THE EFFECT OF TRANSLATION--ROTATION COUPLING ON HELICOPTER GROUND RESONANCE

by

Kenneth B. Amer

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Signature of Author.......Signature redacted.........
Department of Aeronautical Engineering, June, 1947

Signature of Professor
in charge of research.......Signature redacted.........

Signature of Chairman of Depart-
mental Committee on Graduate Students.......Signature redacted

Cambridge, Massachusetts
May 23, 1947

Professor Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Sir:

A thesis entitled, "The Effect of Translation-Rotation Coupling on Helicopter Ground Resonance," is herewith submitted in partial fulfillment of the requirements for the Degree of Master of Science in Aeronautical Engineering.

Respectfully submitted

Signature redacted

Kenneth B. Amer

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I especially wish to acknowledge the excellent supervision which Professor R. H. Miller gave to this thesis and the advice and coöperation of Mr. F. C. Loesch of the Aeronautics Department, who permitted me to use the results of his experimental investigation of this subject.

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SUMMARY

The purpose of this thesis is to expand the Coleman Theory of Ground Resonance (ref. 1) to include the effect of coupling between translational and rotational motion of the helicopter body. This coupling occurs if the elastic center of the tires does not coincide with the rotor axis. The characteristic equation for a helicopter with a three-bladed rotor is derived. A typical characteristic equation for no damping is plotted. It shows that, for a helicopter dynamically similar to the one investigated, the self-excited unstable range is definitely increased by this coupling, and the number of shaft critical speeds is increased to three. A criterion for dynamic similarity is established, based upon the characteristic equation. Lack of time prevented carrying through the investigation to include the effect of blade and body damping.

The result of a series of runs by Mr. F. C. Loesch on a model helicopter is also included. Apparently, the equipment was not sensitive enough to determine accurately the effect of coupling on the self-excited instability range. However, the results definitely show that for a particular value of body damping coefficient, coupling increased considerably the blade damping coefficient necessary to eliminate entirely the self-excited unstable range. Therefore, it is desirable to reduce any coupling between the translational and rotational motion of the helicopter body to a minimum. As no test runs were made at zero damping coefficient, it was not possible to check the theoretical calculation.

I. INTRODUCTION

A rotary wing aircraft may, under certain conditions, be subjected to violent vibrations while warming up on the ground. The term ground resonance is loosely applied to these vibrations, which obtain their energy from the rotational energy of the rotor. These vibrations are of two different types. The first type is similar to the vibration of a rotating shaft with flexible mounting. Any slight unbalance in the shaft will cause a violent vibration when the rotational frequency is equal to the natural frequency of the mounting. A similar situation occurs in a helicopter when the rotational frequency of the rotor is approximately equal to one of the natural frequencies of the body on the ground. These are known as Shaft Critical or Even Frequency Vibrations. The second type is due to a coupling between motions of the blades in the rotational plane and motions of the body on its flexible support (tires and shock absorbers). Rotor blades with lag hinges (hinges which permit freedom of the blade in the rotational plane) are especially susceptible to this type of vibration. Blades fixed at the root but with excessive chordwise flexibility are also susceptible. If, for any reason, a rotor blade starts to oscillate in the rotational plane, the resulting periodic inertia forces and moments cause periodic shears and moments at the hub. These are transmitted to the body of the helicopter. At certain rotor speeds the frequency of these periodic forces may be in resonance with one of the natural frequencies of the airplane on the ground. This will cause the airplane body to begin to oscillate, causing an increase in the amplitude of oscillation of the blades.

The resulting instability is known as Self-Excited or Odd Frequency Oscillation. Either of these types of instability may cause the destruction of the helicopter before it even leaves the ground.

The basic theory of ground resonance has been developed by Coleman and Feingold (refs. 1 and 2). They have treated the phenomenon from purely dynamic considerations. Since their theory is able to predict, qualitatively at least, both types of instability, this appears to be justified. Bennett (ref. 3) has indicated, however, that if an α_2 hinge is used, it is probably necessary to include the aerodynamic effects. The Coleman and Feingold theory indicates that these instabilities can be eliminated by introducing sufficient damping at the lag hinges and at the body supports and/or increasing the natural frequency of the airplane on the ground to a sufficiently high value. For most conventional helicopters today, blade damping is necessary for the elimination of ground resonance. Methods for conducting full scale tests to determine the minimum damping required are described in references 4 and 5. Reference 6 gives a great deal of information relating to the natural frequencies of the airplane on the ground.

The necessity for blade dampers is unfortunate, since they re-introduce some of the chordwise bending moment which the lag hinges were supposed to eliminate. These chordwise bending moments are due to Coriolis forces caused by blade flapping in flight. Thus, the present methods of eliminating ground resonance leave room for a great deal of improvement.

An excellent experimental investigation of the ground resonance theory was conducted by Mr. F. C. Loesch for his Master's Thesis (ref. 7). Mr. Loesch conducted tests on a model helicopter to determine the effect of variation of the blade and body damping coefficients on the instability. Mr. Loesch discovered sizeable discrepancy between the theory and his experimental results. Upon checking his equipment, he discovered he had inadvertently introduced coupling between the translational and rotational motions of the model frame. This was caused by the non-coincidence of the elastic center of the supports and the center of gravity of the model frame. Therefore, an investigation of the effect of this coupling on the ground resonance phenomenon seemed in order.

II. THEORETICAL CONSIDERATIONS

The procedure followed in the derivation of the characteristic equation is similar to Coleman's. Familiarity with his report will be assumed throughout this report. All aerodynamic effects will be neglected but real coördinates will be used. The frame damping and spring constants in the X and Y directions will be assumed equal. The equations of kinetic energy, potential energy, and energy dissipation will be set up. Then, the differential equations of motion will be obtained from the Lagrangian equation. All terms in the Lagrangian equation consist of partial derivatives of the above three equations with respect to the generalized coördinate or its time derivative. Since the derivation will be based on small oscillations, only linear terms need be retained in the differential equations. Therefore, all terms above the second degree in the energy or dissipation equations can be neglected. Also, all sine terms can be replaced by the first term of their series expansion and all cosine terms by the first two terms of their series expansion.

III. SYMBOLS

A_{mn}	element of the characteristic determinant (column m, row n)
a	distance from rotor centerline to lag hinge
b	distance from blade c.g. to lag hinge
B	damping force (moment) per unit linear (angular) velocity
c	$\frac{m\Omega^2 ab}{m(r^2 + b^2)}$ uncoupled natural blade frequency
c_o	$\frac{m\Omega_o^2 ab}{m(r^2 + b^2)}$
$C_1 \ldots\ldots C_7$	factors in the characteristic equation
e	distance from frame c.g. to elastic center of frame support
F	dissipation function
I	moment of inertia of frame about c.g.
I_t	$I + 3m[r^2 + (a + b)^2]$
K	spring force (moment) per unit linear (angular) displacement
M	mass of frame
m	mass of one blade
O	origin of coördinate system
q	generalized coördinate
R	radius of gyration of entire ship about c.g. = $\frac{I_t}{M + 3m}$
$\overline{RS}$	distance from c.g. of deflected blade to rotor ℄
r	radius of gyration of one blade about its own c.g.
t	time
T	kinetic energy
T_{bt}	translational kinetic energy of all three blades (due to velocities of all three c.g.'s
u	$\frac{b}{12}[(\beta_1 + \beta_2)^2 + (\beta_1 + \beta_3)^2]$
U	potential energy
v	$\frac{\sqrt{3}b}{12}[-(\beta_1 + \beta_2)^2 + (\beta_1 + \beta_3)^2]$

x, y	displacements of frame c.g. from origin
x', y'	displacements of c.g. of three blades from frame c.g. in fixed coördinates
z'	complex displacement $(x' + iy')$
α	sum of the three blade deflections
β_1	angular deflection of blade no. 1
$\beta_1 + \beta_2$	angular deflection of blade no. 2
$\beta_1 + \beta_3$	angular deflection of blade no. 3
ζ, η	displacements of c.g. of three blades from frame c.g. in rotating coördinates
δ	non-dimensional damping coefficient
δ_f	$\frac{B_f}{2[(M + 3m) K_f]^{1/2}}$
δ_θ	$\frac{B_\theta}{2[(e^2K_f + K_\theta) I_t]^{1/2}}$
δ_β	$\frac{B_\beta}{2c_o(mr^2 + mb^2)}$
θ	angular deflection of frame (positive clockwise)
λ	$\frac{3[r^2 + b(a + b)]}{r^2 + b^2}$
μ	mass ratio $\frac{3m}{M + 3m}$
ν	$\frac{m[r^2 + b(a + b)]}{I_t}$
Λ_1	$2(1 + r^2/b^2)$
Λ_2	$2(1 + r^2/b^2 - a/b)$
Λ_3	$\frac{B_\beta}{mb^2\omega_n}$
Λ_4	$2(1 + r^2/b^2 + a/b)$
Λ_5	$\frac{ab}{r^2 + b^2}$
ϕ	$\frac{e(M + 3m)}{I_t}$
Ω	rotational speed of rotor (radians per second)(divided by ω_n in application)
Ω_o	= constant = non-dimensional rotor speed at which δ_β and c_o are calculated = 2

Ω_n^2	$\frac{(e^2 + K_\theta/K_f)(M + 3m)}{I_t}$
ω_n^2	$\frac{K_f}{M + 3m}$
ω	non-dimensional system frequency = $\frac{\omega \text{ dimensional}}{\omega_n}$
ξ	complex displacement ($\zeta + i\eta$)

Subscripts:

f	frame (body) of helicopter — translational motion
β	blade
θ	frame (body) of helicopter — angular motion

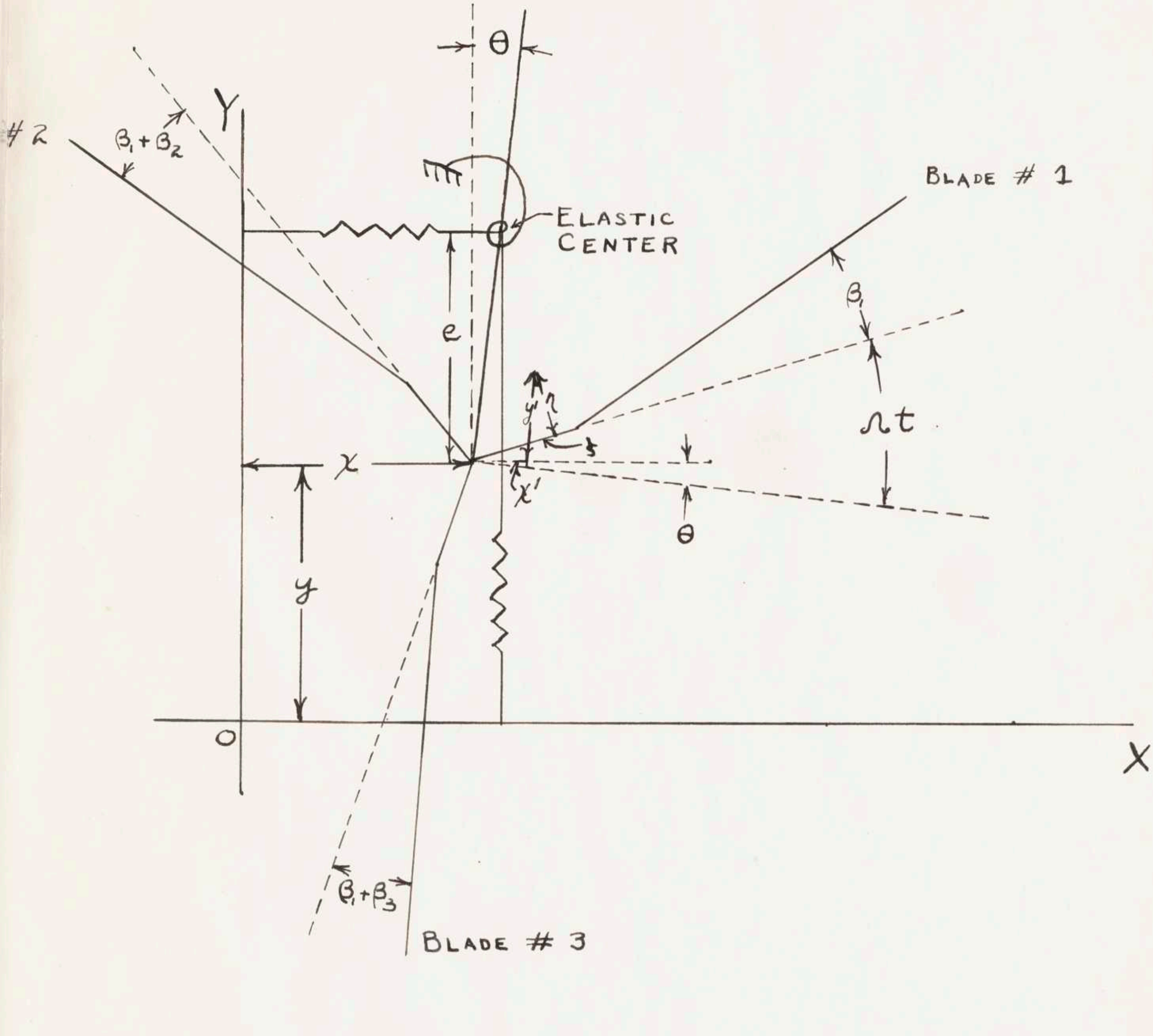


Figure 1

Schematic Diagram Representing Helicopter

IV. MATHEMATICAL DEVELOPMENT

A schematic diagram of the helicopter is shown in Fig. 1. There are six degrees of freedom -- two translational and one rotational for the frame and one for each of the three blades. All elements of the helicopter are assumed to lie in one horizontal plane. The rotor speed will be assumed constant. Some necessary geometric relations will be derived first.

Refer to Fig. 1. Taking moments about the undeflected position of blade number 1:

$$3m\eta = mb\,\sin\beta_1 + mb\left\{\sin[60 - (\beta_1 + \beta_2)] - \sin[60 + (\beta_1 + \beta_3)]\right\}$$

Expand, assuming $\cos\beta = 1 - \frac{\beta^2}{2}$; $\sin\beta = \beta$ (retaining only first and second order terms)

(1a) $$\eta = -\frac{b}{6}[\beta_2 + \beta_3] + v$$

Differentiating:

(1b) $$\dot{\eta} = -\frac{b}{6}[\dot{\beta}_2 + \dot{\beta}_3] + \dot{v}$$

Similarly, by taking moments about an axis normal to the undeflected position of blade number 1:

(2a) $$\zeta = \frac{\sqrt{3}b}{6}[\beta_3 - \beta_2] + u$$

(2b) $$\dot{\zeta} = \frac{\sqrt{3}b}{6}|\dot{\beta}_3 - \dot{\beta}_2| + \dot{u}$$

Solving simultaneously:

(3a, b) $$\beta_3 = \frac{3}{b}\left[\frac{\zeta - u}{\sqrt{3}} - (\eta - v)\right]\ ;\ \beta_2 = -\frac{3}{b}\left[\frac{\zeta - u}{\sqrt{3}} + (\eta - v)\right]$$

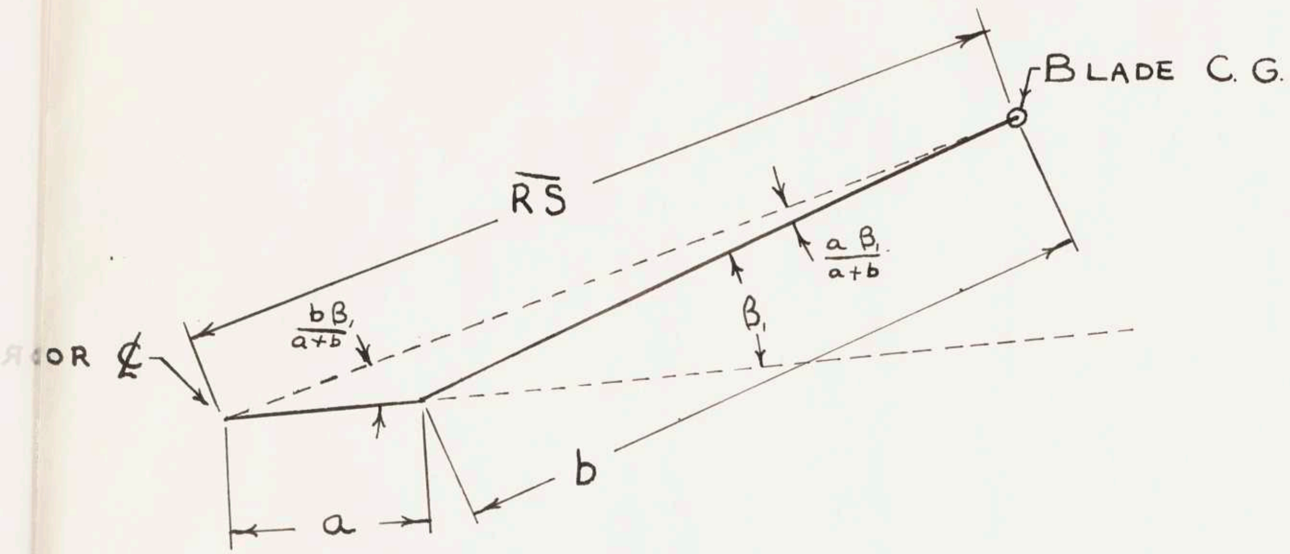


FIGURE 2

BLADE # 1 CONFIGURATION (TYPICAL FOR ALL 3 BLADES)

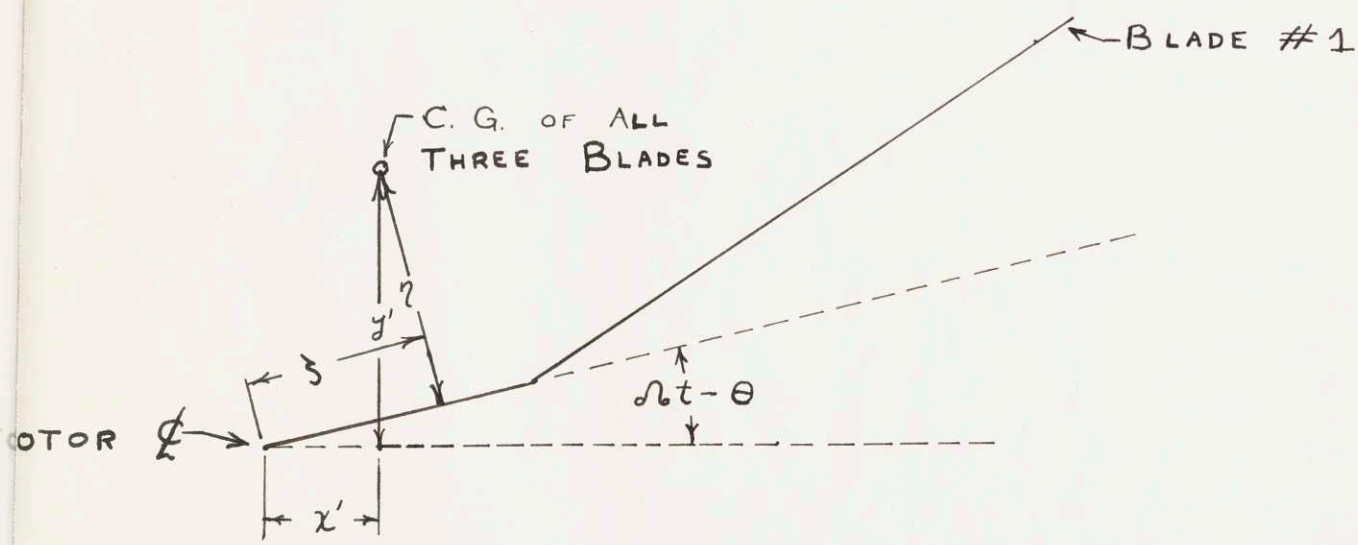


FIGURE 3

DIAGRAM SHOWING RELATION BETWEEN $x'y'$ AND ξ, η

also:

$$\alpha = 3\beta_1 + \beta_2 + \beta_3$$

(4a, b) $\therefore\ \beta_1 = \frac{\alpha}{3} + \frac{2}{b}(\eta - v)\ ; \qquad \dot{\beta}_1 = \frac{\dot{\alpha}}{3} + \frac{2}{b}(\dot{\eta} - \dot{v})$

Notice that both u and v are of the second degree.

Refer to Fig. 3.

(5a) $x' = \xi \cos(\Omega t - \theta) - \eta \sin(\Omega t - \theta)$

(5b) $y' = \xi \sin(\Omega t - \theta) + \eta \cos(\Omega t - \theta)$

Differentiating,

(6a) $\dot{x}' = (\dot{\xi} - \Omega\eta)\cos(\Omega t - \theta) - (\Omega\xi + \dot{\eta})\sin(\Omega t - \theta)$

(6b) $\dot{y}' = (\dot{\xi} - \Omega\eta)\sin(\Omega t - \theta) + (\Omega\xi + \dot{\eta})\cos(\Omega t - \theta)$

(7a) $\ddot{x}' = [\ddot{\xi} - 2\Omega\dot{\eta} - \Omega^2\xi]\cos(\Omega t - \theta) - [\ddot{\eta} + 2\Omega\dot{\xi} - \Omega^2\eta]\sin(\Omega t - \theta)$

(7b) $\ddot{y}' = [\ddot{\xi} - 2\Omega\dot{\eta} - \Omega^2\xi]\sin(\Omega t - \theta) + [\ddot{\eta} + 2\Omega\dot{\xi} - \Omega^2\eta]\cos(\Omega t - \theta)$

(7c) $\xi = z' e^{-i(\Omega t - \theta)}$

(7d) $\therefore\ \bar{\xi} = \bar{z}' e^{i(\Omega t - \theta)}$

Now refer to Fig. 2.

$$\overline{RS_1} = a \cos\left[\frac{b}{(a + b)}\beta_1\right] + b \cos\left[\frac{a}{(a + b)}\beta_1\right]$$

(8a) $= (a + b)\left[1 - \frac{ab\beta_1^2}{2(a + b)^2}\right]$ (neglecting terms above second degree)

Similarly, for the other blades

(8b) $\overline{RS_2} = (a + b)\left[1 - \frac{ab(\beta_1 + \beta_2)^2}{2(a + b)^2}\right]$

(8c) $\overline{RS_3} = (a + b)\left[1 - \frac{ab(\beta_1 + \beta_3)^2}{2(a + b)^2}\right]$

Refer again to Fig. 1. From the basic definition of kinetic energy,

$$T_{bt} = \frac{m}{2}\Big[\dot{x} + (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab}{2(a + b)^2}\beta_1^2\Big\} \sin\Big\{\Omega t - \theta + \frac{b}{(a + b)}\beta_1\Big\}$$

$$- b\dot{\beta}_1 \sin\Big\{\Omega t - \theta + \beta_1\Big\}\Big]^2$$

$$+ \frac{m}{2}\Big[\dot{x} + (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab}{2(a + b)^2}(\beta_1 + \beta_2)^2\Big\} \sin\Big\{\Omega t - \theta + \frac{2\pi}{3}$$

$$+ \frac{b}{(a + b)}(\beta_1 + \beta_2)\Big\} - b(\dot{\beta}_1 + \dot{\beta}_2) \sin\Big\{\Omega t - \theta + \frac{2\pi}{3} + \beta_1 + \beta_2\Big\}\Big]^2$$

$$+ \frac{m}{2}\Big[\dot{x} + (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab}{2(a + b)^2}(\beta_1 + \beta_3)^2\Big\} \sin\Big\{\Omega t - \theta + \frac{4\pi}{3}$$

$$+ \frac{b}{(a + b)}(\beta_1 + \beta_3)\Big\} - b(\dot{\beta}_1 + \dot{\beta}_3) \sin\Big\{\Omega t - \theta + \frac{4\pi}{3} + \beta_1 + \beta_3\Big\}\Big]^2$$

$$+ \frac{m}{2}\Big[\dot{y} - (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab}{2(a + b)^2}\beta_1^2\Big\} \cos\Big\{\Omega t - \theta + \frac{b}{(a + b)}\beta_1\Big\}$$

$$+ b\dot{\beta}_1 \cos\Big\{\Omega t - \theta + \beta_1\Big\}\Big]^2$$

$$+ \frac{m}{2}\Big[\dot{y} - (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab(\beta_1 + \beta_2)^2}{2(a + b)^2}\Big\} \cos\Big\{\Omega t - \theta + \frac{2\pi}{3}$$

$$+ \frac{b}{(a + b)}(\beta_1 + \beta_2)\Big\} + b(\dot{\beta}_1 + \dot{\beta}_2) \cos\Big\{\Omega t - \theta + \frac{2\pi}{3} + \beta_1 + \beta_2\Big\}\Big]^2$$

$$+ \frac{m}{2}\Big[\dot{y} - (\dot{\theta} - \Omega)(a + b)\Big\{1 - \frac{ab(\beta_1 + \beta_3)^2}{2(a + b)^2}\Big\} \cos\Big\{\Omega t - \theta + \frac{4\pi}{3}$$

$$+ \frac{b}{(a + b)}(\beta_1 + \beta_3)\Big\} + b(\dot{\beta}_1 + \dot{\beta}_3) \cos\Big\{\Omega t - \theta + \frac{4\pi}{3} + \beta_1 + \beta_3\Big\}\Big]^2$$

The foregoing expression can be expanded, using the following standard trigonometric and series formulae:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin x = x + \ldots$$

$$\cos x = 1 - \frac{x^2}{2} + \ldots$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos x + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(x + \frac{4\pi}{3}\right) = 0$$

$$\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right) = 0$$

Expanding, simplifying, and substituting previously derived expressions for ξ and η :

(9) $$T_{bt} = \frac{3m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m}{2}\Big\{3(\dot{\theta} - \Omega)^2(a + b)^2 - \Omega^2 ab\Big[3\beta_1^2 + 2\beta_1\Big(\frac{-6\eta}{b}\Big) + \frac{6}{b^2}(\xi^2 + 3\eta^2)\Big] + 3b^2\dot{\beta}_1^2 - 12b\dot{\beta}_1\dot{\eta} + 6\dot{\xi}^2 + 18\dot{\eta}^2 - 2(\dot{\theta} - \Omega)(a + b)(3b\dot{\beta}_1 - 6\dot{\eta} + 6\dot{v}) + \sin(\Omega t - \theta)\Big[-6\Omega\dot{y}\eta - 6\dot{x}\dot{\eta} - 6\xi\dot{x}\Omega + 6\dot{y}\dot{\xi}\Big] + \cos(\Omega t - \theta)\Big[6\xi\dot{y}\Omega + 6\dot{x}\dot{\xi} - 6\Omega\eta\dot{x} + 6\dot{y}\dot{\eta}\Big]\Big\}$$

The remaining kinetic energy of the system is:

(10) $$T - T_{bt} = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mr^2\Big[3(\Omega - \dot{\theta} + \dot{\beta}_1)^2 - \frac{12}{b}(\dot{\eta} - \dot{v})(\Omega - \dot{\theta} + \dot{\beta}_1) + \frac{6}{b^2}(\dot{\xi}^2 + 3\dot{\eta}^2)\Big]$$

The potential energy is given by

(11) $$U = \frac{1}{2}K_f y^2 + \frac{1}{2}K_f(x + e\theta)^2 + \frac{1}{2}K_\theta\theta^2$$

The energy dissipation equation is:

(12a) $$2F = B_f(\dot{x}^2 + \dot{y}^2) + B_\theta\dot{\theta}^2 + B_\beta\big[\dot{\beta}_1^2 + (\dot{\beta}_1 + \dot{\beta}_2)^2 + (\dot{\beta}_1 + \dot{\beta}_3)^2\big]$$

or,

(12b) $$2F = B_f(\dot{x}^2 + \dot{y}^2) + B_\theta\dot{\theta}^2 + B_\beta\Big[3\dot{\beta}_1^2 - \frac{12}{b}\dot{\beta}_1\dot{\eta} + \frac{6}{b^2}(\dot{\xi}^2 + 3\dot{\eta}^2)\Big]$$

where 2F = rate of energy dissipation by damping.

The energy and dissipation equations are in terms of the six generalized coördinates $\underline{x}$, $\underline{y}$, $\underline{\theta}$, $\underline{\xi}$, $\underline{\eta}$, $\underline{\beta}_1$. The six differential equations of motion are obtained from the Lagrangian equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial F}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

If $\underline{q} = \underline{x}$, using equation (6a)

(13a) $$(M + 3m)\ddot{x} + 3m\ddot{x}' + B_f\dot{x} + K_f(x + e\theta) = 0$$

If $\underline{q} = \underline{y}$, using equation (6b)

(13b) $$(M + 3m)\ddot{y} + 3m\ddot{y}' + B_f\dot{y} + K_f y = 0$$

If $\underline{q} = \underline{\beta_1}$, using equations (4a, b)

(13c) $$m(r^2 + b^2)\ddot{\alpha} - 3m[r^2 + b(a + b)]\ddot{\theta} + B_\beta\dot{\alpha} + m\Omega^2 ab\alpha = 0$$

If $\underline{q} = \underline{\theta}$, using equations (4a, b)

(13d) $$\left\{I + 3m[r^2 + (a + b)^2]\right\}\ddot{\theta} - m[r^2 + b(a + b)]\ddot{\alpha} + B_\theta\dot{\theta} + K_f e(x + e\theta) + K_\theta\theta = 0$$

If $\underline{q} = \underline{\eta}$, using equation (4b) and combining with equation (13c)

(13e') $$m\ddot{y}\cos(\Omega t - \theta) - m\ddot{x}\sin(\Omega t - \theta) + 2m\ddot{\eta}\left(1 + \frac{r^2}{b^2}\right) + 2m\Omega^2\frac{a}{b}\eta + \frac{2B_\beta}{b^2}\dot{\eta} = 0$$

If $\underline{q} = \underline{\zeta}$,

(13f') $$m\ddot{y}\sin(\Omega t - \theta) + m\ddot{x}\cos(\Omega t - \theta) + 2m\ddot{\zeta}\left(1 + \frac{r^2}{b^2}\right) + 2m\Omega^2\frac{a}{b}\zeta + \frac{2B_\beta\dot{\zeta}}{b^2} = 0$$

Equations (13e', f') will now be converted to fixed coördinates. Multiplying (13f') by $\underline{i}$ and adding to (13e'),

$$m\ddot{y}e^{i(\Omega t-\theta)} + m\ddot{x}ie^{i(\Omega t-\theta)} + 2im\left(1 + \frac{r^2}{b^2}\right)\ddot{\bar{\xi}} + 2i\frac{B_\beta}{b^2}\dot{\bar{\xi}} + 2im\Omega^2\frac{a}{b}\bar{\xi} = 0$$

Using the relation $\bar{\xi} = \bar{z}' e^{i(\Omega t - \theta)}$ results in:

$$m\ddot{y} + im\ddot{x} + 2im\left(1 + \frac{r^2}{b^2}\right)\left[\ddot{\bar{z}}' + 2i\Omega\dot{\bar{z}}' - \Omega^2\bar{z}'\right]$$

$$+ 2i\frac{B_\beta}{b^2}\left[\dot{\bar{z}}' + i\Omega\bar{z}'\right] + 2im\Omega^2\frac{a}{b}\bar{z}' = 0$$

Setting both real and imaginary parts equal to zero,

(13e) $$m\ddot{y} + 2m\left(1 + \frac{r^2}{b^2}\right)\ddot{y}' + \frac{2B_\beta}{b^2}\dot{y}' - 2m\Omega^2\left(1 + \frac{r^2}{b^2} - \frac{a}{b}\right)y'$$

$$- 4m\Omega\left(1 + \frac{r^2}{b^2}\right)\dot{x}' - \frac{2B_\beta}{b^2}\Omega x' = 0$$

(13f) $$m\ddot{x} + 2m\left(1 + \frac{r^2}{b^2}\right)\ddot{x}' + \frac{2B_\beta}{b^2}\dot{x}' - 2m\Omega^2\left(1 + \frac{r^2}{b^2} - \frac{a}{b}\right)x'$$

$$+ 4m\Omega\left(1 + \frac{r^2}{b^2}\right)\dot{y}' + 2\frac{B_\beta}{b^2}\Omega y' = 0$$

In general, let $q = q_o e^{i\omega t}$. The characteristic equation is obtained by setting the determinant of the coefficients of equations (13a-f) equal to zero.

	x_o	y_o	x_o'	y_o'	α_o	θ_o
a)	A_{11}	0	A_{31}	0	0	A_{61}
b)	0	A_{22}	0	A_{42}	0	0
c)	0	0	0	0	A_{53}	A_{63}
d)	A_{14}	0	0	0	A_{54}	A_{64}
e)	0	A_{25}	A_{35}	A_{45}	0	0
f)	A_{16}	0	A_{36}	A_{46}	0	0

$= 0$

Where

$A_{11} = A_{22} = -\omega^2 + 2i\omega\omega_n\delta_f + \omega_n^2$

$A_{14} = \phi\omega_n^2$

$A_{16} = A_{25} = -\omega^2$

$A_{31} = A_{42} = -\mu\omega^2$

$A_{35} = -A_{46} = -2\Lambda_1\Omega i\omega - 2\Lambda_3\Omega\omega_n$

$A_{36} = A_{45} = -\omega^2\Lambda_1 + 2i\Lambda_3\omega\omega_n - \Lambda_2\Omega^2$

$$A_{53} = -\omega^2 + 2i\delta_\beta c_0\omega + \Omega^2\frac{ab}{(r^2 + b^2)}$$
$$A_{54} = \nu\omega^2$$
$$A_{61} = e\omega_n^2$$
$$A_{63} = \lambda\omega^2$$
$$A_{64} = -\omega^2 + 2i\delta_\theta\Omega_n\omega_n\omega + \Omega_n^2\omega_n^2$$

Expansion of the determinant gives:

$$\text{(14)} \qquad \left[A_{53}A_{64} - A_{54}A_{63}\right]\left[\overline{A_{11}}^2\,\overline{A_{35}}^2 + (A_{11}A_{36} - A_{16}A_{31})^2\right] + A_{14}A_{61}A_{53}\left[-A_{11}(\overline{A_{35}}^2 + \overline{A_{36}}^2) + A_{16}A_{31}A_{36}\right] = 0$$

For the case with $e = 0$

$$A_{14} = A_{61} = 0$$

Therefore the equation factors into two parts. The first part gives the uncoupled equation for the θ-α motion. The second part can be simplified to give Coleman's Equation 32 (ref. 1).

The simplified case of no damping will now be considered. Equation (14) can be manipulated to give a bi-cubic in the variable $(\frac{\Omega}{\omega})$. The result is:

$$\text{(15)} \qquad \Lambda_2^2\Lambda_5C_1\left(\frac{\Omega}{\omega}\right)^6 + \left[-\Lambda_2^2C_2 - 2\Lambda_1\Lambda_4\Lambda_5C_1 - \mu\Lambda_2\Lambda_5C_3C_7\right]\left(\frac{\Omega}{\omega}\right)^4 + \left[\Lambda_1^2\Lambda_5C_1 + 2\Lambda_1\Lambda_4C_2 - \mu\Lambda_1\Lambda_5C_3C_7 + \mu\Lambda_2C_4C_7 + \mu^2\Lambda_5C_6C_7^2\right]\left(\frac{\Omega}{\omega}\right)^2 + \left[-\Lambda_1^2C_2 + \mu\Lambda_1C_4C_7 - \mu^2C_5C_7^2\right] = 0$$

where

$$C_1 = \Omega_n^2 - \omega^2 + \frac{\phi e}{\omega^2 - 1}$$
$$C_2 = \Omega_n^2 - (1 - \lambda\nu)\omega^2 + \frac{\phi e}{\omega^2 - 1}$$
$$C_3 = 2(\Omega_n^2 - \omega^2) + \frac{\phi e}{\omega^2 - 1}$$
$$C_4 = 2\left[\Omega_n^2 - (1 - \lambda\nu)\omega^2\right] + \frac{\phi e}{\omega^2 - 1}$$

$$C_5 = \Omega_n^2 - (1 - \lambda\nu)\omega^2$$

$$C_6 = \Omega_n^2 - \omega^2$$

$$C_7 = \frac{\omega^2}{\omega^2 - 1}$$

The quantities $\underline{\Omega}$ and $\underline{\omega}$ are now non-dimensional. $\underline{\omega}_n$ is the reference frequency.

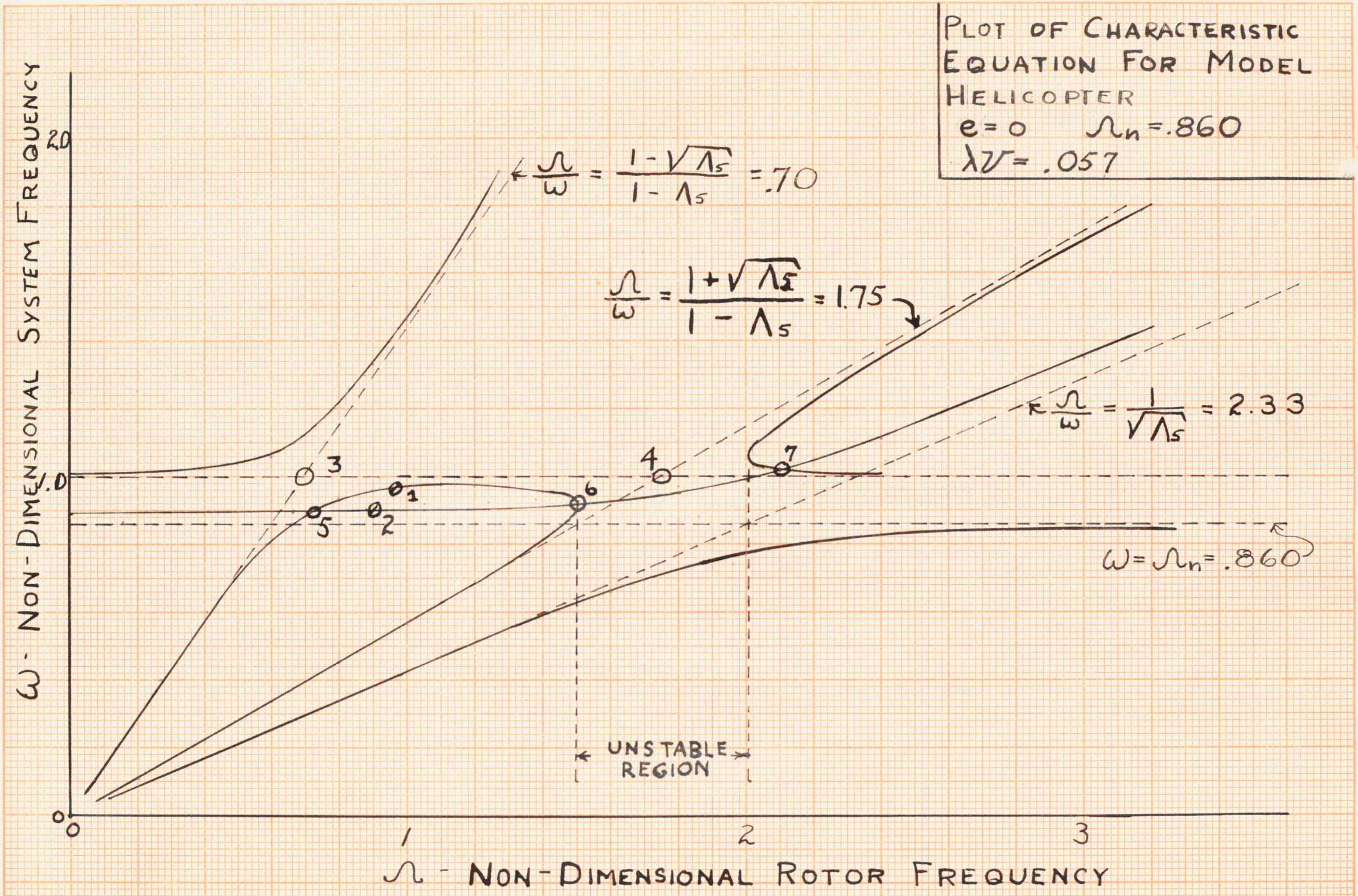
V. METHODS OF APPLYING THEORY

The Λ expressions in equation (15), page 15, depend upon a, b, and r. They are therefore constant for a particular machine. The C expressions depend upon e, a, b, r, K_θ/K_f, M, m, I_t, and ω. For a particular design, these are all constant except ω. Thus, it is possible to plot the relationship between Ω and ω. The procedure consists of assuming values for ω and solving the bi-cubic for Ω. The procedure is quite tedious. The method for solving the bi-cubic used by the author depends upon the fact that for most values of ω, there is at least one value of $(\frac{\Omega}{\omega})^2$ less than 1. A good approximation to this root can be obtained by neglecting the $(\frac{\Omega}{\omega})^6$ term and solving the resulting bi-quadratic equation. Horner's method is then used to determine this root accurately. Once it is determined, it can be factored out of the bi-cubic. The resulting bi-quadratic is then solved by formula.

The numerical calculations of this thesis will be made for Mr. Loesch's model helicopter. Its characteristics are as follows:

$a = .242'$	$M + 3m = 1.15$ slugs
$b = .600'$	$I = 1.429$ slug-ft^2
$r = .654'$	$\frac{K_\theta}{K_f} = .931$ ft^2
$\mu = .0683$	$e = -.208'$ (Negative sign indicates E.C. AFT of rotor ℄. This is unimportant since only e^2 appears in the equation.)
$m = .0262$ slugs	

The completely uncoupled form of equation (15) can be obtained by setting $\mu = e = 0$. The result is the dashed lines of Fig. 4. The line at $\omega = 1$ represents uncoupled X and Y motion. The line



PLOT OF CHARACTERISTIC EQUATION FOR MODEL HELICOPTER
e = 0   Ωn = .860
λV = .057
Ω/ω = (1 - √Λs)/(1 - Λs) = .70
Ω/ω = (1 + √Λs)/(1 - Λs) = 1.75
Ω/ω = 1/√Λs = 2.33
ω = Ωn = .860
UNSTABLE REGION
ω - NON-DIMENSIONAL SYSTEM FREQUENCY
Ω - NON-DIMENSIONAL ROTOR FREQUENCY
2.0
1.0
0
1
2
3
1
2
3
4
5
6
7

FIGURE 4

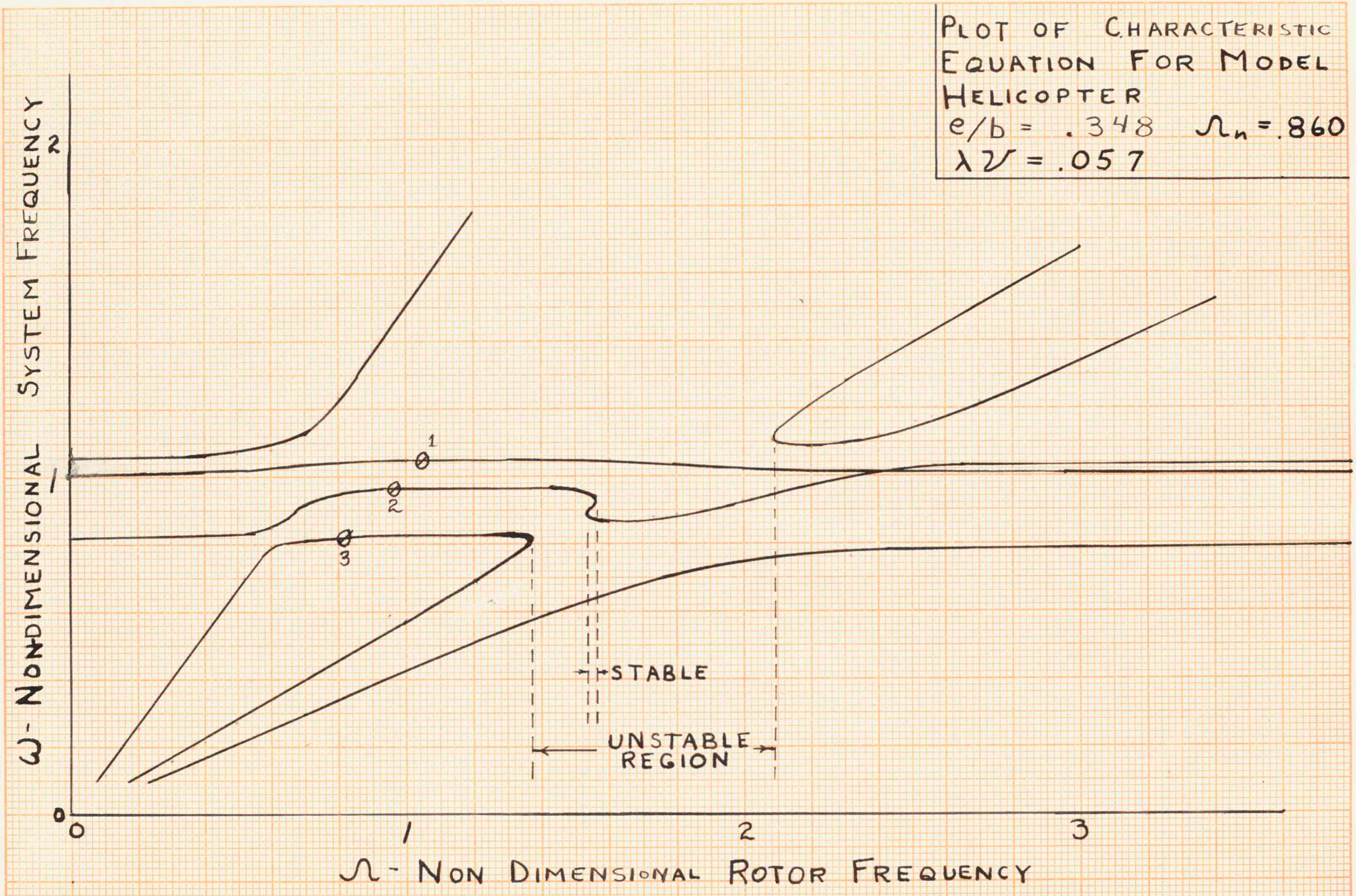

PLOT OF CHARACTERISTIC EQUATION FOR MODEL HELICOPTER
e/b = .348   Ωn = .860
λν = .057
ω - NONDIMENSIONAL SYSTEM FREQUENCY
Ω - NON DIMENSIONAL ROTOR FREQUENCY
0
1
2
3
STABLE
UNSTABLE REGION

FIGURE 5

at $\omega = \Omega_n = .860$ represents uncoupled θ motion. The lines $\frac{\Omega}{\omega} = \left[\frac{1 + \sqrt{\Lambda_5}}{1 - \Lambda_5}\right] = 1.75$ and $\frac{\Omega}{\omega} = \left[\frac{1 - \sqrt{\Lambda_5}}{1 - \Lambda_5}\right] = .70$ represent the X' and Y' motion. The line $\frac{\Omega}{\omega} = \frac{1}{\sqrt{\Lambda_5}} = 2.33$ represents uncoupled α motion. Notice that all the dashed curves depend only on Λ_5 and Ω_n.

If the actual value of μ is used but e is still equal to zero, the result is the solid curves of Fig. 4. Three of the branches are identical with Coleman's curves in Fig. 2 of his report. The biggest change is the breakaway at points 3 and 4. The unstable range is indicated as the region where two values of ω are complex conjugates. Since one of these roots must have a negative imaginary part, this implies a self-excited oscillation. The other two branches of Fig. 4 represent the α-θ motion which, for e equal to zero, is uncoupled. Notice that, theoretically, there are two shaft critical speeds (points 1 and 2, where $\omega = \Omega$).

With e set equal to -.208', the result is shown in Fig. 5. Notice that there are three shaft critical speeds (points 1, 2 and 3) and that the instability range has been extended at both ends. Also notice a small stable region near the center of the unstable region. By comparing with Fig. 4, it can be seen that the increase in the unstable range is caused by the breaking away of the curves at the intersections (points 5, 6 and 7 of Fig. 4).

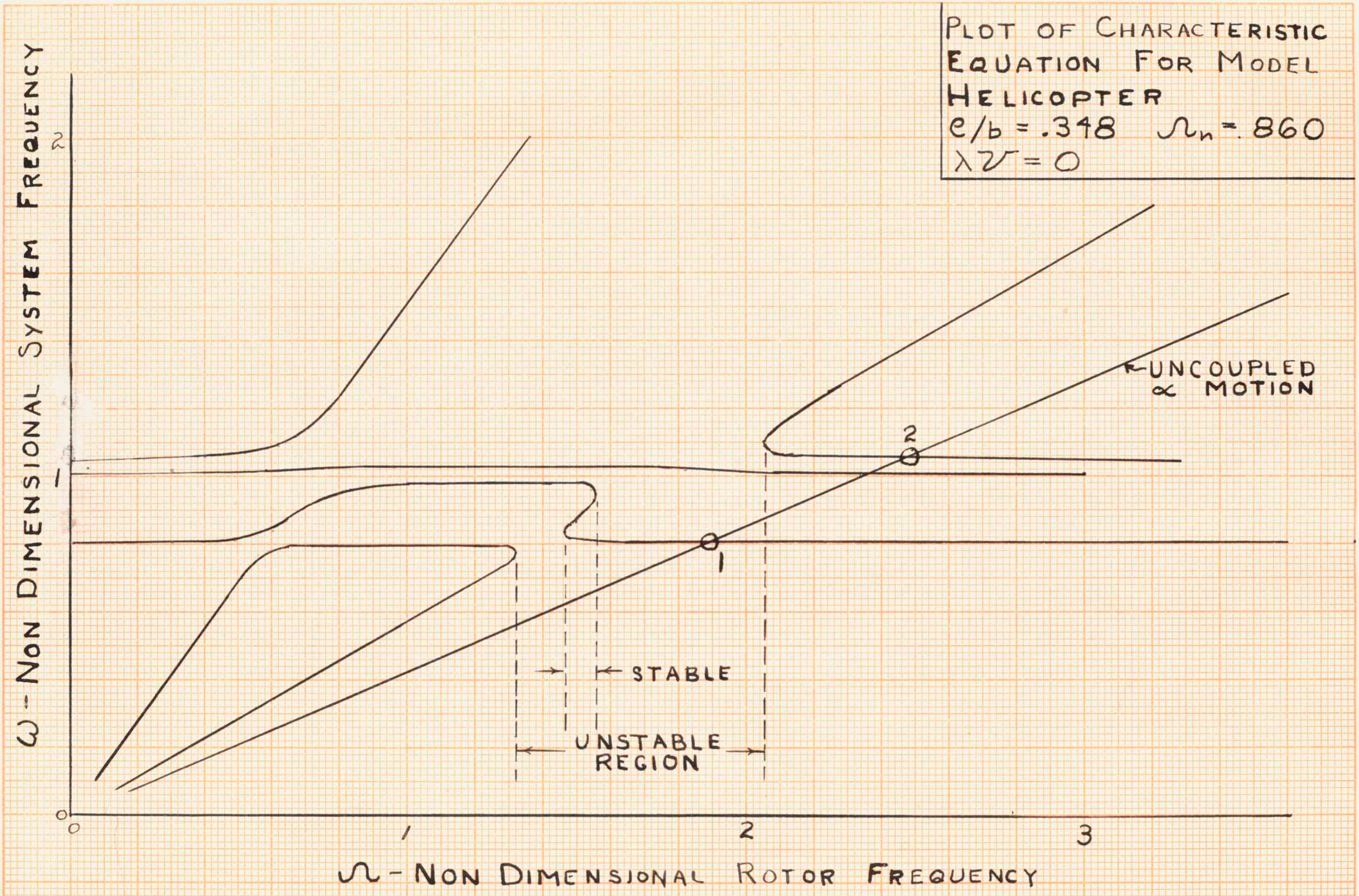

PLOT OF CHARACTERISTIC EQUATION FOR MODEL HELICOPTER
e/b = .348   Ωn = .860
λV = 0
ω - NON DIMENSIONAL SYSTEM FREQUENCY
Ω - NON DIMENSIONAL ROTOR FREQUENCY
UNCOUPLED α MOTION
STABLE
UNSTABLE REGION
0
1
2
3

FIGURE 6

VI. EFFECT OF SETTING $\lambda\nu$ EQUAL TO ZERO

On Mr. Loesch's model helicopter, the motor was mounted on the floor rather than in the machine. The rotor drive shaft was independent of the helicopter frame. Thus, no torques could be transmitted from the rotor to the frame or vice-versa. Thus there was no coupling between the $\underline{\alpha}$ and $\underline{\theta}$ motions. This is equivalent to setting $\lambda\nu = 0$. An investigation of the effect of setting $\lambda\nu = 0$ was felt necessary to determine whether the experimental results were valid. Setting $\lambda\nu = 0$ simplifies the characteristic equation (14), page 15, since it sets $A_{54}A_{63} = 0$. This permits factoring out A_{53}. This means that the $\underline{\alpha}$ motion is no longer coupled with the other motions. For the case of no damping, the characteristic equation can be solved for $\frac{\Omega}{\omega}$. The result is as follows:

$$(16) \qquad \left(\frac{\Omega}{\omega}\right)^4 + \left[-2\frac{\Lambda_1\Lambda_4}{\Lambda_2^2} - \frac{\mu}{\Lambda_2}\frac{C_3C_7}{C_1}\right]\left(\frac{\Omega}{\omega}\right)^2 + \left[\frac{\Lambda_1^2}{\Lambda_2^2} - \mu\frac{\Lambda_1}{\Lambda_2^2}\frac{C_3C_7}{C_1} + \frac{\mu^2}{\Lambda_2^2}\frac{C_6C_7^2}{C_1}\right] = 0$$

Equation (16) is plotted in Fig. 6. Notice that for this particular machine, setting $\lambda\nu = 0$ had only a small effect on the instability range for zero damping. The major difference between Fig. 5 and Fig. 6 is at points 1 and 2 of Fig. 6.

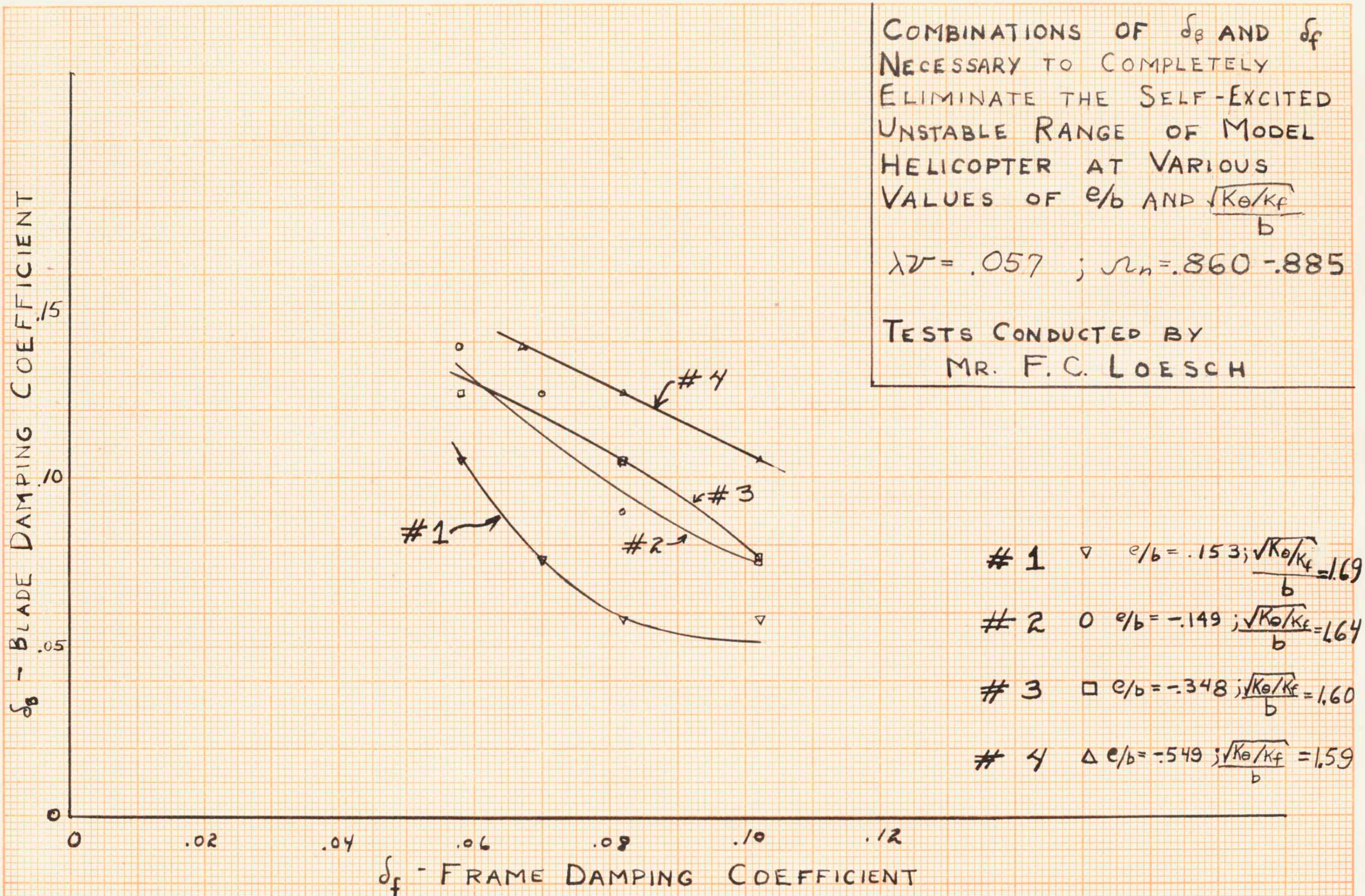

COMBINATIONS OF δ_β AND δ_f NECESSARY TO COMPLETELY ELIMINATE THE SELF-EXCITED UNSTABLE RANGE OF MODEL HELICOPTER AT VARIOUS VALUES OF e/b AND $\frac{\sqrt{K_\theta/K_f}}{b}$
$\lambda\nu = .057$; $\Omega_n = .860 - .885$
TESTS CONDUCTED BY MR. F. C. LOESCH
#1 ▽ $e/b = .153$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.69$
#2 o $e/b = -.149$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.64$
#3 □ $e/b = -.348$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.60$
#4 △ $e/b = -.549$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.59$
#1
#2
#3
#4
δ_β - BLADE DAMPING COEFFICIENT
0
.05
.10
.15
0
.02
.04
.06
.08
.10
.12
δ_f - FRAME DAMPING COEFFICIENT

FIGURE 7

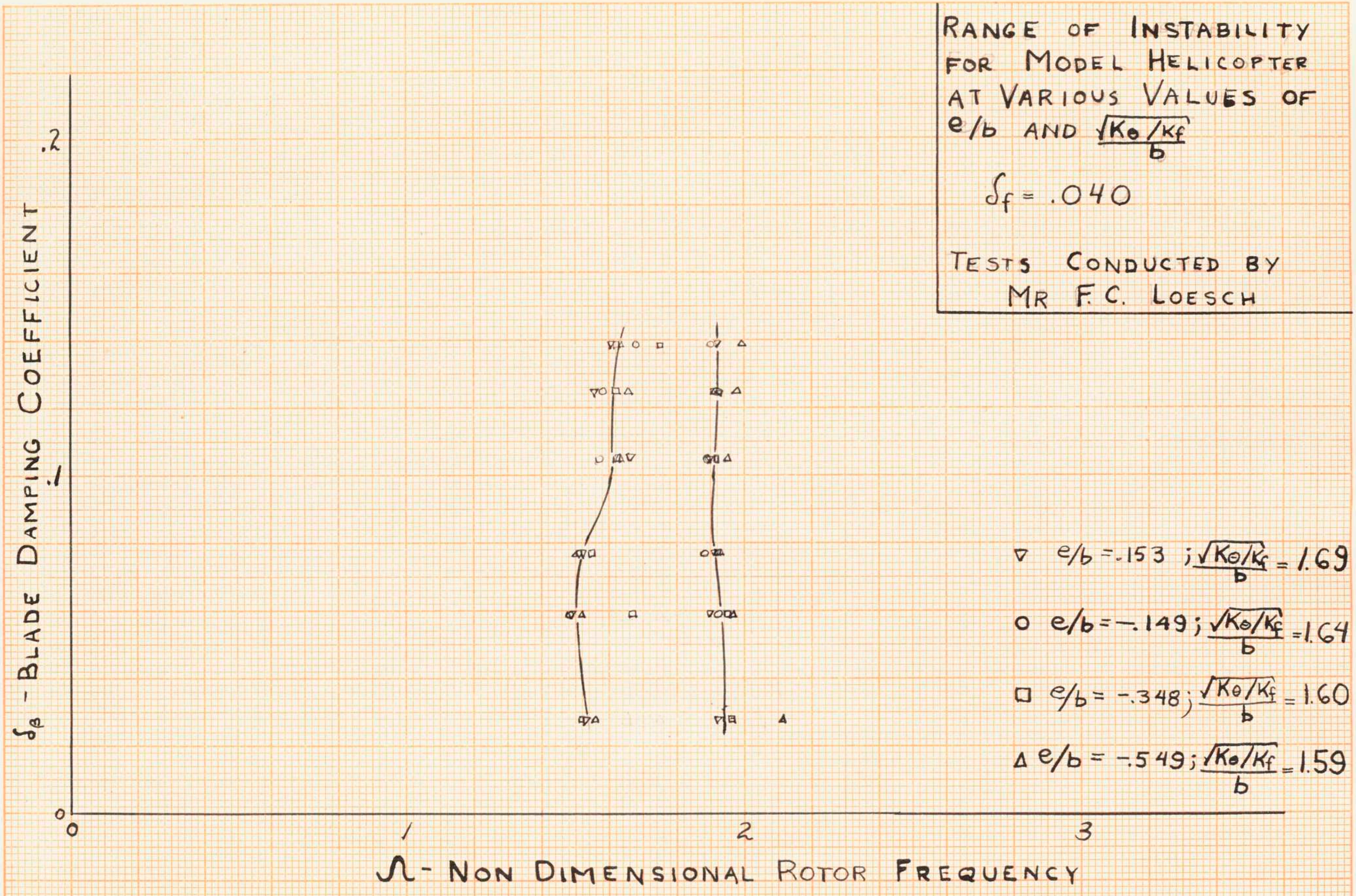

RANGE OF INSTABILITY FOR MODEL HELICOPTER AT VARIOUS VALUES OF e/b AND $\frac{\sqrt{K_\theta/K_f}}{b}$
$\delta_f = .040$
TESTS CONDUCTED BY MR F.C. LOESCH
▽ $e/b = .153$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.69$
○ $e/b = -.149$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.64$
□ $e/b = -.348$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.60$
△ $e/b = -.549$; $\frac{\sqrt{K_\theta/K_f}}{b} = 1.59$
δ_B - BLADE DAMPING COEFFICIENT
.2
.1
0
0
1
2
3
Ω - NON DIMENSIONAL ROTOR FREQUENCY

FIGURE 8

VII. DISCUSSION OF EXPERIMENTAL RESULTS

Mr. Loesch conducted a series of experiments on his model helicopter in which he systematically varied $\underline{e}$, $\underline{\delta}_f$ and $\underline{\delta}_\beta$. Apparently, the test equipment was not sensitive enough to determine accurately the effect of coupling on the instability range. The experimental error was of the same order of magnitude as the effect of coupling. (See Fig. 8). However, an important result of the tests is shown in Fig. 7. It shows that, for a particular value of $\underline{\delta}_f$, coupling definitely increases the value of $\underline{\delta}_\beta$ necessary completely to eliminate the instability. The variation in K_θ/K_f is considered small enough to be neglected. As no runs were made by Mr. Loesch for zero damping, there is no check on the theoretical work at this time.

VIII. DISCUSSION OF THEORETICAL RESULTS

Examination of Figs. 4, 5 and 6 indicates that for a helicopter dynamically similar to the one considered, coupling between the translational and rotational motion of the helicopter increases the range of self-excited instability. Neglecting the coupling between α and θ motion has a negligible effect on the instability range. However, the coupling effects may very possibly depend upon the points of intersection of the uncoupled curves of Fig. 4. Since these uncoupled curves depend only on the values of Ω_n and Λ_5, it is very possible that the importance of the two aforementioned couplings depends a great deal upon the particular values of Ω_n and Λ_5. Offhand, it seems probable that as high as possible a value of Ω_n is desirable to prevent extension of the instability range at the lower end. However, this should be verified by further investigations. The effect of increasing Λ_5 should be very interesting since, when $\Lambda_5 = .250$, $\frac{1}{\sqrt{\Lambda_5}} = \frac{1 + \sqrt{\Lambda_5}}{1 - \Lambda_5}$. That is, two of the uncoupled curves will approach each other and finally coincide. Variations in the values of Ω_n and Λ_5 may alter the magnitude of the effects of the two couplings. In any case, however, it seems desirable to keep the translation-rotation coupling of the helicopter body down to a minimum.

If these suggested investigations reveal that $\lambda\nu$ can be set equal to zero, this theory can be extended to include the effects of damping, using the simplified form of the characteristic equation (14), ($A_{54}A_{63} = 0$). This statement is based on the assumption that if $\lambda\nu$ can be neglected for the case of no damping, it can also be safely neglected for the case with damping. It would probably be a good idea to make a few spot checks to test the validity of this last statement. An investigation of the effects of damping can be carried out using Coleman's method. If it is necessary to use the exact characteristic equation, the investigation will probably be quite tedious.

The existence of a small stable range in the midst of the unstable range of Figs. 5 and 6 is quite interesting. Since the lower unstable range is rather small, it would probably be possible to skip over it entirely in an experimental investigation. This may have happened in Mr. Loesch's work, as a number of test points were rather far off. Thus, checking the theoretical work at only two points (the limits of instability) seems a bit unreliable. A suggested method for checking the entire characteristic equation would be as follows: Mount a vibrator on the frame of the model helicopter; set the rotor rpm; vary the vibrator frequency and record those vibrator frequencies at which the helicopter exhibited large oscillations; repeat for a complete range of rotor rpm; a plot of the vibrator frequencies versus the rotor frequencies should give a reliable check on the theoretically derived characteristic curve. Of course, no data can be obtained in the instability range.

The criterion for dynamic similarity between two helicopters with regard to ground resonance can be deduced from an examination of equation (15), page 15. The coefficients should be the same for both cases. Therefore, for dynamic similarity, the following dimensionless ratios should be the same: $\frac{a}{b}$, $\frac{r}{b}$, $\frac{e}{b}$, $\frac{\sqrt{K_\theta/K_f}}{b}f$, $\frac{R}{b}$, and μ. Also, of course, the non-dimensional damping coefficients, $\underline{\delta}_\beta$, $\underline{\delta}_f$ and $\underline{\delta}_\theta$, should be the same.

IX. CONCLUSIONS

These concluding remarks apply to a helicopter dynamically similar to the one investigated.

1. For the case of no damping, theory shows that coupling between the translational and rotational motions of a helicopter body extends the instability range at both ends and increases the number of shaft critical speeds to three.
2. For the case with damping, the effect of coupling is of the same order of magnitude as the experimental error.
3. For the case of no damping, theory shows that the characteristic equation can be simplified by setting $\lambda\nu = 0$ without appreciably affecting the instability range.
4. For a particular value of frame damping, experimental results show that translation-rotation coupling of the helicopter body increases the value of blade damping necessary to eliminate completely the unstable range.
5. Two helicopters are dynamically similar with regard to their ground resonance characteristics if the following non-dimensional factors are the same for both:

$$\frac{a}{b}, \frac{r}{b}, \frac{e}{b}, \frac{\sqrt{K_\theta/K_f}}{b}, \frac{\bar{R}}{b}, \mu, \delta_f, \delta_\theta, \delta_\beta$$

6. Reduction in the value of $\underline{\Omega}_n$ and/or increase in the value of $\underline{\Delta}_5$ may appreciably change the above conclusions. An investigation of the effect of variation in $\underline{\Omega}_n$ should prove to be very practical.

X. SUMMARY OF TEST RESULTS

$\frac{e}{b} = +.153$ $\qquad \frac{\sqrt{K_\theta/K_f}}{b} = 1.69$

δ_β	δ_f	Instability Range Ω lower	Ω upper
.0283	.040	1.52	1.92
	.058	1.52	1.81
	.082	1.50	1.75
	.102	1.65	1.73
.0593	.040	1.48	1.90
	.058	1.56	1.81
	.082	) none	
	.102	)	
.0773	.040	1.51	1.91
	.058	1.62	1.77
	.070	none	
.105	.040	1.56	1.89
	.051	1.52	1.91
	.058	none	
.125	.040	1.47	1.92
	.051	1.57	1.69
	.058	none	
.139	.040	1.51	1.92
	.051	1.60	1.69
	.058	none	

$\frac{e}{b} = -.149$ $\qquad \frac{\sqrt{K_\theta/K_f}}{b} = 1.64$

δ_θ	δ_f	Instability Range Ω lower	Ω upper
.0283	.102	1.60	1.62
.0	.082	)	
	.058	) No record	
	.040	)	
.0593	.040	1.48	1.92
	.058	1.52	1.81
	.082	1.67	1.75
	.102	1.75	1.62
.0773	.040	1.51	1.88
	.058	1.54	1.82
	.070	1.75	1.70
	.082	1.72	1.73
	.102	none	
.105	.040	1.57	1.89
	.058	1.62	1.83
	.070	1.66	1.74
	.082	none	
.125	.040	1.58	1.91
	.058	1.67	1.67
	.070	possible	
.139	.040	1.68	1.90
	.058	) No record	
	.070	)	
	.082	none	

$\frac{e}{b} = -.348$ $\frac{\sqrt{K_\theta/K_f}}{b} = 1.60$

δ_β	δ_f	Instability Range Ω lower	Instability Range Ω upper
.0283	.040	1.52	1.96
	.058	1.52	1.86
	.082	1.54	1.80
	.102	1.71	1.69
.0593	.040	1.67	1.95
	.058	1.69	1.79
	.082	No record	
	.102	No record	
.0773	.040	1.54	1.92
	.058	1.58	1.74
	.082	1.77	1.67
	.102	none	
.105	.040	1.62	1.91
	.058	1.67	1.75
	.082	none	
.125	.040	1.62	1.91
	.058	none	
.139	.040	1.62	1.75
	.043	1.73	1.75
	.058	none	

$\frac{e}{b} = -.549$ $\frac{\sqrt{K_\theta/K_f}}{b} = 1.59$

δ_θ	δ_f	Instability Range Ω lower	Instability Range Ω upper
.0283	.040	1.56	2.12
.058	.058	1.56	1.89
	.082	1.62	1.80
	.102	1.65	1.75
.0593	.040	1.52	1.96
	.058	1.52	1.86
	.082	1.58	1.73
	.102	1.65	1.75
.0773	.040	1.50	1.92
	.058	1.75	1.86
	.082	1.56	1.71
	.102	1.75	1.71
.105	.040	1.53	1.95
	.058	1.52	1.86
	.082	1.62	1.73
	.102	none	
.125	.040	1.56	1.98
	.051	1.70	1.80
	.058	No record	
.139	.040	1.53	1.99
	.058	1.86	1.75
	.067	none	

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