which neglects the dynamic evolution of $C_{HW}$, and is not based on a sound economic evaluation of the rehabilitation alternative. The availability of predictive models for performance measures, and particularly those to which economic costs can be associated (loss of carrying capacity, breakage) allows one to undertake the required economic evaluations shown in this chapter.

3. The criteria for the selection of the optimal alternative are highly qualitative. Tuberculation necessarily leads to "cleaning and lining", whereas external corrosion requires replacement. While the integration of the engineering qualitative criteria is important in the selection of capital improvement measures, the economic criterion is almost absent.
4. The timing of such statically selected measures is based on a certain qualitative priority scheme, depending on the existent and predicted future budgets. The optimal scheduling of maintenance expenditures based on a sound capital budgeting scheme is the right approach to the global problem.

In this chapter, the systems analysis approach was taken as the data required for building the predictive models for failure rates and loss of carrying capacity is assumed to be available. As was discussed in Chapter 1, when analytical models are not available or well-calibrated, other qualitative criteria such as external corrosion, tuberculosis, etc. become more easily usable for decision-making purposes. However, as discussed in Chapter 2, it is possible to collect the required data for the predictive models, provided that some surveys are conducted. The economic analysis of alternative measures can then be performed as in this chapter, for each single pipe. The possibility of earlier replacement of undersized pipes is still feasible, via the scheduling model of maintenance and expansion alternatives presented in Chapter 5.

In summary, an approach to the economic comparison of the rehabilitation and replacement alternatives at the pipe level was presented. The outcome of this approach is an input to the scheduling model that can ultimately determine the type of alternative selected for each pipe, as well as the time period of the planning horizon in which it is to be implemented. For example, a pipe which was supposed to be rehabilitated based on the separate pipe analysis
FIGURE 3.3: Project Management Approach to Pipe Maintenance Decision-Making
could be replaced by a larger capacity and earlier than expected, due to expansion needs. Even though this is not the case in a maturing system (typical system of interest), it still illustrates the fact that the output of the single pipe analysis is by no means final. Network considerations (changing needs or structure), budget constraints, reliability constraints all can affect the final alternative related to each section of the network.

The rationale for the single pipe level analysis stems from the fact that it generates necessary inputs to the network scheduling model. As was shown in figure 2.2, the single pipe analysis allows the identification of the pipes which are supposed to be replaced or rehabilitated within the planning horizon. These pipes can be constrained to a capital improvement measure in one of the time periods of the horizon, by including the associated constraints in the model. By doing so, the planning model (which minimizes total costs over a 20 to 25 year horizon), does not result in a deferral of major required measures to the next planning horizon. Such a deferral clearly creates inefficiencies and is avoided by recognizing the pipes that should be upgraded (or cancelled) within the limits of the planning horizon.

The scheduling model allocates the available resources optimally and broadens the scope of the possible alternative measures that can be implemented on each section of pipe, under the constraint of required capital improvement measures for specific pipes within the planning horizon.

Besides allowing the scheduling model to generate "economically
sound" outputs, the single pipe level analysis limits the size of the model and its complexity. For a large-scale mixed-integer programming model, this reduction in complexity is quite desirable. For example, if the replacement alternative was chosen in the single pipe analysis, then it might not be necessary to include a decision variable (integer) for the rehabilitation. This latter alternative is definitely dominated by a replacement, expansion, or cancellation alternative, at the network scheduling level given that it is dominated at the single pipe level.

Perhaps the most important result of the single pipe analysis is the estimation of capital improvement and maintenance requirements over a certain planning horizon. This can be used as a basis to define and schedule budgetary requirements in successive time periods. Clearly, the closer the budgets are to the capital requirements of the optimal strategy, the smaller the inefficiencies incurred from the deferral of capital improvement measures due to the deviation between budgets and requirements. The demand-responsive approach to the development maintenance and capital improvement programs can be applied to the estimates of capital requirements obtained from the single pipe analysis. Better resource allocation and debt management would result from this first level exercise. The scheduling of the capital improvement measures throughout the system, based on the resource constraints is still a necessary step, as in many cases capital improvement measures will have to be deferred. The problem is to define the least costly scheduling strategy under the projected budget constraints. In Chapter 5, capital budgeting models are derived, specifically for this purpose.

It follows that the single pipe level analysis is a major part of the decision support system for the scheduling of maintenance and capital improvement measures in the network.
CHAPTER 4: The Prediction of Pipe Failure

I. Causes of Water Main Failure

The causes of water main breaks have been identified by a number of authors. However, the contribution of each possible cause to the failure event still requires significant research. Morris (1966) suggested a number of possible causes for water main breaks, but still underlined that "the cause of water main breaks cannot always be ascertained".

In the 1980's, as failure rates increase in aging mature systems, the identification of the causes of pipe failure and their impacts is necessary for two major reasons:

1. The need to forecast future behavior of different sections of the network, for the purposes of maintenance decision-making. By behavior, we mean in particular failure rates and loss of carrying capacity. The prediction of these variables in the future is necessary to the evaluation of replacement and rehabilitation alternatives.

2. Some of the causes of failure could be "treated". Such remedies could possibly affect the development and the magnitude of the "maintenance problem" by providing guidelines for the layout and operation of new parts of the system, but also the maintenance and operation of old sections. New design and operation criteria would be identified, as explained in Chapters 1 and 2.

The factors affecting water main breaks are internal (strength of the pipe, characteristics such as diameter, ...) and external. External
causes can be divided into four categories, depending on whether they are system specific or not, human or environmental. System specific causes are related to issues such as the type of water (hardness, chemicals carried, etc.) which can influence the evolution of such phenomena as internal corrosion. Some systems would have significantly high breakage rates in general due to their water type. Water type can be considered as having both a human dimension and environmental dimension. Water treatment methods (human) can change the intrinsical characteristics of the water in a system (environmental). The categorization of the major external causes of failure is represented in Table 4.1.

Morris (1966) recognized some of the causes of main breaks. Some of his findings and recommendations are worth mentioning, as they represent the engineer's explanation of the failure mechanism.

1. Inadequate design:
Inadequate design is a cause of failure of the "human" type, which is non system specific. After testing the soil, system pressure and operation procedures should be taken into account to select the adequate type of pipe to be used, as well as its thickness and strength. Available standards should be carefully followed to minimize future undesired consequences of inadequate design.

2. Improper installation:
Improper installation is another human type of cause related to the initial embedment of the pipe. Depending on whether a pipe is rigid, semi-flexible or flexible, breakage vulnerability points can be identified and proper installation undertaken.
<table>
<thead>
<tr>
<th></th>
<th>System Specific</th>
<th>Non System Specific</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human</td>
<td>• Water Type</td>
<td>• Inadequate design</td>
</tr>
<tr>
<td></td>
<td>• Water treatment methods (corrosion control)</td>
<td>• Improper installation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Water hammer</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Peak demand</td>
</tr>
<tr>
<td>Environmental</td>
<td>• Water Type</td>
<td>• Impact</td>
</tr>
<tr>
<td></td>
<td>• Soil movement</td>
<td>• Soil movement</td>
</tr>
<tr>
<td></td>
<td>• Internal corrosion</td>
<td>• External corrosion</td>
</tr>
<tr>
<td></td>
<td>• External corrosion</td>
<td>• Temperature differential</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Location of pipe layout</td>
</tr>
</tbody>
</table>

**TABLE 4.1**: Categorization of External Factors Affecting Main Breaks
3. **Surge and Water Hammer**

An operational cause of failure can be attributed to water hammer which is directly influenced by the correct functioning of check valves. Proper inspection of these valves and adequately trained personnel are necessary to prevent particularly costly breaks that usually result from a "slammed shut" valve.

4. **Soil Movement and Volume Shrinkage**

The type of soil and the ensuing volume changes can be considered as having both a system specific and a non system specific environmental dimension. Shifting soils were suspected to be the cause of excess main breaks in the Fort Worth area (Morris (1966)). According to the same reference, bending stresses caused by the swelling of soils such as clay were found to be three-to-four times greater than such effects as internal pressure.

Also, if a pipe lies in two soil types, a break could occur, as the more expansive soil would tend to elevate the pipe while the pipe would resist such movement, as it lies partly in another soil which constrains these movements.

5. **Internal Corrosion**

Internal corrosion is believed to contribute to the mechanism of main failure. As Morris (1966) points out, "although the final break may be triggered by shifting soils or temperature differential corrosion is often the primary factor." The latter author mentions the ways to control different types of corrosion, primarily based on water treatment methods. The engineering practice (Corless (1982)) seems to
consider external corrosion as the factor affecting the failure mechanism. However internal corrosion can conceivably lead to failures, but is a process that becomes more serious as pipes age. Wall thickness and pipe strength decrease under the timely effect of corrosion and the likelihood of breakage increases at the same time. Because of the relationship of time with such processes as corrosion (internal and external) and material fatigue due to impact and changing stress conditions over time, the derivation of a predictive model for pipe failure taking into account these variables seem possible, provided that the adequate data is available. The issues related to the development of a predictive model are discussed in a further section of this chapter.

Internal corrosion, as well as the means to control it (water treatment), have another detrimental effect on the performance of the system. The loss of carrying capacity due to the drop in the Hazen-Williams coefficient C is a direct effect of internal corrosion. Also, in order to control chemical corrosion, lime is added to the treatment for softening and pH adjustment leading to the formation of a protective film of calcium carbonate. A reduction of the internal diameter of the pipe of about 3% could occur, and a loss of carrying capacity directly results from such reduction, according to the Hazen-Williams formula (chapter 3).

6. **External Corrosion**

External corrosion also leads to main breaks and is usually of these types: galvanic, electrolytic and bacterial. Galvanic
corrosion is usually given the most attention and is activated by moisture in the soil which is the electrolyte of this electrochemical reaction.

External corrosion is an important factor to incorporate in a predictive model for pipe failure for a given system. Unlike internal corrosion, its intensity depends on soil corrosiveness conditions and therefore varies from pipe to pipe. As a predictive model is usually specific to a given system, external corrosion is a major factor or explanatory variable.

7. Temperature Differential

Both high temperatures and low temperatures can "cause" main breaks. Many breaks are recorded in July or August, when facilities are functioning at maximum capacity to meet peak demands. These breaks are due to such excessive pressure, along with other causes such as external stress or corrosion.

Also, in the winter when the water temperature drops very quickly by 20 to 30 degrees, axial stress may result that is added to the internal pressure of the water often bypassing the factor of safety. Additional stress may result from frost effects, especially when pipes are rather shallow.

While "high temperature effects" are human and non system specific (basically peak demand effects), low temperature effects are system specific and depend on the type of weather in the region and the general layout of the system.
8. **Impact**

Other impact effects can be traced back to surface traffic, especially when pipes are shallow and the impact is not absorbed by the pavement on the sub-base, as explained by Morris (1966). It is also important to note the large number of main breaks due to impact caused by the construction and maintenance of other utilities. Morris (1966) pointed out that 9% of 4,000 main breaks recorded in Dallas between 1960 and 1965 occurred in pipes with 12-inch diameter and above. Virtually all of these large main breaks were due to human error occurring in the maintenance of other utilities. Even as such errors can be controlled, it is interesting to note that small diameter pipes (< 8-10 inches) fail rather frequently, while large diameter pipes seem to fail quite rarely, possibly as the effects of the aforementioned causes, and particularly stress related factors, are less critical for large pipes. Higher strength requirements and wall thickness, a better layout make the combined effect of corrosion and stress develop at a much slower pace.

Another impact related factor is the location of the pipe layout. As mentioned by Morris (1966), "rigid pipe is subject to shear breakage at points of entry to pump stations, entrance into bore holes and tunnels, and at points where heavy fittings are installed." Cement pipes seem to be most susceptible to such a breakage mode.

In summary, a break in a water main is a complicated event, resulting from mechanical effects (stress) as well as corrosion effects (mostly electrochemical) often combined together in a single event.
It is also the result of human and environmental causes, some system
specific and some non system specific.

II. Predicting Water Main Failures

II.1 Shamir and Howard's Models

The use of history of water main breaks along with the under-
standing of the causes and mechanisms of failure are the main sources
of information for building a predictive model for pipe failure rates.

Shamir and Howard (1979) circumvented part of the problem by
pooling maintenance history data related to pipes of similar character-
istics (internal and external). They fitted an exponential variation
of the number of failures with time to each pool. The type of models
derived were mentioned in Chapter 3 (B.II). The exponential growth
of the number of breaks with time is a good approximation of reality,
particularly for small diameter pipes where breaks are much more
frequent than large diameter pipes.

The advantage of the "pooling method" is, besides its simplicity,
its ability to develop clear trends of evolution of pipe failure
rates with time. However, it requires large amounts of data which
should be close to evenly distributed among different pools. The
system studied by Shamir and Howard was experiencing about 1,500
breaks/year, with a noticeable trend in different categories of pipe
for exponential growth with time of the number of breaks per year
in a given pipe. Shamir and Howard's simple models were used in the
case example of Chapter 5, illustrating the scheduling model for
maintenance alternatives in a certain grid.
Another approach to forecasting water main failures was through a regression-based model, on a systemwide basis, taking into account most non-system specific explanatory variables, as these vary across different pipes in the network. The first of these models was developed by Clark et al. (1982). The low frequency of breakage in large water mains, and the serious impacts of such breaks makes it necessary to develop a probabilistic model for pipe failure. Such a model is suggested in the section following the description of the Clark model, based on the notion of survival analysis and using the Cox regression techniques for the estimation of its parameters. The necessity of such a probabilistic model for the prediction of the failure of large water mains stems from the fact that while redundancy is inherent in the secondary network, it is often non-existent in the primary network. In other words, if a large feeder main breaks, a shortage or a reduction in service as well as other social costs due to damages to other utilities might result. The actual cost of a break would therefore be the sum of a social cost $S$ and a repair cost $C_b$. Where the value of $S$ is high, the need for accuracy reinforces the fact that a probabilistic model for the category of large pipes is required. Models such as Shamir and Howard's (1979) can generate an approximate number of events per year, which associated cost can be computed as repair costs. An estimation error would not distort the planning exercise, which is the ultimate use of all such predictive models.

Therefore, while simple regression models are useful tools for the category of pipes which failure leads to simple repair costs, a
probabilistic model is necessary for such large pipes which failure is an event with high cost consequences.

II 2 Event Estimating Equations (from Clark et al. (1982))

Two data sets were used by Clark et al., relating to a small and a large utility. Pipe records included information about the diameter length, total number of breaks, type of pipe, as well as the corrosivity level surrounding the pipe, the pressure conditions, the repair history and the age of the pipe. Repair records were available after 1940 on 307 pipes considered in the original data set. However, some of the pipes had been laid out in 1930. Because the first maintenance event did not usually occur until 15 years after the pipe had been laid, the analysis could begin at 1930 instead of 1940 on the assumption that no breaks occurred in the first 10 years. Of the 307 pipes laid between 1930 and 1980, only 108 have been repaired. Figure 4-1 displays this break history between 1940 and 1980.

Examination of the data revealed that two underlying mechanisms seemed to be occurring with those pipes that experienced maintenance events. A lag period occurred between the time the pipe was laid and the first maintenance event. After the first event, the number of events seemed to increase exponentially. Therefore, two equations were developed, the first to estimate the time to the first event and the second, to estimate the number of events occurring after the first event. Equations were developed for both the large and small utility.

First Event Equation - Small Utility

The following equation predicts time from the initial installation
to the first event for the small utility.

$$NY = 2.9 + 0.442D + 0.017P + 0.412I - 0.325RES$$

$$R^2 = 0.34$$  \hspace{1cm} (1)

where

$NY =$ number of years from installation to first repair;

$D =$ diameter of pipe in inches;

$P =$ absolute pressure within a pipe in PSI;

$pounds$ $per$ $square$ $inch$

$I =$ percent of pipe overlain by industrial development in a census tract;

$RES =$ percent of pipe overlain by residential development in a census tract.

The number of pipes that had useable data was 18, and the pipes analyzed were in the ground a relatively short time (13 years on average).

First Event Equation - Large Utility

The equation predicting the first event was as follows:

$$NY = 11.0 + 0.263D - 0.006P - 0.773I - 0.253RES - 0.00006LH$$

$$+ 15.69T \hspace{1cm} R^2 = 0.34$$  \hspace{1cm} (2)

where variables are defined as in equation (1) and where

$LH =$ Length of pipe in highly corrosive soil;

$T =$ pipe type ($1 =$ metallic; $0 =$ reinforced concrete).
FIGURE 4.1. Break History for the Data Set
(from Clark et al (1982))

Cumulative # of Failures

0 1930 1940 1950 1960 1970 1980
Years

Actual
In equation (2) when soil corrosivity and pipe type entered the equation the coefficients of \( p \) and \( i \) switched from positive to negative. The number of pipes considered in equation (2) was 68 and represented 40 years of data.

**First Event Equation - Combined**

Equation (3) represents the combined set of data:

\[
NY = 4.13 + 0.338D - 0.022P - 0.2651 - 0.0983RES - 0.0003I + 13.28T \quad (R^2 = 0.23)
\]  

(3)

The following table contains the partial correlations with \( NY \) for each variable considered in equation (3). Partial correlations (in absolute value) indicate the significance of different predictors. Pipe type was the most significant predictor of the number of years to the first event.

**TABLE 4-2: PARTIAL CORRELATIONS**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Partial Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>.27599</td>
</tr>
<tr>
<td>Absolute pressure</td>
<td>-.02211</td>
</tr>
<tr>
<td>Industrial %</td>
<td>-.37168</td>
</tr>
<tr>
<td>Residential %</td>
<td>-.41525</td>
</tr>
<tr>
<td>Length in highly corrosive soil</td>
<td>-.01996</td>
</tr>
<tr>
<td>Pipe type</td>
<td>.42638</td>
</tr>
</tbody>
</table>
Accumulated Event Equation - Small Utility

The equation that predicts the cumulative number of maintenance events after the first event is as follows:

\[ REP = (0.386)(e^{0.0139^{\text{PRD}}})(e^{0.0602^{\text{A}}})(e^{0.0208^{\text{DEV}}})(SL^{-0.016})(SH^{0.025}) \]

\[(R^2 = 0.49) \] (4)

where

- **REP** = number of repairs up to year considered;
- **PRD** = pressure differential in PSI;
- **A** = age of pipe from first break;
- **DEV** = percent of land, over pipe, which is developed;
- **SL** = surface area of pipe in low corrosive soil;
- **SH** = surface area of pipe in highly corrosive soil.

Accumulated Event Equation - Large Utility

\[ REP = (0.172)(e^{0.7198^{\text{T}}})(e^{0.0040^{\text{PRD}}})(e^{0.0862^{\text{A}}})(0.0189^{\text{DEV}})(SL^{0.014})(SH^{0.069}) \]

\[(R^2 = 0.47) \] (5)

where the variables in equation (5) are the same as previously defined and where

- **T** = type of pipe (1 = metallic, 0 = reinforced concrete).
Accumulated Event Equation Combined

\[ REP = (0.1721)(e^{0.7197})^T (e^{0.0044})^{PRD}(e^{0.0865})^A (e^{0.0121})^{DEV} \]

\[(SL)^{0.014} (SH)^{0.069} \]

\[(R^2 = 0.47) \] (6)

The correlations for the variables in Equation (6) are in Table 4-3. The age from the first repair is the most significant predictor of the total number of repair events followed by the surface area in high corrosive soil.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Partial Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of pipe</td>
<td>.27560</td>
</tr>
<tr>
<td>Pressure differential</td>
<td>.16666</td>
</tr>
<tr>
<td>Age from first repair</td>
<td>.48159</td>
</tr>
<tr>
<td>Percent developed</td>
<td>.25051</td>
</tr>
<tr>
<td>Surface Area in Low and Moderate</td>
<td>.08055</td>
</tr>
<tr>
<td>Surface Area in High</td>
<td>.37972</td>
</tr>
</tbody>
</table>

The predicted events can be compared with actual events as estimated by equation (6) (Figure 4-2). Equation (6) must be applied to each pipe individually to estimate the number of breaks over time. Each of the variables considered in the analysis is discussed in the following sections.
Type of pipe

Type of pipe varies from cast iron to reinforced concrete. Most mains in the data set are cast iron (CI) simply because the system was built at the time cast iron was used. During the 1930's, several steel mains were laid, but they are not prevalent. In the 1950's reinforced concrete pipe (RCP) was used for mains 24 inches (61 cm) or larger. For statistical reasons, pipe type was treated as a dummy variable. The value "one" was assigned to mains of metallic character [steel, CI, DI (ductile iron)] and "0" (zero) for RCP.

The entry of pipe type into the final equation with a positive partial correlation suggests that mains of metallic character are more susceptible to maintenance events than concrete mains. These events may result from many factors including corrosion, that have been discussed earlier in this chapter. Current industry trends are toward concrete, plastic and cast iron wrapped in polyethylene sleeving.

Pressure Differential

Pressure differential measures the maximum difference in absolute pressures within a main in pounds per square inch (psi). If a pipe is subjected to widely differing pressures, then stress may occur at certain points. Pressure differentials cause transients which are more damaging than absolute pressures. The effect of pressure differential is aggravated by metal loss due to external and internal corrosion. That pressure differential was more significant
FIGURE 4.2. Predicted Actual Breaks for Combined Data Set (from Clark et al. (1982))
(higher R) than the mean absolute pressure in a pipe suggests that
pressure differential is a superior measure.

**Age From First Repair**

The time from first event is the most significant variable in
predicting the total number of events for a pipe. A main with
maintenance events early in its life would have a much higher number
of events later. This characteristic is illustrated by the exponen-
tial tendency of the break history in the data base (Figure 4-1).

**Developed Land Percentage**

The percent of developed land is actually a combination of the
four land use percentages (residential, commercial, industrial, and
transportation) found in each census tract. These percentages were
calculated in order to combat the inadequacy of the traffic data.
Inclusion of the developed land percentages in the equation indicates
overhead activity such as live load and drainage influenced the
number of maintenance events for a pipe.

**Surface Area Effects**

Surface area also combines several pieces of data. Size and
length of pipe were combined to yield surface area (the area
actually in contact with the soil). Surface area in low and
moderately corrosive soil were combined into one category. Surface
area was then multiplied by the percentage of its length in low,
moderately and highly corrosive soil.
Although including surface areas for both corrosive categories in Equation (6) may appear contradictory, one must remember that the greater the surface area, regardless of soil type, the more events that should occur. From the table of partial correlations (Table 4-2), one can see that soil corrosivity does influence maintenance events. Surface area in highly corrosive soil has the larger partial correlation between the soil variables.

II.3 A Probabilistic Model for the Prediction of Pipe Failure

Based on a whole school of statistical thinking, a model can be developed using the statistical analysis of failure time data. Many of the major methods within this school are described by Kalbfleisch and Prentice (1980). In particular, the model developed by Cox (1972) is an important contribution and is the cornerstone of the method described in this section.

The Cox regression model is mostly intended for the "low frequency, high outcome" type of pipe, which is usually a large diameter pipe. Its advantages are multiple. First, it is probabilistic in that the probability of failure of a pipe in a given year can be computed.

Furthermore, this probability value is updated, depending on the timely evolution of the performance of the pipe. In other words, the model does not "lock in" the prediction of future failure rate of the pipe, but builds predictions based on the actual realizations of failure rates prior to the periods of interest for prediction purposes.

By comparison to a regression-based model predicting a fractional number of breaks in a large diameter pipe, the evaluation of a probability of failure is much more relevant especially for reliability purposes. Moreover, the Cox regression model can easily incorporate the most recent information about any given pipe. It is a survival analysis
approach and the cornerstone of the model estimation is the hazard function, which is the probability of failure, given that the pipe has "survived" until the concerned time. Unlike the regression-based model which derives future predictions and can only be updated by including new data points in the analyzed data set, the Cox regression model allows to give at any point of time the probability of failure of a pipe, as will be explained in this description of the model.

A. Description of the Model

The model described herein flows directly from the school of thinking related to failure time statistics. Some of the important notions and terminologies are described briefly in the following paragraphs, for the sake of clarity and completeness. The regression method which will be presented is based on the work by Cox (1972). Most of the adopted symbols and terminologies are similar to those used in that reference.

a) Context of the problem

The problem is related to a "population" of pipes, assuming that a pipe is defined as a separate entity. In this first part, related to the application of the Cox regression method to the analysis of pipe failure data, it is assumed that no "overlap effects" or interdependence exist between the behavior of different pipes. It is important to note that a pipe is an "entity" in the sense that it is possible to specify the values of the related major parameters, such as diameter, pressure differential, corrosivity of the surrounding soil, etc. These latter parameters are considered as covariates, in the general terminology of failure time statistics.

Before describing the method as applied to the set of pipes involved in the analysis, some preliminary general definitions
are needed to allow one to set up the general theoretical construct of the Cox regression.

b. Preliminary definitions

Censored data

Within the population of pipes considered, it is supposed that two situations can be observed: the time to "failure" of the pipe or the censoring of the pipe. For the censored pipes, the time to failure is greater than the censoring time.

The censoring mechanism is another major advantage of the Cox regression model over other predictive models for the case of pipe failure. Censoring a pipe at date t, means the loss of information about what happens to it after t. In many systems, insufficient data or missing records would necessitate the use of the censoring concept in order to draw inferences from such imperfect maintenance records.

It is implicitly assumed that "failure" is a terminal state for a given pipe. Therefore, it is important to note how the case of multiple failures is taken into account in the model, as it corresponds to the actual situation where pipes fail more than once throughout their lifetime. Two possible ways can be suggested to handle multiple failures:

1. A generalization of Cox's regression to include direct multiple failures.

2. The consideration of the number of previous failures, as a covariate in the regression. Clearly every single pipe which has experienced more than one failure will generate as many different data "points" or records as there were failures.
This second way is much more convenient because it allows one to use existing Cox regression analysis and diagnosis modules, and was used in the example presented further.

**Survivor and hazard functions**

For a given pipe, denoting by $T$ a random variable representing the failure time, the survivor function $S(t)$ is defined as:

$$ S(t) = \text{Prob}(T \geq t) $$

The hazard or age specific failure rate $(t)$ is defined as:

$$ \lambda(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}(t \leq T \leq t + \Delta t \mid T \geq t)}{\Delta t} $$

If failure times are assumed to be discrete, which often corresponds to the real case in past maintenance data, the hazard rate can be expressed at discrete time points $t$ as:

$$ \lambda(t) = \text{Prob}(T = t \mid T \geq t) $$

**B. The proportional hazards model**

a. **Model formulation**

In the problem of interest, a number of measurements are available say on variables $z_1, z_2, \ldots, z_p$, associated with each single entity or pipe. These variables are called covariates. For example, for the $j$th entity, let the values of the vector of covariates $z$ be $z^j = (z^j_1, \ldots, z^j_p)$. 
The proportional hazards model (Cox, 1972), can be stated as a hazard function conditional on a vector of covariates for a certain entity. The expression of this hazard function is:

$$\lambda(t; z) = \lambda_0(t)e^{\beta z}$$

where $\lambda_0(t)$ is an arbitrary unspecified base-line hazard function for continuous failure time $T$;

and $\beta = (\beta_1, \beta_2, \ldots, \beta_p) = \text{vector of coefficients related to different covariates}$.

b. Estimation of the proportional hazard function

Given a sample of failure time data from the population of entities considered, $\beta$ and $\lambda_0(t)$ can be estimated based on a conditional likelihood related to the occurrence of these failures. The discrete case is assumed herein. A more detailed discussion of both the discrete and continuous cases, as well as further elaboration on the theoretical foundations of failure statistics can be found in Appendix AI.

For a particular discrete observed failure time $t_{(i)}$, the risk set prior to $t_{(i)}$, $R(t_{(i)})$ is defined as the set of entities which have failed or were censored at a time greater or equal to $t_{(i)}$, or have not failed. The number of entities in $R(t_{(i)})$ is denoted by $r_{(i)}$. 
Assuming that a certain number of entities $m_{(i)}$ have failed at $t_{(i)}$, let $s_{(i)}$ be the sum of the covariate vectors $z$ over all these entities. The contribution to the conditional likelihood of these events is:

$$
\exp \{ s_{(i)} \beta \} \prod_{\ell \in R(t_{(i)}, m_{(i)})} \exp \{ s_{(\ell)} \beta \} \tag{7}
$$

The notation in the denominator means that the sum is taken over all distinct sets of $m_{(i)}$ entities drawn from $R(t_{(i)})$.

Assuming $k$ distinct observed failure times, the full conditional likelihood is the product over all these times of expressions of the form (7). The maximum likelihood estimator of $\beta$ is found by solving the equation obtained when the derivative of the log-likelihood is set to zero. The usual technique to evaluate $\beta$ is the Newton-Raphson technique also described in Appendix AI.

Once $\beta$ has been estimated, $\lambda_o(t)$ can be estimated as described in the same Appendix.

As general survival analysis assumes that failure is a terminal state, the case of multiple failures could be done by either considering the previous number of failures as a covariate, i.e., after a failure a pipe is "reborn" with a new identity, or generalize the Cox regression to account for multiple failure. The first method was used in the analysis of the New Haven data set. The latter method is described in Appendix AI.
c. Application of Cox's regression to the pipe failure problem

In the pipe failure problem, the methodology described in part (b) related to the regression method applied to the proportional hazards model can be applied. Thus, the hazard function model is

$$\lambda(t;Z) = \lambda_0(t) e^{\beta Z}$$

where $Z$ = vector of covariates

Such covariates include:

1) The diameter
2) The type of pipe
3) The soil characteristics (corrosivity)
4) The environmental impacts (loading, etc.)
5) The pressure
6) The date of installation of the pipe
7) The number of previous failures

Thus, if: $n$ = total number of pipes in the system (set $\Omega$)

$n_0$ = number of pipes that failed at least once (set $\Omega_0$)

$n - n_0$ = number of pipes that did not fail (set $\tilde{\Omega}_0$ (complementary))

If $j$ is the generic element of $\Omega_0$ (pipe with at least one failure) and $n_j$ is the total number of failures experienced by this pipe throughout its history till the time of the study, then $n_j$ different data points or records are generated by this specific pipe, as the number
of previous failures takes the value 0,1,\ldots,n_j-1.

Thus, the Cox regression will be applied on a sample of data considered of size 
\[ n^1 = \sum_{j=1}^{n_0} n_j + n-n_0. \]

The regression can be conducted in the discrete case or the continuous case. However, it is quite likely, that, given the format of the data base available, the discrete case will be assumed with each time step equal to one year.

Though the consideration of the date of installation of the pipe and the number of previous failures as a "sufficient statistic" is an assumption (with a possible loss of information related to the exact timing of the previous failures), it is a convenient way to evaluate the different alternatives, both at the single pipe level and within the global scheduling model of Chapter 5. In addition to providing the required information for planning purposes, it is very helpful for real-time decision-making, where a decision is required after a break has occurred. The use of the Cox regression model in a real-time context is illustrated in Chapter 7.

\textbf{d. The inclusion of strata}

The proportional hazards model requires that, for any two covariate sets \( z_1 \) and \( z_2 \), the hazard functions are proportional. However, in some cases, other important factors might produce, at different levels, hazard functions which differ markedly from proportionality. Often these factors are too complex to quantify,
or take only a few distinct values over the whole data set, which makes them bad candidates for being selected as covariates.

One way to handle the existence of such factors is to assume that, for the \( j \)th level of the \( q \) levels of this factor, the hazard function for this level (or stratum) is

\[
\lambda_j(t;z) = \lambda_{o_j}(t) \exp(\beta z)
\]

In other words, the basic hazard would be different for different strata thus correcting the problem mentioned above. However the vector of parameters \( \beta \) remains the same for all the strata.

In the case of the New Haven data set, described in Appendix AII, the date of installation of the pipe was considered as a strata instead of a covariate. The reason for this choice is that the date of installation in this case carries more underlying factors with different levels than the simple time effect, such as construction methods, quality of materials used, etc.

e. **Modelling flexibility of Cox's regression**

The proportional hazards model, by not postulating any functional form for the basic hazard \( \lambda_o(t) \), allows one to fit the best exact model to the data at hand. Another major flexibility of the methodology is the inclusion of strata. The idea of categorizing data by strata makes it possible to test whether a variable should be entered in the covariate set. By plotting the logarithm of minus
the logarithm of the survivor function versus time for different strata, it is possible to decide on whether to use the stratum as a covariate. Kalbfleisch and Prentice (1980) mention that if the differences between the curves associated with different strata are fairly constant over time, the stratification scheme can be replaced by adding another covariate. This principle was applied to the New Haven data set (Appendix AII), which was first stratified according to the period of installation of the pipe, then the stratum was replaced by a covariate after the aforementioned test was performed.

III. Evaluation of the Expected Number of Breaks in a Given Year

As an application to the above, the projection of future costs in every given link under different strategies is made possible by using the aforementioned predictive models for future performance (breakage, loss of carrying capacity). In particular, two types of predictive models for failure rate were advocated, based on the break analysis.
1. Regression-based models, predicting the numbers of breaks per pipe in a given year. Shamir and Howard's model (1979) falls in this category, and can be applied to pipes of medium to small diameter which failure cost is simply its repair cost $C_b$. If the expected number of breaks in a link $(i,j)$ in year $t$ is $N_{ij}(t)$, the expected costs are $C_b N_{ij}(t)$.

2. A probabilistic model such as the Cox regression based model described above is particularly suitable for large diameter pipes with a rather "low frequency, high outcome" type of breakage characteristics, including both repair costs and other social costs. The evaluation of the expected costs associated with breaks in year $t$ can be undertaken, using the distributions of probability of failure of the pipe. As the "previous number of failures" is one of the covariates of the probabilistic model, the Cox regression approach provides families of probability distributions which can be used to evaluate as "equivalent number of expected breaks" in a given year $t$.

Let us assume that at the beginning of the time horizon, the analyst is interested in evaluating the breakage costs in a given year $s$ of the planning horizon, in a pipe having experience $(n-1)$ previous breaks. Let us denote by $\psi_n(t_n; z_n)$ the probability density function on the time $t_n$ to the next break. $z_n$ is the vector of covariates and has as components the number of previous failures, the age of the pipe as well as the other physical characteristics of the pipe. Let $e_{s,k}$ be the event $\Xi$ k-th break occurs in year $s$, and $e_{s,k}^c$ the complementary event.
The expected number of breaks in year $s$ is $N(s)$ such that:

$$N(s) = \text{Prob}(e_s^n) + \sum_{r; \ r \leq s} \text{Prob}(e_{n+1}^s / e_n^r) \times \text{Prob}(e_n^r)$$

$$+ \sum_{(q, r); \ q \leq r \leq s} \text{Prob}(e_{n+2}^s / e_{n+1}^q, e_q^r) \times \text{Prob}(e_{n+1}^r / e_n^q) \times \text{Prob}(e_n^q)$$

$$+ \ldots$$

(12)

The computation of $\text{Prob}(e_s^n)$ is carried out using the predictive model for pipe failure. Under the proportional hazard model, the hazard rate can be expressed as

$$\lambda(t) = \lambda^0(t) e^{\beta z}$$

where one of the covariates is the previous number of breaks. The pipe having $(k-1)$ previous breaks has a corresponding hazard rate $\lambda_k(t)$, which can be discretized as:

$$\lambda_k(t) = \sum_u \delta^k(t-u)$$

The evaluation of $\text{Prob}(e_s^n)$ is then obtained as:

$$\text{Prob}(e_s^n) = \text{Prob}(e_s^n, e_{s-1}^n, e_{s-2}^n, \ldots, e_1^n)$$

where $t=0$ corresponds to the present date.

$$\text{Prob}(e_s^n) = \lambda^s(1 - \lambda_{s-1}^n)(1 - \lambda_{s-2}^n) \ldots (1 - \lambda_1^n)$$

also

$$\text{Prob}(e_{n+1}^s / e_n^r) = \lambda^{n+1}_s (1 - \lambda_{s-1}^{n+1}) \ldots (1 - \lambda_r^{n+1})$$

and

$$\text{Prob}(e_{n+2}^s / e_{n+1}^q, e_q^r) = \lambda^{n+2}_s (1 - \lambda_{s-1}^{n+2}) \ldots (1 - \lambda_r^{n+2})$$

The evaluation of the equivalent number of breaks/year $s$ follows therefore from (12).
The expected costs of breakage in a year $s$ can be evaluated as:

$$(C_b + S) N(s) = \bar{C}_b N(s)$$

where $C_b$ = repair cost;

$S$ = social costs associated with a break;

$\bar{C}_b = C_b + S$.

In the next chapters for the sake of simplicity, $C_b$ will represent accordingly $C_b$ or $\bar{C}_b$ and $N(s)$ will represent the equivalent number of expected breaks.

**Reliability constraints for "low frequency, High outcome" event pipes**

The estimation of the social costs $S$ could in many cases be highly subjective, especially as no historical data has usually been recorded on the impact of previous breaks in the pipe. The analysis of such impact would provide a proxy for $S$. This principle is specifically used in the real-time decision-making model of Chapter 7, where a most likely value for $S$ as well as a pessimistic value and an optimistic value are incorporated to evaluate the replacement versus repair trade-offs.

In this latter case, the actual occurrence of a break makes it possible to record its social impacts and come up with a proxy for $S$.

However, faced with the difficulty of estimating the social costs $S$ incurred by a break in a large diameter pipe ($D \geq 6"$), the water utility planner still needs a set of rules to help him make capital improvement decisions related to this category of pipes. As mentioned earlier, these pipes have usually a lower frequency of breakage, but the social impacts of a break are often substantial. Depending on the position of the pipe in the tree network, a break might lead to a shortage that could last a few days as was the case recently in an eastern
city. Besides its position in the network, the flow carried by the pipe is another important factor. For example, a main feeder line carrying a significant part of the output of a water treatment facility is bound to be submitted to stringent reliability constraints.

Such reliability constraints might be expressed in terms of conditions on the hazard rate, setting a threshold beyond which a pipe should be replaced (the hazard rate is an increasing function of time). Denoting the equivalent number of breaks in year \( t \) by \( N(t) \), the pipe should be replaced when \( N(t) \) reaches a value \( N^* \). \( N^* \) can be determined by using the equation translating the adequacy of replacement alternative. Let us denote by: \( -\Delta C_e(t) \) the additional energy costs in year \( t \), due to the usage of the existing deteriorated pipe, expressed in dollars at the beginning of the year \( t \)

\[
\begin{align*}
-C_{REPL} &= \text{replacement cost of the pipe} \\
-N(t) &= \text{equivalent number of breaks for the pipe in year } t \\
-C_b &= \text{repair cost per break event} \\
-S &= \text{social cost of break event} \\
-r &= \text{cost of capital.} \\
-\Delta C_e(t) &= \text{additional operational costs in year } t
\end{align*}
\]

The replacement alternative will be efficient when the \( N(t) \) reaches a value \( N^* \) such that

\[
\frac{(C_b+S)N^* + \Delta C_e(t)}{1 + \frac{r}{2}} \geq \frac{C_{REPL} \times r}{1 + r}
\]  

(13)
In other words, the interest costs in present dollars of the replacement alternative should be smaller than or equal to the expected additional operational costs and maintenance costs in year \( t \). If that was not the case, replacing the pipe in year \( (t+1) \) is more efficient. The failure and operational costs are assumed to be incurred at the middle of the year. \( N(s) \) can be calculated for each year of the planning horizon as in (12).

The optimal replacement time is the smallest \( t \) such that (13) is satisfied. However, in order to find \( N^* \), a proxy for \( S \) should be available. Let us assume that the order to magnitude of \( \Delta C_e(t) \) is by itself important enough to justify a rehabilitation alternative, as evaluated in Chapter 3. The replacement alternative is therefore relevant if:

\[
\frac{(C_b + S)N^*}{(1+\frac{r}{2})} > \frac{(C_{REPL} - C_{REH})r}{1+r}
\]

(14)

where: \( C_{REH} \) = rehabilitation costs of the pipe

**Example:** For a 24" diameter pipe of 1,000 feet, \( C_{REPL} - C_{REH} \) is close to $100,000.

Assuming that \( r = 9\% \) for a municipality

- if the replacement rule is \( N^* = 0.1 \), then:

\[
C_b + S \approx 76,000
\]

i.e., \( S \) is of the order of $74,000 as \( C_b \approx 2,000 \).

This value can be considered, a priori, as a reasonable estimate for the social costs incurred from a break in a critical pipe. Obviously, for some main feeder lines, the order of magnitude of \( S \)
could be much higher than above or equivalently the value of $N^*$ allowed is necessarily lower than 0.1.

In general, different sets of rules can be defined for each size/capacity pool of pipes. Similar rules as those displayed in Table 4.4 for the 24" diameter pipes can be derived depending on the values of S, REPL and REH, and equations (13) and (14) can be used as needed. Capital improvement decisions can therefore be made for the low frequency, high outcome category of pipes either by using such tables and choosing a value for $N^*$, in cases where a direct estimation of S is not possible. However, once the category in which the pipe falls has been determined, an approximate value of S is implicitly assumed and the analysis undertaken in the previous paragraph is possible. In

<table>
<thead>
<tr>
<th>CATEGORY OF PIPES</th>
<th>ORDER OF MAGNITUDE OF S</th>
<th>$N^*$ (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;CRITICAL&quot;</td>
<td>50,000-100,000</td>
<td>0.16-0.08</td>
</tr>
<tr>
<td>MEDIUM IMPACT PIPES</td>
<td>10,000-50,000</td>
<td>0.7-0.16</td>
</tr>
<tr>
<td>LOW IMPACT PIPES</td>
<td>&lt;10,000</td>
<td>&gt;0.7</td>
</tr>
</tbody>
</table>

TABLE 4.4: Categorization of Pipes According to Reliability Standards

other words, as the value of $N^*$ is imposed (in a chance constrained-like manner) for different categories of pipes, then the implicit value of S can be evaluated from (13) or (14) and used directly to evaluate
\( \bar{C}_b = C_b + S \), the opportunity cost of a repair event. As suggested in the previous paragraph \( \bar{C}_b \) can be substituted for \( C_b \) for the purposes of the scheduling model of Chapter 5.

However, a proxy for \( S \) is often quite possible by an impact analysis. Assuming that the major significant impact is the "service impact", one can attempt to evaluate the economic value of this drop in serviceability. For example, if an interruption in service of the main leads to a decreased service to consumers in the vicinity without extreme consequences such as shortage, a possible proxy for \( S \) is the value of the water that was not provided to the consumers during the interruption period. The amount of water which was not provided can be estimated, given the average flow in the pipe. A fraction \( f \) of this water can then be considered as a lost value assuming that consumers increase their consumption after the total serviceability level has been restored. The observation of the pattern of consumption before and after the drop in serviceability due to the interruption of flow in the pipe can provide the necessary information for the estimation of \( f \). However, when shortages are incurred by the break, the social costs are significantly higher as the drop in consumer utility is quite important. This problem is particularly acute if supply is totally cut for a substantial amount of time, which might be the case when a major break occurs in a feeder main.
CHAPTER 5: A Planning Model for the Scheduling of Maintenance and Expansion of Mature Water Distribution Systems

I. A Decision Support System Perspective of Distribution System Planning

In Chapters 1 and 2, the notion of an integrated maintenance management system was presented, along with its interfaces with the expansion/design and operation/control of the water distribution system. It was emphasized that the study of past performance and current status of a system provides significant insights to the capacity planning side as well as the operational modes and practices. The understanding of all these dimensions of water distribution system planning allows the enhancement of design/capacity planning and operational policies. Once the major parameters of these policies have been set to their optimal levels, the problems of assessing maintenance, capital improvement and expansion needs, as well as scheduling these measures under the budgetary constraints still remain to be addressed in an integrated manner, as explained in Chapters 1 and 2.

The major purpose of this chapter is to develop a decision support system for the scheduling of the aforementioned spectrum of measures in a water distribution network. This decision support system is related to the lower level boxes of Figure 2.2, concerned with the derivation of a scheduling plan. In light of the outputs of the economic analysis of capital improvement measures on each section, planning the maintenance and expansion measures becomes a modelling problem integrating this data with planning parameters related to future demands, financial constraints and reliability issues.
It is important to note that the scheduling model should be first used apart from any financial constraints, that is, assuming that such periodic budgets have not been determined. It is specifically to suggest some figures for these budgets that the model should be first used in this manner. Given the demand-responsive approach to maintenance decision-making, mentioned in Chapter 1, the assessment of the macro-needs for the water distribution system for different time periods of the planning horizon is a first necessary step. It helps concretize the requirements for "optimal" decision-making at the network level. Clearly, given the different other concurrent uses of capital, the decided-upon budgets would possibly deviate from the requirements in the ideal case. The outputs of the scheduling model under the new budgetary constraints would be characterized by a deferral of some capital improvement actions as the budgets are expected to be set below the optimal requirement level. The procedure suggested in this section is diagrammed in Figure 5.1. The scheduling model is used for a double purpose as a benchmark for defining the budgets, and based on such financial constraints, as a generator of the required measures in different sections of the network.

Before getting to the details of the scheduling model, a distinction of two elements of decision-making is worth mentioning at this point. Based on the predictive models for system performance represented in terms of economic costs (repair costs, additional energy costs), one major component of the decision support system is the scheduling planning model. However, major breaks in feeder mains, whose
FIGURE 5.1: Iterative Use of the Scheduling Model
impact is not accurately accounted for by simple repair costs, require another "real-time" type of decision-making. The major input to this real-time analysis are a predictive model for major breaks and an assessment of the impacts of a break in the specific main including economic and social costs. Such a probabilistic predictive model for major breaks has been described in Chapter 4. The real-time or crisis decision analysis approach is described in Chapter 7. The two "branches" of decision-making are diagrammed in Figure 5.2. On the "planning" side, the long-term plans generated by the scheduling model are then submitted to a sequencing model that can allocate the different projects on a yearly basis. This latter short-term sequencing model is presented in Chapter 6.

II. Mature Water Distribution System Planning: The Incremental Approach

The scheduling model suggested in this chapter focuses on mature water distribution system maintenance and expansion planning. In the design of a water distribution system, the decision variables are the lengths of the section of pipes of different diameters that are needed, while meeting demand and pressure range constraints. The Optimal Design of water distribution systems was studied by Rasmusen (1976) and Alperovits and Shamir (1977). A real water distribution network is represented in Figure 5.3.

For mature water distribution system planning, the incremental approach is assumed, in the sense that the structural and operational "integrity" of the system are preserved. In other words, sections of
FIGURE 5.2: The Two sides of the Maintenance/Expansion Decision Support System
FIGURE 5.3. SCHEMATIC DIAGRAM OF A REAL NETWORK
(from Alperovitz and Shamir (1977))
pipes of given length and diameters are the links that are candidates for replacement, rehabilitation or expansion. It follows that capital cost functions, usually expressed in terms of the diameter \( D \) of the pipe can be expressed in terms of the capacity \( Q \) of the pipe. For a given pipe of length \( L \), the head loss from one extreme to the other is

\[
\Delta H = J \times L
\]

where \( J \), the hydraulic gradient is given by the Hazen-Williams equation:

\[
J = \alpha \left( \frac{Q}{C} \right)^{1.852} D^{-4.87}
\]

As \( \Delta H \) and \( L \) are part of the system operational mode and structure, \( Q \) and \( D \) are directly related for each section of pipe, which makes it possible to derive the capital cost as a function of the capacity of the pipe. This assumption was made in the scheduling models presented in the next paragraph.

III. The Scheduling Models for Mature Water Distribution Systems

In this section, the scheduling model for future maintenance and expansion measures in a water distribution network is presented. This model is the core of the planning process for dynamic maintenance and expansion decision-making. The design of the multi-period planning process must also respond to a number of required criteria which are delineated in a further section, in order to make the derived strategies relevant in a real context.
A. The Multi-Period Approach

1. The Choice of the Time Horizon and Its Temporal Disaggregation

In order to capture significant economies of scale and make the long-term planning approach worthwhile, time horizons of about twenty to twenty-five years are required.

The optimal timing or scheduling of facilities in order to meet the demands depends also on the disaggregation of the time horizon. From the practical viewpoint, a multi-period approach will lead to the implementation within each given time period, of the maintenance and expansion measures to meet at least the requirements at the end of the same time period; the lag needed for the construction of such facilities justifies overdesigning systems in each time period. As economies of scale are taken into account in the multi-period program, more small time periods rather than fewer long time periods are recommended. A five-year period is recommended, as it is reasonable to assume that only one replacement or expansion measure can be implemented on a given link in a five-year period, because of the disruptions in other services (traffic, etc.) that accompany such actions.

2. The Planning Process

The viability of the multi-period approach depends also on the integration of the uncertainties in the main planning parameters. In particular, the uncertainties in future demands are the major significant ones (Grossman, 1977). Therefore, a single planning exercise undertaken at the beginning of the time horizon that would "statically" derive the scheduling of capacities with no possible further correction of these plans, would be overly conservative because of the high levels
of uncertainties involved. It is therefore quite rational to derive a dynamic planning process that would integrate at the beginning of each time period, the new level of information. Furthermore the existence of some budgetary constraints for a given time period necessitates such an adaptive scheme.

Hence, the development of a planning process that integrates uncertainties in future demands and possible financial constraints on a sound dynamic basis, is of highest priority. Such a planning process should be guided by a well defined set of criteria.

**Reliability:** The scheduling of capacities in the different components of the system over the life of the project should lead to a reliable system in the sense of meeting the requirements in each time period with quasi-certainty.

**Flexibility:** The process should be flexible, allowing for efficient correction of possible overspending in the early phases of implementation of the system. That is, the process should minimize the magnitude of over-spending; the latter being the "price" of the reliability constraint.

**Completeness:** The models used in the process should be capable of integrating different possible constraints and the interactions of the water supply function with other requirements.

**Adaptability:** The adaptability should not generate a single strategy imposed on the planning authority but should rather equip the planners with the complete set of information leading to the formation of the decision, given their risk attitude.
Based on these guidelines, a planning process was designed and is diagrammed in Figure 5.4. At the beginning of a time period, a planning cycle is undertaken, under the current level of information. Thus, this planning model generates a replacement/expansion plan at each time period. The actual implementation of such a plan within the limits of the period is an "operational" problem that is addressed in Chapter 6, where the optimal sequencing of the period projects is derived.

a. **Description of a planning cycle.** Starting from the upper left of the flow-chart of Figure 5.4, the first box represents the model for deriving a scheduling plan under given values of the peak demand parameters. This deterministic module is the scheduling model presented further in this chapter.

The solution provided by this first module under the expected values of peak demands, is a first expansion strategy and can be used as an "anchor" to planning under uncertainty. This first plan has yet to be modified in order to account for the required level of reliability. The safety program represented in module II of the flowchart allows one to generate optimal expansion strategies under different reliability constraints for the first time period of the cycle. For each of these strategies, two indices, which are discussed in a further section, allow the planner to measure the variation of the trade-off between cost and effectiveness. According to his risk attitude, the planner will decide on the level of reliability required, and therefore the associated expansion strategy. The plan related to the first period of the planning cycle is then implemented.
THE PLANNING PROCESS
(A Dynamic Review)

(Module I)

INPUT PROBLEMS IN TERMS OF EXPECTED VALUES OF PEAK DEMANDS

Step One

DETERMINISTIC SCHEDULING

SOLUTION: EXPANSION STRATEGY I

(Module II)

DATA COLLECTION AND MODEL ESTIMATION

VALUE OF INFORMATION

SAFETY PROGRAM

(Module III)

REALIZATION OF SCENARIO UNDERSTANDING/EXPERIENCE

IMPLEMENTATION OF PLANS IN FIRST TIME PERIOD OF THE CYCLE

UPDATING OF FUTURE DEMANDS/FUTURE PERFORMANCE OF SYSTEM

PAST OVERDESIGNING IN FIRST TIME PERIOD?

NO

KEEP RISK ATTITUDE

GO THROUGH A NEW PLANNING SEQUENCE

NEW EXPANSION STRATEGY

YES

REVIEW RISK ATTITUDE

FIGURE 5-4
b. **Transition between two successive cycles.** After the realization of the demands in a given period, new information about the trend of demands is added. Also, the realization of the performance of the system in terms of operating costs and failure rates allows one to update the predictive models for future performance of the system, and to integrate this new information in the next planning cycle. Depending on whether overbuilding did actually occur, the decision-maker might alter his risk attitude for the next planning cycle, because of more confidence in the estimation of the parameters. Under the new set of demands a new predictive model for peak demands in the next periods is derived and can be used to obtain a new expansion strategy through another planning cycle limited to the remaining periods in the time horizon.

In summary, in each planning cycle, the planner is assumed to take a conservative stance for the first time period. However the uncertainties are accounted for only when they are accurately evaluated (by a good estimation of the variances of demands). It seems quite reasonable that rational planning under uncertainty has to start from conservative stances and eventually modify this risk attitude after the implementation of the plan in the first period of the cycle. Implementation leads to a better understanding of the performance of the predictive demand model and provides inputs to the reappraisal of the strategy, and its modification. This is basically the essence of "revolving" planning, continuously monitoring, updating, and correcting strategies. Hence the adaptability and flexibility criteria are directly integrated in the process.
B. The Scheduling Models under Deterministic Peak Demands

The basic assumption of this model is that the planning authority is only concerned about meeting at lowest net present cost certain values of demands at any given time of a planning horizon of the order of twenty years.

Thus, the main objective of the model is to minimize the present value of the costs of the expansion and maintenance of the urban water distribution network.

As a result of the economic analysis undertaken for each section of the watermain in the network (Chapter 3), a candidate set of pipes was identified for early replacement or rehabilitation. However, as this analysis dealt only with the replacement of a pipe with another of equal capacity, the whole issue of expansion still remained possible. For example, if a pipe is suggested for replacement on the basis of a separate analysis, the replacement alternative will remain desirable in the context of the global network. The reason why such a statement is true is that the network optimization will enhance the attractiveness of the replacement alternative, by allowing an expansion of the pipe at a decreasing marginal cost.

Yet the incorporation of the expansion alternative with the other capital improvement measures can be undertaken at two different levels.

1. The network level

A scheduling model for the selection and timing of alternatives in the network can be built as a mixed-integer programming model. Such
a model has a network sub-structure including flow variables and continuity constraints. In this case, the objective function includes capital costs for new pipes, whose capacities are decision variables, in addition to repair costs incurred in the absence of a replacement measure, and rehabilitation costs associated with pipes for which rehabilitation is relevant according to the single pipe analysis. However, due to the continuity constraints and the need for flow variables as well as capacity variables, the size of the optimization problem becomes too large to perform the analysis on real networks. In large systems with 1000-2000 arcs, the number of integer variables makes it impossible to solve the problem with existing mixed integer programming software, without decomposing the network into smaller parts. Major decompositions can be required to make the problem solvable, and the issue of budgetary constraints for each part of the network complicates the problem.

One way to decrease the size of the problem is to generate separately a feasible flow configuration in the network over time and perform the optimization based on that set of flows. The optimization can be reiterated using different flow vectors and retaining only the best solution. Such a scheme would certainly decrease the size of the problem and would be quite implementable in a real-world context, where the flow mappings are available in the network based on demand estimates in the vicinity of the pipes. This allows recognition of where and when in the network an expansion is required. However, even though the size of the problem is reduced, it is still too large a mixed-integer programming model to solve without major decompositions, as two-to-three integer
variables are still needed per arc and per time period. The simplification of the problem comes from the fact that once the network substructure has been eliminated and the expansion alternative is eventually included in the single pipe analysis, the scheduling model can then be built as a capital budgeting model. The advantages of this approach are examined in more detail in the following discussion.

2. The single pipe level

As explained above, the incorporation of the expansion alternative at the pipe level allows for creation of a total separability between pipe projects but for the budgetary constraints. The scheduling model becomes a capital budgeting model. The major advantage of this latter approach is that once the network substructure has been eliminated, different pipes can be pooled into "bundles". The criteria for defining such bundles are partly heuristic and are presented in the section related to the capital budgeting model.

In the next paragraph, the network scheduling model is derived. First, a simplified model is presented where the rehabilitation alternative is excluded. A generalized model including the rehabilitation alternative follows, thus concluding the global network scheduling model. Further, two capital budgeting models, a linear programming and an integer programming model are examined, and the conditions under which each prevails are analyzed.

I. The Network Model

The candidate set for early replacement can be used in generating the expansion and replacement schedules for the network or section of
system considered. In order to derive "optimal" replacement schedules, 
a planning horizon of ten to twenty years disaggregated into time periods 
of about five years, is considered. The objective function in the 
present sum of expansion costs and repair costs on each arc of the 
district. The problem can thus be modelled as:

**1st Formulation (General)**

\[
\text{Min} \sum_{(i,j) \in \Omega} \sum_{t=1}^{p} \frac{C_{ij}(x_{ij}^t)}{(1+r)(t-1)T} - S_{ij}(t)y_{ij}^t
\]

where \(x_{ij}^t\) = capacity increment on arc \((i,j)\) in time period \(t\) 
\(C_{ij}(x_{ij}^t)\) = capital cost function for new pipeline

\(N_{ij}(k)\) = expected number of leakages in year \(k\)

\(y_{ij}^t\) = binary variable \(= 1\) if pipeline is cancelled in time period \(t\) (put out of service) \(= 0\) otherwise

\(C_b\) = repair cost per leakage

\(r\) = real discount rate

\(T\) = number of years per time period

\((i,j)\) = arc index

\(\Omega\) = set of arcs in the district

\(p\) = number of time periods in the planning horizon

\[
R_{ij}(t) = \sum_{s=1}^{(t-1)T} \frac{C_{bN_{ij}(s)}}{(1+r)^s} + \frac{1}{2} \sum_{s=(t-1)T+1}^{tT} \frac{C_{bN_{ij}(s)}}{(1+r)^s}
\]

\[
R_{ij} = \sum_{s=1}^{pT} \frac{C_{bN_{ij}(s)}}{(1+r)^s}
\]

\(R_{ij}\) = net present value of total repair costs over the whole planning horizon, if no replacement measure is undertaken.

\(S_{ij}(t)\) = savings in repair costs (expressed in present dollars) if link \((i,j)\) is replaced in time period \(t\).
**Assumptions:**

1) Time periods of about 5 years are considered and the capacities \( x_{ij}^t \) to implement through the time period \( t \) are supposed to meet the demands at the end of that time period.

2) If a replacement occurs on arc \((i,j)\) in time period \( t \), it is assumed that such replacement will actually be implemented on average in the middle of the time period. This assumption is a good approximation to the real situation.

3) \( C_{ij} \) includes the present value of future replacement costs.

4) While \( R_{ij}(t) \) is the total present cost of repairs until time period \( t \), \( S_{ij}(t) \) represents the savings in current dollars due to a replacement in time period \( t \).

5) The total net present cost of the expansion and replacement strategy is equal to the sum of the value of the objective function and \( \sum_{(i,j) \in \Omega} R_{ij} \).

6) \( C_{ij}(x_{ij}) \) is a concave function of the capacity \( x_{ij} \). A form such as \( \alpha x_{ij}^{0.5} \) is usually accepted as the capital costs for pipelines.

7) The cancellation of the pipe might represent an interruption of service or a replacement.

However, in order to make the problem solvable by a normal optimization method, an approximation of \( C_{ij}(x_{ij}) \) of the form \( a_{ij} z_{ij} + b_{ij} x_{ij} \) (fixed charge-linear form) is considered. Generally, \( z_{ij} \) and \( y_{ij} \) are not related. Thus if \( y_{ij}^t = 1 \) and \( z_{ij}^t = 0 \), the arc \((i,j)\) would be simply "canceled" as of the
time period \( t \), as no repair costs are spent beyond \( t \),
while no replacement or expansion has taken place. \( z_{ij}^t \)
is a binary variable that takes the value 1 only if \( x_{ij}^t > 0 \).

2nd Formulation (Approximate)

Under this approximation, the objective function can be written
as:

\[
\sum_{(i,j) \in \Omega} \left\{ \frac{p}{\sum_{t=1}^p} \frac{a_{ij}^t z_{ij}^t + b_{ij}^t x_{ij}^t}{(t-\delta)^T} - S_{ij}(t) y_{ij}^t \right\}
\]

where \( z_{ij}^t \) = binary variable related to time period \( t \).

This minimization problem is subject to the following constraints.

1) **Continuity constraint at each node**

Denoting the flow in arc \( (i,j) \) in time period \( t \) as \( f_{ij}^t \), the demand or supply at node \( k \) in time period \( t \) is \( d_k^t \) (\( d_k^t > 0 \) if demand, \( d_k^t < 0 \) if supply). The continuity constraint at node \( k \) can be expressed
as:

\[
\sum_i f_{ik}^t - \sum_j f_{kj}^t = d_k^t \quad \forall k, \forall t
\]

2) **Upper bound constraint**

This constraint limits the flow allowable in a given link

\[
f_{ij}^t \leq u_{ij} \quad \forall (i,j) \in \Omega
\]

3) **A technological constraint** limits the allowable capacity increment on arc \( (i,j) \) as the pipe diameters available are limited in size.

\[
x_{ij}^t \leq u'_{ij} \quad \forall (i,j) \in \Omega
\]

4) **Feasibility constraint**

The flow in arc \( (i,j) \) in time period \( t \) cannot exceed the total
capacity of the pipe.

\[ f_{ij}^t \leq \sum_{t'=1}^{t} x_{ij}^t + e_{ij} (1 - \sum_{t'=1}^{t} y_{ij}^t) \]

where \( e_{ij} \) = existing capacity on arc \((i,j)\) at the beginning of the planning horizon.

5) **Exclusivity of replacements or cancellations**

This constraint can be expressed in two different ways:

a) **For the candidate set of pipes** \((i,j)\) that should be cancelled or replaced according to the economic analysis performed on each single pipe, the constraint is:

\[ \sum_{t=1}^{p} y_{ij}^t = 1 \] (i.e., imposes that old capacity \( e_{ij} \) should be cancelled at some point in time during the time horizon)

b) **For all other pipes:**

\[ \sum_{t'=1}^{p} y_{ij}^t \leq 1 \] (the initial capacity could possibly be cancelled)

6) **Consistency constraint**

This constraint imposes that \( z_{ij}^t = 1 \) if \( x_{ij}^t > 0 \)

\[ x_{ij}^t < u_{ij} z_{ij}^t \]

The necessary expansion of some arcs could be undertaken at least cost in one single increment, due to the important economies of scale. Such a case can be encountered when the need for expansion is mostly concentrated in one time period, ordinarily the first of the planning horizon. The approximation of the cost function by the fixed-charge linear function might not be sufficiently representative.
of the level of scale economies. A constraint can then be envisaged, in order to increase the model adequacy, by requiring that a single increment be considered on the concerned arc. Such a constraint can be expressed as:

\[ \sum_{t' = 1}^{p} z_{ij}^{t'} \leq 1 \]

In particular, if in addition to the above conditions, the existing capacity on the arc is inadequate in the first time period, then only \( z_{ij}^1 \) need be included in the model.

7) Resource constraint

The total expenditures in time period \( t \) (a time period = \( T \) years) is the sum of capital costs and repair costs, and cannot exceed the budget figure \( b_t \) for the period

\[ \sum_{(i, j) \in \Omega} \left[ \frac{a_{ij} z_{ij}^t + b_{ij} x_{ij}^t}{(1+r)^{T/2}} \right] + \left[ \sum_{m=1}^{T} \frac{N_{ij} [(t-1)T+m] C_b}{(1+r)^m} \right] \left[ 1 - \sum_{t'=1}^{t-1} y_{ij}^{t'} - \frac{1}{2} y_{ij}^t \right] \leq b_t \]

This resource constraint is consistent with the fact that the scheduling model is a planning model that generates the plans for each time period. The actual yearly or semi-yearly scheduling of the implementation of these plans is another problem that the water utility managers have to tackle, in order to come up with the yearly budgets. It is therefore assumed in the above model that the term \( b_t \) (5-year budget) is an estimate (expressed in "real" dollars of the first year of time period \( t \)) of the maximum amount of funds that could be provided during that period. The yearly scheduling problem would probably require a sequencing of the period plan in order to spread the expenditures as evenly as possible over the five-year period.
It is useful to note, that if the output of this first minimization is an expansion strategy \( (x_{ij}^*)^t, y_{ij}^t \), then the total capacity \( X_{ij}^t \) available on arc \((i,j)\) in time period \(t\) is:

\[
X_{ij}^t = \sum_{t'=1}^{t} (x_{ij}^{t'})^* + e_{ij}(1 - \sum_{t'=1}^{t} y_{ij}^{t'})
\]

As could be seen in the structure of the model, the variables \(y_{ij}^t\) and \(z_{ij}^t\) were not related to each other by any constraint, in order to preserve the generality of the model. This means that the model could possibly yield a solution where in a given time period \(y_{ij}^t = 1\) and \(z_{ij}^t\) as well as \(x_{ij}^t\) are equal to zero. This would represent the "cancellation" of the pipe, with no simultaneous replacement or expansion. Such a case could occur due to a more favorable expansion of another arc (due to economies of scale) which led to the simple "cancellation" of the arc mentioned in this example.

The solution of the scheduling problem can be obtained by using one of the mixed-integer programming software packages. In particular, the branch-and-bound based IBM package MPSX is considered as one of the most powerful mixed-integer programs, and can handle such size problems as those derived from networks with 20 nodes and 40 to 50 arcs. A new version of the program may be released in the near future, where a network algorithm is solved at each step of the branch-and-bound (instead of a simple Linear Programming routine). The existence of such a program makes it possible to get to a near-optimal solution in much less CPU time, assuming that the structure of the scheduling problem is reduced to that of a standard network optimization problem.
2. The Generalized Model

The previous model can be extended to include the rehabilitation alternative when the latter is relevant. Some additional notations are required to explain the following formulation of the generalized model:

\( \Omega = \) set of pipes in the network

\( \Omega_1 = \) set of pipes which are candidates for either replacement or rehabilitation during the planning horizon, as a result of the single pipe analysis (Chapter 3)

\( \Omega_2 = \) set of pipes which are candidates for rehabilitation as a result of the single pipe analysis. \( \Omega_2 \subset \Omega_1 \)

The generalized model can then be formulated as:

\[
\begin{align*}
\text{Min} & \quad \left\{ \sum_{(i,j) \in \Omega} \left[ \sum_{t=1}^{P} \frac{C_{ij}(x_{ij}^t)}{(1+r)^{(t-\frac{1}{2})T}} - S_{ij}(t)y_{ij}^t \right] \\
& + \sum_{(i,j) \in \Omega_2} \left[ \frac{REH_{ij}}{(1+r)^{(t-\frac{1}{2})T}} \cdot w_{ij}^t - s_{ij}(t)w_{ij}^t \right] \right\}
\end{align*}
\]

where:

\( x_{ij}^t \) = capacity increment on arc \((i,j)\) in time period \(t\)

\( C_{ij}(x_{ij}^t) \) = capital cost function for new pipeline in arc \((i,j)\)

\( y_{ij}^t \) = binary variable

\( y_{ij}^t = 1 \) if pipeline is cancelled in time period \(t\)

\( y_{ij}^t = 0 \) otherwise

\( REH_{ij} \) = rehabilitation cost of arc \((i,j)\)

\( w_{ij}^t \) = binary variable

\( w_{ij}^t = 1 \) if pipeline is rehabilitated in time period \(t\)

\( w_{ij}^t = 0 \) otherwise

\( S_{ij}(t) \) = savings in present repair and additional operational costs due to a cancellation of arc \((i,j)\) in period \(t\)
\[ s_{ij}(t) = \text{savings in present operational costs due to a rehabilitation of arc } (i,j) \text{ in period } t \]

\[ O_{ij}(s) = \text{additional operational costs incurred in year } s \text{ if pipe is not rehabilitated, cancelled or replaced} \]

\[ \text{REP}_{ij}(s) = \text{repair costs incurred in year } s \text{ if the pipe is not cancelled or replaced} \]

\[ N_{ij}(s) = \text{expected number of repair events in year } s \text{ in pipe } (i,j) \]

\[ C_b = \text{repair cost per event} \]

\text{Relationships and related notations}

1) Given the notations above, it follows directly that:

\[ \text{REP}_{ij}(s) = C_b N_{ij}(s) \]

2) \[
R_{ij}(t) = \sum_{s=1}^{(t-1)T} \frac{O_{ij}(s) + \text{REP}_{ij}(s)}{(1+r)^s} + \frac{1}{2} \sum_{s=(t-1)T+1}^{tT} \frac{O_{ij}(s) + \text{REP}_{ij}(s)}{(1+r)^s}
\]

3) \[
r_{ij}(t) = \sum_{s=1}^{(t-1)T} \frac{O_{ij}(s)}{(1+r)^s} + \frac{1}{2} \sum_{s=(t-1)T+1}^{tT} \frac{O_{ij}(s)}{(1+r)^s}
\]

4) \[
R_{ij} = \sum_{s=1}^{pT} \frac{O_{ij}(s) + \text{REP}_{ij}(s)}{(1+r)^s}; \quad r_{ij} = \sum_{s=1}^{pT} \frac{O_{ij}(s)}{(1+r)^s}
\]

5) \[
S_{ij}(t) = R_{ij} - R_{ij}(t)
\]

6) \[
s_{ij}(t) = r_{ij} - r_{ij}(t)
\]

\text{Assumptions}

1) As in the simplified model it is assumed that any capital improvement project (rehabilitation, replacement or expansion) associated with time period \( t \), is on average implemented at the middle of the time period.

2) \( R_{ij}(t) \) illustrates the same notion as in the simplified model, with the additional introduction of the operational costs.
3) \( r_{ij}(t) \) represents the total net present cost of additional operational costs until time period \( t \).

4) The total net present cost of the expansion, replacement and rehabilitation strategy is equal to the sum of the value of the objective function (at the optimal solution) and \( \sum_{(i,j) \in \Omega} R_{ij} \)

Constraints

1) The continuity, upper bound, technological and feasibility constraints are unchanged.

2) Exclusivity of cancellations and rehabilitations

a) For the set of pipes \( \Omega_2 \) (recommended for rehabilitation by the single pipe analysis):

\[
\sum_{t'=1}^{P} y_{ij}^t + \sum_{t'=1}^{P} w_{ij}^t = 1
\]

This constraint imposes that pipe \((i,j)\) is cancelled or rehabilitated once and only once in the time horizon.

b) For the pipes \( \Theta \Omega_1-\Omega_2 \) (recommended for replacement by the single pipe analysis):

\[
\sum_{t'=1}^{P} y_{ij}^t = 1 \text{ (i.e., imposes that pipe \((i,j)\) is cancelled at some point during the time horizon)}
\]

c) For all other pipes \( \Theta \Omega_1-\Omega_2 \) the exclusivity constraint is:

\[
\sum_{t'=1}^{P} y_{ij}^t \leq 1 \text{ (cancellation possibly once)}
\]

3) The consistency constraint remains the same, under the same representation for the capital cost functions.
4) The resource constraint can be expressed as follows

\[
\sum_{(i,j) \in \Omega} \left[ \frac{C_{ij}(x_{ij})}{(1+r)^T/2} + \left( \sum_{m=1}^{T} \frac{R_{ij}^T[((t-1)T+m)]}{(1+r)^m} \right) \left( 1 - \sum_{t'=1}^{t-1} y_{ij} - \frac{1}{2} y_{ij} ight) ight] \\
+ \frac{REH_{ij}}{(1+r)^T/2} w_{ij}^t + \left( \sum_{m=1}^{T} \frac{O_{ij}^T[((t-1)T+m)]}{(1+r)^m} \right) \left( 1 - \sum_{t'=1}^{t-1} (y_{ij}^t + w_{ij}'^t) ight) \
- \frac{1}{2}(y_{ij}^t + w_{ij}^t) \right] \leq b_t
\]

where \( b_t \) = budget for time period \( t \) expressed in real dollars of the beginning of the period.

**Analysis of the resource constraint**

The resource constraints presented in different models are based on the assumption that the total maintenance and capital improvement budgets for different time periods (of five years or one year each, depending on the model) were only available for the related period. In the general case, period budgets are insufficient for the required capital improvement measures, which are recommended by the single pipe analysis. Therefore, the resource constraint will usually be binding as the capital budgeting models serve then to minimize the inefficiencies from the deferral of required capital improvement measures. However, some special situations are worth mentioning, along with the way the scheduling model should be applied. All budget figures presented in the different models are in real terms, i.e., are in real dollars of the associated period (inflation adjustment).
a. Insufficient budgets

As budget figures are bound to be lower than the required amount in the associated time periods, the application of the scheduling models under the previously stated resource constraint will not yield any possible solution. One way to solve the capital budgeting problem is to solve the problem while relaxing the resource constraint related to the last period. The result of this modification to the model is a deferral of some of the required measures to the last period. The "deficit" in the last period can then be estimated and used as a benchmark to update the budgets for the whole planning horizon, along with the value of the objective function, which is a measure of the inefficiency of the solution.

b. Uneven requirements

Under the assumption that period budgets grow at a certain rate, the resulting solution might be subject to some "distortions". If the budgeted amounts can "globally" meet the requirements of the whole planning horizon, some capital improvement measures might be suggested for implementation prior to their optimal implementation time. Obviously, this situation is expected to be quite uncommon as most of the measures will have to be deferred rather than implemented earlier than optimally required. However, under uneven requirements some budget figures might be higher than the requirements for the associated period, thus leading to early implementations. The reason why early implementations are a "distortion" is that, contrary to deferrals imposed by budget limits, they can be avoided by allowing for a transfer of funds from a period to the next one, thus saving the
inefficiency due to an early implementation. A way to avoid the possibility of early implementation is to transfer to the next period the costs of early implementation, duly grown to account for the time value of money. As the implementation was "early", such a transfer will generate savings.

The solution might suffer from other distortions when some of the budgets are not saturated, while measures related to further periods are still deferred. Clearly the "unspent" budget is not realistic and in this case inefficient as available resources are not channeled adequately.

The design of a transfer mechanism to handle the problem of distorted solutions can be achieved as follows:

If the expression of the total costs associated with period \( t \) is denoted by \( TC_t \) and the period budget by \( b_t \), then the resource constraint for the first period is:

\[ TC_1 \leq b_1 \]

As at the optimum of the original problem, \( b_1 \) might be strictly superior to \( TC_1 \), the available funds for the next period will be:

\[ (b_1 - TC_1)(1+r)^5 + b_2 \]

where:
- \( T \): length of time period = 5 years
- \( b_2 \): second period budget
- \( r \): real annual discount rate

The resource constraint for the second period is then written as:

\[ TC_2 \leq b_2 + (b_1 - TC_1)(1+r)^5 \]
or
\[ TC_2 + TC_1 (1+r)^5 \leq b_2 + b_1 (1+r)^5 \]

It is easy to see that for period \( t \), the resource constraint would be formulated as:
\[ TC_t + TC_{t-1} (1+r)^5 + \ldots + TC_0 (1+r)^5(t-1) \leq b_t + b_{t-1} (1+r)^5 + \ldots + b_1 (1+r)^5(t-1) \]

The capital budgeting problem can then be solved under the latter "modified" resource constraints, thus solving the problem of uneven requirements.

A more direct way to formulate the unsaturated budget problem is to introduce a slack variable in the budget constraint, which becomes an equality constraint. The budget constraint in the first period can then be written as:
\[ TC_1 + z_1 = b_1 \]
where \( z_1 \) is the slack or surplus fund available in the first period. This surplus can be reinvested and will grow, in real terms, to the value \( z_1 (1+r)^5 \) in the next period, \( r \) being the real annual discount rate (a five-year time period is assumed).

The budget constraint for the second period is the
\[ TC_2 + z_2 = b_2 + z_1 (1+r)^5 \text{ or } TC_2 + z_2 - z_1 (1+r)^5 = b_2 \]
where \( z_2 \) is the surplus in the last period. It can be easily seen that, for time period \( t \) (other than the last time period), the budget constraint can be stated as:
The constraint related to the last time period $p$ is an inequality constraint

$$TC_t + z_t - z_{t-1}(1+r)^5 = b_t$$

The solution of the unsaturated budget problem can therefore be handled by including slack variables in the original model. As a result of this new structure, the integer programming model described next becomes a mixed-integer programming model due to the presence of the slack variables. The introduction of the same slack variables in the budget constraints of the integer programming capital budgeting model discussed further, also changes the latter into a mixed-integer programming model.

Alternatively, when the problem of unsaturated budgets arises without accompanying early implementation, it is possible to correct the distortion iteratively, by "transferring" the unspent portion of the earliest unspent budget to the next period and running the capital budgeting model iteratively as many times as needed, until all modified budgets are saturated. It is easy to see that this process is converging, as at each situation, one more resource constraint is saturated without affecting the binding status of the resource constraints related to the previous periods.

The case of insufficient budgets is the most likely situation, given the present level of capital shortage for the rebuilding and rehabilitation of deteriorated infrastructure. This case will be assumed in the examples studied in a later section of this chapter.
Determination of the budget limits: the proportional rule

It is useful to examine how the values of $b_t$ could be determined, where $b_t$ is the budget limitation on maintenance and capital improvement expenses for a certain district within the global network. In figure 5.1 the determination of the total budget, $\beta_t$, results from balancing the availability of funds and maintenance needs. These needs are evaluated by running the model on each district of the network under no financial constraint, then summing up all the total net present costs across the network. In order to determine $b_t^i$ (for a given district $i$) one could apply the proportional rule to different districts. Assuming that the total costs associated with that district in time period $t$ at the optimum (under no resource constraint) are (in dollars of the beginning of that period), $(b_t^i)^*$. The total costs over the whole network at the optimum are $\sum_i (b_t^i)^*$. 
If the global allocated budget for period $t$ is $B_t$, then the budget limitation in period $t$ and in district $i$ can be suggested using the proportional rule:

$$b_t^i = B_t \times \frac{(b_t^i)^*}{\sum (b_t^i)^*}.$$

In the previous model, the network sub-structure, represented by the continuity constraints, and the flow variables, increased the size of the problem. Large dimensionality would impose the decomposition of the global distribution network into small grids of 10 to 20 arcs. In mature systems, the model can be simplified, making it possible to solve significantly larger problems.

II. **A capital budgeting structure of the scheduling model**

The transition between the design of a new distribution system and the problem of the expansion, rehabilitation or replacement of an existing system was explained in the introduction to the simplified model. However, in mature systems, redundancy is inherently built in the structure of the network. Given demand projections over the time horizon, water utilities utilize computer-based flow models to determine the pipes which would be undersized, as explained by Corless (1982). Therefore, the optimal replacement or rehabilitation time should take into account the possibility of insufficient capacity beyond a certain time. Thus, the expansion issue is solved at the pipe level, after the flow model has shown where and when it is required. The output of
the single pipe analysis is therefore an optimal replacement time (by possibly a different capacity pipe) or an optimal rehabilitation time). This can be used as an input to a capital budgeting model to allow the scheduling of capital expenditure under the budget constraints.

a) The integer programming model

The integer programming capital budgeting model is first presented under the assumption of a project associated with each arc \( l \). However, the model can be applied with no change in its formulation, to projects associated with bundles of pipes. This allows for the reduction of the size of the problem, without affecting the performance of the model.

Model formulation

The same notations as in the model above are adopted, but arcs are indexed by one letter for the sake of simplicity. \( \text{REH}_l \) represents the rehabilitation costs associated with arc \( l \). For arcs requiring expansion (replacement by a larger capacity pipe), the rehabilitation alternative is not considered in the global model.

The model can then be formulated as:

\[
\begin{align*}
\min \sum_{l \in \Omega} \left\{ \left( \sum_{t=1}^{P} \left[ \frac{\text{REPL}_l}{(1+r)(t-\frac{1}{2})T} - s_l(t) \right] y_l^t \right) + \left( \sum_{t=1}^{P} \left[ \frac{\text{REH}_l}{(1+r)(t-\frac{1}{2})T} - s_l(t) \right] w_l^t \right) \right\}
\end{align*}
\]

subject to:
a) **Exclusivity constraint**

\[ \sum_{t=1}^{P} y_{t}^L + \sum_{t=1}^{P} w_{L}^t = 1 \text{ (or } \leq 1, \text{ depending on whether the capital improvement measure is necessary or simply recommendable).} \]  

The variables \( w_{L}^t \) are considered where the rehabilitation alternative is possibly viable.

b) **Budget Constraint**

\[
\sum_{L \in \Omega} \left( \frac{\text{REPL}_L}{(1+r)^{T/2}} y_{L}^t + \frac{T}{m=1} \left( \frac{\text{REPL}_L[(t-1)T+m]}{(1+r)^m} \right) \left( 1 - \sum_{t'=1}^{t-1} y_{L}^{t'} - \frac{1}{2} y_{L}^t \right) \right) \\
+ \frac{\text{REH}_L}{(1+r)^{T/2}} w_{L}^t + \left( \frac{T}{m=1} \frac{0 [(t-1)T+m]}{(1+r)^m} \right) \left( 1 - \sum_{t'=1}^{t-1} (y_{L}^{t'} + w_{L}^t) - \frac{1}{2} (y_{L}^{t'} + w_{L}^t) \right) \right) \leq b_t \quad \forall t = 1, \ldots, P
\]

where \( b_t \) = budget for time period \( t \) in dollars of the beginning of the time period.

**Implementation of the capital budgeting model**

In mature water distribution systems, capital improvement programs can be derived by applying the above model to bundles of arcs, as the implementation of these programs is undertaken through contracting. As contract costs depend on the size of the contract, it is necessary to pool a number of pipes into one contract to capture contract economies of scale. For many reasons, implementing a bundle of projects as one contract within a given time period generates savings to the water utility.

\[ a) \text{ Size of a bundle} \]

The first issue is that of the size of the contract. As it increases, the number and size of participating contractors in
competitive bidding grow, driving down the total contract costs by "competitive efficiency" as mark-ups are lowered to the general market level. Also, pooling different water main sections in one bundle generates cost savings derived from economies of scale in labor costs and equipment utilization. As the ultimate pricing of a contract is the sum of the base cost and the mark-up, both arguments of increased competition (with larger contracting firms bidding for jobs) and economies of scale are relevant.

Significant savings can therefore be gained by simply pooling projects. The issue of the optimal bundle size and contents is still a managerial one, and can be examined by adequately designing the contracts.

b) Contract design: Criteria for bundle selection

Pooling the set of pipes into different bundles has to respond to a number of criteria. While the purpose of the pooling scheme is to capture the aforementioned savings, and to make the scheduling problem more tractable in large-scale systems, it is important to design a bundle or contract without incurring additional costs due to built-in inefficiencies. In other words, the advantages derived from criteria category B of table 5.1 should not be dissipated by neglecting criteria such as in category A. In particular, pipes should be in the same bundle when it is not inefficient to implement their related projects as a whole. For that end, the optimal replacement times should be quite close and the additional maintenance costs should follow identical patterns. The reason why the optimal replacement time criterion is not sufficient for bundle composition, is that two pipes of
TABLE 5.1:
Criteria for Bundle Selection

A. Contract Efficiency Criteria
   1. Optimal replacement/rehabilitation time
   2. Repair costs/additional operations cost patterns
   3. Installation time
   4. Similar characteristics/covariates

B. Contracting Related Criteria
   1. Common technology/equipment utilization for pipe layout
   2. Contiguity
   3. Contract size

C. System constraints

Exclusivity of certain projects
equal replacement or rehabilitation time might have different maintenance cost patterns. In one case delaying the replacement/rehabilitation would lead to much higher costs than in the other. For example, if pipes have repair event trends $N_1(t)$ and $N_2(t)$ such that:

$$
N_1(t) = N_1(t_o) e^{A_1(t-t_o)}
$$

and

$$
N_2(t) = N_2(t_o) e^{A_2(t-t_o)}
$$

$A_1$ and $A_2$ might be significantly different while the optimal replacement times or "maturities" of the pipes are equal.

In practice, after the set of pipes has been decomposed into bundles according to the criteria mentioned in table 5.1, many pipes would differ according to some criteria, as it is almost impossible to replicate pipes with exactly similar trends. The issue of the evaluation of different bundles can be examined in further detail.

In order to evaluate a certain bundle or contract, the optimal maturity date is evaluated and the total net present costs of the pool are compared to the sum of the present costs for each individual pipe, thus leading to an estimate for the savings from pooling the pipes.

In order to illustrate the contract evaluation method, let us consider pipes which are simply experiencing break events. Their predicted number of future breaks is in year $t$, for pipe $i$ ($i=1,\ldots,n$):

$$
N_i(t) = N_i(t_o)e^{A_i(t-t_o)}
$$

The calculation of the optimal replacement time for the bundle, as derived in Chapter 3, would lead to a value $t^*_R$ for the bundle given by:
\[
    t^*_R = t_o + \frac{1}{A} \ln \left[ \frac{R \times \text{CONT}}{\sum_{i} (t'_o \times C_i)} \right]
\]

where the notations are the same as before for $C_i$, $R$ and $A$.

CONT = contract value for the bundle. The net present costs for the replacement strategy of the bundle is then evaluated and compared to the sum of the costs of individual replacement strategies for different pipes. The savings from pooling the pipes in the bundle are then computed. The design of different bundles can therefore be obtained iteratively as shown in figure 5.5, provided that each bundle can be "priced" and that budgetary constraints are not stringent enough to displace entire bundles from one period to another. The case of such bundles can then be studied separately to eliminate the inefficiencies that might possibly emerge as the limit of the budgetary constraint is reached.

In brief, the design of contracts as in figure 5.5 allows one to decompose all the different jobs into a number of contracts which are then scheduled using the capital budgeting model described above. In order to facilitate both the decomposition step and the evaluation step, data management issues should be examined, as certain schemes are particularly suitable for the purposes of maintenance scheduling applications.

**Data management issues**

The need for bundle selection makes it particularly convenient to adopt the relational view of data, which consists of logically representing data in tables where rows are pipe records and columns
FIGURE 5.5

Iterative Decomposition of the Set of Pipes into Bundles

FIGURE 5.6

Major Software Tools for Maintenance Planning
are fields including all relevant information related to the pipe, such as location, internal and external characteristics, optimal replacement time, hazard rate (when necessary), etc.

The availability of data manipulation languages for such data base management structure on microcomputer, as well as financial analysis software (spreadsheet type) and a capital budgeting package makes it possible to formalize the maintenance planning and data management and updating needs of a small water utility. This would enhance significantly the quality of their decision process, both in the assessment of needs and determination of budgets and in the scheduling of capital expenditures for capital improvement purposes. In particular, the design of bundles is highly simplified by using projections of the data base.

Even a large utility can use the same configuration, as computer hardware capacity and power have increased dramatically during the last few years. Significant efforts to computerize data management and maintenance planning should be made given the necessity of having quality input data for performance prediction and maintenance planning purposes (figure 5.6).

III. A linear programming structure of the capital budgeting model

In the previous integer programming approach to the capital budgeting problem, it was assumed that a certain number of bundles could be designed as contracts that can be implemented as a whole. These bundles were then scheduled using the multi-period capital budgeting model described above. Next, a sequencing model (Chapter 6)
allows planning job implementation on a yearly basis. The underlying assumption is obviously that the size of a bundle is such that it can be implemented in one year. At some point, the sizing of the bundles might lead to a deferral of whole bundles as the budget constraint is reached. In general, small-to-medium sized utilities would probably find the underlying assumptions of the integer programming approach quite applicable to their specific situation. 20 to 50 bundles of 10 to 20 pipes each can be separated using the heuristic mechanism described in figure 5.5.

However, in other cases, especially in large systems, the determination of a number of different bundles of 10 to 20 pipes each is rather difficult. Instead, the system is decomposed into larger bundles (>100 pipes). The assumption of a single project representing a bundle used in the integer programming model is not relevant in this case. Also the argument of a bundle project is not viable as a large part of the improvement program will be implemented in-house given the scale of a large system. One way to circumvent the problem is to decompose the bundle into a number of smaller bundles, thus increasing the number of integer variables. A better approach is to assume that a fraction of the bundle (evaluated in units of pipe wall for instance) can be implemented as a separate project. The capital budgeting model is then formulated as a linear programming model. Let us denote by $S = \{s_1, \ldots, s_n\}$ the set of bundles. It is assumed that bundles are designed according to the criteria described in table 5.1. A bundle $s_k$ would therefore be associated with an optimal replacement/rehabilitation time $t^*_{s_k}$ which is denoted by $t^*_{s_k}$. The maintenance and additional operational cost patterns for different pipes in the bundle are assumed to be quite similar as suggested in the bundling selection criteria. Let us
denote by $\alpha_{t, t^*}^{s_k}$ the fraction of $s_k$ to implement at the end of year $t$ and by $\text{TC}_{s_k}$ the total cost of implementing the bundle of projects (rehabilitation or replacement), and by $\text{MOP}_{s_k}$ (u) the additional maintenance and operational costs in year $u$ related to the bundle $s_k$. Let us define $\Delta_{t, t^*}^{s_k} = \text{opportunity cost in present dollars of implementing the bundles } s_k \text{ at time } t > t^*.$

For example, if in year $u > t^*$, the bundle is not implemented, $s_k$ the opportunity loss in dollars of the beginning of year $u$ is:

$$\text{MOP}_{s_k}(u) - (\text{TC}_{s_k} - \frac{\text{TC}_{s_k} x r}{1+r}) = \text{MOP}_{s_k}(u) - \frac{\text{TC}_{s_k} x r}{1+r}$$

The second term of the left-hand side of the previous equality represents the savings in delaying the implementation of $s_k$ by one year (in real terms).

It follows that

$$\Delta_{t, t^*}^{s_k} = \sum_{u=t^*}^{t} \left[ \left( \text{MOP}_{s_k}(u) - \frac{\text{TC}_{s_k} x r}{1+r} \right) x \frac{1}{(1+r)^u} \right]$$

The linear programming model can then be formulated for a planning horizon divided by N years

$$\text{Min} \sum_{s_k} \sum_{t=t^*}^{N} \alpha_{t, t^*}^{s_k} \Delta_{t, t^*}^{s_k}$$

under the constraints:

a) **COMPLETION CONSTRAINT**

$$\sum_{t=1}^{N} \alpha_{t, t^*}^{s_k} = 1$$

If budgets are insufficient for the completion of all bundles, then the budgetary constraint for the last year is dropped.
b) **POSITIVITY CONSTRAINT**

$$\alpha_{s_k} \geq 0 \quad \forall t = t^*_s, \ldots, N$$

$$\forall s_k \in S$$

c) **BUDGETARY CONSTRAINT**

$$\sum_{s_k} \alpha_{s_k} (TC_{s_k}) + \left[ (1 - \sum_{t^*_s < t' < t} \alpha_{t', s_k}) MOP(t) \right] \leq b_t$$

$$\forall t = 1, \ldots, N$$

where \( b_t \) = maintenance and capital improvement budget for year \( t \).

This budgetary constraint can be restated as:

$$\sum_{s_k} \alpha_{s_k} (TC_{s_k}) - \sum_{t^*_s < t' < t} \alpha_{t', s_k} (MOP(t)) \leq b_t - \sum_{s_k} MOP(t)$$

The formal model is bound to be of use for large water utilities where the distribution network is divided into a large number of large-sized bundles (equivalent to a number of contracts). It has many advantages.

1. Its linear programming structure makes it much easier to use in real situations.

2. Because of less dimension limitations, yearly scheduling can be undertaken directly without the need for a sequencing model such as in Chapter 6.

It is important to note that the discounted sum of the yearly budgets is supposed to meet the completion constraints. If that was
not the case, more of the equality constraints should be replaced by an inequality constraint.

In the next section, the incorporation of uncertainties in the general planning process of figure 5.4 is analyzed, followed by an illustrative example of the integer programming model for capital budgeting exposed above.

IV. The Incorporation of Uncertainties in the Planning Process

In the flowchart describing the steps of the planning process, the safety program is a means to include the uncertainties in future demands. Even though the impact of these uncertainties in a mature, often redundant system is limited, the following analysis is presented to illustrate how such uncertainties can be incorporated in the planning process.

1. The Uncertainties in Demands: The Safety Program

The uncertainties in future demands can be decomposed into two elements:

a) the per capita peak period consumption, directly related to the style of life, but also ultimately to pricing policies. It is assumed to be uniformly distributed on an interval \((D_{\text{min}}, D_{\text{max}})\) with mean \(\mu_o\) and variance \(\sigma_o^2\).

\[
\mu_o = \frac{D_{\text{min}} + D_{\text{max}}}{2} \quad \text{and} \quad \sigma_o^2 = \frac{(D_{\text{max}} - D_{\text{min}})^2}{12}
\]

b) the population growth including demographic growth, as well as migrations from and to any demand center.

To combine both population moves and demographic growth, two possible forecasting methods can be considered.
1. A regression over time of the previous population realizations, with particular emphasis on the last ten years of data.

2. A stochastic model of prediction, such as the stochastic residential population forecasting model (Meier (1972)). As mentioned by Grossman (1977), this model can yield better predictions for the next ten years than the previous method, and is basically a more sophisticated regression-based model.

Whatever method is used, the uncertainty in population growth beyond the first time period becomes a composite of estimation error and model error, as the viability of the model decreases with time. That is why only the variance in the peak demands related to the first time period is considered in a given planning cycle.

Denoting the projections of population number in demand center $i$ in time period $t$ by $\hat{n}_i^t$, the actual realization $n_i^t$ is given by

$$n_i^t = \hat{n}_i^t + E_i^t$$

where $E_i^t$ is the estimation error, assumed to be normally distributed with standard deviation $\sigma_i^t$. Hence, the random variable $\tilde{n}_i^t$ is normal with mean $\hat{n}_i^t$ and variance $\sigma_i^t$. This density function is denoted $\psi(n_i^t)$.

**Likelihood of demand**

Under a population realization $n_i^t$, in time period $t$ and in demand node $i$, the central limit theorem leads to a normal distribution for the demand $d_i^t$ with mean $\mu_o n_i^t$ and standard deviation $\sigma \sqrt{n_i^t}$. This distribution is denoted $p(d_i^t/n_i^t)$ and is called the likelihood of $d_i^t$ for a given value of the parameter $n_i^t$. The underlying assumption is that the demands of different standard consumers are independently identically distributed as described above.
Predictive Probability Density of Demand

To obtain the probability distribution $d_1^t$, $f(d_1^t)$, the total probability theorem is applied:

$$f(d_1^t) = \int p(d_1^t/n_1^t) \cdot \psi(n_1^t) \, dn_1^t$$

Using $f(d_1^t)$, both mean and standard deviation of $d_1^t$ are evaluated and are denoted respectively $\mu_1^t$ and $\sigma_1^t$. Only $\mu_1^t$ and $\sigma_1^t$ related to the first time period in a given cycle are jointly used as inputs to the safety program in the associated planning cycle. The reason why uncertainty in the first period is considered is that pipelines are expandable facilities.

2. The Safety Program

The safety program is based on the following principle:

If $\sigma_1^t$ is the standard deviation of the demand node $i$ in the first period, then the planner should be interested in the additional cost $A^*_\alpha$ of meeting peak demands $(\mu_1^t + \alpha \sigma_1^t)$ in the first period, where $\alpha$ is allowed to vary. To support the decision-making process, the notion of resiliency is applied to the water supply system problem under two different perspectives.

Resiliency of the Regional Water Supply System

Resiliency has been the subject of much recent literature (Fiering, 1982; Hashimoto and Loucks, 1982). We use this term to mean the ability of the system to change operating characteristics to accommodate conditions unanticipated in design. Two measures of resiliency are proposed:

- a performance measure
- an economic measure
a) The Performance Definition of Resiliency

Given a certain expansion strategy of the regional system, the system will be considered as "perfectly resilient" if for any conceivable scenario of demands, the system is capable of meeting such levels where and when needed. It is obvious that the expansion strategy derived from the deterministic model will not necessarily lead to "perfectly resilient" systems.

In mathematical terms, the system will be resilient, if there exists a feasible set of flows in the different arcs \((i,j)\) (denoted by \(f_{ij}^t\) in time period \(t\)), for any realization of demands \(d_j^t\) such that:

1. \[ \sum_i f_{ij}^t - \sum_k f_{jk}^t = d_j^t \text{ for sink nodes or intermediary nodes.} \]

2. \[ f_{ij}^t \leq (x_{ij}^t)^*, \text{ where } (x_{ij}^t)^* \text{ is the total capacity recommended by the strategy on arc } (i,j) \text{ in time period } t. \]

Such feasibility constraints could be checked by simply minimizing \[ \sum_{i} \left( \sum_{i,m} f_{im}^t \right) \] where \(i\) are source nodes under the constraints listed above. It is assumed here that the total supply potential can suffice for any level of demand envisaged.

Degree of Resiliency of a Given Expansion Strategy

Resiliency was considered as the fact of being capable of meeting exactly the demands. Thus, given the sequence of stochastic demands, it is possible to evaluate the probability of meeting these demands by checking the feasibility of the constraints aforementioned under simulated occurrences of demands. If this probability is \(p\), the degree of resiliency is \(p\). Example (80% resilient, ...)
Imperfectly Resilient Systems: The Safety Program

As was explained previously, the planner is assumed to take a conservative stance in the first period. The system is supposed to meet "pessimistic" values of demand (for instance $\mu_j^{-t} + \sigma_j^{-t}$). To render the system almost perfectly resilient (capable of meeting $\mu_j^{-t} + \sigma_j^{-t}$), a mathematical programming approach is proposed in the Safety Program.

It is based on the deterministic approach of Module I, with the values of demands in the first time period equal to $\mu_1^{-1} + \alpha \sigma_1^{-1}$ where $\alpha$ is allowed to vary. The additional cost $A_\alpha^*$ to the original expansion strategy is evaluated for a range of values of $\alpha$. If we assume that the system is almost perfectly resilient for $\alpha = 1$ then the additional cost $A_\alpha^*$ is expected to be an opportunity loss from the perfect information case. It is a benchmark for the value of sample information and allows one to estimate the value of a more accurate model of demand projection.

b) Economic Measure of Resiliency

If $(x_{1j}^*)^*$ is the original expansion strategy with total present cost $TC$, and $(x_{1j}^*)^{**}$ is the modified strategy with cost $TC + A^*$, then an economic resiliency index can be defined as $TC/(TC + A^*)$, the ratio of the total cost of the original expansion strategy to the total cost of the modified strategy. However, it is important to note that such a factor merely reflects the "inefficiency" incurred by our imperfect knowledge in the first time period of a given planning cycle.

Hence, using both performance and economic indices of the
FIGURE 5.7

Performance and Economic Resiliency Indices
resiliency, the planner can decide on the strategy to implement. The variation of the performance and resiliency indices with $\alpha$, is illustrated in Figure 5.7.

The issue of uncertainties in demands is important when the evolution of these demands is significant over one time period. In most of the cases, the expansion capital needs are not as important per se as the replacement of deteriorating pipes, as redundancy and extra capacity due to overbuilding are quite common in existing mature systems.

V. Case Examples

V.I. Application of the Linear Programming Model for Capital Budgeting

The problem is related to a large water distribution system which is supposed to be decomposed in ten bundles of either 400,000 or 500,000 feet, as seen in Table 5-1a. These bundles include pipes with similar characteristics and, in particular, equal optimal replacement times and identical repair cost trends. The loss of system performance is measured in terms of repair costs solely, as the problem of loss of carrying capacity is not considered herein.

The predictive model for repair events used in this example is the model by Shamir and Howard (1979) discussed earlier in this work. The base year $t_0$ is 1984 and is set at the value 0. Table 5-1 gives the summary information related to the bundles.

As the earliest replacement time is $t_0^* = 5$, the resource constraints are applied starting in year 5.
Example 1 (insufficient budgets)

It is assumed that a total budget of $20,000,000 for maintenance and capital improvement measures is available in year 5, and is expected to grow at a constant annual rate of 10%. The budget projections are given in table 5-2. Replacement cost and budget units are expressed in $10,000. Budgets are given in real dollars.

A twenty-year planning horizon is considered. The formulation of the linear programming model as well as the output solution are given in appendix B1.

The solution of the model is summarized in table 5-3, which gives the fractions of each bundle that is suggested for implementation in different years.

As can be seen from table 5-3, the bundles in which implementation was most deferred are bundles B, C, and D, as well as bundle J. This can be explained by the fact that the trends in growth of repair costs for these bundles are slower than for the other bundles. Therefore, the opportunity costs of deferral are lower for these bundles than for other bundles.

It can also be noticed that the total requirements for year 20 equal $161,120,000, while the budget for that year is only $83,534,000. This result was obtained as the resource constraint for the last year (year 20) was relaxed, given the fact that the total requirements could not be met by the available budgets. Therefore as of year 20, the gap between requirements and the available budget will be $77,586,000. However, it can be easily seen that, by setting in year 21 a budget of the same order of magnitude as in year 20, the required measures could
<table>
<thead>
<tr>
<th>Bundle</th>
<th>Length (in 1,000 feet)</th>
<th>Total Replacement Cost (in $10,000)</th>
<th>Number of Repair Events in year $t$</th>
<th>Repair Cost (in $)</th>
<th>Optimal Replacement Time $t^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>400</td>
<td>4,000</td>
<td>$600 e^{0.1t}$</td>
<td>2,000</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>400</td>
<td>4,000</td>
<td>$440 e^{0.1t}$</td>
<td>2,600</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>500</td>
<td>5,000</td>
<td>$335 e^{0.1t}$</td>
<td>2,000</td>
<td>15</td>
</tr>
<tr>
<td>D</td>
<td>400</td>
<td>6,000</td>
<td>$1,400 e^{0.05t}$</td>
<td>2,000</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>400</td>
<td>6,000</td>
<td>$1,092 e^{0.05t}$</td>
<td>2,000</td>
<td>10</td>
</tr>
<tr>
<td>F</td>
<td>500</td>
<td>7,500</td>
<td>$1,065 e^{0.05t}$</td>
<td>2,000</td>
<td>15</td>
</tr>
<tr>
<td>G</td>
<td>400</td>
<td>2,800</td>
<td>$292 e^{0.15t}$</td>
<td>2,000</td>
<td>7</td>
</tr>
<tr>
<td>H</td>
<td>400</td>
<td>2,800</td>
<td>$188 e^{0.15t}$</td>
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</tr>
<tr>
<td>I</td>
<td>500</td>
<td>3,500</td>
<td>$110 e^{0.15t}$</td>
<td>2,000</td>
<td>15</td>
</tr>
<tr>
<td>J</td>
<td>400</td>
<td>4,800</td>
<td>$824 e^{0.07t}$</td>
<td>2,000</td>
<td>8</td>
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</table>

**TABLE 5-1a. BUNDLES AND CHARACTERISTICS**

*(LINEAR PROGRAMMING EXAMPLE)*

Note: year 0 is 1984
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<thead>
<tr>
<th>Year</th>
<th>Budget Projections in ($10,000)</th>
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<tr>
<td>5</td>
<td>2,000</td>
</tr>
<tr>
<td>6</td>
<td>2,200</td>
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<tr>
<td>7</td>
<td>2,420</td>
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<tr>
<td>8</td>
<td>2,662.4</td>
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<td>2,928.2</td>
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<tr>
<td>10</td>
<td>3,220.02</td>
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<tr>
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</tr>
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<td>12</td>
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<tr>
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<td>4,287.177</td>
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<td>15</td>
<td>5,187.485</td>
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<td>16</td>
<td>5,708.233</td>
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<tr>
<td>17</td>
<td>6,276.856</td>
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<tr>
<td>18</td>
<td>6,904.541</td>
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<tr>
<td>19</td>
<td>7,594.995</td>
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<tr>
<td>20</td>
<td>8,353.4</td>
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</table>

**TABLE 5-2. BUDGET PROJECTIONS**

**(EXAMPLE 1)**
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<th>Bundle(t*)</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
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<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
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</thead>
<tbody>
<tr>
<td>A(7)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>21</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>B(10)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>17</td>
<td>77</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>C(15)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>100</td>
</tr>
<tr>
<td>D(5)</td>
<td>5</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>8</td>
<td>81</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>E(10)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>72</td>
<td>28</td>
</tr>
<tr>
<td>F(15)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>100</td>
</tr>
<tr>
<td>G(7)</td>
<td>-</td>
<td>-</td>
<td>15</td>
<td>18</td>
<td>22</td>
<td>27</td>
<td>18</td>
<td>-</td>
<td>-</td>
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<tr>
<td>H(10)</td>
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<td>-</td>
<td>-</td>
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<td>62</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>I(15)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>100</td>
</tr>
<tr>
<td>J(8)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>64</td>
<td>-</td>
<td>36</td>
<td>-</td>
</tr>
</tbody>
</table>

**TABLE 5-3: SUMMARY OF THE SOLUTION OF THE LINEAR PROGRAMMING MODEL**

(Example 1)

Fractions of Bundles Recommended in Different Years

(Expression in Percent)
be implemented in year 21.

The issue of contract size is also worth mentioning as a lower limit for the allowable fraction of the bundle might be necessary to gain economies of scale from larger contracts. For example, if that lower limit was about 10% of a bundle such as bundle 10, the implementation of 4.8% of the bundle in year 5 and 5.8% of the bundle in year 6 should be compared to the implementation of 10.6% of the bundle in year 6. If the economies of scale captured through a larger contract are higher than the costs of deferring the implementation of 4.8% of the bundle by one year, then the former alternative should be preferred. It is therefore possible to modify heuristically the solution of the linear programming model when the magnitude of the economies of scale relative to the costs of deferring some measures justify the merger of two or more projects into one contract.

Example 2 (unsaturated budgets)

The same problem as above is solved under other budget projections. It is now assumed that the budget in year 5 is 30,000,000 and grows at a rate of 15% until it reaches 160,000,000 beyond which it remains constant. The new budget schedule is given in table 5-4. The units are unchanged. The original model is first seen, yielding an objective function optimum of 316.92. However, slacks or budget surpluses appear in a number of constraints. Therefore, slack variables are introduced, starting from the earliest unsaturated constraint. The new modified model is then seen, yielding an objective function value of 309.08, thus representing an improvement of the previous problem optimum, as
<table>
<thead>
<tr>
<th>Year</th>
<th>Budget projections in ($10,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3,000</td>
</tr>
<tr>
<td>6</td>
<td>3,450</td>
</tr>
<tr>
<td>7</td>
<td>3,967</td>
</tr>
<tr>
<td>8</td>
<td>4,562</td>
</tr>
<tr>
<td>9</td>
<td>5,246</td>
</tr>
<tr>
<td>10</td>
<td>6,033</td>
</tr>
<tr>
<td>11</td>
<td>6,938</td>
</tr>
<tr>
<td>12</td>
<td>7,979</td>
</tr>
<tr>
<td>13</td>
<td>9,176</td>
</tr>
<tr>
<td>14</td>
<td>10,552</td>
</tr>
<tr>
<td>15</td>
<td>12,135</td>
</tr>
<tr>
<td>16</td>
<td>13,955</td>
</tr>
<tr>
<td>17</td>
<td>16,000</td>
</tr>
<tr>
<td>18</td>
<td>16,000</td>
</tr>
<tr>
<td>19</td>
<td>16,000</td>
</tr>
<tr>
<td>20</td>
<td>16,000</td>
</tr>
</tbody>
</table>

**TABLE 5-4. BUDGET PROJECTIONS (EXAMPLE 2)**
expected. The formulation of the problem treated in Example 2 and its outputs (original problem, followed by modified problem) are given in Appendix BII. The solutions of the original problem and the modified problem are respectively given in tables 5.5 and 5.6.

Both tables 5.5 and 5.6 show a lower deferral of capital improvement measures for different bundles from their optimal replacement time than given in example 1. For instance, Bundle A is completed in year 14 in table 5-3 (example 1) and year 9 in table 5.5 (example 2). This is easily explained by the availability of more resources, as example 2 has higher yearly budgets.

The comparison of tables 5.5 and 5.6 shows a change only in bundle F, thus leading to the slight improvement in the solution mentioned above, after allowing for the resource transfer. The reason why the improvement is so modest is simply that the slack observed in the first case was related to the late budget years of the planning horizon, when resources were already available for the required measures. Therefore, the presence of the slack variables allowed the channeling of most of the unspent budgets to the end of the planning horizon, which allowed to evaluate the cut-off year for funds. In effect, beyond year 15, all the bundles were completed, thus making the remaining resources unnecessary. Obviously, in example 2, all the budgets beyond year 15 are not required.
### Table 5.5: Summary of the Solution of the Linear Programming Model

(Example 2/Without Resource Transfer)
| Year | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Bundle \( (t^*_B) \) |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| A(7) | -   | -   | 2   | 7   | 11  | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   |
| B(10) | -   | -   | -   | -   | -   | -   | -   | 53  | 47  | -   | -   | -   | -   | -   | -   | -   |
| C(15) | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | 100 | -   | -   | -   |
| D(5)  | 23  | .29 | 36  | -   | -   | 12  | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   |
| E(10) | -   | -   | -   | -   | -   | -   | -   | 89  | 11  | -   | -   | -   | -   | -   | -   | -   |
| F(15) | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | 100*| -   | -   | -   |
| G(7)  | -   | -   | -   | 100 | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   |
| H(10) | -   | -   | -   | -   | -   | -   | 100 | -   | -   | -   | -   | -   | -   | -   | -   | -   |
| I(15) | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | -   | 100 | -   | -   |
| J(10) | -   | -   | -   | -   | -   | -   | 82  | 18  | -   | -   | -   | -   | -   | -   | -   | -   |

**TABLE 5.6: SUMMARY OF THE SOLUTION OF THE LINEAR PROGRAMMING MODEL**

(EXAMPLE 2/with resource transfer)

*Change from previous solution
V.2 Application of the Integer Programming Model for Capital Budgeting

This case is related to a small water distribution system. The network has been decomposed into 15 bundles (A to O), and it is assumed that each of these bundles has to be implemented as a single project. The detailed characteristics of these bundles in terms of optimal replacement times, replacement and repair costs, predicted number of repair events, are given in table 5-7. The maintenance costs include only the repair costs, as the problem of the loss of carrying capacity is not considered herein. The predictive model for repair events is the Stämir and Howard model (1979) and the base year \( t_1 \) is 1984, and is set at 0.

The budget projections for the four five-year time periods are given in table 5-7. The budget of the first period is 18,000,000 and is supposed to grow at a rate of 10% per time period (1.925% a year) in real terms.

The mixed integer programming model formulation and its outputs are presented in Appendix BIII. First, the original integer programming model is run without provision for resource transfer from one period to the next. The optimal scheduling strategy is given in the third column of table 5-9. The value of the objective function is then equal to -820.73. It is also noticed that budget constraints are not binding, i.e.
budgets are not saturated. The modified model including slack variables is then used, allowing one to transfer resources from one period to the next period. The new scheduling strategy is given in column 4 of Table 5-8. Bundle F, which implementation period under the first strategy was 4, is moved closer to its optimal replacement time under the new strategy. The other recommended implementation periods are unchanged. The performance of the new solution is better than the previous one as the value of the objective function is -1,285.65.

Therefore, the transfer of resources from one period to the next allowed the enhancement of the scheduling strategy. The same result was also achieved by modifying the original problem iteratively, saturating the budget constraints in sequence.

**Last Period Deficit**

The evaluation of the last period costs under the last modified strategy shows a total estimate of about $32,285,000, thus creating a deficit of $8,327,000 for that period. This would require either a deferral of one or two projects to a further period, or a correction of the last period budget.
<table>
<thead>
<tr>
<th>Bundle</th>
<th>Length (in 1,000 ft)</th>
<th>Total Replacement Cost (in $1,000)</th>
<th>Number of Repair Events in Year t</th>
<th>Repair Cost (in dollars)</th>
<th>Optimal Replacement Time t^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>40</td>
<td>4,000</td>
<td>$60 e^{0.1t}$</td>
<td>2,000</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>40</td>
<td>4,000</td>
<td>$89.51 e^{0.1t}$</td>
<td>2,000</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>50</td>
<td>5,000</td>
<td>$33.5 e^{0.1t}$</td>
<td>2,000</td>
<td>15</td>
</tr>
<tr>
<td>D</td>
<td>50</td>
<td>5,000</td>
<td>$67.5 e^{0.1t}$</td>
<td>2,000</td>
<td>8</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>5,000</td>
<td>$122.9 e^{0.1t}$</td>
<td>2,000</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>40</td>
<td>6,000</td>
<td>$140 e^{0.05t}$</td>
<td>2,000</td>
<td>5</td>
</tr>
<tr>
<td>G</td>
<td>40</td>
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<td>$114.7 e^{0.05t}$</td>
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<td>9</td>
</tr>
<tr>
<td>H</td>
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<td>$106.5 e^{0.05t}$</td>
<td>2,000</td>
<td>15</td>
</tr>
<tr>
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<td>7,500</td>
<td>$91.65 e^{0.05t}$</td>
<td>2,000</td>
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</tr>
<tr>
<td>J</td>
<td>40</td>
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<tr>
<td>K</td>
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<td>2,800</td>
<td>$29.2 e^{0.15t}$</td>
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<td>7</td>
</tr>
<tr>
<td>L</td>
<td>40</td>
<td>2,800</td>
<td>$13.8 e^{0.15t}$</td>
<td>2,000</td>
<td>12</td>
</tr>
<tr>
<td>M</td>
<td>40</td>
<td>2,800</td>
<td>$6.52 e^{0.15t}$</td>
<td>2,000</td>
<td>17</td>
</tr>
<tr>
<td>N</td>
<td>40</td>
<td>2,800</td>
<td>$18.8 e^{0.15t}$</td>
<td>2,000</td>
<td>10</td>
</tr>
<tr>
<td>O</td>
<td>40</td>
<td>2,800</td>
<td>$46.24 e^{0.15t}$</td>
<td>2,000</td>
<td>4</td>
</tr>
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</table>

**TABLE 5-7. BUNDLES AND CHARACTERISTICS (INTEGER PROGRAMMING EXAMPLE)**
<table>
<thead>
<tr>
<th>Time Period</th>
<th>Budget (in $1,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18,000</td>
</tr>
<tr>
<td>2</td>
<td>19,800</td>
</tr>
<tr>
<td>3</td>
<td>21,780</td>
</tr>
<tr>
<td>4</td>
<td>23,958</td>
</tr>
</tbody>
</table>

**TABLE 5-8: BUDGET PROJECTIONS FOR THE PLANNING HORIZON**
<table>
<thead>
<tr>
<th>Bundle</th>
<th>Optimal Replacement Year (Associated Period)</th>
<th>Recommended Implementation Period (Original Model)</th>
<th>Recommended Implementation Period (Model with Resource Transfer)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>7 (2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>3 (1)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>15 (3)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>8 (2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>2 (1)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>5 (1)</td>
<td>4</td>
<td>2*</td>
</tr>
<tr>
<td>G</td>
<td>9 (2)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>H</td>
<td>15 (3)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>I</td>
<td>18 (4)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>J</td>
<td>17 (4)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>K</td>
<td>7 (2)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>L</td>
<td>12 (3)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M</td>
<td>17 (9)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>N</td>
<td>10 (2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>O</td>
<td>4 (1)</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

**Table 5-9:** Comparative Results of Integer Programming Models With and Without Transfer of Resources Between Periods

*Note:* * represents a change in the scheduling strategy from the original to the modified model.
CHAPTER 6. The Short-term Sequencing Model for Capital Improvement Measures

I. Purpose of the Short-term Sequencing Model

The long term scheduling model presented in Chapter 5 was concerned with the derivation of the replacement, rehabilitation and expansion projects over a planning cycle of five years. It was also assumed that these projects were implemented on average at the middle of the time period. However, the exact timing of their implementation within the planning cycle can be optimally "controlled" by minimizing the net present value of the total associated costs, subject to the yearly budgetary constraints. This is the purpose of the short term sequencing model, which is basically a capital budgeting approach to the problem of sequencing of the projects selected for a given five-year time period. It presents the following beneficial features:

1. It makes the outputs of the scheduling model implementable. The long term planning model generates five-year period plans, based on an estimation of the financial constraint for different periods. Still the sequencing of the selected projects over the five years of the time period has to be undertaken, taking into account yearly budgetary constraints. This makes the outputs of the scheduling model readily usable within the specific institutional and economic context of the system under study.

2. By sequencing the selected projects on a yearly basis, the sequencing model is the key to a smooth transition between
the planning level and the project management and/or the contracting level, depending on how the projects are allocated between in-house and outside contracted work. The credibility of the whole planning process is enhanced by adding this sequencing model which is the interface between the long term planning of the system and the yearly job planning.

3. The sequencing model also improves the performance of the long term model. By allocating optimally the resources over the sequence of years, it allows the refinement of the results of the scheduling model, where all projects were assumed to be implemented on average in the middle of a time period.

II. The Sequencing Model

A. Problem Background

The central issue in this problem is deciding on the timing of different replacement, rehabilitation and expansion projects identified for the five year time period by the long term scheduling mode. A forecast of yearly budgetary constraints is available, as well as the set of projects for the period. The objective is to implement all these projects once and only once, while meeting the budget constraints.
In this short-term model, it is assumed that yearly capital improvement budgets are set separately from other maintenance expenditures. While in the long-term models of the previous chapter, total maintenance expenditures were constrained by some upper bound, it is current practice in the actual budgeting procedures to separate funds for capital improvement programs from those associated with routine maintenance. Yearly budget figures for capital improvement measures therefore determine the resource constraint used in the short-term sequencing model.

The projects to be sequenced are either single pipe projects, or bundles of pipes which were recommended for implementation in the time period of interest. In the following discussion a project will be generally referred to a pipe, though the generalization to a bundle is straightforward.

Similar problems have appeared in the literature for some time and were modeled as dynamic programming models. However, because of computational burdens, dynamic programming is not the viable solution method for moderate-to-large sized problems. Butcher et al. (1969)
presented a dynamic programming version of the sequencing problem. This version was later refined by Erlenkotter and Rogers (1977) and Baker and Schrage (1978). The latter authors concluded that dynamic programming was viable as long as the number of projects was relatively small, which would be in the order of 30 for a five time step problem (case of interest). Lately, Neebe and Rao (1983) have presented a Lagrangian relaxation solution method within a branch-and-bound approach, the problem being modelled as a mixed-integer programming problem.

B. Problem Formulation

This discrete-time sequencing problem can be formulated in the following fashion. For \( t = 1, 2, \ldots, T \) (\( T = 5 \) years in the case of interest) let \( b_t \) be the yearly budget. Let \( I \) be the set of projects (number of projects = \( n \)) selected for the time period. For all \( i \in I \) and all \( t = 1, 2, \ldots, T \), the following variables are defined:

\[
x_{it} = \begin{cases} 
1 & \text{if project } i \text{ is implemented in year } t \\
0 & \text{otherwise}
\end{cases}
\]

Let \( c_{it} \) be the costs associated with the implementation of project \( i \) in year \( t \). Assuming that \( c_i \) is the cost at the beginning of the time period, \( c_{it} \) can be defined as \( \frac{c_i}{(1+r)^{t-1}} \), where \( r \) is the "real" discount rate.

Given the above definitions, the general model can be formulated as follows:

\[
\text{minimize } \sum_{t=1}^{T} \sum_{i=1}^{n} c_{it} x_{it}
\]
where

\[ c_{it} = \begin{cases} 
  c' + \frac{t-1}{\sum_{s=1}^{T} \frac{\operatorname{REP}_i(s) + O_i(s)}{(1+r)^s}} + \frac{1}{2} \frac{\operatorname{REP}_i(t) + O_i(t)}{(1+r)^t} & \text{if } i \text{ is a replacement} \\
  c' + \frac{t-1}{\sum_{s=1}^{T} \frac{\operatorname{REP}_i(s)}{(1+r)^s}} + \frac{1}{2} \frac{O_i(t)}{(1+r)^t} & \text{if } i \text{ is a rehabilitation}
\end{cases} \]

\( \operatorname{REP}_i(s) \) = repair costs in year \( s \) for the pipe associated with project \( i \)

\( O_i(s) \) = additional operational costs in year \( s \) for the pipe associated with project \( i \)

subject to the following constraints:

1) **Exclusivity of Implementation**

\[ \sum_{t=1}^{T} x_{it} = 1 \text{ for all } i \in I \]

Every selected project is implemented once and only once within the time period.

2) **Budgetary Constraint**

\[ \sum_{i=1}^{n} c_i x_{it} \leq b_t \quad \forall t = 1, 2, \ldots, T \]

The total costs in year \( t \) are smaller than the early budget for capital improvement measures.

3) **Other Constraints**

\[ x_{it} = 0 \text{ or } 1 \text{ for all } i \text{ and } t \]
C. Solution Procedure Using a Branch-and-Bound Lagrangian Relaxation Approach

The relaxed version of the problem is obtained by augmenting the initial objective function with a multiple of the relaxed constraints (in this case, the budgetary constraints). The vector of associated Lagrange multipliers is \((\lambda_1, \lambda_2, \ldots, \lambda_T)\)

The Lagrangian problem is then given as:

\[
L(\lambda) = \text{minimum} \sum_{t=1}^{T} \sum_{i=1}^{n} (c_{i_t} + \lambda_t c_{i})x_{i_t} - \sum_{t=1}^{T} \lambda_t b_t
\]

subject to

\[
\sum_{t=1}^{T} x_{i_t} = 1 \quad \forall \ i \in I
\]

\[
x_{i_t} = 0 \text{ or } 1
\]

For fixed \(\lambda (\lambda > 0)\) if \(\{x^*_i, t\}\) is an optimal solution to the Lagrangian problem, \(L(\lambda)\) is less than or equal to the optimal objective function value to the original (unrelaxed) problem. Unfortunately \(\{x^*_i, t\}\) might not be a feasible solution to this latter problem as the relaxed constraints might be violated. However, \(\{x^*_i, t\}\) is an optimal solution to the original problem, provided

a) that the relaxed constraints are satisfied:

\[
\sum_{t=1}^{n} c_{i} x_{i_t} \leq b_t \quad \forall t = 1, \ldots, T
\]

b) the complementary slackness conditions are satisfied

\[
\lambda_t \left( \sum_{i=1}^{T} c_{i} x^*_{i_t} - b_t \right) = 0 \quad \forall t = 1, \ldots, T
\]
The problem is therefore to find the right $\lambda$ vector that is associated with an optimal integer solution or at least giving a good large enough value to the lower bound $L(\lambda)$. For that purpose, Neebe and Rao (1983) used the subgradient optimization procedure suggested by Held et al. (1974). The subgradient vector $g = (g_1, g_2, \ldots, g_T)$ is defined as

$$ g_t = \sum_{i=1}^{n} c_i x_{it}^* - b_t \quad \forall t = 1, \ldots, T $$

$g$ is used as a direction to alter $\lambda$. If $g_t > 0$, $\lambda_t$ is increased, which increases the adjusted cost $c_{it} + \lambda_t c_i$. If $g_t < 0$, $\lambda_t$ is decreased.

Denoting by $\lambda_t^k$ the value of $\lambda_t$ at the $k$-th iteration and $\lambda_t^{k+1}$ its value at the $(k+1)$-th iteration, one iteration scheme is:

$$ \lambda_t^{k+1} = \max \{ \lambda_t^k + g_t^k (\bar{L} - L(\lambda_t^k))/4 \sum_{s=1}^{T} g_s^k ; 0 \} $$

where $g_t^k$ is the value of $g_t$ at the $k$-th iteration and $\bar{L}$ is the objective function of the best feasible solution obtained so far and $L(\cdot)$ is as defined earlier.

The procedure is therefore iterative, where the Lagrangian problem is solved at each iteration using an updated vector. The solution of the Lagrangian problem is obtained using a simple inspection routine.
In summary the Lagrangian relaxation procedure can be applied to any node in the branch-and-bound tree and provides a lower bound of objective function value and sometimes terminates with the optimal integer solution.

The computational efficiency of the method was found to be quite high even with problems of the order of \( T = 50 \) and \( n = 50 \), as the CPU time was about 30 seconds. In the type of problem where pipe upgrading or expansion projects are to be selected within a time period, \( n \) is at most in the order of 100 especially when work on different contiguous pipes is grouped into the same contract. However, as \( T = 5 \), the Lagrangian relaxation method seems quite appropriate for the sequencing problem, in terms of solving efficiently the mixed integer programming problem.

III. An Inference into the Organizational Issues of the Upgraded Program Associated with the Maintenance Management System

The necessity of a demand-responsive approach, as the rational way to deal with the rectification of the problem of deterioration of water distribution systems, has to be stressed, on broad economic and social grounds. Not only is it economically efficient to establish a comprehensive maintenance management system coupled with an upgrading and reconstruction program, but it is a social necessity. The reason why social well-being is at stake stems from the fact that water distribution systems, like other infrastructure facilities have finite
service lives which vary depending on a number of parameters external and internal, as explained in Chapter 4. When many components of the system start to fail, the deterioration of the system has taken place, and other failures caused by the same mechanism of reduction of the wall thickness is bound to occur more and more frequently in the same section and other sections with similar general characteristics. By delaying the establishment of an upgrading program which would sort the sections needing capital improvement measures over time, many items of critical social consequences may appear. The following scenario could possibly take place:

1. At some point in time it would seem necessary to replace a major portion of the system. However, capital improvement programs cannot be implemented in one year because of limitations in the amount of specialized labor available, too much disruption in service in other urban activities, and basic system or "structural" limitations of the network.

2. Following the realization of a critically required reshaping of a major portion of the system, the mentioned limitations will add new economic inefficiencies to the economic costs of the previous maintenance history, because of a need to defer most of the required implementations. Reliability might be quite low at that point as well, due to the high proportion of deteriorated system.

It is therefore critical to assess future needs based on past information and new collected data from field surveys as well as the understanding and experience input of maintenance managers. Future
system performance can then be predicted, leading to the estimation of financing requirements for the capital improvement measures scheduled in the future. Major bond issues might be required for each time period in order to upgrade the systems to a level consistent with a well-founded maintenance management system. These costs will have to be ultimately passed to the consumers through higher rates, but increasing rates over different periods in a (still) reasonable trend are less of a social burden than the capital financing needs (and consequently rate escalation) due to a major portion of the system needing immediate upgrading.

Applying the demand-responsive approach in the next decade is the first step towards upgrading the water distribution systems. Its benefits are short term and long term, social and economic. Within this philosophy, the scheduling and sequencing models described in Chapters 5 and 6 help derive the optimal level of spending required to bring about the necessary changes, as well as its allocation throughout the system over time.

Other organizational issues will have to be addressed in order to make the maintenance management system implementable. The allocation of projects between the private sector and the public sector has to be analyzed. Also, the delegation to the states and cities of a large part of the responsibilities of financing capital improvement programs will create more incentives to manage efficiently the resources, through better planning and budgeting procedures.
CHAPTER 7: Real-time Response to Breaks in Major Water Mains

I. The Hidden Costs of Breaks in Major Water Mains

In the previous chapters, the focus of this work was on the planning level of maintenance decision-making. The major concern was to derive the long-term and short-term scheduling of capital expenditures in water distribution systems. The choice and the timing of the capital improvement actions were based on the evolution of the different factors affecting the economic costs associated with maintaining the required levels of serviceability. The major criterion for decision-making was the economic performance under the assumption that only repair costs result from the occurrence of a break (or leak) event. This assumption is quite reasonable in most cases, partly due to the redundancy in the distribution network. However, in some major water mains, breaks are often quite infrequent but have a direct impact on the reliability of water supply. Also, besides the costs of shortages which might occur in the area served by the broken water main, other "hidden costs" can arise from damages to neighboring service facilities such as electric distribution units. An electric shortage that lasted a few days and other sizeable damages resulted from a recently reported major water main break in New York.

Unfortunately, most of these "additional" costs can only be evaluated (and rather roughly) by a direct impact analysis, often quite difficult to perform. However, the best source of data related to these hidden social costs is the actual occurrence of a break, as the impacts can be observed and recorded. The social costs incurred by a break are
then estimated under different scenarios and the alternative of replacing
the pipe can be compared to that of simply repairing the break using
the usual "band-aid" method, within the framework of the Crisis
Decision Analysis.

II. The Importance of a Probabilistic Model for Failure Rate of Major
Water Mains

In Chapter 4, a probabilistic model for the prediction of the
failure rate of a water main based on the Cox regression approach
was derived. However, as most of the computer-based Cox regression
packages are restricted to the survival analysis in the case of a
single "death" or "failure", the number of previous failures was
considered as one of the covariates in the model. Therefore, for each
single pipe, it is possible to associate a family of probability dis-
tributions on the time to failure, indexed by the number of previous
failures. In other words, if the number of previous failures in a pipe
is (n-1), the probability distribution on the time to the next failure
is \( \psi_n(t_n) \), where \( t_n \) is the time to the next failure. The expression
of \( \psi_n \) includes the vector of covariates related to the pipe (land use,
soil corrosivity, age, etc.). When a break occurs at time \( t_n \), the
probability distribution on the time to the next break is \( \psi_{n+1}(t_{n+1}/t_n) \).
At that point, it is necessary to decide on whether to replace or simply
repair the break in a rather short time frame. As shown in Chapter 4,
the analysis of a number of maintenance data sets led to the result
that the time to the next failure become "increasingly short" after
each new failure. Therefore \( \psi_{n+1} \) is more "skewed to the left" than \( \psi_n \),
as the mean interarrival time between breaks decreases as the number of
breaks increases. This result is used to show the following:
In order to compare the repair versus replacement alternatives, it is sufficient to compare the repair now, replace at the next failure alternative to the "replace now" alternatives.

III. Analytical Evaluation of Replacement and Repair Alternatives

When a break occurs in a major main, a social cost \( \tilde{S} \) is incurred from the direct impacts of the break. \( \tilde{S} \) is an uncertain quantity and can be treated as a random variable with most likely value \( S \) which takes different values depending on the associated scenario. If the break is repaired at cost \( C_b \), the most likely total cost incurred is therefore \( \tilde{C}_b = C_b + S \).

If the pipe is replaced, an additional social cost \( \tilde{S}' \) (random variable) is incurred due to the required interruption of traffic, damages to surrounding buildings and infrastructure. The most likely total cost under the replacement alternative is \( \tilde{C}_r = C_r + S' + S \), where \( C_r \) is the replacement cost and \( S' \) is the most likely value of \( S' \). The evaluation of \( S' \) and \( S \) can be undertaken within the framework of the Crisis Decision Analysis, described in the next section, using the templates associated with the pipe under study.

If a pipe with \( (n-1) \) previous failures breaks, the probability distribution of the time to the next failure is \( \psi_n(t_n) \) (the values of the covariates are included in the expression of \( \psi_n \)). Depending on the values of different templates, the decision of repairing versus replacing the pipe has to be made. For example, in a peak demand period, there might be no alternative to repairing immediately the pipe to reestablish the interrupted service. In other situations,
the evaluation of the repair and replacement decisions can be undertaken by simply comparing the "repair now, replace at next failure" alternative to the "replace now" alternative. Theoretically, the repair alternative can be considered as followed by a number of other repairs then a replacement of the pipe. The proof of the statement above can be obtained by induction, and makes the comparison of the repair and replacement alternatives a rather simple task.

Proof: The following notations are assumed:

\[ a_1 = \text{"replace now" alternative} \]
\[ a_2 = \text{"repair now, replace at next failure" alternative} \]
\[ a_3 = \text{"repair now, repair at next failure and replace at second next failure" alternative} \]

\[ \psi_{n+1}(t_{n+1}) = \text{probability distribution on the time to the (n+1)th failure, after the n}^{th} \text{ break.} \]

\[ \psi_{n+1}(t_{n+2}/t_{n+1}) = \text{probability distribution on the time to the (n+2)th failure after the (n+1)th failure, assuming that the latter has occurred at time } t_{n+1}. \quad \psi_{n+2} \text{ is conditional on } t_{n+1} \text{ as } t_{n+1} \text{ determines the age of the pipe at the (n+1)th failure.} \]

The "most likely" net present costs of the alternatives \( a_1, a_2 \) and \( a_3 \) are: under the "continuous" discounting method

\[
\text{NPC}(a_1) = \bar{c}_r
\]
\[
\text{NPC}(a_2) = \bar{c}_b + \int_0^\infty e^{-rt} \bar{c}_r \psi_{n+1}(t) \, dt
\]
\[
\text{NPC}(a_3) = \bar{c}_b + \int_0^\infty e^{-rt} \left[ \bar{c}_b + \int_0^\infty e^{-rt'} \bar{c}_r \psi_{n+2}(t'/t) \, dt' \right] \times \psi_{n+1}(t) \, dt
\]
If $\text{NPC}(a_1) \leq \text{NPC}(a_2)$, then

\[
\bar{c}_r (1- \int_0^\infty e^{-rt} \psi_{n+1}(t) dt) \leq \bar{c}_b \tag{I}
\]

Given the property discussed above concerning the compared skewness of $\psi_{n+1}$ and $\psi_{n+2}$, it is easy to show that

\[
\int_0^\infty e^{-rt'} \psi_{n+2}(t'/t_a) dt' \geq \int_0^\infty e^{-rt}\psi_{n+1}(t) dt \tag{II}
\]

where $t_a$ = actual time from the $n^{th}$ to the $(n+1)^{th}$ failure. The difference $\text{NPC}(a_3) - \text{NPC}(a_2)$ yields the following expressions:

\[
\text{NPC}(a_3) - \text{NPC}(a_2) = \int_0^\infty e^{-rt} \left[ \bar{c}_b + \bar{c}_r \left( \int_0^\infty e^{-rt'} \psi_{n+2}(t'/t_a) dt' - 1 \right) \right] \psi_{n+1}(t) dt
\]

Due to (II)

\[
\text{NPC}(a_3) - \text{NPC}(a_2) \geq \int_0^\infty e^{-rt} \left[ \bar{c}_b + \bar{c}_r \left( \int_0^\infty e^{-rt} \psi_{n+1}(t_a) dt_a - 1 \right) \right] \psi_{n+1}(t) dt
\]

Due to I,

\[
\text{NPC}(a_3) - \text{NPC}(a_2) \geq 0
\]

Thus, if $a_1 >> a_2 (>> = \text{is preferred to})$ then $a_2 >> a_3$ and by induction it can be shown similarly that $a_1 >> a_n$, which proves the required result.

IV. Replacement Versus Repair Decision-making: The Crisis Decision Analysis Framework

When a break occurs in a major water main, it is necessary
to decide in a short time frame whether to replace the pipe (and possibly expand it, if needed), or simply repair it. An approach (Crisis Decision Analysis) was suggested by Uehara, Ashley and Robinson (1981) for the selection of the most appropriate alternative under such time limitations. Originally designed to "provide responsive rational framework for making critical decisions during construction operations," it was noticed that the approach was perfectly suitable for maintenance decision-making such as earth dam maintenance. Crisis Decision Analysis allows one to utilize intuition and experience: the result is a decision which can be implemented quickly and with more confidence.

The required data set per pipe for the purposes of this analysis can be described in a number of templates which are divided into different categories depending on whether these variables are internal or external (human or site related issue), dependent or independent (can or cannot be acted on and altered). A sample of key "templates" is represented in Figure 7.1.

In the previous paragraph, the elicitation of the probability distributions $\psi_{n+1}(t_{n+1})$ and $\psi_{n+2}(t_{n+2}/t_{n+1})$ assuming (n-1) previous breaks in the pipe was derived. However, in order to compare the net present costs of the two alternatives which need to be analyzed it is necessary to estimate the values of the social costs $\tilde{S}'$ and $\tilde{S}$. $\tilde{S}'$ and $\tilde{S}$ depend on the component cost items (damages to other utilities, service interruptions, etc.) and it can be assumed that, as advocated by the Crisis Decision Analysis, three values are available on each variable: a most likely value, an "optimistic" value and
<table>
<thead>
<tr>
<th>Internal</th>
<th>External</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Peak Period at Failure Time</td>
<td>- Soil Data (corrosivity)</td>
</tr>
<tr>
<td></td>
<td>- Land Use</td>
</tr>
<tr>
<td>Independent</td>
<td>- Type and Repair Record of Pipe</td>
</tr>
<tr>
<td></td>
<td>- Vulnerability of System to Pipe Failure</td>
</tr>
<tr>
<td>Dependent</td>
<td></td>
</tr>
<tr>
<td>- Availability of Means (Funds)</td>
<td>Damage to Buildings and Utilities under Different Preventive Measures</td>
</tr>
</tbody>
</table>

Figure 7.1: Sample of Key Parameters

(Crisis Decision Analysis)
a pessimistic value.

By evaluating the two alternatives $a_1$ and $a_2$ under the "most likely" scenarios for $S'$ and $S$, the alternative that "seems" to be more economical is identified. However, it is also possible to find the changes in $S'$ and $S$ which would make the previously dominated alternative switch to the favored position. A closer examination of the break event and the general values of the templates might show such a change, and the decision is therefore adequately updated.

By complementing the analytical capabilities provided in the method of evaluating both alternatives, with the intuition and experiences of water utility planners and engineers related to the values of $S$ and $S'$, a "decision" can be made, with more confidence in its soundness.

**Interfacing the planning mode and the real-time mode**

In Chapter 5, the planning mode of maintenance decision-making was presented. The assumption of a planning cycle at the beginning of each time period was made. In each planning cycle, the expected performance of each single pipe is predicted for the remainder of the planning horizon. The outcome of the planning cycle is primarily a capital expenditure plan for the coming time period, which is then refined through the sequencing model (Chapter 6), given expected yearly budgets over the time period.

In the real-time mode, the issue of an earlier replacement measure is raised, after a break has occurred in a major water main.
The conciliation of such a measure with the budgetary constraint can be obtained by thinking of the candidate "projects" for each year derived from applying the sequencing model to the five-year plan, as possibly deferrable. As the implementation program is under way, some pipes would perform better than expected (less breaks) leading to a possibility of deferring related capital improvement measures. Such a policy of constantly updating the information on pipes would then make it possible to undertake capital measures on pipes that performed worse than expected, therefore requiring earlier replacements, as the difference between the present costs of $a_1$ and $a_2$ (seen in the previous paragraph) widens. The use of the Cox regression model makes the updating mechanism quite convenient.

The need to eventually modify the capital improvement program reinforces the importance of a well designed computerized maintenance system, where the data base on pipes is constantly updated, leading to a possible revisal of the projects that should be funded (as the projects are evaluated under the most recent information).

The interface of the planning mode and the real-time mode is therefore a rationally important part of the implementation of maintenance decision-making. It is highly facilitated by the provision of on-line computerized data management systems which can generate the necessary inputs for decision-making purposes as explained in Chapter 8.
Aging water distribution systems are subject to failures, loss of carrying capacity and other water quality related problems. The problem facing water utilities today is a complex one: how to assess the requirements in capital improvement measures? Given the budgetary constraints which will undoubtedly be imposed on such rebuilding programs, due both to the magnitude of such needs and the concurrent infrastructure programs under way, the question becomes: how to design dynamic capital improvement programs under the existing budgetary, reliability and serviceability constraints.

Even though the estimates of capital requirements for infrastructure programs could be overstated, as they are based on partial criteria such as the age of the system, the capital needs are still considerably high. The recent numbers reported by the *New York Times* (table 8.1) for capital shortages for streets, sewers and waters give a clear idea of the importance of the infrastructure problem.

The first step toward solving the problem is to institute a new approach to maintenance decision-making, which is demand responsive rather than simply "spend the budget". However, in order to assess capital requirements, a better understanding of the system is a critical issue to derive a comprehensive maintenance management system. Such a system would include:

a) A **maintenance and operation policy** consistent with the dynamic preservation of the infrastructure. Obviously, the present level of degradation is a combined result of aging systems with little past
### Streets, Sewers and Water: A Look Toward the Year 2000

1982 Projections, in billions of dollars, of what some states will need and how far each will fall short of cash goals by the turn of the century.

<table>
<thead>
<tr>
<th>State</th>
<th>Needs</th>
<th>Rank</th>
<th>Shortage</th>
<th>Rank</th>
</tr>
</thead>
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<td>Alabama</td>
<td>$12.8</td>
<td>18</td>
<td>$3.2</td>
<td>18</td>
</tr>
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<td>32.3</td>
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<td>4.3</td>
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<td>9.0</td>
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<td>13.4</td>
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<tr>
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<td>9</td>
<td>12.0</td>
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</tr>
<tr>
<td>Missouri</td>
<td>26.4</td>
<td>8</td>
<td>14.3</td>
<td>6</td>
</tr>
<tr>
<td>Montana</td>
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<td>23</td>
<td>1.7</td>
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<td>New Jersey</td>
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<td>15.1</td>
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<td>New Mexico</td>
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<tr>
<td>Washington</td>
<td>6.8</td>
<td>20</td>
<td>4.2</td>
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</tbody>
</table>

Source: Joint Economic Committee of Congress

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**TABLE 8.1:** From the New York Times, February 24, 1984
upgrading effort, but also in many cases insufficient maintenance or neglect. Such past maintenance practices are partly responsible for the large present capital requirements for water distribution infrastructure upgrading.

b) A capital improvement program designed to schedule optimally required capital intensive measure such as replacement, rehabilitation and expansion, given expected future demands, reliability and budgetary constraints.

The complexity of the task lies in designing both major components of the system, as they are interrelated. Clearly, future capital needs will depend significantly on the maintenance and operation policy, as well as the design and layout standards. While this work focuses on the scheduling of capital expenditures for water distribution system upgrading, it is important to note that the existence of an adequate maintenance operation, design and layout policy is assumed as a starting point.

Such a policy would consist of a set of standards. Some of them are already known, but often neglected. For example, operating a water treatment plant above its design capacity leads to deposits on the interior walls of the pipes, hence a loss of carrying capacity. Other standards require more analysis of the status of different systems with different maintenance practices. The cross examination of the past performance records of different systems would allow decomposition of these effects into system specific and pipe specific effects. Many of the system specific effects can be controlled, such as the water treatment methods, the operational practices, the design criteria, etc.
Understanding these effects leads to a possible correction affecting
the future performance of the existing system, such as the adequate control
of water composition, but affecting more directly new equipment layout
in terms of better materials used, better design and installation
criteria. In some poorly maintained systems, leak detection and repair
is a first necessity to limit their increasing deterioration. In any
case, monitoring the performance of the system is a major part of the
maintenance policy. Given a well-conceived maintenance and operation
policy, the question of designing a capital improvement program in a
mature deteriorated system needs to be addressed and is nowadays
getting the attention of both politicians and water utility practitioners.
The need is felt for a decision support model to help assess future
water distribution infrastructure requirements and schedule the capital
expenditures in mature water distribution systems of various sizes,
given prediction of future system performance and projections of future
demands. The major objective of this work is to provide the necessary
framework to build a decision support model that is adapted to the
specific needs of a water utility.

The assessment of future water distribution system requirements
can be achieved by building predictive models of system loss of per-
formance with associated economic costs, i.e. breakage rates and loss of
carrying capacity. Existing predictive models for pipe failure are re-
viewed and a probabilistic model is suggested, based on the notion of
survival analysis and using the Cox regression model. The recommenda-
tion of regression-based models such as presented by Shamir and Howard
(1979) for small diameter pipes, is based on the fact that pipes of this category have a relatively higher failure frequency than large diameter pipes. However, the latter category is often associated with a higher impact break event, in terms of social costs due to possible drops in the serviceability level. This justifies the use of a probabilistic model in general, and the Cox regression model in particular, as the knowledge of the probability of failure of a given pipe at any given time is directly updated under the latest level of information about the pipe. The Cox regression model was therefore recommended as the good way to approach failure statistics in low breakage frequency water mains. While system specific evolution with time of the Hazen-Williams coefficient and consequently of the loss of carrying capacity was analyzed, a predictive model that is more pipe specific could enhance the accuracy of the prediction of additional energy costs due to the loss of carrying capacity.

Based on the predictions of future failure rates and loss of carrying capacity, the economic analysis of different alternatives, replacement, rehabilitation, "do nothing", and possibly expansion, can be undertaken at the pipe level by studying the present values of the streams of cash flows under different scenarios. If replacement or rehabilitation is recommended, an optimal replacement or rehabilitation time is evaluated. The expansion alternative can also be considered at the pipe level, if the future required flows in the pipes are separately obtained from another network flow model which is often the case in deriving an expansion strategy in a real situation.
Two different levels of maintenance decision-making are addressed: the planning level and the real-time operational level:

a) A planning process for the scheduling of capital improvement measures is suggested while the major focus of this work is on mature systems. The possibility of demand uncertainties is considered in the planning process. The process is iterative, as a planning cycle is performed every time period under the most recent level of information on expectations of future system performance and demands. The major module in the process is the scheduling model of capital expenditures at the network level.

A mixed-integer programming model is examined with a network substructure. The underlying assumption is that the expansion alternative has not been considered at the pipe level. Therefore, flow variables, as well as continuity constraints might increase the dimensionality of the problem in a real situation. A closer look at the problem shows that a more practical approach is to evaluate the optimal measure and its timing at the pipe level (including the expansion alternative), and group different pipe projects into bundles. The design of such bundles is discussed and criteria for bundle selection are derived. Two different capital budgeting models are then suggested, operating on these bundles: an integer programming model (but with a significantly reduced problem dimension) and a linear programming model. The former is more applicable to small-to-medium sized systems, while the latter is mostly meant for large-scale systems, with large bundles of projects.
The outputs of the scheduling models are, but for the linear programming model mentioned last, five-year plans which specify the projects to implement within the time period. Projects related to the earliest time period are then sequenced under the year capital improvement budget constraints. The solution approach to this integer programming problem is a Lagrangean relaxation method, particularly efficient for this specific problem structure and dimension.

The previous scheduling/sequencing models can be interfaced with a relational data base management system used to store and update the pipe records and a financial analysis module, to constitute a comprehensive set of tools for the purposes of maintenance planning.

b) The real-time operational level of decision-making is concerned with the reevaluation of the replacement alternative after a break has occurred in a major water main. Under the updated hazard rate, the replacement alternative might seem more attractive as the probability of failure in any given year increases when the number of previous failures increases. The evaluation of the replacement alternatives versus the simple repair alternative is undertaken within the framework of the Crisis Decision Analysis, a technique particularly convenient for analyzing decision in a short time frame.

In summary, this work provides a water distribution planner with a framework for the design of a decision support model including data analysis methods for prediction purposes, as well as decision models for planning and real-time break decision-making. Using this integrated set of tools, the capital improvement needs can be assessed
and the timing of the required measures under the budgetary constraints can be derived for different pipes in the network over the planning horizon.

Recommendations for further research

Many elements of water distribution system maintenance and upgrading still need to be thoroughly investigated. In particular, a model for the prediction of head losses in different pipes is needed. The effect of different system specific and pipe specific variables should be analyzed and a predictive model for the evolution of the head loss per unit length of pipe should be derived for any given system. The economic analysis of both the replacement and rehabilitation alternatives would be significantly enhanced by the availability of such a model. Better decision-making at the single pipe level would result from this added accuracy in the economic analysis.

The definition of a comprehensive maintenance and operation policy aimed at both the preservation of the infrastructure and the provision of reliable water distribution to consumers, is still required. The maintenance standards needed to meet these objectives need to be better articulated, in terms of both the routine monitoring of the system and the preventive measures such as the detection and repair of leaks. Most importantly, the generalization of these standards to poorly maintained systems and those serving small localities would limit the deterioration of the facilities.
More awareness of the means and levels of effort required to meet the aforementioned objectives should be achieved in the near future, as a major step toward instituting a maintenance management system.

On the implementation side, more effort is required to analyze work productivity in both in-house and contracted jobs, new technologies and materials and their cost estimates. This can lead to superior and more economical construction processes. Also, on the contracting side, more research is needed to estimate the economies of scale from different types of bundling, as well as to evaluate the optimal level of bundling needed to capture these economies of scale. Better planning of capital improvement programs would result from this research.

But besides these research directions on both the technical and planning levels, many organizational issues need to be addressed such as the questions of who will finance the capital improvement programs, and what are the best financing mechanisms, and also who will implement these measures and how to allocate the tasks between the public and private sectors. The right organizational and financial structures are necessary to make feasible the task of running a comprehensive maintenance management system and rebuilding the deteriorated water distribution infrastructure.
Appendix AI

Foundations of the Cox Regression Method and Its Application to Pipe Failure

1. Survivor Function

For a given pipe, denoting by $T$ a random variable representing the failure time, the survivor function $S(t)$ is defined as:

$$ S(t) = \text{Prob}(T \geq t) $$

If $f(t)$ is the probability function of the failure time and $F(t)$ the cumulative probability, then, in general:

$$ F(t) = 1 - S(t) $$

2. Hazard Function

Let $\lambda(t)$ be the hazard or age-specific failure rate. That is, $\lambda(t)$ is defined as:

$$ \lambda(t) = \lim_{\Delta t \to 0} \frac{\text{Prob}(t \leq T \leq t + \Delta t | t \leq T)}{\Delta t} $$

It can be shown that in the continuous case

$$ \lambda(t) = \frac{f(t)}{S(t)} $$

The failure time may be continuous or discrete. For the purposes of this problem, there is a need to define the probability of failure in a given year. Thus, the assumption of discrete possible failure times (yearly) is quite convenient for the analysis of the usual data records at hand concerning previous repair events.
If $T$ is discrete, then:

$$\lambda(t) = \sum_{u_j} \lambda_{u_j} \delta(t-u_j)$$

where $\delta(t)$ denotes the Dirac delta function and

$$\lambda_t = \text{Prob}(T = t | T \geq t)$$

Hence, under the above assumption of a discrete failure time, the survivor function is:

$$S(t) = \prod_{u_j < t} (1 - \lambda_{u_j})$$

otherwise, if $\lambda(t)$ is integrable (particularly, if $T$ is continuous) then, the survivor function

$$S(t) = \exp \left\{ - \int_{0}^{t} \lambda(u)du \right\}$$
3. An Example of the Hazard and Survivor Functions

In the case of \( n_0 \) identical and independent entities (with the same hazard and survivor functions), it is interesting to derive the hazard function and survivor function to illustrate the types of insights that the historical data can provide.

Assuming that \( n \) entities have failed \( (n \leq n_0) \) in the past and that the observed failure times are \( t(1), t(2), \ldots, t(k) \), \( t(1) < t(2) \ldots < t(k) \).

The multiplicity of \( t(i) \) is defined as the number of entities that have failed at time \( t(i) \). It is denoted by \( m(i) \).

The set of entities "at risk" at time \( t-0 \), i.e., just prior to \( t \) is denoted by \( R(t) \). It includes the entities which have failed at a time greater or equal to \( t \), or have not failed. The number of entities in \( R(t(i)) \) is defined as \( r(i) \).

Under such previous conditions, it is easy to derive the estimate for the hazard function \( \hat{\lambda}(t) \) as:

\[
\hat{\lambda}(t) = \sum_{i=1}^{k} \frac{m(i)}{r(i)} \delta(t-t(i))
\]

Correspondingly, the survivor function \( \hat{S}(t) \) estimated is:

\[
\hat{S}(t) = \prod_{t(i) < t} \left( 1 - \frac{m(i)}{r(i)} \right)
\]

Such aforementioned functions are maximum likelihood estimates obtained from the available sample. (Kalbfleisch and Prentice (1980)).
4. The Cox Regression Model

a. General model formulation

In the problem of interest, a number of measurements are available say on variables $z_1, z_2, \ldots, z_p$, associated with each single entity or pipe. These variables are called covariates. For example, for the $j^{th}$ entity, let the values of $Z$ be $z_j=(z_{1j}, \ldots, z_{pj})$.

The hazard function model generally assumed is the proportional hazards model (Cox, 1972), which specified that:

$$
\lambda(t; z) = \lambda_0(t) e^{z\beta}
$$

where: $\lambda_0(t)$ is an arbitrary unspecified base-line hazard function for continuous failure time $T$.

$$
z = (z_1, \ldots, z_p) \text{ is the vector of covariates.}
$$

In this model, the covariates act multiplicatively on the hazard function. If $\lambda_0(t) = \lambda(\text{constant})$, the model reduces to the exponential regression model. The Weibull model is another special case when $\lambda_0(t) = \lambda u(\lambda t)^{u-1}$, $u>0$.

The conditional density of $T$ given $z$, under the general model formulation, and in the continuous case is

$$
f(t; z) = \lambda_0(t) e^{z\beta} \exp \left[ -e^{z\beta} \int_0^t \lambda_0(u)du \right]
$$
The conditional survivor function for $T$ given $z$ is:

$$S(t; z) = \left[ S_0(t) \right]^{\exp(z\beta)}$$

where

$$S_0(t) = \exp \left[ - \int_0^t \lambda_0(u) du \right]$$

It is clear that, in the general proportional hazards model where no restriction is imposed on $\lambda_0(.)$, no direct relationship between $z$ and $t$ itself is postulated. This assumption is quite adequate for the purposes of the pipe failure analysis, as the covariates are assumed to bear no direct relationship with the failure time.

However, another possible general model is the accelerated failure time model where the hazard function is

$$\lambda(t; z) = \lambda_0(t e^{-z\beta}) e^{-z\beta}$$

As noted by Kalbfleisch and Prentice (1980), "both the general classes of models described would provide sufficient flexibility for most purposes if methods were available that would not require undue restrictions on the nuisance functions $\lambda_0(.)$". These authors underline the advantage of the proportional hazards model, which stems from the availability of methods of inference about $\beta$. The Cox regression is basically one such method and is based on the partial likelihood notion. Other methods described by Kalbfleisch
and Prentice such as the one related to the marginal likelihood, yield approximately the same estimates. The method proposed by Cox is the most valuable one, for the purposes of this work and is presented hereafter.

b. Estimation of the proportional hazard function of Cox's regression

The purpose of this section is to show how \( \beta \) and \( \lambda_0(.) \) can be estimated, given a sample of failure time data from the population of entities considered. Both the continuous and the discrete analysis cases are described.

**Estimation of the Parameter Set \( \beta \)**

Given that \( \lambda_0(t) \) is arbitrary in the proportional hazards model, no information can be contributed about \( \beta \) by time intervals in which no failures occur, as the component \( \lambda_0(t) \) might conceivably be identically zero in such intervals. Thus, the set of instants \( \{ t_{(i)} \} \) at which failures occur is used to construct a conditional likelihood. In the discrete case, the observed multiplicities \( m_{(i)} \) of \( t_{(i)} \) is also used in the expression of the conditional likelihood.

1. **The Continuous Case**

For a particular failure at time \( t_{(i)} \), the risk set being \( R(t_{(i)}) \), the probability that the failure is on the specific entity with covariates \( z_{(i)} \), conditionally on the risk set, is:

\[
L_{1}(\beta) = \exp \{ z_{(i)} \beta \} / \sum_{1 \in R(t_{(i)})} \exp \{ z_{(i)} \beta \}
\]
Thus, each failure contributes a factor of this type in the conditional likelihood. If \( k \) failures are observed associated with entities having covariates \( z^{(i)} \), the likelihood

\[
L(\beta) = \prod_{i=1}^{k} L^{(i)}(\beta)
\]

The maximum-likelihood estimator of \( L \) is obtained by maximizing the Log-likelihood denoted as \( L(\beta) \).

The expression of \( L(\beta) \) is evaluated as:

\[
L(\beta) = \sum_{i=1}^{k} z^{(i)} \beta - \sum_{i=1}^{k} \log \sum_{\ell \in R(t^{(i)})} \exp \{ z^{(\ell)} \beta \}
\]

If \( p \) covariates are assumed and \( \xi \) and \( \eta \) are running indexes from 1 to \( p \), then the maximization of \( L(\beta) \) requires that the derivative of \( L \) with regard to each \( \beta_{\xi} \) be set to zero.

The expressions of the first order and second order derivatives are given as:

\[
U_{\xi}(\beta) = \frac{\partial L(\beta)}{\partial \beta_{\xi}} = \sum_{i=1}^{k} \{ z^{(i)} - A_{\xi_{1}}(\beta) \}
\]

where

\[
A_{\xi_{1}}(\beta) = \frac{\sum_{\ell \in R(t^{(i)})} \exp(z^{(\ell)} \beta)}{\sum \exp(z^{(\ell)} \beta)}
\]

the sums being over \( \ell \in R(t^{(i)}) \).
Similarly, the second order derivative

\[
I_{E \eta} (\beta) = \frac{\partial^2 L(\beta)}{\partial \beta E \partial \eta} = \sum_{i=1}^{k} C(\xi_{\eta_i}) (\beta),
\]

where

\[
C(\xi_{\eta_i}) (\beta) = \left\{ \frac{\sum_{\ell \in R(t_{(i)}, m_{(i)})} \exp(z_{\ell} \beta) / \sum \exp(z_{\ell} \beta)}{\sum_{\ell \in R(t_{(i)}, m_{(i)})} \exp(s_{(i)} \beta)} - A(\xi_{\eta_i}) (\beta) A(\eta_{\eta_i}) (\beta) \right\}.
\]

The maximum likelihood estimate of \( \beta \) is obtained using the Newton-Raphson technique which will be described after the discrete case is considered.

2. The Discrete Case

In the discrete case, some slight modifications need to be mentioned in the expression of the conditional likelihood. In particular, the contribution to the likelihood related to the failure time \( t_{(i)} \) is

\[
\left\{ \exp \{ s_{(i)} \beta \} / \sum_{\ell \in R(t_{(i)}, m_{(i)})} \exp \{ s_{(\ell)} \beta \} \right\}
\]

where \( s_{(i)} \) is the sum of the covariate vector \( z \) over all the entities failing at \( t_{(i)} \) and the notation in the denominator means that the sum is taken over all distinct sets of \( m_{(i)} \) entities drawn from \( R(t_{(i)}) \).
Thus the total conditional log likelihood now becomes:

\[
\sum_{i=1}^{k} s_1(\beta) = \sum_{i=1}^{k} \log \left[ \sum_{\xi \in \mathcal{R}(t_1 \iota_1) \cap m(\iota_1)} \exp \left\{ s_2(\xi) \right\} \right]
\]

The derivation can be calculated as previously, for instance

\[
\frac{\partial L(\beta)}{\partial \xi} = \sum_{i=1}^{k} s_1(\xi) - \sum_{i=1}^{k} \frac{\exp \left\{ s_2(\xi) \right\} \times s_1(\xi)}{\sum \exp \left\{ s_2(\xi) \right\}}
\]

Following is the description of the Newton-Raphson technique which allows one to derive the maximum likelihood estimator of \( \beta \).

**The Newton-Raphson Technique**

The Newton-Raphson Technique is used to derive the maximum likelihood estimates \( \hat{\beta} \) of the parameter vector \( \beta \). The methodology is based on the first order Taylor series expansion of the Log likelihood \( U(\beta) = \partial \log(L(\beta)) / \partial \beta \), which was denoted previously as \( \frac{\partial L(\beta)}{\partial \beta} \).

Given a trial value \( \beta \), the expansion of \( U(\hat{\beta}) \) can be written as

\[
U(\hat{\beta}) = U(\beta_0) - I(\beta)(\hat{\beta} - \beta_0)
\]

where, \( I(\beta) \) is the "covariance" matrix previously seen and \( \beta \) is "between" \( \beta_0 \) and \( \hat{\beta} \). For \( \beta_0 \) in the vicinity of \( \hat{\beta} \), \( I_0(\beta) \) approximately equals \( I(\beta_0) \), so that setting \( U(\beta) = 0 \) (maximum
likelihood estimator) and solving gives

$$\hat{\beta} = \beta_o + I(\beta_o)^{-1} U(\beta_o)$$

where $I(\beta_o)^{-1}$ is the inverse matrix of $I(\beta_o)$

The right side of the previous equation gives a new trial value for $\beta$ with which the process is repeated until successive $\beta$ estimates are close enough to verify $U(\beta) = 0$ (at convergence of the algorithm).

This method usually leads to the convergence result, unless the likelihood is multimodal.

Thus, using this technique, it is possible to derive the maximum likelihood estimates $\hat{\beta}$ of $\beta$, with enough accuracy. Once $\beta$ is obtained, it is necessary to estimate $\lambda_o(.)$, in order to define completely the hazard function, and subsequently the survivor function.

**Estimation of the hazard and survivor functions**

Once the maximum-likelihood estimator of $\beta$ has been obtained, it is possible to estimate $\lambda_o(.)$. To do this, $\lambda_o(t)$ is taken to be identically zero, except at the points where failures have occurred, and a separate maximum-likelihood estimation is carried out at each such failure point. For the latter it is convenient to write the contribution to $\lambda_o(t)$ at $t_{(i)}$ in the form (Cox, 1972):
\[
\frac{\Pi_{(1)} \exp(-\beta \hat{z}_{(1)})}{1 - \Pi_{(1)} + \Pi_{(1)} \exp(-\beta \hat{z}_{(1)})} \delta(t-t_{(1)})
\]

where \( \hat{z}_{(1)} \) is an arbitrary constant to be chosen. As stated by Cox (1972), it is useful to take \( \hat{z}_{(1)} \) as approximately the mean in the relevant risk set. The maximum-likelihood estimate of \( \Pi_{(1)} \) can then be shown to satisfy:

\[
\hat{\Pi}_{(1)} = \frac{m(i)}{r(i)} - \frac{\hat{\Pi}_{(1)} (1 - \hat{\Pi}_{(1)})}{r(i)} \sum_{j \in R(t_{(1)})} \exp(\beta(z_{(1)} - \hat{z}_{(1)}) - 1 \frac{\exp(\beta(z_{(1)} - \hat{z}_{(1)}) - 1}{1 - \hat{\Pi}_{1} + \hat{\Pi}_{1} \exp(\beta(z_{(1)} - \hat{z}_{(1)})}
\]

where \( m(i) \) = multiplicity of \( t_{(1)} \) and \( r(i) \) = cardinal of \( R(t_{(1)}) \).

This estimate can be solved iteratively. The suggested choice of \( \hat{z}_{(1)} \) is designed to make the number of needed iterations small.

Once \( \lambda_{0}(t) \) has been estimated, the survivor function can be estimated as:

\[
\hat{S}_{o}(t) = \prod_{t_{(1)} < t} \frac{1 - \hat{\Pi}_{(1)} \exp(-\beta \hat{z}_{(1)})}{1 - \hat{\Pi}_{(1)} + \hat{\Pi}_{(1)} \exp(-\beta \hat{z}_{(1)})}
\]

for \( z = 0 \), \( \hat{S}_{o}(t) \) is obtained by replacing \( \exp(-\beta \hat{z}_{(1)}) \) by \( \exp(\beta(z - \hat{z}_{(1)}) \) for \( z \) non zero, which is the case of interest as the vector of covariates can take a number of different values.
c. An alternative method to account for multiple failure

The case of multiple failures for each single entity can be handled directly instead of considering the number of previous failures as a covariate. For this, the failure rate or hazard function for the $k^{th}$ failure is defined as:

$$\lambda^k(t, Z_k(t))$$

where $Z_k(t)$ is the vector of covariates when the $k^{th}$ failure is considered. $\lambda^k(\cdot)$ takes value zero up to $t_{k-1}$.

The same method of conditional likelihood described in the preceding section can be generalized.

Typically, for a given entity which failure times are $t_1, t_2, \ldots, t_r$ the probability of no failure in $(0, t_1)$ is equal to

$$\exp\left[-\int_0^{t_1} \lambda^1(t, Z_1(t)) \, dt\right]$$

Given that there are no failures in $(0, t_1)$, the probability element of a failure in $(t_1, t_1+dt)$ is $\lambda^1(t_1, Z_1(t_1)) \, dt$. Thus the likelihood contribution of the $r$ failures in the continuous case is

$$\prod_{k=1}^{r} \exp\left\{ -\int_{t_{k-1}}^{t_k} \lambda^k(t, Z_k(t)) \, dt \right\} \left[ \lambda^k(t_k, Z_k(t_k)) \delta_k \right]$$

where $\delta_k = 1$ for $k=1, \ldots, r-1$

$\delta_r = 0$ if $t_r$ is a censored failure time

$\delta_r = 1$ otherwise
Appendix AII

AN APPLICATION OF THE COX REGRESSION MODEL

A data set from the New Haven system including 1,428 pipes whose diameters ranged between 6 and 48 inches was analyzed within the context of the Cox regression model, as described in Chapter 4. The frequency distributions of the ages in 1983, the previous number of breaks, the diameter, the length and the pressure over the data set are represented in figures AII-1 to AII-5.

1. General Observations

Diameter: 11 discrete diameters, range 6-48", most frequent diameter 12", 81% of diameters in the range 8" - 16".

Length: Range 100-14,000 feet. Most frequent lengths: 500 ft, 740, 1000, 1500 and 1750.

Breaks: Only 292 pipes had a break (14.5% of pipes) out of 1391 pipes. 202 of them has only one break. Only 3.7% of all pipes had 2 breaks or more. Sparse data beyond 2 breaks.

Year of Pipe Installation: 46% of pipes were installed in 1930-1935. Small clusters in other years were observed.

Type of Pipes: 96% are made from iron (1336 pipes). There are only 20 concrete pipes and 30 "other" pipes with no defined material.
2. **Categorization of the Data Set According to the Period Pipes Installed**

Since pipes were installed in clusters in different time periods, reflecting different construction and maintenance practices and also materials used, it was decided to perform the statistical analysis of every separate set of pipes corresponding to discrete historical periods of pipes' installation. Pipes were divided into 6 groups corresponding to six, more or less discrete, historical periods of installation. These different periods of installation were used as the strata of the Cox regression model derived further.

The six clusters were:

1. **Pipes installed between 1900 – 1930**
   
   Total number of pipes = 136 pipes
   
   . Under corrosive soil (corr = 1) (24 pipes)
     
     63% broke at least once
   
   . Under non-corrosive soil (corr = 0) (112 pipes)
     
     53% broke at least once

2. **Pipes installed between 1930 – 1935**
   
   Total number of pipes = 639 pipes
   
   . Under corrosive soil (165 pipes)
     
     15% broke at least once
   
   . Under non-corrosive soil (474 pipes)
     
     16% broke at least once

3. **Pipes installed between 1935 – 1945**
   
   Total number of pipes = 145 pipes
   
   . Under corrosive soil (54 pipes)
     
     17% broke at least once
   
   . Under non-corrosive soil (91 pipes)
     
     27% broke at least once
4. Pipes installed between 1945 - 1957

Total number of pipes = 152 pipes

- Under corrosive soil (61 pipes)
  18% broke at least once
- Under non-corrosive soil (91 pipes)
  13% broke at least once.

5. Pipes installed between 1958 - 1965

Total number of pipes = 178

- Under corrosive soil (71 pipes)
  11% broke at least once
- Under non-corrosive soil (107 pipes)
  21% broke at least once

6. Pipes installed between 1965 - 1983

Total number of pipes = 178

- Under corrosive soil (69 pipes)
  9% broke at least once
- Under non-corrosive soil (119 pipes)
  19% broke at least once.

As can be seen in these results, corrosivity alone does not explain relatively higher failure rates.
3. Application of the Cox Regression Model to the New Haven Data Set

The sample of data described above was analyzed using the Cox regression approach. The postulated model for the hazard rate was therefore:

\[ \lambda_j(t) = \lambda_{0j}(t)e^{\beta z} \quad j = 1, \ldots, 6 \]

\( j \) being the stratum index

The components of the vector of covariates were:

1) The diameter (DIA) which ranged between 6" and 48"

2) The length (LENGTH) which ranged between 100 and 14,000 feet.
   This variable was expressed in thousands of feet

3) The corrosivity (CORR) which took two values
   \[ \text{CORR} = 1 \quad \text{for corrosive soils} \]
   \[ \text{CORR} = 0 \quad \text{for non-corrosive soils} \]

4) The soil stability/with three values
   \[ S = 0 \quad \text{for unstable soil} \]
   \[ S = 1 \quad \text{for moderately stable soil} \]
   \[ S = 2 \quad \text{for stable soil} \]

5) The internal pressure (PRESSURE) which was divided by 10

6) The percentage of the pipe covered with low development land (LOW) (varying from 0 to 100)

7) The previous number of breaks (PREVBRK) ranging from between 0 and 3.
The descriptive statistics for these covariates are given in Table AII-1.

The analysis of the data yielded estimates for the coefficients $\beta_i$ associated with the different covariates after a number of iterations of the Newton-Raphson algorithm. The values of these coefficients and their exponentials are given in Table AII-2. Also, shown in that table are the standard errors on these coefficients and their ratios to their standard errors. The higher these ratios, the better the estimate.

As can be noticed in that table, the coefficient related to the diameter is negative, showing a lower likelihood of breakage for increasing diameters, other covariates being equal. Length, corrosivity, pressure and previous number of breaks had each a positive coefficient which is quite an expectable result. Also, soil stability and percent of low development land were associated negative coefficients.

The statistical significance of the results shown in Table AII-2 is exhibited in Table AII-3. The p-value, i.e., the smallest $\alpha$ level of confidence for which the hypothesis $\beta_i=0$ vs. $H_1$ can be rejected, increases as more covariates are entered. In particular, the entry of the covariate "Diameter" worsens the p-value, which makes the choice of the diameter as a covariate rather questionable.

The estimated survival function for different strata is plotted in figure AII-6, thus showing the trends of variation of the survival function based on the different basic hazards.
Figures AII-7 and AII-8 are statistical diagnoses plots. AII-7 represents the variation of the Logarithm of minus log (Survival function) with time for different strata. If the resulting curves have approximately constant differences over time, the stratum can be incorporated as a covariate in the general proportional hazards model. In this case, the differences do not fluctuate significantly, which shows that the date of installation can be considered as a covariate.

Figure AII-8 is a plot of the cumulative hazard function of the residuals verses the residuals for different strata. The closer the curve associated with a given stratum is to the 45° diagonal, the better the fit for this specific stratum. Stratum A seems therefore to provide the best fit.

Finally as the date of installation was found to be a good candidate for being in the covariate set, dummy variables related to different installation periods were added as covariates, taking the value 1 if the pipe was installed during that period. Figure AII-9 gives the goodness-of-fit plot related to this unstratified case, which seems to be quite satisfactory.
### TABLE AII-1: DESCRIPTIVE STATISTICS FOR COVARIATES IN THE DATA SET

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>MINIMUM</th>
<th>MAXIMUM</th>
<th>MEAN</th>
<th>STANDARD DEVIATION</th>
<th>SKEWNESS</th>
<th>KURTOSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 DIA</td>
<td>6.0000</td>
<td>48.0000</td>
<td>14.6125</td>
<td>7.6785</td>
<td>2.00</td>
<td>7.12</td>
</tr>
<tr>
<td>3 LENGTH</td>
<td>4.0000</td>
<td>17.3000</td>
<td>7.9899</td>
<td>2.0164</td>
<td>0.67</td>
<td>5.10</td>
</tr>
<tr>
<td>7 CORR</td>
<td>0.0</td>
<td>1.0000</td>
<td>0.2894</td>
<td>0.4536</td>
<td>0.93</td>
<td>1.86</td>
</tr>
<tr>
<td>8 S</td>
<td>0.0</td>
<td>2.0000</td>
<td>0.6363</td>
<td>0.7991</td>
<td>0.75</td>
<td>1.96</td>
</tr>
<tr>
<td>10 PRESSURE</td>
<td>0.0</td>
<td>100.0000</td>
<td>36.5551</td>
<td>46.7830</td>
<td>0.56</td>
<td>1.37</td>
</tr>
<tr>
<td>11 LOW</td>
<td>0.0</td>
<td>3.0000</td>
<td>0.4449</td>
<td>0.8569</td>
<td>1.96</td>
<td>5.76</td>
</tr>
</tbody>
</table>

### TABLE AII-2: ESTIMATES OF THE β-VALUES

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>COEFFICIENT</th>
<th>STANDARD ERROR</th>
<th>COEFF./S.E.</th>
<th>EXP(COEFF.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 DIA</td>
<td>-0.0025</td>
<td>0.0060</td>
<td>-0.4078</td>
<td>0.9975</td>
</tr>
<tr>
<td>3 LENGTH</td>
<td>0.1770</td>
<td>0.0224</td>
<td>7.9012</td>
<td>1.1938</td>
</tr>
<tr>
<td>7 CORR</td>
<td>0.2875</td>
<td>0.1820</td>
<td>1.5801</td>
<td>1.3331</td>
</tr>
<tr>
<td>8 S</td>
<td>-0.3290</td>
<td>0.1129</td>
<td>-2.9141</td>
<td>0.7196</td>
</tr>
<tr>
<td>10 PRESSURE</td>
<td>0.0493</td>
<td>0.0223</td>
<td>2.2103</td>
<td>1.0505</td>
</tr>
<tr>
<td>11 LOW</td>
<td>-0.0039</td>
<td>0.0031</td>
<td>-3.0919</td>
<td>0.9961</td>
</tr>
<tr>
<td>20 PREVBRK</td>
<td>0.6258</td>
<td>0.0470</td>
<td>13.3171</td>
<td>1.8697</td>
</tr>
</tbody>
</table>

### TABLE AII-3: SIGNIFICANCE OF RESULTS

<table>
<thead>
<tr>
<th>STEP NO</th>
<th>VARIABLE ENTERED</th>
<th>IMPROVEMENT CHI-SQUARE</th>
<th>P-VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20 PREVBRK</td>
<td>206.169</td>
<td>0.0</td>
</tr>
<tr>
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TABLE AII-3: SIGNIFICANCE OF RESULTS
FIGURE AII-1: DATA DISTRIBUTION BY AGE IN 1983
FIGURE AII-2: DATA DISTRIBUTION BY NUMBER OF BREAKS
FIGURE AII-3: DATA DISTRIBUTION BY DIAMETER
FIGURE AII-4: DATA DISTRIBUTION BY LENGTH
FIGURE AII-5. DATA DISTRIBUTION BY PRESSURE
Soil Corrosivity: Assuming corrosive soil to be defined by corr=1, we observe that 69% of soil is non-corrosive and 31% corrosive.

Soil Stability: 53% of pipes are installed in unstable soil. 23% are in moderately stable soil.

Pressure: Pressure heads have 400 unique values. Only 13% of pipes had pressure greater than 100.

Land Development:

511 pipes covered 100% by minimal land development
629 pipes covered 100% by moderate land development
115 pipes covered 100% by maximum land development
136 pipes covered by a mixture of the above

Swamp:

95% of pipes are not covered by swamp
1% of pipes is completely covered by swamp

Time to First repair:

For 50% of the pipes which broke, the break occurred after the 20th year
For 4.8% of the pipes which broke, the break occurred after the 1st year
For 10% of the pipes which broke, the break occurred after the 2nd year.

Although the time to subsequent repairs decreases rapidly with the number of breaks, we have very few observations after the third break to derive any substantially significant results after that point.
FIGURE AII-6: BASIC SURVIVAL FUNCTION FOR DIFFERENT STRATA

FIGURE AII-7: LOG MINUS LOG SURVIVAL FUNCTION FOR DIFFERENT STRATA
FIGURE AII-9. GOODNESS-OF FIT TEST FOR THE UNSTRATIFIED CASE
THE PROBLEM IS FORMULATED AS:

\[ \begin{align*}
& \text{MIN} \{ 5.73A7 + 26.85A8 + 63.10A9 + 114.29A10 + 180.23A11 \\
& + 260.78A12 + 355.85A13 + 465.37A14 + 589.33A15 + 727.73A16 \\
& + 860.60A17 + 1048.04A18 + 1230.14A19 + 1427.06A20 \\
& + 3.44B10 + 19.68B11 + 48.51B12 + 89.74B13 + 143.24B14 \\
& + 208.88B15 + 286.58B16 + 376.28B17 + 477.97B18 + 591.64B19 \\
& + 717.32B20 + 3.69C15 + 19.27C16 + 46.52C17 + 85.28C18 \\
& + 135.43C19 + 196.84C20 + 7.41D5 + 27.41D6 + 58.9D7 + 101.16D8 \\
& + 153.44D9 + 215.03D10 + 285.27D11 + 363.53D12 + 449.23D13 \\
& + 541.58D14 + 640.52D15 + 745.33D16 + 855.56D17 + 970.78D18 \\
& + 1090.58D19 + 1214.58D20 + 5.8E10 + 20.91E11 + 44.53E12 \\
& + 76.14E13 + 115.16E14 + 161.08E15 + 213.40E16 + 271.66E17 \\
& + 335.42E18 + 404.28E19 + 477.85E20 + 5.75F15 + 20.03F16 \\
& + 42.23F17 + 71.82F18 + 108.28F19 + 151.11F20 + 2.51G7 \\
& + 21.59G8 + 57.85G9 + 112.0G10 + 184.89G11 + 277.49G12 \\
& + 390.91G13 + 526.4G14 + 685.36G15 + 869.43G16 + 1094.76G17 \\
& + 1334.48G18 + 1605.23G19 + 1909.41G20 + 2.99H10 + 19.91H11 \\
& + 51.25H12 + 97.66H13 + 159.82H14 + 238.56H15 + 323.94H16 \\
& + 438.78H17 + 573.40H18 + 729.14H19 + 907.5H20 + 1.99115 \\
& + 16.81I6 + 44.91I17 + 86.86I18 + 143.31I19 + 158.56I20 \\
& + 5.56J8 + 22.98J9 + 51.70J10 + 91.19J11 + 140.94J12 \\
& + 200.48J13 + 269.37J14 + 347.19J15 + 433.55J16 + 528.09J17 \\
& + 630.47J18 + 740.35J19 + 857.45J20 \} \\
& \text{SUCH THAT :} \end{align*} \]

A) EXCLUSIVITY CONSTRAINTS:

\[ \begin{align*}
A7 + A8 + A9 + A10 + A11 + A12 + A13 + A14 + A15 + A16 + A17 \\
+ A18 + A19 + A20 &= 1 \\
B10 + B11 + B12 + B13 + B14 + B15 + B16 + B18 + B19 + B20 &= 1 \\
C15 + C16 + C17 + C18 + C19 + C20 &= 1 \\
D5 + D6 + D7 + D8 + D9 + D10 + D11 + D12 + D13 + D14 + D15 + D16 \\
+ D17 + D18 + D20 &= 1 \\
E10 + E11 + E12 + E13 + E14 + E15 + E16 + E17 + E18 + E19 + E20 &= 1 \\
F15 + F16 + F17 + F18 + F19 + F20 &= 1 \\
G7 + G8 + G9 + G10 + G11 + G12 + G13 + G14 + G15 + G16 + G17 + G18 \\
+ G19 + G20 &= 1 \]
H10 + H11 + H12 + H13 + H14 + H15 + H16 + H18 + H19 + H20 = 1

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J8 + J9 + J10 + J11 + J12 + J13 + J14 + J15 + J16 + J17 + J18
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B) BUDGET CONSTRAINTS:

5640.475 < 271.0

5622.046 < 377.965 < 311.0

5622.046 < 377.965 < 311.0

5622.046 < 377.965 < 311.0

5882.3D8 - 417.71D7 - 417.71D6 - 417.71D5 - 3732.92A8 - 267.08A7
+ 2606CB - 193.89GB + 4511.5J8 < 389.0

-295.16A8 - 295.16A7 + 2574.73G9 - 225.27G8 - 225.27G7 + 4490.57J9
-309.43J8 < 437.0

5538.36D10 - 461.64D9 - 464.64D8 - 464.64D7 - 464.64D6 - 464.64D5
+ 3673.8A10 - 326.2A9 - 326.2A8 - 326.2A7 + 2538.27G10 - 261.73G9
- 261.73G8 - 261.73G7 + 4468.14J10 - 331.86J9 - 331.86J8 + 3760.8B10
+ 5639.92E10 + 2631.5H10 < 489.0

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- 485.31D5 - 3639.49A11 - 360.51A10 - 360.51A9 - 360.51A8 - 360.51A7
+ 2495.92G11 - 304.08G10 - 304.08C9 - 304.08C8 - 304.08C7 + 4444.1J11
-355.93J10 - 355.93J9 - 355.93J8 + 3735.64B11 - 264.36B10 + 5621.46E11
- 378.54E10 + 2604.22H11 - 195.78H10 < 555.0

- 510.19D6 - 510.19D5 + 3601.57A12 - 398.43A11 - 398.43A10 - 398.43A9
- 398.43A8 - 398.43A7 + 2446.7G12 - 353.3C11 - 353.3C10 - 353.3C9
- 353.3C8 - 353.3C7 + 4418.27J12 - 381.73J11 - 381.73J10 - 381.73J9
- 381.73J8 + 3707.83B12 - 292.17B11 - 292.17B10 + 5602.85E12 - 397.15E11
- 397.15E10 + 25'2.53H12 - 227.47H11 - 227.47H10 < 624.0

5463.65D13 - 536.35D12 - 536.35D11 - 536.35D10 - 536.35D9 - 536.35D8
- 536.35D7 - 536.35D6 - 536.35D5 + 3559.67A13 - 440.33A12 - 440.33A11
- 322.98B12 - 322.98B11 - 322.98B10 + 5581.65E13 - 418.35E12 - 418.35E11
- 418.35E10 + 2535.72H13 - 264.26H12 - 264.26H11 - 264.26H10 < 698.0

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- 563.85D8 - 563.85D7 - 563.85D6 - 563.85D5 + 3513.314 - 486.64A13
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+ 2323.1G14 - 476.9C13 - 476.9G12 - 476.9G11 - 476.9G10 - 476.9G9
- 476.9G8 - 476.9G7 - 4360.9J14 - 439.1J13 - 439.1J12 - 439.1J11
- 356.85B11 - 356.85B10 + 5560.2E14 - 439.8E13 - 439.8E12 - 439.8E11
- 439.8E10 + 2492.95H14 - 307.05H13 - 307.05H12 - 307.05H11 < 776.0
SOLUTION

LP OPTIMUM FOUND AT STEP 65

OBJECTIVE FUNCTION VALUE

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No. Iterations = 65
EXAMPLE II (UNSATURATED BUDGETS)

CASE 1 (WITHOUT SLACK VARIABLES)

\[
\begin{align*}
\text{MIN } & \quad 5.73A7 + 26.85A8 + 63.10A9 + 114.29A10 + 180.23A11 \\
& + 260.78A12 + 355.85A13 + 465.37A14 + 589.33A15 + 727.73A16 \\
& + 880.60A17 + 1048.04A18 + 1230.14A19 + 1427.06A20 \\
& + 3.44B10 + 19.68B11 + 48.51B12 + 89.74B13 + 143.24B14 \\
& + 208.88B15 + 286.58B16 + 376.28B17 + 477.97B18 + 591.64B19 \\
& + 717.32B20 + 3.69C15 + 19.27C16 + 46.52C17 + 85.28C18 \\
& + 135.43C19 + 196.84C20 + 7.41D5 + 27.41D6 + 58.9D7 + 101.16D8 \\
& + 153.44D9 + 215.03D10 + 285.27D11 + 363.53D12 + 449.23D13 \\
& + 541.58D14 + 640.52D15 + 745.33D16 + 855.56D17 + 970.78D18 \\
& + 1090.58D19 + 1214.58D20 + 5.85E10 + 20.91E11 + 44.53E12 \\
& + 76.14E13 + 115.16E14 + 161.06E15 + 213.40E16 + 271.66E17 \\
& + 335.42E18 + 404.28E19 + 477.85E20 + 5.75F15 + 20.03F16 \\
& + 42.23F17 + 71.82F18 + 108.28F19 + 151.11F20 + 2.51G7 \\
& + 21.59G8 + 57.85G9 + 112.0G10 + 184.89G11 + 277.49G12 \\
& + 390.91G13 + 526.40G14 + 685.38G15 + 869.43G16 + 1094.76G17 \\
& + 1334.49G18 + 1605.23G19 + 1909.41G20 + 2.99H10 + 19.91H11 \\
& + 51.25H12 + 97.66H13 + 159.82H14 + 238.56H15 + 323.94H16 \\
& + 438.78H17 + 573.40H18 + 729.14H19 + 907.5H20 + 1.99I15 \\
& + 16.81I6 + 44.91I17 + 86.86I18 + 143.31I19 + 158.56I20 \\
& + 5.56J8 + 22.98J9 + 51.70J10 + 91.19J11 + 140.94J12 \\
& + 200.48J13 + 269.37J14 + 347.19J15 + 433.55J16 + 528.09J17 \\
& + 630.47J18 + 740.35J19 + 857.45J20 \}
\end{align*}
\]

SUCH THAT:

A) EXCLUSIVITY CONSTRAINTS:

\[
\begin{align*}
& + A18 & + A19 & + A20 & = & 1 \\
C15 & + C16 & + C17 & + C18 & + C19 & + C20 & = & 1 \\
D5 & + D6 & + D7 & + D8 & + D9 & + D10 & + D11 & + D12 & + D13 & + D14 & + D15 & + L16 & \\
& + D17 & + D18 & + D20 & = & 1 \\
E10 & + E11 & + E12 & + E13 & + E14 & + E15 & + E16 & + E17 & + E18 & + E19 & + E20 & = & 1 \\
F15 & + F16 & + F17 & + F18 & + F19 & + F20 & = 1 \\
G7 & + G8 & + G9 & + G10 & + G11 & + G12 & + G13 & + G14 & + G15 & + G16 & + G17 & + G18 & + G19 & + G20 & = 1 \\
H10 & + H11 & + H12 & + H13 & + H14 & + H15 & + H16 & + H17 & + H18 & + H19 & + H20 & = 1 \\
I15 & + I16 & + I17 & + I18 & + I19 & + I20 & = 1 \\
\end{align*}
\]
B) BUDGET CONSTRAINTS:

5640.47D5 < 1273.42
5622.04D6 - 377.96D5 < 1561.21
5602.67D7 - 397.34D6 - 397.34D5 + 3758.34A7 + 2633.12G7 < 1899.94
5582.3D8 - 417.71D7 - 417.71D6 - 417.71D5 + 3732.92A8 - 267.08A7
+ 2606C8 - 193.89G8 - 2431.5J8 < 2289.51
-295.16A8 - 295.16A7 + 2574.73G9 - 225.27G8 - 225.27G7 + 4490.57J9
-309.43J8 < 275.46

5538.36D10 - 461.64D9 - 464.64D8 - 464.64D7 - 464.64D6 - 464.64D5
+ 3673.8A10 - 326.2A9 - 326.2A8 - 326.2A7 + 2538.27G10 - 261.73G9
- 261.73G8 - 261.73G7 + 4468.14J10 - 331.86J9 - 331.86J8 + 3760.8B10
+ 5639.92E10 + 2631.5H10 < 3301.53

5514.69D11 - 485.31D10 - 485.31D9 - 485.31D8 - 485.31D7 - 485.31D6
- 485.31D5 + 3639.49A11 - 360.51A10 - 360.51A9 - 360.51A8 - 360.51A7
+ 2495.92G11 - 304.08G10 - 304.08G9 - 304.08G8 - 304.08G7 + 4444.17J11
-355.93J10 - 355.93J9 - 355.93J8 + 3735.64B11 - 264.36B10 + 5621.46E11
- 378.54E10 + 2604.22H11 - 195.78H10 < 3950.0

- 510.19D6 - 510.19D5 + 3601.57A12 - 398.43A11 - 398.43A10 - 398.43A9
- 398.43A8 - 398.43A7 + 2446.7G12 - 353.3G11 - 353.3G10 - 353.3G9
- 353.3G8 - 353.3G7 + 4418.27J12 - 381.73J11 - 381.73J10 - 381.73J9
- 381.73J8 + 3707.83B12 - 292.17B11 - 292.17B10 + 5602.85E12 - 397.15E11
- 397.15E10 + 2572.53H12 - 227.47H11 - 227.47H10 < 4706.63

5463.65D13 - 536.35D12 - 536.35D11 - 536.35D10 - 536.35D9 - 536.35D8
- 536.35D7 - 536.35D6 - 536.35D5 + 3559.67A13 - 440.33A12 - 440.33A11
+ 440.33A10 - 440.33A9 - 440.33A8 - 440.33A7 + 2389.52G13 - 410.48G12
- 322.98B12 - 322.98B11 - 322.98B10 + 5581.65E13 - 418.35E12 - 418.35E11
- 418.35E10 + 2535.72H13 - 264.28H12 - 264.28H10 < 5587.98

5436.15D14 - 563.85D13 - 563.85D12 - 563.85D11 - 563.85D10 - 563.85D9
- 563.85D8 - 563.85D7 - 563.85D6 - 563.85D5 + 3513.36A14 - 486.64A13
- 486.64A12 - 486.64A11 - 486.64A10 - 486.64A9 - 486.64A8 - 486.64A7
+ 2323.10G14 - 476.9G13 - 476.9G12 - 476.9G11 - 476.9G10 - 476.9G9
- 476.9G8 - 476.9G7 + 4360.9J14 - 439.1J13 - 439.1J12 - 439.1J11
- 356.85B11 - 356.85B10 + 5560.2E14 - 439.8E13 - 439.8E12 - 439.8E11
- 439.8E10 + 2492.95H14 - 307.05H13 - 307.05H12 - 307.05H10 < 6613.22

3462.18A15 - 537.82A14 - 537.82A13 - 537.82A12 - 537.82A11 - 537.82A10
- 537.82A9 - 537.82A8 - 537.82A7 + 3605.61B15 - 394.39B14 - 394.39B13
- 394.39B12 - 394.39B11 - 394.39B10 + 4699.73C15 + 5407.24D15 - 592.76D14
+ 592.76D13 - 592.76D12 - 592.76D11 - 592.76D10 - 592.76D9 - 592.76D8
- 592.76D7 - 592.76D6 - 5537.65E15 - 462.35E14 - 462.35E13
- 462.35E12 - 462.35E11 - 462.35E10 + 7049.1F15 + 2245.92G15 - 554.08G14
- 554.08G13 - 554.08G12 - 554.08G11 - 554.08G10 - 554.08G9 - 554.08G8
- 650.02H12 - 650.02H11 - 650.02H10 + 3119.67I19 - 380.33I18 - 380.33I17
- 623.11J9 - 623.11J8 < 9658.8
SOLUTION

LP OPTIMUM FOUND AT STEP  41

OBJECTIVE FUNCTION VALUE

1)  316.918365

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NO. ITERATIONS= 41
EXAMPLE II (WITH SLACK VARIABLES)

THE PROBLEM IS FORMULATED AS:

\[
\text{MIN } \{ \ 5.73A_7 + 26.85A_8 + 63.10A_9 + 114.29A_{10} + 180.23A_{11} \\
+ 260.78A_{12} + 355.85A_{13} + 465.37A_{14} + 589.33A_{15} + 727.73A_{16} \\
+ 880.60A_{17} + 1048.04A_{18} + 1230.14A_{19} + 1427.06A_{20} \\
+ 3.44B_{10} + 19.68B_{11} + 48.51B_{12} + 89.74B_{13} + 143.24B_{14} \\
+ 208.88B_{15} + 286.58B_{16} + 376.28B_{17} + 477.97B_{18} + 591.64B_{19} \\
+ 717.32B_{20} + 3.69C_{15} + 19.27C_{16} + 46.52C_{17} + 85.28C_{18} \\
+ 135.43C_{19} + 196.84C_{20} + 7.41D_{5} + 27.41D_{6} + 58.97D_{7} + 101.16D_{8} \\
+ 153.44D_{9} + 215.03D_{10} + 285.27D_{11} + 363.53D_{12} + 449.23D_{13} \\
+ 541.58D_{14} + 640.52D_{15} + 745.33D_{16} + 855.56D_{17} + 970.78D_{18} \\
+ 1090.58D_{19} + 1214.56D_{20} + 5.85E_{10} + 20.91E_{11} + 44.53E_{12} \\
+ 76.14E_{13} + 115.16E_{14} + 161.08E_{15} + 213.40E_{16} + 271.66E_{17} \\
+ 335.42E_{18} + 404.28E_{19} + 477.85E_{20} + 5.75F_{15} + 20.03F_{16} \\
+ 42.23F_{17} + 71.82F_{18} + 108.28F_{19} + 151.11F_{20} + 2.51G_{7} \\
+ 21.59G_{8} + 57.85G_{9} + 112.00G_{10} + 184.89G_{11} + 277.49G_{12} \\
+ 390.91G_{13} + 526.40G_{14} + 685.38G_{15} + 869.43G_{16} + 1094.76G_{17} \\
+ 1334.48G_{18} + 1605.23G_{19} + 1909.41G_{20} + 2.99H_{10} + 19.91H_{11} \\
+ 51.23H_{12} + 97.66H_{13} + 159.82H_{14} + 238.56H_{15} + 323.94H_{16} \\
+ 438.78H_{17} + 573.40H_{18} + 729.14H_{19} + 907.52H_{20} + 1.99I_{15} \\
+ 16.8I_{16} + 44.91I_{17} + 86.86I_{18} + 143.31I_{19} + 158.56I_{20} \\
+ 5.56I_{8} + 22.98J_{9} + 51.70J_{10} + 91.19J_{11} + 140.94J_{12} \\
+ 200.48J_{13} + 269.37J_{14} + 347.19J_{15} + 433.55J_{16} + 528.09J_{17} \\
+ 630.47J_{18} + 740.35J_{19} + 857.45J_{20} \}
\]

SUCH THAT:

A) EXCLUSIVITY CONSTRAINTS:

\[A_7 + A_8 + A_9 + A_{10} + A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} = 1\]
\[A_{18} + A_{19} + A_{20} = 1\]
\[B_{10} + B_{11} + B_{12} + B_{13} + B_{14} + B_{15} + B_{16} + B_{17} + B_{18} + B_{19} + B_{20} = 1\]
\[C_{15} + C_{16} + C_{17} + C_{18} + C_{19} + C_{20} = 1\]
\[D_{5} + D_{6} + D_{7} + D_{8} + D_{9} + D_{10} + D_{11} + D_{12} + D_{13} + D_{14} + D_{15} + D_{16} + D_{17} + D_{18} + D_{20} = 1\]
\[E_{10} + E_{11} + E_{12} + E_{13} + E_{14} + E_{15} + E_{16} + E_{17} + E_{18} + E_{19} + E_{20} = 1\]
\[F_{15} + F_{16} + F_{17} + F_{18} + F_{19} + F_{20} = 1\]
\[G_{7} + G_{8} + G_{9} + G_{10} + G_{11} + G_{12} + G_{13} + G_{14} + G_{15} + G_{16} + G_{17} + G_{18} + G_{19} + G_{20} = 1\]
\[H_{10} + H_{11} + H_{12} + H_{13} + H_{14} + H_{15} + H_{16} + H_{17} + H_{18} + H_{19} + H_{20} = 1\]
\[I_{15} + I_{16} + I_{17} + I_{18} + I_{19} + I_{20} = 1\]
\[J_{8} + J_{9} + J_{10} + J_{11} + J_{12} + J_{13} + J_{14} + J_{15} + J_{16} + J_{17} + J_{18} + J_{19} + J_{20} = 1\]
B) BUDGET CONSTRAINTS:

\[
5640.47D5 < 1273.42
\]
\[
5622.04D6 - 377.96D5 < 1561.21
\]
\[
5602.7D7 - 397.34D6 - 397.34D5 + 3758.34A7 + 2633.12C7 < 1899.94
\]
\[
5582.3D9 - 417.71D7 - 417.71D6 - 417.71D5 + 3732.92A8 - 267.08A7
+ 2606G8 - 193.89G8 + 4511.5J8 < 2289.51
\]
\[
- 295.16A8 + 295.16A7 + 2574.73C9 - 225.27C8 - 225.27C7 + 4490.57J9
- 309.43J8 < 2755.46
\]
\[
5538.36D10 - 461.64D9 - 464.64D8 - 464.64D7 - 464.67D6 - 464.64D5
+ 3673.8A10 - 326.2A9 - 326.2A8 - 326.2A7 + 2538.27C10 - 261.73G9
- 261.73G8 - 261.73G7 + 4468.14J10 - 331.86J9 - 331.86J8 + 3760.8B10
+ 5639.92E10 + 2631.5H10 < 3301.53
\]
\[
5514.69D11 - 495.31D10 - 485.31D9 - 485.31D8 - 485.31D7 - 485.31D6
- 485.31D5 + 3639.49A11 - 360.51A10 - 360.51A9 - 360.51A8 - 360.51A7
+ 2495.92G11 + 304.08G10 - 304.08G9 - 304.08G8 - 304.08G7 + 4444.1J11
- 355.93J10 - 355.93J9 - 355.93J8 + 3735.64B11 - 264.36B10 + 5621.46E11
- 378.54E10 + 2604.22H11 - 195.76H10 < 3950.0
\]
\[
- 510.19D6 - 510.19D5 - 3601.57A12 - 398.43A11 - 398.43A10 - 398.43A9
- 398.43A8 - 398.43A7 + 2446.7G12 - 353.3G11 - 353.3G10 - 353.3G9
- 353.3G8 - 353.3G7 + 4410.27J12 - 381.73J11 - 381.73J10 - 381.73J9
- 381.73J8 + 3707.83B12 - 292.17B11 - 292.17B10 + 5602.85E12 - 397.15E11
- 397.15E10 + 2572.53H12 - 227.47H11 - 227.47H10 < 4706.63
\]
\[
5463.65D13 - 536.35D12 - 536.35D11 - 536.35D10 - 536.35D9 - 536.35D8
- 536.35D7 - 536.35D6 - 536.35D5 + 3559.67A13 - 440.33A12 - 440.33A11
- 410.48C11 - 410.48C10 - 410.48C9 - 410.48C8 - 410.48C7 + 4390.58J13
- 322.98B12 - 322.98B11 - 322.98B10 + 5581.65E13 - 418.35E12 - 418.35E11
- 418.35E10 + 2535.72H13 - 264.28H12 - 264.28H11 - 264.28H10 + Z0
= 5587.98
\]
\[
5436.15D14 - 563.85D13 - 563.85D12 - 563.85D11 - 563.85D10 - 563.85D9
- 563.85D8 - 563.85D7 - 563.85D6 - 563.85D5 + 3513.36A14 - 486.64A13
+ 486.64A12 - 486.64A11 - 486.64A10 - 486.64A9 - 486.64A8 + 486.64A7 + 2323.1G14 - 476.9G13 - 476.9G12 - 476.9G11 - 476.9G10 + 476.9G9 + 476.9G8 + 476.9G7 + 4360.9J14 - 439.1J13 - 439.1J12 - 439.1J11
- 356.85B11 - 356.85B10 + 5560.2E14 - 439.8E13 - 439.8E12 - 439.8E11
- 439.8E10 + 2492.95H14 - 307.05H13 - 307.05H12 - 307.05H11 - 307.05H10 + Z1 = 1.06Z0 = 6613.22
\]
\[
5462.18A15 - 537.82A14 - 537.82A13 - 537.82A12 - 537.82A11 - 537.82A10
- 537.82A9 - 537.82A8 - 537.82A7 + 3603.61B15 - 394.39B14 - 394.39B13
- 394.39B12 - 394.39B11 - 394.39B10 + 4699.73C15 + 5407.24D15 - 592.76D14
- 592.76D13 - 592.76D12 - 592.76D11 - 592.76D10 - 592.76D9 - 592.76D8
- 592.76D7 - 592.76D6 - 592.76D5 + 5537.65E15 - 462.35E14 - 462.35E13
- 462.35E12 - 462.35E11 - 462.35E10 + 7049.1F15 + 2245.92G15 - 554.08C14
- 554.08C13 - 554.08G12 - 554.08G11 - 554.08G10 - 554.08G9 - 554.08G8
\]
- 650.02H18 - 650.02H17 - 650.02H16 - 650.C2H15 - 650.02H14 - 650.02H13
- 650.02H12 - 650.02H11 - 650.02H10 + 3119.67T19 - 38C.33I18 - 380.33I17
- 623.11J9 - 623.11J8 + Z6 - 1.06Z5 = 9658.8
SOLUTION

LP OPTIMUM FOUND AT STEP 120

OBJECTIVE FUNCTION VALUE

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**NO. ITERATIONS= 120**
APPENDIX BII

INTEGER PROGRAMMING MODEL CASE

I) WITHOUT SLACK VARIABLES

THE PROBLEM IS FORMULATED AS:

\[
\begin{align*}
\min \quad & \left(-143.765 + 139.853 + 805.834 - 1745.53\right) \\
\text{s.t.} \quad & 
\begin{align*}
-1485.26 & B1 - 740.97 & B3 + 492.584 + 1413.59 & C3 \\
+1447.88 & C4 + 163.17 & D2 + 399.94 & D3 + 1087.36 & D4 \\
-2823.05 & E1 - 2357.9 & E2 - 1255.45 & E3 + 498.61 & E4 \\
+729.41 & F1 + 756.66 & F2 + 1061.03 & F3 + 1560.97 & F4 \\
+1322.06 & G2 + 1393.68 & G3 + 1670.32 & G4 + 2224.21 & H3 \\
+2246.37 & H4 - 1003.36 & K2 - 668.64 & K3 + 242.45 & K4 \\
+24.65 & L3 + 524.79 & L4 - 1.79 & N2 + 50.9 & N3 + 515.82 & N4 \\
-2631.22 & O1 - 2450.28 & O2 - 1653.42 & O3 - 205.40 & O4 \\
\end{align*}
\end{align*}
\]

SUCH THAT:

A) EXCLUSIVITY CONSTRAINTS:

\[
\begin{align*}
A2 + A3 + A4 = 1 \\
B1 + B2 + B3 + B4 = 1 \\
C3 + C4 = 1 \\
D2 + D3 + D4 = 1 \\
E1 + E2 + E3 + E4 = 1 \\
F1 + F2 + F3 + F4 = 1 \\
G2 + G3 + G4 = 1 \\
H3 + H4 = 1 \\
K2 + K3 + K4 = 1 \\
L3 + L4 = 1 \\
N2 + N3 + N4 = 1 \\
O1 + O2 + O3 + O4 = 1
\end{align*}
\]

B) BUDGET CONSTRAINTS:

\[
\begin{align*}
2949.62 & B1 + 3624.4 & E1 + 4503.7 & F1 + 2113.0 & O1 < 7072.82 \\
2895.85 & A2 + 2619.97 & B2 - 1675.58 & B1 + 3690.82 & D2
\end{align*}
\]
\[ + 3171.75E2 - 2300.9E1 + 4309.75E2 - 1753.71E2 \\
+ 4466.7E2 + 2009.58E2 + 2155.76E2 + 1770.0502 \\
- 1299.8901 < 2956.96 \]

\[ 2531.11A3 - 1851.8A2 + 2076.49B3 - 2762.54B2 - 2762.54B1 \\
+ 3805.3C3 + 3281.23D3 - 2081.95D2 + 2425.42E3 \\
- 3793.56E2 - 3793.56E1 + 4060.7E3 - 2251.8F2 - 2251.8F1 \\
+ 4262.8G3 - 1843.62G2 + 5626.82H3 + 1551.15K3 - 1737.73K2 \\
+ 2009.57L3 + 1860.6N3 - 1118.8N2 + 1044.07O3 - 2751.85O2 \\
- 2751.85O1 < - 5079.27 \]
### SOLUTION

**INTEGER SOLUTION OF -820.730 AT BRANCH 50 PIVOT 236**

**OBJECTIVE FUNCTION VALUE**

1) \(-820.729919\)

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NO. ITERATIONS = 236
II) WITH SLACK VARIABLES

THE PROBLEM IS FORMULATED AS:

\[
\begin{align*}
\text{MIN} & \quad \{-143.76A_2 + 139.85A_3 + 805.83A_4 - 1745.53B_1 - 1485.26B_2 - 740.97B_3 + 492.5B_4 + 1413.59C_3 + 1447.88C_4 + 163.17D_2 + 399.94D_3 + 1087.36D_4 - 2823.05E_1 - 2357.9E_2 - 1255.45E_3 + 498.61E_4 + 729.41F_1 + 756.66F_2 + 1061.03F_3 + 1560.97F_4 + 1322.06G_2 + 1393.68G_3 + 1670.32G_4 + 2224.21H_3 + 2246.37H_4 - 1003.36K_2 - 668.64K_3 + 242.45K_4 + 24.65L_3 + 524.79L_4 - 1.79N_2 + 50.9N_3 + 515.82N_4 - 2631.22O_1 - 2450.28O_2 - 1653.42O_3 - 205.4O_4\}
\end{align*}
\]

SUCH THAT:

A) EXCLUSIVITY CONSTRAINTS:

\[
\begin{align*}
A_2 + A_3 + A_4 &= 1 \\
B_1 + B_2 + B_3 + B_4 &= 1 \\
C_3 + C_4 &= 1 \\
D_2 + D_3 + D_4 &= 1 \\
E_1 + E_2 + E_3 + E_4 &= 1 \\
F_1 + F_2 + F_3 + F_4 &= 1 \\
G_2 + G_3 + G_4 &= 1 \\
H_3 + H_4 &= 1 \\
K_2 + K_3 + K_4 &= 1 \\
L_3 + L_4 &= 1 \\
N_2 + N_3 + N_4 &= 1 \\
O_1 + O_2 + O_3 + O_4 &= 1
\end{align*}
\]

B) BUDGET CONSTRAINTS:

\[
\begin{align*}
2949.62B_1 + 3624.4E_1 + 4503.7F_1 + 2113.001 + Z_1 &= 7072.82 \\
2895.85A_2 + 2619.97B_2 - 1675.58B_1 + 3690.82D_2 + 3171.75E_2 - 2300.9E_1 + 4309.75F_2 - 1753.71F_2 + 4466.7G_2 + 2009.58K_2 + 2155.76N_2 + 1770.05O_2 - 1299.89O_1 + Z_2 - 1.34Z_1 &= 2956.96 \\
2531.11A_3 - 1851.8A_2 + 2076.49B_3 - 2762.54B_2 - 2762.54B_1 + 3805.3C_3 + 3281.23D_3 - 2081.95D_2 + 2425.42E_3 - 3793.56E_2 - 3793.56E_1 + 4060.7F_3 - 2251.8F_2 - 2251.8F_1 + 4262.8G_3 - 1843.62G_2 + 5626.82H_3 + 1551.15K_3 - 1737.73K_2 + 2009.57L_3 + 1860.6N_3 - 1118.8N_2 + 1044.07O_3 - 2751.85O_2
\end{align*}
\]
- 2751.8501 + Z_3 - 1.34Z_2 = - 5079.27
SOLUTION

INTEGER SOLUTION OF -1285.65 AT BRANCH 11 PIVOT 60

OBJECTIVE FUNCTION VALUE

1) -1285.64990

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**NO. ITERATIONS** = 60  
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References


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