Kazhdan-Laumon Categories and Representations

by

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ABSTRACT

In 1988, D. Kazhdan and G. Laumon constructed the *Kazhdan-Laumon category*, an abelian category $\mathcal{A}$ associated to a reductive group G over a finite field, with the aim of using it to construct discrete series representations of the finite Chevalley group $G(\mathbb{F}_q)$. The well-definedness of their construction depended on their conjecture that this category has finite cohomological dimension. This was disproven by R. Bezrukavnikov and A. Polishchuk in 2001, who found a counterexample for $G = SL_3$.

Since the early 2000s, there has been little activity in the study of Kazhdan-Laumon categories, despite them being beautiful objects with many interesting properties related to the representation theory of G and the geometry of the basic affine space G/U. In the first part of this thesis, we conduct an in-depth study of Kazhdan-Laumon categories from a modern perspective. We first define and study an analogue of the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$ for Kazhdan-Laumon categories and study its combinatorics, establishing connections to Braverman-Kazhdan's Schwartz space on the basic affine space and the semi-infinite flag variety. We then study the braid group action on $D^b(G/U)$ (the main ingredient in Kazhdan and Laumon's construction) and show that it categorifies the *algebra of braids and ties*, an algebra previously studied in knot theory; we then use this to provide conceptual and geometric proofs of new results about this algebra.

After Bezrukavnikov and Polishchuk's counterexample to Kazhdan and Laumon's original conjecture, Polishchuk made an alternative conjecture: though the counterexample shows that the Grothendieck group $K_0(\mathcal{A})$ is not spanned by objects of finite projective dimension, he noted that a graded version of $K_0(\mathcal{A})$ can be thought of as a module over Laurent polynomials and conjectured that a certain localization of this module is generated by objects of finite projective dimension. He suggested that this conjecture could lead toward an alternate proof that Kazhdan and Laumon's construction is well-defined, and he proved this conjecture in Types A_1, A_2, A_3, and B_2. We prove Polishchuk's conjecture for all types and prove that Kazhdan and Laumon's construction is indeed well-defined, giving a new geometric construction of discrete series representations of $G(\mathbb{F}_q)$.

Thesis supervisor: Roman Bezrukavnikov
Title: Professor of Mathematics

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Chapter 1

Introduction

1.1 Historical overview

The perspective and tools provided by geometric representation theory have been remarkably useful in studying the representations of finite groups of Lie type, i.e. groups of the form $G(\mathbb{F}_q)$ for G a reductive Lie group and $\mathbb{F}_q$ a finite field. A crowning achievement was made in 1976 when P. Deligne and G. Lusztig [19] identified irreducible representations of $G(\mathbb{F}_q)$ using the cohomology of Deligne-Lusztig varieties. In 1988, a different and more explicit geometric construction of these representations was proposed by D. Kazhdan and G. Laumon in [32].

1.1.1 Kazhdan and Laumon's construction

In their 1988 paper [32], Kazhdan and Laumon described a gluing construction for perverse sheaves on the basic affine space associated to a semisimple algebraic group G split over a finite field $\mathbb{F}_q$, defining an abelian category $\mathcal{A}$ of "glued perverse sheaves" consisting of certain tuples of perverse sheaves on the basic affine space indexed by the Weyl group. They aimed to use these categories to provide a new geometric construction of representations of the group $G(\mathbb{F}_q)$.

Their proposal was to use $\mathcal{A}$ to construct representations as follows. First, they observed that the discrete series representations they sought to construct arise from characters of the non-split tori $T(w)$ of G, which are indexed by the Weyl group. For each $w \in W$, they defined a category $\mathcal{A}_{w,\mathbb{F}_q}$ in a way such that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$.

They expected that the infinite-dimensional representation

$$K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$$

of $G(\mathbb{F}_q)$ admits a finite-dimensional quotient whose $T(w)$-isotypic components are the discrete series representations they sought to construct. Following the same philosophy as in Grothendieck's sheaf-function dictionary, Kazhdan and Laumon knew that the appropriate

subspace of $K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$ by which one should take the quotient should be the kernel of a certain "Grothendieck-Lefschetz pairing" on $K_0(\mathcal{A}_{w,\mathbb{F}_q})$, which is defined in terms of the Ext groups in the category $\mathcal{A}$. They then made the following conjecture and proved that it implies the well-definedness of their representations.

Conjecture 1.1.1 (Kazhdan-Laumon, [32]). *The category $\mathcal{A}$ has finite cohomological dimension. In other words, for any two objects A and B of $\mathcal{A}$, there is an n for which $\mathrm{Ext}^i(A, B) = 0$ whenever $i > n$.*

1.1.2 A counterexample and a conjecture

More than a decade later, R. Bezrukavnikov and A. Polishchuk found a counterexample to this conjecture in the case $G = \mathrm{SL}_3$.

Proposition 1.1.2 (Bezrukavnikov-Polishchuk, Appendix to [48]). *Conjecture 1.1.1 is false.*

In [48], Polishchuk put forward the idea that although Conjecture 1.1.1 is false as stated in [32], it is not strictly necessary in order to prove the more important assertion that Kazhdan and Laumon's construction of representations is well-defined. He notes that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries the structure of a $\mathbb{Z}[v, v^{-1}]$-module using the formalism of mixed sheaves where v acts by a Tate twist, and then frames Conjecture 1.1.1 as the claim that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ is spanned by objects of finite projective dimension. In this situation, the Grothendieck-Lefschetz pairing defined on $K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$ can be thought of as taking polynomial values in $\mathbb{Z}[v, v^{-1}]$ and then specializing at $v = q^{\frac{1}{2}}$, which is one way to see why Conjecture 1.1.1 would imply the well-definedness of this pairing and therefore the well-definedness of Kazhdan and Laumon's construction.

Although Proposition 1.1.2 shows that this is false, Polishchuk instead proposes that this pairing is still well-defined if one allows it to take values in a certain localization of the ring $\mathbb{Z}[v, v^{-1}]$. Letting $\mathcal{A}_{\mathbb{F}_q} = \mathcal{A}_{e,\mathbb{F}_q}$, i.e. the category of "Weil sheaves" in the Kazhdan-Laumon context, Polishchuk proposes the following more precise conjecture as a first step toward this goal.

Conjecture 1.1.3. *There exists a finite set of polynomials which are nonzero away from roots of unity such that the localization of $K_0(\mathcal{A}_{\mathbb{F}_q})$ at the multiplicative set generated by these polynomials is generated by objects of finite projective dimension.*

In [48], Polishchuk develops a framework toward answering this conjecture, resolving it himself in Types A_2, A_2, A_3, and B_2. He observes that Kazhdan and Laumon's gluing construction relies on an action, defined in [32], of the generalized braid group B_W on the constructible derived category $D^b(G/U)$ of the basic affine space. He suggests that a deeper understanding of this braid action could lead to a proof of his Conjecture 1.1.3 in full generality.

1.1.3 Other work on Kazhdan-Laumon categories

Around the same time, in [16] A. Braverman and Polishchuk showed that there is a way to work around this issue (with the main insight being to use Radon transforms in place of Kazhdan and Laumon's Fourier-Deligne transforms) but only to construct those representations corresponding to characters $T(w)$ which satisfy the strict condition of *quasi-regularity* (c.f. Definition 3 in loc. cit.) At this point, the question of whether Polishchuk's conjecture could be proven and used to complete Kazhdan and Laumon's construction in general was left behind and not revisited in the literature until the results of this thesis.

The only other study of Kazhdan-Laumon categories which appeared in the literature since these results occured in [9]. In this paper, Bezrukavnikov-Braverman-Positselskii study the D-module analogue of the Kazhdan-Laumon construction. They observe that this Kazhdan-Laumon category is equivalent to the category of modules over the ring of global differential operators $\mathcal{D}(G/U)$ on the basic affine space. Their results were used fruitfully to study this ring by Levasseur and Stafford in [35]. This line of research marked the beginning of the study of Kazhdan-Laumon categories for their own sake as interesting objects which encode information about the representation theory of G and the geometry of the basic affine space.

1.2 The definition of Kazhdan-Laumon categories

In this section, we explain Kazhdan and Laumon's definition from [32], which was further expanded on and explained very clearly in [48, Section 1]. We will not need most of the technical details in this construction until Chapter 4. However, since Kazhdan-Laumon categories are the main object of study throughout most of this thesis, we provide the definition here in the introduction to give a sense of how the construction works.

General setup

Let G be a split semisimple group over a finite field $\mathbb{F}_q$. Let T be a Cartan subgroup split over $\mathbb{F}_q$, B a Borel subgroup containing T, and U its unipotent radical. Let $X = G/U$ be the basic affine space associated to G considered as a variety over $\mathbb{F}_q$. Let W be the Weyl group W. We let S denote the set of simple reflections in W. Writing $q = p^m$ for some prime number p, we choose ℓ to be a prime with $\ell \neq p$.

ℓ-adic sheaves, Tate twists, and Grothendieck groups

We work with the category $\mathrm{Perv}(G/U)$ of mixed ℓ-adic perverse sheaves on the basic affine space G/U, and more generally with the constructible derived category $D^b(G/U)$ of mixed ℓ-adic sheaves on G/U. We choose an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and work with $\mathbb{C}$ going forward. Pick a square root $q^{\frac{1}{2}}$ of q in $\mathbb{C}$ once and for all, and define the half-integer Tate twist $(\frac{1}{2})$

on $D^b(G/U)$. We then view $K_0(G/U) = K_0(D^b(G/U))$ as a $\mathbb{Z}[v, v^{-1}]$-module where v^{-1} acts by $(\frac{1}{2})$. When $\mathcal{C}$ is any category for which $K_0(\mathcal{C})$ is a $\mathbb{Z}[v, v^{-1}]$-module, we denote by $K_0(\mathcal{C}) \otimes \mathbb{C}$ the specialization $K_0(\mathcal{C}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ at $v = q^{\frac{1}{2}}$. We use $\mathbb{D}$ to denote the Verdier duality functor. We let ${}^pH^i$ be the perverse cohomology functors for any $i \in \mathbb{Z}$.

We also choose once and for all a nontrivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}_\ell}$, and let $\mathcal{L}_\psi$ be the corresponding Artin-Schrieier sheaf on $\mathbb{G}_a$. The variety G/U comes with a natural Frobenius morphism $\mathrm{Fr} : G/U \to G/U$; we can then consider the category of Weil sheaves $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$ on G/U, i.e. sheaves $K \in \mathrm{Perv}(G/U)$ equipped with a natural isomorphism $\mathrm{Fr}^* K \cong K$.

Elements of G indexed by W

For every simple root α_s of G corresponding to the simple reflection s, we fix an isomorphism of the corresponding one-parameter subgroup $U_s \subset U$ with the additive group $\mathbb{G}_a$. This uniquely defines a homomorphism $\rho_s : SL_2 \to G$ which induces the given isomorphism of $\mathbb{G}_a$ (embedded in SL_2 as upper-triangular matrices) with U_s; then let

$$n_s = \rho_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any $w \in W$, writing a reduced word $w = s_{i_1} \dots s_{i_k}$ we set $n_w = n_{s_{i_1}} \dots n_{s_{i_k}}$, and one can check that this does not depend on the reduced word. We also define for any $s \in S$ the subtorus $T_s \subset T$ obtained from the image of the coroot $\alpha_s^\vee$ and define T_w for any $w \in W$ to be the product of all T_s $(s \in S)$ for which $s \leq w$ in the Bruhat order.

1.2.1 The category $\mathcal{A}$

Fourier transforms on $\mathrm{Perv}(G/U)$

In [32] and [48], to each $w \in W$ the authors assign an element of $D^b(G/U \times G/U)$ which, up to shift, is perverse and irreducible. Following [48], let $X(w) \subset G/U \times G/U$ be the subvariety of pairs $(gU, g'U) \subset (G/U)^2$ such that $g^{-1}g' \in Un_wT_wU$. There is a canonical projection $\mathrm{pr}_w : X(w) \to T_w$ sending $(gU, g'U)$ to the unique $t_w \in T_w$ such that $g^{-1}g' \in Un_wt_wU$. In the case when $w = s \in S$, the morphism $\mathrm{pr}_s : X(s) \to T_s \cong \mathbb{G}_{m,k}$ extends to $\overline{\mathrm{pr}}_s : \overline{X(s)} \to \mathbb{G}_{a,k}$ and we have

$$\overline{K(s)} = (-\overline{\mathrm{pr}}_s)^* \mathcal{L}_\psi$$

and in the case of general $w \in W$

$$\overline{K(w)} = \overline{K(s_{i_1})} * \dots * \overline{K(s_{i_k})} \tag{1.2.1}$$

whenever $w = s_{i_1} \dots s_{i_k}$ is a reduced expression, where $*$ denotes the convolution of sheaves on $G/U \times G/U$ as defined in [32]. One can take this as the definition of Kazhdan-Laumon sheaves, which is well-defined by the proposition below, or refer to the explicit definition of $\overline{K(w)}$ which works for all $w \in W$ at once given in [32] or [48].

Proposition 1.2.1 ([32]). *The Kazhdan-Laumon sheaves $\overline{K(s)}$ for $s \in S$ under convolution satisfy the braid relations (up to isomorphism).*

For any $s \in S$, they note that the endofunctor $K \to K * \overline{K(s)}$ of $D^b(G/U)$ can be identified with a certain "symplectic Fourier-Deligne transform" defined as follows. Let P_s be the parabolic subgroup of G associated to s, and let $Q_s = [P_s, P_s]$. The map $G/U \to G/Q_s$ has all fibers isomorphic to $\mathbb{A}^2 \setminus \{(0,0)\}$, and it is shown in Section 2 of [32] that there exists a natural fiber bundle $\pi : V_s \to G/Q_s$ of rank 2 equipped with a G-invariant symplectic pairing which contains G/U as an open set, with inclusion $j : G/U \to V_s$ with $\pi \circ j$ being the original projection $G/U \to G/Q_s$. There is then a symplectic Fourier-Deligne transform $\tilde{\Phi}_s$ on $D^b(V_s)$ defined by

$$\tilde{\Phi}_s(K) = p_{2!}(\mathcal{L} \otimes p_1^*(K))[2](1), \tag{1.2.2}$$

where the p_i are the projections of the product $V_s \times_{G/Q_s} V_s$ on its factors, and $\mathcal{L} = \mathcal{L}_\psi(\langle , \rangle)$ is a smooth rank-1 $\overline{\mathbb{Q}_\ell}$-sheaf which is the pullback of the Artin—Schreier sheaf $\mathcal{L}_\psi$ under the morphism $\langle , \rangle$, c.f. Section 4 of [48]. We then define the endofunctor Φ_s of $D^b(G/U)$ By

$$\Phi_s(K) = j^* \tilde{\Phi}_s j_! K. \tag{1.2.3}$$

Proposition 1.2.2 ([32], [48]). *The functors Φ_s and $- * \overline{K(s)}$ are naturally isomorphic.*

For any $w \in W$, we let $\Phi_w = \Phi_{s_{i_1}} \dots \Phi_{s_{i_k}}$ where $s_{i_1} \dots s_{i_k}$ is a reduced expression for w as a product of simple reflections. The functors Φ_w are the gluing functors which Kazhdan and Laumon use to define the so-called glued categories $\mathcal{A}$.

Definition 1.2.3. Using the six-functor formalism, one can check that each Φ_s has a right adjoint which we call Ψ_s, following the setup of Section 1.2 of [48]. The functors Ψ_s also form a braid action on $D^b(G/U)$. We then define Ψ_w similarly.

The functors Φ_w (resp. Ψ_w) are each right (resp. left) t-exact on $D^b(G/U)$ with respect to the perverse t-structure. For any $w \in W$, let $\Phi_w^\circ = {}^pH^0\Phi_w$ and $\Psi_w^\circ = {}^pH^0\Psi_w$, noting that $\Phi_w = L\Phi_w^\circ$, $\Psi_w = R\Psi_w^\circ$.

Proposition 1.2.4 ([48], Section 4.1). *For any $s \in S$, there are natural morphisms $c_s : \Phi_s^2 \to \mathrm{Id}$ and $c_s' : \mathrm{Id} \to \Psi_s^2$ satisfying the associativity conditions*

$$\Phi_s \circ c_s = c_s \circ \Phi_s : \Phi_s^3 \to \Phi_s \qquad \Psi \circ c_s' = c_s' \circ \Psi : \Psi_s \to \Psi_s^3. \tag{1.2.4}$$

Corollary 1.2.5. *For any $y', y \in W$, there is a natural transformation $\nu_{y',y} : \Phi_{y'}\Phi_y \to \Phi_{y'y}$.*

Proof. We go by induction on $\ell(y)+\ell(y')$. If $\ell(y'y) = \ell(y')+\ell(y)$, then $\nu_{y',y}$ is the tautological map arising from the fact that the Φ_w form a braid action. If instead $\ell(y'y) < \ell(y') + \ell(y)$, then there exists some $s \in S$ such that $y' = \tilde{y}'s$ and $y = s\tilde{y}$ for some $\tilde{y}', \tilde{y} \in W$ with $\ell(\tilde{y}'s) = \ell(\tilde{y}') + 1$ and $\ell(s\tilde{y}) = \ell(\tilde{y}) + 1$, and so we have maps

$$\Phi_{y'}\Phi_y = \Phi_{\tilde{y}'}\Phi_s^2\Phi_{\tilde{y}} \xrightarrow{\Phi_{\tilde{y}} \circ c_s \circ \Phi_{\tilde{y}}} \Phi_{\tilde{y}'}\Phi_{\tilde{y}} \xrightarrow{\nu_{\tilde{y}',\tilde{y}}} \Phi_{y'y},$$

the former coming from c_s and the latter coming from our induction hypothesis. □

Definition of the Kazhdan-Laumon category

Definition 1.2.6 ([32], [48])**.** The Kazhdan-Laumon category $\mathcal{A}$ has objects which are tuples $(A_w)_{w\in W}$ with $A_w \in \mathrm{Perv}(G/U)$ and equipped with morphisms

$$\theta_{y,w} : \Phi_y^\circ A_w \to A_{yw} \tag{1.2.5}$$

for every $y, w \in W$ such that the diagram

$$\begin{array}{ccc} \Phi_{y'}^\circ\Phi_y^\circ A_w & \xrightarrow{\Phi_{y'}\theta_{y,w}} & \Phi_{y'}^\circ A_{yw} \\ \downarrow{\scriptstyle \nu_{y',y}} & & \downarrow{\scriptstyle \theta_{y',yw}} \\ \Phi_{y'y}^\circ A_w & \xrightarrow{\theta_{y'y,w}} & A_{y'yw} \end{array}$$

commutes for any $y, y', w \in W$.

A morphism f between objects $(A_w)_{w\in W}$ and $(B_w)_{w\in W}$ is a collection of morphisms $f_w : A_w \to B_w$ such that

$$\begin{array}{ccc} \Phi_y^\circ A_w & \xrightarrow{\Phi_y^\circ f_w} & \Phi_y^\circ B_w \\ \downarrow{\scriptstyle \theta^A_{y,w}} & & \downarrow{\scriptstyle \theta^B_{y,w}} \\ A_{yw} & \xrightarrow{f_{yw}} & B_{yw}. \end{array}$$

It is shown in [48] that this category is abelian, and that the functors $j_w^* : \mathcal{A} \to \mathrm{Perv}(G/U)$ defined by $j_w^*((A_w)_{w\in W}) = A_w$ are exact.

Remark 1.2.7. We could have instead asked for morphisms $A_{yw} \to \Psi_y^\circ A_w$, making reference to the functors Ψ_y° rather than the Φ_y°. Later, we will discuss an alternate and more elegant definition of $\mathcal{A}$ as coalgebras over a certain comonad on $\oplus_{w\in W}\mathrm{Perv}(G/U)$ which is assembled from the functors Ψ_y°. In Definition 1.2.6 though, we present the definition of the Kazhdan-Laumon category as it was originally formulated in [32] and later explained in more detail in [48].

Definition of the w-twisted categories $\mathcal{A}_{w,\mathbb{F}_q}$

We note that the category $\mathcal{A}$ carries an action of the Weyl group W as follows. For any $w \in W$, we let $\mathcal{F}_w : \mathcal{A} \to \mathcal{A}$ be the exact functor defined by right translation of the indices in the tuple, i.e. $\mathcal{F}_w((A_y)_{y\in W}) = (A_{yw})_{y\in W}$.

Definition 1.2.8. For any $w \in W$, let $\mathcal{A}_{w,\mathbb{F}_q}$ be the category with objects (A, ψ_A) where $A \in \mathcal{A}$ and $\psi_A : \mathcal{F}_w \mathrm{Fr}^* A \to A$ is an isomorphism. We call these w-twisted Weil sheaves in the Kazhdan-Laumon category.

For any two such objects (A, ψ_A) and (B, ψ_B), we let $\mathrm{Hom}_{\mathcal{A}_{w,\mathbb{F}_q}}(A, B)$ be the set of morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ such that $f \circ \psi_A = \psi_B \circ \mathcal{F}_w \mathrm{Fr}^* f$.

Remark 1.2.9. When $w = e$ is trivial, we will write $\mathcal{A}_{\mathbb{F}_q} = \mathcal{A}_{w,\mathbb{F}_q}$. In this case, it is shown in [48] that $\mathcal{A}_{\mathbb{F}_q}$ is equivalent to the category obtained by applying the Kazhdan-Laumon gluing procedure described in Definition 1.2.6 to the category $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$ of Weil perverse sheaves on G/U, i.e. perverse sheaves K equipped with an isomorphism $\mathrm{Fr}^* K \to K$.

1.3 A summary of this thesis

In this thesis, which comprises the three papers [44], [47], and [46], we revisit Kazhdan-Laumon categories using objects and tools which have appeared in modern geometric representation theory. We now give a brief overview of each chapter and the main results therein, followed by a general summary of the philosophy of (and potential future directions entailed by) the work in this thesis.

Each of the three remaining chapters (Chapter 2, 3, and 4) corresponds to a paper ([44], [47], and [46] respectively). To stay consistent with the conventions of these papers, we include in each chapter a short overview of setup, preliminaries, and notation which will be used in a self-contained way throughout each chapter.

1.3.1 An analogue of Category $\mathcal{O}$

We begin by defining and studying a subcategory of the Kazhdan-Laumon category which we call $\mathcal{A}_B$, the Kazhdan-Laumon Category $\mathcal{O}$. Much like the usual Bernstein-Gelfand-Gelfand Category $\mathcal{O}$ of representations of a semisimple Lie algebra, this category has some beautiful combinatorial properties, which we explore and use to establish novel connections between Kazhdan-Laumon categories and other objects of study in geometric representation theory. In particular,

(i) in Theorem 2.1.1, we explicitly classify the simple objects of the Kazhdan-Laumon Category $\mathcal{O}$ in combinatorial terms.

(ii) In Theorem 2.1.2, we give an injective morphism of Hecke algebra modules from the Grothendieck group $K_0(\mathcal{A}_B)$ of the Kazhdan-Laumon Category $\mathcal{O}$ to Lusztig's periodic Hecke module M_d (thereby giving a combinatorial interpretation and a sort of "Kazhdan-Lusztig theory" for important objects in the Kazhdan-Laumon Category $\mathcal{O}$).

(iii) In Theorem 2.1.3, we exhibit $K_0(\mathcal{A}_B)$ in terms of functions on the basic affine space over a local field, i.e. as a subspace of the Iwahori invariants on Braverman-Kazhdan's Schwartz space of the basic affine space,

(iv) In Theorem 2.1.4, we categorify these results via a functor from $D^b(\mathcal{A}_B)$ to a suitably-defined constructible derived category on the semi-infinite flag variety, which also gives a description of $\mathcal{A}_B$ as a full subcategory of perverse sheaves on the semi-infinite flag variety.

1.3.2 The braid group action on $D^b(G/U)$ and connections to knot theory

We then build on these results to start directly addressing the conjectures of Kazhdan-Laumon and Polishchuk toward the construction of representations. Following Polishchuk's philosophy that there are insights to be found in a deeper understanding of the braid group action on $D^b(G/U)$ via symplectic Fourier-Deligne transforms, we study this action in depth in Chapter 3, which is adapted from the paper [47]. Here, we establish an unexpected connection to knot theory, which both elucidates the behaviour of the braid group action for the sake of studying Kazhdan-Laumon categories, and also allows us to prove some algebraic results which were previously unknown. In particular, we show that symplectic Fourier-Deligne transforms on $D^b(G/U)$ give a categorification of a subalgebra (defined in [42]) of the *algebra of braids and ties* (defined in [3]), which is related to the Yokonuma-Hecke algebra of [53]. This yields

(i) an understanding of all the relations satisfied by this action of the braid algebra, and a concrete explanation for the one cubic relation which had been previously identified by Polishchuk in [48],

(ii) a short, type-independent, geometric proof of the braid relations for Juyumaya's generators of the Yokonuma-Hecke algebra, previously only known by a case-by-case computation across the two papers [28] and [30] (see Corollary 3.1.2),

(iii) a computation of the dimension of Marin's subalgebra from [42] (see Theorem 3.1.4), previously only known in Type A_n for $n \leq 4$, and

(iv) an answer to a conjecture of Marin about a Kazhdan-Lusztig-style canonical basis in the algebra of braids and ties (see Theorem 3.1.3).

1.3.3 The completion of Kazhdan and Laumon's goal via a proof of Polishchuk's conjecture

Finally, we use all of the above results to directly address Polishchuk's conjecture and Kazhdan-Laumon's original goal. These results appear in Chapter 4, which is adapted from the paper [46]. In this section,

(i) we prove Polishchuk's Conjecture 1.1.3 in full generality (see Theorem 4.1.1).

(ii) We use monodromic sheaves to prove a similar conjecture for "twisted" Kazhdan-Laumon categories which correspond to characters of the non-split tori $T(w)$ of $G(\mathbb{F}_q)$ (see Theorem 4.1.2).

(iii) We explain how to use these results to complete Kazhdan-Laumon's goal, showing that their proposed construction indeed gives a well-defined geometric construction of representations of $G(\mathbb{F}_q)$ (see 4.1.3 and Section 4.8).

1.3.4 Conclusion

We emphasize that this thesis has two main purposes. First, to further the study of Kazhdan-Laumon categories and elucidate properties which we hope may be used fruitfully in other areas of representation theory in the future. Secondly, to use the understanding gained from these results to complete Kazhdan and Laumon's goal of constructing representations of $G(\mathbb{F}_q)$.

We hope that both of these goals will lead to interesting future research. In particular, the connection between Kazhdan-Laumon categories and the semi-infinite flag variety situates the Kazhdan-Laumon construction in the context of a *local geometric Langlands correspondence*, and provides connections to other areas of current activity in representation theory. We expect that this relationship should have more ramifications beyond the results already obtained in this thesis. For example, the results of Chapter 2 establish the Kazhdan-Laumon category as a player in the circle of ideas relating the small quantum group, the semi-infinite flag variety, and affine Springer fibers, c.f. [5] and [8].

Finally, Corollary 4.1.3 on the well-definedness of Kazhdan and Laumon's construction of representations of $G(\mathbb{F}_q)$ now opens the door to studying these representations more concretely. A natural future direction would be an in-depth study of the characters of Kazhdan-Laumon representations. In [16], Braverman and Polishchuk explain an explicit correspondence between Kazhdan-Laumon representations and Deligne-Lusztig representations in the special case of a quasi-regular character, and it would be interesting to compute the characters of Kazhdan-Laumon representations explicitly and compare them directly to Deligne-Lusztig characters in case of an arbitrary character.

In summary, we hope that these new connections and possibilities will make the results contained in this thesis the beginning of a new story in addition to the conclusion of an old one.

Chapter 2

Kazhdan-Laumon Category $\mathcal{O}$, Braverman-Kazhdan Schwartz space, and the semi-infinite flag variety

2.1 Introduction

Let $\mathbf{G}$ be a semisimple algebraic group split over a finite field or its algebraic closure, and let $\mathcal{A}$ denote the associated Kazhdan-Laumon category defined in [32]. In this chapter, we study a subcategory $\mathcal{A}_B$ (and its mixed analogue $\mathcal{A}_{\mathcal{P}}$) of $\mathcal{A}$ corresponding to B-equivariant perverse sheaves; this is a natural analogue of Category $\mathcal{O}$ in the context of Kazhdan and Laumon's construction. We begin by studying this category in its own right and describing its simple objects explicitly. To state the following theorem, for any element w in a Weyl group W let $P(w)$ be the standard parabolic subgroup of W generated by all simple reflections s with $\ell(ws) > \ell(w)$.

Theorem 2.1.1. *The Kazhdan-Laumon Category $\mathcal{O}$ associated to the group $\mathbf{G}$ with Weyl group W has*

$$\sum_{w \in W} |P(w)\backslash W| \tag{2.1.1}$$

irreducible objects, described explicitly in Theorem 2.4.7. For any choice of $w \in W$ and coset $P(w)z$, the corresponding simple object can be written as a tuple $(\mathcal{G}_y)_{y \in W}$ of irreducible objects in $\mathrm{Perv}_B(G/U)$ such that

$$\mathcal{G}_y \cong \begin{cases} \mathrm{IC}_w & y \in P(w)z \\ 0 & y \notin P(w)z \end{cases}.$$

The fact that objects of $\mathcal{A}$ are tuples of perverse sheaves indexed by W means that translation of the indices gives a natural action of W on $\mathcal{A}$ (and therefore also on $\mathcal{A}_B$ and $\mathcal{A}_{\mathcal{P}}$) by functors $\mathcal{F}_w$ for $w \in W$. Further, we show that $\mathcal{A}_{\mathcal{P}}$ admits a left action of the Hecke

algebra $\mathcal{H}$ by convolution with mixed B-equivariant perverse sheaves on the flag variety G/B. These actions give $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C}$ the structure of a $\mathcal{H}_q \otimes \mathbb{C}[W]$-module. We then describe this module explicitly in terms of a more familiar combinatorial object, namely Lusztig's periodic Hecke module defined in [36].

Theorem 2.1.2. *There is an isomorphism $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \cong M^0_{d,q}$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules, where $M^0_{d,q}$ is a submodule (introduced in Definition 2.3.8) of Lusztig's periodic Hecke module from [36].*

In [15], Braverman and Kazhdan defined the Schwartz space $\mathcal{S}$ of the basic affine space associated to an algebraic group over a non-Archimedean local field, equipped with a $\mathbb{C}[W]$-action by Fourier transform operators. They, too, found a combinatorial connection between $\mathcal{S}$ and Lusztig's module $M_{d,q}$, proving in loc. cit. that the subspace $\mathcal{S}^{I \times \mathbf{T}(\mathcal{O})}$ of Iwahori-invariants in $\mathcal{S}$ is isomorphic to $M_{d,q}$ as a $\mathcal{H}_q \otimes \mathbb{C}[W]$-module. Combining their result with our Theorem 2.1.2 yields a description of $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C}$ as a certain natural subspace $\mathcal{S}_0^{I \times \mathbf{T}(\mathcal{O})}$ of $\mathcal{S}^{I \times \mathbf{T}(\mathcal{O})}$.

Theorem 2.1.3. *Composing the isomorphism from Theorem 1.2 with the morphism*

$$\mathcal{S}^{I \times \mathbf{T}(\mathcal{O})} \to M_{d,q}$$

from [15] gives an isomorphism

$$K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \cong \mathcal{S}_0^{I \times \mathbf{T}(\mathcal{O})}$$

of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules. Further, the isomorphism can be described directly in terms of Grothendieck's sheaf-function dictionary: the diagram

$$\begin{array}{ccc} K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} & \longrightarrow & \mathcal{S}^{I \times \mathbf{T}(\mathcal{O})} \\ & \searrow \quad \nearrow & \\ & M_{d,q} & \end{array}$$

commutes, where the left and right maps are the morphisms from Theorem 2.1.2 and [15] respectively, and the top map is described in Section 2.6.3 in terms of lifts to the local field setting of functions obtained by taking the trace of a Frobenius endomorphism on objects of $\mathcal{A}_\mathcal{P}$.

In [5], building off of work in [23], [22], and [14], the authors define a certain analogue of a category of perverse sheaves on the semi-infinite flag variety associated to $\mathbf{G}$. In 6.1.8 of [5], the authors discuss how the category of Iwahori-monodromic objects of their category can be viewed as a categorification of $\mathcal{S}^{I \times \mathbf{T}(\mathcal{O})}$. Using their main result which relates such Iwahori-monodromic objects to graded modules over the small quantum group, they define a W-action via functors F_w which categorify the Fourier transforms appearing in [15]. With

this in mind, we conclude this chapter by upgrading Theorem 2.1.3 to an equivalence of categories.

Theorem 2.1.4. *There exists a certain subcategory $\tilde{\mathcal{P}}$ of Iwahori-monodromic perverse sheaves on the semi-infinite flag variety with $K_0(\tilde{\mathcal{P}}) \otimes \mathbb{C} \cong \mathcal{S}_0^{I \times \mathbf{T}(\mathcal{O})}$. There is an equivalence of categories between $\mathcal{A}_{\mathcal{P}}$ and $\tilde{\mathcal{P}}$ which categorifies the isomorphism $K_0(\mathcal{A}_{\mathcal{P}}) \otimes \mathbb{C} \cong \mathcal{S}_0^{I \times \mathbf{T}(\mathcal{O})}$.*

This equivalence is compatible with convolution by perverse sheaves on G/B, which categorifies the $\mathcal{H}_q$-action on each side, and intertwines the W-action by functors $\mathcal{F}_w$ on $\mathcal{A}_{\mathcal{P}}$ and F_w on $\tilde{\mathcal{P}}$.

We hope that Theorem 2.1.4 will serve as a new perspective related to one of the goals stated in [5] to provide a geometric description of Braverman-Kazhdan's Fourier transforms. In loc. cit., Fourier transform functors on perverse sheaves on the semi-infinite flag variety are only defined in the case of Iwahori-monodromic objects, as these functors come from the equivalence in loc. cit. to modules over the small quantum group. If suitable functors could be defined on the full category of perverse sheaves on the semi-infinite flag variety, we hope that Theorem 2.1.4 might generalize to a suitable equivalence of categories to the full Kazhdan-Laumon category $\mathcal{A}$, thereby elevating Kazhdan and Laumon's construction from a tool intended for the study of finite Chevalley groups to an object with interesting connections to current objects of study related to a local geometric Langlands correspondence.

2.2 Preliminaries

Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Z}$. Letting $\kappa = \mathbb{F}_q$ where $q = p^n$ for p prime, we write $G = \mathbf{G}_\kappa$ for the base change to $\mathbb{F}_q$. We assume that G is split, connected, and simply-connected. Let $\mathbf{T}$ be a Cartan subgroup, $\mathbf{B}$ a Borel subgroup containing $\mathbf{T}$, and $\mathbf{U}$ its unipotent radical. We use similar notation T, B, and U for their respective base changes to κ. Let $X = G/U$ be the basic affine space associated to G, considered as a variety over $\mathbb{F}_q$. Let W be the Weyl group, and let $\tilde{W}$ be the affine Weyl group associated to $\mathbf{G}$, with $S \subset W$ and $\tilde{S} \subset \tilde{W}$ their respective subsets of simple reflections. We denote by w_0 the longest element of W. In Section 2.6, we will consider a local field k with residue field κ. In this setting, we will denote by G_k the base change of $\mathbf{G}$ to k, and by X_k the basic affine space over k. Let Π be the set of simple roots, let Γ (resp. $\Gamma^\vee$) be the coroot (resp. weight) lattice of $\mathbf{G}$.

Let $\mathcal{H}$ be the usual Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ generated by $\{T_w\}_{w \in W}$, normalized so that the quadratic relation satisfied by any T_s for $s \in S$ takes the form

$$T_s^2 = (v^2 - 1)T_s + v^2. \tag{2.2.1}$$

For any $w \in W$, let $\tilde{T}_w = v^{-\ell(w)} T_w$, so that for any $s \in S$,

$$(\tilde{T}_s - v)(\tilde{T}_s + v^{-1}) = 0. \tag{2.2.2}$$

Following the conventions of Section 3.5 of [52], recall the canonical basis C_w of $\mathcal{H}$ given in terms of the Kazhdan-Lusztig polynomials $P_{x,w}$ by

$$C_w = \sum_{x \leq w} (-1)^{\ell(w)-\ell(x)} v^{\ell(x)-\ell(w)} P_{x,w} \tilde{T}_{x^{-1}}^{-1}.$$

We will sometimes consider $\mathcal{H}$ as a subalgebra of $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is the affine Hecke algebra generated by $\{\tilde{T}_{\tilde{w}}\}_{\tilde{w} \in W}$ with $\tilde{T}_s$ satisfying the same relations as in (2.2.2) for any $s \in \tilde{S}$. Choosing once and for all a square root $q^{\frac{1}{2}}$ of q, we let $\mathcal{H}_q$ (resp. $\tilde{\mathcal{H}}_q$) be the specialization of $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$) at $v = q^{\frac{1}{2}}$.

Throughout, when $\mathcal{C}$ is an abelian category we will denote by $K_0(\mathcal{C})$ the Grothendieck group of the derived category $D^b(\mathcal{C})$, and we will freely consider the action of left- or right-exact functors on $K_0(\mathcal{C})$ by identifying them with their right- or left-derived functors. When $\mathcal{C}$ is an appropriate highest-weight category of perverse sheaves, we can use Corollary 3.3.2 of [6] to freely identify its bounded derived category with the corresponding underlying constructible derived category.

2.3 Lusztig's periodic Hecke module

In this section, we recall the periodic Hecke module considered in [39] and [36], following the notational conventions of the latter. Our main object of study is the related module M_d defined in loc. cit. The module M_d and its specialization $M_{d,q}$ will appear in every subsequent section, serving as the combinatorial data which underlie all three of the main objects of study in this chapter (the Kazhdan-Laumon Category $\mathcal{O}$, the Schwartz space of the basic affine space, and perverse sheaves on the semi-infinite flag variety).

2.3.1 Background and setup

To begin, we follow the setup of Section 4.2 of [15]. Let Ξ denote the collection of all alcoves for the group $\tilde{W}$ in the real Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathbf{T}$. The set Ξ admits two commuting actions of the group $\tilde{W}$, one on the left and the other on the right. We will follow the conventions of [36] so that the left action of any $s \in \tilde{S}$ takes an alcove A to some neighboring alcove sA, with the right action defined so that each $s \in \tilde{S}$ reflects an alcove around the corresponding affine hyperplane H_s in $\mathfrak{t}_{\mathbb{R}}$, so that A and As are not, in general, neighboring. Let $\mathcal{C}^+ \subset \mathfrak{t}_{\mathbb{R}}$ denote the dominant Weyl chamber. For any $\gamma \in \Gamma^\vee$, let A_γ^+ denote the unique alcove $A \in \gamma + \mathcal{C}^+$ for which $\gamma \in \overline{A}$, and let $A^+ = A_0^+$, the "fundamental alcove." We will also write $A_w = wA^+$ for any $w \in \tilde{W}$.

We also denote by $d : \Xi \times \Xi \to \mathbb{Z}$ (and d_α for $\alpha \in \Pi$) the "relative distance" functions defined in [36], and we use also the definition of $\mathcal{L}(A)$ from 1.2 of loc. cit.; informally, for $A \in \Xi$, $\mathcal{L}(A)$ is the subset of $\tilde{S}$ for which A is "above" sA, where the direction is determined by that of $\mathcal{C}^+$.

Definition 2.3.1. The module M_c is the $\mathbb{Z}[v, v^{-1}]$-span of Ξ with an action of $\tilde{\mathcal{H}}$ defined for $s \in \tilde{S}$ by

$$\tilde{T}_s(A) = \begin{cases} sA & \text{if } s \notin \mathcal{L}(A), \\ sA + (v - v^{-1})A & \text{if } s \in \mathcal{L}(A), \end{cases} \tag{2.3.1}$$

and extended naturally into a left action. It also has an action of Γ commuting with the $\tilde{\mathcal{H}}$-action defined by $\gamma \cdot A = A + \gamma$ for $\gamma \in \Gamma$.

Definition 2.3.2. Let $M_{\geq}$ be the "upward semi-infinite completion" of M_c, consisting of all formal sums $\sum_{A \in \Xi} m_A A$ such that the set $\{A \in \Xi \mid m_A \neq 0\}$ is bounded below (in the sense of 4.13 of [36]).

2.3.2 An action of W

For any $\alpha \in \Pi$, an operator $\theta_\alpha : M_{\geq} \to M_{\geq}$ is defined in [36] as follows. First any $A \in \Xi$, $\alpha \in \Pi$, a sequence A^n, $n \geq 0$ is defined by the conditions that $A^0 = As_\alpha$, and $d_\alpha(A^n, A^{n+1}) > 0$, in addition to the condition that A^n lie in the same "α-strip" as A (c.f. [36] for a more precise definition). Then

$$\theta_\alpha(A) = v^{-1}A^0 + \sum_{n=1}^{\infty} (-1)^{n+1}(v^{-n+1} - v^{-n-1})A^n. \tag{2.3.2}$$

We define for $w \in W$ the operator $\theta_w = \theta_{\alpha_1} \cdots \theta_{\alpha_k}$ for $w = s_{\alpha_1} \cdots s_{\alpha_k}$ a reduced expression, which yields a well-defined action of the Weyl group W on $M_{\geq}$.

2.3.3 The modules M_d and M_d^0

In [36], Lusztig defines a duality operator $\tilde{b}$ on $M_{\geq}$ and exhibits a "canonical basis" for $M_{\geq}$ invariant under this operator.

Proposition 2.3.3 (Theorem 12.2 in [36])**.** *For each $A \in \Xi$, there exists an associated element $A^\sharp \in M_{\geq}$ which satisfies $A^\sharp = A + v^{-1}\sum_B \mathbb{Z}[v^{-1}]B$ and $\tilde{b}(A^\sharp) = A^\sharp$.*

Definition 2.3.4. The module $M_d \subset M_{\geq}$ is the $\mathbb{Z}[v, v^{-1}]$ span of $\{A^\sharp\}_{A \in \Xi}$. It is invariant under the natural actions of $\tilde{\mathcal{H}}$ and Γ obtained by extending the actions defined on M_c. Let $M_{d,q} = M_d \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ be the specialization of M_d at $v = q^{\frac{1}{2}}$.

We now introduce Lusztig's $*$-action of W on Ξ, which we will then use to describe the action of the θ_w on M_d.

Definition 2.3.5 ([39])**.** Let $\mathcal{F}^*$ be the set of $\Gamma^\vee$-translates of the hyperplanes in the boundary of $\mathcal{C}^+$. For any $\gamma \in \Gamma^\vee$, let Π_γ be the unique connected component of $\mathfrak{t}_{\mathbb{R}} \setminus \cup_{H \in \mathcal{F}^*} H$ containing A^+_γ.

For any $s_\alpha \in S$ and $A \in \Xi$, we then define an alcove $s_\alpha * A$ as follows. First, choose the unique $\gamma \in \Gamma^\vee$ such that $A \in \Pi_\gamma$, and choose $w \in W$ so that $A = wA^+_\gamma$. Then define $s_\alpha * A = wA^+_{s_\alpha(\gamma)}$. We then define $w * A$ for any $w \in W$ by induction on $\ell(w)$.

Proposition 2.3.6. *For any $s_\alpha \in S$,*

$$\theta_{s_\alpha}(A^\sharp) = (s_\alpha * A)^\sharp, \tag{2.3.3}$$

Proof. Recall the elements $A^\flat \in M_d$ defined in [36], noting that in our setup they agree with the elements $\tilde{D}_A$ defined in 8.9 of [39]. Additionally, recall the bilinear pairing $(-,-)$ on M_d defined in Section 9 of [36]. Corollary 8.9 of [39] gives that for any $s_\alpha \in S$, $A \in \Xi$, $\theta_{s_\alpha}(A^\flat) = (s_\alpha * A)^\flat$. The elements $A^\sharp$ are defined such that $(A^\flat, B^\sharp) = \delta_{A,B}$. One can check directly that $(\theta_{s_\alpha} A, B) = (A, \theta_{s_\alpha} B)$ for any alcoves A and B. So the continuity property in 3.3 of [36] and the well-definedness of the θ_{s_α} on $A^\sharp$ guaranteed by 12.2 of loc. cit. means $(\theta_s A^\flat, B^\sharp) = (A^\flat, \theta_s B^\sharp)$ for any alcoves A, B. This means

$$\begin{aligned}(A^\flat, \theta_{s_\alpha}(B^\sharp)) &= (\theta_{s_\alpha}(A^\flat), B^\sharp) \\ &= ((A * s_\alpha)^\flat, B^\sharp) \\ &= \delta_{s_\alpha * A, B} \\ &= \delta_{A, s_\alpha * B}.\end{aligned}$$

Since $\theta_{s_\alpha}(B^\sharp)$ satisfies the other defining properties in 12.2 of [36], $\theta_{s_\alpha}(B^\sharp) = (s_\alpha * B)^\sharp$, yielding (2.3.3). □

Definition 2.3.7. For any $z \in W$ and $\tilde{w} \in \tilde{W}$, let $\epsilon_z(\tilde{w})$ be the element of $\tilde{W}$ such that $z * A_{\tilde{w}} = A_{\epsilon_z(\tilde{w})}$.

Definition 2.3.8. Let M^0_d be the $\mathbb{Z}[v, v^{-1}]$-submodule of M_d generated by the finitely-many elements $\{A_w\}_{w \in W}$ and their images under the operators $\{\theta_w\}_{w \in W}$. Let $M^0_{d,q}$ be the specialization of M^0_d at $v = q^{\frac{1}{2}}$.

Lemma 2.3.9. *$M^0_{d,q}$ is a finite-dimensional $\mathcal{H}_q \otimes \mathbb{C}[W]$-submodule of $M_{d,q}$.*

Proof. The fact that $M^0_{d,q}$ is a finite-dimensional $\mathbb{C}[W]$-submodule follows by definition. Since for any $w \in W \subset \tilde{W}$ and any $y \in W$ we have $\tilde{T}_y A_w \in \mathrm{span}\{A_z\}_{z \in W}$ by the definition of the $\mathcal{H}_q$-action. Since this action commutes with the action of $\mathbb{C}[W]$, it follows that $M^0_{d,q}$ is closed under the action of $\mathcal{H}_q$. □

2.3.4 Example in Type $\mathbf{A_1}$

For $\mathbf{G} = \mathbf{SL}_2$, we can identify Ξ with the set of intervals $(n, n+1) \subset \mathbb{R}$ for $n \in \mathbb{Z}$, which we denote by A_n. With this convention, $A_e = A_0$ and $A_{s_1} = A_{-1}$. It follows from the definitions

in [36] that

$$s_1 * A_n = A_{-n} \tag{2.3.4}$$

$$A_n^\sharp = A_n + \sum_{i=1}^{\infty} (-1)^i v^{-i} A_{n+i} \in M_d. \tag{2.3.5}$$

A computation then shows that

$$\tilde{T}_{s_1} A_n = \begin{cases} A_{n+1} & \text{if } n \text{ is odd,} \\ A_{n-1} + (v - v^{-1}) A_n & \text{if } n \text{ is even,} \end{cases}$$

$$\tilde{T}_{s_1} A_n^\sharp = \begin{cases} -v^{-1} A_n^\sharp & \text{if } n \text{ is odd,} \\ v A_n^\sharp + A_{n-1}^\sharp + A_{n+1}^\sharp & \text{if } n \text{ is even.} \end{cases}$$

Further,

$$\theta_{s_1}(A_n^\sharp) = A_{-n}^\sharp.$$

2.4 Kazhdan-Laumon Category $\mathcal{O}$

In this section, we recall the Kazhdan-Laumon construction for gluing perverse sheaves on the basic affine space described in [32]. We define a subcategory $\mathcal{A}_B$ of the Kazhdan-Laumon category $\mathcal{A}$ analogous to Category $\mathcal{O}$ along with a mixed version $\mathcal{A}_{\mathcal{P}}$, and describe its simple objects explicitly.

2.4.1 Category $\mathcal{O}$, mixed sheaves, and the Hecke algebra

In this subsection, we recall the derived category $D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ (for ℓ a prime number not equal to p) of B-equivariant mixed ℓ-adic sheaves on X, with heart $\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ under the perverse t-structure. We will follow the setup of Chapter 7 of [2], adapted to G/U instead of G/B as is treated in loc. cit. Recalling our choice of a square root $q^{\frac{1}{2}}$ of q, we define the half-integer Tate twist $(\frac{1}{2})$ on $D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$. We then view $K_0(\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell))$ as a $\mathbb{Z}[v, v^{-1}]$-module where v^{-1} acts by $(\frac{1}{2})$. When $\mathcal{C}$ is any category for which $K_0(\mathcal{C})$ is a $\mathbb{Z}[v, v^{-1}]$-module, we denote by $K_0(\mathcal{C}) \otimes \mathbb{C}$ the specialization $K_0(\mathcal{C}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ at $v = q^{\frac{1}{2}}$. We use $\mathbb{D}$ to denote the Verdier duality functor and ${}^pH^0$ to denote the zeroth perverse cohomology functor.

Recall that X is stratified by affine cells $\{X_w\}_{w \in W}$, and for each $w \in W$ we let $j_w : X_w \hookrightarrow X$ be the inclusion. We will use the following definitions of standard, costandard, and simple perverse sheaves in $\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ with Tate twists (here our notation differs by

a Tate twist from that of [2]).

$$\begin{aligned}\Delta_w &= j_{w!}(\overline{\mathbb{Q}}_\ell[\ell(w)])(\tfrac{\ell(w)}{2})\\ \nabla_w &= j_{w*}(\overline{\mathbb{Q}}_\ell[\ell(w)])(\tfrac{\ell(w)}{2})\\ \mathrm{IC}_w &= j_{w!*}(\overline{\mathbb{Q}}_\ell[\ell(w)])(\tfrac{\ell(w)}{2})\end{aligned} \tag{2.4.1}$$

The same notation will be used when referring to sheaves on G/B depending on the context. As in 7.2 of [2], we also have left and right convolution operators

$$* : D^b_{B,\mathrm{m}}(G/B, \overline{\mathbb{Q}}_\ell) \times D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell) \to D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell), \tag{2.4.2}$$

$$* : D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell) \times D^b_{B,\mathrm{m}}(G/B, \overline{\mathbb{Q}}_\ell) \to D^b_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell). \tag{2.4.3}$$

Note that for the right convolution operator in (2.4.3), we first identify object lying in $D^b_{B,\mathrm{m}}(G/B, \overline{\mathbb{Q}}_\ell)$ with their pullbacks to $D^b_{B,\mathrm{m}}(G/U, \overline{\mathbb{Q}}_\ell)$ and then take the usual convolution. Convolution gives a ring structure on $K_0(\mathrm{Perv}_{B,\mathrm{m}}(G/B, \overline{\mathbb{Q}}_\ell))$, and it gives the space $K_0(\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell))$ the structure of a bimodule over the former. For $Y = G/B$ or $Y = X$, we make the following definition.

Definition 2.4.1. Let $\mathcal{P}(Y)$ be the full subcategory of $\mathrm{Perv}_{B,\mathrm{m}}(Y, \overline{\mathbb{Q}}_\ell)$ generated by objects of the form $\mathrm{IC}_w(\frac{m}{2})$ for $w \in W$, $m \in \mathbb{Z}$. When $Y = X$, we will write $\mathcal{P} = \mathcal{P}(X)$ for short.

Proposition 2.4.2 (c.f. Ex. 7.4.3 in [2])**.** *There is a unique isomorphism of rings*

$$\mathrm{ch} : K_0(\mathcal{P}(G/B)) \cong \mathcal{H}$$

such that $\mathrm{ch}([\nabla_w]) = \tilde{T}_w$ *for all* $w \in W$*. Moreover, this map satisfies*

$$\begin{aligned}\mathrm{ch}([\mathcal{F}(\tfrac{1}{2})]) &= v^{-1}\mathrm{ch}([\mathcal{F}]), \qquad & \mathrm{ch}([\mathbb{D}(\mathcal{F})]) &= \overline{\mathrm{ch}([\mathcal{F}])},\\ \mathrm{ch}([\mathrm{IC}_w]) &= C_w, & \mathrm{ch}([\nabla_w(-\tfrac{\ell(w)}{2})]) &= T_w.\end{aligned} \tag{2.4.4}$$

Similarly, $K_0(\mathcal{P}(X))$ is isomorphic to $\mathcal{H}$ as a left module over itself via the convolution action of $K_0(\mathcal{P}(G/B))$, where this isomorphism again respects the equations in (2.4.4) for elements of $K_0(\mathcal{P}(X))$.

2.4.2 Kazhdan-Laumon categories

In [32], the authors construct an abelian category $\mathcal{A}$ by "gluing" $|W|$-many copies of the category $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ (which we henceforth denote simply by $\mathrm{Perv}(X)$) via Fourier transforms $F_{s,!}$ indexed by simple reflections $s \in S$. These are referred to in Section 1.2 by the alternate notation Φ°_s, but we refer to them by $F_{s,!}$ in this chapter to stay consistent with the notations of [44], on which this chapter is based. We refer the reader to Section 1.2 (or to [48]) to a detailed explanation of these Fourier transforms.

It is shown in [48] that the functors $\{F_{s,!}\}_{s\in S}$ assemble to give an action of the generalized braid group B_W on $\mathrm{Perv}(X)$. Objects of $\mathcal{A}$ are then tuples indexed by W of elements of $\mathrm{Perv}(X)$ equipped with the additional structure explained in [48]. For each $w \in W$, there is an exact functor $j_w^* : \mathcal{A} \to \mathrm{Perv}(X)$ defined by $j_w^*((\mathcal{G}_{w'})_{w'\in W}) = \mathcal{G}_w$. Each of the functors j_w^* has a left adjoint $j_{w!}$ and a right adjoint j_{w*}, each of which is a functor from $\mathrm{Perv}(X) \to \mathcal{A}$. For a simple object $\mathcal{G}$ of $\mathrm{Perv}(X)$ and a choice of $w \in W$ a simple object $j_{w!*}(\mathcal{G})$ is defined in loc. cit. as the image of the natural map $j_{w!}(\mathcal{G}) \to j_{w*}(\mathcal{G})$ obtained by adjunction. Finally, we also have the exact functors $\{\mathcal{F}_w\}_{w\in W}$, which act by $\mathcal{F}_w((\mathcal{G}_{w'})_{w'\in W}) = (\mathcal{G}_{w'w})_{w'\in W}$.

In [48, Section 9], the same gluing construction is applied to the category $\mathrm{Perv}_{\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ of mixed perverse sheaves on X to construct a category $\mathcal{A}_{\mathrm{m}}$. We will say an object $(\mathcal{G}_w)_{w\in W}$ of $\mathcal{A}_{\mathrm{m}}$ is B-equivariant if each $\mathcal{G}_w$ is B-equivariant.

The following result follows from 6.3 of [48].

Proposition 2.4.3. *The functors $F_{w,!}$ are well-defined on $\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ and on $\mathcal{P}$. Indeed, for any $\mathcal{G} \in \mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$, and $s \in S$,*

$$F_{s,!}(\mathcal{G}) = {}^pH^0(\mathcal{G} * \nabla_s(\tfrac{1}{2})).$$

Definition 2.4.4. Let $\mathcal{A}_{B,\mathrm{m}}$ (resp. $\mathcal{A}_B$) be the Kazhdan-Laumon category obtained by applying the gluing procedure described in [32] and [48] to $\mathrm{Perv}_{B,\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{Perv}_B(X, \overline{\mathbb{Q}}_\ell)$) and the functors $F_{w,!}$. Alternatively, Theorem 1.2.1 in [48] ensures that $\mathcal{A}_B$ is the full subcategory of B-equivariant objects in $\mathcal{A}$.

Similarly, let $\mathcal{A}_{\mathcal{P}}$ be the category obtained in this same way from $\mathcal{P}$.

We now recall that $K_0(\mathcal{P})$ admits an $\mathcal{H}$-action on the left by convolution.

Lemma 2.4.5. *There is a left action of $\mathcal{H}$ on $K_0(\mathcal{P})$ given for $s \in S$ by*

$$\tilde{T}_s \cdot [\mathcal{G}] = [\nabla_s * \mathcal{G}].$$

Since this action commutes with convolution on the right by costandard objects and therefore with the functors $F_{w,!}$, it is straightforward to conclude that this left $\mathcal{H}$-action lifts to $K_0(\mathcal{A}_{\mathcal{P}})$.

Proposition 2.4.6. *There is a left action of $\mathcal{H}$ on $K_0(\mathcal{A}_{\mathcal{P}})$ given for $s \in S$ by*

$$\tilde{T}_s \cdot [(\mathcal{A}_w)_{w\in W}] = [(\nabla_s * \mathcal{A}_w)_{w\in W}]$$

for any $s \in S$.

2.4.3 Simple objects in $\mathcal{A}_{\mathcal{P}}$

We will now classify the simple objects in $\mathcal{A}_{\mathcal{P}}$ up to Tate twist, which correspond to the simple objects in $\mathcal{A}_B$. We first state our main result. To do so, for any $w \in W$, let $P(w)$

denote the standard parabolic subgroup of W generated by the simple reflections $s \in S$ for which $\ell(ws) > \ell(w)$.

Theorem 2.4.7. *Up to Tate twist, any simple object in $\mathcal{A}_{\mathcal{P}}$ is of the form $j_{z!*}(\mathrm{IC}_w)$ for some $w, z \in W$. Two such objects $j_{z_1!*}(\mathrm{IC}_{w_1})$ and $j_{z_2!*}(\mathrm{IC}_{w_2})$ are isomorphic if and only if $w_1 = w_2$ and $z_2 \in P(w_1)z_1$.*

Accordingly, simple objects in $\mathcal{A}_{\mathcal{P}}$ up to Tate twist are in bijection with pairs $(w, \overline{w'})$, where $w \in W$, and $\overline{w'}$ is an element of $P(w)\backslash W$.

To prove Theorem 2.4.7, we will need an intermediate result, Proposition 2.4.9. In the following, we freely use Lemma 1.3.1 in [48], which guarantees that if $\mathcal{G} \in \mathcal{A}$ is simple, then for all $w \in W$, $j_w^*(\mathcal{G})$ is either zero or simple in $\mathrm{Perv}(X)$, and $\mathcal{G} \cong j_{w!*}(j_w^*(\mathcal{G}))$ for any w for which it is nonzero. We begin with a lemma about Kazhdan-Lusztig polynomials $P_{y,w}$ which we will need for our result. This is well-known; one concise proof is given in 4.3.2(ii) of [52].

Lemma 2.4.8. *If $w \in W$ and s is a simple reflection such that $\ell(ws) < \ell(w)$, then $P_{ys,w} = P_{y,w}$ for all $y < w$.*

Proposition 2.4.9. *If $\mathcal{G} \in \mathcal{A}_{\mathcal{P}}$ is simple and $j_z^*\mathcal{G} = \mathrm{IC}_w$ for some $z \in W$ (in other words, $\mathcal{G} \cong j_{z!*}\mathrm{IC}_w$), then for any $y \in W$,*

$$j_{yz}^*\mathcal{G} \cong \begin{cases} \mathrm{IC}_w & \text{if } y \in P(w) \\ 0 & \text{if } y \notin P(w). \end{cases}$$

Proof. Suppose $\mathcal{G} = j_{z!*}\mathrm{IC}_w$. Pick some $s \in P(w)$, and following [48] let ϕ_s denote the action of the convolution $- * \nabla_s(\frac{1}{2})$ on $K_0(\mathrm{Perv}_B(G/U))$. By Lemma 1.3.1 in loc. cit., $j_a^*\mathcal{G} \in \mathcal{P}$ is either simple or zero for every $a \in W$. Writing $[\mathcal{F}] = (c_a)_{a \in W}$ for $c_a \in K_0(\mathcal{P})$, Theorem 5.6.1 of [48] ensures $\phi_s c_z - c_{sz} \in p_s^*(K_0(\mathrm{Perv}(G/Q_s)))$, where $Q_s = [P_s, P_s]$ (for P_s the parabolic subgroup associated to s) and $p_s : G/U \to G/Q_s$ the projection. Further, an element of $K_0(\mathcal{P})$ is in $p_s^*(K_0(\mathrm{Perv}(G/Q_s)))$ if and only if it is in the subspace K_s of $K_0(\mathcal{P})$ spanned by Tate twists of $[\mathrm{IC}_{w'}]$ for w' with $\ell(w's) < \ell(w')$.

Using Lemma 2.4.8, it is easy to check that if $\ell(ws) > \ell(w)$, then

$$\phi_s[\mathrm{IC}_w] - [\mathrm{IC}_w] \in K_s.$$

Since $\phi_s c_z - c_{sz}$ and $\phi_s[\mathrm{IC}_w] - [\mathrm{IC}_w]$ both lie in K_s, it follows that $c_{sz} - [\mathrm{IC}_w] \in K_s$. Since c_{sz} is the class in $K_0(\mathcal{P})$ of a simple object, we must have $c_{sz} = [\mathrm{IC}_w]$, and so $j_{sz}^*\mathcal{F} \cong \mathrm{IC}_w$. By induction on $\ell(y)$, this shows that $j_{yz}^*\mathcal{G} \cong \mathrm{IC}_w$ whenever $y \in P(w)$.

If instead $y \notin P(w)$, we can write a reduced expression $y = y_1 s y_2$ where $y_1 \in P(w)$, $y_2 \in W$ and s is a simple reflection with $\ell(vs) < \ell(v)$. We know that $j_{y_1 s y_2 z}^*\mathcal{G} = j_{s y_2 z}^*\mathcal{G}$ by the previous case, so without loss of generality we can assume y_1 is trivial. We now claim that

$$F_{s y_2,!}(\mathrm{IC}_w) = {}^pH^0(\mathrm{IC}_w * \nabla_{sy}(\tfrac{\ell(s y_2)}{2})) = 0.$$

This will prove the result, since by the definition of $j_{z!*}$, $j^*_{sy_2z}(j_{z!*}(\mathrm{IC}_w))$ is the image of $j^*_{sy_2z}(j_{z!}(\mathrm{IC}_w)) = F_{sy_2,!}(\mathrm{IC}_w)$ in $j^*_{sy_2z}(j_{z*}(\mathrm{IC}_w))$ under the natural map obtained by adjunction.

Indeed, $\mathrm{IC}_w * \nabla_s(\frac{1}{2}) \cong \mathrm{IC}_w(1)[1]$ (see, e.g. 7.2.5 in [2]), and $- * \nabla_{y_2}$ is right t-exact, meaning $\mathrm{IC}_w * \nabla_{sy_2} \cong \mathrm{IC}_w(1)[1] * \nabla_{y_2}$ can lie in perverse degrees at most -1. This means ${}^pH^0(\mathrm{IC}_w * \nabla_{sy_2}(\frac{\ell(sy_2)}{2})) = 0$ as desired. □

Proof of Theorem 2.4.7. The first statement follows from Lemma 1.3.2 of [48] (since twists of IC_w are the simple objects of $\mathcal{P}$), so it remains only to determine when simple objects of the form $j_{w'!*}(\mathrm{IC}_w)$ are isomorphic. Proposition 2.4.9 tells us that $j_{w'!*}(\mathrm{IC}_w)$ and $j_{zw'!*}(\mathrm{IC}_w)$ are isomorphic for any $z \in P(w)$. Further, no other isomorphisms exist among the $\{j_{w'!*}(\mathrm{IC}_w)\}_{w,w' \in W}$, since for any fixed w, the same proposition tells us that the set of $z \in W$ for which $j^*_z(j_{w'!*}(\mathrm{IC}_w))$ is nonzero depends only on the image of w' in $P(w)\backslash W$. □

Example 2.4.10. When $\mathbf{G} = \mathbf{SL}_3$, Table 2.1 is an explicit list of the simple objects in $\mathcal{A}_B$. In this case $w_0 = s_1s_2s_1 = s_2s_1s_2$.

2.4.4 Counting simple objects

We now use Theorem 2.4.7 to give an explicit formula for the number of simple objects in $\mathcal{A}_B$.

Corollary 2.4.11. *The number of simple objects in $\mathcal{A}_B$ is*

$$\sum_{w \in W} |P(w)\backslash W|. \tag{2.4.5}$$

Remark 2.4.12. In Type A_n, we can interpret the quantity in Corollary 2.4.11 as the number of pairs of permutations in the symmetric group S_{n+1} with no common rises in the sense of [1]. This is A000275 in the OEIS:

$$1, 3, 19, 211, 3651, 90921, 3081513, 136407699, \ldots$$

As a result, a generating function for the number of simple objects in $\mathcal{A}_B$ in this case is provided by the coefficients of a Bessel function, i.e. the reciprocal of $J_0(z)$ as in loc. cit.

2.5 Kazhdan-Laumon categories and the module M_d

The Grothendieck group $K_0(\mathcal{A}_\mathcal{P})$ carries a natural W-action along with an action of the Hecke algebra $\mathcal{H}$. In this section, we use the description of simple objects in $\mathcal{A}_\mathcal{P}$ to show that there exists an isomorphism $\eta : K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \to M_d^0$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules.

Table 2.1: The 19 simple objects in $\mathcal{A}_B$ when $G = SL_3$.

	$j_e^*(-)$	$j_{s_1}^*(-)$	$j_{s_2}^*(-)$	$j_{s_1s_2}^*(-)$	$j_{s_2s_1}^*(-)$	$j_{w_0}^*(-)$
$j_{e!*}(\mathrm{IC}_e)$	IC_e	IC_e	IC_e	IC_e	IC_e	IC_e
$j_{e!*}(\mathrm{IC}_{s_1})$	IC_{s_1}	0	IC_{s_1}	0	0	0
$j_{s_1!*}(\mathrm{IC}_{s_1})$	0	IC_{s_1}	0	0	IC_{s_1}	0
$j_{s_1s_2!*}(\mathrm{IC}_{s_1})$	0	0	0	IC_{s_1}	0	IC_{s_1}
$j_{e!*}(\mathrm{IC}_{s_2})$	IC_{s_2}	IC_{s_2}	0	0	0	0
$j_{s_2!*}(\mathrm{IC}_{s_2})$	0	0	IC_{s_2}	IC_{s_2}	0	0
$j_{s_2s_1!*}(\mathrm{IC}_{s_2})$	0	0	0	0	IC_{s_2}	IC_{s_2}
$j_{e!*}(\mathrm{IC}_{s_2s_1})$	$\mathrm{IC}_{s_2s_1}$	0	$\mathrm{IC}_{s_2s_1}$	0	0	0
$j_{s_1!*}(\mathrm{IC}_{s_2s_1})$	0	$\mathrm{IC}_{s_2s_1}$	0	0	$\mathrm{IC}_{s_2s_1}$	0
$j_{s_1s_2!*}(\mathrm{IC}_{s_2s_1})$	0	0	0	$\mathrm{IC}_{s_2s_1}$	0	$\mathrm{IC}_{s_2s_1}$
$j_{e!*}(\mathrm{IC}_{s_1s_2})$	$\mathrm{IC}_{s_1s_2}$	$\mathrm{IC}_{s_1s_2}$	0	0	0	0
$j_{s_2!*}(\mathrm{IC}_{s_1s_2})$	0	0	$\mathrm{IC}_{s_1s_2}$	$\mathrm{IC}_{s_1s_2}$	0	0
$j_{s_2s_1!*}(\mathrm{IC}_{s_1s_2})$	0	0	0	0	$\mathrm{IC}_{s_1s_2}$	$\mathrm{IC}_{s_1s_2}$
$j_{e!*}(\mathrm{IC}_{w_0})$	IC_{w_0}	0	0	0	0	0
$j_{s_1!*}(\mathrm{IC}_{w_0})$	0	IC_{w_0}	0	0	0	0
$j_{s_2!*}(\mathrm{IC}_{w_0})$	0	0	IC_{w_0}	0	0	0
$j_{s_1s_2!*}(\mathrm{IC}_{w_0})$	0	0	0	IC_{w_0}	0	0
$j_{s_2s_1!*}(\mathrm{IC}_{w_0})$	0	0	0	0	IC_{w_0}	0
$j_{w_0!*}(\mathrm{IC}_{w_0})$	0	0	0	0	0	IC_{w_0}

2.5.1 The modules $M_{d,q}^+$ and $\overline{M}_{d,q}^0$

Definition 2.5.1. Let $M_{d,q}^+$ be the $\mathcal{H}_q \otimes \mathbb{C}[W]$-submodule of $M_{d,q}$ generated by $A_{\tilde{w}}^\sharp$ for $\tilde{w} \in \tilde{W}$ with the property that for any $z \in W$, $\epsilon_z(\tilde{w})$ can be written as $w \cdot \lambda$ for some $w \in W, \lambda \geq 0$. It follows from [36] that $M_{d,q}^0 \subset M_{d,q}^+$.

Let $\overline{M}_{d,q}^0$ be the quotient of $M_{d,q}^+$ by the span of the elements $A_{\tilde{w}}^\sharp$ such that there is no pair $z, w \in W$ for which $\epsilon_z(w) = \tilde{w}$. Since it is easy to see that this is a $\mathcal{H}_q \otimes \mathbb{C}[W]$-submodule, we will continue to refer to the $\mathcal{H}_q$-action and the W-action by $\{\theta_y\}_{y \in W}$ on the quotient module $\overline{M}_{d,q}^0$.

We use π to refer to the quotient map from $M_{d,q}^+$ or $M_{d,q}^0$ to $\overline{M}_{d,q}^0$, and we write $\overline{A}^\sharp$ for $\pi(A^\sharp)$ whenever $A^\sharp \in M_{d,q}^+$.

2.5.2 The bijection

By the preceding section, recall that we can consider $K_0(\mathcal{A}_\mathcal{P})$ as a $\mathcal{H} \otimes \mathbb{Z}[v, v^{-1}][W]$ module, with v acting by the half-Tate twist $(-\frac{1}{2})$, $\mathcal{H}$ acting as in Proposition 2.4.6, and W acting by the functors $\mathcal{F}_w$. We will then consider $K_0(\mathcal{A}_\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ where $v \mapsto q^{\frac{1}{2}}$.

Theorem 2.5.2. *There is a well-defined isomorphism $\eta : K_0(\mathcal{A}_\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to M^0_{d,q}$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules such that for any $w \in W$,*

$$\eta([j_{e!}(\Delta_w)]) = A_w. \tag{2.5.1}$$

Further,

$$(\pi \circ \eta)([j_{e!*}(\mathrm{IC}_w)]) = \overline{A}^\sharp_w \tag{2.5.2}$$

in $\overline{M}^0_{d,q}$ for any $w \in W$.

We illustrate in Figure 2.1 the morphism $\pi \circ \eta : K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \to \overline{M}^0_{d,q}$ in the case $\mathbf{G} = \mathbf{SL}_3$. The next subsection will be devoted to the proof of this result.

2.5.3 Proof of Theorem 2.5.2

We begin by defining $\eta' : K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \to \overline{M}^0_{d,q}$ for any $w, y \in W$ by

$$\eta'([j_{y!*}(\mathrm{IC}_w)]) \mapsto \theta_{y^{-1}}\left(\overline{A}^\sharp_w\right). \tag{2.5.3}$$

To prove Theorem 2.5.2, we will first prove that η' is a well-defined and bijective morphism of $\mathbb{C}[W]$-modules onto $\overline{M}^0_{d,q}$. Then we will show that π, too, is an isomorphism of $\mathbb{C}[W]$-modules from $M^0_{d,q}$ to $\overline{M}^0_{d,q}$. Then we will show that $\pi \circ \eta = \eta'$, which gives that η is well-defined and bijective. From this, we will deduce the fact that η is a morphism of $\mathcal{H}_q$-modules.

Lemma 2.5.3. *The map η' is a well-defined isomorphism of $\mathbb{C}[W]$-modules.*

Proof. This follows from the claim that the stabilizer of any $[j_{e!*}(\mathrm{IC}_w)]$ under the action of the $\{\mathcal{F}_w\}_{w\in W}$ is equal to the stabilizer of $A^\sharp_w$ under the action of the $\{\theta_w\}_{w\in W}$. Note that Proposition 2.3.6 shows that the stabilizer of $A^\sharp_w$ under the latter action corresponds to the stabilizer of A_w under the $*$-action of W on Ξ. One can check that this stabilizer is $P(w)$, which is the stabilizer of $[j_{e!*}(\mathrm{IC}_w)]$ under the action of the $\{\mathcal{F}_w\}_{w\in W}$ by Theorem 2.4.7. □

Lemma 2.5.4. *The morphism $\pi : M^0_{d,q} \to \overline{M}^0_{d,q}$ is an isomorphism of $\mathbb{C}[W]$-modules.*

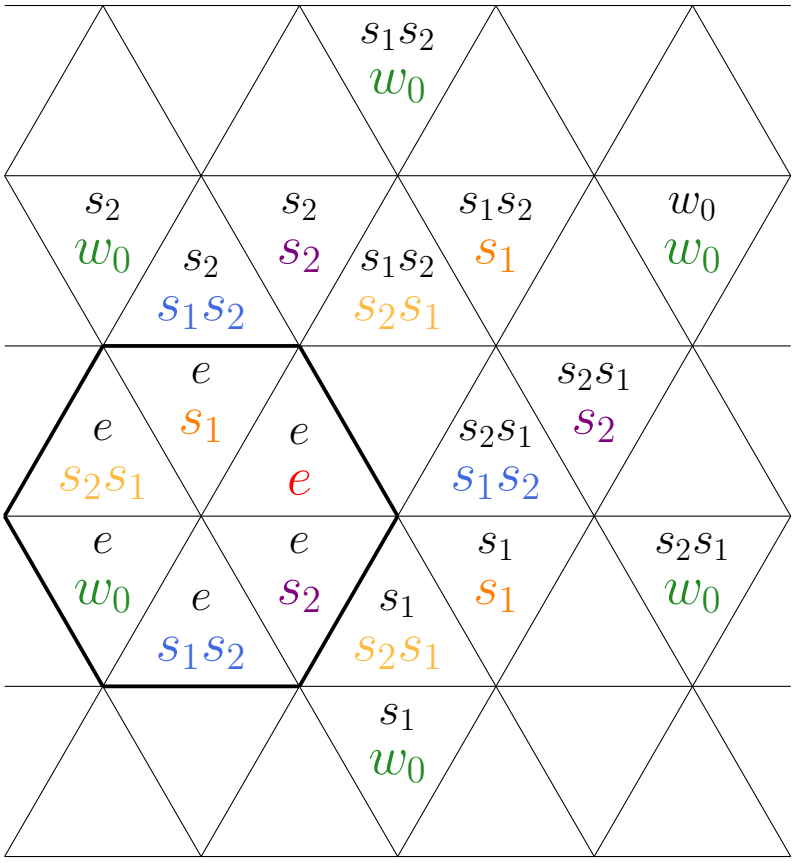


Figure 2.1: A figure illustrating the definition of η' in the case $\mathbf{G} = \mathbf{SL}_3$. For every alcove $A \in \Xi$ in the picture which is labelled with the tuple (v, w), we have $\eta'(j_{v!*}(\mathrm{IC}_w)) = \overline{A^\sharp}$. The colors indicate the orbits of the $A^\sharp$ under the operators θ_w. The set Ξ_{fin} is in bold, and the fundamental alcove A_e has a red label.

Proof. Note first that π is clearly a surjective morphism of $\mathbb{C}[W]$-modules. To show it is an isomorphism, it is sufficient to check that $\dim M^0_{d,q} \leq \dim \overline{M}^0_{d,q}$. By Lemma 2.5.3 and Corollary 2.4.11,

$$\dim \overline{M}^0_{d,q} = \sum_{w \in W} |P(w)\backslash W|. \tag{2.5.4}$$

We now claim that $M^0_{d,q}$ is spanned by elements of the form $\theta_{y^{-1}}(A_w)$ where y is a the minimal-length representative of some coset $P(w)y$ of $P(w)$ in W. Since there are $\sum_{w \in W} |P(w)\backslash W|$ such elements, this will prove the inequality above.

We do this by induction on the Bruhat order: we claim that for any $w \in W$, the space $\mathrm{span}\{\theta_{y^{-1}}(A_{w'})\}_{y \in W, w' \geq w}$ is spanned by the subset of these elements with minimal-length y in the coset $P(w)y$ as above. For $w = w_0$, this is trivial as $P(w) = \varnothing$.

Suppose now that it is true for some w. Then by further induction on the length of y it is enough to show that for any $s \in P(w)$, $\theta_s(A_w)$ is a linear combination of A_{ws}, A_w, and

$\theta_s(A_{ws})$. Indeed, by the definition of θ_s in 2.3.2, one can check that, since $\ell(ws) > \ell(w)$,

$$\theta_s(A_w) = A_w + v^{-1}A_{ws} - v^{-1}\theta_s(A_s), \tag{2.5.5}$$

completing the proof. □

Now we define $\eta = \pi^{-1} \circ \eta'$. By Lemmas 2.5.3 and 2.5.4, we know η is an isomorphism of $\mathbb{C}[W]$-modules. The following proposition will follow from Proposition 2.5.9, which is proved in the next subsection, and so we will delay its proof until then.

Proposition 2.5.5. *For any $w \in W$,*

$$\eta([j_{e!}(\Delta_w)]) = A_w$$

Before we use Proposition 2.5.5 to show that η is a morphism of $\mathcal{H}_q$-modules to prove Lemma 2.5.7 below, we need the following straightforward consequence of Polishchuk's work on "good representations" in [48].

Lemma 2.5.6 ([48], Theorem 11.5.1)**.** *The elements $\{[j_{z!}(\Delta_w)]\}_{z,w\in W}$ generate the space $K_0(\mathcal{A}_\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ whenever v is specialized to a value which is not a root of unity (in particular, in our situation where $v \mapsto q^{\frac{1}{2}}$).*

Proof. In Theorem 11.5.1 of [48], it is shown that $K_0(\mathcal{A}_\mathcal{P})$ must be generated by objects of the form $[j_{z!}(A)]$ for $A \in \mathcal{P}$ so long as the action defined by the $F_{w,!}$ functors factors through the Hecke algebra $\mathcal{H}_q$. Although this is not true for the action of these functors on $K_0(\mathcal{A})$, it holds for $K_0(\mathcal{A}_\mathcal{P})$ by Proposition 2.4.3. Since $D^b(\mathcal{P})$ is generated by standard objects $\{\Delta_w\}_{w\in W}$ and their Tate twists, this proves the lemma. □

Note that it is not true that the elements in Lemma 2.5.6 generate $K_0(\mathcal{A}_\mathcal{P})$ alone before specializing at $q^{\frac{1}{2}}$; this follows from the example provided in Section 2.8 to [48]. The question of whether the Grothendieck group $K_0(\mathcal{A})$ of the full Kazhdan-Laumon category is generated by a similar collection is the subject of much of the work in [48]; in future work, we hope to work beyond the case of Kazhdan-Laumon Category $\mathcal{O}$ to address some of these conjectures in general.

Lemma 2.5.7. *η is a morphism of $\mathcal{H}_q$-modules.*

Proof. By Section 2.4.1,

$$\begin{aligned}
\eta(\tilde{T}_s \cdot [j_{e!}(\Delta_w)]) &= \eta([j_{e!}(\nabla_s * \Delta_w)]) \\
&= \begin{cases} \eta([j_{e!}(\Delta_{sw})]) & \ell(sw) < \ell(w), \\ \eta([j_{e!}(\Delta_{sw})] + [j_{e!}(\Delta_w(-\frac{1}{2})] - [j_{e!}(\Delta_w(\frac{1}{2})]) & \ell(sw) > \ell(w) \end{cases} \\
&= \begin{cases} A_{sw} & \ell(sw) < \ell(w), \\ A_{sw} + (q^{1/2} - q^{-1/2})A_w & \ell(sw) > \ell(w). \end{cases}
\end{aligned}$$

Comparing this with (2.3.1), we conclude

$$\eta(\tilde{T}_s \cdot [j_{e!}(\Delta_w)]) = \tilde{T}_s \cdot \eta([j_{e!}(\Delta_w)]) \tag{2.5.6}$$

for any $w \in W$, $s \in S$.

By Lemma 2.5.6, the elements $[j_{z!}(\Delta_w)]$ for $z, w \in W$ span the space $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C}$. Since the W-action commutes with the action of $\mathcal{H}_q$, the equation (2.5.6) implies that η is a $\mathcal{H}_q$-module homomorphism on all of $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C}$, as desired. □

Combining Lemmas 2.5.3 and 2.5.7 (which, to recall, show together that η is a $\mathbb{C}[W]$-module isomorphism) with Proposition 2.5.5, Theorem 2.5.2 is proved. It remains to prove Proposition 2.5.5; this will be our main focus in the next subsection.

2.5.4 Restriction to fundamental alcoves

Let Ξ_{fin} denote the set of alcoves $\{A_w\}_{w \in W}$ which we call the fundamental alcoves. In this subsection, we define a map $J_e : M^0_{d,q} \to K_0(\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ (which can be interpreted as a sort of restriction to Ξ_{fin}) such that the diagram

$$\begin{array}{ccc} K_0(\mathcal{A}_\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} & \xrightarrow{\eta} & M^0_{d,q} \\ & \searrow^{j_e^*} & \downarrow J_e \\ & & K_0(\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \end{array} \tag{2.5.7}$$

commutes. This gives an interpretation of the functor j_e^* on the M^0_d side of the bijection from Theorem 2.5.2, and will allow us to prove Proposition 2.5.5.

Definition 2.5.8. We define the map $J_e : M^0_{d,q} \to K_0(\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ as the unique $\mathbb{C}$-linear map satisfying

$$J_e(A_w) = \begin{cases} [\Delta_w] & \text{if } w \in W \\ 0 & \text{if } w \in \tilde{W} \setminus W, \end{cases}$$

extending linearly to the domain $M^0_{d,q}$.

Let $J'_e : K_0(\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to M^0_{d,q}$ be the unique section of J_e whose image lies in the subspace of M^0_d spanned by $\{A_w\}_{w \in W}$. Let $\xi = J'_e \circ J_e$ be the projection onto the subspace of $M^0_{d,q}$ spanned by $\{A_w\}_{w \in W}$. Finally, let $\rho = \mathrm{id} - \xi$ be the projection onto the subspace of $M^0_{d,q}$ spanned by $\{A_{\tilde{w}}\}_{\tilde{w} \in \tilde{W} \setminus W}$.

Proposition 2.5.9. *The diagram (2.5.7) commutes, i.e.* $J_e \circ \eta = j_e^*$.

Proof. By the linearity of each of these maps, it is enough to check this statement on the elements $\{[j_{z!*}(\mathrm{IC}_w)]\}_{z,w \in W}$, which span $K_0(\mathcal{P})$ under $\mathbb{Z}[v, v^{-1}]$ by Theorem 2.4.7. By the

same result, we know that

$$j_e^*([j_{z!*}(\mathrm{IC}_w)]) = \begin{cases} [\mathrm{IC}_w] & z \in P(w) \\ 0 & z \notin P(w), \end{cases}$$

while $J_e(\eta([j_{z!*}(\mathrm{IC}_w)])) = J_e(\theta_{z^{-1}}(A_w^\sharp))$. Since $\pi \circ \eta = \eta'$ and it is easy to check that J_e is zero on any objects in the kernel of π by the definition of $\overline{M}_{d,q}^0$, it is enough to show that

$$J_e(\theta_{z^{-1}}(A_w^\sharp)) = \begin{cases} [\mathrm{IC}_w] & z \in P(w) \\ 0 & z \notin P(w). \end{cases}$$

In the case where $z \in P(w)$, we know by the proof of Lemma 2.5.3 that $\theta_z(A_w^\sharp) = A_w^\sharp$, so this reduces to showing $J_e(A_w^\sharp) = [\mathrm{IC}_w]$. By the Kazhdan-Lusztig conjectures, formula (1.5.b) in [33] holds, and so our claim follows from the remark following 13.10 in [39] combined with 11.19 of loc. cit. (for $\epsilon = 0$).

On the other hand, if $z \notin P(w)$, then $\theta_{z^{-1}}(A_w^\sharp) = (z^{-1} * A_w)^\sharp$. It is easy to see that if $a * A_w \neq A_w$ for $a, w \in W$, then $a * A_w$ is not dominated by any alcove in $\{A_{w'}\}_{w' \in W}$ under the partial ordering of alcoves given in [39]. As a result, we cannot have any of the alcoves in $\{A_{w'}\}_{w' \in W}$ occurring with nonzero coefficient in $(z * A_w)^\sharp$, and so $J_e(\theta_{z^{-1}}(A_w^\sharp)) = 0$ in this case, as desired. □

Corollary 2.5.10. *An element $C \in M_d^0$ is zero if and only if $J_e(\theta_z(C)) = 0$ for all $z \in W$.*

Proof. By the previous proposition and the fact that η is an isomorphism of $\mathbb{C}[W]$-modules, this reduces to the fact that $[\mathcal{F}] = 0$ for $\mathcal{F} \in \mathcal{A}_\mathcal{P}$ if and only if $j_z^*(\mathcal{F}) = 0$ for all $z \in W$. □

Now to prove Proposition 2.5.5, we will need the following; it is an easy computation from the definition of θ_s.

Lemma 2.5.11. *For any $w \in W$, $s \in S$,*

$$J_e(\theta_s(A_w)) = [\Delta_w * \nabla_s(\tfrac{1}{2})].$$

We now wish to upgrade this result by induction and show that $J_e(\theta_z(A_w)) = [\Delta_w * \nabla_z(\frac{\ell(z)}{2})]$ for any $w, z \in W$. To do so, we will need to keep track of possible positions of alcoves after applying θ_z to an alcove in Ξ_{fin}. To make this precise, we introduce the following definition and subsequent lemmas.

Definition 2.5.12. For any $v \in W$ and reduced word $\overline{v}$ for v, recall that we can define a subset of positive roots $R(\overline{v})$. If $\overline{v} = s_{i_1} \dots s_{i_k}$, we set

$$R(\overline{v}) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})\}.$$

Note that if $\overline{v}$ is reduced and s is a simple reflection with $\ell(sv) > \ell(v)$, then

$$R(s\overline{v}) = sR(\overline{v}) \cup \{\alpha_s\}.$$

We will sometimes use $R(v)$ to denote $R(\overline{v})$ for some choice of reduced word $\overline{v}$ for $v \in W$.

Definition 2.5.13. Given any subset R of positive roots, we define $\Xi^+(R) \subset \Xi$ to be the set of alcoves A satisfying the following two conditions. First, we require that $A \notin \Xi_{\text{fin}}$. Second, we require that A lies in the same Weyl chamber as some element of R. By this definition, each each $\Xi^+(R)$ will be a union of Weyl chambers but with Ξ_{fin} excised.

One can easily check for any two reduced expressions $\overline{w}$ and $\overline{w}'$ for $w \in W$ that we have $\Xi^+(R(\overline{w})) = \Xi^+(R(\overline{w}'))$.

Definition 2.5.14. For any $w \in W$, let $\Xi^+_w = \Xi^+(R(\overline{w}))$ for a choice of reduced expression $\overline{w}$ for w.

The following is a straightforward computation which follows from the definition of θ_s.

Lemma 2.5.15. *If $v \in W$ and $s \in S$ with $\ell(sv) > \ell(v)$, then*

$$\theta_s(\text{span}(\Xi^+_v) \cup \Xi_{\text{fin}})) \subset \text{span}(\Xi^+_{sv} \cup \Xi_{\text{fin}}),$$
$$\xi(\theta_s(\text{span}(\Xi^+_v))) = 0.$$

Corollary 2.5.16. *For any $w, z \in W$,*

$$\rho(\theta_z(A_w)) \in \text{span}(\Xi^+_z).$$

Proof. We go by induction on $\ell(z)$. For $\ell(z) = 0$, there is nothing to prove. Now suppose the result holds for z, and let $s \in S$ be such that $\ell(sz) > \ell(z)$. Then since $\rho + \xi = \text{id}$,

$$\rho(\theta_{sz}(A_w)) = (\rho \circ \theta_s)[\rho(\theta_z(A_w)) + \xi(\theta_z(A_w))].$$

Clearly $\xi(\theta_z(A_w)) \in \text{span}(\Xi_{\text{fin}})$, and by our induction hypothesis, $\rho(\theta_z(A_w)) \in \text{span}(\Xi^+_z)$. So Lemma 2.5.15 guarantees that $\theta_s[\rho(\theta_z(A_w)) + \xi(\theta_z(A_w))]$ lies in the span of $\Xi^+_{sz} \cup \Xi_{\text{fin}}$, and therefore its image under ρ lies in $\text{span}(\Xi^+_{sz})$. □

With these new definitions and lemmas in hand, we can now provide an inductive proof of Proposition 2.5.5.

Proof of Proposition 2.5.5. By the definition of $j_{e!}$,

$$j_e^*(\mathcal{F}_z(j_{e!}(\Delta_w)]))) = \Delta_w * \nabla_{z^{-1}}(\tfrac{\ell(z)}{2}),$$

so by Corollary 2.5.10, to show that $\eta([j_{e!}(\Delta_w)]) = A_w$ it is enough to show that

$$J_e(\theta_z(A_w)) = [\Delta_w * \nabla_{z^{-1}}(\tfrac{\ell(z)}{2})], \tag{2.5.8}$$

for all $w, z \in W$. We will fix w and proceed by induction on $\ell(z)$. For $\ell(z) = 0$, this is immediate from the definition of J_e.

Suppose then that we know (2.5.8) holds for some z. Suppose $s \in S$ is such that $\ell(sz) > \ell(z)$. Then by this assumption along with Lemma 2.5.11 (extended by linearity, using linearity of the convolution $- * \nabla(\frac{1}{2})$) that

$$\begin{aligned}(J_e \circ \theta_s \circ \xi \circ \theta_z)(A_w) &= J_e(\theta_s(J'_e([\Delta_w * \nabla_{z^{-1}}(\tfrac{\ell(z)}{2})])))\\ &= [\Delta_w * \nabla_{z^{-1}}(\tfrac{\ell(z)}{2}) * \nabla_s(\tfrac{1}{2})]\\ &= [\Delta_w * \nabla_{(sz)^{-1}}(\tfrac{\ell(sz)}{2})].\end{aligned}$$

To complete the proof, it remains to show that

$$J_e(\theta_{sz}(A_w)) = (J_e \circ \theta_s \circ \xi \circ \theta_z)(A_w).$$

By Corollary 2.5.16, we know $\rho(\theta_z(A_w)) \in \mathrm{span}(\Xi_z^+)$. By the first part of Lemma 2.5.15, we then have

$$\begin{aligned}J_e(\theta_{sz}(A_w)) - J_e(\theta_s(\xi(\theta_z(A_w)))) &= (J_e \circ \theta_s)[\theta_z(A_w) - \xi(\theta_z(A_w))]\\ &= (J_e \circ \theta_s)(\rho(\theta_z(A_w)))\\ &\in J_e(\theta_s(\mathrm{span}(\Xi_z^+))),\end{aligned}$$

which is zero by the second part of Lemma 2.5.15. □

2.6 The Schwartz space of the basic affine space

In this section, we recall the definition of the Schwartz space of the basic affine space $\mathcal{S}$ as defined in [15]. We then focus on Iwahori-invariants $\mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ of this space, and revisit a claim made in loc. cit. about its connection to M_d. This will then serve as a bridge to a more direct connection, which we will consider in the final subsection, between $\mathcal{S}$ and the Kazhdan-Laumon category $\mathcal{A}$.

2.6.1 Preliminaries from [15]

In this section, let k be a non-Archimedean local field with residue field $\kappa = \mathbb{F}_q$, and $\mathcal{O}$ its ring of integers. Let π be a uniformizer of k, and suppose the norm on k is chosen so that $||\pi|| = q^{-1}$.

In this section only, we write G_k for the base change of $\mathbf{G}$ to k; we will consider the basic

affine space $\mathbf{X}(k) = (\mathbf{G}/\mathbf{U})(k)$. Let $I \subset \mathbf{G}(k)$ be an Iwahori subgroup. The setup of [15] begins with considering the space $\mathcal{S}_c$ of locally constant compactly supported functions on $\mathbf{X}(k)$; let $\mathcal{S} \supset \mathcal{S}_c$ be the Schwartz space of the basic affine space defined in loc. cit.

Definition 2.6.1. For any $n \in \mathbb{Z}$, let ψ_n be a choice of additive character of conductor n of the field k (i.e. a character for which $\pi^n\mathcal{O} \subset \ker \psi_n$ but $\pi^{n-1}\mathcal{O} \not\subset \ker \psi_n$).

In [15], a W-action on $\mathcal{S}$ is defined via Fourier transforms Φ_α associated to any $s_\alpha \in S$ as follows. Let $P_\alpha(k)$ be the parabolic subgroup of $\mathbf{G}(k)$ naturally corresponding to s_α, and let $Q_\alpha(k) = [P_\alpha(k), P_\alpha(k)]$. Then $\pi_\alpha : \mathbf{X}(k) \to \mathbf{G}(k)/Q_\alpha(k)$ is a fibration with fibers isomorphic to $\mathbb{A}^2_k \setminus \{(0,0)\}$; the Fourier transform Φ_α is defined fiberwise. Because the definition in [15] (c.f. the more explicit definition in [31], which we will now slightly modify for our choice of the character ψ_1) depends on a choice of character and normalization, in the following definition we choose conventions explicitly for the purposes of the present chapter.

Definition 2.6.2. Define the Fourier transform operator $\Phi_\alpha : \mathcal{S} \to \mathcal{S}$ associated to a simple root $s_\alpha \in S$ by

$$(\Phi_\alpha(f))(x) = q \int_{\pi_\alpha^{-1}(\pi_\alpha(x))} \psi_1(\langle x, x' \rangle_{s_\alpha}) f(x') dx', \tag{2.6.1}$$

where we use the identification of the fiber $\pi_\alpha^{-1}(\pi_\alpha(x))$ with $V_s(x) - \{0\}$, where $V_s(x)$ is a two-dimensional vector space with a volume form defined by the pairing $\langle , \rangle_{s_\alpha} : V \times V \to k$ (following the setup and notation of Section 1.1.2 of [31]).

We define Φ_w for any $w \in W$ by $\Phi_w = \Phi_{\alpha_1} \cdots \Phi_{\alpha_k}$ where $w = s_{\alpha_1} \dots s_{\alpha_k}$ is a reduced expression.

Remark 2.6.3. The definition of the Fourier transform endomorphism Φ_α in [15] depends on a choice of additive character of k. From now on, we will always use the character ψ_1 and the normalization (by q) chosen in Definition 2.6.2. This is the only choice of character for which Theorem 2.6.5 will hold as it is written. The usual choice of character in the literature (as is used in Example 3.2 of [15] and throughout [31] and [17], for example) is ψ_0. However, this choice is not compatible with the normalizations chosen in the statement of Theorem 2.6.5, as we explain in Example 2.6.4. The reader should beware that our nonstandard choice of ψ is such that, in the $\mathbf{SL}_2$ case,

$$\begin{aligned} \Phi_\alpha(\chi_{\mathcal{O}\times\pi\mathcal{O}}) &= \chi_{\mathcal{O}\times\pi\mathcal{O}}, \\ \Phi_\alpha(\chi_{\mathcal{O}\times\mathcal{O}}) &= q\chi_{\pi\mathcal{O}\times\pi\mathcal{O}} \neq \chi_{\mathcal{O}\times\mathcal{O}}, \end{aligned}$$

which does not agree with the computation in Example 3.2 of [15] due to the discrepancy between the characters ψ_1 and ψ_0, as well as the constant q by which we multiply the integral in Definition 2.6.2.

The Iwahori subgroup I acts naturally on $\mathbf{X}(k)$ on the left, while $\mathbf{T}(\mathcal{O})$ acts on the right. We consider now the I-invariant subspace $\mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$, as is studied in Section 4 of [15]. Recall,

as explained in loc. cit., that the I-orbits on $\mathbf{X}(k)$ are indexed by $\tilde{W}$; for any $w \in \tilde{W}$ we can associate the I-orbit $Iw\mathbf{U}(k) \subset \mathbf{X}(k)$. We also know that for all $w \in \tilde{W}$ and $f \in \mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$, we can define the convolution $\chi_{IwI} * f$, where χ_{IwI} is the indicator function on the I-orbit $IwI \subset G(k)/I$. This defines an action of the affine Hecke algebra $\tilde{\mathcal{H}}_q$, with T_w acting such that

$$T_{w^{-1}}^{-1}(f) = (-q^{-1/2})^{\ell(w)}\chi_w * f. \tag{2.6.2}$$

2.6.2 Connection to M_d

In Lemma 4.3 and Corollary 4.6 of [15], it was claimed that $M_{d,q} \cong \mathcal{S}(X)^{I\times\mathbf{T}(\mathcal{O})}$ as $\mathcal{H}_q\otimes\mathbb{C}[W]$-modules, where on the Schwartz space side the $\mathcal{H}_q$-action is given by convolution and the W-action on the given by the Φ_w, while on the $M_{d,q}$ side the W-action is given by the θ_w. We reiterate that this result is not true if the "usual" definition of the Fourier transforms using the character ψ_0 is used, as we explain in the following example.

Example 2.6.4. Suppose $\mathbf{G} = \mathbf{SL}_2$, so $\mathbf{X}(k) \cong \mathbb{A}_k^2 \setminus \{(0,0)\}$. For each $w \in \widetilde{A}_1 = \langle s_1, s_0\rangle$, there is an Iwahori-orbit $Iw\mathbf{U}(k) \subset \mathbf{X}(k)$. These sets are either of the form $\pi^n\mathcal{O}\times\pi^n\mathcal{O} \setminus \pi^n\mathcal{O}\times\pi^{n+1}\mathcal{O}$ or $\pi^n\mathcal{O}\times\pi^{n+1}\mathcal{O} \setminus \pi^{n+1}\mathcal{O}\times\pi^{n+1}\mathcal{O}$ for some $n \in \mathbb{Z}$.

Lemma 4.3 in [15] asserts that the map $\gamma : \mathcal{S}(X)^{I\times\mathbf{T}(\mathcal{O})} \to M_{d,q}$ takes some multiple of the indicator function on each Iwahori orbit $Iw\mathbf{U}(k)$ to some alcove $A \in \Xi$. Accordingly, set $A^* = \gamma(-q^{-\frac{1}{2}}\chi_{Is_1\mathbf{U}(k)})$. (Note that $Is_1\mathbf{U}(k) = \mathcal{O}\times\mathcal{O} \setminus \mathcal{O}\times\pi\mathcal{O}$.)

Following the computations in Section 2.3.4, one can compute that we must have that $\gamma(-q^{-\frac{1}{2}}\chi_{\mathcal{O}\times\mathcal{O}}) = (A^*)^\sharp$. Since $\chi_{\mathcal{O}\times\mathcal{O}}$ is $G(\mathcal{O})$-invariant, we know $T_{s_1}(\chi_{\mathcal{O}\times\mathcal{O}}) = -q^{-1/2}\chi_{\mathcal{O}\times\mathcal{O}}$. So if γ is to be an isomorphism of $\mathcal{H}_q$-modules, we must have that $T_{s_1}((A^*)^\sharp) = -q^{-1/2}(A^*)^\sharp$, meaning $A^* = A_n$ for some odd n, by the computations in Section 2.3.4.

This is impossible, however, since if the character ψ_0 is used, then $\Phi_\alpha(\chi_{\mathcal{O}\times\mathcal{O}}) = \chi_{\mathcal{O}\times\mathcal{O}}$, and so we expect $\theta_{s_1}(A_n^\sharp) = A_n^\sharp$. But this only holds for $n = 0$, which is not odd, contradicting the existence of such a γ. This example is intended to justify Definition 2.6.2, and specifically our nonstandard choice of additive character, showing that such a choice is necessary if Corollary 4.6 in [15] is to hold.

We can fix this by our use of the character ψ_1 in Definition 2.6.2, thereby yielding the following result. For ease of notation, we denote by δ_w the function $(-q^{1/2})^{d(A_e,A_w)}\chi_{Iw\mathbf{U}(k)} \in \mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$.

Theorem 2.6.5 ([15])**.** *The map $\Psi : M_c \to \mathcal{S}_c^{I\times\mathbf{T}(\mathcal{O})}$ defined on alcoves A_w by*

$$\Psi(A_w) = \delta_w,$$

extends to an isomorphism between $M_{d,q}$ and $\mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ of $\mathcal{H}_q \otimes \mathbb{C}[\Gamma \rtimes W]$-modules.

Example 2.6.6. We now explain this bijection explicitly in the case of $\mathbf{SL}_2$; its proof in general follows from this computation. In this case, $W = \langle s_1\rangle$; we also denote by s_0 the

simple reflection associated to the affine root. Let A_n be the alcove $(n, n+1)$. Note that for any n, $A_{2n} = A_{(s_1 s_0)^n}$, while $A_{2n-1} = A_{s_1(s_1 s_0)^n}$.

The action of $\mathcal{H}_q$ on $M_{d,q}$ is given by the formulas of Section 2.3.4 specialized at $v = q^{\frac{1}{2}}$. Comparing these with the action of T_{s_1} on $\mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ whose inverse is given by convolution with $-q^{-1/2}\chi_{Is_1I}$, one can check that the actions agree; the same goes for the action of Γ.

It remains to check that the W-actions agree, which is only true when using the character ψ_1 in the definition of Φ_{s_1}. By applying Ψ to the formula (2.3.5),

$$\Psi(A_n^\sharp) = \begin{cases} (-q^{1/2})^n \chi_{(\pi^{n/2}\mathcal{O})\times(\pi^{n/2+1}\mathcal{O})} & \text{if } n \text{ is even,} \\ (-q^{1/2})^n \chi_{(\pi^{(n+1)/2}\mathcal{O})\times(\pi^{(n+1)/2}\mathcal{O})} & \text{if } n \text{ is odd.} \end{cases}$$

Since for $(x, y) \in k \times k$,

$$\begin{aligned} \Phi_{s_1}(\chi_{\pi^k\mathcal{O}\times\pi^{k+1}\mathcal{O}})(x,y) &= q \int_{a\in\pi^k\mathcal{O}} \int_{b\in\pi^{k+1}\mathcal{O}} \psi_1(xb - ya) \\ &= q \left(\int_{a\in\pi^k\mathcal{O}} \psi_1(-ya) \right) \left(\int_{b\in\pi^{k+1}\mathcal{O}} \psi_1(xb) \right) \\ &= q \cdot q^{-2k-1} \chi_{\pi^{-k}\mathcal{O}}(x) \chi_{\pi^{-k+1}\mathcal{O}}(y) \\ &= q^{-2k} \chi_{\pi^{-k}\mathcal{O}\times\pi^{-k+1}\mathcal{O}}(x,y), \end{aligned}$$

which implies that

$$\Phi_{s_1}\Psi(A_n^\sharp) = \Psi(A_{-n}^\sharp) = \Psi(\theta_{s_1}(A_{-n}^\sharp))$$

for n even, while a similar proof works for n odd.

2.6.3 Schwartz space and Kazhdan-Laumon categories

We now translate Theorem 2.5.2 into a statement about the Schwartz space of the basic affine space, since we can relate $M_{d,q}$ to $\mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ by Theorem 2.6.5.

Theorem 2.6.7. *There is an injection $\Theta : K_0(\mathcal{A}_\mathcal{P}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to \mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules given by*

$$\Theta([j_{e!}(\Delta_w)]) = \delta_w \tag{2.6.3}$$

for all $w \in W$.

Proof. This follows directly from Theorems 2.6.5 and 2.5.2. The last part follows in particular by comparing equation (2.5.1) with Theorem 2.6.5 and the definition of δ_w. □

Now we place Theorem 2.5.2 in this context, rephrasing the result in terms of a more direct connection between the Kazhdan-Laumon category $\mathcal{A}$ containing $\mathcal{A}_B$, and the Schwartz space $\mathcal{S} \supset \mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$.

Grothendieck's sheaf-function correspondence tells us that for any $\mathcal{F} \in \mathcal{P}$, we can produce a function $\mathrm{tr}(\mathcal{F}) : (\mathbf{G}/\mathbf{U})(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$. Applying this to the standard sheaves $\Delta_w \in \mathcal{P}$, it follows that $\mathrm{tr}(\Delta_w) = (-q^{-1/2})^{\ell(w)}\chi_w$, where χ_w is the indicator function on the $\mathbb{F}_q$-points of the Bruhat cell $BwU \subset G/U$. Returning to the setting where k is a local field, $\mathcal{O}$ its ring of integers, π the uniformizer, and κ the quotient, one can construct an "Eisenstein map" as follows.

Let $\mathbf{X}(\mathcal{O})_{\mathrm{fin}} \subset \mathbf{X}(\mathcal{O})$ denote the union of the Bruhat cells IwU across all w in the finite Weyl group W. The subset $\mathbf{X}(\mathcal{O})_{\mathrm{fin}}$ has the property that the projection $\mathcal{O} \twoheadrightarrow \kappa$ induces a well-defined and surjective map $\mathbf{X}(\mathcal{O})_{\mathrm{fin}} \to \mathbf{X}(\kappa)$.

We then have the diagram

$$\begin{array}{ccc} & \mathbf{X}(\mathcal{O})_{\mathrm{fin}} & \\ {\scriptstyle \iota}\swarrow & & \searrow{\scriptstyle p} \\ \mathbf{X}(k) & & \mathbf{X}(\kappa) \end{array}$$

which we obtain from the natural maps $k \hookleftarrow \mathcal{O} \twoheadrightarrow \kappa$. One can then consider $\iota_! \circ p^*$ as a map from the space $\mathcal{C}(\mathbf{X}(\kappa))$ of functions on $\mathbf{X}(\kappa)$ to $\mathcal{S}_c = \mathcal{S}_c(\mathbf{X}(k))$.

Definition 2.6.8. Let $\mathcal{S}_0$ be the subspace of $\mathcal{S}$ generated under the Fourier transforms Φ_w by the image of $\iota_! \circ p^*$.

Then (2.6.3) in Theorem 2.6.7 tells us the following.

Proposition 2.6.9. *The map Θ is an isomorphism onto $\mathcal{S}_0^{I\times\mathbf{T}(\mathcal{O})}$. For $\mathcal{F} \in \mathcal{P}$,*

$$\Theta([j_{e!}(\mathcal{F})]) = (\iota_! \circ p^*)(\mathrm{tr}(\mathcal{F})) \in \mathcal{S}_c^{I\times\mathbf{T}(\mathcal{O})}.$$

This concludes the proof of Theorem 2.1.3 from the introduction, giving a direct interpretation of the composition $K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \to M_{d,q} \to \mathcal{S}^{I\times\mathbf{T}(\mathcal{O})}$ (each of which is originally described combinatorially) in terms of Grothendieck's sheaf-function dictionary via the map $\iota_! \circ p^*$.

2.7 Perverse sheaves on the semi-infinite flag variety

2.7.1 Preliminaries

In this section, we follow the technical setup of [5]. In loc. cit., the authors define a category $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$, intended to be an analogue of a category of Iwahori-monodromic perverse sheaves on the semi-infinite flag variety by using the Drinfeld space $\overline{\mathrm{Bun}}_{N^-}$. (In this section, we use the notation N^- for the maximal unipotent subgroup opposite U to be consistent with loc. cit.) This construction builds off of earlier work in [23], [22] and [14].

It is shown in [5] that the category $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ has simple objects $\mathsf{IC}_{\tilde{w}}$ indexed by $\tilde{w} \in \tilde{W}$. For each such $\tilde{w}$, the standard and costandard objects $\boldsymbol{\nabla}_{\tilde{w}}$ and $\boldsymbol{\Delta}_{\tilde{w}}$ are also defined using maps $i_{\tilde{w}}$ corresponding to inclusion of the appropriate stratum indexed by $\tilde{w}$ (we use bold symbols for standard and costandard objects in $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ to distinguish them from the usual standard and costandard objects Δ_w and ∇_w in $\mathrm{Perv}_B(G/B)$; note also that their labelling as standard and costandard sheaves respectively, as opposed to the reverse, is nonstandard but follows the conventions of [5]). One of the main results in loc. cit. is an equivalence between the category of Artinian objects in $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ and a certain category of graded modules over the small quantum group. This result categorifies a known connection between the Grothendieck group of the latter category and Lusztig's periodic Hecke module which appears in [4], c.f. Theorem 17.8. The latter category admits a W-action given in Lemma 1.1.16 of loc. cit. As a result, there is a natural W-action on simple objects in $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ given as follows. We note that equation (55) in Section 6.1.8 of loc. cit. describes this same result in terms of "restricted irreducible" objects, but there is a typo in the indices in the equation referenced, so we reproduce a corrected version here employing instead the conventions and notation of the present chapter.

Proposition 2.7.1 (Section 6.18 in [5]). *The category* $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ *admits a* W*-action by endofunctors* $\{\mathsf{F}_w\}_{w\in W}$. *These functors send simple objects to simple objects, and in particular*

$$\mathsf{F}_s(\mathsf{IC}_{i(\tilde{w})}) = \mathsf{IC}_{i(\epsilon_s(\tilde{w}))}$$

for any $s \in S$, $\tilde{w} \in \widetilde{W}$ *(c.f. Definition 2.3.7), where* $i : \tilde{W} \to \tilde{W}$ *denotes the involution sending* $w \cdot \lambda \to w \cdot (-\lambda)$ *for* $w \in W$, $\lambda \in \Gamma$.

In Section 6.13 of [22], the authors explain a direct connection between perverse sheaves on the semi-infinite flag variety and Lusztig's periodic Hecke module $M_{\geq}$. This is made precise for the category $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ by [5], c.f. Theorem 4.3.6. In 6.1.8 of loc. cit., the authors remark that there exists a category $\mathcal{P}^{\frac{\infty}{2}}$ consisting of "mixed D-modules of Hodge-Tate type in $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$" equipped with a half-Tate twist endofunctor $(\frac{1}{2})$ corresponding to $q^{-1/2}$ generated by the simple objects $\mathsf{IC}_{\tilde{w}}(\frac{m}{2})$ for all $\tilde{w} \in \tilde{W}$ and $m \in \mathbb{Z}$. As before, the corresponding Grothendieck group is then a $\mathbb{Z}[v, v^{-1}]$-module where v^{-1} acts by $(\frac{1}{2})$.

Note that in the original result of [22] loc. cit., the setup and notation of the periodic Hecke module follows the conventions of [50], which differs from those of [39] which we use, hence the presence of the sign change on λ in the results in this section. We also make reference to the $\mathcal{H}$ action on $K_0(\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0})$ discussed in [5] and [22]. Translating the result to our conventions, we arrive the following.

Theorem 2.7.2 ([22]). *There is an isomorphism of* $\mathcal{H}_q \otimes \mathbb{C}[W]$*-modules*

$$K_0\left(\mathcal{P}^{\frac{\infty}{2}}\right) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to M_{d,q}$$

such that for any $\tilde{w} = w \cdot \lambda$ *(for* $w \in W$, $\lambda \in \Gamma$*),*

$$[\mathsf{IC}_{\tilde{w}}] \mapsto A^{\sharp}_{w\cdot(-\lambda)},$$
$$[\boldsymbol{\nabla}_{\tilde{w}}] \mapsto A_{w\cdot(-\lambda)}.$$

Definition 2.7.3. Let $W_{\leq} \subset \tilde{W}$ be the set of $\tilde{w} \in \tilde{W}$ such that for any $y \in W$, $\epsilon_y(\tilde{w})$ can be written as $w \cdot \check{\nu}$ for some $w \in W$, and some coweight $\check{\nu}$ for which $\check{\nu} \leq 0$.

Let $W' \subset W_{\leq} \subset \tilde{W}$ be the set of all $\tilde{w} \in \tilde{W}$ such that there exists $w, z \in W$ with $\mathsf{IC}_{\tilde{w}} = \mathsf{F}_z(\mathsf{IC}_{i(w)})$. In other words, it is the orbit of W in $\tilde{W}$ under the action defined by $i(\tilde{w}) \mapsto \epsilon_w(i(\tilde{w}))$ for $w \in W$.

Finally, let $T = W_{\leq} - W'$ be the subset of $W_{\leq}$ consisting of all elements which do not lie in W'.

Definition 2.7.4. Let $\tilde{\mathcal{P}}_{\leq}$ (resp. $\mathcal{T}$) be the Serre subcategory of $\mathcal{P}^{\frac{\infty}{2}}$ generated by half-integer twists of $\{\mathrm{IC}_{\tilde{w}}\}_{\tilde{w}\in W_{\leq}}$ (resp. $\{\mathrm{IC}_{\tilde{w}}\}_{\tilde{w}\in T}$) under extensions. Let $\mathcal{Q}$ be the Serre quotient $\tilde{\mathcal{P}}_{\leq}/\mathcal{T}$.

Let $\tilde{\mathcal{P}}^{\circ}$, $\mathcal{T}^{\circ}$, and $\mathcal{Q}^{\circ}$ be the non-mixed analogues, e.g. $\tilde{\mathcal{P}}^{\circ}$ is the Serre subcategory of $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ generated by $\{\mathrm{IC}_{\tilde{w}}\}_{\tilde{w}\in W'}$.

Our aim in the subsequent sections will be to categorify Theorem 2.7.2 by showing that $\mathcal{Q}^{\circ}$ and $\mathcal{A}_B$ (resp. $\mathcal{Q}$ and $\mathcal{A}_{\mathcal{P}}$) are equivalent as categories, and then showing that the Serre quotient map $\pi : \tilde{\mathcal{P}}_{\leq} \to \tilde{\mathcal{P}}_{\leq}/\mathcal{T}$ admits a fully faithful right adjoint. By composing this adjoint functor with the equivalence of categories claimed above, we will then get an equivalence of categories between $\mathcal{A}_B$ (resp. $\mathcal{A}_{\mathcal{P}}$) and a full subcategory $\tilde{\mathcal{P}}$ of $\mathcal{P}^{\frac{\infty}{2}}$ as claimed in Theorem 2.1.4.

2.7.2 Setup

We will continue to refer to the setup and notation of [5]. In loc. cit., the authors define the category $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ as the subcategory of sheaves in $\mathrm{Perv}({}^{I^0}_{\infty}\overline{\mathrm{Bun}}_{N^-})$ satisfying certain conditions. For any coweight $\check{\nu}$, they define ${}^{I^0}_{\check{\nu}}\overline{\mathrm{Bun}}_{N^-} \subset {}^{I^0}_{\infty}\overline{\mathrm{Bun}}_{N^-}$, denoting by $\bar{i}_{\check{\nu}}$ the inclusion map. We will use this map in the case where $\check{\nu} = 0$. Similarly, they define ${}^{I^0}_{\leq\check{\nu}}\overline{\mathrm{Bun}}_{N^-} \subset {}^{I^0}_{\infty}\overline{\mathrm{Bun}}_{N^-}$.

For any $\tilde{w} \in W'$, it is easy to see that the strata indexed by $\check{\nu}$ on which $\mathrm{IC}_{\tilde{w}}$ is supported must be such that $\check{\nu} \leq 0$. (This is the statement that $W' \subset W_{\leq}$, which follows purely from the combinatorics of M_d and the definition of the operators θ_w; note that the orderings $\leq$ and $\geq$ are switched when translating from $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0}$ to M_d, as we explain in Section 2.7.1.)

Lemma 2.7.5 ([5], Section 4.1.3)**.** *The closure of* ${}^{I^0}_{0}\overline{\mathrm{Bun}}_{N^-}$ *in* ${}^{I^0}_{\infty}\overline{\mathrm{Bun}}_{N^-}$ *is* ${}^{I^0}_{\leq 0}\overline{\mathrm{Bun}}_{N^-}$.

Corollary 2.7.6. *The functor* $\bar{i}_0^* : \tilde{\mathcal{P}}^{\circ}_{\leq} \to \mathrm{Perv}({}^{I^0}_{0}\overline{\mathrm{Bun}}_{N^-})$ *is exact.*

Proof. By the preceding discussion, any element of $\tilde{\mathcal{P}}^{\circ}_{\leq}$ is supported in $\overset{I^0}{\leq 0}\overline{\mathrm{Bun}}_{N^-}$. It is shown in [5] that $\overset{I^0}{0}\overline{\mathrm{Bun}}_{N^-}$ is an open subset of $\overset{I^0}{\leq 0}\overline{\mathrm{Bun}}_{N^-}$, from which the result follows. □

Now the following result, also shown in [5], will be crucial for our purposes.

Proposition 2.7.7 ([5], Section 4)**.** *There is an equivalence of categories*

$$\mathrm{Perv}_B(G/U) \to {}'\mathrm{Perv}(\overset{I^0}{0}\overline{\mathrm{Bun}}_{N^-}),$$

where ${}'\mathrm{Perv}(\overset{I^0}{0}\overline{\mathrm{Bun}}_{N^-}) \subset \mathrm{Perv}(\overset{I^0}{0}\overline{\mathrm{Bun}}_{N^-})$ *is the subcategory satisfying the same conditions as in the definition of* $\mathrm{Perv}(\mathcal{F}\ell^{\frac{\infty}{2}})^{I^0} \subset \mathrm{Perv}(\overset{I^0}{\infty}\overline{\mathrm{Bun}}_{N^-})$*. This equivalence sends the standard sheaves* Δ_w *to the sheaves* $\boldsymbol{\nabla}_w$ *described in 4.4.3 of [5].*

From now on, by means of the equivalence in this proposition, we will implicitly identify $\mathrm{Perv}_B(G/U)$ with ${}'\mathrm{Perv}(\overset{I^0}{0}\overline{\mathrm{Bun}}_{N^-})$, viewing $\mathrm{Perv}_B(G/U)$ as the target (resp. source) of the functor $\bar{i}_0^*$ (resp. ${}^p\bar{i}_{0!}$).

Proposition 2.7.8. *The functor* $\bar{i}_0^*$ *is zero when restricted to* $\mathcal{T}$*. Further, the category* $\mathcal{T}$ *is closed under the action of the* $\{\mathsf{F}_w\}_{w\in W}$*. As a result, the functor* $\bar{i}_0^*$ *and the functors* $\{\mathsf{F}_w\}_{w\in W}$ *are each well-defined on the quotient category* $\mathcal{Q}$*.*

Proof. By the definition of $\mathcal{T}$, it is generated by a collection of irreducible objects $\mathsf{IC}_{\tilde{w}}$, each of which having the property that $\tilde{w} = w \cdot \check{\nu}$ for $w \in W$ and $\check{\nu}$ some coweight with $\check{\nu} < 0$. This means $\bar{i}_0^*\mathsf{IC}_{\tilde{w}} = 0$ for each such irreducible, and therefore $\bar{i}_0^*(\mathcal{T}) = 0$.

Next, note that by the definition of W', irreducible objects $\mathsf{IC}_{\tilde{w}}$ for $\tilde{w} \in W_{\leq} - W'$ are sent to other such irreducible objects under the functors $\{\mathsf{F}_w\}_{w\in W}$; i.e. these functors send irreducible objects in $\mathcal{T}$ to other irreducible objects in $\mathcal{T}$. Since these functors are exact, this shows that $\mathcal{T}$ is closed under each of them. □

2.7.3 An equivalence of categories

Here, we recall the following result from [48] (specialized to the case of $\mathcal{A}_B$); see also [9] for a more general statement.

Theorem 2.7.9 ([48])**.** *Suppose* $\mathcal{K}$ *is an abelian category equipped with exact functors* $j_w^* : \mathcal{K} \to \mathrm{Perv}_B(G/U)$*, each of which has a left adjoint* $j_{w!}$*. Suppose that these functors satisfy* $j_w^* j_{v!} = F_{wv^{-1},!}$*. Assume also that if an object* $A \in \mathcal{K}$ *satisfies* $j_w^* A = 0$ *for all* $w \in W$*, then* $A = 0$*. Then there is a natural equivalence* $\mathcal{K} \cong \mathcal{A}_B$*.*

By the previous section, we know that $\bar{i}_0^*$ can be considered as a functor from $\mathcal{Q}$ to $\mathcal{P}$. Further, it is exact by Lemma 2.7.6. Accordingly, we make the following definition.

Definition 2.7.10. Let $\mathsf{j}_e^* : \mathcal{Q} \to \mathcal{P}$ be the functor $\bar{i}_0^*$. Further, for any $w \in W$, let

$$\mathsf{j}_w^* := \bar{i}_0^* \circ \mathsf{F}_w,$$
$$\mathsf{j}_{w!} := \mathsf{F}_{w^{-1}} \circ \bar{i}_{0!}$$

Proposition 2.7.11. *Each functor j_w^* has a left adjoint given by*

$$\pi \circ \mathsf{F}_{w^{-1}} \circ {}^p\bar{i}_{0!} \tag{2.7.1}$$

where $\pi : \tilde{\mathcal{P}}_{\leq 0} \to \mathcal{Q}$ is the Serre quotient functor.

Proof. First note that for any $\mathcal{F} \in \mathcal{P}$ and any $Y \in \mathcal{Q}$, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{F}, (i_0^* \circ \mathsf{F}_w)(Y)) \cong \mathrm{Hom}_{\mathcal{P}_{\leq}}((\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F}), Y) \tag{2.7.2}$$

by adjointness of $(\bar{i}_{0!}, \bar{i}_0^*)$. It remains to show that

$$\mathrm{Hom}_{\mathcal{P}_{\leq}}((\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F}), Y) \cong \mathrm{Hom}_{\mathcal{Q}}((\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F}), Y). \tag{2.7.3}$$

But note that $(\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F})$ admits no nontrivial quotient which lies in $\mathcal{T}$, by the adjointness of $(\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!}, \mathsf{F}_w \circ \bar{i}_0^*)$ and by the fact that $\bar{i}_0^*$ is trivial on $\mathcal{T}$. By the definition of morphisms in Serre quotient categories, this means any morphism $\overline{f} \in \mathrm{Hom}_{\mathcal{Q}}((\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F}), Y)$ lifts to a genuine morphism $f : (\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F}) \to Y/Y'$, where Y' is some subobject of Y lying in $\mathcal{T}$. But again by the adjointness of $(\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!}, \mathsf{F}_w \circ \bar{i}_0^*)$ and by the fact that $\bar{i}_0^*$ is trivial on $\mathcal{T}$, any such morphism lifts to a morphism from $(\mathsf{F}_{w^{-1}} \circ \bar{i}_{0!})(\mathcal{F})$ to Y, and this gives the bijection in (2.7.3). □

We know that the functors j_w^* are exact (since the F_w are exact), that $\pi \circ \mathsf{j}_{w!}$ are their left adjoints, and that

$$\mathsf{j}_w^* \mathsf{j}_{v!} = \bar{i}_0^* \circ \mathsf{F}_{wv^{-1}} \circ \bar{i}_{0!}. \tag{2.7.4}$$

So to show that the functors $\{\mathsf{j}_w^*\}_{w \in W}$ as defined here satisfy the conditions in Theorem 2.7.9, it remains to show Lemmas 2.7.12 and 2.7.13 below.

Lemma 2.7.12. *When considered as endofunctors of $\mathcal{P}$, $\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}$ and $F_{w,!}$ are naturally isomorphic.*

Proof. Recall that for any $\mathcal{F} \in \mathcal{P}$, we can write $F_{w,!}(\mathcal{F}) = \mathcal{F} * \nabla_w(\frac{\ell(w)}{2})$. By 4.4.1 of [5], !-convolution on the left (which is the convention for convolution we use) with objects of $\mathrm{Perv}_B(G/U)$ commutes with the functors $\bar{i}_{0!}$ and $\bar{i}_0^*$. By the definition of the F_w in loc. cit., it is also clear that they too commute with convolution on the left (on the level of K_0, this is simply the fact that the W and $\mathcal{H}$-actions on the periodic Hecke module M_d commute with one another). This means

$$\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\mathcal{F}) = \bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\mathcal{F} * \Delta_e) \tag{2.7.5}$$

$$= \mathcal{F} * (\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_e)), \tag{2.7.6}$$

so it remains only to show that $\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_e) \cong \nabla_w(\frac{\ell(w)}{2})$. Note that since the action of F_w on the Grothendieck group agrees with Lusztig's θ_w operator on M_d,

$$[\Delta_w(-\tfrac{\ell(w)}{2}) * (\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_e))] = [\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_w(-\tfrac{\ell(w)}{2}))] = [\Delta_e].$$

Since Δ_e is irreducible and the left-hand side is perverse (since $\bar{i}_{0!}(\Delta_w) = \boldsymbol{\nabla}_w$ and F_w is t-exact), this means $\Delta_w(-\frac{\ell(w)}{2}) * (\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_e)) \cong \Delta_e$, which gives a natural isomorphism $\bar{i}_0^* \circ \mathsf{F}_w \circ \bar{i}_{0!}(\Delta_e) \cong \nabla_w(\frac{\ell(w)}{2})$, as desired. □

Lemma 2.7.13. *If $A \in \mathcal{Q}$ satisfies $\mathsf{j}_w^* A = 0$ for all $w \in W$, then $A = 0$.*

Proof. Note that $K_0(\mathcal{Q}) = \overline{M}^0_{d,q}$. Our discussion in Section 2.5.2 shows that for any nonzero $B \in \overline{M}^0_{d,q}$, there exists at least one $w \in W$ for which $J_e(\theta_w(B)) \neq 0$ which is equivalent to the desired statement under this identification. □

As a result, all of the conditions for Theorem 2.7.9 are satisfied, yielding the following as a corollary.

Proposition 2.7.14. *There is an equivalence of categories between $\mathcal{A}_B$ and $\mathcal{Q}^\circ$ (resp. $\mathcal{A}_\mathcal{P}$ and $\mathcal{Q}$). This categorifies the morphism $\eta' : K_0(\mathcal{A}_\mathcal{P}) \otimes \mathbb{C} \to \overline{M}^0_{d,q}$.*

Now we use the following result to upgrade this result and obtain a fully faithful functor from $\mathcal{A}_B$ to $\mathcal{P}^{\frac{\infty}{2}}$.

Proposition 2.7.15. *The Serre quotient functor $\pi : \tilde{\mathcal{P}}_\leq \to \mathcal{Q}$ admits a right adjoint $\tilde{\sigma}_*$ and a left adjoint $\tilde{\sigma}_!$, each of which is fully faithful.*

Proof. Using Proposition 2.7.14, we can let $\phi : \mathcal{Q} \to \mathcal{A}_B$ be the equivalence of categories described therein. To complete the proof of this proposition, it is enough to show that $\phi \circ \pi$ admits a right adjoint σ_* and a left adjoint $\sigma_!$.

First, we claim that for any $w \in W$, $j_w^* \phi \pi \mathsf{F}_{w^{-1}} = \bar{i}_0^*$ as functors from $\tilde{\mathcal{P}}^\circ_\leq \to \mathcal{P}^\circ$. Indeed, by the definition of ϕ we have $j_w^* \phi = \bar{i}_0^* \circ \mathsf{F}_w$, and so the result follows from Proposition 2.7.8 which says that F_w descends to a map on $\mathcal{Q}$ and therefore commutes with π.

By the dual result to Lemma 0793 in [51], it is enough to show that there exists a subset $\mathcal{J} \subset \mathrm{Ob}(\mathcal{A}_B)$ such that

1. Every object of $\mathcal{A}_B$ is a subobject of an object in $\mathcal{J}$,
2. For every $J \in \mathcal{J}$ there exists a choice of object $\sigma_*(J)$ in $\tilde{\mathcal{P}}_\leq$ such that one has functorial isomorphisms $\mathrm{Hom}_{\tilde{\mathcal{P}}_\leq}(X, \sigma_*(J)) \cong \mathrm{Hom}_{\mathcal{A}_B}(\phi(X), J)$ for every X in $\tilde{\mathcal{P}}_\leq$.

Let $\mathcal{J} \subset \mathrm{Ob}(\mathcal{A}_B)$ be the collection of objects of the form $j_{w*}(\mathcal{F})$ for $w \in W$, $\mathcal{F} \in \mathcal{P}^\circ$. First note that (1) holds because for any $A \in \mathcal{A}_B$, the adjunctions (j_w^*, j_{w*}) give a morphism

$$A \to \bigoplus_{w \in W} j_{w*} j_w^* A \tag{2.7.7}$$

which is easily seen to be an inclusion.

Now to meet condition (2), for any object of the form $j_{w*}(\mathcal{F})$, we let $\sigma_*(j_{w*}(\mathcal{F})) = \mathsf{F}_{w^{-1}}(\bar{i}_{0*}(\mathcal{F}))$. Then for any $X \in \tilde{\mathcal{P}}_{\leq}$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}_{\leq}}(X, \sigma_*(j_{w*}(\mathcal{F}))) &= \mathrm{Hom}_{\mathcal{P}_{\leq}}(X, \mathsf{F}_{w^{-1}}(\bar{i}_{0*}(\mathcal{F}))) \\ &= \mathrm{Hom}_{\mathcal{P}_{\leq}}(\mathsf{F}_w(X), \bar{i}_{0*}(\mathcal{F})) \\ &= \mathrm{Hom}_{\mathcal{P}}(\bar{i}_0^*(\mathsf{F}_w(X)), \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{P}}(j_w^*\phi\pi(X), \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{A}_B}((\phi \circ \pi)(X), j_{w*}(\mathcal{F}))\end{aligned}$$

with the step from the third to the fourth line using the fact that $j_w^*\phi\pi\mathsf{F}_{w^{-1}} = \bar{i}_0^*$ as functors from $\mathcal{P}_{\leq 0}$ to $\mathcal{P}$, with each of the isomorphisms in the above being functorial.

This means the assignment σ_* above extends to a functor $\sigma_* : \mathcal{A}_B \to \tilde{\mathcal{P}}_{\leq}$ which is right-adjoint to the Serre quotient functor π. Since the quotient functor π is a localization functor and σ_* is its right adjoint, we then have that σ is necessarily fully faithful.

Finally, we construct a left adjoint $\sigma_!$ to $\phi\pi$. By the same result as used above, we can repeat the above argument with quotients rather than subobjects, using the fact that for any $A \in \mathcal{A}_B$, there is a surjective morphism

$$\oplus_{w \in W} j_{w!}j_w^*A \to A. \tag{2.7.8}$$

We then define $\sigma_!(j_{w!}(\mathcal{F})) = \mathsf{F}_{w^{-1}}(\bar{i}_{0!}(\mathcal{F}))$. Now to meet the dual version of condition (2) above we proceed with the same argument but dualized, and conclude that indeed $\sigma_!$ is left-adjoint to $\phi\pi$. □

Definition 2.7.16. Let $\tilde{\mathcal{P}}$ be the image of the functor $\tilde{\sigma}_!$, which is a full subcategory of $\mathcal{P}^{\frac{\infty}{2}}$ by Proposition 2.7.15.

As a corollary to Proposition 2.7.15, we get the following elaboration on Theorem 2.1.4. Note that the fact that the equivalence of categories induced by the left derived functor to the fully faithful right-exact functor $\sigma_!$ described in Proposition 2.7.15 categorifies the isomorphism η follows from the fact that $\sigma_!(j_{e!}(\Delta_w)) = \boldsymbol{\nabla}_w$. Its compatibility with convolution follows from the fact that $\bar{i}_{0!}$ commutes with !-convolution by 4.4.1 of [5].

Theorem 2.7.17. *There exists an equivalence of categories between $\mathcal{A}_B$ and $\tilde{\mathcal{P}}^\circ$ (resp. $\mathcal{A}_{\mathcal{P}}$ and $\tilde{\mathcal{P}}$). This categorifies the isomorphism $\eta : K_0(\mathcal{A}_{\mathcal{P}}) \otimes \mathbb{C} \to M_d^0$, and respects convolution on the left by the standard and costandard objects Δ_w and ∇_w of $\mathrm{Perv}_{B,\mathrm{m}}(G/B)$ for all $w \in W$. The W-action on $\mathcal{A}_{\mathcal{P}}$ by functors $\mathcal{F}_w$ is mapped to the W-action on $\tilde{\mathcal{P}}$ by the functors F_w.*

2.8 Illustrations for $\overline{M}_d^0$ and $K_0(\mathcal{A}_B)$ in rank 2

Expanding on Figure 2.1 but suppressing the labels, we now provide illustrations in rank 2 for $\overline{M}_d^0$ and $K_0(\mathcal{A}_B)$. More precisely, in the figures below, the shaded alcoves are the alcoves $A \in \Xi$ for which $\overline{A}^\sharp \in \overline{M}_d^0$. By Theorem 2.5.2, the shaded alcoves index a basis for $K_0(\mathcal{A}_B)$. The colors indicate the orbits of the $\overline{A}^\sharp$ in $\overline{M}_d^0$ under the operators $\{\theta_w\}_{w\in W}$, which correspond to the orbits of simple objects in $\mathcal{A}_B$ under the operators $\{\mathcal{F}_w\}_{w\in W}$. As in Figure 2.1, the alcove A_e is always pictured in red, and the fundamental alcoves $\{A_w\}_{w\in W}$ are outlined with a bold stroke. (To draw these pictures, we used Proposition 2.3.6 which provides a description, in terms of alcoves and reflections along root hyperplanes, of the orbits of $\{A_w^\sharp\}_{w\in W}$ under the operators $\{\theta_w\}_{w\in W}$.)

By Theorem 2.5.2, the number of shaded alcoves in each example is the number of simple objects in $\mathcal{A}_B$. We hope that these rank 2 illustrations demonstrate the underlying alcove geometry behind the formula in Corollary 2.4.11.

Figure 2.2: A schematic for $K_0(\mathcal{A}_B)$ in Type $\mathbf{A}_2$

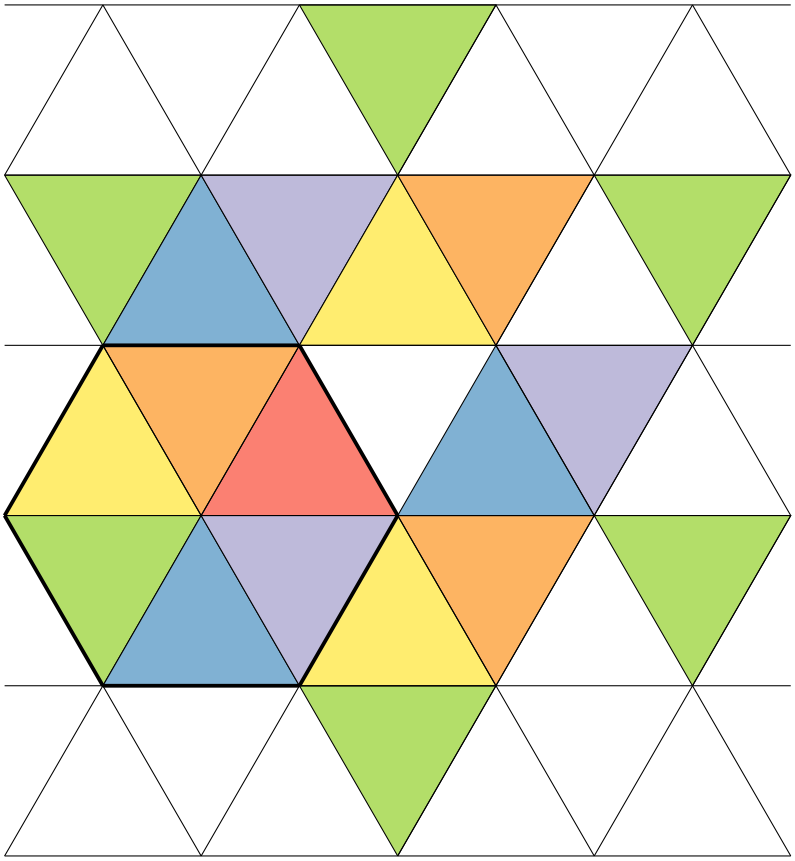


Figure 2.3: A schematic for $K_0(\mathcal{A}_B)$ in Type $\mathbf{B}_2$

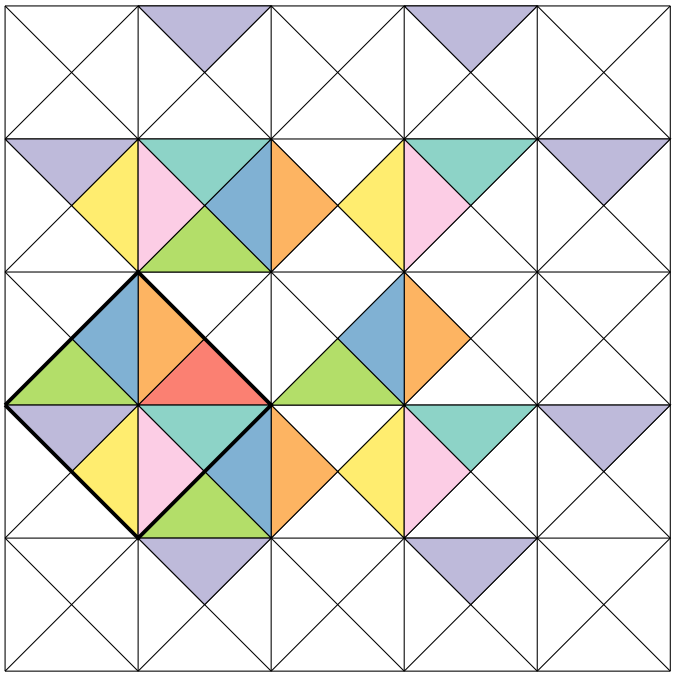
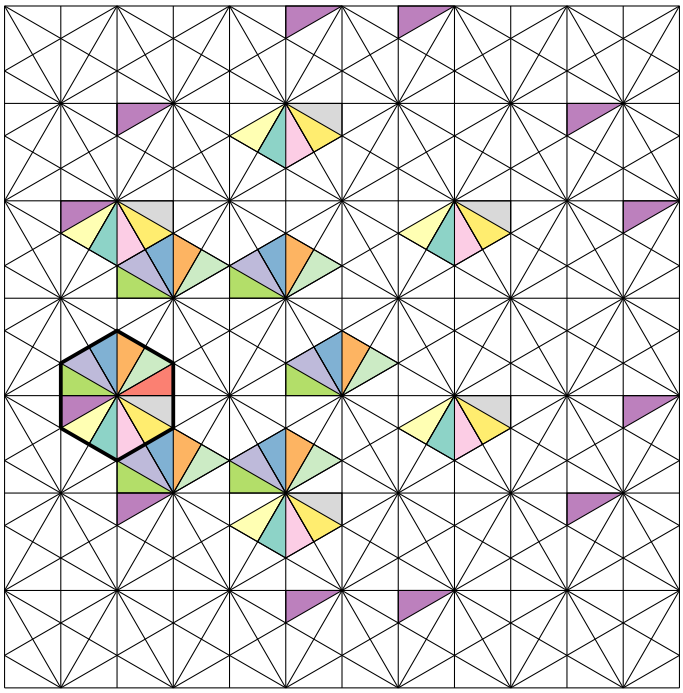


Figure 2.4: A schematic for $K_0(\mathcal{A}_B)$ in Type $\mathbf{G}_2$



Chapter 3

Symplectic Fourier-Deligne transforms on G/U and the algebra of braids and ties

3.1 Introduction

In [32], given a semisimple simply-connected algebraic group G split over a finite field $\mathbb{F}_q$ with associated Weyl group W, the authors define an action of the generalized braid group B_W on the constructible derived category $D_c^b(G/U)$ of mixed ℓ-adic sheaves on the basic affine space associated to G via so-called "symplectic Fourier-Deligne transforms." In [48], this action was studied in further detail, where Polishchuk showed that each of the standard braid generators for B_W satisfies a cubic relation when acting on $K_0(G/U) = K_0(D_c^b(G/U))$. In doing so, Polishchuk shows that the action of B_W factors through the "cubic Hecke algebra" $\mathcal{H}_W^c$ given by the quotient of $\mathbb{C}(v)[B_W]$ by a certain cubic relation in each simple braid generator. In loc. cit., Polishchuk asks which algebra is really acting: are there other relations satisfied by the action of $\mathbb{C}(v)[B_W]$ on $K_0(G/U)$, and can we identify the quotient of $\mathcal{H}_W^c$ by these relations more explicitly? The central result of this chapter is an answer to this question.

Another perspective on symplectic Fourier-Deligne transforms can be given via convolution with "Kazhdan-Laumon sheaves" on $G/U \times G/U$ as follows. In [32], Kazhdan and Laumon define for each $w \in W$ a stratum $X(w) \subset G/U \times G/U$ along with a perverse sheaf $K(w)$ on $X(w)$ and its Goresky-MacPherson extension $\overline{K(w)}$ to the closure $\overline{X(w)}$. They then show that the action of the symplectic Fourier-Deligne transform on $D_c^b(G/U)$ corresponding to any simple reflection s is given by convolution on the right with $\overline{K(s)}$; further, each of these sheaves $\overline{K(w)}$ is a pullback under a multiplication map $G/U \times G/U \to U\backslash G/U$, with their convolution agreeing with a suitably-defined convolution of sheaves on $U\backslash G/U$. As a result, the subalgebra of endomorphisms of $K_0(G/U)$ generated by symplectic Fourier-Deligne transforms is isomorphic to the algebra $\mathrm{KL}(v) \subset K_0(U\backslash G/U)$ generated by classes of Kazhdan-Laumon sheaves under convolution, and this is the perspective we will take on

this algebra for the purposes of this chapter.

In [3], the authors define a certain diagrammatic algebra called the algebra of braids and ties (or sometimes the "bt-algebra") over the field $\mathbb{C}(v)$, generalizing certain properties of the Yokonuma-Hecke algebra of Type A_n to the case of a generic parameter. In [42], the author constructs a generalization (called C_W^R in loc. cit.) of this algebra to arbitrary Coxeter type, which we call the generalized algebra of braids and ties and denote by $\mathcal{E}(v)$. In loc. cit., the author also considers a natural subalgebra which we call $\mathcal{C}(v) \subset \mathcal{E}(v)$ which is the image of a natural homomorphism $\mathbb{C}(v)[B_W] \to \mathcal{E}(v)$. In the next result, we show that Kazhdan-Laumon sheaves under convolution categorify $\mathcal{C}(v)$, answering the aforementioned question asked in [48]. Further, we show that the whole algebra $\mathcal{E}(v)$ can be obtained by the convolution algebra generated by a modest enlargement of this Kazhdan-Laumon algebra, i.e. by Kazhdan-Laumon sheaves in addition to certain other sheaves $E(s)$ introduced in Definition 3.2.3 for each simple reflection s.

Theorem 3.1.1. *Let* $\mathrm{KL}(v)$ *be the algebra sitting inside* $K_0(G/U) \otimes \mathbb{C}(v)$ *generated by the classes of Kazhdan-Laumon sheaves* $\overline{K(w)}$ *for all* $w \in W$ *under convolution. Let* $\widetilde{\mathrm{KL}}(v)$ *be the algebra generated by the classes of Kazhdan-Laumon sheaves in addition to the sheaves* $E(s)$ *for all* $s \in S$.

(a) The algebra $\widetilde{\mathrm{KL}}(v)$ *is isomorphic to the generalized algebra of braids and ties* $\mathcal{E}(v)$*, which has generators and relations given in Definition 3.2.5.*

(b) The algebra $\mathrm{KL}(v)$ *is isomorphic to the distinguished subalgebra* $\mathcal{C}(v)$ *of* $\mathcal{E}(v)$ *generated only by the generators* g_s*,* $s \in S$.

This upgrades Polishchuk's observation that $\mathrm{KL}(v)$ is a quotient of $\mathcal{H}_W^c$ and provides an explicit description of this algebra.

The generalized braids and ties algebra $\mathcal{E}(v)$ was defined as a generic-parameter version of the Yokonuma-Hecke algebra $\mathrm{Y}_{q,k}$ over $\mathbb{F}_{q^k}$, as defined in [53]. In [29] and [30], the authors gave a new presentation of $\mathrm{Y}_{q,k}$ which has since been used fruitfully in the study of this algebra and the algebra of braids and ties (e.g. in [49] and [20]). This presentation uses a generating set $\{L_s\}_{s\in S} \cup \{R_t\}_{t\in T(\mathbb{F}_{q^k})}$ with relations given in Proposition 3.3.5.

We note that in [29] and [30], to complete their proof of the relations satisfied by their new generators for $\mathrm{Y}_{q,k}$, the authors prove case-by-case that the generators L_s satisfy braid relations, proving this for simply-laced types in [29] and then for all other types in [30]. On the other hand, combining our Theorem 3.1.1 with a result from [32] gives a type-independent proof of this fact using the geometry of perverse sheaves on $G/U \times G/U$, avoiding the algebraic calculations done in loc. cit.

Corollary 3.1.2 (First proved in [29], [30])**.** *The generators* $\{L_s\}_{s\in S}$ *of* $\mathrm{Y}_{q,k}$ *introduced by Juyumaya in [29] satisfy braid relations.*

The following is a diagram summarizing the relationship between the algebras discussed so far, where the two vertical isomorphisms are from Theorem 3.1.1, the arrow $\mathcal{E}(v) \to \mathrm{Y}_{q,k}$

is the specialization map at $v = q^{k/2}$, and the rightmost vertical arrow is the Frobenius trace map tr_{q^k} which we will describe in Section 3.3.1.

$$\begin{array}{ccccc} \mathrm{KL}(v) & \hookrightarrow & \widetilde{\mathrm{KL}}(v) & \hookrightarrow & K_0(G/U) \\ \downarrow\wr & & \downarrow\wr & & \\ \mathcal{C}(v) & \hookrightarrow & \mathcal{E}(v) & & \\ & & \downarrow & & \downarrow \\ & & \mathrm{Y}_{q,k} & \hookrightarrow & \mathbb{C}[(G/U)(\mathbb{F}_{q^k})] \end{array}$$

We conclude by explaining two more algebraic consequences, both of which arise from questions posed in [42]. One of the motivating facts about the algebra $\mathcal{C}(v)$ is that it admits a natural quotient map to the Hecke algebra $\mathcal{H}$ by sending $g_s \mapsto -A_s$, where A_s is the standard generator of the Hecke algebra associated to $s \in S$. In the next result, we show that actually this algebra admits a morphism to each monodromic Hecke algebra $\mathbf{H}_{\mathfrak{o}}$ associated to a W-orbit $\mathfrak{o}$ of character sheaves on T as described in [41].

Theorem 3.1.3. *Let $\mathbf{H}_{\mathfrak{o}}$ be the monodromic Hecke algebra associated to an orbit $\mathfrak{o}$.*

(a) For any character sheaf $\mathcal{L}$, convolution with $[\Delta(e)_{\mathcal{L}}]$ induces a morphism $\pi_{\mathcal{L}} : \mathrm{KL}(v) \to \mathbf{H}_{\mathfrak{o}}$. In particular, $\pi = \pi_{\mathcal{L}_{\mathrm{triv}}}$ is the quotient map to the usual Hecke algebra.

(b) There exists a natural lift c_w in $\mathrm{KL}(v)$ of the Kazhdan-Lusztig basis C_w under the map π which satisfies a self-duality condition with respect to a natural involution on $\mathrm{KL}(v)$.

Finally, in [42], the author studies the algebra $\mathcal{C}(v)$ more explicitly in Type A and discusses some connections to the Links-Gould invariant and its defining algebra which was studied in [43]. In Section 4.3 of [42], Marin computes that $\dim \mathcal{C}_{A_1} = 3$, $\dim \mathcal{C}_{A_2} = 20$, and provides some computational evidence that $\dim \mathcal{C}_{A_3}(v) = 217$, $\dim \mathcal{C}_{A_4}(v) = 3364$, noting that this sequence does not exist in the OEIS and asking whether a general formula exists for the dimension. We conclude this chapter by confirming Marin's computational results and extending them by providing a general formula for the dimension of $\mathcal{C}_{A_n}(v)$ (and therefore also of the symplectic Fourier transform algebra $\mathrm{KL}(v)$; note that even its finite-dimensionality is not obvious a priori without Theorem 3.1.1) in Type A_n for all $n \geq 1$.

Theorem 3.1.4. *When considered as an algebra over $\mathbb{C}(v)$, there is an explicit formula for the dimension of the algebra $\mathcal{C}(v)$ given by*

$$\dim \mathcal{C}(v) = |W| \sum_{I \subset S} D_I / N_I \tag{3.1.1}$$

where N_I is the size of the normalizer of the standard parabolic subgroup of W associated to $I \subset S$, and D_I is the number of elements of $w \in W$ such that for each contiguous block $I' \subset I$, there is a simple reflection $s \in I'$ for which $\ell(ws) < \ell(w)$.

This gives a general formula which continues the sequence of dimensions

$$1,\ 3,\ 20,\ 217,\ 3364, \ldots$$

given in [42].

In conclusion, we hope that the results of this chapter will serve two purposes. First, it serves as an explicit algebraic identification of the algebra generated by symplectic Fourier-Deligne transforms in terms of a known finite-dimensional algebra – a project begun by Polishchuk's identification of the cubic relations in $\mathrm{KL}(v)$ more than two decades ago. In the next chapter, we exploit this identification to prove some of the conjectures about representations of the braid group factoring through the symplectic Fourier transform algebra which were originally posed in [48]. Secondly, our main result gives a geometric interpretation for some algebras familiar to experts in the theory of braids and knots with potential for applications to some purely algebraic questions. This chapter contains some basic examples of such algebraic questions which we answer via this geometric perspective, and we hope that further study of this algebra will be enhanced by this interpretation which links the algebraic perspective to the rich set of tools which become available in the context of perverse sheaves and algebraic geometry.

3.2 Preliminaries

Let q be a power of some odd prime p, let $k = \mathbb{F}_q$, and let G be a split semisimple simply-connected algebraic group over k. Let T be a Cartan subgroup split over k, B a Borel subgroup containing T, and U its unipotent radical. Let $X = G/U$ be the basic affine space associated to G considered as a variety over k. Let W be the Weyl group, with $S \subset W$ a set of simple reflections. We let Φ be the set of roots, and we fix a choice of identification of simple reflections $s \in S$ with simple roots $\alpha_s \in S$. For any two simple reflections $s, s' \in S$, we let $m_{ss'}$ be the order of ss' in W.

For every simple root α_s we fix an isomorphism of the corresponding one-parameter subgroup $U_s \subset U$ with the additive group $\mathbb{G}_{a,k}$. This uniquely defines a homomorphism $\rho_s : SL_{2,k} \to G$ which induces the given isomorphism of $\mathbb{G}_{a,k}$ (embedded in $SL_{2,k}$ as upper-triangular matrices) with U_s; then let

$$n_s = \rho_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any $w \in W$, writing a reduced word $w = s_{i_1} \ldots s_{i_k}$ we set $n_w = n_{s_{i_1}} \ldots n_{s_{i_k}}$, and one can check that this does not depend on the reduced word. We also define for any $s \in S$ the subtorus $T_s \subset T$ obtained from the image of the coroot $\alpha_s^\vee$ and define T_w for any $w \in W$ to be the product of all T_s ($s \in S$) for which $s \leq w$ in the Bruhat order.

Once and for all, we pick a square root $q^{\frac{1}{2}}$ of q, and an additive character $\psi : \mathbb{F}_p \to \mathbb{C}^\times$. Note that this induces natural additive characters of $\mathbb{F}_q$ for all powers q of p by composing with the field trace; we will also denote these characters by ψ. Let $\mathcal{L}_\psi$ be the associated Artin-Schreier sheaf on $\mathbb{G}_{a,\mathbb{F}_q}$.

We let $K_0(G/U)$ denote the Grothendieck group of the constructible derived category $D_c^b(G/U)$ of mixed ℓ-adic sheaves on G/U. The "half Tate twist" $(\frac{1}{2})$ corresponding to $q^{\frac{1}{2}}$ gives $K_0(G/U)$ the natural structure of a $\mathbb{Z}[v, v^{-1}]$-module. We choose now an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and choose to work only with the field $\mathbb{C}$ from now on; in doing so, we will then be able to make reference to complex conjugation. By abuse of notation, we will write simply $K_0(G/U)$ when referring to $K_0(G/U) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}(v)$ going forward.

Later we will consider character sheaves $\mathcal{L}$ on the torus, and relatedly multiplicative characters $\theta : T(\mathbb{F}_{q^k}) \to \mathbb{C}^\times$. To either of these objects we can associate a reflection subgroup $W_{\mathcal{L}}^\circ$ or W_θ° of W generated by reflections corresponding to coroots on which $\mathcal{L}$ or θ is trivial; c.f. [41] for more details.

Let $\mathcal{H}$ be the usual Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ (which we sometimes consider over $\mathbb{C}(v)$) generated by $\{A_w\}_{w \in W}$, normalized so that the quadratic relation satisfied by any A_s for $s \in S$ takes the form

$$A_s^2 = (v^2 - 1)A_s + v^2. \tag{3.2.1}$$

For any $w \in W$, let $\tilde{A}_w = v^{-\ell(w)} A_w$, so that for any $s \in S$,

$$(\tilde{A}_s - v)(\tilde{A}_s + v^{-1}) = 0. \tag{3.2.2}$$

There is a natural involution on $\mathcal{H}$ given by $\overline{A_w} \mapsto A_{w^{-1}}^{-1}$ and $\overline{v} = v^{-1}$. Following the conventions of Section 3.5 of [52], recall the canonical basis C_w of $\mathcal{H}$ given in terms of the Kazhdan-Lusztig polynomials $P_{x,w}$ by

$$C_w = \sum_{x \leq w} (-1)^{\ell(w)-\ell(x)} v^{\ell(x)-\ell(w)} P_{x,w} \tilde{A}_{x^{-1}}^{-1}$$

which satisfy $\overline{C_w} = C_w$.

3.2.1 Kazhdan-Laumon sheaves and convolution

In [32] and [48], the authors associate to each $w \in W$ an element of $D_c^b(G/U \times G/U)$ which is perverse up to shift and irreducible.

Following [48], let $X(w) \subset G/U \times G/U$ be the subvariety of pairs $(gU, g'U) \subset (G/U)^2$ such that $g^{-1}g' \in Un_wT_wU$. There is a canonical projection $\mathrm{pr}_w : X(w) \to T_w$ sending $(gU, g'U)$ to the unique $t_w \in T_w$ such that $g^{-1}g' \in Un_wt_wU$. In the case when $w = s \in S$,

the morphism $\mathrm{pr}_s : X(s) \to T_s \cong \mathbb{G}_{m,k}$ extends to $\overline{\mathrm{pr}}_s : \overline{X(s)} \to \mathbb{G}_{a,k}$ and we have

$$\overline{K(s)} = (-\overline{\mathrm{pr}}_s)^* \mathcal{L}_\psi$$

and in the case of general $w \in W$

$$\overline{K(w)} = \overline{K(s_{i_1})} * \cdots * \overline{K(s_{i_k})} \tag{3.2.3}$$

where $w = s_{i_1} \dots s_{i_k}$ is a reduced expression. One can take this as the definition of Kazhdan-Laumon sheaves (as it is well-defined by the proposition below and therefore does not depend on the reduced word given the proposition below) or refer to the explicit definition which works for all $w \in W$ at once given in [32] or [48].

The following was proved by a geometric argument by Kazhdan and Laumon in their original paper on the subject; it will be the main tool we use to (re)prove the braid relations for Juyumaya's generators of the Yokonuma-Hecke algebra once we show that these generators are categorified by Kazhdan-Laumon sheaves.

Proposition 3.2.1 ([32]). *The Kazhdan-Laumon sheaves $\overline{K(s)}$ for $s \in S$ under convolution satisfy the braid relations (up to isomorphism).*

In [21] it is observed that the sheaves $\overline{K(w)}$ can be obtained by pullback from a canonical multiplication map $m : G/U \times G/U \to U\backslash G/U$, and that this equivalence respects convolution. As a result, from now on we will consider $\overline{K(w)}$ as an element of $D_c^b(U\backslash G/U)$ and we will work directly with convolution of sheaves on $U\backslash G/U$ in the sense of Definition 2.3 of loc. cit.

We also refer to Definition 2.4 of loc. cit. for the action of $D_c^b(U\backslash G/U)$ on $D_c^b(G/U)$ by convolution on the right, which we will use going forward when we consider the defining action of Kazhdan-Laumon sheaves on $D_c^b(G/U)$, which for $\overline{K(s)}$ agrees with the corresponding symplectic Fourier-Deligne transform on $D_c^b(G/U)$ as we explain in 3.2.2.

Note that for any $g \in G/U$, we obtain an endomorphism of $D_c^b(G/U)$ by pullback along the natural action of g on G/U by multiplication; it is shown in [32] that convolution by $\overline{K(w)}$ commutes with this action.

Remark 3.2.2. In [32] and [48], the authors study symplectic Fourier-Deligne transforms on the basic affine space corresponding to simple reflections $s \in S$. It is shown in loc. cit. that the symplectic Fourier-Deligne transform of sheaves on G/U associated to $s \in S$ agrees with convolution by the Kazhdan-Laumon sheaf $\overline{K(s)}$, and so the "symplectic Fourier transform algebra" generated by the induced endomorphisms of $K_0(G/U)$ under composition is the same as the subalgebra of $K_0(G/U)$ generated by Kazhdan-Laumon sheaves under convolution which we study at present.

In addition to the Kazhdan-Laumon sheaves $\overline{K(s)}$ introduced above (which in the next section we will show categorify the generators g_s of the generalized algebra and braids and

ties which we will introduce later), it will be useful to introduce another collection of sheaves $E(s)$ which will categorify the other generators e_s of this algebra.

Definition 3.2.3. For any $s \in S$, note that there is a natural inclusion $i_{T_s} : T_s \hookrightarrow G/U$ with $i_{T_s}(t_s) = t_s U$. Let $\underline{\mathbb{C}}_{T_s}$ be the constant sheaf on T_s and define the sheaf $E(s) = i_{T_s!}\underline{\mathbb{C}}_{T_s}$.

Definition 3.2.4. Let $\mathrm{KL}(v)$ be the algebra generated by the images in $K_0(G/U)$ of the classes $[\overline{K(s)}]$ of Kazhdan-Laumon sheaves for all $s \in S$ under convolution. Let $\widetilde{\mathrm{KL}}(v)$ be the subalgebra of $K_0(G/U)$ generated by $\mathrm{KL}(v)$ along with the $[E(s)]$ for all $s \in S$.

We use the notation $\mathsf{a}_w = [\overline{K(w)}]$ for any $w \in W$ and $\mathsf{e}_s = [E(s)]$ for any $s \in S$.

Alternatively, $\mathrm{KL}(v)$ could also be defined as the subalgebra of $K_0(G/U \times G/U)$ generated by the Kazhdan-Laumon sheaves on $G/U \times G/U$ (as originally defined in [32]) under convolution.

3.2.2 The algebras $\mathcal{E}(v)$ and $\mathcal{C}(v)$

We now recall the definition of the "generalized algebra of braids and ties." This was defined in Type A by Aicardi and Juyumaya in [3], and subsequently generalized to all types by Marin in [42]. Here, to unify notation we use $\mathcal{E}(v)$ (originally used in [3] to refer only to the original definition in Type A) to denote the generalized algebra in the sense of Marin (which is called C_W^R in his paper), whose definition is very similar and agrees in Type A with the original algebra of braids and ties.

Definition 3.2.5. The generalized algebra of braids and ties $\mathcal{E}(v)$ is the algebra generated as follows. For each $s \in S$, we have a generator g_s, for each $r \in W$ a reflection, we have a generator e_r. We begin with the relations.

$$\begin{aligned}
\underbrace{g_s g_t g_s \dots}_{m_{st}} &= \underbrace{g_t g_s g_t \dots}_{m_{st}} \quad s,t \in S \\
e_r^2 &= e_r \quad \text{for all reflections } r \in W \\
e_{r_1} e_{r_2} &= e_{r_2} e_{r_1} \\
e_{r_1} e_{r_2} &= e_{r_1} e_{r_1^{-1} r_2 r_1} \quad \text{for all reflections } r_1, r_2 \in W \\
g_s e_r &= e_{srs} g_s \quad \text{for all reflections } r \text{ and all } s \in S \\
g_s^2 &= 1 + (v^2 - 1) e_s (1 + g_s) \quad s \in S
\end{aligned}$$

Now for any $w \in W$ with reduced expression $s_{i_1} \dots s_{i_k}$, the braid relation above shows that $g_w = g_{s_{i_1}} \dots g_{s_{i_k}}$ is well-defined. Further, for any reflection subgroup $J \subset W$, set $e_J = e_{r_1} \dots e_{r_\ell}$ where $\{r_1, \dots, r_\ell\}$ is a generating set of reflections for J. In Section 3.4 of [42], a map p from the set of reflection subgroups of W to the set of closed symmetric root subsystems of Φ is defined. We now impose the additional relation

$$e_J = e_{J'}, \quad p(J) = p(J') \tag{3.2.4}$$

to complete the definition of $\mathcal{E}(v)$. We note that clearly this algebra is also generated only by $\{g_s, e_s\}_{s\in S}$.[1] We then define the algebra $\mathcal{C}(v)$ to be the "braid image" subalgebra of $\mathcal{E}(v)$, i.e. the subalgebra generated by the g_s, $s \in S$.

3.3 Categorification theorems

3.3.1 Sheaves and functions

Since G/U is defined over $\mathbb{F}_q$, any object lying in $D_c^b(G/U)$ comes equipped with a natural Frobenius endomorphism. For any power q^k of q for $k \geq 1$, taking the trace of the kth power of the Frobenius morphism yields a well-defined map

$$\mathrm{tr}_{q^k} : K_0(G/U) \to \mathbb{C}[(G/U)(\mathbb{F}_{q^k})].$$

These assemble to a map

$$\mathrm{tr} : K_0(G/U) \to \prod_{k\geq 1} \mathbb{C}[(G/U)(\mathbb{F}_{q^k})].$$

Proposition 3.3.1 ([34])**.** tr *is injective.*

Definition 3.3.2. For any $k \geq 1$, let $\mathrm{KL}_{q,k}$, an algebra which is a subspace of $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$, be the image of $\mathrm{KL}(v) \subset K_0(G/U)$ under the morphism tr_{q^k}. Similarly, let $\widetilde{\mathrm{KL}}_{q,k}$ be the image of $\widetilde{\mathrm{KL}}(v)$. We let $\overline{\mathsf{a}_s}$ and $\overline{\mathsf{e}_s}$ denote the images in $\mathrm{KL}_{q,k}$ of the corresponding generators.

3.3.2 Symplectic Fourier transforms on functions

As mentioned in Remark 3.2.2, it is shown in [32] and [48] that convolution with $\overline{K(s)}$ agrees with the symplectic Fourier-Deligne transform defined in loc. cit. In particular, one can then deduce the following description.

Given a simple reflection $s \in S$, let P_s be the corresponding parabolic subgroup of G, and let $Q_s = [P_s, P_s]$. As explained in detail in Section 1 of [31], there is a natural $(\mathbb{A}_k^2 \setminus \{(0,0)\})$-fibration $G/U \to G/Q_s$ which can be completed to form a 2-dimensional vector bundle equipped with a G-invariant symplectic form $\langle , \rangle$. With this in mind, the equivalence between $- * \overline{K(s)}$ and the symplectic Fourier-Deligne transform gives that for any $k \geq 1$ and any $f \in \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$, $x \in (G/U)(\mathbb{F}_{q^k})$,

$$\left(\mathrm{tr}_{q^k}(f) * \mathrm{tr}_{q^k}\left(\overline{K(s)}\right)\right)(x) = \frac{1}{q^k} \sum_{y \in xQ_s(\mathbb{F}_{q^k})} \psi(\langle x, y\rangle) f(y). \tag{3.3.1}$$

[1] Note that the algebra generated by the first six relations is called C_W in [42] and our algebra $\mathcal{E}(v)$ obtained by adding in the relation (3.2.4) is called C_W^R in loc. cit. We also note that (3.2.4) is redundant in Type A, and therefore doesn't occur in the original definition of the algebra of braids and ties.

Further, it is a straightforward computation using Grothendieck's sheaf-function dictionary that for any $x \in (G/U)(\mathbb{F}_{q^k})$,

$$\left(\mathrm{tr}_{q^k}(f) * \mathrm{tr}_{q^k}(E(s))\right)(x) = \sum_{x^{-1}y \in T_s(\mathbb{F}_{q^k}) \subset X(\mathbb{F}_{q^k})} f(y). \tag{3.3.2}$$

We include the above formulas to note that the main computations in this chapter (Propositions 3.3.6 and 3.4.1) can be (and indeed were, in a previous version of [47]) done explicitly on the level of functions using the formulas (3.3.1) and (3.3.2) and then appealing to Proposition 3.3.1, but we will instead prove them on the level of sheaves using the six-functor formalism.

3.3.3 Connection to the Yokonuma-Hecke algebra

We will sometimes write elements of $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$ as functions on $(G/U)(\mathbb{F}_{q^k})$, and other times as linear combinations of elements in $(G/U)(\mathbb{F}_{q^k})$. For any $g \in G(\mathbb{F}_{q^k})$, let δ_g be the indicator function on $\overline{g} \in (G/U)(\mathbb{F}_{q^k})$.

Definition 3.3.3. Let $\mathrm{Y}_{q,k}$ (the Yokonuma-Hecke algebra) be defined as the algebra

$$\mathrm{End}_{G(\mathbb{F}_q)}(\mathbb{C}[(G/U)(\mathbb{F}_{q^k})])$$

under composition.

As explained in [30], the Bruhat decomposition $G = \coprod_{n \in N_{G(\mathbb{F}_{q^k})}(T(\mathbb{F}_{q^k}))} UnU$ gives that the standard basis of $\mathrm{Y}_{q,k}$ is parametrized by $n \in N_{G(\mathbb{F}_{q^k})}(T(\mathbb{F}_{q^k}))$. For any $s \in S$, we can define $R_s \in \mathrm{Y}_{q,k}$ (the standard basis element corresponding to s) as in loc. cit., and for any $t \in T(\mathbb{F}_{q^k})$ we can define R_t similarly. In this way, we can identify $\mathrm{Y}_{q,k} \cong \mathbb{C}[U\backslash G/U(\mathbb{F}_{q^k})]$, where the endomorphisms $R_s, R_t \in \mathrm{Y}_{q,k}$ become identified with the indicator functions at s and t respectively, and composition of endomorphisms is identified with convolution.

Following [30], for any $r \in \mathbb{F}_{q^k}^\times$ we define

$$h_s(r) = \rho_s \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},$$

where $\rho_s : SL_2(\mathbb{F}_{q^k}) \to G(\mathbb{F}_{q^k})$ is the morphism corresponding to the simple root α_s. When $t = h_s(r)$ for some $s \in S$, $r \in \mathbb{F}_{q^k}^\times$, we write $H_s(r) = R_t$. We then define for each $s \in S$ the element

$$E_s = \sum_{r \in \mathbb{F}_{q^k}^\times} H_s(r).$$

Now we recall the original presentation of the Yokonuma-Hecke algebra, given by Yokonuma in [53].

Proposition 3.3.4. *The Yokonuma-Hecke algebra $\mathrm{Y}_{q,k}$ is generated as an algebra by the R_s ($s \in S$); and the R_t ($t \in T(\mathbb{F}_{q^k})$) moreover, these generators along with the relations*

$$\begin{aligned} R_s^2 &= qH_s(-1) + R_sE_s \\ \underbrace{R_sR_{s'}R_sR_{s'}\dots}_{m_{ss'}} &= \underbrace{R_{s'}R_sR_{s'}R_s\dots}_{m_{ss'}}, \quad s, s' \in S \\ R_tR_s &= R_sR_{t'}, \quad t' = n_stn_s^{-1}, t \in T(\mathbb{F}_{q^k}) \\ R_{t_1}R_{t_2} &= R_{t_1t_2}, \quad t_1, t_2 \in T(\mathbb{F}_{q^k}) \end{aligned}$$

define a presentation of $\mathrm{Y}_{q,k}$.

In [29] and [30], the authors introduced an alternative set of generators and relations; this new set of generators will be the ones directly categorified by Kazhdan-Laumon sheaves.

For any $s \in S$, let

$$\Psi_s = \sum_{r \in (\mathbb{F}_{q^k})^\times} \psi(r)H_s(r),$$

and let $L_s = q^{-k}(E_s + R_s\Psi_s)$.

Proposition 3.3.5 ([30])**.** *The algebra $\mathrm{Y}_{q,k}$ is generated by L_s, $s \in S$, and R_t, $t \in T(\mathbb{F}_{q^k})$, and a presentation of this algebra is given by imposing the relations*

$$\begin{aligned} L_s^2 &= 1 - q^{-k}(E_s - L_sE_s) \\ \underbrace{L_sL_tL_sL_t\dots}_{m_{st}} &= \underbrace{L_tL_sL_tL_s\dots}_{m_{st}} \\ R_tL_s &= L_sR_{t'}, t' = sts^{-1} \quad (t \in T(\mathbb{F}_{q^k})) \\ R_{t_1}R_{t_2} &= R_{t_1t_2} \quad (t_1, t_2 \in T(\mathbb{F}_{q^k})). \end{aligned}$$

The action by convolution gives rise to a natural morphism

$$\psi_{q,k} : \widetilde{\mathrm{KL}}_{q,k} \to \mathrm{End}(\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]).$$

Convolution of sheaves descends under the map tr_{q^k} to the convolution of functions in the space $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$.

Proposition 3.3.6. *The image of $\psi_{q,k}$ lies in $\mathrm{Y}_{q,k}$. We have*

$$\begin{aligned} \psi_{q,k}(\mathsf{a}_s) &= L_s \\ \psi_{q,k}(\mathsf{e}_s) &= E_s. \end{aligned}$$

Proof. We first note that under the identification $\mathrm{Y}_{q,k} \cong \mathbb{C}[U\backslash G/U(\mathbb{F}_{q^k})]$, we can think of the map $\psi_{q,k}$ as having target $\mathbb{C}[U\backslash G/U(\mathbb{F}_{q^k})]$. In doing so, $\psi_{q,k}$ can be thought of as the natural inclusion $\widetilde{\mathrm{KL}}_{q,k} \hookrightarrow \mathbb{C}[U\backslash G/U]$. Accordingly, it is enough to show that $\mathrm{tr}_{q^k}(\overline{K(s)}) = L_s$ and $\mathrm{tr}_{q^k}(E(s)) = E_s$ as elements of $\mathbb{C}[U\backslash G/U(\mathbb{F}_{q^k})]$.

By the definition of $E(s) = i_{T_s,!}\mathbb{C}$, this is clear for $E(s)$. It remains to show that $\mathrm{tr}_{q^k}(\overline{K(s)}) = L_s$ for any $s \in S$. Since in this computation we only consider a single simple reflection, it is enough to check this in the case where $G = \mathrm{SL}_2$. In this case, let B be the Borel subgroup of upper-triangular matrices with unipotent radical U, and let $p_s : U\backslash \mathrm{SL}_2/U \to \mathbb{G}_a$ be the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto -c. \tag{3.3.3}$$

It follows from the description of $\overline{K(s)}$ in Section 5.2 of [48] that $\overline{K(s)} = p_s^*\mathcal{L}_\psi[2](1)$. Now note that any element of $N_{G(\mathbb{F}_{q^k})}(T(\mathbb{F}_{q^k}))$ can be written as either t or st for some $t \in T(\mathbb{F}_{q^k})$. For any such element, we have that

$$\begin{aligned} L_s(t) &= q^{-k}E_s(t) + q^{-k}R_s\Psi_s(t) && (3.3.4)\\ &= q^{-k} + 0 && (3.3.5)\\ L_s(st) &= q^{-k}E_s(st) + q^{-k}R_s\Psi_s(st) && (3.3.6)\\ &= 0 + q^{-k}\psi(t). && (3.3.7) \end{aligned}$$

Similarly, by our description of $\overline{K(s)}$ and the six-functor formalism for functions, we have that, as elements of $\mathbb{C}[U\backslash G/U]$, $\mathrm{tr}_{q^k}(\overline{K(s)}) = q^{-k}\mathrm{tr}_{q^k}(\mathcal{L}_\psi) \circ p_s$. So, when identifying the target of $\psi_{q,k}$ with $\mathbb{C}[U\backslash G/U]$, for any $t \in T(\mathbb{F}_{q^k})$ we have

$$\begin{aligned} \psi_{q,k}(\mathsf{a}_s)(t) &= q^{-k}\psi(0) = q^{-k}, && (3.3.8)\\ \psi_{q,k}(\mathsf{a}_s)(st) &= q^{-k}\psi(t), && (3.3.9) \end{aligned}$$

which guarantees that $\psi_{q,k}(\mathsf{a}_s) = L_s$. □

3.3.4 Algebras with a generic parameter

We now use Proposition 3.3.6 to prove our main result.

Proof of Theorem 3.1.1. Consider the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{KL}}(v) & & \mathcal{E}(v) \\ & \searrow^{\mathrm{tr}} \quad \swarrow_{i} & \\ & \prod_{k\geq 1}\mathbb{C}[(G/U)(\mathbb{F}_{q^k})] & \end{array},$$

where i is assembled from the composition of the natural maps

$$\mathcal{E}(v) \to \mathcal{E}_{q^k} \to \mathrm{Y}_{q,k} \hookrightarrow \mathbb{C}[(G/U)(\mathbb{F}_{q^k})],$$

where $\mathcal{E}(v) \to \mathcal{E}_{q^k}$ is the specialization map at $v = q^{k/2}$.

The fact that tr is injective follows from Proposition 3.3.1. We now argue that i is injective. In Section 3.4 of [42], which relies on the proof from the end of Section 2.1 of loc. cit., it is stated that the natural map from $\mathcal{E}_{q'}$ (called $\mathcal{C}^R_{W,q}$ in loc. cit.) to $\mathrm{Y}_{q'}$ is an embedding for "generic" q'. Following the proof of this fact in Section 2.1 of loc. cit., one recovers a more precise statement: there exists some r_0 depending on W such that the natural map $\mathcal{E}_{q'} \to \mathrm{Y}_{q'}$ sending $g_s \to -L_s$ is injective whenever $q' - 1$ has a prime factor larger than r_0. For $q = p^k$ any prime power, there exists an a for which $q' = q^a$ satisfies this condition. Indeed, let p' be a prime number greater than r_0 and distinct from p, and set $a = p' - 1$; then by Fermat's little theorem, p' divides $q^a - 1$. Further, we can take large enough a so that the specialization map $v \mapsto q^{a/2}$ taking $\mathcal{E}(v) \to \mathcal{E}_{q^a}$ is itself injective.

Now note that Proposition 3.3.6 (applied to all powers of q at once) shows that the images of tr and i are equal, and that

$$\begin{aligned} \mathrm{tr}(\mathsf{a}_s) &= i(-g_s), \\ \mathrm{tr}(\mathsf{e}_s) &= i(e_s). \end{aligned}$$

Since we have just identified both $\mathrm{KL}(v)$ and $\mathcal{E}(v)$ as the very same subalgebra of $\prod_{k\geq 1} \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$, it follows that the map

$$\psi : \widetilde{\mathrm{KL}}(v) \to \mathcal{E}(v)$$

defined on generators by $\psi(\mathsf{a}_s) = -g_s$, $\psi(\mathsf{e}_s) = e_s$ is a well-defined algebra isomorphism. The same proof works to show that there is a well-defined isomorphism $\psi' : \mathrm{KL}(v) \to \mathcal{C}(v)$ with $\psi'(\mathsf{a}_s) = -g_s$. □

3.3.5 Connection to Polishchuk's cubic relation

As we have explained, since Theorem 3.1.1 gives a full description of the algebra $\mathrm{KL}(v)$ can be thought of as an enhancement of previous results on some relations satisfied by the generators of this algebra.

Namely, in addition to the braid relations shown by Kazhdan and Laumon in [32], Polishchuk found the additional relation

$$(\mathsf{a}_s^2 - 1)(\mathsf{a}_s + v^2) = 0 \tag{3.3.10}$$

for any $s \in S$. We now explain how this relation can be recovered from our description. By Theorem 3.1.1, recalling that a_s can be identified with $-g_s$, it is enough to show that

$(g_s^2-1)(g_s-v^2)=0$ in $\mathcal{E}(v)$, so we can work with the relations in Definition 3.2.5. The last relation shows that $g_s^2-1=(v^2-1)e_s(1+g_s)$, so (3.3.10) becomes equivalent to $(g_s-v^2)(1+g_s)e_s=0$. This is is then obtained by multiplying the last relation in Definition 3.2.5 by e_s.

An informal interpretation of the relation is that (3.3.10) is the least common multiple of the two expressions a_s^2-1 and $(\mathsf{a}_s-1)(\mathsf{a}_2+v^2)$, the former of which is present in the Weyl group, and the latter in the Hecke algebra. Since we expect any relation in $\mathrm{KL}(v)$ to act trivially on $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$, the relation (3.3.10) is then natural since it is the minimal relation among powers of a_s which acts trivially on $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]_\theta$ both when $s\notin W_\theta^\circ$ and when $s\in W_\theta^\circ$ (with the Weyl group and Hecke algebra relations holding in these two situations respectively). In this sense, the algebra $\mathrm{KL}(v)$ can be thought of as the "minimal" algebra containing all a_s, $s\in S$ which encodes the relations of all monodromic Hecke algebras at once.

3.4 Connection to monodromic Hecke categories

3.4.1 The monodromic Hecke algebra

For a W-orbit $\mathfrak{o}$ of character sheaves on T, Lusztig and Yun define in Section 3.14 of [41] a unital associative $\mathbb{Z}[v,v^{-1}]$-algebra with generators A_w (called T_w in loc. cit.) for $w\in W$ and $1_{\mathcal{L}}$ ($\mathcal{L}\in\mathfrak{o}$) satisfying the relations

$$
\begin{aligned}
&1_{\mathcal{L}}1_{\mathcal{L}'}=\delta_{\mathcal{L},\mathcal{L}'}1_{\mathcal{L}}, \quad && \text{for } \mathcal{L},\mathcal{L}'\in\mathfrak{o};\\
&A_wA_{w'}=A_{ww'}, && \text{if } w,w'\in W \text{ and } \ell(ww')=\ell(w)+\ell(w');\\
&A_w1_{\mathcal{L}}=1_{w\mathcal{L}}A_w, && \text{for } w\in W,\mathcal{L}\in\mathfrak{o};\\
&A_s^2=v^2A_1+(v^2-1)\sum_{\mathcal{L};s\in W_{\mathcal{L}}^\circ}A_s1_{\mathcal{L}}, && \text{for simple reflections } s\in W;\\
&A_1=1=\sum_{\mathcal{L}\in\mathfrak{o}}1_{\mathcal{L}}.
\end{aligned}
$$

As in [41], we set $\tilde{A}_w=v^{-\ell(w)}A_w\in\mathbf{H}_{\mathfrak{o}}$. In section 3.14 of [41], the authors discuss a connection between this algebra and $K_0(\oplus_{\mathcal{L},\mathcal{L}'\in\mathfrak{o}}({}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}))$, where the elements $\tilde{A}_w1_{\mathcal{L}}$ become associated with classes in the Grothendieck group of the standard monodromic sheaves $\Delta(w)_{\mathcal{L}}$ defined and studied in loc. cit. As a result, we can identify $\mathbf{H}_{\mathfrak{o}}$ with a subalgebra of $K_0(U\backslash G/U)$ under convolution generated by the $[\Delta(w)_{\mathcal{L}}]$ for $w\in W$, $\mathcal{L}\in\mathfrak{o}$.

3.4.2 Monodromic sheaves from Kazhdan-Laumon sheaves

Proposition 3.4.1. *Convolution with $\Delta(e)_{\mathcal{L}}$ induces a surjection $\pi_{\mathcal{L}}$ from* $\mathrm{KL}(v) \to \mathbf{H}_{\mathfrak{o}}$. *We have*

$$\pi_{\mathcal{L}}(\mathsf{a}_s) = \begin{cases} -v\tilde{A}_s^{-1}1_{\mathcal{L}} & s \in W_{\mathcal{L}}^{\circ} \\ -\tilde{A}_s 1_{\mathcal{L}} & s \notin W_{\mathcal{L}}^{\circ} \end{cases} \tag{3.4.1}$$

Proof. Since the claim only concerns a single simple reflection, it is enough to prove the result for $G = \mathrm{SL}_2$. Let $j_s : U\backslash BsB/U \to U\backslash G/U$ and $j_e : U\backslash B/U \to U\backslash G/U$ be the inclusions of the open and closed Bruhat strata. Recall the projection to T given by $\mathrm{pr}_s : U\backslash BsB/U \to T$ as defined in Section 3.2.1.

Convolving the usual open-closed distinguished triangle for the maps j_s and j_e with $\Delta_{\mathcal{L}}(e)$ gives the distinguished triangle

$$(j_{s!}j_s^*\overline{K(s)}) * \Delta_{\mathcal{L}}(e) \to \overline{K(s)} * \Delta_{\mathcal{L}}(e) \to (j_{e*}j_e^*\overline{K(s)}) * \Delta_{\mathcal{L}}(e). \tag{3.4.2}$$

Since $U\backslash B/U \cong T$ and so $\Delta_{\mathcal{L}}(e) = j_{e!}(\mathcal{L}) = j_{e*}(\mathcal{L})$, proper base change and the projection formula imply

$$(j_{e*}j_e^*\overline{K(s)}) * \Delta_{\mathcal{L}}(e) = (j_{e*}\mathcal{L}) \otimes R\Gamma_c(T, \mathcal{L}), \tag{3.4.3}$$

$$(j_{s!}j_s^*\overline{K(s)}) * \Delta_{\mathcal{L}}(e) = j_{s!}\mathrm{pr}_s^*(\mathcal{L}[2](1)) \otimes R\Gamma_c(T, \mathcal{L} \otimes \mathcal{L}_\psi). \tag{3.4.4}$$

In this rank-one situation, if $s \in W_{\mathcal{L}}^{\circ}$, then $\mathcal{L}$ is constant. In this case, in (3.4.3) we get the tensor product of $\Delta_{\mathcal{L}}(e)$ itself with the compactly supported cohomology of $T \cong \mathbb{G}_{\mathrm{m}}$, whose class in K_0 is (v^2-1). The factor $R\Gamma_c(T, \mathcal{L}\otimes\mathcal{L}_\psi)$ in (3.4.4) is 1-dimensional and concentrated in cohomological degree 1, and has weight zero by Remark 4.4 of [18, Applications de la formule des traces aux sommes trigonométriques]. Taking the class in the Grothendieck group, and then summing the terms obtained from (3.4.3) and (3.4.4), we then get $-v\tilde{A}_s^{-1}1_{\mathcal{L}}$.

Similarly, if $s \notin W_{\mathcal{L}}^{\circ}$, then in this case $\mathcal{L}$ is not constant. So the term (3.4.3) is zero since $R\Gamma_c(T, \mathcal{L}) = 0$. Further, the same remark from [18] gives that $R\Gamma_c(T, \mathcal{L} \otimes \mathcal{L}_\psi)$ is still concentrated in cohomological degree 1, is one-dimensional, but this time has weight 1. This leaves only a single term, whose class in the Grothendieck group gives $-\tilde{A}_s 1_{\mathcal{L}}$ in the monodromic Hecke algebra. □

From now on, let $\pi = \pi_{\mathcal{L}_{\mathrm{triv}}}$ be the surjection $\mathrm{KL}(v) \to \mathcal{H}$.

3.4.3 Lifting the Kazhdan-Lusztig basis of the Hecke algebra

We now recall the Kazhdan-Lusztig basis C_w of the Hecke algebra and its interpretation in terms of simple perverse sheaves $\mathrm{IC}(w)$ on G/U. The Kazhdan-Lusztig basis $\{C_w\}_{w\in W}$ of the Hecke algebra $\mathcal{H}$ is a basis which is uniquely characterized by the conditions that

$C_w \in \tilde{A}_w + \sum_{y \leq w} v\mathbb{Z}[v]\tilde{A}_y$ and $\overline{C}_w = C_w$.

The following lemma is well-known (for instance, c.f. (3.5.3) of [52]):

Lemma 3.4.2. *For any $w, w' \in W$ with $\ell(ww') = \ell(w) + \ell(w')$, we have*

$$C_w C_{w'} - C_{ww'} \in \mathrm{span}_{\mathbb{Z}}\{C_y\}_{y<ww'}$$

There is a well-known geometric interpretation of the generators $\tilde{A}_w$ and C_w of $\mathcal{H}$. Indeed, there are standard sheaves $\Delta(w)$ and $\mathrm{IC}(w)$ on G/U which categorify these generators respectively, c.f. Chapter 7 of [2] for a very clear exposition of this setup in the case of G/B; a version of this with a different normalization relevant to the Kazhdan-Laumon context was explained in Section 4.1 of [45] for G/U. With this in mind, Lemma 3.4.2 shows that the span of $\{[\mathrm{IC}(w)]\}_{w\in W}$ is a subalgebra of $K_0(U\backslash G/U)$.

Proposition 3.4.3 (Proposition 3.8 in [42])**.** *The assignment*

$$\begin{aligned}\overline{g_s} &= g_s^{-1}\\ \overline{e_s} &= e_s\\ \overline{v} &= v^{-1}\end{aligned}$$

extends to well-defined ring automorphism of $\mathcal{E}(v)$. We will use the same bar notation to denote the corresponding involution on $\mathcal{C}(v)$, $\mathrm{KL}(v)$, and $\widetilde{\mathrm{KL}}(v)$ using the isomorphisms of the previous section.

Proposition 3.4.4. *There exists a linearly independent set $\{\mathsf{c}_w\}_{w\in W}$ of $\mathrm{KL}(v)$ satisfying $\overline{\mathsf{c}_w} = \mathsf{c}_w$. It has the property that*

$$\pi(\mathsf{c}_w) = C_w.$$

Proof. We will first show that for any $w \neq 1$, $[\mathrm{IC}(w)] \in K_0(G/U)$ actually lies in $\mathrm{KL}(v)$; once we do that, we will define $\mathsf{c}_1 = 1$ and $\mathsf{c}_w = [\mathrm{IC}(w)]$ for any $w \neq 1$, and then check the properties in the proposition.

We begin in the case where $w = s \in S$. In this case, we define

$$\mathsf{c}_s = \frac{\mathsf{a}_s^2 - 1}{v - v^3}. \tag{3.4.5}$$

It is easy to check by working on the level of functions on $(G/U)(\mathbb{F}_{q^k})$ that c_s as defined above is equal to $[\mathrm{IC}(s)]$.

We have

$$\begin{aligned}\pi(\mathsf{c}_s) &= \frac{v^2\tilde{A}_s^{-2}-1}{v-v^3}\\ &= \frac{v^2((v^{-1}-v)\tilde{A}_s+(v^{-2}-1+v^2))-1}{v-v^3}\\ &= \tilde{A}_s - v = c_s.\end{aligned}$$

Using the relation $(\mathsf{a}_s^2-1)(\mathsf{a}_s+v^2)=0$, it is a straightforward computation that

$$\mathsf{a}_s^{-2}-1 = v^{-4}(1-\mathsf{a}_s^2),$$

which implies

$$\begin{aligned}\overline{\mathsf{c}_s} &= \frac{\mathsf{a}_s^{-2}-1}{v^{-1}-v^{-3}}\\ &= \frac{1-\mathsf{a}_s^2}{v^4(v^{-1}-v^{-3})}\\ &= \mathsf{c}_s.\end{aligned}$$

For the case of general $w \in W$, suppose for induction that $[\mathrm{IC}(y)] \in \mathrm{KL}(v)$ for all y with $\ell(y) < \ell(w)$. Choose some $s \in S$ such that $\ell(sw) < \ell(w)$. By Lemma 3.4.2 and by our induction hypothesis along with the fact that $\mathrm{KL}(v)$ is a subalgebra, this implies $[\mathrm{IC}(sw)] \in \mathrm{KL}(v)$. Now we can define $\mathsf{c}_1 = 1$ and $\mathsf{c}_w = [\mathrm{IC}(w)]$ for all $w \neq 1$, and this clearly defines a linearly independent subset of $\mathrm{KL}(v)$.

We now show that $\overline{\mathsf{c}_w} = \mathsf{c}_w$. Indeed, this is trivial for c_1, and checked this for c_s above for any $s \in S$. By the definition of the duality on $\mathrm{KL}(v)$, it is multiplicative and $\mathbb{Z}$-linear, and so $\overline{\mathsf{c}_w} = \mathsf{c}_w$ follows again by induction on $\ell(w)$ using Lemma 3.4.2

Finally, we show that $\pi(\mathsf{c}_w) = C_w$ for all $w \in W$. First note that clearly the bar involution has the property that it lifts the usual bar involution on the Hecke algebra, so $\overline{\pi(\mathsf{c}_w)} = \pi(\mathsf{c}_w)$. Now once again, Lemma 3.4.2 implies the other defining property of C_w; this means we must have $\pi(\mathsf{c}_w) = C_w$. Alternatively, we could have simply observed that by the definition of c_w, $\pi(\mathsf{c}_w)$ is $[\mathrm{IC}(w)] \in \mathcal{H} \cong K_0(B\backslash G/U)$ which corresponds to the canonical basis element C_w. □

3.5 Dimension formula

In this section, assume that we are in Type A_n, i.e. $G = \mathrm{SL}_{n+1}$. This is the case where $\mathcal{E}(v)$ becomes the usual algebra of braids and ties as originally defined in [3]. The subalgebra $\mathcal{C}_{A_n}(v) \subset \mathcal{E}_{A_n}(v)$ was studied in more detail in this case in Section 4 of [42]. In loc. cit., the author used a computer to compute the sequence of dimensions of $\mathcal{C}_{A_n}(v)$ for a few rational

values of v, obtaining the table below.

n	1	2	3	4
$\dim \mathcal{C}_{A_n}(v)$	3	20	217	3364

In this section, we give an explicit formula for $\dim \mathcal{C}_{A_n}(v)$ for all values of n, extending this table.

3.5.1 A formula in terms of reflection subgroups of W

Definition 3.5.1. We call a reflection subgroup $R' \subset W$ contiguous if its Dynkin type is connected. For any reflection subgroup $R \subset W$, we can write $R = \prod_{i=1}^{\ell} R'_i$ for some contiguous reflection subgroups R'_i uniquely up to permutation of the R'_i; we call these the contiguous components of R.

Definition 3.5.2. For any reflection subgroup R of W, let J_R be the vector subspace of $\mathcal{E}(v)$ spanned by all terms of the form

$$g_w(1 + g_{r_1}) \dots (1 + g_{r_\ell}), \tag{3.5.1}$$

where $r_1, \dots, r_\ell$ are reflections lying in distinct contiguous components of R.

Lemma 3.5.3. *For any reflection subgroup R of W, $J_R e_R \subset \mathcal{C}(v)$.*

Proof. By induction on the number of contiguous components of R, it is enough to show that for any contiguous reflection subgroup R (i.e. one of type A_k for some $k \leq n$) of W and any reflection $r \in R$, we have $(g_r + 1)e_R \in \mathcal{C}(v)$.

Suppose for induction (the case $k = 1$ is tautological) that this result holds for all reflection subgroups of type $A_{k'}$, $k < k$. We will first show that for any reflections $r, r' \in R$ for which $\langle r, r' \rangle$ is itself a reflection subgroup of type A_2 that $(g'_r + 1)(g_r + 1)e_{\langle r,r' \rangle} \in \mathcal{C}(v)$. To do this, using the fact that all reflection subgroups R of type A_2 are conjugate, we can choose a set of reflections $\{r_1, \dots, r_{k-2}\}$ in R such that $\{r_1, \dots, r_{k-2}, r', r\}$ generates R and for which $\{r_1, \dots, r_{k-2}, r'\}$ is of type A_{k-1}. So applying our induction hypothesis, $(g_{r'} + 1)e_{\langle r_1,\dots,r_{k-2},r' \rangle} \in \mathcal{C}(v)$, and so $(g_{r'} + 1)(g_r + 1)e_R$ is too. Similarly we can reverse the roles of r and r' to obtain $(g_r + 1)(g_{r'} + 1)e_R$ as well.

Now using again the conjugacy in R of reflection subgroups of type A_2, let $y \in R$ be such that $yry^{-1} = s_1, yr'y^{-1} = s_2$. Then we can deduce that

$$g_y^{-1}(g_{s_1} + 1)(g_{s_2} + 1)e_{\langle s_1,s_2 \rangle} g_y, g_{y^{-1}}(g_{s_2} + 1)(g_{s_1} + 1)e_{\langle s_1,s_2 \rangle} g_y.$$

Since the $g = g_{s_1} e_{\langle s_1,s_2 \rangle}$ or $g = g_{s_2} e_{\langle s_1,s_2 \rangle}$ satisfy the usual quadratic Hecke relation $(g - v^2)(g +$

$1) = 0$, it is a straightforward computation in the Hecke algebra of type A_2 to observe that

$$\begin{aligned}&(g_{s_1} - 1)e_{\langle s_1,s_2\rangle}, (g_{s_2} - 1)e_{\langle s_1,s_2\rangle}\\ &\in \mathrm{span}\{g_2(g_{s_1} + 1)(g_{s_2} + 1)e_{\langle s_1,s_2\rangle}, g_w(g_{s_2} + 1)(g_{s_1} + 1)e_{\langle s_1,s_2\rangle}\}_{w\in\langle s_1,s_2\rangle}.\end{aligned}$$

By conjugating back by g_y, we deduce that $(g_r + 1)e_{\langle r,r'\rangle}, (g_{r'} + 1)e_{\langle r,r'\rangle} \in \mathcal{C}(v)$.

Now in the general case, note that for any reflection $r \in W$, $(g_r + 1)e_{\langle r\rangle} \in \mathcal{C}(v)$ by the quadratic relation satisfied by g_r. So for the rank k reflection subgroup R in question, we have

$$(g_{r_{i_1}} + 1)e_{r_{i_1}} \dots (g_{r_{i_k}} + 1)e_{r_{i_k}} = (g_{r_{i_1}} + 1) \dots (g_{r_{i_k}} + 1)e_R \in \mathcal{C}(v)$$

for $\{r_1, \dots, r_k\}$ a generating set of reflections for R and $(i_j)_{j=1}^k$ any permutation of the indices. By then applying our rank 2 result to pairs of reflections in this generating set, the result that $(g_r + 1)e_R \in \mathcal{C}(v)$ for any $r \in R$ follows too by induction. □

Lemma 3.5.4. *As vector spaces over $\mathbb{C}(v)$, we have*

$$\mathcal{C}(v) = \bigoplus_{R\subset W} J_R e_R. \tag{3.5.2}$$

where R ranges over all reflection subgroups of W.

Proof. The algebra $\mathcal{C}(v)$ is, by definition, spanned by words in $\{g_s^{\pm 1}\}_{s\in S}$. Using the quadratic relation $g_s^2 = 1 + (v^2 - 1)(1 + g_s)e_s$, we then see that any element of $\mathcal{C}(v)$ is in the $\mathbb{C}(v)$-span of words in the set of terms $\{g_s^{-1}\}_{s\in S} \cup \{(1 + g_s)e_s\}_{s\in S}$. For any $s, t \in S$, we have

$$(1 + g_t)e_t g_s^{-1} = g_s^{-1}(1 + g_s g_t g_s^{-1})e_{tst}.$$

By repeatedly using this relation along with the quadratic relation on a given word to move all g_s terms to the left and reducing any powers g_s^2, $s \in S$ which occur we are left with a spanning set consisting of words of the form

$$g_w^{-1}(1 + b_1 g_{t_1} b_1^{-1})e_{b_1 t_1 b_1^{-1}} \dots (1 + b_\ell g_{t_\ell} b_\ell^{-1})e_{b_\ell t_\ell b_\ell^{-1}}, \tag{3.5.3}$$

where b_i is some product of the $\{g_s\}_{s\in S}$ and each t_i lies in S. We want to reduce this further to show that $\mathcal{C}(v)$ is spanned by the set of elements of the form

$$g_w^{-1}(1 + g_{r_1}) \dots (1 + g_{r_\ell})e_R \tag{3.5.4}$$

for R a reflection subgroup and r_i reflections, each of which lies in a distinct contiguous component of R.

To reduce terms of the form (3.5.3) to those of the form (3.5.4), by induction on the number of terms it is enough to show that each term of the form $(1 + b g_t b^{-1})e_{b^{-1}tb}$ can be

written in the form (3.5.4). This fact follows from induction on the number of terms in b once one shows it for $b = g_s$ for some $s \in S$. In this case,

$$(1 + g_s g_t g_s^{-1}) e_{sts} = (1 + g_{sts}) e_{sts} + (v^{-2} - 1) g_s g_t (1 + g_s) e_{\langle s,t \rangle},$$

which is of the desired form. By following the induction as mentioned and using this relation, one obtains that all elements of $\mathcal{C}(v)$ can be written as a $\mathbb{C}(v)$-linear combination of elements of the form (3.5.4).

To see that this sum is direct, we note that each e_R, when specialized to $\mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$, can be thought of as an idempotent projection to the subspace of functions monodromic with respect to all θ for which $R \subset W_\theta^\circ$. So every R determines a subset M_R of multiplicative characters of $T(\mathbb{F}_{q^k})$ such that $e_R \cdot \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]_\theta = \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]_\theta$, with $e_R \cdot \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]_{\theta'} = 0$ for every $\theta' \notin M_R$. There is a direct sum decomposition

$$\mathbb{C}[(G/U)(\mathbb{F}_{q^k})] = \oplus_\theta \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]_\theta,$$

and recall that we can choose k large enough so the specialization map $\mathcal{C}(v) \to \mathbb{C}[(G/U)(\mathbb{F}_{q^k})]$ is injective by Section 3.3.4. So since each R determines a distinct such subset M_R, there can be no nontrivial linear relations between elements of $\mathcal{C}(v)e_R$ for distinct R. □

Definition 3.5.5. For any subset $I \subset S$, let D_I be the number of elements $w \in W$ such that for each contiguous block $I' \subset I$, there is a simple reflection $s \in I'$ for which $\ell(ws) < \ell(w)$.

Since we are in Type A, any reflection subgroup is conjugate to a parabolic subgroup $P_I \subset W$ corresponding to some $I \subset S$.

Lemma 3.5.6. *Let R be any reflection subgroup of W conjugate to P_I. Then* $\dim J_R e_R = D_I$.

Proof. By invariance of dimension under conjugation, it is enough to prove this for $R = P_I$ for some $I \subset S$. In this case, we recall the fact that for $s \in P_I$, each $g_s P_I$ satisfies a quadratic relation, ie.. $(g_s - v^2)(g_s + 1)e_{P_I} = 0$. This implies that $\dim J_R e_R$ is equal to the dimension of the left ideal of the group algebra $\mathbb{C}[W]$ generated by

$$(1 + r_1) \dots (1 + r_k) \in \mathbb{C}[W], \tag{3.5.5}$$

where each r_i is in a distinct contiguous component of I. If we let J_I' be this ideal, it is straightforward to check that $\mathbb{C}[W]/J_I'$ is spanned by equivalence classes $\overline{w}$ of $w \in W$ such that there exists a contiguous block I' of I for which no reduced expression for w has an element of I' as a suffix. Thus, the dimension of $\mathbb{C}[W]/J_I'$ is exactly the complement of the set described in the statement of the lemma, and so the size of this latter set agrees with $\dim J_I' = \dim J_I e_I = \dim J_R e_R$. □

3.5.2 An explicit formula

Definition 3.5.7. Let $P(n)$ be the set of all partitions $(\lambda_i)_{i=1}^k$ for which

$$\left(\sum_{i=1}^{k} \lambda_i\right) + k - 1 \leq n. \tag{3.5.6}$$

We can identify $P(n)$ with a choice of contiguous blocks $I \subset S$ of the Dynkin diagram of type A_n whose size is decreasing, e.g. when $n = 7$, the partition $(2, 2, 1)$ corresponds to the choice pictured here.

Given any subset $I \subset S$, we can reorder its contiguous blocks so that their size is decreasing, and define λ^I to be the associated partition. Finally, for any $i \geq 1$ let $n_i(I)$ be the number of parts of λ^I which are equal to i.

Lemma 3.5.8 ([27])**.** *Given any subset $I \subset S$, the normalizer of the parabolic subgroup W_I of W generated by the simple reflections in I has order*

$$\left(n + 1 - \sum_i (i+1) n_i(I)\right)! \cdot \prod_i n_i(I)!(i+1)!^{n_i(I)}. \tag{3.5.7}$$

Lemma 3.5.9. *For any $I \subset S$,*

$$D_I = \sum_{T \subset \{\lambda_i^I\}_i} (-1)^{|T|} \frac{(n+1)!}{\prod_{\lambda_i^I \in T} (\lambda_i^I + 1)!} \tag{3.5.8}$$

where the sum is taken over all subpartitions of λ^I, including λ^I itself.

Proof. Indeed, this follows by the principle of inclusion-exclusion, since the quantity

$$\frac{(n+1)!}{\prod_{\lambda_i^I \in T} (\lambda_i + 1)!}$$

is equal to the number of elements $w \in W$ having no reduced expression ending with any of the simple reflections in any of the blocks λ_i^I in a given subset T of blocks. □

Theorem 3.5.10. *For any $I \subset S$, let R_I be the number of reflection subgroups of W conjugate to P_I (which is $|W|$ divided by the quantity in Lemma 3.5.8), and let D_I be as above. Then the dimension of $\mathcal{C}(v)$ in Type A_n is given by*

$$\dim \mathcal{C}(v) = \sum_{I \subset S} R_I \cdot D_I. \tag{3.5.9}$$

3.5.3 Examples and comparison with [42]

We conclude by giving three examples of how these formulas work in practice by verifying the computations done by Marin in Section 4.3 of [42] in types A_2, A_3, A_4. Note that the sequence of dimensions obtained in loc. cit. in these cases is $20, 217, 3364$ (it is clear that $\dim \mathcal{C}_{A_1}(v) = 3$). We will now show how this sequence arises from our computations.

In Figure 3.1, we write N_I for the dimension of the normalizer of W_I, i.e. the quantity given in Lemma 3.5.8. Then R_I is obtained by $R_I = (n+1)!/N_I$ as explained previously, with D_I being given by the formula in Lemma 3.5.9.

I	N_I	R_I	D_I
⊙⊙	6	1	5
⊙•	3	2	3
••	6	1	6

I	N_I	R_I	D_I
⊙⊙⊙	24	1	23
⊙⊙•	6	4	20
⊙•⊙	8	3	6
⊙••	4	6	12
•••	24	1	24

I	N_I	R_I	D_I
⊙⊙⊙⊙	120	1	119
⊙⊙⊙•	24	5	115
⊙⊙•⊙	12	10	50
⊙⊙••	12	10	100
⊙•⊙•	8	15	30
⊙•••	12	10	60
••••	120	1	120

$$\begin{aligned}
\dim \mathcal{C}_{A_2}(v) &= 1 \cdot 5 + 2 \cdot 3 + 1 \cdot 6 \\
&= 20 \\
\dim \mathcal{C}_{A_3}(v) &= 1 \cdot 23 + 4 \cdot 20 + 3 \cdot 6 + 6 \cdot 12 + 1 \cdot 24 \\
&= 217 \\
\dim \mathcal{C}_{A_4}(v) &= 1 \cdot 119 + 5 \cdot 115 + 10 \cdot 50 + 10 \cdot 100 + 15 \cdot 30 + 10 \cdot 60 + 1 \cdot 120 \\
&= 3364
\end{aligned}$$

Table 3.1: Quantities determining the dimension of $\mathcal{C}_{A_n}$ for $n = 2, 3, 4$.

Going beyond these small rank computations, using a computer we were able to compute the dimension of $\mathcal{C}_{A_n}$ for all for $n \leq 43$, and we list the first 12 in Figure 3.2.

n	$\dim \mathcal{C}_{A_n}(v)$
1	3
2	20
3	217
4	3364
5	71098
6	1960867
7	67886033
8	2871659468
9	145498348666
10	8683447971439
11	601843453126056
12	47875219836485209

Table 3.2: The dimension of $\mathcal{C}_{A_n}$ for $1 \leq n \leq 12$.

Chapter 4

Polishchuk's conjecture and Kazhdan-Laumon representations

4.1 Introduction

In this chapter, we return to Kazhdan and Laumon's original goal of using the category $\mathcal{A}$ associated to a semisimple algebraic group G split over $\mathbb{F}_q$ to provide a new geometric construction of discrete series representations of $G(\mathbb{F}_q)$. To do so, recall that although Conjecture 1.1.1 on the finiteness of cohomological dimension fails by [16], Polishchuk suggested in [48] that his Conjecture 1.1.3 could repair this issue as a crucial step toward well-definedness of Kazhdan and Laumon's construction.

In [48], Polishchuk resolves this conjecture himself in Types A_2, A_2, A_3, and B_2. In this chapter's first main theorem, we use this framework along with the algebraic understanding of symplectic Fourier transforms provided in Chapter 3 to prove this conjecture in general.

Theorem 4.1.1. *Conjecture 1.1.3 is true. In particular, the localization of the $\mathbb{Z}[v, v^{-1}]$-module $K_0(\mathcal{A}_{\mathbb{F}_q})$ at the polynomial*

$$p(v) = \prod_{i=1}^{\ell(w_0)} \left(1 - v^{2i}\right) \tag{4.1.1}$$

is generated by objects of finite projective dimension.

As Polishchuk expected, the resolution of Conjecture 1.1.3 brings us very close to showing that Kazhdan and Laumon's construction of discrete series representations is well-defined. The principal result of the present chapter is that by using the formalism of monodromic perverse sheaves, we can prove a similar theorem which indeed completes the necessary technicalities to carry out Kazhdan and Laumon's construction in general.

Theorem 4.1.2. *For any character sheaf $\mathcal{L}$ of T and element $w \in W$, the localization of the $\mathbb{Z}[v, v^{-1}]$-module $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ at $p(v)$ is spanned by classes of objects of finite projective*

dimension in $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$.

This monodromic approach to Kazhdan and Laumon's construction was already successfully carried out in [16], in which Braverman and Polishchuk explain how to carry out a well-defined version of Kazhdan and Laumon's construction in the case where $\mathcal{L}$ corresponds to a *quasi-regular* character. So one can think of the following corollary to Theorem 4.1.2 as a generalization of Braverman and Polishchuk's result to the case of an arbitrary character.

Corollary 4.1.3. *The Kazhdan-Laumon construction proposed in [32] is well-defined for monodromic sheaves corresponding to any character.*

4.1.1 Layout of the chapter

In Section 4.2, we explain some background on Kazhdan-Laumon categories. In Section 4.3, we then explain the monodromic setting required to state Theorem 4.1.2 and discuss the categorical center of the monodromic Hecke category, which will be an important tool in the proof. Then in Section 4.4 we use dg formalism to explain why the derived category of the Kazhdan-Laumon category admits an action of this categorical center. This is followed in Section 4.5 by an explanation of a crucial tool in the study of Kazhdan-Laumon categories proposed by Polishchuk [48] called the canonical complex. We then complete the proofs of our results: in Section 4.6 we prove Theorem 4.1.2, and in Section 4.7 we recall the results of [47] and explain how it, combined with the previous setup, allow us to prove Polishchuk's original conjecture and establish Theorem 4.1.1 independently of the monodromic setting. Finally, in Section 4.8, we explain how to carry out the construction of Kazhdan-Laumon representations explicitly given our theorems.

4.2 Preliminaries

4.2.1 Background and notation

For the rest of this chapter, we recall the setup for derived categories, gluing, and Kazhdan-Laumon categories exactly as it was laid out in Section 1.2. We recall that G is the split semisimple group over $\mathbb{F}_q$ with associated Kazhdan-Laumon category $\mathcal{A}$. We recall also from Section 1.2 the functors Φ_w and Ψ_w along with the W-action on $\mathcal{A}$ used to define the categories $\mathcal{A}_{w,\mathbb{F}_q}$.

Adjoint functors on $D^b(\mathcal{A})$

Definition 4.2.1. For any $w \in W$, let $j_w^* : D^b(\mathcal{A}) \to D^b(G/U)$ be the functor arising from the same-named exact functor $j_w^* : \mathcal{A} \to \mathrm{Perv}(G/U)$ given by $j_w^*((A_y)_{y\in W}) = A_w$.

We then define a functor ${}^{\circ}j_{w!} : \mathrm{Perv}(G/U) \to \mathcal{A}$ by

$$j^{\circ}_{w!}(K) = (\Phi^{\circ}_{yw^{-1}}K)_{y\in W}. \tag{4.2.1}$$

One can check that the morphisms $\nu_{y',y}$ for $y, y' \in W$ introduced in Corollary 1.2.5 endow the tuple $(\Phi^{\circ}_{yw^{-1}}K)_{y\in W}$ with the structure morphisms required to define an object of $\mathcal{A}$. We let $j_{w!}$ be the left-derived functor to $j^{\circ}_{w!}$.

Proposition 4.2.2 (Proposition 7.1.2, [48])**.** *For any $w \in W$, there is an adjunction $(j^{\circ}_{w!}, j^{*}_{w})$. Further, the functor $j_{w!} : D^b(G/U) \to D^b(\mathcal{A})$ has the property that*

$${}^{p}H^{i}(j_{w!}(K)) = ({}^{p}H^{i}\Phi_{yw^{-1}}(K))_{y\in W}, \tag{4.2.2}$$

and there is also an adjunction $(j_{w!}, j^{*}_{w})$ on derived categories.

Analogously, acting instead by the functors $\Psi_{yw^{-1}}$ in (4.2.2) defines a right-adjoint j°_{w*} to j^{*}_{w} and its right-derived functor j_{w*} in the very same way, as is also explained in [48].

The functor ι

Definition 4.2.3. We define an endofunctor ι of $\mathcal{A}$ by

$$\iota((A_w)_{w\in W}) = (\Phi^{\circ}_{w_0}A_{w_0w})_{w\in W} \tag{4.2.3}$$

with the structure morphisms described in 7.2 of [48].

It is shown in loc. cit. that we can also abuse notation and view ι as a functor on $D^b(\mathcal{A})$ (by replacing $\Phi^{\circ}_{w_0}$ with Φ_{w_0}); for our purposes we will only need the functor ι^2, which we will later describe as an endofunctor of $D^b(\mathcal{A})$ more conceptually in Lemma 4.6.2.

Objects of the form $j_{w!}(A)$

Proposition 4.2.4. *For any $B \in D^b(G/U)$ and any $w \in W$, the object $j_{w!}B \in D^b(\mathcal{A})$ has finite projective dimension.*

Proof. By Proposition 7.1.2 of [48], the functor $j_{w!} : D^b(G/U) \to D^b(\mathcal{A})$ is left-adjoint to the restriction functor j^{*}_{w}. Hence for any $A \in \mathcal{A}$,

$$\mathrm{Ext}^{\bullet}_{\mathcal{A}}(j_{w!}(B), A) = \mathrm{Ext}^{\bullet}_{\mathrm{Perv}(G/U)}(B, j^{*}_{w}A), \tag{4.2.4}$$

and the latter is a finite-dimensional vector space because the category $\mathrm{Perv}(G/U)$ has finite cohomological dimension. □

Polynomials and localization

Fix the Weyl group W and choose w_0 its longest element.

Definition 4.2.5. Define $P(x,v), \tilde{P}(x,v) \in \mathbb{Z}[x, v, v^{-1}]$ by

$$P(x,v) = \prod_{i=0}^{\ell(w_0)} (x - v^{2i}) \tag{4.2.5}$$

$$\tilde{P}(x,v) = \prod_{i=1}^{\ell(w_0)} (x - v^{2i}) \tag{4.2.6}$$

and let $p(v) = \tilde{P}(1,v)$.

Definition 4.2.6. For any abelian category $\mathcal{C}$ such that $K_0(\mathcal{C})$ has the structure of a $\mathbb{Z}[v, v^{-1}]$-module, let $V^{\mathrm{fp}} \subset K_0(\mathcal{C})$ be the submodule spanned by all objects of finite projective dimension. Further, let $V^{\mathrm{fp}}_{p(v)}$ denote the localization of this module at $p(v)$, or equivalently at all of the linear factors $(1 - v^{2i})$ for $1 \leq i \leq \ell(w_0)$.

4.3 Monodromic sheaves, character sheaves, and the categorical center

Rank one character sheaves on T

We let $\mathrm{Ch}(T)$ be the category of rank one character sheaves on T; we refer to Appendix A of [54] for a detailed treatment. We note that $\mathrm{Ch}(T)$ carries a natural action of the Weyl group W.

For any $\mathcal{L}$ in $\mathrm{Ch}(T)$, we let $W^\circ_{\mathcal{L}}$ be the normal subgroup of the stabilizer of $\mathcal{L}$ in W which is the Weyl group of the root subsystem of the root system of W on which $\mathcal{L}$ is trivial; see 2.4 of [40] for details.

Monodromic version of the Kazhdan-Laumon category

In [16, Sections 2.1.3, 2.3.4], the authors explain how to define a category $\mathrm{Perv}_{\mathcal{L}}(G/U)$ of monodromic sheaves on G/U with respect to the monodromy $\mathcal{L}$. We then let $D^b_{\mathcal{L}}(G/U) = D^b(\mathrm{Perv}_{\mathcal{L}}(G/U))$ be its derived category.

Remark 4.3.1. We note that in this definition, monodromic perverse sheaves are defined such that the extension of two $\mathcal{L}$-equivariant sheaves may not be $\mathcal{L}$-equivariant, only $\mathcal{L}$-monodromic. In other words, for $\mathcal{L}$ trivial, this reduces to the category of perverse sheaves on G/U with unipotent monodromy (c.f. [13]) on the right rather than simply to $\mathrm{Perv}(G/B)$.

We use this to define the $\mathcal{L}$-monodromic Kazhdan-Laumon category.

Definition 4.3.2. For any $\mathcal{L} \in \mathrm{Ch}(T)$, we define $\mathcal{A}^{\mathcal{L}}$ to be the category obtained by applying the gluing procedure in Definition 1.2.6 to the category $\mathrm{Perv}_{\mathcal{L}}(G/U)$. We further define $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$ in analogy to Definition 1.2.8.

4.3.1 The monodromic Hecke category and its center

The monodromic Hecke category

In [25], the author defines for any $\mathcal{L}$ a category $\mathcal{P}_{\mathcal{L}}$ (called $D^b_{(B)}(G/U)_t$ in loc. cit., where t is a parameter determined by $\mathcal{L}$) such that for $\mathcal{L}$ trivial, this reduces to the familiar derived category $D^b_{(B)}(G/U)$ of B-constructible sheaves on G/U. Elements of $\mathcal{P}_{\mathcal{L}}$ are $\mathcal{L}$-monodromic with respect to the right action of T, while their left monodromies may correspond to any character sheaf in the W-orbit of $\mathcal{L}$.

For any $w \in W$, this category contains monodromic versions ${}_{w\mathcal{L}}\Delta(w)_{\mathcal{L}}$ and ${}_{w\mathcal{L}}\nabla(w)_{\mathcal{L}}$ of standard and costandard sheaves; we emphasize that extensions of such objects in this category may not be $\mathcal{L}$-equivariant with respect to the right T-action even when they remain $\mathcal{L}$-monodromic, so this category also contains monodromic versions of tilting objects $\mathcal{T}(w)_{\mathcal{L}}$ for any $w \in W$.

We let ${}_{\mathcal{L}}\mathcal{P}_{\mathcal{L}}$ be the triangulated subcategory of objects generated by the standard and costandard objects corresponding to $w \in W^\circ_{\mathcal{L}}$, in other words, the subcategory of objects in $\mathcal{P}_{\mathcal{L}}$ which are also left-monodromic with respect to $\mathcal{L}$.

Free-monodromic Hecke categories

In [13], in the unipotent monodromy case where $\mathcal{L}$ is trivial, the authors define a category formed from a certain completion of $D^b_{\mathcal{L}}(G/U)$ called the category of *unipotently free-monodromic* sheaves.

In [25], the case of non-unipotent monodromy was treated carefully. In loc. cit., the author defines a category $\hat{\mathcal{P}}_{\mathcal{L}}$ (which is called $\hat{D}^b_{(B)}(G/U)_t$ in loc. cit. where t is a parameter determined by $\mathcal{L}$), which is a certain completion of the category $\mathcal{P}_{\mathcal{L}}$ defined in Section 4.3.1, equipped with a monoidal structure which we also denote by $*$ in this context.

This category contains elements $\varepsilon_{n,\mathcal{L}}$ and $\hat{\delta}_{\mathcal{L}}$ introduced in Corollary 5.3.3 of [12]. The object $\hat{\delta}_{\mathcal{L}}$ is the monoidal unit for the convolution product on $\hat{\mathcal{P}}_{\mathcal{L}}$, whereas convolution with the objects $\varepsilon_{n,\mathcal{L}}$ can be thought of as a sort of projection to the subcategory of objects whose corresponding "logarithmic monodromy operator" is nilpotent of order at most n; c.f. Appendix A of [13] for an explanation of this perspective.

Finally, we recall that $\hat{\mathcal{P}}_{\mathcal{L}}$ contains for all $w \in W$ free-monodromic versions ${}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}}$, ${}_{w\mathcal{L}}\hat{\nabla}(w)_{w\mathcal{L}}$ of standard and costandard sheaves, c.f. [13] for the unipotent case, [40] for a description of these objects in $\mathcal{P}_{\mathcal{L}}$ for arbitrary $\mathcal{L}$, and [25] for their free-monodromic versions in $\hat{\mathcal{P}}_{\mathcal{L}}$. We define the subcategory ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ analogously to the subcategory ${}_{\mathcal{L}}\mathcal{P}_{\mathcal{L}}$ of $\mathcal{P}_{\mathcal{L}}$.

The center of the monodromic Hecke category

To define the notion of categorical center which will be useful in this setting, we will closely follow the conventions of [11] in this section. We begin by recalling some definitions from loc. cit.

Definition 4.3.3 ([11])**.** Let $\mathcal{Y} = (G/U \times G/U)/T$ where T acts by the right-diagonal multiplication. Let $\mathcal{Y}^{(2)} = (G/U)^4/T^2$ where the right T^2-action on $(G/U)^4$ is defined by

$$(x_1U, x_2U, x_3U, x_4U) \cdot (t, z) = (x_1tU, x_2zU, x_3zU, x_4tU), \tag{4.3.1}$$

and is equipped with a G^2 action by

$$(g, h) \cdot (x_1U, x_2U, x_3U, x_4U) = (gx_1U, gx_2U, hx_3U, hx_4U). \tag{4.3.2}$$

Let $\mathcal{H}^{(1)} = D^b(G\backslash\mathcal{Y})$ and $\mathcal{H}^{(2)} = D^b(G^2\backslash\mathcal{Y}^{(2)})$. It is shown in [11] that $\mathcal{H}^{(2)}$ is equipped with a convolution product which makes it a monoidal category, and that $\mathcal{H}^{(1)}$ is a module category over $\mathcal{H}^{(2)}$ via the "two-sided convolution" described in 2.2 of loc. cit.

Finally we recall the definitions of $\mathcal{H}^{(1)}_{\mathrm{mon}}$ and $\mathcal{H}^{(2)}_{\mathrm{mon}}$ from loc. cit. as the full unipotently monodromic subcategories of $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ with respect to the projections $\mathcal{Y} \to (G/B)^2$ and $\mathcal{Y}^{(2)} \to (G/B)^4$ respectively. For an arbitrary $\mathcal{L} \in \mathrm{Ch}(T)$, we define the categories $\mathcal{H}^{(1)}_{\mathcal{L}}$ and $\mathcal{H}^{(2)}_{\mathcal{L}}$ similarly but replacing unipotent monodromy with $\mathcal{L}$-monodromy.

Definition 4.3.4 ([11], Definition 5.2.1)**.** Let

$$\mathcal{Z}\mathcal{H}^{(1)}_{\mathcal{L}} = \mathrm{Fun}^{\mathrm{fd}}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}}, \mathcal{H}^{(1)}_{\mathcal{L}}),$$

where $\mathrm{Fun}^{\mathrm{fd}}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}}, \mathcal{H}^{(1)}_{\mathcal{L}})$ is the full subcategory of $\mathrm{Fun}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}}, \mathcal{H}^{(1)}_{\mathcal{L}})$ consisting of functors F such that the limit $F(\hat{\delta}_{\mathcal{L}})$ exists in $\mathcal{H}^{(1)}_{\mathcal{L}}$.

Going forward, if $Z \in \mathcal{H}^{(1)}_{\mathcal{L}}$, we will sometimes write $- * Z$ to denote the functor Z, since the functor Z is equivalent to convolution with the object $Z(\hat{\delta}_{\mathcal{L}})$.

The Harish-Chandra functor and the equivalence from the center to character sheaves

Following 3.2 of [11], consider the diagram

$$\begin{array}{ccccc} & & G \times G/B & & \\ & \overset{\pi}{\swarrow} & & \overset{q}{\searrow} & \\ G & & & & \mathcal{Y} \end{array}$$

where π is the projection and q is the quotient of the map $q' : G \times G/U \to G/U \times G/U$ given by $q'(g, xU) = (xU, gxU)$ by the free right T-action, with respect to which q' is equivariant.

Definition 4.3.5. The Harish-Chandra transform is the functor

$$\mathfrak{hc} = q_! \circ \pi^* : D^b(G/_{\mathrm{Ad}}G) \to D^b(G\backslash\mathcal{Y}), \tag{4.3.3}$$

which is monoidal with respect to the natural convolution product on each side by [24], c.f. 2.2 of [11] for a detailed exposition of these convolution products.

Definition 4.3.6. For any $\mathcal{L} \in \mathrm{Ch}(T)$, let $D^b_{\mathfrak{C},\mathcal{L}}(G) \subset D^b(G/_{\mathrm{Ad}}G)$ be the full triangulated subcategory with objects $\mathcal{F}$ satisfying $\mathfrak{hc}(\mathcal{F}) \in \mathcal{H}^{(1)}_{\mathcal{L}}$.

The following proposition is proven for the case of unipotent monodromy in Theorem 5.2.2 of [11], and a version for arbitrary monodromy $\mathcal{L}$ is similar and will appear in a future version of loc. cit.

Proposition 4.3.7 ([11], Theorem 5.2.2)**.** *There is an equivalence $\tilde{a}_{\mathcal{L}}$ of semigroupal categories*

$$\tilde{a}_{\mathcal{L}} : D^b_{\mathfrak{C},\mathcal{L}}(G) \to \mathcal{ZH}^{(1)}_{\mathcal{L}} \tag{4.3.4}$$

such that $\varepsilon \circ \tilde{a}_{\mathcal{L}} = \mathfrak{hc}$ where ε is the evaluation of a functor at $\hat{\delta}$.

Two-sided cells and character sheaves

In this and subsequent sections, we use the notion of two-sided Kazhdan-Lusztig cells in the Weyl group; see e.g. [52] for a clear exposition.

Definition 4.3.8. For any $\mathcal{L}$, let $\underline{C}_{\mathcal{L}}$ denote the set of two-sided cells in $W^\circ_{\mathcal{L}}$. For any $w \in W^\circ_{\mathcal{L}}$, we let $\underline{c}_w$ denote the corresponding cell. We let $\underline{c}_e$ be the top cell which corresponds to the identity element. There is a well-defined partial order $\leq$ on $\underline{C}_{\mathcal{L}}$ for which $\underline{c}_e$ is maximal.

Two-sided Kazhdan-Lusztig cells give a filtration on the category $D^b_{\mathfrak{C},\mathcal{L}}(G)$ (c.f. [37]), and so for each $\underline{c} \in \underline{C}_{\mathcal{L}}$, there are triangulated subcategories $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\leq \underline{c}}$ and $D^b_{\mathfrak{C},\mathcal{L}}(G)_{<\underline{c}}$ of $D^b_{\mathfrak{C},\mathcal{L}}(G)$. We then define $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$ as the quotient category $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\leq \underline{c}}/D^b_{\mathfrak{C},\mathcal{L}}(G)_{<\underline{c}}$, referring to the unipotent case treated in Section 5 of [10] for details. Let G_{ad} be the adjoint quotient of G; the following is a consequence of the classification of irreducible character sheaves in terms of cells given in [37], c.f. Corollary 5.4 of [10].

Proposition 4.3.9. *For any $\mathcal{L} \in \mathrm{Ch}(T)$,*

$$K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})) \cong \bigoplus_{\underline{c} \in \underline{C}_{\mathcal{L}}} K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}}) \tag{4.3.5}$$

as vector spaces. Further, for any $\underline{c} \in \underline{C}_{\mathcal{L}}$, the preimage of the subspace

$$\bigoplus_{\underline{c}' \leq \underline{c}} K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}'}) \tag{4.3.6}$$

in $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ under this isomorphism is a monoidal ideal.

The big free-monodromic tilting object and $\mathbb{K}_{\mathcal{L}}$

In [25] Section 9.4, the author defines free-monodromic tilting sheaves with general monodromy, analogous to the ones appearing in [13] for the case of unipotent monodromy.

Definition 4.3.10. Given $\mathcal{L} \in \mathrm{Ch}(T)$, let $\hat{\mathcal{T}}(w_{0,\mathcal{L}})_{\mathcal{L}}$ be the free-monodromic tilting object in ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ corresponding to the longest element $w_{0,\mathcal{L}}$ of $W_{\mathcal{L}}^{\circ}$. We will denote it simply by $\hat{\mathcal{T}}_{\mathcal{L}}$ for convenience.

In [12], working in the case of unipotent monodromy, the authors construct from $\hat{\mathcal{T}}(w_0)$ an object they call $\mathbb{K}$, defined as $\mathbb{K} = p^* p_! \hat{\mathcal{T}}(w_0)$ where $p : U\backslash G/U \to (U\backslash G/U)/T$ where T acts on $U\backslash G/U$ by conjugation.

They also define a character sheaf Ξ whose details are explained in 1.4 of [12] obtained by averaging the derived pushforward of the constant sheaf on the regular locus of the unipotent variety of G to obtain an element of $D^b(G/_{\mathrm{Ad}}G)$.

Proposition 4.3.11 (Theorem 1.4.1, [12])**.** *There exists an object $\Xi \in D^b(G/_{\mathrm{Ad}}G)$ such that, if $\Xi * \hat{\delta}$ is the projection of Ξ to the derived category of unipotent character sheaves, then $\mathfrak{hc}(\Xi * \hat{\delta}) = \mathbb{K}$.*

We now define an anologue for arbitrary monodromy generalizing the unipotent case.

Definition 4.3.12. Let $\mathbb{K}_{\mathcal{L}} = p^* p_! \hat{\mathcal{T}}_{\mathcal{L}}$. Then by the same argument as in the proof of Theorem 1.4.1 of [12], $\mathfrak{hc}(\Xi * \hat{\delta}_{\mathcal{L}}) = \mathbb{K}_{\mathcal{L}}$ where $\Xi * \hat{\delta}_{\mathcal{L}}$ is the analogous projection to the derived category of character sheaves with monodromy $\mathcal{L}$. As a result, $\mathbb{K}_{\mathcal{L}}$ can be considered as an element of $\mathcal{ZH}_{\mathcal{L}}^{(1)}$.

By abuse of notation, in the future when we use the convolution functor $-*\mathbb{K}_{\mathcal{L}}$, we will often identify $\mathbb{K}_{\mathcal{L}}$ with the character sheaf it comes from under the Harish-Chandra transform, e.g. as in the convolution action of character sheaves which we describe in Proposition 4.4.7.

Fourier transform and convolution with costandard sheaves

Proposition 4.3.13. *Let $\mathcal{F} \in D^b_{\mathcal{L}}(G/U)$ where $\mathcal{L} \in \mathrm{Ch}(T)$. Then for any $s \in S$,*

$$\Phi_s(\mathcal{F}) = \begin{cases} \mathcal{F} * {}_{\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}}(\frac{1}{2}) & s \in W_{\mathcal{L}}^{\circ} \\ \mathcal{F} * {}_{\mathcal{L}}\hat{\nabla}(s)_{s\mathcal{L}} & s \notin W_{\mathcal{L}}^{\circ}. \end{cases} \tag{4.3.7}$$

Proof. In the case where $s \in W_{\mathcal{L}}^{\circ}$, this follows for equivariant sheaves by Proposition 4.3 of [44], c.f. 6.3 of [48], while the second case follows from a similar computation. The proof then generalizes to arbitrary extensions of $\mathcal{L}$-equivariant sheaves, and therefore to all monodromic sheaves as in the claim. □

4.4 Coalgebras and dg enhancements

4.4.1 Coalgebras over comonads and Kazhdan-Laumon categories

The Kazhdan-Laumon category as coalgebras over a comonad

We recall a result from [9] exhibiting the Kazhdan-Laumon category $\mathcal{A}$ as the category of coalgebras over a certain comonad on the underlying category $\mathcal{B} = \mathrm{Perv}(G/U)^{\oplus W}$.

Definition 4.4.1. Let $\Psi : \mathcal{B} \to \mathcal{B}$ be the endofunctor defined for any $A = (A_w)_{w \in W} \in \mathcal{B}$ by

$$(\Psi^\circ A)_w = \oplus_{y \in W} \Psi^\circ_{wy^{-1}} A_y. \tag{4.4.1}$$

It has right-derived functor $\Psi = R\Psi^\circ$ given by

$$(\Psi A)_w = \oplus_{y \in W} \Psi_{wy^{-1}} A_y. \tag{4.4.2}$$

Theorem 4.4.2 ([9])**.** *The Kazhdan-Laumon category $\mathcal{A}$ is equivalent to the category of coalgebras over the left-exact comonad $\Psi^\circ : \mathcal{B} \to \mathcal{B}$.*

Remark 4.4.3. We can just as easily see that the Kazhdan-Laumon category $\mathcal{A}$ is equivalent to the category of algebras over the right-exact monad Φ°, where Φ° is defined as in (4.4.1) but using the Ψ°_w functors.

dg enhancements of derived categories

Given an abelian category $\mathcal{C}$, let $C_{\mathrm{dg}}(\mathcal{C})$ be the dg category with objects complexes of sheaves and morphisms the usual complexes of maps between complexes. We define the dg derived category $D_{\mathrm{dg}}(\mathcal{C})$ to be the dg quotient of $C_{\mathrm{dg}}(\mathcal{C})$ by the full subcategory of acyclic objects; its homotopy category is the usual derived category $D(\mathcal{C})$.

The bounded dg derived category $D^b_{\mathrm{dg}}(\mathcal{C})$ is defined to be the full dg subcategory of $D_{\mathrm{dg}}(\mathcal{C})$ consisting of objects which project to $D^b(\mathcal{C})$ when passing to the homotopy category.

Barr-Beck-Lurie for dg categories applied to the Kazhdan-Laumon category

Note that the comonad $\Psi^\circ : \mathcal{B} \to \mathcal{B}$ can be enhanced to a comonad $\Psi = R\Psi^\circ : D^b_{\mathrm{dg}}(\mathcal{B}) \to D^b_{\mathrm{dg}}(\mathcal{B})$; i.e. the right-derived functor of Ψ° has a dg enhancement (since it arises from the functors Ψ_w which are themselves defined using the six-functor formalism). We can then consider the dg category $D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ of coalgebras over the comonad Ψ. The following is an application of the Barr-Beck-Lurie theorem for dg categories.

Proposition 4.4.4. *The dg categories $D^b_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ})$ and $D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ are equivalent.*

Proof. We first check the conditions of the Barr-Beck-Lurie theorem for the functor $F : D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D_{\mathrm{dg}}(\mathcal{B})$ obtained from the forgetful functor $\mathcal{B}_{\Psi^\circ} \to \mathcal{B}$. This functor has a right

adjoint given by the dg enhancement of the right-derived functor of the cofree coalgebra functor $\mathcal{B} \to \mathcal{B}_{\Psi^\circ}$.

Indeed, we check this adjunction explicitly: this cofree coalgebra functor sends an element $(B_w)_{w\in W} \in D^b_{\mathrm{dg}}(\mathcal{B})$ to the direct sum $\oplus_{w\in W} Rj^\circ_{w*}(A_w)$ where $Rj^\circ_{w*} = j_{w*}$ is the right-adjoint to j^*_w. It is then clear from the adjunction (j^*_w, j_{w*}) that the forgetful functor is its left adjoint. Now notice that in a similar way, we observe that the forgetful functor F has a left adjoint: it follows from the opposite adjunction $(j_{w!}, j^*_w)$ that the functor sending any $(B_w)_{w\in W} \in D^b_{\mathrm{dg}}(\mathcal{B})$ to $\oplus_{w\in W} j_{w!}(B_w)$ (where again $j_{w!} = Lj^\circ_{w!}$) is left adjoint to F.

We are now in the setup of Example 2.6 of [26] (by Remark 4.4.3 we can even work monadically rather than having to dualize the argument in loc. cit.), where F and its right adjoint both preserve small limits since each is itself a right adjoint. Since $D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ})$ and $D_{\mathrm{dg}}(\mathcal{B})$ are compactly-generated, the result of loc. cit. gives that the hypotheses of the Barr-Beck-Lurie theorem will be satisfied as long as F is conservative. It then only remains to check that F is conservative, but this is clear by its exactness and the fact that the forgetful functor $\mathcal{B}_{\Psi^\circ} \to \mathcal{B}$ on the abelian level is faithful.

To conclude, we then note that the equivalence $D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D_{\mathrm{dg}}(\mathcal{B})_\Psi$ provided by the Barr-Beck-Lurie theorem then restricts to an equivalence of bounded derived categories $D^b_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ since cohomology in both categories is computed in the underlying category $D(\mathcal{B})$ after forgetting the coalgebra structure. □

4.4.2 The center of the monodromic Hecke category

dg-enhancement for the monodromic Hecke category

In Section 5.3 of [11], the authors explain how to construct a dg enhancement of $\mathcal{H}^{(1)}_{\mathrm{mon}}$, and a similar construction works for the category $\mathcal{H}^{(1)}_{\mathcal{L}}$; this will be treated in a future version of loc. cit. Going forward, when we consider the convolution action of $\mathcal{H}^{(1)}_{\mathrm{mon}}$ on $D^b(\mathrm{Perv}_{\mathcal{L}}(G/U))$, we will use the fact that this functor has a dg lift and therefore can be considered as a dg functor which preserves distinguished triangles in any of its arguments.

Action of the center on the Kazhdan-Laumon dg-category

For any $\mathcal{L} \in \mathrm{Ch}(T)$, let $\mathcal{B}_{\mathcal{L}} = \oplus_{w\in W} \mathrm{Perv}_{w\mathcal{L}}(G/U)$.

Definition 4.4.5. Given $Z \in \mathcal{ZH}^{(1)}_{\mathcal{L}}$, we consider Z as a functor on the direct sum of categories $\oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$ as follows. Given any object $A = (A_w)$ in $\oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$, let $Z(A) = A * Z \in \oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$ be defined such that $Z(A)_w = (A * Z)_w = A_w * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}}$.

By the definition of $\mathcal{ZH}^{(1)}_{\mathcal{L}}$, for any $B \in {}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$, the functors $- * Z * B$ and $- * B * Z$ on $\mathrm{Perv}_{\mathcal{L}}(G/U)$ are naturally isomorphic. The following lemma is a consequence of this fact.

Lemma 4.4.6. *For any $Z \in \mathcal{ZH}^{(1)}_{\mathcal{L}}$, there is a natural isomorphism $Z \circ \Psi \cong \Psi \circ Z$ of endofunctors of $\mathcal{A}^{\mathcal{L}}$.*

Proof. By Proposition 4.3.13, for any $w \in W$, there exists some Tate twist (d) such that $((Z \circ \Psi)(A))_w(d)$ is isomorphic to

$$\oplus_{y \in W} (\Psi_{wy^{-1}} A_y) * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{4.4.3}$$

$$\cong A_y * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{4.4.4}$$

$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{4.4.5}$$

$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{4.4.6}$$

$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} \tag{4.4.7}$$

$$\cong \oplus_{y \in W} \Psi_{wy^{-1}}(Z(A))(d) \tag{4.4.8}$$

$$\cong ((\Psi \circ Z)(A))_w(d), \tag{4.4.9}$$

and these isomorphisms across all $w \in W$ assemble together to give a natural isomorphism $Z \circ \Psi \cong \Psi \circ Z$. A conceptual explanation for the existence of this isomorphism is explained in Lemma 11.12 of [40]. □

In the following, let $\pi_{\mathrm{ad}} : G \to G_{\mathrm{ad}}$ be the natural map from G to its adjoint quotient.

Proposition 4.4.7. *There is an action of the monoidal category $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on the category $D^b(\mathcal{A}^{\mathcal{L}})$, which we denote by convolution on the right.*

Proof. Given an element $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, we first pull back along π_{ad} to obtain an element in $D^b_{\mathfrak{C},\mathcal{L}}(G)$, then pass to $\mathcal{ZH}^{(1)}_{\mathcal{L}}$ via the isomorphism $\tilde{a}_{\mathcal{L}}$, obtaining a new element $\tilde{Z}$. We can then act by the usual convolution $- * \tilde{Z}$. By abuse of notation, for any element $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ we will define the endofunctor of $\mathrm{Perv}_{\mathcal{L}}(G/U)$ which we just described by the same symbol $- * Z$. We note that $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ has a dg enhancement discussed for the unipotent case in [7] but analogous in general. Further, the functors we use are built from compositions of those occuring in the six-functor formalism and therefore each has a dg enhancement as well.

For any $B \in D^b_{\mathrm{dg}}(\mathcal{B}_{\mathcal{L}})_{\Psi}$ and $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, we define the new object $B * Z$ by $F(B) * Z$ as a tuple of elements of $\mathrm{Perv}_{\mathcal{L}}(G/U)$ where F is the forgetful functor. We then define the coalgebra structure map $B * Z \to \Psi(B * Z)$ by the composition

$$B * Z \to \Psi(B) * Z \cong \Psi(B * Z) \tag{4.4.10}$$

where the first map comes from the coalgebra structure on B and the subsequent isomorphism is the natural isomorphism described in Lemma 4.4.6. One can then check that this defines a coalgebra structure on $B * Z$. □

We note that we get a similar action of $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on $D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ since any object Z here is a sheaf on the variety G_{ad} and therefore has a natural isomorphism $\mathrm{Fr}^* Z \to Z$; this means $- * Z$ commutes with Fr^*, and therefore the action in 4.4.7 also gives an action in the setting of w-twisted Weil sheaves.

4.5 Polishchuk's canonical complex

4.5.1 Parabolic Kazhdan-Laumon categories

Definition of the categories $\mathcal{A}^I_{w,\mathbb{F}_q}$

We will now define certain parabolic analogues of the Kazhdan-Laumon category which correspond to subsets $I \subset S$. We first recall the definition of the categories $\mathcal{A}_{W_I}$ from [48], build off of this definition to define the categories $\mathcal{A}^I$ which we will need in the present work, and finally define $\mathcal{A}^I_{w,\mathbb{F}_q}$ for any $w \in W$.

Definition 4.5.1 ([48], Section 7)**.** For any $I \subset S$, the category $\mathcal{A}_{W_I}$ is the category of tuples of elements of $\mathrm{Perv}(G/U)$ indexed by W_I with morphisms and compatibilities as in Definition 1.2.6 but only for $w \in W_I$, $s \in I$.

For any $J \subset I \subset S$, and any right coset $W_J x$ of W_J in W_I, there is a restriction functor $j^{W_I *}_{W_J x} : \mathcal{A}_{W_I} \to \mathcal{A}_{W_J}$ remembering only the tuple elements and morphisms for $w \in W_J x$, $s \in J$. When $I = S$, we write only $j^*_{W_J x}$, and in this case we omit superscripts similarly for all functors introduced in this section.

Proposition 4.5.2 ([48], Proposition 7.1.1)**.** *For any $J \subset I \subset S$, and any $W_J x$, the functor $j^{W_I *}_{W_J x}$ admits a left adjoint $j^{W_I}_{W_J x !}$.*

Definition 4.5.3. For $I \subset S$, let $\mathcal{A}^I$ be the category whose objects are tuples $(A_w)_{w \in W}$ equipped with morphisms $\Phi^\circ_y A_w \to A_{yw}$ for any $w \in W$, $y \in W_I$ satisfying the same conditions as in Definition 1.2.6 but only for $y \in W_I$. Morphisms in $\mathcal{A}^I$ are morphisms in $\mathrm{Perv}(G/U)^{\oplus W}$ satisfying the compatibilities in Definition 1.2.6 but only for $y \in W_I$.

Alternatively, $\mathcal{A}^I = \oplus_{W_I \backslash W} \mathcal{A}_{W_I}$ with a reindexing by W on the tuples in this category.

Just like $\mathcal{A}$, the category $\mathcal{A}^I$ admits an action of W by functors $\{\mathcal{F}_w\}_{w \in W}$ which are defined by $\mathcal{F}_w((A_y)_{y \in W}) = (A_{yw})_{w \in W}$. We define the category $\mathcal{A}^I_{w,\mathbb{F}_q}$ as in Definition 1.2.8 but with $\mathcal{A}$ replaced by $\mathcal{A}^I$.

Adjunctions between these categories

For any $J \subset I$, there is an obvious restriction functor $j^*_{I,J} : \mathcal{A}^I \to \mathcal{A}^J$ which is the identity on objects but which remembers only the morphisms $\Phi^\circ_y A_w \to A_{yw}$ for $y \in W_J$. As in Proposition 4.2.2, there is an analogous derived version of this morphism and the following adjunction.

Proposition 4.5.4. *For any $J \subset I$, the functor $j^*_{I,J}$ admits a left adjoint $j_{I,J!}$. For any $A \in \mathcal{A}^J$, $j^\circ_{I,J!}(A)$ is the direct sum*

$$\bigoplus_{x \in W_J \backslash W_I} j^{W_I \circ}_{W_J x !}((A_w)_{w \in W_J x}), \tag{4.5.1}$$

where $(A_w)_{w\in W_J x}$ is considered as an element of $\mathcal{A}_{W_J}$. Further, the adjoint pair $(j^\circ_{I,J!}, j^*_{I,J})$ gives also an adjunction between $\mathcal{A}^I_{w,\mathbb{F}_q}$ and $\mathcal{A}^J_{w,\mathbb{F}_q}$.

Proof. The direct sum in (4.5.1) has the natural structure of an object of $\mathcal{A}^I$, as each summand object $j^{W_I\circ}_{W_J x!}(A_w)_{w\in W_J x}$ has such a structure by Proposition 4.5.2. We then note that for any such $A \in \mathcal{A}^J$ and any $B \in \mathcal{A}^I$,

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{A}^I}\left(\bigoplus_{x\in W_J\backslash W_I} j^{I\circ}_{W_J x!}((A_w)_{w\in W_J x}), B\right)\\
\cong &\bigoplus_{x\in W_J\backslash W_I} \mathrm{Hom}_{\mathcal{A}^I}(j^\circ_{W_J x!}((A_w)_{w\in W_J x}), B)\\
\cong &\bigoplus_{x\in W_J\backslash W_I} \mathrm{Hom}_{\mathcal{A}_{W_J x}}((A_w)_{w\in W_J x}, j^*_{W_J x}B)\\
\cong &\,\mathrm{Hom}_{\mathcal{A}^J}((A_w)_{w\in W}, \oplus_{x\in W_J\backslash W_I} j^*_{W_J x}B)\\
= &\,\mathrm{Hom}_{\mathcal{A}^J}((A_w)_{w\in W}, j^*_{I,J}B).
\end{aligned}$$

To see that the adjoint pair $(j^\circ_{I,J!}, j^*_{I,J})$ gives also an adjunction between $\mathcal{A}^I_{w,\mathbb{F}_q}$ and $\mathcal{A}^J_{w,\mathbb{F}_q}$, we note that the morphisms on both sides of the equation above which are compatible with the morphism $\psi_A : \mathcal{F}_w\mathrm{Fr}^*A \to A$ are preserved by these isomorphisms. □

4.5.2 Polishchuk's complex for $\mathcal{A}_{w,\mathbb{F}_q}$

Polishchuk's canonical complex in [48]

For a fixed $A \in \mathcal{A}$ and a choice of $J \subset S$, Polishchuk writes $A(J) = j_{S-J!}j^*_{S-J}A$. In 7.1 of [48], Polishchuk explains that for any $A \in \mathcal{A}$, adjunction of parabolic pushforward and pullback functors $(j_{W_I x!}, j^*_{W_I x})$ gives a canonical morphism $A(J) \to A(J')$ whenever $J \subset J'$. He then defines the complex $C_\bullet(A)$ as

$$C_{n-1} = A(S) \longrightarrow \ldots \longrightarrow C_1 = \bigoplus_{|J|=2} A(J) \longrightarrow C_0 = \bigoplus_{|J|=1} A(J),$$

where $n = |S|$.

Recalling the morphism $\iota : \mathcal{A} \to \mathcal{A}$ sending $(A_w)_{w\in W}$ to $(\Phi^\circ_{w_0}A_{w_0 w})_{w\in W}$, he describes natural morphisms $C_0 \to A$ and $\iota(A) \to C_{n-1}$, and shows that

$$H_i(C_\bullet(A)) = \begin{cases} A & i = 0,\\ 0 & i \neq 0, n-1,\\ \iota(A) & i = n-1. \end{cases} \tag{4.5.2}$$

He then describes a $2n$-term complex $\tilde{C}_\bullet$ formed from attaching $C_\bullet(\iota A)$ to $C_\bullet(A)$ via the

maps $C_0(\iota A) \to \iota A \to C_{n-1}(A)$ with the property that

$$H_i(\tilde{C}_\bullet(A)) = \begin{cases} A & i = 0, \\ 0 & i \neq 0, 2n-1, \\ \iota^2(A) & i = 2n-1. \end{cases} \tag{4.5.3}$$

Compatibility of the complex with w-twisted structure

Proposition 4.5.5. *If $A \in \mathcal{A}_{w,\mathbb{F}_q}$, then the complex $C_\bullet(A)$ is compatible with the w-twisted Weil structures, i.e. is a complex of objects in $\mathcal{A}_{w,\mathbb{F}_q}$.*

Proof. For any k, components of the map $C_k(A) \to C_{k-1}(A)$ are each maps of the form

$$j_{S-J!}j^*_{S-J}A \to j_{S-J'!}j^*_{S-J'}A \tag{4.5.4}$$

where $J' \subset J \subset S$ are such that $|J| = k$, $|J'| = k-1$, which we now describe. By adjunction the data of such a map is equivalent to a map

$$j^*_{S-J}A \to j^*_{S-J}j_{S-J'!}j^*_{S-J'}A, \tag{4.5.5}$$

and compatibility with the w-twisted structure is preserved under this adjunction. By the definition of $j_{S-J'!}$, we have that

$$j^*_{S-J}j_{S-J'!}j^*_{S-J'}A \cong \bigoplus_{x \in W_{S-J}\backslash W} j^*_{S-J}j_{W_{S-J}x!}j^*_{S-J'}A, \tag{4.5.6}$$

and one can check that the map in (4.5.4) appearing in the definition of Polishchuk's complex in [48] comes in (4.5.5) from a natural injection in $\mathcal{A}^{S-J}$ from $j^*_{S-J}A$ into this direct sum defined by sending the yth tuple entry to the yth tuple entry in the direct summand corresponding to the unique x for which $y \in W_{S-J}x$. It is straightforward to check that this injection preserves w-twisted Weil structures on both sides coming from the w-twisted Weil structure on A, and therefore the map in (4.5.4) is a map in $\mathcal{A}_{w,\mathbb{F}_q}$. □

Parabolic canonical complexes

We remark how that the content appearing in 4.5.2 can be generalized to provide complexes in $\mathcal{A}^I_{w,\mathbb{F}_q}$ for any $I \subset S$ with $|I| = k$ and any $w \in W$. Namely, if we fix $I \subset S$, $w \in W$, and $A \in \mathcal{A}^I_{w,\mathbb{F}_q}$, we let $A^I(J) = j^I_{S-J!}j^{I*}_{S-J}A$ whenever $J \supset S - I$, and we define the complex $C^I_\bullet(A)$ by

$$A^I(S) \longrightarrow \ldots \longrightarrow \bigoplus_{|J|=n-k+2} A^I(J) \longrightarrow \bigoplus_{|J|=n-k+1} A^I(J),$$

indexed such that C_{k-1} is the first term and C_0 is the last term in the above. The results in Section 4.5.2 then still hold, giving a version of the canonical complex associated to an object $A \in \mathcal{A}^I_{w,\mathbb{F}_q}$.

Equations in the Grothendieck group

The following is a consequence of the fact that the full twist is central in the braid group, along with the fact that the symplectic Fourier transforms Φ_w form a braid action.

Lemma 4.5.6. *For any $J \subset I \subset S$ there is a natural isomorphism*

$$j^I_{J!} \circ \iota^2 \cong \iota^2 \circ j^I_{J!} \tag{4.5.7}$$

of functors from $D^b(\mathcal{A}^J)$ to $D^b(\mathcal{A}^I)$.

Theorem 4.5.7. *For any $w \in W$ and $A \in \mathcal{A}_{w,\mathbb{F}_q}$ the element*

$$(\iota^2 - 1)^n[A] \in K_0(\mathcal{A}_{w,\mathbb{F}_q}) \tag{4.5.8}$$

lies in V^{fp}, for $n = |S|$.

Proof. By the derived category version of the canonical complex construction in combination with the observation about its homology provided in 4.5.2, for any $A \in \mathcal{A}_{w,\mathbb{F}_q}$ we have the following equation in $K_0(\mathcal{A}_{w,\mathbb{F}_q})$:

$$[\iota A] + (-1)^{|I|-1}[A] = \sum_{\substack{I' \\ I \subset I' \subsetneq S}} (-1)^{|I'|}[j^I_{J!} j^{I*}_J A], \tag{4.5.9}$$

c.f. the proof of Theorem 11.5.1 in [48] where the analogous equation is used in the case where $w = e$, $I = S$. By the "doubled" canonical complex $\tilde{C}_\bullet(A)$ and the description of its homology in 4.5.2, this means that $[\iota^2 A] - [A]$ is a linear combination of elements lying in the image of the functors $j^I_{J!} j^{I*}_J$ for $J \subsetneq I$.

Now by induction on the $|I|$ appearing in the equation above, it follows from Lemma 4.5.6 that $(\iota^2 - 1)^n[A]$ is a linear combination of elements lying in the image of the functors $j_{\varnothing!} j^*_{\varnothing}$. We have that for any $B \in \mathcal{A}^{\varnothing}_{w,\mathbb{F}_q}$,

$$j_{\varnothing!} j^*_{\varnothing} = \oplus_{y \in W} j_{y!} B_y, \tag{4.5.10}$$

and each of these direct summands has finite cohomological dimension by Proposition 4.2.4, completing the proof of the theorem. □

4.6 Central objects and the full twist

4.6.1 Cells, the big tilting object, and the full twist

The full twist

Definition 4.6.1. Let $\mathcal{L} \in \mathrm{Ch}(T)$. Then we define the element

$$\mathrm{FT}_{\mathcal{L}} = {}_{\mathcal{L}}\hat{\nabla}(w_0)_{w_0\mathcal{L}} * {}_{w_0\mathcal{L}}\hat{\nabla}(w_0)_{\mathcal{L}} \tag{4.6.1}$$

of ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$. The object $\mathrm{FT}_{\mathcal{L}}$ admits a central structure and arises comes from a character sheaf in $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ via pullback composed with the Harish-Chandra functor, see [12] for an explicit description of this character sheaf in the unipotent case, whose proof can be adapted similarly for arbitrary monodromy. As with $\mathbb{K}_{\mathcal{L}}$, we will identify $\mathrm{FT}_{\mathcal{L}}$ as an element of $\mathcal{ZH}^{(1)}_{\mathcal{L}}$ with its underlying character sheaf in $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, and by $- * \mathrm{FT}_{\mathcal{L}}$ we will denote the action as described in Proposition 4.4.7.

By the definition of $\mathrm{FT}_{\mathcal{L}}$ combined with Proposition 4.3.13, we obtain the following.

Lemma 4.6.2. *The functors $- * \mathrm{FT}_{\mathcal{L}}$ and ι^2 on $D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ are naturally isomorphic.*

The action of the full twist on cells

Proposition 4.6.3. *For any $\underline{c} \in \underline{C}_{\mathcal{L}}$, there exists an integer $d_{\mathcal{L}}(\underline{c})$ between 0 and $2\ell(w_0)$ such that for any $a \in K_0(D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}})$,*

$$[\mathrm{FT}_{\mathcal{L}}] * a = v^{d_{\mathcal{L}}(\underline{c})} a \tag{4.6.2}$$

Proof. We follow the same argument as in Proposition 4.1 and Remark 4.2 of [10]; in other words, we begin with the observation that $[\mathrm{FT}_{\mathcal{L}}]$ acts trivially on the non-graded version of the Grothendieck group $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}})$. Continuing to follow the argument of loc. cit., we then know that for any object A in the heart of $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$, the object $\mathrm{FT}_{\mathcal{L}} * A \in D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$ is perverse up to shift, and furthermore has the property that $[\mathrm{FT}_{\mathcal{L}} * A] = v^d[A]$ for some d.

To compute the value of d, we can pass back along the Harish-Chandra transform and work in the category $\mathcal{H}^{(1)}_{\mathcal{L}}$. The Grothendieck ring $K_0(\mathcal{H}^{(1)}_{\mathcal{L}})$ is the monodromic Hecke algebra $\mathcal{H}_{\mathcal{L}}$. By [40], this is isomorphic to the usual Hecke algebra associated to the group $W^\circ_{\mathcal{L}} \subset W$, with $\mathrm{FT}_{\mathcal{L}}$ being identified with the usual full-twist $\tilde{T}^2_{w_0,\mathcal{L}}$. By [38, p. 5.12.2], the full twist in the usual Hecke algebra acts on the cell subquotient module of the Hecke algebra corresponding to a cell $\underline{c}$ by the scalar $v^{d(\underline{c})}$, where $d(\underline{c})$ is described in loc. cit. Passing this fact back along the monodromic-equivariant isomorphism from [40], the result follows. □

Definition 4.6.4. For any two-sided cell $\underline{c}$, let $d_{\mathcal{L}}(\underline{c})$ be the integer between 0 and $2\ell(w_0)$ for which the equation in Proposition 4.6.3 holds.

$\mathbb{K}_{\mathcal{L}}$ in the top cell subquotient

Definition 4.6.5. Let $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$ be the submodule of $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))$ spanned by the image of the ideal $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$ under the action described in Proposition 4.4.7.

Definition 4.6.6. For any $\mathcal{L} \in \mathrm{Ch}(T)$, let $q_{\mathcal{L}}(v)$ be the Poincaré polynomial

$$q_{\mathcal{L}}(v) = \sum_{w \in W} (-v^2)^{\ell(w)}$$

of the group $W^\circ_{\mathcal{L}}$. Note by the Chevalley-Solomon formula that $q_{\mathcal{L}}(v)$ can be expressed as the product of some linear factors each of which is a factor of $(v^{2i}-1)$ for some $1 \leq i \leq \ell(w_0)$.[1]

Lemma 4.6.7. *The multiplicity with grading of the irreducible object* $\mathrm{IC}(e)_{\mathcal{L}}$ *in the Jordan-Holder decomposition of* $\hat{\mathcal{T}}_{\mathcal{L}}$ *is* $q_{\mathcal{L}}(v)$.

Proof. In [55], Yun computes the $\mathbb{Z}[v,v^{-1}]$-graded multiplicity of any standard object $\Delta(y)$ in a filtration of $T(w)$, where $w, y \in W$ and $\Delta(y)$ and $T(w)$ are standard and tilting objects respectively in the usual Hecke category. The main equivalence of categories in [40] allows us to extend these results to the monodromic Hecke category by replacing W with $W^\circ_{\mathcal{L}}$, whose combinatorics in terms of tilting, standard, and irreducible objects matches exactly the combinatorics of the completed category ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ (c.f. 9.3.3 of [25] for an explicit description of the standard filtration on a tilting object in the monodromic setting).

Combining Theorem 5.3.1 of [55] with the expression of standard objects in terms of irreducible objects via inverse Kazhdan-Lusztig polynomials, we compute that the multiplicity of $\hat{\mathrm{IC}(e)}_{\mathcal{L}}$ is exactly

$$\sum_{w \in W^\circ_{\mathcal{L}}} (-v^2)^{\ell(w_0)-\ell(w)} = \sum_{w \in W^\circ_{\mathcal{L}}} (-v^2)^{\ell(w)} = q_{\mathcal{L}}(v). \tag{4.6.3}$$

□

For the next proposition, we recall the definition of the character sheaves $\varepsilon_{n,\mathcal{L}}$ from Section 4.3.1.

Proposition 4.6.8. *For any* n, *the element*

$$[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] - (v^2-1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)[\varepsilon_{n,\mathcal{L}}] \tag{4.6.4}$$

of $K_0(D^b_{\mathfrak{C}}(G_{\mathrm{ad}}))$ *lies in the subspace* $K_0(D^b_{\mathfrak{C},<\underline{c}_e}(G_{\mathrm{ad}}))$.

Proof. First note that $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}_e})$ is of rank 1 as a $\mathbb{Z}[v,v^{-1}]$-module. This means we have that the classes of the images of $\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}$ and $\varepsilon_{n,\mathcal{L}}$ under the cell quotient map to $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}_e}$ are scalar multiples, so $[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] - q'(v)[\varepsilon_{n,\mathcal{L}}]$ for some $q'(v) \in \mathbb{Z}[v,v^{-1}]$.

[1] The variable u used throughout [48] is replaced in the present work by v^2.

By [12], $[\mathbb{K}_{\mathcal{L}}] = (v^2-1)^{\mathrm{rank}(T)}[\hat{\mathcal{T}}_{\mathcal{L}}]$ in the full Grothendieck group $K_0(\hat{\mathcal{P}}_{\mathcal{L}})$. Note that in the corresponding top cell subquotient module for the monodromic Hecke algebra $K_0(\mathcal{H}^{(1)}_{\mathcal{L}})$, the equation $[\hat{\mathcal{T}}_{\mathcal{L}}]-q_{\mathcal{L}}(v)[\hat{\delta}_{\mathcal{L}}]$ holds by Lemma 4.6.7. This means that $q'(v) = (v^2-1)^{\mathrm{rank}(T)}q_{\mathcal{L}}(v)$ is the only value for which $[\varepsilon_{n,\mathcal{L}}*\mathbb{K}_{\mathcal{L}}]-q'(v)[\varepsilon_{n,\mathcal{L}}]$ lies in a lower cell submodule of $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$, and therefore $[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] = (v^2-1)^{\mathrm{rank}(T)}q_{\mathcal{L}}(v)[\varepsilon_{n,\mathcal{L}}]$. □

Convolution with $\mathbb{K}_{\mathcal{L}}$ for Kazhdan-Laumon objects

Lemma 4.6.9. *For any $s \in S$, the map*

$$c_s * \mathbb{K}_{\mathcal{L}} : {}_{\mathcal{L}}\hat{\nabla}(s)_{s\mathcal{L}} * {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} \to \mathbb{K}_{\mathcal{L}} \tag{4.6.5}$$

is an isomorphism.

Proof. It is enough to show that the corresponding map $\tilde{c}_s : {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} \to {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}}$ (obtained by convolving c_s with ${}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}}$) is an isomorphism. If $s\mathcal{L} \neq \mathcal{L}$, then by Lemma 3.6 in [40], ${}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} \cong {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}}$, and so this becomes immediate. We now consider the case when $s\mathcal{L} = \mathcal{L}$. Note that $\tilde{c}_s = (i' \circ p) * \mathbb{K}_{\mathcal{L}}$ where i' and p are as in the canonical exact sequences

$$0 \longrightarrow \hat{\mathrm{IC}}(s)_{\mathcal{L}} \xrightarrow{i} {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} \xrightarrow{p} \hat{\mathrm{IC}}(e)_{\mathcal{L}} \longrightarrow 0,$$

$$0 \longrightarrow \hat{\mathrm{IC}}(e)_{\mathcal{L}} \xrightarrow{i'} {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}} \xrightarrow{p'} \hat{\mathrm{IC}}(s)_{\mathcal{L}} \longrightarrow 0.$$

It is then enough to show that $p * \mathbb{K}_{\mathcal{L}}$ and $i' * \mathbb{K}_{\mathcal{L}}$ are each isomorphisms. This follows from the fact that $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} = 0$. Indeed, $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \mathbb{K} = 0$ if and only if $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \hat{\mathcal{T}}_{\mathcal{L}} = 0$, which follows from the fact that its class in the Grothendieck is zero combined with the fact that $\hat{\mathcal{T}}_{\mathcal{L}}$ is convolution-exact, since it is tilting. □

Corollary 4.6.10. *For any $A \in \mathcal{A}_{w,\mathbb{F}_q}$, the object $A * \mathbb{K}_{\mathcal{L}}$ of $\mathcal{A}_{w,\mathbb{F}_q}$ has the property that for any $y, z \in W$, the composition*

$$\theta_{z^{-1},zy} \circ (\Phi^{\circ}_{z^{-1}}\theta_{z,y}) : A_y * \mathbb{K}_{\mathcal{L}} \to A_y * \mathbb{K}_{\mathcal{L}} \tag{4.6.6}$$

is an isomorphism.

Proof. By Proposition 4.3.13, the morphism in question can be written, up to Tate twist, as a morphism from $A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}}$ to $A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}$. In particular, it is a Tate twist of the morphism

$$
\begin{array}{c}
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\downarrow \\
A_y * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\downarrow{\scriptstyle \theta_{z,y*_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}}*\mathbb{K}_{\mathcal{L}}*_{\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}}*_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}}}} \\
A_{zy} * {}_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\downarrow \\
A_{zy} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\downarrow{\scriptstyle \theta_{z^{-1},zy*_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}}*\mathbb{K}_{\mathcal{L}}*_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}}} \\
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},
\end{array}
$$

where the unlabeled arrows are the isomorphisms given by the central structure on $\mathbb{K}_{\mathcal{L}}$; these extend to similar central morphisms for conjugates of of $\mathbb{K}_{\mathcal{L}}$ by standard/costandard sheaves by the same argument as in Lemma 11.12 of [40].

We note that since the second and third morphisms above clearly commute with these central morphisms, the above morphism agrees with the composition

$$
\begin{array}{c}
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\downarrow \\
A_y * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\downarrow{\scriptstyle \theta_{z,y*_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}}*_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}}*\mathbb{K}_{\mathcal{L}}*_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}}} \\
A_y * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\downarrow{\scriptstyle \theta_{z^{-1},zy*_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}}*\mathbb{K}_{\mathcal{L}}*_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}}} \\
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},
\end{array}
$$

where the first morphism is again the isomorphism coming from centrality of $\mathbb{K}_{\mathcal{L}}$ The composition of the last two morphisms in the sequence above must, by the definition of Kazhdan-Laumon categories, be equal to the morphism

$$A_y * c_z * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},$$

where c_z is the morphism ${}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \to \mathrm{Id}$ which is obtained by applying the morphisms c_s successively for every simple reflection s in a reduced expression for z. By the functoriality of the central morphisms discussed above, they also commute with this c_z, and

so the entire composition above actually agrees with the morphism

$$A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * c_z,$$

By our inductive definition of c_z along with the same argument as in Lemma 4.6.9, $\mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * c_z$ is an isomorphism, and therefore $\theta_{z^{-1},zy} \circ (\Phi^{\circ}_{z^{-1}}\theta_{z,y})$ must be, too. □

Proposition 4.6.11. *For any $A \in \mathcal{A}_{w,\mathbb{F}_q}$, $A * \mathbb{K}_{\mathcal{L}}$ has finite projective dimension.*

Proof. We can forget the w-twisted Weil structure on $A * \mathbb{K}_{\mathcal{L}}$ and consider it as an object in $\mathcal{A}$. In $\mathcal{A}$, the adjunction $(j_{e!}, j_e^*)$ gives a morphism

$$a : j_{e!}j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to A * \mathbb{K}_{\mathcal{L}}. \tag{4.6.7}$$

We claim that this is an isomorphism.

It is enough to show that the component morphisms $a_y : j_y^* j_{e!} j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to j_y^*(A * \mathbb{K}_{\mathcal{L}})$ are each isomorphisms. By definition, these are the structure morphisms $\theta_{y,e} : \Phi^{\circ}_y(A * \mathbb{K}_{\mathcal{L}})_e \to (A * \mathbb{K}_{\mathcal{L}})_y$.

By Corollary 4.6.10, the morphisms

$$\theta_{y^{-1},y} \circ (\Phi^{\circ}_{y^{-1}}\theta_{y,e}) : \Phi^{\circ}_{y-1}\Phi^{\circ}_y A_e \to A_e \tag{4.6.8}$$
$$\theta_{y,e} \circ (\Phi^{\circ}_{y^{-1}}\theta_{y^{-1},y}) : \Phi^{\circ}_y\Phi^{\circ}_{y^{-1}} A_y \to A_y \tag{4.6.9}$$

are both isomorphisms. This tells us that $\theta_{y,e}$ is both a monomorphism and an epimorphism.

This means $a : j_{e!}j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to A * \mathbb{K}_{\mathcal{L}}$ is an isomorphism as we claimed. The proposition then follows since all objects in the image of $j_{e!}$ have finite cohomological dimension by Proposition 4.2.4. □

4.6.2 Completing the proof of Theorem 4.1.2

Proof of Theorem 4.1.2 by the action of the full twist on cells

Lemma 4.6.12. *For any $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$, there exists some $r \geq 1$ for which*

$$P^r(\mathrm{FT}_{\mathcal{L}}, v) \cdot a = 0 \tag{4.6.10}$$

in $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$, where P^r is the polynomial for which $P^r(x,v) = P(x,v)^r$ for any x.

Further, if $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})_{<\underline{c}_e}$, then

$$\tilde{P}^r(\mathrm{FT}_{\mathcal{L}}, v) \cdot a = 0. \tag{4.6.11}$$

Proof. By and Propositions 4.4.4 and 4.4.7, the category $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ acts on $D^b_{\mathrm{dg}}(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ in a way which respects distinguished triangles, therefore giving an action of the $\mathbb{Z}[v, v^{-1}]$-

algebra $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on $K_0(\mathcal{A}_{w,\mathbb{F}_q})$. It is then enough to show that there exists r for which $P^r([\mathrm{FT}_{\mathcal{L}}], v) = 0$ in $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ and that $\tilde{P}^r([\mathrm{FT}_{\mathcal{L}}], v)\cdot b = 0$ for any $b \in K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$.

Indeed, since by Proposition 4.6.3 and Proposition 4.3.9, the eigenvalues of the map $[\mathrm{FT}_{\mathcal{L}}] * -$ on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ are each of the form v^{2i} for $0 \leq i \leq \ell(w_0)$ and of the form v^{2i} for $1 \leq i \leq \ell(w_0)$ on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$, we must only choose r to be the maximum multiplicity occuring in the characteristic polynomial of $[\mathrm{FT}_{\mathcal{L}}] * -$ on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$, since each degree 1 term of this characteristic polynomial is a factor of $P(x, v)$ (resp. $\tilde{P}(x, v)$) by definition of the latter. Choosing r in this way, we get that the polynomials in the lemma vanish, as desired. □

Corollary 4.6.13. *For any $a \in K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$, $a \in V^{\mathrm{fp}}_{p(v)}$.*

Proof. Let $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})_{<\underline{c}_e}$. By Theorem 4.5.7, $(\iota^2 - 1)^n \cdot a \in V^{\mathrm{fp}}$, while by Lemmas 4.6.2 and 4.6.12, $\tilde{P}(\iota^2, v) \cdot a = 0$. We claim that this implies that $(\iota^2 - 1)^{n-k} \cdot a \in V^{\mathrm{fp}}_{p(v)}$ for all $0 \leq k \leq n$. Indeed, suppose for induction that $(\iota^2 - 1)^{n-k+1} \cdot a \in V^{\mathrm{fp}}_{p(v)}$, and let $a' = (\iota^2 - 1)^{n-k} \cdot a$. Then $(\iota^2 - 1) \cdot a' \in V^{\mathrm{fp}}_{p(v)}$ and $\tilde{P}(\iota^2, v)^r \cdot a' = 0$, so by the Euclidean algorithm, $p(v)^r \cdot a' = \tilde{P}(1, v)^r \cdot a'$ is in the $\mathbb{Z}[\iota^2, v, v^{-1}]$-span of $(\iota^2 - 1) \cdot a'$, and therefore lies in $V^{\mathrm{fp}}_{p(v)}$. Dividing by $p(v)^r$ gives $a' \in V^{\mathrm{fp}}_{p(v)}$, and so we can proceed by induction until $k = n$ where we conclude that $a \in V^{\mathrm{fp}}_{p(v)}$. □

Proof of Theorem 4.1.2. Let $A \in \mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$. By Proposition 4.6.8,

$$[A * \mathbb{K}_{\mathcal{L}}] - (v^2 - 1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)[A] \tag{4.6.12}$$

is an element of $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$

By Corollary 4.6.13, this means $(v^2 - 1)^{\mathrm{rank}(T)}([A * \mathbb{K}_{\mathcal{L}}] - q_{\mathcal{L}}(v)[A]) \in V^{\mathrm{fp}}_{p(v)}$.

By Proposition 4.6.11, $A * \mathbb{K}_{\mathcal{L}}$ itself has finite projective dimension, and so in combination with equation (4.6.12), this means $(v^2 - 1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)p(v)[A] \in V^{\mathrm{fp}}_{p(v)}$. Note that by Definition 4.6.6 each degree 1 factor of $q_{\mathcal{L}}(v)$ is also a factor of $p(v)$, so we can divide by $q_{\mathcal{L}}(v)$ in the localization $V^{\mathrm{fp}}_{p(v)}$ to get $[A] \in V^{\mathrm{fp}}_{p(v)}$, completing the proof. □

4.7 Polishchuk's rationality conjecture

4.7.1 A general study of $K_0(\mathcal{A}_{\mathbb{F}_q})$

Polishchuk's description of $K_0(\mathcal{A}_{\mathbb{F}_q})$

A crucial tool which we will use in the proof of Theorem 4.1.1 is the following description of $K_0(\mathcal{A}_{\mathbb{F}_q})$ provided by Polishchuk.

Theorem 4.7.1 ([48], Proposition 3.4.1)**.** *The map*

$$K_0(\mathcal{A}_{\mathbb{F}_q}) \to \bigoplus_{w\in W} K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U)) \tag{4.7.1}$$

induced by the functor $\oplus_{w\in W} j_w^*$ *is injective. Its image is the subset*

$$\{(a_w)_{w\in W} \in K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U)) \mid a_{sw} - \Phi_s a_w \in \mathrm{im}(\Phi_s^2 - 1), s \in S, w \in W\}.$$

Recalling [47]

In [47], we study the subalgebra $\mathrm{KL}(v)$ of endomorphisms of $K_0(G/U)$ generated by the symplectic Fourier transforms $\{\Phi_s\}_{s\in S}$. By Section 1.2.1, this is the same as the subalgebra of $K_0(G/U \times G/U)$ generated under convolution by classes of Kazhdan-Laumon sheaves; we denote the generator of $\mathrm{KL}(v)$ corresponding to $w \in W$ by a_w. In this section, we use the monodromic Hecke algebras $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{\mathfrak{o}}$ with the standard generators $\tilde{T}_w$ as defined in [40], see [47] for a more precise outline of our conventions.

In Section 4 of [47], we show that for any character sheaf $\mathcal{L}$ with W-orbit $\mathfrak{o}$, there is a surjection $\pi_{\mathcal{L}} : \mathrm{KL}(v) \to \mathcal{H}_{\mathfrak{o}}$. The following result follows from the main result of loc. cit. which explicitly identifies the algebra $\mathrm{KL}(v)$ as a subalgebra of a generic-parameter version of the Yokonuma-Hecke algebra.

Proposition 4.7.2 ([47])**.** *The following properties are satisfied by the morphisms* $\{\pi_{\mathcal{L}}\}_{\mathcal{L}}$*.*

1. *If* $w_1, w_2 \in W$*, then* $\pi_{\mathcal{L}}(\mathsf{a}_{w_1}\mathsf{a}_{w_2}) = \pi_{w_2\mathcal{L}}(\mathsf{a}_{w_1})\pi_{\mathcal{L}}(\mathsf{a}_{w_2})$ *in* $\mathcal{H}_{\mathfrak{o}}$*.*
2. *If* $w \in W_{\mathcal{L}}^\circ$*, then* $\pi_{\mathcal{L}}(\mathsf{a}_w) = \tilde{T}_w \in \mathcal{H}_{\mathfrak{o}}$*.*
3. *If* $s \in S$ *is not in* $W_{\mathcal{L}}^\circ$*, then* $\pi_{\mathcal{L}}(\mathsf{a}_s^2) = 1$*.*

Finally, the morphism $\prod_{\mathcal{L}} \pi_{\mathcal{L}}$ *is injective, so if* $a \in \mathrm{KL}(v)$ *is such that* $\pi_{\mathcal{L}}(a) = 0$ *for all character sheaves* $\mathcal{L}$*, then* $a = 0$*.*

Lemma 4.7.3. *For any* $\mathcal{L}$ *with* W*-orbit* $\mathfrak{o}$*, the algebra morphism* $\pi_{\mathcal{L}} : \mathrm{KL}(v) \to \mathcal{H}_{\mathfrak{o}}$ *is such that*

$$\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) = \tilde{T}_{w_{0,\mathcal{L}}}^2, \tag{4.7.2}$$

where $w_{0,\mathcal{L}}$ *is the longest element of* $W_{\mathcal{L}}^\circ$*.*

Proof. Recall that if $s \in S$ is such that $s \notin W_{\mathcal{L}}^\circ$, then $\pi_{\mathcal{L}}(\mathsf{a}_s^2) = 1$ in $\mathcal{H}_{\mathfrak{o}}$. For induction, we claim that if $y \in W$ is such that $\ell(y)+\ell(w_{\mathcal{L},0}) = \ell(yw_{\mathcal{L},0})$, and if $s \in S$ such that $\ell(sy) < \ell(y)$, then $s \notin W_{y\mathcal{L}}$, i.e. $y^{-1}sy \notin W_{\mathcal{L}}^\circ$.

Indeed, if we had $y^{-1}sy \in W_{\mathcal{L}}^\circ$, then we since $w_{0,\mathcal{L}}$ dominates all elements of $W_{\mathcal{L}}^\circ$ in the Bruhat order, we could pick $z \in W$ such that $y^{-1}syz = w_{0,\mathcal{L}}$ with $\ell(y^{-1}sy)+\ell(z) = \ell(w_0, \mathcal{L})$.

But then we would have $syz = yw_{0,\mathcal{L}}$ with $\ell(sy) + \ell(z) = \ell(syz) = \ell(yw_{\mathcal{L},0})$. But since $\ell(sy) < \ell(y)$ and $\ell(z) \leq \ell(w_{0,\mathcal{L}})$, this is impossible.

Now choose some y for which $yw_{0,\mathcal{L}} = w_0$ with $\ell(y)+\ell(w_{0,\mathcal{L}})$. We will show that $\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) = \tilde{T}_{w_{0,\mathcal{L}}}^2$ by induction on the length of y. Choosing $s \in S$ such that $\ell(sy) < \ell(y)$, this induction hypothesis along with the fact proved in the previous paragraph gives that

$$\begin{aligned}
\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) &= \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{y^{-1}}\mathsf{a}_y\mathsf{a}_{w_{0,\mathcal{L}}}) && (4.7.3)\\
&= \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}}\mathsf{a}_s^2\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) && (4.7.4)\\
&= \pi_{sy\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}})\pi_{sy\mathcal{L}}(\mathsf{a}_s^2)\pi_{\mathcal{L}}(\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) && (4.7.5)\\
&= \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}}\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) && (4.7.6)\\
&= \pi_{\mathcal{L}}(\mathsf{a}_{syw_{0,\mathcal{L}}}^2) && (4.7.7)\\
&= \tilde{T}_{w_{0,\mathcal{L}}}^2. && (4.7.8)
\end{aligned}$$

□

The action of the full twist on $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$

Proposition 4.7.4. *The endomorphism ι^2 on $K_0(\mathcal{A})$ satisfies $P(\iota^2, v) = 0$.*

Proof. By Theorem 4.7.1, it is enough to show that $P(\Phi_{w_0}^2, v) = 0$ as an endomorphism of the space $K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U))$.

Now recall that the endomorphism $\Phi_{w_0} : K_0(G/U) \to K_0(G/U)$ agrees with right convolution with the Kazhdan-Laumon sheaf $\overline{K(w_0)}$. Since the convolution $D^b(G/U) \times D^b(G/U \times G/U) \to D^b(G/U)$ is a triangulated functor, it is enough to show that $P([\overline{K(w_0)} * \overline{K(w_0)}], v) = 0$ in $K_0(G/U \times G/U)$.

Letting $1_{\mathcal{L}}$ be the idempotent in $\mathcal{H}_{\mathfrak{o}}$ corresponding to the $\mathcal{L}$-monodromic subalgebra, then in loc. cit. we show that

$$\pi_{\mathcal{L}}(\mathsf{a}_s) = \begin{cases} -v\tilde{T}_s^{-1}1_{\mathcal{L}} & s \in W_{\mathcal{L}}^\circ \\ -\tilde{T}_s 1_{\mathcal{L}} & s \notin W_{\mathcal{L}}^\circ. \end{cases} \tag{4.7.9}$$

It is a straightforward calculation from the above to show that $\pi_{\mathcal{L}}([K(w_0) * K(w_0)]) = v^{2\ell(y)}\tilde{T}_y^{-2}1_{\mathcal{L}} \in \mathcal{H}_{\mathcal{L}}$, where y is the longest element of $W_{\mathcal{L}}^\circ$. By the main result of [40], the monodromic Hecke algebra $\mathcal{H}_{\mathcal{L}}$ is isomorphic to $\mathcal{H}_{W_{\mathcal{L}}^\circ}$, with full twists on each side being identified as in Lemma 4.7.3. By 5.12.2 of [38] which identifies the eigenvalues of the full twist in the regular representation, we have that $P(-v^{2\ell(y)}\tilde{T}_y^2, v) = 0$ in $\mathcal{H}_{\mathcal{L}}$.

Finally, we note that by Proposition 4.7.2, if a polynomial is satisfied by $\pi_{\mathcal{L}}([\overline{K(w_0)} * \overline{K(w_0)}])$ in each $\mathcal{H}_{\mathfrak{o}}$, then it is also satisfied in $K_0(G/U \times G/U)$; this completes the proof of the proposition. □

4.7.2 Completing the proof of Theorem 4.1.1

Proof of Theorem 4.1.1

Let $a \in K_0(\mathcal{A}_{\mathbb{F}_q})$. We can write $p(v) = \tilde{P}(1, v) = \tilde{P}(x, v) + r(x, v)(x - 1)$ for some $r(x, v) \in \mathbb{Z}[x, v]$. So if

$$a_0 = \tilde{P}(\iota^2, v)a \tag{4.7.10}$$
$$a_1 = r(\iota^2, v)(\iota^2 - 1)a, \tag{4.7.11}$$

then $a_0 + a_1 = p(v)a$, so it suffices to show that $a_0, a_1 \in V^{\mathrm{fp}}_{p(v)}(\mathcal{A}_{\mathbb{F}_q})$.

First we show this for a_0. We claim that for any $w \in W$ and $s \in S$, $\Phi_s^2((a_0)_w) = (a_0)_w$. Since $a_0 = \tilde{P}(\Phi^2_{w_0}, v)a$, this will follow from the fact that for any $s \in I$, $(\Phi_s^2 - 1)\tilde{P}(\Phi^2_{w_0}, v) = 0$ as an endomorphism of $K_0(G/U)$. By Proposition 4.7.2, this relation holds if and only if it holds after applying $\pi_{\mathcal{L}}$ for any character sheaf $\mathcal{L}$. By Lemma 4.7.3, this reduces to showing that if $\tilde{P}(\tilde{T}^2_{w_0}, v) \neq 0$, then $(\tilde{T}_s^2 - 1)\tilde{P}(\tilde{T}^2_{w_0}, v) = 0$ for any $s \in S$. We can rephrase this as saying that if the full twist acts by the eigenvalue 1, then so does $\tilde{T}_s^2$ for every $s \in S$. Indeed, this follows from the classification of irreducible representations of Hecke algebras, and it is shown directly in 11.5.3 of [48].

Now note that by Theorem 4.7.1, we have $(a_0)_{sw} - \Phi_s(a_0)_w = (\Phi_s^2 - 1)b$ for some b. Using the relation $(\Phi_s + v^2)(\Phi_s^2 - 1) = 0$ from Proposition 6.2.1 of [48] and applying Φ_s to both sides, we then get

$$(\Phi_s^2 - 1)((a_0)_{ws} - (a_0)_w) = (\Phi_s^2 - 1)^2 b \tag{4.7.12}$$
$$= (v^4 - 1)(\Phi_s^2 - 1)b, \tag{4.7.13}$$
$$= (v^4 - 1)((a_0)_{sw} - \Phi_s(a_0)_w), \tag{4.7.14}$$

so $(v^4 - 1)((a_0)_{sw} - \Phi_s(a_0)_w) = 0$, meaning $(a_0)_{sw} = \Phi_s(a_0)_w$. This means $a_0 = j_{w!}j_w^*(a_0)$ in $K_0(\mathcal{A}_{\mathbb{F}_q})$ for any $w \in W$, and so $a_0 \in V^{\mathrm{fp}}_{p(v)}$.

Now it only remains to observe that $a_1 \in V^{\mathrm{fp}}_{p(v)}$. By the definition of a_1, $\tilde{P}(\Phi^2_{w_0}, v)a_1 = 0$. This means we can then apply the exact same argument as in the proof of Corollary 4.6.13 to get that $a_1 \in V^{\mathrm{fp}}_{p(v)}$. Then since a_0 and a_1 both lie in $V^{\mathrm{fp}}_{p(v)}$ and $p(v)a = a_0 + a_1$, we have that $a \in V^{\mathrm{fp}}_{p(v)}$.

4.8 Construction of Kazhdan-Laumon representations

4.8.1 The Grothendieck-Lefschetz pairing

The original proposal in [32]

In Section 3 of [32], the proposed construction of representations is as follows. The authors begin by making Conjecture 1.1.1, which we now know to be false by Bezrukavnikov and Polishchuk's appendix to [48]. However, for objects of finite projective dimension, one can still define the Grothendieck-Lefschetz-type pairing in the manner they describe

First, they define a Verdier duality functor $\mathbb{D} : \mathcal{A}_\psi \to \mathcal{A}_{\psi^{-1}}$, where $\mathcal{A}_\psi = \mathcal{A}$ as we have been using it throughout this chapter, while $\mathcal{A}_{\psi^{-1}}$ is the same category but using the additive character ψ^{-1} instead of ψ (where ψ is the additive character we chose in Section 1.2).

They note that for any $A \in \mathcal{A}_{w,\mathbb{F}_q}$ and $B \in (\mathcal{A}_{\psi^{-1}})_{w,\mathbb{F}_q}$ and any $i \in \mathbb{Z}$, the isomorphisms $\psi_A : \mathcal{F}_w \mathrm{Fr}^* A \to A$ and $\psi_B : \mathcal{F}_w \mathrm{Fr}^* B \to B$ give an endomorphism $\psi^i_{A,B}$ of the vector space $\mathrm{Ext}^i_{\mathcal{A}}(A, \mathbb{D}B)$ given by the composition,

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{A}}(A, \mathbb{D}B) &\to \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{F}_w \mathrm{Fr}^* A, \mathcal{F}_w \mathrm{Fr}^* \mathbb{D}B) \\ &\to \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{F}_w A, \mathcal{F}_w \mathbb{D}B) \to \mathrm{Ext}^i_{\mathcal{A}}(A, \mathbb{D}B)\end{aligned}$$

where the first map arises from the morphisms ψ_A and ψ_B, the next from the canonical isomorphisms $\mathrm{Fr}^* A \to A$ and $\mathrm{Fr}^* B \to B$, and the last from the fact that $\mathcal{F}_w$ is invertible. This map is also described explicitly in 4.3.1 of [16].

We can then define, for A having finite projective dimension and arbitrary B, the value

$$\langle [A], [B] \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}(\psi^i_{A,B}, \mathrm{Ext}^i_{\mathcal{A}}(A, \mathbb{D}B)). \tag{4.8.1}$$

This is clearly well-defined at the level of Grothendieck groups. We will now explain how to use the result in Theorem 4.1.2 to extend Kazhdan and Laumon's pairing, which as of this point is only defined for objects of finite projective dimension, to the full Grothendieck group in the monodromic case.

It is a straightforward computation that for any such A and B, we have

$$\langle [A(-\tfrac{1}{2})], [B] \rangle = q^{\frac{1}{2}} \langle [A], [B] \rangle, \tag{4.8.2}$$

so the pairing is $\mathbb{Z}[v, v^{-1}]$-linear where $\mathbb{Z}[v, v^{-1}]$ acts on the target field such that v is multiplication by $q^{\frac{1}{2}}$.

A pairing on $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$

We can do the same construction on the monodromic Kazhdan-Laumon category $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$ and its Grothendieck group. Then using $\mathbb{Z}[v, v^{-1}]$-linearity, the above definition gives us a pairing

which is well-defined on elements of V^{fp}.

Now we note that the polynomial $p(v)$ evaluated at $v = q^{\frac{1}{2}}$ is nonzero, so we can extend this pairing linearly to the localization $V^{\mathrm{fp}}_{p(v)}$. This then gives that the pairing is well-defined on all of $V^{\mathrm{fp}}_{p(v)} \otimes \mathbb{C}$, where in the tensor product we send $v \mapsto q^{\frac{1}{2}}$. But by Theorem 4.1.2, $V^{\mathrm{fp}}_{p(v)} \otimes \mathbb{C}$ is all of $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$, so we can indeed define the pairing on this entire vector space; we will now explain how to use this to construct the Kazhdan-Laumon representations.

4.8.2 Construction of representations

As Kazhdan and Laumon explain in [32], the category $\mathcal{A}_{w,\mathbb{F}_q}$ is defined so that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$, where $T(w)$ is the (usually non-split) torus of G corresponding to $w \in W$, defined by

$$T(w) = \{t \in T(\overline{\mathbb{F}_q}) \mid \mathrm{Fr}^*(t) = w(t)\}. \tag{4.8.3}$$

We note that $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ then also carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$ where $T(w)$ acts by its character θ which corresponds to the data of the character sheaf $\mathcal{L}$.

As we explained, the pairing $\langle , \rangle$ is well-defined on $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$, and so we can define $K^{\mathcal{L}}_w$ to be its kernel. Then the Kazhdan-Laumon representation corresponding to the pairing $(T(w), \theta)$ which was originally sought in [32] is the vector space $V_{w,\mathcal{L}} = (K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C})/K^{\mathcal{L}}_w$.

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