The algebraic K-theory of the chromatic filtration and the telescope conjecture

by

Ishan Levy

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Authored by:	Ishan Levy Department of Mathematics April 23, 2024
Certified by:	Michael Hopkins Professor of Mathematics, Thesis Supervisor
Accepted by:	Davesh Maulik Professor of Mathematics

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ABSTRACT

We develop tools for understanding the algebraic K-theory of categories such as those coming from the chromatic filtration of the stable homotopy category, and apply these tools to improve our understanding of the large scale structure of stable homotopy theory and understand Ravenel's telescope conjecture.

More specifically, in joint work with Burklund, we prove a general devissage result which in particular identifies the algebraic K-theory of certain coconnective ring spectra satisfying suitable regularity and flatness hypotheses with the K-theory of their π_0. Using this and an extension of the Dundas–Goodwillie–McCarthy theorem to -1-connective ring spectra, we obtain a formula for the algebraic K-theory of the $K(1)$-local sphere in terms of topological cyclic homology of a ring spectrum j_ζ, and in particular find that its algebraic K-groups are not all finitely generated. In joint work with Lee, we extend these computations to understand the algebraic K-theory of the $K(1)$-local sphere in the stable range using THH, where we observe phenomena such as the failure of $\mathbb{Z}_p$ Galois descent for THH for an extension of j_ζ. In joint work with Burklund, Hahn, and Schlank, we show that the failure of $\mathbb{Z}_p$-descent also happens for the $T(2)$-local TC of this extension. Combining this with the cyclotomic redshift result of Ben-Moshe–Carmeli–Schlank–Yanovski, this implies that the $T(2)$-local algebraic K-theory of the $K(1)$-local sphere is not $K(2)$-local, and hence a counterexample to the height 2 telescope conjecture. We also give similar counterexamples to the height n telescope conjecture for all $n \geq 2$ and all primes, and show that $\mathbb{Z}_p$ Galois hyperdescent for chromatically localized algebraic K-theory generically fails.

Thesis supervisor: Michael Hopkins
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

1.1 Stable and chromatic homotopy theory

One of the main goals of stable homotopy theory is to understand the stable homotopy category, Sp, also known as the (∞-)[1]category of spectra. The category of spectra can be viewed as the initial place where linear algebraic constructions in mathematics live, and as such, it plays a foundational role in many areas such as geometric topology, algebraic K-theory, quantum field theory, and Floer theory.

Many classification problems in geometric topology can be described in terms of spectra: for instance cobordism classes of stably framed k-manifolds are in bijection with $\pi_k\mathbb{S}$, the kth homotopy group of the sphere spectrum. By the Freudenthal suspension theorem, this agrees with the homotopy group $\pi_{n+k}S^n$ for $n \geq k+2$. A general spectrum is built out of the sphere spectrum, and so much of the complexity of stable homotopy theory can be viewed as arising from the fact that understanding $\pi_*\mathbb{S}$ is a complicated question.

The stable homotopy groups $\pi_*\mathbb{S}$ are zero for $* < 0$, and are $\mathbb{Z}$ for $* = 0$. Serre proved that $\pi_*\mathbb{S}$ is finite for $* > 0$, but even today the state of the art is an almost complete computation of these groups up to $* < 90$ [IWX23]. There is a lot of structure on the stable homotopy groups of spheres: for example they form a graded commutative ring. However, this ring structure is poorly behaved: by a theorem of Nishida [Nis73], every element of $\pi_i\mathbb{S}$ for $i > 0$ is nilpotent.

Despite the fact that the ring $\pi_*\mathbb{S}$ is $\mathbb{Z}$ modulo its nil-radical, it is not true that the stable homotopy category behaves like the derived category of the integers, up to nilpotents. This was first observed by Adams, who found for $p > 2$ a map $v_1 : \Sigma^{2p-2}\mathbb{S}/p \to \mathbb{S}/p$ such that no power of this map is null, because it induces an isomorphism on topological K-theory.

Because there is a long exact sequence on homotopy groups coming from the cofiber sequence $\mathbb{S} \xrightarrow{p} \mathbb{S} \to \mathbb{S}/p$, given a map $x \in \pi_*\mathbb{S}/p$ that is not v_1-torsion, we obtain a v_1-periodic family of elements in $\pi_{*-1+(2p-2)i}\mathbb{S}$ by applying the boundary map in the long exact sequence to $v_1^i x$. In particular, applying this for $x = 1$, we obtain a family of elements in

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[1] From now on, we omit the '∞-' prefix in our notation

$\pi_{-1+(2p-2)i}\mathbb{S}$ called α_i. These classes account for the p-torsion in the image of the stable J-homomorphism, which is a summand of the stable homotopy groups closely related to Bott periodicity.

In fact, v_1 is not the only map that can be used to generate infinite periodic families of elements in the stable homotopy groups of spheres, as we now explain. From now on, we work p-locally, for a fixed prime p. For each $n \geq 0$, there are associative ring spectra $K(n)$, called Morava K-theories, whose homotopy rings for $n \geq 1$ are $\mathbb{F}_p[v_n^{\pm 1}]$, where v_n is a class in degree $2p^n - 2$. For $n = 0$, we set $K(0)$ to be the Eilenberg–Mac Lane spectrum $\mathbb{Q}$, and set $v_0 = p$ by convention. We say that a finite spectrum X is type $n \in \mathbb{N}$ if $K(i) \otimes X = 0$ for $i < n$ and $K(i) \otimes X \neq 0$ for $i \geq n$. Every nonzero finite spectrum has some type n.

Hopkins and Smith [HS98] showed that given a finite spectrum X of type n, there exists a map $v : \Sigma^{|v|}X \to X$ that induces multiplication by a positive power of v_n after tensoring with $K(n)$, so in particular are non-nilpotent. These maps, which they called v_n-self maps, induce periodicity operators on $\pi_*\mathbb{S}$, similarly to how Adams' self map did. They moreover showed that these v_n-self maps account for all non-nilpotent central self maps of finite spectra up to taking powers of self maps.

One can moreover inductively construct finite spectra of each type n as follows: the sphere spectrum $\mathbb{S}$ is type 0. Given a finite spectrum of type $n-1$, X, we can choose a v_{n-1}-self map $v : \Sigma^{|v|}X \to X$. Then the cofiber of v is a finite spectrum of type n.

Given a type n spectrum X, one can form its telescope $X[v^{-1}]$ by taking the colimit along the self map. This telescope is sometimes denoted $T(n)$, and the map $X \to X[v^{-1}]$ detects the v_n-periodic part of the homotopy groups of X.

Chromatic homotopy theory is the idea that one can study the stable homotopy category by breaking it up into pieces that are v_n-periodic, and then reconstructing it from these pieces. There are two possible meanings of v_n-periodic pieces of the category of spectra, namely the telescopic or $T(n)$-localization $L_{T(n)} : \mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ and the $K(n)$-localization, $L_{K(n)} : \mathrm{Sp} \to \mathrm{Sp}_{K(n)}$. These localizations of spectra are usually constructed using the formalism of Bousfield localization, and can be viewed as subcategories of Sp via the right adjoint of the localization functor, which is fully faithful. The telescopic localization $\mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ captures the v_n-periodic part of homotopy groups: in particular it sends X to $X[v^{-1}]$ for any X of type n.

The $K(n)$-localization functor is less transparent from the point of view of how it changes the homotopy groups of a type n complex, but the homotopy groups of a $K(n)$-local spectrum X are much more computable a priori than that of a $T(n)$-local spectrum, because of a homotopy fixed point spectral sequence with E_2-page $H^*(\mathbb{G}_n; (E_n)^\wedge_* X)$ converging to $\pi_* X$. Here $(E_n)^\wedge_*(X)$ is the completed Morava E-homology of X, an invariant that is often computable, and $H^*(\mathbb{G}_n, -)$ means the continuous group cohomology, with respect to the Morava stabilizer group $\mathbb{G}_n$, which is the profinite completion of the unit group of the division algebra over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$, which acts on completed Morava E-homology.

The $K(n)$-localization, while more understandable from the point of view of computing homotopy groups, doesn't obviously detect all of the v_n-periodic families in homotopy groups of spectra. Indeed, it follows easily definition that the $K(n)$-local category is a further localization of the $T(n)$-local category, so it detects at most as much information as the $T(n)$-local category. Ravenel's telescope conjecture [Rav84] was the optimistic assertion that the $K(n)$-local and $T(n)$-local categories agree.

The $K(n)$-localizations and $T(n)$-localizations glue together as n varies. There are localizations of the category of spectra referred to as L_n and L_n^f-localizations, which fit into pullback squares

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & L_{n-1} X \arrow[d] & L_n^f X \arrow[r] \arrow[d] & L_{n-1}^f X \arrow[d] \\
L_{K(n)} X \arrow[r] & L_{n-1} L_{K(n)} X & L_{T(n)} X \arrow[r] & L_{n-1}^f L_{K(n)} X
\end{tikzcd}$$

The L_n localization glues together the information about the $K(i)$-localizations for $i \leq n$[2], and L_n^f does similarly for the $T(n)$-localizations. The chromatic convergence theorem of Hopkins and Ravenel says that $\mathbb{S} \cong \lim_n L_n \mathbb{S}$, and implies that $\mathbb{S}$ is a retract of $\lim_n L_n^f \mathbb{S}$, so that all information about the stable homotopy groups of spheres can in principal be recovered from these localizations.

1.2 Results

The goal of this work is to use algebraic K-theory as a tool to understand structural information about the chromatic filtration on the stable homotopy category. Higher algebraic K-theory is an invariant of categories defined by Quillen [Qui73], and here we use its modern incarnation, developed by many people, notably Blumberg–Gepner–Tabuada [BGT13], who define it in the setting of small stable categories.

Let $\mathrm{Cat}^{\mathrm{perf}}$ be the category of small idempotent-complete stable categories. A typical example of such a category is the category $\mathrm{Mod}(R)^\omega$ of perfect modules over an associative ring spectrum R, i.e those R-modules that are generated by R under finite colimits and limits, and retracts. Algebraic K-theory is a functor $K : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ equipped with a natural transformation

$$[-]_{\mathcal{C}} : \mathcal{C}^{\cong} \implies \Omega^\infty K(\mathcal{C})$$

where $\Omega^\infty K(\mathcal{C})$ is the underlying space of the spectrum $K(\mathcal{C})$, and $\mathcal{C}^{\cong}$ is the space of objects of $\mathcal{C}$. It is the initial such functor such that it is *localizing*, i.e for a sequence of categories in $\mathrm{Cat}^{\mathrm{perf}}$

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

where F is fully faithful and $\mathcal{D}/\mathcal{C}$ is the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$, K sends this to a cofiber sequence of spectra.

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[2] Here $K(0)$-localization must be interpreted as rationalization.

For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, $\pi_0 K(\mathcal{C})$ is the abelian group that agrees with Grothendieck's K_0, in that the natural transformation $[-]_{\mathcal{C}}$ on π_0 is the universal Euler characteristic sending an object X in $\mathcal{C}$ to its K-theory class $[X]$. For an associative ring spectrum R, The notation $K(R)$ is used to refer to $K(\mathrm{Mod}(R)^\omega)$.

In this thesis, we study $K(R)$ and $K(\mathcal{C})$ for $R, \mathcal{C}$ chromatically interesting ring spectra and categories. Our main examples of interest are algebraic K-theory of $\mathrm{Sp}^\omega_{\geq n}$ the category of type at least n finite spectra, and $\mathrm{Sp}^\omega_{T(n)}, \mathrm{Sp}^\omega_{K(n)}$, the categories of compact $K(n)$-local and $T(n)$-local spectra. The K-theory of such categories was first considered in the work of Waldhausen [Wal84], who made fundamental observations about cofiber sequences relating $K(\mathrm{Sp}^\omega_{\geq n})$ and $K(\mathrm{Sp}_{T(n)})$.

Studying such algebraic K-theory spectra allows us to learn about the chromatic filtration in two ways. One one hand, algebraic K-theory of a category captures structural information about a category, so we learn structural information about the chromatic filtration by studying its algebraic K-theory. This is the subject of Chapter 3, which computes the algebraic K-theory of $\mathrm{Sp}^\omega_{K(1)}$ and $\mathrm{Sp}^\omega_{\geq 2}$ in low degrees, and interprets the K-theory classes in terms of explicit spectra and automorphisms. On the other hand, K-theory produces spectra which are often themselves very interesting, and can be used to detect information about the stable homotopy category. This is the subject of Chapter 5, which uses the algebraic K-theory of certain ring spectra $\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}$ to detect the failure of the telescope conjecture at heights $n \geq 1$.

The main difficulty in studying the algebraic K-theory of categories that are relevant to chromatic homotopy theory is that they are usually not the module categories of connective ring spectra. For example, Morita theory implies that $\mathrm{Sp}^\omega_{\geq n}$ equivalent to the category $\mathrm{Mod}(\mathrm{End}(X))^\omega$ of perfect modules over the endomorphism ring of a type n spectrum X, but such endomorphism rings can never be connective unless $n = 0$. In the setting of connective rings, trace methods often allow one to understand algebraic K-theory in terms of that of classical rings, and in terms of topological cyclic homology, or TC.

In Chapter 2, which is joint with Robert Burklund, we explain how it is often possible to understand the K-theory of *coconnective* ring spectra, under suitable regularity and flatness hypotheses, that can be checked at the level of homotopy groups. More generally, we prove a version of Quillen's devissage theorem in the setting of stable ∞-categories, that shows that K-theory is invariant under 'unipotent' extensions of categories with bounded t-structure.

In Chapter 3, we show that the use of trace methods extends beyond the realm of connective rings to -1-connective rings. This allows for the computation of the algebraic K-theory of -1-connective ring spectra in terms of TC and the K-theory of discrete rings.

We use this to obtain a formula for the K-theory of the $K(1)$-local category, and more generally for $\mathbb{Z}$-fixed points of certain connective ring spectra. We find that the TC of a commutative ring spectrum j_ζ, whose underlying spectrum is the -1-connective cover of the $K(1)$-local sphere, is the main term controlling the K-theory of the $K(1)$-local category.

We use this formula to understand the homotopy groups of $K(\mathrm{Sp}_{K(1)})$ and $K(\mathrm{Sp}_{\geq 2})$ in low degrees. Surprisingly, we find that for $p = 2$, $K_0(\mathrm{Sp}_{\geq 2}) \cong \mathbb{Z} \oplus \bigoplus_0^\infty \mathbb{Z}/2\mathbb{Z}$. We also give explicit descriptions of type 2 spectra representing these classes in K_0.

In Chapter 4, which is joint with David Lee, we study the THH of j_ζ and its variants. This can be viewed as partial progress in the direction of studying $\mathrm{TC}(j_\zeta)$ the most interesting part of the formula for the K-theory of the $K(1)$-local sphere. In our study, we find two new phenomena. j_ζ is produced as the $\mathbb{Z}$-fixed points of Adams operations acting on a certain connective ring spectrum (either ko or ℓ), and we find that at the level of THH modulo (p, v_1), the THH behaves as if this action was trivial. The other phenomenon we observe is that the functor THH doesn't commute with taking the $\mathbb{Z}$-fixed points, and the failure corresponds in a precise way to the failure of $B\mathbb{Z}_p$ to be its own free loop space.

In Chapter 5, which is joint with Robert Burklund, Jeremy Hahn, and Tomer Schlank, we find that the $T(2)$-local TC of j_ζ is not $K(2)$-local. The key ingredient is to prove a version of the Lichtenbaum–Quillen conjecture for the TC of certain variants of j_ζ corresponding to large finite Galois extensions of the $K(1)$-local sphere. This Lichtenbaum–Quillen conjecture is then used to show that the $T(2)$-local TC 'asymptotically' behaves as if the action was trivial. This $T(2)$-local TC doesn't commute with taking the $\mathbb{Z}$-fixed points in the case of a trivial action, and by $K(2)$-local descent results of Ben-Moshe–Carmeli–Schlank–Yanovski, this is enough to show that it is not $K(2)$-local.

We also produce ring spectra $\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}$ at higher heights, and use their algebraic K-theory to exhibit counterexample to the height $n+1$ telescope conjecture using similar methods.

In Chapter 6, we describe the results of three forthcoming works related to the topics of this thesis. In joint work with Robert Burklund, we prove that the algebraic K-theory of $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$ agree after inverting the prime p, and completely compute these algebraic K-theory spectra after inverting p. We more generally show that this is true for any filtered colimit preserving additive invariant replacing K-theory, which we dub the K-theoretic telescope conjecture away from p.

In joint work with Vova Sosnilo, we generalize the results of Chapter 3, and exhibit techniques to describe the algebraic K-theory of a much larger class of categories in terms of topological cyclic homology and the K-theory of discrete rings. In particular, our results are applicable to understand the algebraic K-theory of many algebraic stacks, and to understanding the algebraic K-theory of many ring spectra that are bounded below. In another forthcoming work, I use these techniques to obtain formulas for the integral algebraic K-theory spectra of the $K(n)$ and L_n-local categories in terms of topological cyclic homology. The key idea is to use the results of Chapter 2 to reduce studying the K-theory of such categories to the K-theory of certain synthetic deformations of these categories. The K-theory of these deformations can then be studied using trace methods since they fit into the framework above.

Finally, we note to the reader that Chapter 2 has been published in Selecta Math., and Chapter 3, Chapter 4, Chapter 5 have appeared on arXiv, but are not yet published. We also expect that the results described in Chapter 6 will appear and be published elsewhere.

Chapter 2

On the K-theory of regular coconnective rings (with Robert Burklund)

We show that for a coconnective ring spectrum satisfying regularity and flatness assumptions, its algebraic K-theory agrees with that of its π_0. We prove this as a consequence of a more general devissage result for stable infinity categories. Applications of our result include giving general conditions under which K-theory preserves pushouts, generalizations of $\mathbb{A}^n$-invariance of K-theory, and an understanding of the K-theory of categories of unipotent local systems.

2.1 Introduction

In this paper we examine the relationship between coconnectivity, regularity, and algebraic K-theory. As a consequence of our investigation we prove that the K-theory of a large collection of coconnective rings agrees with that of their π_0.

Theorem 2.1.1. *Given a coconnective $\mathbb{E}_1$-algebra R such that*

1. *$\pi_0 R$ is left regular coherent and*
2. *$\tau_{\leq -1} R$ has Tor amplitude in $[-\infty, -1]$ as a right $\pi_0 R$-module,*

the natural map in connective K-theory

$$K(\pi_0 R) \to K(R)$$

is an equivalence and both $\pi_0 R$ and R have vanishing K_{-1}.

Although not immediately clear, Theorem 2.1.1 is a devissage theorem. The core step in the proof is an application of Quillen's devissage theorem [Qui73, Theorem 4] and condition (1) is exactly what is needed for the canonical t-structure on $\pi_0 R$-modules to restrict to a bounded t-structure on perfect $\pi_0 R$-modules with heart finitely presented, discrete $\pi_0 R$-modules. The essential novelty in Theorem 2.1.1 comes from condition (2) as a simple condition, easily checked in practice[1], which guarantees a K-equivalence. As a demonstration we work through the prototypical example of devissage.

Example 2.1.2. From the localization sequence

$$\mathrm{Perf}(\mathbb{Z})^{p-\mathrm{nil}} \hookrightarrow \mathrm{Perf}(\mathbb{Z}) \twoheadrightarrow \mathrm{Perf}(\mathbb{Z}[1/p])$$

we obtain a cofiber sequence on non-connective K-theory

$$K^{\mathrm{nc}}\left(\mathrm{Perf}(\mathbb{Z})^{p-\mathrm{nil}}\right) \to K^{\mathrm{nc}}(\mathrm{Perf}(\mathbb{Z})) \to K^{\mathrm{nc}}(\mathrm{Perf}(\mathbb{Z}[1/p])).$$

Identifying $\mathbb{F}_p$ as a generator of $\mathrm{Perf}(\mathbb{Z})^{p-\mathrm{nil}}$ we have an identification

$$\mathrm{Perf}(\mathbb{Z})^{p-\mathrm{nil}} \cong \mathrm{Perf}(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p)).$$

Devissage can then be phrased as the assertion that $K(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p)) \cong K(\mathbb{F}_p)$ and K_{-1} vanishes, from which we obtain a cofiber sequence on connective algebraic K-theory

$$K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Z}[1/p]).$$

In order to prove this using Theorem 2.1.1 we compute the homotopy groups of $\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p)$, which are

$$\pi_s \,\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & s = 0, -1 \\ 0 & \text{otherwise} \end{cases}$$

and observe that conditions (1) and (2) are satisfied. ◁ ◁

Proving this localization sequence using Quillen's devissage theorem directly involves checking that p-torsion perfect $\mathbb{Z}$-modules admits a bounded t-structure and that every p-torsion abelian group has a filtration whose associated graded consists of $\mathbb{F}_p$-modules. For a more general $\mathbb{E}_1$-algebra R constructing t-structures and filtrations in order to apply devissage involves contemplating the behavior of a generic perfect R-module, which can become unwieldy. By contrast, the Tor amplitude condition in Theorem 2.1.1 lives at the level of the homotopy groups of R and is therefore quite concrete. This change represents a considerably gain in practical usability.

Before moving on, let us point out that this example also highlights another key feature of devissage: namely, while localization sequences occur at the level of noncommutative motives, devissage does not[2]. It is rather a property K-theory satisfies in addition to being localizing. Similarly, Theorem 2.1.1 does not hold in general for localizing (or additive) invariants.

We prove Theorem 2.1.1 as a corollary of our main result, Theorem 2.1.3, which is a more general devissage result taking place at the level of stable categories.

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[1] Condition (2) is satisfied if $\pi_{-i} R$ has Tor dimension $< i$ as a right $\pi_0 R$-module.

[2] For example $\mathrm{THH}(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p))$ is the fiber of the map $\mathrm{THH}(\mathbb{Z}) \to \mathrm{THH}(\mathbb{Z}[1/p])$, which does not agree with $\mathrm{THH}(\mathbb{F}_p)$.

Theorem 2.1.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between small, stable, idempotent complete categories and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded t-structure on $\mathcal{C}$. If we assume that*

(A) the image of F generates $\mathcal{D}$ under finite colimits and retracts and

(B) F is fully faithful when restricted to $\mathcal{C}^\heartsuit$

then there exists a unique bounded t-structure on $\mathcal{D}$ for which F is t-exact. Moreover, the induced maps on connective K-theory

$$\begin{tikzcd} K(\mathcal{C}^\heartsuit) \arrow[r] \arrow[d] & K(\mathcal{D}^\heartsuit) \arrow[d] \\ K(\mathcal{C}) \arrow[r] & K(\mathcal{D}) \end{tikzcd}$$

are all equivalences and K_{-1} of each term vanishes.

Theorem 2.1.3 is proved in Section 2.2 and the key step is the construction of the t-structure on $\mathcal{D}$. This is the most technical point in the proof and it uses all of the conditions of the theorem in an essential way. In fact, as a byproduct of this argument we obtain relatively fine-grained control over the abelian category $\mathcal{D}^\heartsuit$. Specifically, we find that the inclusion $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{D}^\heartsuit$ satisfies the hypotheses of Quillen's devissage theorem [Qui73, Theorem 4] from which we obtain the K-equivalence between $\mathcal{C}^\heartsuit$ and $\mathcal{D}^\heartsuit$. The proof ends with applying Barwick's theorem of the heart [Bar15] to identify the K-theories of $\mathcal{C}$ and $\mathcal{D}$ with that of their heart and the main results of [Sch06] and [AGH19] to obtain the vanishing of K_{-1}.

Examining the relation between $\mathcal{C}^\heartsuit$ and $\mathcal{D}^\heartsuit$ we see that conditions (A) and (B) together can be thought of as asking that $\mathcal{D}$ behave like a category of unipotent local systems with coefficients in $\mathcal{C}$. Indeed, (B) is analogous to the fact that maps between trivial representations can be computed on underlying and (A) is analogous to the fact that unipotent representations are generated from trivial representations under extensions. For this reason we say that a map is *unipotent* if it satisfies these conditions. With this reformulation we can now introduce the key slogan of this paper:

Devissage is the invariance of K-theory under unipotent maps.

In Section 2.3 we deduce Theorem 2.1.1 from Theorem 2.1.3 and discuss several points which are complementary to Theorem 2.1.3. A subtlety worth noting here is that up to this point we have been working entirely with *connective* K-theory as this is the setting where Barwick's theorem of the heart and Quillen's devissage are applicable. In fact, assuming $\mathcal{C}^\heartsuit$ is Noetherian, we show in Lemma 2.3.12 that $\mathcal{D}^\heartsuit$ is Noetherian as well. This lets us extend Theorem 2.1.3 to non-connective K-theory in the Notherian setting for the simple reason that negative K-group vanish in the Noetherian setting [AGH19]. One might wonder whether Theorem 2.1.3 holds for negative K-groups in the non-Noetherian settting and since we are not aware of any counter-example we are led to ask[3]:

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[3] See also [AGH19, Conjecture C], which is part of Question 2.1.4.

Question 2.1.4. Do the theorem of the heart and devissage hold for negative K-theory?

In Sections 2.4 and 2.5 we turn to applications of our main theorem. Combining our work with the work of Land and Tamme on the K-theory of pullbacks and pushouts [LT19; LT23] we provide general conditions under which K-theory preserves pushouts.

Theorem 2.1.5. *Suppose $C \xleftarrow{g} A \xrightarrow{f} B$ is a span of discrete rings where A is left regular coherent and both f and g are right faithfully flat. Then connective K-theory preserves the pushout of this span.*

Theorem 2.1.5 allows us to generalize Waldhausen's theorems on the K-theory of generalized free products [Wal78a] to the setting of non-discrete rings. Using similar techniques, we then obtain an $\mathbb{A}^n$-invariance result for K-theory.

Theorem 2.1.6 ($\mathbb{A}^n$-invariance of algebraic K-theory)**.** *Let $\mathcal{C}$ be a small, stable, idempotent complete category equipped with a bounded t-structure. Then $K_i(\mathcal{C}) \cong K_i(\mathcal{C}[x_1, \dots, x_n])$ for $i \geq n-1$.*

In the $n = 1$ case, for a regular, Noetherian ring this recovers Quillen's fundamental theorem of algebraic K-theory [Qui73, Theorem 8]. An alternative proof extending the result to regular coherent rings was given by Waldhausen, again in the $n = 1$ case [Wal78a]. If $\mathcal{C}^\heartsuit$ is in addition Noetherian, then through a combination of [AGH19, Proposition 3.14], Barwick's theorem of the heart and the extension of Quillen's argument to abelian categories (see [MS13]) we in fact know that $K^{\mathrm{nc}}(\mathcal{C}) \cong K^{\mathrm{nc}}(\mathcal{C}[x_1, \dots, x_n])$ and that the negative K-groups vanish. The case of non-Noetherian regular coherent rings and $n > 1$ is more difficult, because $R[x]$ may not even be coherent, so one cannot induct on n in the obvious way. Nevertheless, the $n > 1$ case was already known when R is a discrete ring as a corollary of the Farrell-Jones conjecture for the groups $\mathbb{Z}^n$ and the $n = 1$ case[4] (see for example [Dav08, Corollary 2]). The degree bounds in the above theorem ultimately come from our use of connective K-theory and a positive answer to Question 2.1.4 would allow us to remove these restrictions.

In Section 2.5.2 we examine the category of unipotent local systems on a connected space X with coefficients in a category $\mathcal{C}$ with a bounded t-structure. As one might expect, we find that the K-theory of unipotent local systems agrees with the K-theory of $\mathcal{C}$, generalizing [AGH19, Theorem 4.8]. In the final pair of subsections we work through a collection of examples which demonstrate that the conditions in Theorem 2.1.1 cannot be weakened.

Notations and Conventions

In order to preserve the brevity of this paper we assume the reader is generally familiar with higher algebra and algebraic K-theory. We also make use of the following notations and conventions throughout.

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[4] While the Farrell-Jones conjecture is about nonconnective K-theory, $\mathbb{A}^1$-invariance is only known for connective K-theory, and for this reason this alternative approach arrives at the same degree bound, $i \geq n-1$, for $\mathbb{A}^n$-invariance.

- The term category will refer to an ∞-category as developed by Joyal and Lurie.
- $\mathrm{Map}(a,b)$ will denote the space of maps from a to b (in some ambient category).
- In a stable category $\mathrm{map}(a,b)$ will denotes the mapping spectrum between a and b.
- For an $\mathbb{E}_1$-algebra R, $\mathrm{Mod}(R)$ will refer to its category of left modules and $\mathrm{Perf}(R)$ will refer to the category of perfect left R-modules, i.e. the compact objects in $\mathrm{Mod}(R)$.
- We use $\mathcal{C}, \mathcal{D}$ to denote small, idempotent complete stable categories, and use $\mathrm{Cat}^\sharp$ to denote the category of such categories and exact functors.
- Given an exact functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}^\sharp$, $F^* : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ denotes $\mathrm{Ind}(F)$, and $F_* : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$ denotes the right adjoint of F^*.
- We use $\mathcal{U}_{\mathrm{loc}}$ for the noncommutative motive (or just nc motive for short) functor of Blumberg–Gepner–Tabuada [BGT13].
- We use $K(-)$ for connective K-theory and $K^{\mathrm{nc}}(-)$ for non-connective K-theory.
- We use x_n for a polynomial generator in degree n and ϵ_n for an exterior generator in degree n. As an example, $\mathbb{S}[x_n]$ is the free $\mathbb{E}_1$-algebra on a class in degree n.
- We use $\mathcal{C}[\epsilon_n]$ as notation for $\mathcal{C} \otimes \mathrm{Perf}(\mathbb{S}[\epsilon_n])$ and similarly for polynomial generators.

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2.2 The Main Theorem

In this section we prove our main theorem, Theorem 2.1.3, whose statement we reproduce below.

Theorem 2.2.1 (Theorem 2.1.3)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between small, stable, idempotent complete categories and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded t-structure on $\mathcal{C}$. If we assume that*

(A) the image of F generates $\mathcal{D}$ under finite colimits and retracts and

(B) F is fully faithful when restricted to $\mathcal{C}^\heartsuit$

then there exists a unique bounded t-structure on $\mathcal{D}$ for which F is t-exact. Moreover, the induced maps on connective K-theory

$$\begin{tikzcd}
K(\mathcal{C}^\heartsuit) \arrow[r] \arrow[d] & K(\mathcal{D}^\heartsuit) \arrow[d] \\
K(\mathcal{C}) \arrow[r] & K(\mathcal{D})
\end{tikzcd}$$

are all equivalences and K_{-1} of each term vanishes.

Before proceeding, we give a sketch of the strategy we follow in proving this theorem. The final step is applying Barwick's theorem of the heart and Quillen's devissage theorem to produce K-theory equivalences, along with the vanishing results on K_{-1} due to Schlichting and Antieau–Gepner–Heller. In order to apply these results we need to produce a bounded t-structure on $\mathcal{D}$ which is relatively well behaved. The key idea is that after passing to categories of ind-objects it is in fact quite easy to produce such a t-structure. Condition (B) is then rigged so that we have the control necessary to restrict this t-structure to compact objects in $\mathrm{Ind}(\mathcal{D})$ (i.e. $\mathcal{D}$). For the remainder of this section the notation from the statement of Theorem 2.1.3 will remain in place and we assume F satisfies conditions (A) and (B).

Passing to ind-completions gives us an induced commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook] & \mathcal{D} \arrow[d, hook] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "F^*"] & \mathrm{Ind}(\mathcal{D})
\end{tikzcd}$$

where F^* is a left adjoint and the vertical arrows are each the inclusion of the full subcategory of compact objects[5].

As promised, we begin by producing a t-structure on the level of ind-objects. This is rather easy since the category of ind-objects is presentable.

Lemma 2.2.2 ([Lur17, Proposition 1.4.4.11])**.** *Let $\mathcal{A}$ be a presentable, stable category. If $\{X_\alpha\}$ is a small collection of objects in $\mathcal{A}$, then there is an accessible*[6] *t-structure, $(\mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$, on $\mathcal{A}$ such that $\mathcal{A}_{\geq 0}$ is the smallest full subcategory of $\mathcal{A}$ containing each X_α and closed under colimits and extensions. The full subcategory of coconnective objects is characterized by the condition $Y \in \mathcal{A}_{\leq 0}$ if and only if $\mathrm{Map}(\Sigma X_\alpha, Y) = 0$ for each X_α.*

We equip $\mathrm{Ind}(\mathcal{C})$ with the t-structure whose connective part is generated by $\mathcal{C}_{\geq 0}$ and we equip $\mathrm{Ind}(\mathcal{D})$ with the t-structure whose connective part is generated by $F(\mathcal{C}_{\geq 0})$. Applying Ind to the split localization sequence

$$\mathcal{C}_{\geq 0} \to \mathcal{C} \to \mathcal{C}_{<0}$$

---

[5] We will suppress any further mention of these inclusions.

[6] A t-structure on a presentable category is accessible if the full subcategory of connective objects is accessible.

we can then read off that $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \subseteq \mathrm{Ind}(\mathcal{C})_{\geq 0}$ and $\mathrm{Ind}(\mathcal{C}_{<0}) \subseteq \mathrm{Ind}(\mathcal{C})_{<0}$which in turn implies that $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \cong \mathrm{Ind}(\mathcal{C})_{\geq 0}$ and $\mathrm{Ind}(\mathcal{C}_{<0}) \cong \mathrm{Ind}(\mathcal{C})_{<0}$.

Lemma 2.2.3. *F^* is t-exact.*

Proof. F^* sends connective objects to connective objects by construction. To show that F^* preserves coconnectivity we need to check that for every $c \in \mathcal{C}_{\geq 1}$ and $x \in \mathrm{Ind}(\mathcal{C})_{\leq 0}$ the mapping space $\mathrm{Map}(F^*(c), F^*(x))$ is contractible. Since the t-structure on $\mathrm{Ind}(\mathcal{C})$ restricts to compact objects we can write x as a filtered colimit of compact, coconnective objects. This implies (since F^* is a left adjoint) that it suffices to prove $\mathrm{Map}(F^*(c), F^*(x)) = 0$ when x is compact. Via the boundedness of the t-structure on $\mathcal{C}$ this follows from condition (B). $\square$

At this point we are now ready to prove that the t-structure on $\mathrm{Ind}(\mathcal{D})$ restricts to a bounded t-structure on $\mathcal{D}$. The main idea in proving this is that on the one hand, (A) guarantees that every object in $\mathcal{D}$ is only finitely many steps away from being in the image of F, while on the other hand, the t-structure on $\mathcal{C}$ can be used to produce a rich collection of compact objects in $\mathrm{Ind}(\mathcal{D})^{\heartsuit}$.

Proposition 2.2.4. *The t-structure on $\mathrm{Ind}(\mathcal{D})$ restricts to a bounded t-structure on $\mathcal{D}$. Each $d \in \mathcal{D}^{\heartsuit}$ has a finite filtration with associated graded in the image of $F|_{\mathcal{C}^{\heartsuit}} : \mathcal{C}^{\heartsuit} \to \mathcal{D}^{\heartsuit}$.*

Proof. In order to prove the proposition it will suffice to show that every object $d \in \mathcal{D}$ satisfies the following condition:

($*$) d is t-bounded and each $\pi_i^{\heartsuit}(d)$ has a finite filtration whose associated graded lies in $\mathcal{C}^{\heartsuit} \subseteq \mathrm{Ind}(\mathcal{D})^{\heartsuit} \cap \mathcal{D}$.

Note that if d satisfies ($*$), then the finite filtrations on the homotopy groups implies that each $\pi_i^{\heartsuit}(d)$ lies in $\mathcal{D} \subseteq \mathrm{Ind}(\mathcal{D})$ and similarly the t-boundedness implies $d \in \mathcal{D}$.

Since F^* is t-exact $F(c)$ satisfies ($*$) for each $c \in \mathcal{C}^{\heartsuit}$. Using hypothesis (A) it will now suffice to show that the full subcategory of objects of $\mathcal{D}$ satisfying ($*$) is thick. As the condition ($*$) is stated entirely in terms of homotopy groups, it will suffice to show that the corresponding condition ($**$) on the level of the heart cuts out a subcategory closed under kernels, cokernels and extensions[7].

($**$) $d \in \mathrm{Ind}(\mathcal{D})^{\heartsuit}$ has a finite filtration whose associated graded lies in $\mathcal{C}^{\heartsuit}$.

Given two objects $A, B \in \mathcal{D}$, each with a finite filtration in $\mathcal{D}$ and a map $r : A \to B$, we can paste the filtrations to form a filtration on the cofiber $\mathrm{cof}\, r$ as follows. Let us view our filtration on A (and similarly on B) as a functor $\mathrm{Fil}_* A : \mathbb{Z} \to \mathcal{D}$ so that $\mathrm{Fil}_i A$ denotes the i^{th} filtered pieces of A, $\mathrm{gr}_i A = \mathrm{cof}(\mathrm{Fil}_{i-1} A \to \mathrm{Fil}_i A)$. To be a finite filtration on A means that $\lim_i \mathrm{Fil} A = 0, \mathrm{colim}_i \mathrm{Fil} A = A$ and $\mathrm{gr}_* A$ is nonzero only finitely many times. Because

---

[7] This uses that fact that these operations suffice to describe how homotopy groups change under cofiber sequences and idempotents

the filtrations are finite, by shifting we can suppose that our filtrations on A and B are such that whenever $\mathrm{gr}_i B, \mathrm{gr}_j A$ are nonzero, $j < i$. r then canonically lifts to a map of filtered objects $\mathrm{Fil}_* A \to \mathrm{Fil}_* B$ because all the maps of $\mathrm{Fil}_* B$ in the range where $\mathrm{Fil}_* A$ is nonzero are isomorphisms. Taking the cofiber of this map of filtered objects, we obtain a finite filtration on the cofiber. The associated graded of this pasted filtration agrees with $\mathrm{gr}_* B$ at first and switches over to being $\Sigma \mathrm{gr}_* A$.

By pasting filtrations, we learn that the collection of objects satisfying $(**)$ is closed under extensions, since extensions of A by B are cofibers of maps of the form $\Sigma^{-1} A \to B$. We will handle kernels and cokernels simultaneously. Suppose $A, B \in \mathrm{Ind}(\mathcal{D})^\heartsuit$ satisfy $(**)$ and r is a map between them. We paste the filtrations on A and B to form a filtration on the cofiber, $\mathrm{cof}(r)$. In the spectral sequence associated to this filtered object, the E_1-page is the homotopy groups of the associated graded of the filtration. By the previous paragraph has a copy of the associated graded of B in topological degree 0, and a copy of the associated graded of A in topological degree 1. This spectral sequence converges to the associated graded of a finite filtration on the $\pi_*^\heartsuit$ of the cofiber. By hypothesis, the E_1-page of this spectral sequence involves only objects of $\mathcal{C}^\heartsuit$. Now, since F is t-exact and fully faithful on $\mathcal{C}^\heartsuit$, kernels and cokernels of maps between objects in the image of $F|_{C^\heartsuit}$ remain in the image of $F|_{C^\heartsuit}$. Consequently, as we run the differentials in this spectral sequence the terms remain in the essential image of $\mathcal{C}^\heartsuit$. Since the spectral sequence has only finitely many pages we learn that it abuts to a filtration of the desired type on the kernel and cokernel of r (which appears as $\pi_1^\heartsuit(\mathrm{cof}(r))$ and $\pi_0^\heartsuit(\mathrm{cof}(r))$ respectively).

The second conclusion follows from knowing that $(*)$ applies to the objects of $\mathcal{D}^\heartsuit \subseteq \mathcal{D}$. ☐

We now recall Quillen's devissage and the theorem of the heart, which we use to finish the proof of the main theorem.

Theorem 2.2.5 ([Bar15] Barwick's theorem of the heart)**.** *Let $\mathcal{C}$ be a stable category with bounded t-structure. Then the inclusion $\mathcal{C}^\heartsuit \to \mathcal{C}$ induces an equivalence on connective K-theory.*

Theorem 2.2.6 ([Qui73, Theorem 4] Quillen's devissage)**.** *Let $\mathcal{A} \subset \mathcal{B}$ be an exact fully faithful inclusion of abelian categories with $\mathcal{A}$ closed in $\mathcal{B}$ under subobjects, and such that every object of $\mathcal{B}$ has a finite filtration with associated graded in $\mathcal{A}$. Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces an equivalence on connective K-theory.*

Proof (of Theorem 2.1.3). At this point we have already shown that $\mathcal{D}$ admits a bounded t-structure (Proposition 2.2.4) for which F is t-exact (Lemma 2.2.3). Next we argue that this t-structure on $\mathcal{D}$ is unique. We start by observing that the filtration condition from Proposition 2.2.4 implies that the extension closure of $F(\mathcal{C}_{\geq 0})$ is $\mathcal{D}_{\geq 0}$ and the extension closure of $F(\mathcal{C}_{<0})$ is $\mathcal{D}_{<0}$. Now suppose we have another t-structure $(\mathcal{D}_{\succeq 0}, \mathcal{D}_{\prec 0})$ on $\mathcal{D}$ for which F is t-exact. Using the assumption that F is t-exact the above lets us conclude that $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}_{\succeq 0}$ and $\mathcal{D}_{<0} \subseteq \mathcal{D}_{\prec 0}$. Using the orthogonality of positive and negative parts of t-structures in turn implies that in fact $\mathcal{D}_{\geq 0} = \mathcal{D}_{\succeq 0}$ and $\mathcal{D}_{<0} = \mathcal{D}_{\prec 0}$ giving the desired uniqueness.

Finally, we examine the square on K-theory,

$$\begin{tikzcd}
K(\mathcal{C}^{\heartsuit}) \arrow[r] \arrow[d] & K(\mathcal{D}^{\heartsuit}) \arrow[d] \\
K(\mathcal{C}) \arrow[r, "K(F)"] & K(\mathcal{D}).
\end{tikzcd}$$

Because the t-structures on $\mathcal{C}$ and $\mathcal{D}$ are bounded, we can use Theorem 2.2.5 to see that the vertical maps are equivalences. Moreover, the results of [Sch06] and [AGH19] show that K_{-1} of $\mathcal{C}$ and $\mathcal{D}$ vanish.

In order to finish the proof it suffices to show that the top horizontal map is an equivalence, which we show by applying Theorem 2.2.6. The map $f : \mathcal{C}^{\heartsuit} \to \mathcal{D}^{\heartsuit}$ is fully faithful and exact by construction, and we showed in Proposition 2.2.4 that the filtration condition is satisfied.

It remains to check that if $d \in \mathcal{D}^{\heartsuit}$ is a subobject of $c \in \mathcal{C}^{\heartsuit}$, then $d \in \mathcal{C}^{\heartsuit}$. Using the exactness of the inclusion it will suffice to instead show that $\operatorname{coker}(d \to c) \in \mathcal{C}^{\heartsuit}$. Using Proposition 2.2.4 we can equip d with a finite filtration with associated graded in $\mathcal{C}^{\heartsuit}$. The cokernel $\operatorname{coker}(d \to c)$ can be produced by successively quotienting c by the pieces in the associated graded of the filtration on d, thus we only need to know that quotients by subobjects coming from $\mathcal{C}^{\heartsuit}$ stay in $\mathcal{C}^{\heartsuit}$. This last statement follows from the fact that f is fully faithful and exact.

□

Remark 2.2.7. The assumption that $\mathcal{C}$ and $\mathcal{D}$ be idempotent complete in Theorem 2.1.1 can be removed. For $\mathcal{C}$ this does not provide additional generality as any stable category with a bounded t-structure is automatically idempotent complete.[8] If $\mathcal{D}$ is not idempotent complete, then we can instead apply Theorem 2.1.1 to the map $\mathcal{C} \to \mathcal{D}^{\mathrm{idem}}$ to learn that the composite

$$K_0(\mathcal{C}) \to K_0(\mathcal{D}) \to K_0(\mathcal{D}^{\mathrm{idem}})$$

is an isomorphism. This means that $K_0(\mathcal{D}) \to K_0(\mathcal{D}^{\mathrm{idem}})$ is surjective, which implies that $\mathcal{D}$ is idempotent complete by [Tho97]. ◁

Remark 2.2.8. We end this section by noting that the following converse to Theorem 2.1.3 holds: if we have $F : \mathcal{C} \to \mathcal{D}$ a map which we can use the theorem of the heart and Quillen's devissage to prove is a K-equivalence, then (A) and (B) must hold.

To see this, first note that to apply the theorem of the heart we need bounded t-structures on $\mathcal{C}$ and $\mathcal{D}$ so that F is t-exact. To apply Quillen's devissage, the induced functor on hearts should by fully faithful and its image should generate $\mathcal{D}^{\heartsuit}$ under extensions. The condition

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[8] It suffices to split idempotents on the heart and abelian categories are automatically idempotent complete.

on generating $\mathcal{D}^\heartsuit$ implies (A). The t-exactness of F, plus fully faithfulness on the heart implies (B).

Simply put, this is saying that Theorem 2.1.3 is essentially equivalent to the combination of Quillen's devissage and the theorem of the heart. $\triangleleft$ $\triangleleft$

2.3 Complements

In this section we discuss a couple points which are complementary to Theorem 2.1.3. We begin by introducing some ideas from noncommutative geometry which provide a convenient language for thinking about our main theorem. Then, we discuss variants of the main theorem and prove Theorem 2.1.1 from the introduction. In the third subsection we consider improvements to Theorem 2.1.3 which are possible when $\mathcal{C}^\heartsuit$ is Noetherian. We end the section by briefly discussing negative K-groups.

2.3.1 Some nc geometry

For us noncommutative geometry refers to thinking about small idempotent complete stable categories equipped with a "positive half" closed under finite colimits and extensions. This is quite close to established notions of noncommutative geometry such as in [Orl16], with the notable difference being that we work relative to the sphere rather than relative to a discrete base ring k. We explore this setting in some depth in [BL24] and in this section we build on the groundwork from that paper[9]. Before proceeding we remind the reader of the main definitions.

Definition 2.3.1. We use $\mathrm{Cat}^\sharp$ to denote the category of small idempotent complete stable categories. Our main objects of study are objects of $\mathrm{Cat}^\sharp_{\geq 0}$. This is the category of $\mathcal{C} \in \mathrm{Cat}^\sharp$ equipped with an idempotent complete prestable[10] full subcategory $\mathcal{C}_{\geq 0}$ that generates $\mathcal{C}$. $\triangleleft$

Being prestable amounts to asking that $\mathcal{C}_{\geq 0}$ be closed under finite colimits and extensions. Often, we abuse notation by writing $\mathcal{C} \in \mathrm{Cat}^\sharp_{\geq 0}$, leaving the subcategory of positive objects, $\mathcal{C}_{\geq 0}$, implicit.

Example 2.3.2. Given an $\mathbb{E}_1$-algebra R, the category compact R-modules, $\mathrm{Mod}(R)^\omega$, naturally lives in $\mathrm{Cat}^\sharp_{\geq 0}$. The positive objects are those built from R via extensions, finite colimits and retracts. $\triangleleft$ $\triangleleft$

Given $\mathcal{C} \in \mathrm{Cat}^\sharp_{\geq 0}$, the subcatgory $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \subset \mathrm{Ind}(\mathcal{C})$ determines a t-structure on $\mathrm{Ind}(\mathcal{C})$ (see Lemma 4.2.3). In fact, $\mathcal{C}_{\geq 0}$ can be recovered from the t-structure as $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \cap \mathcal{C}$.

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[9] Even though we cite results in [BL24], we do not use anything particularly difficult from there, and so the results here can be considered independent of that paper.

[10] As introduced in [Lur18b, Appendix C]

Example 2.3.3. In the t-structure associated to Example 2.3.2, a connective object is one built out of copies of R under colimits and extensions, and a coconnective object is one whose underlying spectrum is coconnective. ◁ ◁

Definition 2.3.4. Given $\mathcal{C} \in \mathrm{Cat}^{\sharp}_{\geq 0}$,

- $\mathcal{C}$ is *regular* if the t-structure on $\mathrm{Ind}(\mathcal{C})$ restricts to $\mathcal{C}$ (i.e truncations of compact objects are compact),
- $\mathcal{C}$ is *bounded* if each $c \in \mathcal{C}$ is bounded as an object of $\mathrm{Ind}(\mathcal{C})$,
- a functor $\mathcal{C} \to \mathcal{D}$ is *quasi-affine* if its image generates $\mathcal{D}$ under finite colimits and retracts and
- a quasi-affine functor $\mathcal{C} \to \mathcal{D}$ is *unipotent* if it induces a fully faithful functor on $\mathrm{Ind}(\mathcal{C})^{\heartsuit}$. ◁

◁

Example 2.3.5. We say a ring R is regular if $\mathrm{Mod}(R)^{\omega}$ is regular. If R is a discrete ring then, as a result of well-known arguments, R is regular in this sense iff R is left regular coherent. For a proof stating things this way, see [BL24, Proposition 2.4]. ◁ ◁

Theorem 2.1.1 gives sufficient conditions for a coconnective ring R to be regular, but they are not necessary. Moreover for a general regular coconnective ring, $K(R)$ may not agree with $K(\pi_0 R)$. See Example 2.5.14 for an example of such a regular coconnective ring.

Example 2.3.6. We show in [BL24, Proposition 2.16] that if $\mathcal{C}$ is regular, and $n \neq 0$, then $\mathcal{C}[x_n]$ is regular. ◁ ◁

Remark 2.3.7. If we think in terms of categories of quasicoherent sheaves, the reasoning behind the term quasi-affine is relatively transparent.

The term unipotent bears more explanation. The key identifying features of a unipotent group are that maps between trivial representations can be computed on underlying and every representation is built out of extensions of trivial reps. Our definition takes these properties as the definition of unipotent.

Note that conditions (A) and (B) of Theorem 2.1.3 are equivalent to saying that F is unipotent. This lets us reinterpret Theorem 2.1.3 as saying that bounded regularity can be transferred along unipotent maps, and that such maps induce K-equivalences. This reinterpretation is a precise form of the slogan in the introduction. ◁ ◁

2.3.2 Other forms of the main theorem

In practice unipotence can be difficult to check so we recall an equivalent condition which is often more transparent. The functor $F^* : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ has a right adjoint F_*, which preserves colimits since F^* preserves compact objects.

Lemma 2.3.8 ([BL24, Corollary 4.12])**.** *For a map $F : \mathcal{C} \to \mathcal{D}$ as in Theorem 2.1.3 condition (B) is equivalent to:*

(B′) For every $c \in \mathcal{C}^\heartsuit$, the cofiber of the unit map $c \to F_*F^*(c)$ is in $\mathrm{Ind}(\mathcal{C})_{\leq -1}$.

Proof sketch. Unraveling (B′) gives the statement that the cofiber of

$$\mathrm{map}(d, c) \to \mathrm{map}(Fd, Fc)$$

is coconnected for all $c \in \mathcal{C}^\heartsuit$ and $d \in \mathcal{C}_{\geq 0}$. This visibly implies (B). The key point in proving the reverse implication is using the fact that $\mathcal{C}^\heartsuit$ is closed under extensions in $\mathrm{Ind}(\mathcal{C})$[11]. ☐

We now provide a version of Theorem 2.1.3 for categories of modules over an $\mathbb{E}_1$-algebra from which Theorem 2.1.1 will follow.

Proposition 2.3.9. *Let $f : A \to B$ be a map of $\mathbb{E}_1$-algebras such that*

1. $\mathrm{Mod}(A)^\omega$ *is bounded and regular and*

2. $\mathrm{cof}(f)$ *has Tor amplitude in $[-\infty, -1]$ as a right A-module,*

then $\mathrm{Mod}(B)^\omega$ is bounded and regular, the base-change functor $(-) \otimes_A B$ is a t-exact K-equivalence, and K_{-1} of A and B vanish.

Proof. We apply Theorem 2.1.3 to the base-change functor

$$B \otimes_A - : \mathrm{Mod}(A)^\omega \to \mathrm{Mod}(B)^\omega.$$

This functor is quasi-affine since A is sent to B which is a generator. Condition (B′) of Lemma 2.3.8 asks that for every $N \in \mathrm{Mod}_A^\heartsuit$ the A-module $\mathrm{cof}(f) \otimes_A N$ be coconnected. This is equivalent to the given Tor amplitude bound on $\mathrm{cof}(f)$. ☐

Proof (of Theorem 2.1.1). We apply Proposition 2.3.9 to the connective cover map $f : \pi_0 R \to R$. From Example 2.3.5 we know that $\pi_0 R$ is left regular coherent iff $\mathrm{Mod}(\pi_0 R)^\omega$ is regular. Boundedness is automatic. Since $\tau_{<0} R \cong \mathrm{cof}(f)$, the Tor amplitude bounds in Proposition 2.3.9 and Theorem 2.1.1 match up. ☐

Remark 2.3.10. Amplifying Remark 2.2.8, we note that the conditions of Theorem 2.1.1 are actually *equivalent* to (A) and (B′) for the connective cover map, implying a converse to this theorem. ◁ ◁

Remark 2.3.11. The reader might wonder when the the abelian categories $\mathcal{C}^\heartsuit$ and $\mathcal{D}^\heartsuit$ appearing in Theorem 2.1.3 are equivalent. We remark that this will be the case as soon as F satisfies:

(C) for every $c \in \mathcal{C}^\heartsuit$, the cofiber of the unit map $c \to F_*F^*(c)$ is ≤ -2 in the t-structure on $\mathrm{Ind}(\mathcal{C})$[12].

◁ ◁

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[11] In this paper we only use that (B′) implies (B) and not the reverse implication.

[12] See [BL24, Proposition 7.2] for details.

2.3.3 The Noetherian case

In situation of Theorem 2.1.3 if we further assume that the heart of $\mathcal{C}$ is Noetherian, then we can draw stronger conclusions about K-theory and the induced t-structure on $\mathcal{D}$.

Lemma 2.3.12. *In situation of Theorem 2.1.3, if $\mathcal{C}^\heartsuit$ is Noetherian, then the heart of the induced t-structure on $\mathcal{D}$ is Noetherian as well. Moreover, in the square of nonconnective K-theories,*

$$\begin{tikzcd}
K^{\mathrm{nc}}(\mathcal{C}^\heartsuit) \arrow[r] \arrow[d] & K^{\mathrm{nc}}(\mathcal{D}^\heartsuit) \arrow[d] \\
K^{\mathrm{nc}}(\mathcal{C}) \arrow[r] & K^{\mathrm{nc}}(\mathcal{D})
\end{tikzcd}$$

all maps are equivalences.

Proof. We would first like to show that every $d \in \mathcal{D}^\heartsuit$ is Noetherian. By Proposition 2.2.4, d has a finite filtration with associated graded in $F(\mathcal{C}^\heartsuit)$, so since Noetherian objects are closed under extensions, it suffices to show that $F(c)$ is Noetherian for each $c \in \mathcal{C}^\heartsuit$. As argued in the proof of Theorem 2.1.3, F is fully faithful with image closed under passing to subobjects. This implies that the lattice of subobjects of $F(c)$ agrees with that of c, so $F(c)$ is Noetherian since c is.

Now by applying the vanishing theorems of [Sch06] and [AGH19], the negative K-groups of categories with a bounded t-structure with Noetherian heart vanish, so the square of equivalences in Theorem 2.1.3 extends to negative K-theory. ☐

Next we examine the interpretation of condition (B) as "unipotence" more closely. Let $f : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ denote the restriction of F to $\mathcal{C}^\heartsuit$, as well as its Ind-completion. At the moment we know that

1. $f : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is fully faithful.
2. Each $d \in \mathcal{D}^\heartsuit$ has a finite filtration with associated graded in the image of f.
3. Every subobject of $f(c)$ comes from a subobject of c.

Note that the finite filtrations are not guaranteed to be functorial[13], but when $\mathcal{D}$ is Noetherian, functoriality comes for free in the form of the *socle filtration*:

Construction 2.3.13. At the level of Ind-categories the functor f has right adjoint given by $g := \tau_{\geq 0} G(-)$. Since f is a fully faithful left adjoint it exhibits $\mathrm{Ind}(\mathcal{C})^\heartsuit$ as a coreflective subcategory of $\mathrm{Ind}(\mathcal{D})^\heartsuit$. We let $\mathrm{soc}_0(M)$ denote $fg(M)$, which sits as a subobject of M via the counit map.

We define $\mathrm{soc}_n(-)$ inductively via the pullback

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[13] Quillen's devissage, doesn't require a functorial filtration, but merely an object-wise one.

$$\begin{tikzcd}
 & \mathrm{soc}_n \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{soc}_0(\mathrm{Id}/\mathrm{soc}_{n-1}) \arrow[d, hook] \\
\mathrm{soc}_{n-1} \arrow[r, hook] & \mathrm{Id} \arrow[r, two heads] & \mathrm{Id}/\mathrm{soc}_{n-1}.
\end{tikzcd}$$

The key property of the socle filtration is that $\mathrm{soc}_n/\mathrm{soc}_{n-1}$ is in the image of f for every n. Since the maps $\mathrm{Id}/\mathrm{soc}_{n-1} \to \mathrm{Id}/\mathrm{soc}_n$ become zero after applying g and since $g = \tau_{\geq 0}G$ and G detects coconnectivity we learn that

$$\operatorname*{colim}_n \mathrm{soc}_n \to \mathrm{Id}$$

is an equivalence, i.e. the socle filtration is exhaustive.

Now, using our Noetherian-ness hypothesis we know that arbitrary subobjects of compact objects are compact in $\mathrm{Ind}(\mathcal{D})^\heartsuit$, therefore the socle filtration restricts to a functorial filtration on $\mathcal{D}^\heartsuit$ that is finite on each object. $\triangleleft$ $\triangleleft$

Remark 2.3.14. The construction of a bounded t-structure together with socle filtrations can be interpreted as a generalization of [KN13, Theorem 8.1] where a similar result is proven for dg-algebras with strong finiteness assumptions. $\triangleleft$ $\triangleleft$

It is important to note that outside of the Noetherian setting the socle filtration, while still existent at the level of Ind-categories, need not restrict to compact objects.

Example 2.3.15. Let $\mathcal{C}$ denote the category of pairs (V_0, V_1) where V_0 is a $k[x_1, \dots]$-module and V_1 is a k-module. Since the infinite dimensional affine space is regular coherent, $\mathcal{C}$ has a bounded t-structure. Let $\mathcal{D}$ denote the category of triples $(V_0,\ V_1,\ V_0 \otimes_{k[x_1,\dots]} k \to V_1)$.

The natural functor $\mathcal{C} \to \mathcal{D}$ which uses the zero map is fully faithful on the heart, therefore the hypotheses of Theorem 2.1.3 are satisfied. On the other hand, the socle of

$$(k[x_1, \dots],\ k,\ k[x_1, \dots] \twoheadrightarrow k)$$

is (I, k) and the augmentation ideal of $k[x_1, \dots]$ is not compact. $\triangleleft$ $\triangleleft$

2.3.4 Negative K-theory

It would be desirable to extend Theorem 2.1.3 to negative K-theory, however both the theorem of the heart and Quillen's devissage only apply to connective K-theory in their current form. For that reason we ask the following question (which we hope has a positive answer):

Question 2.3.16. Do the theorem of the heart and devissage hold for negative K-theory?

As discussed above, if $\mathcal{C}^\heartsuit$ is Noetherian, then $\mathcal{D}^\heartsuit$ is Noetherian and so the negative K-theory vanishes. This might suggest that one should approach this question by proving a vanishing statement for negative K-theory. However, the example from [Nee21] shows that in general regularity does not imply the vanishing of negative K-groups.

In order to probe question of this type more closely we examine the relation between *stable coherence* and the vanishing of negative K-theory in [BL24, Section 3.2][14]. We reproduce the statements proved therein for the convenience of the reader interested in thinking about Question 2.3.16.

Definition 2.3.17. Given a category $\mathcal{C}$ with bounded t-structure we say that $\mathcal{C}$ is $\mathbb{A}^n$-*coherent* if the finitely presented $\mathbb{Z}[t_1, \dots, t_n]$-modules in $\mathrm{Ind}(\mathcal{C}^\heartsuit)$ form an abelian category[15]. If $\mathcal{C}$ is $\mathbb{A}^n$-coherent for all n, then we say it is *stably coherent.* ◁ ◁

Example 2.3.18. In [Gla89, Example 7.3.13], it is shown that an infinite product of copies of the ring $\mathbb{Q}[\![x, y]\!]$ is regular coherent, but this ring is not $\mathbb{A}^1$-coherent (demonstrating that this is a non-trivial condition on a regular coherent ring). ◁ ◁

The following lemma uses the vanishing results of [AGH19].

Lemma 2.3.19 ([BL24, Corollary 3.14, Lemma 3.17])**.** *If $\mathcal{C}$ is regular and $\mathbb{A}^n$-coherent, then the first $n + 1$ negative K-groups of $\mathcal{C}$ vanish.*

Proposition 2.3.20 ([BL24, Proposition 3.18])**.** *If we are given a functor $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^\sharp$ and a bounded $\mathbb{A}^n$-coherent t-structure on $\mathcal{C}$ such that the conditions of Theorem 2.1.3 are satisfied, then the induced t-structure on $\mathcal{D}$ is $\mathbb{A}^n$-coherent as well.*

2.4 K-theory of pushouts

In this section we prove Theorem 2.4.11 (a corollary of which is Theorem 2.1.5 from the introduction) which says that K-theory preserves pushouts of (well-behaved) regular presta -ble categories. This theorem arises from the examining the interaction of our main theorem with the Land–Tamme $\odot$-product. In fact, the idea that a result like Theorem 2.1.1 should be true was suggested to the authors by Markus Land and Georg Tamme with the intention of using such a result to compute the K-theory of pushouts.

A review on the $\odot$-product

The Land–Tamme $\odot$-product is a relatively new operation on $\mathbb{E}_1$-algebras (and categories more generally) first introduced in [LT19], with generalizations appearing in [BKRS20] and the forthcoming [LT23]. Here we roughly follow [LT23] in our formulation of this operation.

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[14] see also [AGH19, Section 3.5] for a similar discussion

[15] If $\mathcal{C}$ is the category of perfect R-modules for R a discrete ring, then this is equivalent to asking that $R[t_1, \dots, t_n]$ be left coherent.

Construction 2.4.1. Given a pair of categories $\mathcal{B}$ and $\mathcal{C}$ in $\mathrm{Cat}^\sharp$ and an arrow $f \in \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{B}))$ we can form the oplax limit of f, which we denote $\mathcal{B}\vec{\times}_f\mathcal{C} \in \mathrm{Cat}^\sharp$. This is the category of triples (b, c, r), where $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $r : b \to f(c)$ is a map. For our purposes, the key observation about $\mathcal{B}\vec{\times}_f\mathcal{C}$ is that the forgetful map $\mathcal{B}\vec{\times}_f\mathcal{C} \to \mathcal{B} \times \mathcal{C}$ induces an equivalence at the level of nc motives

$$\mathcal{U}_{\mathrm{loc}}(\mathcal{B}\vec{\times}_f\mathcal{C}) \cong \mathcal{U}_{\mathrm{loc}}(\mathcal{B} \times \mathcal{C}) \cong \mathcal{U}_{\mathrm{loc}}(\mathcal{B}) \oplus \mathcal{U}_{\mathrm{loc}}(\mathcal{C}).$$

Adding another layer, associated to each square

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{B} \arrow[r] \arrow[ur, Rightarrow] & \mathrm{Ind}(\mathcal{B}).
\end{tikzcd}$$

we have an induced map $\mathcal{A} \to \mathcal{B}\vec{\times}_f\mathcal{C}$ in $\mathrm{Cat}^\sharp$ and we define the Land–Tamme $\odot$-product $\mathcal{B} \odot^f_{\mathcal{A}} \mathcal{C}$ to be the cofiber of this map. This cofiber sequence in $\mathrm{Cat}^\sharp$ provides a pushout in nc motives

$$\begin{tikzcd}
\mathcal{U}_{\mathrm{loc}}(\mathrm{im}\,\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathcal{U}_{\mathrm{loc}}(\mathcal{C}) \arrow[d] \\
\mathcal{U}_{\mathrm{loc}}(\mathcal{B}) \arrow[r] & \mathcal{U}_{\mathrm{loc}}(\mathcal{B} \odot^f_{\mathcal{A}} \mathcal{C})
\end{tikzcd}$$

where $\mathrm{im}\,\mathcal{A}$ denotes the image of $\mathcal{A}$ inside $\mathcal{B}\vec{\times}_f\mathcal{C}$. ◁ ◁

Remark 2.4.2. If we specialize to the case where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are module categories of $\mathbb{E}_1$-algebras A, B and C, then we can move down a categorical level:

1. The category $\mathrm{Fun}^L(\mathrm{Mod}(C), \mathrm{Mod}(B))$ can be identified with $\mathrm{Mod}(B \otimes C^{\mathrm{op}})$, meaning the bonding map is just a choice of $(B,\, C)$-bimodule M.

2. $\mathrm{im}\,\mathcal{A}$ is generated by the image of A, meaning $\mathrm{im}(\mathcal{A}) \cong \mathrm{Mod}(\mathrm{im}\,A)^\omega$ where $\mathrm{im}\,A$ is the endomorphism algebra of the image of A.

3. If the functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{C}$ came from $\mathbb{E}_1$-algebra maps $A \to B$ and $A \to C$, then $\mathcal{B} \odot^f_{\mathcal{A}} \mathcal{C}$ is generated by the image of B (which is equivalent to the image of C). We let $B \odot^M_A C$ denote the ring of endomorphisms of this object where M is the bimodule used as the bonding map.

◁ ◁

Speaking practically, the fundamental difficulty in working with the $\odot$-product lies in identifying the categories $\mathrm{im}(\mathcal{A})$ and $\mathcal{B} \odot^f_{\mathcal{A}} \mathcal{C}$. A fundamental insight of Land and Tamme is that in many cases of interest these categories are surprisingly computationally accessible.

Example 2.4.3. In [LT19], where the $\odot$-product was introduced, the following example of Construction 2.4.1 is analyzed. Suppose we are given a pullback square of $\mathbb{E}_1$-algebras

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d] \\
B \arrow[r] & D.
\end{tikzcd}$$

Using D as our (B, C)-bimodule, $\mathrm{Mod}(A)$ for $\mathcal{A}$ and the map $B \to D$ of (B, A)-bimodules for the natural transformation, we can construct a $\odot$-product $B \odot^D_A C$. In this situation they prove that $\mathrm{im}(A) \cong A$ and the spectrum underlying $B \odot^D_A C$ is equivalent to $B \otimes_A C$ as a (B, C)-bimodule. Furthermore, they show in [LT19, Proposition 1.13] that the underlying C-bimodule of $B \odot^D_A C$ is the cofiber of the map $I \otimes_A C \to C$, where I is the fiber of $C \to D$. $\triangleleft$ $\triangleleft$

In the forthcoming [LT23] another, somewhat dual, situation is analyzed.

Theorem 2.4.4 ([LT23])**.** *Given a span* $\mathcal{B} \xleftarrow{b} \mathcal{A} \xrightarrow{c} \mathcal{C}$ *in* $\mathrm{Cat}^\sharp$ *there is a square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "c"] \arrow[d, "b"] & \mathcal{C} \arrow[d, "b^* c_*"] \\
\mathcal{B} \arrow[r] \arrow[ur, Rightarrow, "b^* \eta_c"'] & \mathrm{Ind}(\mathcal{B})
\end{tikzcd}$$

and an equivalence of the associated $\odot$*-product with the pushout of the span,*

$$\mathcal{B} \odot^{b^* c_*}_{\mathcal{A}} \mathcal{C} \cong \mathcal{B} \coprod_{\mathcal{A}} \mathcal{C}.$$

Corollary 2.4.5 ([LT23])**.** *Given a span* $B \leftarrow A \to C$ *of* $\mathbb{E}_1$*-algebras we have an equivalence*

$$B \odot^{B \otimes_A C}_A C \cong B \coprod_A C$$

of the $\odot$*-product with the pushout of the span in* $\mathbb{E}_1$*-algebras.*

Remark 2.4.6. In Corollary 2.4.5 if we have a span of commutative algebras instead, then the base-change equivalence $B \otimes_A C \otimes_C - \cong B \otimes_A -$ allows us to recognize that we are actually in the situation of Example 2.4.3 for the cospan $B \to B \otimes_A C \leftarrow C$. The benefit of making this identification is that we can identify $\mathrm{im}(A)$ as the pullback of this cospan. $\triangleleft$ $\triangleleft$

Lemma 2.4.7. *The* $\odot$*-product is compatible with base-change, i.e.*

$$(\mathcal{B} \odot^f_{\mathcal{A}} \mathcal{C}) \otimes \mathcal{D} \cong (\mathcal{B} \otimes \mathcal{D}) \odot^{f \otimes \mathcal{D}}_{\mathcal{A} \otimes \mathcal{D}} (\mathcal{C} \otimes \mathcal{D}).$$

Proof. This follows from the fact that $-\otimes\mathcal{B}$ preserves fully faithful maps and localization sequences (see for example [AGH19, Lemma 3.3, Corollary 3.5]) and commutes with pullbacks, lax pullbacks and oplax limits of arrows. $\square$

The $\odot$-product allows us to produce examples of equivalences of nc motives which do not arise from equivalences of categories. We end our recollection by working through a pair of examples which illustrate this flexibility phenomenon.

Definition 2.4.8. Let $\mathbb{S}[x_n]$ denote the polynomial algebra on a generator in degree n. We let N_n denote the cofiber

$$\mathrm{N}_n \coloneqq \mathrm{cof}\left(\mathcal{U}_{\mathrm{loc}}(\mathbb{S}) \to \mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_n])\right)$$

and use N to mean N_0. $\triangleleft$ $\triangleleft$

If we think about N as a homology theory on nc motives it is the NK-theory of Bass which measures the failure of $\mathbb{A}^1$-invariance.

Example 2.4.9. Consider the span of commutative algebras $\mathbb{S} \leftarrow \mathbb{S}[x_n] \to \mathbb{S}$. Applying Corollary 2.4.5 we obtain a pullback of nc motives

$$\begin{tikzcd}
\mathcal{U}_{\mathrm{loc}}(\mathrm{im}\,\mathbb{S}[x_n]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}_{\mathrm{loc}}(\mathbb{S}) \arrow[d] \\
\mathcal{U}_{\mathrm{loc}}(\mathbb{S}) \arrow[r] & \mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_{n+1}]).
\end{tikzcd}$$

Using Remark 2.4.6 we can identify $\mathrm{im}(\mathbb{S}[x_n])$ as $\mathbb{S}[\epsilon_n]$ (the exterior algebra on a class in degree n) since $\mathbb{S}\otimes_{\mathbb{S}[x_n]}\mathbb{S} \cong \mathbb{S}[\epsilon_{n+1}]$ and the pullback moves the exterior generator down a degree. Since the diagram above is diagonally symmetric we have a splitting

$$\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[\epsilon_n]) \cong \mathbb{1} \oplus \Sigma^{-1}\mathrm{N}_{n+1}.$$

$\triangleleft$ $\triangleleft$

We learned of next example, which allows us to turn a copy of the coordinate axes in the plane into a polynomial algebra, from Markus Land and Georg Tamme.

Example 2.4.10. Consider the algebra $R \coloneqq \mathbb{S}[x_a, x_b]/(x_a x_b)$ which is built from the pullback square on the left below (where x_a is in degree a and x_b is in degree b).

$$\begin{tikzcd}
\mathbb{S}[x_a, x_b]/(x_a x_b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}[x_a] \arrow[d] \\
\mathbb{S}[x_b] \arrow[r] & \mathbb{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_a, x_b]/(x_a x_b)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_a]) \arrow[d] \\
\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_b]) \arrow[r] & \mathcal{U}_{\mathrm{loc}}\left(\mathbb{S}[x_a] \coprod_{\mathbb{S}[x_a, x_b]} \mathbb{S}[x_b]\right)
\end{tikzcd}$$

Applying Corollary 2.4.5 and Remark 2.4.6 we then obtain the pullback of nc motives on the right. In fact, we can simplify this by exhibiting an equivalence of $\mathbb{E}_1$-algebras

$$\mathbb{S}[x_a] \coprod_{\mathbb{S}[x_a,x_b]} \mathbb{S}[x_b] \cong \mathbb{S}[x_{a+b+2}].$$

To show this, suppose first that $a, b > 0$. Then using the fact that

$$\mathrm{Spc}_* \xrightarrow{\Sigma^\infty_+\Omega-} \mathrm{Alg}(\mathrm{Sp})$$

is a left adjoint and so preserves pushouts, we reduce to the pushout square

$$\begin{tikzcd}
S^{a+1} \times S^{b+1} \arrow[r] \arrow[d] & S^{a+1} \arrow[d] \\
S^{b+1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^{a+b+3}.
\end{tikzcd}$$

We use a trick in order to extend this equivalence to the case where a, b are not strictly positive. First, we lift the pushout above to a pushout in graded rings where x_a and x_b are in grading 1. Since the functor forgetting the grading on an $\mathbb{E}_1$-algebra preserves colimits, it will suffice to compute the pushout in the graded setting. Next we use the $\mathbb{E}_2$-monoidal shearing functor which suspends by $2n$ in grading n constructed in [Lur15] to reduce to the case where a, b are positive.

In the graded setting the generator x_{a+b+2} is in grading 2 and as a consequence of the fact that $\mathbb{S}[x_a]$ is the free graded algebra on the class x_a in grading 1 we obtain a factorization $\mathbb{S}[x_a] \to \mathbb{S} \to \mathbb{S}[x_{a+b+2}]$. With control over the maps in the square above we now obtain an equivalence of nc motives

$$\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_a, x_b]/(x_a x_b)) \cong \mathbb{1} \oplus \mathrm{N}_a \oplus \mathrm{N}_b \oplus \Sigma^{-1}\mathrm{N}_{a+b+2}.$$

◁ ◁

K-theory of pushouts

There is a sharp contrast between the ideas behind the Land–Tamme $\odot$-product and our main theorem. The $\odot$-product arises from 2-categorical maneuvers and essentially operates at the level categories and nc motives. Meanwhile our main theorem is specific to K-theory, exploiting additive but non-exact operations (such as truncation) in an essential way. The complementary nature of these approaches allows us to combine them to surprising effect.

Theorem 2.4.11. *Suppose we are given a span $\mathcal{B} \xleftarrow{b} \mathcal{A} \xrightarrow{c} \mathcal{C}$ in $\mathrm{Cat}^\sharp$ where $\mathcal{A}$ is equipped with a bounded t-structure. If we assume that*

(D) the induced functor $\mathcal{A}^\heartsuit \to \mathcal{B} \vec{\times}_{b^* c_*} \mathcal{C}$ is fully faithful,

then connective K-theory preserves the pushout of the span, i.e the diagram below is a pushout square.

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & K(\mathcal{C}) \arrow[d] \\
K(\mathcal{B}) \arrow[r] & K\left(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C}\right)
\end{tikzcd}$$

Proof. From Theorem 2.4.4, we have a pushout square

$$\begin{tikzcd}
K^{\mathrm{nc}}(\operatorname{im}\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & K^{\mathrm{nc}}(\mathcal{C}) \arrow[d] \\
K^{\mathrm{nc}}(\mathcal{B}) \arrow[r] & K^{\mathrm{nc}}\left(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C}\right).
\end{tikzcd}$$

Condition (D) implies that the functor $\mathcal{A} \to \operatorname{im}(\mathcal{A})$ satisfies the hypotheses of Theorem 2.1.3, so we have an equivalence $K(\mathcal{A}) \cong K(\operatorname{im}\mathcal{A})$, and K_{-1} of each vanishes. This implies that the square above remains a pushout when we take connective covers and replace $K(\operatorname{im}\mathcal{A})$ by $K(\mathcal{A})$. □

In order to make this theorem easier to apply we give a simpler condition which implies (D) and is more natural to check in practice.

Lemma 2.4.12. *In the situation of Theorem 2.4.11 condition (D) is implied by*

(D′) The functors $\mathcal{A}^\heartsuit \to \mathcal{B}$ and $\mathcal{A}^\heartsuit \to \mathcal{C}$ are faithful.

Proof. Let F denote the functor $\mathcal{A} \to \mathcal{B}\vec{\times}_{b^*c_*}\mathcal{C}$. Using Lemma 2.3.8 it suffices to show that $\operatorname{cof}(a \to F_*F^*(a))$ is ≤ -1 for each $a \in \mathcal{A}^\heartsuit$. In order to proceed we'll give a formula for F_*F^*. From the pullback square

$$\begin{tikzcd}
\operatorname{Map}_{\mathcal{B}\vec{\times}_{b^*c_*}\mathcal{C}}(F^*x, F^*y) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \operatorname{Map}_{\mathcal{B}}(b^*x, b^*x) \arrow[d] \\
\operatorname{Map}_{\mathcal{C}}(c^*x, c^*y) \arrow[r, "b^*c_*"] & \operatorname{Map}_{\mathcal{B}}(b^*c_*c^*x, b^*c_*c^*y) \arrow[r, "b^*\eta_c\circ-"] & \operatorname{Map}_{\mathcal{B}}(b^*x, b^*c_*c^*y)
\end{tikzcd}$$

natural in both x and y we learn that F_*F^* sits in a pullback square

$$\begin{tikzcd}
F_*F^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & b_*b^* \arrow[d, "b_*b^*\circ\eta_c"] \\
c_*c^* \arrow[r, "\eta_b\circ c_*c^*"] & b_*b^*c_*c^*.
\end{tikzcd}$$

We can then read off that

$$\operatorname{cof}(\mathrm{Id} \to F_*F^*) \cong \Sigma^{-1}\operatorname{cof}(\mathrm{Id} \to b_*b^*) \circ \operatorname{cof}(\mathrm{Id} \to c_*c^*)$$

where $\circ$ is the composition monoidal structure on $\mathrm{Fun}^L(\mathrm{Mod}(A), \mathrm{Mod}(A))$. Using [BL24, Remark 4.9] (which is a variant of Lemma 2.3.8) and compatibility with colimits we can reformulate the faithfulness hypothesis as saying that $\mathrm{cof}(\mathrm{Id} \to c_*c^*)$ and $\mathrm{cof}(\mathrm{Id} \to b_*b^*)$ preserve coconnectivity. Composing and desuspending we obtain the desired coconnectivity bound on $\mathrm{cof}(\mathrm{Id} \to F_*F^*)$. □

For discrete rings condition (D′) has a simple interpretation: A map $A \to B$ is faithful on the heart exactly when B is right faithfully flat as an A-module (see [BL24, Lemma 4.7]). Consequently, we obtain the following corollary, which appeared in the introduction as Theorem 2.1.5.

Corollary 2.4.13. *Suppose $B \xleftarrow{f} A \xrightarrow{g} C$ is a span of discrete rings where A is left regular coherent and both f and g are right faithfully flat. Then connective K-theory preserves the pushout of this span.*

Remark 2.4.14. Note that (D) does not imply (D′). For example if X and Y are (well-behaved) smooth varieties which form a Zariski covering of Z, then (D) is satisfied for the span

$$\mathrm{QCoh}(X) \leftarrow \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Y)$$

while (D′) need not be satisfied. ◁ ◁

Remark 2.4.15. Corollary 2.4.13 (and in turn Lemma 2.4.12 and Theorem 2.4.11) can be viewed as a generalization of [Wal78a, Theorems 1 and 4] where the stronger condition that $f : A \to B$ and $g : A \to C$ are pure inclusions[16] was imposed. ◁ ◁

2.5 Applications and Examples

In this section we use Theorem 2.1.1 in a collection of applications and examples. Of particular note are

(Prop.2.5.1) which proves $\mathbb{A}^1$-invariance for regular categories.

(Prop.2.5.2) which proves $\mathbb{A}^n$-invariance for regular categories in high degrees.

(Prop.2.5.10) which analyzes the K-theory of unipotent local systems.

(Exm.2.5.13, 2.5.15 and 2.5.16) which show that the conditions of Theorem 2.1.1 are sharp.

---

[16] This asks that B have a splitting $B \cong f(A) \oplus I$ as an A-bimodule where I is a projective right A-module.

2.5.1 Invariance theorems

We give a short proof of $\mathbb{A}^1$-invariance of K-theory for categories with a bounded t-structure. This result was first proven for regular Noetherian rings by Quillen in his foundational paper [Qui73]. Building on this we then prove that $K_j(-)$ is $\mathbb{A}^n$-invariant once $j \geq n-1$ (again for categories with a bounded t-structure). Using Theorem 2.4.11 we then extend $\mathbb{A}^1$-invariance to the case of adjoining free variables generalizing the main results of [Ger74].

Proposition 2.5.1 ($\mathbb{A}^1$-invariance for regular categories)**.**
If $\mathcal{C} \in \mathrm{Cat}^\sharp$ admits a bounded t-structure, then $K(\mathcal{C}) \cong K(\mathcal{C}[x_0])$.

Proof. In order to prove this we must show that $K^{\mathrm{nc}}(\mathrm{N}\otimes\mathcal{C})$ vanishes in non-negative degrees. Applying Theorem 2.1.3 to the map $\mathcal{C} \to \mathcal{C}[\epsilon_{-1}]$ and applying the equivalence of motives $\mathcal{U}_{\mathrm{loc}}(\mathcal{C}[\epsilon_{-1}]) \cong \mathcal{U}_{\mathrm{loc}}(\mathcal{C}) \oplus \Sigma^{-1}\mathrm{N}\otimes\mathcal{U}_{\mathrm{loc}}(\mathcal{C})$ from Example 2.4.9 we learn that $K^{\mathrm{nc}}(\mathrm{N}\otimes\mathcal{C})$ vanishes in non-negative degrees as desired[17]. ☐

Just as N controls $\mathbb{A}^1$-invariance, $\mathbb{A}^n$-invariance is controlled by tensor-powers of N. Using the same ideas we can show that K-theory is $\mathbb{A}^n$-invariant in sufficiently large degrees as well.

Proposition 2.5.2 ($\mathbb{A}^n$-invariance for regular categories)**.** *Suppose $\mathcal{C} \in \mathrm{Cat}^\sharp$ admits a bounded t-structure. Then $\tau_{\geq n-1}K(\mathcal{C}) \cong \tau_{\geq n-1}K(\mathcal{C}[x_1,\dots,x_n])$, where $|x_i| = 0$.*

Proof. From the equivalence

$$\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x_1,\dots,x_n]) \cong \mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x]^{\otimes n}) \cong (\mathcal{U}_{\mathrm{loc}}(\mathbb{S}[x]))^{\otimes n} \cong (\mathbb{1}\oplus\mathrm{N})^{\otimes n}$$

we can read off that the obstructions to $\mathbb{A}^n$-invariance in degree j are $K_j(\mathrm{N}^{\otimes k}\otimes\mathcal{C})$ for $1 \leq k \leq n$. Using Example 2.4.9 we can find $\Sigma^{-k}\mathrm{N}^{\otimes k}\otimes\mathcal{C}$ as a summand in $\mathcal{U}_{\mathrm{loc}}(\mathcal{C}[\epsilon_1,\dots,\epsilon_k])$ (where each exterior generator is in degree -1). Applying Theorem 2.1.3 to the map $\mathcal{C} \to \mathcal{C}[\epsilon_1,\dots,\epsilon_k]$ we learn that $K^{\mathrm{nc}}(\Sigma^{-k}\mathrm{N}^{\otimes k}\otimes\mathcal{C})$ vanishes in degrees ≥ -1, which lets us conclude. ☐

Corollary 2.5.3. *Let R be a left regular coherent ring. Then $K_i(R) \cong K_i(R[x_1,\dots,x_n])$ for $i \geq n-1$.*

As mentioned in the introduction, the above corollary, which is more subtle for $n > 1$, was already known when R is a discrete ring, where it follows from the Farrell–Jones conjecture for the groups $\mathbb{Z}^n$.

Proposition 2.5.4 (Free generator invariance)**.** *Let $\mathcal{C} \in \mathrm{Cat}^\sharp$ have a bounded t-structure. Then $K(\mathcal{C}) \cong K(\mathcal{C}\{x_1,\dots,x_n\})$, where $|x_i| = 0$.*

Proof. We proceed by induction on n with base-case given by Proposition 2.5.1. If we consider the pushout of categories

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[17] In degree zero this uses that K_{-1} of $\mathcal{C}$ and $\mathcal{C}[\epsilon_{-1}]$ both vanish.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[d, "j"] & \mathcal{C}\{x_1, \ldots, x_{n-1}\} \arrow[d] \\
\mathcal{C}[x] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{C}\{x_1, \ldots, x_n\}.
\end{tikzcd}$$

then condition (D′) holds since the arrows labeled i and j are t-exact at the level of ind-completions and each have a section (sending all the x's to zero). As a consequence we can apply Theorem 2.4.11 and conclude. ☐

Corollary 2.5.5. *Let R be a left regular coherent ring, then $K(R) \cong K(R\{x_1, \ldots, x_n\})$.*

In fact, Corollary 2.5.5 is a special case of the next example, which allows for a more general module in place of the indeterminants $x_1, \ldots, x_n$.

Example 2.5.6. Applying Corollary 2.4.5 to the span of rings

$$R \leftarrow R\{\Sigma^{-1}M\} \rightarrow R$$

we get a pullback square

$$\begin{tikzcd}
K^{nc}(R \oplus \Sigma^{-1}M) \arrow[r] \arrow[d] & K^{nc}(R) \arrow[d] \\
K^{nc}(R) \arrow[r] & K^{nc}(R\{M\}).
\end{tikzcd}$$

Applying Theorem 2.1.1 to the section $R \rightarrow R \oplus \Sigma^{-1}M$ we learn that the top horizontal map is an equivalence on -1-connective covers, so via the pullback square we learn that $K(R) \cong K(R\{M\})$.

◁

◁

There are many more invariance-type results that can be proven using a combination of Theorem 2.1.3 and the Land-Tamme $\odot$-product and we end this subsection with a more generic example.

Example 2.5.7. Suppose we are given a map of discrete rings $R \rightarrow S$ with R left regular coherent and an S-bimodule M which is right flat over R. We can form the pullback of $\mathbb{E}_1$-algebras

$$\begin{tikzcd}
R \oplus \Sigma^{-1}M \arrow[r] \arrow[d] & S \arrow[d] \\
R \arrow[r] & S \oplus M
\end{tikzcd}$$

where $S \oplus M$ is the trivial square-zero extension of S by M and likewise for $R \oplus \Sigma^{-1}M$. From Theorem 2.1.1 we know that $K(R \oplus \Sigma^{-1}M) \cong K(R)$ and therefore

$$K(S) \cong K\left(R \odot^{S\oplus M}_{R\oplus\Sigma^{-1}M} S\right).$$

We can think of this as a relative version of $\mathbb{A}^1$-invariance for the map $R \to S$ and bi-module M. The underlying (R, S)-bimdoule of the $\odot$-product is given by $R \otimes_{R\oplus\Sigma^{-1}M} S$ which is equivalent to $R\{M\} \otimes_R S$. The free algebra $R\{M\}$ is discrete and right flat as an R-module since M is, so $R\{M\} \otimes_R S$ is discrete as well.

We now will finish completely describing its ring structure. By the previous example, which is the case $R = S$, the $\odot$-product receives ring maps from both $R\{M\}$ and S. It remains then to determine the left multiplication of an element of S by one of M. This can be read off using Example 2.4.3, which gives a cofiber sequence of S-bimodules

$$S \to R \odot^{S\oplus M}_{R\oplus\Sigma^{-1}M} S \to M \otimes_{R\oplus\Sigma^{-1}M} S,$$

showing that the left multiplication of S on M is the one coming from the left S-module structure. ◁ ◁

Note that in Example 2.5.7 the ring S is not required to be regular! For example, we can let $R = k$ be a field, take $S = k[\epsilon]/\epsilon^2$ and let M be k thought of as an S-bimodule via the augmentation. In this case we obtain an equivalence

$$K(k[\epsilon]/\epsilon^2) \cong K(k\{\epsilon, y\}/(\epsilon^2, \epsilon y)).$$

2.5.2 *K*-theory of unipotent representations

Next we analyze the K-theory of categories of local systems with values in a regular category. The following generalizes the discussion in [AGH19, Section 4.3], in which they analyze the K-theory of cochain algebras of finite, connected spaces with coefficients in commutative Noetherian rings using the Koszul dual description of the module categories in [Mat16, Proposition 7.8] as ind-unipotent representations of the loopspace.

Definition 2.5.8. Given $\mathcal{C} \in \mathrm{Cat}^\sharp$ and a connected $X \in \mathrm{Spc}_*$, let $\mathrm{Rep}(\Omega X, \mathcal{C})$ denote the category of local systems on X with values in $\mathcal{C}$ (this is just $\mathrm{Fun}(X, \mathcal{C})$)[18]. Pullback along the map $X \to *$ provides a functor

$$(-)^{\mathrm{triv}} : \mathcal{C} \to \mathrm{Rep}(\Omega X, \mathcal{C})$$

which associates to $c \in \mathcal{C}$ the constant local system at c. Let $\mathrm{Rep}(\Omega X, \mathcal{C})^{\mathrm{uni}}$ denote $\mathrm{im}((-)^{\mathrm{triv}})$ (where the image is taken as an idempotent complete stable category). We refer to this as the category of unipotent local systems valued in $\mathcal{C}$. ◁ ◁

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[18] There is a subtlety here, which is that in general $\mathrm{Rep}(\Omega X, \mathcal{C})$ and $\mathrm{Fun}(X, \mathrm{Ind}(\mathcal{C}))^\omega$ differ. It is this which motivated us to use Rep as notation when Fun would appear to suffice.

Remark 2.5.9. One can make similar definitions for $\mathcal{A}$ a small abelian category. Namely, we let $\mathrm{Rep}(\Omega X, \mathcal{A})$ denote $\mathrm{Fun}(X, \mathcal{A})$, and let $\mathrm{Rep}(\Omega X, \mathcal{A})^{\mathrm{uni}}$ denote unipotent representations, i.e the category generated under extensions, kernels and cokernels by the image of $(-)^{\mathrm{triv}}$. ◁ ◁

Proposition 2.5.10. *If* $\mathcal{C} \in \mathrm{Cat}^{\sharp}_{\geq 0}$ *is bounded and regular, and* X *is connected, then*

1. *Truncation on* $\mathcal{C}$ *provides* $\mathrm{Rep}(\Omega X, \mathcal{C})$ *a bounded* t*-structure with heart* $\mathrm{Rep}(\pi_1 X, \mathcal{C}^{\heartsuit})$.
2. *The* t*-structure on* $\mathrm{Rep}(\Omega X, \mathcal{C})$ *restricts to* $\mathrm{Rep}(\Omega X, \mathcal{C})^{\mathrm{uni}}$, *with heart* $\mathrm{Rep}(\pi_1 X, \mathcal{C}^{\heartsuit})^{\mathrm{uni}}$.
3. $(-)^{\mathrm{triv}}$ *induces an equivalence* $K(\mathcal{C}) \cong K(\mathrm{Rep}(\Omega X, \mathcal{C})^{\mathrm{uni}})$.

Proof. For (1), in order to check that $\tau_{\geq 0}$ and $\tau_{<0}$ determine a t-structure on $\mathrm{Rep}(\Omega X, \mathcal{C})$ we just need to check that the space of maps from $\tau_{\geq 0}c$ to $\tau_{<0}d$ is contractible. To do this we use the formula

$$\mathrm{Map}_{\mathrm{Rep}(\Omega X, \mathcal{C})}(\tau_{\geq 0}c, \tau_{<0}d) \cong \mathrm{Map}_{\mathcal{C}}(\tau_{\geq 0}c, \tau_{<0}d)^{h\Omega X}$$

where ΩX acts on the space of maps in $\mathcal{C}$ by conjugation. Since $\mathrm{Map}_{\mathcal{C}}(\tau_{\geq 0}c, \tau_{<0}d)$ is contractible, so is the limit under the action. Boundedness is inherited from $\mathcal{C}$ since the underlying object functor $\mathrm{Rep}(\Omega X, \mathcal{C}) \to \mathcal{C}$ is t-exact and conservative. The heart is clearly $\mathrm{Rep}(\Omega X, \mathcal{C}^{\heartsuit})$, and since $\mathcal{C}$ is a 1-category, this is the same as $\mathrm{Rep}(\Omega \tau_{\leq 1} X, \mathcal{C}^{\heartsuit})$. We conclude since $\tau_{\leq 1} X \cong B\pi_1 X$.

For (2) and (3), we check that the functor $(-)^{\mathrm{triv}}$ is unipotent (see Definition 2.3.4), so that we can apply Theorem 2.1.3 to conclude. Quasi-affineness follows from construction, and fully faithfulness on the heart follows from the fact that equivariant maps between objects in $\mathcal{C}^{\heartsuit}$ with trivial $\pi_1(X)$-action are just given by the underlying maps in $\mathcal{C}^{\heartsuit}$ since it is a 1-category. ☐

Remark 2.5.11. If R is an $\mathbb{E}_1$-algebra, then the R-module R (with trivial action) is a generator of $\mathrm{Rep}(\Omega X, \mathrm{Mod}(R))^{\mathrm{uni}}$, therefore we may identify this category with

$$\mathrm{Mod}(C^*(X; R))$$

, the category of modules over the cochain algebra of X with values in R. If R is a regular coherent discrete ring, Proposition 2.5.10(3) then provides an equivalence[19]

$$K(R) \cong K(C^*(X; R)).$$

When X is additionally compact and R Noetherian and commutative, the above result combined with Lemma 2.3.12 and vanishing of negative K-theory coincides with [AGH19, Theorem 4.8]. However, as pointed out to us by Markus Land, their proof is not quite correct, since they claim that the heart of the t-structure on $C^*(X; R)$ agrees with that of R,

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[19] One could also have deduced the equivalence by a direct application of Theorem 2.1.1: to do this, one can use the fact that a product of flat right modules over a left coherent ring is flat. In particular, the t-structure on $\mathrm{Mod}(C^*(X; R))$ is the standard one.

which is not true if for example $X = S^1$. The hearts will agree exactly when all unipotent representations of $\pi_1 X$ are trivial, which is the same as asking that $\mathrm{cof}(R \to \mathrm{End}_{R[\pi_1 X]}(R, R))$ has Tor amplitude in $[-\infty, -2]$ as a right R-module. Despite this, the hearts are in general sufficiently similar that Quillen's devissage provides an equivalence on K-theory. $\triangleleft$ $\triangleleft$

2.5.3 Testing the limits of Theorem 2.1.1

In the next sequence of examples we probe the limits of Theorem 2.1.1. Summarizing what we find: the conditions of Theorem 2.1.1 are sharp. To see that regularity of $\pi_0 R$ is necessary we look at an example where $\mathbb{A}^1$-invariance fails.

Example 2.5.12. We consider the exterior algebra $k[\epsilon_0, \epsilon_{-1}]$ over a field k. We have an equivalence from Example 2.4.9 of non-connective K-theories

$$K^{\mathrm{nc}}(k[\epsilon_0, \epsilon_{-1}]) \cong K^{\mathrm{nc}}(k[\epsilon_0]) \oplus \Sigma^{-1} K^{\mathrm{nc}}(\mathrm{N}_0 \otimes k[\epsilon_0]).$$

Since $\mathbb{A}^1$-invariance fails for $k[\epsilon_0]$ (see [HM01]), both terms in the sum are non-trivial. On the other hand Theorem 2.1.1 predicts only the first term. $\triangleleft$ $\triangleleft$

Now we turn to the tor condition of Theorem 2.1.1. Essentially the simplest example of an algebra which violates it is the trivial square zero extension $S \coloneqq \mathbb{F}_p[x] \oplus \Sigma^{-1}\mathbb{F}_p$ where x is in degree zero and acts by zero on $\mathbb{F}_p$. In a conversation with Markus Land and Georg Tamme we determined that $K_1(S)$ differs from K_1 of $\mathbb{F}_p[x]$ by using the $\odot$-product to reduce to a connective ring and then using trace methods[20].

Example 2.5.13. The algebra S fits into the pullback square on the left.

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{F}_p[x] \arrow[d] \\ \mathbb{F}_p[x] \arrow[r] & \mathbb{F}_p[x] \oplus \mathbb{F}_p \end{tikzcd} \qquad \begin{tikzcd} K^{\mathrm{nc}}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K^{\mathrm{nc}}(\mathbb{F}_p[x]) \arrow[d] \\ K^{\mathrm{nc}}(\mathbb{F}_p[x]) \arrow[r] & K^{\mathrm{nc}}\left(\mathbb{F}_p[x]\{\mathbb{F}_p\}\right) \end{tikzcd}$$

Writing the $\mathbb{F}_p[x]$-bimodule $\mathbb{F}_p[x] \oplus \mathbb{F}_p$ as the tensor product[21] $\mathbb{F}_p[x] \otimes_{\mathbb{F}_p[x]\{\Sigma^{-1}\mathbb{F}_p\}} \mathbb{F}_p[x]$ we can apply Corollary 2.4.5 to identify $\mathbb{F}_p[x] \odot^{\mathbb{F}_p[x] \oplus \mathbb{F}_p}_{\mathbb{F}_p[x]\{\Sigma^{-1}\mathbb{F}_p\}} \mathbb{F}_p[x]$ with $\mathbb{F}_p[x]\{\mathbb{F}_p\}$. From this we obtain the pullback square of K-theories on the right.

Using $\mathbb{A}^1$-invariance we have an isomorphism of relative K-theories

$$\mathrm{cof}(K^{\mathrm{nc}}(\mathbb{F}_p) \to K^{\mathrm{nc}}(S)) \cong \Sigma^{-1} \mathrm{cof}(K^{\mathrm{nc}}(\mathbb{F}_p) \to K^{\mathrm{nc}}(\mathbb{F}_p[x]\{\mathbb{F}_p\})$$

To conclude that $K_1(S)$ differs from $K_1(\mathbb{F}_p)$ we will argue that $K_2(\mathbb{F}_p[x]\{\mathbb{F}_p\})$ is not even finitely generated.

Let $R \coloneqq \mathbb{F}_p[x]\{\mathbb{F}_p\}$. We can construct a DGA model for R which is $\mathbb{F}_p[x]\{y, z\}$ with $|y| = 0, |z| = 1$, $d(z) = xy$. From this we can compute that

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[20] A similar analysis also works for $S = \mathbb{Z} \oplus \Sigma^{-1}\mathbb{F}_p$

[21] Here $\{M\}$ denotes the free algebra on a bimodule.

- $\pi_0 R \cong \mathbb{F}_p[x,y]/xy$,
- x acts by zero on $\pi_1 R$ and
- $\pi_1 R$ is a free $\mathbb{F}_p$-vector space on the classes $y^{a_0}[z,y]y^{a_1}$ with $a_0, a_1 \geq 0$.

As a consequence of Waldhausen's calculation of the first nonzero vanishing homotopy group of the fiber of $K(A) \to K(\pi_0 A)$ for a connective simplicial ring A ([Wal85, Proposition 1.2]), we learn that the fiber of $K(R) \to K(\pi_0 R)$ is 1-connected, and has second homotopy group given by

$$\mathrm{HH}_0(\mathbb{F}_p[x,y]/xy; \pi_1 R) \cong \mathbb{F}_p\{y^a[z,y] \mid a \geq 0\}.$$

Since $K_3(\mathbb{F}_p[x,y]/xy)$ is finitely generated (see [Hes07]) we learn that $K_2(R)$ is not finitely generated as promised. ◁ ◁

In the example above, although the K-theory differs from that of the connective cover, if we think in terms of Theorem 2.1.3 it is not immediately clear at what point things broke down. Possibilities include:

- The ring failed to be regular (in the sense of Definition 2.3.4).
- The base-change functor from the connective cover failed to be t-exact.
- The base-change failed to be fully faithful on the heart.

In view of this we now proceed to give several more geometric examples where we have better control over how things break down.

Example 2.5.14. Consider the quasi-affine variety $X := \mathbb{A}^n_k \setminus \{0\}$ over a field k for $n \geq 2$. Since this scheme is quasi-affine, its category of quasicoherent sheaves is equivalent to the category of modules over the ring of global sections, R. This is a commutative k-algebra whose homotopy groups are the coherent cohomology groups of $\mathbb{A}^n_k \setminus \{0\}$.

In this case we have

$$\pi_s R \cong \begin{cases} k[x_1, \dots, x_n] & s = 0 \\ (\prod_i x_i^{-1})k[x_1^{-1}, \dots, x_n^{-1}] & s = 1-n \\ 0 & \text{otherwise} \end{cases}.$$

The divisible module which shows in degree $1-n$ has tor dimension n and therefore violates condition (2) in Theorem 2.1.1. There is a localization sequence

$$\mathrm{Mod}(k[x_1, \dots, x_n])^{x_i-\mathrm{nil}}) \longrightarrow \mathrm{Mod}(k[x_1, \dots, x_n]) \longrightarrow \mathrm{Mod}(R)$$

coming from the fact that $\mathbb{A}^n_k - 0$ is an open subset of $\mathbb{A}^n_k$. $\mathrm{Mod}(k[x_1, \dots, x_n])^{x_i-\mathrm{nil}})$ is generated by $\otimes_{i=1}^n \mathrm{cof}(x_i)$, whose endomorphism is an exterior algebra over k on n classes in degree -1. Thus by Theorem 2.1.1, $K(\mathrm{Mod}(k[x_1, \dots, x_n])^{x_i-\mathrm{nil}})) \cong K(k)$, and so the map $K(\mathbb{A}^n_k) \to K(R)$ is not an equivalence, as its fiber is $K(k)$. ◁ ◁

In Example 2.5.14, the tor condition fails and the K-theories differ, but R is regular anyway. What happens here is that the map $\mathrm{Mod}(\pi_0 R)^\heartsuit \to \mathrm{Mod}(R)^\heartsuit$ isn't faithful because the module k supported at the origin is sent to zero. This example also exhibits another more subtle behavior. In [Wal85, Proposition 1.1] (which is extended to general connective ring spectra by [LT19, Lemma 2.4]), Waldhausen shows that an n-connective map of connective algebras induces an $(n+1)$-connective map on K-theory. A similar phenomenon does not occur our setting. In Example 2.5.14 the first degree where R differs from its $\pi_0 R$ is $1-n$ while the K-theory first differs in degree 1, which is independent of the parameter n.

Since Example 2.5.14 isn't tight with respect to the tor condition we now provide another family of examples which, although more geometrically degenerate, do show that the tor condition is tight.

Example 2.5.15. For $n \geq 1$, consider $\mathbb{A}^n$ with a doubled origin over the same field k, that is to say we look at the pullback below

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma(\mathbb{A}^n) \arrow[d] \\ \Gamma(\mathbb{A}^n) \arrow[r] & \Gamma(\mathbb{A}^n - 0). \end{tikzcd}$$

From our examination of $\Gamma(\mathbb{A}^n - 0)$ in Example 2.5.14[22] we know that $\pi_{-n} R$ has tor dimension n. Since $\Gamma(\mathbb{A}^n) \to \Gamma(\mathbb{A}^n - 0)$ is a localization the induced square on K-theory is a pullback (see [Tam18]). From this we can read off that

$$K(R) \cong K(\mathbb{A}^n_k) \oplus K(k)$$

where again the comparison map $\pi_0 R \to R$ induces the inclusion of the left summand. ◁ ◁

In the previous two examples the K-theory of R and its connective cover differed, but R was still regular (in the sense of Definition 2.3.4). Our next example will show that it is possible for $\pi_0 R$ to be regular while R is non-regular. To do this we use an affine nodal cubic curve C over a field k, which has non-vanishing K_{-1} (see [Wei13, **page** III.4.4]). The main result of [AGH19] then implies that the category of perfect coherent sheaves on C is not regular.

Example 2.5.16. Consider the nodal cubic curve $C \coloneqq \mathrm{Spec}(k[x,y]/y^2 - x^2(x-1))$ and let R denote the pullback below

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & k[x^{\pm 1}] \arrow[d] \\ C \arrow[r] & C[x^{-1}]. \end{tikzcd}$$

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[22] the same analysis works for $n = 1$, which isn't covered in Example 2.5.14

Geometrically, the bottom horizontal arrow corresponds to removing the nodal point from Spec(C) and the right vertical arrow corresponds to the quotient of C minus the node by the C_2 action that sends y to $-y$. The homotopy groups of R are

$$\pi_s R = \begin{cases} k[x] & s = 0 \\ (yx^{-1})k[x^{-1}] & s = -1 \\ 0 & \text{otherwise} \end{cases} .$$

As in the previous example, since the top horizontal arrow is a localization the induced square on K-theory is a pullback [Tam18]. Since $C[x^{-1}]$ is regular and Noetherian its negative K-groups vanish. This implies that $K_{-1}(R) \cong K_{-1}(C) \neq 0$. On the other hand $\pi_0 R = k[x]$ has vanishing K_{-1}. Since $K_{-1}(R)$ is non-zero, the category of compact R-modules cannot be regular. ◁ ◁

This example demonstrates that when the tor condition is violated, $\mathrm{Mod}(R)^\omega$ can fail to be regular in addition to the K-theories of R and $\pi_0 R$ differing. In essence what we have done in Example 2.5.16 is taken a singularity and hidden it in degree -1. The fact that this can be done implies that the tor condition in Theorem 2.1.1 and the condition that $\pi_0 R$ be left regular coherent cannot be disentangled—an idea we explore further in the next example.

Example 2.5.17. Suppose that R is a regular, discrete, Noetherian, commutative algebra. Using Theorem 2.1.3, Example 2.4.9, Example 2.4.10 and Proposition 2.5.1 we obtain K-theory equivalences

$$\begin{aligned} K(R[x_0, x_{-1}]/(x_0 x_{-1})) &\cong K(R) \oplus K(\mathrm{N}_0 \otimes R) \oplus K(\mathrm{N}_{-1} \otimes R) \oplus \Sigma^{-1} K(\mathrm{N}_1 \otimes R) \\ &\cong K(R[x_0]) \oplus K(R[\epsilon_0]) \cong K(R[\epsilon_0]). \end{aligned}$$

◁ ◁

What distinguishes this example is that $R[x_0, x_{-1}]/(x_0 x_{-1})$ is coconnective, has Noetherian, regular π_0, but violates the tor condition, while $R[\epsilon_0]$ is discrete (and therefore satisfies the tor condition), but is non-regular. This suggests that at the level of nc motives regularity of $\pi_0 R$ and the tor condition are not individually particularly meaningful. Instead we should think of the combination of these two conditions, i.e. unipotence, as a meaningful single condition.

2.5.4 An example we do not cover

We end the paper by giving a simple example of a ring R such that the connective cover map $\pi_0 R \to R$ induces an equivalence on nc motives, but R does not satisfy the hypotheses of Theorem 2.1.1.

Example 2.5.18. Let R be the ring $\mathrm{End}_{\mathbb{F}_p[x,y]}((x,y))^{op}$. As a module over $\mathbb{F}_p[x,y]$, (x,y) has three cells, two in degree 0, and one in degree 1 with attaching maps x and y. From this we can compute the homotopy groups of R

$$\pi_s R \cong \begin{cases} k[x,y] & s = 0 \\ (x,y)/(x,y)^2 & s = -1 \\ 0 & \text{otherwise} \end{cases} .$$

$\mathbb{F}_p[x,y]$ is regular, but $(x,y)/(x,y)^2$ has tor dimension 2 over $\mathbb{F}_p[x,y]$, so R does not satisfy the conditions of Theorem 2.1.1.

Now we proceed to show that R is regular and the connective cover map induces an equivalence of nc motives. In working with perfect R-modules we identify this category with the thick subcategory of $\mathrm{Mod}(\mathbb{F}_p[x,y])^{\omega}$ generated by (x,y). To see that R is regular, first observe that $\mathbb{F}_p$ is connective in the standard t-structure for R since it is a retract of $(x,y)\otimes_{\mathbb{F}_p[x,y]}\mathbb{F}_p$. From the extension $(x,y) \to \mathbb{F}_p[x,y] \to \mathbb{F}_p$ we can then conclude that $\mathbb{F}_p[x,y]$ is connective in this t-structure as well. This then implies that $\mathrm{Mod}(R)^{\omega}_{\geq 0}$ is equivalent to $\mathrm{Mod}(\mathbb{F}_p[x,y])^{\omega}_{\geq 0}$. Since (x,y) represents the class 1 in $K_0(\mathbb{F}_p[x,y])$, base-change along the map $\pi_0 R = \mathbb{F}_p[x,y] \to R$, which can be identified with the functor $\otimes_{\mathbb{F}_p[x,y]}(x,y)$, induces multiplication by 1 on the nc motive of $\mathrm{Mod}(\mathbb{F}_p[x,y])^{\omega}$. $\triangleleft$ $\triangleleft$

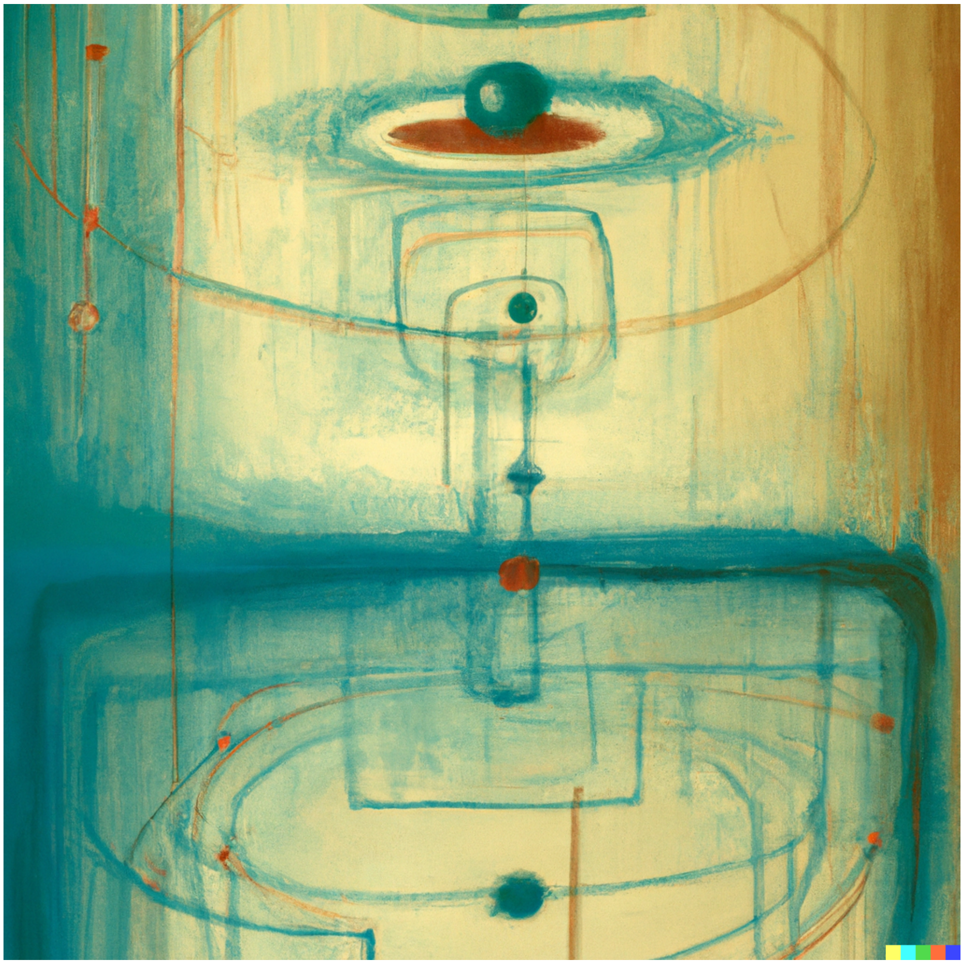
In this example the natural map $K(\pi_0 R) \to K(R)$ is an equivalence despite the fact that R doesn't satisfy the conditions of Theorem 2.1.1. The essential issue here is that (x,y) is not flat, i.e. the base-change functor $\pi_0 R \to R$ is not t-exact. Since, at its core Theorem 2.1.1 operates using Quillen's devissage theorem it cannot be used for examples of this type. The equivalence of K-theories in this example arises because R is Morita equivalent to its connective cover, which is a different (and less interesting) reason for them to agree. As noted above, this implies that the connective cover map induces an equivalence of nc motives in this case, something which rarely happens for rings to which one can apply Theorem 2.1.1.

Another point contrasting with Theorem 2.1.1 is the fact that the equivalence $K(\pi_0 R) \cong K(R)$ is not visible at the level of the homotopy ring of R. Indeed, if R' is the trivial square zero extension of $\mathbb{F}_p[x,y]$ by $\Sigma^{-1}(x,y)/(x,y)^2$, then its homotopy ring agrees with R, but $K(\pi_0 R') \to K(R')$ is not an equivalence, because the map $\pi_0 R' \to R'$ has the map $\mathbb{F}_p[x] \to \mathbb{F}_p[x] \oplus \Sigma^{-1}\mathbb{F}_p$ as a retract, which was shown in Example 2.5.13 to not be a K-theory equivalence.

Chapter 3

The algebraic K-theory of the K(1)-local sphere via TC

We describe the algebraic K-theory of the $K(1)$-local sphere and the category of type 2 finite spectra in terms of K-theory of discrete rings and topological cyclic homology. We find an infinite family of 2-torsion classes in the K_0 of type 2 spectra at the prime 2, and explain how to construct representatives of these K_0 classes.


[1]

3.1 Introduction

Fix a prime p and let $\mathrm{Sp}_p^{\diamondsuit}$ be the category of dualizable p-complete spectra. One of the fundamental results in stable homotopy theory is the thick subcategory theorem of Hopkins and Smith [HS98], which says that every nonzero thick subcategory of $\mathrm{Sp}_p^{\diamondsuit}$ is one of the categories $\mathrm{Sp}^{\omega}_{\geq n}$[2] of finite spectra of type at least n for some $n \geq 0$. A modern interpretation of this result is the statement that the Balmer spectrum of $\mathrm{Sp}_p^{\diamondsuit}$ agrees with the Zariski spectrum of $\mathcal{M}_{\mathrm{fg},p}$, the moduli stack of p-typical formal groups.

In this paper, we study a subtle additional structure on the Balmer spectrum of $\mathrm{Sp}_p^{\diamondsuit}$, namely its sheaf of algebraic K-theory.[3] To the open set corresponding to the height $\leq n$ locus, this sheaf takes the value $K(L_n^f \mathbb{S}_p)$, and on global sections, it is $K(\mathbb{S}_p)$. This sheaf was

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[1] image generated by OpenAI DALL·E 2

[2] when $n = 0$, this denotes the category $\mathrm{Sp}_p^{\diamondsuit}$.

[3] It is not important that we work in a p-completed setting, it is just convenient, as the chromatic localizations L_n^f don't affect the rationalization or ℓ-adic completions.

first considered in [Wal84], where fundamental localization sequences relating $K(L_n^f\mathbb{S}_p)$ to the K-theory of the monochromatic layers $K(\mathrm{Sp}^{\omega}_{T(n)})$ were observed. Thomason in [Tho97] showed that understanding the homotopy groups of the sheaf in low degrees would allow one to refine the thick subcategory theorem, and classify *stable* subcategories[4] of $\mathrm{Sp}_p^{\diamondsuit}$.

The only case in which $K(L_n^f\mathbb{S}_p)$ is well understood is the case $n = 0$, where it is $K(\mathbb{Q}_p)$.[5] For $n \geq 1$, essentially the only thing previously known about $K(L_n^f\mathbb{S}_p)$ was its chromatic height, because of redshift [AR02; CMNN23; LMMT20; HW22; Yua21]. Further information was previously out of reach: for example, no K group was previously known.

In contrast, $K(\mathbb{S}_p)$ is now well understood. The reason is that $\mathbb{S}_p$ is a connective ring, and the Dundas–Goodwillie–McCarthy (or DGM) theorem [DGM13; Ras18] gives a pullback square for any connective ring R of the form

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(R) \arrow[d] \\
K(\pi_0 R) \arrow[r] & \mathrm{TC}(\pi_0 R)
\end{tikzcd}$$

where the horizontal maps are the cyclotomic trace. This largely reduces the computation of K-theory to understanding TC and the K-theory of discrete rings, both of which are usually more tractable invariants. Finally one must analyse the cyclotomic trace and reconstruct the K-theory of a connective ring from its constituent pieces in the pullback square. For the sphere, this is carried out in [Rog03; BM19][6].

The rings $L_n^f\mathbb{S}_p$ are not connective, so DGM cannot directly be applied to compute their K-theory. Here we show nevertheless that $K(L_1^f\mathbb{S}_p)$ can be described in terms of TC and K-theory of discrete rings, answering Problem 2.6 of [Ant15]. To state our result, we need to introduce the ring j_ζ below.

Definition A. *Let j_ζ be the $\mathbb{E}_\infty$-ring $\ell_p^{h\mathbb{Z}}$ for $p > 2$, and $ko_2^{h\mathbb{Z}}$ if $p = 2$. Here ℓ_p is the p-completed Adams summand of connective topological K-theory and ko_2 is 2-completed connective real topological K-theory, and the $\mathbb{Z}$ action comes from the Adams operation Ψ^{1+p}.*

The underlying spectrum of j_ζ can be described as the -1-connective cover of the $K(1)$-local sphere.

Theorem A. *$K(L_1^f\mathbb{S}_p) \cong K(L_{K(1)}\mathbb{S})$, there is a cofibre sequence split on π_**

$$K(j_\zeta) \longrightarrow K(L_{K(1)}\mathbb{S}) \longrightarrow \Sigma K(\mathbb{F}_p)$$

and a pullback square

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[4] In contrast to thick subcategories, stable subcategories may not be closed under retracts.

[5] see [Wei05] for a discussion of what is known.

[6] to understand the homotopy groups of the K-theory of the sphere from the pullback square, a finite generation result of Dwyer as well as an analysis of the arithmetic fracture square are used in [BM19].

$$\begin{tikzcd}
K(j_\zeta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(j_\zeta) \arrow[d] \\
K(\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

Let F be the fibre of the map $\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})$. Then $F[\frac{1}{p}] = 0$. For $p > 2$, F is $(2p-2)$-connective and $\pi_{2p-2}(F/p) \cong \bigoplus_0^\infty \mathbb{F}_p$. For $p = 2$, F is 1-connective and $\pi_1 F \cong \bigoplus_0^\infty \mathbb{F}_2$.

In particular, even for the ring $L_1^f\mathbb{S}$, whose localizations at the primes other than p agree with that of the sphere, its K-theory is not degree-wise finitely generated! This is in sharp contrast to $K(\mathbb{S})$ and $K(L_0^f\mathbb{S}) = K(\mathbb{S}[\frac{1}{p}])$, which are degree-wise finitely generated.

The proof of the cofibre sequence in Theorem A, which is carried out in section 3.2, comes from analysing the localization sequence

$$\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2} \longrightarrow \mathrm{Mod}(j_\zeta)^\omega \longrightarrow \mathrm{Mod}(L_1^f j_\zeta)^\omega$$

and showing that on K-theory, it induces the desired cofibre sequence. Because $L_1^f j_\zeta = L_{K(1)}\mathbb{S}$, the only substantial claim is that $K(\mathbb{F}_p) \cong K(\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2})$. To obtain this we choose a particularly good generator of the category $\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2}$, namely $j_\zeta \otimes Z$, where for $p > 2$, Z is the Smith–Toda complex $\mathbb{S}/(p, v_1)$ constructed by Toda [Tod71], and for $p = 2$ it is $\mathbb{S}/(2, \eta, v_1)$, the type 2 spectrum constructed by Davis and Mahowald [DM81]. We then show that the endomorphism ring of this generator is coconnective with $\pi_0 = \mathbb{F}_p$, so that we can conclude by applying the devissage result of [BL23] that $K(\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2}) \cong K(\mathbb{F}_p)$.

The other main claim in Theorem A is the pullback square, which allows us to understand $K(j_\zeta)$. This is almost an immediate consequence of Theorem B below, which extends the DGM theorem to include the map $\ell_p^{h\mathbb{Z}} \to \mathbb{Z}_p^{h\mathbb{Z}}$. Recall that a *truncating invariant* E is a localizing invariant for which the map $E(R) \to E(\pi_0 R)$ is an equivalence for any connective ring R. In this language, DGM says that the fibre of the cyclotomic trace is truncating.

Theorem B. *Let $f : R \to S$ be a map of connective $\mathbb{E}_1$-rings with a $\mathbb{Z}$-action such that f is 1-connective. Then for any truncating invariant E, $E(R^{h\mathbb{Z}}) \to E(S^{h\mathbb{Z}})$ is an equivalence. Moreover, if f is n-connective, then $\mathrm{TC}(R^{h\mathbb{Z}}) \to \mathrm{TC}(S^{h\mathbb{Z}})$ is too.*

We also obtain the following variant:

Theorem C. *Let $R \to S$ be a 1-connective map of -1-connective rings such that $\pi_{-1}R$ is a finitely generated $\pi_0 R$-module. Then for any truncating invariant E, $E(R) \to E(S)$ is an equivalence.*

The proof of Theorem B, which can be found in section 3.3 is an application of the work of Land–Tamme on the K-theory of pullbacks. Namely, one has a pullback diagram

$$\begin{tikzcd}
R^{h\mathbb{Z}} \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & R \times R
\end{tikzcd}$$

Applying the main result of [LT19], one obtains a pullback square after applying any localizing invariant, where $R \times R$ is replaced by the ring $R \odot_{R^{h\mathbb{Z}}}^{R\times R} R$. The latter ring is connective, and comparing with the analogous construction for S and using the pullback square and the fact that the invariant is truncating, one obtains the result.

In addition to Theorem A, we also obtain a similar formula for $K(\mathrm{Sp}^{\omega}_{T(1)})$ in Theorem 3.4.2 , which we use in section 3.7 to answer [HS99, Problem 16.4] at height 1. For $K(\mathrm{Sp}^{\omega}_{\geq 2})$ we obtain the result below.

Theorem D. *There is a fibre sequence* $X \to K(\mathrm{Sp}^{\omega}_{\geq 2}) \to K(\mathbb{F}_p)$ *split on* π_**, where* X *is the total fibre of the square*

$$\begin{tikzcd}
\mathrm{TC}(\mathbb{S}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathbb{Z}_p) \arrow[d] \\
\mathrm{TC}(j_\zeta) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

- *For* $p > 2$*,* X *is* $(2p-3)$*-connective, so* $K_0(\mathrm{Sp}^{\omega}_{\geq 2}) = \mathbb{Z}$ *with generator* $[\mathbb{S}/(p, v_1)]$.
- *For* $p = 2$*,* X *is connective with* $\pi_0 X \cong \bigoplus_0^\infty \mathbb{Z}/2$*, and the torsion free quotient of* $K_0(\mathrm{Sp}^{\omega}_{\geq 2})$ *is generated by* $[\mathbb{S}/(2, \eta, v_1)]$.

In particular we find, contrary to our initial expectations that at the prime 2 there are infinitely many 2-torsion classes in $K_0(\mathrm{Sp}^{\omega}_{\geq 2})$! As a corollary, we obtain a classification of dense stable subcategories of type 2 spectra. A full stable subcategory $C' \subset C$ is *dense* if the inclusion is an equivalence on idempotent completions.

Corollary 3.1.1. *(Corollary 3.5.2) The dense stable subcategories of* $\mathrm{Sp}^{\omega}_{\geq 2}$ *for* $p > 2$ *are in bijection with subgroups of* $\mathbb{Z}$*, and the dense stable subcategories of* $\mathrm{Sp}^{\omega}_{\geq 2}$ *at the prime* 2 *are in bijection with subgroups of* $\mathbb{Z} \oplus \bigoplus_0^\infty \mathbb{F}_2$.

In section 3.6, we explain how to construct explicit spectra representing all of the 2-torsion classes, but we briefly explain the first one here. We first choose a self map $v_1^4 : \Sigma^8\mathbb{S}/2 \to \mathbb{S}/2$. Because $\eta\sigma$ is 2-torsion in $\pi_8\mathbb{S}_2$, we can produce an extension of it to a map $\Sigma^8\mathbb{S}/2 \to \mathbb{S}$. Let $\overline{\eta\sigma}$ be the composite

$$\overline{\eta\sigma} : \Sigma^8\mathbb{S}/2 \to \mathbb{S} \to \mathbb{S}/2$$

Then $[\mathbb{S}/(2, v_1^4 + \overline{\eta\sigma})] - [\mathbb{S}/(2, v_1^4)]$ represents the first 2-torsion class in $K_0(\mathrm{Sp}_{\geq 2})$.

We ask open questions throughout the paper related to this work. A particularly important one is the following:

Question 3.1.2. What can be said about $\mathrm{TC}(j_\zeta)$? For example, can its homotopy groups be computed, at least mod (p, v_1) or (p, v_1, v_2)?

As a first step to the above question, in forthcoming work with David Lee, we compute $\mathrm{THH}(j_\zeta)$ mod (p, v_1) at odd primes.

Theorem 3.1.3 (Lee–Levy [LL23]). *For $p > 2$, there is an isomorphism of graded rings*

$$\pi_* \operatorname{THH}(j_\zeta)/(p, v_1) \cong \pi_* \operatorname{THH}(\ell)/(p, v_1) \otimes \pi_* \mathrm{HH}(C^*(S^1; \mathbb{F}_p)/\mathbb{F}_p)$$

We don't know the extent to which the methods of this paper are capable of understanding higher height phenomena. Despite this, in forthcoming work with Robert Burklund, we completely compute the K-theory sheaf after inverting the prime p.

Theorem 3.1.4 (Burklund–Levy [BL25]). *For $n \geq 1$, there are isomorphisms*

$$K(L_n\mathbb{S}_p)[\frac{1}{p}] \cong K(L_n^f\mathbb{S}_p)[\frac{1}{p}] \cong K(\mathbb{Z}_p)[\frac{1}{p}] \oplus \Sigma K(\mathbb{F}_p)[\frac{1}{p}]$$

Moreover $K(\mathrm{Sp}_{\geq n})[\frac{1}{p}] \cong K(\mathbb{F}_p)[\frac{1}{p}]$, *and a generator of* $K_0(\mathrm{Sp}_{\geq n})[\frac{1}{p}]$ *is given by the class of a generalized Moore spectrum* $[\mathbb{S}/(p, v_1^{p^{i_1}}, \dots, v_{n-1}^{p^{i_{n-1}}})]$.

Conventions

We assume the reader is familiar with higher algebra and algebraic K-theory. Some conventions we use are:

- The term category refers to an ∞-category as developed by Joyal and Lurie.
- $\mathrm{Map}_C(a, b)$ denotes the space of maps from a to b in a category C. C is omitted from the notation when it is clear from context.
- Similarly, in a stable category C, $\mathrm{map}_C(a, b)$ denotes the mapping spectrum.
- For an $\mathbb{E}_1$-algebra R, $\mathrm{Mod}(R)$ refers to its category of left modules.
- For an $\mathbb{E}_1$-algebra R and a stable category C, $R \otimes C$ is shorthand for either $\mathrm{Mod}(R) \otimes C$ if C is presentable or $\mathrm{Mod}(R)^\omega \otimes C$ if C is small.[7]
- We use $K(-)$ for *nonconnective* K-theory. For a compactly generated stable category C, $K(C)$ will mean $K(C^\omega)$.
- $\mathcal{U}_{\mathrm{loc}}$ and $\mathcal{U}_{\mathrm{add}}$ denote the versions of the universal localizing and additive invariants of [BGT13] that do not preserve any kind of filtered colimits.
- We use x_n for a polynomial generator in degree n and ϵ_n for an exterior generator in degree n. As an example, $\mathbb{S}[x_n]$ is the free $\mathbb{E}_1$-algebra on a class in degree n.

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[7] there is no ambiguity because if C is stable, presentable, and small, it must be zero, in which case $\mathrm{Mod}(R) \otimes C$ and $\mathrm{Mod}(R)^\omega \otimes C$ are both the zero category.

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3.2 Localization sequences and devissage

The main goal of this section is to prove the parts of Theorem A that come from localization sequences and devissage, namely Proposition 3.2.2, Proposition 3.2.3, and Lemma 3.2.4 below. These results allow us to reduce the study of objects such as $K(L_1^f\mathbb{S})$, $K(\mathrm{Sp}_{\geq 2}^{\omega})$, and $K(\mathrm{Sp}_{K(1)}^{\omega})$ to the study of $K(j_\zeta)$. The key tool here is devissage in the form given in [BL23].

Theorem 3.2.1 ([BL23])**.** *If R is a coconnective ring with π_0 regular, and π_{-i} has tor dimension $< i$ over π_0, then the connective cover map $\pi_0 R \to R$ is an equivalence on K-theory.*

To begin, recall that there is a localization sequence

$$\mathrm{Sp}_{\geq n+1} \to \mathrm{Sp} \to L_n^f\mathrm{Sp} \tag{3.1}$$

Our propositions are obtained by tensoring this with the rings in question.

Proposition 3.2.2. *The natural map $K(L_1^f\mathbb{S}_p) \to K(L_{K(1)}\mathbb{S})$ is an equivalence.*

Proof. Tensoring the localization sequence (3.1) for $n = 0$ with the map $L_1^f\mathbb{S}_p \to L_{K(1)}\mathbb{S}$, we get a map of localization sequences

$$\begin{tikzcd}
\mathrm{Mod}(L_1^f\mathbb{S}_p)^{p-\mathrm{nil}} \arrow[r] \arrow[d] & \mathrm{Mod}(L_1^f\mathbb{S}_p) \arrow[r] \arrow[d] & \mathrm{Mod}(L_0^f L_1^f\mathbb{S}_p) \arrow[d] \\
\mathrm{Mod}(L_{K(1)}\mathbb{S})^{p-\mathrm{nil}} \arrow[r] & \mathrm{Mod}(L_{K(1)}\mathbb{S}) \arrow[r] & \mathrm{Mod}(L_0^f L_{K(1)}\mathbb{S})
\end{tikzcd}$$

The category in the top left is the category of $T(1)$-local spectra, whereas the one in the bottom left is the category of $K(1)$-local spectra. By the telescope conjecture at height 1 [Mil81; Mah81], these two categories agree, so the left vertical map is an equivalence.

L_0^f is just inverting p, and $L_0^f L_1^f\mathbb{S}_p = L_0^f\mathbb{S}_p = \mathbb{Q}_p$. $L_0^f L_{K(1)}\mathbb{S}$ is the ring $\mathbb{Q}_p[\epsilon_{-1}]$, ($\epsilon_{-1}$ is usually called ζ). It follows that the right vertical map is a connective cover map, and so applying Theorem 3.2.1, it is an equivalence on K-theory.

Since K-theory is a localizing invariant, the middle vertical map is also an equivalence on K-theory. □

The next proposition is somewhat more subtle, because the ring j_ζ is not regular in the sense of [BL24; BL23]. Nevertheless, a formal neighborhood of its height ≥ 2 locus is regular, which is all that is needed.

Proposition 3.2.3. *There is a cofibre sequence $K(\mathbb{F}_p) \to K(j_\zeta) \to K(L_{K(1)}\mathbb{S})$.*

Proof. Tensoring the localization sequence 3.1 for $n = 1$ (relative to Sp^ω) with $\mathrm{Mod}(j_\zeta)^\omega$, we get a cofibre sequence

$$\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2} \longrightarrow \mathrm{Mod}(j_\zeta)^\omega \longrightarrow \mathrm{Mod}(L_1^f j_\zeta)^\omega$$

We claim that the natural map $L_1^f j_\zeta \to L_{K(1)}\mathbb{S}$ is an equivalence. Indeed, it is clear that $L_{K(1)} j_\zeta = L_{K(1)}\mathbb{S}$, and j_ζ and $L_{K(1)}\mathbb{S}$ are rationally both $\mathbb{Q}_p[\epsilon_{-1}]$, so this follows.

It remains to identify $K(\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2})$ with $K(\mathbb{F}_p)$. By the thick subcategory theorem, $\mathrm{Sp}_{\geq 2}$ is generated by any type 2 spectrum Z, so $\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2}$ is generated by $j_\zeta \otimes Z$ for such Z.

Let Z denote the type 2 spectrum which for $p > 2$ is the Smith-Toda complex $\mathbb{S}/(p, v_1)$ constructed in [Tod71] and for $p = 2$, is the type 2 complex $\mathbb{S}/(2, \eta, v_1)$[8]constructed in [DM81]. The key property of Z is that $Z \otimes \mathrm{ko} \cong \mathbb{F}_2$ at the prime 2 and $Z \otimes \ell \cong \mathbb{F}_p$ for $p > 2$. It follows that $Z \otimes j_\zeta \cong \mathbb{F}_p^{h\mathbb{Z}}$, which is in particular coconnective with $\pi_0 = \mathbb{F}_p$.

Since Z has only one cell in dimension 0 and the rest in positive degrees, $\mathrm{End}(Z \otimes j_\zeta) \cong Z \otimes Z^\vee \otimes j_\zeta$ is also coconnective with $\pi_0 = \mathbb{F}_p$. We learn from Morita theory and Theorem 3.2.1 that $K(j_\zeta \otimes \mathrm{Sp}_{\geq 2}) \cong K(\mathrm{End}_{j_\zeta}(j_\zeta \otimes X)) \cong K(\mathbb{F}_p)$.

□

The following lemma is necessary to get that the cofibre sequences in Theorem A and Theorem 3.4.2 are split on π_*.

Lemma 3.2.4. *$K(\mathbb{Z}_p^{h\mathbb{Z}} \otimes \mathrm{Sp}^\omega_{\geq 1}) \cong K(\mathbb{F}_p)$ and the composite*

$$\mathrm{Mod}(\mathbb{F}_p)^\omega \to j_\zeta \otimes \mathrm{Sp}^\omega_{\geq 2} \to j_\zeta \otimes \mathrm{Sp}^\omega_{\geq 1} \to \mathbb{Z}_p^{h\mathbb{Z}} \otimes \mathrm{Sp}^\omega_{\geq 1}$$

is null after applying $\mathcal{U}_{add}$.

Proof. $\mathrm{cof}\, p \in \mathrm{Mod}(\mathbb{Z}_p^{h\mathbb{Z}})^{p-\mathrm{nil}}$ has a coconnective endomorphism ring with $\pi_0 = \mathbb{F}_p$, so $K(\mathrm{Mod}(\mathbb{Z}_p^{h\mathbb{Z}})^{p-\mathrm{nil}}) = K(\mathbb{F}_p)$ by Theorem 3.2.1. It remains to prove the second claim.

Let R be ℓ_p or ko_2 depending on the prime so that $R^{h\mathbb{Z}} = j_\zeta$ by definition. Then there are $\mathbb{Z}$-equivariant $\mathbb{E}_\infty$-maps $\pi : R \to \mathbb{F}_p$ and $f : R \to \mathbb{Z}_p$, and $\mathbb{F}_p$ is perfect over R as it is $R \otimes Z$, where Z is as in the proof of Proposition 3.2.3. We use the same names to denote the induced maps on $\mathbb{Z}$-homotopy fixed points. There is also the connective cover map $g : \mathbb{F}_p \to \mathbb{F}_p^{h\mathbb{Z}}$.

Taking $\mathbb{Z}$-homotopy fixed points, we obtain a diagram

---

[8] There are actually 8 distinct v_1 self maps on $\mathbb{S}/(2, \eta)$ and 4 nonisomorphic $\mathbb{S}/(2, \eta, v_1)$s, but this is irrelevant here: for example all choices of $\mathbb{S}/(2, \eta, v_1)$ become isomorphic after basechange to j_ζ. To see this, one can see that there is no obstruction to producing an isomorphism over j_ζ of any two of these. This comes from the fact that $Z \otimes j_\zeta$ is coconnective, but is built out of cells which after the bottom one are in positive degree.

$$\begin{tikzcd} \mathrm{Mod}(\mathbb{Z}_p^{h\mathbb{Z}})^{\omega,p-\mathrm{nil}} & \mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}^\omega_{\geq 2} \arrow[l, "f^*"] \arrow[r, shift left, "\pi^*"] & \mathrm{Mod}(\mathbb{F}_p^{h\mathbb{Z}})^\omega \arrow[l, shift left, "\pi_*"] \end{tikzcd}$$

Here π_* is the right adjoint of π^*, which exists since $\mathbb{F}_p^{h\mathbb{Z}}$ is perfect over j_ζ. We also have the map $g^* : \mathrm{Mod}(\mathbb{F}_p)^\omega \to \mathrm{Mod}(\mathbb{F}_p^{h\mathbb{Z}})^\omega$, and we can rephrase our lemma as saying that $f^* \circ \pi_* \circ g^*$ is null on $\mathcal{U}_{\mathrm{add}}$.

We will in fact just show that $f^* \circ \pi_*$ is null on $\mathcal{U}_{\mathrm{add}}$. To do this, that composite is given by tensoring with the $\mathbb{Z}_p^{h\mathbb{Z}} - \mathbb{F}_p^{h\mathbb{Z}}$-bimodule $\mathbb{Z}_p^{h\mathbb{Z}} \otimes_{j_\zeta} \mathbb{F}_p^{h\mathbb{Z}}$. We have a chain of equivalences

$$\mathbb{Z}_p^{h\mathbb{Z}} \otimes_{j_\zeta} \mathbb{F}_p^{h\mathbb{Z}} \cong \mathbb{Z}_p^{h\mathbb{Z}} \otimes Z \cong (\mathbb{Z}_p \otimes Z)^{h\mathbb{Z}} \cong (\mathbb{Z}_p \otimes_R \mathbb{F}_p)^{h\mathbb{Z}}$$

Thus the Postnikov filtration on the $\mathbb{Z}_p - \mathbb{F}_p$-bimodule $\mathbb{Z}_p \otimes_R \mathbb{F}_p$ gives a finite $\mathbb{Z}$-equivariant filtration with associated graded $\mathbb{F}_p[\epsilon_{|v_1|+1}]$ for $p > 2$ and $\mathbb{F}_2[\epsilon_{|\eta|+1}, \epsilon_{|v_1|+1}]$ for $p = 2$. Taking the $\mathbb{Z}$-fixed points of this filtration, we get a finite filtration of the bimodule $\mathbb{Z}_p^{h\mathbb{Z}} \otimes_{j_\zeta} \mathbb{F}_p^{h\mathbb{Z}}$ whose associated graded is $\mathbb{F}_p^{h\mathbb{Z}}[\epsilon_{|v_1|+1}]$ for $p > 2$ and $\mathbb{F}_2^{h\mathbb{Z}}[\epsilon_{|\eta|+1}, \epsilon_{|v_1|+1}]$ for $p = 2$. As a $\mathbb{Z}_p^{h\mathbb{Z}} - \mathbb{F}_p^{h\mathbb{Z}}$-bimodule, $\mathbb{F}_p^{h\mathbb{Z}}$ corresponds to the functor $\mathrm{Mod}(\mathbb{F}_p^{h\mathbb{Z}})^\omega \to \mathrm{Mod}(\mathbb{Z}_p^{h\mathbb{Z}})^\omega$ that is right adjoint to the base change from $\mathbb{Z}_p^{h\mathbb{Z}}$ to $\mathbb{F}_p^{h\mathbb{Z}}$. Since $\epsilon_{|v_1|+1}$ is in odd degree, $\mathcal{U}_{\mathrm{add}}$ splits finite filtrations and sends suspension to -1, we learn that after applying $\mathcal{U}_{\mathrm{add}}$, the $f^* \circ \pi_*$ becomes null. □

3.3 Topological cyclic homology

In this section we prove Theorem 3.3.5 and Theorem 3.3.7, which refine Theorem B and Theorem C, and in particular extend the Dundas-Goodwillie-McCarthy theorem to certain -1-connective rings. This allows us to understand $K(j_\zeta)$ in terms of the cyclotomic trace.

Given a ring R giving R a $\mathbb{Z}$ action is the same as giving an automorphism ϕ of R. $R^{h\mathbb{Z}}$ is then the pullback of the diagonal map $\Delta : R \to R \times R$ along the twisted diagonal $(1, \phi) : R \to R \times R$. The idea for extending DGM to the nonconnective ring $R^{h\mathbb{Z}}$ is to use the work of Land–Tamme on the K theory of pullbacks to relate the K-theory and TC of $R^{h\mathbb{Z}}$ to that of connective rings.

Recall that for any $\mathbb{E}_1$-ring R, there is a standard t-structure on $\mathrm{Mod}(R)$, where a module is connective iff it is generated under colimits and extensions by R, and coconnective iff its underlying spectrum is.

Lemma 3.3.1. *Let R be a -1-connective $\mathbb{E}_1$-ring. Then for any R-module M which is connective as a spectrum, M is connective in the standard t-structure on* $\mathrm{Mod}(R)$. *In particular, for any right R-module N whose underlying spectrum is connective, $M \otimes_R N$ is connective.*

Proof. Using the t-structure on R-modules, we obtain a cofibre sequence $\tau_{\geq 0}M \to M \to \tau_{<0}M$. $\tau_{\geq 0}M$ is -1-connective as an underlying spectrum since R is, and it is built from R via colimits and extensions. $\tau_{<0}M$ is coconnected as an underlying spectrum. Since M is connective as a spectrum and $\tau_{\geq 0}M$ is -1-connective as a spectrum, $\tau_{<0}M$ is connective as well, so must be 0. It follows that $M = \tau_{\geq 0}M$ is connective in the t-structure. $M \otimes_R N$ is connective since it is built from $R \otimes_R N = N$ out of colimits and extensions and N is connective. ☐

Lemma 3.3.2. *Suppose that $R \to R'$ is an i-connective map of -1-connective $\mathbb{E}_1$-rings for $i \geq -1$, M, N are right and left R'-modules that are connective in the standard t-structure. Then the map $M \otimes_R N \to M \otimes_{R'} N$ is $(i+1)$-connective.*

Proof. M, N are built out of R' under colimits and extensions, so it suffices to assume $M \cong N \cong R'$. Then we are trying to show that $R' \otimes_R R' \to R'$ is i-connective. This map has a section given by the left unit, so its fibre is the cofibre of the section. The cofibre of the unit map, M', is $(i+1)$-connective by assumption. $M' \otimes_R R'$ is an extension of $M' \otimes_R R = M'$ by $M' \otimes_R M'$. $M' \otimes_R M'$ is $(2i+2)$-connective by Lemma 3.3.1, so the result follows. ☐

The following proposition is due to Waldhausen [Wal78b, Proposition 1.2][9], except he stated it for $\mathbb{Z}$-algebras, though the general proof is identical. We reproduce the proof below for convenience and future reference. The proposition is a precursor to trace methods.

Proposition 3.3.3 (Waldhausen)**.** *Let $f : R \to S$ be an i-connective map of connective $\mathbb{E}_1$-algebras for $i \geq 1$. Then $\mathrm{fib}(K(f))$ is $(i+1)$-connective, with*

$$\pi_{i+i} \operatorname{fib}(K(f)) \cong \mathrm{HH}_0(\pi_0 S; \pi_i \operatorname{fib} f)$$

.

Proof. Since $i+1 \geq 2$, it suffices by the Hurewicz theorem to show that $\mathrm{fib}(K(f))$ is $(i+1)$-connective and that $H_{i+1} \operatorname{fib}(K(f)) \cong \mathrm{HH}_0(\pi_0 R; \pi_i \operatorname{fib} f)$. The nonpositive K-theory only depends on $\pi_0 R$ for a connective ring [BGT13, Theorem 9.53], so by the Hurewicz theorem, it suffices to show the connectivity statement at the level of homology of BGL^+.

Consider the map of homology Serre spectral sequences computing the homologies of $\mathrm{BGL}(R)$ and $\mathrm{BGL}(R)^+$ via the vertical maps in the diagram below:

$$\begin{tikzcd}
\mathrm{BGL}(R) \arrow[r] \arrow[d] & \mathrm{BGL}(R)^+ \arrow[d] \\
\mathrm{BGL}(S) \arrow[r] & \mathrm{BGL}(S)^+
\end{tikzcd}$$

The signature of the E_2-term of the Serre spectral sequence for the left vertical map is

$$H_p(\mathrm{BGL}(S); H_q(\operatorname{fib} \mathrm{BGL}(f))) \implies H_{p+q}(\mathrm{BGL}(R))$$

The first nonzero term in this spectral sequence is $H_0(\mathrm{BGL}(S); H_{i+1}(\operatorname{fib} \mathrm{BGL}(f)))$. Because GL and the infinite matrix ring M only disagree on π_0 and f is i-connective for $i \geq 1$, $\operatorname{fib} \mathrm{GL} f$ can be identified with $\operatorname{fib} M f$. By the Hurewicz theorem, $H_{i+1}(\operatorname{fib} \mathrm{BGL} f)$ then agrees with

---

[9] see also [LT19, Lemma 2.4].

$\pi_i \operatorname{fib} Mf = \pi_i M \operatorname{fib} f$, where we view $\operatorname{fib} f$ as a nonunital ring in order to make sense of Mf. Under this identification, the action of $\pi_0\mathrm{GL}(S)$ is identified with conjugation action of $\pi_0\mathrm{GL}(R)$ on $\pi_i M \operatorname{fib} f$. The trace then gives an isomorphism $\mathrm{tr} : H_0(\mathrm{BGL}(S); \pi_i(M \operatorname{fib} f)) \to \mathrm{HH}_0(\pi_0 S; \pi_i \operatorname{fib} f)$.

The E_2-term of the Serre spectral sequence for the right vertical map on the other hand is $H_*(\mathrm{BGL}(S)^+; H_*(\operatorname{fib} K(f)))$. Since $H_*(BGL(R)) \cong H_*(BGL(R))^+$, the map of Serre spectral sequences yields an isomorphism on abutments. $\mathrm{BGL}(R)^+ \to \mathrm{BGL}(S)^+$ comes from a map of spectra (the 1-connective cover of K-theory), so the coefficient system for homology is trivial, and in the lowest degree s in which $\operatorname{fib}(K(f))$ is nonzero, its homology is $H_s(\operatorname{fib} K(f))$, which must survive to the E_∞-page for degree reasons. Since in the left vertical map Serre spectral sequence, there are no terms contributing to H_s for $s \leq i$, we must then have that $H_s(\operatorname{fib} K(f)) = 0$ for $s \leq i$, giving the connectivity statement. By looking at the lowest nonvanishing terms in the spectral sequences and comparing the two spectral sequences, we then learn that $H_{i+1}(\operatorname{fib} K(f))) \cong H_0(\mathrm{BGL}(S); \pi_i(M \operatorname{fib} f)) \cong \mathrm{HH}_0(\pi_0 S; \pi_i \operatorname{fib} f)$, proving the proposition. ☐

The following proposition is likely well known:

Proposition 3.3.4. *Let S be a set of primes. Then* $\operatorname{fib}(\mathrm{TC}(R)[S^{-1}] \to \mathrm{TC}(R[S^{-1}]))$ *is a truncating invariant.*

Proof. By [DGM13] it is equivalent to prove that $\operatorname{fib}(K(R)[S^{-1}] \to K(R[S^{-1}]))$ is a truncating invariant. Since K-theory is a filtered colimit preserving localizing invariant, it is a sheaf with respect to localizing at primes. Thus we can assume that our rings are p-local, in which case TC is already p-local. Thus we can assume that $S = \{p\}$. All in all, we have reduced the problem to showing that for p-local connective rings R, the map $R \to \pi_0 R$ is an equivalence after applying the functor $\operatorname{fib}(\mathrm{TC}(R) \otimes \mathbb{Q} \to \mathrm{TC}(R \otimes \mathbb{Q}))$

Now since rationalization is t-exact, and because of Proposition 3.3.3's connectivity result, it suffices to show the functor is an equivalence for each of the maps $\tau_{\leq n} R \to \tau_{\leq n-1} R$. These are square zero extensions, so now the result follows from [Ras18, Theorem 5.15.1]. ☐

By taking the cospan $R \to R \times R \leftarrow R$ associated to $R^{h\mathbb{Z}}$ as discussed above, we see that the following result refines Theorem B:

Theorem 3.3.5. *Suppose we are given a map of cospans of connective $\mathbb{E}_1$-rings that is levelwise i-connective for $i \geq 1$. Then for any truncating invariant E, the map on the pullbacks induces an E-equivalance, and an i-connective map on* TC.

Proof. Let

$$\begin{tikzcd}
R_0 \arrow[r] \arrow[d] & R_1 \arrow[d] & R_2 \arrow[l] \arrow[d] \\
S_0 \arrow[r] & S_1 & S_2 \arrow[l]
\end{tikzcd}$$

be the map of cospans in consideration, and let R_3, S_3 denote the pullbacks. Applying [LT19], we obtain a pullback square:

$$\begin{tikzcd}
\mathcal{U}_{\mathrm{loc}}(R_3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}_{\mathrm{loc}}(R_0) \arrow[d] \\
\mathcal{U}_{\mathrm{loc}}(R_2) \arrow[r] & \mathcal{U}_{\mathrm{loc}}(R_0 \odot^{R_1}_{R_3} R_2)
\end{tikzcd}$$

where the underlying spectrum of $R_0 \odot^{R_1}_{R_3} R_2$ is $R_0 \otimes_{R_3} R_2$, which is connective by Lemma 3.3.1. Moreover one has a corresponding pullback square for the S_i. The maps $R_i \to S_i$ are i-connective for $i \geq 1$, so they induce an equivalence on E, and also the map on pullbacks is $(i-1)$-connective. The map $R_0 \otimes_{R_3} R_2 \to S_0 \otimes_{R_3} S_2$ is i-connective by Lemma 3.3.1, and the map $S_0 \otimes_{R_3} S_2 \to S_0 \otimes_{S_3} S_2$ is i-connective by applying both Lemma 3.3.1 and Lemma 3.3.2 so the composite, which on underlying spectra agrees with $R_0 \odot^{R_1}_{R_3} R_2 \to S_0 \odot^{S_1}_{S_3} S_2$ is i-connective. It thus induces an equivalence on E and a $(i+1)$-connective map on TC.

From the pullback square above, we then learn that $E(R_3) \to E(S_3)$ is also an equivalence, and that $\mathrm{TC}(R_3) \to \mathrm{TC}(S_3)$ is n-connective. ☐

We now apply Theorem B to j_ζ:

Corollary 3.3.6. *Let $R = ko_2$ for $p = 2$ and ℓ_p for $p > 2$. There are pullback squares*

$$\begin{tikzcd}
K(R^{h\mathbb{Z}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(R^{h\mathbb{Z}}) \arrow[d] \\
K(\mathbb{Z}_p^{h\mathbb{Z}}) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

$$\begin{tikzcd}
K(ko_2^{h\mathbb{Z}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(ko_2^{h\mathbb{Z}}) \arrow[d] \\
K(\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}) \arrow[r] & \mathrm{TC}(\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}})
\end{tikzcd}$$

where for R, the vertical maps are $(2p-2)$-connective for $p > 2$ and 1-connective for $p = 2$, and for the second pullback square for ko_2, the vertical maps are 4-connective. The vertical fibres are p-nil.

Proof. The pullback squares and connectivity statements follow from Theorem B and the fact that $\tau_{\leq 3}ko_2 \cong \tau_{\leq 2}\mathbb{S}_2$, $\tau_{\leq 2p-3}\ell_2 \cong \mathbb{Z}_p$ which implies that the actions on those truncations are trivial.

We now show that the vertical fibres are p-nil for the first square, as the proof for the second square is identical. To do this, we apply Proposition 3.3.4 and Theorem B to learn that the vertical fibre agrees after inverting p with $\mathrm{fib}(\mathrm{TC}(R^{h\mathbb{Z}}[\frac{1}{p}]) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}[\frac{1}{p}]))$. But $R^{h\mathbb{Z}}[\frac{1}{p}] \to \mathbb{Z}_p^{h\mathbb{Z}}[\frac{1}{p}]$ is an equivalence so the vertical fibre is p-nil. ☐

We now prove a variant of Theorem 3.3.5, where the ring R in question is -1-connective, but doesn't have to come from a pullback square. The idea is the same as before: to resolve $\mathrm{Mod}(R)$ by module categories of connective rings, only this time instead of the resolution coming to us from a pullback square, we construct one by hand. The result below is a refinement of Theorem C.

Theorem 3.3.7. *Let $R \to S$ be an 1-connective map of -1-connective rings. Then for any truncating invariant E, $E(R) \to E(S)$ is an equivalence. Moreover, if f is n-connective, then $\mathrm{TC}(R) \to \mathrm{TC}(S)$ is $(n-1)$-connective.*

Proof. Choose generators $x_\alpha \in \pi_{-1}R, \alpha \in A$ as a $\pi_0 R$-module. We will build an R-module X whose cells correspond to the free monoid on the set of x_α such that its endomorphism ring is connective. We will then embed $\mathrm{Mod}(R)$ fully faithfully into $\mathrm{Mod}(\mathrm{End}(\oplus_A X))$, and show that the cofibre is also the module category of a connective ring. Doing the same for S, and comparing, we will obtain the result.

To construct X, set $X_0 = R$, and choose a free module on A in degree -1 to hit the generators x_i of X in degree -1, and let X_1 be the cofibre. $\pi_{-1}X_1$ is canonically identified with a sum over A copies of $\pi_{-1}X_0$ indexed on the x_αs. We can then repeat this process, constructing X_i as the cofibre of a free module on A^i hitting the generators of X_{i-1}, which are indexed on words of length i in the x_i. Let $X = \mathrm{colim}_i X_i$. Note that X is connective because the map $X_i \to X_{i+1}$ is zero on negative homotopy groups by construction. We claim that $\mathrm{End}(\bigoplus_A X)$ is connective. More generally, we will show that $\mathrm{map}(\bigoplus_A X, Y) \cong \lim_i \mathrm{map}(\bigoplus_A X_i, Y)$ is connective for any Y that is connective as a spectrum. The individual terms $\mathrm{map}(\bigoplus_A X_i, Y)$ are connective because Y is connective as a spectrum and X_i are built out of finitely many cells of degree 0. Thus it suffices to show that any map $X_i \to Y$ can be extended to X_{i+1}, so that the $\lim^1$-term that could potentially contribute to $\pi_{-1}(\mathrm{map}(X, Y))$ vanishes. But this follows since the obstructions to making an extension lives in $\pi_{-1}Y$, which vanishes.

We now show that the thick subcategory generated by X contains R. The X_i filtration makes X into a filtered R-module, with associated graded a free module on the free monoid generated by the x_α. We will construct a filtered self map $\sigma x_\alpha : X \to X$ such that on the associated graded, x_α is left multiplication by x_α[10]. The obstruction to extending a filtered map defined until X_{k-1} to X_k at the kth step in the filtration is the map θ in the diagram below:

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[10] As pointed out to me by Robert Burklund, there is a universal example, the trivial square zero extension $\mathbb{S} \oplus \bigoplus_\alpha \Sigma^{-1}\mathbb{S}$, which one can show by obstruction theory admits an $\mathbb{E}_1$-map to R for any -1-connective R sending the classes in degree -1 to the classes x_α. This gives an alternate way to construct the module X and self maps σx_α via basechange from the universal example.

$$\begin{tikzcd}
 & \Sigma^{-1}R^{n^k} \arrow[d] \arrow[dddr, "\theta"] & & \\
\cdots \arrow[r] & X_{k-1} \arrow[r] \arrow[d] & X_k \arrow[r] \arrow[d, dashed] & \cdots \\
\cdots \arrow[r] & X_k \arrow[r] & X_{k+1} \arrow[r] \arrow[d] & \cdots \\
 & & R^{n^{k+1}} &
\end{tikzcd}$$

θ has to be null since the map $X_k \to X_{k+1}$ is 0 on π_{-1} by construction. Thus we can produce the dashed arrow in the diagram. Since π_0 of the space of nulhomotopies for an individual component $\Sigma^{-1}R$ is exactly $\pi_0 R^{n^{k+1}}$, we can choose the nulhomotopy so that the map on the associated graded is as desired.

Now note that the cofibre of X by all of the σx_αs is just R, so that R is in the thick subcategory generated by X. Let $\langle X\rangle$ be the thick subcategory of $\mathrm{Mod}(R)$ generated by X, and let $\langle X\rangle/\langle R\rangle$ denote the localization of $\langle X\rangle$ away from the thick subcategory generated by R. Morita theory gives an equivalence $\langle X\rangle = \mathrm{Mod}(\mathrm{End}(X))^\omega$, and we thus have a localization sequence

$$\mathrm{Mod}(R)^\omega \to \mathrm{Mod}(\mathrm{End}(X))^\omega \to \mathrm{Mod}(\mathrm{End}_{\langle X\rangle/\langle R\rangle}(X))^\omega$$

We claim that $\mathrm{End}_{\langle X\rangle/\langle R\rangle} X$ is also connective. To understand this endomorphism ring, we observe that the X_i are cofinal among perfect R-modules mapping to X. This means that $\mathrm{End}_{\langle X\rangle/\langle R\rangle}(X)$ is computed as $\mathrm{colim}_i \operatorname{map}(X, X/X_i)$, so it suffices to show $\operatorname{map}(X, X/X_i)$ is connective. But X/X_i is connective as a spectrum, and as we have shown, $\operatorname{map}(X, Y)$ is connective whenever Y is connective as a spectrum.

Finally, we note that $X \otimes_R S$ has exactly the same properties as an S-module, and in fact $X \to X \otimes_R S$ is i-connective by Lemma 3.3.1. However due to the possibility of $\lim^1$, this only guarantees that $\mathrm{End}(X) \to \mathrm{End}(X \otimes_R S)$ as well as the maps of endomorphisms in the cofibres are $(i-1)$-connective. Nevertheless, the contribution of the $\lim^1$-term to π_0 is square zero, so since truncating invariants are nil-invariant [LT19, Theorem B], we learn that $E(\mathrm{End}(X)) \to E(\mathrm{End}(X \otimes_R S))$ is an equivalence, and similarly for the localized ring. By the localization sequence, the $E(R) \to E(S)$ is also an equivalence. Since $\mathrm{End}(X) \to \mathrm{End}(X \otimes_R S)$ is $(i-1)$-connective, it induces an i-connective map on TC, so via the localization sequence, the original map $R \to S$ induces an $(i-1)$-connective map on TC. □

Remark 3.3.8. The connectivity bound for TC may not be optimal in the above theorem, and the finiteness hypothesis might not be necessary. Also, if one of the x_i is chosen to be zero, $\mathrm{End}(X)$ vanishes on every additive invariant by the Eilenberg swindle. This means that we have really proven that the suspension of $\mathcal{U}_{\mathrm{loc}}(R)$ is $\mathcal{U}_{\mathrm{loc}}$ of a connective ring. ◁

Question 3.3.9. To what extent do Theorem 3.3.5 and Theorem 3.3.7 generalize?

For instance, can Theorem 3.3.5 be generalized to other finite limits of sufficiently connective ring maps? The results proven here are certainly not the most general: for example the methods of this section are capable of proving that for a sufficiently connective map of rings $R \to S$ with a $\mathbb{Z}^n$-action that is trivial in low degrees, $E(R^{h\mathbb{Z}^n}) \to E(S^{h\mathbb{Z}^n})$ is an equivalence for any truncating invariant E.

3.4 The main theorems

We now put together the results so far to prove the main theorems A and D, as well as Theorem 3.4.2, which are stated where they are proven for convenience.

Theorem A. *$K(L_1^f\mathbb{S}_p) \cong K(L_{K(1)}\mathbb{S})$, there is a cofibre sequence split on π_**

$$K(j_\zeta) \longrightarrow K(L_{K(1)}\mathbb{S}) \longrightarrow \Sigma K(\mathbb{F}_p)$$

and a pullback square

$$\begin{tikzcd}
K(j_\zeta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(j_\zeta) \arrow[d] \\
K(\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

Let F be the fibre of the map $\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})$. Then $F[\frac{1}{p}] = 0$. For $p > 2$, F is $(2p-2)$-connective and $\pi_{2p-2}(F/p) \cong \bigoplus_0^\infty \mathbb{F}_p$. For $p = 2$, F is 1-connective and $\pi_1 F \cong \bigoplus_0^\infty \mathbb{F}_2$.

Proof. The first statement is just Proposition 3.2.2, and the cofibre sequence follows from Proposition 3.2.3. We now show that the cofibre sequence is split on π_*. After inverting p, $K(j_\zeta) \to K(\mathbb{Z}_p^{h\mathbb{Z}})$ is an equivalence by Corollary 3.3.6, so the map is null by Lemma 3.2.4. Thus the cofibre sequence splits on π_* after inverting p. Because $K_*(\mathbb{F}_p) \cong K_*(\mathbb{F}_p)[\frac{1}{p}]$ in positive degrees and is torsion, we obtain the desired result in degrees $\neq 1$. On π_1, one observes that $j_\zeta/(p, v_1)$ and $j_\zeta/(2, \eta, v_1)$ are zero in $K_0(j_\zeta)$, so that the cofibre sequence is a short exact sequence on π_1, so it splits since $K_0(\mathbb{F}_p) \cong \mathbb{Z}$ is projective.

Corollary 3.3.6 gives a pullback square

$$\begin{tikzcd}
K(j_\zeta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(j_\zeta) \arrow[d] \\
K(\mathbb{Z}_p^{h\mathbb{Z}}) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

and gives the connectivity claims for F. By applying Theorem 3.2.1 to $\mathbb{Z}_p^{h\mathbb{Z}}$, we learn that $K(\mathbb{Z}_p) \cong K(\mathbb{Z}_p^{h\mathbb{Z}_p})$, so that this pullback square agrees with the one in the theorem statement.

It remains to prove the claims about the first nonvanishing homotopy group of F.

To compute π_1 of the vertical fibre for $p = 2$, we first observe from the second pullback square in Corollary 3.3.6 that it s the same as π_1 of the fibre of $K(\tau_{\leq 2}\mathbb{S}^{h\mathbb{Z}}) \to K(\mathbb{Z}_2^{h\mathbb{Z}})$, where the action on $\tau_{\leq 2}\mathbb{S}$ is (necessarily) trivial. From Land–Tamme (see the proof of Theorem 3.3.5), we get a pullback square

$$\begin{tikzcd}
\mathcal{U}_{\mathrm{loc}}(\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}_{\mathrm{loc}}(\tau_{\leq 2}\mathbb{S}_2) \arrow[d] \\
\mathcal{U}_{\mathrm{loc}}(\tau_{\leq 2}\mathbb{S}_2) \arrow[r] & \mathcal{U}_{\mathrm{loc}}(\tau_{\leq 2}\mathbb{S}_2 \odot^{\tau_{\leq 2}\mathbb{S}_2 \times \tau_{\leq 2}\mathbb{S}_2}_{\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}} \tau_{\leq 2}\mathbb{S}_2)
\end{tikzcd}$$

We claim that there is an equivalence of $\mathbb{E}_1$-algebras $\tau_{\leq 2}\mathbb{S}_2 \odot^{\tau_{\leq 2}\mathbb{S}_2 \times \tau_{\leq 2}\mathbb{S}_2}_{\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}} \tau_{\leq 2}\mathbb{S}_2 \cong \tau_{\leq 2}\mathbb{S}_2[z]$. To see this, we first apply the formula in [LT23] (see [BL23, Example 4.9]), which gives an equivalence of $\mathbb{E}_1$-algebras $\tau_{\leq 2}\mathbb{S}_2 \odot^{\tau_{\leq 2}\mathbb{S}_2[\epsilon_0]}_{\tau_{\leq 2}\mathbb{S}_2[\epsilon_{-1}]} \tau_{\leq 2}\mathbb{S}_2 \cong \tau_{\leq 2}\mathbb{S}_2[z]$. It suffices then to show that

$$\tau_{\leq 2}\mathbb{S}_2 \odot^{\tau_{\leq 2}\mathbb{S}_2[\epsilon_0]}_{\tau_{\leq 2}\mathbb{S}_2[\epsilon_{-1}]} \tau_{\leq 2}\mathbb{S}_2 \cong \tau_{\leq 2}\mathbb{S}_2 \odot^{\tau_{\leq 2}\mathbb{S}_2 \times \tau_{\leq 2}\mathbb{S}_2}_{\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}} \tau_{\leq 2}\mathbb{S}_2$$

To do this, we use the observation in [LT23] (see also [BL23, Section 4]) that $R_1 \odot^{R_4}_{R_2} R_3$ just depends on R_1, R_3, and the unital $R_1 - R_3$-bimodule R_4. Thus it suffices to show that $\tau_{\leq 2}\mathbb{S}[\epsilon_0]$ and $\tau_{\leq 2}\mathbb{S} \times \tau_{\leq 2}\mathbb{S}$ define isomorphic unital $\tau_{\leq 2}\mathbb{S} - \tau_{\leq 2}\mathbb{S}$-bimodules. But indeed, they are both symmetric bimodules, and $\tau_{\leq 2}\mathbb{S}[\epsilon_0]$ is a free $\mathbb{E}_0$-$\tau_{\leq 2}\mathbb{S}$-algebra on ϵ_0, so by sending ϵ_0 to $(1, 0) \in \pi_0(\tau_{\leq 2}\mathbb{S} \times \tau_{\leq 2}\mathbb{S})$ we get an isomorphism.

The same argument for $\mathbb{Z}_2$ instead of $\tau_{\leq 2}\mathbb{S}_2$ shows that $\mathbb{Z}_2 \odot^{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\mathbb{Z}_2^{h\mathbb{Z}}} \mathbb{Z}_2 \cong \mathbb{Z}_2[z]$. From Proposition 3.3.3, we then learn that $K_2(\tau_{\leq 2}\mathbb{S}_2[z], \mathbb{Z}_2[z]) \cong \mathrm{HH}_0(\mathbb{Z}_2[z]; \mathbb{Z}/2[z]) \cong \mathbb{Z}/2[z]$. By comparing the Land–Tamme pullback squares for $\tau_{\leq 2}\mathbb{S}^{h\mathbb{Z}}$ and $\mathbb{Z}_2^{h\mathbb{Z}}$, since $K_2(\tau_{\leq 2}\mathbb{S}_2, \mathbb{Z}_2) = \mathbb{Z}/2$, this shows that $K_1(\tau_{\leq 2}\mathbb{S}_2^{h\mathbb{Z}}, \mathbb{Z}_2^{h\mathbb{Z}})$ is infinitely many copies of $\mathbb{Z}/2$.

At odd primes, as before, it suffices to show that $\pi_{2p-1}K(\ell_p \odot^{\ell_p \times \ell_p}_{\ell_p^{h\mathbb{Z}}} \ell_p, \mathbb{Z}_p[z])/p$ is countably generated. To do this, we will first study the underlying spectrum of $\ell_p \odot^{\ell_p \times \ell_p}_{\ell_p^{h\mathbb{Z}}} \ell_p$, which is the tensor product $\ell_p \otimes_{\ell_p^{h\mathbb{Z}}} \ell_p$. We claim that the natural map $j_\zeta \otimes_{j_\zeta^{h\mathbb{Z}}} \ell^{h\mathbb{Z}} \to \ell_p$ is a p-completion, where we consider $j_\zeta \to \ell$ as a $\mathbb{Z}$-equivariant map with a trivial action on j_ζ. To see this, we consider the commutative diagram below, where the horizontal arrows are given by 1 minus the action of $1 \in \mathbb{Z}$.

$$\begin{tikzcd}
j_\zeta \arrow[r] \arrow[d] & j_\zeta \arrow[r] \arrow[d] & j_\zeta \arrow[r] \arrow[d] & \cdots \\
\ell_p \arrow[r] & \ell_p \arrow[r] & \ell_p \arrow[r] & \cdots
\end{tikzcd}$$

The horizontal maps are ${j_\zeta}^{h\mathbb{Z}}$ and $\ell_p^{h\mathbb{Z}}$ module maps respectively. Since the action of $1 \in \mathbb{Z}$ on $\pi_*\ell_p$ is 1 mod p, the horizontal maps are zero on π_* mod p, so the colimit along the horizontal maps are zero p-adically. Moreover the fibres of each horizontal map is ${j_\zeta}^{h\mathbb{Z}}$ and $\ell_p^{h\mathbb{Z}}$ respectively. Thus we have produced a filtration of j_ζ as a ${j_\zeta}^{h\mathbb{Z}}$-module that basechanges p-adically to a filtration of ℓ_p as a $\ell_p^{h\mathbb{Z}}$ module, proving the claim.

As a consequence, we obtain that the map $\ell_p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta \to \ell_p \otimes_{\ell_p^{h\mathbb{Z}}} \ell_p$ is an equivalence after p-completion. We now filter j_ζ via the homotopy fixed point filtration, i.e the filtration $(\tau_{\geq *}\ell_p)^{h\mathbb{Z}_p}$. The map $j_\zeta \to \ell_p$ is then a filtered map, where ℓ_p is given the Postnikov filtration.

In what follows, $C^*(\mathbb{Z}_p; R) = \mathrm{colim}_i(C^*(\mathbb{Z}/p^i; R))$ for R an $\mathbb{E}_\infty$-algebra denotes the algebra of continuous cochains on $\mathbb{Z}_p$ with coefficients in R. Up to p-completion, it is also given by the tensor product $R \otimes_{R^{h\mathbb{Z}}} R$, where R has the trivial $\mathbb{Z}$-action. Taking the tensor product $j_\zeta/p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta$ in filtered rings gives a spectral sequence converging to the homotopy of the tensor product. We know that the tensor product is $(j_\zeta \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta)/p \cong C^*(\mathbb{Z}_p; j_\zeta/p)$. The associated graded of j_ζ is a $\mathbb{Z}$-algebra since it is the homotopy fixed points of the associated graded of ℓ_p, which has the Postnikov filtration. We thus have isomorphisms

$$\mathrm{gr}j_\zeta/p \otimes_{\mathrm{gr}{j_\zeta}^{h\mathbb{Z}}} \mathrm{gr}j_\zeta \cong \mathrm{gr}j_\zeta/p \otimes_{\mathrm{gr}{j_\zeta/p}^{h\mathbb{Z}}} \mathrm{gr}j_\zeta/p \cong C^*(\mathbb{Z}_p; \mathrm{gr}j_\zeta/p)$$

Thus we learn that the spectral sequence for $(j_\zeta \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta)/p$ degenerates. $\pi_*\mathrm{gr}j_\zeta/p \cong \mathbb{F}_p[\zeta, v_1]$ since the Adams operations act trivially on $\pi_*\ell_p$ mod p, so the associated graded of the homotopy ring of this tensor product is $\mathbb{F}_p[\zeta, v_1] \otimes C^*(\mathbb{Z}_p; \mathbb{F}_p)$.

The spectral sequence for $j_\zeta/p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta$ maps to the one coming from the tensor product of filtered rings $\ell_p/p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta$. The E_1-page of the spectral sequence for $\ell_p/p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta$ is the homotopy ring of $\mathrm{gr}\ell_p/p \otimes_{\mathrm{gr}{j_\zeta}^{h\mathbb{Z}}} \mathrm{gr}j_\zeta \cong \mathrm{gr}\ell_p/p \otimes_{\mathrm{gr}{\ell_p/p}^{h\mathbb{Z}}} \mathrm{gr}\ell_p/p \cong \mathrm{gr}\ell_p/p \otimes_{\mathbb{F}_p} C^*(\mathbb{Z}_p; \mathbb{F}_p)$. Thus the E_1-page is $\mathbb{F}_p[v_1] \otimes_{\mathbb{F}_p} C^*(\mathbb{Z}_p; \mathbb{F}_p)$, so the map of spectral sequences is surjective and thus both spectral sequences degenerate. We also learn that $j_\zeta/p \otimes_{{j_\zeta}^{h\mathbb{Z}}} j_\zeta \to \ell_p/p \otimes_{j_\zeta^{h\mathbb{Z}}} j_\zeta$ is an isomorphism on π_* in even degrees because at the level of E_1-pages it is the map $\mathbb{F}_p[\zeta, v_1] \otimes_{\mathbb{F}_p} C^*(\mathbb{Z}_p; \mathbb{F}_p) \to \mathbb{F}_p[v_1] \otimes_{\mathbb{F}_p} C^*(\mathbb{Z}_p; \mathbb{F}_p)$

Using the formula for $\odot$ in the case the action is trivial as we did for $p = 2$, we obtain an equivalence of $\mathbb{E}_1$-rings $j_\zeta \odot_{j_\zeta^{h\mathbb{Z}}}^{j_\zeta \times j_\zeta} j_\zeta \cong j_\zeta[z]$. Thus we have an $\mathbb{E}_1$-algebra map

$$j_\zeta[z] \cong j_\zeta \odot_{j_\zeta^{h\mathbb{Z}}}^{j_\zeta \times j_\zeta} j_\zeta \to \ell_p \odot_{\ell_p^{h\mathbb{Z}}}^{\ell_p \times \ell_p} \ell_p$$

Since $\odot$ is the tensor product on underlying spectra, we learn that mod p, this map is an isomorphism in even degrees, so we learn that $\pi_*(\ell_p \odot_{\ell_p^{h\mathbb{Z}}}^{\ell_p \times \ell_p} \ell_p/p) = \mathbb{F}_p[v_1, z]$ as a ring. It follows that $\mathrm{HH}_0(\mathbb{Z}_p[z]; \pi_{2p-2}\ell_p \odot_{\ell_p^{h\mathbb{Z}}}^{\ell_p \times \ell_p} \ell_p/p) = \mathbb{F}_p[z]$ so using Proposition 3.3.3, we learn that $\pi_{2p-1}K(\ell_p \odot_{\ell_p^{h\mathbb{Z}}}^{\ell_p \times \ell_p} \ell_p, \mathbb{Z}_p[z])/p$ is $\mathbb{F}_p[z]$, which indeed is countably generated.

□

Question 3.4.1. Is the boundary map $K(\mathbb{F}_p) \to K(j_\zeta)$ in Theorem A null?

We saw in the above proof that the map is null after inverting p, so that Question 3.4.1 is essentially a p-adic question.

Next, we give a formula for $K(\mathrm{Sp}^\omega_{T(1)})$:

Theorem 3.4.2. *There is a cofibre sequence split on* π_*

$$K(j_\zeta \otimes \mathrm{Sp}^\omega_{\geq 1}) \longrightarrow K(\mathrm{Sp}^\omega_{T(1)}) \longrightarrow \Sigma K(\mathbb{F}_p)$$

and a pullback square

$$\begin{tikzcd}
K(j_\zeta \otimes \mathrm{Sp}^{\omega}_{\geq 1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(j_\zeta) \arrow[d] \\
K(\mathbb{F}_p) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

Proof. First note that

$$K(j_\zeta \otimes \mathbb{Q}) \longrightarrow K(L_{K(1)}\mathbb{S} \otimes \mathbb{Q}) \longrightarrow 0$$

is a cofibre sequence since $j_\zeta \otimes \mathbb{Q} \cong L_{K(1)}\mathbb{S} \otimes \mathbb{Q}$. Thus combining this with cofibre sequence from Theorem A via the localization sequences for rationalization, we get the claimed cofibre sequence.

To obtain the pullback square, we again consider what happens when we rationalize. Then $j_\zeta \otimes \mathbb{Q} \cong \mathbb{Q}_p^{h\mathbb{Z}}$, so

$$\begin{tikzcd}
K(j_\zeta \otimes \mathbb{Q}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}(j_\zeta \otimes \mathbb{Q}) \arrow[d] \\
K(\mathbb{Q}_p) \arrow[r] & \mathrm{TC}(\mathbb{Q}_p^{h\mathbb{Z}})
\end{tikzcd}$$

is not only a pullback square, but the vertical fibres vanish. By combining this pullback square with the one for j_ζ in Theorem A, we obtain the claimed pullback square.

Finally, the splitting of the cofibre sequence at the level of π_* follows exactly as in the proof of Theorem A. Namely, after inverting p, the cofibre sequence becomes a split cofibre sequence by Lemma 3.2.4, which shows that the cofibre sequence is split on π_* in degrees $\neq 1$ since $K(\mathbb{F}_p)$ is p'-torsion in degrees $\neq 0$. In degree 1, one gets a short exact sequence on homotopy groups since $K_0(\mathbb{F}_p) \to K_0(j_\zeta \otimes \mathrm{Sp}^{\omega}_{\geq 1})$ is null, and this short exact sequence splits since $K_0(\mathbb{F}_p) \cong \mathbb{Z}$ is projective. ☐

Theorem D. *There is a fibre sequence $X \to K(\mathrm{Sp}_{\geq 2}) \to K(\mathbb{F}_p)$ split on π_*, where X is the total fibre of the square*

$$\begin{tikzcd}
\mathrm{TC}(\mathbb{S}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathbb{Z}_p) \arrow[d] \\
\mathrm{TC}(j_\zeta) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

- *For $p > 2$, X is $(2p-3)$-connective, so $K_0(\mathrm{Sp}_{\geq 2}) = \mathbb{Z}$ with generator $[\mathbb{S}/(p, v_1)]$.*
- *For $p = 2$, X is connective with $\pi_0 X \cong \bigoplus_0^\infty \mathbb{Z}/2$, and the torsion free quotient of $K_0(\mathrm{Sp}_{\geq 2})$ is generated by $[\mathbb{S}/(2, \eta, v_1)]$.*

Proof. Consider the diagram of cofibre sequences given by tensoring the first localization sequence of rings with j_ζ and applying K-theory. We use Proposition 3.2.3 to identify the lower sequence.

$$\begin{tikzcd}
K(\mathrm{Sp}_{\geq 2}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K(\mathbb{S}_p) \arrow[r] \arrow[d] & K(L_1^f \mathbb{S}_p) \arrow[d] \\
K(\mathbb{F}_p) \arrow[r] & K(j_\zeta) \arrow[r] & K(L_{K(1)}\mathbb{S})
\end{tikzcd}$$

The right vertical map is an equivalence by Theorem A, so the left square is a pullback square. Thus the fibre $K(\mathrm{Sp}_{\geq 2}) \to K(\mathbb{F}_p)$ is the fibre $K(\mathbb{S}) \to K(j_\zeta)$ which by comparing the DGM squares for $\mathbb{S}$ with the pullback square in Theorem A yields the pullback square. The claims about X come from the claims about the vertical fibres in the pullback square of Theorem A. Namely, $\mathrm{fib}(\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p))$ is $(2p-2)$-connective by Proposition 3.3.3, so combining this with the connectivity bound in Theorem A gives the claim.

The splitting on π_* of the cofibre sequence follows from a similar argument as in the proof of Theorem A. Namely, the map $K(\mathbb{S}_p)[\frac{1}{p}] \to K(\mathbb{Z}_p)[\frac{1}{p}]$ is an equivalence, because $\mathrm{fib}(\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p))[\frac{1}{p}] \cong \mathrm{fib}(\mathrm{TC}(\mathbb{S}_p[\frac{1}{p}]) \to \mathrm{TC}(\mathbb{Z}_p[\frac{1}{p}])) = 0$ by Proposition 3.3.4, and $\mathrm{fib}(\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}))[\frac{1}{p}] = 0$ by Corollary 3.3.6. In particular, the cofibre sequence is split after inverting p, and since $\pi_i K(\mathbb{F}_p) = \pi_i K(\mathbb{F}_p)[\frac{1}{p}]$ in nonzero degrees, the cofibre sequence is split on homotopy in nonzero degrees. In degree 0, it is a short exact sequence homotopy groups since the map $K_0(\mathbb{F}_p) \to K_0(j_\zeta)$ is null, and must be split since $K_0(\mathbb{F}_p)$ is projective.

The remaining thing to justify is that $\mathbb{S}/(p, v_1)$ for $p > 2$ and $\mathbb{S}/(2, \eta, v_1)$ for $p = 2$ are generators of the torsion free quotient of K_0. The map $K(\mathrm{Sp}_{\geq 2}) \to K(\mathbb{F}_p)$ is on K_0 exactly this torsion free quotient, and the generator of K_0 was seen to be as claimed in the proof of Proposition 3.2.3. □

3.5 The K-theory sheaf

In this section we define the K-theory sheaf K^Δ, and explain how it classifies stable tensor ideals, refining the classification of thick tensor ideals of the Balmer spectrum. After doing so, we extract some consequences of our main theorems.

Recall that if C is a small rigid symmetric monoidal stable category[11], the Balmer spectrum of C, $\mathrm{Spec}^\Delta(C)$, allows one to classify thick tensor ideals of C [Bal05]. To each open set O of the Balmer spectrum, there is a finite localization $L_O(C)$ given by localizing away from objects whose support doesn't intersect O. By associating to each O the algebraic K-theory of $L_O(C)$, we obtain the *K-theory sheaf* $K^\Delta(C)$ on $\mathrm{Spec}^\Delta(C)$, which is a sheaf of $\mathbb{E}_\infty$-rings.

---

[11] it is sufficient for C to be monoidal, in which case $K^\Delta(C)$ is a sheaf of $\mathbb{E}_1$-rings

Knowing $K^{\Delta}(C)$ allows one to in particular refine the classification of thick tensor ideals to a classification of stable tensor ideals. This is due to the following elementary result of Thomason:

Proposition 3.5.1 (Thomason [Tho97])**.** *Given a small stable category C, there is a bijection between dense stable subcategories $C' \subset C$ and subgroups of $K_0(C)$, given by the assignment $C' \mapsto K_0(C') \subset K_0(C)$.*

Here, a dense subcategory of a stable category is one such that the inclusion induces an equivalence on idempotent completions. It follows from Proposition 3.5.1 that every stable tensor ideal is given by a thick subcategory $C' \subset C$ and a submodule of $K_0(C')$ as a $K_0(C)$-module. $K_0(C')$ can be extracted from $K^{\Delta}(C)$ as follows: since K-theory commutes with filtered colimits, we can assume that C' is a compact stable tensor ideal. Its support is then a closed set $Z_{C'}$ of the Balmer spectrum, and we let $O_{C'}$ denote the open complement. We have a cofibre sequence $K(C') \to K(C) \to K(L_{O_{C'}}(C))$, allowing us to extract $K_0(C')$.

We essentially ran the above process of extracting $K(C')$ in the proof of Theorem D to obtain a description of $K(\mathrm{Sp}_{\geq 2})$ as the fibre of $K(\mathbb{S}_p) \to K(L_1^f \mathbb{S}_p)$. The following is a consequence of Theorem D.

Corollary 3.5.2. *The dense stable subcategories of $\mathrm{Sp}^{\omega}_{\geq 2}$ for $p > 2$ are in bijection with subgroups of $\mathbb{Z}$, and the dense stable subcategories of $\mathrm{Sp}^{\omega}_{\geq 2}$ at the prime 2 are in bijection with subgroups of $\mathbb{Z} \oplus \bigoplus_0^{\infty} \mathbb{F}_2$.*

The following corollary is a consequence of the fact that the map $K_0(\mathrm{Sp}_{\geq 1}) \to K_0(\mathrm{Sp}^{\omega}_{K(1)})$ is surjective (in fact it is an isomorphism).

Corollary 3.5.3. *Any compact $K(1)$-local spectrum is the $K(1)$-localization of a type 1 spectrum.*

Proof. One observes that the $K(1)$-local spectra which are $K(1)$-localizations of type 1 spectra are a stable subcategory of $\mathrm{Sp}^{\omega}_{K(1)}$ corresponding to the subgroup of K_0 that is the image of $K_0(\mathrm{Sp}_{\geq 1})$. ☐

Note that apriori all that is clear is that a compact $K(1)$-local spectrum is a retract of the $K(1)$-localization of a finite type 1 spectrum.

Question 3.5.4. Given a compact $K(1)$-local spectrum, is there a way of finding a lift of it to $\mathrm{Sp}_{\geq 1}$?

Next we interpret Theorem 3.1.4 in the corollary below. We can say that a stable subcategory $C' \subset C$ is *p-saturated* if $\oplus_1^p X \in C' \implies X \in C'$.

Corollary 3.5.5 (Burklund–Levy)**.** *The nonzero p-saturated stable subcategories of Sp_p are specified by a type n and a $\mathbb{Z}[\frac{1}{p}]$-submodule of $\mathbb{Z}[\frac{1}{p}]$.*

3.6 Constructing type 2 spectra

In Theorem D, it was shown that $K_0(\mathrm{Sp}_{\geq 2}) \cong \mathbb{Z} \oplus \bigoplus_0^\infty \mathbb{F}_2$ at the prime 2. Here we explain how to construct type 2 spectra representing the 2-torsion classes. The key point is understanding the boundary maps in the K-theory of a localization sequence, which is the purpose of the Lemma 3.6.2 below.

Construction 3.6.1. Let $C \xrightarrow{i} D \xrightarrow{\pi} E$ be a localization sequence, and let $f : d \to d$ for $d \in D$ be a map. Suppose f has the property that its cofibre is in C. Then if $\mathrm{Map}(d,d)_f$ denotes the connected component containing f, taking the cofibre gives a map $\mathrm{cof} : \mathrm{Map}(d,d)_f \to \mathrm{B}\,\mathrm{Aut}(\mathrm{cof}\, f)$.

On the other hand, since $\mathrm{cof}\, f$ vanishes in E, composing with π gives a map $\pi : \mathrm{Map}(d,d)_f \to \mathrm{Aut}(\pi d)$. ◁

Recall also that given an object $c \in C$, there is a canonical map $\mathrm{B}\,\mathrm{Aut}(c) \to C^{\cong} \to \Omega^\infty K(C)$

Lemma 3.6.2. *In the situation of Construction 3.6.1, the diagram below commutes up to a sign, where δ is the boundary map associated to the localization sequence.*

$$\begin{tikzcd}
\mathrm{Map}(d,d)_f \arrow[r, "\pi"] \arrow[dr, "\mathrm{cof}"'] & \mathrm{Aut}(\pi d) \arrow[r] & \Omega^{\infty+1}K(E) \arrow[d, "\delta"] \\
 & \mathrm{B}\,\mathrm{Aut}(\mathrm{cof}\, f) \arrow[r] & \Omega^\infty K(C)
\end{tikzcd}$$

Proof. Recall (eg see [HLS22, Definition 3.4]) that $\Omega^\infty K(C)$ can be modeled via the Q-construction $\Omega|\mathrm{Span}(C)|$, where $\mathrm{Span}(C)$ is the category of spans of objects in C, with composition given by pulling back. The natural map $C^{\cong} \to \Omega^\infty K(C)$ sends x to the span $0 \leftarrow x \to 0$.

Consider the diagram below in $|\mathrm{Span}(C)|$ where an arrow with a perpendicular line indicates the direction of a span as a map in $\mathrm{Span}(C)$. All of the 2-cells are the obvious ones, except for the one labeled πf^{-1}, where πf^{-1} is the difference between the 2-cell and the obvious one.

$$\begin{tikzcd}
& & 0 & & \\
& 0 \arrow[ur] \arrow[dl, "|" marking] & \Sigma\pi d \arrow[u] \arrow[d, "|" marking] & 0 \arrow[ul] \arrow[dr, "|" marking] & \\
0 & 0 \arrow[l, "|" marking] \arrow[r] & \Sigma\pi d & 0 \arrow[l] \arrow[r, "|" marking] & 0 \\
& \pi d \arrow[ul, "|" marking] \arrow[dr] & 0 \arrow[u, "|" marking] \arrow[d] & \pi d \arrow[ur, "|" marking] \arrow[dl] & \\
& & 0 & & \\
& & \xRightarrow[\pi f^{-1}]{} & & \\
& & 0 \arrow[uuuull, bend left=60] \arrow[uuuurr, bend right=60, "|" marking] & &
\end{tikzcd}$$

Because the boundary is sent to 0, this diagram represents a map $S^2 \to |\operatorname{Span}(C)|$. Since it is given by conjugating πf^{-1} by the triangles in the diagram above, it represents the image of f in $\Omega^{\infty+1}K(E)$. To compute δ of this, consider the lift of the diagram below to $|\operatorname{Span}(D)|$:

$$\begin{tikzcd}
& & 0 & & \\
& \operatorname{cof} f \arrow[ur] \arrow[dl, "|" marking] & \Sigma d \arrow[u] \arrow[d, "|" marking] & \operatorname{cof} f \arrow[ul] \arrow[dr, "|" marking] & \\
\operatorname{cof} f & \operatorname{cof} f \arrow[l, "|" marking] \arrow[r] & \Sigma d & \operatorname{cof} f \arrow[l] \arrow[r, "|" marking] & 0 \\
& d \arrow[ul, "|" marking] \arrow[dr] & 0 \arrow[u, "|" marking] \arrow[d] & d \arrow[ur, "|" marking] \arrow[dl] & \\
& & 0 & & \\
& & \xRightarrow[f^{-1}]{} & & \\
& & 0 \arrow[uuuull, bend left=60] \arrow[uuuurr, bend right=60, "|" marking] & &
\end{tikzcd}$$

Here f^{-1} is the map of spans identifying d with the fibre of $d \to \operatorname{cof} f$. Note there is a subtlety about making the diagram lift the previous one: the maps $\Sigma d \to \Sigma\pi d$ are given by multiplication by f, and the maps $d \to \pi d$ are the canonical ones. This lift is a diagram in the shape of D^2 and has the property that its boundary S^1 lives in $|\operatorname{Span}(C)|$. Thus the boundary S^1 represents δ applied to the map $S^2 \to |\operatorname{Span}(E)|$. But by composing the composable maps in the boundary, we find that the boundary is canonically homotopic to (up to a sign) $\{\operatorname{cof} f\} \to C^{\cong} \to \Omega^\infty K(C)$. By the naturality of this construction in f, we have shown the diagram commutes as claimed. $\square$

Given the lemma, we now find explicit generators for $K_1(\mathbb{S}^{h\mathbb{Z}})$. There is an equivalence $\mathbb{S}^{h\mathbb{Z}} \cong \mathbb{S}[\epsilon_{-1}]$, and a localization sequence

$$\mathrm{Mod}(\mathbb{S}[\epsilon_{-1}]) \longrightarrow \mathrm{Mod}(\mathbb{S}[x]) \longrightarrow \mathrm{Mod}(\mathbb{S}[x^{\pm 1}])$$

where we identify $\mathbb{S}[\epsilon_{-1}] \cong \mathrm{End}_{\mathbb{S}[x]}(\mathbb{S}[x]/x)$. From Proposition 3.3.3, and examining the long exact sequence on homotopy groups, we learn that $K_1(\mathbb{S}[\epsilon_{-1}]) \cong K_1(\mathbb{S}) \oplus \mathrm{coker}(\mathbb{Z}/2[x] \to \mathbb{Z}/2[x^{\pm 1}])$ where $\mathbb{Z}/2[x]$ and $\mathbb{Z}/2[x^{\pm 1}]$ are $\mathrm{HH}_0(\pi_0 R; \pi_1 R)$, where $R = \mathbb{S}[x], \mathbb{S}[x^{\pm}]$. In the proof of Proposition 3.3.3, the Hochschild homology term is coming from

$$H_0(\mathrm{BGL}(\pi_0 R); \pi_2(\mathrm{BGL}(R))$$

, which is isomorphic to $H_0(\mathrm{BGL}_1(\pi_0 R); \pi_2(\mathrm{BGL}_1(R))$ since the bimodule is symmetric. Thus those K_2 classes come from the classes $\eta x^i \in \pi_2 \mathrm{B}\,\mathrm{Aut}(R)$. Thus to find representatives of the K_1 classes, we need to compute the boundary map $K_2(\mathbb{S}[x^{\pm 1}]) \to K_1(\mathbb{S}[\epsilon_{-1}])$ on these classes, but this exactly what Lemma 3.6.2 is made to do. Namely, note that $\eta x^{-i} \in \pi_1 \mathrm{Aut}(\mathbb{S}[x^{\pm 1}])$ lifts to π_1 of the component of $\mathrm{Map}_{\mathbb{S}[x]}(\mathbb{S}[x], \mathbb{S}[x])$ containing the map x^i, by composing with the automorphism η on the target.

Taking the cofibre, we get a nontrivial element, which we call g_i of $\pi_1(\mathrm{B}\,\mathrm{Aut}(\mathrm{cof}\, f)) = \pi_0(\mathrm{Aut}(\mathrm{cof}\, f))$. g_i is the map obtained as the horizontal cofibre of the diagram

$$\begin{tikzcd}
\mathbb{S}[x] \arrow[r, "x^i"] \arrow[d, equal] & \mathbb{S}[x] \arrow[d, equal] \\
\mathbb{S}[x] \arrow[r, "x^i"'] \arrow[ur, Rightarrow, shorten=12pt, "\eta"] & \mathbb{S}[x]
\end{tikzcd}$$

Since g_i is nontrivial, for $i = 1$ the only possibility is that $g_1 = 1 + \beta_x \eta$. In general, we can use the fact that ηx^{-i} is in the image of the analogous localization sequence for $\mathbb{S}[x^i]$ to learn that $g_i = 1 + \beta_{x^i} \eta$.

We can describe these maps in terms of $\mathbb{S}^{h\mathbb{Z}}$. Let X_i be the module corresponding to $\mathbb{S}[x]/x^i$. It has a cellular filtration by the other X_j, defined inductively by observing that X_i can be constructed as the cofibre of the map $\zeta_{i-1} : \Sigma^{-1}\mathbb{S}^{h\mathbb{Z}} \to X_{i-1}$ given by hitting the generator in π_{-1}.

ζ_i extends to a self map of $\zeta_i : \Sigma^{-1} X_i \to X_i$ corresponding to β_{x^i}[12], which we give the same name. Thus we have proven:

Lemma 3.6.3. *$K_1(\mathbb{S}^{h\mathbb{Z}}, \mathbb{S})$ is generated by $[g_n]$, where g_n is the automorphisms $g_n = 1 + \eta\zeta_i : X_i \to X_i$.*

---

[12] Explicitly, this map is the composite $X_i \to \Sigma X_{2i} \to \Sigma X_i$, where we view X_i as the first and last i cells of X_{2i}.

The map $K_1(\mathbb{S}^{h\mathbb{Z}}, \mathbb{S}) \to K_1(j_\zeta, \mathbb{Z}_2^{h\mathbb{Z}})$ is an isomorphism (see Corollary 3.3.6), so the latter is generated by automorphisms of the same name. By Theorem D, the image of these class along the map $K_1(j_\zeta) \to K_1(L_{K(1)}\mathbb{S}_2) \cong K_1(L_1^f\mathbb{S}_2) \to K_0(\mathrm{Sp}^\omega_{\geq 2})$ are the generators of 2-torsion classes in K_0.

To actually compute the representatives in K_0, we first observe that these 2-torsion classes are in the kernel of the map $K_1(L_{K(1)}\mathbb{S}) \to K_1(L_{K(1)}\mathbb{S} \otimes \mathbb{Q})$, because η is. It follows from the localization sequence that these classes lift to $K_1(\mathrm{Sp}^\omega_{T(1)})$. To actually produce lifts, we observe that the diagram below (thought of in $\mathrm{Mod}(L_{K(1)}\mathbb{S}^\omega)$) commutes up to homotopy:

$$\begin{tikzcd}
X_i \arrow[r, "1"] \arrow[d, "2"] & X_i \arrow[d, "2"] \\
X_i \arrow[r, "g_i"] & X_i
\end{tikzcd}$$

Let $\bar{g}_i$ denote an automorphism of $X_i/2$ obtained by taking vertical cofibres. Since 1 represents the trivial element of K_1 and the class is 2-torsion, $\bar{g}_i$ represents the same K_1-class as $\bar{g}_i$ by additivity of K-theory, but constitutes a lift to $K_1(\mathrm{Sp}^\omega_{T(1)})$. Note that $\bar{g}_i$ is *not* $g_i \otimes \mathrm{cof}\, 2$, as the latter has trivial K_1 class, as $\mathrm{cof}\, 2$ is 0 in $K_0(L_{K(1)}\mathbb{S})$.

The desired K_0 classes are then obtained as the image via the map $K_1(\mathrm{Sp}^\omega_{T(1)}) \to K_0(\mathrm{Sp}_{\geq 2})$ from the localization sequence $\mathrm{Sp}_{\geq 2} \to \mathrm{Sp}_{\geq 1} \to \mathrm{Sp}^\omega_{T(1)}$. This can be again computed by Lemma 3.6.2, but we need to use a trick to account for the fact that the self map we need to cofibre by is no longer in degree 0. This trick is an instantiation of the rotation invariance phenomenon studied in [Lur15].

Lemma 3.6.4. *Let C be a stable category. Then the map $\mathcal{U}_{loc}(C) \to \mathcal{U}_{loc}(C[x_2^{\pm 1}])$ is naturally a split inclusion.*

Proof. There is a localization sequence $C[x_2]^{x_2-\mathrm{nil}} \to C[x_2] \to C[x_2^{\pm 1}]$. Because C is canonically a retract of $C[x_2]$, it will suffice to show that the map $F : C[x_2]^{x_2-\mathrm{nil}} \to C[x_2] \to C$ is naturally null on $\mathcal{U}_{\mathrm{loc}}$. There is a cofibre sequence of functors in $\mathrm{Fun}(C[x_2]^{x_2-\mathrm{nil}})$

$$\Sigma^2 U \xrightarrow{x_2} U \longrightarrow F$$

where U is the underlying functor, and x_2 is the natural transformation given by multiplication by x_2. Since $\mathcal{U}_{\mathrm{loc}}$ is additive and $\Sigma^2 U$ and U induce the same map after applying $\mathcal{U}_{\mathrm{loc}}$, it follows that F is null upon applying $\mathcal{U}_{\mathrm{loc}}$.

□

Out next goal is to lift $X_i \otimes \mathrm{cof}\, 2$ to $\mathrm{Sp}_{\geq 1}$. Recall that X_i was constructed as the cofibre of a map $\zeta_{i-1} : \Sigma^{-1}L_{K(1)}\mathbb{S} \to X_{i-1}$. Let us fix a v_1-self map on $\mathrm{cof}\, 2$, so that a power of v_1 will indicate a power of that particular self map. After tensoring with $\mathrm{cof}\, 2$, we can lift maps from $\mathrm{Sp}^\omega_{T(1)}$ to $\mathrm{Sp}_{\geq 1}$ after sufficient composition with v_1. Thus we can inductively construct finite type 1 spectra $\tilde{X}_i$ such that its $T(1)$-localization is $X_i \otimes \mathrm{cof}\, 2$ and it has a lift of $v_1^? \zeta_i$ so that the cofibre is $\tilde{X}_{i+1}$.

Choose a v_1 self map on $\tilde{X}_{i+1}$, and note that $\bar{g}_i v_1^{j_i}$ lifts to a self map $\tilde{g}_i$ of $\tilde{X}_{i+1}$ for j_i sufficiently large.

Proposition 3.6.5. *The boundary map $K_1(\mathrm{Sp}^{\omega}_{T(1)}) \to K_0(\mathrm{Sp}_{\geq 2})$ sends $\bar{g}_i$ to $[\mathrm{cof}\, \tilde{g}_i] - [\mathrm{cof}\, v_1^{j_i}]$, where the cofibres are taken as self maps of $\tilde{X}_i$. Thus these are a basis of the 2-torsion of $K_0(\mathrm{Sp}_{\geq 2})$, and these along with $\mathrm{cof}(2, \eta, v_1)$ generate $K_0(\mathrm{Sp}_{\geq 2})$.*

Proof. The later statements follow from the claim about the boundary map by applying Theorem D, so we will just prove the claim about the boundary map. By Lemma 3.6.4, it suffices to do so after tensoring the localization sequence with $\mathbb{S}[x_2^{\pm 1}]$. Let u_1 denote the v_1-self map of $X_i \otimes \mathrm{cof}\, 2$, except shifted into degree 0. It is an automorphism so we have $[\bar{g}_i] = [\bar{g}_i u_1^{j_i}] - [u_1^{j_i}]$. But $\bar{g}_i u_1^{j_i}$ and $u_1^{j_i}$ lift to self maps of $\tilde{X}$, so by applying Lemma 3.6.2 and observing that x_2 is an automorphism so can be ignored when taking cofibres, we learn that the boundary is $[\mathrm{cof}\, \tilde{g}_i] - [\mathrm{cof}\, v_1^{j_i}]$. ☐

We now run this consruction explicitly for g_1. Here $\mathrm{cof}\, 2$ has a v_1^4-self map, g_1 is $1 + \eta\zeta$, and σ is a lift of ζv_1^4. since $\eta\sigma$ is 2-torsion, so we can form the map $\overline{\eta\sigma} : \Sigma^8 \mathbb{S}/2 \to \mathbb{S} \to \mathbb{S}/2$, where the first map in the composite is given by a nulhomotopy of $2\eta\sigma$. $\tilde{g}_1$ is then given by the automorphism of $\mathrm{cof}\, 2$ named $v_1^4 + \overline{\eta\sigma}$, so the first 2-torsion K_0 class is $[\mathbb{S}/(2, v_1^4 + \overline{\eta\sigma})] - [\mathbb{S}/(2, v_1^4)]$.

Even though we can explicitly construct representing K_0 classes, the computation of $K_0(\mathrm{Sp}_{\geq 2})$ remains somewhat inexplicit. For example, given a type 2 spectrum X, where $p > 2$, it does not obviously give a way of building X out of $\mathbb{S}/(p, v_1)$ via cofibre sequences. On the other hand, any type 1 spectrum has an explicit way of being built out of $\mathbb{S}/p$: namely its cell decomposition naturally decomposes into Moore spectra. Thus we ask:

Question 3.6.6. Is there an explicit way, given a type 2 spectrum, to build it out of representatives of the generating K_0 classes via cofibre sequences?

Question 3.6.7. At the prime 2, given a type 2 spectrum, is there a way of computing its K_0 class?

One possible approach to Question 3.6.7 would be to try to understand the isomorphisms $K_0(\mathrm{Sp}_{\geq 2}) \cong \mathrm{im}(K_1(\mathrm{Sp}^{\omega}_{K(1)})) \cong K_1(\mathrm{Sp}^{\omega}_{K(1)})/K_1(\mathrm{Sp}_{\geq 1})$.

3.7 Euler characteristics

We turn to studying the torsion free part of $K_0(\mathrm{Sp}_{\geq n})$, and use Theorem 3.4.2 an answer to [HS99, Problem 16.4] at height 1. We study natural homomorphisms out of this torsion free part called Euler characteristics. We explain how the image of the Euler characteristic $\chi_{\mathrm{BP}\langle n\rangle}$ is an obstruction to small type $n+1$ spectra such as Smith–Toda complexes existing. Using the existence of spectra such as ko and tmf, we compute the image of $\chi_{\mathrm{BP}\langle n\rangle}$ at heights ≤ 2, and conjecture what the answer is in general.

Recall from [MR99] that a spectrum X is said to be fp of type at most n if it is bounded below, p-complete, and $X \otimes Y$ is π-finite for any $Y \in \mathrm{Sp}_{\geq n+1}$. Let fp_n denote the full subcategory of Sp consisting of fp spectra of type at most n. Note that fp_{-1} is the category of p-torsion π-finite spectra.

Lemma 3.7.1. *$K(\mathrm{fp}_{-1}) \cong K(\mathbb{F}_p)$ and $K(\mathrm{fp}_0) \cong K(\mathbb{Z}_p)$.*

Proof. The t-structure on spectra is bounded on fp_0 and fp_{-1} with hearts finitely generated discrete $\mathbb{Z}_p$-modules and p-nil discrete $\mathbb{Z}_p$-modules respectively. The result then follows from the nonconnective theorem of the heart [AGH19] (which is a refinement of [Bar15]), and Quillen's devissage [Qui73]. □

Remark 3.7.2. We could alternatively have used Theorem 3.2.1 to prove Lemma 3.7.1, but the method above seems more direct. ◁

Tensoring sets up a pairing $\chi : K(\mathrm{fp}_n) \otimes K(\mathrm{Sp}_{\geq n+1}) \to K(\mathrm{fp}_{-1}) \cong K(\mathbb{F}_p)$, which we call the *Euler characteristic*. If we fix a class $[X] \in K_0(\mathrm{fp}_n)$, we obtain a map $\chi_X : K(\mathrm{Sp}_{\geq n+1}) \to K(\mathbb{F}_p)$. On π_0, identifying $K_0(\mathbb{F}_p) \cong \mathbb{Z}$, this map takes Y to $\Sigma_i(-1)^i \log_p |\pi_i(X \otimes Y)|$.

Euler characteristics are in general nontrivial homomorphisms. For example, if $\mathrm{BP}\langle n\rangle$ is the (p-completed) truncated Brown–Peterson spectrum, then it is easy to see that on the generalized Moore spectrum $\mathbb{S}/(p^{i_0}, v_1^{i_1}, \dots v_n^{i_n})$, $\chi_{\mathrm{BP}\langle n\rangle}$ takes the value Πi_j. In fact $\chi_{\mathrm{BP}\langle n\rangle}$ is shown in [BL25] to give an equivalence $K(\mathrm{Sp}_{\geq n+1})[\frac{1}{p}] \to K(\mathbb{F}_p)[\frac{1}{p}]$. In particular, the image of the map on π_0 integrally is generated by some power of p.

Euler characteristics can be used to obstruct the existence of small type n spectra. For example, because $\mathrm{ko}_2/\eta \cong \mathrm{ku}_2$ and ko_2 and ku_2 are in fp_1, the image of χ_{ku_2} is a multiple of 2 (in fact it is exactly 2). It follows that the Smith–Toda complex $\mathbb{S}/(2, v_1)$ cannot exist, because $\chi_{\mathrm{ku}_2}(\mathbb{S}/(2, v_1))$ would be 1. This example shows the image of $\chi_{\mathrm{BP}\langle n\rangle}$ is an obstruction to Smith–Toda complexes and other small type $n+1$ spectra from existing. Below we compute the image of $\chi_{\mathrm{BP}\langle n\rangle}$ at low heights.

Proposition 3.7.3. *The table below lists the image of $\chi_{\mathrm{BP}\langle n\rangle} : K_0(\mathrm{Sp}_{\geq n+1}) \to \mathbb{Z}$ at some low heights. In all these cases, there exists $X \in \mathrm{fp}_n$ such that χ_X has image $\mathbb{Z}$.*

	prime			
n	2	3	5	>5
$0, -1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
2	$8\mathbb{Z}$	$3\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
3	*?*	*?*	*?*	$\mathbb{Z}$

Proof. In all the cases in the table where the answer is $\mathbb{Z}$, there exists a Smith–Toda complex $V(n)$ [Tod71], which has the property that $V(n) \otimes BP\langle n\rangle = \mathbb{F}_p$, so 1 is indeed in the image.

For $n = 1, p = 2$, we observe that ko_2 is a type 1 fp spectrum such that $ko_2/\eta = BP\langle 1\rangle$ and $ko_2 \otimes \mathbb{S}/(2, \eta, v_1) \cong \mathbb{F}_2$. It follows that $2[ko_2] = [BP\langle 1\rangle]$ in $K_0(\mathrm{fp}_1)$ and that the image of χ_{ko_2} is 1. Thus the image of $\chi_{\mathrm{BP}\langle 1\rangle} = 2\chi_{ko_2}$ is 2.

For $n = 2, p = 2$, we use [LN12], which shows that 2-adically there is an $\mathbb{E}_\infty$-map $\mathrm{tmf} \to \mathrm{tmf}_1(3) \cong BP\langle 2\rangle$ realizing at the level of mod 2 cohomology the quotient $\mathcal{A} /\!\!/ E(2) \to \mathcal{A} /\!\!/ \mathcal{A}(2)$. It follows that $\mathrm{tmf}_1(3) \otimes_{\mathrm{tmf}} \mathbb{F}_2$ is the dual of $\mathcal{A}(2) /\!\!/ E(2)$, which is an exterior algebra on three generators in even degrees. Since tmf is connective with $\pi_0(\mathrm{tmf}) = \mathbb{Z}_2$, the map $K_0(\mathrm{tmf}) \to K_0(\mathbb{F}_2) \cong \mathbb{Z}$ is an isomorphism, and we learn that $\mathrm{tmf}_1(3)$ is perfect over tmf and $[\mathrm{tmf}_1(3)]$ is the class $8 \in K_0(\mathrm{tmf})$. tmf is fp of type 2: in [BE20], a type 2 spectrum Z was constructed with a v_2^1-self map with the property that $Z/v_2 \otimes \mathrm{tmf} \cong \mathbb{F}_2$. It follows that χ_{tmf} has image $\mathbb{Z}$, and that $\chi_{\mathrm{BP}\langle 2\rangle} \cong 8\chi_{\mathrm{tmf}}$.

For $n = 2, p = 3$, [BP04] constructed a v_2^1 self map on the spectrum $Y(2) \otimes V(1)$, where $Y(2)$ is the 8-skeleton of $\mathbb{S} /\!\!/ \alpha_1$. Moreover, there is an equivalence $Y(2) \otimes \mathrm{tmf} \cong \mathrm{BP}\langle 2\rangle \oplus \Sigma^8 \mathrm{BP}\langle 2\rangle$ (see Remark 2.3 of ibid.). It follows that $(Y(2) \otimes V(1))/v_2$ has χ_{tmf} equal to 2, and $3[\mathrm{tmf}] = 2[\mathrm{BP}\langle 2\rangle]$. Thus the image of $\chi_{\mathrm{BP}\langle 2\rangle}$ must be $3\mathbb{Z}$, and the image of $\chi_{\mathrm{BP}\langle 2\rangle \oplus \Sigma \mathrm{tmf}}$ is $\mathbb{Z}$. ☐

Question 3.7.4. What is the generator of the image of $\chi_{\mathrm{BP}\langle n\rangle} : K_0(\mathrm{Sp}_{\geq n+1}) \to \mathbb{Z}$? Is it the size of the maximal finite p-subgroup of the height n Morava stabilizer group?

Indeed, in the cases studied in Proposition 3.7.3, the result agrees with the size of the maximal p-subgroup of the Morava stabilizer group at that height (see [Hew95]).

Question 3.7.5. Does there always exist an X fp of type n such that $\chi_X : K_0(\mathrm{Sp}_{\geq n+1}) \to \mathbb{Z}$ is surjective?

The truth of Question 3.7.5 would suggest that fp spectra and finite spectra are dual in the sense that the extent of the failure of small type $n+1$-spectra to exist corresponds exactly to that of the failure of $\mathrm{BP}\langle n\rangle$ to be a regular fp type n ring closest to the sphere.

Now we turn to answer a question of Hovey–Strickland [HS99]. They considered two homomorphisms, χ, ξ out of $K_0(\mathrm{Sp}^\omega_{K(n)})$. At height n, prime p, suppose that M_* is a graded p-torsion graded abelian group of periodicity $|v_n|p^i$. Then $\mathrm{len}M_*$ is defined to be $\frac{1}{p^i}\sum_0^{|v_n|p^i-1} \log_p |M_*|$. $\chi : K_0(\mathrm{Sp}_{K(n)}) \to \mathbb{Z}$ is defined as $[X] \mapsto \mathrm{len}\pi_{\mathrm{even}}(E(n)_* \otimes X) - \mathrm{len}\pi_{\mathrm{odd}}(E(n)_* \otimes X)$ and $\xi : K_0(\mathrm{Sp}_{K(n)}) \to \mathbb{Z}[\frac{1}{p}]$ is defined as $[X] \mapsto \mathrm{len}\pi_{\mathrm{even}}(X) - \mathrm{len}\pi_{\mathrm{odd}}(X)$. They then ask:

Problem 3.7.6 ([HS99, Problem 16.4])**.** What is the relationship between χ and ξ? Is χ an isomorphism? If not, is $\mathbb{Q} \otimes \chi$ an isomorphism? Can one say anything about the higher K-theory of $\mathrm{Sp}^\omega_{K(n)}$?

In Theorem 3.4.2, we described $K(\mathrm{Sp}^\omega_{K(1)})$ as a spectrum. We now answer the rest of Problem 3.7.6 for $n = 1$.

Proposition 3.7.7. *For $n = 1$, $\xi = 0$, and χ is an isomorphism.*

Proof. The generator of $K_0(\mathrm{Sp}^\omega_{K(1)}) \cong \mathbb{Z}$ is $L_{K(1)}\mathbb{S}/p$, so we need merely check the result on the generator. $\pi_* L_{K(1)}/p$ is an exterior algebra on $\pi_* K(1)$ on ζ, which is an odd degree class, so we see that ξ evaluates to 0. $E(1)/p$ is $K(1)$, so $\chi(\mathbb{S}/p) = 1$. ☐

In [BL25], we answer most of Problem 3.7.6 at all heights.

Proposition 3.7.8 ([BL25])**.** *χ is an isomorphism after inverting p and ξ is 0.*

Remark 3.7.9. In fact χ is closely related to $\chi_{\mathrm{BP}\langle n-1\rangle}$: there is a commutative diagram

$$\begin{tikzcd}
K_0(\mathrm{Sp}_{\geq n}) \arrow[r, "\chi_{\mathrm{BP}\langle n-1\rangle}"] \arrow[d] & \mathbb{Z} \arrow[d, equal] \\
K_0(\mathrm{Sp}_{K(n)}) \arrow[r, "\chi"] & \mathbb{Z}
\end{tikzcd}$$

Indeed, it suffices to check this rationally, where it can be easily checked on a generalized Moore spectrum by the results of [BL25]. ◁

Question 3.7.10. Is the map $K_0(\mathrm{Sp}_{\geq n}) \to K_0(\mathrm{Sp}^{\omega}_{K(n)})$ an isomorphism after quotienting by p-torsion?[13]

Because of Remark 3.7.9, a positive answer to Question 3.7.10 would imply that the remaining part of Problem 3.7.6, determining the image of χ, is equivalent to Question 3.7.4.

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[13] One can also ask: is this true with $T(n)$ replacing $K(n)$?

Chapter 4

Topological Hochschild homology of the image of j (with David Lee)

We compute the mod (p, v_1) and mod $(2, \eta, v_1)$ THH of many variants of the image-of-J spectrum. In particular, we do this for j_ζ, whose TC is closely related to the K-theory of the $K(1)$-local sphere. We find in particular that the failure for THH to satisfy $\mathbb{Z}_p$-Galois descent for the extension $j_\zeta \to \ell_p$ corresponds to the failure of the p-adic circle to be its own free loop space. For $p > 2$, we also prove the Segal conjecture for j_ζ, and we compute the K-theory of the $K(1)$-local sphere in degrees $\leq 4p - 6$.

4.1 Introduction

The algebraic K-theory of the $K(1)$-local sphere, or $K(L_{K(1)}\mathbb{S})$, is an object capturing fundamental structural information about the $K(1)$-local category. Part of Ausoni–Rognes' original vision of chromatic redshift was that it could be understood, at least $T(2)$-locally, via Galois hyperdescent. More specifically, they conjectured [AR02, pg 4] that the map

$$K(L_{K(1)}\mathbb{S}) \otimes V \to K(\mathrm{KU}_p)^{h\mathbb{Z}_p^\times} \otimes V$$

is an equivalence in large degrees when V is a type 2 finite spectrum. The $T(n+1)$-local K-theory of Morava E-theory has been shown in [CMNN20, Theorem 1.10] to have Galois descent for finite subgroups of the Morava stabilizer group. Moreover, recent work of Ben Moshe–Carmeli–Schlank–Yanovski [BMCSY23] shows that

$$L_{K(2)}K(L_{K(1)}\mathbb{S}) \to L_{K(2)}K(\mathrm{KU}_p)^{h\mathbb{Z}_p^\times}$$

is an equivalence, i.e that Galois hyperdescent is satisfied $K(2)$-locally.

Recent work of the second author [Lev22] has made $K(L_{K(1)}\mathbb{S})$ an integrally accessible object. If we consider the connective Adams summand ℓ_p (or ko_2 for $p = 2$) as a $\mathbb{Z}$-equivariant

$\mathbb{E}_\infty$-ring via the Adams operation Ψ^{1+p}, then j_ζ is defined to be its $\mathbb{Z}$-homotopy fixed points. Then it is shown that there is a cofiber sequence

$$K(j_\zeta) \to K(L_{K(1)}\mathbb{S}) \to \Sigma K(\mathbb{F}_p)$$

split on π_*. It is also shown that the Dundas–Goodwillie–McCarthy square

$$\begin{tikzcd} K(j_\zeta) \arrow[r] \arrow[d] & \mathrm{TC}(j_\zeta) \arrow[d] \\ K(\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}}) \end{tikzcd}$$

is a pullback square. The three spectra $K(\mathbb{F}_p), K(\mathbb{Z}_p)$, and $\mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})$[1] are understood, so understanding $K(L_{K(1)}\mathbb{S})$ is essentially reduced to understanding $\mathrm{TC}(j_\zeta)$.

The primary goal of this paper is to understand $\mathrm{THH}(j_\zeta)$ modulo (p, v_1) and $(2, \eta, v_1)$, which is the first step in understanding $\mathrm{TC}(j_\zeta)$.

Theorem 4.1.1. *For $p > 2$, there is an isomorphism of rings*

$$\pi_* \mathrm{THH}(j_\zeta)/(p, v_1) \cong \pi_* \mathrm{THH}(\ell_p)/(p, v_1) \otimes_{\mathbb{F}_p} \mathrm{HH}_*(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p).$$

For $p = 2$, there is an isomorphism of rings

$$\pi_* \mathrm{THH}(j_\zeta)/(2, \eta, v_1) \cong \pi_* \mathrm{THH}(\mathrm{ko}_2)/(2, \eta, v_1) \otimes_{\mathbb{F}_2} \mathrm{HH}_*(\mathbb{F}_2^{B\mathbb{Z}}/\mathbb{F}_2).$$

Each of the terms on the right hand side of the equivalences is well understood. The ring $\pi_* \mathrm{THH}(\ell_p)/(p, v_1)$ can be found in [MS93b] or Example 4.4.3, and $\pi_* \mathrm{THH}(\mathrm{ko}_2)/(2, \eta, v_1)$ can be found in [AR05] or Example 4.4.16.

The last tensor factor is given in Lemma 4.4.6 as

$$\mathrm{HH}_*(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p) \cong \Lambda[\zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$$

where $|\zeta| = -1$ and $C^0(\mathbb{Z}_p; \mathbb{F}_p)$ denotes the ring of continuous functions from $\mathbb{Z}_p$ to $\mathbb{F}_p$.

The $C^0(\mathbb{Z}_p; \mathbb{F}_p)$ appearing can be viewed as the failure of descent at the level of THH for the $\mathbb{Z}_p$-Galois extension coming from the $\mathbb{Z}_p$-action on ℓ_p and ko_2. More precisely, at the level of π_*, the map

$$\mathrm{THH}(\ell_p^{h\mathbb{Z}})/(p, v_1) \to \mathrm{THH}(\ell_p)^{h\mathbb{Z}}/(p, v_1)$$

is induced by the base change along $C^0(\mathbb{Z}_p; \mathbb{F}_p) \to \mathbb{F}_p$ that sends a continuous function to its value at 0 (Remark 4.4.13).

This phenomenon can be explained by interpreting THH in terms of free loop spaces. If X is a pro-p-finite space, then the $\mathbb{F}_p$-Hochschild homology of the cochain algebra $C^*(X; \mathbb{F}_p)$ is computed as

$$\mathrm{HH}(C^*(X; \mathbb{F}_p)/\mathbb{F}_p) = C^*(LX; \mathbb{F}_p)$$

---

[1] $\mathbb{Z}_p^{B\mathbb{Z}}$ denotes the cochains on the circle $B\mathbb{Z}$ with coefficients in $\mathbb{Z}_p$. Its TC is essentially the nil-TC of $\mathbb{Z}_p$ by [LT23, Corollary 4.5], which is studied in [HM04].

where LX is the free loop space of X. Since $C^*(B\mathbb{Z}_p;\mathbb{F}_p) \cong \mathbb{F}_p^{B\mathbb{Z}}$, the failure of the descent

$$\mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p) \ncong \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p)^{B\mathbb{Z}}$$

is explained by the fact that $B\mathbb{Z}_p$ is not $LB\mathbb{Z}_p \cong B\mathbb{Z}_p \times \mathbb{Z}_p$. For any p-complete $\mathbb{E}_\infty$-ring R with a trivial $\mathbb{Z}$-action, this completely accounts for the failure of p-complete THH to commute with $\mathbb{Z}$-fixed points (Corollary 4.4.8). The content of Theorem A is that the same phenomenon happens for $\mathrm{THH}(j_\zeta)$ on π_* mod (p, v_1) or $(2, \eta, v_1)$, even though the action is no longer trivial. In particular, Theorem A implies that there is an isomorphism of rings

$$\pi_* \mathrm{THH}(j_\zeta)/(p,v_1) \cong \pi_* \mathrm{THH}(\ell_p^{B\mathbb{Z}})/(p,v_1)$$

where $\ell_p^{B\mathbb{Z}}$ is the homotopy fixed points of ℓ_p by a *trivial* $\mathbb{Z}$-action.

The key idea in our proof of Theorem A is to run the spectral sequence for THH obtained by filtering j_ζ via the homotopy fixed point filtration, and showing that the differentials in the associated spectral sequence behave similarly enough to the case of a trivial action. To understand the associated graded algebra of the homotopy fixed point filtration, we further filter it by the p-adic filtration. At the level of the associated graded of both filtrations, j_ζ is indistinguishable from the fixed points by a trivial action, and we show that mod (p, v_1) and $(2, \eta, v_1)$ this remains true at the level of homotopy rings after running the spectral sequences for THH of those filtrations.

The phenomenon that the $\mathbb{Z}$-action on ℓ_p behaves like the trivial one is shown in [BHLS23] to asymptotically hold even at the level of cyclotomic spectra. More precisely, it is shown there that given any fixed type 3 finite spectrum V, for all sufficiently large k,

$$\mathrm{THH}(\ell_p^{hp^k\mathbb{Z}}) \otimes V \cong \mathrm{THH}(\ell_p^{B\mathbb{Z}}) \otimes V$$

as cyclotomic spectra.

It is shown then that the failure of descent we observe on THH continues at the level of the $T(2)$-local TC. Combining this with the aforementioned hyperdescent result of the $K(2)$-local K-theory and the formula for the K-theory of the $K(1)$-local sphere, this implies that $L_{T(2)}K(L_{K(1)}\mathbb{S})$ is not $K(2)$-local and hence is a counterexample to the height 2 telescope conjecture. In particular, this implies that the map

$$K(L_{K(1)}\mathbb{S}) \otimes V \to K(\mathrm{KU})^{h\mathbb{Z}_p^\times} \otimes V$$

considered by Ausoni–Rognes is *not* an equivalence in large degrees.

The ring $C^0(\mathbb{Z}_p;\mathbb{F}_p)$ that appears in our formula for $\mathrm{THH}(j_\zeta)$ is a key ingredient in [BHLS23] to maintain asymptotic control over $\mathrm{THH}(j_{\zeta,k})$ as a cyclotomic spectrum, and is one of the advantages of j_ζ versus the usual connective image-of-J spectrum $j = \tau_{\geq 0} j_\zeta$. If one was only interested in understanding $L_{T(2)}K(L_{K(1)}\mathbb{S})$, there are isomorphisms

$$L_{T(2)}K(L_{K(1)}\mathbb{S}) \cong L_{T(2)}\mathrm{TC}(j) \cong L_{T(2)}\mathrm{TC}(j_\zeta)$$

so one can in principle approach the telescopic homotopy via $\mathrm{TC}(j)$ instead of $\mathrm{TC}(j_\zeta)$.

However, j is not as well behaved as j_ζ is, as we now explain. We extend our methods for computing $\mathrm{THH}(j_\zeta)$ in Sections 4.5 and 4.6 to compute THH of j, giving a relatively simple proof of the result below due to Angelini-Knoll and Höning [AK21; Hön21].

Theorem 4.1.2. *For $p > 3$*[2]*, the ring $\pi_* \mathrm{THH}(j)/(p, v_1)$ is the homology of the CDGA*

$$\mathbb{F}_p[\mu_2] \otimes \Lambda[\alpha_1, \lambda_2, a] \otimes \Gamma[b], \quad d(\lambda_2) = a\alpha_1$$

$$|b| = 2p^2 - 2p,\ |a| = 2p^2 - 2p - 1,\ |\lambda_2| = 2p^2 - 1,\ |\mu_2| = 2p^2$$

For $k \geq 1$ and any $p > 2$, we have an isomorphism of rings

$$\pi_* \mathrm{THH}(\tau_{\geq 0}(\ell_p^{hp^k\mathbb{Z}}))/(p, v_1) \cong \pi_* \mathrm{THH}(\ell_p)/(p, v_1) \otimes \mathrm{HH}_*(\tau_{\geq 0}\mathbb{F}_p[v_1]^{Bp^k\mathbb{Z}}/\mathbb{F}_p[v_1])/v_1.$$

The ring $\mathrm{HH}_*(\tau_{\geq 0}\mathbb{F}_p[v_1]^{Bp^k\mathbb{Z}}/\mathbb{F}_p[v_1])/v_1$[3] is described in Proposition 4.5.1: it is isomorphic to $\Gamma[d\alpha_{1/p^k}] \otimes \Lambda_{\mathbb{F}_p}[\alpha_{1/p^k}]$ where α_{1/p^k} is a class in degree $2p-3$ and $d\alpha_{1/p^k}$ is a divided power generator in degree $2p-2$.

In the above theorem, $\pi_* \mathrm{THH}(j)/(p, v_1)$ is *not* what one would expect in the case of the trivial action: there are two more differentials in the spectral sequence for the filtration we use to prove Theorem 4.1.2 than what one would find for the trivial action. The differentials witness the fact that $\lambda_1, \lambda_2 \in \pi_* \mathrm{THH}(\ell_p)/(p, v_1)$ (see Example 4.4.3) don't lift to $\mathrm{THH}(j)/(p, v_1)$. Whereas most computations of THH in this paper use Bökstedt's computation of $\mathrm{THH}(\mathbb{F}_p)$ as their fundamental input, these differentials ultimately come from the Adams–Novikov spectral sequence.

A key difference between the THH of j_ζ and j is that the ring $C^0(\mathbb{Z}_p; \mathbb{F}_p)$ that appeared in $\pi_* \mathrm{THH}(j_\zeta)/(p, v_1)$ is replaced by a divided power algebra for j. The advantage of the ring $C^0(\mathbb{Z}_p; \mathbb{F}_p)$ over a divided power algebra is that up to units, it consists entirely of idempotents, which decompose $\mathrm{THH}(j_\zeta)$ as an S^1-equivariant spectrum into a continuous $\mathbb{Z}_p$-indexed family of spectra. This decomposition is not evidently present in $\mathrm{THH}(j)$.

Another advantage of j_ζ over j is that j_ζ satisfies the THH Segal conjecture while j doesn't, which we show for $p > 2$ in Section 4.8:

Theorem 4.1.3. *For $p > 2$, the cyclotomic Frobenius map*

$$\mathrm{THH}(j_\zeta)/(p, v_1) \to \mathrm{THH}(j_\zeta)^{tC_p}/(p, v_1)$$

has $(2p-3)$-coconnective fiber, but the fiber of the cyclotomic Frobenius map

$$\mathrm{THH}(j)/(p, v_1) \to \mathrm{THH}(j)^{tC_p}/(p, v_1)$$

is not bounded above.

---

[2] We also compute an associated graded ring $\mathrm{THH}(j)/(p, v_1)$ for $p = 3$ (see Theorem 4.5.5), but are unable to solve multiplicative extension problems coming from the fact that $j/(p, v_1)$ is not an associative algebra for $p = 3$. Nonassociative multiplicative extensions aren't considered in [AK21], so the results of that paper also only compute an associated graded ring for $p = 3$.

[3] By rescaling, there is an isomorphism $p^k\mathbb{Z} \cong \mathbb{Z}$, so this ring doesn't depend on k.

The Segal conjecture for a ring j is a necessary condition [AN21, Proposition 2.25] for the Lichtenbaum–Quillen conjecture to hold, i.e for $\mathrm{TR}(j)\otimes V$ to be bounded above for any finite type 3 spectrum V. Thus Theorem 4.1.3 implies that j doesn't satisfy the Lichtenbaum–Quillen conjecture. On the other hand, Theorem 4.1.3 is a key ingredient in proving the Lichtenbaum–Quillen conjecture for j_ζ as carried out in [BHLS23]. This Lichtenbaum–Quillen conjecture can be viewed as the part of Ausoni–Rognes's conjecture that is true. Namely, it implies that the map

$$K(L_{K(1)}\mathbb{S})\otimes V \to K(L_{K(1)}\mathbb{S})\otimes V[v_2^{-1}]$$

is an equivalence in large degrees for V a type 2-complex. The Lichtenbaum–Quillen conjecture for $\ell^{hp^k\mathbb{Z}}$ for $k \gg 0$ is a key ingredient in [BHLS23] to show that the telescope conjecture fails at height 2.

In Section 4.7, we show how THH computations can give information about TC in the stable range. For a map of $\mathbb{E}_1$-rings $f : R \to S$, recall that the $\mathbb{E}_1$-cotangent complex $L_{S/R}$ is the fiber of the multiplication map $S\otimes_R S \to S$ as an S-bimodule. We prove the following result:

Theorem 4.1.4. *Given a map of $\mathbb{E}_1$-ring spectra $f : R \to S$, there is a natural map*

$$\mathrm{fib}\,\mathrm{TC}(f) \to \mathrm{THH}(S; L_{S/R}).$$

If f is an n-connective map of (-1)-connective rings for $n \geq 1$, this natural map is $(2n+1)$-connective.

A consequence of Theorem 4.1.4 is that the natural map above can be identified with the linearization map in the sense of Goodwillie calculus for the functor $\mathrm{fib}\,\mathrm{TC}(f) : \mathrm{Alg}(\mathrm{Sp})_{/S} \to \mathrm{Sp}$ when S is (-1)-connective and $f : R \to S$ is 1-connective.

In the case the map f is a trivial square zero extension of connective rings, a K-theory version of the result was obtained as [DM94, Theorem 3.4], and a TC version is essentially [Ras18, Theorem 4.10.1][4]. The point of Theorem 4.1.4 is to have a version of the result that works for arbitrary maps of $\mathbb{E}_1$-rings rather than trivial square-zero extensions, and for (-1)-connective rings instead of connective rings.

We use Theorem 4.1.4 to reprove basic facts about TC, such as the understanding of the map $\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p)$ on π_{2p-1}. This is an ingredient in the computation of $\mathrm{TC}(\mathbb{Z}_p)$ as a spectrum (see [BM93, Section 9]).

We also apply Theorem 4.1.4 to compute the fiber of the map $\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})$ in the stable range, giving information about $K(L_{K(1)}\mathbb{S})$:

Theorem 4.1.5. *For $p > 2$, there are isomorphisms*

$$\tau_{\leq 4p-6}\,\mathrm{fib}(\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})) \cong \Sigma^{2p-2}C^0(\mathbb{Z}_p;\mathbb{F}_p)$$

and

$$K_*L_{K(1)}\mathbb{S} \cong K_{*-1}\mathbb{F}_p \oplus K_*\mathbb{S}_p \oplus \pi_*\Sigma^{2p-2}C^0(\mathbb{Z}_p;\mathbb{F}_p)/\mathbb{F}_p, \quad * \leq 4p-6.$$

---

[4] See also [Hes94] and [LM12].

In particular, for $p > 2$, the infinite family of classes in the fiber of $\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})$ found in [Lev22] are simple p-torsion, and completely account for all the classes in the stable range.

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Notations and conventions

- The term category will refer to an ∞-category as developed by Joyal and Lurie.
- We refer the reader to [NS18] for basic facts about THH, which we freely use.
- For $\mathcal{C}$ a monoidal category, and R an $\mathbb{E}_1$-algebra, we use $\mathrm{THH}_{\mathcal{C}}(R)$ to denote the THH of R in $\mathcal{C}$. for $\mathcal{C} = \mathrm{Mod}(S)$ for an $\mathbb{E}_2$-algebra S, we also denote this by $\mathrm{THH}(R/S)$.
- $\mathrm{Map}(a, b)$ will denote the space of maps from a to b (in some ambient category).
- Tensor products and THH are implicitly p-completed.
- We use $\Lambda[x]$ and $\Gamma[x]$ to denote exterior and divided power algebras in homotopy rings.
- In an $\mathbb{F}_p$-vector space, we use $a \doteq b$ to mean that $a = cb$ for some unit $c \in \mathbb{F}_p^\times$, and $a \dot{\mapsto} b$ to mean that a is sent to b up to a unit in $\mathbb{F}_p^\times$.
- Conventions about filtrations and spectral sequences are addressed in Section 4.2.
- For a pro-finite set A, we use $C^0(A; \mathbb{F}_p)$ to denote continuous functions from A to $\mathbb{F}_p$.
- Let $\mathcal{D}$ be a monoidal category acting on a category $\mathcal{C}$. Given objects $X \in \mathcal{C}, Z \in \mathcal{D}$ with a self map $f : X \otimes Z \to X$, we use X/f to denote the cofibre of this map. We use $X/(f_1, \ldots, f_n)$ to denote $(\ldots(X/f_1)/\ldots)/f_n$, where each f_i is a self map of $X/(f_1, \ldots, f_{i-1})$.

4.2 Filtrations

In this section, we set up notation for working with filtered objects and explain how to put filtrations on ℓ_p, ko_2, j_ζ, and j, as well as for finite extensions. Our constructions amount to the filtration coming from the homotopy fixed point spectral sequences computing those objects, which in all cases except for j_ζ, is also the Adams–Novikov filtration.

4.2.1 Filtered objects and spectral sequences

Let $\mathcal{C}$ be a presentably symmetric monoidal stable category with accessible t-structure compatible with the symmetric monoidal structure. Let $\mathrm{Fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_{\leq}, \mathcal{C})$ be the category of decreasingly filtered objects, and let $\mathrm{gr}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ be the category of graded objects, so that both are symmetric monoidal via Day convolution. Basic properties of these categories are developed in [Lur15] and [BHS0 , Appendix B]. Given an object $x \in \mathrm{Fil}(\mathcal{C})$ or $\mathrm{gr}(\mathcal{C})$, we write x_i for the value at $i \in \mathbb{Z}$. The left adjoint of the functor $(-)_i$ in the case of $\mathrm{Fil}(\mathcal{C})$ is the functor $(-)^{0,i}$, defined for $c \in \mathcal{C}$ by

$$(c^{0,i})_j = \begin{cases} c & (j \leq i) \\ 0 & (j > i). \end{cases}$$

We also use the notation

$$c^{k,n} := \Sigma^k c^{0,n+k}$$

$$\pi^{\heartsuit}_{k,n} x := \pi^{\heartsuit}_k x_{n+k}$$

$$\pi_{k,n} x := \pi_k x_{n+k} = \pi_0 \operatorname{Map}(\overset{k,n}{\mathbb{1}}, x)$$

and use c to also denote $c^{0,0}$. There is class $\tau \in \pi_{-1,0}\, \mathbb{1}^{0,0}$ called the filtration parameter because the map $x_i \to x_{i-1}$ giving the filtration is obtained level-wise from tensoring with τ.

The functor $(-)^{0,0} : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ is a symmetric monoidal fully faithful functor, which we refer to as the *trivial filtration*. We often identify an object $c \in \mathcal{C}$ with the trivial filtered object in $\mathrm{Fil}(\mathcal{C})$. In fact, $\mathrm{gr}(\mathcal{C})$ can be identified with $\mathrm{Mod}_{\mathrm{cof}\,\tau}(\mathrm{Fil}(\mathcal{C}))$, so that taking associated graded amounts to base changing to $\mathrm{cof}\,\tau$. Given an object $x \in \mathrm{Fil}(\mathcal{C})$, we let $\mathrm{gr} x \in \mathrm{gr}(\mathcal{C})$ denote the associated graded object, so that $(\mathrm{gr} x)_i = \mathrm{gr}_i x = \mathrm{cof}(x_{i+1} \xrightarrow{\tau} x_i)$.

On the other hand, there is an identification $\mathrm{Fil}(\mathcal{C})[\tau^{-1}] \cong \mathcal{C}$, so that given a filtered object $x \in \mathrm{Fil}(\mathcal{C})$, its underlying object $ux \in \mathcal{C}$, given by $\mathrm{colim}_i\, x_i$, is identified with $x[\tau^{-1}]$. Under the assumption that the t-structure is compatible with filtered colimits, we have an isomorphism $\pi^{\heartsuit}_{**} x[\tau^{-1}] \cong \pi^{\heartsuit}_{*} ux \otimes \mathbb{Z}[\tau^{\pm 1}]$.

Construction 4.2.1. Given a filtered object $x \in \mathrm{Fil}(\mathcal{C})$, there is a spectral sequence which we refer to as *the spectral sequence associated with* x.

$$E_1^{s,t} = \pi^{\heartsuit}_{t-s,s} \mathrm{gr} x = \pi^{\heartsuit}_{t-s} (\mathrm{gr} x)_t \implies \pi^{\heartsuit}_{t-s}(ux)$$

The d_r-differential is a map from $E_r^{s,t}$ to $E_r^{s+r+1,t+r}$, which is a page off from the usual Adams convention, i.e. our d_r differential would be the d_{r+1} differential in the Adams convention. We shall say *Adams weight* and *filtration degree* to refer to the bidegrees s and t, respectively. ◁

In addition to the spectral sequence associated with x, there is also the τ-Bockstein spectral sequence, which has signature

$$E_1^{**} = (\pi^{\heartsuit}_{**} \mathrm{gr} x)[\tau] \implies \pi^{\heartsuit}_{**} x$$

We do not use the following lemma, but we state it as an exercise to help acquaint the unfamiliar reader with filtered objects. The τ-inverted τ-Bockstein spectral sequence refers to the spectral sequence obtained from the τ-Bockstein spectral sequence by inverting τ on each page.

Lemma 4.2.2. *Let $x \in \mathrm{Fil}(\mathcal{C})$. For each $r \geq 1$, the E_r-page of the τ-inverted τ-Bockstein spectral sequence for x is isomorphic to $\mathbb{Z}[\tau^{\pm}]$ tensored with the E_r-page of the spectral sequence associated with x. Moreover, the d_r differential on the former is given by τ^r times the d_r differential on the latter. The filtration on $\pi^{\heartsuit}_{**}x[\tau^{\pm 1}]$ coming from the spectral sequence agrees with the filtration on $\pi^{\heartsuit}_{*}x \otimes \mathbb{Z}[\tau^{\pm}]$ coming from the filtration on x.*

Proof. These statements can be checked for example by using explicit formulas for the pages and differentials. See, for example, [Lur17, Construction 1.2.2.6]. □

4.2.2 *t*-structures

We turn to studying t-structures on categories of filtered objects. Our ability to produce t-structures comes from the following general result.

Lemma 4.2.3 ([Lur17, Proposition 1.4.4.11])**.** *Let $\mathcal{C}$ be a presentable stable category. If $\{X_\alpha\}$ is a small collection of objects in $\mathcal{C}$, then there is an accessible t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ such that $\mathcal{C}_{\geq 0}$ is the smallest full subcategory of $\mathcal{C}$ containing each X_α and closed under colimits and extensions. The full subcategory of coconnective objects is characterized by the condition that $Y \in \mathcal{C}_{\leq 0}$ if and only if $\mathrm{Map}(\Sigma X_\alpha, Y) = 0$ for each X_α.*

Definition 4.2.4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. Define a t-structure $(\mathrm{Fil}(\mathcal{C})^f_{\geq 0}, \mathrm{Fil}(\mathcal{C})^f_{\leq 0})$ on the underlying category $\mathrm{Fil}(\mathcal{C})$ be the t-structure whose connective objects are generated by the objects $\Sigma^{f(i)} c^{0,i}$ for $c \in \mathcal{C}_{\geq 0}$. We let $\tau^f_{\geq i}$ and $\tau^f_{\leq i}$ denote the associated truncation functors. We similarly define a t-structure $(\mathrm{gr}(\mathcal{C})^f_{\geq 0}, \mathrm{gr}(\mathcal{C})^f_{\leq 0})$ by taking the image of those objects under the functor gr to be the generators. ◁

Lemma 4.2.5. *Let $x \in \mathrm{Fil}(\mathcal{C})$.*

1. *$x \in \mathrm{Fil}(\mathcal{C})^f_{\leq 0}$ if and only if x_i is $f(i)$-coconnective in $\mathcal{C}$ for each i.*
2. *If f is nondecreasing, then $x \in \mathrm{Fil}(\mathcal{C})^f_{\geq 0}$ iff x_i is $f(i)$-connective for each i. In this case, the truncation functor $\tau^f_{\geq 0}$ is given by $(\tau^f_{\geq 0}x)_i = \tau_{\geq f(i)}(x_i)$.*
3. *The same results hold for $(\mathrm{gr}(\mathcal{C})^f_{\geq 0}, \mathrm{gr}(\mathcal{C})^f_{\leq 0})$.*

Proof. We prove the result for $\mathrm{Fil}(\mathcal{C})$, as the result for $\mathrm{gr}(\mathcal{C})$ is similar but easier. Coconnectivity can be checked by mapping in the generators of $\mathrm{Fil}(\mathcal{C})^f_{\leq 0}$. Because of the adjunction defining the functor $(-)^{0,n}$, the condition for coconnectivity follows.

Now suppose f is nondecreasing. To prove the claims, It suffices to show that if $x \in \mathrm{Fil}(\mathcal{C})$ has $x_i \in \mathcal{C}_{\geq f(i)}$, then x admits no maps to a coconnected object. If y is a coconnected object, then x_i admits no maps to y_j for $j \leq i$ because y_j is $f(j)$-coconnected, and since f is nondecreasing, it is $f(i)$-coconnected. It follows that there are no nonzero maps of filtered objects $x \to y$. ☐

Lemma 4.2.6. *The t-structures* $\mathrm{Fil}(\mathcal{C})^f, \mathrm{gr}(\mathcal{C})^f$ *are compatible with the symmetric monoidal structure if* $f(0) = 0$ *and* $f(i) + f(j) \geq f(i + j)$.

Proof. The condition $f(0) = 0$ guarantees that the unit is connective. One needs to check that the tensor product of any pair of generators of $\mathrm{Fil}(\mathcal{C})^f_{\geq 0}$ is still in $\mathrm{Fil}(\mathcal{C})^f_{\geq 0}$. But the tensor product of $\Sigma^{f(i)}c^{0,i}$ and $\Sigma^{f(j)}d^{0,j}$ is $\Sigma^{f(i)+f(j)}(c \otimes d)^{0,i+j}$, which is in $\mathrm{Fil}(\mathcal{C})^f_{\geq 0}$ because $c \otimes d$ is in $\mathcal{C}_{\geq 0}$ and so the assumption on f shows that this is connective. ☐

The functor gr is right t-exact with respect to the t-structure corresponding to a nondecreasing function f, but not in general t-exact. In the following situation it preserves $\tau_{\geq 0}$.

Lemma 4.2.7. *Suppose that* $c \in \mathrm{Fil}(\mathcal{C})$, $f : \mathbb{Z} \to \mathbb{Z}$ *is nondecreasing,* $\pi^{\heartsuit}_{k,i-k}c = 0$ *for* $f(i-1) \leq k < f(i)$, *and* $\pi^{\heartsuit}_{f(i)-1,i-f(i)+2}$ *contains no simple* τ*-torsion. Then* $\tau^f_{\geq 0}\mathrm{gr}(c) \cong \mathrm{gr}(\tau^f_{\geq 0}(c))$ *and* $\tau^f_{\leq 0}\mathrm{gr}(c) \cong \mathrm{gr}(\tau^f_{\leq 0}(c))$.

Proof. It suffices to prove the statement for $\tau_{\geq 0}$ since gr is exact. There is a cofiber sequence $c_{i+1} \xrightarrow{\tau} c_i \to \mathrm{gr}_i c$. By Lemma 4.2.5 we would like $\tau_{\geq f(i+1)}c_{i+1} \to \tau_{\geq f(i)}c_i \to \tau_{\geq f(i)}\mathrm{gr}_i c$ to remain a cofiber sequence. From the exact sequence of homotopy groups, we see that we would like $\tau_{\geq f(i+1)}c_{i+1} = \tau_{\geq f(i)}c_{i+1}$ and $\pi^{\heartsuit}_{f(i)-1}c_{i+1} \to \pi^{\heartsuit}_{f(i)-1}c_i$ to be injective. This is exactly the condition that $\pi^{\heartsuit}_k c_i = \pi^{\heartsuit}_{k,i-k}c$ vanish when $f(i-1) \leq k < f(i)$ and $\pi^{\heartsuit}_{f(i)-1}c_{i+1} = \pi^{\heartsuit}_{f(i)-1,i-f(i)+2}c$ has no simple τ-torsion. ☐

Example 4.2.8. Let $f(i) = \lceil ai \rceil$ where $a \geq 0$. This gives rise to the *slope* $\frac{1-a}{a}$ *t-structure*, whose truncation functors we denote $\tau^{/a}_{\geq 0}, \tau^{/a}_{\leq 0}$. ◁

Example 4.2.9. Let $f(i) = 0$ for $i \leq 0$ and $f(i) = \lceil \frac{i}{2} \rceil$ for $i > 0$. This gives rise to the v *t-structure*, whose truncation functors we denote $\tau^{v}_{\geq 0}, \tau^{v}_{\leq 0}$. ◁

The slope $\frac{1-a}{a}$ and v t-structures satisfy the conditions of Lemma 4.2.6 and Lemma 4.2.5, so are compatible with the symmetric monoidal structure, and can be computed by truncating level-wise. The reason for the name slope is that in the Adams grading, the homotopy groups of objects in the heart of this t-structure lie along a line of slope $\frac{1-a}{a}$. The v t-structure is named so because the curve it describes is the vanishing curve on the homotopy groups of the BP-synthetic sphere at the prime 2.

Example 4.2.10. We now specialize Example 4.2.8 to obtain two t-structures we use here.

Taking $a = 0$, we get the *constant* t-structure, whose connective cover functor $\tau^{\mathrm{const}}_{\geq 0}$ just takes connective cover on each filtered piece.

Taking $a = 1$, we get the *diagonal* t-structure, whose connective cover functor $\tau^{d}_{\geq 0}$ is given by taking the i^{th}-connective cover on the i^{th} filtered piece. ◁

The functor $(-)^{\mathrm{const}} : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ is the symmetric monoidal functor given by the constant filtered object.

4.2.3 Filtrations on rings of interest

We now specialize to the case $\mathcal{C} = \mathrm{Sp}$ with its standard symmetric monoidal structure. We begin by constructing j_ζ as a filtered ring. We use $\tau_{\geq *}(-)$ to denote the composite functor $\tau^d_{\geq 0}((-)^{\mathrm{const}})$. Indeed, $\tau_{\geq i}(-)$ is the i^{th} filtered piece of this functor.

We now use $\tau_{\geq *}(-)$ to obtain a filtration on $\ell_p, j_\zeta, \mathrm{ko}_2$, and j for $p > 2$. We use R^{fil} to denote these rings equipped with these filtrations, and R^{gr} to denote the associated graded algebras.

Definition 4.2.11. Let $\mathbb{Z}_p^{\mathrm{fil}}$ be the ring of p-adic integers with the p-adic filtration. It is a filtered $\mathbb{E}_\infty$-ring since it is a commutative ring in the heart of the constant t-structure. Its associated graded ring is $\mathbb{F}_p[v_0]$, where $v_0 \in \pi_{0,1}\mathbb{Z}_p^{\mathrm{gr}}$. We write $\widetilde{v_0} \in \pi_{0,1}\mathbb{Z}_p^{\mathrm{fil}}$ for the class of filtration 1 detecting $p \in \mathbb{Z}_p$, which projects to v_0 in the associated graded. ◁

Definition 4.2.12. For $p > 2$, consider ℓ_p, viewed as an $\mathbb{E}_\infty$-ring equipped with the $\mathbb{Z}$-action given by the Adams operation Ψ^{1+p}, and for $p = 2$, consider it with the $\mathbb{Z} \times C_2$-action given by the Adams operations Ψ^3, Ψ^{-1}.

We now define most of our filtered $\mathbb{E}_\infty$-rings of interest:

- $\ell_p^{\mathrm{fil}} := \tau_{\geq *}\ell_p$
- $\mathrm{ko}_2^{\mathrm{fil}} := \tau^v_{\geq 0}((\ell_2^{\mathrm{fil}})^{hC_2})$
- $j_{\zeta,k}^{\mathrm{fil}} := (\ell_p^{\mathrm{fil}})^{hp^k\mathbb{Z}}$ for $p > 2$ and $(\mathrm{ko}_2^{\mathrm{fil}})^{h2^k\mathbb{Z}}$ for $p = 2$
- $ju_{\zeta,k}^{\mathrm{fil}} := (\ell_2^{\mathrm{fil}})^{h2^k\mathbb{Z}}$
- $j_k^{\mathrm{fil}} := \tau_{\geq 0}^{\mathrm{const}}(j_{\zeta,k}^{\mathrm{fil}})$ for $p > 2$.

In the case $k = 0$, we just write $j_\zeta^{\mathrm{fil}}, ju^{\mathrm{fil}}, j^{\mathrm{fil}}$, and we remove fil from the notation if we want to denote the underlying $\mathbb{E}_\infty$-ring. For example, we write $j_{\zeta,k} = \ell_p^{hp^k\mathbb{Z}}$. ◁

Remark 4.2.13. The filtrations of Definition 4.2.12 aren't as 'fast' as they can possibly be. Namely, the spectra in the filtrations only change every multiple of $2p-2$ filtrations. Speeding up the filtration doesn't affect very much related to the filtration in any case. ◁

Remark 4.2.14. For $p > 2$, it is also possible to use variants of the *Adams* filtration on the various rings of study, as in [HW22, Section 4.3], which would avoid the use of two filtrations. However this doesn't work as well at the prime 2, since the Adams filtration is poorly suited to studying ko_2's THH. ◁

The key properties of these filtrations that we use is that the associated graded algebras mod p are easy to describe.

Lemma 4.2.15. *The associated graded algebras of filtered rings defined in Definition 4.2.12 are $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras.*

Proof. The 0'th piece of every associated graded algebra is coconnective with $\pi_0 = \mathbb{Z}_p$, so the unit map from $\mathbb{S}^{0,0}$ factors canonically through $\mathbb{Z}$, giving it a canonical $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra structure. $\square$

Lemma 4.2.16. *For $p > 2$, there are isomorphisms of graded $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras*

$$\ell_p^{\mathrm{gr}}/p \cong \mathbb{F}_p[v_1]$$
$$j_{\zeta,k}^{\mathrm{gr}}/p \cong \mathbb{F}_p[v_1] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{B\mathbb{Z}}$$

and for $p = 2$, there are isomorphisms of graded $\mathbb{E}_\infty$-$\mathbb{F}_2$-algebras

$$j_{\zeta,k}^{\mathrm{gr}}/2 \cong (\mathrm{ko}_2^{\mathrm{gr}}/2) \otimes_{\mathbb{F}_2} \mathbb{F}_2^{B\mathbb{Z}}$$
$$ju_{\zeta,k}^{\mathrm{gr}}/2 \cong \mathbb{F}_2[v_1] \otimes_{\mathbb{F}_2} \mathbb{F}_2^{B\mathbb{Z}}$$
$$\mathrm{ko}_2^{\mathrm{gr}}/2 \cong \tau_{\geq 0}^{v}(\mathbb{F}_2^{BC_2} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_1]).$$

Proof. ℓ_p^{gr} is the associated graded of the Postnikov filtration, which is $\mathbb{Z}_p[v_1]$, where the grading of v_1 is its topological degree, namely $2p-2$. Reducing mod p, we get the claim about ℓ^{gr}/p. The $\mathbb{Z}$-action on ℓ_p^{gr} is the action of Ψ^{1+p} on the homotopy of ℓ_p. It is a ring automorphism sending v_1 to $(1+p)^{p-1}v_1$, which in particular is trivial modulo p. Since ℓ_p^{gr} is a discrete object (it is in the heart of the diagonal t-structure), it follows that the action on ℓ_p^{gr}/p is trivial, giving the claimed identification of $j_{\zeta,k}^{\mathrm{gr}}$ for $p > 2$ and $ju_{\zeta,k}^{\mathrm{gr}}$ for $p = 2$.

For $p = 2$, we first recall that the in the homotopy fixed point spectral sequence for $\mathrm{KO}_2 \cong \mathrm{KU}_2^{hC_2}$, all differentials are generated under the Leibniz rule by the differential $d_3 v_1^2 = \eta^3$, where η is represented by the class in $H^1(C_2; \pi_2\mathrm{KU}_2)$. The spectral sequence for $\ell_2^{hC_2} = \mathrm{ku}_2^{hC_2}$, displayed in Figure 4.1, embeds into this, after a page shift. Thus, we see that everything in $\pi_{**}(\mathrm{ku}^{\mathrm{fil}})^{hC_2}$ above the line of slope 1 intercept zero is either in negative underlying homotopy or doesn't have τ-multiples on or below the line of slope 1 intercept 2. We learn that the bigraded homotopy ring of $(\ell_2^{\mathrm{fil}})^{hC_2}$ is

$$\mathbb{Z}_2[x, \eta, \tau, b, v_1^4]/(b^2 - 4v_1^4, \eta^3\tau^2, 2\eta, 2x, x\eta\tau^2, v_1^4 x - \eta^4, \eta b),$$

where x represents $v_1^{-4}\eta^4$, and b represents $2v_1^2$.

By applying Lemma 4.2.7, we learn that the connective cover $\tau_{\geq 0}^{v}$ can be computed the level of associated graded, and that this even holds after taking the cofiber by 2. The C_2-action on $\mathrm{ku}_2^{\mathrm{gr}}/2$ is trivial, so indeed $\mathrm{ko}_2^{\mathrm{gr}}/2 \cong \mathrm{gr}(\tau_{\geq 0}^{v}(\mathbb{F}_2^{BC_2} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_1]))$. For $j_{\zeta,k}^{\mathrm{gr}}/2$, we just observe that the residual $\mathbb{Z}$-action is also trivial. $\square$

Remark 4.2.17. At the prime 2, it is possible to define j as a filtered $\mathbb{E}_\infty$-ring, but we do not study this in this paper. One can define its underlying $\mathbb{E}_\infty$-ring as the pullback

$$\begin{tikzcd}
j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{ko}_2^{h\mathbb{Z}} \arrow[d] \\
\tau_{\leq 2}\mathbb{S}_2 \arrow[r] & (\tau_{\leq 2}\mathrm{ko}_2)^{h\mathbb{Z}}
\end{tikzcd}$$

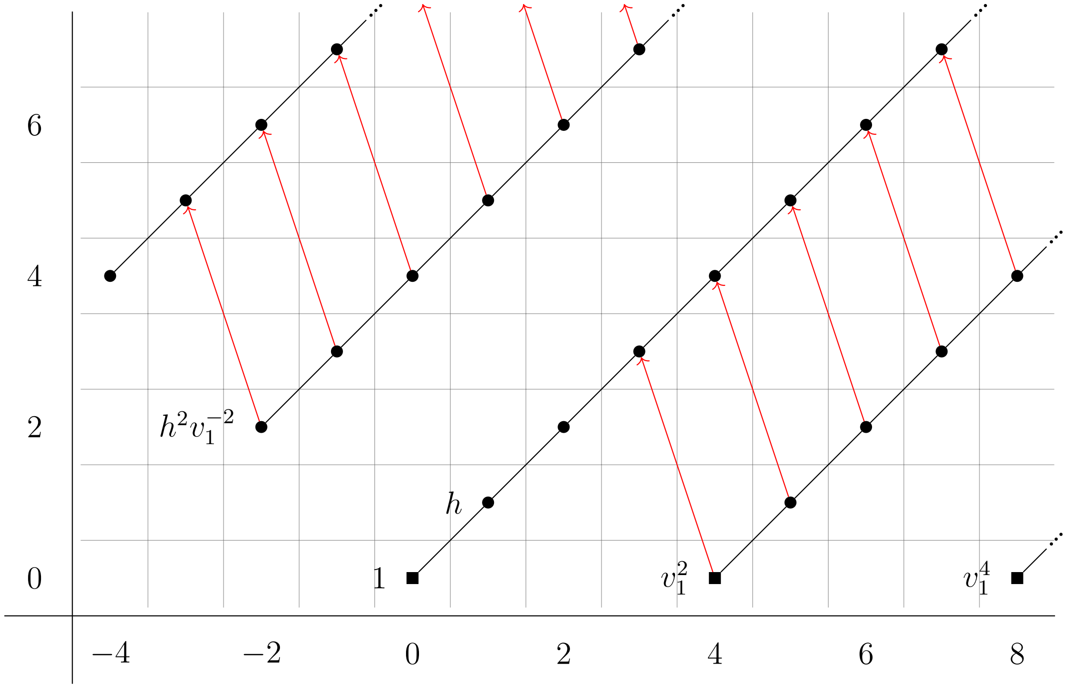

Figure 4.1: Above is the E_2-page of the spectral sequence associated with the filtered ring $(\tau_{\geq *}\ell_2)^{hC_2}$, which embeds into the homotopy fixed point spectral sequence for KO_2. The dots indicate a copy of $\mathbb{F}_2$, the rectangles indicate a copy of $\mathbb{Z}_2$, and the diagonal lines indicate multiplication by h. The spectral sequence collapses at the E_3-page.

and then consider the underlying filtered $\mathbb{E}_\infty$-ring of $\nu_{BP}(j)$ where ν_{BP} is the synthetic analogue functor of [Pst22]. ◁

Finally, we show convergence properties of our THH applied to the filtrations we use. Given a filtered spectrum $X \in \mathrm{Fil}(\mathrm{Sp})$, the spectral sequence associated with X converges conditionally if and only if $\lim_i X_i = 0$. This is equivalent to asking that X is τ-complete, where τ is in $\pi_{0,-1}\mathbb{S}^{0,0}$.

The following lemma shows completeness for THH with respect to all of the filtrations constructed in this section.

Lemma 4.2.18. *Suppose that R is a filtered ring such that the i-th filtered piece R_i is $(-1+ci)$-connective for every i and some fixed $c > 0$. Then, the i-th filtered piece of* $\mathrm{THH}(R)$ *is also $(-1+ci)$-connective, so in particular the filtration on* $\mathrm{THH}(R)$ *is complete.*

Proof. Note that $\overline{R} = \mathrm{cof}(\mathbb{S}^{0,0} \to R)$ satisfies the same conditions of the statement. The filtration from the cyclic bar construction gives us an increasing filtration on $\mathrm{THH}(R)$ with k-th associated graded piece $\Sigma^k R \otimes \overline{R}^{\otimes k}$. The i-th filtered piece of $\Sigma^k R \otimes \overline{R}^{\otimes k}$ is $(-1+ci)$-connective since it is a colimit of spectra of the form $\Sigma^k R_{j_0} \otimes \overline{R}_{j_1} \otimes \cdots \otimes \overline{R}_{j_k}$ with $j_0 + \cdots + j_k \geq$

i, which has connectivity of at least

$$k + \sum_{s=0}^{k} (-1 + cj_s) \geq -1 + ci. \qquad \square$$

The other filtration we use is the p-adic filtration on $\mathbb{Z}_p$, which we call $\mathbb{Z}_p^{\mathrm{fil}}$, whose associated graded algebra is $\mathbb{F}_p[v_0]$. We call $\tilde{v_0}$ the element in $\pi_{0,1}\mathbb{Z}_p$ that is a lift of p to filtration 1, and projects to v_0 in the associated graded.

Lemma 4.2.19. *Let R be a (possibly graded) $\mathbb{E}_1$-$\mathbb{Z}_p$-algebra. Then, the filtration on the filtered ring*

$$\mathrm{THH}(R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}.$$

is complete and its associated graded ring is concentrated in two filtration degrees $t = 0, 1$. Informally, the filtration is of the form

$$\cdots \to 0 \to 0 \to I \to \mathrm{THH}(R)/p$$

for some (possibly graded) spectrum I. In particular, the associated spectral sequence collapses at the E_2-page.

Proof. By using the symmetric monoidality of THH and the fact that $p = 0$ in $\mathbb{Z}_p^{\mathrm{fil}}/\tilde{v}_0$, we obtain an equivalence

$$\mathrm{THH}(R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}})/\tilde{v}_0 \cong (\mathrm{THH}(R)/p) \otimes_{(\mathrm{THH}(\mathbb{Z}_p)/p)} \mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\tilde{v}_0.$$

Since the conclusion of the statement is stable under base-change along trivially filtered rings, the statement reduces to the case $R = \mathbb{Z}_p$.

For $R = \mathbb{Z}_p$, the associated graded is $\mathrm{THH}(\mathbb{F}_p[v_0])/v_0$, which has homotopy ring $\mathbb{F}_p[\sigma^2 p] \otimes \Lambda[dv_0]$ (see Example 4.4.2), which is indeed in filtrations ≤ 1. It remains to see that $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\tilde{v}_0 = \mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}}; \mathbb{F}_p)$ has a complete filtration. It suffices to show that

$$\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}}; \mathbb{F}_p) \otimes_{\mathrm{THH}(\mathbb{Z}_p)} \mathbb{Z}_p \cong \mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}}/\mathbb{Z}_p; \mathbb{F}_p)$$

has a complete filtration, since $\mathrm{THH}(\mathbb{Z}_p)$ is built from $\mathbb{Z}_p$ via extensions and limits that are finite in each degree, and completeness of the filtration can be checked degreewise. The nth associated graded term of the cyclic bar construction computing this is

$$\Sigma^n (\mathbb{Z}_p^{\mathrm{fil}})^{\otimes_{\mathbb{Z}_p} n} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong \Sigma^n (\mathbb{Z}_p^{\mathrm{fil}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{\otimes_{\mathbb{F}_p} n}$$

$\mathbb{Z}_p^{\mathrm{fil}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is complete since it is $\mathbb{F}_p$ in each nonnegative degree, with transition maps 0, or in other words, it is a direct sum $\mathbb{F}_p \oplus \bigoplus_1^\infty \Sigma^{0,i}\mathbb{F}_p/\tau$. It follows that its tensor powers over $\mathbb{F}_p$ are also sums of $\mathbb{F}_p$ in each degree with transition maps 0 in positive filtration, so are complete. Since only finitely many terms in the cyclic bar complex contribute to each degree of THH, we learn that the THH is complete. $\square$

4.3 Tools for understanding THH

In this section, we explain some general tools which we use in understanding THH.

4.3.1 Suspension operation in THH

We begin by reviewing and proving some basic facts about the suspension maps, which are studied in [HW22, Section A]. Let R be an $\mathbb{E}_1$-algebra in a presentably symmetric monoidal stable category $\mathcal{C}$. By [HW22, Section A], there are natural maps

$$\sigma : \Sigma \operatorname{fib}(1_R) \to R \otimes R \tag{4.1}$$
$$\sigma^2 : \Sigma^2 \operatorname{fib}(1_R) \to \mathrm{THH}(R) \tag{4.2}$$

where 1_R is the unit map of R. Note that the first map is defined by the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d, "1_R"] & 0 \arrow[d] \\
R \arrow[r, "\mathrm{id}\otimes 1_R - 1_R \otimes \mathrm{id}"] & R \otimes R
\end{tikzcd}$$

and that it factors through $\operatorname{fib}(\mu) \to R \otimes R$ where $\mu : R \otimes R \to R$ is the multiplication map.

Let I be an object of $\mathcal{C}$ with a map $I \to R \otimes R$ and nullhomotopies of the composites

$$I \to R \otimes R \xrightarrow{\mu} R$$
$$I \to R \otimes R \xrightarrow{\mu \circ T} R,$$

where $T : R \otimes R \to R \otimes R$ is the exchange map. Then, we obtain a map

$$\Sigma I \to \mathrm{THH}(R)$$

by the commutative diagram

$$\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[d] \\
& R \otimes R \arrow[r, "\mu"] \arrow[d, "\mu \circ T"] & R \arrow[d, "1\otimes \mathrm{id}"] \\
0 \arrow[r] & R \arrow[r, "\mathrm{id}\otimes 1"] & R \otimes_{R \otimes R^{op}} R.
\end{tikzcd} \tag{4.3}$$

By the proof of [HW22, Lemma A.3.2], if $I = \Sigma \operatorname{fib}(1_R)$ and the map $I \to R \otimes R$ is given by (4.1), then the induced map $\Sigma I \to \mathrm{THH}(R)$ is the map (4.2).

Definition 4.3.1. Let X be a spectrum. Given a class $x \in \pi_*(X \otimes 1)$ and a lift $\widetilde{x} \in \pi_*(X \otimes \mathrm{fib}(1_R))$ we shall write $\sigma x \in \pi_{*+1}(X \otimes R \otimes R)$ and $\sigma^2 x \in \pi_{*+2}(X \otimes \mathrm{THH}(R))$ for the image of $\widetilde{x}$ under the maps (4.1) and (4.2). The notation is ambiguous since we need to choose a lift $\widetilde{x}$, but these lifts will often be well-defined.

We shall write d for
$$\pi_*(X \otimes R) \to \pi_{*+1}(X \otimes \mathrm{THH}(R))$$
induced by the map of spectra $\Sigma R \to \Sigma^2 \mathrm{fib}(1_R) \to \mathrm{THH}(R)$. ◁

Remark 4.3.2. If R is homotopy commutative in addition to being an $\mathbb{E}_1$-algebra, then we can set $I = \mathrm{fib}(\mu)$ in (4.3) and obtain a map
$$\sigma : \Sigma \,\mathrm{fib}(\mu) \to \mathrm{THH}(R), \tag{4.4}$$
which is functorial on R and the homotopy[5] $\mu \cong \mu \circ T$. Then, the map (4.2) is the composite
$$\Sigma^2 \mathrm{fib}(1_R) \to \Sigma \,\mathrm{fib}(\mu) \to \mathrm{THH}(R)$$
of (4.1) and (4.4) up to sign.

If X is a spectrum, given a class $y \in \pi_*(X \otimes R \otimes R)$ and a lift $\widetilde{y} \in \pi_*(X \otimes \mathrm{fib}(\mu))$, we shall write $\sigma y \in \pi_{*+1}(X \otimes \mathrm{THH}(R))$ for the image of $\widetilde{y}$ under the map (4.4). Then, we have $dx = \sigma((\eta_L - \eta_R)x)$ for $x \in \pi_*(X \otimes R)$, where η_L and η_R are the left and right units of $R \otimes R$, respectively. ◁

Lemma 4.3.3 ([AR05, Prop. 5.10])**.** *Let X be a homotopy unital ring spectrum and R be an $\mathbb{E}_2$-algebra in $\mathcal{C}$. Then, d satisfies the Leibniz rule*
$$d(xy) = d(x)y + (-1)^{|x|} x d(y)$$
for any $x, y \in \pi_*(X \otimes R)$.

Proof. By [HW22, Example A.2.4], the map d can be identified with the map
$$S^1_+ \otimes R \to \mathrm{THH}(R)$$
induced by the unit map $R \to \mathrm{THH}(R)$ and the S^1-action on $\mathrm{THH}(R)$. Since the map $R \to \mathrm{THH}(R)$ is a map of $\mathbb{E}_1$-rings, the S^1-action on the target gives an S^1-family of ring maps, and so we obtain a map of $\mathbb{E}_1$-rings
$$R \to \lim_{S^1} \mathrm{THH}(R) = DS^1_+ \otimes \mathrm{THH}(R) = \mathrm{THH}(R) \oplus \Sigma^{-1} \mathrm{THH}(R) \tag{4.5}$$

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[5] The same construction is studied in [HW22, Variant A.2.2], but we believe that additional hypotheses are required to make sense of their argument. For example, R is only assumed to be an $\mathbb{E}_1$-ring in their generality, but an assumption such as homotopy commutativity of R is needed to ensure that the composite
$$\mathrm{fib}(\mu) \to R \otimes R \xrightarrow{\mu \circ T} R$$
is nullhomotopic. In their notation, we would need to assume, for example, that there is a homotopy $1_k \cong 1_k^\tau$. This does not affect any other part of their work since they only use rings that have enough structure.

given by the sum of the identity map and d. Here, DS^1_+ is the Spanier-Whitehead dual of S^1 with the algebra structure given by the diagonal map of S^1.

The homotopy ring of DS^1_+ is given by

$$\pi_*(DS^1_+) = (\pi_* S^0)[t]/(t^2)$$

with $|t| = -1$. Since (4.5) is a ring map, taking the X-homology, we have

$$1 \otimes xy + t \otimes d(xy) = (1 \otimes x + t \otimes dx)(1 \otimes y + t \otimes dy)$$

for $x, y \in \pi_*(X \otimes R)$. Expanding it using $t^2 = 0$ gives us the desired Leibniz rule. □

Our use of the symbol d recovers the use in the HKR theorem. Recall that a strict Picard element $\mathscr{L}$ of a symmetric monoidal category $\mathcal{C}$ is a map of spectra $\mathbb{Z} \to \mathrm{pic}(\mathcal{C})$. Given such a strict Picard element, viewing it as a symmetric monoidal functor $\mathbb{Z} \to \mathcal{C}$, the colimit of the composite

$$\mathbb{N} \to \mathbb{Z} \to \mathcal{C}$$

is an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$ which we denote $\mathbb{1}[x]$, where x is a class in the Picard graded homotopy in the degree of $\mathscr{L}$.

Lemma 4.3.4 (HKR isomorphism)**.** *Let C be a presentably symmetric monoidal stable category with a strict Picard element $\mathscr{L}$. Let $\mathbb{1}[x]$ denote the polynomial algebra on a class x in degree $\mathscr{L}$. Then $\mathrm{HH}(\mathbb{1}[x])$ is a free $\mathbb{1}[x]$-module on 1 and dx.*

Proof. The universal example of such a C is graded spectra, where $\mathbb{1}[x]$ is the graded polynomial algebra $\Sigma^\infty_+\mathbb{N}$, so it suffices to prove it there. But now this follows from from the Kunneth spectral sequence computing $\pi_* \mathrm{THH}(\mathbb{S}[x]) = \pi_*\mathbb{S}[x] \otimes_{\mathbb{S}[x_1,x_2]} \mathbb{S}[x]$, since dx is $\sigma((\eta_L - \eta_R)(x))$. □

We now explain some basic THH computations involving the suspension map.

Example 4.3.5 (Bökstedt periodicity)**.** The fundamental computation of Bökstedt states that the ring $\pi_* \mathrm{THH}(\mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p[\sigma^2 p]$. ◁

Lemma 4.3.6. *Let $R \in \mathrm{Fil}(\mathrm{Sp})$ be a filtered $\mathbb{E}_1$-ring and $X \in \mathrm{Sp}$ a spectrum. Let $y \in \pi_{k,r-k}(R \otimes X)$, $x \in \pi_k X$ be classes such that $\tau^r y = x \in \pi_{k,-k}(R \otimes X)$.*

Then there is a choice of nullhomotopy of x in $\mathrm{THH}(\mathrm{gr}R) \otimes X$ such that in the spectral sequence for $\mathrm{THH}(R) \otimes X$, the corresponding element $\sigma^2 x$ on the E_1-page survives to the E_r-page and has d_r-differential $d_r(\sigma^2 x) = \pm dy$.

Proof. A choice of homotopy $\tau^r y \sim x$ in $R \otimes X$ becomes in $\mathrm{cof}(\mathbb{S}^{0,0} \to R) \otimes X$ a choice of nullhomotopy of the image of $\tau^r y$, which corresponds to a map $\Sigma^{|y|}\mathrm{cof}(\tau^r) \to \mathrm{cof}(\mathbb{S}^{0,0} \to R) \otimes X$. This map of filtered spectra gives a map of the associated spectral sequences, and in the spectral sequence for $\mathrm{cof}(\tau^r)$, there is a d_r-differential between the two spheres on the associated graded.

We claim the image of the two shifts of $\operatorname{cof}\tau$ in the map

$$\Sigma^{|y|}(\operatorname{cof}(\tau)\oplus\Sigma^{1,-(r+1)}\operatorname{cof}(\tau))\cong\Sigma^{|y|}\operatorname{cof}(\tau^r)\otimes\operatorname{cof}(\tau)\to\operatorname{cof}(\mathbb{S}^{0,0}\to R)\otimes X\otimes\operatorname{cof}(\tau)$$

correspond to the image of y and the suspension of a nullhomotopy of x under the map $\mathbb{S}^{0,0}\to \mathrm{gr}R$.

The claim that the first $\operatorname{cof}\tau$ is sent to y is clear by construction, and the claim that the second $\operatorname{cof}\tau$ is sent to the suspension of a nullhomotopy of x follows since on associated graded our original homotopy $\tau^r y\sim x$ becomes a nullhomotopy of x.

It then follows that there is a d_r differential between these two classes.

Composing with the filtered map

$$\Sigma\operatorname{cof}(\mathbb{S}^{0,0}\to R)\otimes X\cong\Sigma^2\operatorname{fib}(\mathbb{S}^{0,0}\to R)\otimes X\xrightarrow{\sigma^2}\mathrm{THH}(R)\otimes X$$

of Equation (4.2), y gets sent to dy and the nullhomotopy of x gets sent to σ^2x (up to a possible sign), giving the desired differential in the spectral sequence for $\mathrm{THH}(R)\otimes X$. Therefore, it is enough to prove that the connecting map sends $\widetilde{x}$ to y, and since the map $\pi_*(Z\otimes X_1)\to\pi_*(Z\otimes X_0)$ is injective, it is enough to prove that $\widetilde{x}$ is sent to $\eta_*(x)$ by the composite $F\to X_1\to X_0$. This composite is homotopic to $F\to\mathbb{S}\to X_0$ since the connecting map $F\to X_1$ is given by the nullhomotopy □

4.3.2 THH in the stable range

Throughout this subsection, let S be a connective $\mathbb{E}_\infty$-algebra and R be a connective $\mathbb{E}_1$-S-algebra.

In this section, we show that in the situation that the unit map $S\to R$ is highly connective, $\mathrm{THH}(R/S)$ in low degrees becomes relatively straightforward to understand. This is used later in Section 4.5 to understand $\mathrm{THH}(j)$. Let Δ_n denote the subcategory of Δ consisting of ordinals of size $\leq n$.

Lemma 4.3.7. *If the unit map $S\to R$ is i-connective, then the natural map*

$$\operatorname*{colim}_{\Delta_n^{op}} R^{\otimes_S *+1}\to\operatorname*{colim}_{\Delta^{op}} R^{\otimes_S *+1}\cong\mathrm{THH}(R/S)$$

is $(n+1)(i+2)-1$*-connective.*

Proof. Let $\overline{R}=\operatorname{cof}(S\to R)$ be the cofiber of the unit map. The m^{th} term of the associated graded of the filtration coming from the cyclic bar construction is $\Sigma^m R\otimes_S\overline{R}^{\otimes_S m}$, which is $m(i+2)$-connective because R is connective and $\overline{R}$ is $(i+1)$-connective. It follows that the cofiber of the map in question has an increasing filtration whose associated graded pieces are $m(i+2)$-connective for $m>n$. This implies the result. □

The above lemma gives a simple description of THH in low degrees.

Proposition 4.3.8. *If the unit map $S \to R$ is i-connective, then the map*

$$\Sigma^2 \operatorname{fib}(1_R) \oplus R \xrightarrow{\sigma^2 \oplus 1} \mathrm{THH}(R/S)$$

is $(2i+2)$-connective, where σ^2 is defined as in (4.2).

Proof. Consider the case $n = 1$ in Lemma 4.3.7. Then, we have an equivalence

$$\operatorname*{colim}_{\Delta_1^{op}} R^{\otimes_S *+1} \cong \operatorname{colim} \left(\begin{tikzcd} R \otimes_S R \arrow[r, "\mu"] \arrow[d, "\mu \circ T"] & R \\ R & \end{tikzcd} \right)$$

(see [MV15, Theorem 9.4.4]), where T is the exchange map, and this colimit maps into $\mathrm{THH}(R/S)$ by a $(2i+3)$-connective map.

Therefore, it is enough to prove that the map

$$\operatorname{colim} \left(\begin{tikzcd} \Sigma \operatorname{fib}(1_R) \oplus R \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_2"] & R \\ R & \end{tikzcd} \right) \to \operatorname{colim} \left(\begin{tikzcd} R \otimes_S R \arrow[r, "\mu"] \arrow[d, "\mu \circ T"] & R \\ R & \end{tikzcd} \right)$$

is $(2i+2)$-connective, where the map $\Sigma \operatorname{fib}(1_R) \oplus R \to R \otimes_S R$ is $\sigma \oplus (1_R \otimes \mathrm{id})$ and the two maps $R \to R$ are the identities. The fiber of this map is

$$\Sigma \operatorname{fib}(\Sigma \operatorname{fib}(1_R) \oplus R \xrightarrow{\sigma \oplus 1} R \otimes_S R)$$

which is $(2i+2)$-connective by the next lemma. □

Lemma 4.3.9. *If the unit map $1_R : S \to R$ is i-connective, then the map*

$$\Sigma \operatorname{fib}(1_R) \oplus R \xrightarrow{\sigma \oplus 1} R \otimes_S R$$

is $(2i+1)$-connective.

Proof. This is equivalent to asking that the total cofiber of the following diagram

$$\begin{tikzcd} S \otimes_S S \arrow[r] \arrow[d] & S \otimes_S R \arrow[d] \\ R \otimes_S S \arrow[r] & R \otimes_S R \end{tikzcd}$$

is $(2i+2)$-connective. This follows from the assumption since the total cofiber is $\Sigma^2 \operatorname{fib}(1_R) \otimes_S \operatorname{fib}(1_R)$, which is $(2i+2)$-connective since $\operatorname{fib}(1_R)$ is i-connective. □

Corollary 4.3.10. *The group* $\pi_{2p-1}\,\mathrm{THH}(\mathbb{Z}_p)$ *is isomorphic to* $\mathbb{Z}/p$ *and is generated by* $\sigma^2\alpha_1$*.*

Proof. Since $\mathbb{S}_p \to \mathbb{Z}_p$ is $(2p-3)$-connective, the result follows from Proposition 4.3.8, which implies that σ^2 induces an isomorphism

$$\mathbb{Z}/p = \pi_{2p-3}\,\mathrm{fib}(\mathbb{S}_p \to \mathbb{Z}_p) \cong \pi_{2p-1}\,\mathrm{THH}(\mathbb{Z}_p). \qquad \square$$

Corollary 4.3.11. *For* $p > 2$*, the map*

$$j \oplus \Sigma^2\,\mathrm{fib}(\mathbb{S}_p \to j) \xrightarrow{1\oplus\sigma^2} \mathrm{THH}(j)$$

is $(4p^2-4p-2)$*-connective.*

Proof. For $p > 2$, $\mathbb{S}_p \to j$ is $2p^2-2p-2$-connective. This is because the first element of the fiber is β_1 (see for example [Rav86, Theorem 4.4.20]) which is in that degree. $\square$

4.4 The THH of j_ζ

In this section, we compute $\mathrm{THH}(j_\zeta)/(p, v_1)$ using the filtration constructed in Section 4.2. Let us first assume that p is an odd prime. We shall discuss the case $p = 2$ later in the section.

4.4.1 THH of $\mathbb{Z}_p$ and ℓ_p

Before computing the THH of j_ζ, we shall compute the THH of $\mathbb{Z}_p$ modulo p and the THH of ℓ_p modulo (p, v_1) in this section, as a warm-up. They will be computed using the spectral sequences associated with $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$ and $\mathrm{THH}(\ell_p^{\mathrm{fil}})$. Later, we show that the computation of the spectral sequence for $\mathrm{THH}(j_\zeta^{\mathrm{fil}})$ looks the same. We note that the computations for $\mathbb{Z}_p$ and ℓ_p are well-known (see for example [AR05, Theorem 5.12]).

Lemma 4.4.1. *Let* k *be a discrete commutative ring and let* R *be a* $\mathbb{Z}^m$*-graded* $\mathbb{E}_2$*-*k*-algebra such that the homotopy groups of* R *form a polynomial* k*-algebra*

$$\pi_*R = k[x_1, \ldots, x_n]$$

on even degree generators $x_1, \ldots, x_n$*. Then, there is an equivalence of* $\mathbb{Z}^m$*-graded* $\mathbb{E}_1$*-*$\mathrm{THH}(k)$*-algebras*

$$\mathrm{THH}(R) \cong \mathrm{THH}(k) \otimes_k \mathrm{HH}(k[x_1, \ldots, x_n]/k).$$

Proof. Let $\mathbb{S}[x_1, \ldots, x_n]$ be the $\mathbb{Z}^m$-graded $\mathbb{E}_2$-ring spectrum of [Lur15]. Then, by [HW22, Prop. 4.2.1], there is an equivalence of $\mathbb{Z}^m$-graded $\mathbb{E}_2$-k-algebras

$$R \cong k \otimes \mathbb{S}[x_1, \ldots, x_n].$$

Therefore, since THH is a symmetric monoidal functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$, there is an equivalence of $\mathbb{Z}^m$-graded $\mathbb{E}_1$-k-algebras

$$\mathrm{THH}(R) \cong \mathrm{THH}(k) \otimes \mathrm{THH}(\mathbb{S}[x_1, \ldots, x_n]),$$

and the statement follows by base changing the second tensor factor on the right hand side along $\mathbb{S} \to k$. □

Example 4.4.2. Consider the filtered spectrum $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\tilde{v}_0$. Its associated graded spectrum is $\mathrm{THH}(\mathbb{F}_p[v_0])/v_0$ and its underlying spectrum is $\mathrm{THH}(\mathbb{Z}_p)/p$. The E_1-page of the associated spectral sequence is $\mathbb{F}_p[\sigma^2 p] \otimes \Lambda[dv_0]$ by Lemma 4.4.1. Note that $\sigma^2 p$ and dv_0 are in filtrations 0 and 1, respectively.

By Lemma 4.3.6, we have a differential $d_1(\sigma^2 p) \doteq dv_0$ in the spectral sequence associated with the filtered ring $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$. Then, mapping to $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}$ and using the Leibniz rule, we can determine all differentials, and the E_2-page is isomorphic to $\mathbb{F}_p[(\sigma^2 p)^p] \otimes \Lambda[(\sigma^2 p)^{p-1} dv_0]$. There are no differentials in later pages by Lemma 4.2.19.

Therefore, the homotopy ring $\pi_* \mathrm{THH}(\mathbb{Z}_p)/p$ is isomorphic to $\mathbb{F}_p[\mu] \otimes \Lambda[\lambda_1]$ with $|\mu| = 2p$ and $|\lambda_1| = 2p-1$. By [HW22, Propsition 6.1.6], μ can be identified with $\sigma^2 v_1$[6], where $v_1 \in \pi_{2p-2}\mathbb{S}_p$ and λ_1 can be identified with σt_1, in the sense of Remark 4.3.2, where $t_1 \in \pi_*(\mathbb{Z} \otimes \mathbb{Z})$ is the image of $t_1 \in \pi_*(\mathrm{BP} \otimes \mathrm{BP})$ under the map $\mathrm{BP} \to \mathbb{Z}$. By Corollary 4.3.10, we have $\lambda_1 \doteq \sigma^2 \alpha_1$[7]. ◁

Example 4.4.3. Consider the filtered spectrum $\mathrm{THH}(\ell_p^{\mathrm{fil}})/(p, \widetilde{v_1})$, where $\widetilde{v_1} \in \pi_* \ell_p$ is the class of filtration $(2p-2)$. Its associated graded spectrum is $\mathrm{THH}(\mathbb{Z}[v_1])/(p, v_1)$ and its underlying spectrum is $\mathrm{THH}(\ell_p)/(p, v_1)$. By Lemma 4.4.1, the E_1-page of the associated spectral sequence is $\mathbb{F}_p[\sigma^2 v_1] \otimes \Lambda[\lambda_1, dv_1]$. Note that the for degree reasons, the first and last page a differential can happen is the E_{2p-2}-page.

Applying Lemma 4.3.6, there is a differential $d_{2p-2}\sigma^2 v_1 \doteq dv_1$ in the spectral sequence associated with the filtered spectrum $\mathrm{THH}(\ell_p^{\mathrm{fil}})/p$. Mapping to $\mathrm{THH}(\ell_p^{\mathrm{fil}})/(p, \widetilde{v_1})$ and using the Leibniz rule, we can determine the d_{2p-2}-differentials on powers of $\sigma^2 v_1$. The class λ_1 is a permanent cycle for degree reasons. Therefore, the E_{2p-1}-page is isomorphic to $\mathbb{F}_p[(\sigma^2 v_1)^p] \otimes \Lambda[\lambda_1, (\sigma^2 v_1)^{p-1} dv_1]$. The classes $(\sigma^2 v_1)^p, (\sigma^2 v_1)^{p-1} dv_1$ are permanent cycles for degree reasons, so the spectral sequence degenerates at the E_{2p-1}-page.

We let λ_2 denote a class detecting $(\sigma^2 v_1)^{p-1} dv_1$, and μ denote a class detecting $(\sigma^2 v_1)^p$. To check that there are no multiplicative extensions, we need to check $\lambda_1^2 = \lambda_2^2 = 0$, which follows for degree reasons. The homotopy ring $\pi_* \mathrm{THH}(\ell_p)/(p, v_1)$ is thus isomorphic to $\mathbb{F}_p[\mu_1] \otimes \Lambda[\lambda_1, \lambda_2]$ where λ_1 and λ_2 can be identified with σt_1 and σt_2 as in the case of $\mathrm{THH}(\mathbb{Z}_p)/p$. For $p > 2$, μ_2 can be identified with $\sigma^2 v_2$. ◁

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[6] v_1 is not well defined at the prime 2, but still exists: it is just not a self map of cof(2). It is generally defined as any element of $\pi_{2p-2}\mathbb{S}/p$ whose BP-Hurewicz image is v_1.

[7] Alternatively, if one knows that the p-Bockstein on μ is $\doteq \lambda_1$, one learns that $\sigma^2 \alpha \doteq \lambda_1$ from the fact that the p-Bockstein on v_1 is α_1 and the fact that σ^2 is compatible with the p-Bockstein (since it comes from a map of spectra).

4.4.2 The associated graded

We further filter the associated graded ring j_ζ^{gr} by the p-adic filtration to ultimately reduce the computation to our understanding of $\mathrm{THH}(\mathbb{F}_p)$. In running the spectral sequences to obtain the THH mod (p, v_1), we find that they are close enough to the spectral sequences of $\ell_p^{B\mathbb{Z}}$, the fixed points of ℓ_p with the trivial $\mathbb{Z}$-action.

Definition 4.4.4. We define the p-adic filtration on j_ζ^{gr} to be $j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}}$. This is an $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra object in the category of filtered graded spectra.

By taking the associated graded, we obtain $j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0]$, which is an $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra object in the category of bigraded spectra. We shall write *hfp grading* for the grading on j_ζ^{gr} if we need to distinguish it from the p-adic grading on $\mathbb{F}_p[v_0]$. For example, in $j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0]$, v_1 has hfp degree $2p-2$ and p-adic degree 0, and v_0 has hfp degree 0 and p-adic degree 1. $\triangleleft$

Lemma 4.4.5. *For $p > 2$, there is an isomorphism of bigraded $\mathbb{E}_1$-$\mathrm{THH}(\mathbb{F}_p)$-algebras for*

$$\mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0]) \cong \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{HH}(\mathbb{F}_p[v_0, v_1]/\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p)$$

Proof. First note that $j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0] \cong j_\zeta^{\mathrm{gr}}/p \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_0]$, which by Lemma 4.2.16 is equivalent to $\mathbb{F}_p[v_1, v_0] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{B\mathbb{Z}}$. Then, the statement follows from Lemma 4.4.1. $\square$

We next study the behavior of fixed points by trivial $\mathbb{Z}$-actions on THH. We use the spherical Witt vectors adjunction [BSY22, Proposition 2.2] [Lur18a, Section 5.2] between perfect $\mathbb{F}_p$-algebras and p-complete $\mathbb{E}_\infty$-rings. For a perfect $\mathbb{F}_p$-algebra A, $\mathbb{W}(A)$ is an $\mathbb{E}_\infty$-ring that is (p-completely) flat under $\mathbb{S}_p$, and whose $\mathbb{F}_p$ homology is A. The right adjoint is $\pi_0^\flat$ which is defined to be the inverse limit perfection of the $\mathbb{F}_p$-algebra $\pi_0(R)/p$.

Lemma 4.4.6. *There is an equivalence of $\mathbb{E}_\infty$-$\mathbb{S}_p^{B\mathbb{Z}}$-algebras*

$$\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}}) \cong \mathbb{S}_p^{B\mathbb{Z}} \otimes \mathbb{W}(C^0(\mathbb{Z}_p; \mathbb{F}_p))$$

. The restriction map $\mathbb{S}_p^{B\mathbb{Z}} \to \mathbb{S}_p^{Bp\mathbb{Z}}$ on $\pi_0^\flat$ is the map $C^0(\mathbb{Z}_p; \mathbb{F}_p) \to C^0(p\mathbb{Z}_p; \mathbb{F}_p)$ that restricts a function to $p\mathbb{Z}_p$.

Proof. There is a natural map $\mathbb{S}_p^{B\mathbb{Z}} \otimes \mathbb{W}(\pi_0^\flat(\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}})) \to \mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}})$, and so for the first claim it suffices to show that this is an equivalence and that $\pi_0^\flat(\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}})) \cong C^0(\mathbb{Z}_p; \mathbb{F}_p)$. Both of these can be checked after base change to $\mathbb{F}_p$. Note that $\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}})_p \otimes \mathbb{F}_p \cong \mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p)$.

Since $\mathbb{F}_p^{B\mathbb{Z}} = \mathrm{colim}_n\, \mathbb{F}_p^{\mathrm{B}\mathbb{Z}/p^n\mathbb{Z}}$ and $\mathrm{B}\mathbb{Z}/p^n\mathbb{Z}$ is p-finite, we have, by [Lur11, Corollary 1.1.10],

$$\mathrm{HH}(\mathbb{F}_p^{\mathrm{B}\mathbb{Z}/p^n\mathbb{Z}}/\mathbb{F}_p) \cong \mathbb{F}_p^{\mathrm{B}\mathbb{Z}/p^n\mathbb{Z}} \otimes_{\mathbb{F}_p^{(\mathrm{B}\mathbb{Z}/p^n\mathbb{Z})^2}} \mathbb{F}_p^{\mathrm{B}\mathbb{Z}/p^n\mathbb{Z}} \cong \mathbb{F}_p^{\mathrm{B}\mathbb{Z}/p^n\mathbb{Z} \times_{(\mathrm{B}\mathbb{Z}/p^n\mathbb{Z})^2} \mathrm{B}\mathbb{Z}/p^n\mathbb{Z}}.$$

We have equivalences of spaces natural in n

$$\mathrm{B}\mathbb{Z}/p^n\mathbb{Z} \times_{(\mathrm{B}\mathbb{Z}/p^n\mathbb{Z})^2} \mathrm{B}\mathbb{Z}/p^n\mathbb{Z} \cong \mathrm{LB}\mathbb{Z}/p^n\mathbb{Z} = \mathrm{B}\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}.$$

where L denotes the free loop space. Then, via the Künneth isomorphism and taking the colimit over n, we get

$$\mathrm{HH}(\mathbb{F}_p^{B\mathbb{Z}}/\mathbb{F}_p) \cong \mathbb{F}_p^{B\mathbb{Z}} \otimes \operatorname*{colim}_n \mathbb{F}_p^{\mathbb{Z}/p^n\mathbb{Z}}.$$

Since $\operatorname{colim}_n \mathbb{F}_p^{\mathbb{Z}/p^n\mathbb{Z}}$ is $C^0(\mathbb{Z}_p;\mathbb{F}_p)$, so we obtain the desired equivalence.

To see the claim about $\pi_0^\flat$, we note the natural map $\mathbb{F}_p^{B\mathbb{Z}} \to \mathbb{F}_p^{Bp\mathbb{Z}}$ is the colimit of $\mathbb{F}_p^{hB\mathbb{Z}/p^n\mathbb{Z}} \to \mathbb{F}_p^{hB\mathbb{Z}/p^{n-1}\mathbb{Z}}$, where the map is given by the inclusion $\mathbb{Z}/p^{n-1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. At the level of the π_0, $LB\mathbb{Z}/p^{n-1}\mathbb{Z} \to LB\mathbb{Z}/p^n\mathbb{Z}$ is also the inclusion $\mathbb{Z}/p^{n-1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$, so induces the restriction map at the level of $C^0(-;\mathbb{F}_p)$. Taking the colimit over n gives the claim. □

Remark 4.4.7. Lemma 4.4.6 can be interpreted as saying that the failure of p-adic THH to commute with taking $\mathbb{Z}$-homotopy fixed points in the universal case is measured by $\pi_0^\flat$. In particular, the map $\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}}) \xrightarrow{f} \mathrm{THH}(\mathbb{S}_p)^{B\mathbb{Z}}$ on $\pi_0^\flat$ is the map $C^0(\mathbb{Z}_p;\mathbb{F}_p) \xrightarrow{\pi_0^\flat f} \mathbb{F}_p$ evaluating at 0, and the comparison map is base changed along $\mathbb{W}(\pi_0^\flat f)$. ◁

Corollary 4.4.8. *Let R be a p-complete $\mathbb{E}_\infty$-ring. Then there is an equivalence of p-complete $\mathbb{E}_\infty$-R-algebras $\mathrm{THH}(R^{B\mathbb{Z}}) \cong \mathrm{THH}(R)^{B\mathbb{Z}} \otimes \mathbb{W}(C^0(\mathbb{Z}_p;\mathbb{F}_p))$.*

Combining Corollary 4.4.8 with Lemma 4.4.5 and the HKR isomorphism, we get the following.

Corollary 4.4.9. *For $p > 2$, we have an isomorphism of rings*

$$\pi_* \mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0]) \cong \mathbb{F}_p[\sigma^2 p, v_0, v_1] \otimes \Lambda[dv_0, dv_1, \zeta] \otimes C^0(\mathbb{Z}_p;\mathbb{F}_p)$$

4.4.3 Spectral sequences

Let us first run the spectral sequence for the p-adic filtration.

Proposition 4.4.10. *For $p > 2$, we have an isomorphism of rings*

$$\pi_* \mathrm{THH}(j_\zeta^{\mathrm{gr}})/p \cong \pi_* \mathrm{THH}(\mathbb{Z}_p)/p \otimes \mathbb{F}_p[v_1] \otimes \Lambda[dv_1, \zeta] \otimes C^0(\mathbb{Z}_p;\mathbb{F}_p).$$

Proof. As in Example 4.4.2, the spectral sequence associated with $\mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes \mathbb{Z}_p^{\mathrm{fil}})/\tilde{v}_0$ has E_1-page isomorphic to $\pi_* \mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0])/v_0 \cong \mathbb{F}_p[\sigma^2 p, v_1] \otimes \Lambda[dv_0, dv_1] \otimes H^*_{\mathrm{ct}}(S^1 \times \mathbb{Z}_p;\mathbb{F}_p)$ and converges to $\pi_* \mathrm{THH}(j_\zeta^{\mathrm{gr}})/p$.

Because there is a map of filtered rings $j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}} \to \mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}})$, we see that the classes v_1, ζ are permanent cycles. The class dv_1 is a permanent cycle since it detects the suspension dv_1 of $v_1 \in \pi_* j_\zeta^{\mathrm{gr}}/p$. The elements of $C^0(\mathbb{Z}_p;\mathbb{F}_p)$ are permanent cycles since there are no elements of negative topological degree and positive filtration.

From the map of filtered rings

$$\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}}) \to \mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{fil}}),$$

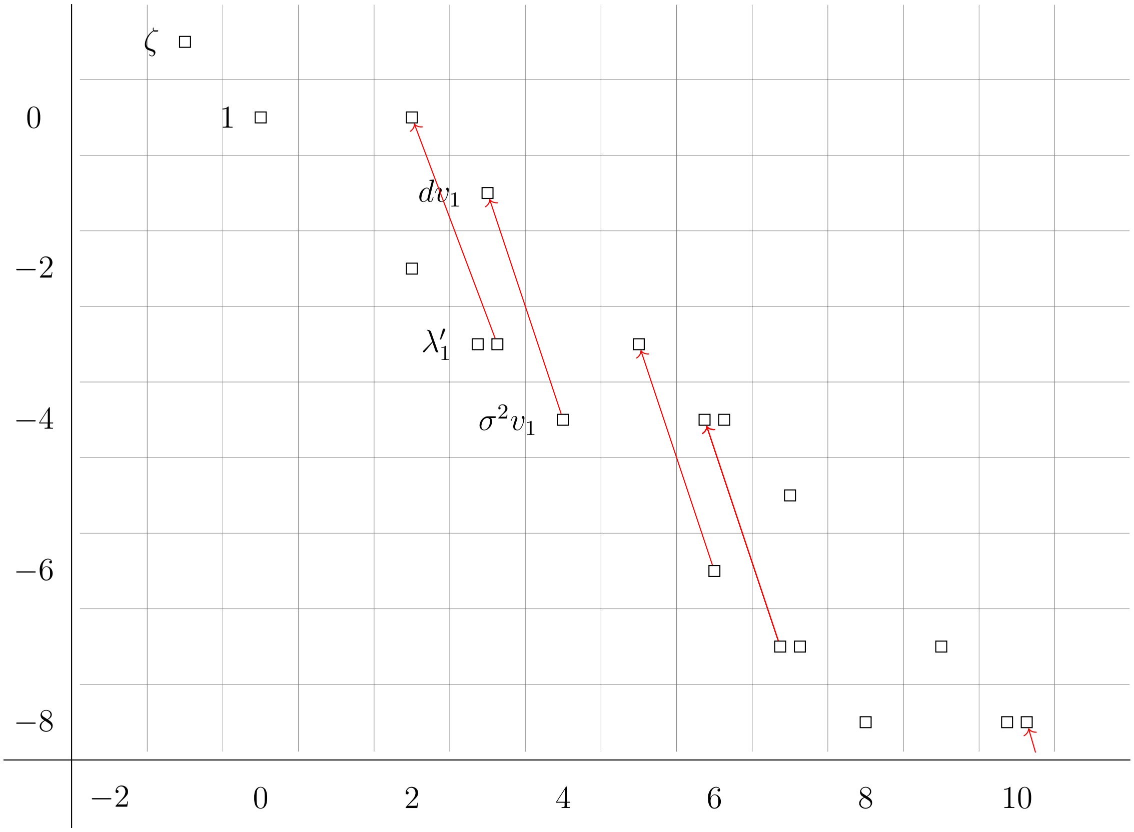

Figure 4.2: Above is the spectral sequence associated with the filtered ring $\mathrm{THH}(ju^{\mathrm{fil}})/(2, v_1)$. This spectral sequence is the $p = 2$ version of the spectral sequence in Theorem 4.4.11 (see Theorem 4.6.2), and only has d_2 differentials. Each square represents a copy of $C^0(\mathbb{Z}_2; \mathbb{F}_2)$.

there is a d_1-differential $\sigma^2 p \mapsto \sigma v_0$ by Example 4.4.2, and $(\sigma^2 p)^p$ and $(\sigma^2 p)^{p-1} dv_0$ are permanent cycles detecting images of classes in $\mathrm{THH}(\mathbb{Z}_p)$. It follows that after the d_1-differential, the E_2-page is $\mathbb{F}_p[(\sigma^2 p)^p, v_1] \otimes \Lambda[(\sigma^2 p)^{p-1} dv_0, dv_1, \zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$, so the spectral sequence collapes at the E_2-page. There are no multiplicative extensions since every class comes from either j_ζ^{gr}, $\mathrm{THH}(\mathbb{Z}_p)$, or $\mathrm{THH}(\mathbb{S}_p^{h\mathbb{Z}})$. □

Our next goal is to compute mod (p, v_1) the spectral sequence $\mathrm{THH}(j_\zeta^{\mathrm{gr}}) \implies \mathrm{THH}(j_\zeta)$. Before doing so, we run the analogous spectral sequence for computing $\mathrm{THH}(\ell_p)/(p, v_1)$, as a warm up. We consider the $\mathbb{E}_\infty$-ring $\mathbb{Z}_\zeta := \mathbb{Z}_p^{B\mathbb{Z}}$ with the trivial filtration.

Theorem 4.4.11. *For $p > 2$, $\pi_*(\mathrm{THH}(j_\zeta))/(p, v_1) \cong \mathbb{F}_p[\sigma^2 v_2] \otimes \Lambda[\lambda_1, \lambda_2, \zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$ with $|\lambda_i| = 2p^i - 1$ and $|\sigma^2 v_2| = 2p^2$.*

Proof. As in Example 4.4.3, we consider the spectral sequence associated with the filtered spectrum $\mathrm{THH}(j_\zeta^{\mathrm{fil}})/(p, \tilde{v}_1)$. The analogous spectral sequence in the case $p = 2$ is displayed in Figure 4.2 above. The underlying spectrum is $\mathrm{THH}(j_\zeta)/(p, v_1)$ and the associated graded spectrum is $\mathrm{THH}(j_\zeta^{\mathrm{gr}})/(p, v_1)$. By Proposition 4.4.10, the E_1-page is isomorphic to $\mathbb{F}_p[\sigma^2 v_1] \otimes \Lambda[\lambda_1, dv_1, \zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$.

The classes in $C^0(\mathbb{Z}_p; \mathbb{F}_p)$ are permanent cycles by the Leibniz rule, since they are all their own p^{th}-power. The class $\zeta \in H^1(S^1; \mathbb{F}_p)$ is a permanent cycle because it detects a class in the image of $j_\zeta \to \mathrm{THH}(j_\zeta)$.

By Lemma 4.3.6, there is a differential $d_{2p-2}(\sigma^2 v_1) \doteq dv_1$, and the Leibniz rule determines the differentials on powers of $\sigma^2 v_1$.

Similarly, by Lemma 4.3.6, there must be a d_{2p-2} differential $\lambda_1 \doteq \sigma^2\alpha_1 \mapsto d\alpha_1$ in the spectral sequence $\mathrm{THH}(j_\zeta^{\mathrm{gr}}) \implies \mathrm{THH}(j_\zeta)$ mod p. By Lemma 4.3.3, we have

$$d\alpha_1 = d(v_1\zeta) = v_1 d\zeta - \zeta dv_1,$$

so that we have the differential $d_{2p-2}(\lambda_1) \doteq \zeta dv_1$ mod (p, v_1). By using the previous paragraph and replacing λ_1 with

$$\lambda_1' = \lambda_1 - \epsilon\zeta\mu$$

for some $\epsilon \in \mathbb{F}_p^\times$, we may assume that $d_{2p-2}(\lambda_1') = 0$.

This completely determines the spectral sequence up to the E_{2p-2}-page, and we learn the E_{2p-1}-page is isomorphic to $\mathbb{F}_p[(\sigma^2 v_1)^p] \otimes \Lambda[\lambda_1', (\sigma^2 v_1)^{p-1} dv_1, \zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$. There are no more differentials since there is no class outside filtration degree 0 and $2p-2$. There are no multiplicative extension problems since the multiplicative generators in nonzero degree are free generators as a graded ring.

Finally, let us show that the polynomial generator μ_2 is the class $\sigma^2 v_2$. Let us consider the map $j_\zeta^{\mathrm{fil}} \to \mathbb{Z}_\zeta$ induced by applying $(\tau_{\geq *}(-))^{B\mathbb{Z}}$ to the $\mathbb{Z}$-equivariant truncation map $\ell_p \to \mathbb{Z}_p$. This induces a map of spectral sequences for THH. Since $\mathbb{Z}_\zeta$ has the trivial filtration, its THH does too, so has no differentials in its associated spectral sequence. By Corollary 4.4.8 and Example 4.4.2, $\mathrm{THH}(\mathbb{Z}_\zeta)_p \cong \mathrm{THH}(\mathbb{Z}_p)^{B\mathbb{Z}} \otimes \mathbb{W}(C^0(\mathbb{Z}_p; \mathbb{F}_p))$, so

$$\pi_* \mathrm{THH}(\mathbb{Z}_\zeta)/(p, v_1) \cong \mathbb{F}_p[\sigma^2 v_1, \lambda_1, \lambda] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p).$$

$v_2 \in \pi_{2p^2-2}\mathbb{S}/(p, v_1)$ has a canonical nullhomotopy in $j_\zeta/(p, v_1) \cong \mathbb{F}_p^{B\mathbb{Z}}$ and $\mathbb{Z}_\zeta/(p, v_1) \cong \mathbb{F}_p[\sigma v_1]^{B\mathbb{Z}}$, so there is a canonical element

$$\sigma^2 v_2 \in \pi_{2p^2} \mathrm{THH}(j_\zeta)/(p, v_1)$$

and $\pi_{2p^2} \mathrm{THH}(\mathbb{Z}_\zeta)/(p, v_1)$, which we claim is detected in the spectral sequence for $\mathrm{THH}(j_\zeta^{\mathrm{fil}})$ by $(\sigma^2 v_1)^p$. To see this, it suffices to show this in $\mathrm{THH}(\mathbb{Z}_\zeta)$ because the map is injective in degree $2p^2-2$. But now it is the image of $\sigma^2 v_2$ from the map $\ell_p^{\mathrm{fil}} \to \mathbb{Z}_\zeta$, and in ℓ_p^{fil}, which we know by Example 4.4.3 is detected by $(\sigma^2 v_1)^p$. □

Remark 4.4.12. In the proof of the previous theorem, a reader might wonder why λ_1 supports a differential while $\sigma^2\alpha_1$ is still well-defined in $\mathrm{THH}(j_\zeta)$. This can be explained by the fact that $\sigma^2\alpha_1$ is not well-defined in $\mathrm{THH}(\mathbb{Z}_\zeta)/(p, v_1)$ since

$$\pi_{2p-3}(\mathrm{fib}(\mathbb{S} \to \mathbb{Z}_\zeta)/(p, v_1)) \to \pi_{2p-3}(\mathbb{S}/(p, v_1))$$

is not injective. The class $\sigma^2\alpha_1$ is well-defined in $\mathrm{THH}(\mathbb{Z})/(p, v_1)$ and $\mathrm{THH}(j_\zeta)/(p, v_1)$, but their images in $\mathrm{THH}(\mathbb{Z}_\zeta)/(p, v_1)$ are different. The class λ_1 in the E_1-page represents the former and λ_1' represents the latter. ◁

Remark 4.4.13. We can carry out the same computation for $\mathrm{THH}(\ell_p)^{h\mathbb{Z}}/(p, v_1)$ using the same filtrations ℓ_p^{fil} and $\ell_p^{\mathrm{gr}} \otimes \mathbb{Z}_p^{\mathrm{fil}}$. Then, we obtain an isomorphism of rings

$$\pi_* \mathrm{THH}(\ell_p)^{h\mathbb{Z}}/(p, v_1) \cong \mathbb{F}_p[\sigma^2 v_2] \otimes \Lambda[\lambda_1, \lambda_2, \zeta].$$

Furthermore, by keeping track of the map

$$\mathrm{THH}(j_\zeta)/(p, v_1) \to \mathrm{THH}(\ell_p)^{h\mathbb{Z}}/(p, v_1)$$

at every stage, we see that on homotopy groups, this map is the base-change along

$$C^0(\mathbb{Z}_p; \mathbb{F}_p) \to \mathbb{F}_p$$

that evaluates a function at $0 \in \mathbb{Z}_p$. The key point is in the proof of Lemma 4.4.6, where one uses the fact that the natural map $B\mathbb{Z}/p^n \to LB\mathbb{Z}/p^n$ coming from constant loops is the inclusion of the component 0 for each $n \geq 0$. ◁

4.4.4 The prime 2

We next turn to the prime 2. We first need to run the analogous analysis as in Example 4.4.3 for ko_2. We consider $\mathrm{ko}_2^{\mathrm{gr}}/2[v_0]$ as the bigraded ring given as the associated graded of $\mathrm{ko}_2^{\mathrm{gr}} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2^{\mathrm{fil}}$. To understand this, we need the following lemma.

Lemma 4.4.14. *There is an isomorphism of bigraded rings*

$$\pi_* \mathrm{THH}(\mathrm{ko}_2^{\mathrm{gr}}/2[v_0])/\eta \cong \mathbb{F}_2[v_0, v_1, \sigma^2 2, d\eta]/((d\eta)^2 + v_1 d\eta) \otimes \Lambda[dv_0, dwv_1]$$

Proof. The associated graded of $\mathrm{ko}_2^{\mathrm{gr}}/2[v_0]$ with respect to the Posnikov filtration is

$$\mathbb{F}_2[v_0, v_1, \eta]$$

By symmetric monoidality of THH, we have an equivalence

$$\mathrm{THH}(\mathbb{F}_2[v_0, v_1, \eta]) \cong \mathrm{THH}(\mathbb{F}_2[v_0, v_1]) \otimes_{\mathrm{THH}(\mathbb{F}_2)} \mathrm{THH}(\mathbb{F}_2[\eta])$$

Since the argument of Lemma 4.4.5 works at the prime 2, we learn that the first tensor factor has homotopy ring $\mathbb{F}_2[\sigma^2 p, v_0, v_1] \otimes \Lambda[dv_0, dv_1]$.

For the second tensor factor, we note that $\mathrm{THH}(\mathbb{F}_2[\eta]) \otimes_{\mathrm{THH}(\mathbb{F}_2)} \mathbb{F}_2 \cong \mathrm{HH}(\mathbb{F}_2[\eta]/\mathbb{F}_2)$, whose homotopy ring is $\mathbb{F}_2[\eta] \otimes \Lambda[d\eta]$. Since the map $\mathrm{THH}(\mathbb{F}_2) \to \mathbb{F}_2$ is the cofiber of $\sigma^2 p$, we can run a $\sigma^2 p$-Bockstein spectral sequence to recover $\mathrm{THH}(\mathbb{F}_2[\eta])$. In the spectral sequence, $\eta, d\eta$

are permanent cycles since they are in the image of the unit map and the map d. We also see that there are no multiplicative extensions mod η for degree reasons, i.e. we have

$$\pi_* \operatorname{THH}(\mathbb{F}_2[\eta])/\eta = \Lambda(d\eta) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\sigma^2 2].$$

In the spectral sequence computing $\operatorname{THH}(\mathrm{ko}_2^{\mathrm{gr}}/2[v_0])$ from this, everything is a permanent cycle since all classes are generated either from the image of the unit map, the map from $\operatorname{THH}(\mathbb{F}_2)$, or the map d.

Now we turn to the multiplicative extensions, which we compute by mapping to the $\sigma^2 2$-completion of $\operatorname{THH}(\mathbb{F}_2^{BC_2}[v_0, v_1])$. As before, we can compute this via the $\sigma^2 2$-Bockstein spectral sequence whose E_1-page is $\operatorname{HH}(\mathbb{F}_2^{BC_2}[v_0, v_1]/\mathbb{F}_2)[\sigma^2 2]$.

We have an isomorphism $\operatorname{HH}(\mathbb{F}_2^{BC_2}[v_0, v_1]/\mathbb{F}_2) \cong \operatorname{HH}(\mathbb{F}_2^{BC_2}/\mathbb{F}_2) \otimes_{\mathbb{F}_2} \operatorname{HH}(\mathbb{F}_2[v_0, v_1])$. Moreover, $\operatorname{HH}_*(\mathbb{F}_2[v_0, v_1]) \cong \mathbb{F}_2[v_0, v_1] \otimes \Lambda[dv_0, dv_1]$, and $\operatorname{HH}(\mathbb{F}_2^{BC_2})$ is $\mathbb{F}_2^{BC_2} \times \mathbb{F}_2^{BC_2}$, since the free loop space of BC_2 is $BC_2 \times C_2$. If h is the generator of $\pi_{-1}\mathbb{F}_2^{BC_2}$, then a nontrivial idempotent in $\pi_0 \operatorname{HH}(\mathbb{F}_2^{BC_2})$ is given by dh. By the Leibniz rule (Lemma 4.3.3), $d\eta = v_1 dh + h dv_1$, so $(d\eta)^2 = v_1^2 dh = v_1 d\eta + \eta dv_1$. This this relation happens in $\operatorname{THH}(\mathrm{ko}_2)/\sigma^2 2$, but for degree reasons, this forces it to happen in $\operatorname{THH}(\mathrm{ko}_2)/\eta$ as well.

To see that the classes dv_0 and dv_1 square to 0, we note that this is true in

$$\operatorname{HH}(\mathbb{F}_2[v_0, v_1]/\mathbb{F}_2)$$

, and that we have a map

$$\operatorname{THH}(\mathbb{F}_2) \otimes_{\mathbb{F}_2} \operatorname{HH}(\mathbb{F}_2[v_0, v_1]/\mathbb{F}_2) \cong \operatorname{THH}(\mathbb{F}_2[v_0, v_1]) \to \operatorname{THH}(\mathbb{F}_2[v_0, v_1]^{BC_2})$$

using the isomorphism of Lemma 4.4.1. ☐

Lemma 4.4.15. *There is an isomorphism of graded rings*

$$\pi_*(\operatorname{THH}(\mathrm{ko}_2^{\mathrm{gr}})/(2, \eta)) \cong \mathbb{F}_2[v_1, \sigma^2 v_1, d\eta]/((d\eta)^2 + v_1 d\eta) \otimes \Lambda[\sigma^2\eta, dv_1]$$

Proof. We now understand the spectral sequence computing $\pi_*(\operatorname{THH}(\mathrm{ko}_2^{\mathrm{gr}})/(2, \eta))$ by running the 2-adic filtration spectral sequence on $\operatorname{THH}(\mathrm{ko}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2^{\mathrm{fil}})/(v_0, \eta)$. By Example 4.4.2, there is a differential from $\sigma^2 v_0$ to dv_0, $\sigma^2\eta$ is a class squaring to zero detected by $\sigma^2 v_0 dv_0$, and $\sigma^2 v_1$[8] detects $(\sigma^2 v_0)^2$. The remaining classes are either in the image of the unit map or the image of d, so are permanent cycles. The relation $(d\eta)^2 + v_1 d\eta = 0$ occurs because it does on associated graded, and because there are no classes in topological degree 4 and positive p-adic filtration. The class dv_1 squares to zero since there are no classes of weight -2, topological degree 6, and positive p-adic filtration. ☐

We now compute $\operatorname{THH}(\mathrm{ko}_2)/(2, \eta, v_1)$, which was also computed in [AR05, Theorem 8.14].

---

[8] The element $v_1 \in \pi_2\mathbb{S}/2$ exists, even though it does not extend to a self map.

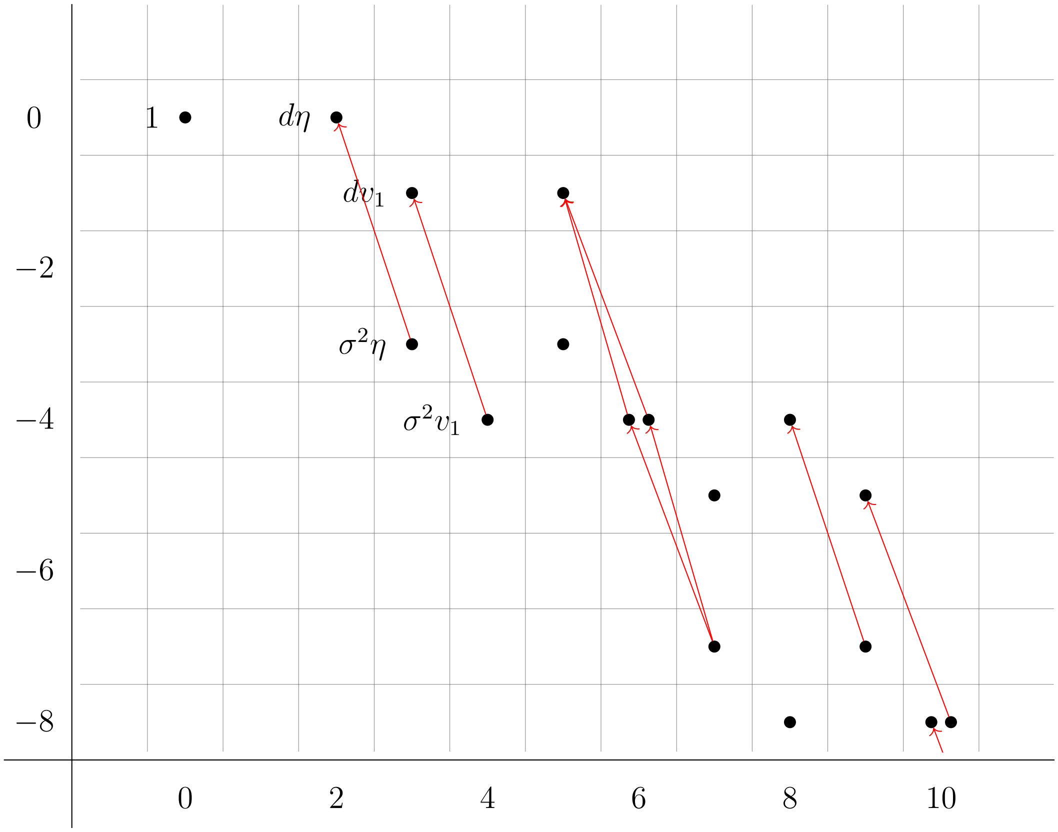

Figure 4.3: Above is the spectral sequence for the filtered ring $\mathrm{THH}(\mathrm{ko}_2^{\mathrm{fil}})/(2, \eta, v_1)$.

Example 4.4.16. We now can run the spectral sequence

$$\mathrm{THH}(\mathrm{ko}_2^{\mathrm{gr}})/(2, \eta, v_1) \implies \mathrm{THH}(\mathrm{ko}_2)/(2, \eta, v_1)$$

, which is a spectral sequence associated with a filtered $\mathbb{E}_\infty$-ring since $\mathbb{F}_2 \cong \mathrm{ko}_2^{\mathrm{fil}}/(2, \eta, v_1)$, where η and v_1 are taken in filtration 2. This spectral sequence is displayed in Figure 4.3. The first page of this spectral sequence by Lemma 4.4.15 is $\mathbb{F}_2[\sigma^2 v_1] \otimes \Lambda[dv_1, d\eta, \sigma^2\eta]$. It follows as in Example 4.4.3 that there are differentials from $\sigma^2\eta$ to $d\eta$ and $\sigma^2 v_1$ to dv_1. What remains after these differentials are $\mathbb{F}_2[(\sigma^2 v_1)^2] \otimes \Lambda[\sigma^2 v_1 dv_1, \sigma^2\eta d\eta]$. For degree reasons, there can be no further differentials. the classes in odd degree square to 0 because there are no classes in degrees 2 or 6 mod 8. ◁

We now run the analogous analysis to compute $\mathrm{THH}(j_\zeta)/(2, \eta, v_1)$.

Lemma 4.4.17. *There is an isomorphism of graded rings*

$$\pi_* \mathrm{THH}(j_\zeta^{\mathrm{gr}})/(2, \eta, v_1) \cong \pi_* \mathrm{THH}(\mathrm{ko}_2^{\mathrm{gr}})/(2, \eta, v_1) \otimes \pi_*(\mathrm{HH}(\mathbb{F}_2^{B\mathbb{Z}}/\mathbb{F}_2))$$

Proof. Since $\mathrm{ko}_2^{\mathrm{gr}}/2 \otimes_{\mathbb{F}_2} \mathbb{F}_2^{B\mathbb{Z}} \cong j_\zeta^{\mathrm{gr}}/2$, we learn from Lemma 4.4.14 that

$$\pi_*(\mathrm{THH}(j_\zeta^{\mathrm{gr}}/2[v_0])/(v_0, \eta, v_1) \cong \mathbb{F}_2[\sigma^2 v_0] \otimes \Lambda[d\eta, dv_0, dv_1] \otimes \mathrm{HH}_*(\mathbb{F}_2^{B\mathbb{Z}}/\mathbb{F}_2)$$

where $\mathrm{HH}(\mathbb{F}_2^{B\mathbb{Z}}/\mathbb{F}_2)$ is computed via Corollary 4.4.8 as $\mathbb{F}_2^{B\mathbb{Z}} \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$.

Exactly as in Lemma 4.4.15, in the spectral sequence for the 2-adic filtration, there is a differential from $\sigma^2 v_0$ to dv_0, $\sigma^2\eta$ is a class squaring to zero detected by $\sigma^2 v_0 dv_0$, and $\sigma^2 v_1$ is a class detecting $(\sigma^2 v_0)^2$. The rest of the classes are permanent cycles because they are either in the unit map, come from d, or are permanent cycles by the Leibniz rule. □

Theorem 4.4.18. *There is an isomorphism of rings for $p = 2$*

$$\pi_* \mathrm{THH}(j_\zeta)/(2, \eta, v_1) \cong \mathbb{F}_2[\mu] \otimes \Lambda[\lambda_2, x, \zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$$

where $|x| = 5$, $|\lambda_2| = 7, |\mu| = 8$.

Proof. We run the spectral sequence $\mathrm{THH}(\mathrm{ko}_2^{\mathrm{gr}})/(2, \eta, v_1) \implies \mathrm{THH}(\mathrm{ko}_2)/(2, \eta, v_1)$. As in Example 4.4.16, there are differentials from $\sigma^2\eta$ to $d\eta$ and $\sigma^2 v_1$ to dv_1. For degree reasons, $(\sigma^2 v_1)^2$ is a permanent cycle, as are $\sigma^2\eta d\eta$, $\sigma^2 v_1 dv_1$, and ζ. $C^0(\mathbb{Z}_2; \mathbb{F}_p)$ is a permanent cycle by the Leibniz rule. If we let λ_2 and x denote classes detecting $\sigma^2 v_1 dv_1$ and $\sigma^2\eta d\eta$ respectively, then $\lambda_2^2 = 0$ and $x^2 = 0$ for degree reasons. □

4.5 The THH of j

We now consider $\mathrm{THH}(j)/(p, v_1)$ for $p > 2$. We first compute the Hochschild homology of the $\mathbb{F}_p$-algebra j^{gr}/p, which is isomorphic to $\tau_{\geq 0}(\mathbb{F}_p[v_1]^{B\mathbb{Z}})$ by Lemma 4.2.16.

Proposition 4.5.1. *Let $p > 2$. $\mathrm{HH}_*((j^{\mathrm{gr}}/p)/\mathbb{F}_p) \cong \mathrm{HH}_*(\tau_{\geq 0}(\mathbb{F}_p[v_1]^{B\mathbb{Z}})/\mathbb{F}_p)$ is isomorphic as a ring to*

$$\Lambda[dv_1, \alpha_1] \otimes \mathbb{F}_p[v_1, x_0, x_1, \ldots]/(x_i^p = v_1^{p^{i+1}-p^i} x_i + v_1^{p^{i+1}-p^i-1}\alpha_1(\prod_{j=0}^{i-1} x_j^{p-1})dv_1;\ i \geq 0)$$

where $|x_i| = p^i(2p-2)$, and x_i is in grading $p^i(2p-2)$.

Proof. Define a graded ring $R = \tau_{\geq 0}\mathbb{Z}_p[v_1]^{B\mathbb{Z}}$ so that $R/p \cong j^{\mathrm{gr}}/p$. We shall show that $\pi_* \mathrm{HH}(R/\mathbb{Z}_p)$ is the $\mathbb{Z}_p$-algebra generated by v_1, dv_1, α, and a set of generators $x_0, x_1, \ldots$ with $|x_i| = p^i(2p-2)$ having relations

$$x_i^p = px_{i+1} + v_1^{p^{i+1}-p^i} x_i + v_1^{p^{i+1}-p^i-1}\alpha(\prod_{j=0}^{i-1} x_j^{p-1})dv_1.$$

Then, the statement follows by the base-change $\mathbb{Z}_p \to \mathbb{F}_p$.

Let $R_\zeta = \mathbb{Z}_p[v_1]^{B\mathbb{Z}}$ and let $\eta : R \to R_\zeta$ denote the connective cover map. To compute the Hochschild homology, we shall show that the map $\eta_* : \pi_* \mathrm{HH}(R) \to \pi_* \mathrm{HH}(R_\zeta)$ is injective and describe the image. Note that $\pi_* R_\zeta = \mathbb{Z}_p[v_1, \zeta]$ and $\pi_* R = \mathbb{Z}_p[v_1, \alpha]$ where $\eta_*(\alpha) = v_1\zeta$.

Let us consider the Künneth spectral sequence

$$E_2(\mathrm{HH}(R)) = \mathrm{Tor}^{\pi_*(R\otimes_{\mathbb{Z}}R)}(\pi_*R, \pi_*R) \implies \pi_*\mathrm{HH}(R). \tag{4.6}$$

Since $\pi_*R = \mathbb{Z}_p[v_1] \otimes \Lambda[\alpha]$, the E_2-page can be computed as

$$E_2(\mathrm{HH}(R)) = \mathbb{Z}_p[v_1] \otimes \Lambda[dv_1, \alpha] \otimes \Gamma[d\alpha].$$

Similarly, there is a spectral sequence

$$E_2(\mathrm{HH}(R_\zeta)) = \mathbb{Z}_p[v_1] \otimes \Lambda[dv_1, \zeta] \otimes \Gamma[d\zeta] \implies \pi_*\mathrm{HH}(R_\zeta) \tag{4.7}$$

up to p-completion.

We claim that $E_2(\mathrm{HH}(R)) \to E_2(\mathrm{HH}(R_\zeta))$ is injective. By Lemma 4.3.3, we have

$$d\alpha \mapsto -\zeta dv_1 + v_1 d\zeta.$$

To prove the injectivity, it is enough to prove it after taking the associated graded group with respect to the (dv_1)-adic filtration. Then, we may assume that $d\alpha$ maps to $v_1 d\zeta$, and since $E_2(\mathrm{HH}(R_\zeta))$ is torsion-free, the divided power $\gamma_n(d\alpha)$ maps to $v_1^n\gamma_n(d\zeta)$. Therefore, we have the desired injectivity. Note also that the map is injective mod p.

The spectral sequence (4.7) degenerates at the E_2-page using the symmetric monoidality of HH, Corollary 4.4.8, and Lemma 4.3.4. We then see that (4.6) also degenerates at the E_2-page and that $\eta_* : \pi_*\mathrm{HH}(R) \to \pi_*\mathrm{HH}(R_\zeta)$ is injective, even after mod p.

Let us describe the Künneth filtration on

$$\pi_*\mathrm{HH}(R_\zeta) = \mathbb{Z}_p[v_1] \otimes \Lambda[dv_1, \zeta] \otimes W(C^0(\mathbb{Z}_p; F_p))$$

in more detail. Here, the ring

$$W(C^0(\mathbb{Z}_p; \mathbb{F}_p)) = \lim_k C^0(\mathbb{Z}_p; \mathbb{Z}_p/p^k)$$

is the ring of all continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$. It can also be described, up to completion, as the algebra generated by $y_0, y_1, \ldots$ with relations

$$y_i^p = py_{i+1} + y_i.$$

Here, the element y_0 is the identity function $\mathbb{Z}_p \to \mathbb{Z}_p$ and the y_i's for $i > 0$ can be defined with the above formula since $y^p \equiv y \pmod{p}$ for any $y \in W(C^0(\mathbb{Z}_p; \mathbb{F}_p))$. In $\pi_*\mathrm{HH}(R_\zeta)$, the element y_0 equals $d\zeta$, and the y_i's represent the p^i-th divided power of $d\zeta$ in the Künneth spectral sequence (4.7).

To determine $\pi_*\mathrm{HH}(R)$, we need to find the classes x_i's representing the divided powers $\gamma_{p^i}(d\alpha) \in E_2(\mathrm{HH}(R))$ up to a p-adic unit. The first divided power $d\alpha \in E_2(\mathrm{HH}(R))$ has a canonical lift $x_0 := d\alpha \in \pi_*\mathrm{HH}(R)$ and its image under η_* is $v_1y_0 - \zeta dv_1$. Inductively, suppose that we have chosen $x_0, \ldots, x_i$ in a way that the image of x_j is

$$\eta_*(x_j) = v_1^{p^j} y_j - v_1^{p^j-1}\Big(\prod_{k=0}^{j-1} y_k^{p-1}\Big)\zeta dv_1$$

for $0 \leq j \leq i$. Let x_{i+1} be any class representing $\gamma_{p^{i+1}}(d\alpha)$. Then, after scaling by a unit, we must have

$$x_i^p = px_{i+1} + c$$

for some class $c \in \pi_* \operatorname{HH}(R)$ with Künneth filtration $< p^{i+1}$. Applying η_*, we have

$$\eta_*(c) \equiv \eta_*(x_i)^p \equiv v_1^{p^{i+1}} y_i^p \equiv v_1^{p^{i+1}} y_i \pmod p.$$

Let $d \in \pi_* \operatorname{HH}(R)$ be the class $v_1^{p^{i+1}-p^i-1}(v_1 x_i + \alpha(\prod_{k=0}^{i-1} x_k^{p-1})dv_1)$, having Künneth filtration p^i. Then, we can compute that $\eta_*(d) = v_1^{p^{i+1}} y_i$ so that $\eta_*(c) \equiv \eta_*(d) \pmod p$. Since η_* is injective mod p, we have $c \equiv d \pmod p$, so by replacing x_{i+1} with $x_{i+1} - (c-d)/p$, we can assume that $c = d$. Then, we have

$$\begin{aligned}\eta_*(x_{i+1}) &= p^{-1}\eta_*(x_i^p - c)\\ &= p^{-1}\left(v_1^{p^{i+1}} y_i - pv_1^{p^{i+1}-1}(y_i \cdots y_0)^{p-1}\zeta dv_1 - v_1^{p^{i+1}} y_i\right)\\ &= v_1^{p^{i+1}} y_{i+1} - v_1^{p^{i+1}-1}(y_i \cdots y_0)^{p-1}\zeta dv_1.\end{aligned}$$

The desired ring structure of $\pi_* \operatorname{HH}(R)$ can now be read off from the ring structure on $\pi_* \operatorname{HH}(R_\zeta)$. □

Lemma 4.5.2. *There is an isomorphism of bigraded $\mathbb{E}_1$-$\mathrm{THH}(\mathbb{F}_p)$-algebras for $p > 2$*

$$\mathrm{THH}(j^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0]) \cong \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \operatorname{HH}(\mathbb{F}_p[v_0]/\mathbb{F}_p) \otimes_{\mathbb{F}_p} \operatorname{HH}(\tau_{\geq 0}\mathbb{F}_p[v_1]^{B\mathbb{Z}}/\mathbb{F}_p)$$

Proof. We run the strategy of Lemma 4.4.5 with appropriate modifications. First, we have the isomorphism $j^{\mathrm{gr}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p[v_0] \cong j^{\mathrm{gr}}/p \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_0]$, which by Lemma 4.2.16 is equivalent to $\tau_{\geq 0}\mathbb{F}_p[v_1, v_0] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{B\mathbb{Z}}$. As an $\mathbb{E}_2$-ring, we claim this is equivalent to the tensor product of $\mathbb{F}_p \otimes \mathbb{S}[v_0]$ with the pullback of the cospan

$$\begin{tikzcd} & \mathbb{S}[v_1] \otimes \mathbb{S}^{B\mathbb{Z}} \arrow[d] \\ \mathbb{S} \arrow[r] & \mathbb{S}^{B\mathbb{Z}} \end{tikzcd}$$

where the vertical map is the augmentation sending v_1 to 0.

This isomorphism is a consequence of the isomorphism of Lemma 4.4.5 and the pullback square

$$\begin{tikzcd} j^{\mathrm{gr}}/p \arrow[r] \arrow[d] & j_\zeta^{\mathrm{gr}}/p \arrow[d] \\ \mathbb{F}_p \arrow[r] & \mathbb{F}_p^{B\mathbb{Z}} \end{tikzcd}$$

Given this equivalence, we conclude by arguing exactly as in Lemma 4.4.5. □

Proposition 4.5.3. *Let $p > 2$. Then*

$$\pi_* \operatorname{THH}(j^{\mathrm{gr}})/p \cong \pi_* \operatorname{THH}(\mathbb{Z}_p)/p \otimes \pi_* \operatorname{HH}(\tau_{\geq 0}\mathbb{F}_p[v_1]^{B\mathbb{Z}}/\mathbb{F}_p)$$

Proof. We follow the strategy in Proposition 4.4.10, running the spectral sequence corresponding to the p-adic filtration

$$\pi_* \operatorname{THH}(j^{\mathrm{gr}}/p[v_0])/p \implies \pi_* \operatorname{THH}(j^{\mathrm{gr}})/p.$$

The E_1-page is understood via Lemma 4.5.2 to be

$$\mathbb{F}_p[\sigma^2 p, v_0] \otimes \Lambda[dv_0] \otimes \pi_* \operatorname{HH}(\tau_{\geq 0}\mathbb{F}_p[v_1]^{B\mathbb{Z}}/\mathbb{F}_p)$$

where the last tensor factor is described in Proposition 4.5.1. There is a differential $d_1\sigma^2 p = dv_0$, coming from the map from $\mathbb{Z}_p^{\mathrm{fil}} \to \mathbb{Z}_p^{\mathrm{fil}} \otimes j^{\mathrm{fil}}$ and Example 4.4.2.

We need to show that the remaining classes are permanent cycles. The classes v_1, α are permanent cycles because they are in the image of the unit map, and dv_1 is a permanent cycle because it is in the image of the map σ^2. The classes x_i are permanent cycles for degree reasons, as everything of positive p-adic filtration is in nonnegative degree, and the differentials respect the hfp grading. One also sees for degree reasons and the map from $\operatorname{THH}(\mathbb{Z}_p)/p$ that there are no multiplicative extension problems. ☐

We now run the spectral sequence $\operatorname{THH}(j^{\mathrm{gr}})/(p, v_1) \implies \operatorname{THH}(j)/(p, v_1)$ associated with the filtered spectrum $\operatorname{THH}(j^{\mathrm{fil}})/(p, \widetilde{v_1})$ where $\widetilde{v_1} \in \pi_* j/p$ is the class of filtration $2p-2$. The following lemma guarantees the multiplicativity of the spectral sequences.

Lemma 4.5.4. *$j^{\mathrm{fil}}/(p, \widetilde{v_1})$ admits a homotopy commutative $\mathcal{A}_{p-1}$-multiplication for $p > 2$, and in particular is homotopy associative for $p > 3$.*

Proof. By [Ang08, Example 3.3], it follows that $\mathbb{S}/p$ is an $\mathcal{A}_{p-1}$-algebra, and it is easy to see that there is no obstruction to its multiplication being homotopy commutative for $p > 2$. We conclude by observing that $j^{\mathrm{fil}}/(p, \widetilde{v_1}) \cong \tau_{\leq 2p-3} j^{\mathrm{fil}} \otimes \mathbb{S}/p$.

Note that by loc. cit., the multiplication is not $\mathcal{A}_p$, the obstruction being α_1. ☐

Theorem 4.5.5. *For $p > 3$, $\pi_* \operatorname{THH}(j)/(p, v_1)$ is the homology of the CDGA*

$$\mathbb{F}_p[\mu_2] \otimes \Lambda[\alpha_1, \lambda_2, a] \otimes \Gamma[b], \quad d(\lambda_2) = a\alpha_1$$

$$|b| = 2p^2 - 2p,\ |a| = 2p^2 - 2p - 1,\ |\lambda_2| = 2p^2 - 1,\ |\mu_2| = 2p^2$$

and for $p = 3$, the above result is true after taking an associated graded ring.

Proof. The E_1-page of the spectral sequence

$$E_1 = \pi_* \operatorname{THH}(j^{\mathrm{gr}})/(p, v_1) \implies \pi_* \operatorname{THH}(j)/(p, v_1)$$

is isomorphic to

$$\mathbb{F}_p[\mu_1] \otimes \Lambda[\sigma^2\alpha_1, dv_1, \alpha_1] \otimes \Gamma[d\alpha_1].$$

by Proposition 4.5.3. By Lemma 4.3.6, there are d_{2p-2}-differentials

$$\sigma^2\alpha_1 \mapsto d\alpha_1$$
$$\sigma^2 v_1 \mapsto dv_1.$$

The class α_1 is a permanent cycle since it must represent the image of $\alpha_1 \in \pi_* j/(p, v_1)$ along the unit map, and the divided power classes $(d\alpha_1)^{(k)}$ are permanent cycles because they are in weight 0, and there are no classes of weight > 1. Therefore, by the Leibniz rule, the E_{2p-1}-page is isomorphic to

$$\mathbb{F}_p[\mu_2] \otimes \Lambda[\lambda_2, a, \alpha_1] \otimes \Gamma[\gamma_p(d\alpha_1)]$$

where μ_2, λ_2 and a represent $(\mu_1)^p, (\sigma^2 v_1)^{p-1} dv_1$ and $(\sigma^2\alpha_1)\gamma_{p-1}(d\alpha_1)$, respectively.

For degree reasons, the only possible further nonzero differential is

$$d_{p-1}(\lambda_2) \doteq \alpha_1$$

To prove that this differential actually happens, it is enough to show that

$$\pi_{2p^2-2}\operatorname{THH}(j)/(p, v_1) = 0.$$

By Corollary 4.3.11, there is a $(4p^2 - 4p - 2)$-connective map $j \oplus \Sigma^2 \operatorname{fib}(\mathbb{S}_p \to j) \to \operatorname{THH}(j)$, so it suffices to show that

$$\pi_{2p^2-2}(j/(p, v_1)) = \pi_{2p^2-2}(\Sigma^2 \operatorname{fib}(1_j)/p, v_1) = 0.$$

The former group is clearly 0. The latter is 0 from the computation of the Adams–Novikov E_2-page for $\mathbb{S}/(p, v_1)$ in low degrees (see the discussion after [Rav86, Theorem 4.4.9] and Theorem 4.4.8 of op. cit.).

The last nontrivial differential of the spectral sequence is displayed for $p = 3$ in Figure 4.4.

We now check for $p \geq 5$ that there are no multiplicative extension problems in our description of the commutative ring structure on $\pi_* \operatorname{THH}(j)/(p, v_1)$. If we choose $\gamma_{p^i} b$ to be detected by $(\gamma_{p^{i+1}}(d\alpha_1))$, the relations $\gamma_{p^i}(b)^p = 0$ follow since there is nothing of higher filtration in that degree. Let μ_2 be any lift of $(\sigma^2 v_1)^p$. The homology of the CDGA $\Lambda_{\mathbb{F}_p}[\alpha_1, \lambda_2, a], d(\lambda_2) = a\alpha_1$ is 6-dimensional over $\mathbb{F}_p$, given by

$$\{1, a, \alpha_1, \lambda_2 a, \lambda_2\alpha_1, \lambda_2 a\alpha_1\}$$

Let α_1, x, y, z denote lifts of the classes $\alpha_1, a, \lambda_2 a, \lambda_2\alpha_1$ respectively (so that $\alpha_1 y$ is a lift of $\lambda_2 a\alpha_1$). The relation $\alpha_1 y = -xz$ holds because it is true on the associated graded and there is nothing of higher filtration in that degree. The classes $\alpha_1 z, yz, x\alpha_1$ are 0 because there are no nonzero classes in degree $(p+1)(2p-2), 2p^2 - 1 + 2(2p-3), 2(2p^2-1) + (2p-3) + p(2p-2) + 1$ respectively. The only remaining relation, $xy = 0$, occurs because it happens on the associated graded, and there is nothing of higher filtration. $\square$

Remark 4.5.6. For $p = 3$, it is more complicated to figure out the multiplicative extensions, since the homotopy ring is not necessarily associative. Many of the multiplicative extensions can be ruled out using the Postnikov filtration on $j/(3, v_1)$, but not all of them: for example this doesn't rule out the possible non-associative extension $x(x\mu_2^2) = zb^2$ in degree 62. $\triangleleft$

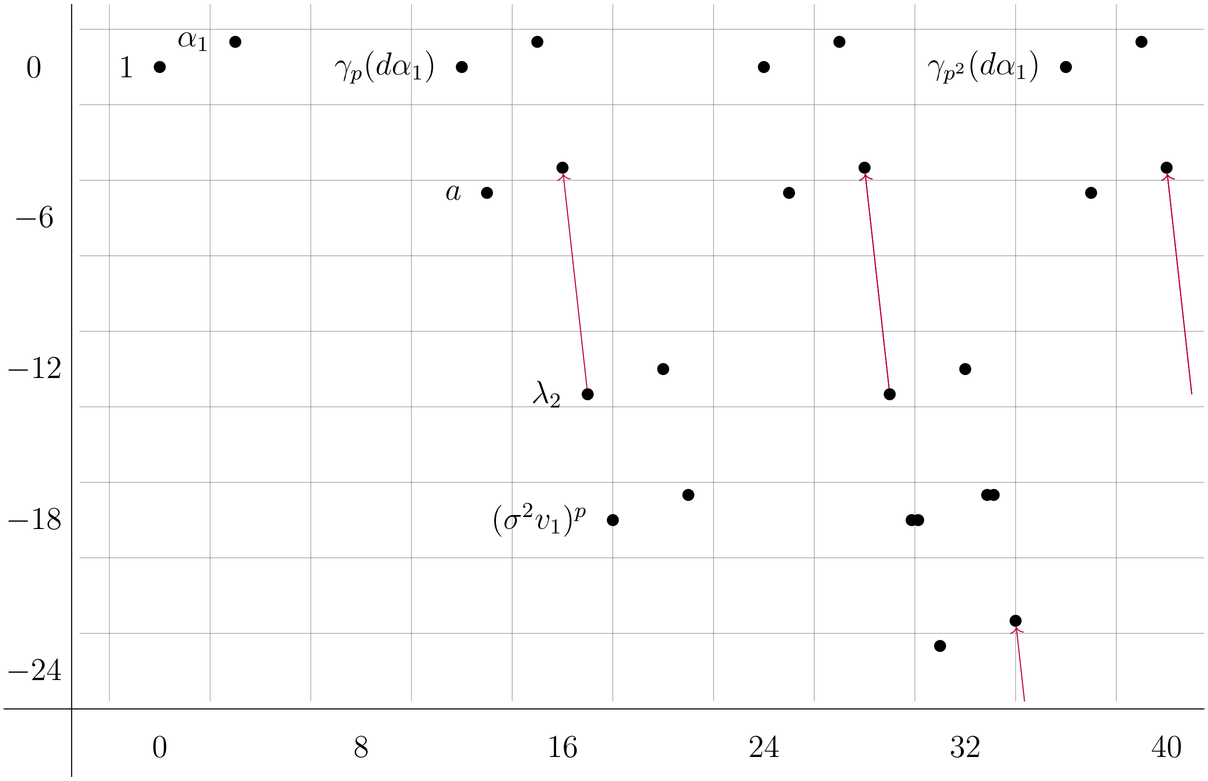

Figure 4.4: Above is the E_8-page of the spectral sequence associated with the filtered ring $\mathrm{THH}(j^{\mathrm{fil}})/(3, v_1)$. The spectral sequence collapses at the E_9-page.

4.6 THH of finite extensions

In this section, we shall make the analogous computations for the THH of $j_{\zeta,k} := \ell_p^{hp^k\mathbb{Z}}$, $ju_{\zeta,k}$, and and also of $j_k := \tau_{\geq 0} j_{\zeta,k}$ for $p > 2$, which are introduced as filtered rings in Definition 4.2.12. $j_{\zeta,k}$ is a $\mathbb{Z}/p^k$ Galois extension of j_ζ in Sp_p. The computations are very similar to the cases of j_ζ and j, so we shall only point out the differences from the proofs of those cases.

Theorem 4.6.1. *There is an isomorphism of rings for $p > 2$*

$$\pi_* \mathrm{THH}(j_{\zeta,k})/(p, v_1) \cong \pi_*(\mathrm{THH}(\ell_p)/(p, v_1)) \otimes \Lambda[\zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$$

and for $p = 2$

$$\pi_* \mathrm{THH}(j_{\zeta,k})/(2, \eta, v_1) \cong \pi_*(\mathrm{THH}(\mathrm{ko}_2)/(2, \eta, v_1)) \otimes \Lambda[\zeta] \otimes C^0(\mathbb{Z}_2; \mathbb{F}_p)$$

The maps $\mathrm{THH}(j_{\zeta,k})/(p, v_1) \to \mathrm{THH}(j_{\zeta,k+1})/(p, v_1)$ on π_* are the identity on the

$$\mathrm{THH}(\ell_p)/(p, v_1)$$

component, send ζ to 0, and are the restriction map $C^0(\mathbb{Z}_p; \mathbb{F}_p) \to C^0(p\mathbb{Z}_p; \mathbb{F}_p) \cong C^0(\mathbb{Z}_p; \mathbb{F}_p)$.

Proof. The proof strategy is the same as in Theorem 4.4.11 and Theorem 4.4.18. One difference is that for $k \geq 1$, the class λ_1 in the spectral sequence

$$\mathrm{THH}(j^{\mathrm{gr}}_{\zeta,k})/(p, v_1) \implies \mathrm{THH}(j_{\zeta,k})/(p, v_1)$$

is a permanent cycle, which can be seen from the Leibniz rule. In particular, Remark 4.4.12 doesn't apply for $k \geq 1$, and as noted in the remark, this difference doesn't affect the final answer.

The claim about the maps $\pi_* \mathrm{THH}(j_{\zeta,k})/(p, v_1) \to \mathrm{THH}(j_{\zeta,k+1})/(p, v_1)$ can be deduced at the level of associated graded of the filtrations. For example, by choosing elements $\lambda_1, \lambda_2, \sigma^2 v_2$ in $\mathrm{THH}(j_\zeta)/(p, v_1)$, one sees that their images in $\mathrm{THH}(j_{\zeta,k})/(p, v_1)$ are valid generators of the corresponding classes. To see what the transition maps do on $\Lambda[\zeta] \otimes C^0(\mathbb{Z}_p; \mathbb{F}_p)$, we can use Lemma 4.4.6 since these classes are in the image of $\mathrm{THH}(\mathbb{S}_p^{B\mathbb{Z}})$. It then follows that map sends $C^0(\mathbb{Z}_p; \mathbb{F}_p) \to C^0(p\mathbb{Z}_p; \mathbb{F}_p)$ given by restriction of functions, and ζ goes to $p\zeta = 0$ because that is what happens on the level of mod p cohomology of the p-fold cover map $S^1 \to S^1$. □

We next explain the computation for $ju_{\zeta,k}$, which is nearly identical to that of $j_{\zeta,k}$

Theorem 4.6.2. *For each $k \geq 0$, there is an isomorphism of rings*

$$\pi_* \mathrm{THH}(ju_{\zeta,k})/(2, v_1) \cong \pi_*(\mathrm{THH}(\ell_2)/(2, v_1)) \otimes \Lambda[\zeta] \otimes C^0(\mathbb{Z}_2; \mathbb{F}_p)$$

The maps $\mathrm{THH}(ju_{\zeta,k})/(p, v_1) \to \mathrm{THH}(ju_{\zeta,k+1})/(p, v_1)$ on π_* are the identity on the $\mathrm{THH}(\ell_2)/(2, v_1)$ component, send ζ to 0, and are the restriction map

$$C^0(\mathbb{Z}_2; \mathbb{F}_p) \to C^0(2\mathbb{Z}_2; \mathbb{F}_p) \cong C^0(\mathbb{Z}_2; \mathbb{F}_p)$$

Proof. The proof is nearly exactly as the proof of Theorem 4.6.1 for $p > 2$. The only difference is that in checking multiplicative extension problems in spectral sequences, one must check that odd degree classes square to zero (since we are at the prime 2). This always follows because the square lands in a zero group; see Figure 4.2 for a chart. □

Our argument to compute $\mathrm{THH}(j_k)$ for $k \geq 1$ uses Dyer–Lashof operations to produce permanent cycles, so we first give $j_k/(p, v_1)$ an $\mathbb{E}_\infty$-structure.

Proposition 4.6.3. *For $k \geq 1$, $j_k/(p, v_1)$ admits the structure of an $\mathbb{E}_\infty$-algebra under j_k that is a trivial square zero extension of $\mathbb{F}_p$ by $\Sigma^{2p-2}\mathbb{F}_p$.*

Proof. To construct $j_k/(p, v_1)$ as an $\mathbb{E}_\infty$-ring, we first begin with $\tau_{\leq 2p-3} j_k$, whose homotopy groups are $\mathbb{Z}_p$ in degree 0 and $\mathbb{Z}/p^{k+1}$ in degree $2p-3$, where α_1 is a p-torsion class in degree $2p-3$.

By [Lur17, Corollary 7.4.1.28] this is a square zero extension of $\mathbb{Z}_p$ by $\Sigma^{2p-3}\mathbb{Z}/p^{k+1}$, i.e it fits into a pullback square

$$\begin{tikzcd}
\tau_{\leq 2p-3} j_k \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\mathbb{Z}_p \arrow[r] & \mathbb{Z}_p \oplus \Sigma^{2p-2}\mathbb{Z}/p^{k+1}
\end{tikzcd}$$

By using the map $\mathbb{Z}/p^{k+1} \to \mathbb{Z}/p$ that kills every multiple of p (including α_1 since $k \geq 1$), we can produce an $\mathbb{E}_\infty$-algebra R under $\tau_{\leq 2p-3} j_k$ defined as the pullback

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\mathbb{Z}_p \arrow[r] & \mathbb{Z}_p \oplus \Sigma^{2p-2}\mathbb{Z}/p
\end{tikzcd}$$

We claim that R is a trivial square zero extension of $\mathbb{Z}_p$. To see this, square zero extensions of $\mathbb{Z}_p$ by $\Sigma^{2p-1}\mathbb{F}_p$ are classified by maps of $\mathbb{Z}_p$-modules $L_{\mathbb{Z}_p/\mathbb{S}_p} \to \Sigma^{2p-1}\mathbb{F}_p$, where $L_{\mathbb{Z}_p/\mathbb{S}_p}$ denotes the $\mathbb{E}_\infty$ relative cotangent complex. By [Lur17, Theorem 7.4.3.1], since $\mathbb{S}_p \to \mathbb{Z}_p$ is $2p-3$-connective, there is a $4p-4$-connective map

$$\mathbb{Z}_p \otimes_{\mathbb{S}_p} \mathrm{cof}(\mathbb{S}_p \to \mathbb{Z}_p) \to L_{\mathbb{S}_p/\mathbb{Z}_p}$$

showing that $\pi_{2p-2} L_{\mathbb{Z}_p/\mathbb{S}_p}$ is $\mathbb{F}_p$. It follows that up to isomorphism, there is a unique nontrivial square zero extension of $\mathbb{Z}_p$ by $\Sigma^{2p-3}\mathbb{F}_p$. But $\tau_{\leq 2p-3}\mathbb{S}_p$ must be this nontrivial extension, since $\alpha_1 \neq 0$ there. Since $\alpha_1 = 0$ in R, it follows that R is the trivial square zero extension $\mathbb{Z}_p \oplus \Sigma^{2p-3}\mathbb{F}_p$. Thus $\tau_{\leq 2p-3}(R \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ is an $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra under it that is a trivial square zero extension of $\mathbb{F}_p$ by $\Sigma^{2p-2}\mathbb{F}_p$. Since R, and $v_1 = 0$ modulo p, there is a unital map $j_k/(p, v_1) \to R$, which is an equivalence because it is on homotopy groups. ☐

Theorem 4.6.4. *For $k \geq 1, p > 2$, there is an isomorphism*

$$\pi_* \mathrm{THH}(j_k)/(p, v_1) \cong \pi_* \mathrm{THH}(\ell_p)/(p.v_1) \otimes \Lambda[\alpha_{1/p^k}] \otimes \Gamma[d\alpha_{1/p^k}]$$

where $|\alpha_{1/p^k}| = 2p-2$ and $|\sigma\alpha_{1/p^k}| = 2p-1$.

Proof. The proof of Proposition 4.5.3 carries over exactly for j_k to give an isomorphism

$$\pi_* \mathrm{THH}(j_k^{\mathrm{gr}})/(p, v_1) \cong \pi_* \mathrm{THH}(\mathbb{Z}_p)/p \otimes \pi_* \mathrm{HH}(\tau_{\geq 0}\mathbb{F}_p[v_1]^{Bp^k\mathbb{Z}}/\mathbb{F}_p)/v_1$$

The second tensor factor on the right hand side by Proposition 4.5.1 is $\Lambda[\alpha_{1/p^k}, dv_1] \otimes \Gamma[d\alpha_{1/p^k}]$[9].

In the spectral sequence for $\mathrm{THH}(j_k^{\mathrm{gr}})/(p, v_1) \implies \mathrm{THH}(j_k)/(p, v_1)$, there is a differential $d_{2p-2}\sigma^2 v_1 = dv_1$ arising as in Theorem 4.5.5, but the target of the differential from $\sigma^2\alpha_1$, which is $\sigma\alpha_1$, is zero since $\alpha_1 = 0$ in $j_k/(p, v_1)$. In fact, the class $\sigma^2\alpha_1$ is a permanent cycle

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[9] As an algebra this doesn't depend on k, but we have given names depending on k to indicate that the exterior class α_{1/p^k} is sent to 0 in $\mathrm{THH}(j_{k+1}^{\mathrm{gr}})/(p, v_1)$.

since it can be constructed using a nullhomotopy of α_1. Let λ_1 be a class in $\mathrm{THH}(j_k)/(p, v_1)$ detecting this.

By Proposition 4.6.3, $j_k/(p, v_1)$ is an $\mathbb{E}_\infty$-algebra under j_k that is an $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra, so $\mathrm{THH}(j_k)/(p, v_1) \cong \mathrm{THH}(j_k) \otimes_{j_k} j_k/(p, v_1)$ is an $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra with Dyer–Lashof operations. We define λ_2 to be the $\mathbb{E}_2$-Dyer–Lashof operation on λ_1. In $\mathrm{THH}(\ell_p)/(p, v_1)$, this operation on the class λ_1 gives the class λ_2 in $\pi_{2p^2-1} \mathrm{THH}(\ell_p)/(p, v_1)$ [AR02, Section 2], which is detected by $\sigma^2 v_1^{p-1} dv_1$ in the spectral sequence for $\mathrm{THH}(\ell_p^{\mathrm{fil}})/(p, v_1)$ by Example 4.4.3. Since maps of filtered objects can only increase filtrations in which elements are detected, it follows that λ_2 must also be detected by $\sigma^2 v_1^{p-1} dv_1$ in $\mathrm{THH}(j_k)/(p, v_1)$, so that class is a permanent cycle. The class α_{1/p^k} is a permanent cycle since it is in the image of the unit map, and the classes in $\Gamma[d\alpha_{1/p^k}]$ must be permanent cycles for degree reasons, so there are no further differentials. There are no even degree classes of positive weight, so classes representing the divided powers of $d\alpha_{1/p^k}$ have zero p^{th}-power for degree reasons. For degree reasons there can be no further multiplicative extensions. ☐

4.7 TC in the stable range

TC is an important invariant of rings, partially because of the Dundas–Goodwillie-McCarthy theorem [DGM13], which says that for nilpotent extensions of rings, the relative K-theory is the relative TC.

Theorem 4.7.1 (Dundas–Goodwillie–McCarthy)**.** *Let $f : R \to S$ an i-connective map of connective $\mathbb{E}_1$-rings, for $i \geq 1$. Then there is a pullback square*

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] & K(S) \arrow[d] \\
\mathrm{TC}(R) \arrow[r] & \mathrm{TC}(R)
\end{tikzcd}$$

A precursor to this theorem is a result of Waldhausen[10], which computes the first nonvanishing homotopy group of $\mathrm{fib}\, \mathrm{TC}(f) \cong \mathrm{fib}\, K(f)$ in terms of Hochschild homology.

Proposition 4.7.2 (Waldhausen [Wal78b, Proposition 1.2])**.** *Let $f : R \to S$ be an i-connective map of connective $\mathbb{E}_1$-algebras for $i \geq 1$. Then $\mathrm{fib}(K(f)) \cong \mathrm{fib}(\mathrm{TC}(f))$ is $(i+1)$-connective, with $\pi_{i+1} \mathrm{fib}(K(f)) \cong \mathrm{HH}_0(\pi_0 S; \pi_i \mathrm{fib}\, f)$.*

Our goal in this section is to refine Proposition 4.7.2 to compute the spectrum $\mathrm{fib}(K(f))$ in the stable range in terms of THH. We use this to understand the maps $K(\mathbb{S}_p) \to K(\mathbb{Z}_p)$ and $K(j_\zeta) \to K(\mathbb{Z}_p^{B\mathbb{Z}})$ in the stable range.

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[10] Although Waldhausen proves this result for $\mathbb{E}_1$-$\mathbb{Z}$-algebras, the proof works equally well for any $\mathbb{E}_1$-algebra: see for example [Lev22, Proposition 3.3].

Given a map of $\mathbb{E}_1$-rings, $R \to S$, the relative $\mathbb{E}_1$-cotangent complex $L_{S/R}$ is the S-bimodule given by the fiber of the multiplication map $S \otimes_R S \to S$[11]. Our result is as follows:

Theorem 4.7.3. *Given a map of ring spectra $f : R \to S$, there is a natural map $\operatorname{fib} \mathrm{TC}(f) \to \mathrm{THH}(S; L_{S/R})$. If f is an n-connective map of -1-connective rings for $n \geq 1$, this natural map is $2n+1$-connective.*

Remark 4.7.4. In fact the map of Theorem 4.7.3 is the linearization map in the sense of Goodwillie calculus, of the functor $f \mapsto \operatorname{fib}(\mathrm{TC}(f))$. See [Hes94; DM94] for a variant of this, where one considers only trivial square-zero extensions of S rather than arbitrary $\mathbb{E}_1$-ring maps. ◁

We first construct the natural transformation using the following lemma.

Lemma 4.7.5. *Let $f : R \to S$ be a map of $\mathbb{E}_1$-rings. Then there is a natural equivalence $\mathrm{THH}(R; S) \cong \mathrm{THH}(S; S \otimes_R S)$ making the diagram below commute.*

$$\begin{tikzcd}
\mathrm{THH}(R;S) \arrow[rr] \arrow[dr] & & \mathrm{THH}(S;S) \\
& \mathrm{THH}(S; S \otimes_R S) \arrow[ur] &
\end{tikzcd}$$

Proof. Consider the map $f^* : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ and its right adjoint $f_* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$. The composite $f^* f_*$ corresponds to the S-bimodule $S \otimes_R S$, and the composite $f_* f^*$ corresponds to the R-bimodule S. Since THH of a bimodule is the trace of the bimodule as an endomorphism in presentable stable categories, cyclic invariance of the trace gives the desired equivalence $\mathrm{THH}(R; S) \cong \mathrm{THH}(S; S \otimes_R S)$. There is a diagram

$$\begin{tikzcd}
\mathrm{Mod}(R) \arrow[r, "f^*"] \arrow[d, shift right, "f^*"'] & \mathrm{Mod}(S) \arrow[d, shift right, "1_S"'] \\
\mathrm{Mod}(S) \arrow[r, "1_S"] \arrow[u, shift right, "f_*"'] & \mathrm{Mod}(S) \arrow[u, shift right, "1_S"']
\end{tikzcd}$$

where we use the natural transformation $\epsilon : f^* f_* \to 1_S$ and 1_{f^*} to fill in the 2-morphisms in the diagram. The horizontal maps in the diagram induce at the level of bimodules the maps $f^* f_* \implies 1_S$ and $f_* f^* \implies 1_S$ which induce the maps $\mathrm{THH}(R; S), \mathrm{THH}(S; S \otimes_R S) \to \mathrm{THH}(S; S)$ in the triangle of the lemma statement. The C_2-action on $\mathrm{THH}(S)$ coming from writing 1_S as $1_S \circ 1_S$ corresponds to restricting the S^1-action on $\mathrm{THH}(S)$ to $C_2 \subset S^1$. It follows that the claimed diagram naturally commutes because S^1 is connected, so the rotation by π action on $\mathrm{THH}(S)$ is homotopic to the identity. □

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[11] See for example [Lur17, Remark 7.4.1.12].

Construction 4.7.6. We now construct a map $\mathrm{fib}(\mathrm{TC}(f)) \to \mathrm{THH}(S; L_{S/R})$, which is a natural transformation when viewed as a map between functors $\mathrm{Alg}(\mathrm{Sp})^{\Delta^1} \to \mathrm{Sp}$.

The map $f : R \to S$ gives a natural map as follows: composing the map $\mathrm{TC}(R) \to \mathrm{THH}(R)$ with $\mathrm{THH}(R) \to \mathrm{THH}(R; S)$, we obtain a commutative square

$$\begin{tikzcd} \mathrm{TC}(R) \arrow[r] \arrow[d] & \mathrm{TC}(S) \arrow[d] \\ \mathrm{THH}(R; S) \arrow[r] & \mathrm{THH}(S; S) \end{tikzcd}$$

Taking horizontal fibers and using the isomorphism of Lemma 4.7.5, we obtain the desired natural transformation. ◁

We will first prove Theorem 4.7.3 in the case $R \to S$ is a square-zero extension with ideal M. To do this, we consider the square-zero extension as a filtered $\mathbb{E}_1$-ring with underlying R and associated graded $S \oplus M[1]$. Then $\mathrm{THH}(R)$ is a filtered S^1-equivariant spectrum, and the Frobenius maps $\Phi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ send filtration i to filtration ip, so in particular can be thought of as filtration preserving maps, since the filtration is only in nonnegative degrees.

The key input we use is the computation of THH of a trivial square-zero extension as an S^1-equivariant spectrum:

Proposition 4.7.7 ([Ras18, Proposition 4.5.1])**.** *For $S \oplus M$ the trivial square-zero extension of an $\mathbb{E}_1$-ring R by a bimodule M, there is an S^1-equivariant graded equivalence* $\mathrm{THH}(S \oplus M) \cong \mathrm{THH}(S) \oplus \bigoplus_{m=1}^{\infty} \mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{S^1} \mathrm{THH}(S; (\Sigma M)^{\otimes m})$

Here $\mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{S^1}$ is the right adjoint of the forgetful functor from S^1-equivariant spectra to $\mathbb{Z}/m\mathbb{Z}$-spectra, and the $\mathbb{Z}/m\mathbb{Z}$-action on $\mathrm{THH}(S; (\Sigma M)^{\otimes m})$ comes from cyclically permuting the tensor factors.

We also record a key property of the THH of -1-connective rings that we use:

Lemma 4.7.8. *Let $R \to S$ be an n-connective map of -1-connective rings, and M a connective S-bimodule. Then* $\mathrm{THH}(S; M)$ *is connective, and the map*

$$\mathrm{THH}(R; M) \to \mathrm{THH}(S; M)$$

is $n + 1$-connective.

Proof. Both of these follow from examining the associated graded coming from the cyclic bar complex computing $\mathrm{THH}(R; M)$ and $\mathrm{THH}(S; M)$. For the latter is given by $\Sigma^m S^{\otimes m} \otimes M$ which indeed is connective, and $\Sigma^m S^{\otimes m} \otimes M \to \Sigma^m R^{\otimes m} \otimes M$ is $n + m$-connective for $m \geq 1$ and an isomorphism for $m = 0$. □

Proposition 4.7.9. *Let $f : R \to S$ be an n-connective square-zero extension of -1-connective $\mathbb{E}_1$-rings for $n \geq 0$. Then the map* $\mathrm{fib}\,\mathrm{TC}(f) \to \mathrm{THH}(S; L_{S/R})$ *is $2n + 1$-connective.*

Proof. We consider the map $\mathrm{fib}\,\mathrm{TC}(f) \to \mathrm{fib}\,\mathrm{THH}(f) \to \mathrm{THH}(S; L_{S/R})$ as a map of filtered spectra, viewing S as a filtered $\mathbb{E}_1$-ring with associated graded $R \oplus M$. By Proposition 4.7.7, $\mathrm{gr}(\mathrm{fib}\,\mathrm{THH}(R)) \cong \bigoplus_{m=1}^{\infty} \mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{S^1} \mathrm{THH}(S; (\Sigma M)^{\otimes m})$ as an S^1-spectrum. Since The Frobenius map is zero on associated graded since it takes filtration i to ip, so we learn that

$$\mathrm{gr}_m(\mathrm{fib}\,\mathrm{TC}(f)) \cong (\Sigma \mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{S^1} \mathrm{THH}(S; (\Sigma M)^{\otimes m}))_{hS^1}$$

[12]. In particular, since S is -1-connective and $n \geq 0$, the connectivity of these terms goes to ∞ as $m \to \infty$ via Lemma 4.7.8 so the filtration on TC is complete. Since $\mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{S^1}$ decreases connectivity by 1, we learn that $\mathrm{gr}_m(\mathrm{fib}\,\mathrm{TC}(f))$ is $(n+1)m - 1$-connective. In particular, the map $\mathrm{fib}\,\mathrm{TC}(f) \to \mathrm{gr}_1\,\mathrm{fib}\,\mathrm{TC}(f)$ is $2n+1$-connective.

To finish, it suffices to show the following two claims:

1. $\mathrm{THH}(S; L_{S/R}) \to \mathrm{gr}_1\,\mathrm{THH}(S; L_{S/R})$ is $2n+2$-connective.
2. $\mathrm{gr}_1\,\mathrm{fib}\,\mathrm{TC}(f) \to \mathrm{gr}_1\,\mathrm{fib}\,\mathrm{THH}(S; L_{S/R})$ is an isomorphism.

The claim (1) follows from the fact that $\mathrm{gr}L_{S/R} \cong L_{S/S\oplus M} \cong \bigoplus_{m=1}^{\infty} (\Sigma M)^{\otimes s m}$, and $(\Sigma M)^{\otimes s m}$ is $2n+2$-connective for $m \geq 2$.

For claim (2), we see that

$$\mathrm{gr}_1\,\mathrm{fib}\,\mathrm{TC}(f) \cong \Sigma(\mathrm{Ind}_{\mathbb{Z}/1\mathbb{Z}}^{S^1} \mathrm{THH}(S; \Sigma M))_{hS^1} \cong (\mathrm{Ind}_{\mathbb{Z}/1\mathbb{Z}}^{S^1} \mathrm{THH}(S; \Sigma M))^{hS^1} \cong \mathrm{THH}(S; \Sigma M)$$

ΣM is exactly $\mathrm{gr}_1 L_{S/R}$, and $\mathrm{THH}(S; \mathrm{gr}_1 L_{S/R}) \cong \mathrm{gr}_1\,\mathrm{THH}(S; L_{S/R})$ since S is entirely in grading 0, so we are done. ☐

We prove Theorem 4.7.3 by reducing to the case of a square-zero extension. First, we produce a natural way to factor a map of $\mathbb{E}_1$-rings through a square-zero extension. We recall that given a S'-S-bimodule M with a unit map $\mathbb{S} \to M$, the pullback $S' \times_M S$ admits an $\mathbb{E}_1$-algebra structure where the maps $S' \to M$ and $S \to M$ are the S'-module and S-module maps adjoint to the unit map. This ring structure can be constructed as the endomorphism ring of the triple $(S', S, S \to S' \otimes'_S M)$ viewed as an object of the oplax limit $\mathrm{Mod}(S)\vec{\times}M\mathrm{Mod}(S')$ (see [LT23, Construction 2.5] and [**burklund2021k**]). When M comes from a cospan of ring maps $S' \to R \leftarrow S$, this agrees with the pullback of the span of rings by [LT19, Lemma 1.7].

Construction 4.7.10. Given a map $f : R \to S$, we consider $S \otimes_R S$ as an S-S-bimodule with unit 1. We define $R_{f,2}$ to be the $\mathbb{E}_1$-ring given by $S \times_{S \otimes_R S} S$. ◁

Lemma 4.7.11. *We have natural maps $R \xrightarrow{h} R_{f,2} \xrightarrow{g} S$. If $R \to S$ is an n-connective map of connective rings for $n \geq 0$, then h is $2n$-connective, g is n-connective, and g is a square-zero extension.*

Proof. The fiber of $h : R \to R_{f,2}$ is the total fiber of the square

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[12] See also [Ras18, Theorem 4.10.1].

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
S \arrow[r] & S \otimes_R S
\end{tikzcd}$$

which is $\operatorname{fib} f \otimes_R \operatorname{fib} f$, which is $2n$-connective. Since f is n-connective, it follows that g is too. It remains to show that g is a square-zero extension, which will follow if we identify $S \otimes_R S$ as an S-bimodule with unit with the associated structure on $S \oplus L_{S/R}$ coming from the cospan of rings $S \to S \oplus L_{S/R} \leftarrow S$ corresponding to the universal derivation. But since R maps into the pullback of this cospan (since it is the universal square-zero extension of S under R) we have a square of ring maps

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
S \arrow[r] & S \oplus L_{S/R}
\end{tikzcd}$$

which defines an isomorphism of unital S-bimodules $S \otimes_R S \to S \oplus L_{S/R}$. ☐

Proof of Theorem 4.7.3. We consider the maps h, g, f as in Lemma 4.7.11, giving us the diagram

$$\begin{tikzcd}
\operatorname{fib} \mathrm{TC}(h) \arrow[r] \arrow[d] & \operatorname{fib} \mathrm{TC}(f) \arrow[r] \arrow[d] & \operatorname{fib} \mathrm{TC}(g) \arrow[d] \\
\mathrm{THH}(R_{f,2}; L_{R_{f,2}/R}) \arrow[r] & \mathrm{THH}(S; L_{R/S}) \arrow[r] & \mathrm{THH}(S; L_{R_{2,f}/S})
\end{tikzcd} \tag{4.8}$$

To produce a nullhomotopy of the composite of the lower horizontal maps, we identify them with the vertical fibers of the following cofiber sequence using Lemma 4.7.5:

$$\begin{tikzcd}
\mathrm{THH}(R; R_{f,2}) \arrow[r] \arrow[d] & \mathrm{THH}(R; S) \arrow[r] \arrow[d] & \mathrm{THH}(R_{f,2}; S) \arrow[d] \\
\mathrm{THH}(R_{f,2}) \arrow[r] & \mathrm{THH}(S) \arrow[r] & \mathrm{THH}(S)
\end{tikzcd}$$

The map $\mathrm{THH}(R_{f,2}) \to \mathrm{THH}(S)$ lifts to $\mathrm{THH}(R_{f,2}; S)$, and this lifting provides the desired nullhomotopy. Moreover, we see that the fiber of the map $\mathrm{THH}(R_{f,2}; L_{R_{f,2}/R}) \to \mathrm{THH}(S; \operatorname{fib} L_{R/S} \to L_{R_{2,f}/S})$ is identified with the total fiber of the square

$$\begin{tikzcd}
\mathrm{THH}(R; R_{f,2}) \arrow[r] \arrow[d] & \mathrm{THH}(R; S) \arrow[d] \\
\mathrm{THH}(R_{f,2}) \arrow[r] & \mathrm{THH}(R_{f,2}; S)
\end{tikzcd}$$

which is the fiber of the map $\mathrm{THH}(R; \mathrm{fib}\, g) \to \mathrm{THH}(R_{f,2}; \mathrm{fib}\, g)$. By [Lev22, Lemma 3.2], since h is $2n$-connective and $\mathrm{fib}\, g$ is n-connective, we see that this map is $3n+1$-connective.

We next observe that in the right square of diagram (4), we know all maps except possibly the vertical map which we want to show is $2n+1$-connective. Indeed, $\mathrm{fib}\, \mathrm{TC}(h)$ is $2n+1$-connective by Lemma 4.7.11 and Proposition 4.7.2, the right vertical map is $2n+1$-connective by Proposition 4.7.9, and the lower horizontal map is $2n+1$-connective since the map $S \otimes_R S \to S \otimes_{R_{f,2}} S$ is $2n+1$-connective by [Lev22, Lemma 3.2]. It follows that the middle vertical map in diagram (4) is $2n$-connective. But since f is an arbitrary n-connective map and h is $2n$-connective, we learn that the left vertical map is $4n$-connective. It follows that the middle vertical map is $2n+1$-connective since it is an extension of a $2n+1$-connective map and a $4n$-connective map since $n \geq 1$. □

Remark 4.7.12. There is a version of Theorem 4.7.3 for a 0-connective map of connective rings, but one must ask that $\pi_0 R \to \pi_0 S$ has a nilpotent kernel. ◁

4.7.1 Applications to the sphere and the $K(1)$-local sphere

We now apply Theorem 4.7.3 to the map $\mathbb{S}_p \to \mathbb{Z}_p$ for $p \geq 2$ to understand the map $\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p)$ in the stable range. The proposition below contains a key ingredient of [BM93, Section 9] used to understand the homotopy type of $\mathrm{TC}(\mathbb{Z}_p)$.

Proposition 4.7.13. *For $p > 2$, the map $\pi_*\mathrm{TC}(\mathbb{S}_p) \to \pi_*\mathrm{TC}(\mathbb{Z}_p)$ in degrees $\leq 4p-6$ is an isomorphism in all degrees except $2p-1$, where it is the map $p\mathbb{Z}_p \to \mathbb{Z}_p$.*

Proof. By Theorem 4.1.4 we have a $4p-5$-connective map

$$\mathrm{fib}(\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p)) \to \mathrm{fib}(\mathrm{THH}(\mathbb{S}_p; \mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p))$$

The target of the map is $\mathrm{fib}(\mathbb{Z}_p \to \mathrm{THH}(\mathbb{Z}_p))$, which after applying $\tau_{\leq 4p-5}$ is $\Sigma^{2p-2}\mathbb{F}_p$. Thus it follows that there is a cofiber sequence

$$\Sigma^{2p-2}\mathbb{F}_p \to \tau_{\leq 4p-4}\mathrm{TC}(\mathbb{S}_p) \to \tau_{\leq 4p-4}\mathrm{TC}(\mathbb{Z}_p)$$

Recall that $\mathrm{TC}(\mathbb{S}_p) \cong \mathbb{S}_p \oplus \Sigma(\mathcal{C}\mathbb{P}^\infty_{-1})_p$ [BHM93][13], and that $\pi_*\mathrm{TC}(\mathbb{Z}_p)/(p, v_1)$ is $\mathbb{F}_p$ in odd degrees between -1 and $2p-1$, and in degrees $0, 2p-2$, and 0 in all other degrees [BM93][14]. From this description, it follows that both $\mathrm{TC}(\mathbb{S}_p)/(p, v_1)$ and $\mathrm{TC}(\mathbb{Z}_p)/(p, v_1)$ are $\mathbb{F}_p$ in degrees $2p-2, 2p-1$. Thus in the cofiber sequence above mod (p, v_1), the class in degree $2p-2$ must go to 0 and the class in degree $2p-1$ must go to the generator. It follows that integrally, the class must go to 0, and that it maps to the $\mathbb{Z}_p$ in $\mathrm{TC}_{2p-1}(\mathbb{S}_p)$ via the p-Bockstein, giving the conclusion. □

Corollary 4.7.14. *For $p > 2$, Proposition 4.7.13 also holds for j. In particular, the obstruction to lifting $\lambda_1 \in \mathrm{TC}(\mathbb{Z}_p)$ to $\mathrm{TC}(j)$ is up to a unit in $\mathbb{F}_p$ the class*

$$\sigma\alpha_1 \in \mathrm{THH}(\mathbb{Z}_p; L_{\mathbb{Z}_p/j})$$

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[13] see also [KN18, Theorem 8.4]

[14] This argument is not circular, because $\mathrm{TC}(\mathbb{Z}_p)/(p, v_1)$ is computed without knowing this proposition.

Proof. Since the map $\mathbb{S}_p \to j$ is $2p^2 - 2p - 2$-connective (see Corollary 4.3.11), the map $\mathbb{S}_p \to \mathbb{Z}_p$ agrees with the map $j_p \to \mathbb{Z}_p$ in the stable range, so the analysis in Proposition 4.7.13 applies for j. In particular, the obstruction to lifting the class $\lambda_1 \in \mathrm{TC}(\mathbb{Z}_p)$ to j is nonzero in $\mathrm{THH}(\mathbb{Z}_p; L_{\mathbb{Z}_p/j})$, so must be $\sigma\alpha_1$ up to a unit in $\mathbb{F}_p$, since $\pi_{2p-2}\mathrm{THH}(\mathbb{Z}_p; L_{\mathbb{Z}_p/j}) \cong \mathbb{F}_p$ is generated by this class. ☐

We now apply Theorem 4.7.3 to the map $j_\zeta \to \mathbb{Z}_\zeta$, and then make deductions about $K(L_{K(1)}\mathbb{S})$ in the stable range.

Lemma 4.7.15. *There is an isomorphism $\Sigma^{2p-2}\mathbb{F}_p \cong L_{\mathbb{Z}_\zeta/j_\zeta}$, where the generator is $\sigma(\alpha_1)$.*

Proof. In fact, we claim that $L_{\mathbb{Z}_\zeta^{\mathrm{gr}}/j_\zeta^{\mathrm{gr}}} \cong \Sigma^{2p-2,0}\mathbb{F}_p$ on the class $\sigma(\alpha_1)$ which implies the result, since this is the associated graded of $L_{\mathbb{Z}_\zeta/j_\zeta}$. To see this, we note that $L_{\mathbb{Z}_\zeta^{\mathrm{gr}}/j_\zeta^{\mathrm{gr}}}/p \cong L_{\mathbb{Z}_\zeta^{\mathrm{gr}}/p/j_\zeta^{\mathrm{gr}}/p}$. Since $j_\zeta^{\mathrm{gr}}/p \to \mathbb{Z}_\zeta/p$ is the augmentation of a polynomial algebra over the target on the class v_1, $L_{\mathbb{Z}_\zeta^{\mathrm{gr}}/p/j_\zeta^{\mathrm{gr}}/p} \cong \Sigma^{2p-1}\mathbb{Z}_\zeta^{\mathrm{gr}}/p$, where the generating class is $\sigma(v_1)$. In j_ζ^{gr}, there is a p-Bockstein differential $d_1 v_1 = v_1\zeta = \alpha_1$, so applying the map σ, we get that $\sigma(v_1)$ has a p-Bockstein d_1-differential hitting $\zeta\sigma(v_1) = \sigma(\alpha_1)$. Thus we can conclude. ☐

The following proposition gives a way in which $\mathrm{TC}(j_\zeta)$ does not behave as if the action on ℓ_p is trivial.

Proposition 4.7.16. *For $p > 2$, the image of $\lambda_1 \in \mathrm{TC}(\mathbb{Z}_p)/(p, v_1)$ in $\mathrm{TC}(\mathbb{Z}_\zeta)/(p, v_1)$ does not lift to $\mathrm{TC}(j_\zeta)/(p, v_1)$. The same statement is true for K-theory replacing* TC*.*

Proof. The result for K-theory is equivalent to the one for TC by [Lev22]. We have a commutative square of maps

$$\begin{tikzcd}
\mathrm{fib}(\mathrm{TC}(j) \to \mathrm{TC}(\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{fib}(\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})) \arrow[d] \\
\mathrm{THH}(\mathbb{Z}_p; L_{\mathbb{Z}_p/j}) \arrow[r] & \mathrm{THH}(\mathbb{Z}_\zeta; L_{\mathbb{Z}_\zeta/j_\zeta})
\end{tikzcd}$$

where the vertical maps are $4p - 5$-connective by Theorem 4.7.3.

The lower horizontal map sends $\sigma(\alpha_1)$ to $\sigma(\alpha_1)$, the generator of $\pi_{2p-2}\mathrm{THH}(\mathbb{Z}_\zeta; L_{\mathbb{Z}_\zeta/j_\zeta})$. But $\sigma(\alpha_1)$ since the class is the obstruction to lifting λ_1 from $\mathrm{TC}(\mathbb{Z}_p)$ to $\mathrm{TC}(\mathbb{S}_p)$, we learn that the obstruction to lifting λ_1 from $\mathrm{TC}(\mathbb{Z}_p^{B\mathbb{Z}})$ to $\mathrm{TC}(j_\zeta)$ is nontrivial. We also see that this obstruction is nonzero modulo (p, v_1). ☐

Theorem 4.7.17. *For $p > 2$, there are isomorphisms*

$$\tau_{\leq 4p-6}\,\mathrm{fib}(\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_\zeta)) \cong \Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)$$

and

$$K_* L_{K(1)}\mathbb{S} \cong K_{*-1}\mathbb{F}_p \oplus K_*\mathbb{S}_p \oplus \pi_*\Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)/\mathbb{F}_p, \quad * \leq 4p - 6$$

Proof. The map $f : j_\zeta \to \mathbb{Z}_\zeta$ is $2p-3$-connective, so we learn that

$$\operatorname{fib}\mathrm{TC}(f) \to \mathrm{THH}(\mathbb{Z}_\zeta; L_{\mathbb{Z}_\zeta/j_\zeta})$$

is $4p-5$-connective using Theorem 4.7.3. For the first statement, it suffices to show that $\tau_{\leq 4p-4}\mathrm{THH}(\mathbb{Z}_\zeta; L_{\mathbb{Z}_\zeta/j_\zeta}) \cong \Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)$. But using Corollary 4.4.8 and Lemma 4.7.15, we learn

$$\mathrm{THH}(\mathbb{Z}_\zeta; L_{\mathbb{Z}_\zeta/j_\zeta}) \cong \mathrm{THH}(\mathbb{Z}_\zeta; \Sigma^{2p-2}\mathbb{Z}_\zeta/p \otimes_{\mathbb{S}_p^{B\mathbb{Z}}} \mathbb{S}_p)$$
$$\cong \Sigma^{2p-2}\mathrm{THH}(\mathbb{Z}_\zeta)/p \otimes_{\mathbb{S}_p^{B\mathbb{Z}}} \mathbb{S}_p \cong \Sigma^{2p-2}\mathrm{THH}(\mathbb{Z}_p)/p \otimes_{\mathbb{F}_p} C^0(\mathbb{Z}_p; \mathbb{F}_p)$$

Since $\pi_*\mathrm{THH}(\mathbb{Z}_p)/p$ is by Example 4.4.2 $\mathbb{F}_p[\sigma^2\alpha_1, \sigma^2 v_1]$, we indeed learn the claim.

To get the statement about K-theory, by [Lev22], we have $K_*(L_{K(1)}\mathbb{S}) \cong K_*(j_\zeta) \oplus K_{*-1}(\mathbb{F}_p)$ and a map of cofiber sequences

$$\begin{tikzcd}
\operatorname{fib}(\mathrm{TC}(\mathbb{S}_p) \to \mathrm{TC}(\mathbb{Z}_p)) \arrow[r] \arrow[d] & K(\mathbb{S}_p) \arrow[r] \arrow[d] & K(\mathbb{Z}_p) \arrow[d] \\
\operatorname{fib}(\mathrm{TC}(j_\zeta) \to \mathrm{TC}(\mathbb{Z}_\zeta)) \arrow[r] & K(j_\zeta) \arrow[r] & K(\mathbb{Z}_p)
\end{tikzcd}$$

where the fiber terms are 0 after inverting p. The third terms after p-completion are $K_{2p-1}(\mathbb{Z}_p)_p \cong \mathbb{Z}_p$, generated by λ_1.

We can split $\pi_*\Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)$ into $\pi_*\Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)/\mathbb{F}_p \oplus \mathbb{F}_p$ via the augmentation to $\mathbb{F}_p$ coming from evaluation at 0. Note that the image of the boundary map of the cofiber sequence on mod p homotopy groups is the $\mathbb{F}_p$-summand by Proposition 4.7.16. It follows that after quotienting out by $\Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)/\mathbb{F}_p$ in the homotopy groups of the first two terms in the lower cofiber sequence, it agrees with that of the upper cofiber sequence in degrees $\leq 4p-6$.

It follows that there is a short exact sequence for $* \leq 4p-6$:

$$0 \to \pi_*\Sigma^{2p-2}C^0(\mathbb{Z}_p; \mathbb{F}_p)/\mathbb{F}_p \to K_*(j_\zeta) \to K_*(\mathbb{S}_p) \to 0$$

But the map $K(\mathbb{S}_p) \to K(j_\zeta)$ clearly splits this sequence, giving the result. □

4.8 The Segal conjecture

The Segal conjecture for a cyclotomic spectrum X is the statement that the cyclotomic Frobenius map $X \to X^{tC_p}$ is an isomorphism in large degrees. Knowing the Segal conjecture for $\mathrm{THH}(R)\otimes V$ where V is a finite spectrum is a key step in proving the Lichtenbaum–Quillen conjecture for X, i.e the fact that $\mathrm{TR}(X)$ (and hence $\mathrm{TC}(X)$) is bounded (see [HW22]).

Asking that the Segal conjecture hold for $\mathrm{THH}(R) \otimes V$ is a regularity and finiteness condition on R: for example it holds when V is p-torsion and R is a p-torsion free excellent regular noetherian ring with the Frobenius on R/p a finite map [Mat21, Corollary 1.5]. In this section, we show that the Segal conjecture does hold for j_ζ for $p > 2$ as well as the extensions $j_{\zeta,k}$, but doesn't hold for the connective covers j and j_k. In particular the Lichtenbaum–Quillen conjecture doesn't hold for j_k, and our result is used in [BHLS23] to show that it does hold for $j_{\zeta,k}$ for $p > 2$.

A related regularity phenomenon was noted in [Lev22], namely that j_ζ is regular[15] at the height 2-locus: i.e the t-structure on $\mathrm{Mod}(j_\zeta)$ restricts to a bounded t-structure on $\mathrm{Mod}(j_\zeta)^\omega \otimes \mathrm{Sp}_{\geq 2}$. This t-structure is the key point in relating j_ζ's algebraic K-theory to that of the $K(1)$-local sphere. On the other hand, j is not regular at the height 2-locus which is why its integral K-theory is not closely related to that of the $K(1)$-local sphere.

Our first goal is to show that for odd p, $j_{\zeta,k}$ satisfies the Segal conjecture. A key input is the following proposition, the proof of which is the same as in the reference, though the statement is somewhat more general.

Proposition 4.8.1. *[HW22, Proposition 4.2.2] Let R be an $\mathbb{E}_1$-ring, and consider the $\mathbb{Z}^m$-graded polynomial algebra $R[a_1,\dots,a_n] := R \otimes \bigotimes_1^n \mathbb{S}[a_i]$, where each a_i has positive weight*[16] *and is even topological degree and $\mathbb{S}[a_i]$ is the free $\mathbb{E}_1$-algebra. The map*

$$\varphi : L_p \operatorname{THH}(R[a_1,\dots,a_n]) \to \operatorname{THH}(R[a_1,\dots,a_n])^{tC_p}$$

at the level of π_* is equivalent to the map

$$\pi_* \operatorname{THH}(R)[a_i] \otimes \Lambda[da_i] \to \pi_* \operatorname{THH}(R)^{tC_p}[a_i] \otimes \Lambda[da_i]$$

where the a_i, da_i are sent to themselves. If R is an $\mathbb{E}_2$-algebra and $\mathbb{S}[a_i]$ are given the $\mathbb{E}_2$-algebra structures coming from [Lur15], this is a homomorphism of rings.

The following lemma is used to reduce showing the Segal conjecture is true to the associated graded of a filtration on the ring.

Lemma 4.8.2. *Let C be a presentably symmetric monoidal stable category with a complete t-structure compatible with filtered colimits, and suppose that $f : R^{\mathrm{fil}} \to R'^{\mathrm{fil}}$ is a map of homotopy associative filtered rings in C, where the filtration on the source and target is complete.*

If there is an element $x \in \pi_* R := \pi_* \mathrm{map}(\mathbb{1}, R), * > 0$ such that the associated graded map $R^{\mathrm{gr}} \to R'^{\mathrm{gr}}$ is n-coconnective in the constant t-structure and sends a class detecting x to a unit, then the map $R \to R'$ is also n-coconnective, and is equivalent to the map

$$R \to R[x^{-1}]$$

Proof. First, since the filtrations are complete and the map f is n-coconnective on associated graded, we learn that the fiber is n-coconnective on associated graded, and complete, so the underlying object is n-coconnective.

Let $\tilde{x}$ be an element in $\pi_{**} R^{\mathrm{fil}}$ whose underlying element is x that is sent to a unit in R'^{gr}. Since the filtration on R' is complete, it follows that $\tilde{x}$ is sent to a unit, which allows us to build a map $R^{\mathrm{fil}}[\tilde{x}^{-1}] \to R'^{\mathrm{fil}}$ via the colimit of the diagram

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[15] See [BL24] for a discussion of regularity in the setting of prestable ∞-categories.

[16] i.e it is nonnegative weight in each copy of $\mathbb{Z}$ in $\mathbb{Z}^m$, and positive weight in some copy of $\mathbb{Z}$.

$$\begin{tikzcd}
\Sigma^{|x|} R^{\mathrm{fil}} \arrow[r] \arrow[d, "x"] & \Sigma^{|x|} R'^{\,\mathrm{fil}} \arrow[d, "x"] \\
R^{\mathrm{fil}} \arrow[r] & R'^{\,\mathrm{fil}} \\
\vdots & \vdots
\end{tikzcd}$$

Note that the horizontal maps become more and more coconnective and the right vertical maps are all equivalences. Then because the t-structure is complete and compatible with filtered colimits, we learn that in the colimit the map is an equivalence. We also learn that the filtration on $R^{\mathrm{fil}}[x^{-1}]$ is complete, allowing us to conclude. □

Before proceeding to prove the Segal conjecture, we recall as in [HW22, Section C.5] that given a filtered $\mathbb{Z}^m$-graded $\mathbb{E}_1$-ring R^{fil}, the cyclotomic Frobenius map refines to a filtered map

$$\varphi : L_p \operatorname{THH}(R^{\mathrm{fil}}) \to \operatorname{THH}(R^{\mathrm{fil}})^{tC_p}$$

where L_p is the operation on filtered spectra scaling the filtration and the gradings on R by p.

Theorem 4.8.3 (Segal conjecture for $j_{\zeta,k}$)**.** *For $p > 2$ and $k \geq 0$, the map*

$$\operatorname{THH}(j_{\zeta,k})/(p, v_1) \to \operatorname{THH}(j_{\zeta,k})^{tC_p}/(p, v_1)$$

has $2p-3$-coconnective fiber, and is equivalent to the map

$$\operatorname{THH}(j_{\zeta,k})/(p, v_1) \to \operatorname{THH}(j_{\zeta,k})[\mu^{-1}]/(p, v_1)$$

where $\mu \in \pi_{2p^2} \operatorname{THH}(j_{\zeta,k})$.

Proof. Using the filtration on $j_{\zeta,k}$ constructed in Section 4.2, we get a filtered map

$$\varphi : L_p \operatorname{THH}(j_{\zeta,k})/(p, \tilde{v_1}) \to \operatorname{THH}(j_{\zeta,k})^{tC_p}/(p, \varphi\tilde{v_1})$$

By the proof of Theorem 4.4.11 and Theorem 4.6.1, the class μ is detected in the spectral sequence for $\operatorname{THH}(j_{\zeta,k})/(p, v_1)$ by $(\sigma^2 v_1)^p$. Thus by applying Lemma 4.8.2 for $C = \mathrm{Sp}$ and $R^{\mathrm{fil}} \to R'^{\,\mathrm{fil}}$ the maps in question, it suffices to show

(a) The filtration on the source and target are complete.

(b) The associated graded map inverts the class $\sigma^2 v_1$ and is $2p-3$-coconnective.

To see (a), the source is complete by Lemma 4.2.18. The Tate construction $(-)^{tC_p}$ sits in a cofiber sequence up to shifts between the orbits $(-)_{hC_p}$ and fixed points $(-)^{hC_p}$, so it suffices to show each of those is complete. The orbits are complete for connectivity reasons: in any finite range of degrees, the orbits are computed via a finite colimit. The fixed points are complete because complete objects are closed under limits.

We turn to proving (b). We further filter $j^{\mathrm{gr}}_{\zeta,k}$ by the p-adic filtration as $j^{\mathrm{gr}}_{\zeta,k} \otimes \mathbb{Z}_p^{\mathrm{fil}}$ and consider the map of filtered graded $\mathbb{E}_\infty$-rings $L_p \operatorname{THH}(j_{\zeta,k})/(\tilde{p}, v_1) \to \operatorname{THH}(j_{\zeta,k})^{tC_p}/(\varphi\tilde{p}, \varphi v_1)$. We claim:

(i) The filtration on the source and target are complete.

(ii) The associated graded map inverts the class $\sigma^2 p$ and is $2p-3$-coconnective.

Given these claims, the proof is complete, since $\sigma^2 v_1$ is detected in the spectral sequence by $(\sigma^2 p)^p$ (see Example 4.4.2), so claim (b) follows from Lemma 4.8.2.

(i) follows from an argument identical to the argument for (a), the only difference being that we use Lemma 4.2.19 to see that the filtration on $\mathrm{THH}(j_\zeta^{\mathrm{gr}} \otimes \mathbb{Z}_p^{\mathrm{fil}})/(\tilde{p}, v_1)$ is complete. To see (ii), by Lemma 4.2.16 the associated graded algebra is $\mathbb{F}_p[v_0, v_1]^{B\mathbb{Z}}$, where the action is trivial. By Lemma 4.4.5 we have $\pi_* \mathrm{THH}(\mathbb{F}_p[v_0,v_1]^{B\mathbb{Z}})/(v_0,v_1) \cong C^0(\mathbb{Z}_p;\mathbb{F}_p) \otimes \Lambda[dv_0, dv_1, \zeta] \otimes \mathbb{F}_p[\sigma^2 p]$, where $|dv_0| = 1, |\zeta| = -1, |dv_1| = 2p-1$. It follows that if the Frobenius map mod (v_0, v_1) inverts $\sigma^2 p$, it is $2p-3$-coconnective, since it is injective on π_*, and an element in the cokernel of largest degree is $(\sigma^2 p)^{-1} \sigma v_1 \sigma v_0$, which is in degree $2p-2$.

Thus it remains to see that the Frobenius map mod (v_0, v_1) on π_* inverts the class $\sigma^2 p$. Since THH is a localizing invariant and $\mathbb{S}^{B\mathbb{Z}}$ is a trivial square-zero extension as an $\mathbb{E}_1$-algebra, by [LT23, Theorem 4.1] we have a pullback square of bigraded $\mathrm{THH}(\mathbb{F}_p)$-modules in cyclotomic spectra

$$\begin{tikzcd}
\mathrm{THH}(\mathbb{F}_p[v_0,v_1]^{B\mathbb{Z}}) \arrow[r] \arrow[d] & \mathrm{THH}(\mathbb{F}_p[v_0,v_1]) \arrow[d] \\
\mathrm{THH}(\mathbb{F}_p[v_0,v_1]) \arrow[r] & \mathrm{THH}(\mathbb{F}_p[v_0,v_1][x_0])
\end{tikzcd}$$

where x_0 is a polynomial generator in degree 0. It thus suffices to show that for

$$\mathrm{THH}(\mathbb{F}_p[v_0,v_1][x_0]), \mathrm{THH}(\mathbb{F}_p[v_0,v_1])$$

the cyclotomic Frobenius map inverts $\sigma^2 p$. These statements follow from Proposition 4.8.1 with $R = \mathbb{F}_p, \mathbb{F}_p[x_0]$, using the Segal conjecture for these discrete rings which is well known: for example [Mat21, Corollary 1.5] implies the Frobenius is an isomorphism in large degrees, but since it sends $\sigma^2 p$ to a unit [NS18, Corollary IV.4.13], it must just invert $\sigma^2 p$. ☐

Remark 4.8.4. The bound $2p-3$ in Theorem 4.8.3 is optimal: the map is injective on π_*, and a class of largest degree not in the image is $\mu^{-1}\lambda_1\lambda_2$, in degree $2p-2$, ◁

Now we show that the Segal conjecture fails for $\mathrm{THH}(j_k)$.

Theorem 4.8.5. *For $p > 2$ and $k \geq 0$, the fiber of the Frobenius map $\mathrm{THH}(j_k)/(p, v_1) \to \mathrm{THH}(j_k)^{tC_p}/(p, v_1)$ is not bounded above. Thus j_k does not satisfy the Lichtenbaum–Quillen conjecture, i.e $\mathrm{TR}(j_k) \otimes V$ is not bounded above for V a finite type 3 spectrum.*

Proof. First we note that the failure of the Segal conjecture implies the failure of the Lichtenbaum–Quillen conjecture by [AN21, Proposition 2.25], so we show that the Segal conjecture fails.

We first show that $\mu \in \mathrm{THH}(j_k)/(p, v_1)$ is sent to a unit in $\mathrm{THH}(j_k)^{tC_p}/(p, v_1)$. It follows from the spectral sequences used to calculate $\mathrm{THH}(j_k)/(p, v_1)$ that the image of μ in $\mathrm{THH}(\mathbb{F}_p)$ is $(\sigma^2 p)^{p^2}$ up to a unit, which is sent under the Frobenius map to a class detected up to a unit by t^{-p^2} in the Tate spectral sequence for $\mathrm{THH}(\mathbb{F}_p)^{tC_p}/(p, v_1)$ by [NS18, Corollary IV.4.13]. This is the lowest filtration of the Tate spectral sequence in π_{2p^2}, so since in that filtration the map $\mathrm{THH}(j_k)/(p, v_1) \to \mathrm{THH}(\mathbb{F}_p)/(p, v_1)$ is the map $\mathbb{Z}_p \to \mathbb{F}_p$, we learn that the image of μ must be detected by a unit multiple of t^{-p^2} in the Tate spectral sequence for $\mathrm{THH}(j_k)^{tC_p}$ and hence be a unit.

If the Frobenius map has an element x in the kernel, then $x\mu^i$ is also in the kernel for each i, so the fiber isn't bounded above. On the other hand, if the Frobenius map is injective, then the classes $\varphi(\mu)^{-1}\varphi((\sigma\alpha_{1/p^k})^{(pi)})$ are an infinite family of classes of increasing degree in $\mathrm{THH}(j_k)^{tC_p}$ that are not in the image of φ, so in this case too, we learn that the fiber is not bounded above. □

Remark 4.8.6. In fact, $\pi_* \mathrm{THH}(j_k)^{tC_p}/(p, v_1)$ under the Frobenius map is the completion of $\pi_* \mathrm{THH}(j_k)[\mu^{-1}]/(p, v_1)$ at the ideal generated by $(\sigma\alpha_{1/p^k}^{(pi)})$ for each i, and the map is in particular injective on π_*. ◁

Chapter 5

K-theoretic counterexamples to Ravenel's telescope conjecture (with Robert Burklund, Jeremy Hahn, Tomer Schlank)

At each prime p and height $n+1 \geq 2$, we prove that the telescopic and chromatic localizations of spectra differ. Specifically, for $\mathbb{Z}$ acting by Adams operations on $\mathrm{BP}\langle n\rangle$, we prove that the $T(n+1)$-localized algebraic K-theory of $\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}$ is not $K(n+1)$-local. We also show that Galois hyperdescent, $\mathbb{A}^1$-invariance, and nil-invariance fail for the $K(n+1)$-localized algebraic K-theory of $K(n)$-local $\mathbb{E}_\infty$-rings. In the case $n = 1$ and $p \geq 7$ we make complete computations of $T(2)_*\mathrm{K}(R)$, for R certain finite Galois extensions of the $K(1)$-local sphere. We show for $p \geq 5$ that the algebraic K-theory of the $K(1)$-local sphere is asymptotically L_2^f-local.

Contents

5.1 Introduction

Chromatic homotopy theory can be described as a surprising and intimate relationship between stable homotopy theory and the theory of 1-dimensional commutative formal groups. Morava and Ravenel laid out a vision for this relationship which can be summarized as giving a natural bijection between the moduli stack of formal groups and the "primes" in stable homotopy theory.

The key ideas of this point of view were summarized by the Ravenel conjectures in [Rav84], most of which were proven by Devinatz–Hopkins–Smith in [DHS88] and [HS98]. The main conjecture from [Rav84] that remained unresolved was the telescope conjecture, which suggests that two natural competing definitions for monochromatic spectra agree.

More precisely, let $K(n+1)$[1] denote height $n+1$ Morava K-theory, and let U be a finite p-local spectrum of type $n+1$. By [HS98], there is a self map $v\colon \Sigma^d U \to U$ for $d > 0$ inducing an isomorphism after tensoring with $K(n+1)$. We define $T(n+1) \coloneqq U[v^{-1}]$, and there is a fundamental inclusion of localized categories

$$\mathrm{Sp}_{K(n+1)} \subseteq \mathrm{Sp}_{T(n+1)}.$$

The category $\mathrm{Sp}_{T(n+1)}$ is of interest because it detects the v_{n+1}-periodic part of the stable homotopy groups of spheres. On the other hand, computations in $\mathrm{Sp}_{K(n+1)}$ are a priori much more tractable, being closely connected to the algebraic cohomology of the moduli of formal groups.

The telescope conjecture postulates that the inclusion between these two categories is in fact an equality, and as such was originally favored by Occam's razor. For $n+1=0$, the conjecture is trivial. For $n+1=1$, the telescope conjecture was proved by Mahowald at $p=2$ [Mah81], using *bo*-resolutions, and by Miller [Mil81] for $p>2$, using a localized Adams

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[1] The reader might question why we use $n+1$ rather than n. Indeed, when stating the telescope conjecture, this may seem somewhat unorthodox. However, as suggested by the paper's title, we employ algebraic K-theory to develop counterexamples to the telescope conjecture. Taking into account the redshift phenomena of [Rog14], a significant portion of our analysis pertains to objects with a height that is one less than that for which we are disproving the telescope conjecture. Thus, the inclusion of the "+1."

spectral sequence. Both proofs proceed by explicit computation of the homotopy groups on the telescopic side.

Parts of the analogous computations for heights $n+1 \geq 2$ led Ravenel to believe that the telescope conjecture is in fact *false* in these cases [Rav92]. Subsequent work, based on a certain family of Thom spectra $y(n+1)$, outlined a general strategy for a disproof, suggesting a concrete description for the gap between the telescopic and $K(n+1)$-local sides [Rav95]. Variations on this strategy have been considered by experts in the field [BBBCX19; RBBBCX17], but have not resulted in a disproof. The main result of this paper is a disproof of the telescope conjecture, at all primes p and at all heights $n+1 \geq 2$.

To do this, we construct for each height n and prime p a family of associative ring spectra $\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}$, indexed by integers $k \geq 0$, obtained by taking fixed points of Adams operations on the truncated Brown–Peterson spectrum $\mathrm{BP}\langle n\rangle$. We then show:

Theorem A. *Let p be any prime and $n+1 \geq 2$. Then, for all $k \geq 0$,*

$$L_{T(n+1)}\mathrm{K}\left(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}\right)$$

is not $K(n+1)$-local. In particular,

$$\mathrm{Sp}_{K(n+1)} \neq \mathrm{Sp}_{T(n+1)}.$$

If Theorem A is proved for a particular $k = k_1$, it automatically holds for all $0 \leq k \leq k_1$. Thus, it will suffice to prove Theorem A for $k \gg 0$, and we will often assume $k \gg 0$ below.

5.1.1 Cyclotomic hyperdescent

Before explaining how we prove Theorem A, we indicate what our proof says about *how* the telescope conjecture fails.

A key tool in studying $\mathrm{Sp}_{K(n+1)}$ is the fact, due to Devinatz–Hopkins [DH04], that there is an equivalence $\mathbb{S}_{K(n+1)} \cong E_{n+1}(\overline{\mathbb{F}}_p)^{h\mathbb{G}_{n+1}}$. Here $E_{n+1}(\overline{\mathbb{F}}_p)$ is a Lubin–Tate spectrum attached to $\overline{\mathbb{F}}_p$, and $\mathbb{G}_{n+1}$ is the extended Morava stabilizer group [GH04]. Since $E_{n+1}(\overline{\mathbb{F}}_p)$ is well understood at the level of homotopy rings, this gives an approach to studying the $K(n+1)$-local sphere via a homotopy fixed point spectral sequence.

Using the Galois theory of Rognes [Rog08], $E_{n+1}(\overline{\mathbb{F}}_p)$ can be interpreted as the algebraic closure (see [BR08]) of $L_{K(n+1)}\mathbb{S}$, with Galois group $\mathbb{G}_{n+1}$, so that the Devinatz–Hopkins result allows the $K(n+1)$-local category to be studied via Galois descent. From this perspective, two reasons we have lacked computational control over the $T(n+1)$-local category are:

1. We know few explicit Galois extensions of $\mathbb{S}_{T(n+1)}$.
2. We do not know that descent is valid for the infinite Galois extensions that exist.

In [CSY21], Carmeli, Yanovski and the fourth author show that all of the *abelian* Galois extensions of $\mathbb{S}_{K(n+1)}$ may be lifted to $\mathbb{S}_{T(n+1)}$, by constructing them as "chromatic cyclotomic extensions." The most interesting of these extensions are the p-cyclotomic extensions

$$\mathbb{S}_{T(n+1)} \to \mathbb{S}_{T(n+1)}[\omega_{p^k}^{(n+1)}],$$

which have Galois group $(\mathbb{Z}/p^k)^\times$ and are concretely realized as summands of $T(n+1)$-localized suspended Eilenberg–MacLane spaces $L_{T(n+1)}\Sigma^\infty_+ \mathrm{K}(\mathbb{Z}/p^k, n+1)$. These telescopic p-cyclotomic extensions generalize the classical p-cyclotomic extensions $\mathbb{Q} \to \mathbb{Q}[\zeta_{p^k}]$ from the case $n+1=0$. The filtered colimit of the p-cyclotomic extensions, denoted $\mathbb{S}_{T(n+1)}[\omega_{p^\infty}^{(n+1)}]$, is a $\mathbb{Z}_p^\times$-pro-Galois extension, and, in the case $n+1=1$, is the extension $L_{T(1)}\mathbb{S} \to \mathrm{KU}_p$.

Though the maps

$$\mathbb{S}_{T(n+1)} \to \mathbb{S}_{T(n+1)}[\omega_{p^k}^{(n+1)}]^{h(\mathbb{Z}/p^k)^\times}$$

are equivalences for each $k \geq 0$, this *does not* guarantee that the map

$$\mathbb{S}_{T(n+1)} \to \mathbb{S}_{T(n+1)}[\omega_{p^\infty}^{(n+1)}]^{h\mathbb{Z}_p^\times}$$

is an equivalence.[2] Put differently, localization with respect to $\mathbb{S}_{T(n+1)}[\omega_{p^\infty}^{(n+1)}]$ yields a third category of *cyclotomically complete* $T(n+1)$-*local spectra*:

$$\mathrm{Sp}_{K(n+1)} \subseteq (\mathrm{Sp}_{T(n+1)})^\wedge_{\mathrm{cyc}} \subseteq \mathrm{Sp}_{T(n+1)},$$

where both inclusions are potentially strict (see [BCSY22, Question 7.36]).

Our proof of Theorem A actually shows that the second inclusion is strict: i.e., we prove that $L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ is not cyclotomically complete for $n \geq 1$ (see Theorem 5.6.25).

5.1.2 The proof

The first ingredient in our proof is the cyclotomic redshift result [BMCSY23] of Ben-Moshe, Carmeli, Yanovski and the fourth author. Cyclotomic redshift states that chromatic cyclotomic extensions are compatible with algebraic K-theory in the sense that, for any $T(n)$-local $\mathbb{E}_1$-ring R, there are natural $(\mathbb{Z}/p^i)^\times$-equivariant equivalences:

$$L_{T(n+1)}K(R[\omega_{p^i}^{(n)}]) \cong L_{T(n+1)}K(R)[\omega_{p^i}^{(n+1)}].$$

The $n=0$ case of this theorem is due to Bhatt–Clausen–Mathew [BCM20].

Using cyclotomic redshift, and the fact that the $(p^k\mathbb{Z}_p)$-pro-Galois extension

$$L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}} \to L_{T(n)}\mathrm{BP}\langle n\rangle$$

---

[2] Asking that this map be an equivalence is equivalent to asking that $\mathbb{S}_{T(n+1)}[\omega_{p^\infty}^{(n+1)}]$ be a faithful module over $\mathbb{S}_{T(n+1)}$. It is also equivalent to asking that the sheaf of finite $\mathbb{Z}_p^\times$-sets defined by the finite Galois sub-extensions be a hypersheaf [BMCSY23, **page** 6.2].

is closely related to a cyclotomic extension, we deduce that there is an equivalence

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \cong L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})^{\wedge}_{\mathrm{cyc}}$$

for each $k \geq 0$. Thus, in order to prove Theorem A, it suffices to show that the map

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}$$

is not an equivalence for $k \gg 0$. We refer to this map as the K-theoretic coassembly map, as it measures the failure of $T(n+1)$-local K-theory to commute with the limit $(-)^{hp^k\mathbb{Z}}$.

Our way of accessing these $T(n+1)$-local K-theory spectra is via trace methods. It follows from the landmark works [CMNN23; LMMT20; DGM13; Mit90] that, for any $\mathbb{E}_1$-ring R and $n+1 \geq 2$, there is a natural equivalence

$$L_{T(n+1)}K(R) \cong L_{T(n+1)}\mathrm{TC}(\tau_{\geq 0}R).$$

In principle, we may therefore replace the K-theoretic coassembly map with the map

$$L_{T(n+1)}\mathrm{TC}(\tau_{\geq 0}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})) \to L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}.$$

However, this is not the replacement for the K-theoretic coassembly map that we use, because the rings $\tau_{\geq 0}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ lack regularity properties that make TC easy to access.[3] Instead we use a variant of the Dundas–Goodwillie–McCarthy theorem, due to the third author [Lev22], that applies to (-1)-connective rings. This allows us to replace the K-theoretic coassembly map with the TC coassembly map

$$L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}},$$

reducing Theorem A to the claim that this TC coassembly map is not an isomorphism when $k \gg 0$.

The key to analyzing the TC coassembly map is the following result, which allows us to replace the $\mathbb{Z}$-action by Adams operations on $\mathrm{BP}\langle n\rangle$ with the *trivial* action:

Theorem B (Asymptotic constancy for $\mathrm{BP}\langle n\rangle$)**.** *Fix a telescope $T(n+1)$ of a type $n+1$ p-local finite spectrum. Then for all $k \gg 0$ there is a commuting square*

$$\begin{tikzcd}
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}) \arrow[r] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{B\mathbb{Z}},
\end{tikzcd}$$

where the horizontal maps are TC *coassembly maps.*

---

[3] Using language introduced later in this introduction, they do not satisfy the height n Lichtenbaum–Quillen property. See [LL23] for a discussion and proof of this in the case $n = 1$.

Because of Theorem B, it suffices to show that the TC coassembly map is not an isomorphism for the trivial $\mathbb{Z}$-action on $\mathrm{BP}\langle n\rangle$ (whose homotopy fixed points we write as $\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}$ to emphasize the triviality of the action). To do this we use the following general fact:

Proposition 5.1.1. *For any p-complete $\mathbb{E}_1$-ring R, the p-completion of $\mathrm{TC}(R)$ is in the thick subcategory generated by the p-completion of the fiber of the coassembly map $\mathrm{TC}(R^{B\mathbb{Z}}) \to \mathrm{TC}(R)^{B\mathbb{Z}}$.*

This proposition is proven by analyzing the universal case $R = \mathbb{S}$, and showing after p-completion that $\mathrm{THH}(\mathbb{S})$ is in the thick subcategory generated by the fiber of

$$\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$$

in the category CycSp of cyclotomic spectra.[4] The general case is obtained by tensoring with $\mathrm{THH}(R)$ in CycSp and applying the p-completed TC functor. The phenomenon in Proposition 5.1.1 is closely related to the failure of hyperdescent, nil-invariance, and $\mathbb{A}^1$-invariance in $K(n+1)$-local K-theory of $T(n)$-local rings for $n \geq 1$ (see also Theorem 5.3.22).

Applying Proposition 5.1.1, and tensoring with $T(n+1)$, we learn that the TC coassembly map is not an equivalence so long as $T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle) \neq 0$. This follows from the redshift result of the second author and Wilson [HW22], allowing us to finish the proof of Theorem A.

Remark 5.1.2. Let U denote a type $n+1$ finite complex. To build intuition for Theorem B, the reader might first contemplate the simpler statement that, when $k \gg 0$, the $p^k\mathbb{Z}$-action on $U \otimes \mathrm{BP}\langle n\rangle$ is trivial. This follows from the fact that $U \otimes \mathrm{BP}\langle n\rangle$ is π-finite and has a unipotent $\mathbb{Z}$-action on its homotopy groups.

In [LL23], Lee and the third author computed that, for all $k \geq 0$, the tensor product $\mathbb{S}/(p, v_1) \otimes \mathrm{THH}(\mathrm{BP}\langle 1\rangle^{hp^k\mathbb{Z}})$ has the same homotopy ring as the tensor product $\mathbb{S}/(p, v_1) \otimes \mathrm{THH}(\mathrm{BP}\langle 1\rangle^{\mathrm{B}\mathbb{Z}})$. This is another simpler analog of Theorem B, and at its core rests on the triviality of the $\mathbb{Z}$-action on $\mathrm{BP}\langle 1\rangle/(p, v_1)$. ◁

5.1.3 The Lichtenbaum–Quillen property

The key finiteness property of $\mathrm{BP}\langle n\rangle$ used to prove Theorem B is the *height n Lichtenbaum–Quillen property*:

Definition. We say that an $\mathbb{E}_1$-ring spectrum R satisfies the height n LQ property if $\mathrm{THH}(R)$ is bounded below, and, for any p-local finite type $n+2$ complex V, $V \otimes \mathrm{TR}(R)$ is bounded.

Equivalently, R satisfies the height n LQ property if $V \otimes \mathrm{THH}(R)$ is bounded in the t-structure on cyclotomic spectra of Antieau–Nikolaus [AN21].

The height 0 LQ property has substantial history, being closely related to the Lichtenbaum–Quillen conjectures describing K-theory of discrete rings in terms of étale cohomology

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[4] We remind the reader that the use of the word 'cyclotomic' differs when referring to cyclotomic extensions and cyclotomic spectra.

[HM03; HM04; Mat21]. Ausoni and Rognes proved the height 1 LQ property for $\mathrm{BP}\langle 1\rangle$ at primes $p \geq 5$ [AR02], and had the further vision to highlight height n LQ properties as central to their redshift philosophy.

Work of the second author and Wilson [HW22] gave additional techniques for proving LQ properties, which with Raksit were connected to the Bhatt–Morrow–Scholze interpretation of prismatic cohomology [HRW22; BMS19; Pst23]. Using these techniques, the height n LQ property for $\mathrm{BP}\langle n\rangle$ was proved for all n and p [HW22].

Here, we establish a tool for descending LQ properties through fixed points by unipotent $\mathbb{Z}$-actions. Our proof of Theorem B comes from setting $R = \mathrm{BP}\langle n\rangle$ in the following theorem:

Theorem C (Cyclotomic asymptotic constancy)**.** *Suppose that R is a connective p-complete $\mathbb{E}_1 \otimes \mathbb{A}_2$[5]-ring of fp-type n,[6] equipped with a locally unipotent $\mathbb{Z}$-action,[7] and let V be a finite p-local spectrum of type $n+2$.*

If R satisfies the height n LQ property, then $R^{hp^k\mathbb{Z}}$ satisfies the height n LQ property for all $k \gg 0$. Furthermore, for $k \gg 0$ there is a commutative diagram of cyclotomic spectra

$$\begin{tikzcd}
V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & V \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
V \otimes \mathrm{THH}(R^{B\mathbb{Z}}) \arrow[r] & V \otimes \mathrm{THH}(R)^{B\mathbb{Z}},
\end{tikzcd}$$

where the horizontal maps are the coassembly maps.

Since the above is a diagram of cyclotomic spectra, we also obtain a corresponding square after replacing THH with TC.

The height n LQ property for an $\mathbb{E}_1$-ring R implies, for any type $n+1$ spectrum U with v_{n+1}-self map v, that the map

$$U \otimes \mathrm{TC}(R) \to U[v^{-1}] \otimes \mathrm{TC}(R) = T(n+1) \otimes \mathrm{TC}(R)$$

has bounded above fiber. In short, knowledge of $U_*\mathrm{TC}(R)$ through a finite range of degrees is enough to completely determine the periodic homotopy groups $T(n+1)_*\mathrm{TC}(R)$. This property is exactly the opposite of what makes the homotopy groups of $T(n+1)$ so inaccessible: for $n+1 \geq 2$, there is no bounded range of degrees in which all classes in $\pi_*T(n+1)$ lift to π_*U.[8]

To exemplify the approachability of telescopic homotopy in the presence of LQ properties, Ausoni and Rognes were able to completely calculate $T(2)_*\mathrm{TC}(\mathrm{BP}\langle 1\rangle)$ for primes $p \geq 5$ and $T(2) = v_2^{-1}\mathbb{S}/(p, v_1)$ [AR02]. However, it was only recently in [HRW22] that $T(2) \otimes \mathrm{TC}(\mathrm{BP}\langle 1\rangle)$ was proved to be $K(2)$-local.

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[5] An $\mathbb{E}_1 \otimes \mathbb{A}_2$-ring is a unital algebra in the category of $\mathbb{E}_1$-rings.

[6] A connective p-complete spectrum X is fp-type n if, for any type $n+1$ finite complex U, $U \otimes X$ is π-finite [MR99].

[7] This is the same as asking that the $\mathbb{Z}$-action on π_*R/p be locally unipotent.

[8] This follows from (2) in the forthcoming work section below, along with Serre's finiteness theorem

5.1.4 Height 2 and the K(1)-local sphere

Though we need not calculate much about $\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ to prove Theorem A, it is sometimes possible to make complete calculations. We do so for $n = 1, p \geq 7$, and $k \gg 0$ in Section 5.7. By restricting to primes $p \geq 7$, we are able to fix as a preferred type 2 spectrum a homotopy commutative and associative Smith–Toda complex $V(1) = \mathbb{S}/(p, v_1)$, with corresponding telescope $T(2) = v_2^{-1}V(1)$.

We study the connective Adams summand ℓ of $\mathrm{ku}_{(p)}$, which is a form of $\mathrm{BP}\langle 1\rangle$. There is a $\mathbb{Z}$ action on ℓ by classical Adams operations, such that the p-completion of $\ell^{h\mathbb{Z}}$ is the (-1)-connective cover of the $K(1)$-local sphere.

Trace theorems of the third author [Lev22, Theorem B], along with devissage results of the first and third author [BL23], were used in [Lev22] to show that $\mathrm{TC}(\ell^{h\mathbb{Z}})$ is closely related to the algebraic K-theory of the $K(1)$-local sphere.[9] In particular, trace methods were used to completely compute $\pi_*K(L_{K(1)}\mathbb{S})[\frac{1}{p}]$, and to make computations of the integral homotopy groups $\pi_*K(L_{K(1)}\mathbb{S})$ in low degrees [Lev22]. Furthermore, $V(1)_*\,\mathrm{THH}(\ell^{hp^k\mathbb{Z}})$ was studied extensively by Lee and the third author, for all $k \geq 0$ [LL23].

As Theorem 5.7.1 here, we present for $k \gg 0$ a complete computation of $V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$, as well as of the TC coassembly map

$$V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \to V(1)_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}.$$

In particular, we deduce the following corollary:

Theorem D. *Let $p \geq 7$ be a prime, and let $\mathbb{Z}$ act on the Adams summand L of $\mathrm{KU}_{(p)}$ via the $\mathbb{E}_\infty$ Adams operation Ψ^{1+p}. Then, for all $k \gg 0$, the $\mathbb{F}_p[v_2^{\pm 1}]$-module map*

$$T(2)_*\mathrm{K}(L^{hp^k\mathbb{Z}}) \to T(2)_*\left(L_{K(2)}\mathrm{K}(L^{hp^k\mathbb{Z}})\right)$$

may be identified with the direct sum of the maps enumerated below. The degrees of classes are determined from their names via the facts that $|t| = -2, |\lambda_i| = 2p^i - 1, |\partial| = -1, |\zeta| = -1$, and the degree of any continuous function (see Notation (17)) is 0.

1. *The projection $\mathbb{F}_p\{1, \partial\} \oplus \overline{C^0(\mathbb{Z}_p^\times)}\{\partial\zeta\} \to \mathbb{F}_p\{1, \partial\}$ onto the first factor, tensored over $\mathbb{F}_p$ with the inclusion $\mathbb{F}_p[v_2^{\pm 1}]\langle\lambda_1, \lambda_2\rangle \to \mathbb{F}_p[v_2^{\pm 1}]\langle\lambda_1, \lambda_2, \zeta\rangle$.*
2. *The map $\mathbb{F}_p[v_2^{\pm 1}]\langle\zeta\rangle \otimes C^0(p\mathbb{Z}_p) \to \mathbb{F}_p[v_2^{\pm 1}]\langle\zeta\rangle$ evaluating a continuous function at 0, tensored over $\mathbb{F}_p$ with the graded $\mathbb{F}_p$-vector space on basis elements enumerated below:*
   - *(a) $t^d\lambda_1$, for each $0 < d < p$, in degree $2p - 1 - 2d$.*
   - *(b) $t^{pd}\lambda_2$, for each each $0 < d < p$, in degree $2p^2 - 1 - 2pd$*
   - *(c) $t^d\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2d$.*
   - *(d) $t^{pd}\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2pd$.*

---

[9] This is true for $p > 2$, and when $p = 2$ there is an analogous statement with $\mathrm{ko}_{(2)}$ replacing ℓ.

At height 1, Theorem A is closely related to the failure of part of Ausoni–Rognes' original vision of chromatic redshift. Namely, they conjectured [AR02, pg 4] that the map

$$V(1) \otimes K(L_{K(1)}\mathbb{S}) \to V(1) \otimes K(\mathrm{KU})^{h\mathbb{Z}_p^\times}$$

should have bounded above fiber, and so in particular be a $T(2)$-local equivalence. Our results imply that this map is not an equivalence $T(2)$-locally, but rather is the cyclotomic completion map.

Nevertheless, using [LL23] we prove for $p \geq 5$ and *all* $k \geq 0$ that $\ell^{hp^k\mathbb{Z}}$ satisfies the height 1 LQ property. When combined with [Lev22] this implies the following result, which can be considered a replacement for the Ausoni–Rognes conjecture:

Theorem E. *For $p \geq 5$, the map*

$$V(1) \otimes K(L_{K(1)}\mathbb{S}) \to V(1)[v_2^{-1}] \otimes K(L_{K(1)}\mathbb{S})$$

has bounded above fiber.

Remark 5.1.3. Many of our results about $V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ suggest approachable lines of further investigation. For example, one might study $\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ modulo larger powers of p and v_1, for small k, or at small primes. ◁

5.1.5 Forthcoming work

Although this paper disproves the telescope conjecture at all heights at least 2 and all primes p, many parts of the argument simplify in the case $n = 1, p \geq 7$. For example, the full strength of Theorem C is not needed, and can be replaced by a more elementary π_*-level statement for a Smith–Toda complex $V(2)$. Moreover, the application of cyclotomic redshift is more direct in this case, and the construction of Adams operations is classical. We intend to write a shorter paper expositing a more efficient approach to this simplest case of our theorem.

In forthcoming work of Shachar Carmeli, Lior Yanovski, and the four authors, we aim to explore additional consequences that our disproof of the telescope conjecture has for stable homotopy theory. In particular, we intend to prove the following three statements for $n \geq 1$:

1. The kernel of $\mathrm{Pic}(\mathrm{Sp}_{T(n+1)}) \to \mathrm{Pic}(\mathrm{Sp}_{K(n+1)})$ is infinite.
2. For any nonzero telescope $T(n+1)$, there exists some integer k such that $\pi_k T(n+1)$ is not a finitely generated abelian group.
3. For every non-zero finite p-local spectrum X (e.g., $X = \mathbb{S}_{(p)}$), there exists some $C > 0$ such that

$$\lim_{M\to\infty} \sum_{i=M}^{M+C} \dim_{\mathbb{F}_p}(\pi_i(X) \otimes \mathbb{F}_p) = \infty.$$

5.1.6 Contents

We now briefly describe the contents of the paper. In Section 5.2, we begin by reviewing the modern approach to the theory of cyclotomic and polygonic spectra, as developed in the foundational papers [NS18], [AN21], and [KMN23]. We then study finiteness and boundedness properties of cyclotomic spectra relevant to understanding the Lichtenbaum–Quillen property. In Section 5.3, we prove Proposition 5.1.1 and study the cyclotomic spectrum $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$, a key universal ring over which the cyclotomic spectra $\mathrm{THH}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ are modules. We also prove Proposition 5.1.1 and discuss how it is related to the failure of $\mathbb{A}^1$-invariance, nilinvariance, and hyperdescent in $K(n+1)$-local K-theory for $n+1 \geq 2$. In Section 5.4, we prove Theorem 5.4.30, which in particular implies Theorem C. We do this by first proving a version of Theorem C at the level of spectra with Frobenius (rather than cyclotomic spectra). We then use the results of the previous sections to bootstrap this to the level of cyclotomic spectra. In Section 5.5, we construct Adams operations on $\mathrm{BP}\langle n\rangle$ as $\mathbb{E}_1 \otimes \mathbb{A}_2$-algebra automorphisms. The key ingredients here are the $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n\rangle$ of [HW22] and the stable Adams conjecture [Fri80; Cla11; BK22]. In Section 5.6, we combine the main theorems of the previous sections, cyclotomic redshift, and other results to disprove the telescope conjecture. In Section 5.7, we completely compute the TC coassembly map for the $p^k\mathbb{Z}$-action on ℓ, mod (p, v_1) and for $p \geq 7$, $k \gg 0$, in particular proving Theorem D. We also prove the height 1 LQ property for $\ell^{hp^k\mathbb{Z}}$ when $p \geq 5$, for all $k \geq 0$, and prove Theorem E. Finally, in Section 5.8, we include material much of which is well known but not treated to the extent needed in the literature.

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Notations and Conventions

1. We fix a prime p, and use n to refer to a chromatic height.

2. We use $\mathrm{Map}_{\mathcal{C}}$ to denote mapping spaces in a category $\mathcal{C}$, and $\mathrm{map}_{\mathcal{C}}$ to denote mapping spectra. We drop the $\mathcal{C}$ when it is clear from context.

3. We use $\mathbb{S}_{T(n)}$ to refer to the $T(n)$ local sphere.

4. We define constants $m_p^{\mathbb{A}_2}$, m_p^{hc} and $m_p^{\mathbb{E}_1}$ by

$$m_p^{\mathbb{A}_2} := \begin{cases} 2 & p=2 \\ 1 & p \geq 3 \end{cases} \quad m_p^{hc} := \begin{cases} 3 & p=2 \\ 2 & p=3 \\ 1 & p \geq 5 \end{cases} \quad m_p^{\mathbb{E}_1} := \begin{cases} 3 & p=2 \\ 2 & p \geq 3 \end{cases}.$$

   The significance of $m_p^{\mathbb{A}_2}$ (resp. m_p^{hc}, $m_p^{\mathbb{E}_1}$) is that for $k \geq m_p^{\mathbb{A}_2}$ (m_p^{hc}, $m_p^{\mathbb{E}_1}$) the Moore spectrum $\mathbb{S}/p^k$ admits an $\mathbb{A}_2$-algebra (hcring, $\mathbb{E}_1$-algebra) structure [Oka84] [Bur22].

5. If R is a commutative ring, we write $R[x]$ for the polynomial algebra over R with generator x.

6. If R is a commutative ring, we write $R\langle\epsilon\rangle$ for the exterior algebra over R with generator ϵ.

7. Unless stated otherwise, spectra we consider are implicitly p-completed. We denote the category of p-complete spectra by Sp. We denote the p-complete sphere by $\mathbb{S} \in \mathrm{Sp}$.

8. We use Sp^{ω} and $\mathrm{Sp}^{\diamondsuit}$ to denote the categories of compact and dualizable objects in Sp, respectively.

9. If G is an $\mathbb{E}_1$-monoid in spaces, then we use the term G-spectra for the category $\mathrm{Fun}(BG, \mathrm{Sp})$ of functors from the classifying category of G to the category of p-complete spectra.

10. We write $\mathbb{T}$ for the compact Lie group $U(1)$.

11. We write $(w) : \mathbb{T} \to \mathbb{C}^{\times}$ for the weight w character of $\mathbb{T}$, $\mathbb{S}^{(w)}$ for the associated representation sphere, and $a_{(w)} : \mathbb{S}^0 \to \mathbb{S}^{(w)}$ for the Euler class.

12. We denote by
$$\mathrm{CycSp} := \mathrm{LEq}\left(\mathrm{Sp}^{B\mathbb{T}} \rightrightarrows \mathrm{Sp}^{B\mathbb{T}}\right)$$
    the p-completion of the category of p-typical cyclotomic spectra of [NS18].

13. Similarly the functors
$$\mathrm{THH}\colon \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{CycSp}$$
$$\mathrm{TC}, \mathrm{TC}^{-}, \mathrm{TP}, \mathrm{TR}\colon\ \mathrm{CycSp} \to \mathrm{Sp}$$
    are all implicitly p-completed, as is the functor $K : \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$.

14. For $\mathcal{C}$ a presentably symmetric monoidal category, we use $\mathrm{THH}_{\mathcal{C}} : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ to denote the THH relative to $\mathcal{C}$ functor. For $R \in \mathrm{CAlg}(\mathcal{C})$ and $A \in \mathrm{Alg}(\mathrm{Mod}_{\mathcal{C}}(R)))$, we use $\mathrm{THH}(A/R)$ to mean $\mathrm{THH}_{\mathrm{Mod}(\mathcal{C};R)}(A)$.

15. We write Perf for the category of perfect commutative $\mathbb{F}_p$-algebras.

16. Following [Lur18a, Sec. 5.2] and [BSY22, Prop. 2.2, Cons. 2.33], there is an adjunction

$$\mathbb{W}(-) \colon \mathrm{Perf} \rightleftarrows \mathrm{CAlg}(\mathrm{Sp}) : \pi_0^\flat(-)$$

where the right adjoint $\pi_0^\flat$ can be computed as the inverse limit along Frobenius on the commutative ring $\pi_0(-)/p$.

17. Given a topological space X, we write $C^0(X)$ for the ring of continuous (that is, locally constant) functions from X to $\mathbb{F}_p$.

18. We write $\overline{C^0(X)}$ for the quotient $C^0(X)/\mathbb{F}_p$, where we view $\mathbb{F}_p$ as a subset of $C^0(X)$ via the inclusion of constant functions.

19. Given a continuous map $f : Y \to X$ of topological spaces, restriction of functions gives a corresponding ring homomorphism $C^0(X) \to C^0(Y)$. We denote this by res_f, or just res if f is clear, and call it the restriction map.

20. In the case where $f : Y \to X$ is the inclusion of a subspace we write $(-)_{|Y}$ for base change along the map $\mathbb{W}(\mathrm{res}_f)$.

21. For $n \geq 1$, $\mathrm{BP}\langle n\rangle$ refers to an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra form of the truncated Brown–Peterson spectrum as constructed in [HW22, §2].

22. E_n refers to the height n Lubin–Tate theory constructed by Goerss–Hopkins–Miller [Lur18a, Theorem 5.0.2], associated to the (unique up to isomorphism) height n formal group over $\overline{\mathbb{F}}_p$.

23. $\mathbb{G}_n$ refers to the height n extended Morava stabilizer group, which acts on E_n and fits into a short exact sequence

$$1 \to \mathcal{O}_D^\times \to \mathbb{G}_n \to \mathrm{Gal}(\mathbb{F}_p) \to 1.$$

Here, $\mathcal{O}_D^\times$ is the units in the ring of integers of the division algebra over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$, and $\mathrm{Gal}(\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ is the absolute Galois group of $\mathbb{F}_p$.

5.2 Cyclotomic spectra

Throughout this paper we use cyclotomic spectra heavily. We follow the modern approach to cyclotomic spectra developed in [NS18], and, in Section 5.4.2, we also use the theory of p-polygonic spectra of [KMN23]. In this section, we begin by recalling the basics of these theories. The bulk of the section is then dedicated to relating boundedness in the cyclotomic t-structure of [AN21] to other "boundedness properties" for cyclotomic spectra. In particular, we introduce quantitative versions of the properties studied by the second author and Wilson in [HW22] in order to prove the Lichtenbaum–Quillen property for $\mathrm{BP}\langle n\rangle$. The main result of Section 5.2.4 is that being almost compact (with respect to the cyclotomic t-structure) is a relatively mild condition on a cyclotomic spectrum: for p-nilpotent cyclotomic spectra, it is equivalent to the underlying spectrum being almost compact.

5.2.1 Preliminaries

Equivariant spectra

Expanding on the conventions section above, we set out notation for working in the category $\mathrm{Sp}^{B\mathbb{T}}$ of p-complete spectra with $\mathbb{T}$-action that we will use throughout the paper.

For w a non-negative integer, we denote by $\mathbb{T}(w)$ the circle with action given by the quotient $\mathbb{T}/C_w$, where $C_w \to T$ is the cyclic subgroup of order w. The projection $\mathbb{T}(w) \to *$ gives rise to a cofiber sequence $\mathbb{S}[\mathbb{T}/C_w] = \Sigma^\infty_+\mathbb{T}(w) \to \mathbb{S} \to \mathbb{S}^{(w)}$. The underlying spectrum of $\mathbb{S}^{(w)}$ is $\mathbb{S}^2$, and $\mathbb{S}^{(w)} \in \mathrm{Sp}^{B\mathbb{T}}$ is therefore tensor invertible. For $k \in \mathbb{Z}$, we denote by $\Sigma^{k(w)}(-)$ the functor of tensoring with $\mathbb{S}^{k(w)} := (\mathbb{S}^{(w)})^{\otimes k}$. Applying $\Sigma^{-(w)}$ to the final map in the above cofiber sequence yields an Euler class

$$a_{(w)} \colon \mathbb{S}^{-(w)} \to \mathbb{S}.$$

We next consider the following filtered $\mathbb{T}$-spectrum:

$$\cdots \to \mathbb{S}^{-m(1)} \xrightarrow{a_{(1)}} \cdots \xrightarrow{a_{(1)}} \mathbb{S}^{-2(1)} \xrightarrow{a_{(1)}} \mathbb{S}^{-(1)} \xrightarrow{a_{(1)}} \mathbb{S}.$$

For any $X \in \mathrm{Sp}^{\mathrm{B}\mathbb{T}}$, the tensor product of X with the above filtered spectrum gives the *$a_{(1)}$-Bockstein filtration* on X. Taking $\mathbb{T}$-fixed points then gives rise to the homotopy fixed point spectral sequence computing $\pi_* X^{h\mathbb{T}}$.[10] Similarly, the usual $\mathbb{T}$-Tate spectral sequence may be identified with the $a_{(1)}$-inverted $a_{(1)}$-Bockstein spectral sequence.

Additionally, for $k \geq 0$ we will often use the category $\mathrm{Sp}^{BC_{p^k}}$ of p-complete C_{p^k}-spectra. We view $C_{p^k} \subset \mathbb{T}$ as the cyclic subgroup of order p^k and will often study $X^{hC_{p^k}}$, $X_{hC_{p^k}}$, and $X^{tC_{p^k}}$ when X is a $\mathbb{T}$-equivariant spectrum. When we use these notations with $k = \infty$, we mean $\mathrm{Sp}^{B\mathbb{T}}$, $X^{h\mathbb{T}}$, $X_{h\mathbb{T}}$, and $X^{t\mathbb{T}}$, respectively. When working in $\mathrm{Sp}^{BC_{p^k}}$ for $k < \infty$, we write

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[10] To compare this construction of the $\mathbb{T}$-homotopy fixed point spectral sequence with another the reader prefers it may help to note that the cofiber of $a^m_{(1)} \colon \mathbb{S}^{-m(1)} \to \mathbb{S}$ is isomorphic to $\mathbb{S}^{S^{2m-1}}$, where the action on S^{2m-1} is as the unit sphere in $\mathbb{C}^m$.

$\mathbb{S}^{(w)}$, $\Sigma^{k(w)}$, and $a_{(w)}$ for the images of the corresponding objects under the restriction map $\mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{BC_{p^k}}$.

Given $1 \leq k \leq \infty$, the C_p-Tate construction is a functor

$$(-)^{tC_p} \colon \mathrm{Sp}^{BC_{p^k}} \to \mathrm{Sp}^{B(C_{p^k}/C_p)} = \mathrm{Sp}^{BC_{p^{k-1}}}.$$

In the case $k = \infty$, we identify $\mathbb{T}/C_p$ with $\mathbb{T}$ by rescaling, and so consider $(-)^{tC_p}$ as an endofunctor of $\mathrm{Sp}^{B\mathbb{T}}$.

We record an elementary lemma:

Lemma 5.2.1. *For any $X \in \mathrm{Sp}^{B\mathbb{T}}$, $1 \leq j \leq \infty$, we have $(\mathbb{S}^{\mathbb{T}} \otimes X)^{tC_{p^j}} \cong \Sigma^\infty_+ \mathbb{T}^{tC_{p^j}} \otimes X \cong 0$, so the functors $(-)^{tC_{p^j}}$ take $a_{(1)} \colon X \to \Sigma^{(1)} X$ to an isomorphism.*

Proof. This follows because $\mathbb{T}$ has a finite equivariant cell decomposition by free C_{p^j} cells. □

Definition 5.2.2. We recall that any $\mathbb{T}$-spectrum X has a *Connes operator*, which is a degree 1 self-map of non-equivariant spectra

$$\sigma \colon \Sigma X \to X.$$

Viewing X as a module over the group ring $\mathbb{S}[\mathbb{T}]$ the Connes operator is given by multiplication by the class $\sigma \in \pi_1(\mathbb{S}[\mathbb{T}])$ coming from the (pointed) identity map $S^1 \to \mathbb{T}$. We also use σ to refer to induced operator $\sigma \colon \pi_i X \to \pi_{i+1} X$. ◁

Cyclotomic spectra

We work with the p-complete p-typical variants of cyclotomic spectra, defined in [NS18].

Definition 5.2.3. The category of cyclotomic spectra we use is defined as the lax equalizer

$$\mathrm{CycSp} \coloneqq \mathrm{LEq}\left(\mathrm{Sp}^{B\mathbb{T}} \rightrightarrows \mathrm{Sp}^{B\mathbb{T}} \right)$$

of the identity functor and the functor $(-)^{tC_p}$. We use CycSp_+ to denote the full subcategory of bounded below objects. In other words, to give a cyclotomic spectrum is to give an $X \in \mathrm{Sp}^{B\mathbb{T}}$ with a map $\varphi_X : X \to X^{tC_p}$ in $\mathrm{Sp}^{B\mathbb{T}}$. ◁

Definition 5.2.4. Given $X \in \mathrm{CycSp}$, we let $\mathrm{TC}^-(X) \coloneqq X^{h\mathbb{T}}$, $\mathrm{TP}(X) \coloneqq X^{t\mathbb{T}}$ and

$$\mathrm{TC}(X) \coloneqq \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, X).$$

We moreover have (see [AN21, Example 3.4, Construction 3.18]):

$$\mathrm{TR}^{k+1}(X) \coloneqq X^{hC_{p^k}} \times_{(X^{tC_p})^{hC_{p^{k-1}}}} \cdots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X.$$

and further define $\mathrm{TR}(X) \coloneqq \lim_k \mathrm{TR}^k(X)$.

For $X \in \mathrm{CycSp}_+$, we recall that the natural map $X^{tC_{p^{k+1}}} \to (X^{tC_p})^{hC_{p^k}}$ is an equivalence [NS18, Lemma II.4.1], simplifying the formula above. ◁

$\mathrm{TC}(X)$ is computed for $X \in \mathrm{CycSp}$ by the equalizer

$$\begin{tikzcd} \mathrm{TC}(X) \arrow[r] & \mathrm{TC}^{-}(X) \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\mathrm{can}"'] & (X^{tC_p})^{h\mathbb{T}} \end{tikzcd}$$

where the maps are obtained by applying $(-)^{h\mathbb{T}}$ to the maps $\varphi : X \to X^{tC_p}$ and the map $\mathrm{can} : X^{hC_p} \to X^{tC_p}$. When $X \in \mathrm{CycSp}_+$, the map $\mathrm{TP}(X) \to (X^{tC_p})^{h\mathbb{T}}$ is an equivalence [NS18, Lemma II.4.2], and can can be identified with the natural map $\mathrm{TC}^- \to \mathrm{TP}$.

There is a natural Frobenius endomorphism [AN21, Construction 3.18] $F : \mathrm{TR} \to \mathrm{TR}$ such that for $X \in \mathrm{CycSp}$, $\mathrm{TC}(X)$ is also computed as the equalizer

$$\begin{tikzcd} \mathrm{TC}(X) \arrow[r] & \mathrm{TR}(X) \arrow[r, shift left, "F"] \arrow[r, shift right, "1"'] & \mathrm{TR}(X) \end{tikzcd}$$

Example 5.2.5. We define a cyclotomic spectrum $L_{\langle p^\infty \rangle}\mathbb{S}$[11]. Its underlying spectrum is

$$\bigoplus_{j \geq 0} \mathbb{S}[\mathbb{T}/C_{p^j}],$$

and the Frobenius is the isomorphism (by the Segal conjecture) given by the sum of the composites[12]

$$\mathbb{S}[\mathbb{T}/C_{p^j}] \to (\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{hC_p} \to (\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{tC_p}. \qquad \triangleleft$$

The following can be compared to [BM16, Theorem 6.5]:

Lemma 5.2.6. *The cyclotomic spectrum $L_{\langle p^\infty \rangle}\mathbb{S}$ co-represents $\mathrm{TR}(-)$ in CycSp_+*

Proof. The formula for mapping out of $L_{\langle p^\infty \rangle}\mathbb{S}$ is given as the equalizer

$$\begin{tikzcd} \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(L_{\langle p^\infty \rangle}\mathbb{S}, X) \arrow[r, shift left, "(-)^{tC_p} \circ \varphi"] \arrow[r, shift right, "\varphi \circ (-)"'] & \mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(L_{\langle p^\infty \rangle}\mathbb{S}, X^{tC_p}) \end{tikzcd}$$

Since for any $Y \in \mathrm{Sp}^{B\mathbb{T}}$ there is a natural isomorphism

$$\mathrm{map}_{\mathrm{Sp}^{B\mathbb{T}}}(\mathbb{S}[\mathbb{T}/C_{p^k}], Y) \cong Y^{hC_{p^k}}$$

, we can identify the above equalizer with the equalizer

$$\begin{tikzcd} \prod_0^\infty X^{hC_{p^i}} \arrow[r, shift left, "(-)^{tC_p} \circ \varphi"] \arrow[r, shift right, "\varphi \circ (-)"'] & \prod_0^\infty (X^{tC_p})^{hC_{p^i}} \end{tikzcd}$$

The map $(-)^{tC_p} \circ \varphi$ can be identified with the canonical maps because of the description of the Frobenius map on $L_{\langle p^\infty \rangle}\mathbb{S}$. Then this equalizer agrees with $\mathrm{TR}(X)$. $\square$

---

[11] Our notation is meant to suggest it comes from applying the functor $L_{\langle p^\infty \rangle}$ between p^∞-polygonic and cyclotomic spectra of [KMN23] to $\mathbb{S}$. However we do not use this fact.

[12] Note that the coassembly map pulling the sum inside the $(-)^{tC_p}$ is an isomorphism by [Yua23, Corollary 6.7].

We will also use the following recognition criterion for $L_{\langle p^\infty\rangle}\mathbb{S}$:

Lemma 5.2.7. *Let R be bounded below commutative algebra for which the canonical map $R \to R^{tC_p}$ is an isomorphism. Suppose we are given an R-module in cyclotomic spectra X together with*

1. *a splitting of X as $X \cong \oplus_{k\geq 0} X_k$ in $\mathrm{Mod}(R)^{B\mathbb{T}}$,*
2. *a collection of isomorphisms $q_k : X_k \cong R \otimes \mathbb{S}[\mathbb{T}/C_{p^k}]$ in $\mathrm{Mod}(R)^{B\mathbb{T}}$ and*
3. *a collection of isomorphisms $\varphi_k : X_k \to X_{k+1}^{tC_p}$ in $\mathrm{Mod}(R)^{B\mathbb{T}}$*

such that the cyclotomic Frobenius on X is given by the sum of the map φ_k as a map of $\mathbb{T}$-equivariant R-modules. Then, there is an isomorphism of R-modules in cyclotomic spectra

$$X \cong R \otimes L_{\langle p^\infty\rangle}\mathbb{S}.$$

Proof. Let $\mathrm{Sp}^{B\mathbb{T}}_{sc} \subseteq \mathrm{Sp}^{B\mathbb{T}}$ be the full subcategory on those Y for which the natural transformation $c : Y \to (p^*Y)^{tC_p}$ is an isomorphism at Y. Note that this property depends only on the underlying spectrum and by exactness of $(-)^{tC_p}$ all of the objects we will consider are in this subcategory.

Let $f_0 := q_0^{-1}$. By induction on $k \geq 1$ we will construct $\mathbb{T}$-equivariant isomorphisms of R-modules $f_k : R \otimes \mathbb{S}[T/C_{p^k}] \to X_k$ and homotopies making the following diagram of $\mathbb{T}$-equivariant R-modules commute

$$\begin{tikzcd}
R \otimes \mathbb{S}[\mathbb{T}/C_{p^{k-1}}] \arrow[r, "f_{k-1}"] \arrow[d, "c"'] & X_{k-1} \arrow[d, "\varphi"] \\
R \otimes \mathbb{S}[\mathbb{T}/C_{p^k}])^{tC_p} \arrow[r, "f_k^{tC_p}"'] & X_k^{tC_p}.
\end{tikzcd}$$

Let $g_k := q_k^{tC_p} \circ \varphi_{k-1} \circ f_{k-1}$ and let $f_k := q_k^{-1} \circ p^*(c^{-1} \circ g_k)$. Then we have

$$\begin{aligned}
f_k^{tC_p} \circ c &= \left(q_k^{-1} \circ p^*(c^{-1} \circ g_k)\right)^{tC_p} \circ c = (q_k^{-1})^{tC_p} \circ (p^*(c^{-1} \circ g_k))^{tC_p} \circ c \\
&= (q_k^{-1})^{tC_p} \circ c \circ c^{-1} \circ g_k = (q_k^{-1})^{tC_p} \circ (q_k^{tC_p} \circ \varphi_{k-1} \circ f_{k-1}) \\
&= \varphi_{k-1} \circ f_{k-1}
\end{aligned}$$

The sum of the maps f_k is the desired isomorphism in $\mathrm{Mod}(R)^{B\mathbb{T}}$ and sum of the commuting squares lifts it to an isomorphism of R-modules in cyclotomic spectra. □

p-polygonic spectra

In Section 5.4.2, we make use of the p-polygonic spectra of [KMN23], which we review here:

Definition 5.2.8. The category of p-polygonic spectra is defined as the oplax limit of the functor $(-)^{tC_p} : \mathrm{Sp}^{BC_p} \to \mathrm{Sp}$, i.e

$$\mathrm{PgcSp}_{\langle p \rangle} := \mathrm{Sp}\vec{\times}_{(-)^{tC_p}}\mathrm{Sp}^{BC_p} \qquad \triangleleft$$

In particular, we see that $\mathrm{PgcSp}_{\langle p \rangle}$ can be identified with the category of p-complete genuine C_p-equivariant spectra. Given $X = (X_0, X_1, X_0 \to (X_1)^{tC_p}) \in \mathrm{PgcSp}_{\langle p \rangle}$, we sometimes use the following notations from genuine equivariant homotopy theory:

$$X^{\Phi C_p} := X_0, X^{\Phi e} := X_1, X^{hC_p} := X_1^{hC_p}, X^{tC_p} := X_1^{tC_p},$$

and X^{C_p} for the pullback

$$\begin{tikzcd}
X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Phi C_p} \arrow[d, "\mathrm{ev}_1"] \\
X^{hC_p} \arrow[r, "(-)^{tC_p}"'] & X^{tC_p}.
\end{tikzcd}$$

Definition 5.2.9. We denote by $\mathrm{res}_\varphi \colon \mathrm{PgcSp}_{\langle p \rangle} \to \mathrm{Sp}^{\Delta^1}$ the lax symmetric monoidal functor sending $(X, Y, X \to Y^{tC_p})$ to $X \to Y^{tC_p}$. We also define $\mathrm{res}_{\pentagon} \colon \mathrm{CycSp} \to \mathrm{PgcSp}_{\langle p \rangle}$ to be the functor sending a cyclotomic spectrum X to the triple $(X, X, X \xrightarrow{\varphi} X^{tC_p})$. We abuse notation by writing $\mathrm{res}_\varphi := \mathrm{res}_\varphi \circ \mathrm{res}_{\pentagon} \colon \mathrm{CycSp} \to \mathrm{Sp}^{\Delta^1}$. ◁

t-structures

The category of spectra admits a t-structure such that $\mathrm{Sp}^\heartsuit$ is the category of (derived) p-complete abelian groups. For every $1 \leq k \leq \infty$, this induces a pointwise t-structure on $\mathrm{Sp}^{BC_{p^k}}$, where connectivity and coconnectivity are detected on the underlying non-equivariant spectrum. We shall denote truncation functors with respect to this t-structure by $\tau_{>b}$ and $\tau_{\leq b}$.

In [AN21], Antieau and Nikolaus define the cyclotomic t-structure on CycSp. In this t-structure an $X \in \mathrm{CycSp}$ is connective if and only if it is connective as a spectrum[13]. We shall denote truncation functors with respect the cyclotomic t-structure by $\tau_{>b}^{\mathrm{cyc}}$ and $\tau_{\leq b}^{\mathrm{cyc}}$.

Notation 5.2.10. Given any of the categories $\mathcal{C} = \mathrm{Sp}^{BC_{p^k}}, \mathrm{CycSp}$, we use $\mathcal{C}_{\leq b}, \mathcal{C}_{>b}, \mathcal{C}_{[c,b]}$ to denote the collection of appropriately bounded objects with respect to the aforementioned t-structures. ◁

Lemma 5.2.11 ([AN21, Theorem 9])**.** *The functor* $\mathrm{TR} \colon \mathrm{CycSp}_+ \to \mathrm{Sp}$ *is conservative and reflects connectivity and coconnectivity.*

Proof. To reconcile the statement with [AN21, Theorem 9], which isn't for the p-complete category, we observe that the forgetful functors from the p-completed categories to the uncompleted categories are conservative and reflect connectivity and coconnectivity, reducing the statement to the uncompleted case. ☐

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[13] We note that since our categories are p-completed, our definitions don't exactly agree.

Lemma 5.2.12. *A filtered colimit of cyclotomic spectra bounded in the range* $[a, b]$ *is itself bounded in the range* $[a, b + 3]$.

Proof. Recall from [AN21, Thm. 9] that there is a t-exact isomorphism between the category of bounded below cyclotomic spectra and the category of V-complete bounded below topological Cartier modules. The t-structure on topological Cartier modules is compatible with filtered colimits, so to prove the lemma it will suffice to show that the p-complete V-completion functor on bounded topological Cartier modules has t-amplitude $[0, 3]$.

Given an $M \in \mathrm{TCart}_p^\heartsuit$ we have a formula for the V-completion as $\lim_n M/V^n$ [AN21, Prop. 3.22]. Doing this in the p-complete category gives $\lim_{n,k}(M/p^k)/V^n$. From this formula we can easily read off that the functor has t-amplitude $[0, \infty]$. A closer analysis of this formula in fact yields the desired conclusion, that V-completion has t-amplitude $[0, 3]$ (see [Mat21, Prop. 6.5] and use the fact that M/p^k is p-complete). □

Topological Hochschild homology

Many objects in CycSp and $\mathrm{PgcSp}_{\langle p\rangle}$ are constructed via topological Hochschild homology.

Definition 5.2.13. We denote by

$$\mathrm{THH}\colon \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{CycSp}$$

the p-completed version of topological Hochschild homology, as defined in [NS18]. This functor is symmetric monoidal [NS18, **page** IV.2]. ◁

We will often similarly consider $\mathrm{TC}^-, \mathrm{TP}, \mathrm{TC}$, and TR^k as functors from Alg(Sp) by precomposing with THH.

Definition 5.2.14. We let

$$\mathrm{THH}_{\pentagon}(-;-)\colon \mathrm{BiMod} \to \mathrm{PgcSp}_{\langle p\rangle}$$

be the p-completed version of the functor of [KMN23, Theorem 6.31].

This is a lax symmetric monoidal functor because every step of the construction of [KMN23, Theorem 6.31] is lax symmetric monoidal as a functor from BiMod. ◁

The following is a restatement of [NS18, Remark III.1.5.], where $\mathrm{Nm}_e^{C_p}$ refers to the Hill–Hopkins–Ravenel norm:

Lemma 5.2.15. *For* $V \in \mathrm{Sp}$ *there is a natural isomorphism* $\mathrm{THH}_{\pentagon}(\mathbb{S}; V) \cong \mathrm{Nm}_e^{C_p}(V)$ *in* $\mathrm{PgcSp}_{\langle p\rangle}$.

We will denote by $\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}}$ the category of functors to Alg(Sp) from the category

$$\mathrm{EQ} \coloneqq \cdot \rightrightarrows \cdot$$

in Alg(Sp). Such a diagram $R \rightrightarrows S$ gives S the structure of an R-bimodule, and so there is a functor

$$\mathrm{THH}_{\pentagon}(-;-)\colon \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \to \mathrm{PgcSp}_{\langle p\rangle}.$$

There is the following basic compatibility:

Lemma 5.2.16. *There is a commuting diagram of lax symmetric monoidal functors*

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{Sp}) \arrow[r, "\mathrm{THH}"] \arrow[d] & \mathrm{CycSp} \arrow[r, "\mathrm{res}_\varphi"] \arrow[d, "\mathrm{res}_{\pentagon}"] & \mathrm{Sp}^{\Delta^1} \\
\mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \arrow[r, "\mathrm{THH}_{\pentagon}"] & \mathrm{PgcSp}_{\langle p \rangle} \arrow[ur, "\mathrm{res}_\varphi"'] &
\end{tikzcd}$$

where the left vertical map sends an $\mathbb{E}_1$*-algebra* R *to the constant diagram* $R \rightrightarrows R$.

Proof. The commutation of the square is immediate from the definitions of the cyclotomic [NS18, III.2, Corollary III.3.8] and polygonic [KMN23, Proposition 6.30] Frobenius maps, upon noting that the inclusion $\Delta^{op} \to \Lambda_\infty^{op}$ is cofinal [NS18, Proposition B.5], and that the composite

$$\Delta^{op} \to \Lambda_\infty^{op} \to \Lambda_p^{op} \to \mathrm{Free}(C_p) \times_{\mathrm{Fin}} (\mathrm{Sp}^{\otimes})_{\mathrm{act}}$$

where the third map is as in [NS18, **page** III.2], is the map of [KMN23, Construction 6.30].

The triangle commutes by the definition of $\mathrm{res}_{\pentagon}$. ☐

Lemma 5.2.17. *Suppose we are given* $(A, B) \in \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}}$ *and* $V \in \mathrm{Alg}(\mathrm{Sp})$ *such that the underlying spectrum of* V *is a dualizable* $\mathbb{S}$*-module. The natural map*

$$\mathrm{res}_\varphi \mathrm{THH}_{\pentagon}(\mathbb{S}; V) \otimes \mathrm{res}_\varphi \mathrm{THH}_{\pentagon}(A; B) \to \mathrm{res}_\varphi \mathrm{THH}_{\pentagon}(A; V \otimes B),$$

coming from the lax symmetric monoidal structure on $\mathrm{res}_\varphi \mathrm{THH}_{\pentagon}(-;-)$*, is an isomorphism.*

Proof. Using the fact that $\mathrm{THH}(-;-)$ is symmetric monoidal we can reduce to showing that the natural map

$$\mathrm{THH}_{\pentagon}(\mathbb{S}; V)^{tC_p} \otimes \mathrm{THH}_{\pentagon}(A; B)^{tC_p} \to (\mathrm{THH}_{\pentagon}(\mathbb{S}; V) \otimes \mathrm{THH}_{\pentagon}(A; B))^{tC_p}$$

is an isomorphism. Applying the identification $\mathrm{THH}_{\pentagon}(\mathbb{S}; V) \cong V^{\otimes p} \in \mathrm{Sp}^{BC_p}$ (Lemma 5.2.15) and denoting $X \coloneqq \mathrm{THH}_{\pentagon}(A; B) \in \mathrm{Sp}^{BC_p}$ we can a rewrite the above map as

$$(V^{\otimes p})^{tC_p} \otimes X^{tC_p} \to (V^{\otimes p} \otimes X)^{tC_p}.$$

Fixing X, we can consider the map as a natural transformation of functors $\mathrm{Sp}^{\diamondsuit} \to \mathrm{Sp}$. We claim that both the source and target of this natural transformation are exact. Indeed for the source it follows from [NS18, Proposition III.1.1], and for the target we may use the proof of [NS18, Proposition III.1.1].

Since $\mathrm{Sp}^{\diamondsuit}$ is the thick subcategory generated by $\mathbb{S}$, this reduces the statement to the case $V = \mathbb{S}$, where the claim follows from the Segal conjecture. ☐

5.2.2 Quantitative forms of boundedness

In this subsection we explore a collection of "boundedness" conditions that a cyclotomic spectrum X might satisfy. The results proved here are primarily used in Section 5.4 to prove that certain cyclotomic spectra are bounded in the cyclotomic t-structure. This subsection can be viewed as a quantitative version of [HW22, Section 3.5].

Canonical vanishing

Definition 5.2.18. Suppose we are given an $X \in \mathrm{Sp}^{B\mathbb{T}}$ which is bouned below. We say that:

1. X satisfies weak canonical vanishing with parameter b if for each $1 \leq j \leq \infty$ and $m > b$ the canonical maps
$$\pi_m X^{hC_{p^j}} \to \pi_m X^{tC_{p^j}}$$
are zero. We abbreviate this by saying that X satisfies $\mathrm{WCV}(\leq b)$.

2. X satisfies strong canonical vanishing with parameter b if there exists some $d \geq 0$ for which the composition
$$\tau_{>b}X \to X \xrightarrow{a_{(1)}^d} \Sigma^{d(1)}X$$
is $\mathbb{T}$-equivariantly null. We abbreviate this by saying that X satisfies $\mathrm{SCV}(\leq b)$. ◁

Note that $\mathrm{WCV}(\leq b) \implies \mathrm{WCV}(\leq b+1)$ and similarly $\mathrm{SCV}(\leq b) \implies \mathrm{SCV}(\leq b+1)$.

Lemma 5.2.19. *Let $X \in \mathrm{Sp}^{B\mathbb{T}}$ be bounded below. For all $b \in \mathbb{Z}$, if X satisfies* $\mathrm{SCV}(\leq b)$ *then X satisfies* $\mathrm{WCV}(\leq b)$*.*

Proof. As X satisfies $\mathrm{SCV}(\leq b)$ we may find a d such that the composite
$$\tau_{>b}X \to X \xrightarrow{a_{(1)}^d} \Sigma^{d(1)}X$$
is $\mathbb{T}$-equivariantly null. We learn that X satisfies $\mathrm{WCV}(\leq b)$ from the following diagram:
$$\begin{tikzcd}
\tau_{>b}(X^{hC_{p^j}}) \arrow[r] & (\tau_{>b}X)^{hC_{p^j}} \arrow[r] \arrow[d] & (\tau_{>b}X)^{tC_{p^j}} \arrow[d] \arrow[dr, "0"] & \\
 & X^{hC_{p^j}} \arrow[r, "\mathrm{can}"] & X^{tC_{p^j}} \arrow[r, "\cong", "(a_{(1)}^d)^{tC_{p^j}}"'] & (\Sigma^{d(1)}X)^{tC_{p^j}},
\end{tikzcd}$$
where the bottom left horizontal map is an isomorphism by Lemma 5.2.1. □

Exponent of Nilpotence

We will also need control over exponents of nilpotence in the sense of [Mat18].

Recollection 5.2.20. In [Mat18], in the process of analyzing descent, Mathew sets out the basic properties of the *exponent of nilpotence* of an object with respect to a ring. We briefly recall some of these properties.

Let $\mathcal{C}$ be a stable, exactly[14] symmetric monoidal category. Let B be an $\mathbb{E}_1$-algebra in $\mathcal{C}$, and let $I_B \to \mathbb{1} \to B$ be the associated fiber sequence. Given an $X \in \mathcal{C}$, the exponent of nilpotence of X with respect to B, denoted $\exp_B(X)$, is the smallest m such that either of following (equivalent) conditions hold:

---

[14] i.e the tensor product is exact in each variable

1. X is a retract of an object with an m-step resolution by B-modules, or

2. The map $I_B^{\otimes m} \to \mathbb{1}$ becomes null upon tensoring with X.

By [Mat18, Proposition 2.3] the function $\exp_B(-)$ is sub-additive in cofiber sequences, e.g. given a cofiber sequence $X \to Y \to Z$ we have

$$\exp_B(Y) \leq \exp_B(X) + \exp_B(Z). \qquad \triangleleft$$

Remark 5.2.21. In the category of $\mathbb{T}$-spectra we have a cofiber sequence

$$\mathbb{S}^{-(1)} \xrightarrow{a_{(1)}} \mathbb{S} \to \mathbb{S}^{\mathbb{T}},$$

and so we find for any $\mathbb{T}$-spectrum X that $\exp_{\mathbb{S}^{\mathbb{T}}}(X) \leq t$ if and only if $a_{(1)}^t \otimes X$ is null. $\triangleleft$

Lemma 5.2.22.

1. *Let $\mathcal{C}$ be a stable, exactly symmetric monoidal category, and let R and B be $\mathbb{E}_1$-algebras in $\mathcal{C}$. Then, for every R-module M,*

$$\exp_B(M) \leq \exp_B(R).$$

2. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact lax symmetric monoidal functor between stable categories. Let $A \in \mathrm{Alg}(\mathcal{C})$ and $B \in \mathrm{Alg}(\mathcal{D})$ be $\mathbb{E}_1$-algebras. Then, for each $M \in \mathcal{C}$,*

$$\exp_B(F(M)) \leq \exp_A(M) \cdot \exp_B(F(A)).$$

3. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal category with compatible t-structure, and $\iota\colon \mathrm{Sp} \to \mathcal{C}$ the unique functor in $\mathrm{CAlg}(\mathrm{Pr}^L)$. If $X \in \mathcal{C}_{[c,b]}$ and p^m acts by zero on X, then*

$$\exp_{\iota\mathbb{F}_p}(X) \leq (b - c + 1)m.$$

Proof. Part (1). Using the R-module structure we write M as a retract of $M \otimes R$. Now we observe that, if $R \otimes \left(I_B^{\otimes m} \to \mathbb{1}\right)$ is null, then $M \otimes R \otimes \left(I_B^{\otimes m} \to \mathbb{1}\right)$ is null as well.

Part (2). Resolving M by A-modules, and using exactness of F and subadditivity of $\exp_B(-)$, we reduce to the case where M is an A-module. Using the lax symmetric monoidal structure on F we can give $F(M)$ an $F(A)$-module structure. Using the conclusion of (1), it now suffices to prove (2) in the case where $M = A$, but in this case the lemma reduces to the equality $\exp_B(F(A)) = \exp_B(F(A))$.

Part (3). Resolving X by its t-structure homotopy groups we reduce to the case X is in the heart. Using the fact that $\mathcal{C}^\heartsuit$ is a 1-category, we can give X the structure of a $\iota(\mathbb{Z}/p^m)$-module. By (1) we are then reduced to the case $X = \iota(\mathbb{Z}/p^m)$. By (2) we learn that

$$\exp_{\iota\mathbb{F}_p}(\iota(\mathbb{Z}/p^m)) \leq \exp_{\mathbb{F}_p}(\mathbb{Z}/p^m) \cdot \exp_{\iota\mathbb{F}_p}(\iota\mathbb{F}_p) \leq m \cdot 1. \qquad \square$$

Definition 5.2.23. Suppose we are given an $X \in \mathrm{Sp}^{B\mathbb{T}}$ which is bounded below.

- X satisfies Tate nilpotence of exponent d if $\exp_{\mathbb{S}^{\mathbb{T}}}(X^{tC_p}) \leq d$, where X^{tC_p} is given its residual $\mathbb{T}/C_p$-action.
- X satisfies $\mathbb{F}_p$ nilpotence of exponent d if $\exp_{\mathbb{F}_p}(X) \leq d$. $\triangleleft$

Lemma 5.2.24. *Given a bounded below* $X \in \mathrm{Sp}^{B\mathbb{T}}$ *we have*

$$\exp_{\mathbb{S}^{\mathbb{T}}}(X^{tC_p}) \leq \exp_{\mathbb{F}_p}(X).$$

Proof. Applying Lemma 5.2.22(2) to the functor

$$(-)^{tC_p} \colon \mathrm{Sp}_p^{B\mathbb{T}} \to \mathrm{Sp}_p^{B\mathbb{T}},$$

it will suffice to show that

$$\exp_{\mathbb{S}^{\mathbb{T}}}(\mathbb{F}_p^{tC_p}) = 1.$$

Per Remark 5.2.21 (and using that $\mathbb{F}_p^{tC_p}$ is an algebra) it will suffice for us to show that the image of $a_{(1)}$ in $\pi^{\mathbb{T}}_{-(1)}(\mathbb{F}_p^{tC_p})$ is zero. But the map $a_{(1)} : \mathbb{F}_p^{tC_p} \to \Sigma^{(1)}\mathbb{F}_p^{tC_p}$ is obtained by applying $(-)^{tC_p}$ to the map $a_{(p)} : \mathbb{F}_p \to \Sigma^{(p)}\mathbb{F}_p$, so it will suffice to argue that $a_{(p)} \otimes \mathbb{F}_p$ is zero. The map $a_{(p)}$ can be written as the composite

$$\mathbb{S} \xrightarrow{a_{(1)}} \mathbb{S}^{(1)} \xrightarrow{\underline{p}} \mathbb{S}^{(p)},$$

where $\underline{p}$ denotes the pointed suspension of the p^{th} power map on $\mathbb{C}^{\times}$ (with the weight 1 $\mathbb{T}$-action on the source and the weight p $\mathbb{T}$-action on the target). There are orientations $\mathbb{F}_p \otimes \mathbb{S}^2 \cong \mathbb{F}_p \otimes \mathbb{S}^{(1)}$ and $\mathbb{F}_p \otimes \mathbb{S}^2 \cong \mathbb{F}_p \otimes \mathbb{S}^{(p)}$, under which the map $\underline{p}$ is identified with p (i.e zero). $\square$

Boundedness of cyclotomic spectra

We now consider the above properties in the context of cyclotomic spectra, and use the notations $\mathrm{WCV}(\leq b)$ and $\mathrm{SCV}(\leq b)$, and refer to Tate and $\mathbb{F}_p$ nilpotence exponent to indicate properties of the underlying $\mathbb{T}$-spectrum.

Definition 5.2.25. For $X \in \mathrm{CycSp}$, and $b \in \mathbb{Z}$. We say that X satisfies the Segal condition[15] with parameter b if the fiber of the map $X \xrightarrow{\varphi} X^{tC_p}$ is b-truncated. We abbreviate this by saying that X satisfies $\mathrm{Segal}(\leq b)$. $\triangleleft$

Lemma 5.2.26. *Let* $X \in \mathrm{CycSp}_+$. *If* X *satisfies* $\mathrm{Segal}(\leq b)$ *and* $\exp_{\mathbb{S}^{\mathbb{T}}}(X^{tC_p}) \leq d$ *then* X *satisfies* $\mathrm{SCV}(\leq b + 2d)$.

---

[15] This is because this condition is related to the Segal conjecture.

Proof. Consider the following diagram of $\mathbb{T}$-spectra

$$\begin{tikzcd}
 & & \Sigma^{d(1)} \operatorname{fib}(\varphi) \arrow[d] \\
\tau_{>b+2d}X \arrow[r] \arrow[urr, dashed] & X \arrow[r, "a^d_{(1)}"] \arrow[d, "\varphi"] & \Sigma^{d(1)}X \arrow[d, "\varphi"] \\
 & X^{tC_p} \arrow[r, "a^d_{(1)}"] & \Sigma^{d(1)}X^{tC_p}.
\end{tikzcd}$$

Using Remark 5.2.21 and the hypothesis that $\exp_{\mathbb{S}^{\mathbb{T}}}(X^{tC_p}) \leq d$ we know the bottom horizontal arrow is null and therefore that the dashed lift exists. On the other hand $\Sigma^{d(1)} \operatorname{fib}(\varphi)$ is $(b+2d)$-truncated since X satisfies Segal($\leq b$). The dashed lift is therefore null. ☐

Lemma 5.2.27. *Let* $X \in \mathrm{CycSp}_+$. *If* X *satisfies* WCV($\leq b$) *and* Segal($\leq b$), *then* $X \in \mathrm{CycSp}_{\leq b}$.

Proof. This lemma is a quantitative version of [HW22, Thm. 3.3.2(f)] and we follow the argument there closely. A similar argument can be found in [Mat21].

We have $\mathrm{TR}(X) \cong \lim_k \mathrm{TR}^k(X)$, where for every k there is a pullback square

$$\begin{tikzcd}
\mathrm{TR}^{k+1}(X) \arrow[rr] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & & X^{hC_{p^k}} \arrow[d, "\mathrm{can}"] \\
\mathrm{TR}^k(X) \arrow[r] & X^{hC_{p^{k-1}}} \arrow[r, "\varphi^{hC_{p^{k-1}}}"] & X^{tC_{p^k}}.
\end{tikzcd}$$

Thus, by induction on k we learn that the fiber of the map $\mathrm{TR}^{k+1}(X) \to X^{hC_{p^k}}$ has a finite filtration with associated graded given by

$$\left\{\operatorname{fib}\left(\varphi^{hC_{p^j}} : X^{hC_{p^j}} \to X^{tC_{p^{j+1}}}\right)\right\}_{j<k}.$$

Since X satisfies Segal($\leq b$), these are all b-truncated and we learn that the maps

$$\mathrm{TR}^{k+1}(X) \to X^{hC_{p^k}}$$

induce an injection on $\pi_m(-)$ for $m > b$.[16] Additionally, as X satisfies Segal($\leq b$) the Frobenius map $X^{hC_{p^{k-1}}} \to (X^{tC_p})^{hC_{p^{k-1}}} \cong X^{tC_{p^k}}$ induces an injection on $\pi_m(-)$ for $m > b$.

Thus both horizontal maps in the square are injective on $\pi_m(-)$ for $m > b$. Now, since X satisfies WCV($\leq b$), the right vertical map in zero on $\pi_m(-)$ for $m > b$. We deduce that the map $\mathrm{TR}^{k+1}(X) \to \mathrm{TR}^k(X)$ induces the zero map on $\pi_m(-)$ for $m > b$. Thus, the limit is b-truncated, proving the result. ☐

---

[16] See also [NS18, Cor. II.4.9] for a weaker bound.

Lemma 5.2.28. *Suppose we are given an $X \in \mathrm{CycSp}_{[c,b]}$ on which p^m acts by zero. Then $\exp_{\mathbb{F}_p}(X) \leq (b-c+1)m$.*

Proof. Applying Lemma 5.2.22(3) in the case $\mathcal{C} = \mathrm{CycSp}$ and $X = X$ we learn that X is $\mathbb{F}_p$-nilpotent of exponent $(b-c+1)m$ as a cyclotomic spectrum. Using that the forgetful functor down to $\mathbb{T}$-spectra is symmetric monoidal we can apply Lemma 5.2.22(2) to conclude that the underlying $\mathbb{T}$-spectrum of X is $\mathbb{F}_p$-nilpotent with the same bound. ☐

We also recall the following result:

Proposition 5.2.29 ([AN21, Prop 2.25])**.** *Any $X \in \mathrm{CycSp}_{\leq b}$ satisfies* Segal($\leq b$).

We collect the previous results to conveniently refer to later:

Proposition 5.2.30. *Let $X \in \mathrm{CycSp}$ be a bounded below cyclotomic spectrum.*

1. *If X satisfies* SCV($\leq b$)*, then X satisfies* WCV($\leq b$).
2. *If X satisfies* WCV($\leq b$) *and* Segal($\leq b$)*, then $X \in \mathrm{CycSp}_{\leq b}$.*
3. *If $X \in \mathrm{CycSp}_{\leq b}$, then X satisfies* Segal($\leq b$).
4. *If $X \in \mathrm{CycSp}_{[c,b]}$ and p^m acts by zero on X, then X satisfies* SCV($\leq b+2(b-c+1)m$).

Proof. (1) is a restatement of Lemma 5.2.19. (2) is a restatement of Lemma 5.2.27. (3) is a restatement of Proposition 5.2.29. (4) is obtained by combining Lemmas 5.2.28, 5.2.24, 5.2.29 and 5.2.26. ☐

5.2.3 The Bökstedt class

In this subsection we explore another boundedness condition, specific to rings in cyclotomic spectra: having a Bökstedt class. As it turns out, a bounded below R has a Bökstedt class exactly when it is cyclotomically bounded and we use this as a hook for proving boundedness in later sections.[17]

Definition 5.2.31. Let R be an $h\mathbb{A}_2$-ring (Definition 5.8.1) in cyclotomic spectra satisfying Segal($\leq b$). We say that an element $\mu \in \pi_{2p^k} R$ for $k \geq 0$ is a Bökstedt class if

(a) both μ and $\varphi(\mu)$ are hcentral (Definition 5.8.2),

(b) μ is in the image of the $\mathbb{T}$-transfer map $\Sigma R_{h\mathbb{T}} \to R$

(c) $\varphi(u) \in \pi_{2p^k}(R^{tC_p})$ is a unit. ◁

---

[17] As a warning, the notion of Bökstedt class we use in this paper is quite strong: for example, $\mathrm{THH}(\mathbb{Z}_p)/p$ does not have a Bökstedt class because $\mathrm{TR}(\mathbb{Z}_p)/p$ is not bounded.

Lemma 5.2.32. *Let $i\colon R \to S$ be an hcentral map of $h\mathbb{A}_2$-rings in cyclotomic spectra satisfying* Segal$(\leq b)$*. If $\mu \in \pi_{2p^k}R$ is a Bökstedt class, then the class $i(\mu) \in \pi_{2p^k}R$ is a Bökstedt class.*

Proof. First, as i is hcentral by hypothesis, by Remark 5.8.3, both $i(\mu)$ and $\varphi(i(\mu))$ are hcentral. Since $\varphi(\mu)$ is a unit, $\varphi(i(\mu)) = i^{tC_p}(\varphi(\mu))$ is a unit as well. Condition (b), that μ be in the image of the transfer, follows from naturality of the transfer map. ☐

Lemma 5.2.33. *Let R be an $h\mathbb{A}_2$-ring in cyclotomic spectra satisfying* Segal$(\leq b)$ *and let $\mu \in \pi_{2p^k}R$ be a Bökstedt class.*

(1) R/μ is $(b+1+2p^k)$-truncated.

(2) The induced map $\varphi : R[\mu^{-1}] \to R^{tC_p}$ is an isomorphism

Proof. Using that $\varphi(\mu)$ is a unit we have an isomorphism $\mathrm{fib}(\varphi_R)/\mu \cong R/\mu$ and we can then use our hypothesis that R satisfies Segal$(\leq b)$ to conclude that $R/\mu \cong \mathrm{fib}(\varphi_R)/\mu$ is $(b+1+2p^m)$-truncated. This verifies (1). As $\mathrm{fib}(\varphi_R)$ is b-truncated the action of μ on this fiber is locally nilpotent. It follows that $\mathrm{fib}(\varphi_R)[\mu^{-1}] = 0$ and we obtain (2). ☐

Lemma 5.2.34. *Let $b \geq 0$ and let $m \geq 2m_p^{\mathbb{A}_2}$. The cyclotomic spectrum $R = \mathbb{S}/p^m \otimes \tau_{\leq b}^{\mathrm{cyc}}\mathbb{S}$ admits an hcring structure in* CycSp*, satisfies* Segal$(\leq b+1)$ *and has a Bökstedt class $\mu \in \pi_{2p^k}R$ for $k = (\lfloor b/2 \rfloor + 1)m$.*

Proof. Let $R \coloneqq \mathbb{S}/p^m \otimes \tau_{\leq b}^{\mathrm{cyc}}\mathbb{S}$. The truncation of the unit, $\tau_{\leq b}^{\mathrm{cyc}}\mathbb{S}$, is a commutative algebra and, since $m \geq m_p^{hc}$, $\mathbb{S}/p^m$ admits an hcring structure, therefore R admits an hcring structure. Using that $R \in \mathrm{CycSp}_{[0,b+1]}$ and p^m acts by zero on R we learn from Proposition 5.2.30 that R satisfies Segal$(\leq b+1)$ and WCV$(\leq b+1+2(b+2)m)$. Moreover, the spectrum $\mathrm{TC}(R)$ is bounded in the range $[-1, b+1]$.

Give $\mathrm{TC}(R)$ the trivial $\mathbb{T}$-action and consider the $\mathbb{T}$-tate sseq which has signature

$$\pi_*\mathrm{TC}(R)[t^{\pm 1}] \cong E_2^{s,t} \Longrightarrow \pi_s\mathrm{TC}(R)^{t\mathbb{T}}$$

where $|t| = (-2, 2)$. Note that this spectral sequence degenerates at the E_{b+4}-page for degree reasons and the powers of t can only (possibly) support a d_r-differential for $r \leq b+2$ and even. Using that $\mathrm{TC}(R)$ is an hcring, the Leibniz rule for differentials implies (inductively) that for every $i \in \mathbb{Z}$

$$d_{2r}((t^{i\cdot p^{(r-1)m}})^{p^m}) = p^m \cdot d_{2r}(t^{i\cdot p^{(r-1)m}}) = 0$$

Put together, we obtain a unit $u \in \pi_{2p^{b'm}}\mathrm{TC}(R)^{t\mathbb{T}}$ detected by the permanent cycle $t^{-p^{b'm}}$ where $b' \coloneqq \lfloor b/2 \rfloor + 1$.

The natural $\mathbb{T}$-equivariant hcring map $\mathrm{TC}(R) \to R$ gives us an hcring map $\mathrm{TC}(R)^{t\mathbb{T}} \to R^{t\mathbb{T}}$ and we let v denote the image of u under this map. From the fact that R satisfies Segal$(\leq b+1)$, we learn that $\varphi^{h\mathbb{T}}\colon R^{h\mathbb{T}} \to R^{t\mathbb{T}}$ is an isomorphism on homotopy groups starting in degree $b+3$. Similarly, examining the cofiber sequence

$$\Sigma R_{h\mathbb{T}} \xrightarrow{\mathrm{Nm}} R^{h\mathbb{T}} \xrightarrow{\mathrm{can}} R^{t\mathbb{T}}$$

the fact that R satisfies $\mathrm{WCV}(\leq b+1+2(b+2)m)$ implies that the map Nm is surjective on homotopy groups in degrees larger than $b+1+2(b+2)m$. Since $2p^k = 2p^{b'm} > b+1+2(b+2)m \geq b+3$ we may lift v along $\varphi^{h\mathbb{T}}$ and Nm to a class $\hat{\mu} \in \pi_{2p^k}\Sigma R_{h\mathbb{T}}$.

Let $\mu := \mathrm{tr}(\hat{\mu}) \in \pi_{2p^k}R$. We will show that μ is a Bökstedt class. Condition (a) is automatic since R is an hcring. Condition (b) follows from the fact that $\mu = \mathrm{tr}(\hat{\mu})$. Finally we wish to prove that $\varphi(\mu)$ is a unit. From the commutative diagram

$$\begin{tikzcd}
\Sigma R_{h\mathbb{T}} \arrow[r, "\mathrm{Nm}"] \arrow[dr, "\mathrm{tr}"] & R^{h\mathbb{T}} \arrow[r, "\varphi^{h\mathbb{T}}"] \arrow[d, "i"] & R^{t\mathbb{T}} \arrow[d, "i'"] \\
& R \arrow[r, "\varphi"] & R^{tC_p}
\end{tikzcd}$$

we can read off that $\varphi(\mu) = i'(v)$ and, since v is a unit and i' an hcring map, that $\phi(\mu)$ is a unit as well. ☐

Corollary 5.2.35. *Let $c \leq b$ and $m \geq m_p^{\mathbb{A}_2}$. Let R be an $h\mathbb{A}_2$ ring in cyclotomic spectra with $p^m = 0$ such that $R \in \mathrm{CycSp}_{[c,b]}$. Then R admits a Bökstedt class $\mu \in \pi_{2p^k}R$ for $k = (b-2c+2)m$.*

Proof. Because R is cyclotomically $\leq b$, it satisfies $\mathrm{Segal}(\leq b)$ by Proposition 5.2.29.

Let $e := b - 2c$. We claim that the boundedness hypothesis on R implies it can naturally be refined to an $h\mathbb{A}_2$-ring in $\mathrm{Mod}(\mathrm{CycSp}; \tau_{\leq e}^{\mathrm{cyc}}\mathbb{S})$. Indeed, the underlying cyclotomic spectrum uniquely lifts to $\mathrm{Mod}(\mathrm{CycSp}; \tau_{\leq e}^{\mathrm{cyc}}\mathbb{S})$ because it is bounded in the range $[c, b]$, and the multiplication map $R \otimes R \to R$ uniquely factors through $\tau_{\leq b}(R \otimes R)$, which is bounded in the range $[2c, b]$ and agrees with $\tau_{\leq b}(R \otimes_{\tau_{\leq e}^{\mathrm{cyc}}\mathbb{S}} R)$.

Using Lemma 5.8.5 we now obtain an hcentral ring map

$$i : \mathbb{S}/p^{2m} \otimes \tau_{\leq e}^{\mathrm{cyc}}\mathbb{S} \to R.$$

Applying Lemma 5.2.34 to $\mathbb{S}/p^{2m} \otimes \tau_{\leq e}^{\mathrm{cyc}}\mathbb{S}$ we learn that it has a Bökstedt class μ in degree $2p^{k'}$ where $k' = (\lfloor e/2 \rfloor + 1)2m$. Replacing μ by its p^{th} power (if necessary) we may replace k' by $k = (e+2)m$. Applying Lemma 5.2.32 to i we learn that $i(\mu) \in \pi_{2p^k}R$ is a Bökstedt class. ☐

Lemma 5.2.36. *Let R be p-nilpotent $h\mathbb{A}_2$-algebra in cyclotomic spectra and let $\mu \in \pi_k R$ be a class such that*

1. *R/μ is b-truncated and*

2. *μ is in the image of the transfer map $\Sigma R_{h\mathbb{T}} \to R$.*

Then R satisfies $\mathrm{SCV}(\leq b)$.

Proof. Recall that the norm and transfer fit into a diagram

$$\begin{tikzcd}
\mathrm{colim}_d(\Sigma^{-1+d(1)}\mathbb{S}/a_{(1)}^d \otimes R)^{h\mathbb{T}} \arrow[r] \arrow[d, equal] & R^{h\mathbb{T}} \arrow[r] \arrow[d, equal] & (\mathbb{S}^{\mathbb{T}} \otimes R)^{h\mathbb{T}} \arrow[d, equal] \\
\Sigma R_{h\mathbb{T}} \arrow[r, "\mathrm{Nm}"] & R^{h\mathbb{T}} \arrow[r] & R.
\end{tikzcd}$$

Let $\hat{\mu}$ be a lift of μ along the transfer. Since R is p-nilpotent the colimit in the top left commutes with $\pi_*(-)$[18] and we learn that there exists some $d \gg 0$ such that $a_{(1)}^d \cdot \hat{\mu} = 0$. Using this we construct a diagram of $\mathbb{T}$-spectra witnessing the desired nullhomotopy.

$$\begin{tikzcd}
 & \tau_{>b}R \arrow[dl, dashed] \arrow[d] \arrow[dr, "0"] & \\
\Sigma^{2p^k}R \arrow[r, "\mathrm{Nm}(\hat{\mu})\cdot -"] \arrow[dr, "0"'] & R \arrow[r] \arrow[d, "a_{(1)}^d"] & R/\mathrm{Nm}(\hat{\mu}) \\
 & \Sigma^{d(1)}R &
\end{tikzcd}$$

□

5.2.4 Almost compact cyclotomic spectra

Although the category of cyclotomic spectra has very few compact objects, when equipped with the cyclotomic t-structure it has a rich collection of almost compact objects (in the sense of Section 5.8.2). The main result of this subsection, Proposition 5.2.42, shows that a cyclotomic spectrum is almost compact whenever its underlying spectrum is almost compact. Before we can prove this proposition we will need some preparatory lemmas.

Lemma 5.2.37. *Give $\mathrm{Sp}^{B\mathbb{T}}$ the pointwise t-structure. If the underlying spectrum of $X \in \mathrm{Sp}^{B\mathbb{T}}$ is almost compact, then X is almost compact as an object of $\mathrm{Sp}^{B\mathbb{T}}$.*

Proof. Let $F : K \to \mathrm{Sp}^{B\mathbb{T}}_{[c,b]}$ be a filtered diagram. Then we have isomorphisms

$$\begin{tikzcd}
\mathrm{colim}_{k\in K}\,\mathrm{map}_{\mathbb{T}}(X, F(k)) \arrow[r] \arrow[d, "\cong"] & \mathrm{map}_{\mathbb{T}}(X, \mathrm{colim}_{k\in K} F(k)) \arrow[d, "\cong"] \\
\mathrm{colim}_{k\in K}\,\mathrm{map}(X, F(k))^{h\mathbb{T}} \arrow[r] \arrow[d, "\cong"] & \mathrm{map}(X, \mathrm{colim}_{k\in K} F(k))^{h\mathbb{T}} \arrow[d, "\cong"] \\
(\mathrm{colim}_{k\in K}\,\mathrm{map}(X, F(k)))^{h\mathbb{T}} \arrow[r, "\cong"] & (\mathrm{colim}_{k\in K}\,\mathrm{map}(X, F(k)))^{h\mathbb{T}}\,.
\end{tikzcd}$$

---

[18] Recall we are working with p-complete spectra, so homotopy groups in general do not compute with filtered colimits.

The key observation here is that the objects $\mathrm{map}(X, F(k))$ are uniformly bounded above and $(-)^{h\mathbb{T}}$ commutes with filtered colimits of spectra that are uniformly bounded above. ☐

Construction 5.2.38. Let $c \leq b$, $m \geq m_p^{\mathbb{E}_1}$ and $e \geq b + 2(b-c+1)m$ be integers. Let j be the truncation natural transformation

$$j\colon (-) \Rightarrow \tau_{\leq e}(-)\colon \mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m) \to \mathrm{Sp}^{B\mathbb{T}}.$$

whiskering along

$$(-)^{tC_p}\colon \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{B\mathbb{T}}$$

We get a natural transformation

$$j^{tC_p}\colon (-)^{tC_p} \Rightarrow (\tau_{\leq e}(-))^{tc_p}.$$

We will construct a natural $\mathbb{T}$-equivariant retraction r of j^{tC_p}. ◁

Details. Let i be the corresponding covering natural transformation $i\colon \tau_{>e}(-) \Rightarrow (-)$ obtained as the fiber of j. We will construct r by providing a nullhomotopy of i^{tC_p}.

Let $R := \tau_{\leq b-c}^{\mathrm{cyc}}(\mathbb{S}/p^m)$ and let $d := (b-c+1)m$. Using Lemmas 5.2.28, 5.2.24 and Remark 5.2.21 we pick a nullhomotopy ϵ of the $\mathbb{T}$-equivariant R^{tC_p}-linear map

$$a^d_{(1)}\colon \Sigma^{-d(1)} R^{tC_p} \to R^{tC_p}$$

. As any object of $\mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m)$ is naturally an R-module in cyclotomic spectra we may construct the following diagram of $\mathbb{T}$-spectra which is natural in

$$X \in \mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m)$$

$$\begin{tikzcd}
\tau_{>e} X \arrow[r, "i"] & X \arrow[r, "\varphi"] \arrow[d, "a^d_{(1)}"] & X^{tC_p} \arrow[r] \arrow[d, "a^d_{(1)}"] & R^{tC_p} \otimes X^{tC_p} \arrow[d, "a^d_{(1)} \otimes X^{tC_p}"', ""{name=A}] \arrow[d, bend left=70, "0", ""'{name=B}] \\
& \Sigma^{d(1)} X \arrow[r, "\varphi"] & \Sigma^{d(1)} X^{tC_p} & \Sigma^{d(1)} R^{tC_p} \otimes X^{tC_p} \arrow[l, "\mu"]
\arrow[from=A, to=B, Leftrightarrow, "\epsilon"']
\end{tikzcd}$$

From this diagram we obtain a nullhomotopy of the natural transformation $\varphi \circ a^d_{(1)} \circ i$.

On the other hand, since every $X \in \mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m)$ satisfies $\mathrm{Segal}(\leq b)$ (by Proposition 5.2.29) and the source of i is pointwise $(e+1)$-connective it follows that the nullhomotopy of $\varphi \circ a^d_{(1)} \circ i$ can be uniquely lifted to a nullhomotopy of $a^d_{(1)} \circ i$. Apply $(-)^{tC_p}$ to this nullhomotopy, and using the fact that $a_{(1)}^{tC_p}$ is an isomorphism, we obtain the desired nullhomotopy of i^{tC_p}. ☐

Lemma 5.2.39. *The functor*

$$(-)^{tC_p} : \mathrm{CycSp} \to \mathrm{Sp}^{B\mathbb{T}}$$

commutes with colimits of uniformly bounded filtered diagrams of cyclotomic spectra.

Proof. Pick an $m \geq m_p^{\mathbb{E}_1}$, As forgetting to Sp and tensoring with $\mathbb{S}/p^m$ is conservative and commutes with colimits it will suffice to instead show that

$$X \mapsto (\mathbb{S}/p^m \otimes X)^{tC_p}$$

from CycSp to Sp commutes with uniformly bounded filtered colimits.

Let $F\colon K \to \mathrm{CycSp}_{[c,b]}$ be a filtered diagram with colimit[19] Y. As the $F(k)$ are all cyclotomically bounded in therange $[c, b]$, using Lemma 5.2.12 we see that Y is bounded in the range $[c, b+3]$. Using the retraction from Construction 5.2.38 with $e \gg 0$ we learn that the coassembly map

$$\operatorname*{colim}_{k\in K}(\mathbb{S}/p^m \otimes F(k))^{tC_p} \to (\mathbb{S}/p^m \otimes Y)^{tC_p}$$

in Sp is a retract of

$$\operatorname*{colim}_{k\in K}(\tau_{\leq e}(\mathbb{S}/p^m \otimes F(k)))^{tC_p} \to (\tau_{\leq e}(\mathbb{S}/p^m \otimes Y))^{tC_p}.$$

The latter map is an isomorphism since $(-)^{tC_p}$ commutes with uniformly bounded filtered colimits of spectra (for $(-)^{hC_p}$ this is clear and $(-)_{hC_p}$ is a colimit). As a retract of an isomorphism is an isomorphism we may conclude. □

Construction 5.2.40. Let $c \leq b$, $m \geq m_p^{\mathbb{E}_1}$ and $e \geq b + 2(b - c + 1)m$ be integers. Using the retraction r from Construction 5.2.38 we construct the following diagram of functors $\mathrm{CycSp}^{\mathrm{op}} \times \mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m) \to \mathrm{Sp}$

$$\begin{tikzcd}
\mathrm{map}_{\mathbb{T}}(-,-) \arrow[r, "g\mapsto g^{tC_p}\circ\varphi"] \arrow[d, "g\mapsto j\circ g"] & \mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \arrow[d, "g\mapsto j^{tC_p}\circ g"] \arrow[dr, equal] & \\
\mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \arrow[r, "g\mapsto g^{tC_p}\circ\varphi"] & \mathrm{map}_{\mathbb{T}}(-,(\tau_{\leq e}(-))^{tC_p}) \arrow[r, "g\mapsto r\circ g"] & \mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}).
\end{tikzcd}$$

Let J denote the composite map along the bottom row and let $\overline{J}$ denote the further composite

$$\mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \xrightarrow{J} \mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \to \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}((-)^{tC_p})). \qquad \triangleleft$$

Lemma 5.2.41. *Let $c \leq b$, $m \geq m_p^{\mathbb{E}_1}$ and $e \geq b + 2(b - c + 1)m$ be integers. There is an isomorphism of functors*

$$\mathrm{CycSp}^{\mathrm{op}} \times \mathrm{Mod}(\mathrm{CycSp}_{[c,b]}; \mathbb{S}/p^m) \to \mathrm{Sp}$$

between the functor sending (X, Y) to $\mathrm{map}_{\mathrm{CycSp}}(X, Y)$ and the functor sending (X, Y) to the equalizer

$$\mathrm{Eq}\left(\mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e}Y) \underset{\overline{J}}{\overset{g\mapsto(\tau_{\leq e}\varphi)\circ g}{\rightrightarrows}} \mathrm{map}_{\mathbb{T}}(X, \tau_{\leq e}(Y^{tC_p})) \right)$$

where $\overline{J}$ is the natural transformation from Construction 5.2.40.

---

[19] Note that the colimit is taken in CycSp.

Proof. Let $Z(-,-)$ denote the functor that we wish to compare with $\mathrm{map}_{\mathrm{CycSp}}(-,-)$. We have a commuting square

$$\begin{tikzcd}
\mathrm{map}_{\mathbb{T}}(-,-) \arrow[r] \arrow[d, "\varphi\circ-"] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \arrow[d, "(\tau_{\leq e}\varphi)\circ-"] \\
\mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \arrow[r] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}((-)^{tC_p}))
\end{tikzcd}$$

and Construction 5.2.40 provides us with a commuting square

$$\begin{tikzcd}
\mathrm{map}_{\mathbb{T}}(-,-) \arrow[r] \arrow[d, "g\mapsto g^{tC_p}\circ\varphi"] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \arrow[d, "\overline{J}"] \\
\mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \arrow[r] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}((-)^{tC_p})).
\end{tikzcd}$$

Note that the horizontal arrows in the two squares agree. Taking differences of the vertical maps we construct a 3x3 diagram of fiber sequences which includes the desired comparison map

$$\begin{tikzcd}
D(-,-) \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{CycSp}}(-,-) \arrow[r] \arrow[d] & Z(-,-) \arrow[d] \\
\mathrm{map}_{\mathbb{T}}(-,\tau_{>e}(-)) \arrow[r] \arrow[d, "H"] & \mathrm{map}_{\mathbb{T}}(-,-) \arrow[r] \arrow[d, "g\mapsto(\varphi\circ g)-(g^{tC_p}\circ\varphi)"] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \arrow[d, "((\tau_{\leq e}\varphi)\circ-)-\overline{J}"] \\
\mathrm{map}_{\mathbb{T}}(-,\tau_{>e}((-)^{tC_p})) \arrow[r] & \mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \arrow[r] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}((-)^{tC_p}))
\end{tikzcd}$$

We show the comparison map is an isomorphism by showing that the map H is an isomorphism.

Taking fibers in the two squares above we can read off that H is the difference of two maps H_1 and H_2 coming from the respective squares. We analyze H_1 and H_2 separately. First we show H_1 is an isomorphism. Examining the first square we see that $H_1 \cong ((\tau_{>e}\varphi)\circ -)$. From Proposition 5.2.29 we know that the target satisfies Segal($\leq b$) and therefore $(\tau_{>e}\varphi\circ -)$ is an isomorphism since $e > b$.

Next we show H_2 is null. From the diagram constructing J in Construction 5.2.40 and the definition of $\overline{J}$ we obtain a refinement of the second square to a diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathbb{T}}(-,-) \arrow[r] \arrow[d, "g\mapsto g^{tC_p}\circ\varphi"'] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}(-)) \arrow[d, "\overline{J}"] \arrow[dl, "J"] \\
\mathrm{map}_{\mathbb{T}}(-,(-)^{tC_p}) \arrow[r] & \mathrm{map}_{\mathbb{T}}(-,\tau_{\leq e}((-)^{tC_p})).
\end{tikzcd}$$

It follows that the induced map on horizontal fibers is nullhomotopic. The difference $H = H_1 - H_2$ is thus an isomorphism and we may conclude that the term $D(-,-)$ is zero. □

Proposition 5.2.42. *Let X be a bounded below, p-nilpotent cyclotomic spectrum. If the underlying spectrum of X is almost compact, then X is almost compact as a cyclotomic spectrum.*

Proof. Without loss of generality we may assume that X is connective. In order to prove the proposition we must show that for every filtered diagram $F : K \to \mathrm{CycSp}_{[c,b]}$ with colimit[20] Y the natural comparison map

$$\operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathrm{CycSp}}(X, F(k)) \to \mathrm{Map}_{\mathrm{CycSp}}(X, Y)$$

is an isomorphism. As the $F(k)$ are all cyclotomically bounded in the range $[c, b]$, using Lemma 5.2.12 we see that Y is bounded in the range $[c, b + 3]$.

Pick an $m \geq m_p^{\mathbb{E}_1}$ such that p^m acts by zero on X. Then we have $(\mathbb{S}/p^m)^\vee \otimes X \cong X \oplus \Sigma X$ and as a consequence it suffices to prove the proposition for $(\mathbb{S}/p^m)^\vee \otimes X$. Dualizing the copy of $(\mathbb{S}/p^m)^\vee$ over to the other side we return to considering X, but are now free to assume that $F^\triangleright$ takes values in $\mathrm{Mod}(\mathrm{CycSp}_{[c,b+4]}; \mathbb{S}/p^m)$.

Using Lemma 5.2.37 and the hypothesis that the underlying spectrum of X is almost compact we learn that the underlying $\mathbb{T}$-spectrum of X is almost compact (with respect to the pointwise t-structure on $\mathbb{T}$-spectra).

We analyze the comparison map using the formulas for maps of cyclotomic spectra from Lemma 5.2.41. Taking $e = b+4+2(b-c+5)m$ we obtain the following diagram where each column is a fiber sequence:

$$\begin{tikzcd}
\mathrm{colim}_{k\in K} \mathrm{Map}_{\mathrm{CycSp}}(X, F(k)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CycSp}}(X, Y) \arrow[d] \\
\mathrm{colim}_{k\in K} \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e} F(k)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e} Y) \arrow[d] \\
\mathrm{colim}_{k\in K} \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e}(F(k)^{tC_p})) \arrow[r] & \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e}(Y^{tC_p})).
\end{tikzcd}$$

We will prove the proposition by showing that the lower two horizontal maps are isomorphisms.

Using the facts that (i) X is connective, (ii) X is almost compact as a $\mathbb{T}$-spectrum and (iii) the truncation functors on $\mathrm{Sp}^{B\mathbb{T}}$ commute with filtered colimits of $\mathbb{S}/p^m$-modules,[21] we have isomorphisms

$$\begin{aligned}\operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e} F(k)) &\cong \operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathbb{T}}(X, \tau_{[0,e]} F(k)) \cong \mathrm{Map}_{\mathbb{T}}(X, \operatorname*{colim}_{k\in K} \tau_{[0,e]} F(k)) \\ &\cong \mathrm{Map}_{\mathbb{T}}(X, \tau_{[0,e]} \operatorname*{colim}_{k\in K} F(k)) \cong \mathrm{Map}_{\mathbb{T}}(X, \tau_{[0,e]} Y) \cong \mathrm{Map}_{\mathbb{T}}(X, \tau_{\leq e} Y).\end{aligned}$$

---

[20] Note that the colimit is taken in CycSp, not $\mathrm{CycSp}_{[c,b]}$.

[21] The restriction to torsion $\mathbb{T}$-spectra may seem odd, but recall our convention that $\mathrm{Sp}^{B\mathbb{T}}$ consists of p-complete spectra with $\mathbb{T}$-action.

Similarly, we have isomorphisms

$$\begin{aligned}\operatorname*{colim}_{k\in K}\mathrm{Map}_{\mathbb{T}}(X,\tau_{\leq e}(F(k)^{tC_p}))&\cong\operatorname*{colim}_{k\in K}\mathrm{Map}_{\mathbb{T}}(X,\tau_{[0,e]}(F(k)^{tC_p}))\\&\cong\mathrm{Map}_{\mathbb{T}}(X,\operatorname*{colim}_{k\in K}\tau_{[0,e]}(F(k)^{tC_p}))\cong\mathrm{Map}_{\mathbb{T}}(X,\tau_{[0,e]}(\operatorname*{colim}_{k\in K}F(k)^{tC_p}))\\&\cong\mathrm{Map}_{\mathbb{T}}(X,\tau_{[0,e]}((\operatorname*{colim}_{k\in K}F(k))^{tC_p}))\cong\mathrm{Map}_{\mathbb{T}}(X,\tau_{\leq e}(Y^{tC_p}))\end{aligned}$$

where the key step is using Lemma 5.2.39 to move the filtered colimit past the $(-)^{tC_p}$. ☐

The converse to Proposition 5.2.42 is more straightforward to prove.

Lemma 5.2.43. *Let X be a cyclotomic spectrum. If X is almost compact as a cyclotomic spectrum, then its underlying spectrum is almost compact as well.*

Proof. Let $\mathbb{S}^{\mathbb{T}}\otimes -$ denote the functor sending a spectrum $Y\in\mathrm{Sp}$ to the cyclotomic spectrum

$$\varphi:\mathbb{S}^{\mathbb{T}}\otimes Y\to 0$$

using that $(\mathbb{S}^{\mathbb{T}}\otimes Y)^{tC_p}=0$ (Lemma 5.2.1). Computing maps out to such objects we find

$$\mathrm{Map}_{\mathrm{CycSp}}(X,\mathbb{S}^{\mathbb{T}}\otimes Y)\cong\mathrm{Map}_{\mathbb{T}}(X,\mathbb{S}^{\mathbb{T}}\otimes Y)\cong\mathrm{Map}(X,Y)$$

that $\mathbb{S}^{\mathbb{T}}\otimes -$ is right adjoint to the forgetful functor CycSp $\to$ Sp. Using the formula for mapping out to $\mathbb{S}^{\mathbb{T}}\otimes Y$ and the fact that $\mathbb{S}^{\mathbb{T}}$ is (-1)-connective we can read off that the functor $\mathbb{S}^{\mathbb{T}}\otimes -$ has t-amplitude $[-1,0]$.

If the right adjoint to a functor F commutes with colimits and has bounded t-amplitude, then F sends almost compact objects to almost compact objects. Applying this to the forgetful functor CycSp $\to$ Sp we may now conclude. ☐

We end the section by using the theory we have developed to prove that $L_{\langle p^\infty\rangle}\mathbb{S}$, the object corepresenting TR, has bounded cyclotomic t-amplitude.

Lemma 5.2.44. *For any set S the functor*

$$(\oplus_S L_{\langle p^\infty\rangle}\mathbb{S})\otimes -:\mathrm{CycSp}_+\to\mathrm{CycSp}_+$$

has t-amplitude bounded in the range $[0,4]$.

Proof. Let $Y=\oplus_S L_{\langle p^\infty\rangle}\mathbb{S}$. The lower bound in the lemma follows from the fact that the underlying spectrum of Y is ≥ 0. We prove the upper bound in several stages beginning with the following subclaim:

(A) Given an $X\in(\mathrm{Sp}^{B\mathbb{T}})_{\leq b}$ on which p^m acts by zero, $Y\otimes X\in(\mathrm{Sp}^{B\mathbb{T}})_{\leq b+1}$.

As the forgetful functor $\mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}$ is t-exact, conservative, and symmetric monoidal, it suffices to prove this t-amplitude bound in Sp. As an object of Sp, we have (Example 5.2.5)

$$Y = \bigoplus_{S} \Big((\bigoplus_{i=0}^{\infty} \mathbb{S}) \bigoplus \Sigma (\bigoplus_{i=0}^{\infty} \mathbb{S}) \Big).$$

It will thus suffice to show that $(\oplus_I \mathbb{S}) \otimes X$ is b-truncated. This follows from the fact that a sum of p^m-torsion objects is p^m-torsion (and therefore p-complete).

(B) If X is cyclotomically bounded, then $\mathrm{fib}(\varphi_{Y\otimes X}) \cong Y \otimes \mathrm{fib}(\varphi_X)$.

The cyclotomic Frobenius of $Y \otimes X$ is given by the composite of maps in $\mathrm{Sp}^{B\mathbb{T}}$

$$Y \otimes X \xrightarrow{Y \otimes \varphi_X} Y \otimes X^{tC_p} \xrightarrow{\varphi_Y \otimes X^{tC_p}} Y^{tC_p} \otimes X^{tC_p} \to (Y \otimes X)^{tC_p}.$$

From Example 5.2.5, we know that the second map is an equivalence and the underlying C_p-spectrum of Y is isomorphic to $\bigoplus_S \big(\mathbb{S} \otimes \mathbb{T} \oplus \bigoplus_{i=1}^{\infty} (\mathbb{S}^0 \oplus \mathbb{S}^1)\big)$. Using Lemma 5.2.1 and [Yua23, Corollary 6.7], we may rewrite the third map above as

$$\left(\bigoplus_{S\times\mathbb{N}} (\mathbb{S}^0 \oplus \mathbb{S}^1)\right) \otimes X^{tC_p} \to \left(\left(\bigoplus_{S\times\mathbb{N}} (\mathbb{S}^0 \oplus \mathbb{S}^1)\right) \otimes X\right)^{tC_p}.$$

Since X is cyclotomically bounded, after writing the infinite sum as a filtered colimit over the finite sub-sums it follows from Lemma 5.2.39 that this map is an isomorphism. We now obtain the desired isomorphism.

(C) Suppose we are given an $X \in \mathrm{CycSp}^{\heartsuit}$ on which p acts by zero, then $Y \otimes X$ is cyclotomically 3-truncated.

From Proposition 5.2.30(3,4) we learn that X satisfies SCV($\leq$ 2) and Segal($\leq$ 0). In particular there exist some d such that the composite

$$\tau_{>2} X \to X \xrightarrow{a^d_{(1)}} \Sigma^{d(1)} X$$

is null. Using (A) we learn that $Y \otimes \tau_{\leq 2} X \in (\mathrm{Sp}^{B\mathbb{T}})_{\leq 3}$. The diagram below now witnesses that $Y \otimes X$ satisfies SCV($\leq$ 3).

$$\begin{tikzcd}
 & Y \otimes \tau_{>2} X \arrow[d] \arrow[dr, "0"] & \\
\tau_{>3}(Y \otimes X) \arrow[ur, dashed] \arrow[r] \arrow[dr, "0"'] & Y \otimes X \arrow[r] \arrow[d] & \Sigma^{d(1)} Y \otimes X \\
 & Y \otimes \tau_{\leq 2} X &
\end{tikzcd}$$

Using (A) again we find that $Y \otimes \mathrm{fib}(\varphi_X) \in (\mathrm{Sp}^{B\mathbb{T}})_{\leq 1}$. Rewriting this with the isomorphism from (B) we conclude that $Y \otimes X$ satisfies satisfies Segal(≤ 1). We learn by applying Proposition 5.2.30(1,2) that $Y \otimes X$ is cyclotomically 3-truncated as desired.

We are now ready to complete the proof. Suppose we are given a cyclotomic spectrum $X \in [c, b]$, we would like to show that $Y \otimes X$ is cyclotomically $(b+4)$-truncated. Without loss of generality we may reduce to the case where X is in the heart. As we work in p-complete cyclotomic spectra we have an isomorphism $Y \otimes X \cong \lim(Y \otimes X/p^k)$. Using this, and the fact that X/p^k is a k-fold extension of copies of X/p, we may reduce to showing that $Y \otimes X/p$ is cyclotomically 4-truncated. Finally, splitting X/p up using the t-structure we reduce to claim (C). □

5.3 THH of cochains on the circle

Given a p-complete ring spectrum with a $\mathbb{Z}$-action $R \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}}$, the first half of this paper is devoted to studying the system of cyclotomic spectra

$$\mathrm{THH}(R^{h\mathbb{Z}}) \longrightarrow \mathrm{THH}(R^{hp\mathbb{Z}}) \longrightarrow \mathrm{THH}(R^{hp^2\mathbb{Z}}) \longrightarrow \cdots \longrightarrow \mathrm{THH}(R).$$

In this section specifically, we analyze the initial example, where $R = \mathbb{S}$ with the trivial $\mathbb{Z}$-action. In other words, we look at the system of commutative algebras in cyclotomic spectra $\mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$. The key idea governing our analysis is that, since $\mathbb{S}^{B\mathbb{Z}}$ is the $\mathbb{S}$-cochains on $B\mathbb{Z}_p$, $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ is controlled by the geometry the free loop space of $B\mathbb{Z}_p$. We highlight that the underlying cyclotomic spectrum of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ was studied in [Mal17], and the underlying commutative algebra was studied in [LL23, Lemma 4.6].

In the final subsection we prove Proposition 5.1.1 from the introduction, by analyzing the fiber of the coassembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$. This proposition is the key statement we use to show that various TC coassembly maps are not an isomorphisms.

We make significant use of the spherical Witt vectors functor, which is a way of producing canonical lifts of perfect $\mathbb{F}_p$-algebras to formally étale p-complete commutative algebras. We refer the reader to Conventions (17)-(20) for relevant notation and recall the main properties of this construction:

Proposition 5.3.1 ([Lur18a, Sec. 5.2], [BSY22, Prop. 2.2, Cons. 2.33]). *There is an adjunction*

$$\mathbb{W}(-) \colon \mathrm{Perf} \rightleftarrows \mathrm{CAlg}(\mathrm{Sp}) : \pi_0^\flat(-)$$

where the right adjoint $\pi_0^\flat$ can be computed as the inverse limit along Frobenius on the commutative ring $\pi_0(-)/p$.

This adjunction witnesses Perf *as a colocalization of* CAlg(Sp)*. The essential image of the (fully faithful) functor $\mathbb{W}$ consists of those connective $R \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\mathbb{F}_p \otimes R$ is a discrete perfect $\mathbb{F}_p$-algebra. In this situation, we have $R \cong \mathbb{W}(R \otimes \mathbb{F}_p)$.*

5.3.1 As a commutative algebra

We begin by analyzing the system of commutative algebras $\mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$, saving discussion of $\mathbb{T}$-equivariance and cyclotomic structure for later subsections. In fact, it will be simpler to analyze $\mathrm{THH}(\mathbb{S}^{BM})$ for any (discrete) finite projective $\mathbb{Z}_p$-module M, together with its functoriality in M. Our primary example of interest is then obtained by specializing to the system $0 \to \cdots \to p^2\mathbb{Z}_p \to p\mathbb{Z}_p \to \mathbb{Z}_p$.

Remark 5.3.2. We remind the reader that, for any free finite rank $\mathbb{Z}$-module M, the map $\mathbb{S}^{BM_p} \to \mathbb{S}^{BM}$ is an equivalence. Indeed, the map $BM \to BM_p$ induces an equivalence on p-complete suspension spectra since it is an equivalence on $\mathbb{F}_p$-homology, so this follows from taking duals. ◁

Definition 5.3.3. Let $\mathrm{Latt}_{\mathbb{Z}_p}$ be the category of discrete, finite, projective $\mathbb{Z}_p$-modules. ◁

Construction 5.3.4. Let $\mathcal{D}$ be a category with all finite limits. Given a $G \in \mathrm{Mon}(\mathcal{D})^{\mathrm{gp}}$, we construct the following diagram in $\mathcal{D}$

$$\begin{tikzcd}
G \arrow[r, "\triangle"] \arrow[d, "\mathrm{id}"] & G \times G \arrow[d, "(a,b)\mapsto(a,b\cdot a^{-1})"] & G \arrow[l, "\triangle"'] \arrow[d, "\mathrm{id}"] \\
G \arrow[r, "a\mapsto(a,1)"] & G \times G & G, \arrow[l, "a\mapsto(a,1)"']
\end{tikzcd}$$

natural in G. Taking the pullback of these spans and remembering the maps out to the left column, we obtain a diagram

$$\begin{tikzcd}
\mathcal{L}G \arrow[rr, "\mathrm{sh}", "\cong"'] \arrow[dr] & & G \times \Omega_e G \arrow[dl] \\
& G. &
\end{tikzcd}$$

Herel the left vertical map is restriction along the inclusion of the basepoint $* \to \mathbb{T}$, and the right vertical map is the projection to the first factor. ◁

Lemma 5.3.5. *The counit of the $\mathbb{W} \vdash \pi_0^\flat$ adjunction, applied to the natural assembly map, yields an identification of commutative algebras*

$$\begin{tikzcd}
\mathbb{W}C^0(A) \arrow[r, "i"] \arrow[d] & \mathbb{W}C^0(A^\delta) \arrow[d] \\
\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S} \arrow[r] & \mathbb{S}^{\Omega_e BA}
\end{tikzcd}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$. Here, A^δ denotes A with the discrete topology, and the top horizontal map is the inclusion of continuous functions into all functions.

Proof. First, we claim that $\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}$ and $\mathbb{S}^{\Omega_e BA}$ are bounded below. For the former this follows by writing A as $\mathbb{Z}_p^j$ for some j and expanding $\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}$ as $(\mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}_p}} \mathbb{S})^{\otimes j}$, which is connective by [Lev22, Lemma 3.1]. For the latter it is clear.

By Proposition 5.3.1 it will now suffice to prove that the indicated arrows in the diagram below are isomorphisms.

$$\begin{tikzcd}
\operatorname{colim}_k \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA/p^k}} \mathbb{F}_p \arrow[r, "\cong", "(1)"'] \arrow[d, "\cong"', "(3)"] & \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA}} \mathbb{F}_p \arrow[d] & (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \otimes \mathbb{F}_p \arrow[l, "\cong"', "(2)"] \arrow[d] \\
\operatorname{colim}_k \mathbb{F}_p^{\Omega_e BA/p^k} \arrow[r] \arrow[d, "\cong"', "(5)"] & \mathbb{F}_p^{\Omega_e BA} \arrow[d, "\cong"', "(6)"] & \mathbb{S}^{\Omega_e BA} \otimes \mathbb{F}_p \arrow[l, "\cong"', "(4)"] \\
C^0(A) \arrow[r, "i"] & C^0(A^\delta) &
\end{tikzcd}$$

The top left square is constructed from the assembly map for the colimit over k and the assembly map for the suspension of augmented commutative algebras. The map labeled (3) is an isomorphism by convergence of the Eilenberg–Moore spectral sequence (see [Lur11, Corollary 1.1.10]). The map labeled (1) is an isomorphism since $H_c^*(A; \mathbb{F}_p) \cong H^*(A; \mathbb{F}_p)$.

The top right square is constructed using the unit map $\mathbb{S} \to \mathbb{F}_p$, $\mathbb{F}_p$-linearization and the assembly map for the suspension of augmented commutative algebras. The maps (2) and (4) are isomorphisms since $\mathbb{F}_p \otimes \mathbb{S}^{BA} \cong \mathbb{F}_p^{BA}$ and $\mathbb{F}_p \otimes \mathbb{S}^{\Omega_e BA} \cong \mathbb{F}_p^{\Omega_e BA}$. This follow from the fact that $\mathbb{F}_p \otimes -$ commutes with finite limits and arbitrary products that are uniformly bounded below.

The bottom square is contructed from the natural identifications and the fact that continuous functions on an $A \in \mathrm{Latt}_{\mathbb{Z}_p}$ with values in $\mathbb{F}_p$ are locally constant. ☐

Lemma 5.3.6. *The counit of the* $\mathbb{W} \vdash \pi_0^\flat$ *adjunction, the natural map* $\mathbb{S}^{BA} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{A}})$, *and the assembly map for the* $\mathbb{T}$*-shaped colimit* THH *along the cochains functor* $\mathbb{S}^{(-)}$ *together induce the following identification of commutative algebras*

$$\begin{tikzcd}
\mathbb{S}^{BA} \otimes \mathbb{W}C^0(A) \arrow[r] \arrow[d, "\cong"] & \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A^\delta) \arrow[d, "\cong"] \\
\mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] & \mathbb{S}^{\mathcal{L}BA}
\end{tikzcd}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$.

Proof. The functor $\mathbb{S}^{(-)} \colon \mathrm{Spc} \to \mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}}$ preserves products, and therefore takes grouplike commutative monoids to grouplike commutative monoids. Using naturality along the functor $\mathbb{S}^{(-)} : \mathrm{Spc} \to \mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}}$, Construction 5.3.4 provides us with identifications of $\mathbb{S}^{BA}$-algebras

$$\begin{tikzcd}
\mathbb{S}^{BA} \otimes (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \arrow[r] \arrow[d, "\cong"] & \mathbb{S}^{BA \times \Omega_e BA} \arrow[d, "\cong"] \\
\mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] & \mathbb{S}^{\mathcal{L}BA},
\end{tikzcd}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$. The lemma now follows from Lemma 5.3.5 and the fact that $\pi_0^\flat(\mathbb{S}^{BA} \otimes R) \cong \pi_0^\flat(R)$ for any $R \in \mathrm{CAlg}(\mathrm{Sp})$ ($\pi_0^\flat$ is a right adjoint with values in a 1-category). ☐

Lemma 5.3.7. *The coassembly map for the limit over BA and the counit of the $\mathbb{W} \vdash \pi_0^\flat$ adjunction fit into a pushout square of commutative algebras*

$$\begin{tikzcd} \mathbb{W}(C^0(A)) \arrow[r, "(-)_{|0}"] \arrow[d] & \mathbb{W}(\mathbb{F}_p) \arrow[d] \\ \mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{S}^{BA} \end{tikzcd}$$

natural in $A \in \mathrm{Latt}_{\mathbb{Z}_p}$. Moreover, the lower horizontal map coincides with the map

$$\operatorname*{colim}_{\mathbb{T}} \mathbb{S}^{B\mathbb{A}} \to \operatorname*{colim}_{*} \mathbb{S}^{B\mathbb{A}}$$

induced by the map $\mathbb{T} \to *$, where the colimit is taken in $\mathrm{CAlg}(\mathrm{Sp})$*.*

Proof. As $\pi_0^\flat$ is a right adjoint with values in a 1-category, we have $\pi_0^\flat \mathbb{S}^{BA} \cong \mathbb{F}_p$. It now follows from Lemma 5.3.6 and the fact that the composite

$$\mathbb{S}^{BA} \to \mathrm{THH}(\mathbb{S}^{BA}) \to \mathbb{S}^{BA}$$

is the identity that the square is a pushout.

To see the claim about the map induced by $\mathbb{T} \to *$, it suffices to consider the square

$$\begin{tikzcd} \mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] \arrow[d] & \mathrm{THH}(\mathbb{S})^{B\mathbb{A}} \arrow[d, "\cong"] \\ \mathbb{S}^{B\mathbb{A}} \arrow[r, "\mathrm{Id}"] & \mathbb{S}^{B\mathbb{A}}, \end{tikzcd}$$

whose horizontal maps are the BA coassembly maps for the identity and THH functors, and whose vertical maps are induced by the natural transformation $\mathrm{THH}(-) \to -$ arising from the projection $\mathbb{T} \to *$. ☐

The following notation will be useful in the sequel:

Construction 5.3.8. Let $\epsilon \in \pi_{-1}\mathbb{S}^{B\mathbb{Z}}$ be the class corresponding to the element 1 in $H^1(B\mathbb{Z};\mathbb{Z})$, and let $\zeta \in \pi_{-1}\,\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ be the image of ϵ under the natural map $\mathbb{S}^{B\mathbb{Z}} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$. ◁

5.3.2 As a $\mathbb{T}$-equivariant commutative algebra

Having understood the commutative algebra structure of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$, we turn to describing the circle action. Under the assembly map to $\mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$, the circle action is compatible with the circle action on the free loop space $\mathcal{L}B\mathbb{Z}_p$, and it is through this that we gain control over the situation.

Definition 5.3.9. For $w \in \mathbb{Z}_p$, we let $B\mathbb{Z}_p(w)$ denote $B\mathbb{Z}_p$ with the action of $\mathbb{T} = B\mathbb{Z}$ coming from left multiplication via the homomorphism $w\colon B\mathbb{Z} \to B\mathbb{Z}_p$.

Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Taking R-valued cochains on $B\mathbb{Z}_p(w)$ we obtain a $\mathbb{T}$-action on the commutative R-algebra $R^{B\mathbb{Z}_p}$, which we denote $R^{B\mathbb{Z}_p(w)} \in \mathrm{CAlg}(\mathrm{Sp})^{B\mathbb{T}}_{R/-}$. Note that when $w \in \mathbb{Z} \subseteq \mathbb{Z}_p$, we have $R^{B\mathbb{Z}_p(w)} \cong R^{\mathbb{T}/C_w}$. ◁

Example 5.3.10. Consider the $\mathbb{T}$-action on $\mathcal{L}B\mathbb{Z}_p$, the free loop space of the p-adic circle. On the connected component of the degree w map $w\colon B\mathbb{Z} \to B\mathbb{Z}_p$, the rotation action on the source circle becomes the 'w-speed' rotation action on the target. Altogether, we obtain a $\mathbb{T}$-equivariant isomorphism

$$\mathcal{L}B\mathbb{Z}_p \cong \coprod_{w \in \mathbb{Z}_p} B\mathbb{Z}_p(w). \qquad ◁$$

Lemma 5.3.11. *In $\pi_0 \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ we have $\sigma(\zeta) = (1 + \eta\zeta) \cdot \mathrm{Id}_{\mathbb{Z}_p}$.*[22]

Proof. From Lemma 5.3.6 we can read off that the assembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \to \mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$ is injective on homotopy groups, and it will therefore suffice to compute $\sigma(\zeta)$ in the target. Breaking things up accross the coproduct in Example 5.3.10 and using the fact that $\mathbb{Z} \subset \mathbb{Z}_p$ is dense we reduce to computing $\sigma(\epsilon)$ in $\pi_0(\mathbb{S}^{B\mathbb{Z}(w)})$ where $w \in \mathbb{Z}$. Writing $\mathbb{S}^{B\mathbb{Z}(w)}$ as $w^*\mathbb{S}^{\mathbb{T}}$ and using that the degree w map $\mathbb{T} \to \mathbb{T}$ sends σ to $w\sigma$ we find that it will suffice to compute that $\sigma(\epsilon) = 1 + \eta\epsilon$ in $\mathbb{S}^{\mathbb{T}}$. After rationalization this is straightforward, the general case follows from the fact that $\sigma \circ \sigma = \eta\sigma$ (see [AN21, Sec. 3.5]). □

Construction 5.3.12. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Tensoring the p-fold covering map $p\colon \mathbb{T} \to \mathbb{T}/C_p$ with the commutative algebra R we obtain a map of $\mathbb{T}$-equivariant commutative algebras

$$\psi_p\colon\ \mathrm{THH}(R) \to p^* \mathrm{THH}(R)$$

natural in R where p^* denotes the restriction map $\mathrm{Sp}^{B\mathbb{T}/C_p} \to \mathrm{Sp}^{B\mathbb{T}}$.

As the map $p\colon \mathbb{T} \to \mathbb{T}/C_p$ preserves the base-point ψ_p also carries the structure of a commutative R-algebra map (but not compatibly with the circle action). ◁

Lemma 5.3.13. *The map ψ_p refines to a $\mathbb{T}$-equivariant isomorphism*

$$\psi_p\colon\ \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p} \xrightarrow{\cong} p^* \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$$

which induces the map $\mathrm{res}_p\colon C^0(p\mathbb{Z}_p) \to C^0(\mathbb{Z}_p)$ on $\pi_0^\flat$. In particular, the C_p action on $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|p\mathbb{Z}_p}$ is trivializable.

---

[22] Note that $\pi_0 C^0(\mathbb{Z}_p)$ is the ring of cont. functions from $\mathbb{Z}_p$ to itself. By $\mathrm{Id}_{\mathbb{Z}_p} \in \pi_0\mathbb{W}C^0(\mathbb{Z}_p)$ we mean the identity function.

Proof. As the assembly maps to $\mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$ are injective on homotopy groups by Lemma 5.3.6 and the construction of ψ_p is natural it will suffice to prove the corresponding claim for $\mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$. Here we observe that precomposition with the degree p map of S^1 produces the map $\mathcal{L}B\mathbb{Z}_p \to \mathcal{L}B\mathbb{Z}_p$ which sends the circle at component $a \in \mathbb{Z}_p$ to the circle at component pa isomorphically. □

Lemma 5.3.14. *For each $k \geq 0$ there is an isomorphism of $\mathbb{W}C^0(p^k\mathbb{Z}_p^\times)$-modules in $\mathrm{Sp}^{B\mathbb{T}}$*

$$\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|p^k\mathbb{Z}_p^\times} \cong \mathbb{W}C^0(p^k\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}/C_{p^k}].$$

In particular, if M is a $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}$-module, then $M^{tC_{p^j}} = 0$ for all $1 \leq j \leq \infty$.

Proof. Using Lemma 5.3.13 it will suffice to handle the case where $k = 0$.

Using the restriction of ζ to $\pi_{-1}\,\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}$ we construct an induced $\mathbb{T}$-equivariant map of $\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules

$$z : \mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}] \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}.$$

On homotopy groups this gives a map of $\pi_*\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules

$$z : (\pi_*\mathbb{W}C^0(\mathbb{Z}_p^\times))\{[*], [\mathbb{T}]\} \to C^0(\mathbb{Z}_p^\times)\{1, \zeta\}$$

with $z([*]) = \zeta$. The Connes operator and Lemma 5.3.11 now let us compute that

$$z([\mathbb{T}]) = z(\sigma([*])) = \sigma(z([*])) = \sigma(\zeta) = \mathrm{Id}_{\mathbb{Z}_p}(1 + \eta\zeta)$$

The first claim follows from that fact that, when restricted to $\mathbb{Z}_p^\times$, $\mathrm{Id}_{\mathbb{Z}_p}$ is a unit.

The second claim follows from the first since we now know $a_{(1)} = 0$ as a $\mathbb{T}$-equivariant self-map of $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}$ and hence that $a_{(1)}$ acts by zero on any $\mathbb{T}$-equivariant $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}_p})_{|\mathbb{Z}_p^\times}$-module M. □

Lemma 5.3.15. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})^{BC_p}$ be bounded below. In the associated spain*

$$\pi_0^\flat R \leftarrow \pi_0^\flat R^{hC_p} \to \pi_0^\flat R^{tC_p}$$

the left arrow is an isomorphism if the action on $\pi_0 R$ is trivial (e.g. when the C_p actions extends to a $\mathbb{T}$-action) and the right arrow is an isomorphism if the C_p-action is trivial.

Proof. The first claim follows from $\pi_0^\flat(-)$ being a right adjoint that factors through the functor $\pi_0(-)$ and has target a 1-category.

We now prove the second claim. Using the fact that (i) the Postnikov tower refines the map $\mathbb{S} \to \mathbb{Z}_p$ to an ω-indexed tower of square-zero extensions, (ii) the fact that $(-)^{hC_p}$ and $(-)^{tC_p}$ are exact and commute with uniformly bounded below limits and (iii) nil-invariance of $\pi_0^\flat$ [BSY22, Lem. 2.16] we reduce to proving the lemma for $\mathbb{Z}_p \otimes R$. In this case we may identify the map $\pi_*(\mathbb{Z}_p \otimes R)^{hC_p} \to \pi_*(\mathbb{Z}_p \otimes R)^{tC_p}$ with the map $\pi_*(\mathbb{F}_p \otimes R)[\![t]\!] \to \pi_*(\mathbb{F}_p \otimes R)(\!(t)\!)$ where $|t| = -2$. As R is bounded below and t is in a negative degree this is a nil-extension in degree 0 and we may conclude. □

Remark 5.3.16. In the setting of Lemma 5.3.15, a trivialization of the C_p-action on R gives a section $R \to R^{hC_p}$ of the map $R^{hC_p} \to R$ and through this we see that the map

$$\pi_0^\flat R \to \pi_0^\flat R^{tC_p}$$

associated to a trivialization of the C_p-action on R will agree with the isomorphism from Lemma 5.3.15. ◁

Construction 5.3.17. Using Lemmas 5.3.14, 5.3.13 and 5.3.15 we fix an identifcation

$$\begin{tikzcd}
\pi_0^\flat \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}}) \arrow[d, "\cong"] & \pi_0^\flat \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}})^{hC_p} \arrow[l] \arrow[r] \arrow[d, "\cong"] & \pi_0^\flat \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}})^{tC_p} \arrow[d, "\cong"] \\
C^0(\mathbb{Z}_p) & C^0(\mathbb{Z}_p) \arrow[l, "\mathrm{Id}"'] \arrow[r, "(-)_{|p\mathbb{Z}_p}"] & C^0(p\mathbb{Z}_p)
\end{tikzcd}$$

which on the left column agrees with the one from Lemma 5.3.6. ◁

5.3.3 As a cyclotomic spectrum

The cyclotomic Frobenius on $\operatorname{THH}(\mathbb{S}^{B\mathbb{Z}})$ is an isomorphism and it turns out that this cyclotomic spectrum is closely related to the one corepresenting TR.

Proposition 5.3.18. *The cyclotomic Frobenius map* $\varphi\colon \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \to \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p}$ *is an isomorphism and under the identifications from Construction 5.3.17* $\pi_0^\flat(\varphi)$ *is given by* $\mathrm{res}_{1/p}$ *where* $1/p$ *is the map* $1/p\colon p\mathbb{Z}_p \to \mathbb{Z}_p$*.*

Proof. We will construct the following diagram

$$\begin{tikzcd}
\mathbb{S}^{B\mathbb{Z}_p} \arrow[r, "\Delta_p"] \arrow[d] & ((\mathbb{S}^{B\mathbb{Z}_p})^{\otimes p})^{tC_p} \arrow[r, "m"] \arrow[d, "(\mu_p^*)^{tC_p}"] & (\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} \arrow[d] & \mathbb{S}^{B\mathbb{Z}_p} \arrow[l, "\mathrm{can}", "\cong"'] \arrow[d] \\
\operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \arrow[r, "\varphi"] & \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} \arrow[r, "\psi_p^{tC_p}"] & (p^* \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p}))^{tC_p} & \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}_p}) \arrow[l, "\cong"']
\end{tikzcd}$$

of shape $\Delta^3 \times (* \to B\mathbb{T})$. The left square is from [NS18, Cor. IV.2.3]. The middle square is obtained by tensoring the commutative algebra $\mathbb{S}^{B\mathbb{Z}_p}$ with the square associated to the short exact sequence $C_p \to \mathbb{T} \to \mathbb{T}/C_p$ and applying $(-)^{tC_p}$. The right square is constructed using a trivialization of the C_p action on $* \to \mathbb{T}/C_p$. The indicated isomorphisms are from [Yua23, corollary 6.7].

The map $\psi_p^{tC_p}$ is an isomorphism by Lemmas 5.3.13 and 5.3.14. Using the universal property of THH as a commutative algebra we will show that the composite along the bottom row is the identity by showing that the composite along the top row is the identity. The composite $m \circ \Delta_p$ is the Tate-valued Frobenius. As the Tate-valued Frobenius composed with can^{-1} is the identity for $\mathbb{S}$, naturality in the limit over $B\mathbb{Z}_p$ implies it is the identity for $\mathbb{S}^{B\mathbb{Z}_p}$ as well.

The identification of $\pi_0^\flat\varphi$ now follows from applying Lemma 5.3.13, Lemma 5.3.15 and Construction 5.3.17. □

Proposition 5.3.19. *Let* $L_{\langle p^\infty \rangle}\mathbb{S} \in \mathrm{CycSp}$ *denote the object corepresenting* $\mathrm{TR}(-)$. *There is a fiber sequence of cyclotomic spectra*

$$\mathbb{W}(C^0(\mathbb{Z}_p^\times)) \otimes \Sigma^{-1} L_{\langle p^\infty \rangle}\mathbb{S} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$$

where the second map is the natural coassembly map.

Proof. Let F denote the fiber of the coassembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$. As it is the fiber of a map of commutative algebras, F is naturally a non-unital commutative algebra in CycSp.

Using Lemma 5.3.7 we can read off that as a $\mathbb{T}$-equivariant nonunital commutative algebra F is isomorphic to the direct sum $\oplus_{k\geq 0} F_{|p^k\mathbb{Z}_p^\times}$. By Lemma 5.3.14 we have isomorphisms of $\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules in $\mathrm{Sp}^{B\mathbb{T}}$

$$F_{|p^k\mathbb{Z}_p^\times} \cong \mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}/C_{p^k}]$$

and by Proposition 5.3.18 the cyclotomic Frobenius on F breaks up as a sum[23] of isomorphisms $F_{|p^k\mathbb{Z}_p^\times} \cong F^{tC_p}_{|p^{k+1}\mathbb{Z}_p^\times}$ of $\mathbb{W}C^0(\mathbb{Z}_p^\times)$-modules in $\mathrm{Sp}^{B\mathbb{T}}$.[24] The proposition now follows from Lemma 5.2.7. ☐

Corollary 5.3.20. *The cyclotomic spectrum* $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ *has* t*-amplitude in the range* $[-1, 3]$ *as an object of* CycSp_+.

Proof. Using the cofiber sequence from Proposition 5.3.19 and the fact that $\mathrm{THH}(\mathbb{S})$ is the unit of cyclotomic spectra we can read off that it will suffice to show that $\mathbb{W}(C^0(\mathbb{Z}_p^\times)) \otimes \Sigma^{-1} L\mathbb{S}_{\langle p^\infty \rangle}$ has t-amplitude in the range $[-1, 3]$. Writing $\mathbb{W}(C^0(\mathbb{Z}_p^\times))$ as a sum of spheres, we conclude by applying Lemma 5.2.44. ☐

5.3.4 The failure of hyperdescent and $\mathbb{A}^1$-invariance

Having studied $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ as a cyclotomic spectrum, we briefly pause to extract some consequences. The following implies Proposition 5.1.1 of the introduction.

Corollary 5.3.21. *Let* $R \in \mathrm{Alg}(\mathrm{Sp})$ *be connective. Then* $\mathbb{W}C^0(\mathbb{Z}_p^\times) \otimes \mathrm{TC}(R)$ *is in the thick subcategory generated by the fiber of the coassembly map*

$$\mathrm{TC}(R^{B\mathbb{Z}}) \to \mathrm{TC}(R)^{B\mathbb{Z}}.$$

Proof. From the fiber sequence $\mathrm{TC} \to \mathrm{TR} \xrightarrow{1-F} \mathrm{TR}$, we deduce that there is a corresponding cofiber sequence of corepresenting objects $L_{\langle p^\infty \rangle}\mathbb{S} \to L_{\langle p^\infty \rangle}\mathbb{S} \to \mathbb{S}$ in CycSp. Combining this with Proposition 5.3.19 we learn that $\mathbb{W}C^0(\mathbb{Z}_p^\times)$ is in the thick subcategory of CycSp generated by the fiber of the coassembly map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S})^{B\mathbb{Z}}$.

Using the fact that THH is symmetric monoidal, the fact that $\mathbb{W}(C^0(\mathbb{Z}_p^\times))$ is a p-adic sum of spheres and the fact that $\mathrm{TC} : \mathrm{CycSp}_{\geq 0} \to \mathrm{Sp}$ preserves colimits [CMM21, Theorem 2.7], we conclude. ☐

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[23] By this we mean the sum of the maps composed with the assembly map for tC_p for the infinite sum.

[24] Note that here we have freely rescaled $p^k\mathbb{Z}_p^\times$ to make the same algebra act on all objects.

The remainder of this section is dedicated to discussing consequences of Corollary 5.3.21 for our understanding of $K(n+1)$-local K-theory of $T(n)$-local rings. Note that the material in in the remainder of the section is not cited in later sections, though the argument of Theorem 5.3.22 is used in the proof of Theorem 5.6.25.

Recall that $K(1)$-local K-theory of connective commutative $\mathbb{Q}$-algebras is an incredibly well behaved invariant.

1. It satisfies étale hyperdescent under mild finiteness conditions [Tho85] (see [CM21]).
2. It satisfies nil-invariance [LMMT20, Corollary 4.23].
3. It satisfies $\mathbb{A}^1$-invariance: $L_{K(1)}K(R) \cong L_{K(1)}K(R[t])$ [LMMT20, Corollary 4.24].

Unfortunately, as a consequence of Corollary 5.3.21, $K(n+1)$-local K-theory of $T(n)$-local algebras does not share any of these properties when $n \geq 1$. This follows from the following result, setting $X = L_{K(n+1)}\mathbb{S}$.

Theorem 5.3.22. *Let R be a $T(n)$-local $\mathbb{E}_1$-ring for $n \geq 1$ and let X be a spectrum. If $F(R) \coloneqq L_{T(n+1)}(X \otimes K(R)) \neq 0$, then none of the maps*

$$\begin{aligned} F(R) &\to F(R[t]) \\ F(R) &\to F(R\langle \epsilon_{-1} \rangle) \\ F(R^{B\mathbb{Z}}) &\to F(R)^{B\mathbb{Z}} \end{aligned}$$

are isomorphisms.

Remark 5.3.23. Recall that $R\langle \epsilon_{-1} \rangle$ denotes a trivial square-zero extension by a class in degree -1. Note also that $L_{T(n+1)}K(R[t]) \cong L_{T(n+1)}K(L_{T(n)}(R[t]))$ by the purity results of [LMMT20]. ◁

Proof. We first observe that the first two claims are equivalent to each other. Indeed, it follows from [LT23, Theorem 4.1][25] that $K(R[\epsilon_{-1}]) \cong K(R) \oplus \Sigma^{-1}NK(R)$ and $K(R[t]) \cong K(R) \oplus NK(R)$, so that both claims are equivalent to the nonvanishing of $L_{T(n+1)}(X \otimes NK(R))$. By Theorem 5.6.1, this is equivalent to asking that $L_{T(n+1)}(X \otimes N\mathrm{TC}(\tau_{\geq 0}R)) \neq 0$.

By [McC23], the cofiber of $\mathbb{S}_p \to \mathrm{THH}(\mathbb{S}[t])_p$ corepresents TR_p in $(\mathrm{CycSp})_p$, where $(\mathrm{CycSp})_p$ is the p-completion of the category of integral cyclotomic spectra as opposed to the category of p-typical p-complete cyclotomic spectra. Thus $\mathbb{S}_p$ is in the thick subcategory it generates in $(\mathrm{CycSp})_p$. Tensoring this with $\mathrm{THH}(R)$ and applying $L_{T(n+1)}(X \otimes \mathrm{TC})$, we learn that $L_{T(n+1)}(X \otimes \mathrm{TC}(R)) \neq 0$ is in the thick subcategory generated by $L_{T(n+1)}(X \otimes N\mathrm{TC}(\tau_{\geq 0}R))$, allowing us to conclude that the latter is nonzero.

The last statement about

$$F(R^{B\mathbb{Z}}) \to F(R)^{B\mathbb{Z}}$$

not being an equivalence follows similarly, one only need apply Corollary 5.3.21 and Corollary 5.6.3 instead of [McC23] and Theorem 5.6.1. ☐

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[25] see also the discussion in [BL23, Example 4.9]

Remark 5.3.24. The first map in Theorem 5.3.22 tests $\mathbb{A}^1$-invariance. The second map tests nil-invariance.[26] The third map tests $\mathbb{Z}_p$-Galois hyperdescent along the p-adic $\mathbb{Z}_p$-Galois extension $R^{B\mathbb{Z}} \to R$. ◁

The results of [BSY22] imply that Theorem 5.3.22 applies to any $T(n)$-local commutative algebra for $X = \mathbb{S}$ or $X = L_{K(n+1)}\mathbb{S}$.

Remark 5.3.25. The failure of $\mathbb{Z}_p$-Galois hyperdescent for $T(n+1)$-local K-theory for $n \geq 1$ in the case of a trivial action leads to an expectation that hyperdescent should also fail for $\mathbb{Z}_p$-Galois extensions that sufficiently "close" to the trivial action. The next section is devoted to studying one instance where this expectation is correct.

Indeed, we disprove the telescope conjecture by showing that hyperdescent fails $T(n+1)$-locally for the $\mathbb{Z}_p$-Galois extension $L_{T(n)}\mathrm{BP}\langle n\rangle^{h\mathbb{Z}} \to L_{T(n)}\mathrm{BP}\langle n\rangle$ (it behaves enough like a trivial action) then constrasting this with the results of [BMCSY23] which imply that this extension *does* satisfy hyperdescent $K(n+1)$-locally. ◁

5.4 Locally unipotent $\mathbb{Z}$-actions & Lichtenbaum–Quillen

In this section we prove our cyclotomic asymptotic constancy theorem (Theorem C of the introduction). This theorem says that under strong finiteness hypotheses a ring spectrum R with a $\mathbb{Z}$-action will behave as though the $p^k\mathbb{Z}$-action obtained by restriction is trivial when computing THH and TC. We begin by introducing the precise finiteness hypotheses we will need.

Definition 5.4.1. Let $n \geq -1$. Following [MR99] we say that a p-complete bounded below spectrum X is of fp-type n if, for each finite spectrum U of type $n+1$, $U \otimes X$ is π-finite.[27] ◁

We will also need a cyclotomic version of being of fp-type n.

Definition 5.4.2. Let $n \geq -1$. We say that an $\mathbb{E}_1$-ring R satisfies the height n Lichtenbaum–Quillen property if THH(R) is bounded below and, for each finite spectrum V of type $n+2$, the cyclotomic spectrum $V \otimes \mathrm{THH}(R)$ is bounded.[28] ◁

Remark 5.4.3. If R satisfies the height n LQ property, then for any finite spectrum U of type $n+1$ the localization map

$$U \otimes \mathrm{TC}(R) \to L_{T(n)}(U \otimes \mathrm{TC}(R))$$

induces an isomorphism on $\pi_*(-)$ for $* \gg 0$. which can be viewed as a TC version of the Lichtenbaum–Quillen conjecture for R. ◁

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[26] Note that in the $T(n)$-local setting many examples are periodic and there can be no particular distinction between square-zero extension on classes in positive degree and square-zero extensions on classes in negative degree.

[27] Note that by the thick subcategory theorem, it suffice to check this property for a single finite, type $n+1$ spectrum U with $K(n+1) \otimes U \neq 0$.

[28] As above.

The main theorem of the section is then the following:

Theorem 5.4.4. *Suppose we are given an* $R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$ *that is connective, of fp-type* $n \geq -1$*, and satisfies the height* n *LQ property. Then, for all* $k \gg 0$ *sufficiently large,* $R^{hp^k\mathbb{Z}}$ *satisfies the height* n *LQ property as well.*

Furthermore, if V *is a finite spectrum of type* $n+2$*, then for* $k \gg 0$ *sufficiently large there is an isomorphism*

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong V \otimes \mathrm{THH}(R^{Bp^k\mathbb{Z}})$$

of $\mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$*-modules in cyclotomic spectra.*

5.4.1 THH of algebras with locally unipotent $\mathbb{Z}$-actions

In this subsection we begin our investigation of the THH of algebras with unipotent $\mathbb{Z}$-actions. Here we focus on some elementary observations which follow from the work in Section 5.3 in a straightforward way.

Convention 5.4.5. Throughout this section we let $W_k := \mathrm{THH}(\mathbb{S}^{Bp^k\mathbb{Z}})$ and $W := W_0$. The system of natural maps

$$\mathbb{S}^{B\mathbb{Z}} \to \mathbb{S}^{Bp\mathbb{Z}} \to \mathbb{S}^{Bp^2\mathbb{Z}} \to \cdots \to \mathbb{S}$$

induces a corresponding sequence of maps of commutative algebras in cyclotomic spectra

$$W \to W_1 \to W_2 \to \cdots W_\infty \cong \mathbb{S}.$$

Given an $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$ we will use the natural W_k-modules structure on the cyclotomic spectrum $\mathrm{THH}(R^{hp^k\mathbb{Z}})$ throughout the section.

For a W_k-module X, we sometimes use $X_{|0}$ to refer to the cyclotomic spectrum $X \otimes_{W_k} \mathbb{S}^{Bp^k\mathbb{Z}}$, where the map $W_k \to \mathbb{S}^{Bp^k\mathbb{Z}}$ is the coassembly map, which on underlying is given by restriction to 0 by Lemma 5.3.7. ◁

Remark 5.4.6. Note that given $j < k$ in fact $\mathbb{S}^{Bp^j\mathbb{Z}} \cong \mathbb{S}^{Bp^k\mathbb{Z}}$ and therefore $W_j \cong W_k$. However, the natural map $W_j \to W_k$ is not an isomorphism. ◁

Lemma 5.4.7. *Let* $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$*. The induced* $\mathbb{Z}$ *action on* $\mathrm{THH}(R) \in \mathrm{CycSp}$ *is locally unipotent (as is the action on the underlying spectrum of* $\mathrm{THH}(R)$*).*

If R *is connective, then the action on* $\mathrm{TC}(R)$ *is locally unipotent as well.*

Proof. The forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}$ is a conservative left adjoint, so by Lemma 5.8.26 it suffices to show that the action on the spectrum $\mathrm{THH}(R)$ is locally unipotent. For this we observe that by Lemma 5.8.18 $\mathrm{Sp}^{B\mathbb{Z},u}$ is closed under tensor products and colimits and use the tensor product formula for THH.

The final claim follows by applying Lemma 5.8.26 to $\mathrm{TC}\colon \mathrm{Sp} \otimes \mathrm{CycSp}_{\geq 0} \to \mathrm{Sp}$ using [CMM21, Theorem 2.7]. □

Lemma 5.4.8. *Let $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$. The natural map*

$$W_k \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}) \xrightarrow{\cong} \mathrm{THH}(R^{hp^k\mathbb{Z}})$$

is an isomorphism of W_k-modules in CycSp.

Proof. Since THH is a symmetric monoidal functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{CycSp}$, it will suffice to show that the map $R^{h\mathbb{Z}} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathbb{S}^{Bp^k\mathbb{Z}} \to R^{hp^k\mathbb{Z}}$ is an isomorphism. This follows from Lemma 5.8.21 and Lemma 5.4.7. ☐

Lemma 5.4.9. *The coassembly map fits into a commuting diagram of lax symmetric monoidal functors from* $\mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$ *to* CycSp,

$$\begin{tikzcd}
 & \mathrm{THH}((-)^{h\mathbb{Z}}) \arrow[dl] \arrow[dr] & \\
\mathrm{THH}((-)^{h\mathbb{Z}})_{|0} \arrow[rr, "\cong"] & & \mathrm{THH}(-)^{h\mathbb{Z}}.
\end{tikzcd}$$

Proof. We begin with the lax symmetric monoidal natural transformation

$$\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R)^{h\mathbb{Z}}$$

given by the coassembly map. We know from Lemma 5.3.7 that when $R = \mathbb{S}$ the coassembly map agrees with the restriction map $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})_{|0}$ so by base-change we obtain a lax symmetric monoidal natural transformation under $\mathrm{THH}(R^{h\mathbb{Z}})$:

$$\epsilon\colon \mathrm{THH}(R^{h\mathbb{Z}})_{|0} \to \mathrm{THH}(R)^{h\mathbb{Z}}.$$

In order to prove ϵ is an isomorphism we use Remark 5.8.23 to reduce to checking it is an isomorphism after applying $\mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} -$. Now we have isomorphisms

$$\begin{aligned}\mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathrm{THH}(R^{h\mathbb{Z}})_{|0} &\cong \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathrm{THH}(R^{h\mathbb{Z}}) \otimes_{\mathbb{W}C^0(\mathbb{Z}_p)} \mathbb{W}(\mathbb{F}_p) \cong W_\infty \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}) \\ &\cong \mathrm{THH}(R) \cong \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathrm{THH}(R)^{h\mathbb{Z}}\end{aligned}$$

where we have used Lemma 5.3.6 in the second step, Lemma 5.4.8 in the third step and Lemma 5.8.21 and Lemma 5.4.7 in the final step. ☐

5.4.2 The Dehn twist trivialization

Our next goal is to prove a weak version of Theorem 5.4.4 at the level of the underlying spectrum of THH together with the Frobenius map (rather than at the level of cyclotomic spectra).

We will work with rings $R \in \mathrm{Alg}(\mathrm{Sp}^{B\mathbb{Z},u})$ equipped with a finite spectrum $V \in \mathrm{Sp}^{\diamondsuit}$ and a trivialization of the $\mathbb{Z}$-action on $V \otimes R$. It will be convenient to also assume that V has an associative algebra structure and encode this as a part of the data, that is we consider the following category:

Definition 5.4.10. We let UAlg be the presentably symmetric monoidal category defined by the pullback below

$$\begin{tikzcd}
\mathrm{UAlg} \arrow[r, "{u,v}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u} \times \mathrm{Alg}(\mathrm{Sp}^{\diamondsuit}) \arrow[d, "{(R,V)\mapsto(R\otimes V)}"] \\
\mathrm{Alg}(\mathrm{Sp}) \arrow[r, "\mathrm{triv}"'] & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u}.
\end{tikzcd}$$

We will typically denote objects of UAlg by pairs (R, V) where R is the algebra with locally unipotent $\mathbb{Z}$-action and V is the finite algebra (leaving all other data implicit). $\triangleleft$

Our goal is now to study the functor

$$(R, V) \mapsto V \otimes \mathrm{THH}(R^{h\mathbb{Z}}).$$

More precisely, we will prove the following theorem which serves as a replacement for $\mathbb{Z}_p$-hyperdescent for THH.

Theorem 5.4.11. *There is a natural transformation of lax symmetric monoidal functors*

$$\eta : \mathbb{W}(C^0(\mathbb{Z}_p)) \otimes V \otimes \mathrm{res}_\varphi \left(\mathrm{THH}(R)^{h\mathbb{Z}}\right) \Rightarrow V \otimes \mathrm{res}_\varphi \mathrm{THH}(R^{h\mathbb{Z}}) : \mathrm{UAlg} \to \mathrm{Sp}^{\Delta^1}$$

such that

1. *η becomes an isomorphism after composing with pullback along $i_0 : * \to \Delta^1$.*
2. *η becomes an isomorphism upon restricting to the subcategory of those (R, V) for which the assembly map*

$$((V \otimes \mathrm{THH}(R))^{tC_p})^{\oplus\infty} \to ((V \otimes \mathrm{THH}(R))^{\oplus\infty})^{tC_p}$$

is an isomorphism.

The setup

Before proceeding we will need to set up a certain amount of notation for the functors we will use for manipulating objects of UAlg.

- Let EQ be the category $0 \rightrightarrows 1$.
- Let $\tau : \mathrm{EQ} \to \Delta^1$ be the functor identifying the two arrows.
- Recall that $B\mathbb{Z}^{\triangleright}$ sits in a pushout square

$$\begin{tikzcd}
B\mathbb{Z} \arrow[r, "i_1"] \arrow[d, "\pi"] & B\mathbb{Z} \times \Delta^1 \arrow[d, "\pi'"] \\
* \arrow[r, "i_1"] & B\mathbb{Z}^{\triangleright}. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

- Let ρ be the essentially surjective functor $\mathrm{EQ} \to B\mathbb{Z}^{\triangleright}$.
- Let c be the functor $B\mathbb{Z}^{\triangleright} \to *$.
- Let e be the functor $\mathrm{EQ} \to *$.
- By abuse of notation we will also use
  τ for the map $\mathrm{EQ} \times B\mathbb{Z} \to \Delta^1 \times B\mathbb{Z}$,
  π for the map $\mathrm{EQ} \times B\mathbb{Z} \to \mathrm{EQ}$,
  π for the map $\Delta^1 \times B\mathbb{Z} \to \Delta^1$ and
  e for the map $\mathrm{EQ} \times B\mathbb{Z} \to B\mathbb{Z}$.

Construction 5.4.12. We may construct a commuting diagram of symmetric monoidal functors

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u} \times \mathrm{Alg}(\mathrm{Sp}^{\Diamond}) \arrow[r, "(R,V)\mapsto(R\otimes V)"'] \arrow[d, "(R,V)\mapsto(R\to V\otimes R)"] & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z},u} \arrow[d] & \mathrm{Alg}(\mathrm{Sp}) \arrow[l, "\mathrm{triv}"'] \arrow[d, equal] \\
(\mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}})^{\Delta^1} \arrow[r, "\mathrm{ev}_1"'] & \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}} & \mathrm{Alg}(\mathrm{Sp}) \arrow[l, "\mathrm{triv}"']
\end{tikzcd}$$

and upon taking pullbacks of the horizontal cospans this gives us a symmetric monoidal functor

$$\mathrm{cn} : \mathrm{UAlg} \to \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$$

and a symmetric monoidal natural isomorphism

$$\rho^*\mathrm{cn}(R, V) \cong \pi^*\left(\mathbb{S} \rightrightarrows V\right) \otimes e^*R.$$

$\triangleleft$

Construction 5.4.13. Given a diagram $R \rightrightarrows S \in \mathrm{Alg}(\mathcal{C})^{\mathrm{EQ}}$, S is naturally an S-S-bimodule, so by using the two different maps $R \to S$, we can forget down to the structure of an R-R-bimodule. We thus obtain a symmetric monoidal functor

$$\mathrm{bm} : \mathrm{Alg}(\mathrm{Sp})^{\mathrm{EQ}} \to \mathrm{Bimod}\,.$$

and a symmetric monoidal natural isomorphism between $\mathrm{bm} \circ e^*$ and the functor sending $R \in \mathrm{Alg}(\mathrm{Sp})$ to $(R, R) \in \mathrm{Bimod}$ where R is viewed as an R-R-bimodule in the standard way. $\triangleleft$

- Let $\mathrm{THH}^{\mathrm{EQ}} := \mathrm{THH} \circ \mathrm{bm}$ be the symmetric monoidal functor sending a diagram $R \rightrightarrows S$ to the THH of R with coefficients in the R-bimodule S.
- Let $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon} := \mathrm{THH}_{\pentagon} \circ \mathrm{bm}$ be the symmetric monoidal functor obtained by composing bm with the p-polygonic THH functor discussed in Section 5.2.1. Note that there is a natural isomorphism of symmetric monoidal functors $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(-)^{\Phi C_p} \cong \mathrm{THH}^{\mathrm{EQ}}(-)$ (see Section 5.2.1 and [KMN23, Theorem D]).

The tilt of the unit

Our next task will be to determine the tilt of $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})$.

Construction 5.4.14. If we write the unit $\mathbb{S} \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$ as $c^*\mathbb{S}$ we may use the identification $\pi_*\rho^*c^*\mathbb{S} \cong e^*\mathbb{S}^{B\mathbb{Z}}$ and Lemma 5.2.16 to obtain isomorphisms

$$\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S}) \cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(e^*\mathbb{S}^{B\mathbb{Z}}) \cong \mathrm{res}_{\pentagon} \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}).$$

Expanding this out into an isotropy seperation square using Lemma 5.2.16 we obtain an identification of the diagrams of commutative algebras below.

$$\begin{tikzcd}
\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{\Phi C_p} \arrow[d] \\
\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{hC_p} \arrow[r] \arrow[d] & \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{tC_p} \\
\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{\Phi e} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) \arrow[d, "\varphi"] \\
\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})^{hC_p} \arrow[r, "\mathrm{can}"] \arrow[d] & \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})^{tC_p} \\
\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}}) &
\end{tikzcd}$$

Applying $\pi_0^{\flat}(-)$ to the diagrams above and using Construction 5.3.17 and Proposition 5.3.18 to identify the objects and morphisms we obtain the following diagram:

$$\begin{tikzcd}
C^0(p^{-1}\mathbb{Z}_p) \arrow[r, "(-)_{|\mathbb{Z}_p}"] \arrow[d, "\mathrm{res}_{1/p}"] & C^0(\mathbb{Z}_p) \arrow[d, "\mathrm{res}_{1/p}"] \\
C^0(\mathbb{Z}_p) \arrow[r, "(-)_{|p\mathbb{Z}_p}"] \arrow[d, "\cong"] & C^0(p\mathbb{Z}_p) \\
C^0(\mathbb{Z}_p). &
\end{tikzcd}$$

Through this we obtain a distinguished map of commutative algebras

$$\mathbb{W}C^0(p^{-1}\mathbb{Z}_p) \to \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{C_p}.$$

Since $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*(-))$ is lax symmetric monoidal, it canonically refines to a lax symmetric monoidal functor

$$\begin{aligned}
\mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}} &\xrightarrow{\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*(-))} \mathrm{Mod}(\mathrm{PgcSp}_{\langle p\rangle}; \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})) \\
&\to \mathrm{Mod}(\mathrm{PgcSp}_{\langle p\rangle}; \mathbb{W}C^0(p^{-1}\mathbb{Z}_p)).
\end{aligned}$$

which we, by abuse of notation, also denote $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*(-))$. ◁

The key point in the construction above is that it gives us access to restriction operations indexed by $p^{-1}\mathbb{Z}_p$ on the output of $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*(-))$. The main way we will use this is in the following lemma:

Lemma 5.4.15. *For any $A \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$ we have*

$$\mathrm{res}_{\varphi}\left(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*A)_{|p^{-1}\mathbb{Z}_p^{\times}}\right) = 0.$$

Proof. $\mathrm{res}_{\varphi}(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*A)_{|p^{-1}\mathbb{Z}_p^{\times}})$ is naturally a $\mathrm{res}_{\varphi}(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})_{|p^{-1}\mathbb{Z}_p^{\times}})$-module and it will therefore suffice to prove that the latter object is zero. Examining the third diagram in Construction 5.4.14 we can read off that

$$\pi_0^{\flat}\left(\mathrm{res}_{\varphi}(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})_{|p^{-1}\mathbb{Z}_p^{\times}})\right) \cong \left(C^0(\mathbb{Z}_p)_{|p^{-1}\mathbb{Z}_p^{\times}} \xrightarrow{\mathrm{res}_{1/p}} C^0(p\mathbb{Z}_p)_{|\mathbb{Z}_p^{\times}}\right) \cong (0 \to 0)\,. \qquad \square$$

Constructing the Dehn twist

We now upgrade the functor $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*(-))$ to a functor taking values in $\mathrm{PgcSp}^{*/\!\!/\mathbb{Z},u}_{\langle p\rangle}$. We refer to this additional $\mathbb{Z}$-action as the Dehn twist and its existence is instrumental in the proof of Theorem 5.4.11.

Construction 5.4.16. We start by considering the functor

$$\sigma_0 \colon \mathrm{EQ} \to B\mathbb{Z}$$

sending the top map to 0 and the bottom map to 1. Using σ_0 we construct an automorphism of $\mathrm{EQ} \times B\mathbb{Z}$ via the formula $\sigma(a,b) := (a, \sigma_0(a) + b)$. The automorphism σ naturally fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{EQ} \times B\mathbb{Z} \arrow[r, "\sigma"] \arrow[d, "\pi"] & \mathrm{EQ} \times B\mathbb{Z} \arrow[d, "\pi"] \\
\mathrm{EQ} \arrow[r, "\mathrm{Id}"] & \mathrm{EQ}.
\end{tikzcd}$$

Viewing this automorphism as a $\mathbb{Z}$-action and taking homotopy orbits we obtain a functor[29]

$$\bar{\pi} \colon (\mathrm{EQ} \times B\mathbb{Z})/\!\!/\mathbb{Z} \to \mathrm{EQ} \times */\!\!/\mathbb{Z}.$$

Similarly, the map $\rho : \mathrm{EQ} \times B\mathbb{Z} \to B\mathbb{Z}^{\triangleright}$ is naturally equivariant for this $\mathbb{Z}$-action and so we obtain a corresponding functor

$$\bar{\rho} \colon (\mathrm{EQ} \times B\mathbb{Z})/\!\!/\mathbb{Z} \to B\mathbb{Z}^{\triangleright} \times */\!\!/\mathbb{Z}.$$

Finally, we let $\underline{\bar{\rho}}$ be the composite of $\bar{\rho}$ with the projection $B\mathbb{Z}^{\triangleright} \times */\!\!/\mathbb{Z} \to B\mathbb{Z}^{\triangleright}$. $\triangleleft$

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[29] Note that $B\mathbb{Z} \cong */\!\!/\mathbb{Z}$. We maintain this notational distinction purely to help the reader to keep in mind the role played by the different circles we see.

Remark 5.4.17. Associated to the pullback squares

$$\begin{tikzcd}
B\mathbb{Z}^{\triangleright} \arrow[d, "t"] & \mathrm{EQ} \times B\mathbb{Z} \arrow[l, "\rho"'] \arrow[r, "\pi"] \arrow[d, "t"] & \mathrm{EQ} \arrow[d, "t"] \\
B\mathbb{Z}^{\triangleright} \times */\!\!/\mathbb{Z} & (\mathrm{EQ} \times B\mathbb{Z})/\!\!/\mathbb{Z} \arrow[l, "\bar{\rho}"'] \arrow[r, "\bar{\pi}"] & \mathrm{EQ} \times (*/\!\!/\mathbb{Z})
\end{tikzcd}$$

we have an isomorphism

$$t^*\bar{\pi}_*\bar{\varrho}^*A \cong \pi_* t^* \bar{\varrho}^* A \cong \pi_*\rho^*A$$

natural in $A \in \mathcal{C}^{B\mathbb{Z}^{\triangleright}}$. In other words, $\bar{\pi}_*\bar{\varrho}^*A$ equips $\pi_*\rho^*A$ with a $\mathbb{Z}$-action. ◁

Definition 5.4.18. Given an $A \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$ we refer to the $\mathbb{Z}$-action on $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)$ as the Dehn twist. ◁

Remark 5.4.19. The reason we refer to this action as a Dehn twist is that the classifying space of $\mathrm{EQ} \times B\mathbb{Z}$ is a torus and the map σ is a Dehn twist on this torus. The map π is then the map witnessing that if we view a torus as a trivial S^1-bundle on a circle, then the Dehn twist can be made into a bundle automorphism. The map $\bar{\varrho}$ can be described as witnessing that the Dehn twist acts trivially on the associated nodal curve. ◁

The Dehn twist on the unit

Proposition 5.4.20. *Under the isomorphism*

$$\pi_0^{\flat}(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{\Phi C_p}) \cong C^0(\mathbb{Z}_p)$$

from Construction 5.4.14 the Dehn twist action is identified with restriction along the map $+1 : \mathbb{Z}_p \to \mathbb{Z}_p$*.*

The idea behind this proposition is that $\mathrm{THH}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{\Phi C_p} \cong \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ is a continuous version of the cochains of the free loop space on the p-adic circle, which can be thought of as the space of sections of the projection $B\mathbb{Z}_p \times B\mathbb{Z} \to B\mathbb{Z}$. The Dehn twist acts as an automorphism of $B\mathbb{Z}_p \times B\mathbb{Z}$ which by construction induces the automorphism on the space of sections that on π_0 is the map $+1 : \mathbb{Z}_p \to \mathbb{Z}_p$.

Before proving the proposition we will need a lemma simplifying the computation of $\mathrm{THH}^{\mathrm{EQ}}$ for commutative algebras.

Lemma 5.4.21. *There is a commutative diagram of symmetric monoidal functors*

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C})^{\mathrm{EQ}} \arrow[r] \arrow[d, "\mathrm{colim}"'] & \mathrm{Alg}(\mathcal{C})^{\mathrm{EQ}} \arrow[d, "\mathrm{THH}^{\mathrm{EQ}}"] \\
\mathrm{CAlg}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

where $\mathrm{THH}^{\mathrm{EQ}}$ *denotes the* THH *relative to* $\mathcal{C}$*.*

Proof. For a diagram $A \rightrightarrows B$ in $\mathrm{CAlg}(C)^{\mathrm{EQ}}$, the colimit can be computed as $A \otimes_{A \otimes A} B$, which is also a formula for $\mathrm{THH}^{\mathrm{EQ}}$. $\square$

Proof of Proposition 5.4.20. As explained in Section 5.4.2 we have an $\mathbb{Z}$-equivariant isomorphism of commutative algebras

$$\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*\mathbb{S})^{\Phi C_p} \cong \mathrm{THH}^{\mathrm{EQ}}(\bar{\pi}_*\bar{\varrho}^*\mathbb{S})$$

so it will suffice to focus on the latter. Next we note that the operations $\bar{\pi}_*$ and $\bar{\varrho}_*$ commute with the forgetful functor from commutative algebras to algebras, so we are free to work at the level of commutative algebras By Lemma 5.4.21, we can replace $\mathrm{THH}^{\mathrm{EQ}}$ with the colimit in $\mathrm{CAlg}(\mathrm{Sp})$. This lets us rewrite our object of interest as $\mathrm{colim}_{\mathrm{EQ}}\, \bar{\pi}_*\bar{\varrho}^*\mathbb{S}$.

The operations $\bar{\pi}_*$ and $\bar{\varrho}_*$ are also compatible with the symmetric monoidal, limit preserving, functor $\mathbb{S}^{(-)} : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp})$. This gives us, for each space $X \in (\mathrm{Spc}^{\mathrm{op}})^{B\mathbb{Z}^{\triangleright}}$, a Dehn twist-equivariant assembly map

$$\underset{\mathrm{EQ}}{\mathrm{colim}}\, \bar{\pi}_*\bar{\varrho}^*\mathbb{S}^X \to \mathbb{S}^{\lim_{\mathrm{EQ}} \bar{\pi}_*\bar{\varrho}^* X}.$$

Taking X to be the constant diagram on a point, $\lim_{\mathrm{EQ}} \pi_*\rho^* X$ is the free loop space of $B\mathbb{Z}$, and it follows from Lemma 5.3.6 that this assembly map is (on $\pi_0^\flat$) the map $C^0(\mathbb{Z}_p) \to \mathbb{F}_p^{\mathbb{Z}}$ taking a continuous function $\mathbb{Z}_p \to \mathbb{F}_p$ to its restriction to $\mathbb{Z} \subset \mathbb{Z}_p$. The map $C^0(\mathbb{Z}_p) \to \mathbb{F}_p^{\mathbb{Z}}$ is injective since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, so to verify the claim of the proposition, it will suffice to show that the Dehn twist action on $\mathbb{F}_p^{\mathbb{Z}} \cong \pi_0^\flat \mathbb{S}^{\lim_{\mathrm{EQ}} \bar{\pi}_*\bar{\varrho}^* \bullet}$ is given by the automorphism coming from pre-composition with the map $+1 : \mathbb{Z} \to \mathbb{Z}$. This in turn reduces to determining the Dehn twist action on π_0 of $\lim_{\mathrm{EQ}} \bar{\pi}_*\bar{\varrho}^* \bullet$.

Unrolling definitions and using the fact that $\pi : \mathrm{EQ} \times B\mathbb{Z} \to \mathrm{EQ}$ is a cartesian fibration we identify $\lim_{\mathrm{EQ}} \pi_*\rho^* \bullet$ with the space of sections of π and identify the Dehn twist action with the action on the space of sections generated by the map σ from Construction 5.4.16. The π_0 of the space of sections of π can be identified with pairs of integers (i, j) (recording the values of the top and bottom arrow respectively) modulo the equivalence relation generated by $(i, j) = (i+1, j+1)$ (coming from the automorphism of the source (or target) object). The autormorphism σ acts by sending a section (i, j) to $(i, j + 1)$. The proposition follows. $\square$

Corollary 5.4.22. *There is an isomorphism of $\mathbb{Z}$-equivariant algebras*

$$\pi_0^\flat(\mathrm{THH}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*\mathbb{S})^{C_p}) \cong C^0(\overrightarrow{\mathbb{Z}_p}) \times (C^0(p^{-1}\mathbb{Z}_p^\times), \psi)$$

lying over the isomorphism from Construction 5.4.14 for some continuous automorphism ψ of $p^{-1}\mathbb{Z}_p^\times$.

Proof. In Construction 5.4.14 we identified the natural map

$$\pi_0^\flat(\mathrm{THH}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{C_p}) \to \pi_0^\flat(\mathrm{THH}_{\pentagon}(\pi_*\rho^*\mathbb{S})^{\Phi C_p})$$

with the map

$$C^0(p^{-1}\mathbb{Z}_p) \xrightarrow{(-)_{|\mathbb{Z}_p}} C^0(\mathbb{Z}_p).$$

Let ψ denote the Dehn twist actions on the source. From Stone duality we know that ψ acts as precomposition by some continuous automorphism $p^{-1}\mathbb{Z}_p \to p^{-1}\mathbb{Z}_p$. The fact that the map above is compatible with the Dehn twist action implies that for $c \in \mathbb{Z}_p$ we have $\psi(c) = c+1$. It follows that ψ preserves $p^{-1}\mathbb{Z}_p^\times$ as well and we obtain the desired splitting. $\square$

The trivialization

Lemma 5.4.23. *Let $A \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^\triangleright}$. The Dehn twist action on $\mathrm{THH}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)$ is locally unipotent.*

Proof. Per Corollary 5.8.28 it will suffice to argue that the $\mathbb{Z}$-actions on $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)^{\Phi e}$ and $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)^{\Phi C_p}$ are each locally unipotent. From the formulas

$$\begin{aligned}
\mathrm{THH}_{\pentagon}(A;M)^{\Phi C_p} &\cong \mathrm{THH}(A;M) \\
\mathrm{THH}_{\pentagon}(A;M)^{\Phi e} &\cong \mathrm{THH}(A;M^{\otimes_A p}) \\
\mathrm{THH}(A;M) &\cong A \otimes_{A\otimes A} M \cong \operatorname{colim} A \otimes (A \otimes A)^{\otimes \bullet} \otimes M \\
M^{\otimes_A p} &\cong \operatorname*{colim}_{(\Delta^{\mathrm{op}})^{\times p-1}} \big(M \otimes A^{\otimes \bullet} \otimes M \otimes \cdots \otimes A^{\otimes \bullet} \otimes M\big)
\end{aligned}$$

and the fact that $\mathrm{Sp}^{B\mathbb{Z},u}$ is closed under colimits and $\otimes$ we learn that it will suffice for us to argue that the Dehn twist action on $\bar{\iota}^*\bar{\pi}_*\bar{\varrho}^*A$ is trivial where $\bar{\iota} : (\bullet \coprod \bullet) \times *\mathbin{/\!/}\mathbb{Z} \to \mathrm{EQ} \times *\mathbin{/\!/}\mathbb{Z}$ is the natural embedding.

Consider the following diagram

$$\begin{tikzcd}
 & B\mathbb{Z}^{\triangleright} & \\
B\mathbb{Z} \coprod B\mathbb{Z} \arrow[ur, "j"] \arrow[d, "q"'] & (B\mathbb{Z} \coprod B\mathbb{Z}) \times *//\mathbb{Z} \arrow[l, "\bar{\varpi}"'] \arrow[r, "\bar{i}"] \arrow[d, "\bar{q}"] & (\mathrm{EQ} \times B\mathbb{Z})//\mathbb{Z} \arrow[ul, "\bar{\varrho}"'] \arrow[d, "\bar{\pi}"'] \\
\bullet \coprod \bullet & (\bullet \coprod \bullet) \times *//\mathbb{Z} \arrow[l, "\varpi"'] \arrow[r, "\bar{\iota}"] & \mathrm{EQ} \times *//\mathbb{Z}
\end{tikzcd}$$

Where $q, \bar{q}, \varpi, \bar{\varpi}$ are projections and j embed the first copy of $B\mathbb{Z}$ into $B\mathbb{Z}^\triangleright$ and send the second copy to the cone point. Further, q and $\bar{\pi}$ are cartesian fibrations and and both squares are pullback squares.[30]

We now have isomorphisms

$$\bar{\iota}^*\bar{\pi}_*\bar{\varrho}^*A \cong \bar{q}_*\bar{i}^*\bar{\varrho}^*A \cong \bar{q}_*\bar{\varpi}^*j^*A \cong \varpi^*q_*j^*A$$

from which we can read off that the Dehn twist action is trivial as desired. $\square$

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[30] All these assertions are easily verifiable by hand as all the categories that appear in the diagram are 1-categories with only two objects.

Proposition 5.4.24. *There is a natural isomorphism of lax symmetric monoidal functors* $\mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}} \to \mathrm{PgcSp}_{\langle p\rangle}$ *between the functor sending* $A \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$ *to* $\mathrm{THH}_{\pentagon}(\pi_*\rho^*A)$ *and the functor sending* $A \in \mathrm{Alg}(\mathrm{Sp})^{B\mathbb{Z}^{\triangleright}}$ *to*

$$\left(\mathbb{W}C^0(\mathbb{Z}_p) \otimes \mathrm{THH}_{\pentagon}(\pi_*\rho^*A)_{|0}\right) \oplus \mathrm{THH}_{\pentagon}(\pi_*\rho^*A)_{|p^{-1}\mathbb{Z}_p^\times}.$$

Proof. By lax symmetric monoidality, $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*(-))$ naturally lands in $\mathbb{Z}$-equivariant modules over $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*\mathbb{S})$. Restricting along the counit of the spherical Witt vector, tilt adjunction after applying $(-)^{C_p}$ we obtain a refinement of $\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*(-))$ landing in

$$\mathrm{Mod}\left(\mathrm{PgcSp}^{*/\!\!/\mathbb{Z}}_{\langle p\rangle};\ \mathbb{W}\pi_0^\flat(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*\mathbb{S})^{C_p})\right).$$

The $\mathbb{Z}$-equivariant identification from Corollary 5.4.22 now gives us a natural $\mathbb{Z}$-equivariant splitting

$$\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A) \cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)_{|\mathbb{Z}_p} \oplus \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)_{|p^{-1}\mathbb{Z}_p^\times}$$

where the first term is a $\mathbb{Z}$-equivariant $\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p})$-module. Applying Proposition 5.8.35 (the necessary unipotence hypotheses having been checked in Lemma 5.4.23) we now obtain the desired lax symmetric monoidal identification

$$\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\bar{\pi}_*\bar{\varrho}^*A)_{|\mathbb{Z}_p} \cong \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \otimes \left(\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*A)_{|\mathbb{Z}_p}\right)_{|0}. \qquad \square$$

Lemma 5.4.25. *There is a symmetric monoidal natural isomorphism of functors from* UAlg *to* $\mathbb{W}C^0(p^{-1}\mathbb{Z}_p)$*-modules in* $\mathrm{PgcSp}_{\langle p\rangle}$

$$\mathrm{Nm}(V) \otimes \mathrm{res}_{\pentagon}\,\mathrm{THH}(R^{h\mathbb{Z}}) \cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V)).$$

Proof. Using Lemma 5.2.15, Lemma 5.2.17 and Lemma 5.2.16, we obtain isomorphisms

$$\begin{aligned}\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V)) &\cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}\left(\pi_*\left(\pi^*(\mathbb{S} \rightrightarrows V) \otimes e^*R\right)\right) \cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}\left((\mathbb{S} \rightrightarrows V) \otimes \pi_*e^*R\right)\\ &\cong \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}\left((\mathbb{S} \rightrightarrows V) \otimes e^*R^{h\mathbb{Z}}\right) \cong \mathrm{Nm}(V) \otimes \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}\left(e^*R^{h\mathbb{Z}}\right)\\ &\cong \mathrm{Nm}(V) \otimes \mathrm{res}_{\pentagon}\,\mathrm{THH}\left(R^{h\mathbb{Z}}\right). \qquad \square\end{aligned}$$

Proof (of Theorem 5.4.11). Using Lemma 5.4.25, Proposition 5.4.24 and Lemma 5.4.15 we obtain symmetric monoidal natural isomorphisms

$$\begin{aligned}V \otimes \mathrm{res}_\varphi\,\mathrm{THH}(R^{h\mathbb{Z}}) &\cong \mathrm{res}_\varphi\,\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V))\\ &\cong \mathrm{res}_\varphi\left(\left(\mathbb{W}(C^0(\mathbb{Z}_p)) \otimes \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V))_{|0}\right) \oplus \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V))_{|p^{-1}\mathbb{Z}_p}\right)\\ &\cong \mathrm{res}_\varphi\left(\mathbb{W}(C^0(\mathbb{Z}_p)) \otimes \mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V))_{|0}\right).\end{aligned}$$

Using Lemma 5.4.25 a second time together with the identification of zero fibers from Lemma 5.4.9 we can simplify the inner part of this last term further

$$\begin{aligned}\mathrm{THH}^{\mathrm{EQ}}_{\pentagon}(\pi_*\rho^*\mathrm{cn}(R,V))_{|0} &\cong \left(\mathrm{Nm}(V)\otimes \mathrm{res}_{\pentagon}\mathrm{THH}(R^{h\mathbb{Z}})\right)_{|0} \\ &\cong \mathrm{Nm}(V)\otimes \mathrm{res}_{\pentagon}\left(\mathrm{THH}(R^{h\mathbb{Z}})\right)_{|0} \cong \mathrm{Nm}(V)\otimes \mathrm{res}_{\pentagon}\mathrm{THH}(R)^{h\mathbb{Z}}.\end{aligned}$$

Plugging this into the above and using the fact that $\mathrm{res}_\varphi(\mathrm{Nm}(V)) \cong V$ and $\mathrm{res}_\varphi(-)$ is symmetric monoidal when one of the inputs is a compact object of Sp^{C_p} we arrive at a natural isomorphism

$$V\otimes \mathrm{res}_\varphi \mathrm{THH}(R^{h\mathbb{Z}}) \cong V\otimes \mathrm{res}_\varphi\left(\mathbb{W}(C^0(\mathbb{Z}_p))\otimes \mathrm{res}_{\pentagon}(\mathrm{THH}(R)^{h\mathbb{Z}})\right).$$

Using the assembly map

$$\mathbb{W}(C^0(\mathbb{Z}_p))\otimes \mathrm{res}_\varphi\left(\mathrm{res}_{\pentagon}(\mathrm{THH}(R)^{h\mathbb{Z}})\right) \to \mathrm{res}_\varphi\left(\mathbb{W}(C^0(\mathbb{Z}_p))\otimes \mathrm{res}_{\pentagon}(\mathrm{THH}(R)^{h\mathbb{Z}})\right)$$

together with Lemma 5.2.16 we now obtain the desired lax symmetric monoidal natural transformation

$$\mathbb{W}(C^0(\mathbb{Z}_p))\otimes V\otimes \mathrm{res}_\varphi(\mathrm{THH}(R)^{h\mathbb{Z}}) \to V\otimes \mathrm{res}_\varphi(\mathrm{THH}(R^{h\mathbb{Z}})).$$

For the final claims we must analyze when the assembly map above is an isomorphism. As the pullback along $0\coprod 1\to\Delta^1$ is conservative it suffices to analyze the assembly maps for $(-)^{\Phi C_p}$ and $(-)^{tC_p}$ separately. The former is colimit preserving (proving (1)), so we are reduced to the case of $(-)^{tC_p}$ where this follows from the given hypothesis (here we use that the condition that the assembly map be an isomorphism is stable under cofiber sequences and that the underlying spectrum of $\mathbb{W}(C^0(\mathbb{Z}_p))$ is a sum of spheres). □

5.4.3 Bootstrapping trivializations to CycSp

The previous parts of this section show give conditions under which $V\otimes\mathrm{THH}(R^{h\mathbb{Z}})$ as a spectrum with Frobenius map is constant in that it is isomorphic to $V\otimes\mathbb{W}(C^0(\mathbb{Z}_p))\otimes\mathrm{THH}(R)^{h\mathbb{Z}}$. The goal of this subsection is to provide tools to upgrade this spectrum level statement to the level of cyclotomic spectra in order to prove Theorem 5.4.30. We begin with a tool that helps us prove boundedness properties for W-modules:

Lemma 5.4.26. *Let M be a p-nilpotent W-module in cyclotomic spectra. If $M_{|p^k\mathbb{Z}_p}$ satisfies* $\mathrm{WCV}(\leq b)$ *and M satisfies* $\mathrm{Segal}(\leq b)$*, then M is cyclotomically b-truncated.*

Proof. Let $U_{i,j}$ be the subset of $\mathbb{Z}_p$ consisting of elements $a\in\mathbb{Z}_p$ such that $i\leq v_p(a)\leq j$. Let

$$X_{i,j} := \mathrm{Eq}\left(\prod_{r\geq 0}(M_{|U_{i,j}})^{hC_{p^r}} \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{r\geq 1}(M_{|U_{i+1,j+1}})^{tC_{p^r}}\right)$$

for $i < j$ where we have used the splitting of M along idempotents in $\pi_0 W$ to pick out components of can and φ.

Using the fact that $\varphi_{|p^k\mathbb{Z}_p}$ factors through $M_{|p^{k+1}}\mathbb{Z}_p$ and the fact that $M^{tC_p} = M^{tC_p}|_{p\mathbb{Z}_p}$ (see Proposition 5.3.18), we construct a natural finite filtration

$$X_{0,1} \to X_{0,2} \to \cdots \to X_{0,k} \to \mathrm{TR}(X)$$

with associated graded given by $X_{0,1}, X_{1,2}, \ldots, X_{k-1,k}, X_{k,\infty}$. We will show that $\mathrm{TR}(X)$ is b-truncated by proving that each of these terms b-truncated.

First we show that $X_{i-1,i}$ is b-truncated. The localized canonical map used in $X_{i-1,i}$ is zero since the target is 0 when restricted to $U_{i-1,i}$. Again using the fact that $\varphi_{|p^k\mathbb{Z}_p}$ factors through $M_{|p^{k+1}\mathbb{Z}_p}$, this means that $X_{i-1,i}$ is isomorphic to fib $\left(\prod_{j\geq 0}(\varphi_{|U_{i-1,i}})^{hC_{p^j}}\right)$. The latter object is b-truncated since it is a product of homotopy fixed points of retracts of the fiber of φ, which is b-truncated by hypothesis.

Now we show that $X_{k,\infty}$ is b-truncated as well. This time we use the weak canonical vanishing hypothesis which tells us that the localized canonical maps used in constructing $X_{k,\infty}$ are zero on homotopy groups for $* \geq b+1$. As above, it follows that $X_{k,\infty}$ is b-truncated since the Frobenius maps induce as isomorphism on $\pi_s(-)$ for $s > b+1$ and an injection for $s = b+1$. □

Proposition 5.4.27. *Let $m \geq m_p^{\mathbb{A}_2}$ and let R be a $h\mathbb{A}_2$-ring in W-modules in cyclotomic spectra with $p^m = 0$. Suppose that*

1. *$R_{|0}$ is cyclotomically bounded in the range $[c, b]$.*
2. *There is an isomorphism of graded $h\mathbb{A}_2$-rings*

$$\pi_* R \cong \pi_0(\mathbb{W}C^0(\mathbb{Z}_p)) \otimes \pi_* R_{|0}.$$

such that the map $R \to R_{|0}$ is given by restriction to 0 at the level of π_*.

3. *R satisfies* Segal$(\leq b')$.

Then R is cyclotomically bounded in the range $[c, e]$ where $e = b' + 1 + 2p^{(b-2c+2)m}$.

Proof. Let $k := (b - 2c + 2)m$ and let $e := (b' + 1 + 2p^k)$. From assumptions (1) and (2) it follows that the underlying spectrum of R is c-connective. What remains is to show that R is e-truncated as a cyclotomic spectrum.

Using the boundedness of $R_{|0}$ and Corollary 5.2.35 we pick a Bökstedt class $\mu \in \pi_{2p^k} R_{|0}$. Assumption (2) now allows us to now consider the associated class $1 \otimes \mu \in \pi_{2p^k}(R)$ whose restriction to 0 is μ.

As μ is a Bökstedt class, we may in particular fix a constant d and a class

$$\alpha \in \pi_{2p^k}(\Sigma^{-1+d(1)}\mathbb{S}/a_{(1)}^d \otimes R_{|0})^{h\mathbb{T}}$$

such that the image of α under the partial transfer map

$$(\Sigma^{-1+d(1)}\mathbb{S}/a^d_{(1)} \otimes R_{|0})^{h\mathbb{T}} \to R_{|0}^{h\mathbb{T}}$$

is μ. Writing $R_{|0}$ as $\mathrm{colim}_j\, R_{|p^j\mathbb{Z}_p}$ and using the fact that $\pi_{2p^k}(\Sigma^{-1+d(1)}\mathbb{S}/a^d_{(1)} \otimes -)^{h\mathbb{T}}$ commutes with filtered colimits of uniformly p-nilpotent objects, we can, for $j \gg 0$, we pick a lift $\widetilde{\alpha}$ of α to $\pi^{\mathbb{T}}_{2p^k}(\Sigma^{-1+d(1)}\mathbb{S}/a^d_{(1)} \otimes R_{|p^j\mathbb{Z}_p})$. Again using the filtered colimit we can read off that, since both $1 \otimes \mu$ and the partial transfer of $\widetilde{\alpha}$ restrict to μ at the zero fiber, these classes must become equal at some finite stage. In other words, after restriction to $p^j\mathbb{Z}_p$ for $j \gg 0$ the class $1 \otimes \mu$ is in the image of the d-partial transfer (and in particular in the image of the transfer).

Since μ is a Bökstedt class, by Lemma 5.2.33, $(R_{|0})/\mu$ is e-truncated. From this we learn that the map $\pi_*(\Sigma^{2p^k} R_{|0}) \xrightarrow{\mu\cdot -} \pi_*(R_{|0})$ is an isomorphism for $* > e$ and injective for $* = e$. Assumption (2) now lets us conclude that multiplication by $1 \otimes \mu$ on R satisfies the same property and therefore that $\mathrm{cof}((1 \otimes \mu) \cdot -)$ is bounded in the range $[c, e]$. The same is true after restriction to $p^j\mathbb{Z}_p$ as this is given by inverting an idempotent on the W-module $\mathrm{cof}((1 \otimes \mu) \cdot -)$.

Applying Lemma 5.2.36 and Proposition 5.2.30(1) to $R_{|p^j\mathbb{Z}_p}$ (together with the class $1 \otimes \mu$) we learn that $R_{|p^j\mathbb{Z}_p}$ satisfies WCV($\leq e$) for $j \gg 0$. Finally, we apply Lemma 5.4.26 using assumption (3) in order to conclude that R is cyclotomically e-truncated. □

Proposition 5.4.27 pairs nicely with the next proposition.

Proposition 5.4.28. *Let X be a p-nilpotent W-module in cyclotomic spectra and let $X_k \coloneqq W_k \otimes_W X$. If we suppose that*

1. *The X_k are uniformly cyclotomically bounded.*
2. *There is an isomorphism of W-modules in spectra*

$$X \cong W \otimes X_\infty.$$

3. *X_∞ is almost compact as a cyclotomic spectrum.*

Then, for all $k \gg 0$ sufficiently large, there is an isomorphism of W_k-modules in cyclotomic spectra

$$X_k \cong W_k \otimes X_\infty.$$

Proof. The first step is producing a comparison map. Since the X_k are uniformly cyclotomically bounded, using the almost compactness of X_∞ we learn that there exists a $k_1 \gg 0$ and a lift ψ of the indentity map $X_\infty \to X_\infty$ through X_{k_1} as display below.

$$\begin{tikzcd}
& X_{k_1} \arrow[dr] & \\
X_\infty \arrow[ur, dashed, "\psi"] \arrow[rr, "p"] & & X_\infty
\end{tikzcd}$$

Using the W_k-module structure on X_k for $k \geq k_1$ we can linearize ψ to produce a compatible system of maps of W_k-modules

$$\psi_k : W_k \otimes X_\infty \to X_k$$

for $k \geq k_1$ which recovers the identity map $X_\infty \to X_\infty$ in the colimit as $k \to \infty$.

We now show that ψ_k is an isomorphism for $k \gg 0$. Both the X_k and the $W_k \otimes X_\infty$ are uniformly bounded (for the latter see Corollary 5.3.20) so it will suffice to instead show that for every $s \in \mathbb{Z}$ there exists a $k \gg 0$ such that the map $\pi_s(\psi_k)$ is an isomorphism.

Using assumption (1) we can read off that

$$\pi_* X_k \cong \pi_*(W_k \otimes X_\infty) \cong \pi_0(\mathbb{W}C^0(p^k\mathbb{Z}_p))\langle\zeta\rangle \otimes \pi_* X_\infty.$$

Let $N_{k,s} := \pi_0(\mathbb{W}C^0(p^k\mathbb{Z}_p))\langle\zeta\rangle \otimes \pi_s X_\infty$ and let $\psi_{k,s} : N_{k,s} \to N_{k,s}$ be the endomorphism corresponding to $\pi_s(\psi_k)$. Recall from above that $\psi_{\infty,s}$ is an isomorphism. Let $\overline{N}_{k,s}$ and $\overline{\psi}_{k,s}$ be the reduction mod ζ of $N_{k,s}$ and $\psi_{k,s}$ respectively. From the expression above we can read off that

(a) $\mathrm{Tor}^1_{\mathbb{Z}_p\langle\zeta\rangle}(\mathbb{Z}_p, N_{k,*}) = 0$ and therefore that $\psi_{k,s}$ is an isomorphism if both $\overline{\psi}_{k,s}$ and $\overline{\psi}_{k,s+1}$ are isomorphisms and

(b) $\overline{N}_{k,s} \cong \pi_0(\mathbb{W}C^0(p^k\mathbb{Z}_p)) \otimes \pi_s X_\infty$ with the variance in k given by restriction.

Using assumption (3) we know that $\pi_s X_\infty$ is finitely generated, p-nilpotent and in view of (b) and the fact that $\overline{\psi}_{\infty,s}$ is an isomorphism we may apply Lemma 5.8.15(1) (with $\mathcal{C} = \mathrm{Sp}$, the R_α the system $\mathbb{W}C^0(p^k\mathbb{Z}_p)$ and $X = Y = \overline{N}_{k,s}$) and conclude that $\overline{\psi}_{k,s}$ is an isomorphism for $k \gg 0$. Using (a) this in turn implies that $\psi_{k,s}$ is an isomorphism for $k \gg 0$. This means $\pi_s\psi_k$ is an isomorphism and the proposition follows. □

5.4.4 Finishing the argument

In this subsection, we piece together the ingredients from earlier subsections to prove this section's main theorem.

We begin with a lemma that allows us to replace our finite spectra with connective rings:

Lemma 5.4.29. *Let X be finite spectrum of type w and let $q \geq 0$. There exists a connective $\mathbb{E}_k$-algebra V whose underlying spectrum is finite and of type n such that X is a retract of $V \otimes X$.*

Proof. From [Bur22, Thm. 1.3] we know that there exists a connective, finite spectrum V' of type w with an $\mathbb{E}_1$-algebra structure. For $i \geq 1$, let $V^{(i)}$ denote the cofiber of the map $I^{\otimes i} \to \mathbb{S}$ where $I \to \mathbb{S} \to V'$ is a cofiber sequence. $V^{(i)}$ is of type w and when $i \geq q+1$ it admits an $\mathbb{E}_q$-algebra structure by [Bur22, Theorem 1.5]. By Recollection 5.2.20 X is a retract of $X \otimes V^{(i)}$, for $i \gg 0$. Taking $V = V^{(i)}$ for i sufficiently large satisfies the conditions of the lemma. □

We now prove the main theorem of this section:

Theorem 5.4.30 (Cyclotomic asymptotic constancy)**.** *Let $R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$ be connective of fp-type $n \geq -1$, and let X be a finite spectrum of type $n+2$. Suppose that $X \otimes \mathrm{THH}(R)$ is bounded in the range $[c, b]$.*

There exists a function $k(R, X, n)$ such that for all $k \geq k(R, X, n)$, $X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ is bounded in the range $[c-1, b+3]$ and there is an isomorphism of W_k-modules in cyclotomic spectra

$$X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong X \otimes W_k \otimes \mathrm{THH}(R).$$

Proof. The proof of this theorem mostly consists of combining the piecemeal results proved throughout this section. For ease of hypothesis-tracking we subdivide the proof into a sequence of labelled sub-claims.

We apply Lemma 5.4.29 with $q = 2$ to obtain a connective, finite spectrum V of type $n+2$ equipped with an $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-algebra structure such that X is a retract of $X \otimes V$. For most of the proof we will work with V, returning to X at the end.

(0) $V \otimes \mathrm{THH}(R)$ and $X \otimes \mathrm{THH}(R)$ are almost compact in CycSp.

Since R is fp-type n, R/p is bounded below and has level-wise finite homotopy groups. By Lemma 5.8.10 the underlying spectra of $V \otimes \mathrm{THH}(R)$ and $X \otimes \mathrm{THH}(R)$ are almost compact. Proposition 5.2.42 now implies (0).

Part 1: Locally unipotent actions

(1a) For each $k \geq 0$, the $p^k\mathbb{Z}$-action on $\mathrm{THH}(R)$ is unipotent.

(1b) For each $k \geq 0$, there is an isomorphism of W_k-modules

$$\mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong W_k \otimes_W \mathrm{THH}(R^{h\mathbb{Z}}).$$

(1c) For each $k \geq 0$, there is an isomorphism

$$\mathrm{THH}(R^{hp^k\mathbb{Z}})_{|0} \cong \mathrm{THH}(R)^{hp^k\mathbb{Z}}.$$

(1a), (1b) and (1c) follow form Lemmas 5.4.7, 5.4.8 and 5.4.9 respectively.

(1d) The $p^k\mathbb{Z}$-action on $V \otimes R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$ is trivial for $k \gg 0$.

The $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-algebra $V \otimes R$ is connective, π-finite (implies p-nilpotent, bounded and almost compact) and has a locally unipotent $\mathbb{Z}$-action, therefore we may apply Lemma 5.8.43 to obtain (1d).

(1e) The $p^k\mathbb{Z}$-action on $V \otimes \mathrm{THH}(R)$ in $\mathrm{Alg}_{\mathbb{A}_2}(\mathrm{CycSp}^{B\mathbb{Z},u})$ is trivial for $k \gg 0$.

The $\mathbb{A}_2$-algebra in cyclotomic spectra $V \otimes \mathrm{THH}(R)$ is connective, bounded, p-nilpotent, almost compact (by (0)) and has a locally unipotent $\mathbb{Z}$-action (by (1a)), therefore we may apply Lemma 5.8.43 to obtain (1e).

Part 2: The Dehn twist trivialization

(2a) There are isomorphisms of $\mathbb{A}_2$-algebras in W_k-modules in spectra,

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong V \otimes \mathbb{W}C^0(p^k\mathbb{Z}_p) \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}} \cong V \otimes W_k \otimes \mathrm{THH}(R)$$

for $k \gg 0$ sufficiently large.

(2b) The cyclotomic spectrum $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ satisfies Segal($\leq b$) for $k \gg 0$ sufficiently large.

The second isomorphism in (2a) follow from (1e). Using (1d) we lift the pair (R, V) to an $\mathbb{A}_2$-algebra object of UAlg. (2a) now follows form Theorem 5.4.11(1).

The cyclotomic spectrum $V \otimes \mathrm{THH}(R)$ satisfies Segal($\leq b$) by Proposition 5.2.29. It follows that we may prove (2b) using Theorem 5.4.11(2). The additional hypothesis needed for Theorem 5.4.11(2) follows from Lemma 5.2.39 using the fact that $V \otimes \mathrm{THH}(R)$ is cyclotomically bounded.

Part 3: Bootstrapping the trivialization to CycSp

(3a) There is a $k' \gg 0$ sufficiently large so that the cyclotomic spectra $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ are uniformly bounded for $k \geq k'$.

For each k, $V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ is an h$\mathbb{A}_2$-ring in W_k-modules. We wish to prove that $V \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}})$ is uniformly bounded in the cyclotomic t-structure as k varies. To do this we will verify the hypotheses of Proposition 5.4.27 for the h$\mathbb{A}_2$-ring in $W = W_k$-modules in cyclotomic spectra $\mathrm{THH}(R^{hp^k\mathbb{Z}})$ for $k \gg 0$. We will do this where the constants c, b, b' are independent of k so that the bound is uniform in k.

Hypothesis (1) is satisfied because $(V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}))_{|0} \cong V \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}}$ by (1c), which is bounded in the range $[c-1, b]$ if $V \otimes \mathrm{THH}(R)$ is bounded in the range $[c, b]$. Hypothesis (2) follows from (1a) and (2a). Hypothesis (3) is (2b).

(3b) There is an isomorphism of W_k-modules in cyclotomic spectra

$$V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong V \otimes W_k \otimes \mathrm{THH}(R)$$

for all $k \gg 0$ sufficiently large.

We prove this using Proposition 5.4.28 with $X = V \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}}$ where k is large enough for both (2a) and (3a) to hold. Hypothesis (1) follows from (3a) and (1b). Hypothesis (2) follows from (2a) and (1b). Hypothesis (3) follows form (0) and (1b).

(3c) There is an isomorphism of W_k-modules in cyclotomic spectra

$$X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong X \otimes W_k \otimes \mathrm{THH}(R)$$

for all $k \gg 0$ sufficiently large.

Recall that V was constructed so that X is a retract of $X \otimes V$. This means $X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ is a retract of $X \otimes V \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ as a W_k-module in cyclotomic spectra. Applying Lemma 5.8.15(2) along the system of the W_k using Corollary 5.3.20, (0), (1b), (3b) we conclude.

To finish the proof, we note that Corollary 5.3.20 along with (3c) implies the claimed bound in the t-structure for $k \gg 0$. □

Corollary 5.4.31. *Let $R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$ be connective of fp-type $n \geq -1$, and let X be a finite spectrum of type $n+2$. Suppose that R satisfies the height n LQ property.*

Then, there exists a $k_{R,X,n}$ such that for all $k \geq k_{R,X,n}$, there is a commutative square in cyclotomic spectra as below, where the horizontal maps are the coassembly maps:

$$\begin{tikzcd}
X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & X \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
X \otimes \mathrm{THH}(R^{B\mathbb{Z}}) \arrow[r] & X \otimes \mathrm{THH}(R)^{B\mathbb{Z}}
\end{tikzcd}$$

Proof. We apply Theorem 5.4.30 to learn that for all $k \gg 0$ is an isomorphism of W_k-modules in cyclotomic spectra

$$X \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \cong X \otimes W_k \otimes \mathrm{THH}(R)$$

Since the coassembly map is given by base change along the map $W_k \to \mathrm{THH}(\mathbb{S})^{Bp^k\mathbb{Z}} \cong \mathbb{S}^{B\mathbb{Z}}$ by Lemma 5.4.9, Lemma 5.4.7, we are done. □

We now extract another corollary, which in particular allows us to understand coassembly maps telescopically in terms of the trivial action. Applying this to $R = \mathrm{BP}\langle n\rangle$ and inverting v_{n+1} (which is done in Theorem 5.6.25), we recover Theorem B from the introduction. The key lemma lets us access $T(n+1)$-homology by working modulo a power of v_{n+1}:

Lemma 5.4.32. *Let X be a spectrum and let $v : \Sigma^k X \to X$ be a self map such that X/v is bounded in the range $[c, b]$. There are isomorphisms*

1. *$\pi_s(-) \cong \pi_s(-/v^a)$ for $s < c + ak$ and*
2. *$\pi_s(-) \cong \pi_{s-k}(-)$ for $s > b$.*

Moreover, these isomorphisms are natural in maps compatible with the self maps.

We note in particular that for $a \gg 0$ we have $b < c + ak$ and therefore the homotopy groups of an X as in Lemma 5.4.32 are determined by $\pi_*(X/v^a)$ for $a \gg 0$.

Proof. The first claim follows from examining the cofiber sequence $\Sigma^{ak}X \xrightarrow{v^a} X \to X/v^a$ and noting that $\Sigma^{ak}X$ is $(c+ak)$-connective. The second claim follows from examining the cofiber sequence $\Sigma^k X \xrightarrow{v} X \to X/v$ and noting that X/v is b-truncated. ☐

Corollary 5.4.33 (Telescopic asymptotic constancy)**.** *Let $R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{B\mathbb{Z},u})$ be connective of fp-type $n \geq 0$ and satisfy the height n LQ property. Fix a $V \in \mathrm{Sp}^\omega$ of type $n+1$ with v_{n+1}-self map v.*

Then, there exists a function $k(R, V, v, n)$ such that for all $k \geq k(R, V, v, n)$, there is a commutative diagram of $\mathbb{Z}[v]$-modules as below, where the horizontal maps are the coassembly maps

$$\begin{tikzcd}
V_*\mathrm{TC}(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & V_*\mathrm{TC}(R)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
V_*\mathrm{TC}(R^{B\mathbb{Z}}) \arrow[r] & V_*\mathrm{TC}(R)^{B\mathbb{Z}}
\end{tikzcd}$$

Proof. Since R satisfies the height n LQ property, by applying Corollary 5.3.20, we learn that $V/v \otimes \mathrm{THH}(R^{B\mathbb{Z}}) \cong V/v \otimes W \otimes \mathrm{THH}(R)$ is bounded in the cyclotomic t-structure. This means that $V/v \otimes \mathrm{THH}(R^{B\mathbb{Z}})$ and $V/v \otimes \mathrm{THH}(R)^{B\mathbb{Z}}$ are bounded in the range $[b+1, c]$ for some b, c. Applying Corollary 5.4.31 we learn that for $k \gg 0$, $V/v \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}})$ is bounded in the same range.

Choose an $a \gg 0$, we now apply Corollary 5.4.31 again, this time to obtain a square

$$\begin{tikzcd}
V/v^a \otimes \mathrm{THH}(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & V/v^a \otimes \mathrm{THH}(R)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
V/v^a \otimes \mathrm{THH}(R^{B\mathbb{Z}}) \arrow[r] & V/v^a \otimes \mathrm{THH}(R)^{B\mathbb{Z}}
\end{tikzcd}$$

for $k \gg 0$. Expanding this to a cube using the self map v and applying Lemma 5.4.32 we obtain the desired conclusion. ☐

Warning 5.4.34. Note that the isomorphism in Corollary 5.4.33 is of **homotopy groups only** and generally cannot be upgraded to a spectra level isomorphism. Indeed in the key example of this paper the assembly map will become an equivalence after $K(n+1)$ localization for the Adams action but not for the trivial action. ◁

5.5 Adams operations on BP$\langle n \rangle$

This section, which use no inputs from the other sections of the paper, is devoted to the construction of well-behaved *Adams operations* on BP$\langle n \rangle$. We will need such Adams operations to disprove the telescope conjecture at height $n+1$ in Section 5.6. We note that another disproof of the of the telescope conjecture at height 2 and primes at least 7 is given in Section 5.7 which is independent of this section.

Convention 5.5.1. In this section, we do *not* implicitly p-complete objects as in much of the rest of the paper.

For the remainder of this section, we let $\ell := m_p^{\mathbb{E}_1}$

As they are important for this section we remind the reader of conventions (21)-(23):

- $\mathrm{BP}\langle n\rangle$ refers to an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra form of the truncated Brown–Peterson spectrum as constructed in [HW22, §2].
- E_n refers to the height n Lubin–Tate theory constructed by Goerss–Hopkins–Miller [Lur18a, Theorem 5.0.2], associated to the (unique up to isomorphism) height n formal group over $\overline{\mathbb{F}}_p$. ◁
- $\mathbb{G}_n$ refers to the height n extended Morava stabilizer group, which acts on E_n and fits into a short exact sequence

$$1 \to \mathcal{O}_D^\times \to \mathbb{G}_n \to \mathrm{Gal}(\mathbb{F}_p) \to 1$$

  where $\mathcal{O}_D^\times$ is the units in the maximal order of the division algebra over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$, and $\mathrm{Gal}(\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ is the absolute Galois group of $\mathbb{F}_p$.

Our first goal will be to construct an Adams operation Ψ^ℓ on the Lubin–Tate theory E_n, and the p-localized complex bordism spectrum $\mathrm{MU}_{(p)}$.

Construction 5.5.2. Let $\Psi^\ell \in \mathbb{G}_n$ be the commutative algebra automorphism

$$\Psi^\ell \colon E_n \to E_n,$$

arising from the action of $\mathbb{Z}_p^\times$ on the chosen formal group of height n over $\overline{\mathbb{F}}_p$, that acts on $\pi_{2k}E_n$ by multiplication by ℓ^k.

We write $E_n^\Psi \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)}^{B\mathbb{Z}})$ for E_n equipped with the $\mathbb{Z}$-action whose generating automorphism is Ψ^ℓ. ◁

Next we construct an Adams operation on MU.

Construction 5.5.3. The stable Adams conjecture, as proved in [Fri80],[31] provides us with a commuting diagram of infinite loop maps

$$\begin{tikzcd}
\mathrm{BU}_{(p)} \arrow[rr, "\Psi^\ell"] \arrow[dr, "J"'] & & \mathrm{BU}_{(p)} \arrow[dl, "J"] \\
& \mathrm{BSL}_1(\mathbb{S}_{(p)}). &
\end{tikzcd}$$

Thomifying this diagram we obtain a commutative algebra automorphism

$$\Psi^\ell \colon \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$$

---

[31] See also [Cla11, Lemma 2.2], [BH20, Section 16.2], [BK22, Appendix A]. The exact statement we use here is [BK22, Theorem 1.8].

that we call the Adams operation on $\mathrm{MU}_{(p)}$. Note that Ψ^ℓ depends on the choice of homotopy in the diagram of infinite loop spaces above. We fix a choice of this homotopy for the remainder of the paper.

We write $\mathrm{MU}^{\Psi}_{(p)} \in \mathrm{CAlg}(\mathrm{Sp}^{B\mathbb{Z}})$ for $\mathrm{MU}_{(p)}$ equipped with the $\mathbb{Z}$-action whose generating automorphism is Ψ^ℓ. ◁

We now turn to the main goal of the section: constructing an Adams operation Ψ^ℓ on each of the truncated Brown–Peterson spectrum $\mathrm{BP}\langle n\rangle$.

Theorem 5.5.4. *The* $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$*-algebra underlying the* $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$*-algebra* $\mathrm{BP}\langle n\rangle$ *admits a lift to an object*

$$\mathrm{BP}\langle n\rangle^{\Psi} \in \mathrm{Alg}_{\mathbb{E}_1\otimes\mathbb{A}_2}\left(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}^{\Psi}_{(p)})\right)$$

such that

1. *there is a map* $\iota\colon \mathrm{BP}\langle n\rangle^{\Psi} \to E_n^{\Psi}$ *in* $\mathrm{Alg}_{\mathbb{E}_1}\left(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}^{\Psi}_{(p)})\right)$,
2. *an identification*

$$\begin{tikzcd} L_{T(n)}\mathrm{BP}\langle n\rangle^{\Psi} \arrow[r, "\iota"] \arrow[d, "\cong"] & E_n^{\Psi} \arrow[d, "\cong"] \\ (E_n^{\Psi})^{h\mu_{p^n-1}\rtimes\hat{\mathbb{Z}}} \arrow[r] & E_n^{\Psi} \end{tikzcd}$$

in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{B\mathbb{Z}})$ *where* $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}} \subseteq \mathbb{G}_n$ *fits into a map of exact sequences*

$$\begin{tikzcd} \mu_{p^n-1} \arrow[r, hook] \arrow[d, hook] & \mu_{p^n-1}\rtimes\hat{\mathbb{Z}} \arrow[r, two heads] \arrow[d, hook] & \hat{\mathbb{Z}} \arrow[d, "\cong"] \\ \mathcal{O}_D^{\times} \arrow[r, hook] & \mathbb{G}_n \arrow[r, two heads] & \mathrm{Gal}(\mathbb{F}_p), \end{tikzcd}$$

3. *and the underlying* $\mathbb{Z}$*-action on* $\mathrm{BP}\langle n\rangle$ *is locally unipotent in* p*-complete spectra after* p*-completion.*

This theorem is the only result from this section used in later sections of the paper.

Remark 5.5.5. It would be interesting to know whether the Adams operation constructed in Theorem 5.5.4 underlies an $\mathbb{E}_2$-algebra or $\mathbb{E}_3$-algebra automorphism. If so, one could equip $\mathrm{THH}(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}})$ with an $\mathbb{E}_1$-algebra or $\mathbb{E}_2$-algebra structure. As it is, Theorem 5.5.4 equips $\mathrm{THH}(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}})$ with a unital multiplication map, but provides no guarantee of either homotopy associativity or commutativity. ◁

Remark 5.5.6. Complex conjugation on $\mathrm{MU}_{(p)}$ can be viewed as the action of the Adams operation Ψ^{-1}. This is not the same as the action of Ψ^ℓ we study here, even at $p = 2$, but it may be noteworthy that the interaction between complex conjugation, Morava E-theory, and $\mathrm{BP}\langle n\rangle$ has received a great deal of prior attention. This interaction is at the heart of the Hill–Hopkins–Ravenel approach to Kervaire invariant 1 and the computation of C_{2^k} fixed points of Morava E-theory [HK01; HHR16; HS20; BHSZ21]. From the point of view of Real homotopy theory, $\mathrm{BP}\langle n\rangle$ with its action by Ψ^{-1} is expected to be an $\mathbb{E}_{2\sigma+1}$-$\mathrm{MU}_{\mathbb{R}}$-algebra, and in particular an $\mathbb{E}_{2\sigma+1}$-algebra [HW22, Remark 1.0.14]. Since $2\sigma+1$ contains only one copy of the trivial representation, this would not imply a C_2-action on $\mathrm{BP}\langle n\rangle$ by $\mathbb{E}_2$-algebra maps. However, $\mathbb{Z}$-equivariantly $S^{2\sigma} \cong S^2$, so Ψ^{-1} itself would be an $\mathbb{E}_3$-algebra map. ◁

5.5.1 An $\mathbb{E}_4$ complex orientation of Lubin–Tate theory

Here, we explain how the $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on $\mathrm{BP}\langle n\rangle$ can be used to consrtruct a map of $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebras $\mathrm{BP}\langle n\rangle \to E_n$. We use this along with the self centrality of E_n to construct an $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$.

Construction 5.5.7. In [HW22, Proposition 2.6.2], it is shown that $\mathrm{BP}\langle n\rangle$ admits the structure of an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}[y]$-algebra, where y in degree $2p^n-1$ acts by v_n. Here, $\mathrm{MU}_{(p)}[y]$ is the Thom spectrum of a composite of commutative monoid maps

$$\mathbb{N} \xrightarrow{p^n-1} \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathrm{MU}_{(p)}).$$

Let $\mathrm{MU}_{(p)}[y^{1/(p^n-1)}]$ denote the Thom spectrum of the composite

$$\mathbb{N} \xrightarrow{1} \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathrm{MU}_{(p)}),$$

the natural commutative $\mathrm{MU}_{(p)}$-algebra map $\mathrm{MU}_{(p)}[y] \to \mathrm{MU}_{(p)}[y^{1/p^n}]$ allows us to construct an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra

$$\mathrm{BP}\langle n\rangle\left[v_n^{1/(p^n-1)}\right] \coloneqq \mathrm{MU}_{(p)}[y^{1/(p^n-1)}] \otimes_{\mathrm{MU}_{(p)}[y]} \mathrm{BP}\langle n\rangle. \qquad ◁$$

Lemma 5.5.8. *The commutative algebra map*

$$\mathbb{W}(\mathbb{F}_p) \otimes \mathrm{MU}_{(p)}[y^{\pm 1}] \to \operatorname*{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathrm{MU}_{(p)}[y^{\pm 1/p^n}]$$

obtained from the map from Construction 5.5.7 by inverting y and tensoring up to $\mathbb{W}(\overline{\mathbb{F}}_p)$ on the target is a $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}}$ pro-Galois extension.

Proof. Let $A \coloneqq \mathbb{W}(\mathbb{F}_p) \otimes \mathrm{MU}_{(p)}[y^{\pm 1}]$, and let $B \coloneqq \operatorname{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathrm{MU}_{(p)}[y^{\pm 1/p^n}]$. Using the fact that $\pi_0\mathbb{W}(\mathbb{F}_{p^n})$ has a primitive $(p^n-1)^{\mathrm{st}}$ root of unity we can read off that the map $\pi_*A \to \pi_*B$ is a graded $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}}$-pro-Galois extensions of graded commutative rings. Using [Rog08, Corollary 10.1.5] (as k varies) we lift this to the structure of a $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}}$-pro-Galois extension on the map $A \to B$. □

The following proposition was first observed as [ABM23, Example 8.7]:

Proposition 5.5.9. *There is an identification of underlying $\mathbb{E}_3$-algebras*

$$\begin{tikzcd}
L_{T(n)}\mathrm{BP}\langle n\rangle \arrow[r] \arrow[d, "\cong"] & L_{T(n)}\left(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes \mathrm{BP}\langle n\rangle \left[v_n^{1/(p^n-1)}\right]\right) \arrow[d, "\cong"] \\
E_n^{h\mu_{p^n-1} \rtimes \hat{\mathbb{Z}}} \arrow[r] & E_n,
\end{tikzcd}$$

where $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}} \subseteq \mathbb{G}_n$ fits into a map of exact sequences

$$\begin{tikzcd}
\mu_{p^n-1} \arrow[r, hook] \arrow[d, hook] & \mu_{p^n-1} \rtimes \hat{\mathbb{Z}} \arrow[r, two heads] \arrow[d, hook] & \hat{\mathbb{Z}} \arrow[d, "\cong"] \\
\mathcal{O}_D^\times \arrow[r, hook] & \mathbb{G}_n \arrow[r, two heads] & \mathrm{Gal}(\mathbb{F}_p).
\end{tikzcd}$$

Proof. Let $A \coloneqq L_{T(n)}\mathrm{BP}\langle n\rangle$ and let $B \coloneqq L_{T(n)}\left(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes \mathrm{BP}\langle n\rangle \left[v_n^{1/(p^n-1)}\right]\right)$. The homotopy ring of A is the completion of the graded ring $\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ at the Landweber ideal $(p, \dots, v_{n-1})$. It follows that the homotopy ring of B is obtained from this by adjoining a $(p^n-1)^{\mathrm{st}}$-root of v_n that lives in degree 2, base changing to $W(\overline{\mathbb{F}}_p)$, and re-completing at $(p, \dots, v_{n-1})$.

Since B is an $\mathrm{MU}_{(p)}$-algebra with homotopy ring satisfying Landweber's criterion, the natural map

$$\pi_*(B) \otimes_{\pi_*(\mathrm{MU}_{(p)})} \mathrm{MU}_{(p),*}(X) \to B_*(X)$$

is an isomorphism, so that B is a Landweber exact homotopy associative and commutative ring. Because its homotopy ring agrees with that of a Lubin–Tate theory, it follows from [Ram23, Corollary 4.53] that it agrees as an $\mathbb{E}_3$-algebra with a Lubin–Tate theory, which is E_n since there is one formal group over $\overline{\mathbb{F}}_p$ of height n up to isomorphism.

Let $G = \mu_{p^n-1} \rtimes \hat{\mathbb{Z}}$. From Lemma 5.5.8 we obtain an $\mathbb{E}_3$-algebra $\mu_{p^n-1} \rtimes \hat{\mathbb{Z}}$-action on B over A such that $A \cong B^{hG}$. Examining the action on π_*B we read off that the map $G \to \pi_0 \operatorname{Aut}_{\mathbb{E}_3}(B)$ is injective. Using [Ram23, Corollary 4.53] again we know that the $\mathbb{E}_3$-automorphisms of E_n are the discrete set $\mathbb{G}_n$. Viewing G as a subgroup of $\mathbb{G}_n$ through the identification $B \cong E_n$ and examining the action of $\hat{\mathbb{Z}}$ on $\mathbb{W}(\overline{\mathbb{F}}_p) \subset \pi_0 E_n$ gives the map of exact sequences. ☐

Construction 5.5.10. As a consequence of Proposition 5.5.9, we obtain a refinement of the underlying $\mathbb{E}_3$-algebra structure on E_n to an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure and a map of $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebras

$$\iota \colon \mathrm{BP}\langle n\rangle \to E_n.$$

We note that this map is injective on π_* by the proof of Proposition 5.5.9. ◁

Recall that for an $\mathbb{E}_m$-algebra R, its center $\mathcal{Z}_{\mathbb{E}_m}(R)$ is the terminal $\mathbb{E}_{m+1}$-algebra A equipped with a lift of R to an $\mathbb{E}_m$-A-algebra [Lur17, Section 5.3] [Fra13]. In particular, the $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n corresponds to an $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to \mathcal{Z}_{\mathbb{E}_3}E$.

Proposition 5.5.11. *Let $m \geq 2$. The natural $\mathbb{E}_m$-algebra map $\mathcal{Z}_{\mathbb{E}_m}(E_n) \to E_n$ is an isomorphism, and the $\mathbb{E}_{m+1}$-algebra structure on $\mathcal{Z}_{\mathbb{E}_m}(E_n)$ agrees with the restriction of the commutative algebra structure on E_n.*

Proof. It follows from [Fra13, Proposition 3.16] that $\mathcal{Z}_{\mathbb{E}_m}(E_n)$ can be computed as

$$\mathrm{End}_{\int_{\mathbb{R}^m - 0} E_n} E_n,$$

Using the fact that E_n is $K(n)$-local we obtain an isomorphism

$$\mathrm{map}_{\int_{\mathbb{R}^m - 0} E_n}(E_n, E_n) \cong L_{K(n)} \mathrm{map}_{L_{K(n)} \int_{\mathbb{R}^m - 0} E_n}(E_n, E_n),$$

so for the first claim it suffices to show that the map $\int_{\mathbb{R}^m - 0} E_n \to E_n$ is a $K(n)$-local isomorphism. Using the isomorphism

$$\int_{\mathbb{R}^m - 0} E_n \cong \operatorname*{colim}_{S^{m-1}} E_n,$$

from [Fra13, Corollary 3.27], where the colimit is taken in commutative algebras, it will suffice to argue that E_n is codiscrete as a $K(n)$-local commutative algebra. In order to check that E_n is codiscrete it suffices to show that the map $\mathrm{colim}_{S^1} E_n \cong \mathrm{THH}(E_n) \to E_n$ is $K(n)$-locally an isomorphism. This follows from the fact that E_n is a $K(n)$-local pro-Galois extension of $L_{K(n)}\mathbb{S}$ [Rog08, **page** 5.4.6].

It remains to check that the $\mathbb{E}_{m+1}$-structure on $\mathcal{Z}_{\mathbb{E}_m}(E_n) \cong E_n$ is the usual one. The commutative algebra structure on E_n restricts to an $\mathbb{E}_m$-E_n-algebra structure on E_n. This gives us an $\mathbb{E}_{m+1}$-algebra map $E_n \to \mathcal{Z}_{\mathbb{E}_m}(E_n)$ whose composite with the $\mathbb{E}_m$-algebra map $\mathcal{Z}_{\mathbb{E}_m}(E_n) \to E_n$ is the identity $\mathbb{E}_m$-algebra map. This means that the $\mathbb{E}_{m+1}$-algebra map $E_n \to \mathcal{Z}_{\mathbb{E}_m}(E_n)$ is an isomorphism. ☐

The following corollary is immediate from Proposition 5.5.11 and Proposition 5.5.9.

Corollary 5.5.12. *The $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n constructed via Proposition 5.5.9 arises from an $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$.*

5.5.2 A $\mathbb{Z}$-equivariant $\mathbb{E}_3$ complex orientation of Lubin–Tate theory

In this subsection, we show that the underlying $\mathbb{E}_3$-algebra map of the $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$ constructed in the previous subsection is compatible with the action of the Adams operation Ψ^ℓ.

Proposition 5.5.13. *The underlying $\mathbb{E}_3$-algebra map of any $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$ can be refined to a map of $\mathbb{Z}$-equivariant $\mathbb{E}_3$-algebras $\mathrm{MU}_{(p)}^{\Psi} \to E_n^{\Psi}$.*

Before embarking on the proof of Proposition 5.5.13, we set some notation.

Definition 5.5.14. Suppose we are given an integer $a > 0$ and a $k \in \mathbb{Z}_{(p)}^{\times}$.

- We let $S^{(a,[k])} \in \mathrm{Spc}_*^{B\mathbb{Z}}$ be the $\mathbb{Z}$-equivariant pointed space consisting of $S^a_{(p)}$ together with a degree k self-map.
- We let $\Omega^{a,[k]}$ be the right adjoint to $S^{(a,[k])} \wedge -$. This is given by a $\mathbb{Z}$-equivariant mapping space out of $S^{a,[k]}$.
- We let $\mathbb{S}^{(a,[k])} := \Sigma^\infty S^{(a,[k])} \in \mathrm{Sp}^{B\mathbb{Z}}_{(p)}$.
- Given a p-local $\mathbb{Z}$-equivariant infinite loop space $M = \Omega^\infty m$, we let $B^{a,[k]}M$ be the $\mathbb{Z}$-equivariant infinite loop space $\Omega^\infty(\mathbb{S}^{(a,[k])} \otimes m)$. In particular, $\Omega^{a,[k]}B^{a,[k]}M \cong M$. $\triangleleft$

Note that smash products of $\mathbb{Z}$-equivariant spheres are given by the formula

$$S^{(a_1,[k_1])} \wedge S^{(a_2,[k_2])} \cong S^{(a_1+a_2,[k_1k_2])}.$$

Proof of Proposition 5.5.13. The space of $\mathbb{E}_4$-algebra maps $\mathrm{MU}_{(p)} \to E_n$ is isomorphic to the space of nullhomotopies of the composite

$$\mathrm{BU}_{(p)} \xrightarrow{J} \mathrm{BSL}_1(\mathbb{S}_{(p)}) \to \mathrm{BSL}_1(E_n)$$

in the category of four-fold loop maps [ACB19, Theorem 3.5].[32] Equivalently, this is the space of nullhomotopies in pointed spaces of the composite

$$\mathrm{B}^4\mathrm{BU}_{(p)} \to \mathrm{B}^5\mathrm{SL}_1(\mathbb{S}_{(p)}) \to \mathrm{B}^5\mathrm{SL}_1(E_n)$$

We will prove the theorem by showing that every nullhomotopy of this composite can be refined to a $\mathbb{Z}$-equivariant nullhomotopy of the composite

$$\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)} \to \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(\mathbb{S}_{(p)}) \to \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^\Psi). \tag{5.1}$$

Here, both $\mathrm{BU}_{(p)}$ and $\mathrm{SL}_1(E_n^\Psi)$ are equipped with $\mathbb{Z}$ actions via Ψ^ℓ, and $\mathrm{SL}_1(\mathbb{S}_{(p)})$ is equipped with trivial $\mathbb{Z}$-action. The $\mathbb{Z}$-equivariant infinite loop map $\mathrm{BU}_{(p)} \to \mathrm{BSL}_1(\mathbb{S})$ is given by the solution to the stable Adams conjecture fixed in Construction 5.5.3.

To see that this is sufficient, note that the functor $\Omega^{4,[\ell^2]}(-)\colon \mathrm{Spc}_*^{B\mathbb{Z}} \to \mathrm{Spc}_*^{B\mathbb{Z}}$ takes values in triple loop spaces. This is because $S^{4,[\ell^2]} \cong S^{3,[0]} \wedge S^{1,[\ell^2]}$, where first smash factors has trivial $\mathbb{Z}$-action.

First, we show that the composite in (5.1) is $\mathbb{Z}$-equivariantly nullhomotopic. Let $X, Y \in \mathrm{Spc}_*^{B\mathbb{Z}}$ be the equivariant mapping spaces

$$\mathrm{Map}_{\mathrm{Spc}_*}(\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^\Psi)) \text{ and } \mathrm{Map}_{\mathrm{Spc}_*}\left(\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^\Psi)_{\mathbb{Q}}\right)$$

respectively (where the $\mathbb{Z}$-action is by conjugation) so that we have an isomorphism

$$\mathrm{Map}_{\mathrm{Spc}_*^{B\mathbb{Z}}}(\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^\Psi)) \cong X^{h\mathbb{Z}},$$

---

[32] Recall that SL_1 is the 1-connective cover of GL_1.

between the space of $\mathbb{Z}$-equivariant maps and the fixed points of X. The obstruction to (5.1) being null is now a class in $\pi_0(X^{h\mathbb{Z}})$. By Lemma 5.5.15 below, the homotopy groups of X are torsion free, concentrated in odd degrees, the $\mathbb{Z}$-action on π_1 is trivial and the natural map $X \to Y$ is a π_*-injection. Altogether this implies that the natural map $\pi_0(X^{h\mathbb{Z}}) \to \pi_0(Y^{h\mathbb{Z}})$ is injective. To prove that (5.1) is null it now suffices to prove that the further composite

$$\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)} \to \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(\mathbb{S}_{(p)}) \to \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^{\Psi}) \to \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^{\Psi})_{\mathbb{Q}}$$

is $\mathbb{Z}$-equivariantly nullhomotopic. This composite factors through $\mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(\mathbb{S}_{(p)})_{\mathbb{Q}}$ which is contractible.

It remains to show that one can find a $\mathbb{Z}$-equivariant nullhomotopy of (5.1) lifting any given non-equivariant nullhomotopy. Homotopy classes of equivariant and nonequivariant nulhomotopies are torsors over $\pi_1(X^{h\mathbb{Z}})$ and $\pi_1(X)$ respectively. Using Lemma 5.5.15 again we see that these groups agree since $\pi_2(X)$ is trivial, and $\pi_1(X)$ has a trivial action. Thus any given non-equivariant nullhomotopy can be refined to an equivariant nullhomotopy. ☐

Lemma 5.5.15. *The homotopy groups of the pointed mapping space*

$$\mathrm{Map}_{\mathrm{Spc}_*}(\mathrm{B}^{4,[\ell^2]}\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\mathrm{SL}_1(E_n^{\Psi}))$$

are torsion free, concentrated in odd degrees, have Ψ^ℓ act by ℓ^k on π_{2k+1}, and embed into the homotopy groups of the mapping space to the rationalization

$$\mathrm{Map}_{\mathrm{Spc}_*}(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})_{\mathbb{Q}}).$$

Proof. Let $X, Y \in \mathrm{Spc}_*^{B\mathbb{Z}}$ be the equivariant mapping spaces

$$\mathrm{Map}_{\mathrm{Spc}_*}\left(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})\right) \text{ and } \mathrm{Map}_{\mathrm{Spc}_*}\left(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}, \mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})_{\mathbb{Q}}\right)$$

respectively (where the $\mathbb{Z}$-action is by conjugation). We begin by analyzing the underlying (non-equivariant) spaces of X and Y. Filtering the target spaces

$$\mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi}) \text{ and } \mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})_{\mathbb{Q}}$$

by their Postnikov towers we obtain spectral sequences of signature

$$\begin{tikzcd}
H^s(\mathrm{BU}\langle 6\rangle_{(p)};\ \pi_t(\mathrm{B}^5\,\mathrm{SL}_1(E_n))) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(X) \arrow[d] \\
H^s(\mathrm{BU}\langle 6\rangle_{(p)};\ \pi_t(\mathrm{B}^5\,\mathrm{SL}_1(E_n)_{\mathbb{Q}})) \arrow[r, Rightarrow] & \pi_{t-s}(Y)
\end{tikzcd}$$

and a map between them (recall that the underlying space of $\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}$ is $\mathrm{BU}\langle 6\rangle_{(p)}$).

E_n has torsion free homotopy concentrated in even degrees and the integral cohomology of $\mathrm{BU}\langle 6\rangle$ is a finite sum of copies of $\mathbb{Z}$ concentrated only in even degrees [Sin68] (cf. [AHS01, Corollary 4.7]). Therefore, on the E_1-page these spectral sequences are concentrated entirely in degrees with $t-s$ odd and the map between them is injective. In particular, both spectral sequence degenerate at the E_1-page and converge strongly. From this we learn that π_*X and π_*Y are concentrated in odd degrees and the map $\pi_*X \to \pi_*Y$ is injective.

In order to complete to proof it will now suffice (using the injectivity proved above) to analyze the $\mathbb{Z}$-action on $\pi_*(Y)$. The Postnikov tower of $\Sigma^{5,[\ell^2]}\mathrm{sl}_1(E_n^{\Psi})) \otimes \mathbb{Q}$ splits $\mathbb{Z}$-equivariantly (here we use that the group ring $\mathbb{Q}[\mathbb{Z}]$ has projective dimension 1). As a consequence we obtain a $\mathbb{Z}$-equivariant isomorphism,

$$\pi_{2k+1}(Y) \cong \prod_{t-s=2k+1} H^s\left(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}; \mathbb{Q} \otimes \pi_t(\mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi}))\right).$$

We now analyze the $\mathbb{Z}$-actions on each term in this product. Using the fact that the cohomology of $\mathrm{BU}\langle 6\rangle$ is finitely generated in each degree we obtain a $\mathbb{Z}$-equivariant isomorphism

$$H^s\left(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}; \mathbb{Q} \otimes \pi_t(\mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi}))\right) \cong H^s(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}; \mathbb{Q}) \otimes \pi_t(\mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})).$$

Returning to Construction 5.5.2 we now determine that

(a) Ψ^ℓ acts on $\pi_{2k}E_n^{\Psi}$ by ℓ^k,

(b) Ψ^ℓ acts on $\pi_{2k+1}\left(\mathrm{B}^{5,[\ell^2]}\,\mathrm{SL}_1(E_n^{\Psi})\right)$ by ℓ^k,

(c) Ψ^ℓ acts on $\pi_{2k}\mathrm{bu}_{(p)}$ by ℓ^k,

(d) Ψ^ℓ acts on $\pi_{2k}\,\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}$ by ℓ^k,

Using that the rational homology of $\mathrm{BU}\langle 6\rangle$ is a polynomial algebra generated off of the image of the homotopy groups we can now work out the action on the rational homology (and cohomology) of $\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}$,

(e) Ψ^ℓ acts on $H_{2k}(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}; \mathbb{Q})$ by ℓ^k and

(f) Ψ^ℓ acts on $H^{2k}(\mathrm{B}^{4,[\ell^2]}\,\mathrm{BU}_{(p)}; \mathbb{Q})$ by ℓ^{-k}.

Using (b), (f) and the isomorphism above we determine that Ψ^ℓ acts on $\pi_{2k+1}(Y)$ by ℓ^k. □

Corollary 5.5.16. *The $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n constructed via Proposition 5.5.9 refines to a lift of E_n^{Ψ} to a $\mathbb{Z}$-equivariant $\mathbb{E}_2$-$\mathrm{MU}_{(p)}^{\Psi}$-algebra E_n^{Ψ}.*

Proof. This follows by applying Corollary 5.5.12, Proposition 5.5.13 and Proposition 5.5.11. □

5.5.3 Digression: $\mathbb{E}_1$-cells

The goal of this digression is to set up a theory of $\mathbb{E}_1$-cells that works well when applied to both the category of $\mathrm{MU}_{(p)}$-modules and the category of $\mathbb{Z}$-equivariant $\mathrm{MU}_{(p)}$-modules. In Section 5.5.4, we use this to first upgrade $\mathrm{BP}\langle n\rangle$ to a $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebra, and then to a $\mathbb{Z}$-equivariant $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$-algebra. Our treatment here takes inspiration from [BL21].

Convention 5.5.17. Throughout this subsection $\mathcal{C}$ will denote a fixed stable, presentably symmetric monoidal category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ compatible with the monoidal structure. ◁

Recollection 5.5.18. Recall from [Lur17, Corollary 1.4.4.5] that the universal colimit preserving functor from a presentable category $\mathcal{D}$ to a stable presentable category is the stabilization,

$$\Sigma^\infty_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{Sp}(\mathcal{D}).$$

By [Fra08, Chapter 4, Theorem 4.2], for A an $\mathbb{E}_1$-algebra in $\mathcal{C}$, the stabilization of $\mathrm{Alg}(\mathcal{C})_{/A}$ can be identified with the functor sending $B \to A$ to the A-bimodule $\mathrm{fib}(A \otimes A \to A \otimes_B A)$. The right adjoint is given by the trivial square zero extension functor. Note that in the case $A = 1_{\mathcal{C}}$, the category of bimodules is just $\mathcal{C}$. ◁

Definition 5.5.19. We say that an algebra A in $\mathcal{C}$ is connected if it is connective and the unit map induces an isomorphism $\pi_0 \mathbb{1} \cong \pi_0 A$.[33] Let $\mathrm{Alg}(\mathcal{C})^{\geq 1} \subseteq \mathrm{Alg}(\mathcal{C})$ denote the full subcategory of connected algebras. ◁

Note that $\mathrm{Alg}(\mathcal{C})^{\geq 1} \subseteq \mathrm{Alg}(\mathcal{C})$ is closed under colimits. Given a connected algebra A, its base change to $\pi_0 A$ acquires a canonical augmentation $A \otimes \pi_0 \mathbb{1} \to \pi_0 \mathbb{1}$ via the 0-truncation map.

Construction 5.5.20. We define $\mathbb{H}$ to be the composite

$$\mathrm{Alg}(\mathcal{C})^{\geq 1} \xrightarrow{\otimes \pi_0 \mathbb{1}} \mathrm{Alg}(\mathrm{Mod}(\mathcal{C}; \pi_0 \mathbb{1}))_{-/\pi_0 \mathbb{1}} \xrightarrow{\Sigma^\infty} \mathrm{Mod}(\mathcal{C}; \pi_0 \mathbb{1})$$

where the second functor is stabilization. Concretely, $\mathbb{H}$ can be computed by the formula

$$A \mapsto \mathrm{fib}(\pi_0 \mathbb{1} \to \pi_0 \mathbb{1} \otimes_{A \otimes \pi_0 \mathbb{1}} \pi_0 \mathbb{1}).$$

By construction, $\mathbb{H}$ preserves colimits. The stabilization adjunction gives a natural map

$$A \to A \otimes \pi_0 \mathbb{1} \to \pi_0 \mathbb{1} \oplus \mathbb{H}(A). \qquad ◁$$

Example 5.5.21. Let us suppose that $\mathbb{1} = \tau_{\leq 0} \mathbb{1}$. Then for a free algebra $\mathbb{1}\{X\}$, we have $\mathbb{H}(\mathbb{1}\{X\}) = X$, and the natural map $\bigoplus_0^\infty X^{\otimes i} \cong \mathbb{1}\{X\} \to \mathbb{H}(\mathbb{1}\{X\}) \cong \mathbb{1} \oplus X$ can be identified with the projection onto the arity 1 summand. ◁

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[33] Compare with "connected" (graded) Hopf algebras.

Example 5.5.22. Take $\mathcal{C} = \mathrm{Mod}(\mathrm{MU}_{(p)})$, then $\pi_*\mathbb{H}(\mathrm{BP}\langle n\rangle)$ is torsion free, finitely generated and concentrated in positive odd degrees. ◁

Details. Expanding out the formula for $\mathbb{H}$ we obtain

$$\mathbb{Z}_{(p)} \oplus \Sigma\mathbb{H}(\mathrm{BP}\langle n\rangle) \cong \mathbb{Z}_{(p)} \otimes_{\mathrm{BP}\langle n\rangle\otimes_{\mathrm{MU}_{(p)}}\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}.$$

Applying the Tor spectral sequence and using the fact that the map $\pi_*\mathrm{MU}_{(p)} \to \pi_*\mathrm{BP}\langle n\rangle$ is the quotient by a regular sequence of even classes, we learn that $\pi_*(\mathrm{BP}\langle n\rangle \otimes_{\mathrm{MU}_{(p)}} \mathbb{Z}_{(p)})$ is an exterior algebra over $\mathbb{Z}_{(p)}$ on the suspensions of the classes in the regular sequence. There are no multiplicative extensions between these generators for filtration reasons (see, e.g., [Ang08, Proposition 3.6]). Applying the Tor spectral sequence again, we obtain a spectral sequence computing $\pi_*\mathbb{H}(\mathrm{BP}\langle n\rangle)$ whose E_2-page is a divided power algebra over $\mathbb{Z}_{(p)}$ on classes in positive even degrees. The spectral sequence degenerates at E_2 for bidegree reasons, and we obtain the desired conclusion. ☐

A key tool of this subsection is the following analog of the Hurewicz theorem.

Lemma 5.5.23. *Let $f\colon A \to B$ be a map in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$ and $k \geq 1$.*

1. *$\mathrm{fib}(f)$ is k-connective if and only if $\mathrm{fib}(\mathbb{H}(f))$ is k-connective.*
2. *If $\mathrm{fib}(f)$ is k-connective, then the natural map $\mathrm{fib}(f) \to \mathrm{fib}(\mathbb{H}(f))$ is $(k+1)$-connective.*

Proof. First, note that for $M \in \mathcal{C}$ which is bounded below M is s-connective iff $M \otimes \pi_0\mathbb{1}$ is s-connective. Thus, without loss of generality we may replace f with the map $f \otimes \pi_0\mathbb{1}$ of augmented $\pi_0\mathbb{1}$-algebras and replace $\mathcal{C}$ with $\mathrm{Mod}(\mathcal{C};\pi_0\mathbb{1})$.

Part (2). Using the simplicial resolution of f coming from the free augmented algebra, augmentation ideal monad we reduce to the case where f is of the form $\mathbb{1}\{X\} \to \mathbb{1}\{Y\}$, the map induced by a k-connective map $g\colon X \to Y$, where X, Y are 1-connective. In this situation we may identify (see Example 5.5.21) the fiber sequence $F \to \mathrm{fib}(f) \to \mathrm{fib}(\mathbb{H}(f))$ with the fiber sequence

$$\oplus_{j\geq 2}\,\mathrm{fib}(g^{\otimes j}) \to \oplus_{j\geq 1}\,\mathrm{fib}(g^{\otimes j}) \to \mathrm{fib}(g).$$

Using that $\mathrm{fib}(g)$ is k-connective and X, Y are 1-connective it follows that $\mathrm{fib}(g^{\otimes j})$ is $(k-1+j)$-connective. This suffices to conclude.

Part (1). Since both A and B are connected algebras $\mathrm{fib}(f)$ is 0-connective. Applying (2) inductively we find that $\mathrm{fib}(f)$ is k-connective if and only if $\mathrm{fib}(\mathbb{H}(f))$ is k-connective. ☐

Lemma 5.5.24. *Let $f\colon A \to B$ be a map in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$ and let*

$$\mathbb{H}(A) = M_0 \xrightarrow{m_1} M_1 \xrightarrow{m_2} M_2 \to \cdots \to M_\infty = \mathbb{H}(B)$$

be an increasing filtration such that

(a) *m_i is k_i-connective for a non-decreasing sequence k_i with $k_1 \geq 0$ and*

(b) *for each $i \geq 1$ there are objects $X_i \in \mathcal{C}_{\geq k_i}$ such that $X_i \otimes \pi_0 \mathbb{1} \cong \mathrm{fib}(m_i)$ and $[X_i, Y] = 0$ for any $(k_i + 2)$-connective Y.*

There exists a filtration in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$

$$A := R_0 \xrightarrow{r_1} R_1 \to \cdots \to R_\infty \to B$$

such that

1. *the map $R_i \to B$ is k_{i+1}-connective,*
2. *the functor $\mathbb{H}(-)$ sends the filtration $\{R_i\}$ to the filtration $\{M_i\}$*
3. *each map r_i fits into a pushout square in* $\mathrm{Alg}(\mathcal{C})$

$$\begin{tikzcd}
\mathbb{1}\{X_i\} \arrow[r, "\mathrm{aug}"] \arrow[d] & \mathbb{1} \arrow[d] \\
R_{i-1} \arrow[r, "r_i"] & R_i,
\end{tikzcd}$$

4. *the map $R_\infty \to B$ is ∞-connective.*

Proof. We construct a filtration between A and B satisfying (1), (2) and (3) by induction on i. For the base case $i = 0$, we take $A = R_0$, which clearly satisfies (2), and we consider condition (3) to be vacuous in this case. Condition (1) in this case and in the inductive case follows from condition (2) by applying Lemma 5.5.23 and using the hypothesis that k_j are non-decreasing. Thus we will only focus on (2) and (3) for the inductive step.

For the inductive step, suppose we are given $A \to R_{i-1} \to B$ and an identification of $\mathbb{H}(A) \to \mathbb{H}(R_{i-1}) \to \mathbb{H}(B)$ with $M_0 \to M_{i-1} \to M_\infty$. We will construct an R_i satisfying the same condition using pushout of the form in (3).

Consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd}
X_i \arrow[r, dashed, "g_i"] \arrow[d] & \mathrm{fib}(R_{i-1} \to B) \arrow[d] \\
\mathrm{fib}(m_i) \arrow[r, "h_i"] & \mathrm{fib}(\mathbb{H}(R_{i-1}) \to \mathbb{H}(B))
\end{tikzcd}$$

where the right vertical map is $(k_i + 1)$-connective by condition (1) and Lemma 5.5.23. As X_i has no nonzero maps to a $(k_i + 2)$-connective objects, the obstructions to producing the dashed lift g_i vanishes and we fix a choice of lift. Using g_i we can define R_i to be the pushout of $\mathbb{E}_1$-algebras that fits into the following diagram

$$\begin{tikzcd}
\mathbb{1}\{X_i\} \arrow[r, "\mathrm{aug}"] \arrow[d, "\overline{g}_i"] & \mathbb{1} \arrow[r] \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] & \mathbb{1} \arrow[d] \\
R_{i-1} \arrow[r, "r_i"] & R_i \arrow[r] & B.
\end{tikzcd}$$

$\mathbb{1}\{X_i\}$ is connective since X_i is, so after applying $\tau_{\leq 0}$ (which preserves colimits), the pushout square defining R_i becomes the pushout square in $\mathrm{Alg}(\mathcal{C}^\heartsuit)$:

$$\begin{tikzcd}
\tau_{\leq 0}\mathbb{1}\{X_i\} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\
\mathbb{1} \arrow[r] & \tau_{\leq 0}R_i
\end{tikzcd}$$

which shows that R_i is connected, since this pushout is the free algebra on the suspension of X_i, which is 0 since $\mathcal{C}^\heartsuit$ is discrete.

Applying $\mathrm{fib}(\mathbb{H}(-))$ to rows of the square above we obtain a $\pi_0\mathbb{1}$-linear map u_i fitting into the following diagram

$$\begin{tikzcd}
X_i \arrow[r] \arrow[dr, "g_i"'] & \mathrm{fib}(\mathbb{1}\{X_i\} \to \mathbb{1}) \arrow[r] \arrow[d] & X_i \otimes \pi_0\mathbb{1} \arrow[d, "u_i"] \\
& \mathrm{fib}(R_{i-1} \to B) \arrow[r] & \mathrm{fib}(\mathbb{H}(R_{i-1}) \to \mathbb{H}(B)).
\end{tikzcd}$$

Comparing this diagram with the one above we observe they are both examples of $\pi_0\mathbb{1}$-linearizations and therefore obtain an isomorphism $X_i \otimes \pi_0\mathbb{1} \cong \mathrm{fib}(m_i)$ that identifies u_i with h_i. Condition (2) follows.

In order to complete the proof of the lemma we must check that the filtration we have constructed satisfies condition (4). Using the compatibility of $\mathbb{H}(-)$ together with condition (2) we see that the map $\mathbb{H}(R_\infty) \to \mathbb{H}(B)$ is an isomorphism. It now follows from Lemma 5.5.23 that the map $R_\infty \to B$ is ∞-connective. ☐

We will also need to understand how $\mathbb{H}$ interacts with tensor products.

Lemma 5.5.25. *Let $A, B \in \mathrm{Alg}(\mathcal{C})^{\geq 1}$. There is a natural identification*

$$\mathrm{fib}\left(\mathbb{H}(A \coprod B) \to \mathbb{H}(A \otimes B)\right) \cong \mathbb{H}(A) \otimes_{\pi_0\mathbb{1}} \mathbb{H}(B).$$

Proof. Without loss of generality, we may replace $\mathcal{C}$ with $\mathrm{Mod}(\mathcal{C}; \pi_0\mathbb{1})$, A with $A \otimes \pi_0\mathbb{1}$ and B with $B \otimes \pi_0\mathbb{1}$. For any $A \in \mathrm{Alg}(\mathcal{C})_{-/\mathbb{1}}$, there is a natural identification $\mathbb{1} \otimes_A \mathbb{1} \cong \mathbb{1} \oplus \Sigma\mathbb{H}(A)$. Using this we then compute that

$$\begin{aligned} \mathbb{1} \otimes_{A \otimes B} \mathbb{1} &\cong (\mathbb{1} \otimes_A \mathbb{1}) \otimes (\mathbb{1} \otimes_B \mathbb{1}) \cong (\mathbb{1} \oplus \Sigma\mathbb{H}(A)) \otimes (\mathbb{1} \oplus \Sigma\mathbb{H}(B)) \\ &\cong \mathbb{1} \oplus \Sigma\mathbb{H}(A) \oplus \Sigma\mathbb{H}(B) \oplus \Sigma^2\mathbb{H}(A) \otimes \mathbb{H}(B) \end{aligned}$$ ☐

By keeping track of these isomorphisms via the map from $A \coprod B \to A \otimes B$, we conclude.

Example 5.5.26.

1. If $\mathcal{C}$ is the category of p-local spectra, then $\pi_0\mathbb{1} \cong \mathbb{Z}_{(p)}$, whose modules are $\mathbb{Z}_{(p)}$-modules.

2. If $\mathcal{C}$ is the category of $\mathrm{MU}_{(p)}$-modules, then $\pi_0\mathbb{1} \cong \mathbb{Z}_{(p)}$, whose modules are $\mathbb{Z}_{(p)}$-modules.

3. If $\mathcal{C}$ is the category of $\mathbb{Z}$-equivariant $\mathrm{MU}^{\Psi}_{(p)}$-modules, then $\pi_0\mathbb{1} \cong \mathbb{Z}_{(p)}$ (with trivial action), whose modules are $\mathbb{Z}$-equivariant $\mathbb{Z}_{(p)}$-modules. $\triangleleft$

Definition 5.5.27. In the category $\mathrm{Sp}^{B\mathbb{Z}}_{(p)}$ of $\mathbb{Z}$-equivariant $\mathbb{S}_{(p)}$-modules we let $\mathbb{S}_{(p)}(j)$ be the invertible object given by $\mathbb{S}_{(p)}$ with Ψ^ℓ acting by $\ell^j \in (\pi_0\mathbb{S}_{(p)})^\times$.

Given a $\mathbb{Z}$-equivariant commutative $\mathbb{S}_{(p)}$-algebra R we let $R(j) \in \mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; R)$ be the corresponding invertible object obtained by base change. Note that $\mathbb{Z}_{(p)}(j) \in \mathrm{Mod}(\mathbb{Z}_{(p)})^{B\mathbb{Z}}$ lies in the heart. $\triangleleft$

Next we prove a lemma useful for checking the vanishing condition in Lemma 5.5.24(b). This starts with the observation that in the category of spectra $[\mathbb{S}, Y] = 0$ for any 1-connective Y. Similarly, if R is a connective commutative algebra, then in R-modules $[R, Y] = 0$ for any 1-connective R-module Y.

Lemma 5.5.28. *Let a $\mathbb{Z}$-equivariant commutative $\mathbb{S}_{(p)}$-algebra R. If $Y \in \mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; R)$ is 2-connective, then for all j we have $[R(j), Y] = 0$.*

Proof. As $R(j)$ is obtained by base change from $\mathbb{S}_{(p)}(j)$ and restriction preserves connectivity it will suffice to prove the lemma in the case $R = \mathbb{S}_{(p)}$. If we let $\Psi_Y : Y \to Y$ be the action on Y then we have

$$\mathrm{map}_{\mathrm{Sp}^{B\mathbb{Z}}}(\mathbb{S}_{(p)}(j), Y) \cong \mathrm{fib}(Y \xrightarrow{\ell^j - \Psi_Y} Y). \qquad \square$$

5.5.4 A $\mathbb{Z}$-equivariant $\mathbb{E}_1 \otimes \mathbb{A}_2$ form of $\mathrm{BP}\langle n\rangle$

We now prove an $\mathbb{E}_1$-version of Theorem 5.5.4 which we then upgrade to the full statement.

Proposition 5.5.29. *The underlying $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebra of $\mathrm{BP}\langle n\rangle$ lifts to a*

$$\mathrm{BP}\langle n\rangle^\Psi \in \mathrm{Alg}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}^{\Psi}_{(p)})$$

in such a way that there is a $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}^{\Psi}_{(p)}$-algebra map $\iota\colon \mathrm{BP}\langle n\rangle^\Psi \to E_n^\Psi$ to the underlying $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}^{\Psi}_{(p)}$-algebra of the E_n^Ψ from Corollary 5.5.16.

Proof. The $\mathbb{Z}$-action on $\mathrm{MU}^{\Psi}_{(p)}$ gives rise to a $\mathbb{Z}$-action on $\mathrm{Mod}(\mathrm{MU}_{(p)})$ and there is a symmetric monoidal isomorphism

$$\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}^{\Psi}_{(p)}) \cong \mathrm{Mod}(\mathrm{MU}_{(p)})^{h\mathbb{Z}}.$$

As consequence of this identification, to prove the proposition it will suffice for us to construct a square of $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebras

$$\begin{tikzcd}
\mathrm{BP}\langle n\rangle \arrow[r] \arrow[d, "\iota"] & (\Psi^\ell)_*(\mathrm{BP}\langle n\rangle) \arrow[d, "(\Psi^\ell)_*(\iota)"] \\
E_n \arrow[r, "\Psi^\ell"] & (\Psi^\ell)_*(E_n).
\end{tikzcd}$$

where the map on the bottom row comes from Corollary 5.5.16 and the vertical maps are obtained via Proposition 5.5.9.

We will construct this square by obstruction theory using a presentation of $\mathrm{BP}\langle n\rangle$ by $\mathbb{E}_1$-cells coming from Lemma 5.5.24. Using a splitting of the Postnikov tower of $\mathbb{H}(\mathrm{BP}\langle n\rangle)$ and the concentration in odd degrees from Example 5.5.22 we construct a filtration

$$\begin{aligned}\mathbb{H}(\mathrm{MU}_{(p)}) = 0 &\to \Sigma^1\pi_1(\mathbb{H}(\mathrm{BP}\langle n\rangle)) \to \Sigma^1\pi_1(\mathbb{H}(\mathrm{BP}\langle n\rangle)) \oplus \Sigma^3\pi_3(\mathbb{H}(\mathrm{BP}\langle n\rangle))\\ &\to \cdots \to \oplus_{i>0}\Sigma^{2i-1}\pi_{2i-1}(\mathbb{H}(\mathrm{BP}\langle n\rangle)) \cong \mathbb{H}(\mathrm{BP}\langle n\rangle).\end{aligned}$$

As each $\pi_{2i-1}(\mathbb{H}(\mathrm{BP}\langle n\rangle))$ is a finite sum of copies of $\mathbb{Z}_{(p)}$ by Example 5.5.22 we may take X_i to be a sum of copies of $\Sigma^{2i-2}\mathrm{MU}_{(p)}$ and apply Lemma 5.5.24. From this we obtain a filtration

$$\mathrm{MU}_{(p)} = R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} R_2 \to \cdots \to R_\infty \xrightarrow{\cong} \mathrm{BP}\langle n\rangle$$

of $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebras and pushouts

$$\begin{tikzcd}
\mathrm{MU}_{(p)}\{\oplus\Sigma^{2i-2}\mathrm{MU}_{(p)}\} \arrow[r, "\mathrm{aug}"] \arrow[d] & \mathrm{MU}_{(p)} \arrow[d] \\
R_{i-1} \arrow[r, "r_i"] & R_i. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We now proceed by induction constructing a commuting diagram of $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebras

$$\begin{tikzcd}
\mathrm{MU}_{(p)}\{\oplus\Sigma^{2i-2}\mathrm{MU}_{(p)}\} \arrow[r] \arrow[d, "\mathrm{aug}"] & R_{i-1} \arrow[d, "r_i"] \arrow[ddr, bend left, "f_{i-1}"] & \\
\mathrm{MU}_{(p)} \arrow[r] & R_i \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d] \arrow[dr, dashed, "f_i"'] & \\
& \mathrm{BP}\langle n\rangle \arrow[d] & (\Psi^\ell)_*(\mathrm{BP}\langle n\rangle) \arrow[d] \\
& E_n \arrow[r, "\Psi^\ell"] & (\Psi^\ell)_*(E_n).
\end{tikzcd}$$

Let x denote a map $\oplus\Sigma^{2i-2}\mathrm{MU}_{(p)} \to R_{i-1}$ along which the cell producing R_i is attached. We will first check that $f_{i-1}\circ x$ is nullhomotopic. Since the source of x is a free $\mathrm{MU}_{(p)}$-module and $\pi_*(\Psi^\ell)_*\mathrm{BP}\langle n\rangle \to \pi_*(\Psi^\ell)_*E_n$ is injective (Construction 5.5.10), it will suffice to check that the further composite $(\Psi^\ell)_*(\iota) \circ f_{i-1} \circ x$ is nullhomotopic. This is implied by the fact that the diagram of solid arrows commutes and x maps to zero in along the augmentation.

Any nullhomotopy of $f_{i-1} \circ x$ produces a map f_i together with a homotopy filling the triangle within the desired diagram. It remains to pick a homotopy making the trapezoid below the triangle commute. This is equivalent to checking that two nullhomotopies of the composite

$$\oplus\Sigma^{2i-2}\mathrm{MU}_{(p)} \xrightarrow{x} R_{i-1} \to (\Psi^\ell)_*(E_n)$$

are compatible. Since the source is a free $\mathrm{MU}_{(p)}$-module and $\pi_{2i-1}E_n \cong 0$, the space of such nullhomotopies is connected. $\square$

Lemma 5.5.30. *Ψ^ℓ acts on $\pi_{2k}\mathrm{MU}_{(p)}^\Psi$ via multiplication by ℓ^k.*

Proof. As the homotopy groups of $\mathrm{MU}_{(p)}$ are torsion free it suffices to prove the lemma after rationalization. After rationalizing $\mathrm{BSL}_1(\mathbb{S}_{(p)})$ becomes trivial and so we have a canonical $\mathbb{Z}$-equivariant identification

$$\mathbb{Q} \otimes \mathrm{MU}_{(p)}^\Psi \cong \mathbb{Q} \otimes \Sigma^\infty_+ \mathrm{BU}\,.$$

The lemma now follows from the fact that the rational homology of BU is a polynomial algebra on classes b_i in the Hurewicz image and the formula for the action of the Adams operation Ψ^ℓ on the homotopy of BU. $\square$

Corollary 5.5.31. *Ψ^ℓ acts on $\pi_{2k}\mathrm{BP}\langle n\rangle^\Psi$ via multiplication by ℓ^k.*

Proof. The unit map $\mathrm{MU}_{(p)}^\Psi \to \mathrm{BP}\langle n\rangle^\Psi$ is surjective on homotopy groups. $\square$

Lemma 5.5.32. *$\pi_*^\heartsuit\mathbb{H}(\mathrm{BP}\langle n\rangle^\Psi)$ is concentrated in odd degrees, and each homotopy group $\pi_{2i-1}^\heartsuit\mathbb{H}(\mathrm{BP}\langle n\rangle^\Psi)$ has a finite filtration whose associated graded consists of copies of $\mathbb{Z}_{(p)}(j)$ for $j < i$.*

Proof. The proof of this lemma proceeds by keeping track of actions in Example 5.5.22. As there we have an isomorphism of $\mathbb{Z}$-equivariant $\mathbb{Z}_{(p)}$-modules

$$\mathbb{Z}_{(p)} \oplus \Sigma\mathbb{H}(\mathrm{BP}\langle n\rangle^\Psi) \cong \mathbb{Z}_{(p)} \otimes_{\mathrm{BP}\langle n\rangle^\Psi \otimes_{\mathrm{MU}_{(p)}^\Psi} \mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}.$$

Examining the Tor spectral sequence we find that $\pi_*^\heartsuit(\mathrm{BP}\langle n\rangle^\Psi \otimes_{\mathrm{MU}_{(p)}^\Psi} \mathbb{Z}_{(p)})$ is an exterior algebra over $\mathbb{Z}_{(p)}$ on a collection of classes $\Sigma^{2i+1}\mathbb{Z}_{(p)}(i)\{\sigma b_i\}$ where $i \in \{i > 0 \mid i \neq p-1, \dots, p^n-1\}$. The classes σb_i are suspensions of polynomial algebra generators of $\mathrm{MU}_{(p)}^\Psi$ and the action comes from Lemma 5.5.30.

Applying the Tor spectral sequence again we obtain a spectral sequence computing $\pi_*^\heartsuit(\mathbb{Z}_{(p)} \oplus \Sigma\mathbb{H}(\mathrm{BP}\langle n\rangle^\Psi))$ whose E_2-page is a divided power algebra over $\mathbb{Z}_{(p)}$ on classes $\Sigma^{2i+2}\mathbb{Z}_{(p)}(i)\{b_i\}$. This spectral sequence degenerates at E_2 and so we obtain a filtration on $\pi_*^\heartsuit(\mathbb{H}(\mathrm{BP}\langle n\rangle^\Psi))$ with the desired properties. $\square$

Next we upgrade the $\mathbb{E}_1$-algebra from Proposition 5.5.29 to an $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-algebra.

Lemma 5.5.33. *The underlying $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$-algebra of $\mathrm{BP}\langle n\rangle$ lifts to a*

$$\mathrm{BP}\langle n\rangle^\Psi \in \mathrm{Alg}_{\mathbb{E}_1\otimes\mathbb{A}_2}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}_{(p)}^\Psi)).$$

Proof. Consider the $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}_{(p)}^\Psi$-algebra $\mathrm{BP}\langle n\rangle^\Psi$ from Proposition 5.5.29. We will lift $\mathrm{BP}\langle n\rangle^\Psi$ to a $\mathbb{Z}$-equivariant $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}^\Psi$-algebra by equipping it with a unital multiplication in the category of $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}^{\Psi}_{(p)}$-algebras. To do this we produce an extension as indicated in the diagram below.

$$\begin{tikzcd}
\mathrm{BP}\langle n\rangle^{\Psi} \coprod \mathrm{BP}\langle n\rangle^{\Psi} \arrow[rr, "\nabla"] \arrow[dr, "s"] & & \mathrm{BP}\langle n\rangle^{\Psi} \\
& \mathrm{BP}\langle n\rangle^{\Psi} \otimes_{\mathrm{MU}^{\Psi}_{(p)}} \mathrm{BP}\langle n\rangle^{\Psi} \arrow[ur, dashed] &
\end{tikzcd}$$

We will do this using a cellular obstruction theory argument. To do this we apply Lemma 5.5.24 to place a filtration on the map s. Using Lemma 5.5.25 we know that $\mathrm{fib}(\mathbb{H}(s))$ is isomorphic to $\mathbb{H}(\mathrm{BP}\langle n\rangle^{\Psi}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{H}(\mathrm{BP}\langle n\rangle^{\Psi})$. From Lemma 5.5.32 we can now conclude that the homotopy groups of $\mathrm{fib}(\mathbb{H}(s))$ are concentrated in positive even degrees and that $\pi^{\heartsuit}_{2i}\,\mathrm{fib}(\mathbb{H}(s))$ has a finite filtration with associated graded given by copies of $\mathbb{Z}_{(p)}(j)$ with $j < i$. Even-ness and Lemma 5.5.28 together imply that the Postnikov tower of $\mathrm{fib}(\mathbb{H}(s))$ splits (all k-invariants vanish). Splicing a splitting of the Postnikov tower together with the filtrations on the homotopy groups and picking the X_i to be of the form $\Sigma^{2k_i}\mathrm{MU}^{\Psi}_{(p)}(j)$ (see Lemma 5.5.28) we may apply Lemma 5.5.24 to obtain a filtration

$$\mathrm{BP}\langle n\rangle^{\Psi} \coprod \mathrm{BP}\langle n\rangle^{\Psi} = R_0 \xrightarrow{r_1} R_1 \to \cdots \to R_{\infty} \xrightarrow{\cong} \mathrm{BP}\langle n\rangle^{\Psi} \otimes_{\mathrm{MU}^{\Psi}_{(p)}} \mathrm{BP}\langle n\rangle^{\Psi}$$

where R_i is obtained from R_{i-1} by a pushout along the augmentation of a free algebra on $\Sigma^{2k+1}\mathrm{MU}^{\Psi}_{(p)}(j)$ with $j < k$.

Thus, the obstructions to making the extension are a sequence of $\mathbb{Z}$-equivariant maps

$$\Sigma^{2k}\mathrm{MU}^{\Psi}_{(p)}(j) \to \mathrm{BP}\langle n\rangle^{\Psi}$$

where $j < k$. As the homotopy of $\mathrm{BP}\langle n\rangle$ is even, these maps are determined by their value on $\pi^{\heartsuit}_{2k}(-)$. On the other hand, any map $\mathbb{Z}_{(p)}(j) \to \pi^{\heartsuit}_{2k}\mathrm{BP}\langle n\rangle^{\Psi}$ is null since $\pi^{\heartsuit}_{2k}\mathrm{BP}\langle n\rangle^{\Psi}$ is a sum of copies of $\mathbb{Z}_{(p)}(k)$ by Corollary 5.5.31 and $j < k$.

Given a nullhomotopy, we note that forgetting down to a non-equivariant nullhomotopy must agree with the one coming from the underlying $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-structure of $\mathrm{BP}\langle n\rangle$, since there is a unique such nulhomotopy in that case (as $\mathrm{BP}\langle n\rangle$ has even homotopy groups). Thus the $\mathbb{Z}$-equivariant $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$-algebra structure refines the nonequivariant one. □

We are now ready to put all the pieces together and conclude.

Proof (of Theorem 5.5.4). The $\mathbb{Z}$-equivariant lift $\mathrm{BP}\langle n\rangle^{\Psi}$ of the $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$-algebra structure on $\mathrm{BP}\langle n\rangle$ was constructed in Lemma 5.5.33. The $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}^{\Psi}_{(p)}$-algebra map $\iota\colon \mathrm{BP}\langle n\rangle^{\Psi} \to E^{\Psi}_n$ was constructed in Proposition 5.5.29. The identification of the underlying $\mathbb{E}_1$-algebras comes from Proposition 5.5.9 and this upgrades to an identification of $\mathbb{Z}$-equivariant $\mathbb{E}_1$-algebras because $\Psi^{\ell} \in \mathbb{G}_n$ is central and therefore commutes with $\mathbb{F}_p^{\times} \rtimes \hat{\mathbb{Z}}$. The $\mathbb{Z}$-action on the p-completion of $\mathrm{BP}\langle n\rangle$ is locally unipotent in p-complete spectra by Corollary 5.5.31, the fact that $\pi_*\mathrm{BP}\langle n\rangle$ is concentrated in degrees divisible by $2p-2$ and Corollary 5.8.27. □

5.6 Disproving the telescope conjecture

In this section, we study the algebraic K-theory of the fixed points of the $\mathbb{Z}$-action on $\mathrm{BP}\langle n\rangle$ constructed in Section 5.5. For $n \geq 1$ and $k \geq 0$, we prove that the $T(n+1)$-localized coassembly map

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}$$

is not an equivalence, but becomes an equivalence after $K(n+1)$-localization. Thus, the telescope conjecture fails.

We do this by looking at the coassembly map from two highly divergent perspectives, which are connected via trace theorems:

1. From the perspective of locally unipotent $\mathbb{Z}$-actions on ring spectra, the results of Section 5.4 tell us that the coassembly map cannot be an isomorphism.
2. From the perspective of *cyclotomic redshift* of [BMCSY23], the map
$$L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}} \to L_{T(n)}\mathrm{BP}\langle n\rangle$$
splits after base change to the maximal abelian extension of the $K(n)$-local sphere, and therefore the coassembly map is a $K(n+1)$-local isomorphism.

We begin in Section 5.6.1 with a discussion of the connection between TC, which has been the subject of the paper so far, and algebraic K-theory. In Section 5.6.2, we set up abstract machinery relating coassembly maps and descent. In Section 5.6.3, we apply this machinery to show that cyclotomic redshift implies that $L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}$ becomes an equivalence after cyclotomic completion. Finally, with all requisite inputs assembled, we exhibit a counterexample to the telescope conjecture in Section 5.6.4.

5.6.1 *K*-theory and TC

In this subsection, we explain why $T(n+1)$-localized K-theory and $T(n+1)$-localized TC coincide for our examples of interest. We begin with connective rings, where we have the following theorem proved in [CMNN23] and [LMMT20]:

Theorem 5.6.1 ([LMMT20, Purity Theorem, Cor. 4.29], [CMNN23, Cor. 4.11])**.** *Let R be a connective $\mathbb{E}_1$-algebra. For $n \geq 1$, the $(T(n) \oplus T(n+1))$-localization map and the cyclotomic trace induce isomorphisms*

$$L_{T(n+1)}K(L_{T(n)\oplus T(n+1)}R) \overset{\cong}{\leftarrow} L_{T(n+1)}K(R) \overset{\cong}{\to} L_{T(n+1)}\mathrm{TC}(R).$$

To disprove the telescope conjecture, we will need to understand the topological cyclic homologies of fixed points of $\mathbb{Z}$-actions on connective $\mathbb{E}_1$-algebras. Such fixed points are (-1)-connective, but often not 0-connective, so the above theorem does not apply. In order to surmount this obstacle, we use the following theorem, which is phrased in terms of the notion of truncating invariant from [LT19, Def. 3.1]:

Theorem 5.6.2 ([Lev22, Theorem B]). *Let R and S be connective $\mathbb{E}_1$-algebras with $\mathbb{Z}$-action. Let $f\colon R \to S$ be a 1-connective, $\mathbb{Z}$-equivariant $\mathbb{E}_1$-algebra map. For any truncating invariant E, the induced map*

$$E(R^{h\mathbb{Z}}) \to E(S^{h\mathbb{Z}})$$

is an isomorphism.

As a corollary of this theorem we prove an analog of Theorem 5.6.1 for (-1)-connective rings.

Corollary 5.6.3. *For $n \geq 1$, let R be a $T(n+1)$-acyclic, connective $\mathbb{E}_1$-algebra with $\mathbb{Z}$-action. The coassembly map, $T(n)$-localization map, and cyclotomic trace fit into a commuting diagram*

$$\begin{tikzcd}
L_{T(n+1)}K(L_{T(n)}R^{h\mathbb{Z}}) \arrow[d] & L_{T(n+1)}K(R^{h\mathbb{Z}}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & L_{T(n+1)}\mathrm{TC}(R^{h\mathbb{Z}}) \arrow[d] \\
L_{T(n+1)}K(L_{T(n)}R)^{h\mathbb{Z}} & L_{T(n+1)}K(R)^{h\mathbb{Z}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & L_{T(n+1)}\mathrm{TC}(R)^{h\mathbb{Z}},
\end{tikzcd}$$

where each horizontal map is an isomorphism.

Proof. The lower horizontal maps are isomorphisms by Theorem 5.6.1. The upper left map is an isomorphism by the purity theorem from [LMMT20]. We now focus our attention on the upper right map.

Let $K_{\mathrm{inv}}(-)$ be the fiber of the cyclotomic trace map, so that our goal is now to prove that $L_{T(n+1)}K_{\mathrm{inv}}(R^{h\mathbb{Z}}) = 0$. $L_{T(n+1)}K_{\mathrm{inv}}(-)$is a truncating invariant by the Dundas–Goodwillie–McCarthy theorem [DGM13]. Applying Theorem 5.6.2 to the 1-connective $\mathbb{Z}$-equivariant map $R \to \pi_0 R$, we obtain an isomorphism

$$L_{T(n+1)}K_{\mathrm{inv}}(R^{h\mathbb{Z}}) \cong L_{T(n+1)}K_{\mathrm{inv}}((\pi_0 R)^{h\mathbb{Z}}).$$

Since $n \geq 1$, Mitchell's vanishing theorem [Mit90] implies, for any $\mathbb{Z}$-algebra A, that $L_{T(n+1)}K_{\mathrm{inv}}(A) = 0$. Since the $\mathbb{Z}^{B\mathbb{Z}}$-algebra structure on $(\pi_0 R)^{h\mathbb{Z}}$ restricts to a $\mathbb{Z}$-algebra structure, we learn that $L_{T(n+1)}K_{\mathrm{inv}}((\pi_0 R)^{h\mathbb{Z}}) = 0$, which completes the proof. □

5.6.2 Descent and coassembly

In this subsection, we recall and develop general abstract machinery relating coassembly and descent. At the end, this machinery is applied to prove Lemma 5.6.15, which is a statement specifically about coassembly maps for $T(n+1)$-localized algebraic K-theory.

Convention 5.6.4. Throughout this subsection $\mathcal{C}$ will denote a category with finite limits and colimits, such that finite coproducts are disjoint and universal. We suppose $F\colon \mathcal{C}^{op} \to \mathcal{D}$ is a product preserving functor into some other category $\mathcal{D}$. ◁

Example 5.6.5. Suppose $\mathcal{C}_0$ is a presentably symmetric monoidal additive category. Then $\mathcal{C} = \mathrm{CAlg}(\mathcal{C}_0)^{op}$ satisfies the conditions of Convention 5.6.4. Indeed, the fact that coproducts are disjoint follows from the fact that products in $\mathrm{CAlg}(\mathcal{C}_0)$ are computed on underlying, with the projections to each factor given by inverting orthogonal idempotents. The fact that coproducts are universal follows from the fact that base change preserves finite products, since finite products preserve finite coproducts.

In particular, the main example of interest will be $\mathcal{C} = \mathrm{CAlg}\left(\mathrm{Sp}_{T(n)}\right)^{op}$, with F the functor $L_{T(n+1)}K \colon \mathrm{CAlg}\left(\mathrm{Sp}_{T(n)}\right) \to \mathrm{Sp}_{T(n+1)}$. ◁

F-covers

Definition 5.6.6. An F-cover in $\mathcal{C}$ is a morphism $f \colon A \to B$ in $\mathcal{C}$ that is a universal F-descent morphism in the sense of [LZ12, Definition 3.1.1]. In other words for every map g that is a base change of f, F satisfies descent for the Čech nerve of g.

We recall [LZ12, Lemma 3.1.2] the collection of F-covers is closed under composition and base change, and if $f \circ g$ is an F-cover, so is f. ◁

We do not need to use Proposition 5.6.7 below, but rather include it in order to explain our terminology: F-covers are exactly the covers in the universal F-descent topology.

Proposition 5.6.7. *There exists a Grothendieck topology on $\mathcal{C}$ called the universal F-descent topology where a sieve on an object $C \in \mathcal{C}^{op}$ is a cover iff it contains a finite collection of maps $C_i \to C$ such that the map $\coprod_i C_i \to C$ in $\mathcal{C}$ is an F-cover. In particular, the sieve corresponding to a finite collection of map $C_i \to \mathcal{C}$ is a cover in this topology iff the map $\coprod C_i \to \mathcal{C}$ is an F-cover.*

Moreover, for any functor $\mathscr{F} : \mathcal{C}^{op} \to \mathcal{D}$, $\mathscr{F}$ is a sheaf iff it preserves finite products and satisfies descent for the Čech nerve of any F-cover. In particular, F is a universal F-descent sheaf.

Proof. To construct such a topology, we wish to apply [Lur18b, Proposition A.3.2.1] to $\mathcal{C}$ equipped with the collection of F-covers. Conditions (a) and (b) follow from [LZ12, Lemma 3.1.2], and condition (d) follows by assumption on $\mathcal{C}$.

It remains to check that (c) is satisfied, i.e that if $f_i \colon a_i \to b_i$ is a universal F-descent morphism for a finite collection of maps, then so is $\coprod_i f_i$. Since coproducts are universal, it is enough to prove that F is an F-descent morphism. Because coproducts are disjoint and F is product preserving, the Čech complex for $\coprod_i f_i$ is the product of the Čech complexes for each f_i, so the result follows.

For the statement in the second paragraph, we apply [Lur18b, Proposition A.3.1.1], where condition (e) is satisfied by hypothesis on $\mathcal{C}$. □

Lemma 5.6.8. *Let $\mathcal{D}$ be a category with a Grothendieck topology, and suppose that the square below is a pullback square, and f' is a cover.*

$$\begin{tikzcd}
c \arrow[r, "g"] \arrow[d, "f"] & c' \arrow[d, "f'"] \\
d \arrow[r, "g'"] & d'
\end{tikzcd}$$

Then g is an cover iff g' is an cover.

Proof. If g' is a cover, g is too since covers are closed under pullbacks. Conversely, if g is a cover, then $g \circ f' = g' \circ f$, is too, which implies that g' is. ☐

Remark 5.6.9. By Proposition 5.6.7, the above lemma applies to any $\mathcal{D} = \mathcal{C}$ such as in Convention 5.6.4. We note that in this case, the lemma could have been proven directly using the closure properties of F-covers [LZ12, Lemma 3.1.2]. ◁

The following lemmas lets us transfer the notions of F-covers along functors:

Lemma 5.6.10. *Suppose we have functors*

$$\mathcal{C}^{op} \xrightarrow{F} \mathcal{D} \xrightarrow{B} \mathcal{D}'$$

where B preserves finite products and totalizations which exist, and F preserves finite products. Then if $f \in \mathcal{C}$ is a map,

1. *If f is an F-cover, it is a $B \circ F$-cover*
2. *If B is conservative and f is a $B \circ F$-cover, it is an F-cover.*

Proof. For (1) it is enough to show that any F-descent morphism is also a $B \circ F$-descent morphism, but this follows from the fact that F preserves totalizations. (2) follows since conservative totalization preserving functors reflect totalizations. ☐

Lemma 5.6.11. *Let $\mathcal{C}, \mathcal{C}'$ be as in Convention 5.6.4, and suppose we have functors*

$$\mathcal{C}'^{op} \xrightarrow{A^{op}} \mathcal{C}^{op} \xrightarrow{F} \mathcal{D}$$

where F and $F \circ A^{op}$ preserve finite products and A preserves pullbacks. Then if $A(f)$ is an F-cover, f is an $F \circ A^{op}$-cover.

Proof. Since A preserves finite limits, it preserves the Čech complex and base change, from which the result follows.

☐

The following lemma will be useful in giving examples of F-covers:

Lemma 5.6.12. *Let $G = \Omega X$ be a loop space. Suppose that a is a G-equivariant object in $\mathcal{C}$, such that the map $F(-) \to F(a \times -)^{hG}$ is an isomorphism. Then the map $a \to *$ is an F-cover.*

Proof. Since the functor $(-)^{hG} \colon \mathcal{C}^{BG} \to \mathcal{C}$ preserves limits and the forgetful functor $\mathcal{C}^{BG} \to \mathcal{C}$ is conservative, by Lemma 5.6.10 it is enough to show that f is a $F(a \times -)$-cover. Using Lemma 5.6.11, it is enough to show that $a \times f$ is an F-cover. But this is true because the map $a \times a \to a$ admits a section given by the diagonal map. ☐

We now turn to comparing $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-descent with coassembly maps.

Lemma 5.6.13. *Let $R \in \mathrm{Alg}(\mathrm{Sp})$, and let S be a Stone topological space. There is a natural isomorphism*

$$K(\mathbb{W}(C^0(S)) \otimes R) \cong \mathbb{W}(C^0(S)) \otimes K(R).$$

Proof. Write S as a cofiltered limit $\varprojlim S_\alpha$ of finite discrete spaces S_α. Using the fact that each of the functors $K(-)$, $\mathbb{W}(-)$ and $-\otimes R$ commute with filterd colimits and finite products (for $\mathbb{W}(-)$ see [BSY22, Prop. 2.2, Lem. 2.10]) the composite $K(\mathbb{W}(-) \otimes R)$ does as well. We now construct a chain of isomorphisms

$$\begin{aligned} K(\mathbb{W}(C^0(S)) \otimes R) &\cong K\left(\mathbb{W}\left(\varinjlim_{\alpha} \mathbb{F}_p^{\times S_\alpha}\right) \otimes R\right) \cong \varinjlim_{\alpha} K\left(\mathbb{W}\left(\mathbb{F}_p\right) \otimes R\right)^{\times S_\alpha} \\ &\cong \left(\varinjlim_{\alpha} \mathbb{S}^{\times S_\alpha}\right) \otimes K\left(\mathbb{S} \otimes R\right) \cong \left(\varinjlim_{\alpha} \mathbb{W}\left(\mathbb{F}_p\right)^{\times S_\alpha}\right) \otimes K(R) \\ &\cong \mathbb{W}\left(\varinjlim_{\alpha} \mathbb{F}_p^{\times S_\alpha}\right) \otimes K(R) \cong \mathbb{W}(C^0(S)) \otimes K(R). \end{aligned}$$

☐

Lemma 5.6.14. *For $n \geq 1$, Let $R \in \mathrm{Alg}(\mathrm{Sp}_{T(n)})$ and Let S be a Stone topological space. There is a natural isomorphism*

$$L_{T(n+1)}K(L_{T(n)}(\mathbb{W}(C^0(S)) \otimes R)) \cong L_{T(n+1)}(\mathbb{W}(C^0(S)) \otimes K(R)).$$

Proof. As R is $T(n)$-local, the map $\mathbb{W}(C^0(S)) \otimes R \to L_{T(n)}(\mathbb{W}(C^0(S)) \otimes R)$ is a $(T(n) \oplus T(n+1))$-local isomorphism and therefore we have a natural isomorphism

$$L_{T(n+1)}K(L_{T(n)}(\mathbb{W}(C^0(S)) \otimes R)) \cong L_{T(n+1)}K(\mathbb{W}(C^0(S)) \otimes R)$$

by the purity theorem of [LMMT20]. The lemma now follows from Lemma 5.6.13. ☐

Lemma 5.6.15. *For $n \geq 1$, let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)}^{B\mathbb{Z},u})$, and $F = L_{T(n+1)}K$. There is a commuting triangle, natural in R,*

$$\begin{tikzcd} & F(R^{h\mathbb{Z}}) \arrow[dl] \arrow[dr] & \\ F(R)^{h\mathbb{Z}} \arrow[rr, dash, "\cong"] & & \lim_{\Delta} F(R^{\otimes_{R^{h\mathbb{Z}}} \bullet+1}) \end{tikzcd}$$

identifying the coassembly map with the Čech nerve of $R^{h\mathbb{Z}} \to R$. In particular, the coassembly map for R is an isomorphism if and only if R satisfies F-descent.

Proof. We begin by constructing the following square with Čech covers and coassembly maps

$$\begin{tikzcd}
F(R^{h\mathbb{Z}}) \arrow[r] \arrow[d] & \lim_\Delta F\left(R^{h\mathbb{Z}} \otimes_{R^{h\mathbb{Z}}} R^{\otimes_{R^{h\mathbb{Z}}}\bullet}\right) \arrow[d] \\
F(R)^{h\mathbb{Z}} \arrow[r] & \left(\lim_\Delta F\left(R \otimes_{R^{h\mathbb{Z}}} R^{\otimes_{R^{h\mathbb{Z}}}\bullet}\right)\right)^{h\mathbb{Z}} .
\end{tikzcd}$$

The bottom horizontal arrow is an isomorphism since the map $R \to R \otimes_{R^{h\mathbb{Z}}} R$ admits a retraction. It will now suffice for us to argue that for any commutative R-algebra A, the coassembly map

$$F(R^{h\mathbb{Z}} \otimes_{R^{h\mathbb{Z}}} R \otimes_R A) \to F(R \otimes_{R^{h\mathbb{Z}}} R \otimes_R A)^{h\mathbb{Z}}$$

is an isomorphism.

Using Lemma 5.8.21, Remark 5.8.23 and Lemma 5.3.6 we construct a cube

$$\begin{tikzcd}
\mathbb{S}^{B\mathbb{Z}} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{S} \arrow[dr] \arrow[dd] & \\
& R^{h\mathbb{Z}} \arrow[rr] \arrow[dd] & & R \arrow[dd] \\
\mathbb{S} \arrow[rr] \arrow[dr] & & \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \arrow[dr] & \\
& R \arrow[rr] & & R \otimes_{R^{h\mathbb{Z}}} R
\end{tikzcd}$$

where every face is a pushout of commutative algebras and terms on the bottom face have $\mathbb{Z}$-actions. The identification of the $\mathbb{Z}$-action on the back right comes from the proof of part (3) of Proposition 5.8.35. From the right face this cube we can read off that there is an isomorphism of $\mathbb{Z}$-equivariant commutative R-algebras

$$R \otimes_{R^{h\mathbb{Z}}} R \cong \mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \otimes R$$

where on the left hand side the $\mathbb{Z}$-action is via the action on the left tensor factor and the R-algebra structure is via the right tensor factor. This lets rewrite the coassembly map above as

$$F(A) \to F(\mathbb{W}C^0(\overrightarrow{\mathbb{Z}_p}) \otimes A)^{h\mathbb{Z}}.$$

This map is an isomorphism by Lemma 5.6.14 and Proposition 5.8.35 (3). □

5.6.3 Cyclotomic hyperdescent

In this subsection we introduce the final key idea in giving a counter-example to the telescope conjecture: cyclotomic redshift. This result, proven in [BMCSY23], is about the compatibility of chromatically localized K-theory with the cyclotomic extensions. We use cyclotomic redshift to show that the map $\mathbb{S}_{T(n)} \to \mathbb{S}^{ab}_{T(n)}$ is a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover, where $\mathbb{S}^{ab}_{T(n)}$ is the lift of the maximal abelian extension of the $K(n)$-local sphere to the $T(n)$-local sphere constructed in [CSY21]. We show that since the extensions $L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}} \to L_{T(n)}\mathrm{BP}\langle n\rangle$ admits a retraction after base change to $\mathbb{S}^{ab}_{T(n)}$, they are also $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-covers.

We refer the reader to the introduction for notation regarding the chromatic cyclotomic extensions of [CSY21].

Recollection 5.6.16. We recall that the Galois group $\mathbb{Z}_p^{\times}$ of the infinite p-cyclotomic extension $\mathbb{S}_{T(n)}[\omega_{p^\infty}]$ is isomorphic via the p-adic logarithm to the product $T_p \times \mathbb{Z}_p$, where T_p is the torsion subgroup that is $\{\pm 1\}$ for $p = 2$ and $\mathbb{F}_p^{\times}$ for $p > 2$.

We let $T_p \times \mathbb{Z}$ be the subgroup obtained via the inclusion $\mathbb{Z} \to \mathbb{Z}_p$.

In [BCSY22, Proposition 6.19] it is proven that the functor $(-)^{\wedge}_{\mathrm{cyc}} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ given by

$$X \mapsto (\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]^{h(T_p\times\mathbb{Z})}) \otimes X$$

is a smashing, symmetric monoidal localization, where the $\mathbb{Z}$ action comes from restriction of the $\mathbb{Z}_p$-action to a generator. We call the local objects for this localization cyclotomically complete $T(n)$-local spectra and there are natural inclusions

$$\mathrm{Sp}_{K(n)} \subset (\mathrm{Sp}_{T(n)})^{\wedge}_{\mathrm{cyc}} \subset \mathrm{Sp}_{T(n)}.$$

◁

Definition 5.6.17. We define $\mathbb{S}^{\mathrm{ab}}_{T(n)}$ to be $\mathbb{W}(\overline{\mathbb{F}}_p) \otimes \mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}] \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. By [CSY21], this is a lift of the maximal abelian Galois extension of the $K(n)$-local sphere to a pro-Galois extension of the $T(n)$-local sphere, and we equip it with a $\mathbb{Z}\times(T_p\times\mathbb{Z})$-action via the $\mathbb{Z}$-action on $\mathbb{W}(\overline{F}_p)$ coming from the Frobenius and the $T_p \times \mathbb{Z}$-action on $\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]$. ◁

Cyclotomic redshift can be phrased as the following hyperdescent result for the K-theory of the p-cyclotomic extensions.

Theorem 5.6.18 ([BMCSY23, Theorem 5.11, Proposition 5.17])**.** *Let $n \geq 0$, and $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. The natural lax symmetric monoidal transformations*

$$L_{T(n+1)}K(R) \to L_{T(n+1)}K(\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]^{hT_p} \otimes R)^{h\mathbb{Z}}$$

$$L_{T(n+1)}K(R) \to L_{T(n+1)}K(\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}] \otimes R)^{h(T_p\times\mathbb{Z})}$$

exhibit the target as the cyclotomic completion of the source, where the tensor products are taken in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$.

Corollary 5.6.19. *For $n \geq 0$, the map $\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]$ is a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover.*

Proof. This follows directly from Lemma 5.6.12 and Theorem 5.6.18, as we can identify $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$ with $(L_{T(n+1)}K(\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}] \otimes R)^{\wedge}_{\mathrm{cyc}})^{h(T_p\times\mathbb{Z})}$ as a functor. □

Next, we extend Corollary 5.6.19 to apply to $\mathbb{S}^{\mathrm{ab}}_{T(n)}$. To do this, we show TC satisfies hyperdescent along the π_0-étale extension $\mathbb{W}(\mathbb{F}_p) \to \mathbb{W}(\overline{\mathbb{F}}_p)$.

Lemma 5.6.20. *Let $R \in \mathrm{Alg}(\mathrm{Sp})$ and give $\mathbb{W}(\overline{\mathbb{F}}_p)$ the $\mathbb{Z}$-action by Frobenius. The natural map*

$$\mathrm{TC}(R) \xrightarrow{\cong} \mathrm{TC}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R)^{h\mathbb{Z}}$$

is an isomorphism.

Proof. The equivalence follows from applying TC to the equivalence of cyclotomic spectra:

$$\begin{aligned}\mathrm{THH}(R) &\cong \mathbb{W}(\overline{\mathbb{F}}_p)^{h\mathbb{Z}} \otimes \mathrm{THH}(R) \cong \mathrm{THH}(\mathbb{W}(\overline{\mathbb{F}}_p))^{h\mathbb{Z}} \otimes \mathrm{THH}(R) \\ &\cong (\mathrm{THH}(\mathbb{W}(\overline{\mathbb{F}}_p)) \otimes \mathrm{THH}(R))^{h\mathbb{Z}} \cong \mathrm{THH}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R)^{h\mathbb{Z}}.\end{aligned}$$ □

Corollary 5.6.21. *For $n \geq 1$, the map $\mathbb{S}_{T(n)} \to \mathbb{W}(\overline{\mathbb{F}}_p) \otimes \mathbb{S}_{T(n)}$ is a $L_{T(n+1)}K$-cover, and $\mathbb{S}_{T(n)} \to \mathbb{S}^{\mathrm{ab}}_{T(n)}$ is a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover.*

Proof. By the definition of $\mathbb{S}^{\mathrm{ab}}_{T(n)}$, Corollary 5.6.19, and the fact that $L_{T(n+1)}K$-covers are also $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-covers (Lemma 5.6.10), it is enough to show the first statement.

$\tau_{\geq 0}R$ (resp. $\mathbb{W}(\overline{\mathbb{F}}_p) \otimes \tau_{\geq 0}R$) is a connective, $T(n+1)$-acyclic $\mathbb{E}_1$-algebra whose $T(n)$-localization is R (resp. $L_{T(n)}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R)$). Applying Corollary 5.6.3 we obtain a commuting diagram whose horizontal maps are isomorpshism.

$$\begin{tikzcd}
L_{T(n+1)}K(R) \arrow[r, dash, "\cong"] \arrow[d] & L_{T(n+1)}\mathrm{TC}(\tau_{\geq 0}R) \arrow[d] \\
L_{T(n+1)}K(L_{T(n)}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R))^{h\mathbb{Z}} \arrow[r, dash, "\cong"] & L_{T(n+1)}\mathrm{TC}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes \tau_{\geq 0}R)^{h\mathbb{Z}}
\end{tikzcd}$$

The right vertical map is an isomorphism by Lemma 5.6.20 so we conclude by Lemma 5.6.12. □

The following lemma will be useful for knowing that the target of the coassembly map is cyclotomically complete:

Lemma 5.6.22. *For $n \geq 1$, if $R \in \mathrm{CAlg}(\mathbb{S}_{T(n)})$ and there is a map*

$$\mathbb{S}[\omega_{p^\infty}]^{hT_p} \to L_{T(n)}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R)$$

in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then $L_{T(n+1)}K(R)$ is cyclotomically complete.

Proof. Because $\mathbb{S}_{T(n)} \to \mathbb{W}(\overline{\mathbb{F}}_p) \otimes \mathbb{S}_{T(n)}$ is a $L_{T(n+1)}K$-cover by Corollary 5.6.21, it is enough to show that for $R' := L_{T(n)}(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes R)$, $L_{T(n+1)}K(R')$ is cyclotomically complete. By Theorem 5.6.18, it is enough to show that the map $L_{T(n+1)}K(R') \to L_{T(n+1)}K(\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]^{hT_p} \otimes R')$ has a retraction, but such a retraction can be produced using the map $\mathbb{S}_{T(n)}[\omega^{(n)}_{p^\infty}]^{hT_p} \to R'$ from the hypothesis. □

Lemma 5.6.23. *Given maps $A \to B \to C \to D$ where $A \to B$ is a finite Galois extension and $B \to C$ is a pro-Galois extension, then the map $B \otimes_A D \to C \otimes_A D$ admits a retraction.*

Proof. Without loss of generality, we can assume $C = D$, since if the map has a retraction after base change to C it also does for D.

Let G be the Galois group of $A \to C$ and let H be the Galois group of $A \to B$. Then the map $B \otimes_A C \to C \otimes_A C$ is isomorphic to the map $C^{G/H} \to C^G$ induced by the surjection of profinite sets $G \to G/H$. But this map has a section since G/H is finite. ☐

Proposition 5.6.24. *For $n \geq 1$, consider $R = L_{T(n)}\mathrm{BP}\langle n\rangle$ with the $\mathbb{E}_\infty$-$\mathbb{Z}$-action from Theorem 5.5.4. Then for each $k \geq 0$, the map $f_k : R^{hp^k\mathbb{Z}} \to R$ is a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover, and moreover $L_{T(n+1)}K(R)$ is cyclotomically complete.*

Proof. We begin with a few recollections and some notation.

- Let ℓ be as in Convention 5.5.1.
- Let $\Psi^\ell \in Z(\mathbb{G}_n) \subseteq \mathbb{G}_n$ be as in Convention 5.5.1, and let $\mathbb{Z} \subset \mathbb{G}_n$ be the subgroup generated by Ψ^ℓ.
- Let $C \subseteq \mathbb{G}_n$ be the cyclic subgroup of order $p^n - 1$ from Theorem 5.5.4.
- Let $\mathrm{cyc} : \mathbb{G}_n \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character of [CSY21, Definition 5.6].
- Let $\overline{\mathrm{cyc}}$ be the further quotient $\mathbb{G}_n \xrightarrow{\mathrm{cyc}} \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{Z}_p$.
- Recall that $\mathbb{G}_n$ has a subgroup $\mathcal{O}_D^\times \subseteq \mathbb{G}_n$ where $\mathcal{O}_D^\times$ is the units in the ring of integers of the division algebra D over $\mathbb{Q}_p$ with Hasse invariant $\frac{1}{n}$.
- Let $j = v_p(\ell^{np^k} - 1)$.

Theorem 5.5.4 implies that the map f_k after base change to $L_{T(n)}\mathbb{W}(\overline{\mathbb{F}}_p)$ becomes isomorphic to the map $(E_n^{hC})^{hp^k\mathbb{Z}} \to E_n^{hC}$, which we denote $\tilde{f}_k$. We claim there is a pushout square of $T(n)$-local commutative algebras

$$\begin{tikzcd}
\mathbb{S}_{T(n)}[\omega_{p^j}^{(n)}]^{hT_p} \arrow[r] \arrow[d] & \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]^{hT_p} \arrow[d] \\
(E_n^{hC})^{hp^k\mathbb{Z}} \arrow[r, "\tilde{f}_k"] & E_n^{hC}.
\end{tikzcd}$$

To see that this claim lets us finish the argument, we first note that the right vertical map along with Lemma 5.6.22 lets us conclude $L_{T(n+1)}K(R)$ is cyclotomically complete.

Next, by applying the pushout square and Lemma 5.6.23, we learn that $\tilde{f}_k \otimes \mathbb{S}[\omega_{p^\infty}^{(n)}] = f_k \otimes \mathbb{S}^{\mathrm{ab}}_{T(n)}$ admits a retraction, and hence is a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover. By Remark 5.6.9 and Corollary 5.6.21, we find that f_k is also a $L_{T(n+1)}K^{\wedge}_{\mathrm{cyc}}$-cover.

We turn to proving the claim. By [CSY21, Theorem 5.8], the p-adic cyclotomic character restricts to the determinant map on $\mathcal{O}_D^\times$. This lets us compute that

$$\mathrm{cyc}(\Psi^\ell) = \ell^n.$$

We observe that since C is cyclic and $\mathbb{Z}_p$ is torsion-free $\overline{\mathrm{cyc}}$ factors through $\mathbb{G}_n/C$. Together these facts allows us to construct a pullback square of quotients of $\mathbb{G}_n$

$$\begin{tikzcd}
\mathbb{G}_n/C \arrow[r] \arrow[d, "\overline{\mathrm{cyc}}"] & (\mathbb{G}_n/C)/\Psi^{\ell^{p^k}} \arrow[d] \\
\mathbb{Z}_p \arrow[r] & \mathbb{Z}_p/p^j.
\end{tikzcd}$$

Applying $E_n^{h(\ker(-))}$ to this square we obtain the desired pushout square of $T(n)$-local commutative algebras after $K(n)$-localization. The pushout square holds before $K(n)$-localization too since $K(n)$-localization is smashing in $\mathrm{Sp}_{T(n)}$ and the lower rings in the square are $K(n)$-local. □

5.6.4 The failure of the telescope conjecture

We are now ready to prove the following refinement of Theorem A from the introduction.

Theorem 5.6.25. *Let* $\mathrm{BP}\langle n\rangle$ *be as in Theorem 5.5.4. For every prime* p*, height* $n \geq 1$ *and* $k \geq 0$*, there is a diagram*

$$\begin{tikzcd}
L_{T(n+1)}K(L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[d] & L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[d] \\
L_{T(n+1)}K(L_{T(n)}\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} & L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}
\end{tikzcd}$$

where the vertical maps are coassembly maps. The vertical maps are not isomorphisms, but rather exhibit the target as the cyclotomic completion of the source. In particular, this gives a counterexample to the height $n+1$ *telescope conjecture.*

Proof. By applying Corollary 5.6.3 to $R = \mathrm{BP}\langle n\rangle$, we get the diagram as in the theorem statement, where the vertical maps are coassembly maps. By applying Proposition 5.6.24 and Lemma 5.6.15, the left vertical map is an equivalence after cyclotomic completion, and the target is cyclotomically complete. Thus the vertical maps are the cyclotomic completion maps.

It remains to show that the vertical maps are not isomorphisms, or equivalently, the source of the coassembly map is not cyclotomically complete. Since cyclotomic completion is smashing and the maps between $L_{T(n+1)}K(L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ as k varies are $\mathbb{E}_\infty$-maps

(Theorem 5.5.4), we learn that if the result holds for k, i.e $L_{T(n+1)}K(L_{T(n)}\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$ is not cyclotomically complete, then it also holds for all $k' < k$.

Thus it suffices to prove that the right coassembly map is not an isomorphism for all $k \gg 0$.

Choose U to be a finite spectrum of type $n+1$ with a v_{n+1}-self map v, so that $T(n+1) := U[v^{-1}] \neq 0$. $\mathrm{BP}\langle n\rangle$ is fp-type n, has a locally unipotent $p^k\mathbb{Z}$-action (Theorem 5.5.4, Lemma 5.8.31), is an $\mathbb{E}_1 \otimes \mathbb{A}_2$-algebra, and satisfies the height n Lichtenbaum–Quillen property by [HW22], so we map apply Corollary 5.4.33 and invert v to obtain for all $k \gg 0$ a square

$$\begin{tikzcd}
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}) \arrow[r] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{B\mathbb{Z}}
\end{tikzcd}$$

Thus we are reduced to showing that the lower coassembly map is not an isomorphism. We know from [HW22] that $T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is non-zero, so since by Corollary 5.3.21 $T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is in the thick subcategory generated by the fiber of the coassembly map

$$T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}) \to T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)^{B\mathbb{Z}}$$

we can conclude the proof. □

5.7 Explicit $T(2)$-local TC of some height 1 local fields

In Section 5.6, we showed that the telescope fails by proving that the coassembly map

$$T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to T(n+1)\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}$$

is not an isomorphism for $n \geq 1, k \gg 0$. The goal of this section is to completely compute π_* of a version of this coassembly map when $n = 1, k \gg 0$, and $p \geq 7$.

Specifically, we study the connective Adams summand ℓ, which is a form of $\mathrm{BP}\langle 1\rangle$.[34] The $\mathbb{Z}$-action on ℓ is generated by the classical Adams operation Ψ^{1+p}. We restrict to primes $p \geq 7$ in order to make use of Smith–Toda complexes $V(0) = \mathbb{S}/p$, $V(1) = \mathbb{S}/(p, v_1)$, and $V(2) = \mathbb{S}/(p, v_1, v_2)$, which exist as hcrings for such primes [YY77, Proposition 3.3]. We take $T(2)$ to be $v_2^{-1}V(1)$.

Our main result is true for all k larger than a fixed positive integer, which depends only on the prime p:[35]

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[34] In this section, we follow historical convention and use ℓ to denote the Adams summand. In previous sections ℓ was used to denote an element of $\mathbb{Z}_p^\times$, and we trust the reader may distinguish these uses from context.

[35] We expect that this integer does not in fact depend on p, and may be taken to be as small as 2 or 3.

Theorem 5.7.1. *Let $p \geq 7$ be a prime, and let $\mathbb{Z}$ act on the connective Adams summand ℓ via the $\mathbb{E}_\infty$ Adams operation Ψ^{1+p}. Then, for all $k \gg 0$, the $\mathbb{F}_p[v_2]$-module map*

$$V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \to V(1)_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}$$

may be identified with the direct sum of the maps enumerated below. The degrees of classes are determined from their names via the facts that $|t| = -2, |\lambda_i| = 2p^i - 1, |\partial| = -1, |\zeta| = -1$, and the degree of any locally constant function is 0.

1. *The projection $\mathbb{F}_p\{1, \partial\} \oplus \overline{C^0(\mathbb{Z}_p^\times)}\{\partial\zeta\} \to \mathbb{F}_p\{1, \partial\}$ onto the first factor, tensored over $\mathbb{F}_p$ with the inclusion $\mathbb{F}_p[v_2]\langle\lambda_1, \lambda_2\rangle \to \mathbb{F}_p[v_2]\langle\lambda_1, \lambda_2, \zeta\rangle$.*
2. *The map $\mathbb{F}_p[v_2]\langle\zeta\rangle \otimes C^0(p\mathbb{Z}_p) \to \mathbb{F}_p[v_2]\langle\zeta\rangle$ evaluating a continuous function at 0, tensored over $\mathbb{F}_p$ with the graded $\mathbb{F}_p$-vector space on basis elements enumerated below:*
    - *(a) $t^d\lambda_1$, for each $0 < d < p$, in degree $2p - 1 - 2d$.*
    - *(b) $t^{pd}\lambda_2$, for each each $0 < d < p$, in degree $2p^2 - 1 - 2pd$*
    - *(c) $t^d\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2d$.*
    - *(d) $t^{pd}\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2pd$.*

As we will explain, by inverting v_2 we recover the following corollary, which in turn implies Theorem D from the introduction:

Corollary 5.7.2. *Let $p \geq 7$ be a prime, and let $\mathbb{Z}$ act on the Adams summand L of $\mathrm{KU}_{(p)}$ via the $\mathbb{E}_\infty$ Adams operation Ψ^{1+p}. Then, for all $k \geq 0$, the map*

$$T(2) \otimes \mathrm{K}(L^{hp^k\mathbb{Z}}) \to T(2) \otimes \mathrm{K}(L)^{hp^k\mathbb{Z}}$$

is both the cyclotomic completion map and the $K(2)$-localization map.

At the level of π_* as an $\mathbb{F}_p[v_2^{\pm 1}]$-module, this map for $k \gg 0$ may be identified with the direct sum of the maps enumerated below. The degrees of classes are determined from their names via the facts that $|t| = -2, |\lambda_i| = 2p^i - 1, |\partial| = -1, |\zeta| = -1$, and the degree of any locally constant function is 0.

1. *The projection $\mathbb{F}_p\{1, \partial\} \oplus \overline{C^0(\mathbb{Z}_p^\times)}\{\partial\zeta\} \to \mathbb{F}_p\{1, \partial\}$ onto the first factor, tensored over $\mathbb{F}_p$ with the inclusion $\mathbb{F}_p[v_2^{\pm 1}]\langle\lambda_1, \lambda_2\rangle \to \mathbb{F}_p[v_2^{\pm 1}]\langle\lambda_1, \lambda_2, \zeta\rangle$.*
2. *The map $\mathbb{F}_p[v_2^{\pm 1}]\langle\zeta\rangle \otimes C^0(p\mathbb{Z}_p) \to \mathbb{F}_p[v_2^{\pm 1}]\langle\zeta\rangle$ evaluating a continuous function at 0, tensored over $\mathbb{F}_p$ with the graded $\mathbb{F}_p$-vector space on basis elements enumerated below:*
    - *(a) $t^d\lambda_1$, for each $0 < d < p$, in degree $2p - 1 - 2d$.*
    - *(b) $t^{pd}\lambda_2$, for each each $0 < d < p$, in degree $2p^2 - 1 - 2pd$*
    - *(c) $t^d\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2d$.*
    - *(d) $t^{pd}\lambda_1\lambda_2$, for each $0 < d < p$, in degree $2p^2 + 2p - 2 - 2pd$.*

Remark 5.7.3. We think of $\ell^{hp^k\mathbb{Z}}$ as a bounded below model (analogous to an order in the ring of integers) of the height 1 local field obtained as a finite subextension of the Iwasawa extension $\mathbb{S}_{T(1)}[\omega^{(1)}_{p^\infty}]^{hT_p}$. ◁

We now proceed with the proofs of Theorem 5.7.1 and Corollary 5.7.2. After these are proved, we discuss in Section 5.7.4 some qualitative results about $\mathrm{K}(L_{K(1)}\mathbb{S})$ itself, which do not require passage to a finite Galois extension to prove.

5.7.1 Recollections regarding $V(2) \otimes \mathrm{THH}(\ell)$

In this subsection, we recall a few classical results about the cyclotomic spectrum $\mathrm{THH}(\ell)$. Most importantly we have the following:

Theorem 5.7.4 (Ausoni–Rognes)**.** *For $p \geq 5$, the cyclotomic spectrum*

$$V(2) \otimes \mathrm{THH}(\ell)$$

is bounded in the cyclotomic t-structure. In other words, ℓ satisfies the height 1 LQ property.

Proof. This follows from [AR02, Theorem 8.5] (see also [HW22, Theorem G]). □

From now on let $p \geq 7$ unless stated otherwise.

Next, we recall the explicit identification of $V(1)_*\mathrm{TC}(\ell)$ by Ausoni and Rognes, together with additional algebraic identifications of the objects

$$V(2)_* \,\mathrm{THH}(\ell), V(2)_* \,\mathrm{THH}(\ell)^{tC_p}, V(2)_*\mathrm{TC}^-(\ell), V(2)_* \,\mathrm{TP}(\ell)$$

. An alternative perspective on some of these calculations, with spectral sequence diagrams, may be found in [HRW22, Section 6].

Recollection 5.7.5. Ausoni and Rognes [AR02, Definitions 1.3 and 1.8], following work of Bökstedt and Madsen, constructed classes

$$\lambda_1, \lambda_2 \in V(0)_*K(\ell).$$

Following these authors, we will also use λ_1 and λ_2 to refer to the images of these classes in trace invariants under $V(0)_*K(\ell)$. In particular, under this convention both

$$\varphi^{h\mathbb{T}} : V(2)_*\mathrm{TC}^-(\ell) \to V(2)_* \,\mathrm{TP}(\ell), \text{ and}$$

$$\mathrm{can} : V(2)_*\mathrm{TC}^-(\ell) \to V(2)_* \,\mathrm{TP}(\ell),$$

are $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module maps. ◁

There is an isomorphism of $\mathbb{F}_p$-algebras

$$V(1)_* \,\mathrm{THH}(\ell) \cong \mathbb{F}_p[\mu] \otimes \Lambda(\lambda_1, \lambda_2),$$

due to McClure–Staffeldt [MS93a] [AR02, Proposition 2.6]. Since v_2 is trivial in this ring, killing it introduces an exterior generator and we arrive at the following:

Convention 5.7.6. We fix a preferred isomorphism of $\mathbb{F}_p$-algebras

$$V(2)_* \operatorname{THH}(\ell) \cong \mathbb{F}_p[\mu]\langle \lambda_1, \lambda_2, \epsilon \rangle,$$

such that λ_1 and λ_2 are as in Recollection 5.7.5 and ϵ corresponds to a choice of nullhomotopy of v_2 in $V(1)_* \operatorname{THH}(\ell)$. Here, $|\mu| = 2p^2, |\lambda_1| = 2p-1$, and $|\lambda_2| = |\epsilon| = 2p^2-1$. ◁

Remark 5.7.7. We will soon see that μ is a Bökstedt class in the sense of Section 5.2.3. ◁

Proposition 5.7.8. *The $\mathbb{T}$-Tate spectral sequence*

$$\mathrm{E}_2 = \mathbb{F}_p[\mu, t^{\pm 1}]\langle \lambda_1, \lambda_2, \epsilon \rangle \implies V(2)_* \operatorname{TP}(\ell)$$

has differentials

$$d_2(\epsilon) = t\mu,$$
$$d_{2p}(t) = t^{p+1}\lambda_1,$$
$$d_{2p^2}(t^p) = t^{p^2+p}\lambda_2,$$

with all other differentials determined by multiplicative structure and the facts that λ_1 and λ_2 are permanent cycles. The E_∞-page is isomorphic to $\mathbb{F}_p[t^{\pm p^2}]\langle \lambda_1, \lambda_2 \rangle$.

Proof. This follows from work of Ausoni–Rognes [AR02, Section 6], who compute the much more difficult $\mathbb{T}$-Tate spectral sequence for $V(1)_* \operatorname{TP}(\ell)$. More precisely, the map of $\mathbb{T}$-Tate spectral sequences induced by the $\mathbb{T}$-equivariant map $V(1)_* \operatorname{THH}(\ell) \to V(2)_* \operatorname{THH}(\ell)$ induces the differentials on t and t^p. The d_2 differential on ϵ follows from [AR02, Proposition 4.8]. □

Note that the $\mathbb{T}$-homotopy fixed point spectral sequence

$$\mathrm{E}_2 = \mathbb{F}_p[\mu, t]\langle \lambda_1, \lambda_2, \epsilon \rangle \implies V(2)_* \mathrm{TC}^-(\ell)$$

is formally determined by the $\mathbb{T}$-Tate spectral sequence through truncation.

Definition 5.7.9. Let N denote the submodule of the $\mathbb{F}_p\langle \lambda_1, \lambda_2 \rangle$-module $V(2)_*\mathrm{TC}^-(\ell)$ that consists of classes simultaneously in the kernel of can and of positive Nygaard filtration (i.e., of positive filtration in the $\mathbb{T}$-homotopy fixed point spectral sequence). ◁

Examining the canonical map between the $\mathbb{T}$-homotopy fixed point spectral sequence and the $\mathbb{T}$-Tate spectral sequence, we arrive at the following result:

Corollary 5.7.10. *The canonical map*

$$\mathrm{can} : V(2)_*\mathrm{TC}^-(\ell) \to V(2)_* \operatorname{TP}(\ell)$$

is trivial in degrees $* \geq 2p^2$. The subspace N of $V(2)_*\mathrm{TC}^-(\ell)$ is a $4(p-1)$ dimensional $\mathbb{F}_p$-vector space, with basis elements detected in the $\mathbb{T}$-homotopy fixed point spectral sequence by:

1. $t^d\lambda_1$*, for each* $0 < d < p$*, in degree* $2p - 1 - 2d$

2. $t^{pd}\lambda_2$*, for each each* $0 < d < p$*, in degree* $2p^2 - 1 - 2pd$

3. $t^d\lambda_1\lambda_2$*, for each* $0 < d < p$*, in degree* $2p^2 + 2p - 2 - 2d$*.*

4. $t^{pd}\lambda_1\lambda_2$*, for each* $0 < d < p$*, in degree* $2p^2 + 2p - 2 - 2pd$*.*

Classes in (3) *are obtained from classes in* (1) *by multiplication by* λ_2*, and classes in* (4) *are obtained from classes in* (2) *by multiplication by* λ_1*.*

We next turn to the cyclotomic Frobenius map.

Proposition 5.7.11. *The cyclotomic Frobenius*

$$\varphi : V(2)_* \operatorname{THH}(\ell) \to V(2)_* \operatorname{THH}(\ell)^{tC_p}$$

is the universal $\mathbb{F}_p$*-algebra map that inverts* μ*. In other words,*

$$V(2)_* \operatorname{THH}(\ell)^{tC_p} \cong \mathbb{F}_p[\varphi(\mu)^{\pm 1}]\langle \lambda_1, \lambda_2, \varphi(\epsilon)\rangle.$$

Proof. It suffices to identify

$$\varphi : V(1)_* \operatorname{THH}(\ell) \to V(1)_* \operatorname{THH}(\ell)^{tC_p}$$

with the map

$$\mathbb{F}_p[\mu]\langle \lambda_1, \lambda_2\rangle \to \mathbb{F}_p[\varphi(\mu)^{\pm 1}]\langle \lambda_1, \lambda_2\rangle.$$

This is [AR02, Theorem 5.5]. □

Corollary 5.7.12. *The map*

$$\varphi^{h\mathbb{T}} : V(2)_*\mathrm{TC}^-(\ell) \to V(2)_* \operatorname{TP}(\ell)$$

is an isomorphism in degrees $* \gg 0$*. It is trivial on all classes of positive Nygaard filtration.*

Proof. The first claim follows from the fact that the fiber of φ is bounded above, and so the fiber of $\varphi^{h\mathbb{T}}$ is as well. To see the second claim, we must check that the E_∞-page of the $\mathbb{T}/C_p$-homotopy fixed point spectral sequence for $\left(\operatorname{THH}(\ell)^{tC_p}\right)^{h\mathbb{T}/C_p}$ is concentrated on the 0-line. This $\mathbb{T}/C_p$-homotopy fixed point spectral sequence begins with

$$E_2 = \mathbb{F}_p[\varphi(\mu)^{\pm 1}, t]\langle \varphi(\epsilon), \lambda_1, \lambda_2\rangle.$$

We may calculate the d_2 differential using the φ map from the $\mathbb{T}$-homotopy fixed point spectral sequence determined by Proposition 5.7.8. After running the d_2 differential, we find that the E_3-page is concentrated on the 0-line. □

Corollary 5.7.13. *As a graded* $\mathbb{F}_p\langle \lambda_1, \lambda_2\rangle$*-module,* $V(2)_*\mathrm{TC}(\ell)$ *is isomorphic to the direct sum of* N *and* $\mathbb{F}_p\langle \lambda_1, \lambda_2, \partial\rangle$*, where* $|\partial| = -1$*.*

Proof. This follows immediately from [AR02, Theorem 0.3], but we indicate some details that will generalize to the study of $V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$. We must calculate the equalizer and coequalizer of the map

$$\varphi^{h\mathbb{T}} - \mathrm{can} : V(2)_*\mathrm{TC}^{-}(\ell) \to V(2)_*\,\mathrm{TP}(\ell).$$

Since the $\varphi^{h\mathbb{T}}$ is trivial on classes of positive Nygaard filtration, we may form the diagram

$$\begin{tikzcd}
\ker_1 \arrow[r] \arrow[d] & \mathrm{Nyg}_{\geq 1}\left(V(2)_*\mathrm{TC}^{-}(\ell)\right) \arrow[r, "0-\mathrm{can}"] \arrow[d] & \mathrm{Nyg}_{\geq 1}\left(V(2)_*\,\mathrm{TP}(\ell)\right) \arrow[r] \arrow[d] & \mathrm{coker}_1 \arrow[d] \\
\ker \arrow[r] \arrow[d] & V(2)_*\mathrm{TC}^{-}(\ell) \arrow[r, "\varphi-\mathrm{can}"] \arrow[d] & \mathbb{F}_p[t^{\pm p^2}]\langle\lambda_1,\lambda_2\rangle \arrow[r] \arrow[d] & \mathrm{coker} \arrow[d] \\
\ker_2 \arrow[r] & \mathrm{Nyg}_{=0}\left(V(2)_*\mathrm{TC}^{-}(\ell^{\mathrm{B}\mathbb{Z}})\right) \arrow[r, "\overline{\varphi}-\overline{\mathrm{can}}"] & \left(V(2)_*\,\mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}})\right)/\mathrm{Nyg}_{\geq 1} \arrow[r] & \mathrm{coker}_2.
\end{tikzcd}$$

and apply the snake lemma. From Proposition 5.7.8 and Corollary 5.7.10 we see that $\ker_1$ is exactly N, while coker_1 is zero. Furthermore, since can is trivial in degrees $* \geq 2p^2$, we identify the map

$$\overline{\mathrm{can}} : \mathbb{F}_p[\mu]\langle\lambda_1,\lambda_2\rangle \to \mathbb{F}_p[\varphi(\mu)]\langle\lambda_1,\lambda_2\rangle$$

as the map that kills μ. All of this proves that $V(2)_*\mathrm{TC}(\ell)$ admits a filtration with associated graded the direct sum of:

- $\ker_1 = N$
- $\ker_2 \cong \mathbb{F}_p\langle\lambda_1,\lambda_2\rangle$, and
- $\Sigma^{-1}\mathrm{coker}_1 \cong \mathbb{F}_p\langle\lambda_1,\lambda_2\rangle\{\partial\}$.

To conclude, we observe that $\mathbb{F}_p\langle\lambda_1,\lambda_2\rangle$ extension problems in the above filtration are ruled out by the facts that $\ker_2$ and coker_1 are free modules. □

Finally, Ausoni and Rognes prove that the v_2-Bockstein spectral sequence converging to $V(1)_*\mathrm{TC}(\ell)$ degenerates, with no differentials or extension problems.

Theorem 5.7.14 (Theorem 0.3 of [AR02])**.** *As a graded $\mathbb{F}_p[v_2]\langle\lambda_1,\lambda_2\rangle$-module, $V(1)_*\mathrm{TC}(\ell)$ is isomorphic to the direct sum of:*

1. $N \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2]$,
2. $\mathbb{F}_p[v_2]\langle\lambda_1,\lambda_2,\partial\rangle$.

5.7.2 Calculations regarding $V(2) \otimes \mathrm{THH}(\ell^{B\mathbb{Z}})$

Using Theorem 5.7.4, Theorem 5.4.30, and the fact that ℓ is a $\mathbb{Z}$-equivariant $\mathbb{E}_\infty$-ring of fp-type 1, we deduce that the coassembly map

$$V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \to V(1)_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}$$

is, as an $\mathbb{F}_p[v_2]$-module map for $k \gg 0$, the same as the coassembly map

$$V(1)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(1)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}$$

associated to the trivial $\mathbb{Z}$ action on ℓ.

In this subsection we study the cyclotomic spectrum $V(2) \otimes \mathrm{THH}(\ell^{B\mathbb{Z}})$ at a fixed prime $p \geq 7$, obtaining analogs of many of the results and proofs from the previous subsection.

First we observe that the natural map

$$V(2)_* \mathrm{THH}(\ell) \to V(2)_* \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}})$$

defines elements $\mu, \zeta, \epsilon, \lambda_1$, and λ_2 in the codomain, according to Convention 5.7.6. Furthermore, since λ_1 and λ_2 lift to $V(2)_*\mathrm{TC}(\ell)$, they also lift to $V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$. Additionally, we use the natural map

$$\mathrm{THH}(\mathbb{S}^{\mathrm{B}\mathbb{Z}}) \to \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}})$$

to equip with $V(2)_* \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}})$ with the struture of a $C^0(\mathbb{Z}_p)$-algebra. In terms of these elements, we identify the coassembly map in THH as follows:

Proposition 5.7.15. *The coassembly map*

$$V(2)_* \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_* \mathrm{THH}(\ell)^{\mathrm{B}\mathbb{Z}}$$

may be identified with the $\mathbb{F}_p$-algebra map

$$C^0(\mathbb{Z}_p)[\mu]\langle\zeta, \epsilon, \lambda_1, \lambda_2\rangle \to \mathbb{F}_p[\mu]\langle\zeta, \epsilon, \lambda_1, \lambda_2\rangle$$

that evaluates a continuous function at 0.

Proof. We apply Lemma 5.3.6 to the isomorphism of $\mathbb{E}_\infty$-algebras

$$\mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}}) \cong \mathrm{THH}(\ell) \otimes \mathrm{THH}(\mathbb{S}^{\mathrm{B}\mathbb{Z}}).$$

which is compatible with the coassembly map. □

We next turn to the $\mathbb{T}$-homotopy fixed point and $\mathbb{T}$-Tate spectral sequences converging to $V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}})$ and $V(2)_* \mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}})$, respectively.

Theorem 5.7.16. *The $\mathbb{T}$-Tate spectral sequence*

$$\mathrm{E}_2 = C^0(\mathbb{Z}_p)[\mu, t^{\pm 1}]\langle \lambda_1, \lambda_2, \epsilon, \zeta\rangle \implies V(2)_* \operatorname{TP}(\ell^{B\mathbb{Z}})$$

has differentials

$$d_2(\epsilon) = t\mu,$$
$$d_2(\zeta) = ft\zeta,$$
$$d_{2p}(t) = t^{p+1}\lambda_1,$$
$$d_{2p^2}(t^p) = t^{p^2+p}\lambda_2,$$

where $f \in C^0(\mathbb{Z}_p)$ is a function vanishing exactly on $p\mathbb{Z}_p$. All other differentials are determined by multiplicative structure and the facts that λ_1, λ_2, and elements of $C^0(p\mathbb{Z}_p)\langle\zeta\rangle$ are permanent cycles. The E_∞-page is isomorphic to $C^0(p\mathbb{Z}_p)[t^{\pm p^2}]\langle \zeta, \lambda_1, \lambda_2\rangle$.

Proof. We are studying here the $\mathbb{T}$-Tate spectral sequence associated to the $\mathbb{T}$-equivariant spectrum

$$V(2) \otimes \operatorname{THH}(\ell) \otimes \operatorname{THH}(\mathbb{S}^{B\mathbb{Z}}).$$

By [Mal17, Corollary 1.3], this is additively isomorphic to

$$V(2) \otimes \operatorname{THH}(\ell) \oplus \bigoplus_{n \geq 1} V(2) \otimes \operatorname{THH}(\ell) \otimes \mathbb{S}^{\mathbb{T}/C_n},$$

and as in Lemma 5.3.6 and Example 5.3.10 the projection map to the summand indexed by n is given by evaluation of locally constant functions at n.

We will now calculate the $\mathbb{T}$-Tate spectral sequence associated to

$$V(2) \otimes \operatorname{THH}(\ell) \otimes \mathbb{S}^{\mathbb{T}/C_n}.$$

As n ranges over all positive integers, these spectral sequences determine the spectral sequences of Theorem 5.7.16.

The $\mathbb{T}$-Tate spectral sequence associated to

$$V(2) \otimes \operatorname{THH}(\ell) \otimes \mathbb{S}^{\mathbb{T}/C_n}$$

is of signature

$$\mathrm{E}_2 = \mathbb{F}_p[\mu, t^{\pm 1}]\langle \lambda_1, \lambda_2, \epsilon, \zeta\rangle \implies V(2)_* \operatorname{THH}(\ell)^{tC_n}.$$

Using the restriction map from the $\mathbb{T}$-Tate spectral sequence associated to $V(2)\otimes\operatorname{THH}(\ell)$, as calculated in Proposition 5.7.8, one sees that there is a d_2 differential $d_2(\epsilon) = t\mu$, while

$$d_2(\lambda_1) = d_2(\lambda_2) = d_2(\mu) = d_2(t) = 0.$$

If n has p-adic valuation 0, then $V(2) \otimes \operatorname{THH}(\ell) \otimes \mathbb{S}^{\mathbb{T}/C_n}$ is induced, and so the spectral sequence becomes trivial at the E_3-page after a non-zero differential on ζ.

If n has positive p-adic valuation, it follows from the calculations of Ausoni–Rognes [AR02, Section 6] that $V(2)_* \operatorname{THH}(\ell)^{tC_n}$ is non-trivial, so the d_2 differential on ζ must be trivial. In this case the E_3-page is isomorphic to

$$\mathbb{F}_p[t^{\pm 1}]\langle \lambda_1, \lambda_2, \zeta \rangle.$$

If n is of p-adic valuation 1, then the spectral sequence must converge to $V(2)_* \operatorname{THH}(\ell)^{tC_p}$, which was calculated in Proposition 5.7.11. The only differentials consistent with both Proposition 5.7.8 and Proposition 5.7.11 are

$$d_{2p}(t) = t^{p+1}\lambda_1, \text{ and}$$

$$d_{2p^2}(t^p) = t^{p^2+p}\lambda_2,$$

as induced from Proposition 5.7.8. In particular, ζ must be a permanent cycle in order to obtain an E_∞-page of size at least that of $V(2)_* \operatorname{THH}(\ell)^{tC_p}$, and one is left with an E_∞-page of $\mathbb{F}_p[t^{\pm p^2}]\langle \lambda_1, \lambda_2, \zeta \rangle$.

For n of p-adic valuation greater than 1, the transfer map $\mathbb{S}^{\mathbb{T}/C_{(n/p)}} \to \mathbb{S}^{\mathbb{T}/C_n}$ sends ζ to ζ, so ensures that ζ is a permanent cycle. Now the only differentials compatible with [AR02, Proposition 6.3] and the restriction maps $\mathbb{S}^{\mathbb{T}} \to \mathbb{S}^{\mathbb{T}/C_n}$ are the same pattern of differentials as in the p-adic valuation 1 case. □

The $\mathbb{T}$-homotopy fixed point spectral computing $V(2)_*\mathrm{TC}^-(\ell^{B\mathbb{Z}})$ is formally determined as a truncation of the above $\mathbb{T}$-Tate spectral sequence. In particular, we have the following corollary:

Corollary 5.7.17. *When restricted to classes of positive Nygaard filtration, the* can *map*

$$\mathrm{Nyg}_{\geq 1}V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}}) \to \mathrm{Nyg}_{\geq 1}V(2)_* \operatorname{TP}(\ell^{\mathrm{B}\mathbb{Z}})$$

has trivial cokernel. The kernel of this restricted can *map is isomorphic to*

$$C^0(p\mathbb{Z}_p)\langle \zeta \rangle \otimes_{\mathbb{F}_p} N.$$

Proof. The subspace $C^0(p\mathbb{Z}_p)\langle \zeta \rangle \otimes_{\mathbb{F}_p} N$ is clearly in the kernel of the restricted can map. At the level of homotopy fixed point spectral sequences, we can see that can is an isomorphism on the remaining quotient of $V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}})$ □

Proposition 5.7.18. *The cyclotomic Frobenius*

$$\varphi : V(2)_* \operatorname{THH}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_* \operatorname{THH}(\ell^{\mathrm{B}\mathbb{Z}})^{tC_p}$$

is the universal map that inverts μ. In other words, there is a preferred isomorphism

$$V(2)_* \operatorname{THH}(\ell^{B\mathbb{Z}})^{tC_p} \cong C^0(p\mathbb{Z}_p)[\varphi(\mu)^{\pm 1}]\langle \lambda_1, \lambda_2, \varphi(\epsilon), \varphi(\zeta) \rangle.$$

Under this isomorphism, φ carries a locally constant function $x \mapsto f(x)$ to the function $x \mapsto f(x/p)$.

Proof. This follows by combining Proposition 5.7.11 and Proposition 5.3.18. □

Lemma 5.7.19. *The cyclotomic Frobenius map*

$$\varphi^{h\mathbb{T}} : V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_*\,\mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}})$$

annihilates all classes of positive Nygaard filtration.

Proof. It will suffice to prove that the E_∞-page of the $(\mathbb{T}/C_p)$-homotopy fixed point spectral sequence

$$C^0(p\mathbb{Z}_p)[\varphi(\mu)^{\pm 1}, t]\langle \lambda_1, \lambda_2, \varphi(\zeta), \varphi(\epsilon)\rangle \implies V(2)_*(\mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}})^{tC_p})^{h\mathbb{T}/C_p}$$

is trivial above the 0-line. As in the proof of Corollary 5.7.12, this is already true at the E_3-page, because of d_2 differentials induced from the $\mathbb{T}$-homotopy fixed point spectral sequence for $V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}})$. □

Theorem 5.7.20. *The graded $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module map*

$$V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}$$

is isomorphic to the direct sum of the following three $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$ module maps:

1. *The map*
$$C^0(p\mathbb{Z}_p)\langle\zeta\rangle \otimes_{\mathbb{F}_p} N \to \mathbb{F}_p\langle\zeta\rangle \otimes_{\mathbb{F}_p} N$$
evaluating a function at 0.
2. *The inclusion*
$$\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle \to \mathbb{F}_p\langle\lambda_1, \lambda_2, \zeta\rangle.$$
3. *The projection*
$$\overline{C^0(\mathbb{Z}_p^\times)}\{\partial\zeta\} \oplus \mathbb{F}_p\{\partial\} \to \mathbb{F}_p\{\partial\}$$
onto the second factor, tensored over $\mathbb{F}_p$ with the inclusion from (2).

Proof. Our first goal will be to compute the equalizer and coequalizer of the maps

$$\mathrm{can}, \varphi^{h\mathbb{T}} : V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_*\,\mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}}),$$

thereby computing $V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$.

On classes of positive Nygaard filtration, $\varphi^{h\mathbb{T}}$ is trivial. It follows that $\varphi^{h\mathbb{T}} - \mathrm{can}$ takes classes of positive Nygaard filtration to classes of positive Nygaard filtration. As a result, we may consider the following diagram and apply the snake lemma:

$$\begin{tikzcd}
\ker_1 \arrow[r] \arrow[d] & \mathrm{Nyg}_{\geq 1}\left(V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}})\right) \arrow[r, "0-\mathrm{can}"] \arrow[d] & \mathrm{Nyg}_{\geq 1}\left(V(2)_*\,\mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}})\right) \arrow[r] \arrow[d] & \mathrm{coker}_1 \arrow[d] \\
\ker \arrow[r] \arrow[d] & V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}}) \arrow[r, "\varphi^{h\mathbb{T}}-\mathrm{can}"] \arrow[d] & C^0(p\mathbb{Z}_p)\langle \zeta, \lambda_1, \lambda_2\rangle[t^{\pm p^2}] \arrow[r] \arrow[d] & \mathrm{coker} \arrow[d] \\
\ker_2 \arrow[r] & \mathrm{Nyg}_{=0}\left(V(2)_*\mathrm{TC}^-(\ell^{\mathrm{B}\mathbb{Z}})\right) \arrow[r, "\overline{\varphi}-\overline{\mathrm{can}}"] & \left(V(2)_*\,\mathrm{TP}(\ell^{\mathrm{B}\mathbb{Z}})\right)/\mathrm{Nyg}_{\geq 1} \arrow[r] & \mathrm{coker}_2.
\end{tikzcd}$$

By Corollary 5.7.17, we identify $\ker_1$ with $C^0(p\mathbb{Z}_p)\langle\zeta\rangle \otimes_{\mathbb{F}_p} N$, and observe that coker_1 is trivial. We next check that $\ker_2$ and coker_2 are the other two summands. To do, it is helpful to observe that $\overline{\varphi}$ must be an isomorphism in large degrees, because $\varphi^{h\mathbb{T}}$ is and $\ker_1$ is bounded above.

By construction, the domain of $\overline{\varphi} - \overline{\mathrm{can}}$ can be identified with the 0-line of the E_∞-page of the homotopy fixed point spectral sequence, which is calculated by Theorem 5.7.16 to be

$$C^0(p\mathbb{Z}_p)[\mu]\langle\zeta, \lambda_1, \lambda_2\rangle \oplus C^0(\mathbb{Z}_p^\times)[\mu]\langle\lambda_1, \lambda_2, \epsilon\rangle \subset V(2)_*\,\mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}}).$$

We next note that $\overline{\varphi}$ carries μ to a class detected by a unit multiple of t^{-p^2}, and indeed otherwise φ could not be an isomorphism in large degrees. Theorem 5.7.16 then implies the codomain of $\overline{\varphi} - \overline{can}$ can be identified with

$$C^0(p\mathbb{Z}_p)[(\overline{\varphi} - \overline{\mathrm{can}})(\mu)]\langle\zeta, \lambda_1, \lambda_2\rangle,$$

since there is a map from this ring into $V(2)_*\,\mathrm{TP}(\ell^{B\mathbb{Z}})$ which is an isomorphism on the associated graded groups after projecting to the quotient by positive Nygaard filtration.

As in Proposition 5.3.18 and Proposition 5.7.18, $\overline{\varphi}$ carries a function $f \in C^0(p\mathbb{Z})$ to the function $x \mapsto f(x/p)$, where $f(x/p)$ is interpreted as zero if $x/p \in \mathbb{Z}_p^\times$. Similarly, Proposition 5.3.18 determines $\overline{\varphi}$ on ζ. Furthermore, $\overline{\varphi}$ is an $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$ module map, leaving only $\overline{\varphi}(\epsilon)$ undetermined.

Now, the fact that $\overline{\varphi}$ is an isomorphism in large degrees is enough to prove that $\overline{\varphi}(\epsilon)$ is sent to ζ times a function non-trivial on elements of p-adic valuation 1, and so $\ker_2$ and coker_2 must be $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$ and $\overline{C^0(\mathbb{Z}_p^\times)}\{\zeta\} \oplus \mathbb{F}_p$, respectively.

All of this describes a filtration on $V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$, with associated graded the direct sum of

- $\ker_1 \cong C^0(p\mathbb{Z}_p)\langle\zeta\rangle \otimes_{\mathbb{F}_p} N$,
- $\ker_2 \cong \mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$, and
- $\Sigma^{-1}\mathrm{coker}_2 \cong \Sigma\overline{C^0(\mathbb{Z}_p^\times)}\{\partial\zeta\} \oplus \mathbb{F}_p\{\partial\}$

This filtration is by construction compatible with that of the proof of Corollary 5.7.13 above, along the coassembly map. We therefore see that the coassembly map is as indicated, up to possible filtration jumps. Such filtration jumps are precluded by $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module structure. □

5.7.3 Coassembly for $V(1)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$

In the previous subsection, we computed the coassembly map

$$V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(2)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}. \tag{5.2}$$

To finish the proof of Theorem 5.7.1, it remains only to prove that the $\mathbb{F}_p[v_2]$-module map

$$V(1)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}) \to V(1)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}$$

is given by $\mathbb{F}_p[v_2] \otimes_{\mathbb{F}_p} (5.2)$.

Proof of Theorem 5.7.1. Examine the map of v_2-Bockstein spectral sequences

$$\begin{tikzcd}
\mathrm{E}_1 = V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})[v_2] \arrow[r, Rightarrow] \arrow[d] & V(1)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}) \arrow[d] \\
\mathrm{E}_1 = V(2)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}[v_2] \arrow[r, Rightarrow] & V(1)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}.
\end{tikzcd}$$

When analyzing these Bockstein spectral sequences, we emphasize again that the construction (see Recollection 5.7.5) of elements $\lambda_1, \lambda_2 \in V(1)_*\mathrm{TC}(\ell)$ equips both $V(1)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$ and $V(1)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}$ with $\mathbb{F}_p[v_2]\langle\lambda_1, \lambda_2\rangle$-module structure. Both v_2-Bockstein spectral sequences, and the map between them, respect $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module structure.

Examining $V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}})$, as determined by Theorem 5.7.20, we see that this $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module is concentrated in degrees $-2 \leq * \leq 2p^2 + 2p - 2$. It follows immediately that the only potential v_2-Bockstein differentials are d_1 differentials. In fact, the d_1 differential is also trivial, as is forced by $\mathbb{F}_p\langle\lambda_1, \lambda_2\rangle$-module structure. Specifically, all classes of degrees larger than $2p^2 - 2$ are multiples of λ_2, and all classes of degrees larger than $2p^2 - 2p - 1$ are multiples of either λ_1 or λ_2. On the other hand, no element of degree less than $2p - 3$ is a multiple of λ_1, and no element of degree less than $2p^2 - 3$ is a multiple of λ_2.

The v_2-Bockstein spectral sequence for $V(1)_*\mathrm{TC}(\ell)^{\mathrm{B}\mathbb{Z}}$ similarly degenerates, because the Bockstein spectral sequence for $V(1)_*\mathrm{TC}(\ell)$ does by the direct computations of Ausoni and Rognes [AR02, Theorem 0.3].

Finally, when analyzing the map of Bockstein spectral sequences, all possible filtration jumps are ruled out by $\Lambda(\lambda_1, \lambda_2)$ module structure. □

To finish the subsection, we prove Theorem 5.7.21 below, which along with Theorem 5.7.1 immediately implies Corollary 5.7.2.

Theorem 5.7.21. *Let p be a prime, and let $\mathbb{Z}$ act on the connective Adams summand ℓ and periodic Adams summand L via Ψ^{1+p}. Let $T(2)$ denote the telescope of a type 2 p-local finite complex. Then, for any $k \geq 0$, there is a commuting diagram*

$$\begin{tikzcd}
L_{T(2)}K(L^{hp^k\mathbb{Z}}) \arrow[d] & L_{T(2)}K(\ell^{hp^k\mathbb{Z}}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & L_{T(2)}\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \arrow[d] \\
L_{T(2)}K(L)^{hp^k\mathbb{Z}} & L_{T(2)}K(\ell)^{hp^k\mathbb{Z}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & L_{T(2)}\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}
\end{tikzcd}$$

where the vertical maps are the coassembly maps. Moreover, the coassembly maps agree with both the cyclotomic completion and $K(2)$-localization maps.

Proof. The diagram of equivalences above comes from applying Corollary 5.6.3, since $L = L_{T(1)}\ell$. By Lemma 5.6.15, to show the left map is an equivalence after cyclotomic completion, it is enough to show that the maps $f_k : L^{hp^k\mathbb{Z}} \to L$, are $L_{T(2)}K^{\wedge}_{\mathrm{cyc}}$-covers.

Recall that there is a $\mathbb{Z}_p^{\times}$-equivariant equivalence $\mathrm{KU} \cong L_{T(1)}\mathbb{S}[\omega^{(1)}_{p^\infty}]$ [CSY21, Example 5.11]. This shows that f_k is isomorphic to the map $\mathbb{S}_{T(1)}[\omega^{(1)}_{p^k}]^{hT_p} \to \mathbb{S}_{T(1)}[\omega^{(1)}_{p^\infty}]^{hT_p}$. Since the source is a finite Galois extension of the $T(1)$-local sphere, this map admits a retraction after tensoring with KU by Lemma 5.6.23. Then, by Corollary 5.6.19 and Remark 5.6.9, we learn $L^{hp^k\mathbb{Z}} \to L$ is a $L_{T(2)}K^{\wedge}_{\mathrm{cyc}}$-cover.

The target of the right coassembly map is $K(2)$-local by [HRW22, Theorem 1.3.6], so we conclude the result.

□

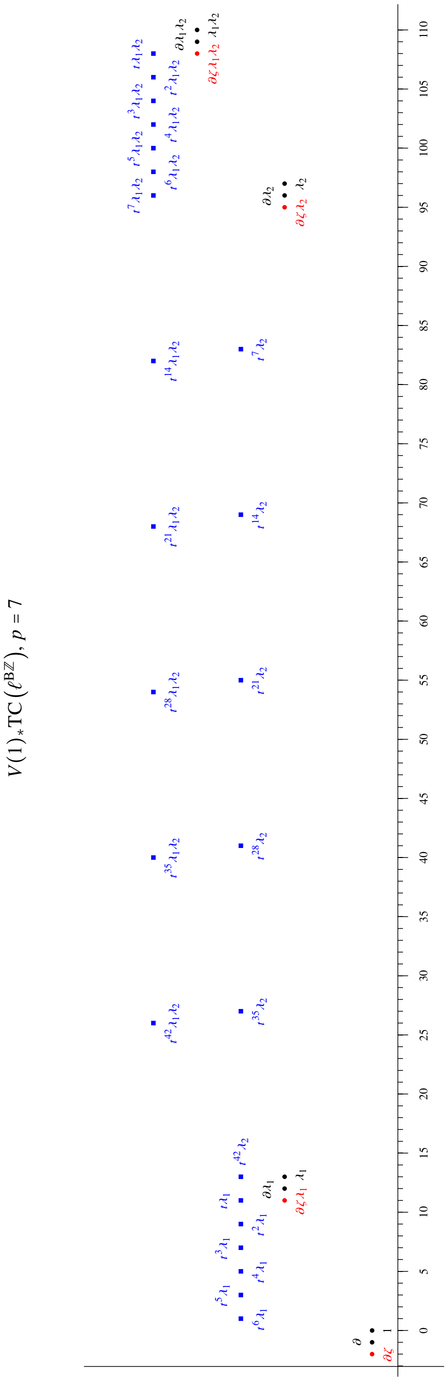

Figure 1: A depiction of the homotopy groups of $V(1)\otimes \mathrm{TC}\left(\ell^{B\mathbb{Z}}\right)$ for $p=7$. This is isomorphic to $V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ for $k \gg 0$, where the $\mathbb{Z}$ action on ℓ is by an Adams operation ψ^m with m a topological generator of $1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. The labeled axis records the degree $*$. The unlabeled axis is not meaningful, and is present only to declutter the display. A black dot represents a copy of $\mathbb{F}_p[v_2]$. A red dot represents a copy of $\overline{C^0(\mathbb{Z}_p^\times)}[v_2]$. A blue square represents a copy of $C^0(p\mathbb{Z}_p)\langle\zeta\rangle[v_2]$.

5.7.4 Lichtenbaum–Quillen for the $K(1)$-local sphere

In this subsection, we apply the tools of Section 5.4 to prove the height 1 LQ property for $\ell^{hp^k\mathbb{Z}}$ for each $k \geq 0, p \geq 5$, where ℓ is the connective Adams summand with $\mathbb{Z}$-action generated by the Adams operation Ψ^{1+p}. The rings $\ell^{hp^k\mathbb{Z}}$ were studied in work of Lee and the third author [LL23], where a constancy result for their THH mod (p, v_1) was proven. The Segal conjecture for their THH was also proven there, which we use here as an ingredient to prove the height 1 LQ property. By combining this with the work of the third author [Lev22], we show that for $p \geq 5$ that the algebraic K-theory of the $K(1)$-local sphere is asymptotically L_2^f-local, in the sense of Definition 5.7.23 below.

The following lemma was communicated to us by Tristan Yang, and will appear in his work on effective LQ properties:

Lemma 5.7.22. *Let X be a p-local spectrum and $n \geq 0$. The following conditions are equivalent:*

1. *The map*
$$X \to L_n^f X$$
has bounded above fiber.

2. *For any type at least n finite p-local spectrum V with v_{n+1}-self map v, the map*
$$V \otimes X \to V[v^{-1}] \otimes X$$
has bounded above fiber.

3. *For any type at least $n+1$ finite p-local spectrum V, $V \otimes X$ is bounded above.*

Proof. (1) implies (2), since L_n^f-localization is smashing and $L_n^f V \cong V[v^{-1}]$. (3) follows from (2) since $V[v^{-1}] = 0$ for V of type at least $n+1$.

We now show that (3) implies (1). It will suffice to show by descending induction on k that $V \otimes F$ is bounded above for each V of type k, where F is the fiber of $X \to L_n^f X$. By hypothesis we know the result for $k \geq n+1$.

Suppose that we know the result for $k+1$ and let V be a finite type k spectrum with v_k-self map v. $V[v^{-1}] \otimes F = 0$ because F is L_n^f-acyclic. Then it is enough to show the fiber of $V \otimes F \to V[v^{-1}] \otimes F$ is bounded above. But this fiber is built from extensions, negative suspensions, and filtered colimits from $V/v \otimes F$, which is bounded above by the inductive hypothesis, so we may conclude. □

Definition 5.7.23. We say that X is asymptotically L_n^f-local if it satisfies the equivalent conditions of Lemma 5.7.22. ◁

Note that the above definition can be considered a weakening of Mahowald–Rezk's notion of an fp-type n spectrum [MR99].

We now prove the two theorems of this subsection, beginning with the height 1 LQ property for the $K(1)$-local sphere:

Theorem 5.7.24. *For a fixed prime $p \geq 5$ and type 3 complex V, $V \otimes \ell^{hp^k\mathbb{Z}}$ is bounded in the cyclotomic t-structure, uniformly for all $k \geq 0$.*

Proof. Fix a prime $p \geq 5$. Let $V = \mathbb{S}/(p, v_1)$, which is an $\mathbb{A}_2$-ring by [Oka84].

To prove the theorem, it will suffice to verify the conditions of Proposition 5.4.27 with parameters uniform in k for $X_k = V \otimes \mathrm{THH}(\ell^{hp^k\mathbb{Z}})$. Condition (3) follows from [LL23, Theorem 8.3], and condition (2) follows from [LL23, Theorem 6.1, Remark 4.13]

By Theorem 5.7.4, ℓ satisfies the height 1 LQ property. The $\mathbb{Z}$-action on $\pi_*\ell/p$ is trivial and in particular unipotent, so the action on ℓ is unipotent by Corollary 5.8.27. By applying Lemma 5.4.9, we learn that $(X_k)_{|0} \cong V \otimes \mathrm{THH}(\ell)^{hp^k\mathbb{Z}}$ verifying condition (1). ☐

$\mathrm{TC}(\ell^{h\mathbb{Z}})$ for $p > 2$ was shown in [Lev22] to be closely related to the K-theory of the $K(1)$-local sphere. By combining this with Theorem 5.7.24, we learn that the K-theory of the $K(1)$-local sphere is asymptotically L_2^f local for $p \geq 5$, proving in particular Theorem E in the introduction:

Theorem 5.7.25. *For $p \geq 5$, $K(L_{K(1)}\mathbb{S})$ is asymptotically L_2^f-local.*

Proof. The category of asymptotically L_2^f-local spectra is clearly a thick subcategory.

By [Lev22], there is a cofiber sequence

$$K(\mathbb{F}_p) \to K(\ell^{h\mathbb{Z}}) \to K(L_{K(1)}\mathbb{S})$$

and a pullback square

$$\begin{tikzcd}
K(\ell^{h\mathbb{Z}}) \arrow[r] \arrow[d] & \mathrm{TC}(\ell^{h\mathbb{Z}}) \arrow[d] \\
K(\mathbb{Z}_p^{h\mathbb{Z}}) \arrow[r] & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})
\end{tikzcd}$$

Since $K(\mathbb{F}_p)$ p-adically is $\mathbb{Z}_p$, it is asymptotically L_2^f-local. It thus suffices to show that $\mathrm{TC}(\ell^{h\mathbb{Z}})$ as well as $\mathrm{fib}(K(\mathbb{Z}_p^{h\mathbb{Z}}) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}))$ are too. By Theorem 5.7.24, $\mathrm{TC}(\ell^{h\mathbb{Z}})$ is asymptotically L_2^f-local.

By [LT23, Theorem 4.1], to show that $\mathrm{fib}(K(\mathbb{Z}_p^{h\mathbb{Z}}) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}))$ is asymptotically L_2^f-local, it is equivalent to show that $\mathrm{fib}(K(\mathbb{Z}_p[x]) \to \mathrm{TC}(\mathbb{Z}_p[x]))$ is. But this now follows from [CM21, Theorem 1.1, 1.2]. ☐

5.8 Appendix

5.8.1 Homotopy ring structures

Definition 5.8.1. The category of *hcrings* in a symmetric monoidal presentable category $\mathcal{C}$ is the category of commutative algebras in $h\mathcal{C}$, the homotopy 1-category of $\mathcal{C}$. Similarly, an *hring* is an associative algebra in $h\mathcal{C}$, and a $h\mathbb{A}_2$*ring* is an $\mathbb{A}_2$-algebra in $h\mathcal{C}$. ◁

Definition 5.8.2. Let R be an h$\mathbb{A}_2$ring in $\mathcal{C}$. A map $z : S \to R$ is *hcentral* if the following diagrams in $\mathcal{C}$ commute

$$\begin{tikzcd} S \otimes R \arrow[d, bend right, "z\cdot -"'] \arrow[d, bend left, "-\cdot z"] \\ R \end{tikzcd} \qquad \begin{tikzcd} S \otimes R \otimes R \arrow[r, "S\otimes\mu"] \arrow[d, "(z\cdot -)\otimes R"] & S \otimes R \arrow[d, "z\cdot -"] \\ R \otimes R \arrow[r, "\mu"] & R \end{tikzcd} \qquad \begin{tikzcd} S \otimes R \otimes R \arrow[r, "S\otimes\mu"] \arrow[d, "R\otimes(-\cdot z)"] & S \otimes R \arrow[d, "-\cdot z"] \\ R \otimes R \arrow[r, "\mu"] & R. \end{tikzcd}$$

We say $z \in \pi_j R$ is hcentral if it is as a map $\Sigma^j \mathbb{1} \to R$. ◁

Remark 5.8.3. Given an hcentral $X \to R$ and a map $Y \to X$ the composite $Y \to X \to R$ is an hcentral as well. As a consequence of this, If $i : R_1 \to R_2$ is an hcentral ring map then the elements of $\pi_* R_1$ naturally land in the subring of hcentral elements of $\pi_* R_2$. ◁

Remark 5.8.4. A map of hcrings (or of hrings) is the same data as a map of the underlying h$\mathbb{A}_2$rings. For this reason we will sometimes abuse notation, refer to these as simply "ring maps". ◁

Lemma 5.8.5. *Let R be an h$\mathbb{A}_2$ring in a stable, exactly symmetric monoidal category $\mathcal{C}$. If p^k acts by zero on R and $k \geq m_p^{\mathbb{A}_2}$, then there is an hcentral ring map*

$$\mathbb{1}_{\mathcal{C}}/p^{2k} \to R.$$

Proof. Let R' be a h$\mathbb{A}_2$-ring with $p^k = 0$. Consider the diagram

$$\begin{tikzcd} R' \arrow[r, "p^{2k}"] \arrow[d, "p^k"] & R' \arrow[r] \arrow[d, equal] & R'/p^{2k} \arrow[d] \\ R' \arrow[r, "p^k"] & R' \arrow[r] & R'/p^k. \end{tikzcd}$$

After picking a nullhomotopy of the endomorphism p^k of R' we can use this nullhomotopy to give splittings $R'/p^{2k} \cong R' \oplus \Sigma R'$ and $R'/p^k \cong R' \oplus \Sigma R'$ which are compatible in the sense that the reduction map $R'/p^{2k} \to R'/p^k$ factors as

$$R' \oplus \Sigma R' \twoheadrightarrow R' \hookrightarrow R' \oplus \Sigma R'.$$

Using the fact that $p^k = 0$ in R we construct a unital map $i : \mathbb{1}/p^k \to R$. In order to prove the lemma it will suffice to show that the composite map $\mathbb{1}/p^{2k} \to \mathbb{1}/p^k \to R$ is hcentral and a ring map. For this we must check that each of the maps

$$\mathbb{1}/p^k \otimes R \xrightarrow{x\otimes y \mapsto (i(x)\cdot y)-(y\cdot i(x))} R$$
$$\mathbb{1}/p^k \otimes R \otimes R \xrightarrow{x\otimes y\otimes z \mapsto ((i(x)\cdot y)\cdot z)-(i(x)\cdot(y\cdot z))} R$$
$$R \otimes R \otimes \mathbb{1}/p^k \xrightarrow{y\otimes z\otimes x \mapsto (y\cdot(z\cdot i(x)))-((y\cdot z)\cdot i(x))} R$$
$$\mathbb{1}/p^k \otimes \mathbb{1}/p^k \xrightarrow{x\otimes y \mapsto (i(x)\cdot i(y))-i(x\cdot y)} R$$

becomes null upon precomposing with $r : \mathbb{S}/p^{2k} \to \mathbb{S}/p^k$.

Using the splittings proven above, applied to $R' = R$ in the first three equations and applied to $R' = \mathbb{S}/p^k$ for the fourth (since $k \geq m_p^{\mathbb{A}_2}$), and the fact that i is unital we can rewrite these maps as follows

$$\begin{array}{ccccc} R \oplus \Sigma R & \xrightarrow{(0,a)} R & \qquad & (R \otimes R) \oplus \Sigma(R \otimes R) & \xrightarrow{(0,b)} R \\ (R \otimes R) \oplus \Sigma(R \otimes R) & \xrightarrow{(0,c)} R & \qquad & \mathbb{1}/p^k \oplus \Sigma \mathbb{1}/p^k & \xrightarrow{(0,d)} R \end{array}$$

The factorization of the reduction map through the inclusion of the first summand now completes the proof of the lemma. ☐

5.8.2 Almost compact objects

It will be useful for us to have a finiteness condition weaker than compactness which is sensitive to the choice of a t-structure on a stable category. For this reason we introduce the notion of *almost compact objects.*

Definition 5.8.6. Let $\mathcal{C}$ be a stable presentable category equipped with a t-structure. Given an $X \in \mathcal{C}$ which is bounded below we say that X is *almost compact* if for any filtered diagram $F : K \to \mathcal{C}$ such that the set of objects $\{F(k)\}_{k \in K}$ are uniformly bounded in the range $[c, b]$, the assembly map

$$\operatorname*{colim}_{k \in K} \mathrm{Map}_{\mathcal{C}}(X, F(k)) \to \mathrm{Map}_{\mathcal{C}}(X, \operatorname*{colim}_{k \in K} F(k))$$

is an isomorphism. ◁

Example 5.8.7. A (not necessarily p-complete) spectrum X is almost compact iff it is bounded below and $\pi_k X$ is finitely generated for all k. A p-complete spectrum X is almost compact under the additional condition that π_k is p-nil for each k. ◁

As the filtered colimits appearing in the definition of almost compact are a subset of those appearing in the definition of compact we have the following lemma:

Lemma 5.8.8. *Let $\mathcal{C}$ be a stable presentable category equipped with a t-structure. If an object $X \in \mathcal{C}$ is compact, then it is almost compact.*

Almost compact objects enjoy stronger closure properties than compact objects. Specifically, they are closed under geometric realizations which are uniformly bounded below.

Lemma 5.8.9. *Let $\mathcal{C}$ be a stable presentable category equipped with a t-structure. The full subcategory of almost compact objects is closed under finite (co)limits and geometric realizations of simplicial diagrams which are uniformly bounded below.*

Proof. If X almost compact, then by using suspensions of the diagram we map out to we can upgrade the isomorphism of mapping spaces in the definition of almost compact to an isomorphism of mapping spectra. Exactness of the mapping spectrum then makes it clear that the collection of almost compact objects is closed under finite (co)limits.

For closure under geometric realizations that are uniformly bounded bounded below we follow the argument in [Lur18b, Prop. C.6.4.4].[36] Without loss of generality let $X_\bullet$ be a simplicial object in $\mathcal{C}_{\geq 0}$ and let $K \to \mathcal{C}$ be a filtered diagram uniformly bounded in the range $[0, a]$. Then, we have isomorphisms

$$\begin{aligned}
&\operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathcal{C}}\left(\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} X_\bullet, k\right) \cong \operatorname*{colim}_{k\in K} \lim_{\bullet\in\Delta} \mathrm{Map}_{\mathcal{C}}(X_\bullet, k) \cong \operatorname*{colim}_{k\in K} \lim_{\bullet\in\Delta_{s,\leq a+1}} \mathrm{Map}_{\mathcal{C}}(X_\bullet, k) \\
&\cong \lim_{\bullet\in\Delta_{s,\leq a+1}} \operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathcal{C}}(X_\bullet, k) \cong \lim_{\bullet\in\Delta} \operatorname*{colim}_{k\in K} \mathrm{Map}_{\mathcal{C}}(X_\bullet, k) \cong \lim_{\bullet\in\Delta} \mathrm{Map}_{\mathcal{C}}\left(X_\bullet, \operatorname*{colim}_{k\in K} k\right) \\
&\cong \mathrm{Map}_{\mathcal{C}}\left(\operatorname*{colim}_{\Delta^{\mathrm{op}}} X_\bullet, \operatorname*{colim}_{k\in K} k\right).
\end{aligned}$$

The first step pulls the colimit out. The second step uses the fact that $X_\bullet$ is ≥ 0 and k is $\leq a$ to conclude that $\mathrm{Map}_{\mathcal{C}}(X_\bullet, k)$ is an a-truncated space and therefore we may replace the limit over Δ by a limit over $\Delta_{s,\leq a+1}$, which is a finite diagram. The third step uses the fact that $\Delta_{s,\leq a+1}$ is finite and K is filtered to exchange the limit and colimit. The fourth step uses the fact that colimits in space preserve a-truncated objects and so that we may replace the limit over $\Delta_{s,\leq a+1}$ with a limit over Δ. The fifth step uses the hypothesis that each $X_\bullet$ is almost compact. The final step pulls the limit over Δ back inside. □

Lemma 5.8.10. *Suppose that* $R \in \mathrm{Alg}(\mathrm{Sp}_{\geq 0})$ *and* R/p *is bounded below with finitely generated homotopy groups in each degree. Then* $\mathrm{THH}(R) \otimes V$ *is almost compact in* Sp *for any* $V \in \mathrm{Sp}$ *that is almost compact.*

Proof. We first claim that the tensor product of R with any almost compact object $V \in \mathrm{Sp}$ is almost compact. Our assumption on R along with Example 5.8.7 implies that for all $k \geq 0$, R/p^k is almost compact. By choosing a cell decomposition of V made up of increasingly suspended copies $\mathbb{S}/p^k$, we see that $V \otimes R$ is almost compact by Example 5.8.7 again.

Next, we observe that by induction, $V \otimes R^{\otimes n+1}$ is almost compact for all n. Since it is also uniformly bounded below (using connectivity of R), we learn by applying Lemma 5.8.9 that $V \otimes \mathrm{THH}(R)$ is almost compact. □

Example 5.8.11. Let $\mathrm{Mod}(\mathbb{Z})_p$ be the category of p-complete $\mathbb{Z}$-modules with its standard t-structure.

- $\mathbb{Z}/p$ is compact and therefore almost compact.
- $\mathbb{Z}_p$ is neither compact nor almost compact.
- $\oplus_{i\geq 0}\Sigma^i\mathbb{Z}/p$ is almost compact, but not compact. ◁

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[36] For a discussion of why we cannot simply cite this proposition see Remark 5.8.12 below.

Remark 5.8.12. In [Lur18b, Dfn. C.6.4.1], Lurie defines an object X in a presentable category $\mathcal{D}$ to be almost compact if $\tau_{\leq n}X$ is a compact object of $\tau_{\leq n}\mathcal{D}$. In order to compare our definition with Lurie's it is useful to note that Lurie only applies this definition when $\mathcal{D} = \mathcal{C}_{\geq 0}$ is the connective objects of a t-structure on a stable category $\mathcal{C}$ and filtered colimits preserve coconnectivity in $\mathcal{C}$. In this situation our definitions agree.

Without the assumption that filtered colimits in $\mathcal{C}$ preserve coconnectivity these definitions are not equivalent. As an example, note in Example 5.8.11 $\mathbb{Z}/p$ is compact and almost compact (according to Definition 5.8.6), but that $\mathbb{Z}/p \in (\mathrm{Mod}(\mathbb{Z})_p)_{\geq 0}$ is not almost compact (according to Lurie's definition).

This subtlety is not an idle curiosity and is indeed relevant to the present paper: Filtered colimits do not preserve coconnectivity in the cyclotomic t-structure. ◁

Definition 5.8.13. Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories equipped with t-structures and let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between them.

We say that F has *t-amplitude* $\geq a$ if $F(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq a}$. We say that F has *t-amplitude* $\leq b$ if $F(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq b}$. If F has t-amplitude $\geq a$ and $\leq b$, then we say that F has *t-amplitude in the range* $[a, b]$. ◁

Remark 5.8.14. Exactness implies that if a functor $F : \mathcal{C} \to \mathcal{D}$ has t-amplitude $[a_0, a_1]$, then for every interval $[b_0, b_1]$ we have $F(\mathcal{C}_{[b_0,b_1]}) \subseteq \mathcal{D}_{[b_0+a_0,b_1+a_1]}$. ◁

Lemma 5.8.15. *Let $\mathcal{C}$ be a stable, presentably symmetric monoidal category with a compatible t-structre and let $R_\bullet : I \to \mathrm{CAlg}(\mathcal{C})$ be a filtered diagram of commutative algebras in $\mathcal{C}$ with uniformly bounded t-amplitude and colimit $\mathbb{1}_{\mathcal{C}}$.*

1. *Given an almost compact object $X \in \mathcal{C}$ and a bounded object $Y \in \mathcal{C}$ there is an isomorphism*
$$\operatorname*{colim}_{\alpha\in I} \mathrm{Map}_{\mathrm{Mod}(\mathcal{C};R_\alpha)}(R_\alpha \otimes X, R_\alpha \otimes Y) \cong \mathrm{Map}_{\mathcal{C}}(X, Y).$$
2. *Given a bounded, almost compact object $X \in \mathcal{C}$ and an $\alpha \in I$, if Y is a retract of $R_\alpha \otimes X$ in $\mathrm{Mod}(\mathcal{C}; R_\alpha)$, then there is an arrow $\alpha \to \beta \in I$ and an isomorphism of R_β-modules*
$$R_\beta \otimes_{R_\alpha} Y \cong R_\beta \otimes (\mathbb{1} \otimes_{R_\alpha} Y).$$

Proof. Part (1). Using the assumptions that X is almost compact and the $R_\alpha \otimes Y$ are uniformly bounded we have isomorphisms

$$\begin{aligned}\operatorname*{colim}_{\alpha\in I} \mathrm{Map}_{\mathrm{Mod}(\mathcal{C};R_\alpha)}(R_\alpha \otimes X, R_\alpha \otimes Y) &\cong \operatorname*{colim}_{\alpha\in I} \mathrm{Map}_{\mathcal{C}}(X, R_\alpha \otimes Y) \\ &\cong \mathrm{Map}_{\mathcal{C}}(X, \operatorname*{colim}_{\alpha\in I} R_\alpha \otimes Y) \cong \mathrm{Map}_{\mathcal{C}}(X, Y).\end{aligned}$$

Part (2). Let $e : R_\alpha \otimes X \to R_\alpha \otimes X$ be an idempotent map of R_α-modules such that $(R_\alpha \otimes X)[e^{-1}] \cong Y$. Then after base change to $\mathbb{1}$ we obtain an idempotent e_∞ such that $X[e_\infty^{-1}] \cong \mathbb{1} \otimes_{R_\alpha} Y$. Our goal is now to compare the two idempotents e and $1 \otimes e_\infty$ which agree after base change to $\mathbb{1}$. Using the assumption that X is bounded and almost compact it follows from part (1) that e and $1 \otimes e_\infty$ agree after base change to R_β for some arrow $\alpha \to \beta \in I$. □

5.8.3 Locally unipotent $\mathbb{Z}$-actions

In this appendix we study *locally unipotent* $\mathbb{Z}$-actions. The material contained here is used heavily in Section 5.4. Unlike in the body of the paper we work integrally in this appendix.

Notation 5.8.16. Throughout this appendix if an object X has a $\mathbb{Z}$-action we will write $\psi \colon X \to X$ for the automorphism associated to the generator $1 \in \mathbb{Z}$. ◁

Definition 5.8.17. Let $\mathcal{C}$ be a stable, presentable category. We define $\mathcal{C}^{B\mathbb{Z},u} \subseteq \mathcal{C}^{B\mathbb{Z}}$ to be the localizing subcatgory of $\mathcal{C}^{B\mathbb{Z}}$ generated by objects with trivial action. We also write $\iota : \mathcal{C}^{B\mathbb{Z},u} \to \mathcal{C}^{B\mathbb{Z}}$ for the defining inclusion and $(-)^u$ for its right adjoint. ◁

Lemma 5.8.18. *Let $\mathcal{C}$ be a stable, presentably symmetric monoidal category. The subcategory $\mathcal{C}^{B\mathbb{Z},u} \subseteq \mathcal{C}^{B\mathbb{Z}}$ is closed under the tensor product.*

Proof. As the tensor product on $\mathcal{C}$ commutes with colimits seperately in each variable, it suffices to observe that if $X, Y \in \mathcal{C}^{B\mathbb{Z}}$ have a trivial $\mathbb{Z}$-action, then $X \otimes Y \in \mathcal{C}^{B\mathbb{Z}}$ also has a trivial $\mathbb{Z}$-action. □

Remark 5.8.19. As a consequence of Lemma 5.8.18, when $\mathcal{C}$ is symmetric monoidal we may equip $\mathcal{C}^{B\mathbb{Z},u}$ and the defining inclusion ι with symmetric monoidal structures by restriction. This in turn equips $(-)^u$ with a lax symmetric monoidal structure. ◁

Before we proceed further with the general theory it will be profitable to study the case $\mathcal{C} = \mathrm{Sp}$ in more detail.

Construction 5.8.20. Let $\mathbb{S}[t^{\pm 1}]$ denote the spherical group ring of $\mathbb{Z}$, let i denote the commutative algebra map $\mathbb{S}[t^{\pm 1}] \to \mathbb{S}$ obtained from the map $\mathbb{Z} \to *$ and let $j : \mathbb{S}[t^{\pm 1}] \to \mathbb{S}[t^{\pm 1}, (t-1)^{-1}]$ denote the localization of $\mathbb{S}[t^{\pm 1}]$ obtained by inverting $(t-1)$. Using j we construct a recollement

$$\begin{tikzcd}
\mathrm{Mod}(\mathbb{S}[t^{\pm 1}])^{\wedge}_{(t-1)} \arrow[r, hook, bend left] \arrow[r, bend right] & \mathrm{Mod}(\mathbb{S}[t^{\pm 1}]) \arrow[l, "(-)^{\wedge}_{(t-1)}"'] \arrow[r, bend left, "j^*"] \arrow[r, bend right] & \mathrm{Mod}(\mathbb{S}[t^{\pm 1}][(t-1)^{-1}]) \arrow[l, hook, "j_*"]
\end{tikzcd}$$

based on inverting and completing at $(t-1)$.

Writing the $\mathbb{S}[t^{\pm 1}]$-module $\mathbb{S}$ as $\mathbb{S}[t^{\pm 1}]/(t-1)$ we find that the functor i_* (restriction of scalars along i) factors through the left adjoint to $(t-1)$-adic completion. We can then describe $\mathrm{Mod}(\mathbb{S}[t^{\pm 1}])^{\wedge}_{(t-1)}$ as the localizing subcategory of $\mathrm{Mod}(\mathbb{S}[t^{\pm 1}])$ generated under colimits by the image of i_*.

If we identify $\mathrm{Sp}^{B\mathbb{Z}}$ with $\mathbb{S}[t^{\pm 1}]$-modules, then we obtain the following identifications:

- The trivial action functor $(-)^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{Sp}^{B\mathbb{Z}}$ can be identified with i_*.
- The right adjoint to i_* can be identified with $(-)^{h\mathbb{Z}}$.

- The subcategory of $(t-1)$-nilpotent $\mathbb{S}[t^{\pm 1}]$-modules (the image of the left adjoint to $(t-1)$-adic completion) can be identified with $\mathrm{Sp}^{B\mathbb{Z},u}$.

Note that since $\mathrm{Sp}^{B\mathbb{Z},u}$ is generated under colimits by the image of the trivial action functor, its right adjiont $(-)^{h\mathbb{Z}} : \mathrm{Sp}^{B\mathbb{Z},u} \to \mathrm{Sp}$ is conservative. ◁

Lemma 5.8.21. *There is a symmetric monoidal isomorphism*

$$\mathrm{Mod}(\mathbb{S}^{B\mathbb{Z}}) \cong \mathrm{Sp}^{B\mathbb{Z},u}$$

under which ι *is given by the formula* $X \mapsto \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} X$ *and* $(-)^u$ *is given by* $Y \mapsto Y^{h\mathbb{Z}}$*. This identification is natural in restriction along the maps* $n\mathbb{Z} \subseteq \mathbb{Z}$*.*

Proof. We begin by considering the cocontinuous symmetric monoidal functor $(-)^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{Sp}^{B\mathbb{Z},u}$ and its conservative right adjiont $(-)^{h\mathbb{Z}}$ from Construction 5.8.20. As $(-)^{h\mathbb{Z}}$ is a finite limit, this right adjoint is cocontinuous. It now follows from [BHS0 , Prop. A.4] that the induced adjunction

$$\mathrm{Mod}(\mathrm{Sp}; \mathbb{S}^{B\mathbb{Z}}) \to \mathrm{Sp}^{B\mathbb{Z},u}$$

is the desired symmetric monoidal equivalence. Naturality in endomorphisms of $\mathbb{Z}$ follows by functoriality of the construction of the comparison functor. □

We now return to the general case. The next lemma allows us to extend most results from the case $\mathcal{C} = \mathrm{Sp}$ to the general case using base-change.

Lemma 5.8.22. *Let* $\mathcal{C}$ *be a stable presentable category. There is a natural identification of* $\mathcal{C}^{B\mathbb{Z},u} \subseteq \mathcal{C}^{B\mathbb{Z}}$ *with* $\left(\mathrm{Sp}^{B\mathbb{Z},u} \subseteq \mathrm{Sp}^{B\mathbb{Z}}\right) \otimes \mathcal{C}$*.*

Proof. Given a spectrum with a $\mathbb{Z}$-action and an object of $\mathcal{C}$ their tensor product naturally lives in $\mathcal{C}^{B\mathbb{Z}}$ and so we obtain a comparison functor

$$\mathrm{Sp}^{B\mathbb{Z}} \otimes \mathcal{C} \to \mathcal{C}^{B\mathbb{Z}}.$$

Using the fact that Pr^{L} is ambidextrous for all spaces-shaped colimits and the Lurie tensor product commutes with colimits seperately in each variable we can verify that the map above is an isomorphism via the chain of isomorphisms

$$\mathcal{C}^{B\mathbb{Z}} \cong \mathcal{C}_{B\mathbb{Z}} \cong (\mathrm{Sp} \otimes \mathcal{C})_{B\mathbb{Z}} \cong \mathrm{Sp}_{B\mathbb{Z}} \otimes \mathcal{C} \cong \mathrm{Sp}^{B\mathbb{Z}} \otimes \mathcal{C}.$$

As fully faithful left adjoints are closed under basechange in Pr^{L} (this follows for example from [Lur17, Proposition 4.8.1.17]) it will now suffice for us to identify the two full subcategories $\mathcal{C}^{B\mathbb{Z},u}$ and $\mathrm{Sp}^{B\mathbb{Z},u} \otimes \mathcal{C}$ of $\mathcal{C}$. The former is generated under colimits by objects of the form X^{triv} with $X \in \mathcal{C}$. The latter is generated under colimits by objects of the form $X \otimes Y^{\mathrm{triv}}$ with $X \in \mathcal{C}$ and $Y \in \mathrm{Sp}$. The conclusion follows. □

Remark 5.8.23. In view of Lemma 5.8.22 tensoring Lemma 5.8.21 with $\mathcal{C}$ gives a version of this identification for a general category. ◁

Proposition 5.8.24. *Let $\mathcal{C}$ be a stable presentable category. The $\iota \dashv (-)^u$ adjunction extends to a recollement*

$$\begin{tikzcd}
\mathcal{C}^{B\mathbb{Z},u} \arrow[r, hook, bend left, "\iota"] \arrow[r, bend right] & \mathcal{C}^{B\mathbb{Z}} \arrow[l, "(-)^u"'] \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \mathrm{Mod}(\mathbb{S}[t^{\pm 1}][(t-1)^{-1}]) \otimes \mathcal{C} \arrow[l, hook', "j^*"']
\end{tikzcd}$$

whose gluing sequences take the form

$$\iota(X^u) \to X \to X[(\psi - 1)^{-1}].$$

In particular, $X^u = 0$ if and only if the map $\psi - 1 : X \to X$ is an isomorphism.

Proof. The case of $\mathrm{Sp} = \mathcal{C}$ follows from Construction 5.8.20. The general case now follows by tensoring the case of Sp with $\mathcal{C}$ by Lemma 5.8.22. ☐

We end the section with a collection of lemmas useful for checking whether a $\mathbb{Z}$-action is locally unipotent, and examples.

Example 5.8.25. Given an $M \in \mathrm{Mod}(\mathbb{Z})^{\heartsuit}$ with a $\mathbb{Z}$-action we can read off from the gluing sequence in Proposition 5.8.24 that the action on M is locally unipotent if and only if for all $m \in M$ we have $(\psi - 1)^{\circ N}(m) = 0$ for $N \gg 0$. ◁

Lemma 5.8.26. *Let $F : \mathcal{C} \to \mathcal{D}$ be a filtered colimit preserving, additive functor between stable presentable categories. F preserves local unipotence and commutes with the functors ι, $(-)^u$ and $(-)[(\psi - 1)^{-1}]$. If F is conservative, then it detects local unipotence.*

Proof. As F is additive and preserves filtered colimits we have

$$F(X[(\psi - 1)^{-1}]) \cong F(X)[(\psi - 1)^{-1}].$$

The conclusion now follows from the fact the that gluing sequences in $\mathcal{C}^{B\mathbb{Z}}$ as found in Proposition 5.8.24 are sent to the corresponding gluing sequences in $\mathcal{D}^{B\mathbb{Z}}$. ☐

Corollary 5.8.27. *Let $\mathcal{C}$ be a presentable stable category with a compact generator $V \in \mathcal{C}$. An $X \in \mathcal{C}^{B\mathbb{Z}}$ is locally unipotent if and only if the $\mathbb{Z}$-action on the homotopy groups $\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^k V, X))$ is locally unipotent for all k.*

Proof. The functor $\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^* V, -)) : \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})^{\mathrm{Gr}}$ is additive, filtered colimit preserving (since V is compact) and conservative (since V generates $\mathcal{C}$). ☐

Corollary 5.8.28. *A $\mathbb{Z}$-action on an object $X \in \mathrm{PgcSp}_{\langle p \rangle}$ is locally unipotent if and only if the associated $\mathbb{Z}$-actions on $X^{\Phi C_p}$ and $X^{\Phi e}$ are locally unipotent.*

Proof. The functor $(\Phi C_p, \Phi e) : \mathrm{PgcSp}_{\langle p \rangle} \to \mathrm{Sp} \times \mathrm{Sp}$ is conservative and colimit preserving. ☐

Corollary 5.8.29. *Suppose $\mathcal{C}$ is equipped with a t-structure compatible with filtered colimits. The truncation functors $\tau_{\geq 0}$ and $\tau_{\leq 0}$ preserve local unipotence.*

Proof. $\tau_{\geq 0}$ and $\tau_{\leq 0}$ are additive and commute with filtered colimits (by hypothesis). ☐

Lemma 5.8.30. *Suppose $\mathcal{C}$ is equipped with a t-structure compatible with filtered colimits. If $X \in \mathcal{C}_p^{B\mathbb{Z},u}$, then $\tau_{\geq 0}X \in \mathcal{C}_p^{B\mathbb{Z},u}$ as well.*

Proof. From Proposition 5.8.24 we know that an object in $X \in \mathcal{C}_p^{B\mathbb{Z}}$ has a locally unipotent action iff the p-completion of $X[(\psi - 1)^{-1}]$ is zero (here the colimit is evaluated in $\mathcal{C}$). Equivalently, this is the same as saying that $X[(\psi - 1)^{-1}]$ is a $\mathbb{Q}$-module.

As the functor $\tau_{\geq 0}(-)$ commutes with filtered colimits by assumption it suffices to observe that $\tau_{\geq 0}(-)$ is additive and therefore sends $\mathbb{Q}$-modules to $\mathbb{Q}$-modules. ☐

Lemma 5.8.31. *Let $\mathcal{C}$ be a stable, p-complete, presentable category. The functor $(-)^u$ commutes with restriction to $p\mathbb{Z} \subset \mathbb{Z}$. In particular, an $X \in \mathcal{C}^{B\mathbb{Z}}$ has a locally unipotent action if and only if the action is locally unipotent after restricting to $p\mathbb{Z} \subseteq \mathbb{Z}$.*

Proof. Using the p-completeness assumption we have an isomorphism

$$X[(\psi^p - 1)^{-1}] \cong X[(\psi - 1)^{-1}]$$

since $(\psi - 1)^p = \psi^p - 1 + p\phi$ for some endomorphism ϕ. The conclusion now follows from Proposition 5.8.24. ☐

Lemma 5.8.32. *Let $\mathcal{C}$ be a stable, p-complete, presentable category. The natural map $\operatorname{colim}_k X^{hp^k\mathbb{Z}} \to X^u$ is an isomorphism. In particular, an $X \in \mathcal{C}^{B\mathbb{Z}}$ has a locally unipotent action if and only if the natural map $\operatorname{colim}_k X^{hp^k\mathbb{Z}} \to X$ is an isomorphism.*

Proof. From Lemma 5.8.31 we know that $(-)^u$ does not change upon restriction to the subgroups $p^k\mathbb{Z}$. The natural map is now obtained as the colimit of the maps $X^{hp^k\mathbb{Z}} \to X^u$.

Recall that $(\operatorname{colim}_k \mathbb{S}^{Bp^k\mathbb{Z}})_p \cong \mathbb{S}_p$. Using the p-completeness assumption on $\mathcal{C}$, the formula for $(-)^u$ from Remark 5.8.23 and Lemma 5.8.31 we obtain isomorphisms

$$\begin{aligned}\operatorname*{colim}_k X^{hp^k\mathbb{Z}} &\cong \mathbb{S}_p \otimes_{\mathbb{S}_p} (\operatorname*{colim}_k X^{hp^k\mathbb{Z}}) \cong (\operatorname*{colim}_k \mathbb{S}_p) \otimes_{\operatorname{colim}_k \mathbb{S}_p^{Bp^k\mathbb{Z}}} (\operatorname*{colim}_k X^{hp^k\mathbb{Z}}) \\ &\cong \operatorname*{colim}_k \left(\mathbb{S}_p \otimes_{\mathbb{S}_p^{Bp^k\mathbb{Z}}} X^{hp^k\mathbb{Z}}\right) \cong \operatorname*{colim}_k X^u \cong X^u.\end{aligned}$$

☐

Lemma 5.8.33. *Let $\mathcal{C}$ be a stable, p-complete, presentable category. Suppose we are given a compact object $X \in \mathcal{C}$ with a locally unipotent $\mathbb{Z}$-action. For $k \gg 0$ the action of $p^k\mathbb{Z}$ on X is trivializable.*

Proof. The action of $p^k\mathbb{Z}$ on X is trivializable iff the natural map $X^{hp^k\mathbb{Z}} \to X$ admits a section. From Lemma 5.8.32 we have a colimit diagram

$$X^{h\mathbb{Z}} \to X^{hp\mathbb{Z}} \to X^{hp^2\mathbb{Z}} \to \cdots \to X.$$

Now the hypotheses imply that the identity map $X \to X$ factors through a finite stage of this diagram. ☐

Lemma 5.8.34. *Let $\mathcal{C}$ be a stable, p-complete, presentable category equipped with a t-structure. Suppose we are given an $X \in \mathcal{C}$ which is both bounded and almost compact and a locally unipotent $\mathbb{Z}$-action on X. For $k \gg 0$ the action of $p^k\mathbb{Z}$ on X is trivializable.*

Proof. The proof is the same as the proof of Lemma 5.8.33, but using (uniform) boundedness of the $X^{hp^k\mathbb{Z}}$ and almost compactness of X in place of compactness of X. □

Finally, we discuss the unipotent algebra that corresponds to the ring map $\mathbb{S}_p^{B\mathbb{Z}} \to \mathbb{S}_p$ coming from the map $* \to B\mathbb{Z}$ via the equivalence $\mathrm{Mod}_{\mathrm{Sp}_p}(\mathbb{S}_p^{B\mathbb{Z}}) \cong \mathrm{Sp}_p^{B\mathbb{Z},u}$ of Remark 5.8.23:

Proposition 5.8.35. *Let $\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p})) \in \mathrm{CAlg}(\mathrm{Sp}_p^{B\mathbb{Z}})$ be the spherical Witt vectors of $C^0(\mathbb{Z}_p)$ endowed with the $\mathbb{Z}$-action coming from the map $+1 : \mathbb{Z}_p \to \mathbb{Z}_p$. Let $\mathcal{C}$ be a stable p-complete presentable category.*

1. *The $\mathbb{Z}$-action on $\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p}))$ is locally unipotent.*
2. *The unit map $\mathbb{S}_p \to \mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p}))^{h\mathbb{Z}}$ is an isomorphism.*
3. *There is an symmetric monoidal isomorphism*

$$\mathcal{C} \cong \mathrm{Mod}(\mathcal{C}^{B\mathbb{Z},u}; \mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p})))$$

   given by sending X to $\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p})) \otimes X$ with inverse given by $Y \mapsto Y^{h\mathbb{Z}}$.

4. *For each $a \in \mathbb{Z}_p$ there is a isomorphism of symmetric monoidal functors between the functor $(-)^{h\mathbb{Z}}$ and the functor $(-)_{|a}$ given by the composite*

$$\mathrm{Mod}(\mathrm{Sp}_p^{B\mathbb{Z},u}; \mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p}))) \to \mathrm{Mod}(\mathbb{W}(C^0(\mathbb{Z}_p))) \xrightarrow{(-)_{|a}} \mathrm{Sp}_p.$$

If $\mathcal{C}$ is presentably symmetric monoidal, then the isomorphisms in (3) and (4) can be made symmetric monoidal.

Proof. Since $\mathbb{W}(-)$ and $C^0(-)$ commute with colimits, we have isomorphisms $\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p})) \cong \mathrm{colim}_k\, \mathbb{W}(C^0(\overrightarrow{\mathbb{Z}/p^k}))$. The $+1$ action on $\mathbb{W}(C^0(\mathbb{Z}/p^k))$ is trivial after restricting to $p^k\mathbb{Z} \subseteq \mathbb{Z}$ is thus unipotent by Lemma 5.8.31. Passing to the colimit along k we obtain (1).

For (2) note that since both the source and target are p-complete and bounded below it will suffice to verify the claim after tensoring with $\mathbb{F}_p$. The claim now follows from the short exact sequence of abelian groups

$$0 \to \mathbb{F}_p \to C^0(\mathbb{Z}_p) \xrightarrow{\psi-\mathrm{id}} C^0(\mathbb{Z}_p) \to 0.$$

For claims (3) and (4) the base-change formula from Lemma 5.8.22 and the recollement from Proposition 5.8.24 reduce to proving the symmetric monoidal versions of (3) and (4) for $\mathcal{C} = \mathrm{Sp}_p$.

Part (3) follows from (1), (2) and Remark 5.8.23 upon taking modules over the commutative algebra $\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p}))$ under the symmetric monoidal equivalence $\operatorname{Mod}(\mathbb{S}_p^{B\mathbb{Z}}) \cong \operatorname{Sp}_p^{B\mathbb{Z},u}$. Part (4) follows from (3) via the symmetric monoidal isomorphisms

$$(-)_{|a} \cong \left(\mathbb{W}(C^0(\overrightarrow{\mathbb{Z}_p})) \otimes (-)^{h\mathbb{Z}}\right)_{|a} \cong (\mathbb{W}(C^0(\mathbb{Z}_p))_{|a} \otimes (-)^{h\mathbb{Z}} \cong (-)^{h\mathbb{Z}}. \qquad \square$$

5.8.4 Connectivity for $\mathcal{O}$-algebras

In this subsection, let $\mathcal{C}$ be a presentably symmetric monoidal stable category with t-structure compatible with the symmetric monoidal structure, and let $\mathcal{O}$ be a unital operad in spaces. Here we prove some connectivity statements that are helpful in studying the deformation theory of $\mathcal{O}$-algebras. The only result directly used in the main part of the paper is Lemma 5.8.43, which we use to trivialize powers of locally unipotent automorphisms of π-finite connective $\mathbb{E}_1 \otimes \mathbb{A}_2$-algebras.

Lemma 5.8.36. *Let $A, B \in \mathcal{C}^{\Delta^1 \times \Delta^1}$ be two squares of connective objects such that the total fiber is k_1+k_2-connective, the horizontal morphisms are k_1-connective, the vertical morphisms are k_2-connective. Then $A \otimes B \in \mathcal{C}^{\Delta^1 \times \Delta^1}$ is also of this form.*

Proof. We can expand $A \otimes B$ as the composite of squares

$$\begin{tikzcd}
A_{00} \otimes B_{00} \arrow[r] \arrow[d] & A_{01} \otimes B_{00} \arrow[r] \arrow[d] & A_{01} \otimes B_{01} \arrow[d] \\
A_{10} \otimes B_{00} \arrow[r] \arrow[d] & A_{11} \otimes B_{00} \arrow[r] \arrow[d] & A_{11} \otimes B_{01} \arrow[d] \\
A_{10} \otimes B_{10} \arrow[r] & A_{11} \otimes B_{10} \arrow[r] & A_{11} \otimes B_{11}
\end{tikzcd}$$

From this it is easy to verify that the total fiber of each square is $k_1 + k_2$-connective by the assumptions, and that the horizontal and vertical arrows above are k_1, k_2-connective respectively. These then imply the claims about $A \otimes B$. $\square$

The following is a version of the Blakers–Massey theorem for $\mathcal{O}$-algebras that establishes a stable range:

Lemma 5.8.37. *Suppose we are given a pushout square*

$$\begin{tikzcd}
A \arrow[r, "b"] \arrow[d, "c"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

of connective $\mathcal{O}$-algebras in $\mathcal{C}$ such that the maps b, c are k_1, k_2 connective respectively. Then the natural map $A \to B \times_D C$ is $k_1 + k_2$-connective and the map $B \coprod_A C \to D$ is $k_1 + k_2 + 1$-connective, where $\coprod_A$ denotes the pushout along A in $\mathcal{C}$ (as opposed to $\operatorname{Alg}_{\mathcal{O}}\mathcal{C}$).

Proof. We first note that the free $\mathcal{O}$-algebra functor takes k-connective maps of connective objects to k-connective maps, and geometric realization commutes with pullbacks and preserve connectedness. Thus, by taking the monadic resolution of the diagram and geometric realizing, we can assume that the diagram is obtained from applying the free functor to a pushout square

$$\begin{tikzcd}
W \arrow[r, "x"] \arrow[d, "y"] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

in $\mathcal{C}_{\geq 0}$ with x, y k_1, k_2-connective respectively.

The total fiber of the square then breaks up into a direct sum over i of the functor $\mathcal{O}(i) \otimes_{\Sigma_i} (-)$ (which is exact and preserves connectivity) applied to the total fibers of the square

$$\begin{tikzcd}
W^{\otimes i} \arrow[r] \arrow[d] & X^{\otimes i} \arrow[d] \\
Y^{\otimes i} \arrow[r] & Z^{\otimes i}
\end{tikzcd}$$

which are is $k_1 + k_2$ by iteratively applying Lemma 5.8.36.

The last statement about $k_1 + k_2 + 1$-connectivity follows from this since the cofiber of the map $B \coprod_A C \to D$ is the total cofiber of the square, which is the double suspension of the total fiber. □

Corollary 5.8.38. *Let R be a connective $\mathcal{O}$-algebra in $\mathcal{C}$. Let $R' \to R$ be an m-connective map of $\mathcal{O}$-algebras for $m \geq 0$. Consider the pushout square in* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & SR'
\end{tikzcd}$$

Then the induced map $R' \to \omega SR' := R \times_{SR'} R$ is $2m$-connective.

We next prove a version of [Lur17, Theorem 7.4.1.23] that holds for $\mathcal{O}$-algebras.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,b]}$ denote the full subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R}$ consisting of maps $f : R' \to R$ such that $\mathrm{fib}\, f$ is $[a, b]$ in the t-structure.

Lemma 5.8.39. *Let $m \geq 0$. Suppose $R = \tau_{\leq m} R$ is a connective $\mathcal{O}$-algebra in $\mathcal{C}$. For each a, b with $m \leq a, b \leq 2a - 1$, the trivial square-zero functor $\Omega^\infty : \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R}) \to (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R})_*$ restricts to an equivalence onto $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,b]})_*$ when restricted to those objects whose image lies in that subcategory.*

Moreover the functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[m,2m-1]} \to (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[m+1,2m]})_{R \coprod R/}$$

sending $R' \to R$ to $\tau_{\leq 2m} SR'$ equipped with the two sections from R is an equivalence.

Proof. The functor Ω^∞ when restricted to objects whose image is in $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,b]})_*$, is the projection of the inverse limit

$$\ldots(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a+2,b+2]})_* \xrightarrow{\Omega} (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a+1,b+1]})_* \xrightarrow{\Omega} (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,b]})_*$$

Note that $[a+1,b+1]$ satisfies the inequalities in the hypothesis if $[a,b]$ does. We claim that the composite

$$\tau_{\leq b+1}\Sigma : (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,b]})_* \to (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a,\infty]})_* \xrightarrow{\tau_{\leq b+1}\Sigma} (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[a+1,b+1]})_*$$

is an inverse to Ω. Indeed, the natural map $\mathrm{id} \to \Omega\tau_{\leq b+1}\Sigma$ is an equivalence by applying Corollary 5.8.38, so that $\tau_{\leq b+1}\Sigma$ is a fully faithful left adjoint. But its right adjoint Ω is conservative, so this is an adjoint equivalence.

The second statement is similar, namely it follows from Corollary 5.8.38 that ω gives an inverse to the functor $\tau_{\leq 2m}S$. □

The following lemma gives us control on the Postnikov tower of a connective $\mathcal{O}$-algebra.

Lemma 5.8.40. *Let $m \geq 0$ and suppose $R = \tau_{\leq m}R$ is a connective $\mathcal{O}$-algebra in $\mathcal{C}$. Then the functor $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[m+1,m+1]})_* \to \Sigma^{m+1}\mathcal{C}^\heartsuit$ sending an algebra to the fiber of the augmentation is faithful.*

Proof. First let R be any connective $\mathcal{O}$-algebra, and let $R \oplus M \to R$ be a trivial square-zero extension with $M \in \Sigma^{m+1}\mathcal{C}^\heartsuit$.

We claim that the following square becomes a pushout square in $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R})_*$ after applying $\tau_{\leq m+1}$, and that the vertical maps are $m+1$-connective.

$$\begin{tikzcd}
R \coprod \mathbb{1}\{\mathrm{fib}\,\epsilon\} \arrow[r] \arrow[d] & R \coprod \mathbb{1}\{M\} \arrow[d, "\epsilon"] \\
R \arrow[r] & R \oplus M
\end{tikzcd}$$

All functors involved commute with sifted colimits, so by resolving R by free augmented algebras via the monadic resolution, we can reduce to the case of $R = \mathbb{1}\{X_i\}$.

Now the right vertical map breaks into homogeneous components, which in degree $0, 1$ are isomorphisms, and in degrees $j \geq 2$ is the $m+1$-connective map $(X_i \oplus M)^{\otimes j} \to X_i^{\otimes j}$ composed with the connectivity preserving exact functor $\mathcal{O}(j)\otimes_{\Sigma_j}$. It follows that the left vertical map is also $m+1$-connective since it is a sum of the maps $(X_i \oplus \mathrm{fib}\,\epsilon)^{\otimes j} \to (X_i)^{\otimes j}$ composed with $\mathcal{O}(j)\otimes_{\Sigma_j}$. The pushout in $\mathcal{O}$-algebras maps to the pushout in $\mathcal{C}$, and this map is $2m+2$-connective using Lemma 5.8.37, so in particular induces an equivalence after applying $\tau_{\leq m+1}$.

Thus it suffices to see that the diagram is a pushout in $\mathcal{C}$ after applying $\tau_{\leq m+1}$, which amounts to observing the following exact sequence:

$$\pi^\heartsuit_{m+1}(R \coprod \mathbb{1}\{\mathrm{fib}\,\epsilon\}) \to \pi^\heartsuit_{m+1}(R \coprod \mathbb{1}\{M\}) \to \pi^\heartsuit_{m+1}M \to 0$$

Indeed, by construction, the composite of the first two maps is 0 the second map is an epimorphism with kernel generated by $\mathrm{fib}\,\epsilon$, proving the exact sequence.

We finish the proof by observing that in the situation of the lemma, the pushout square above shows that the map $\tau_{\leq m+1}(R \coprod \mathbb{1}\{M\}) \to R \oplus M$ is an epimorphism in

$$(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/R,[m+1,m+1]})_*$$

, which means that maps out of $R \oplus M$ are determined by their maps on the underlying object in $\mathcal{C}$. ☐

We have enough tools now to do basic deformation theory of $\mathcal{O}$-algebras.

Definition 5.8.41. We define the category of $\mathcal{O}$-modules of an $\mathcal{O}$-algebra B to be

$$\mathrm{Mod}_{\mathcal{O}}(B) := \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/B})$$

. We define L_B to be the image of B under $\Sigma^\infty_B : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/B} \to \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/B})$. ◁

Proposition 5.8.42. *Let R be a connective unital $\mathcal{O}$-algebra in $\mathcal{C}$ with an automorphism ϕ that is the identity on $\pi_*^{\heartsuit}$. Suppose that $p^k = 0$ on $\pi_*^{\heartsuit} R$ and that $R = \tau_{\leq m} R$. Then $\phi^{p^{km}}$ is equivalent to the identity.*

Proof. We prove the result inductively on m. For $m = 0$, the result is clear. For the inductive step, by replacing ϕ with $\phi^{p^{k(m-1)}}$, we can assume that ϕ is the identity on $\tau_{\leq m-1} R$, and it suffices to show that ϕ^{p^k} is equivalent to the identity on R.

Let $B = \tau_{\leq m-1} R$. By applying Lemma 5.8.39, since $R \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/B,[m,2m-1]}$, we learn that R is canonically a square-zero extension of B by $\Sigma^n \pi_n R \in \mathrm{Mod}_{\mathcal{O}}(B) := \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/B})$. This means that its group of automorphisms can be computed in the category $\mathrm{Mod}_{\mathcal{O}}(B)_{L_B/}$. By considering the conservative forgetful functor $\mathrm{Mod}_{\mathcal{O}}(B)_{/L_B} \to \mathrm{Mod}_{\mathcal{O}}(B)$, we obtain that the automorphism group fits into a fiber sequence of groups

$$\Omega\,\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}(B)}(L_B, \Sigma^{m+1}\pi_m R) \to \mathrm{Aut}_{\mathrm{Mod}_{\mathcal{O}}(B)}(B) \to \mathrm{Aut}_{\mathrm{Mod}_{\mathcal{O}}(B)}(\Sigma^{m+1}\pi_m R)$$

By Lemma 5.8.40, ϕ has trivial image under the first map since it acts trivially on $\pi_n R$. Thus it suffices to see that $p^k = 0$ in $\Omega\,\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}(B)}(L_B, \Sigma^{m+1}\pi_n R)$. But this mapping spectrum is a module over the endomorphism ring of $\Sigma^{m+1}\pi_n R$ in $\mathrm{Mod}_{\mathcal{O}}(B)$, which has $p^k = 0$ using Lemma 5.8.40 again and the fact that $p^k = 0$ on $\pi_m R$. ☐

Lemma 5.8.43. *Let $R \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{B\mathbb{Z},u})$ be connective, almost compact, bounded, and p-nilpotent. Then the $p^k\mathbb{Z}$ action on R is trivializable for all $k \gg 0$.*

Proof. Since R is almost compact and bounded, applying Lemma 5.8.34, we learn that the $p^k\mathbb{Z}$-action on R is trivializable as an action in $\mathcal{C}$ for $k \gg 0$. Since R is bounded, connective, and p-nilpotent, by applying Proposition 5.8.42, we learn that after possibly increasing k, the action becomes trivializable as an $\mathcal{O}$-algebra. ☐

Chapter 6

Forthcoming work

To conclude this thesis, we describe the results of projects related to the topics of this thesis that have not been completed yet, and describe some open questions. We also point the reader to Section 3.7, which also has some open problems.

6.1 The K-theoretic telescope conjecture away from p (with Robert Burklund)

In this project, we describe how to compute the K-theory of categories related to the chromatic filtration after inverting the prime p. In particular, they end up being fairly simply described in terms of the algebraic K-theory of discrete rings.

The main theorem of this project is the following:

Theorem 6.1.1. *Let $\mathcal{U}_{add}$ be the universal filtered colimit preserving additive invariant. Then the map*

$$\mathcal{U}_{add}(\mathrm{Sp}^{\omega}_{T(n)})[\frac{1}{p}] \to \mathcal{U}_{add}(\mathrm{Mod}(\mathrm{MU})^{\omega}_{T(n)})[\frac{1}{p}]$$

admits a retraction.

The proof of Theorem 6.1.1 proceeds by interpolating between the sphere and MU using Ravenel's $X(n)$ filtration, and showing that each map

$$\mathcal{U}_{\mathrm{add}}(\mathrm{Mod}(X(n-1))^{\omega}_{T(n)})[\frac{1}{p}] \to \mathcal{U}_{\mathrm{add}}(\mathrm{Mod}(X(n))^{\omega}_{T(n)}[\frac{1}{p}]$$

admits a retraction. Results of Robert Burklund show that compact $T(n)$-local modules over $X(n)$ admit $X(n-1)$-linear self maps whose cofiber is a compact $T(n)$-local $X(n-1)$-module. These self maps are unique up to pth powers similarly to v_n-self maps [HS98], and we use these to construct the desired retraction.

Since MU satisfies the telescope conjecture, the following is an immediate consequence of Theorem 6.1.1, which we dub the K-theoretic telescope conjecture away from p:

Corollary 6.1.2. *The natural map* $\mathcal{U}_{add}(\mathrm{Sp}^{\omega}_{T(n)})[\frac{1}{p}] \to \mathcal{U}_{add}(\mathrm{Sp}^{\omega}_{K(n)})[\frac{1}{p}]$ *is an equivalence.*

Since the homotopy ring of MU is regular, it is not difficult to use the results of Chapter 2 and truncating-ness of p-inverted K-theory on p-nilpotent connective rings to show that

$$K(\mathrm{Mod}(\mathrm{MU})^{\omega}_{T(n)})[\frac{1}{p}] \cong (K(\mathbb{F}_p) \oplus \Sigma K(\mathbb{F}_p))[\frac{1}{p}]$$

We know that $K(\mathrm{Sp}^{\omega}_{T(n)})[\frac{1}{p}]$ is a retract of this from Theorem 6.1.1, but it is in fact a $K(\mathbb{F}_p)[\frac{1}{p}]$-module retract due to the following result:

Theorem 6.1.3. *Suppose that* $\mathcal{C}$ *is a stable category such that* $\mathcal{C}[\frac{1}{p}] = 0$*. Then* $K(\mathcal{C})[\frac{1}{p}]$ *canonically has the structure of a* $K(\mathbb{F}_p)[\frac{1}{p}]$*-module.*

Using the above results, it is fairly straightforward to conclude that $K(\mathrm{Sp}^{\omega}_{T(n)})[\frac{1}{p}]$ and $K(\mathrm{Mod}(\mathrm{MU})^{\omega}_{T(n)})[\frac{1}{p}]$ agree. Using the localization sequences relating these to $K(L_n^f\mathbb{S})[\frac{1}{p}]$, and $K(\mathrm{Sp}_{\geq n})$, we obtain the following result, which amounts to a complete description of the K-theory of the chromatic filtration after inverting p.

Theorem 6.1.4. *There are equivalences for* $n \geq 1$*,*

$$K(L_n^f\mathbb{S}_p)[\frac{1}{p}] \cong K(L_n\mathbb{S}_p)[\frac{1}{p}] \cong (K(\mathbb{Z}_p) \oplus \Sigma K(\mathbb{F}_p))[\frac{1}{p}]$$

$$K(\mathrm{Sp}_{T(n)})[\frac{1}{p}] \cong K(\mathrm{Sp}_{K(n)})[\frac{1}{p}] \cong (K(\mathbb{F}_p) \oplus \Sigma K(\mathbb{F}_p))[\frac{1}{p}]$$

$$K(\mathrm{Sp}_{\geq n})[\frac{1}{p}] \cong K(\mathbb{F}_p)[\frac{1}{p}]$$

6.2 Localizing invariants for $-n$-connective heart categories (with Vova Sosnilo)

In this project, we propose a general setting in which tools in stable homotopy theory and homological algebra that work in connective situations should extend, which behave $-n$-connective. Specifically, we define the notion of a $-n$-connective heart category for $n \geq 1$, and prove that its universal localizing invariant is isomorphic up to a $-n$-fold suspension, to that of a category of modules over a connective ring. This isomorphism is suitably functorial, so that it allows one to extend the notion of truncating invariants to this setting, and prove analogs of the Dundas–Goodwillie–McCarthy theorem.

This can be viewed as a generalization of the results of Chapter 3 about -1-connective rings, which corresponds to the case $n = 1$. The results here are used in the project below to obtain formulas for the algebraic K-theory of the chromatic filtration in terms of TC.

The key definition is the notion of a $-n$-connective heart category, which is a certain kind of bounded heart category in the sense of Saunier [Sau23].

Definition 6.2.1. A $-n$-connective heart category is a stable category $\mathcal{C}$ with two subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ such that:

1. $\mathcal{C}_{\geq 0}$ is closed under finite colimits and extensions, and $\mathcal{C}_{\leq 0}$ is closed under finite limits and extensions.
2. For each $x \in \mathcal{C}$ there is a cofiber sequence $x_{\leq 0} \to x \to x_{>0}$ where $x_{\leq 0} \in \mathcal{C}_{\leq 0}$ and $\Sigma^{-1}x_{>0} \in \mathcal{C}_{\geq 0}$.
3. $\mathcal{C} = \bigcup_i \Sigma^{-i}\mathcal{C}_{\geq 0} = \bigcup_i \Sigma^{i}\mathcal{C}_{\leq 0}$
4. For $x \in \mathcal{C}_{\leq 0}, y \in \mathcal{C}_{\geq 0}$, $\mathrm{map}(x, y)$ is $-n$-connective.

◁

(1) + (2) of the above definition is Saunier's notion of a heart structure, and (3) is the condition that this heart structure is bounded. (4) can be viewed as a weakening of the orthogonality condition of a weight structure in the sense of Bondarko [Bon10]. Indeed, in the case $n = 0$, Definition 6.2.1 agrees exactly with the notion of a bounded weight structure.

Key examples of categories with $-n$-connective heart structures include: coherent sheaves on qc quasi-affine schemes, and on quotient stacks of such schemes by actions of pro-unipotent groups schemes of finite cohomological dimension.

Given a $-n$-connective heart structure, we can produce a t-structure on $\mathrm{Ind}(\mathcal{C})$ such that the connective objects are those objects X such that $\mathrm{map}(y, X)$ is connective for $y \in \mathcal{C}_{\leq 0}$. We say that a functor $\mathcal{C} \to \mathcal{D}$ is *resolvable* if $F(\mathcal{C}_{\geq 0}) \subset \mathcal{D}_{\geq 0}$, $F(\mathcal{C}_{\leq 0}) \subset \mathcal{D}_{\leq 0}$, and $\mathrm{Ind}(F)$ is right t-exact.

We prove the following theorem:

Theorem 6.2.2. *Given a $-n$-connective heart category $\mathcal{C}$, there is a boundedly weighted category $\mathcal{C}_n$ equipped with an isomorphism $\eta_n(\mathcal{C}) : \Sigma^n\mathcal{U}_{loc}(\mathcal{C}) \cong \mathcal{U}_{loc}(\mathcal{C}_n)$.*

The assignment $\mathcal{C} \mapsto \mathcal{C}_n$ can be made functorial in resolvable morphisms for all $\mathcal{C}$ which are κ-small for a regular cardinal κ, and η_n can be made into a natural transformation. Moreover, $\mathcal{C}_n$ is generated by a single object.

The category $\mathcal{C}_n$, as a boundedly weighted category generated by a single object, is equivalent to the perfect module category of some connective ring R. Theorem 6.2.2 can be read as saying that understanding localizing invariants on $-n$-connective heart categories can be reduced to understanding localizing invariants for connective rings. This allows many tools, such as truncating invariants, to be transported from the setting of connective rings to that of $-n$-connective heart categories.

Recall that a truncating invariant E can be defined as a localizing invariant such that $E(R) \to E(S)$ is an equivalence whenever $R \to S$ is a map of connective rings that is surjective on π_0 with nilpotent kernel. [1]

---

[1] In fact this is not the usual definition, but is equivalent to the usual one via [LT19, Theorem B]

We can extend the collection of maps for which truncating invariants are isomorphisms via the notion of a nilpotent extension of $-n$-connective heart categories. For $\mathcal{C}$ a $-n$-connective heart category, we put a t-structure on $\mathrm{End}_{\mathrm{Pr}^L} \mathrm{Ind}(\mathcal{C})$ by declaring the connective objects to be the right t-exact endofunctors.

Definition 6.2.3. We say that a resolvable functor $F : \mathcal{C} \to \mathcal{D}$ between $-n$-connective heart categories is a *nilpotent extension* if

- $F(\mathcal{C}_{\leq 0})$ generates $\mathcal{D}_{\leq 0}$ under finite limits and extensions.
- $F(\mathcal{C}_{\geq 0})$ generates $\mathcal{D}_{\geq 0}$ under finite colimits and extensions.
- The unit map of the adjunction $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \to \mathrm{Ind}(F)^R \mathrm{Ind}(F)$ is a surjection on $\pi_0^{\heartsuit}$ with kernel a nilpotent ideal.

$\triangleleft$

In the case $n = 0$, this agrees with the notion of nilpotent extension of [ES22].

We prove the following result:

Theorem 6.2.4. *The functor $(-)_n$ of Theorem 6.2.2 takes nilpotent extensions to nilpotent extensions, i.e maps induced from maps $R \to S$ of connective rings which are surjective on π_0 with nilpotent kernel. In particular, any truncating invariant E sends such a nilpotent extension to an equivalence.*

For example, because the fiber of the cyclotomic trace $\mathrm{fib}(K \to \mathrm{TC})$ is a truncating invariant, the above theorem often allows one to apply trace methods to compute the algebraic K-theory of such categories in terms of topological cyclic homology, and the K-theory of classical exact categories in the sense of Quillen [Qui73]. The latter can often be reduced to understanding K-theory of discrete rings via devissage.

6.3 The algebraic K-theory of the chromatic filtration via TC

In this project, we use the results from Section 6.2 to obtain formulas for the algebraic K-theory of the L_n and $K(n)$-local categories in terms of topological cyclic homology.

We construct a category $A_n = \mathrm{Syn}_{\mathrm{MU}}^{\mathrm{flat}}(L_{K(n)}\mathrm{Sp})^{\omega}$ of compact 'flat' MU-synthetic $K(n)$-local spectra, which consists of objects having a horizontal vanishing line on their synthetic homotopy groups.

We show that A_n admits the structure of a $-n^2 - n$-connective heart category, and show that the map $A_n \to \mathrm{Mod}_{\mathrm{cof}\,\tau}(A_n)$ is a nilpotent extension in the sense of Definition 6.2.3. We have a localization sequence

$$A_n^{\tau-\mathrm{nilp}} \to A_n \to \mathrm{Sp}_{K(n)}$$

and prove an identification $K(A_n^{\tau-\mathrm{nilp}}) \cong K(\mathrm{Mod}_{\mathrm{cof}\,\tau}(A_n)) \cong \oplus_0^{2p^n-3} K(\mathbb{F}_p)$ coming from devissage. All together we use these to obtain the following theorem:

Theorem 6.3.1. *There are fiber sequence for $n \geq 1$:*

$$F \to K(L_{K(n)}\mathrm{Sp}^\omega) \to K(\mathbb{F}_p) \oplus \Sigma K(\mathbb{F}_p)$$
$$F \to \mathrm{TC}(A_n) \to \mathrm{TC}(\mathrm{Mod}_{\mathrm{cof}\,\tau}(A_n))$$

We similarly obtain a result for the L_n-local categories, where A_n' is a similarly defined category of L_n-local synthetic spectra:

Theorem 6.3.2. *There are fiber sequence for $n \geq 1$:*

$$F \to K(L_n\mathrm{Sp}_p^\omega) \to K(\mathbb{Z}_p) \oplus \Sigma K(\mathbb{F}_p)$$
$$F \to \mathrm{TC}(A_n') \to \mathrm{TC}(\mathrm{Mod}_{\mathrm{cof}\,\tau}(A_n'))$$

Upon inverting the prime p, the terms F in the above two theorems vanish, showing compatibility with the computation of Theorem 6.1.4.

6.4 Further questions

There are many important questions related to the K-theory of the chromatic filtration that remain unanswered. We list here a few of them:

Question 6.4.1. Is there a way to computationally access the integral algebraic K-theory of the $T(n)$-local category?

The above question can be viewed equivalently as asking whether we can access the algebraic K-theory of the category $\mathrm{Sp}^\omega_{\geq n}$ of type $\geq n$ spectra for arbitrary n.

Question 6.4.2. To what extent do the algebraic K-theory spectra of $K(n)$-local ring spectra detect $T(n)$-local information? For example, is $(-) \otimes_{T(n+1)} L_{T(n+1)}K(L_{K(n)}\mathbb{S})$ a conservative functor on $\mathrm{Sp}_{T(n)}$? What about $(-) \otimes_{T(n+1)} L_{T(n+1)}K(L_{T(n)}\mathbb{S})$?

A positive answer to the above question would suggest that K-theory can be used as a tool to completely probe the $T(n)$-local categories.

The Lichtenbaum–Quillen property played a key role in being able to access the telescopic algebraic K-theory of chromatically ring spectra. We offer a conjecture which relates this property to another notion of regularity, as considered in Chapter 2.

Conjecture 6.4.3 (LQ from regularity)**.** *Let R be a connective $\mathbb{E}_2$-ring such that $\pi_0 R$ is commutative and smooth over a DVR with perfect residue field. If the standard t-structure on $\mathrm{Mod}(R)$ restricts to a bounded t-structure on $\mathrm{Mod}(R)^\omega \otimes \mathrm{Sp}^\omega_{\geq n+1}$ for $n \geq -1$, then R satisfies the height n Lichtenbaum–Quillen property.*

In the case $n \leq 0$, the result is known to be true, and follows from a result of Hesselholt and Madsen [HM03; HM04]. Beyond this case, the result is not known.

There are certainly stronger statements than Conjecture 6.4.3 that one could expect to be true, but we have opted in favor of making a statement that seems very likely to be true rather than one that is the most general. In particular, in light of the LQ results of Chapter 5, it seems likely that the connectivity hypothesis can be weakened to asking for a stable category with a $-n$-connective heart structure in the sense of Definition 6.2.1. We also believe that $\mathbb{E}_1$ should also suffice in Conjecture 6.4.3. However, the condition on $\pi_0 R$ must be replaced with an appropriate (mild) finiteness property, which we are not sure how to formulate.

Ausoni–Rognes conjectured that $K(R)$ should be fp type $n+1$ in the sense of Mahowald–Rezk when R is fp type n. One condition missing from their condition is a regularity condition: there are easy counterexamples such as $R = \mathbb{F}_p\langle \epsilon_0 \rangle$ otherwise. Furthermore, the results of Chapter 3 show that one also needs connectivity of R for the result to hold. We offer the following version of their conjecture:

Conjecture 6.4.4 (Finiteness)**.** *Let R be a connective $\mathbb{E}_1$-ring of fp type n such that the standard t-structure on* $\mathrm{Mod}(R)$ *restricts to a bounded t-structure on* $\mathrm{Mod}(R)^\omega \otimes \mathrm{Sp}^\omega_{\geq n+1}$ *for $n \geq -1$. Then* $\mathrm{TC}(R)$ *is fp type $n+1$.*

We note that Conjecture 6.4.4 and Conjecture 6.4.3 are closely related: if the conclusion of Conjecture 6.4.3 holds for a ring R as in Conjecture 6.4.4, then this and the fact that R is fp type n can be used to prove the conclusion of Conjecture 6.4.4.

Finally, since by Chapter 5, K-theory is a situation in which one can systematically do computations of telescopic homotopy groups, we wonder whether there is a way to systematically leverage this to be able to learn about the $T(n)$-local category. One way to do this would be to have a well behaved Adams filtration on TC, analogous to the motivic filtration of [HRW22].

Question 6.4.5. Is there a well–behaved, natural, and computable filtration on TC of ring spectra which naturally obtains a map from the Adams filtration on the sphere spectrum?

The above question is not precisely phrased, but a convincing positive answer would be a filtration that for example allows one to learn about differentials in the localized Adams spectral sequence of a telescope via comparison to TC.

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