ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW ALTITUDE SATELLITES

by

Andrew Joseph Green

A.S., Mohawk Valley Community College
(1970)

B.S., United States Military Academy
(1974)

M.S., Massachusetts Institute of Technology
(1977)

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Signature of Author _____
Department of Aeronautics and Astronautics
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Approved by _____
Chairman of Thesis Committee

Approved by _____
Thesis Advisor

Approved by _____
Thesis Advisor

Approved by _____
Thesis Advisor

Accepted by _____
Chairman, Departmental Graduate Committee

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ABSTRACT

This thesis combines the semianalytical satellite theory with current estimation procedures such that orbit determination and prediction processes for low altitude satellites are improved. A general formulation for the first order short periodic variations due to any type of perturbation, conservative or non-conservative, is presented. The short periodic formulation is given in terms of a Fourier series in the mean-mean longitude and permits both analytical and numerical implementation. Based upon the first order short periodic formulation, the averaged equations of motion are rigorously extended to second order for both single and multiple perturbing forces. This formalism is applied to the satellite orbit perturbed by both oblateness and atmospheric drag. The resulting second order drag theory includes a complete treatment of the oblateness/drag coupling that retains the flexibility of the numerical averaging approach, that is, an arbitrarily complex density model may be included. A semianalytical satellite theory for the partial derivatives of perturbed motion is presented. The semianalytical partial derivatives are composed of averaged partial derivatives and short periodic partial derivatives. The semianalytical satellite theory is combined with a batch filter (least squares differential correction algorithm) such that the epoch mean elements and various parameters within the drag formulation can be determined. The solve-for parameters in the drag formulation include the drag coefficient and parameters within an Adaptive Modified Harris-Priester density model. An algorithmic structure for coupling the semianalytical satellite theory with the Kalman filter is also presented. Numerical results for several test cases are presented. It is found that the semianalytical satellite theory can be used to accurately determine and predict satellite ephemerides.

Thesis Supervisor: Walter M. Hollister
Title: Associate Professor of Aeronautics and Astronautics

Thesis Supervisor: John P. Vinti
Title: Lecturer of Aeronautics and Astronautics

Thesis Supervisor: Richard H. Battin
Title: Adjunct Professor of Aeronautics and Astronautics
Associate Department Head, C. S. Draper Laboratory

Thesis Supervisor: Paul J. Cefola
Title: Section Manager, C. S. Draper Laboratory

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TABLE OF CONTENTS

TABLE OF CONTENTS

(cont.)

TABLE OF CONTENTS

(cont.)

Chapter 1

INTRODUCTION

1.1 Background

The Space Age is entering an era in which satellites will be utilized increasingly by the civilian and military sectors. The civilian sector will utilize satellites for both scientific and commercial endeavors, while in the military sector, satellites will become an integral part of the national defense. In order to meet the needs of both sectors, the number and quality of space systems will steadily increase. Space surveillance and the determination of accurate ephemerides are essential for the success of individual space missions and for national security.

Satellite ephemerides are determined by an orbit determination (OD) system. In space surveillance applications, the OD system must be capable of accurately and efficiently generating the ephemerides of a large number of satellites. A high percentage of these satellites are in the near-Earth environment due to the desirability and necessity of satellite missions within this regime. Low altitude satellites are predominantly perturbed by the non-sphericity of the Earth's gravitational field and by atmospheric drag. Failure to accurately account for these perturbations on near-Earth satellites can

significantly degrade the orbit determination and prediction process.

OD systems in operation today utilize special perturbation satellite theories (SP theories) or general perturbation satellite theories (GP theories) for orbit generation. SP theories involve numerical integration of the equations of motion expressed either in Cowell, Encke or Variation of Parameter (VOP) format. SP theories allow flexibility in considering the number and type of disturbing forces while maintaining the desired accuracy. The major disadvantage with SP theories is that they are usually computationally expensive. GP theories permit analytical solutions to the equations of motion that are usually based upon canonical transformation theory. Existing GP theories usually employ restricted force models and therefore are not as accurate as SP theories. GP theories are generally computationally less costly than SP theories.

Drag perturbed satellites are frequently handled with SP theories in operational OD systems because of the inability of analytical methods to accurately handle this perturbation. SP theories will permit an accurate handling of the drag perturbation but will significantly increase the computational costs. An additional advantage of SP theories is that upper atmosphere density models can be changed very easily as new models are determined or old models updated. GP theories do

exist which can handle atmospheric drag. However, they are based upon restrictive density models that can vary significantly from the true density of the upper atmosphere. In order to change density models in a GP theory, an almost complete reformulation of the analytical solution commonly is required.

All OD systems use tracking data (usually from ground based radars) in the orbit determination and prediction process. This is done by use of a batch filter (least squares differential correction algorithm) or a sequential filter (Kalman filter). The batch filter takes all the tracking data over a certain observation span and determines the best estimate of the epoch conditions. The sequential filter determines the best estimate of the current conditions from the tracking data by processing the observations recursively. Knowing the best estimate for either the epoch conditions (batch filter) or the current conditions (sequential filter), the orbit generator can then be used to predict the satellite ephemeris. Some of the variables that can be solved for in the estimation process are the state vectors (position and velocity or an orbital element set), force model parameters, tracking station locations, measurement bias, time bias, etc. The state vector usually solved for in current OD systems is the position and velocity of the spacecraft with respect to an inertial Cartesian coordinate frame (the estimation process is coupled with a SP theory). In the drag perturbed case, the only drag parameter

that is usually solved for in operational OD systems is a multiplicative factor in the drag force, i.e., the drag coefficient or ballistic coefficient. Due to the many modeling errors associated with the drag formulation, filtering (batch or sequential) becomes a very important part of any OD system designed for near-Earth satellites.

Operating procedures for the major OD systems are basically similar. Certain throughput and accuracy requirements must be met. In order to meet these requirements, certain trade-offs are made between accuracy and computer run time. This is typically done by assigning priorities to the various satellites whose ephemerides must be updated. A precision SP theory would be used on satellites with the highest priorities. Then, depending upon the assigned priority, varying degrees of a SP or GP theory would be used upon the other satellites such that all requirements are met. This is especially true for a space computation center which must keep track of very large numbers of space objects.

It is noted that a major portion of any OD system is the satellite theory used for orbit generation. The orbit generator is used in both the estimation and prediction process. An orbit generator, which accurately and efficiently handles all perturbations, is essential to the performance of an OD system.

1.2 Problem Description

Because of the need for space surveillance and accurate ephemeris generation on an ever-increasing number of satellites, it becomes necessary to enhance the throughput requirements of present-day OD systems while maintaining equivalent or even increased accuracy requirements. In order to meet these future requirements, changes will have to be made to the computer hardware and/or orbit generators of current OD systems. Advances in computer hardware may eventually allow OD systems to employ precision SP methods to all satellites and still allow the computer run times to be dramatically decreased (Ref. 1). However, there are many system difficulties and large software costs associated with implementing major computer hardware changes (for example, see Ref. 2). It therefore appears that developing improved orbit generators, which are compatible with current computer hardware and system constraints, is a viable means of improving the performance of current OD systems. Because of the large number of near-Earth satellites and the importance placed upon these satellites, the improved OD system must be able to accurately and efficiently generate the ephemerides of low altitude satellites.

A satellite theory which has the potential to be an improved orbit generator is the semianalytical satellite theory as described in References 3 through 15. Semianalytical satellite theory is based upon the generalized method of averaging

which basically separates the short periodic motion from the long periodic and secular motion. Canonical perturbation theory, which is used in most GP theories, is a specific case of the generalized method of averaging (Ref. 16). Semianalytical satellite theory makes a transformation from osculating element space to mean element space. The transformation equations involve short periodic functions associated with each disturbing force. The mean orbital element rates contain only long periodic and secular motion. Therefore, a much larger numerical integration time step can be used on the mean equations of motion (typically 1 day) than can be used with the osculating equations of motion. The basic procedure for a semianalytical satellite theory is to transform to mean element space, numerically integrate the equations of motion for the mean elements and then transform back to osculating element space. Semianalytical satellite theory has the potential of greatly increasing the speed of current OD systems while retaining accuracy equivalent to SP methods.

This research is therefore concerned with examining semianalytical satellite theory and combining it with current estimation procedures in order to improve orbit determination and prediction processes for low altitude satellites.

The development of a complete semianalytical OD system can be broken into two phases. The first phase involves making

enhancements to the semianalytical satellite theory such that it can accurately and efficiently handle all perturbations. The second phase of the development is the coupling of the semianalytical satellite theory with current estimation procedures.

This study is basically concerned with low altitude satellites. As previously indicated, the major perturbations experienced by a near-Earth satellite are the non-sphericity of the Earth's gravitational field and atmospheric drag. If semianalytical satellite theory is to be applicable for low altitude satellites, it must be able to handle these perturbations. The effects of the non-sphericity of the Earth's gravitational field (conservative force) has been considered successfully in previous investigations (see Section 1.3). Therefore, the present investigation is mainly interested in making enhancements to the semianalytical satellite theory such that atmospheric drag can be handled accurately and efficiently.

Drag is modeled by the following formula

$$\underline{D} = - B \rho(\underline{r},t) V_{rel} \underline{V}_{rel} \qquad (1\text{-}1)$$

where

$\underline{D}$ = perturbing acceleration due to drag

$$B = \frac{C_D A}{2m} \equiv \text{ballistic coefficient}$$

C_D = drag coefficient

A = projected area of the satellite

m = mass of the satellite

$\rho(\underline{r},t)$ = density as a function of position and time

$\underline{V}_{rel}$ = relative velocity vector = $\underline{V} - \underline{V}_a$

$\underline{V}$ = true (inertial) velocity of the satellite

$\underline{V}_a$ = velocity of the atmosphere

The ballistic coefficient, velocity profile of the upper atmosphere and the density of the upper atmosphere are potential sources of errors in the drag formulation for OD systems (Ref. 17).

The determination of the ballistic coefficient entails knowing the projected area and drag coefficient of the satellite. Both of these parameters vary as the attitude and geometry of the satellite vary. The drag coefficient also varies somewhat with altitude. The drag coefficient usually has to be determined experimentally. Since it is impossible to reproduce the upper atmospheric conditions in a wind tunnel, the determination of C_D is not very accurate. C_D is usually taken to be between 1.9 and 2.5 (Ref. 17). It must therefore be concluded that the ballistic coefficient is an averaged value

that can be a serious source of error in trying to accurately predict the orbits of drag perturbed satellites.

The velocity profile of the upper atmosphere is generally modeled as if the upper atmosphere rotates with an angular rate equal to Earth rate. Various studies have shown that the actual upper atmosphere approximately rotates with the Earth and has both east-west and north-south winds (Ref. 17). King-Hele has shown that these winds have daily and seasonal variations (Ref. 18). It appears that modeling the velocity profile of the upper atmosphere as a rigid rotation with the Earth is a feasible real-world model but that modeling errors are still present.

The density of the upper atmosphere has a dynamic behavior that appears to be caused by variations in the Sun's behavior (see Section 1.3). The upper atmospheric density can change erratically and in an unpredictable manner, particularly during geomagnetic storms. Density models are a major source of modeling errors in any OD system.

It becomes apparent from the above discussion that the drag formulation has inherent uncertainties associated with it. Therefore, in order to improve orbit determination and prediction processes for low altitude satellites, estimation of drag and/or density parameters appears necessary. This is called adaptive estimation.

1.3 Prior Work

Previous investigations of the motion of artificial satellites, that are applicable to this research, fall into the following categories: semianalytical satellite theory, density modeling, drag theories and adaptive estimation. Each area is reviewed in this section.

Semianalytical Satellite Theory

In Reference 19, Uphoff suggested a numerical averaging theory. The approach taken by Uphoff was to determine the mean (averaged) element rates by numerically averaging all perturbations (gravitational, drag, etc.) over one revolution of the satellite. Uphoff included tesseral terms of the Earth's gravitational field in the mean element rates. This introduced frequency components in the mean element rates that restricted the numerical integration stepsize. For a large enough tesseral gravitational field, a numerical averaging satellite theory could be more costly than a SP theory. Lutsky and Uphoff (Ref. 20) extended the numerical averaging theory of Uphoff to include short periodic variations. The approach of Reference 20 is to assume a Fourier series expansion in "mean true anomaly" for the derivatives of the osculating elements with respect to the "mean true anomaly." The coefficients in this Fourier expansion are then constructed via numerical

quadratures. Reference 20 gives numerical results for the gravitational, luni-solar and atmospheric drag perturbations.

Cefola, Long and Holloway (Ref. 10) introduced the basic concept of the current semianalytical satellite theory, i.e., analytical averaging for conservative perturbations and numerical averaging for non-conservative perturbations. Reference 12 recommended that tesseral non-resonance terms be considered as short periodic variations. This removed the restricting frequency components from the mean element rates and therefore allowed large numerical integration time steps. Equinoctial elements (Ref. 15) were used in the above semianalytical developments becuase of the nonsingular nature of this element set (see Section 2.1). References 11 and 13 expressed the gravitational potential (central body and third body) in terms of equinoctial elements. This allows for the direct analytical determination of the partial derivatives of the potential with respect to the orbital elements in the Lagrangian VOP equations (see Appendix A). These results are necessary in order to obtain analytical expressions for the mean element rates due to these conservative perturbations. The analytical expressions are written in terms of mathematical special functions (Jacobi polynomials and Hansen coefficients) so that recursions are available. The resulting computer code employs these recursions; thus the semianalytical theory has the same flexibility as SP theories. Kaula (Ref. 21) obtained similar expressions

for the gravitational potential in terms of Keplerian elements. References 9 and 12 give some numerical results of the semi-analytical orbit generator without the short periodic variations on the output grid.

The theoretical foundation of the semianalytical satellite theory and how it is derived from the generalized method of averaging was presented by McClain (Refs. 3 and 4). McClain also developed a general functional formulation of the short periodic functions due to any type of perturbation, conservative or non-conservative. For the non-conservative forces, McClain's functional formulation becomes very difficult to implement.

Cefola and McClain (Ref. 5) developed an analytical formulation for the first order short periodic variations due to central body gravitational and lunar-solar point mass effects. The short periodic functions for the zonal terms of the central body gravitational field were expressed in terms of the mean true longitude and were closed form in the eccentricity. The tesseral m-daily terms were expressed in terms of the Greenwich hour angle and were closed form in the eccentricity. The short periodic variations for the tesseral terms were expressed in terms of the mean-mean longitude and truncation on powers of the eccentricity was necessary. Lunar-solar short periodic variations were derived in terms of the mean

eccentric longitude and were closed form in the eccentricity. The lunar-solar short periodic functions were derived using a time-independent assumption, i.e., the third body was held fixed over the anaytical averaging integral. As with the analytical expressions for the mean element rates, the short periodic analytical expressions are written in terms of mathematical special functions which have recursive relations available.

Zeis (Refs. 6 and 7) developed a computer program in the MACSYMA language (Ref. 22) whose goal was to construct automatically the corresponding explicit formulas for the mean element rates and short periodic functions. Zeis also developed a methodology for constructing the J_2^2 averaged equations of motion and short periodic functions. Zeis' expressions were good for only direct orbits (see Section 2.1.1) and were truncated at zeroth order in the eccentricity. Two of Zeis' J_2^2 short periodic expressions were later found to be incorrect and the correct versions are given in Appendix E.

Density Modeling

In order to accurately predict the ephemeris of a drag perturbed satellite, it is necessary to accurately model the density of the upper atmosphere (90 km to 1200 km). The density of the upper atmosphere has many variations. A brief summary of these variations, based upon References 23 through

26, is given in this section followed by a discussion of some of the density models that are used in orbit generators.

The upper atmosphere has an oblate shape due to the oblateness of the earth. This should not be surprising but must be considered for accurate density determination.

Another major variation in the upper atmospheric density is the diurnal variation. Radiation from the Sun is maximum at the sub-solar point (local noon). This causes atmospheric heating which results in an increase in the density. This increase in density causes a bulge in the density distribution which is called the diurnal bulge. The bulge follows the path of the Sun but lags the sub-solar point by approximately 30 degrees (2 P.M. local time) because of the rotation of the Earth and transients within the atmosphere. The minimum value of density occurs at approximately 4 A.M. local time. The diurnal effect also has seasonal variations because the bulge is alternately located in the northern and southern hemispheres (ecliptic and equatorial planes are not aligned).

The density of the upper atmosphere also exhibits a 27 day variation which is caused by the 27 day cycle of the Sun's extreme ultraviolet radiation. The decimetric flux from the Sun varies in the same manner as the extreme ultraviolet radiation and therefore serves as an indication of this variation.

The decimetric flux is measured by Earth observatories and given the symbol $F_{10.7}$ (10^{-22} watts m^{-2} Hz^{-1}).

The upper atmospheric density also has a semiannual variation. The cause of this variation has not been determined precisely but it is believed to be caused by solar and/or interstellar corpuscular emissions. Maximum density occurs around the first week in April and October and minimum density occurs during January and July.

The density of the upper atmosphere shows a definite increase during periods of geomagnetic storms. These storms are erratic and usually cannot be predicted. K_p is the geomagnetic index and is a reliable indication of the changes in the density of the upper atmosphere due to geomagnetic storms. The changes in density lag behind changes in the geomagnetic index by about 6 hours. It is also noted that after a strong geomagnetic storm, the density falls to a lower value than it was prior to the storm.

The solar wind has a 11 year cycle and this is paralleled by a corresponding 11 year variation in the density of the upper atmosphere. It has been found that the smoothed monthly mean of the decimetric flux ($F_{10.7}$) is correlated with the long term 11 year solar cycle. $\bar{F}_{10.7}$ denotes the monthly mean $F_{10.7}$.

The helium concentration in the upper atmosphere greatly affects the density. A seasonal increase in the concentration of helium above the winter pole has also been observed. The mechanism of the seasonal migration of helium has not yet been fully explained but empirical equations have been developed that adequately describe the density variations associated with this phenomenon.

Another constituent of the upper atmosphere that can significantly affect the density is the amount of neutral hydrogen. The concentration of neutral hydrogen in the upper atmosphere depends upon its production and escape rates. It has been found that the amount of hydrogen in the upper atmosphere is proportional to the exospheric temperature. This fact has allowed empirical equations to be developed that adequately predict the hydrogen concentration in the upper atmosphere.

Density waves in the upper atmosphere have been observed aboard various instrumented satellites. There waves are usually observed at the higher latitudes and generally propagate from north to south in the northern hemisphere or vice versa in the southern hemisphere. It is believed that these waves are the mechanism by which energy deposited in the auroral zones is conveyed to lower latitudes. The effects of these waves on the density are usually neglected in the orbit determination problem.

From the above discussion it becomes clear that the upper atmospheric density has a dynamic nature. Most, if not all, of the density variations are caused by variations in the Sun's behavior. Better prediction of the Sun's behavior ($F_{10.7}$, $\bar{F}_{10.7}$, and K_p) should improve prediction of the density of the upper atmosphere. This appears to be the approach that some of the Russian scientists are taking (Refs. 27-29).

Density models for the upper atmosphere have been and will be an area of intense investigation. A few density models that are applicable to this investigation will be briefly discussed in this section. Reference 17 gives an excellent discussion of the various density models and the interested reader is directed there for further information.

A density model which is static and does not take into account any of the upper atmospheric variations discussed above is the exponential model. The model is written as

$$\rho = \rho_o \exp\left[\frac{r_o - r}{H}\right] \qquad (1\text{-}2)$$

where

ρ = density at radius r

ρ_o = reference density at reference radius

r_o = reference radius

H = scale height

The scale height is a function of the altitude but is generally taken as a constant. The exponential density model is used in many GP drag theories due to its simple analytical form. Variations in $F_{10.7}$, $\bar{F}_{10.7}$ and K_p can be partially reflected by changes in ρ_o, r_o and H. These changes can become unwieldy and are usually not adequate because the diurnal and oblateness effects are not modeled.

Another static density model is the power law density model. This is represented as

$$\rho = \rho_o \left[\frac{r_o - s}{r - s}\right]^{\tau} \qquad (1\text{-}3)$$

where ρ_o, τ and s are constants depending upon the reference radius, r_o. Lane (Ref. 30) introduced this model for use in his analytical drag development. Again some of the variations in the upper atmosphere can be partially reflected by changes in r_o, ρ_o and s. As in the exponential density model, these changes are usually not adequate.

Cook and King-Hele (Ref. 31) have extended the exponential density model to account for the diurnal effect of the upper atmosphere. Chen (Ref. 32) modified the model of Cook

and King-Hele such that the oblateness effects were also included. This model, which will be called the diurnal-oblate exponential density model, can be written as

$$\rho = \rho_o \exp\left[\frac{h_o - h}{H}\right](1 + F \cos \psi) \qquad (1\text{-}4)$$

where

ρ = density at point in question

ρ_o = density at reference point

h = height of point in question above the oblate Earth

h_o = height of reference point above the oblate Earth

F = diurnal factor of the atmosphere

H = scale height of the atmosphere

ψ = angle between satellite's position and diurnal bulge

Obviously the maximum density (ρ_{max}) occurs at $\psi = 0°$ and the minimum density (ρ_{min}) occurs at $\psi = 180°$. Therefore, the ratio of ρ_{max} to ρ_{min} for a given altitude, h, can be written as (Ref. 32)

$$f \equiv \frac{\rho_{max}}{\rho_{min}} = \frac{1 + F}{1 - F} \qquad (1\text{-}5)$$

which implies that

$$F = \frac{f - 1}{f + 1} \tag{1-6}$$

If the determination of ρ_{max}, ρ_{min}, ρ_o and H depend upon the solar activity ($F_{10.7}$, $\bar{F}_{10.7}$, and K_p), then the diurnal-oblate exponential density model attempts to account for most of the upper atmospheric variations.

Fominov (Ref. 33) suggested a density model which has the following general form

$$\rho = f(t) \left[\rho_1 + \rho_\phi + \rho_\psi\right] \tag{1-7}$$

where

$f(t)$ = function that reflects the time variations of the density, i.e., $F_{10.7}$, $\bar{F}_{10.7}$, K_p

ρ_1 = density of a spherically symmetric atmosphere

ρ_ϕ = factor which represents the density dependence upon latitude

ρ_ψ = factor which represents the density dependence upon the Sun's position

ρ_ϕ and ρ_ψ are expressed in terms of Legendre polynomials. This model is interesting because it uses spherical harmonics much in the same manner that they are used in expressions for the Earth's gravitational field.

The Jacchia 1971 density model (Ref. 34) is generally considered the "state of the art" in density predictions. In the Jacchia 1971 model the altitude and exospheric temperature are calculated first. Jacchia's model for the exospheric temperature represents most of the variations in the upper atmopshere. Knowing the exospheric temperature and altitude, a series of tables allows the determination of the density. Next, some density corrections are determined and added to the density obtained from the tables. Roberts (Ref. 35) modified Jacchia's density model such that the density tables were not necessary. This density model is called the Analytical Jacchia-Roberts density model and is discussed in detail in Reference 17.

Elyasberg (Ref. 36) introduced a density model which is generally called the U.S.S.R. model. This model was used on the U.S. - U.S.S.R. Apollo-Soyuz mission. The general form of the model is

$$\rho = \rho_h K_1 K_2 K_3 K_4 \qquad (1\text{-}8)$$

where

ρ_h = night-time vertical profile of density

K_1 = factor representing solar radiation density variations ($F_{10.7}$ and $\bar{F}_{10.7}$)

K_2 = factor representing diurnal variations

K_3 = factor representing semiannual variation

K_4 = factor representing geomagnetic density variations

Reference 17 gives an excellent discussion of this model. Mueller (Ref. 37) has suggested some modifications to the U.S.S.R. model that apparently improve its performance.

The mass spectrometer and incoherent scatter (MSIS) density model is a recently developed model that appears promising (Ref. 38). Using measurements of the densities of the upper atmospheric constituents (made from mass spectrometers on near-Earth satellites) and neutral temperature measurements inferred from incoherent scatter measurements (made from various ground stations), a model for the densities of the upper atmospheric constituents (N_2, O, He, Ar, H) is developed. Like the Jacchia model, the MSIS model is based upon an exospheric temperature profile. The exospheric temperature and other quantities are represented by expansions in terms of spherical harmonics. The coefficients of these expansions are determined from the data measurements.

The Modified Harris-Priester density model is the density model used in this research (see Appendix C). The Modified Harris-Priester density model consists of a set of 10

tables each associated with a different smoothed decimetric flux ($\bar{F}_{10.7}$). The range of $\bar{F}_{10.7}$ is from 65 to 275. Each table contains the diurnal maximum and minimum densities at various altitudes from 100 km to 1000 km. The density is then determined by an exponential interpolation algorithm (see Appendix C). The Modified Harris-Priester density model is derived from the Harris-Priester density model and this relationship is described in Reference 17. The Modified Harris-Priester atmosphere (also the Harris-Priester atmosphere) is symmetric with respect to the diurnal bulge. This is not actually the case and an Asymmetric Modified Harris-Priester density model is presented in Reference 17. The Analytical Jacchia-Roberts, MSIS and U.S.S.R. density models account for the asymmetric nature of the diurnal variation. While the Modified Harris-Priester density model was employed in this research primarily for computer software reasons, it is a sufficiently realistic model. In addition, the analysis presented in this work is general with respect to the density model.

Drag Theories

Much research has been done on the motion of drag perturbed satellites. These investigations can generally be classified as analytical or numerical averaging drag theories. Drag theories which consider oblateness and drag perturbations separately and add their effects (i.e., assume superposition

is valid) are called first order drag theories. Those drag theories which consider the effects of oblateness/drag coupling are called second order drag theories. Since the oblateness and drag perturbations are sufficiently large for a low altitude satellite, the coupling between oblateness and drag does exist and must be considered for accurate orbit prediction.

A series of papers by King-Hele, Cook and Walker (Refs. 31 and 39 through 43) studied the effects of atmospheric drag on the semimajor axis, eccentricity and argument of perigee. This work was basically done in order to determine the lifetimes of drag perturbed satellites. Initially the density model was an exponential model but it was subsequently improved until it took into account diurnal effects and the scale height dependence upon altitude. A rotating atmosphere was also assumed. This analysis can be classified as a first order analytical drag theory (J_2 perturbations were not considered). Santora (Ref. 44) basically extended the work of King-Hele, Cook and Walker to include diurnal and Earth oblateness effects such that orbits with small eccentricities ($e = 0$ to 0.01) could be better handled.

A first order analytical drag theory was introduced by Zee (Ref. 45) which basically added the effects of J_2 and drag perturbations. Zee assumed a non-rotating atmosphere with an exponential density. Zee's results were good only for small eccentricity.

Another first order drag theory of interest was developed by Fominov (Ref. 33). Fominov used the density model as given in Equation (1-7) which results in a drag theory with a recursive nature. This can be very desirable from a computational point of view.

A first order drag theory derived via computerized symbolic algebra (FORMAC) was presented by Barry and Rowe (Ref. 46). The Jacchia 1970 density model was approximated by a Fourier series in the true anomaly of the satellite. A rotating atmosphere was assumed.

Reference 47 is a recent first order drag theory. This work suggests a formulation which is applicable to both the drag perturbed satellite and reentry vehicle. A rotating atmosphere with an exponential density was assumed.

Brouwer and Hori (Ref. 48) introduced a second order analytical drag theory based upon canonical perturbation theory. The method of Brouwer and Hori was an extension of Brouwer's earlier work (Ref. 49) and as such was in terms of Delaunay variables. The drag theory of Brouwer and Hori assumed a non-rotating atmosphere with an exponential density model. In order to get an analytical solution, Brouwer and Hori expanded the exponential density function in terms of a series of the

eccentricity to fifth power. For satellites with a low perigee height, the series approximation was not sufficient. Lane (Ref. 50) introduced the power law density model such that the equations of motion could be integrated analytically without the power series expansion in eccentricity. This gave improved performance for satellites with a low perigee height. Lane also assumed a non-rotating atmosphere. Reference 51 extended the work of Lane by assuming a quadratic scale height (Lane's power law density model is representative of a linear scale height variation).

At the same time that Brouwer introduced his satellite theory, Vinti introduced an alternative satellite theory (Ref. 52). Sherrill (Ref. 53) extended Vinti's satellite theory to include drag perturbations. Sherrill assumed a non-rotating oblate spheriodal density with exponential height variations. Sherrill's approach was similar to the approach of Brouwer and Hori and is also a second order analytical drag theory. Watson, Mistretta and Bonavito (Ref. 54) also extended Vinti's theory to account for drag perturbations. The approach of Reference 54 was to use an iteration scheme to correct the Vinti theory until the drag perturbations were accounted for.

Scheifele, Mueller and Stark (Ref. 55) introduced a second order analytical drag theory which was based upon the

previous work of Scheifele (Ref. 56). Reference 55 (also Ref. 56) used non-singular Poincare-Similar elements. The analysis was done in extended phase space with true anomaly as the independent variable. The density model used took into account the oblateness and diurnal effects.

Chen (Ref. 32) suggested a second order drag theory which was based upon the diurnal-oblate exponential density model. A rotating atmosphere was assumed. Chen used Keplerian orbital elements and his solutions were derived by the method of two-variable asymptotic expansions.

Many of the analytical drag theories (King-Hele, Zee, Brouwer-Hori, Lane, Sherrill, etc.) were completed by the late 1960's. These analytical drag theories were very complicated for relatively simple density models. At about this same time, new and more complicated density models were being developed. These facts made numerical averaging drag theories attractive because of the capability to allow complete drag models and to permit different density models to be inserted with relative ease.

A numerical averaging satellite theory is similar to a semianalytical satellite theory except that mean element rates and short periodic variations are all determined numerically. The mean element rates of a first order numerical averaging

drag theory are determined by simply performing a numerical quadrature on the drag disturbing force over a revolution of the satellite. A first order drag theory is usually not sufficient for low altitude satellites because it neglects the coupling of oblateness and drag perturbations.

A second order numerical averaging drag theory was suggested by Pimm (Ref. 57) and Reference 58. The approach proposed in References 57 and 58 was to apply Izsak's J_2 short periodic radial correction (Ref. 59) in the density calculation at each of the abscissas of the quadrature algorithm. This is a good approximation to the effects of oblateness/drag coupling (see Section 2.4.3). Lutsky and Uphoff (Ref. 20) suggested the use of Izsak's J_2 short periodic velocity corrections as well as the height corrections in the density calculation at each quadrature point. This improves the approximation of the oblateness/drag coupling but still does not include all of it.

Another second order numerical averaging drag theory was developed by Liu (Refs. 60 and 61) in terms of Keplerian elements. Liu also used a modified form of Izsak's J_2 short periodic height correction in the density determination. Liu assumed further that drag perturbations affected only the mean element rates for the semimajor axis and eccentricity.

Adaptive Estimation

As stated previously, the drag formulation has many uncertainties associated with it. It appears that adaptive estimation of the drag model may improve orbit determination and prediction processes for low altitude satellites. Adaptive estimation can generally be classified as either adaptive parameter estimation or adaptive noise estimation. Adaptive parameter estimation is the process of trying to improve the force model by solving for uncertain constants within the formulation. Adaptive noise estimation is where characteristics about the noise (system or observational) in the formulation are solved for. This research will be concerned with only adaptive parameter estimation.

A previous investigation that is directly applicable to this investigation was done by Dowd in Reference 17. Dowd used adaptive parameter estimation with an Extended Kalman Filter and showed improved performance in the orbit determination and prediction process. Dowd used a SP theory (Cowell) with Cartesian position and velocity as his state. An Asymmetric Modified Harris-Priester atmosphere was developed and used. Dowd found that adaptive parameter estimation of the density model was a viable means of improving the estimation and prediction of drag perturbed satellites.

Other investigations into adaptive estimation of the drag perturbed satellite problem are given in References 62 through 66. These investigations, like Dowd's work, used sequential filters. Adaptive estimation (parameter and noise) of the drag formulation was done. Reference 17 gives a good overview of these previous investigations.

1.4 Research Objectives

The objective of this research is to combine semianalytical satellite theory with current estimation procedures such that orbit determination and prediction processes for low altitude satellites are improved.

From Section 1.3 it is seen that much of the theoretical foundation already exists for the semianalytical orbit generator. This is especially true for conservative forces for which analytical expressions exist for both the mean element rates and short periodic variations. For non-conservative forces the semianalytical satellite theory uses numerical averaging techniques to obtain the mean element rates to first order. To obtain an approximation to the second order drag effects, Izsak's J_2 short periodic height corrections can be applied to the density calculation at each of the quadrature points. Enhancements that need to be made to the semianalytical satellite theory such that drag perturbed satellites can be better handled are:

1. a short periodic development that adequately handles drag perturbations
2. a more complete second order numerical averaging drag theory.

These developments are presented in Chapter 2. The short periodic development presented is applicable to all perturbations. For conservative forces, the development of Chapter 2 permits analytical expressions.

As previously indicated, the drag formulation has many inherent uncertainties associated with it. This is especially true in the density model. It is therefore necessary to couple the semianalytical satellite theory with current estimation procedures. These current estimation procedures should also be capable of adaptive estimation. A semianalytical batch filter with numerical results is presented in Chapter 3. A development of the semianalytical partial derivatives of perturbed motion is included in Chapter 3. In Chapter 4, the theoretical formulation of a Semianalytical Linearized Kalman Filter is presented.

This development (except the Semianalytical Linearized Kalman Filter) has been implemented into the Research and Development version of the Goddard Trajectory and Determination System (GTDS-RD) at the Charles Stark Draper Laboratory in

Cambridge, Massachusetts. Further details about the implementation are given in Appendix F.

Chapter 2

SEMIANALYTICAL SATELLITE THEORY FOR THE PREDICTION OF PERTURBED MOTION

Semianalytical satellite theory offers significant advantage for precision orbital computations. In comparison with the integration of osculating equations of motion by numerical methods (SP theories), the semianalytical satellite theory gives much greater computational speed while retaining equivalent accuracy. In comparison with conventional GP theories, the semianalytical satellite theory makes it possible to include more complete physical models and to simplify the computations that must be done at each output point.

A semianalytical satellite theory makes a transformation from osculating element space to mean element space. The mean elements are obtained by removing the short periodic motion from the osculating elements by applying the generalized method of averaging. Since the short periodic motion has been removed, the averaged equations of motion allow a much larger numerical integration time step. A semianalytical satellite theory is composed of an Averaged Orbit Generator (AOG) and a Short Periodic Generator (SPG). The AOG integrates the averaged equations of motion and determines the mean elements on the integration grid. The SPG determines the short periodic variations on the output grid. Because of the slowly varying

nature of the mean elements, an interpolator can be used to obtain the mean elements at the output times.

As was previously indicated, the semianalytical orbit generator (AOG and SPG) must be capable of accurately and efficiently handling all types of perturbations, conservative or non-conservative. This research is especially concerned with improving the performance of the semianalytical orbit generator in the presence of atmospheric drag.

This chapter presents a short periodic development and a second order drag theory. The short periodic development includes time-independent and weak time-dependent formulations that are applicable to conservative and non-conservative disturbing forces. For conservative perturbations, analytical expressions are possible. The second order drag theory is a complete treatment of all second order drag effects including the oblateness/drag coupling. The second order drag theory is modular with respect to the atmospheric density model. In this case, modularity means that alternate density models may be inserted with the same ease that occurs in first order numerical averaging drag theories.

In Section 2.1 some mathematical preliminaries are presented. First, the equinoctial system and the transformation from the observational coordinate frame to the inertial coor-

dinate frame are discussed. The basic concepts of the generalized method of averaging and how it applies to the semianalytical satellite theory are presented next. The difference between a time-independent and time-dependent formulation is also discussed.

A first order short periodic theory is discussed in Section 2.2. Time-independent and weak time-dependent formulations are presented. A time-independent formulation assumes that the physical problem has only the satellite's phase angle or that the second phase angle is independent of the satellite's phase angle. Perturbations which generally can be expressed in terms of a single phase angle are zonal terms of the gravitational field, atmospheric drag and solar radiation pressure. Due to the presence of a second phase angle when considering tesseral terms of the gravitational field (Greenwich hour angle) or third body perturbations (mean longitude of the third body), it becomes necessary to introduce a time-dependent short periodic formulation. If the angular rate of the second phase angle is very small in comparison with the satellite's mean motion, it is possible to use a weak time-dependent formulation which is presented in this section. The formulations presented in Section 2.2 are expressed in terms of a Fourier series representation in the mean-mean longitude.

A complete second order drag theory for the semianalytical orbit generator is developed in Section 2.3. For low altitude satellites, the oblateness/drag coupling terms in the AOG can be very important. Having a representation of the first order short periodic variations for J_2 and drag effects allows a rigorous treatment of the secular and long periodic oblateness/drag coupling terms.

Numerical results are presented in Section 2.4. The problem of initial conditions is also discussed in this section. A low altitude circular test case and a low altitude eccentric test case are used to demonstrate the time-independent short periodic formulation in the SPG and the second order drag theory in the AOG. A medium altitude circular test case is used to demonstrate the weak time-dependent formulation.

2.1 Mathematical Preliminaries

2.1.1 The Equinoctial System

The equinoctial elements are defined as follows (Ref. 3):

$$\begin{aligned} a &= a \\ h &= e \sin(\omega + I\Omega) \\ k &= e \cos(\omega + I\Omega) \end{aligned} \qquad (2\text{-}1)$$

$$p = \tan^{I}(i/2) \sin \Omega$$

$$q = \tan^{I}(i/2) \cos \Omega$$

$$\lambda = M + \omega + I\Omega \qquad (2\text{-}1)$$

where a, e, i, Ω, ω, and M are the classical Keplerian element set and where I is the retrograde factor. The retrograde factor has to be +1 when i = 0° and -1 when i = 180° in order to avoid a singularity in the Variation of Parameter equations. When i is any value between 0° and 180°, either set of equinoctial elements can be used, but due to mathematical difficulties the switch over should occur prior to getting very close to the limiting values. Since i seldom equals 180°, the direct equinoctial element set (I = +1) is generally used.

The equinoctial reference frame is shown in Figure 2-1. The equinoctial coordinate frame can be obtained from the Earth centered inertial frame (mean of 1950) by the following rotations: a rotation of Ω° about the inertial z direction, a rotation of i° about the new x axis followed by a rotation of $-I\Omega$° about the new z axis. The unit vectors in the equinoctial coordinate frame are $\hat{f}$, $\hat{g}$ and $\hat{w}$. The $\hat{f}$ and $\hat{g}$ vectors are in the orbital plane. The $\hat{w}$ vector is normal to the orbital plane.

The transformation from the inertial frame to the equinoctial frame, based upon the concept of the osculating ellipse, can be expressed as

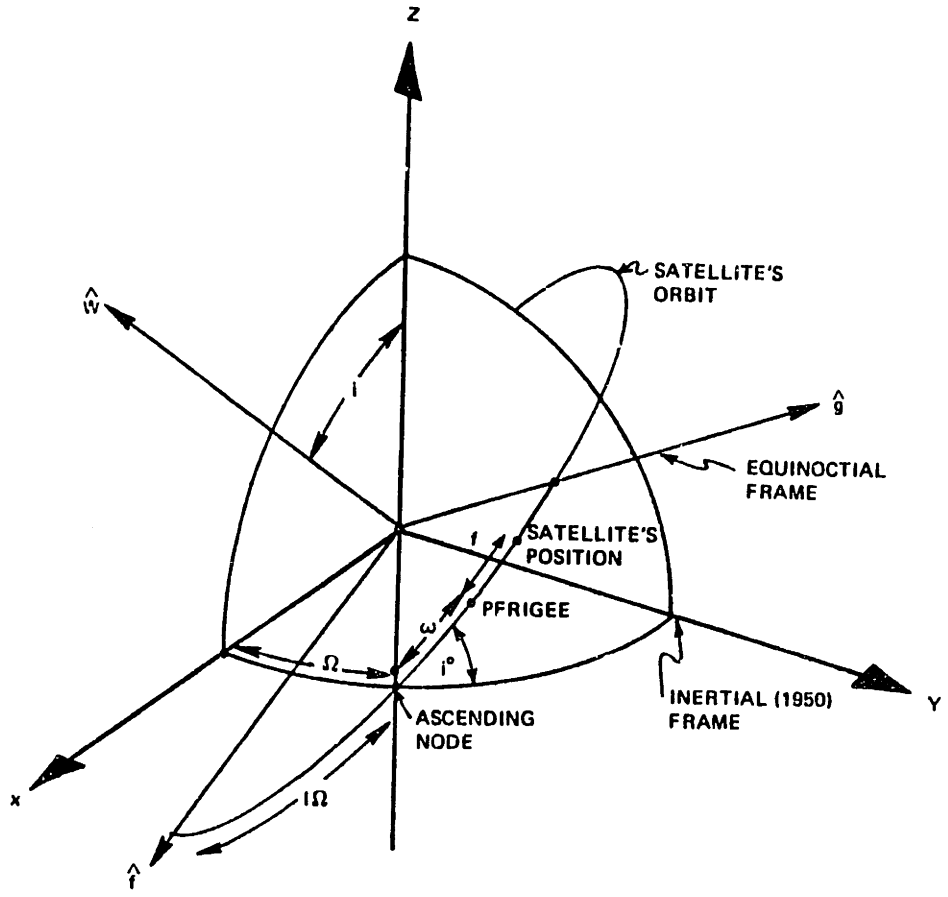

Figure 2-1. Equinoctial Coordinate Frame

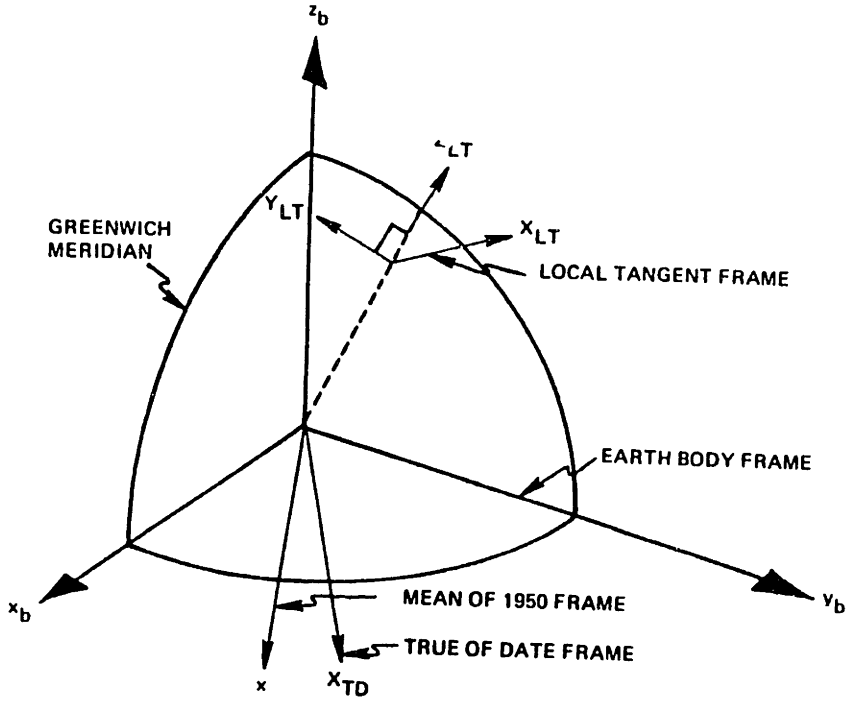

Figure 2-2. Coordinate Frames

$$\underline{r}_{eq} = D\,\underline{R}$$
$$\dot{\underline{r}}_{eq} = D\,\dot{\underline{R}} \qquad (2\text{-}2)$$

where

$\underline{r}_{eq}$, $\dot{\underline{r}}_{eq}$ ≡ position and velocity in the equinoctial frame

$\underline{R}$, $\dot{\underline{R}}$ ≡ position and velocity in the inertial frame

D ≡ rotation matrix

The rotation matrix can be expressed in terms of Keplerian or equinoctial elements as (Ref. 3)

$$D = \begin{bmatrix} \cos^2\Omega + I\sin^2\Omega\cos i & \cos\Omega\sin\Omega(1-I\cos i) & -I\sin\Omega\sin i \\ I\cos\Omega\sin\Omega(1-I\cos i) & I(\sin^2\Omega + I\cos^2\Omega\cos i) & \cos\Omega\sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \qquad (2\text{-}3a)$$

or

$$D = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 1 - p^2 + q^2 & 2pq & -2pI \\ 2pqI & (1 + p^2 - q^2)I & 2q \\ 2p & -2q & (1 - p^2 - q^2)I \end{bmatrix} \qquad (2\text{-}3b)$$

The mean, eccentric, and true longitudes in the equinoctial reference system are defined respectively as

$$\lambda = M + \omega + I\Omega \tag{2-4a}$$

$$F = E + \omega + I\Omega \tag{2-4b}$$

$$L = f + \omega + I\Omega \tag{2-4c}$$

where M, E and f are respectively the mean, eccentric and true anomalies associated with the Keplerian element set.

The radius in a Keplerian ellipse is expressed as

$$r = \frac{a(1 - e^2)}{1 + e \cos f}$$

It can be shown that

$$e \cos f = k \cos L + h \sin L$$

and

$$e^2 = h^2 + k^2$$

Therefore

$$r = \frac{a(1 - h^2 - k^2)}{1 + k \cos L + h \sin L} \tag{2-5}$$

In a similar fashion it can be shown that

$$r = a(1 - k \cos F - h \sin F) \tag{2-6}$$

From Kepler's second law and Equation (2-4c) the following two-body relation results

$$\frac{dL}{dt} = \frac{na^2\sqrt{1 - h^2 - k^2}}{r^2} \tag{2-7}$$

Equating the derivatives of Equations (2-5) and (2-6) and rearranging yields another two-body relation

$$\frac{dF}{dt} = \frac{na}{r} \tag{2-8}$$

Position and velocity in the equinoctial frame can be expressed as

$$\underline{r}_{eq} = \begin{bmatrix} X_1 \\ Y_1 \\ 0 \end{bmatrix} \qquad \dot{\underline{r}}_{eq} = \begin{bmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ 0 \end{bmatrix} \tag{2-9a}$$

where

$$X_1 = r \cos L \tag{2-9b}$$

$$Y_1 = r \sin L \tag{2-9c}$$

$$\dot{X}_1 = \frac{-na(h + \sin L)}{\sqrt{1 - h^2 - k^2}} \tag{2-9d}$$

$$\dot{Y}_1 = \frac{na(k + \cos L)}{\sqrt{1 - h^2 - k^2}} \tag{2-9e}$$

By use of the two-body relations that relate true anomaly to eccentric anomaly, it can be shown that

$$\cos L = \frac{a}{r}\left[(1 - h^2\beta)\cos F + hk\beta \sin F - k\right] \tag{2-10a}$$

$$\sin L = \frac{a}{r}\left[(1 - k^2\beta)\sin F + hk\beta \cos F - h\right] \tag{2-10b}$$

where

$$\beta = \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \tag{2-10c}$$

It follows from Equations (2-9) and (2-10) that

$$X_1 = a\left[(1 - h^2\beta)\cos F + hk\beta \sin F - k\right] \tag{2-11a}$$

$$Y_1 = a\left[(1 - k^2\beta)\sin F + hk\beta \cos F - h\right] \tag{2-11b}$$

$$\dot{X}_1 = \frac{na^2}{r}\left[hk\beta \cos F - (1 - h^2\beta)\sin F\right] \tag{2-11c}$$

$$\dot{Y}_1 = \frac{na^2}{r}\left[(1 - k^2\beta)\cos F - hk\beta \sin F\right] \tag{2-11d}$$

Kepler's equation is given as

$$M = E - e \sin E$$

Adding $\omega + I\Omega$ to each side yields

$$\lambda = F - e \sin[F - (\omega + I\Omega)]$$

Expanding the above Equation and using Equation (2-1) yields Kepler's equation in equinoctial elements

$$\lambda = F - k \sin F + h \cos F \qquad (2\text{-}12)$$

Since the overall goal of this research is to combine the semianalytical satellite theory with estimation procedures, it is necessary to develop the transformation from the local tangent frame (frame in which the measurement/observations are usually expressed) to the equinoctial frame (frame in which the semianalytical satellite theory is developed). The coordinate frames of interest are the mean of 1950, true of date, Earth body fixed and local tangent coordinate frames. The mean of 1950 frame is an inertial frame whose x-y plane is in the mean 1950 equatorial plane with the x axis along the 1950 node of the mean equatorial and ecliptic planes (mean implies that nutations are neglected). The true of date frame is an inertial frame whose x-y axis is in the true equatorial plane with the x axis along the node of the true equatorial and ecliptic planes (true means that nutations are considered and that the frame is with respect to a certain specified date). The Earth body frame is a coordinate frame whose x-y axis is in the equatorial plane with the x axis along the Greenwich meridian. The local tangent frame has its origin at the observation sight on the

surface of the Earth with its x axis pointing east, y axis pointing north and z axis up. These coordinate frames are shown in Figure 2-2.

The transformation of position and velocity in the mean of 1950 coordinate frame to the Earth body coordinate frame can be expressed as (Ref. 67)

$$\underline{r}_b = B(t)\ C\ \underline{R}$$

$$\dot{\underline{r}}_b = \dot{B}(t)\ C\ \underline{R} + B(t)\ C\ \dot{\underline{R}} \qquad (2\text{-}13)$$

where

$\underline{r}_b$, $\dot{\underline{r}}_b$ ≡ position and velocity with respect to the Earth body frame

$\underline{R}$, $\dot{\underline{R}}$ ≡ position and velocity with respect to the mean of 1950 inertial frame

$B(t)$ ≡ rotation matrix from true of date to Earth body coordinate frame. It is time dependent due to the rotation of the Earth.

C ≡ rotation matrix from mean of 1950 to true of date coordinate frame

The transformation of position and velocity in the Earth body coordinate frame to the local tangent frame can be expressed as (Ref. 67)

$$\underline{r}_{LT} = M_{LT}[\underline{r}_b - \underline{r}_s]$$
$$\dot{\underline{r}}_{LT} = M_{LT}[\dot{\underline{r}}_b] \qquad (2\text{-}14)$$

where

$\underline{r}_{LT}$, $\dot{\underline{r}}_{LT}$ ≡ position and velocity with respect to the local tangent frame

$\underline{r}_s$ ≡ Measurement/observation station location with respect to the Earth body frame

M_{LT} ≡ rotation matrix from Earth body to local tangent frame

Combining Equation (2-13) and (2-14) yields

$$\underline{r}_{LT} = M_{LT}[B(t)\ C\ \underline{R} - \underline{r}_s]$$
$$\dot{\underline{r}}_{LT} = M_{LT}[\dot{B}(t)\ C\ \underline{R} + B(t)\ C\ \dot{\underline{R}}] \qquad (2\text{-}15)$$

Inverting Equation (2-15) and realizing that the inverse of an orthogonal matrix equals the transpose

$$\underline{R} = C^T B^T [M_{LT}^T\ \underline{r}_{LT} + \underline{r}_s]$$
$$\dot{\underline{R}} = C^T B^T [M_{LT}^T\ \dot{\underline{r}}_{LT} - \dot{B}\ B^T (M_{LT}^T\ \underline{r}_{LT} + \underline{r}_s)] \qquad (2\text{-}16)$$

Substitution of Equation (2-16) into Equation (2-2) gives the transformation between the local tangent coordinate frame and the equinoctial frame.

$$\begin{aligned} \underline{r}_{eq} &= DC^T B^T [M_{LT}^T \underline{r}_{LT} + \underline{r}_s] \\ \dot{\underline{r}}_{eq} &= DC^T B^T [M_{LT}^T \dot{\underline{r}}_{LT} - \dot{B}\, B^T (M_{LT}^T \underline{r}_{LT} + \underline{r}_s)] \end{aligned} \tag{2-17}$$

Inverting Equation (2-17) yields

$$\begin{aligned} \underline{r}_{LT} &= M_{LT} [BCD^T \underline{r}_{eq} - \underline{r}_s] \\ \dot{\underline{r}}_{LT} &= M_{LT} [BCD^T \dot{\underline{r}}_{eq} + \dot{B}CD^T \underline{r}_{eq}] \end{aligned} \tag{2-18}$$

Equations (2-17) and (2-18) allow transformation between the local tangent frame and the equinoctial frame and vice versa. The rotation matrices used are defined in Reference 67.

It is usually sufficient to work in the inertial true of date coordinate frame. If this is the case, the additional transformation from the true of date coordinate frame to the mean of 1950 coordinate frame is not necessary. The above expressions will still apply in this case except that the C matrix has to be replaced with the Identity matrix.

2.1.2 A Review of the Semianalytical Satellite Theory

The Variation of Parameter (VOP) equations of motion, for a single disturbing force, can be expressed in either Gaussian or Lagrangian form (see Appendix A). Either form can be expressed functionally in terms of osculating equinoctial variables as

$$\begin{aligned} \dot{a}_i &= \frac{da_i}{dt} = \varepsilon F_i(\underline{a},\lambda) \qquad (i = 1,2,3,4,5) \\ \dot{\lambda} &= \frac{d\lambda}{dt} = n + \varepsilon F_6(\underline{a},\lambda) \end{aligned} \tag{2-19}$$

where

$\underline{a}$ = vector of the five slowly varying osculating equinoctial elements, $\underline{a}^T = [a,h,k,p,q] = [a_1,a_2,a_3,a_4,a_5]$

λ = osculating mean longitude of the satellite (fast variable)

n = mean motion = $\mu^{1/2} a_1^{-3/2}$

ε = small parameters (i.e., J_2, J_3, Ballistic coefficient, etc.)

The presence of the small parameter is basic to the method of averaging (Ref. 3). The formulation of Equation (2-19) does not include a second phase angle such as the Greenwich hour angle (Θ) or the mean longitude of a third body. Therefore,

Equation (2-19) implicitly assumes that λ is independent of any other phase angle associated with the physical problem. This is called a time-independent formulation (Ref. 4). References 4, 68 and 69 address the problem of multiple phase angles in the VOP formulation.

In the method of averaging, a transformation is made from osculating element space to mean element space. The quantities $\bar{a}_i$ and $\bar{\lambda}$ are called the mean elements. The transformation to second order in the small parameter is assumed to be of the form

$$a_i = \bar{a}_i + \varepsilon \eta_{i,1}(\underline{\bar{a}},\bar{\lambda}) + \varepsilon^2 \eta_{i,2}(\underline{\bar{a}},\bar{\lambda})$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}20)$$

where a_6 equals λ. Equation (2-20) is called the near-identity transformation because it contains powers of the small parameter. The trigonometric components of the short periodic functions will contain only satellite dependent frequencies since it was these frequencies that were removed in order to obtain the mean elements. Assuming that the five slowly varying mean elements are constant over one satellite revolution, it is reasonable to require that the short periodic variations be periodic over 2π in $\bar{\lambda}$ and have zero mean value

$$\int_0^{2\pi} \eta_{i,j}(\bar{\underline{a}},\bar{\lambda})\, d\bar{\lambda} = 0 \qquad (2\text{-}21)$$

$$(i = 1,2,3,4,5,6)$$
$$(j = 1,2)$$

It is also assumed that the equations of motion for the mean elements can be written to second order in the small parameter as

$$\dot{\bar{a}}_i = \frac{d\bar{a}_i}{dt} = \varepsilon A_{i,1}(\bar{\underline{a}}) + \varepsilon^2 A_{i,2}(\bar{\underline{a}})$$

$$(i = 1,2,3,4,5) \qquad (2\text{-}22)$$

$$\dot{\bar{\lambda}} = \frac{d\bar{\lambda}}{dt} = \bar{n} + \varepsilon A_{6,1}(\bar{\underline{a}}) + \varepsilon^2 A_{6,2}(\bar{\underline{a}})$$

where the functions $A_{i,j}(\bar{\underline{a}})$ depend only on the five slowly varying mean elements and where the mean motion, $\bar{n}$, equals $\mu^{1/2}\,(\bar{a}_1)^{-3/2}$. The fact that the right hand side of Equation (2-22) contains only the slowly varying mean elements allows the numerical integration time step to be greatly increased over a Special Perturbation technique. This is equivalent to saying that the mean elements contain only the secular and long periodic motion.

Substitution of the near-identity transformation into the right hand side of Equation (2-19) and expanding in a Taylor series about the mean elements yields the following to second order in the small parameter

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \varepsilon\left[F_i(\bar{\underline{a}},\bar{\lambda}) - \frac{3\bar{n}\,\delta_{i6}}{2\,\bar{a}_1}\,\eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\right]$$

$$+ \varepsilon^2\left[\sum_{k=1}^{6}\eta_{k,1}(\bar{\underline{a}},\bar{\lambda})\,\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_k} + \frac{15\bar{n}\,\delta_{i6}}{8\bar{a}_1^{\,2}}\,\eta_{1,1}^2(\bar{\underline{a}},\bar{\lambda}) - \frac{3\bar{n}\,\delta_{i6}}{2\bar{a}_1}\,\eta_{1,2}(\bar{\underline{a}},\bar{\lambda})\right]$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}23)$$

where δ_{ij} is the Kroenecker Delta function which is defined as

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad (2\text{-}24)$$

Equation (2-23) is very useful in subsequent constructions because it expresses the osculating element rates in terms of the mean elements.

Taking the time derivative of Equation (2-20) and using Equation (2-22) results in

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \varepsilon\left[A_{i,1}(\bar{\underline{a}}) + \bar{n}\,\frac{\partial\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{\lambda}}\right]$$

$$+ \varepsilon^2\left[A_{i,2}(\bar{\underline{a}}) + \bar{n}\,\frac{\partial\,\eta_{i,2}(\bar{\underline{a}},\bar{\lambda})}{\partial\lambda} + \sum_{k=1}^{6}A_{k,1}(\bar{\underline{a}})\,\frac{\partial\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{a}_k}\right]$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}25)$$

which is an alternate expression for the osculating element rates expressed in terms of the mean elements.

Equating equal powers of the small parameter in Equations (2-23) and (2-25), that is, matching the alternate approximations for the osculating element rates, results in the following expressions

$$A_{i,1}(\bar{\underline{a}}) + \bar{n}\ \frac{\partial\ \eta_{i,1}(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{\lambda}} = F_i(\bar{\underline{a}},\bar{\lambda}) - \frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\ \eta_{1,1}(\bar{\underline{a}},\lambda) \tag{2-26}$$

$$A_{i,2}(\bar{\underline{a}}) + \bar{n}\ \frac{\partial\ \eta_{i,2}(\bar{\underline{a}},\lambda)}{\partial\bar{\lambda}} + \sum_{k=1}^{6} A_{k,1}(\bar{\underline{a}})\ \frac{\partial\ \eta_{i,1}(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{a}_k}$$

$$= \sum_{k=1}^{6} \eta_{k,1}(\bar{\underline{a}},\bar{\lambda})\ \frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{a}_k} + \frac{15\bar{n}\ \delta_{i6}}{8\bar{a}_1^{\,2}}\ \eta_{1,1}^2(\bar{\underline{a}},\bar{\lambda})$$

$$- \frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\ \eta_{1,2}(\bar{\underline{a}},\bar{\lambda}) \tag{2-27}$$

The first and second order mean element rates are obtained by averaging the above equations with respect to the mean-mean longitude. Assuming that the slowly varying mean elements are constant over one orbit of the satellite and that the yet-to-be determined short periodic functions are "well-behaved" and 2π periodic [see Equation (2-21)], it follows

that the first and second order mean element rates are given by

$$A_{i,1}(\underline{\bar{a}}) = \frac{1}{2\pi}\int_0^{2\pi} F_i(\underline{\bar{a}},\bar{\lambda})\, d\bar{\lambda}$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}28)$$

$$A_{i,2}(\underline{\bar{a}}) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=1}^{6}\left[\eta_{k,1}(\underline{\bar{a}},\bar{\lambda})\ \frac{\partial F_i(\underline{\bar{a}},\bar{\lambda})}{\partial \bar{a}_k}\right] + \frac{15\bar{n}\,\delta_{i6}}{8\bar{a}_1^2}\ \eta_{1,1}^2\ (\underline{\bar{a}},\bar{\lambda})\right\} d\bar{\lambda} \qquad (2\text{-}29)$$

It is interesting to note that the second order mean element rates require the first order short periodic functions. For conservative disturbing forces (central and third body gravitational effects), the mean element rates can usually be obtained in analytical form. For non-conservative disturbing forces (drag and solar radiation pressure) a numerical approach usually has to be used. The real difficulty is associated with expressing the equations of motion for non-conservative perturbations in terms of orbital elements. The empirical nature of many realistic atmospheric density models is noted.

Knowing $A_{i,1}(\bar{\underline{a}})$ and $A_{i,2}(\bar{\underline{a}})$, Equations (2-26) and (2-27) represent a partial differential equation for the first and second order short periodic functions. Equations (2-26) and (2-27) can therefore be solved to within an arbitrary function of the five slowly varying mean elements. The requirement that the short periodic functions be 2π periodic with respect to the satellite frequency necessitates that this arbitrary function be zero. Equations (2-26), (2-27) and (2-29) also indicate the coupling of the mean fast variable ($\bar{\lambda}$) with the mean semimajor axis ($\bar{a}_1$) that occurs in the short periodic functions and in the second and higher order mean element rates.

From Equations (2-26) and (2-21), it is seen that the first order short periodic variation can be expressed as

$$\eta_{i,1}(\bar{\underline{a}},\bar{\lambda}) = \frac{1}{\bar{n}}\int\left\{F_i(\bar{\underline{a}},\bar{\lambda}) - A_{i,1}(\bar{\underline{a}}) - \left(\frac{3\bar{n}}{2\dot{a}_1}\delta_{\underline{i6}}\right)\eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\right\} d\bar{\lambda} \qquad (2\text{-}30)$$

References 5 and 6 use Equation (2-30) to obtain analytical expressions for the short periodics due to gravitational effects. When non-conservative disturbing forces are considered, Equation (2-30) usually becomes very difficult to solve.

The development of a semianalytical satellite theory based upon the generalized method of averaging proceeds in a very systematic manner. The Nth order mean element rates, $A_{i,N}(\bar{a})$, depend upon the N-1 previously determined short periodic functions. Using McClain's short periodic formulation (Ref. 3), the Nth order short periodic functions will depend upon the N previously determined mean element rates and the N-1 previously determined short periodic functions.

When more than one disturbing force is considered, the coupling between the various perturbations must be considered. Again, the generalized method of averaging permits a systematic approach to the development of a semianalytical satellite theory. To first order in the small parameters, the averaged equations of motion and the short periodic functions are obtained by summing, respectively, the first order averaged equations of motion and the first order short periodic functions for each separate perturbation (Ref. 3). For higher order terms, mixed (coupled) terms appear in both the averaged equations of motion and the short periodic functions. These higher order mixed terms will depend upon previously determined mean element rates and short periodic functions. For example, the averaged equations of motion to second order in all the small parameters require all the first order short periodic functions. A detailed development of the semianaly-

tical satellite theory for multiple disturbing forces is given in Chapter 3 of Reference 3.

In applying the generalized method of averaging to the problem of an additional phase angle (ϕ) in the disturbing function [$\varepsilon F_i(\bar{\underline{a}}, \bar{\lambda}, \phi)$], it is necessary to carefully consider the relationship between the two phase angles ($\bar{\lambda}$ and ϕ). If the phase angle ϕ can be considered independent of $\bar{\lambda}$, which is equivalent to being independent of time, the time-independent formulation, as given above, is valid. In other words, a time-independent formulation requires that the phase angles be independent of each other. Therefore the averaging operations, as given in Equations (2-21), (2-28) and (2-29), are performed holding ϕ constant. If ϕ and $\bar{\lambda}$ cannot be considered independent of each other, a time-dependent formulation must be used. A time-dependent formulation must therefore take into account the relationship between ϕ and $\bar{\lambda}$ when the generalized method of averaging is applied.

An an example of the difference between a time-independent and a time-dependent formulation, consider the case of a low altitude satellite perturbed by the Sun. In this case, the two phase angles would be the mean-mean longitude of the satellite ($\bar{\lambda}$) and the mean longitude of the Sun (λ'). It is interesting to note that changes in the Sun's position over a revolution of a low altitude satellite will cause variations

in the solar point mass effects, the atmospheric drag and the solar radiation pressure. Variations in the solar point mass effects and solar radiation pressure will be caused directly by changes in the Sun's position. The variation in the drag perturbation will be caused by the changes in the density that results from the Sun's motion. These variations will naturally be very small. The two-body equation of motion for λ' is

$$\dot{\lambda}' = n' \tag{2-31}$$

where n' is the mean motion of the Sun's apparent motion about the Earth which will be assumed to be constant. Equation (2-31) has a solution of the following form

$$\lambda' = n'(t - t_o) + \lambda'_o \tag{2-32}$$

where t_o and λ'_o are the initial conditions. Similarly the two-body solution for $\bar{\lambda}$ can be written as

$$\bar{\lambda} = \bar{n}(t - t_o) + \bar{\lambda}_o \tag{2-33}$$

where $\bar{\lambda}_o$ is the initial value of $\bar{\lambda}$ at time t_o. Solving for $(t - t_o)$ in both Equations (2-32) and (2-33), equating the results and rearranging yields

$$\lambda' = \frac{n'}{\bar{n}} (\bar{\lambda} - \bar{\lambda}_o) + \lambda'_o \tag{2-34a}$$

This can be written in terms of the period of the Sun (P') and the period of the satellite (P) as

$$\lambda' = \frac{P}{P'} (\bar{\lambda} - \bar{\lambda}_o) + \lambda'_o \qquad (2\text{-}34b)$$

Equations (2-34a) and (2-34b) show the two-body relationship between $\bar{\lambda}$ and λ'. For a low altitude satellite with a period of 1.5 hours, the ratio $\frac{P}{P'}$ is small, i.e., approximately 0.00017. It is reasonable to assume that λ' is independent of time ($\dot{\lambda}' \sim 0$) over a revolution of $\bar{\lambda}$ and therefore apply the time-independent formulation. Physically this assumption is equivalent to holding the Sun's position fixed over the averaging operation. It is also noted that in the generalized method of averaging, the five slowly varying mean elements ($\bar{\underline{a}}$) are also assumed to be constant over the averaging operation.

As the altitude of the satellite is increased or if the third body becomes the moon, the factor $\frac{P}{P'}$ increases. At some point, the assumption of holding λ' constant over a revolution of the satellite is no longer valid. At this point a time-dependent formulation is needed.

In order to be complete in this section, the two-body relation between $\bar{\lambda}$ and the Greenwich hour angle (θ) is presented. Assuming the angular rate of the Earth to be constant

and equal to ω_e, the two-body relation between θ and $\bar{\lambda}$ is derived in a similar manner to the derivation of Equation (2-34a). The result is

$$\theta = \frac{\omega_e}{\bar{n}} (\bar{\lambda} - \bar{\lambda}_o) + \theta_o \qquad (2\text{-}35)$$

Equations (2-34a) and (2-35) are useful in deciding what type of formulation to use.

2.2 A First Order Short Periodic Theory

A first order short periodic theory based upon a Fourier series analysis is presented in this section. A time-independent formulation and a weak time-dependent formulation are developed. These formulations are applicable to perturbations due to zonal terms of the gravitation potential, third body point mass effects for low and medium altitude satellites, atmospheric drag and solar radiation pressure. The short periodic development presented allows a numerical implementation for all applicable perturbations and an analytical implementation for conservative perturbations.

2.2.1 A Time-Independent Formulation

First order short periodic functions could be obtained from the McClain formulation as given in Equation (2-30). As noted previously, this formulation usually becomes very

difficult for non-conservative forces such as drag and solar radiation pressure. This section develops another method of obtaining the short periodic functions which is based upon a Fourier series analysis.

Assume that the osculating element rates can be expressed in a Fourier series in the mean fast variable. Since the VOP equations of motion, $F_i(\underline{a},\lambda)$, are of bounded variations over one revolution of the satellite, the assumed Fourier series will converge to $\dot{a}_i$ (Ref. 70). This assumption is expressed as follows to first order in the small parameter

$$\dot{a}_i = X_{io} + \varepsilon \sum_{\sigma=1}^{\infty} [X_{i\sigma} \cos(\sigma\bar{\lambda}) + Z_{i\sigma} \sin(\sigma\bar{\lambda})]$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}36a)$$

where

$$X_{io} = \frac{1}{2\pi} \int_0^{2\pi} \dot{a}_i \, d\bar{\lambda} \qquad (2\text{-}36b)$$

$$\varepsilon X_{i\sigma} = \frac{1}{\pi} \int_0^{2\pi} \dot{a}_i \cos(\sigma\bar{\lambda}) \, d\bar{\lambda} \qquad (2\text{-}36c)$$

$$\varepsilon Z_{i\sigma} = \frac{1}{\pi} \int_0^{2\pi} \dot{a}_i \sin(\sigma\bar{\lambda}) \, d\bar{\lambda} \qquad (2\text{-}36d)$$

Substitution of Equation (2-23), taken only to first order in the small parameter, into Equation (2-36b) yields

$$X_{io} = \frac{1}{2\pi} \int_0^{2\pi} \left[\bar{n}\, \delta_{i6} + \varepsilon\, F_i(\bar{\underline{a}}, \bar{\lambda}) - \left(\frac{3\bar{n}\, \delta_{i6}}{2\, \bar{a}_1} \right) \varepsilon \eta_{1,1}\, (\bar{\underline{a}}, \bar{\lambda}) \right] d\bar{\lambda}$$

where δ_{i6} is the Kroenecker Delta function as defined in Equation (2-24). Assuming that the five slowly varying mean elements are constant over one revolution of the satellite and applying Equation (2-21) results in

$$X_{io} = \bar{n}\, \delta_{i6} + \frac{1}{2\pi} \int_0^{2\pi} \varepsilon\, F_i(\bar{\underline{a}}, \bar{\lambda})\, d\bar{\lambda} \qquad (2\text{-}37)$$

From Equations (2-22) and (2-28), it is seen that X_{io} is the mean element rate, i.e.,

$$X_{io} = \dot{\bar{a}}_i \qquad (2\text{-}38)$$

Equation (2-36a) can therefore be written as

$$\dot{a}_i = \dot{\bar{a}}_i + \varepsilon \sum_{\sigma=1}^{\infty} [X_{i\sigma} \cos(\sigma\bar{\lambda}) + Z_{i\sigma} \sin(\sigma\bar{\lambda})] \qquad (2\text{-}39)$$

Likewise, Equation (2-25) can be rewritten to first order as

$$\dot{a}_i = \bar{\dot{a}}_i + \varepsilon \left[\frac{\partial \eta_{i,1}(\bar{\underline{a}}, \bar{\lambda})}{\partial \bar{\lambda}} \right] \bar{n} \qquad (2\text{-}40)$$

Equating Equations (2-39) and (2-40) and cancelling the mean element rates yields the following partial differential equations for the short periodic functions

$$\bar{n} \left[\frac{\partial \varepsilon \eta_{i,1}(\bar{\underline{a}}, \bar{\lambda})}{\partial \bar{\lambda}} \right] = \sum_{\sigma=1}^{\infty} [\varepsilon X_{i\sigma} \cos(\sigma\bar{\lambda}) + \varepsilon Z_{i\sigma} \sin(\sigma\bar{\lambda})]$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}41)$$

Integrating Equation (2-41) with respect to $\bar{\lambda}$ results in

$$\varepsilon \eta_{i,1}(\bar{\underline{a}}, \bar{\lambda}) = \sum_{\sigma=1}^{\infty} \left[\frac{\varepsilon X_{i\sigma}}{\sigma \bar{n}} \sin(\sigma\bar{\lambda}) - \frac{\varepsilon Z_{i\sigma}}{\sigma \bar{n}} \cos(\sigma\bar{\lambda}) \right] + E_i(\bar{\underline{a}})$$

where $E_i(\bar{\underline{a}})$ is an arbitrary function of the five slowly varying elements. Applying the requirement that the short periodic variations be 2π periodic and have zero mean value, as given in Equation (2-21), results in $E_i(\bar{\underline{a}})$ being equal to zero. Therefore

$$\varepsilon \eta_{i,1}(\bar{\underline{a}}, \bar{\lambda}) = \sum_{\sigma=1}^{\infty} \left[\frac{\varepsilon X_{i\sigma}}{\sigma \bar{n}} \sin(\sigma\bar{\lambda}) - \frac{\varepsilon Z_{i\sigma}}{\sigma \bar{n}} \cos(\sigma\bar{\lambda}) \right] \qquad (2\text{-}42)$$

For convenience the following definitions are made

$$\varepsilon\, C_{i\sigma} \equiv \frac{\varepsilon\, X_{i\sigma}}{\sigma\bar{n}}$$
$$\varepsilon\, D_{i\sigma} \equiv \frac{\varepsilon\, Z_{i\sigma}}{\sigma\bar{n}} \qquad (2\text{-}43)$$

From Equations (2-23), (2-36), and (2-43), the coefficients $\varepsilon\, C_{i\sigma}$ and $\varepsilon\, D_{i\sigma}$ can be written as

$$\varepsilon\, C_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon\, F_i(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\, d\bar{\lambda} + \frac{\delta_{i6}}{\sigma\pi}\int_0^{2\pi}\cos(\sigma\bar{\lambda})\, d\lambda - \frac{3\,\delta_{i6}}{2\,\sigma\bar{a}_1\pi}\int_0^{2\pi}\varepsilon\,\eta_{1,1}(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\, d\bar{\lambda}$$

$$\varepsilon\, D_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon\, F_i(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda})\, d\bar{\lambda} + \frac{\delta_{i6}}{\sigma\pi}\int_0^{2\pi}\sin(\sigma\bar{\lambda})\, d\bar{\lambda} - \frac{3\,\delta_{i6}}{2\,\sigma\bar{a}_1\pi}\int_0^{2\pi}\varepsilon\,\eta_{1,1}(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda})\, d\bar{\lambda}$$

Noting that

$$\frac{1}{\pi}\int_0^{2\pi}\varepsilon\,\eta_{1,1}(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\, d\bar{\lambda} = -\varepsilon\, D_{1\sigma}$$

$$\frac{1}{\pi}\int_0^{2\pi} \varepsilon\, \eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\, \sin(\sigma\bar{\lambda})\, d\bar{\lambda} = \varepsilon\, C_{1\sigma}$$

results in the following expression for the short periodic coefficients

$$\varepsilon\, C_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon\, F_i(\bar{\underline{a}},\bar{\lambda})\, \cos(\sigma\bar{\lambda})\, d\bar{\lambda} + \left(\frac{3\,\varepsilon\, D_{1\sigma}}{2\,\sigma\bar{a}_1}\right)\delta_{i6}$$

$$\varepsilon\, D_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon\, F_i(\bar{\underline{a}},\bar{\lambda})\, \sin(\sigma\bar{\lambda})\, d\lambda - \left(\frac{3\,\varepsilon\, C_{1\sigma}}{2\,\sigma\bar{a}_1}\right)\delta_{i6} \qquad (2\text{-}44)$$

where i goes from 1 to 6 and δ_{i6} is the Kroenecker Delta function [Equation (2-24)]. The short periodic functions can therefore be written as

$$\varepsilon\, \eta_{i,1}(\bar{\underline{a}},\bar{\lambda}) = \sum_{\sigma=1}^{\infty} [\varepsilon\, C_{i\sigma}\, \sin(\sigma\bar{\lambda}) - \varepsilon\, D_{i\sigma}\, \cos(\sigma\bar{\lambda})] \qquad (2\text{-}45)$$

Equations (2-44) and(2-45) represent the first order short periodic variation. This is summarized in Table 2-1.

As was stated earlier, Equations (2-44) and (2-45) were derived assuming that λ is independent of any other phase angle. Such an assumption results in a time-independent formulation. A time-independent formulation basically assumes that the Earth or third body is not rotating or moving during the averaging interval. It turns out that this formulation is applicable to zonal terms of the gravitational potential, drag, solar-radiation pressure and third-body effects for low altitude satellites. For high precision orbit generation, tesseral terms of the gravitational potential and third body effects for medium and high altitude satellites are not amenable to the above formulation due to the presence of a second phase angle which cannot be considered constant over the averaging interval.

From Equation (2-44) it can be seen that the short periodic coefficients, $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$, are functions of the five slowly varying mean elements. The short periodic coefficients are therefore slowly varying. This should make it possible to evaluate the coefficients at only a few points during a specified time span and then interpolate in order to determine the coefficients at any time within the time span. At the present time it appears that the short periodic coefficients need only be evaluated on the integration grid of the mean elements. Since the integration grid of the mean elements is on the order of one day, the computational cost

Table 2-1

FIRST ORDER SHORT PERIODIC FUNCTIONS

$$\varepsilon\, \eta_{i,1}(\bar{\underline{a}},\bar{\lambda}) = \sum_{\sigma=1}^{\infty} [\varepsilon\, C_{i\sigma} \sin(\sigma\bar{\lambda}) - \varepsilon\, D_{i\sigma} \cos(\sigma\bar{\lambda})]$$

$$\varepsilon\, C_{i\sigma} = \frac{1}{\sigma\bar{n}\pi} \int_0^{2\pi} \varepsilon\, F_i(\bar{\underline{a}},\bar{\lambda}) \cos(\sigma\bar{\lambda})\, d\bar{\lambda} + \left(\frac{3\, \varepsilon\, D_{1\sigma}}{2\, \sigma\, \bar{a}_1}\right) \delta_{i6}$$

$$\varepsilon\, D_{i\sigma} = \frac{1}{\sigma\bar{n}\pi} \int_0^{2\pi} \varepsilon\, F_i(\bar{\underline{a}},\bar{\lambda}) \sin(\sigma\bar{\lambda})\, d\bar{\lambda} - \left(\frac{3\, \varepsilon\, C_{1\sigma}}{2\, \sigma\, \bar{a}_1}\right) \delta_{i6}$$

$i = 1,2,3,4,5,6$

associated with the determination of the short periodic functions should be moderate.

For a numerical averaging satellite theory, the computation of the short periodic coefficients will be determined by a quadrature algorithm. The major portion of the integrand of Equation (2-44), $F_i(\bar{\underline{a}},\bar{\lambda})$, will have already been calculated at the abscissas of the quadrature algorithm in the determination of the functions $A_{i,1}(\bar{\underline{a}})$. By saving the integrand of Equation (2-28) at the abscissas of the quadrature algorithm, multiplying by $\sin(\sigma\bar{\lambda})$ or $\cos(\sigma\bar{\lambda})$, and then applying the same

quadrature algorithm, the evaluation of the short periodic coefficients can be obtained at minimum costs. Lutsky and Uphoff (Ref. 20) have stated that the use of recursion relations for $\sin(\sigma\bar{\lambda})$ and $\cos(\sigma\bar{\lambda})$ will also reduce the computer costs. Multiplication of $\epsilon F_i(\bar{\underline{a}},\bar{\lambda})$ by $\sin(\sigma\bar{\lambda})$ may introduce frequency components that could require a higher order quadrature algorithm. In other words, the number of coefficients and quadrature order must be considered jointly.

Another interesting fact about the formulation of the short periodics as given in Table 2-1 is the coupling between the fast variable and the semimajor axis. The coupling is through the short periodic coefficient ($C_{6\sigma}$ and $D_{6\sigma}$) and is rather easy to implement. It appears that Reference 20 neglected this term. This is also the term that Kaula described as a second order short periodic term in the mean anomaly [see Equation (3.115) in Ref. 21]. Neglecting this term significantly degrades the accuracy of the semianalytical orbit generator.

When the disturbing force is derivable from a potential, analytical expressions are usually possible for the short periodic coefficients. The Lagrangian VOP equations of motion can be written as

$$\varepsilon F_i(\underline{a},\lambda) = - \sum_{\ell=1}^{6} (a_i, a_\ell) \frac{\partial R}{\partial a_\ell}$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}46)$$

where

(a_i, a_ℓ) = Poisson brackets

$R = R(\underline{a},\lambda)$ = disturbing potential

With the use of Leibnitz's rule and the properties of Poisson brackets, Equation (2-28) can be written as

$$\varepsilon A_{i,1}(\bar{\underline{a}}) = - \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \bar{R}}{\partial \bar{a}_\ell}$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}47)$$

where

$$\bar{R} = \frac{1}{2\pi} \int_0^{2\pi} R(\bar{\underline{a}},\bar{\lambda})\, d\bar{\lambda} \qquad (2\text{-}48)$$

Substitution of Equation (2-46) into Equation (2-44) results in the following expressions for the short periodic coefficients

$$\varepsilon\, C_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi}\left\{-\sum_{\ell=1}^{5}(\bar{a}_i,\bar{a}_\ell)\frac{\partial}{\partial\bar{a}_\ell}\left[R(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\right]\right.$$

$$\left.-(\bar{a}_i,\bar{\lambda})\left[\frac{\partial R(\underline{\bar{a}},\bar{\lambda})}{\partial\bar{\lambda}}\cos(\sigma\bar{\lambda})\right]\right\}d\bar{\lambda}$$

$$+\left(\frac{3\,\varepsilon\,D_{1\sigma}}{2\,\sigma\bar{a}_1}\right)\delta_{i6} \qquad (2\text{-}49)$$

$$\varepsilon\, D_{i\sigma} = \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi}\left\{-\sum_{\ell=1}^{5}(\bar{a}_i,\bar{a}_\ell)'\frac{\partial}{\partial\bar{a}_\ell}\left[R(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda})\right]\right.$$

$$\left.-(\bar{a}_i,\bar{\lambda})\left[\frac{\partial R(\underline{\bar{a}},\bar{\lambda})}{\partial\bar{\lambda}}\sin(\sigma\bar{\lambda})\right]\right\}d\bar{\lambda}$$

$$-\left(\frac{3\,\varepsilon\,C_{1\sigma}}{2\,\sigma\bar{a}_1}\right)\delta_{i6}$$

Note that

$$\left[\frac{\partial R(\underline{\bar{a}},\bar{\lambda})}{\partial\bar{\lambda}}\right]\cos(\sigma\bar{\lambda}) = \frac{\partial\left[R(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\right]}{\partial\bar{\lambda}}$$

$$+\,\sigma R(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda}) \qquad (2\text{-}50)$$

$$\left[\frac{\partial R(\underline{\bar{a}},\bar{\lambda})}{\partial\bar{\lambda}}\right]\sin(\sigma\bar{\lambda}) = \frac{\partial\left[R(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda})\right]}{\sigma\bar{\lambda}}$$

$$-\,\sigma R(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})$$

Substitution of Equation (2-50) into Equation (2-49) and applying Leibnitz's rule and the properties of Poisson Brackets, the short periodic coefficients can be written as

$$\varepsilon C_{i\sigma} = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R \cos(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} - \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R \sin(\sigma\bar{\lambda})]} + \left(\frac{3 \varepsilon D_{1\sigma}}{2 \sigma \bar{a}_1}\right) \delta_{i6}$$

$$\varepsilon D_{i\sigma} = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R \sin(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} + \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R \cos(\sigma\bar{\lambda})]} - \left(\frac{3 \varepsilon C_{1\sigma}}{2 \sigma \bar{a}_1}\right) \delta_{i6} \qquad (2\text{-}51)$$

where

$$\overline{[R \cos(\sigma\bar{\lambda})]} = \frac{1}{\pi} \int_0^{2\pi} R(\bar{\underline{a}}, \bar{\lambda}) \cos(\sigma\bar{\lambda}) \, d\bar{\lambda}$$

$$\overline{[R \sin(\sigma\bar{\lambda})]} = \frac{1}{\pi} \int_0^{2\pi} R(\bar{\underline{a}}, \bar{\lambda}) \sin(\sigma\bar{\lambda}) \, d\bar{\lambda} \qquad (2\text{-}52)$$

The disturbing potential due to the non-sphericity of the Earth's gravitational field and third body point mass effects are given in terms of a complex disturbing potential (Refs. 3, 4, 11, 13 and 68). Functionally this can be expressed as

$$R(\underline{\bar{a}},\bar{\lambda}) = \text{Real}\{U^*(\underline{\bar{a}},\bar{\lambda})\} \tag{2-53}$$

where $U^*(\underline{\bar{a}},\bar{\lambda})$ is the complex disturbing potential. Substitution of Equation (2-53) into (2-52) and interchanging the order of the operations yields

$$\overline{[R\cos(\sigma\bar{\lambda})]} = \text{Real}\left\{\frac{1}{\pi}\int_0^{2\pi} U^*(\underline{\bar{a}},\bar{\lambda})\cos(\sigma\bar{\lambda})\,d\bar{\lambda}\right\} \tag{2-54}$$

$$\overline{[R\sin(\sigma\bar{\lambda})]} = \text{Real}\left\{\frac{1}{\pi}\int_0^{2\pi} U^*(\underline{\bar{a}},\bar{\lambda})\sin(\sigma\bar{\lambda})\,d\bar{\lambda}\right\}$$

It is noted that

$$\cos(\sigma\bar{\lambda}) = \frac{\exp(j\sigma\bar{\lambda}) + \exp(-j\sigma\bar{\lambda})}{2}$$

$$\sin(\sigma\bar{\lambda}) = \frac{\exp(j\sigma\bar{\lambda}) - \exp(-j\sigma\bar{\lambda})}{2j}$$

$$\text{Real}\{-jx\} = \text{Im}\{x\}$$

where

$$\exp(j\phi) = \cos(\phi) + j\sin(\phi)$$

$$\text{Real}\{x\} \equiv \text{Real part of } x$$

$$\text{Im}\{x\} \equiv \text{Imaginary part of } x$$

Applying the above expressions to Equation (2-54) gives the following

$$\overline{[R\cos(\sigma\bar{\lambda})]} = \text{Real}\left\{\frac{1}{2\pi}\int_0^{2\pi}[U^*(\bar{\underline{a}},\bar{\lambda})\exp(j\sigma\bar{\lambda}) + U^*(\bar{\underline{a}},\bar{\lambda})\exp(-j\sigma\bar{\lambda})]\,d\bar{\lambda}\right\} \tag{2-55}$$

$$\overline{[R\sin(\sigma\bar{\lambda})]} = \text{Im}\left\{\frac{1}{2\pi}\int_0^{2\pi}[U^*(\bar{\underline{a}},\bar{\lambda})\exp(j\sigma\bar{\lambda}) - U^*(\bar{\underline{a}},\bar{\lambda})\exp(-j\sigma\bar{\lambda})]\,d\bar{\lambda}\right\}$$

A short periodic coefficient generator, S_x, is defined as follows

$$S_x = \frac{1}{2\pi}\int_0^{2\pi} U^*(\bar{\underline{a}},\bar{\lambda})\exp(jx\bar{\lambda})\,d\bar{\lambda} \tag{2-56}$$

With the use of the short periodic coefficient generator, Equation (2-55) can be rewritten as

$$\begin{aligned}\overline{[R\cos(\sigma\bar{\lambda})]} &= \text{Real}\{S_\sigma + S_{-\sigma}\}\\ \overline{[R\sin(\sigma\bar{\lambda})]} &= \text{Im}\{S_\sigma - S_{-\sigma}\}\end{aligned} \tag{2-57}$$

The analytical form of the short periodic coefficients due to a conservative disturbing force is summarized in Table

2-2. Naturally the analytical short periodic coefficients are also functions of the five slowly varying mean elements. Therefore, as discussed before, it should be possible to evaluate the analytical short periodic coefficients only on the integration grid and use an interpolator to obtain the coefficients at the required output times. The coupling of the fast variable with the semimajor axis is still present.

Analytical expressions for the short periodic coefficient generator, S_x, can usually be obtained. In this section S_x will be obtained for zonal terms of the gravitational field and for third body effects.

Zonal Terms

The complex disturbing potential for the non-sphericity of the Earth's gravitational field can be expressed in terms of equinoctial variables as (Refs. 3, 4, 11 and 13)

$$U^*(\underline{a},\lambda,\Theta) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} U^*_{nm}(\underline{a},\lambda,\Theta) \qquad (2\text{-}58a)$$

where $U^*_{nm}(\underline{a},\lambda,\Theta)$ is the disturbing potential due to a spherical harmonic pair of the gravitational field. $U^*_{nm}(\underline{a},\lambda,\Theta)$ is given as

Table 2-2

Analytical Short Periodic Coefficients
for Conservative Disturbing Forces

$$\varepsilon C_{i\sigma} = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R \cos(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} - \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R \sin(\sigma\bar{\lambda})]}$$

$$+ \left(\frac{3 \varepsilon D_{1\sigma}}{2 \sigma \bar{a}_1}\right) \delta_{i6}$$

$$\varepsilon D_{i\sigma} = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R \sin(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} + \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R \cos(\sigma\bar{\lambda})]}$$

$$- \left(\frac{3 \varepsilon C_{1\sigma}}{2 \sigma \bar{a}_1}\right) \delta_{i6}$$

$(\bar{a}_i, \bar{a}_\ell)$ = Poisson Bracket

$\overline{[R \cos(\sigma\bar{\lambda})]} = \text{Real}\{S_\sigma + S_{-\sigma}\}$

$\overline{[R \sin(\sigma\bar{\lambda})]} = \text{Im}\{S_\sigma - S_{-\sigma}\}$

$$S_x = \frac{1}{2\pi} \int_0^{2\pi} U^*(\underline{\bar{a}}, \bar{\lambda}) \exp(jx\bar{\lambda}) \, d\bar{\lambda}$$

$R = R(\underline{a}, \lambda)$ = disturbing potential = $\text{Real}\{U^*(\underline{a}, \lambda)\}$

$$U^*_{nm}(\underline{a},\lambda,\Theta) = \left(\frac{\mu}{a}\right)\left(\frac{R_e}{a}\right)^n C^*_{nm} \sum_{s=-n}^{n} V^m_{n,s}\, S^{(m,s)}_{2n}(p,q) \cdot$$

$$\cdot \sum_{\tau=-\infty}^{\infty} Y^{-n-1,s}_{\tau}(h,k)\, \exp[j(\tau\lambda - m\Theta)] \qquad (2\text{-}58b)$$

where

μ	=	Earth's gravitational constant
R_e	=	Earth's equatorial radius
C^*_{nm}	=	$C'_{nm} - jS'_{nm}$ = complex spherical pair, i.e., $C'_{no} = -J_n$
Θ	=	Greenwich hour angle
$V^m_{n,s}$	=	coefficient which is not a function of the orbital elements
$S^{(m,s)}_{2n}(p,q)$	=	functions introduced by the rotation from the equatorial frame to the orbital frame
$Y^{-n-1,s}_{\tau}(h,k)$	=	Modified Hansen coefficients introduced when the potential is expressed in terms of the mean longitude

The disturbing potential due to an arbitrary zonal term ($m = 0$) does not depend upon Θ and is therefore written as

$$U^*_{no}(\underline{a},\lambda) = -\sum_{s=-n}^{n} \sum_{\tau=-\infty}^{\infty} \left(\frac{\mu}{a}\right)\left(\frac{R_e}{a}\right)^n J_n V^0_{n,s} \cdot$$

$$\cdot\ S^{(0,s)}_{2n}(p,q)\ Y^{-n-1,s}_{\tau}(h,k)\ \exp(j\tau\lambda) \qquad (2\text{-}59)$$

Substituting Equation (2-59) into Equation (2-56) allows the determination of the short periodic coefficient generator due to an arbitrary zonal term. This can be written as

$$S_x = -\sum_{s=-n}^{n} \sum_{\tau=-\infty}^{\infty} \left(\frac{\mu}{\bar{a}}\right)\left(\frac{R_e}{\bar{a}}\right)^n J_n V^0_{n,s}\ S^{(0,s)}_{2n}(\bar{p},\bar{q}) \cdot$$

$$\cdot\ Y^{-n-1,s}_{\tau}(\bar{h},\bar{k}) \left\{\frac{1}{2\pi}\int_0^{2\pi} \exp[j(\tau\bar{\lambda}+x\bar{\lambda})]\ d\bar{\lambda}\right\}$$

The integral in the above expression will vanish for all values of τ and x except when $\tau = -x$. Therefore, the short periodic coefficient generator due to an arbitrary zonal term is the following

$$S_x = -\sum_{s=-n}^{n} \left(\frac{\mu}{\bar{a}}\right)\left(\frac{R_e}{\bar{a}}\right)^n J_n V^0_{n,s}\ S^{(0,s)}_{2n}(\bar{p},\bar{q}) \cdot$$

$$\cdot\ Y^{-n-1,s}_{-x}(\bar{h},\bar{k}) \qquad (2\text{-}60)$$

It is noted that for a small eccentricity (e), the modified Hansen coefficient, $Y_{-x}^{-n-1,s}(\bar{h},\bar{k})$, is proportional to $e^{|s+x|}$. This should therefore allow a mechanism for determining the number of coefficients (σ_{max}) required in the short periodic functions.

Third Body Effects

The complex disturing potential for third body point mass effects can be expressed in terms of equinoctial variables as (Refs. 4, 11 and 68)

$$U^*(\underline{a},\lambda) = \frac{\mu'}{a'} \sum_{n=2}^{\infty} \left(\frac{a}{a'}\right)^n \sum_{m=0}^{n} K_m V_{n,m} \sum_{r=-n}^{n} V_{n,r}^{m} \cdot$$

$$\cdot\ S_{2n}^{(m,r)}(\alpha,\beta,\gamma) \sum_{s=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} Y_s^{-n-1,r}(h',k') \cdot$$

$$\cdot\ Y_\tau^{n,-m}(h,k)\ \exp[j(\tau\lambda + s\lambda')] \qquad (2\text{-}61)$$

where

μ' = third body gravitational constant

$\underline{a}',\lambda'$ = third body equinoctial orbital elements

α,β,γ = direction cosines of third body orbital frame with respect to satellite orbital frame

$K_m, V_{n,m}, V^m_{n,r}$ = coefficients which are not functions of the orbital element

$S^{(m,r)}_{2n}(\alpha,\beta,\gamma)$ = functions introduced by the rotation between the third body frame and the satellite frame

$Y^{n-1,r}_s(h',k'), Y^{n,-n}_\tau(h,k)$ = Modified Hansen coefficients

References 4, 11 and 68 explain the third body potential in much more detail.

Substitution of the above potential into Equation (2-56) yields the following short periodic coefficient generator for third body effects

$$S_x = \frac{\mu'}{a'} \sum_{n=2}^{\infty} \left(\frac{\bar{a}}{a'}\right)^n \sum_{m=0}^{n} K_m V_{n,m} \sum_{r=-n}^{n} V^m_{n,r} S^{(m,r)}_{2n}(\alpha,\beta,\gamma) \cdot$$

$$\cdot \sum_{s=-\infty}^{\infty} Y^{-n-1,r}_s(h',k') \; Y^{n,-m}_{-x}(\bar{h},\bar{k}) \exp(js\lambda') \qquad (2\text{-}62)$$

It should be noted that Equation (2-62) was derived assuming a time-independent formulation. In other words, the third body was held fixed during the averaging operation. For

medium and high altitude satellites this assumption can introduce significant errors. Section 2.2.2 will address this problem.

Again it is noted that for small eccentricities, the modified Hansen coefficients can be used as a means of determining the maximum number of short periodic coefficient required.

2.2.2 A Weak Time-Dependent Formulation*

When the disturbing function has more than one rapidly varying quantity, it may be necessary to apply a time-dependent formulation. In a time-dependent formulation the interaction of the different fast variables is taken into account. If the time rate of change of the additional fast variable is small in comparison with the satellite's mean motion [see Equations (2-34) and (2-35)], it is possible to develop a weak time-dependent formulation which takes advantage of the previously developed time-independent formulation. A weak time-dependent formulation is applicable to third body perturbations on medium altitude satellites. Tesseral terms of the Earth's gravitational field and third body perturbations on high altitude satellites probably cannot employ this

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* This concept originated in discussions with P.J. Cefola, W.D. McClain and L.W. Early of the CSDL in August 1979.

development because the time rate of change of the additional phase angle (Greenwich hour angle or mean longitude of the third body) would be too large in comparison to the satellite's mean motion.

The osculating VOP equations of motion can be written in functional form as

$$\dot{a}_i = n\,\delta_{i6} + \varepsilon\, F_i(\underline{a},\lambda,\phi) \qquad (i = 1,2,3,4,5,6) \tag{2-63}$$

where ϕ is the second phase angle* (or fast variable) associated with the VOP formulation. The near-identity transformation is assumed to be

$$a_i = \bar{a}_i + \varepsilon\, \eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi) \qquad (i = 1,2,3,4,5,6) \tag{2-64}$$

where $\varepsilon\, \eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi)$ is the short periodic function which will naturally depend upon both phase angles. The assumed form of the first order averaged equations of motion are

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \varepsilon\, A_{i,1}(\underline{\bar{a}},\phi) \tag{2-65}$$

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* In this development ϕ will be referred to as a phase angle but could also represent a fast variable.

where $\epsilon A_{i,1}(\bar{\underline{a}},\phi)$ are the mean element rates which will depend upon the five slowly varying mean element rates and also the phase angle ϕ. The mean element rates will not contain the satellite frequency.

Substitution of Equation (2-64) into the right hand side of Equation (2-63) and expanding in a Taylor series about the mean elements yields the following to first order

$$\dot{a}_i = \bar{n}\,\delta_{i6} + \epsilon\left[F_i(\bar{\underline{a}},\bar{\lambda},\phi) - \left(\frac{3\bar{n}\,\delta_{i6}}{2\bar{a}_1}\right)\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)\right] \tag{2-66}$$

An alternate expression for the osculating element rates can be obtained by taking the time derivative of Equation (2-64) and substituting Equation (2-65). This results in

$$\dot{a}_i = \bar{n}\,\delta_{i6} + \epsilon\left\{A_{i,1}(\bar{\underline{a}},\phi) + \bar{n}\left[\frac{\partial\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\bar{\lambda}}\right] + \dot{\phi}\left[\frac{\partial\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\phi}\right]\right\} \tag{2-67}$$

Equating equal powers of the small parameter in Equation (2-66) and (2-67) results in

$$\varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi) - \left(\frac{3\,\bar{n}\,\delta_{i6}}{2\bar{a}_1}\right)\varepsilon\,\eta_{1,1}(\bar{\underline{a}},\bar{\lambda},\phi) = \varepsilon\,A'_{i,1}(\bar{\underline{a}},\phi)$$

$$+\,\bar{n}\left[\frac{\partial\ \varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\bar{\lambda}}\right] + \dot{\phi}\left[\frac{\partial\ \varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\phi}\right] \qquad (2\text{-}68)$$

The mean element rates can be obtained from Equation (2-68) by applying the weak time-dependent assumption.

In this development the weak time-dependent assumption will specifically mean that ϕ can be approximated by ϕ_o over the averaging interval of $\bar{\lambda}$ (ϕ_o is the value of ϕ at the mid-point of the averaging interval). In other words, ϕ and the five slowly varying mean elements ($\bar{\underline{a}}$) are assumed constant over the averaging interval. In addition, it will also be assumed that the short periodic functions are 2π periodic with zero mean when the weak time-dependent assumption is applied. This is represented as

$$\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi_o)\,d\bar{\lambda} = 0 \qquad (2\text{-}69)$$

where ϕ_o is the value of ϕ when $\bar{\lambda}$ equals $\bar{\lambda}_o$.

Averaging Equation (2-68) with respect to $\bar{\lambda}$ yields

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, F_i(\underline{\bar{a}},\bar{\lambda},\phi)d\bar{\lambda} - \frac{1}{2\pi}\int_0^{2\pi}\left\{\left(\frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\right)\varepsilon\eta_{1,1}(\underline{\bar{a}},\bar{\lambda},\phi)\right\}d\bar{\lambda}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, A_{i,1}(\underline{\bar{a}},\phi)d\bar{\lambda} + \frac{1}{2\pi}\int_0^{2\pi}\left\{\bar{n}\left[\frac{\partial\ \varepsilon\eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi)}{\partial\bar{\lambda}}\right]\right.$$

$$\left. + \dot{\phi}\left[\frac{\partial\ \varepsilon\eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi)}{\partial\phi}\right]\right\}d\bar{\lambda}$$

To first order in the small parameter, this can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, F_i(\underline{\bar{a}},\bar{\lambda},\phi)d\bar{\lambda} - \frac{1}{2\pi}\int_0^{2\pi}\left\{\left(\frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\right)\varepsilon\eta_{1,1}(\underline{\bar{a}},\bar{\lambda},\phi)\right\}d\bar{\lambda}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, A_{i,1}(\underline{\bar{a}},\phi)\ d\bar{\lambda}$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d[\varepsilon\eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi)]}{dt}\right\}d\bar{\lambda}$$

Since the yet to be determined $\varepsilon\eta_{i,1}(\underline{\bar{a}},\bar{\lambda},\phi)$ is assumed to be a "well behaved" function, the order of the integration and differentiation can be interchanged in the above expression- This results in

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi)d\bar{\lambda} - \frac{1}{2\pi}\int_0^{2\pi} \left\{\left(\frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\right)\varepsilon\eta_{1,1}(\bar{\underline{a}},\bar{\lambda},\phi)\right\}d\bar{\lambda}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varepsilon A_{i,1}(\bar{\underline{a}},\phi)\ d\bar{\lambda}$$

$$+ \frac{d}{dt}\left\{\frac{1}{2\pi}\int_0^{2\pi} \varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)\ d\bar{\lambda}\right\}$$

Invoking the weak time-dependent assumption to the above expression results in

$$\frac{1}{2\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)d\bar{\lambda} - \left(\frac{3\bar{n}\ \delta_{i6}}{2\bar{a}_1}\right)\left\{\frac{1}{2\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon\eta_{1,1}(\bar{\underline{a}},\bar{\lambda},\phi_o)d\bar{\lambda}\right\}$$

$$= \varepsilon A_{i,1}(\bar{\underline{a}},\phi_o) + \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi_o)d\bar{\lambda}\right\}$$

Applying Equation (2-69) it is seen that

$$\varepsilon A_{i,1}(\bar{\underline{a}},\phi) = \frac{1}{2\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)\ d\bar{\lambda} \qquad (2\text{-}70)$$

It is noted that the weak time-dependent mean element rates are the same as would be obtained with a time-independent formulation. The difference between a time-independent formulation and a weak time-dependent formulation occurs only (to

first order) in the short periodic functions. This is due to the fact that in the weak time-dependent formulation, $\dot{\phi}$, does not equal zero [see Equation (2-66)].

As in the time-independent formulation, it is assumed that the osculating element rates can be expressed in a Fourier series in $\bar{\lambda}$. It is further assumed that the coefficients of the Fourier series depend upon the five slowly varying mean elements and ϕ. This is expressed as

$$\dot{a}_i = X_{io}(\bar{\underline{a}},\phi) + \varepsilon \sum_{\sigma=1}^{\infty} [X_{i\sigma}(\bar{\underline{a}},\phi) \cos(\sigma\bar{\lambda}) + Z_{i\sigma}(\bar{\underline{a}},\phi) \sin(\sigma\bar{\lambda})]$$

$$(i = 1,2,3,4,5,6) \qquad (2\text{-}71a)$$

where

$$X_{io}(\bar{\underline{a}},\phi) = \frac{1}{2\pi}\int_0^{2\pi} \dot{a}_i \, d\bar{\lambda} \qquad (2\text{-}71b)$$

$$\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi) = \frac{1}{\pi}\int_0^{2\pi} \dot{a}_i \cos(\sigma\bar{\lambda}) \, d\bar{\lambda} \qquad (2\text{-}71c)$$

$$\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi) = \frac{1}{\pi}\int_0^{2\pi} \dot{a}_i \sin(\sigma\bar{\lambda}) \, d\bar{\lambda} \qquad (2\text{-}71d)$$

Substitution of Equation (2-66) into Equation (2-71b) and using the weak time-dependent assumption, it is seen that

$$X_{io}(\bar{\underline{a}},\phi) = \bar{n}\,\delta_{i6} + \epsilon\, A_{i,1}(\bar{\underline{a}},\phi) \qquad (2\text{-}72a)$$

where $\epsilon\, A_{i,1}(\bar{\underline{a}},\phi)$ is defined in Equation (2-70). From Equation (2-65) it is seen that

$$X_{io}(\bar{\underline{a}},\phi) = \dot{\bar{a}}_i \qquad (2\text{-}72b)$$

Equating Equations (2-67) and (2-71a) and cancelling like terms yields the following partial differential equations for the short periodic functions

$$\bar{n}\left[\frac{\partial\, \epsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\bar{\lambda}}\right] + \dot{\phi}\left[\frac{\partial\, \epsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)}{\partial\phi}\right]$$

$$= \sum_{\sigma=1}^{\infty} [X_{i\sigma}(\bar{\underline{a}},\phi)\, \cos(\sigma\bar{\lambda})$$

$$+ Z_{i\sigma}(\bar{\underline{a}},\phi)\, \sin(\sigma\bar{\lambda})] \qquad (2\text{-}73)$$

The weak time-dependent short periodic functions are assumed to be of the following form

$$\epsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi) = \sum_{\sigma=1}^{\infty} [\epsilon U_{i\sigma}(\bar{\underline{a}},\phi)\, \sin(\sigma\bar{\lambda})$$

$$- \epsilon V_{i\sigma}(\bar{\underline{a}},\phi)\, \cos(\sigma\bar{\lambda})] \qquad (2\text{-}74)$$

where $\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)$ and $\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)$ are functions to be determined. The assumed form of the Equation (2-74) is not unreasonable since it was shown that the time-independent short periodic functions are of a similar form (see Table 2-1).

Substitution of the assumed form of $\varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi)$ into the left hand side of Equation (2-73) and rearranging yields

$$\sum_{\sigma=1}^{\infty}\left\{\left[\sigma\bar{n}\ \varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) - \dot{\phi}\left[\frac{\partial\ \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right]\right]\cos(\sigma\bar{\lambda})\right.$$

$$\left. + \left[\sigma\bar{n}\ \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) + \dot{\phi}\left[\frac{\partial\ \varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right]\right]\sin(\sigma\bar{\lambda})\right\}$$

$$= \sum_{\sigma=1}^{\infty}\left[\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)\cos(\sigma\bar{\lambda})\right.$$

$$\left. + \varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)\sin(\sigma\bar{\lambda})\right] \qquad (2\text{-}75)$$

Since the above expression must be true for all values of $\bar{\lambda}$, the following must be true

$$\sigma\bar{n}\ \varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) - \dot{\phi}\left[\frac{\partial\ \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right] = \varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)$$

$$\sigma\bar{n}\ \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) + \dot{\phi}\left[\frac{\partial\ \varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right] = \varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)$$

This can be rewritten as

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \frac{\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right]$$

(2-76)

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)}{\partial\phi}\right]$$

Equation (2-76) is a set of coupled partial differential equations for the time-dependent short periodic coefficients, $\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)$ and $\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)$. By substituting the expressions for $\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)$ and $\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)$ respectively into the right-hand side of Equation (2-76) it is seen that

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \frac{\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\left[\frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi}\right] - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\left[\frac{\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi^2}\right] + \cdots$$

(2-77)

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\left[\frac{\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi}\right] + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\left[\frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi^2}\right] - \cdots$$

Equation (2-77) shows that the time-dependent short periodic coefficients are represented by an alternating sign series. For a weak time-dependency, $\dot{\phi}$ will be much smaller than $\bar{n}$ which implies that the $\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^k$ terms tend rapidly towards zero as k becomes large.

The functions $\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)$ and $\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)$ are obtained by substitution of Equation (2-66) into Equations (2-71c) and (2-71d) and applying the weak time-dependent assumption. Noting that

$$\frac{1}{\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon\eta_{1,1}(\bar{\underline{a}},\bar{\lambda},\phi_o)\cos(\sigma\bar{\lambda})d\bar{\lambda} = -\varepsilon V_{1\sigma}(\bar{\underline{a}},\phi_o)$$

$$\frac{1}{\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon\eta_{1,1}(\bar{\underline{a}},\bar{\lambda},\phi_o)\sin(\sigma\bar{\lambda})d\bar{\lambda} = \varepsilon U_{1\sigma}(\bar{\underline{a}},\phi_o)$$

results in the following

$$\frac{\varepsilon \dot{X}_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} = \frac{1}{\sigma\bar{n}\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)\cos(\sigma\bar{\lambda})d\bar{\lambda} + \left(\frac{3\,\delta_{i6}}{2\,\sigma\,\bar{a}_1}\right)\varepsilon V_{1\sigma}(\bar{\underline{a}},\phi_o) \qquad (2\text{-}78)$$

$$\frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} = \frac{1}{\sigma\bar{n}\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)\sin(\sigma\bar{\lambda})d\bar{\lambda} - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\varepsilon U_{1\sigma}(\bar{\underline{a}},\phi_o) \tag{2-78}$$

Substitution of Equation (2-77) into Equation (2-78) results in

$$\frac{\varepsilon X_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} = \frac{1}{\sigma\bar{n}\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)\cos(\sigma\bar{\lambda})d\bar{\lambda} + \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\frac{\varepsilon Z_{1\sigma}(\bar{\underline{a}},\phi_o)}{\sigma\bar{n}} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\left[\frac{\varepsilon X_{1\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi}\right]_{\phi_o} + \ldots\right\} \tag{2-79}$$

$$\frac{\varepsilon Z_{i\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}} = \frac{1}{\sigma\bar{n}\pi}\int_{\bar{\lambda}_o-\pi}^{\bar{\lambda}_o+\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda},\phi_o)\cos(\sigma\bar{\lambda})d\bar{\lambda} - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\frac{\varepsilon X_{1\sigma}(\bar{\underline{a}},\phi_o)}{\sigma\bar{n}} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\left[\frac{\varepsilon Z_{1\sigma}(\bar{\underline{a}},\phi)}{\sigma\bar{n}}\right]}{\partial\phi}\right]_{\phi_o} - \ldots\right\}$$

For convenience the following definitions are made

$$\varepsilon C_{i\sigma} = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \varepsilon F_i(\underline{\bar{a}}, \bar{\lambda}, \phi_o) \cos(\sigma \bar{\lambda}) \, d\bar{\lambda}$$

$$+ \left(\frac{3 \delta_{i6}}{2 \sigma \bar{a}_1} \right) \varepsilon D_{1\sigma} \qquad (2\text{-}80)$$

$$\varepsilon D_{i\sigma} = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \varepsilon F_i(\underline{\bar{a}}, \bar{\lambda}, \phi_o) \sin(\sigma \bar{\lambda}) \, d\bar{\lambda}$$

$$- \left(\frac{3 \delta_{i6}}{2 \sigma \bar{a}_1} \right) \varepsilon C_{1\sigma}$$

Substitution of Equation (2-80) into Equation (2-79) and rearranging results in

$$\frac{\varepsilon X_{i\sigma}(\underline{\bar{a}}, \phi)}{\sigma \bar{n}} = \varepsilon C_{i\sigma} - \left(\frac{3 \delta_{i6}}{2 \sigma \bar{a}_1} \right) \left\{ \left(\frac{\dot{\phi}}{\sigma \bar{n}} \right) \left[\frac{\partial \varepsilon C_{1\sigma}}{\partial \phi} \right]_{\phi_o} \right.$$

$$\left. - \left(\frac{\dot{\phi}}{\sigma \bar{n}} \right)^2 \left[\frac{\partial^2 \varepsilon D_{1\sigma}}{\partial \phi^2} \right]_{\phi_o} + \ldots \right\} \qquad (2\text{-}81)$$

$$\frac{\varepsilon Z_{i\sigma}(\underline{\bar{a}}, \phi)}{\sigma \bar{n}} = \varepsilon D_{i\sigma} - \left(\frac{3 \delta_{i6}}{2 \sigma \bar{a}_1} \right) \left\{ \left(\frac{\dot{\phi}}{\sigma \bar{n}} \right) \left[\frac{\partial \varepsilon D_{1\sigma}}{\partial \phi} \right]_{\phi_o} \right.$$

$$\left. - \left(\frac{\dot{\phi}}{\sigma \bar{n}} \right)^2 \left[\frac{\partial^2 \varepsilon C_{1\sigma}}{\partial \phi^2} \right]_{\phi_o} + \ldots \right\}$$

The time-dependent short periodic coefficients are obtained by substitution of Equation (2-81) into Equation (2-77), viz.

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon C_{i\sigma} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon D_{i\sigma}}{\partial\phi}\right] - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\ \varepsilon C_{i\sigma}}{\partial\phi^2}\right] + \ldots - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon C_{1\sigma}}{\partial\phi}\right] + \ldots\right\} \quad (2\text{-}82)$$

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon D_{i\sigma} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon C_{i\sigma}}{\partial\phi}\right] + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\ \varepsilon D_{i\sigma}}{\partial\phi^2}\right] - \ldots - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon D_{1\sigma}}{\partial\phi}\right] + \ldots\right\}$$

Equation (2-82) is an expression for the weak time-dependent short periodic coefficients. The weak time-dependent short periodic functions are given in Equation (2-74). The coupling between the fast variable and the semimajor axis in Equation (2-82) was taken only to first power in $\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)$. Using the chain rule, Equation (2-82) can be rewritten as

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon C_{i\sigma} + \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon D_{i\sigma}}{\partial t}\right]$$

$$- \left(\frac{1}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\ \varepsilon C_{i\sigma}}{\partial t^2}\right] + \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon C_{1\sigma}}{\partial t}\right] + \cdots\right\} \quad (2\text{-}83)$$

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon D_{i\sigma} - \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon C_{i\sigma}}{\partial t}\right]$$

$$+ \left(\frac{1}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2\ \varepsilon D_{i\sigma}}{\partial t^2}\right] - \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial\ \varepsilon D_{1\sigma}}{\partial t}\right] + \cdots\right\}$$

The weak time-dependent short periodic formulation is summarized in Table 2-3. The weak time-dependent formulation in Table 2-3 is interesting in that it uses the previously defined time-independent short periodic coefficients ($\varepsilon C_{i\sigma}$, $\varepsilon D_{i\sigma}$) to define the weak time-dependent short periodic coefficients [$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)$, $\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)$]. The formulation as given in Table 2-3 also indicates that there is an additional coupling between the fast variable and the semimajor axis beyond the time-independent coupling as given in Equations (2-44) and (2-80).

Table 2-3

Weak Time-Dependent Short Periodic Functions

$$\varepsilon\eta_{i,1}(\bar{\underline{a}},\bar{\lambda},\phi) = \sum_{\sigma=1}^{\infty} [\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi)\sin(\sigma\bar{\lambda}) - \varepsilon V_{i\sigma}(\bar{\underline{a}},\phi)\cos(\sigma\bar{\lambda})]$$

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon C_{i\sigma} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{i\sigma}}{\partial \phi}\right] - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2 \varepsilon C_{i\sigma}}{\partial \phi^2}\right] + \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\bar{a}_1}\right)\left\{\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{1\sigma}}{\partial \phi}\right] + \cdots\right\}$$

$$= \varepsilon C_{i\sigma} + \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{i\sigma}}{\partial t}\right] - \left(\frac{1}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2 \varepsilon C_{i\sigma}}{\partial t^2}\right] + \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\bar{a}_1}\right)\left\{\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{1\sigma}}{\partial t}\right] + \cdots\right\}$$

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon D_{i\sigma} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{i\sigma}}{\partial \phi}\right] + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2 \varepsilon D_{i\sigma}}{\partial \phi^2}\right] + \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{1\sigma}}{\partial \phi}\right] + \cdots\right\}$$

$$= \varepsilon D_{i\sigma} - \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{i\sigma}}{\partial t}\right] + \left(\frac{1}{\sigma\bar{n}}\right)^2\left[\frac{\partial^2 \varepsilon D_{i\sigma}}{\partial t^2}\right] + \cdots$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left\{\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{1\sigma}}{\partial t}\right] + \cdots\right\}$$

Table 2-3

Weak Time-Dependent Short Periodic Functions
(cont.)

$$\varepsilon C_{i\sigma} = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \varepsilon F_i(\underline{\bar{a}}, \bar{\lambda}, \phi_o) \cos(\sigma\bar{\lambda})\, d\bar{\lambda} + \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \varepsilon D_{1\sigma}$$

$$\varepsilon D_{i\sigma} = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \varepsilon F_i(\underline{\bar{a}}, \bar{\lambda}, \phi_o) \sin(\sigma\bar{\lambda})\, d\bar{\lambda} - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \varepsilon C_{1\sigma}$$

The weak time-dependent short periodic coefficients will be a function of the five slowly varying mean elements and the phase angle ϕ. Since the weak time-dependent assumption requires that ϕ also be slowly varying, the time-dependent short periodic coefficients should be slowly varying. Therefore, as with the time-independent short periodic coefficients, it should be possible to determine the time-dependent short periodic coefficients on the averaged integration grid and interpolate to obtain them at the output time. Numerical results presented in Section 2.4 verify this point.

One of the requirements of the weak time-dependent assumption was that the short periodic functions be 2π periodic with zero mean when ϕ is held constant over the averaging interval [see Equation (2-69)]. This is equivalent to requiring that the short periodic functions do not contain m-daily or s-monthly/yearly frequencies [$(m\theta)$ or $(s\lambda')$ frequencies that appear in the disturbing potentials for tesserals or third body, see Equations (2-58b) or (2-61)]. Therefore, the m-daily or s-monthly/yearly frequencies must be in the mean element rates. In the case of tesseral perturbations this can require the stepsize of the AOG to be considerably less than one day. It is basically for this reason that the weak time-dependent formulation is not applicable to tesseral perturbations. In the case of third body perturbations on a medium

altitude satellite, the AOG should adequately determine the major s-monthly/yearly effects with a one to two day numerical integration time step. In this investigation a medium altitude satellite will refer to satellites with periods not greater than one day.

For a weak time-dependency, the term $\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2$ should be very small and the second derivative of the short periodic coefficients with respect to ϕ should also be small. It should therefore be possible to neglect $\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)^2$ terms in the expressions for $\varepsilon U_{i\sigma}(\underline{\bar{a}},\phi)$ and $\varepsilon V_{i\sigma}(\underline{\bar{a}},\phi)$. This results in

$$\varepsilon U_{i\sigma}(\underline{\bar{a}},\phi) = \varepsilon C_{i\sigma} + \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{i\sigma}}{\partial \phi}\right] - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{1\sigma}}{\partial \phi}\right] \tag{2-84a}$$

$$\varepsilon V_{i\sigma}(\underline{\bar{a}},\phi) = \varepsilon D_{i\sigma} - \left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{i\sigma}}{\partial \phi}\right] - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left(\frac{\dot{\phi}}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{1\sigma}}{\partial \phi}\right]$$

or equivalently

$$\varepsilon U_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon C_{i\sigma} + \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{i\sigma}}{\partial t}\right] - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{1\sigma}}{\partial t}\right] \tag{2-84b}$$

$$\varepsilon V_{i\sigma}(\bar{\underline{a}},\phi) = \varepsilon D_{i\sigma} - \left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon C_{i\sigma}}{\partial t}\right] - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right)\left(\frac{1}{\sigma\bar{n}}\right)\left[\frac{\partial \varepsilon D_{1\sigma}}{\partial t}\right]$$

The determination of the time-dependent short periodic coefficients requires the calculation of the time-independent short periodic coefficients and the various partial derivative of these coefficients with respect to time or ϕ. The calculation of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ would be done as previously described in Section 2.2.1 (numerical quadrature or analytically). The determination of the various partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ with respect to ϕ or t can be done numerically or analytically.

A numerical method of determining the time derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ is to use a finite-differencing technique (see Appendix B). Functionally this can be written as

$$\left[\frac{\partial \varepsilon C_{i\sigma}}{\partial t}\right] \sim \frac{\varepsilon C_{i\sigma}(t_o + \Delta t) - \varepsilon C_{i\sigma}(t_o)}{\Delta t} \tag{2-85}$$

$$\left[\frac{\partial \varepsilon D_{i\sigma}}{\partial t}\right] \sim \frac{\varepsilon D_{i\sigma}(t_o + \Delta t) - \varepsilon D_{i\sigma}(t_o)}{\Delta t}$$

The advantage of the finite-differencing technique is that much of the same software that is used to calculate $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ can be used to obtain their time derivatives. Once the time derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ are known, Equation (2-84b) can be used to determine the time-dependent short periodic coefficients.

Another numerical method that can be used to obtain the partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ with respect to ϕ is the quadrature approach. Taking the kth partial derivative of Equation (2-80) results in

$$\left[\frac{\partial^k \varepsilon C_{i\sigma}}{\partial \phi^k}\right] = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \left[\frac{\partial^k \varepsilon F_i(\bar{a}, \bar{\lambda}, \phi)}{\partial \phi^k}\right]_{\phi_o} \cos(\sigma \bar{\lambda}) \, d\bar{\lambda}$$

$$+ \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon D_{1\sigma}}{\partial \phi^k}\right] \tag{2-86}$$

$$\left[\frac{\partial^k \varepsilon D_{i\sigma}}{\partial \phi^k}\right] = \frac{1}{\sigma \bar{n} \pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \left[\frac{\partial^k \varepsilon F_i(\bar{a}, \bar{\lambda}, \phi)}{\partial \phi^k}\right]_{\phi_o} \sin(\sigma\bar{\lambda}) d\bar{\lambda}$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon C_{1\sigma}}{\partial \phi^k}\right] \qquad (2\text{-}86)$$

A quadrature algorithm would then be applied to Equation (2-86) in order to determine the partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ with respect to ϕ. Equations (2-82) or (2-84a) could then be used to obtain the time-dependent short periodic coefficients.

For conservative disturbing forces, an analytical approach is usually possible for determining the partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ with respect to ϕ. Taking the kth partial derivatives of Equation (2-51) and realizing that the Poisson Brackets to not depend upon ϕ and that the order of differentiation can be interchanged, the following expression results

$$\left[\frac{\partial^k \varepsilon C_{i\sigma}}{\partial \phi^k}\right] = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R^{(k)} \cos(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell}$$

$$- \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R^{(k)} \sin(\sigma\bar{\lambda})]} + \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon D_{1\sigma}}{\partial \phi^k}\right] \qquad (2\text{-}87)$$

$$\left[\frac{\partial^k \varepsilon D_{i\sigma}}{\partial \phi^k}\right] = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R^{(k)} \sin(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell}$$

$$+ \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R^{(k)} \cos(\sigma\bar{\lambda})]} - \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon C_{1\sigma}}{\partial \phi^k}\right] \tag{2-87}$$

where from Equation (2-52) it is seen that

$$\overline{[R^{(k)} \cos(\sigma\bar{\lambda})]} = \frac{1}{\pi} \int_0^{2\pi} \left[\frac{\partial^k R(\underline{\bar{a}}, \bar{\lambda}, \phi)}{\partial \phi^k}\right] \cos(\sigma\bar{\lambda})\ d\bar{\lambda} \tag{2-88}$$

$$\overline{[R^{(k)} \sin(\sigma\bar{\lambda})]} = \frac{1}{\pi} \int_0^{2\pi} \left[\frac{\partial^k R(\underline{\bar{a}}, \bar{\lambda}, \phi)}{\partial \phi^k}\right] \sin(\sigma\bar{\lambda})\ d\bar{\lambda}$$

When a complex potential, as given in Equation (2-53), is considered, it can be shown that [see the derivation of Equation (2-57)]

$$\overline{[R^{(k)} \cos(\sigma\bar{\lambda})]} = \text{Real}\{S_\sigma^{(k)} + S_{-\sigma}^{(k)}\} \tag{2-89}$$

$$\overline{[R^{(k)} \sin(\sigma\bar{\lambda})]} = \text{Im}\{S_\sigma^{(k)} - S_{-\sigma}^{(k)}\}$$

where

$$S_x^{(k)} = \frac{1}{2\pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \left[\frac{\partial^k U^*(\underline{\bar{a}}, \bar{\lambda}, \phi)}{\partial \phi^k}\right]_{\phi_o} \exp(jx\bar{\lambda})\ d\bar{\lambda} \tag{2-90}$$

Table 2-4 summarizes these results.

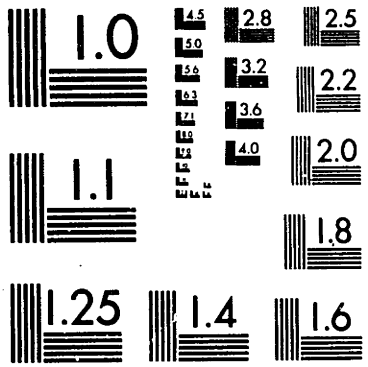

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Table 2-4

Analytical Partial Derivatives of the Short Periodic Coefficients for Conservative Disturbing Forces

$$\left[\frac{\partial^k \varepsilon C_{i\sigma}}{\partial \phi^k}\right] = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R^{(k)} \cos(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} - \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R^{(k)} \sin(\sigma\bar{\lambda})]}$$

$$+ \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon D_{1\sigma}}{\partial \phi^k}\right]$$

$$\left[\frac{\partial^k \varepsilon D_{i\sigma}}{\partial \phi^k}\right] = \frac{-1}{\sigma \bar{n}} \sum_{\ell=1}^{5} (\bar{a}_i, \bar{a}_\ell) \frac{\partial \overline{[R^{(k)} \sin(\sigma\bar{\lambda})]}}{\partial \bar{a}_\ell} + \frac{(\bar{a}_i, \bar{\lambda})}{\bar{n}} \overline{[R^{(k)} \cos(\sigma\bar{\lambda})]}$$

$$- \left(\frac{3\ \delta_{i6}}{2\ \sigma\ \bar{a}_1}\right) \left[\frac{\partial^k \varepsilon C_{1\sigma}}{\partial \phi^k}\right]$$

$$\overline{[R^{(k)} \cos(\sigma\bar{\lambda})]} = \text{Real}\{S_\sigma^{(k)} + S_{-\sigma}^{(k)}\}$$

$$\overline{[R^{(k)} \sin(\sigma\bar{\lambda})]} = \text{Im}\{S_\sigma^{(k)} - S_{-\sigma}^{(k)}\}$$

$$S_x^{(k)} = \frac{1}{2\pi} \int_{\bar{\lambda}_o - \pi}^{\bar{\lambda}_o + \pi} \frac{\partial^k U^*(\bar{a}, \bar{\lambda}, \phi)}{\partial \phi^k} \exp(jx\bar{\lambda})\ d\bar{\lambda}$$

$R = R(\underline{a}, \lambda, \phi)$ = disturbing potential = $\text{Real}\{U^*(\underline{a}, \lambda, \phi)\}$

For third body effects, the complex disturing potential was given in Equation (2-61). The additional phase angle in this case is the mean longitude of the third body (λ'). Taking the kth derivative of the complex potential with respect to ϕ and substituting into Equation (2-90) results in

$$S_x^{(k)} = \frac{\mu'}{a'} \sum_{n=2}^{\infty} \left(\frac{\bar{a}}{a'}\right)^n \sum_{m=0}^{n} K_m V_{n,m} \sum_{r=-n}^{n} V_{n,r}^{m} S_{2n}^{(m,r)}(\alpha,\beta,\gamma) \cdot$$

$$\cdot \sum_{s=-\infty}^{\infty} Y_s^{-n-1,r}(h',k') Y_{-x}^{n,-m}(\bar{h},\bar{k})(js)^x \cdot$$

$$\cdot \exp(js\lambda'_o) \qquad (2\text{-}91)$$

Equations (2-62) and (2-91) along with Tables 2-2, 2-3 and 2-4 will yield complete analytical expressions for the third body weak time-dependent short periodic coefficients.

2.3 A Second Order Drag Theory in the Averaged Equations of Motion

A semianalytical satellite theory to first order in the small parameter (the AOG and SPG) may not be sufficient for highly accurate ephemeris generation. This is especially true for low altitude satellites in which the second order long periodic and secular motion caused by oblateness and drag effects may be significant. Since the artificial satellite

problem is a non-linear dynamical system, the coupling between oblateness and drag must also be considered in the second order effects (Refs. 48 and 50).

This section extends the AOG to second order in the small parameters when two perturbing forces are considered. The general functional form of a second order AOG for two perturbations is developed first. Next, the oblateness/drag coupling is formulated in a general form. The general form of the second order AOG is simplified by use of the first order short periodic variations as given in Table 2-1. Finally, algorithmic considerations for the oblateness/drag coupling are discussed.

A Second Order AOG

The osculating VOP equations of motion for two perturbing forces can be expressed functionally as

$$\dot{a}_i = n\,\delta_{i6} + \varepsilon\, F_i(\underline{a},\lambda) + \nu\, G_i(\underline{a},\lambda) \qquad (2\text{-}92)$$

where ε and ν are the small parameters.

As was stated previously, the second order averaged equations of motion require the first order short periodic functions [see Equation (2-29)]. The transformation equations

from mean element space to osculating element space can therefore be written as

$$a_i = \bar{a}_i + \varepsilon \eta_{i,1}(\underline{\bar{a}},\bar{\lambda}) + \nu \psi_{i,1}(\underline{\bar{a}},\bar{\lambda}) \qquad (2\text{-}93)$$

where $\varepsilon \eta_{i,1}(\underline{\bar{a}},\bar{\lambda})$ are the first order short periodic functions due to one perturbation (oblateness) and where $\nu \psi_{i,1}(\underline{\bar{a}},\bar{\lambda})$ are the first order short periodic functions due to the other perturbation (drag).

The mean element rates are assumed to be a function of the five slowly varying mean elements and must include the mixed or coupling terms that result due to multiple perturbations. To second order in both small parameters, this can be written as (Ref. 3)

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \varepsilon B_{i,1,0}(\underline{\bar{a}}) + \nu B_{i,0,1}(\underline{\bar{a}}) + \varepsilon \nu B_{i,1,1}(\underline{\bar{a}}) + \varepsilon^2 B_{i,2,0}(\underline{\bar{a}}) + \nu^2 B_{i,0,2}(\underline{\bar{a}}) \qquad (2\text{-}94)$$

The coupling term is $\varepsilon \nu B_{i,1,1}(\underline{\bar{a}})$.

The functions $B_{i,j,k}(\underline{\bar{a}})$ are determined in an analagous manner to the determination of the functions $A_{i,j}(\underline{\bar{a}})$ [see

Equations (2-28) and (2-29)]. First, expressions for the osculating element rates in terms of the functions $B_{i,j,k}(\bar{\underline{a}})$ and the first order short periodic functions are obtained by taking the time derivative of Equation (2-93) and substituting Equation (2-94). Next, alternative expressions for the osculating element rates are obtained in terms of the first order short periodic functions and the VOP disturbing functions, [$\varepsilon F_i(\bar{\underline{a}},\bar{\lambda})$ and $\nu G_i(\bar{\underline{a}},\bar{\lambda})$] by substituting the near-identity transformation into the right hand side of the osculating VOP equations of motion [Equation (2-92)] and expanding in a Taylor series. Equating equal powers of the small parameters in these two expressions for the osculating element rates yields expressions for the functions $B_{i,j,k}(\bar{\underline{a}})$. Averaging these expressions with respect to the mean-mean longitude over one revolution and assuming that the five slowly varying mean elements are constant over the averaging interval, the component functions of the mean element rates to second order in the small parameters are

$$B_{i,1,0}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} F_i(\bar{\underline{a}},\bar{\lambda})\, d\bar{\lambda} \qquad (2\text{-}95a)$$

$$B_{i,0,1}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} G_i(\bar{\underline{a}},\bar{\lambda})\, d\bar{\lambda} \qquad (2\text{-}95b)$$

$$B_{i,1,1}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{\ell=1}^{6}\left\{\eta_{\ell,1}(\bar{\underline{a}},\bar{\lambda})\left[\frac{\partial G_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] + \psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda})\left[\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right]\right\} + \frac{15\,\bar{n}\,\delta_{i6}}{4\,\bar{a}_1^{\,2}}\,\eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\,\psi_{1,1}(\bar{\underline{a}},\bar{\lambda})\right] d\bar{\lambda} \qquad (2\text{-}95c)$$

$$B_{i,2,0}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{\ell=1}^{6}\eta_{\ell,1}(\bar{\underline{a}},\bar{\lambda})\left[\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] + \frac{15\,\bar{n}\,\delta_{i6}}{8\,\bar{a}_1^{\,2}}\,\eta_{1,1}^2(\bar{\underline{a}},\bar{\lambda})\right\} d\bar{\lambda} \qquad (2\text{-}95d)$$

$$B_{i,0,2}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{\ell=1}^{6}\psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda})\left[\frac{\partial G_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] + \frac{15\,\bar{n}\,\delta_{i6}}{8\,\bar{a}_1^{\,2}}\,\psi_{1,1}^2(\bar{\underline{a}},\bar{\lambda})\right\} d\bar{\lambda} \qquad (2\text{-}95e)$$

Comparing the above expressions with Equations (2-28) and (2-29) it is seen that the second order effects of multiple perturbations cannot be considered separately. The coupling term, $\varepsilon\,\nu\,B_{i,1,1}(\bar{\underline{a}})$, can be a major part of the mean element rates. Although this section has considered only two perturbations,

the extension to an arbitrary number of perturbating forces is straightforward but very tedious.

Oblateness/Drag Coupling

For the case of oblateness/drag coupling, the mixed term, $\varepsilon \nu B_{i,1,1}(\bar{a})$, includes both the J_2-drag and drag-J_2 coupling. The J_2-drag coupling is the effect of the first order J_2 short periodic variations $[\varepsilon \eta_{i,1}(\bar{a},\bar{\lambda})]$ on the drag equations of motion $[\nu G_i(\bar{a},\bar{\lambda})]$. Similarly, the drag-$J_2$ coupling is the effect of the first order drag short periodic variations $[\nu \psi_i(\bar{a},\bar{\lambda})]$ on the J_2 equations of motion $[\varepsilon F_i(\bar{a},\bar{\lambda})]$. Functionally this can be written as

$$B_{i,1,1}(\bar{a}) = B_{i,J_2DR}(\bar{a}) + B_{i,DRJ_2}(\bar{a}) \qquad (2\text{-}96)$$

where $B_{i,J_2DR}(\bar{a})$ represents the J_2-drag coupling and $B_{i,DRJ_2}(\bar{a})$ represents the drag-J_2 coupling. From Equation (2-95c) it is seen that

$$B_{i,J_2DR}(\bar{a}) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{\ell=1}^{6}\eta_{\ell,1}(\bar{a},\bar{\lambda})\left[\frac{\partial G_i(\bar{a},\bar{\lambda})}{\partial \bar{a}_\ell}\right] + \frac{15\,\bar{n}\,\delta_{i6}}{8\,\bar{a}_1^{\,2}}\eta_{1,1}(\bar{a},\bar{\lambda})\,\psi_{1,1}(\bar{a},\bar{\lambda})\right\} d\bar{\lambda} \qquad (2\text{-}97a)$$

$$B_{i,DRJ_2}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{\ell=1}^{6}\psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda})\left[\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] + \frac{15\,\bar{n}\,\delta_{i6}}{8\,\bar{a}_1^{\,2}}\psi_{1,1}(\bar{\underline{a}},\bar{\lambda})\,\eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\right\} d\bar{\lambda}$$

(2-97b)

By using the first order short periodic formulation as given in Table 2-1, the above expressions can be simplified. From Table 2-1 the J_2 short periodic variations are expressed as

$$\varepsilon\,\eta_{i,1}(\bar{\underline{a}},\bar{\lambda}) = \varepsilon\sum_{\sigma=1}^{\infty}\left[C_{i\sigma}\sin(\sigma\bar{\lambda}) - D_{i\sigma}\cos(\sigma\bar{\lambda})\right]$$

(2-98)

and the drag short periodic variations are expressed as

$$\nu\,\psi_{i,1}(\bar{\underline{a}},\bar{\lambda}) = \nu\sum_{k=1}^{\infty}\left[G_{ik}\sin(k\bar{\lambda}) - H_{ik}\cos(k\bar{\lambda})\right]$$

(2-99)

From the orthogonality of sines and cosines, it is a simple matter to show that

$$\frac{1}{\pi}\int_0^{2\pi}\eta_{1,1}^2(\bar{\underline{a}},\bar{\lambda})\,d\bar{\lambda} = \sum_{\sigma=1}^{\infty}\left[C_{1\sigma}{}^2 + D_{1\sigma}{}^2\right] \qquad (2\text{-}100a)$$

$$\frac{1}{\pi}\int_0^{2\pi}\psi_{1,1}^2(\bar{\underline{a}},\bar{\lambda})\,d\bar{\lambda} = \sum_{\sigma=1}^{\infty}\left[G_{1\sigma}{}^2 + H_{1\sigma}{}^2\right] \qquad (2\text{-}100b)$$

$$\frac{1}{\pi}\int_0^{2\pi} \eta_{1,1}(\bar{\underline{a}},\bar{\lambda})\ \psi_{1,1}(\bar{\underline{a}},\bar{\lambda})d\bar{\lambda} = \sum_{\sigma=1}^{\infty} [C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma}]$$

(2-100c)

Therefore, from Equations (2-95) through (2-100), it is seen that the J_2-drag and drag-J_2 coupling can be written as

$$B_{i,J_2DR}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{\ell=1}^{6} \eta_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial G_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma}]$$

(2-101a)

$$B_{i,DRJ_2}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{\ell=1}^{6} \psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma}]$$

(2-101b)

In a similar manner it can be shown that the second order effects for J_2 and drag are

$$B_{i,2,0}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{\ell=1}^{6} \eta_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial F_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [C_{1\sigma}^{\,2} + D_{1\sigma}^{\,2}] \qquad (2\text{-}102)$$

$$B_{i,0,2}(\underline{\bar{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{\ell=1}^{6} \psi_{\ell,1}(\underline{\bar{a}},\bar{\lambda}) \left[\frac{\partial G_i(\underline{\bar{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [G_{1\sigma}^{\,2} + H_{1\sigma}^{\,2}] \qquad (2\text{-}103)$$

It is possible to obtain a functionally simplified form of the AOG as given in Equations (2-101) through (2-103). Define the following terms

$$a_i' \equiv \bar{a}_i + \varepsilon\, \eta_{i,1}(\underline{\bar{a}},\bar{\lambda}) + \nu\, \psi_{i,1}(\underline{\bar{a}},\bar{\lambda})$$
$$(i = 1,2,3,4,5,6) \qquad (2\text{-}104)$$

By use of Taylor's theorem it is seen that to second order in both small parameters

$$\varepsilon\, F_i(\underline{a}',\lambda') = \varepsilon\, F_i(\underline{\bar{a}},\bar{\lambda})$$

$$+ \nu\, \varepsilon \sum_{\ell=1}^{6} \psi_{\ell,1}(\underline{\bar{a}},\bar{\lambda}) \left[\frac{\partial F_i(\underline{\bar{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right]$$

$$+ \varepsilon^2 \sum_{\ell=1}^{6} \eta_{\ell,1}(\underline{\bar{a}},\bar{\lambda}) \left[\frac{\partial F_i(\underline{\bar{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] \qquad (2\text{-}105a)$$

$$\nu\, G_i(\underline{a}',\lambda') = \nu\, G_i(\bar{\underline{a}},\bar{\lambda})$$

$$+ \varepsilon\,\nu \sum_{\ell=1}^{6} \eta_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial G_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right]$$

$$+ \nu^2 \sum_{\ell=1}^{6} \psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial G_i(\bar{\underline{a}},\bar{\lambda})}{\partial \bar{a}_\ell}\right] \qquad (2\text{-}105b)$$

Comparing the above expressions with the integrands in Equations (2-95a), (2-95b), (2-101), (2-102) and (2-103), it is seen that the complete second order mean element rates can be written as

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \frac{1}{2\pi}\int_0^{2\pi} [\varepsilon\, F_i(\underline{a}',\lambda') + \nu\, G_i(\underline{a}',\lambda')]d\bar{\lambda}$$

$$+ \left(\frac{15\,\bar{n}\,\delta_{i6}}{16\,\bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [\varepsilon^2(C_{1\sigma}^{\,2} + D_{1\sigma}^{\,2})$$

$$+ 2\,\varepsilon\,\nu(C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma}) + \nu^2(G_{1\sigma}^{\,2} + H_{1\sigma}^{\,2})] \qquad (2\text{-}106)$$

where a' and λ' are defined in Equation (2-104).

The complete second order drag effects can be obtained numerically from Equation (2-106). The definite integral in

Equation (2-106) would be solved by a quadrature algorithm in which the elements $\underline{a}'$ and λ' would be determined by Equations (2-104), (2-98) and (2-99). The five slowly varying mean elements ($\underline{\bar{a}}$) and the short periodic coefficients ($\varepsilon C_{i\sigma}$, $\varepsilon D_{i\sigma}$, νG_{ik}, νH_{ik}) can be assumed to be constant over the averaging interval. This assumption would be valid through second order in the small parameter.

In a similar manner the various components of a second order drag theory can be determined. The drag and J_2-drag effects can be represented by the following

$$\nu B_{i,0,1}(\underline{\bar{a}}) + \varepsilon\, \nu B_{i,J_2DR}(\underline{\bar{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \nu G_i(\underline{a}'',\lambda'')d\bar{\lambda} \tag{2-107a}$$

where

$$a''_i = \bar{a}_i + \varepsilon\, \eta_{i,1}(\underline{\bar{a}},\bar{\lambda}) \qquad (i = 1,2,3,4,5,6) \tag{2-107b}$$

There is no coupling between the semimajor axis and the fast variable in Equation (2-107a) because only the J_2 short periodic coefficients are computed in Equation (2-107b). The J_2 and drag-J_2 effects are given as

$$\varepsilon B_{i,1,0}(\bar{\underline{a}}) + \varepsilon\, \nu B_{i,DRJ_2}(\bar{\underline{a}})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, F_i(\underline{a}''', \lambda''')\, d\bar{\lambda} \qquad (2\text{-}108a)$$

where

$$a_i''' = \bar{a}_i + \nu\, \psi_{i,1}(\bar{\underline{a}}, \bar{\lambda})$$
$$(i = 1,2,3,4,5,6) \qquad (2\text{-}108b)$$

Again, there is no coupling term in Equation (2-108a) because only drag short periodic coefficients are computed in Equation (2-108b). The drag, J_2-drag and drag-squared effects are

$$\nu B_{i,0,1}(\bar{\underline{a}}) + \varepsilon\, \nu B_{i,J_2DR}(\bar{\underline{a}}) + \nu^2 B_{i,0,2}(\bar{\underline{a}})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \nu G_i(\underline{a}', \lambda')\, d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [\varepsilon\nu(C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{i\sigma})$$

$$+ \nu^2(G_{1\sigma}^{\,2} + H_{1\sigma}^{\,2})] \qquad (2\text{-}109)$$

where $\underline{a}'$ and λ' is defined in Equation (2-104). Similarly, the J_2, drag-J_2 and $J_2^{\,2}$ effects are represented as

$$\varepsilon B_{i,1,0}(\bar{\underline{a}}) + \varepsilon\, \nu B_{i,DRJ_2}(\bar{\underline{a}}) + \varepsilon^2 B_{i,2,0}(\bar{\underline{a}})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varepsilon F_i(\underline{a}', \lambda')\, d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1}\right) \sum_{\sigma=1}^{\infty} [\varepsilon\nu(C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma})$$

$$+ \varepsilon^2(C_{1\sigma}{}^2 + D_{1\sigma}{}^2)] \qquad (2\text{-}110)$$

Algorithmic Considerations

For a numerical averaging satellite theory, Equation (2-106) will allow the calculation of the complete second order averaged equations of motion. Assuming that the effects of oblateness and drag perturbations are calculated with separate quadratures, the above formulation would require four quadratures each time the integrator requires a function evaluation: two for the determination of the J_2 and drag short periodic coefficients and two for the determination of the J_2 and drag mean element rates as given in Equation (2-106). The implementation of Equations (2-106) through (2-110) into a numerical averaging satellite theory is easy since most of the same software that is used to obtain the first order mean element rates can be used. It should be noted that including the short periodic variations in the integration of Equations (2-106)

through (2-110) may introduce frequency components that would require a higher order quadrature algorithm than a first order theory would need. In other words a complete second order numerical averaging satellite theory is more than twice as computationally costly than a first order numerical averaging satellite theory.

Zeis (Refs. 6 and 7) has developed explicit expressions for the second order mean element rates due to J_2. Zeis used MACSYMA computer language (Ref. 22) to solve Equation (2-95d) explicitly for $B_{i,2,0}(\bar{a})$. The expressions obtained are for direct orbits and are good to zeroth order in the eccentricity.

Equation (2-101b) and the MACSYMA approach of Zeis has been used in this investigation to obtain analytical expressions for the drag-J_2 coupling in terms of mean elements and drag short periodic coefficients ($G_{i\sigma}$ and $H_{i\sigma}$). The expressions obtained are for direct orbits and are good to zeroth order in the eccentricity. These expressions and the MACSYMA approach used are presented in Appendix D. The drag short periodic coefficients in these expressions have to be determined by a numerical quadrature. In other words, $B_{i,DRJ_2}(\bar{a})$ can be obtained in analytical form based upon numerically determined coefficients. This shows that the semianalytical satellite theory is truly semi-analytical in nature.

Izsak (Ref. 59) has determined analytical expressions for the radial and velocity first order short periodic variations due to J_2. Long and McClain (Ref. 12) have shown that applying Izsak's height (radial) correction in the determination of the density significantly improves the accuracy of the AOG. Using Izsak's height correction in the density determination can be viewed as an approximation to the J_2-drag effect, $B_{i,J_2DR}(\bar{a})$. Numerical results will show that this is a very good approximation.

2.4 Numerical Results

This section presents some of the numerical results obtained during the present investigation. These results extend and improve the numerical results presented in References 9 and 12. The present implementation effort has been concerned basically with demonstrating the accuracy of the semianalytical satellite theory. Current development efforts are improving the computational efficiency as well as the accuracy of the semianalytical satellite theory. Some of the goals and approaches of the current development effort will also be indicated in this section.

A review of the software implementation into the Research and Development version of the Goddard Trajectory Determination

System (GTDS-RD) is presented first in this section. Since the results obtained depend upon the choice of initial mean elements, a brief discussion on how the initial conditions are determined is presented. The orbits considered in this section are a low altitude near circular orbit, a low altitude elliptical orbit and a medium altitude circular orbit. Each of these test cases is discussed separately.

2.4.1 Software Overview

The numerical short periodic variations (SPG) and the oblateness/drag coupling capabilities (AOG) have been implemented in a specially modified version of GTDS-RD that operates on an Amdahl 470 V6. The initial software development has been concerned with testing the accuracy of these conceptual developments. Later development efforts will optimize the software with respect to efficiency. The SPG and AOG are each discussed separately.

The Short-Periodic Generator (SPG)

The SPG is called at each point on the output grid of the semianalytical satellite theory.* In the initial software development, the short periodic coefficients also are computed at each point on the output grid. The numerical results

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* The call to the SPG occurs in subroutine 'ORBITV' of GTDS-RD just before the call to the output subroutines (see Appendix F).

indicate that the short periodic coefficients can be computed on the averaging integration grid and an interpolator used to determine them at the output times. A development effort presently underway incorporates this option (Ref. 71).

The present SPG determines the short periodic variations by time-independent and weak time-dependent numerical techniques for several types of perturbations. Perturbations are broken into three separate types: conservative, drag and solar radiation pressure (the AOG handles numerical averaging perturbations in the same fashion). Conservative perturbations include central body non-sphericity and third body effects. At the present time, the explicit expressions (good to zeroth order in the eccentricity and for direct orbits) that Zeis (Ref. 6) obtained for J_2^2 short periodic variations have been included in the SPG. The formulas for the Δq and $\Delta\lambda$ due to J_2^2 were rederived via the MACSYMA capability available on the MIT Lab for Computer Science/Artificial Intelligence Group's LISP computer during the present effort. These results differ from the results given in Reference 6. The new results are given in Appendix E. Eventually the SPG will be able to handle all conservative perturbations analytically; in particular, tesseral harmonics of the gravitational potential will be included analytically (Ref. 5).

The SPG is as flexible as the AOG with respect to the handling of perturbations. The SPG is not coupled to the AOG, i.e., a perturbation could be handled one way in the AOG and another way in the SPG. It is also possible to have a different perturbation model in the AOG and SPG. In other words, the AOG could have an 8 x 0 gravitational field with atmospheric drag and lunar-solar effects and the SPG could have a simple 2 x 0 field. Although all conservative perturbations are presently considered in one numerical quadrature, it will eventually be possible to handle some conservative perturbations analytically and others numerically if that is desired. Drag and solar radiation pressure will be handled numerically in the present effort. This provides great flexibility with respect to including new density models or complicated spacecraft solar radiation pressure models.

The present SPG allows up to 10 pairs of time-independent short periodic coefficients [the C_{ik} and D_{ik} in Equation (2-44)]. In other words, the short periodic functions are expanded through the 10 $\bar{\lambda}$ terms. This upper limit on the number of time-independent short periodic coefficients was imposed because a 10 $\bar{\lambda}$ term should be handled at the very minimum with a 48 point numerical quadrature and this was the maximum quadrature order that the present SPG has available. As was stated previously, the number of coefficients and the quadrature order must be considered jointly. It should also

be noted that although the determination of the short periodic variations can conceptually require as much as 120 quadrature operations (20 coefficients for the 6 elements), it is in effect only one numerical quadrature operation since the software can be implemented such that the perturbing accelerations (right hand side of the VOP equations of motion) have to be evaluated only once at each of the abscissas of the quadrature algorithm. The determination of the time-independent short periodic coefficients due to each type of perturbation (conservative, drag and solar-pressure) requires a separate numerical quadrature.

The present SPG also has the capability to include the weak time-dependent corrections to the time-independent short periodic coefficients (see Table 2-3). These corrections are computed by a single-sided finite-differencing technique as given in Equations (2-84b) and (2-85). The weak time-dependent correction is applicable to conservative, drag and solar radiation pressure perturbations.

All quadratures in GTDS-RD (AOG and SPG) use the eccentric longitude, $\bar{F}$, as the fast variable. Transformation from mean-mean longitude, $\bar{\lambda}$, to mean eccentric longitude, $\bar{F}$, is accomplished by the following two-body relation

$$d\bar{\lambda} = \left(\frac{\bar{r}}{\bar{a}_1}\right) d\bar{F} \qquad (2\text{-}111)$$

The advantages of such a transformation is that Kepler's equation does not need to be solved whenever the accelerations need to be evaluated and there is a generally desirable smoothing of the perturbations around perigee.

The Averaged Orbit Generator (AOG)

The second order oblateness/drag coupling effects have been implemented into GTDS-RD.* The present AOG does J_2 and J_2^2 mean element rates analytically. There are seven (7) options that can be used to handle the coupled oblateness/drag effects. These options are listed below.

1. First order drag effects only. This is done by performing a numerical quadrature on Equation (2-95b).

2. First order drag effects with Izsak's J_2 height correction used in the determination of the density. This option obtains the first order drag effects and an approximation to the J_2-drag effects $[\varepsilon \nu B_{i,J_2DR}(\bar{a})]$. This option is done by performing a numerical quadrature on Equation (2-95b) and adding

* These improvements to the AOG occur in subroutine 'HODRAG' that is called from subroutine 'AVRAGE' of GTDS-RD (see Appendix F).

Izsak's height correction to the radial distance when a density calculation is made at the abscissas of the quadrature algorithm.

3. Drag and J_2-drag effects. This is done by performing a numerical quadrature on Equation (2-107). This option requires two quadratures: one to obtain the J_2 short periodic coefficients and one to solve Equation (2-107).

4. Drag, J_2-drag and drag-squared effects. This option is represented by Equation (2-109). This option presently requires three quadratures: two in order to obtain the J_2 and drag short periodic coefficients and one in order to solve Equation (2-109).

5. Drag, J_2-drag, drag-J_2 and drag-squared effects. Since the J_2 and J_2^2 effects are done analytically, this option is represented as

$$\nu B_{i,0,1}(\bar{\underline{a}}) + \varepsilon\nu B_{i,J_2DR}(\bar{\underline{a}}) + \varepsilon\nu B_{i,DRJ_2}(\bar{\underline{a}}) + \nu^2 B_{i,0,2}(\bar{\underline{a}}) =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \{\varepsilon F_i(\underline{a}''',\lambda''') + \nu G_i(\underline{a}',\lambda')\}d\bar{\lambda} - \frac{1}{2\pi}\int_0^{2\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda})d\bar{\lambda}$$

$$+ \left(\frac{15\ \bar{n}\ \delta_{i6}}{16\ \bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [2\varepsilon\nu(C_{1\sigma}G_{1\sigma} + D_{1\sigma}H_{1\sigma}) + \nu^2(G_{1\sigma}^{\,2} + H_{1\sigma}^{\,2})] \qquad (2\text{-}112)$$

where $\underline{a}'$ and λ' are defined in Equation (2-104) and where $\underline{a}''''$ and λ'''' are defined in Equation (2-108b). This option presently requires five quadratures: two in order to obtain the J_2 and drag short periodic coefficients and three in order to solve Equation (2-112). The last quadrature in Equation (2-112) could have been eliminated if J_2 and J_2^2 effects were not done analytically. This option determines the complete mean element rates to second order in both small parameters.

6. Drag, J_2-drag and drag-squared effects done numerically and drag-J_2 effects done analytically with the expressions obtained in Appendix D. This option is accomplished by adding the analytically determined drag-J_2 rates to the rates obtained from option 4. This option is good for near-circular, direct orbits. The advantage of this option is that it gives a very good approximation of the complete second order oblateness/drag effects with only three numerical quadratures.

7. First order drag effects with Izsak's J_2 height correction and analytical drag-J_2 terms. This option obtains the first order drag effects and approximates the J_2-drag and drag-J_2 effects (for an

orbit with zero mean eccentricity, the drag-J_2 effects would be exact). The drag-squared effects are neglected in this option. This option requires two numerical quadratures: one in order to determine the drag short periodic coefficients that are used in the drag-J_2 expressions and one that is used to determine the first order drag terms with Izsak's correction (see option 2).

The number of short periodic coefficients that are used in the above options must be considered jointly with the quadrature order of the averaging operation. This is because including short periodic variations in the above averaging operations may introduce frequency components that could necessitate a higher order quadrature algorithm. The maximum quadrature order that is available in the present AOG is a 48-point quadrature. It is also possible to have a different number of short periodic coefficients in the SPG and in the AOG.

The present AOG also has Zeis' explicit expressions for the second order mean element rates due to J_2 programmed (Refs. 6 and 7). It is possible to use either the expressions for J_2^2 that GTDS-RD has or the Zeis expressions for J_2^2. Zeis'

expressions are for direct orbits* and good to zeroth order in the eccentricity whereas the J_2^2 expressions in GTDS-RD are good for any orbit. As was discussed in References 6 and 7, Zeis' expressions differ from the GTDS-RD expressions for the zero eccentricity case. The numerical results have indicated that for near-circular orbits, Zeis' J_2^2 expressions for the mean element rates give better results than the present GTDS-RD expressions.

One additional comment is appropriate. When the analytical J_2 short periodic expressions are available in GTDS-RD, one numerical quadrature will be eliminated from options 3, 4, 5 and 6.

2.4.2 Initial Conditions

As was stated in the Mathematical Preliminaries section, semianalytical satellite theory employs a transformation from osculating element space to mean element space. Since errors in the initial conditions result in a secularly-growing prediction error, it is very important to obtain accurate initial mean elements. Three methods of obtaining the initial mean elements that were considered in this investigation are: Numerical Osculating to Mean, Precise Conversion of Elements and Epoch Point Conversion.

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* Truncated expressions valid for both direct and retrograde cases have recently (August 1979) been programmed into GTDS-RD at CSDL.

The Numerical Osculating to Mean (NOM) method is a present capability of GTDS-RD. The NOM procedure generates a file of the osculating orbital elements over a revolution of the satellite by a Cowell (Special Perturbation) technique. The mean elements are obtained by averaging the osculating elements over one mean period of the satellite, i.e.,

$$\bar{a}_i(t_o) = \frac{1}{\bar{P}} \int_{t_o - \frac{\bar{P}}{2}}^{t_o + \frac{\bar{P}}{2}} a_i(t)\, dt \qquad (2\text{-}113)$$

where $\bar{P}$ stands for the mean period and t_o stands for the epoch time. Since the mean period depends solely on the mean semimajor axis, it is necessary to first perform a Newton-Raphson iteration on Equation (2-113) for the mean semimajor axis. Once the mean period is known, the other mean elements can be obtained by a direct application of Equation (2-113).

The Precise Conversion of Elements (PCE) method is also presently in GTDS-RD. The PCE procedure is a differential correction algorithm that can be applied to the semianalytical satellite theory (AOG and SPG) with a Cowell-generated orbit file used as the observational data. The PCE can be performed over any time span. For this research the PCE was generally done over a two hour time span. The PCE requires an accurate SPG such that the observational residuals are small enough to allow convergence. The initial guess of the epoch mean elements for the PCE is obtained by a NOM procedure.

The Epoch Point Conversion (EPC) procedure is a single point conversion method. The algorithm would employ the technique of successive substitution into the near-identity transformation until a specified agreement is reached. This can be represented as

$$\bar{a}_i^{[1]}(t_o) = a_i(t_o) \tag{2-114a}$$

$$\bar{a}_i^{[k+1]}(t_o) = a_i(t_o) - \Delta a_i[\underline{\bar{a}}^{[k]}(t_o), \bar{\lambda}^{[k]}(t_o)] \tag{2-114b}$$

where $\Delta a_i[\underline{\bar{a}}^{[k]}(t_o), \bar{\lambda}^{[k]}(t_o)]$ represents all the short periodic variations (J_n, drag, J_2^2, etc.) and is dependent upon the epoch mean elements determined from the previous (kth) iteration. Equation (2-114a) indicates that the initial guess of the epoch mean elements is the epoch osculating elements. The EPC method is not presently in GTDS-RD. If the initial mean elements are assumed known, it is possible to obtain the initial osculating elements by a direct substitution into the near-identity transformation. Functionally this can be written as

$$a_i(t_o) = \bar{a}_i(t_o) + \Delta a_i[\underline{\bar{a}}(t_o), \bar{\lambda}(t_o)] \tag{2-115}$$

where $\bar{a}_i(t_o)$ is assumed to be known. This can be thought of as the inverse of the EPC method and will be denoted as $(EPC)^{-1}$.

The $(EPC)^{-1}$ procedure is available in the present GTDS-RD and rquires only an evaluation of the SPG at the epoch time.

The mean elements obtained by a NOM, a PCE or an EPC method are dependent upon the perturbation model chosen. It is therefore necessary to have a consistent set of initial mean elements with each ephemeris generation. In other words, the orbit generation should use the same perturbation model as the NOM, PCE or EPC used in generating the initial conditions.

2.4.3 Low Altitude Circular Test Case

The Low Altitude Circular Test Case (Table 2-5) serves as a good test of the SPG and the AOG. The orbit has a perigee height of 200 kilometers and never exits the atmosphere. The apogee of the orbit is 333 kilometers. The atmospheric density model used was the Modified Harris-Priester atmospheric model. The life time of this satellite is less than 21 days.

Table 2-5. Osculating Elements and Parameters for the Low Altitude Circular Test Case

a = 6644.586 km	Epoch = 10 hrs. 24 min. 21 Oct. 1974
e = .01	Reference Frame = True of Date
i = 67.98538419°	C_D = 2.0
Ω = 91.99738419°	Area = 1.86 m^2
ω = 200.6741688°	Mass = 677 kg
M = 164.3173126°	Density = Modified Harris-Priester with $\bar{F}_{10.7}$ = 150 10^{-22} watts m^{-2} Hz^{-1}
Period = 1.492785 hrs.	

Within this test case, the following issues are investigated:

1. The importance of the coupling of the semimajor axis in the short periodic function for the fast variable.
2. The various AOG options (described previously).
3. The PCE initial condition procedure versus the NOM initial condition procedure.
4. The use of the semianalytical satellite theory, consisting of the AOG and SPG, to produce accurate ephemerides.
5. The slowly varying nature of the short periodic coefficients.

The following conventions are adopted in this section and will also apply to future sections. First, whenever drag is considered in the AOG, one of the seven options, as given in Section 2.4.1, will be specified. The number of short periodic coefficients and quadrature order will be specified by the following notation (# short periodic coefficients/quadrature order). For example, (7/48) indicates that the summation in Table 2-1 goes to seven and that a 48 point Gaussian quadrature is used to determine the coefficients (for a drag option, it also indicates the quadrature order used in determining the

element rates). Finally, unless otherwise indicated, the integrator used in both the AOG and in the Cowell orbit generator is a 12th order Adams-Bashforth Predictor/Adams-Moulton Corrector.

Figures 2-3 and 2-4 show the effects of the coupling between the fast variable and the semimajor axis as given in Equation (2-44). Obviously this coupling term, as given in Equations (2-30) and (2-44), is of major importance. As discussed in previous sections, these are the terms that appear to be neglected in Reference 20. The irregular shape of Figure 2-3 is due to the selection effect of one point every 15 minutes. The major cause of the near-linear increase in the along track error in Figure 2-4 is due to an incorrect initial mean semimajor axis. This point will be discussed subsequently.

In Table 2-6, a 25 hour comparison is made between ephemerides generated by the semianalytical satellite theory with the various drag options and by the Cowell orbit generator. This table should be viewed with a certain amount of caution since the sampling rate is nearly commensurable with the satellite's period. This means that the six points sampled per orbit are nearly the same for every orbit during the 25 hour compare. The result of this fact is that the actual maximum error could be slightly larger. It should also be noted that Table 2-6 does not consider the J_2^2 short periodic effects.

Table 2-6. 25 Hour Orbital Comparison†/Semianalytical versus Cowell for the Low Altitude Circular Test Case

Drag* Options	Max Sampled Radial Difference (m)	Max Sampled Cross Track Difference (m)	Max Sampled Along Track Difference (m)	Max Sampled Δr (m)
1	442.65	15.38	18572.41	18577.69
2	8.30	11.02	21.31	22.54
3	8.52	11.02	21.40	22.64
4	8.17	11.02	23.42	24.38
5	8.33	11.15	12.10	15.82
6	8.35	11.16	11.40	15.52
7	8.46	11.18	7.20	15.32

† The sampling rate was once every 15 minutes.

* The drag options used a 48 point quadrature with 4 short periodic coefficients when needed (4/48).

Perturbations:

Cowell: 6x0 gravitational field and drag; 30 second numerical integration time step.

AOG: 1st order analytical expressions for the 6x0 gravitational field, Zeis' expressions for J_2^2 effects and drag options as indicated; 1 day numerical integration time step.

SPG: 6x0 gravitational field (7/48) and drag (7/48), both to 1st order (time-independent).

Initial Conditions: 2 hour PCE with consistent perturbation models. Option 1 used the same initial mean elements as Option 2 (PCE for Option 1 would not converge).

Figures 2-5, 2-6 and 2-7 are the radial, cross track and along track error signatures over the first orbit of drag option 6 in Table 2-6. In Figures 2-5, 2-6 and 2-7, the sampling rate is about one point per minute. From Figure 2-7, it is seen that the sampling rate of Table 2-6 will sample the maximum along track difference. From Figures 2-5 and 2-6 is is seen that a 15 minute sampling rate would yield maximum radial and cross track differences which could be 2 to 5 meters less than the actual maximum differences.

Table 2-6 also seems to indicate some small discrepancies. First of all, option 2 (Izsak's height correction) appears to be better than option 3 (numerical J_2-drag effects) even though option 2 should be an approximation to option 3. Option 4 (numerical J_2-drag and drag-squared effects) should give better results than option 3 since it is a more complete model. Similarly, option 6 (numerical J_2-drag and drag-squared effects with analytical drag-J_2 effects) and option 7 (Izsak's height correction and analytical drag-J_2 effects) are approximations to option 5 (numerical J_2-drag, drag-J_2 and drag-squared effects), yet they give slightly better results.

It is felt that these discrepancies may be due to any or all of the following reasons: the sampling rate, lack of J_2^2 e terms in the AOG, quadrature noise, and possible small instabilities within the integrator. Changing the drag option

changes the perturbation model slightly and this would result in a different error signature over each orbit. The sampling rate used in generating Table 2-6 may be causing some of the observed discrepancies. The use of 4 short periodic coefficients in the second order AOG is in effect a zero eccentricity assumption (Ref. 6). Applying Izsak's J_2 height correction (valid for any eccentricity) into the density calculation may be a better approximation than the implicit zero eccentricity approximation used in options 3, 4 and 5 of Table 2-6. Since most of the discrepancies in options 2 through 8 of Table 2-6 are less than a meter, the possibility of quadrature noise as a source of some of the error must also be considered. For a circular orbit, a 48 point numerical quadrature should be sufficient for 4 short periodic coefficients in the second order AOG. It was found that the 12th order Adams-Bashforth Predictor/Adams-Moulton Corrector integrator had some instabilities associated with it when drag was considered with a Modified Harris-Priester atmosphere. It is felt that these instabilities are caused by the tabular nature of the Modified Harris-Priester density model which has a discontinuous derivative of the density with respect to altitude. Therefore, integrator noise could be causing some or all of the discrepancies noted in Table 2-6.

It is also observed from Table 2-6 that the cross track difference remains approximately 11 meters in options 2 through

7. In options 5, 6 and 7, the cross track error is a major component of the total position error. In Figure 2-6, one can begin to see the secular growth of the cross track error. It is felt that this secular error growth may be due to the lack of J_2^2 e terms in Zeis' explicit expressions (Ref. 6) for the mean element rates due to J_2^2 effects. Since the rotation of the line of nodes is strongly affected by oblateness, the lack of J_2^2 e terms in the AOG may be causing the secular growth of the cross track error (note that the osculating eccentricity of this test case is .01). This fact could also explain why the cross track error remains at 11 meters in Table 2-6 for options 2 through 7.

From Table 2-6 it can be concluded that second order drag effects are very important for low altitude satellites. The major second order drag effect is the J_2-drag coupling. It also appears that applying Izsak's J_2 height correction to the density calculation (option 2) is an adequate and efficient means of representing the J_2-drag effect. For high precision low altitude orbit prediction, drag-J_2 is an important effect and must be considered. It is interesting to note that option 7 (Izsak's J_2 height correction and the analytical drag-J_2 expressions) gives very good results. The additional computational cost of option 7 over option 2 is moderate. The small difference between options 3 and 4 seem to indicate that the drag-squared effect can be neglected in this case.

A comparison of Figures 2-4 and 2-7 indicates that the PCE procedure for obtaining initial mean elements is significantly superior to the NOM procedure. This was expected since the PCE used NOM initial conditions as the first guess. From Table 2-7 it is seen that the PCE procedure used in Figure 2-7 increased the initial mean semimajor axis by 5.5 meters over the initial mean semimajor axis obtained by the NOM procedure used in Figure 2-4.

Table 2-7. Comparison of NOM and PCE Initial Mean Equinoctial Elements for Drag Option 6 of Table 2-6

	NOM	PCE
$\bar{a}$ =	6636.3742041 km	6636.3797106 km
$\bar{h}$ =	$-9.81788006 \times 10^{-3}$	$-9.81801561 \times 10^{-3}$
$\bar{k}$ =	$4.00771339 \times 10^{-3}$	$4.00785279 \times 10^{-3}$
$\bar{p}$ =	$6.73729951 \times 10^{-1}$	$6.73730536 \times 10^{-1}$
$\bar{q}$ =	$-2.34672347 \times 10^{-2}$	$-2.34672726 \times 10^{-2}$
$\bar{\lambda}$ =	1.692578400 (rad)	1.69257015 (rad)

Figures 2-8, 2-9 and 2-10 are radial, cross track and along track differences, respectively, between the semianalytical orbit generator and Cowell orbit generator after 23 hours from epoch. Drag option 7 was used in the AOG and Zeis' J_2^2 expressions (with corrections as given in Appendix E) were

used in the SPG. The maximum position error over the time span shown was 12.6 meters. A comparison of this with drag option 7 of Table 2-6 indicates that J_2^2 short periodic variations significantly improve the radial, along track and maximum position errors. The cross track difference is still about 11 meters and is the major source of error.

It is also interesting to note that applying the weak time-dependent corrections to the time-independent drag short periodic coefficients made no noticeable change in Figures 2-8, 2-9 and 2-10, or, for that matter, in the drag short periodic coefficients themselves. This should be expected since the Sun only moves about $\frac{1}{5670}$ th of a degree during a complete revolution of this satellite. It can therefore be concluded that drag perturbations can adequately be handled with a time-independent formulation. This may not be the case if the time variations of $F_{10.7}$, $\bar{F}_{10.7}$ and/or K_p are taken into account in the density model.

A five-day comparison was made between the semianalytical orbit generator and the Cowell orbit generator with the same initial conditions and perturbations as given in Figure 2-8. The maximum position error was 715.5 meters. A similar five-day comparison of the semianalytical and Cowell orbit generator was made with the initial conditions and perturbations as in drag option 6 of Table 2-6. Zeis' expressions for

J_2^2 short periodic effects were also included in the PCE and the comparison. The maximum position error in this case was 460.5 meters. The along track error was the major error in both cases. Obviously drag option 6 is superior to drag option 7. This is not surprising since drag option 6 includes all second order drag effects whereas drag option 7 approximates the J_2-drag effect and neglects drag-squared effects.

Figures 2-11 and 2-12 show, respectively, a one orbit comparison of the osculating semimajor axis and eccentricity as generated by the semianalytical theory and a Cowell orbit generator. The plus (+) will overwrite the star (*) whenever the two orbit generators are the same. The discretizing of data in the plot routine is the cause for the few apparent differences between the semianalytical satellite theory and the Cowell orbit generator that is seen in Figures 2-11 and 2-12. The semianalytical satellite theory, as presented in this paper, can generate ephemerides with precision approaching that of a Cowell orbit generator.

Figure 2-13 shows the effect of a drag perturbation only on the semimajor axis for the Low Altitude Circular Test Case. Perigee for this orbit occurs at about 49 minutes after epoch. Even though this orbit has an eccentricity of .01 and an apogee height of 333 kilometers, the major effect of drag still occurs close to perigee.

Figures 2-14, 2-15 and 2-16 are time histories of the mean semimajor axis, mean h and mean k respectively, over a 10 day time span. Figures 2-14, 2-15 and 2-16 also show that the mean elements contain only the secular and long periodic motion. The advantage of numerical integration in mean element space over numerical integration in osculating element space is vividly seen by comparing Figures 2-14, 2-15 and 2-16 to Figures 2-11 and 2-12 (note the different time scales).

The first order zonal (J_2 through J_8) short periodic variations for the semimajor axis, h and k, are shown in Figures 2-17, 2-18 and 2-19, respectively. Figure 2-17 indicates that the first order semimajor axis short periodic variation for this test case can be an 8 kilometer effect.

The first order drag short periodic variation for the semimajor axis, h, k and λ are shown respectively in Figures 2-20, 2-21, 2-22 and 2-23. Perigee passage for this orbit occurs at approximately 49 minutes. Figure 2-20 shows that the drag short periodic variation for the semimajor axis can be a 35 meter effect for this test case. As the perigee height is lowered, this effect will become much larger. Drag short periodic variations must be considered for low altitude, high accuracy, orbit generation.

Figures 2-24 through 2-33 show some of the zonal and drag short periodic coefficients over a 10 day time span for the Low Altitude Circular Test Case. The slowly varying nature of these coefficients is evident. Obviously the time-independent short periodic coefficients do not need to be determined at every output time. The time-independent short periodic coefficients can be determined on the integration grid of the semianalytical satellite theory (typically 1 day) and an interpolator used to obtain them efficiently at the output times. Even for the most rapidly varying short periodic coefficients as given in Figures 2-25 and 2-31, this procedure will be possible. It is interesting to note that the more rapidly varying short periodic coefficients are one to two orders of magnitude less than the dominant short periodic coefficients for that element and perturbation (Figures 2-24 and 2-30).

Figures 2-14 through 2-16 and 2-24 through 2-33 also suggest that the formulation of the osculating elements as given by Equation (2-98) may be an efficient means of representing and storing ephemerides. The mean elements and short periodic coefficients can both be interpolated over long time spans. This may be advantageous for spacecraft on-board computation since the on-board computer would only need a relatively low order interpolator.

The above figures also indicate that estimation procedures in the mean element space may have distinct advantages over current estimation procedures in the osculating element space. The near linear nature of the mean elements indicates that the linearization assumptions used in a batch and sequential estimation process will be better satisfied.

2.4.4 Low Altitude Eccentric Test Case

The Low Altitude Eccentric Test Case (Table 2-8) is a demanding test of the semianalytical satellite theory. The orbit has a perigee height of 115 kilometers and an apogee height of 4100 kilometers. The satellite is therefore entering and exiting the atmosphere. The drag effect is concentrated in a very narrow region around perigee.

Table 2-8. Osculating Elements and Parameters for the Low Altitude Eccentric Test Case

a = 8520.320139 km	Epoch = 10 hrs. 24 mins. 26 Feb. 74
e = .237768825	Reference Frame = True of Date
i = 68.06458843°	C_D = 2.0
Ω = 91.56426845°	Area = 1.86 m^2
ω = 93.81101481°	Mass = 677 kg
Period = 2.174163268 hrs.	Density = Modified Harris-Priester density model with $\bar{F}_{10.7}$ = 150 (10^{-22} watts m^{-2} Hz^{-1})

Table 2-9 gives the initial condition procedure and perturbations used in this test case. It was found that drag option 7 gave the best results. This was expected because for larger eccentricities, the required number of short periodic coefficients in the second order AOG for drag options 3 through 6 introduce frequency components that bring the 48-point numerical quadrature into question.

Table 2-9. Initial Conditions and Perturbations for the Low Altitude Eccentric Test Case

Perturbations

Cowell: 4x0 gravitational field and drag: 10 second numerical integration time step.

AOG: 1st order analytical expression for the 4x0 gravitational field. Zeis expressions for J_2^2 effects and drag option 7 (4/48): 1 day numerical integration time step.

SPG: 6x0 gravitational field (8/48) and drag (10/48) both to first order (time-independent)

Initial Conditions: 2 hour PCE

$\bar{a}$ = 8514.8171053

$\bar{e}$ = .23791153768

$\bar{I}$ = 68.056536888°

$\bar{\Omega}$ = 91.553565358°

$\bar{\omega}$ = 93.860226464°

$\bar{M}$ = 275.28693265°

Figures 2-34 through 2-36 are the radial, cross track and along track difference over the first orbit of the Low Altitude Eccentric Test Case. The maximum position difference is 211 meters. One cause for the error between the semi-analytical satellite theory and Cowell is probably the small eccentricity assumptions made in both the AOG and SPG.

Figures 2-37 through 2-41 are comparisons of the osculating Keplerian elements, a, e, i, Ω, and ω, as obtained by the semianalytical satellite theory and the Cowell orbit generator. Perigee and apogee passage occur respectively at 30 minutes and 96 minutes into the orbit. The osculating elements produced by the semianalytical theory track the 'true' Cowell elements very closely.

Figure 2-42 shows the effects of only the drag perturbation on the Low Altitude Eccentric Test Case. This figure shows that the drag effect occurs around perigee only. This figure also shows that modeling the short periodic variations by a Fourier series can lead to Gibb's phenomenon when the perturbation is discontinuous and concentrated in one area. Gibb's phenomenon can be reduced by increasing the number of time-independent short periodic coefficients and the quadrature order. Another solution to the Gibb's phenomenon is to reformulate the short periodic variations in another geometric variable such that the perturbation effect is smoothed around

perigee. Figure 2-42 also indicates that the semianalytical satellite theory is slightly biased after perigee passage. This is probably due to the fact that the semianalytical satellite theory did not properly model the drag perturbation around perigee. Increasing the quadrature order and number of short periodic coefficients will probably reduce this bias.

2.4.5 Medium Altitude Circular Test Case

The Medium Altitude Circular Test Case (Table 2-10) serves as a good test of the weak time-dependent formulation. The orbit is near circular and has a mean period of 12 hours. This test case is representative of a Global Positioning Satellite (GPS).

Table 2-10. Initial Mean Elements and Parameters for the Medium Altitude Circular Test Case

$\bar{a}$ = 26559.5 km	Epoch = 0.0 hrs., 0.0 mins., 1 July 1979
$\bar{e}$ = .001	Reference Frame = True of Date
$\bar{i}$ = 63.0°	Reflectivity Constant = 1.5
$\bar{\Omega}$ = 0°	Area = 9.091 m^2
$\bar{\omega}$ = 0°	Mass = 454.217 kg
$\bar{M}$ = 0°	
$\overline{\text{Period}}$ = 12 hours	

A comparison of Figures 2-43 and 2-44 shows the importance of the weak time-dependent formulation. In Figure 2-43 it is seen that the mean along track error (compared to Cowell) is about 32 meters in 24 hours when a time-independent formulation is used. When a weak time-dependent formulation is used (Figure 2-44), the mean along track error is reduced to about one meter in 24 hours. It is felt that the remaining error in Figure 2-44 is due to the single-sided finite-differencing technique used to approximate the time derivatives of the short periodic coefficients [see Equation (2-85) or Appendix B] and/or the neglect of $\left(\frac{\dot{\phi}}{\sigma n}\right)^2$ terms in Equation (2-83). The perturbing time increment used in the finite differencing technique was five minutes for the lunar-solar point mass effects (about .01% of the Lunar period) and one hour for solar radiation pressure effects (about .01% of the apparent Solar period). Table 2-11 gives a comparison of the $[EPC]^{-1}$ initialization procedure for both the time-independent and weak time-dependent formulation. As can be seen from Table 2-11, the weak time-dependent formulation increased the initial osculating semimajor axis by 1.32 meters over the time-independent formulation. Figures 2-45 and 2-46 are the radial and cross track differences corresponding to Figure 2-44.

Table 2-11. Initial Osculating Elements from an $[EPC]^{-1}$ Procedure

Time-independent formulation with perturbations as given in Figure 2-43	Weak time-dependent formulation with perturbations as given in Figure 2-44
a = 26561.56435 km	a = 26561.56567 km
e = .0010482547	e = .0010484201
i = 63.00112426°	i = 63.00112407°
Ω = 359.9999661°	Ω = 359.9999657°
ω = 359.8607354°	ω = 359.8560915°
M = .1390436311	M = 0.1436848842°

It was also noted that neglect of the weak time-dependent formulation for solar radiation pressure effects made minor changes in the above results. This indicates that the time-independent formulation for the lunar-solar point mass effects is the major source of error in Figure 2-43. It therefore appears that lunar-solar point mass effects on medium altitude satellites cannot be adequately handled with a time-independent formulation for high precision semianalytical orbit generation. The weak time-dependent formulation is a means of obtaining accurate ephemeris for medium altitude satellites.

Figures 2-47, 2-48 and 2-49 are comparisons of the osculating semimajor axis, eccentricity and inclination respectively over the first orbit of the satellite. The (+) denotes the semianalytical orbit generator and the (*) denotes

the Cowell orbit generator. The (+) will overwrite the (*) when the values for the element coincide to within the discretization of the plot. Obviously, the semianalytical satellite theory can generate precision ephemerides.

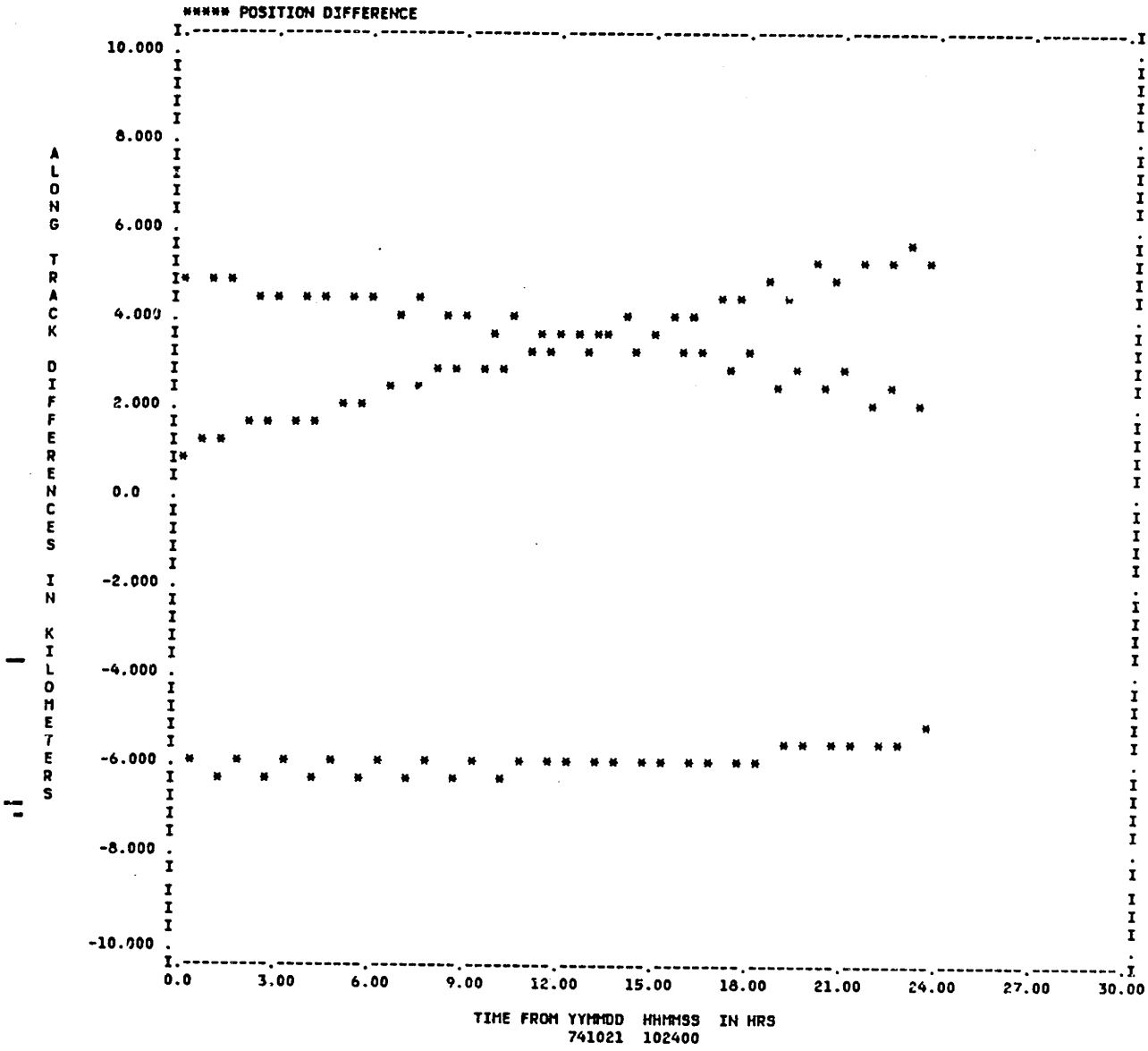
Figure 2-50 is a plot of the first order short periodic variation due to solar radiation pressure for the Medium Altitude Test Case. The solar radiation pressure short periodic variation is a 10 meter effect for this test case.

A one orbit plot of the first order lunar semimajor axis short periodic variation is presented in Figure 2-51. Only lunar perturbations were considered in both the AOG and SPG. The numerical weak time-dependent short periodic formulation was used. As can be seen from Figure 2-51, the lunar short periodic variation on a medium altitude satellite is a 150 meter effect. The lunar short periodic variation must be considered for accurate orbit generation on medium and high altitude satellites.

Figures 2-52 through 2-58 show some of the lunar weak time-dependent short periodic coefficients over a 12 day time span. The weak time-dependent short periodic coefficients show more of a time variation than the time-independent short periodic coefficients (see Section 2.4.3). Evaluation of the weak time-dependent coefficients on the integration grid and

use of an interpolator to obtain them at the output times, still appears to be possible. A higher order interpolator algorithm may be needed for the weak time-dependent short periodic coefficients than for the time-independent short periodic coefficients. It is also noted that the more rapidly varying weak time-dependent short periodic coefficients are one to two orders of magnitude less than the dominant coefficients (Figures 2-53 and 2-54).

Figure 2-3. Along Track Difference/Semianalytical* minus Cowell for the Low Altitude Circular Test Case



* The coupling terms in the fast variable short periodic coefficients were not included.

Perturbations

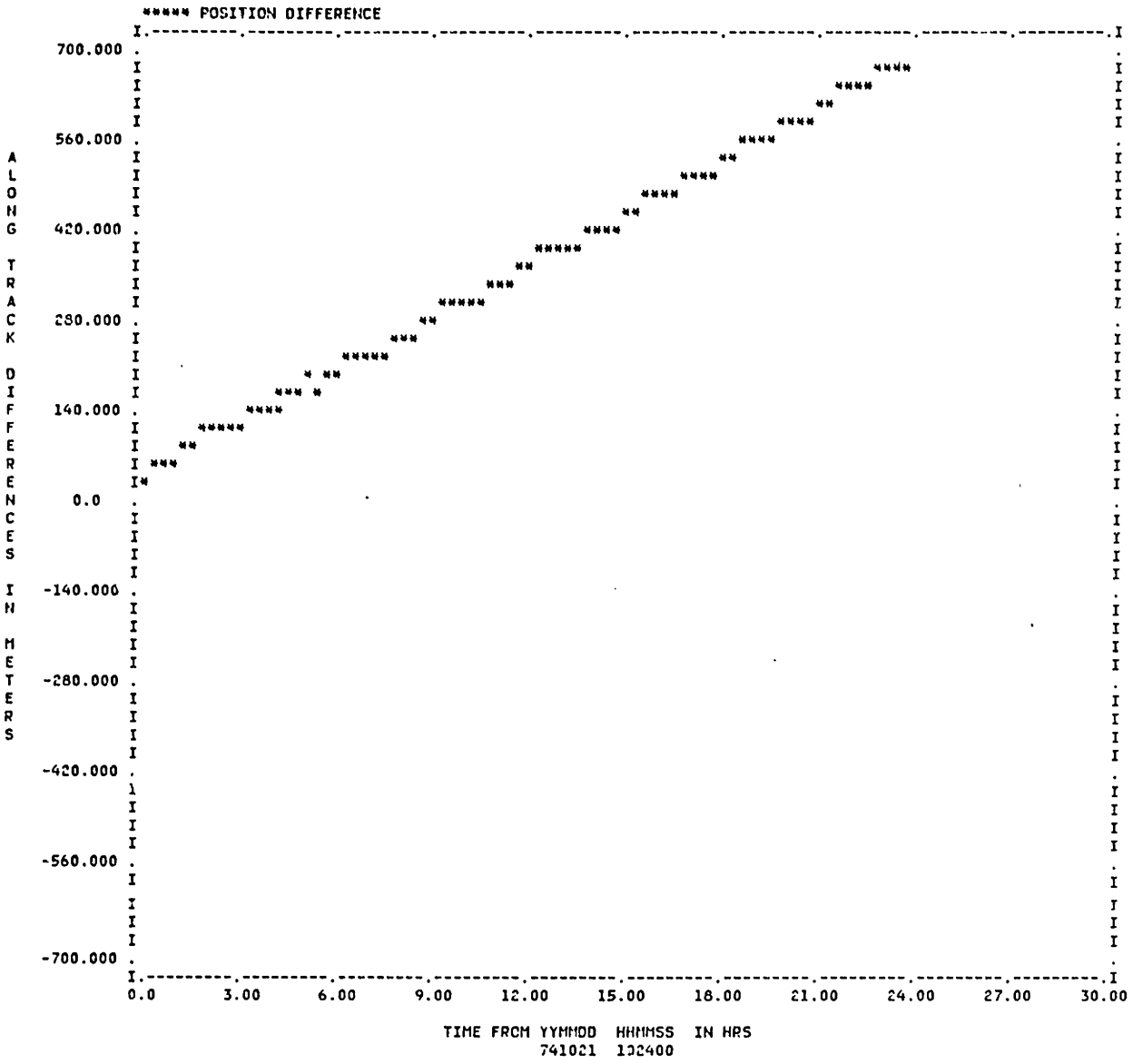
COWELL: 6x0 gravitational field and drag: 30 sec time step

AOG: 1st order analytical expressions for the 6x0 gravitational field, Zeis' expressions for J_2^2 effects and option 6 for drag (4/48): 1 day numerical integration time step

SPG: 6x0 gravitational field (7/48) and drag (7/48), both to 1st order (time-independent)

Initial Conditions: NOM

Figure 2-4. Along Track Difference/Semianalytical* minus Cowell for the Low Altitude Circular Test Case



* The coupling terms in the fast variable short periodic coefficients were included.

Perturbations

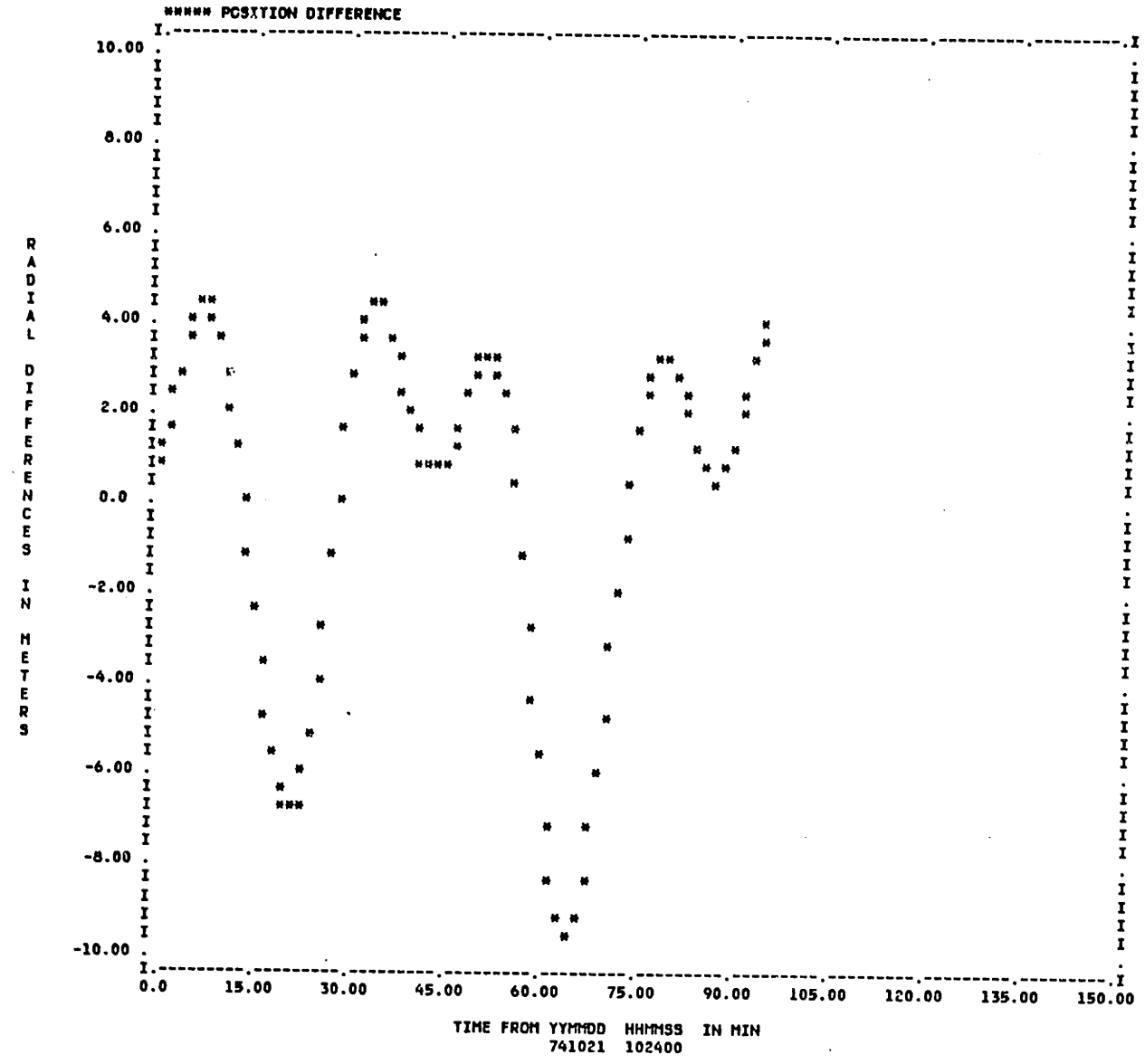
COWELL: 6x0 gravitational field and drag: 30 sec numerical integration time step

AOG: 1st order analytical expressions for the 6x0 gravitational field, Zeis' expressions for J_2^2 effects and option 6 for drag (4/48): 1 day numerical integration time step

SPG: 6x0 gravitational field (7/48) and drag (7/48), both to 1st order (time-independent)

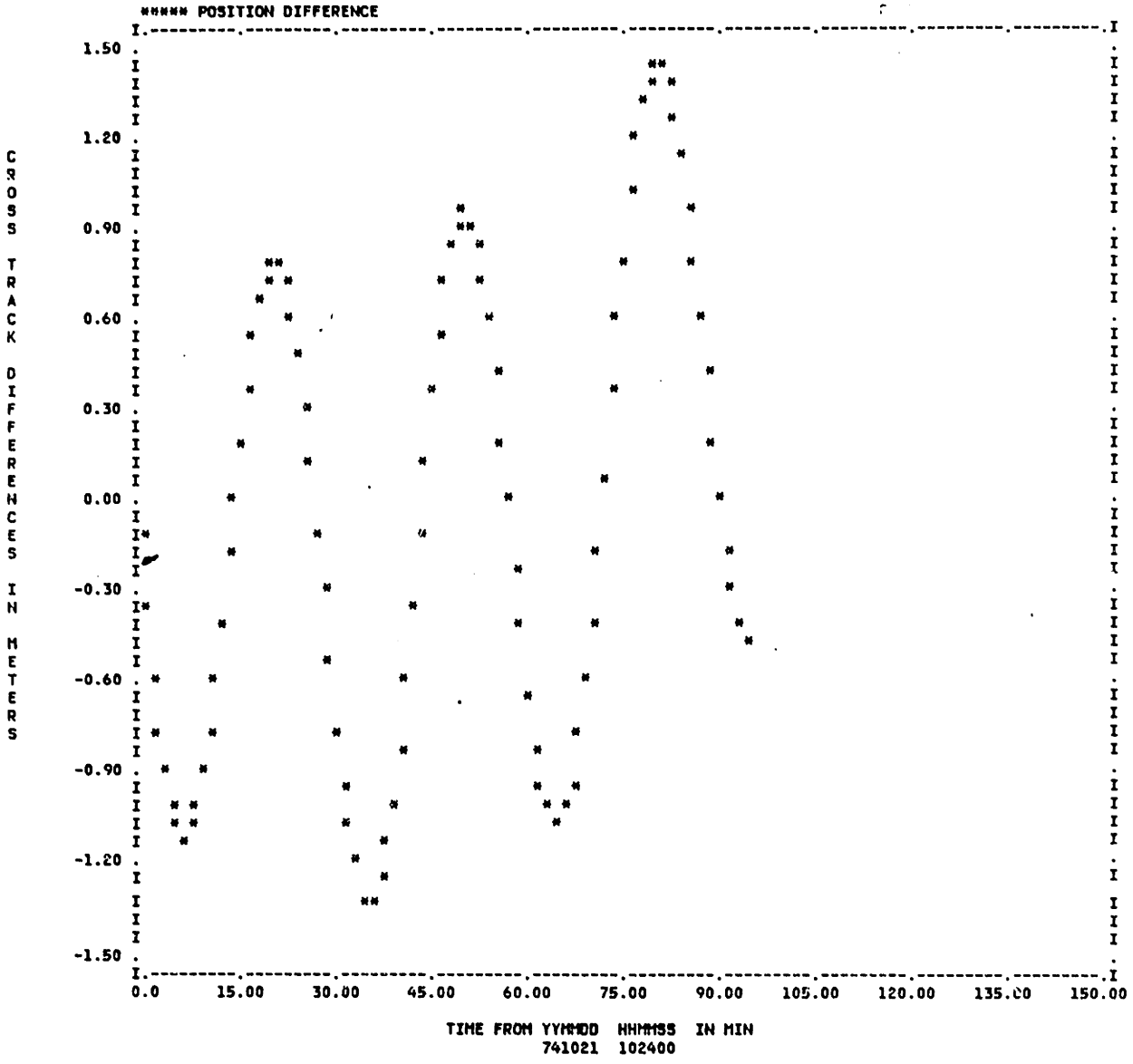
Initial Conditions: NOM

Figure 2-5. Radial Difference/Semianalytical minus Cowell for the Low Altitude Circular Test Case



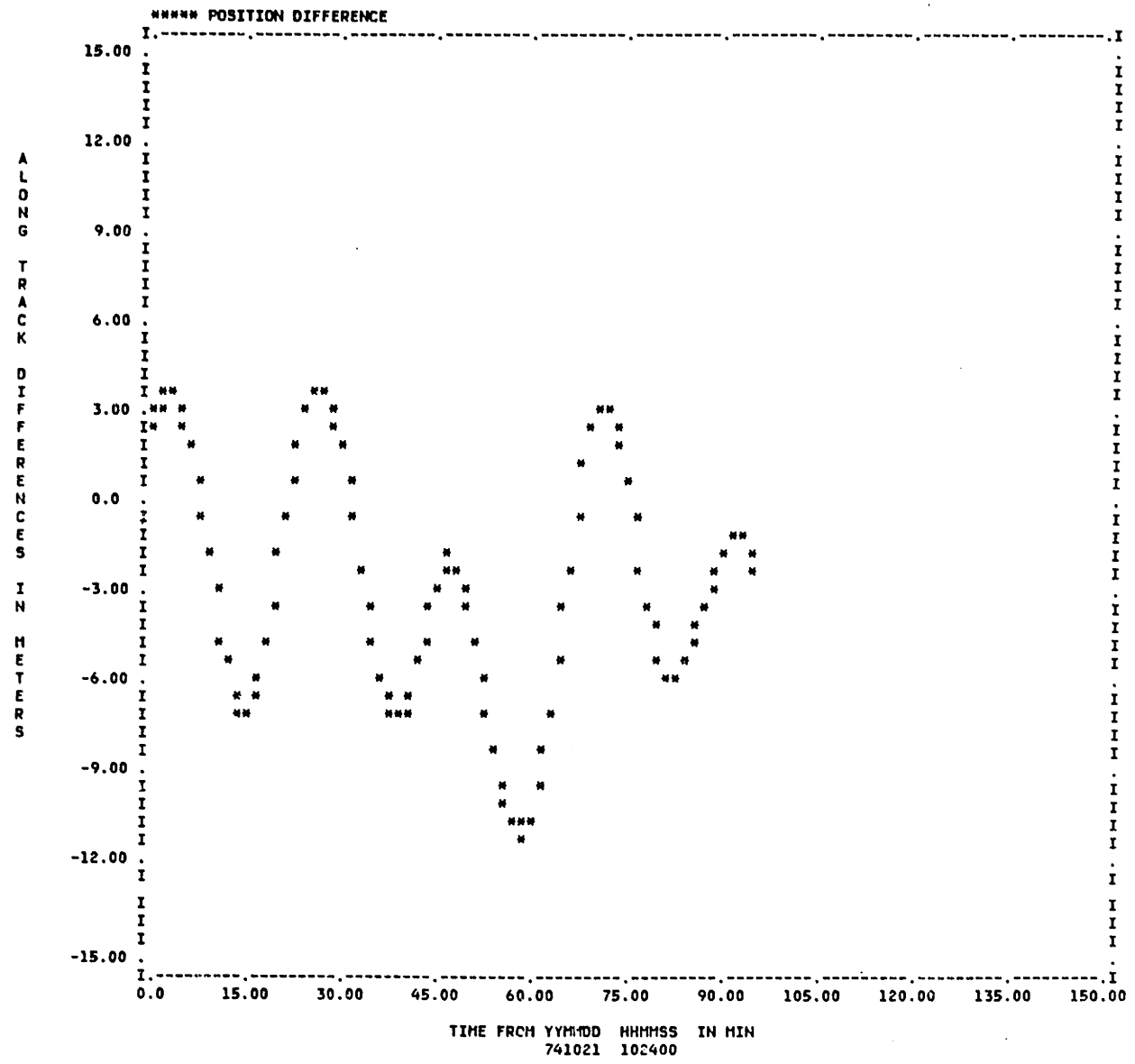
Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-6. Cross Track Difference/Semianalytical minus Cowell for the Low Altitude Circular Test Case



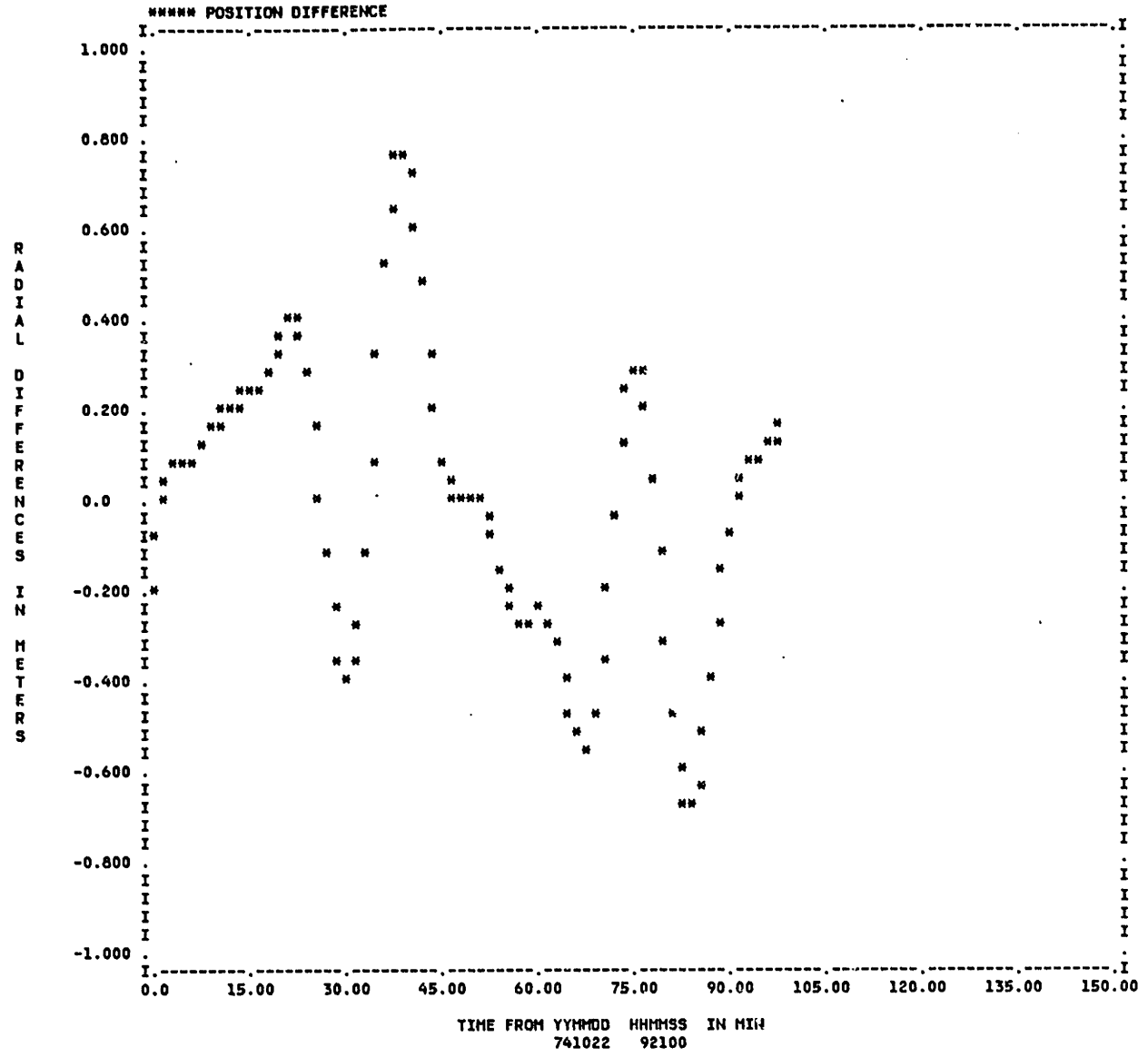
Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-7. Along Track Difference/Semianalytical minus Cowell for the Low Altitude Circular Test Case



Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-8. Radial Difference after 23 hours from Epoch/Semianalytical minus Cowell for the Low Altitude Circular Test Case



Perturbations

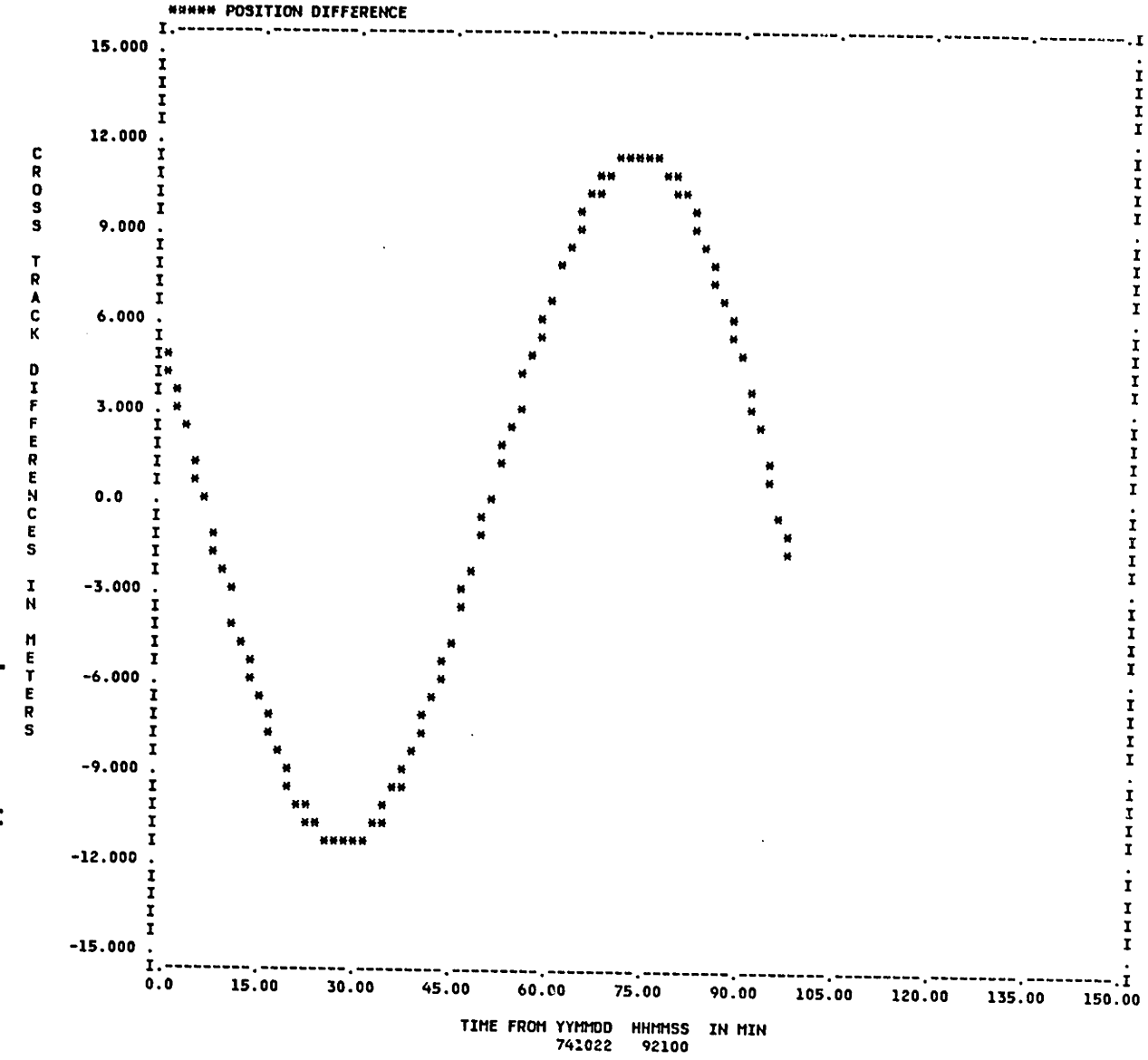
Cowell: 6x0 gravitational field and drag: 30 sec numerical integration time step.

AOG: 1st order analytical expressions for the 6x0 gravitational field, Zeis' expressions for J_2^2 effects and option 7 for drag (48 pt quadrature order); 1 day numerical integration time step.

SPG: 6x0 gravitational field (7/48) and drag (7/48) both to first order (time-independent), Zeis' expressions for J_2^2 effects.

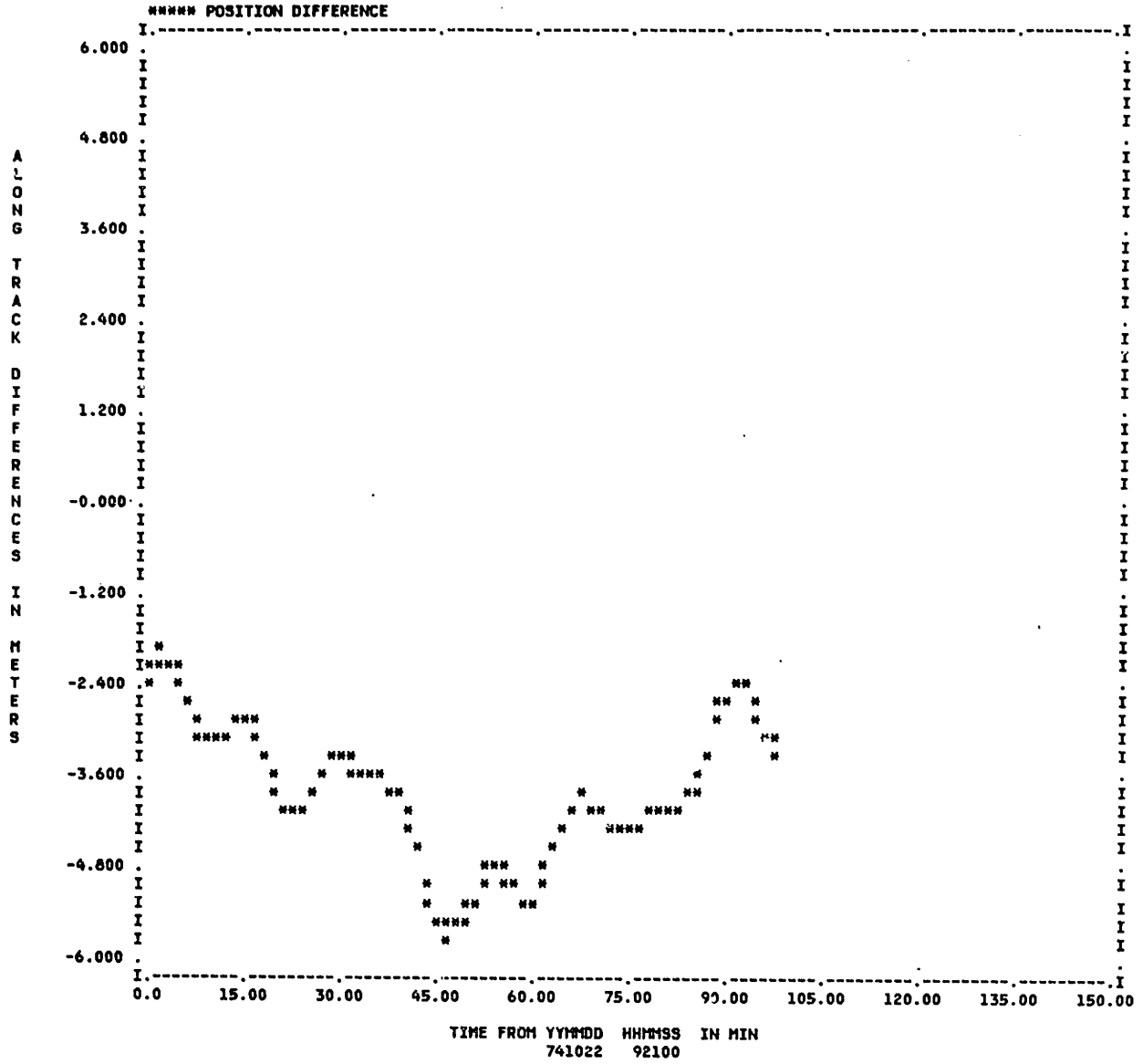
Initial Conditions: 2 hour PCE

Figure 2-9. Cross Track Difference after 23 Hours from Epoch/Semianalytical minus Cowell for the Low Altitude Circular Test Case



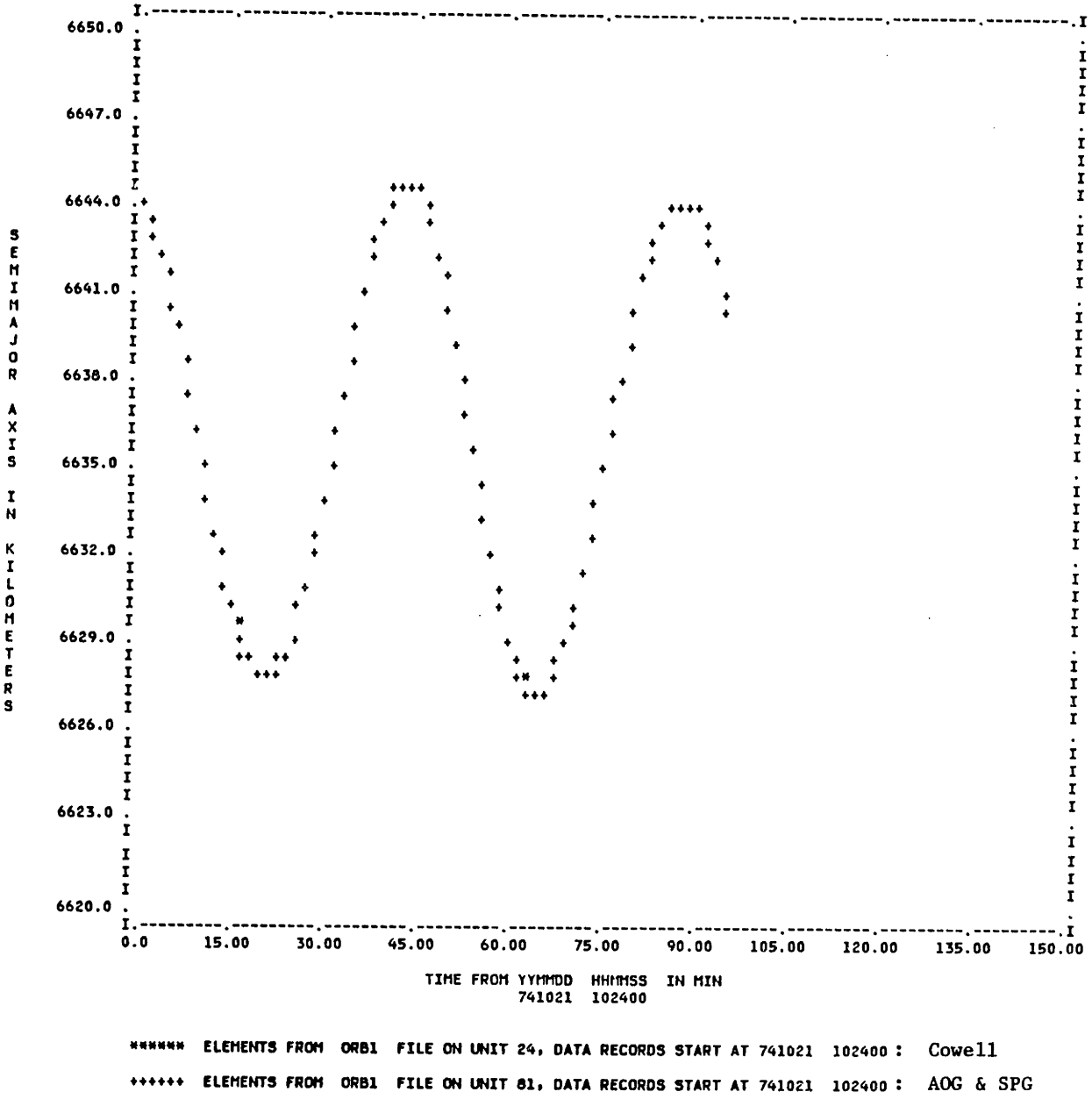
Same initial conditions and perturbations as in Figure 2-8.

Figure 2-10. Along Track Difference after 23 Hours from Epoch/Semianalytical minus Cowell for the Low Altitude Circular Test Case



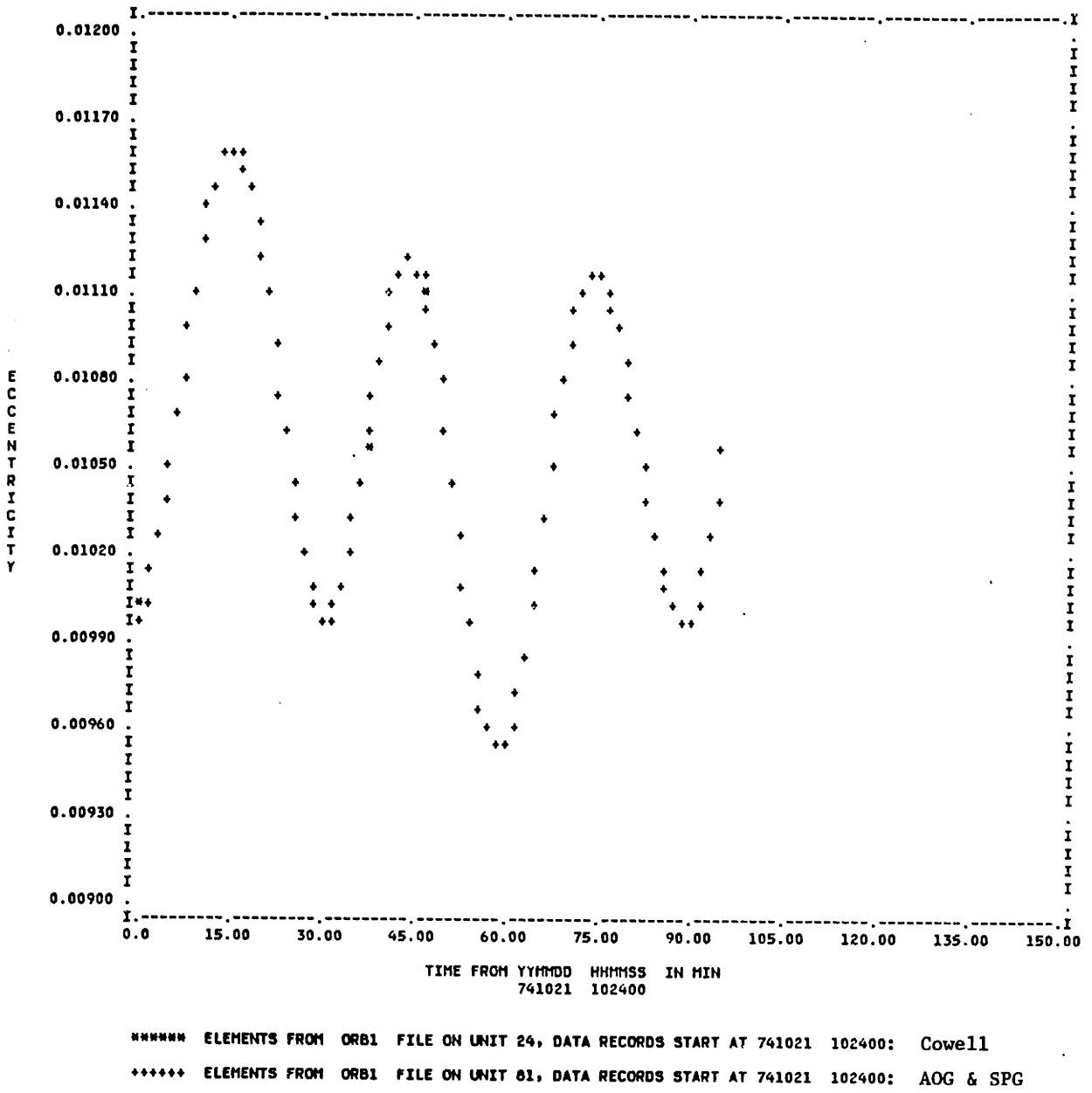
Same initial conditions and perturbations as in Figure 2-8.

Figure 2-11. Osculating Semimajor Axis Comparison/Semianalytical versus Cowell for the Low Altitude Circular Test Case



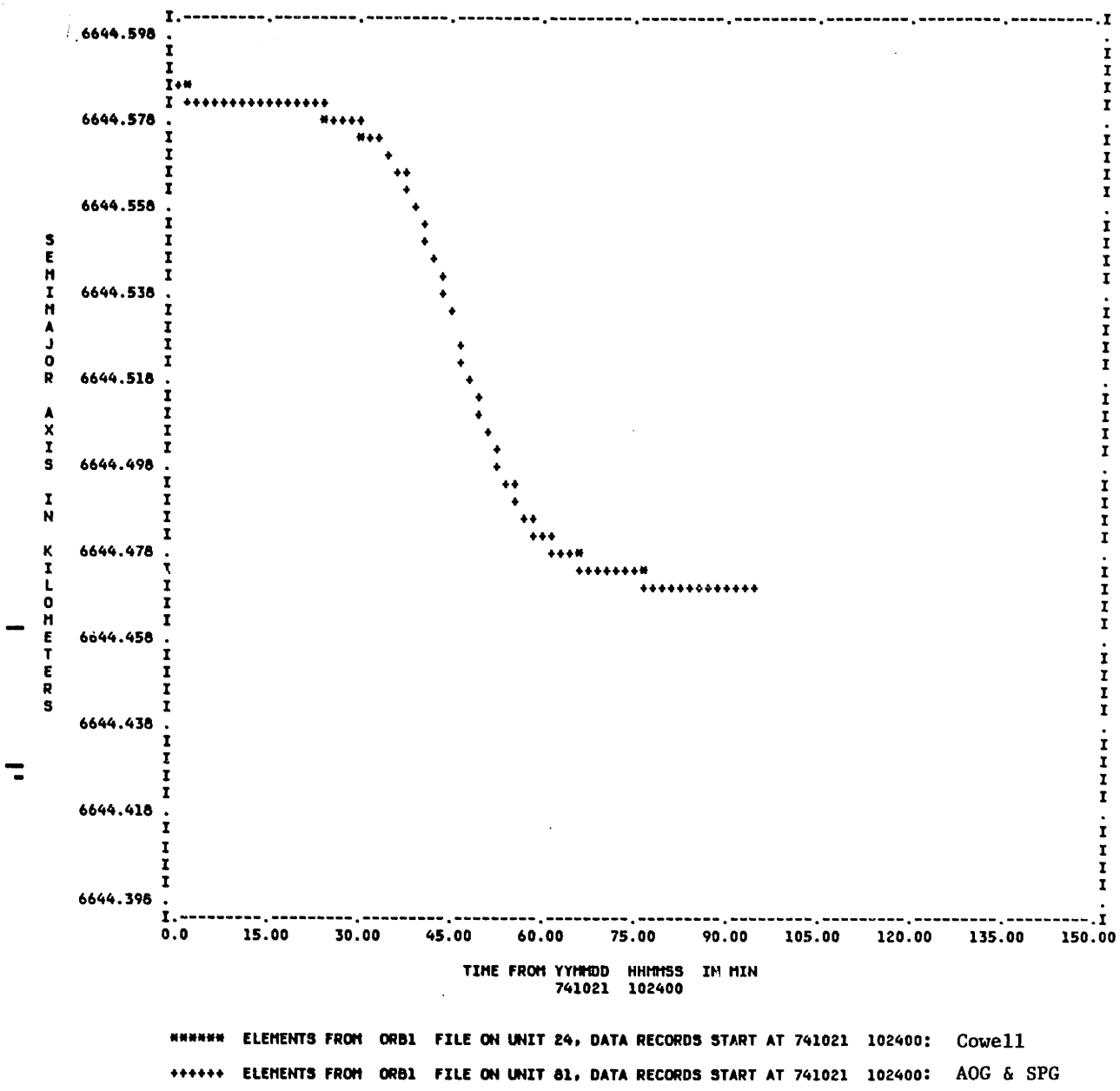
Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-12. Osculating Eccentricity Comparison/Semianalytical versus Cowell for the Low Altitude Circular Test Case



Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-13. Osculating Semimajor Axis Comparison/Semianalytical versus Cowell for the Low Altitude Circular Test Case



Perturbations

Cowell: drag only (30 sec numerical integration time step)

AOG: drag only, option 1 (48 pt quadrature)
1 day numerical integration time step

SPG: 1st order drag only (10/48), time-independent formulation

Initial Conditions: NOM

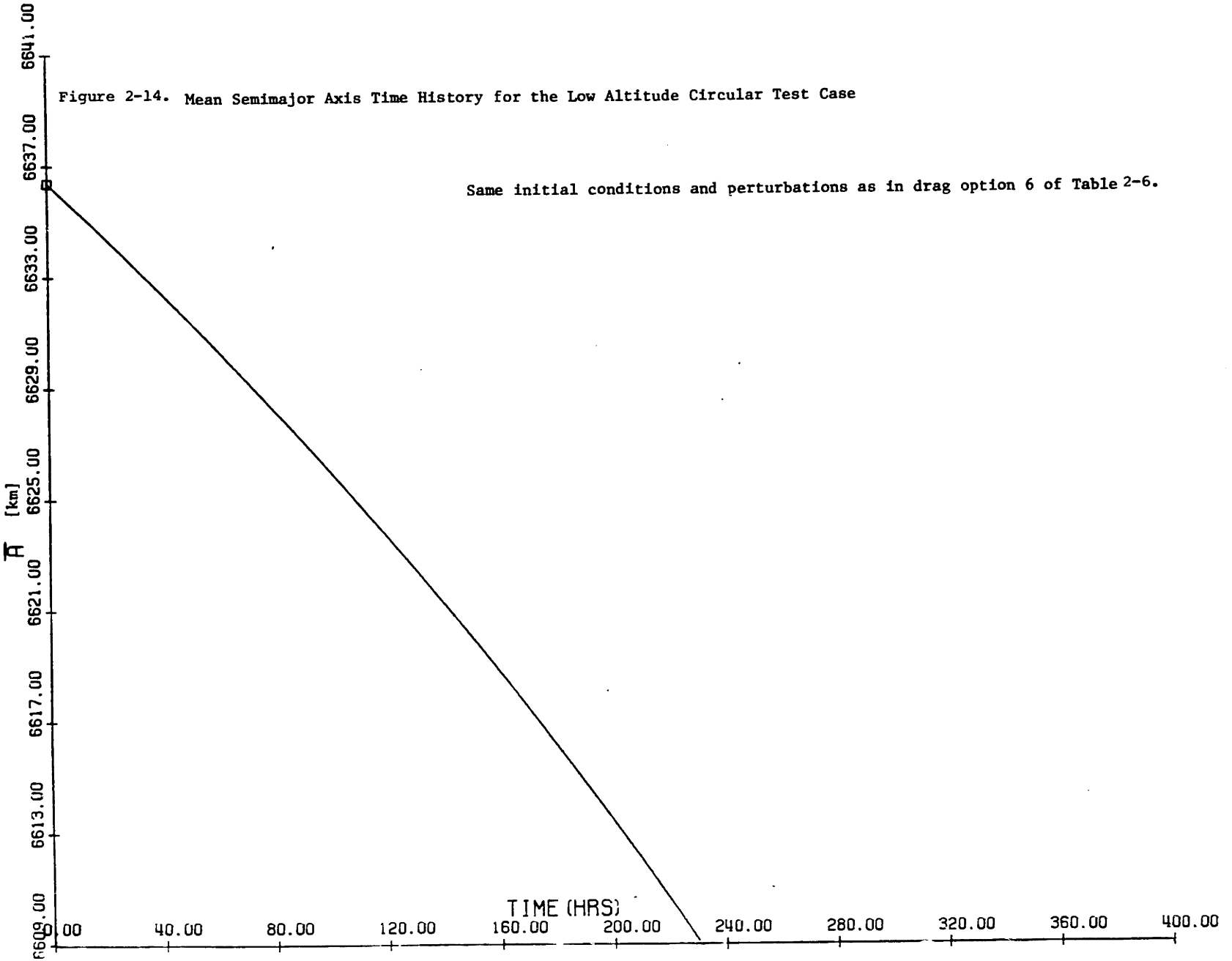


Figure 2-14. Mean Semimajor Axis Time History for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

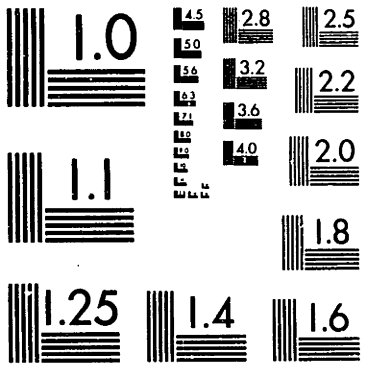


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Figure 2-15. $\bar{h}$ Time History for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

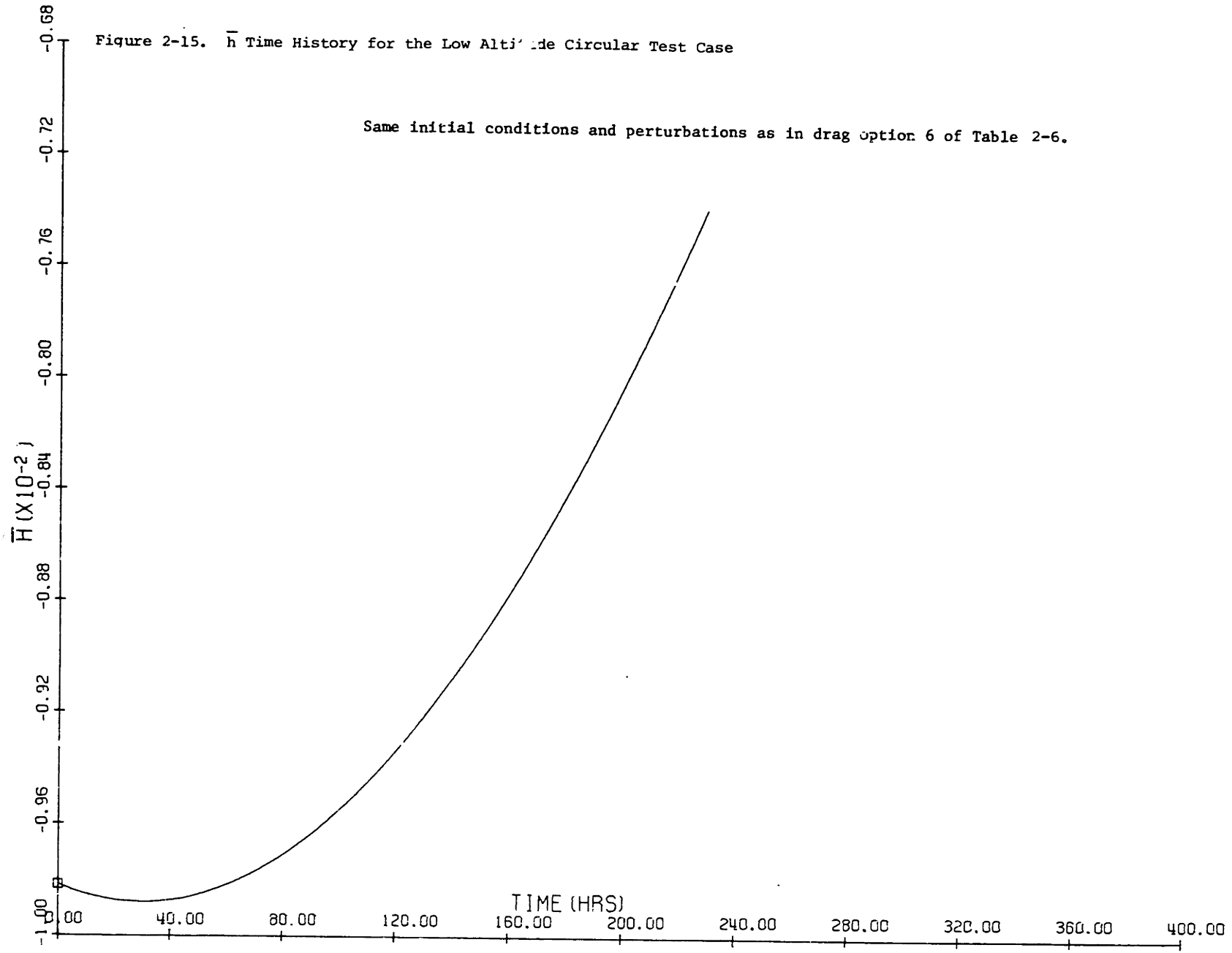


Figure 2-16. $\bar{k}$ Time History for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in diag option 6 of Table -2-6.

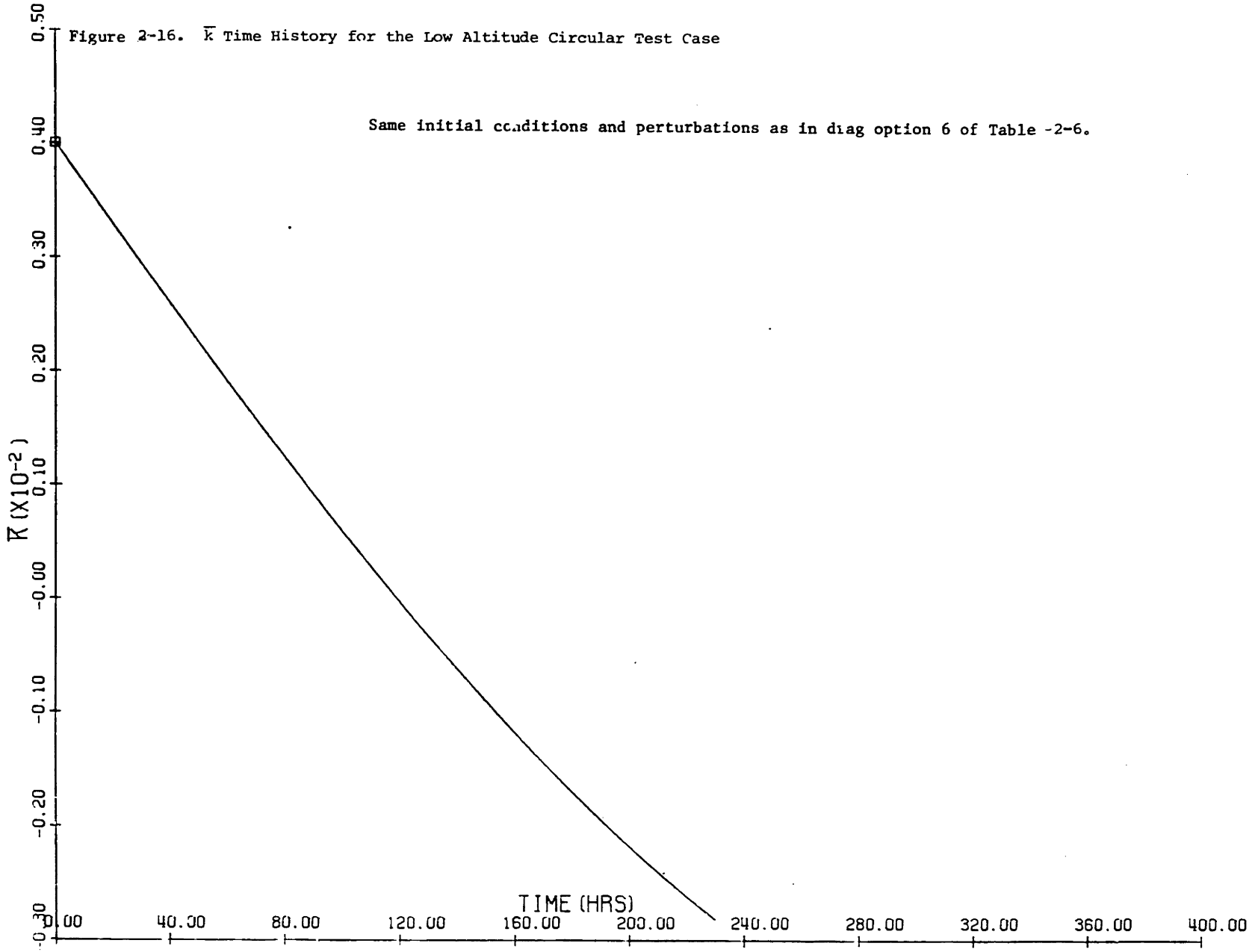


Figure 2-17. One Orbit Time History of the Zonal Semimajor Axis First Order Short Periodic Variation [$\varepsilon \eta_{1,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

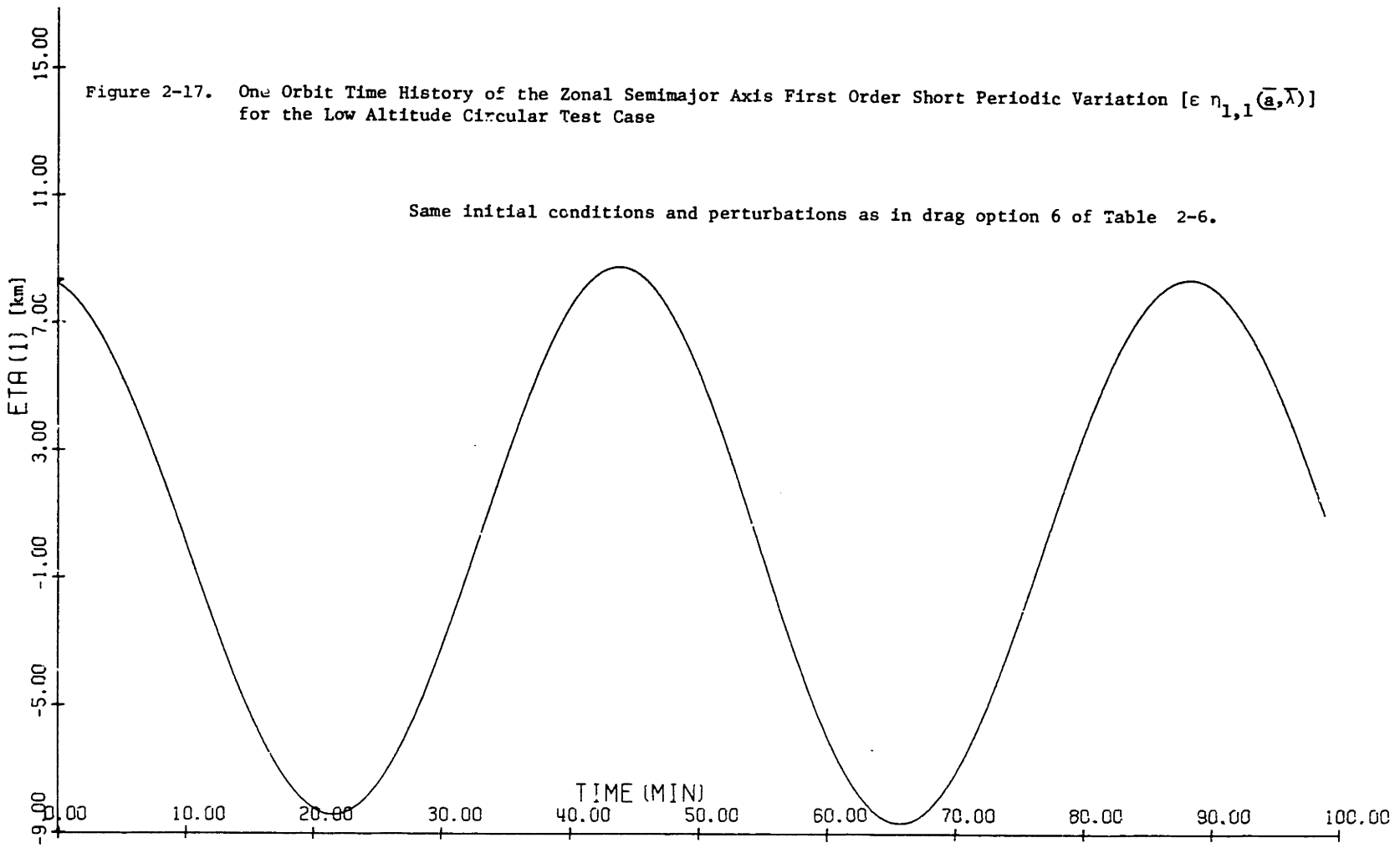
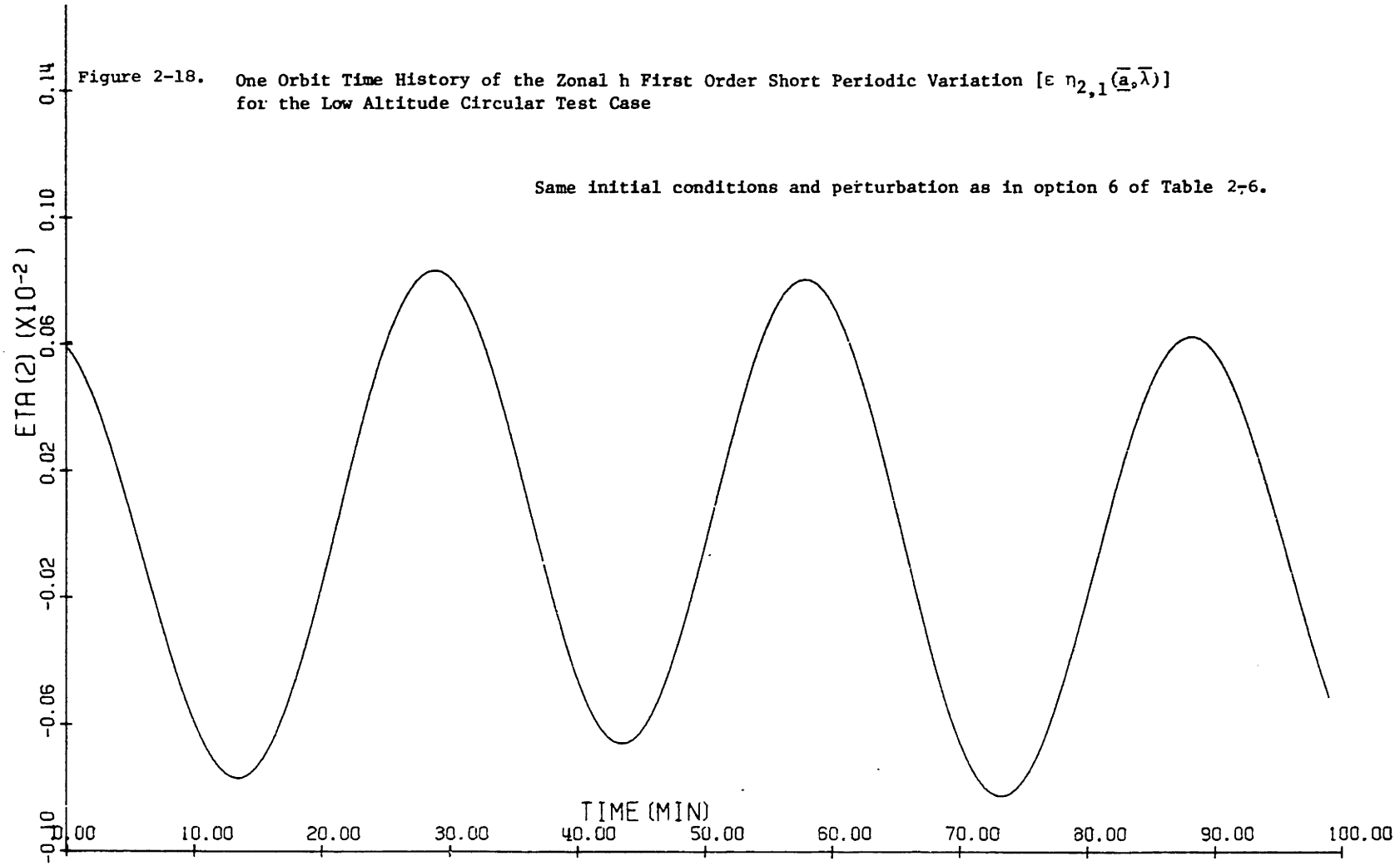


Figure 2-18. One Orbit Time History of the Zonal h First Order Short Periodic Variation [$\varepsilon\, \eta_{2,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case

Same initial conditions and perturbation as in option 6 of Table 2-6.



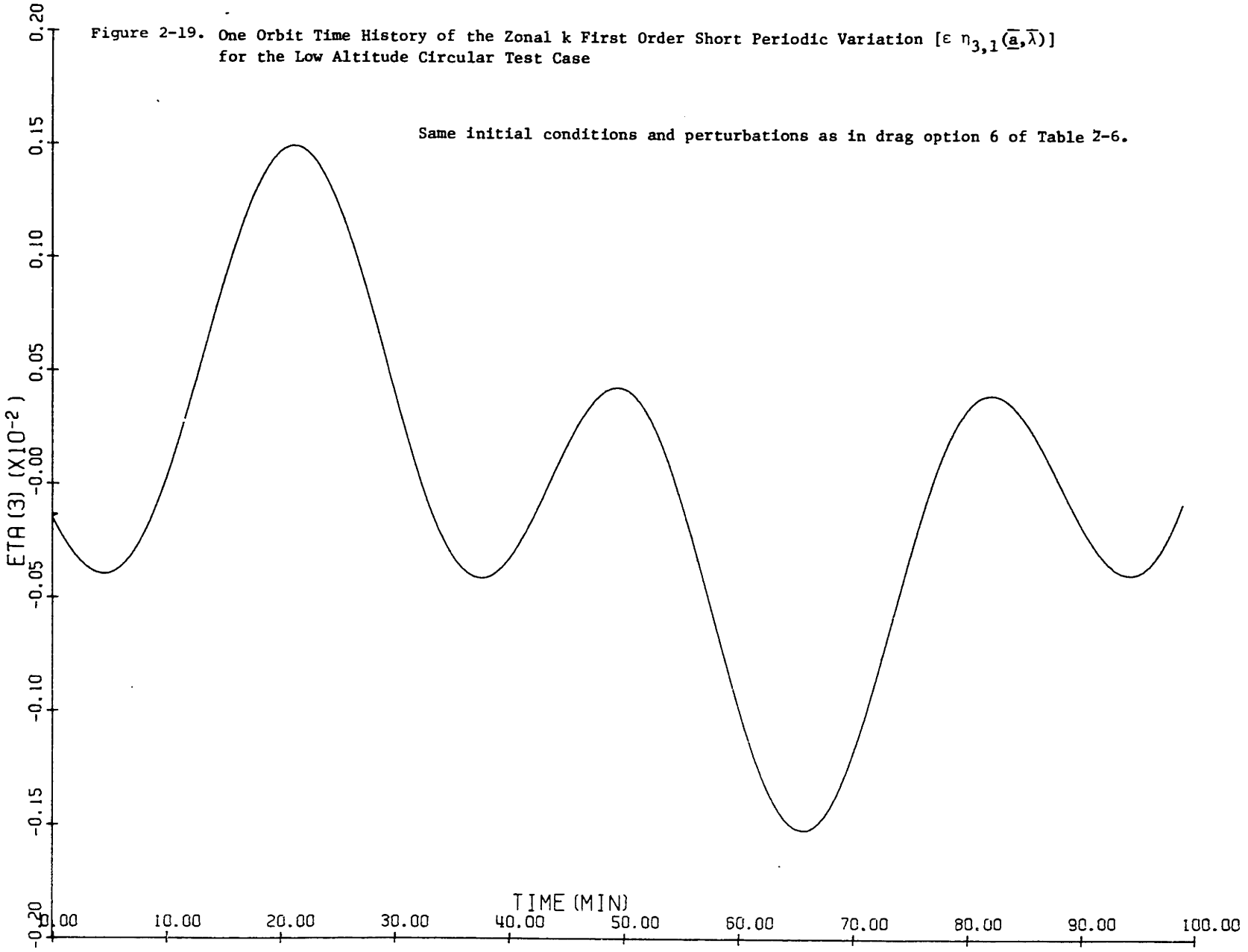


Figure 2-19. One Orbit Time History of the Zonal k First Order Short Periodic Variation [$\varepsilon \eta_{3,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-20. One Orbit Time History of the Drag Semimajor Axis First Order Short Periodic Variation $[\nu \psi_{1,1}(\underline{a},\bar{\lambda})]$ for the Low Altitude Circular Test Case

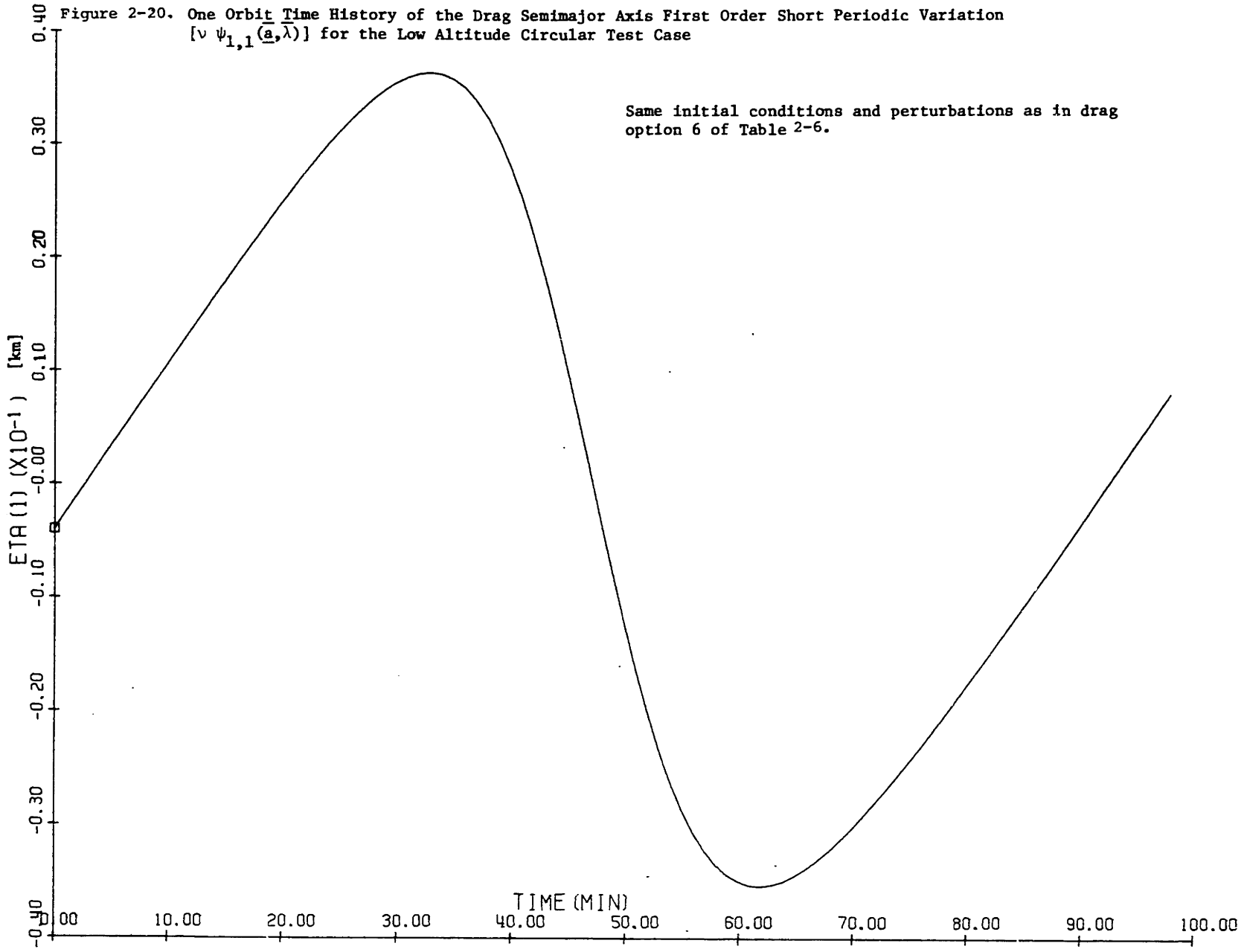


Figure 2-21. One Orbit Time History of the Drag h First Order Short Periodic Variation [$\nu \psi_{2,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case.

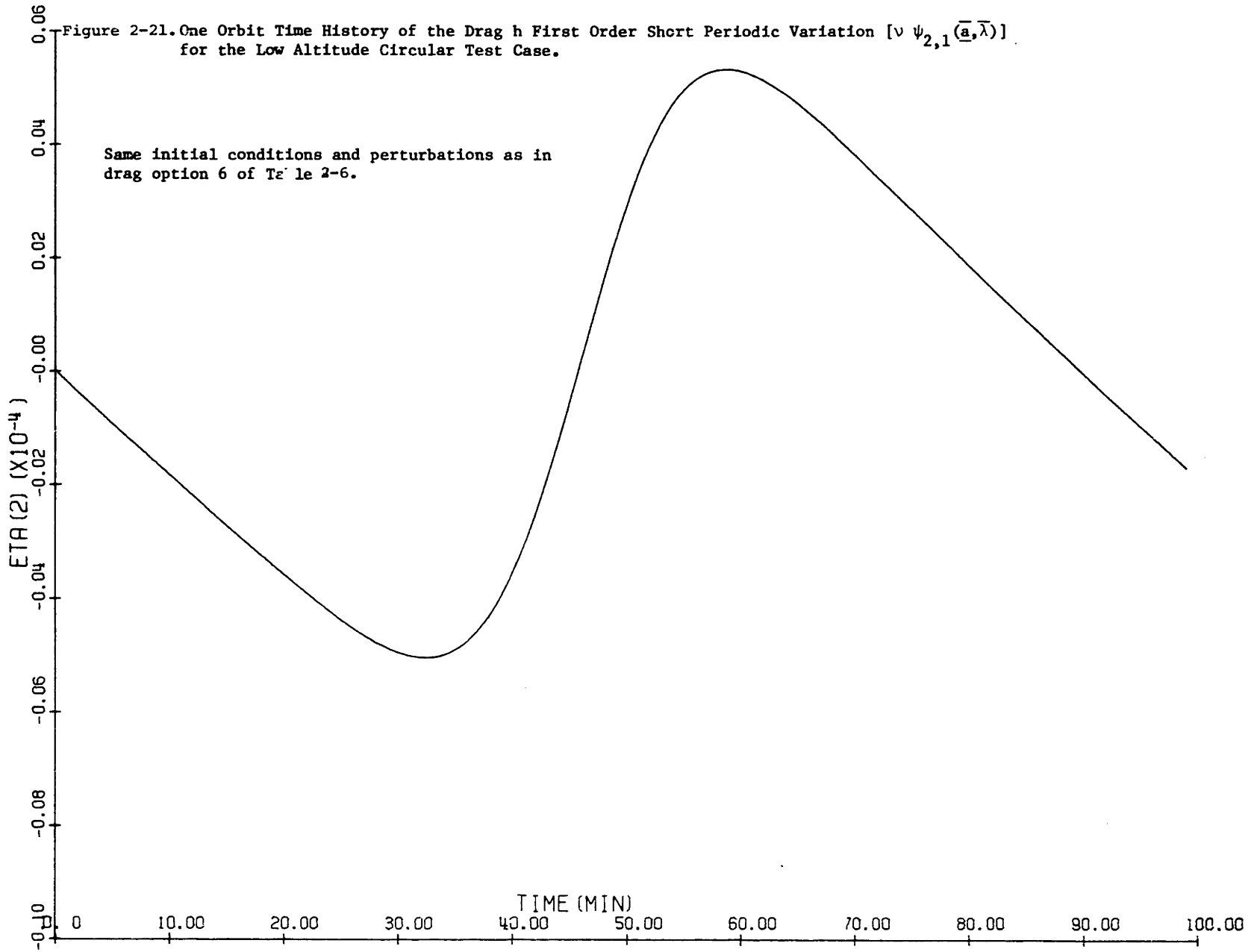


Figure 2-22. One Orbit Time History of the Drag k First Order Short Periodic Variation [$\nu\ \psi_{3,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case

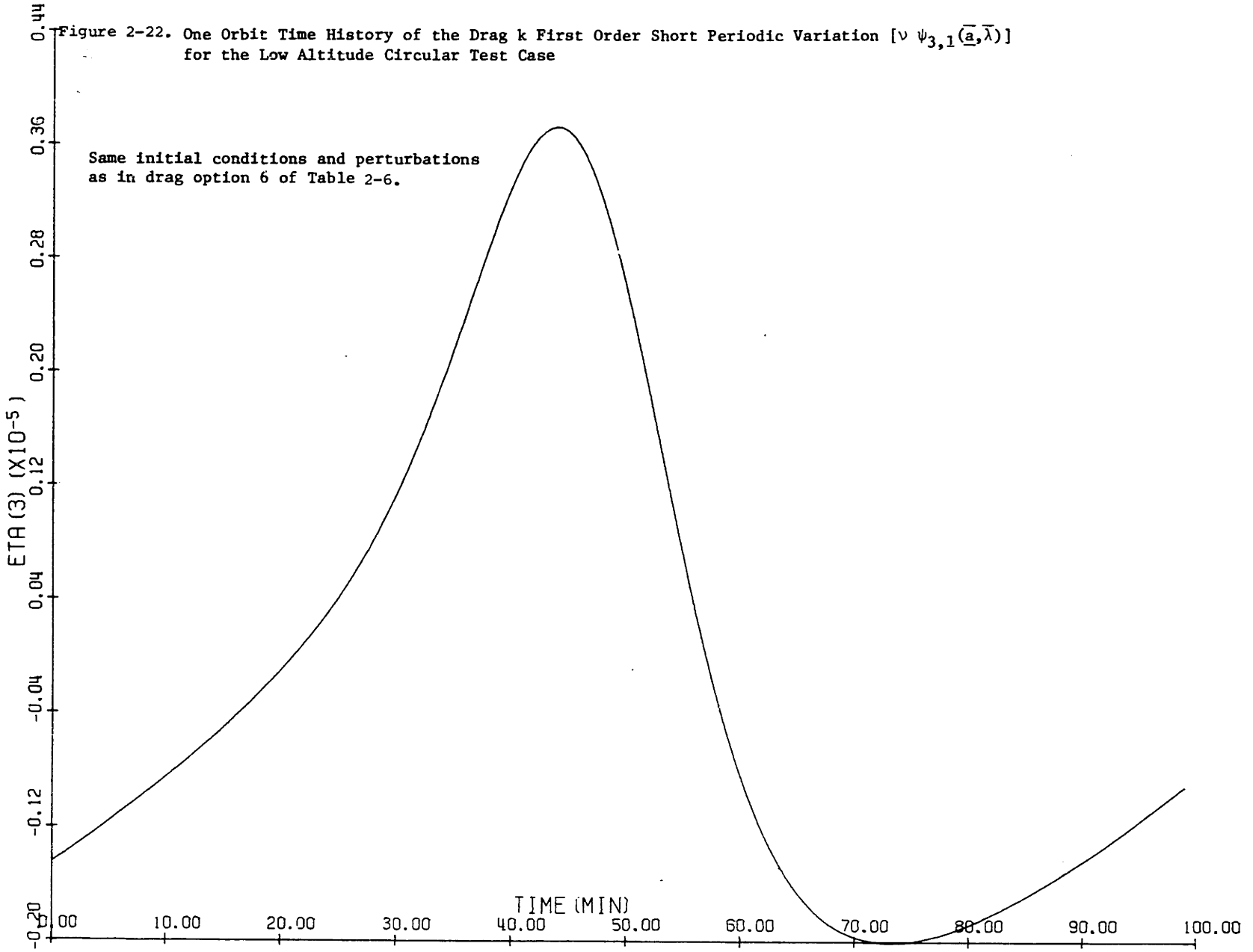


Figure 2-23. One Orbit Time History of the Drag λ First Order Short Periodic Variation [$\nu\ \psi_{6,1}(\bar{a},\bar{\lambda})$] for the Low Altitude Circular Test Case

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

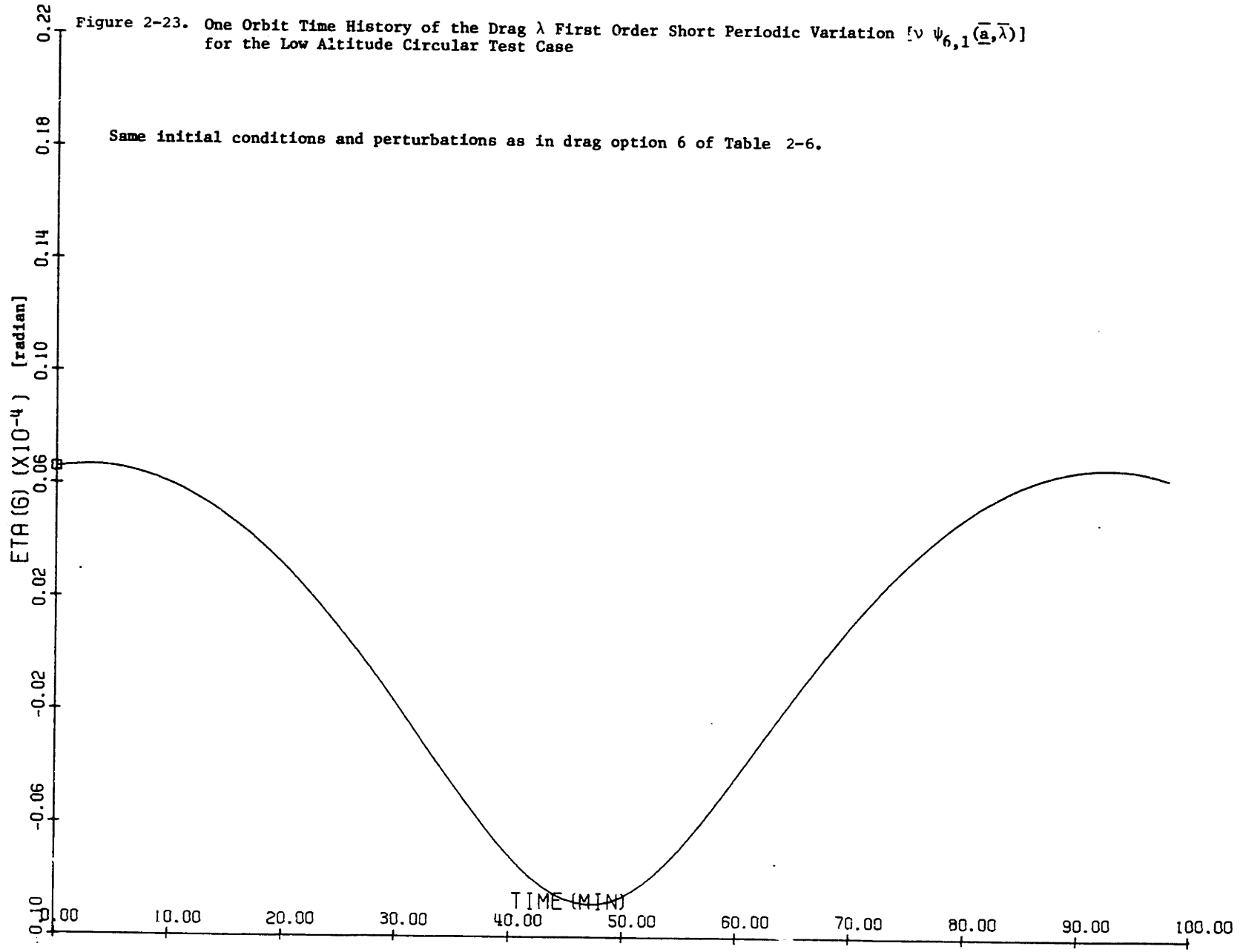
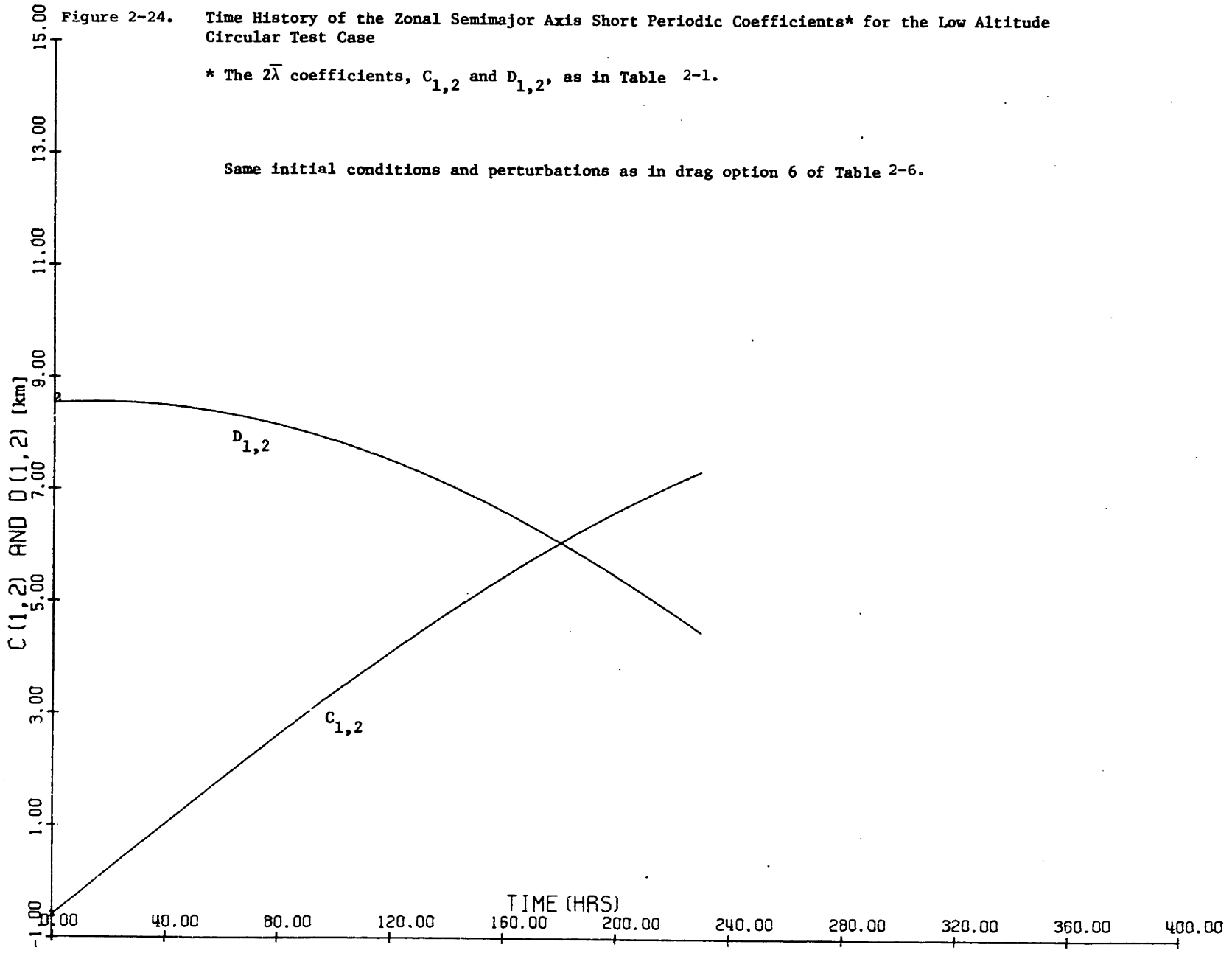


Figure 2-24. Time History of the Zonal Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $2\overline{\lambda}$ coefficients, $C_{1,2}$ and $D_{1,2}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.



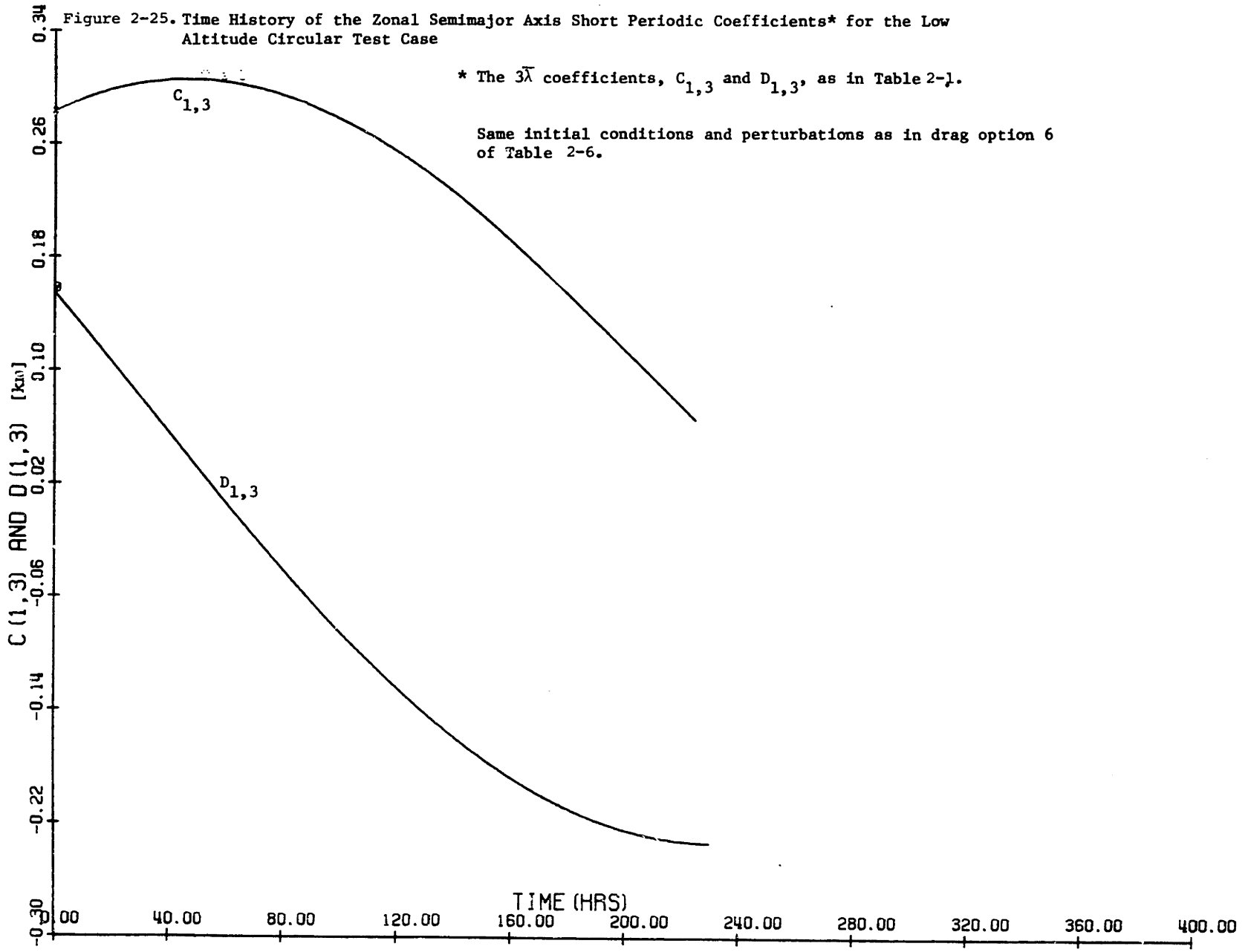


Figure 2-25. Time History of the Zonal Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $3\bar{\lambda}$ coefficients, $C_{1,3}$ and $D_{1,3}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-26. Time History of the Zonal h Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $2\bar{\lambda}$ coefficients, $C_{2,2}$ and $D_{2,2}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

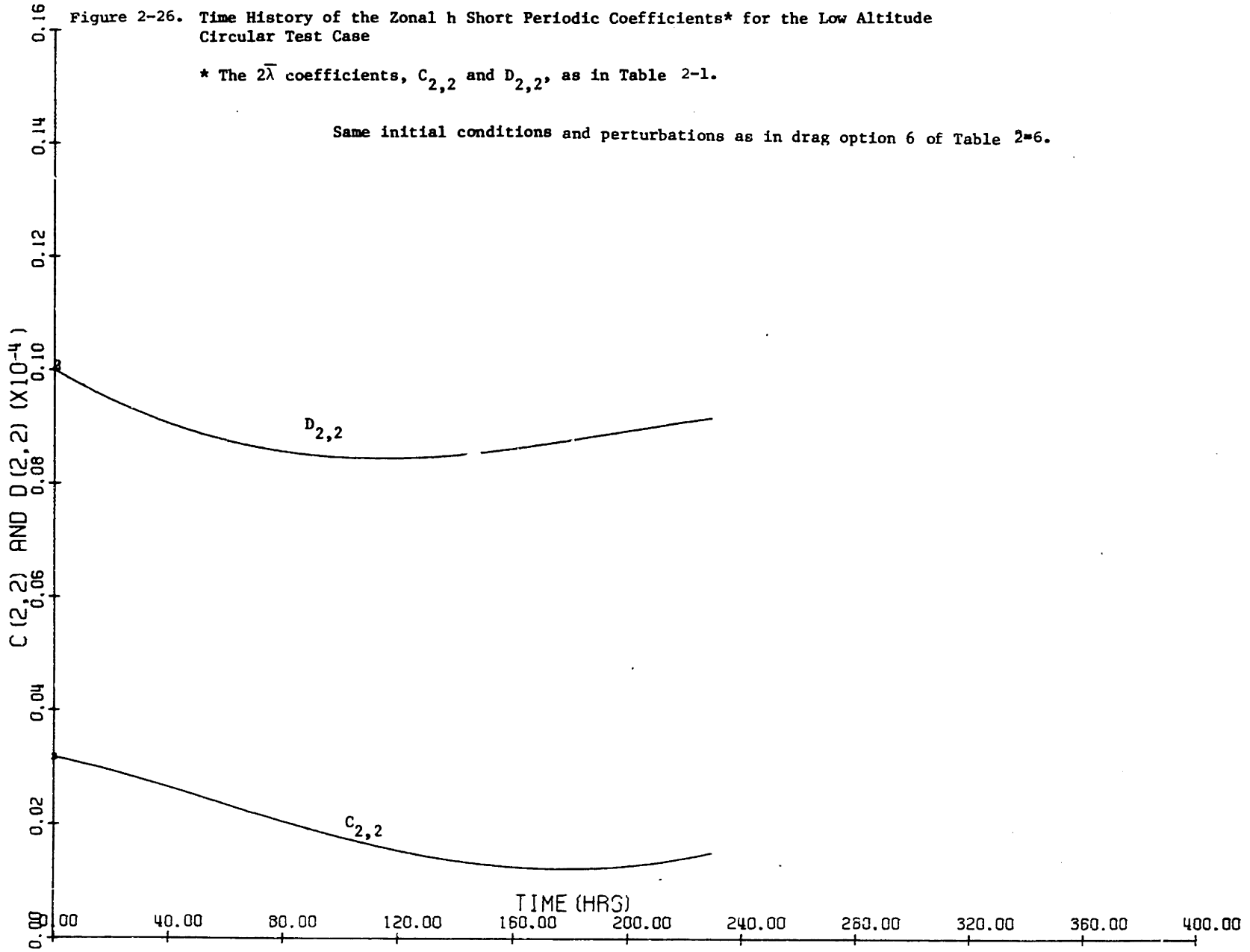
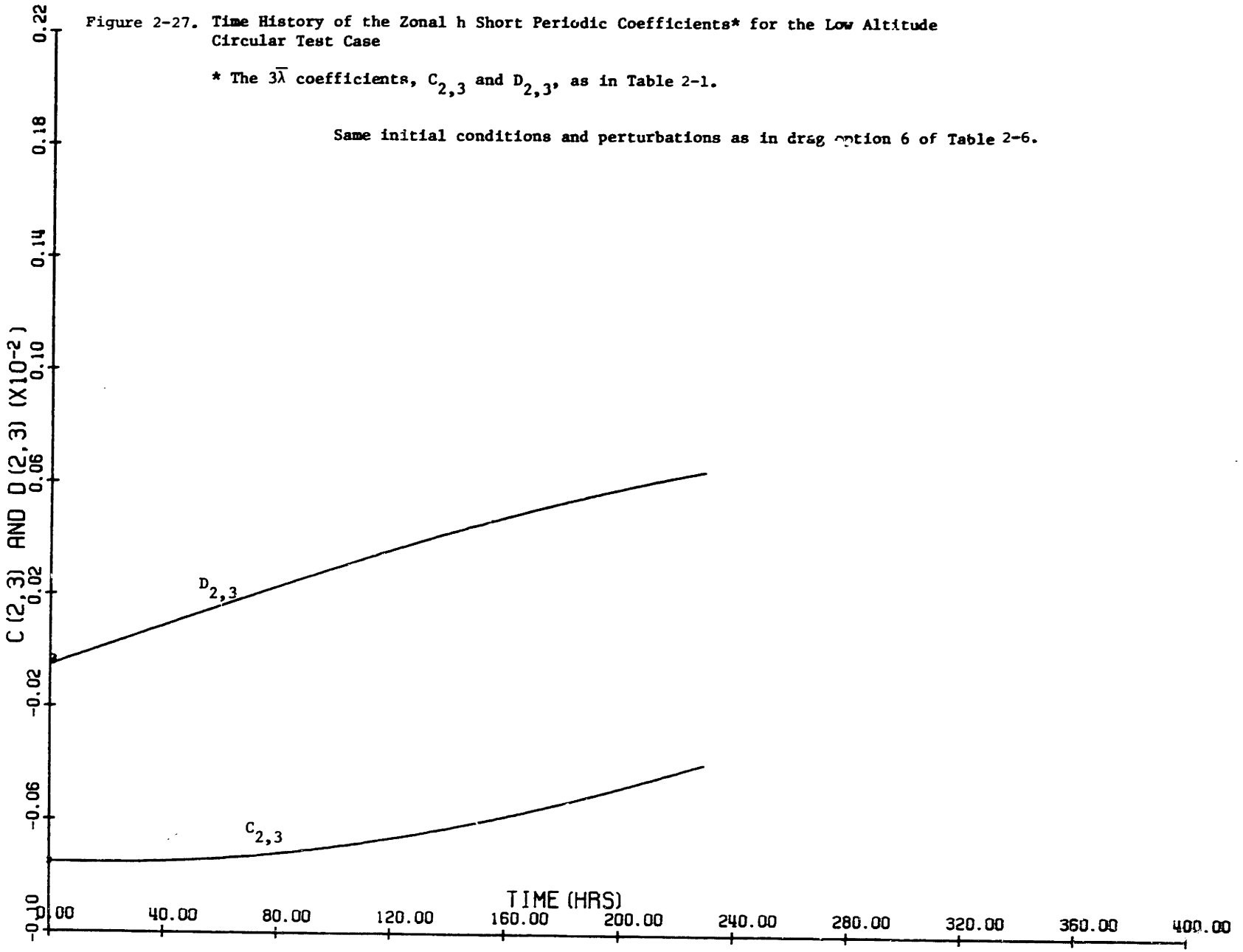


Figure 2-27. Time History of the Zonal h Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $3\bar{\lambda}$ coefficients, $C_{2,3}$ and $D_{2,3}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.



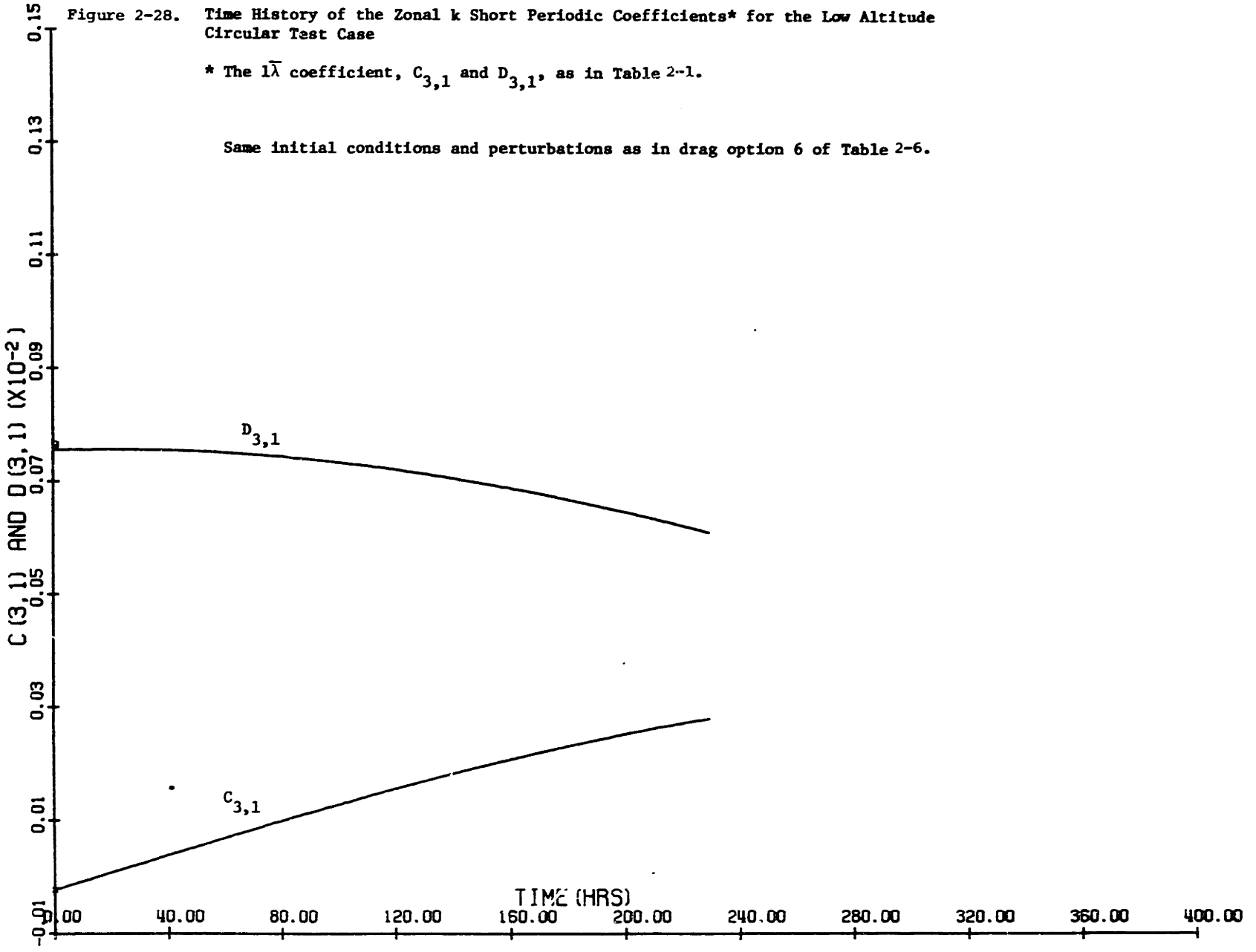


Figure 2-28. Time History of the Zonal k Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficient, $C_{3,1}$ and $D_{3,1}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-29. Time History of the Zonal λ Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $2\bar{\lambda}$ coefficients, $C_{6,2}$ and $D_{6,2}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

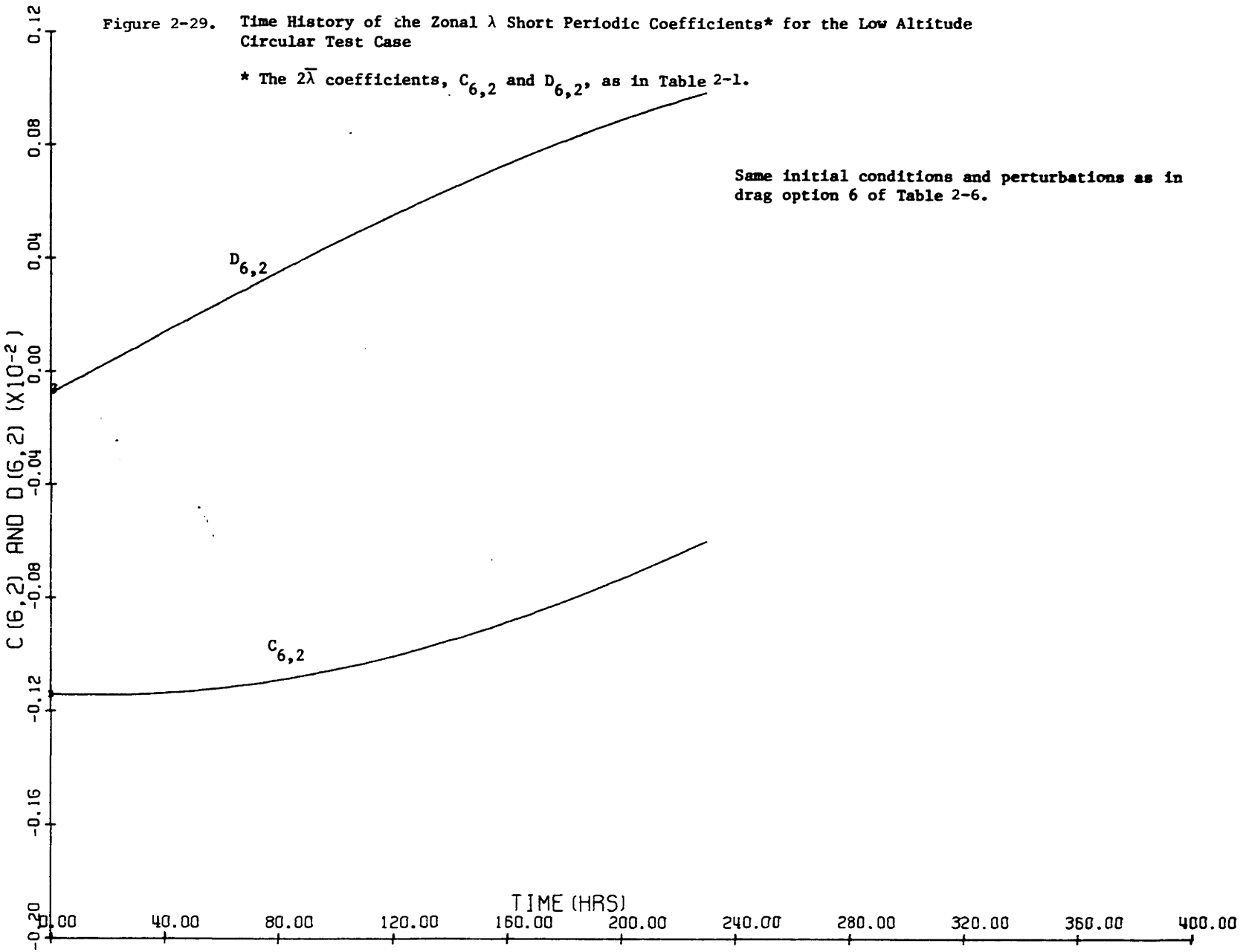
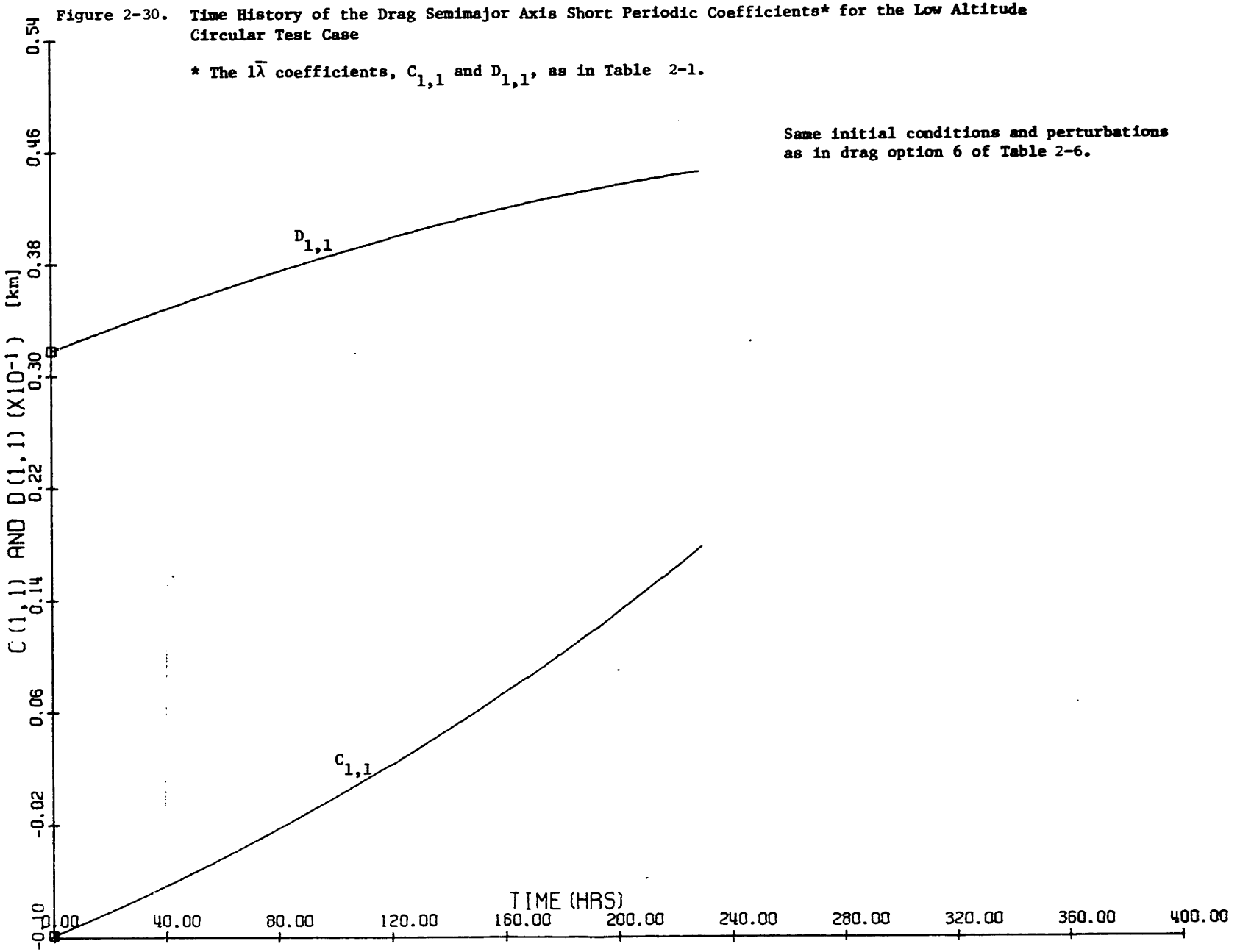


Figure 2-30. Time History of the Drag Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients, $C_{1,1}$ and $D_{1,1}$, as in Table 2-1.



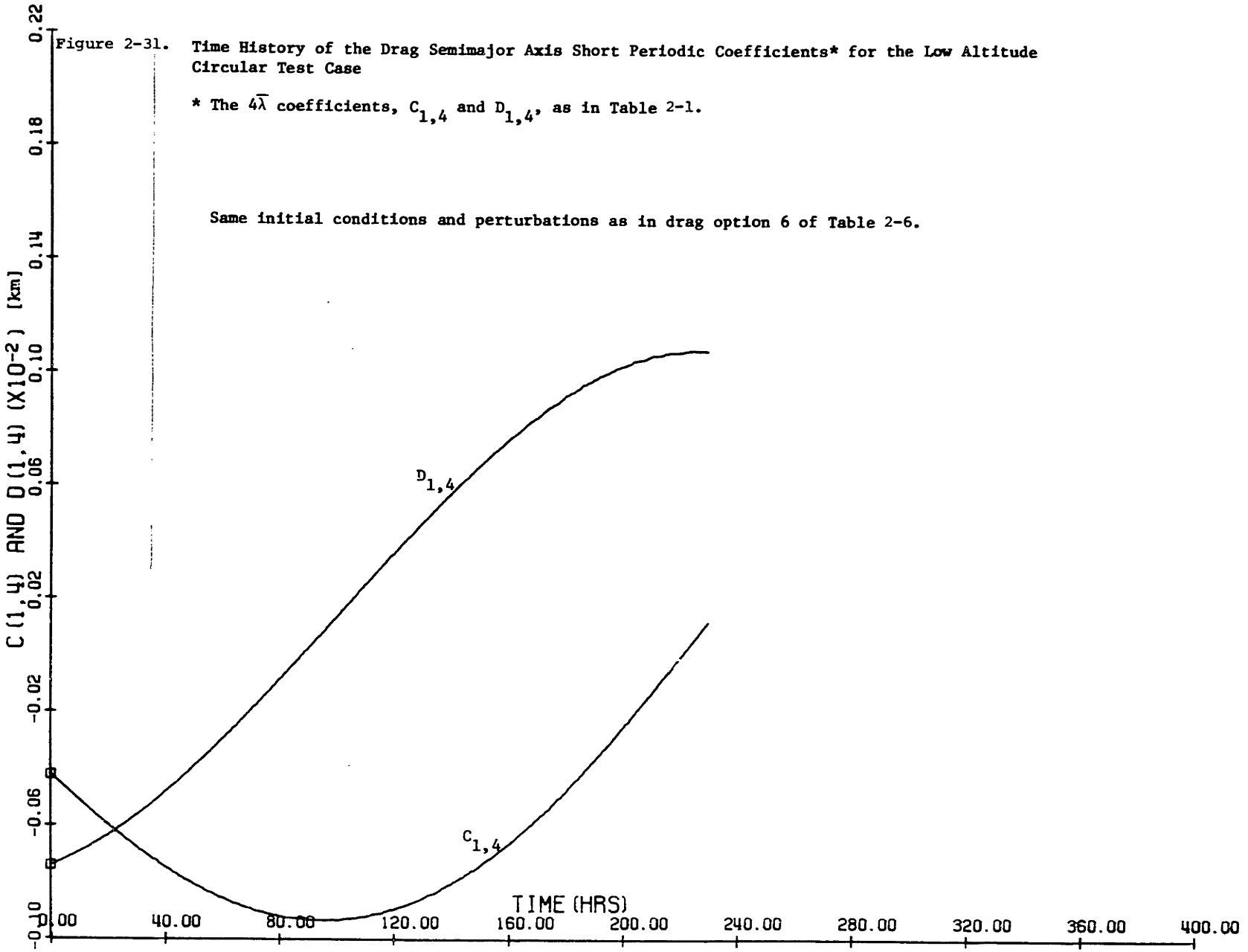


Figure 2-31. Time History of the Drag Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $4\bar{\lambda}$ coefficients, $C_{1,4}$ and $D_{1,4}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

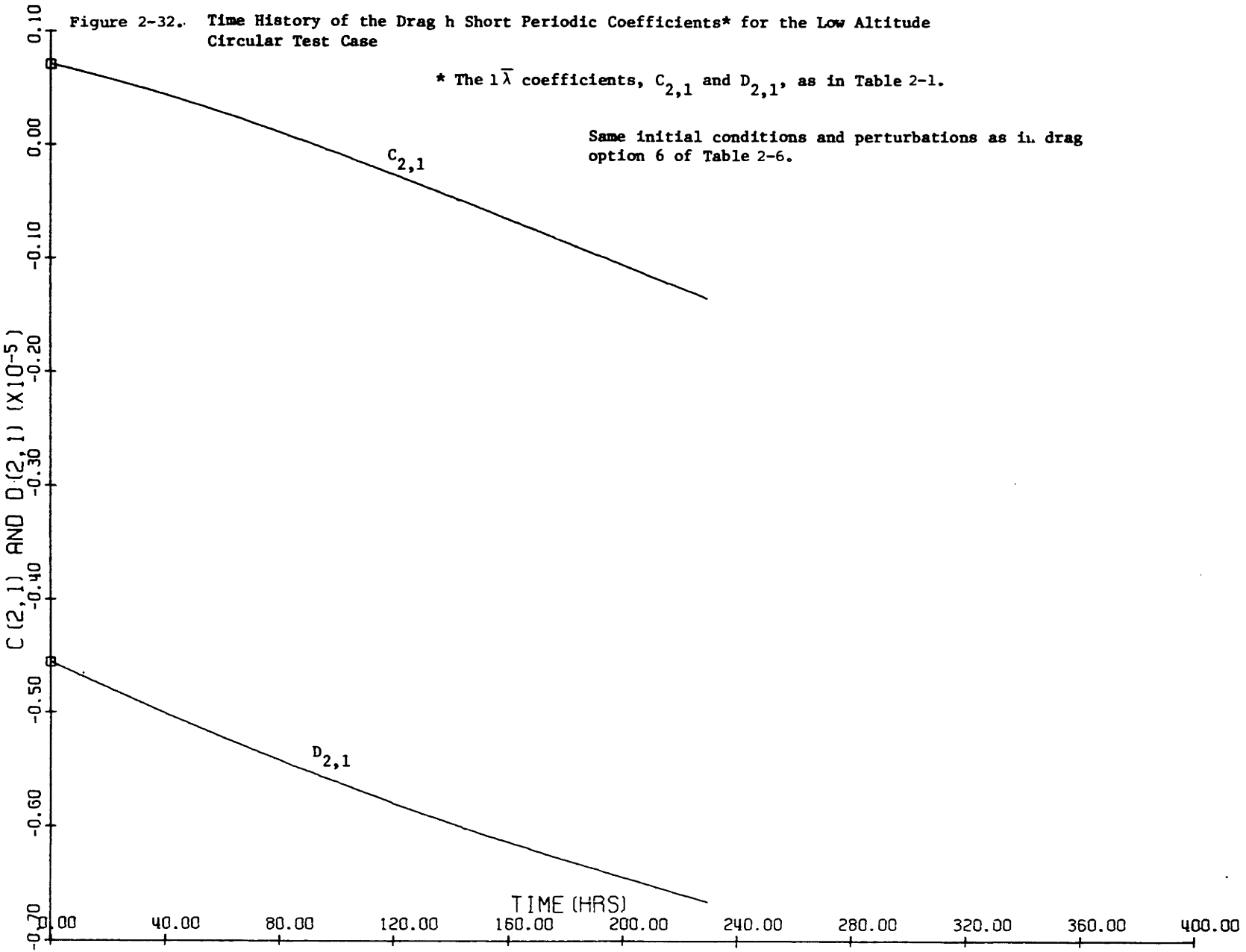


Figure 2-32. Time History of the Drag h Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients, $C_{2,1}$ and $D_{2,1}$, as in Table 2-1.

Same initial conditions and perturbations as in drag option 6 of Table 2-6.

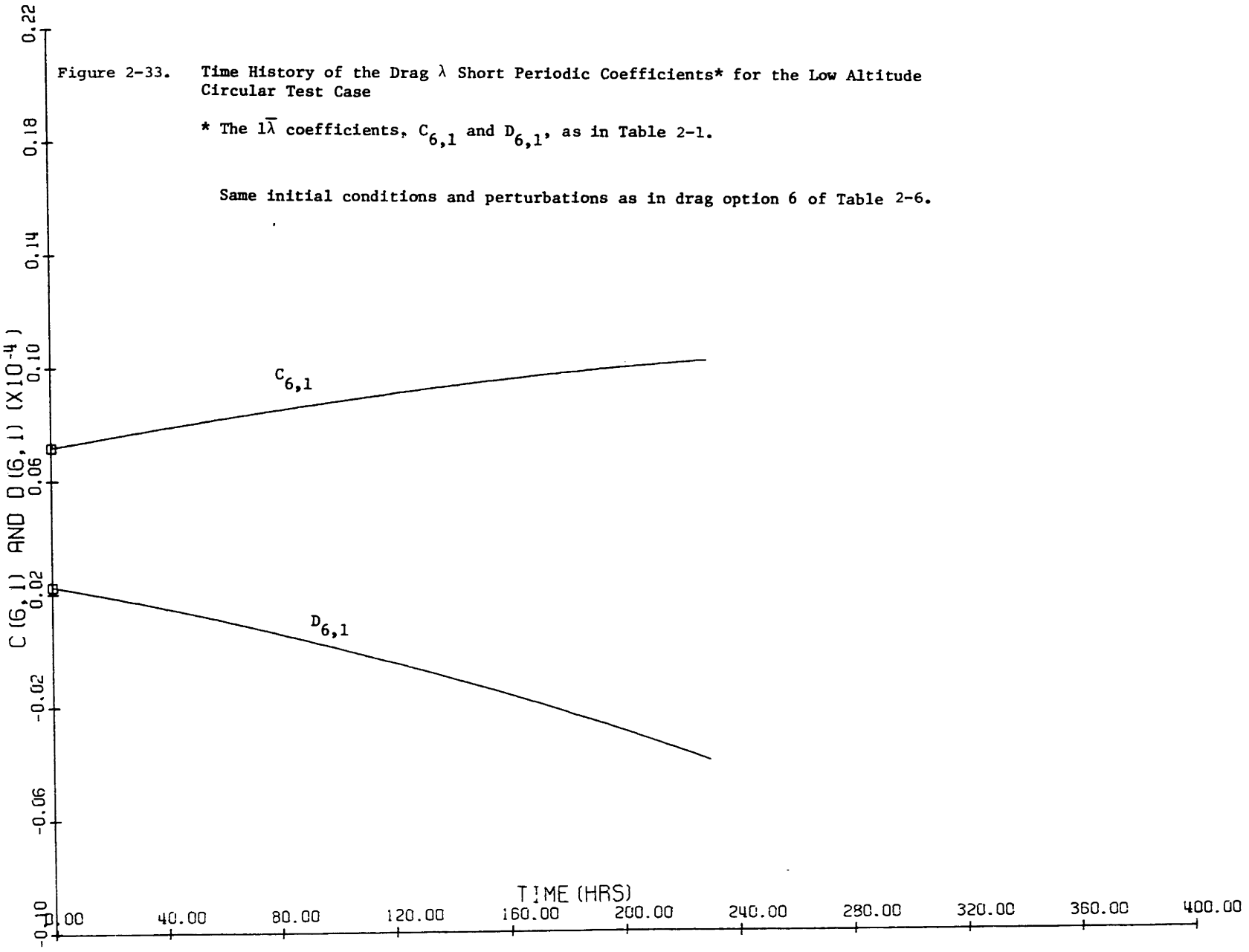
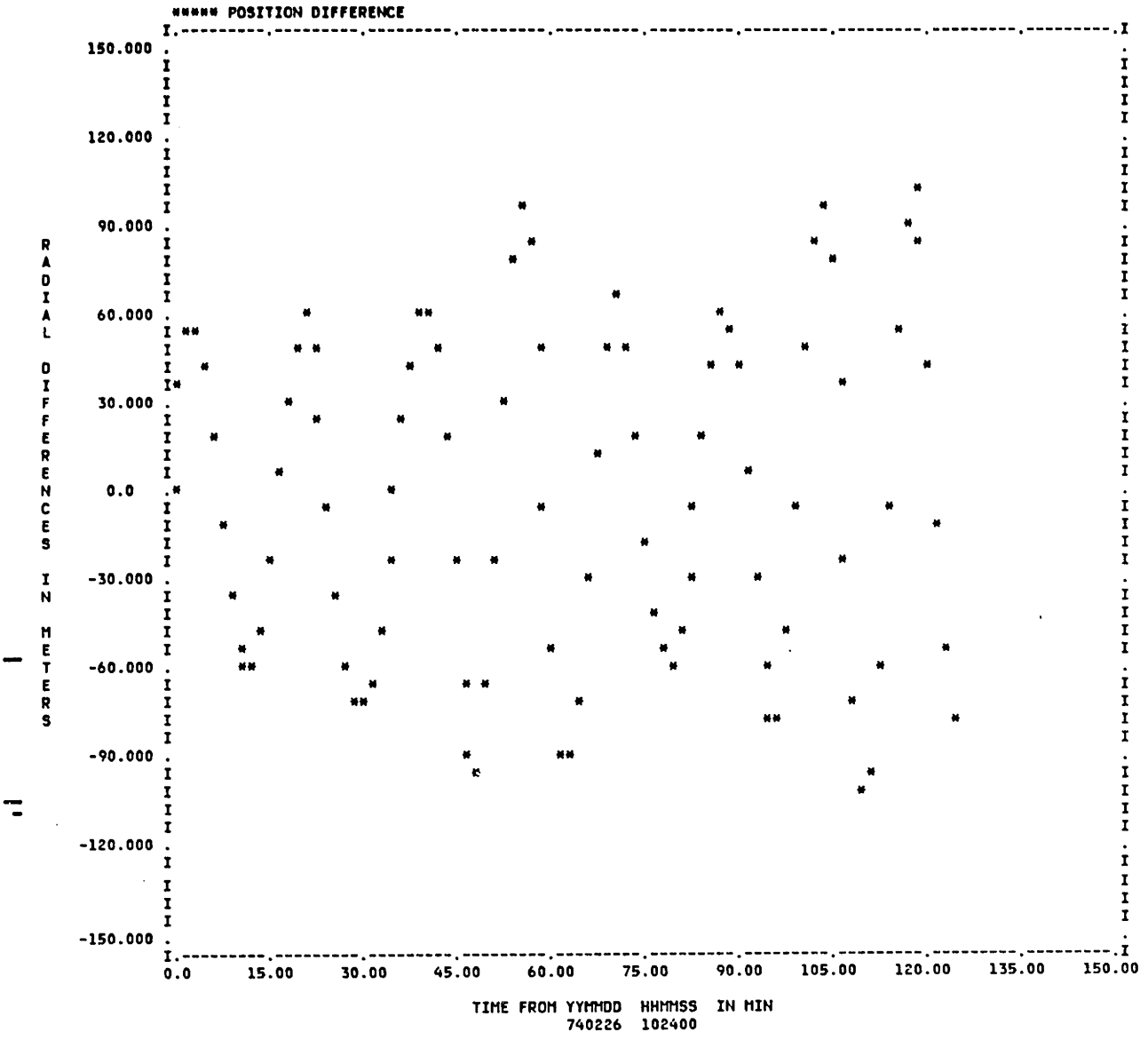


Figure 2-33. Time History of the Drag λ Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients, $C_{6,1}$ and $D_{6,1}$, as in Table 2-1.

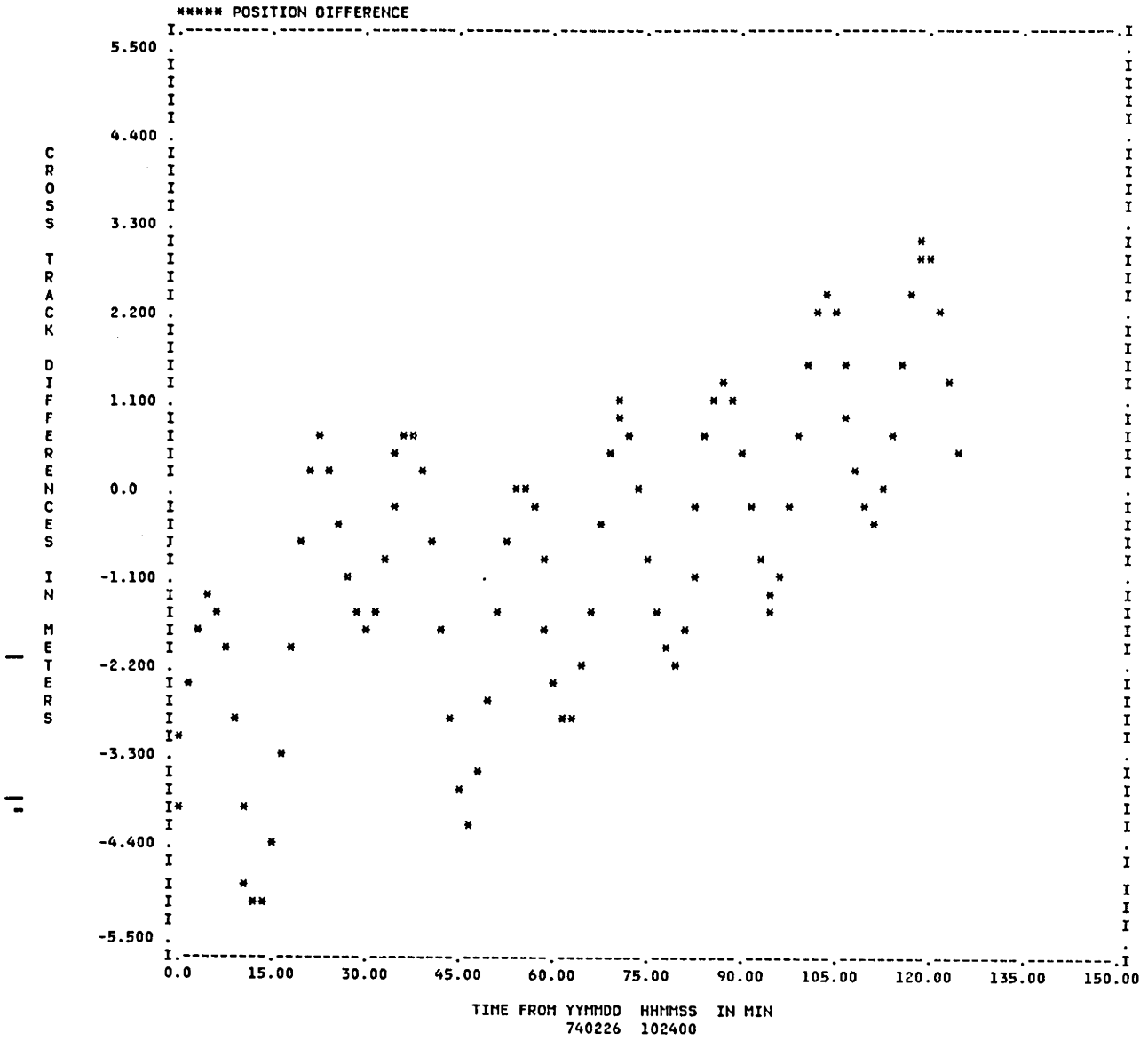
Same initial conditions and perturbations as in drag option 6 of Table 2-6.

Figure 2-34. Radial Difference/Semianalytical minus Cowell for the Low Altitude Eccentric Test Case



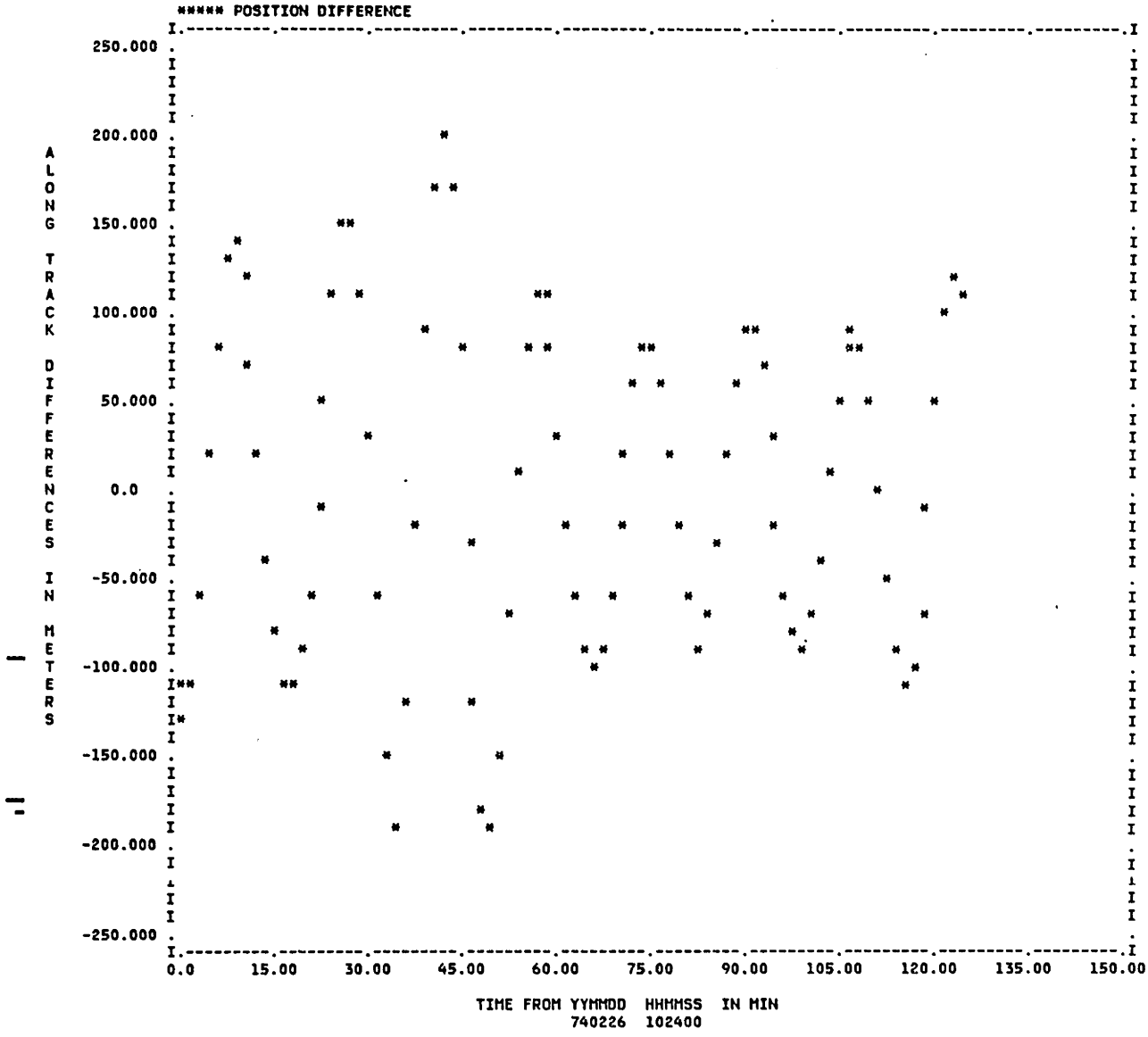
Initial conditions and perturbations as in Table 2-9.

Figure 2-35. Cross Track Difference/Semianalytical minus Cowell for the Low Altitude Eccentric Test Case



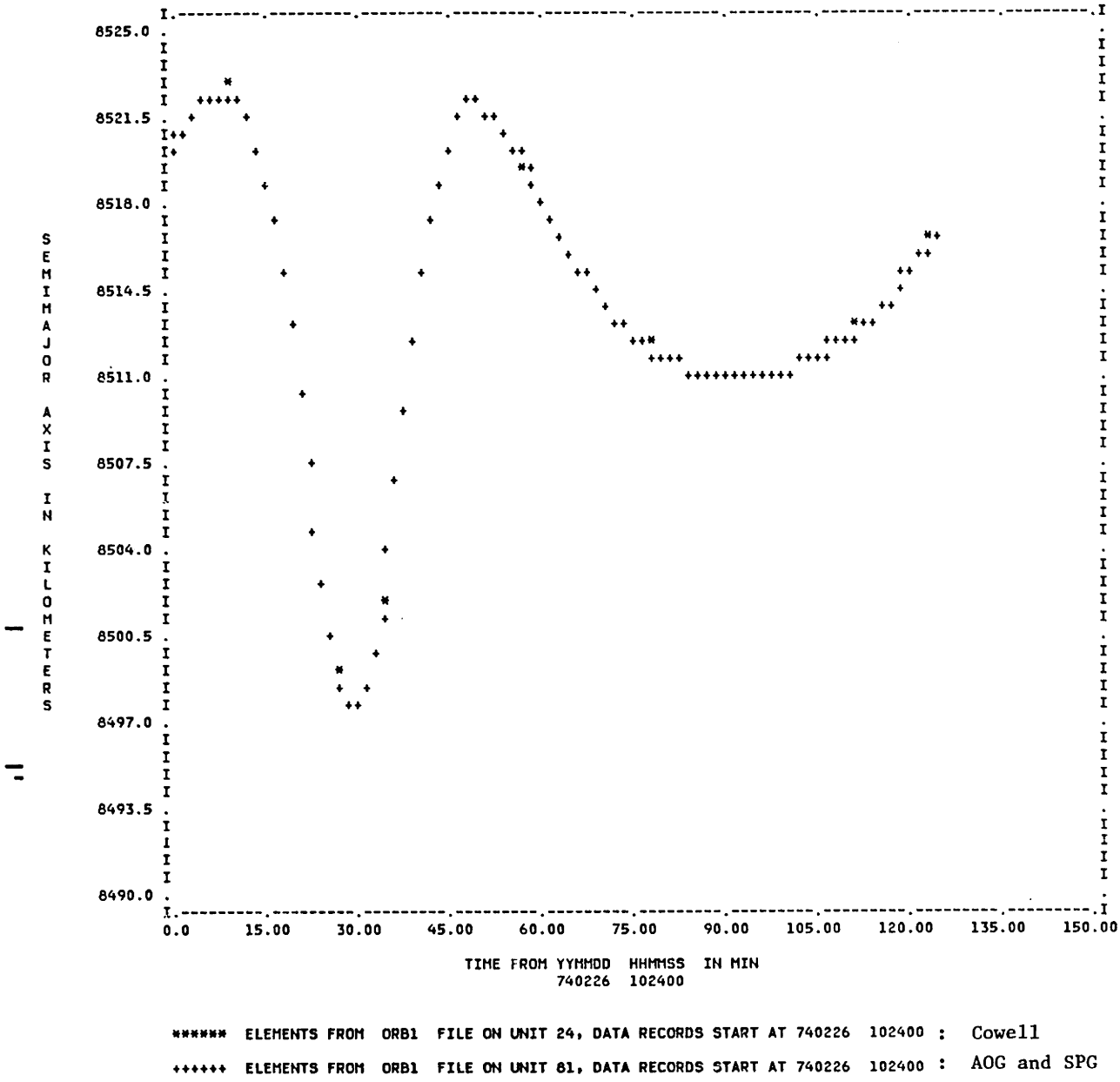
Initial conditions and perturbations as in Table 2-9.

Figure 2-36. Along Track Difference/Semianalytical minus Cowell for the Low Altitude Eccentric Test Case



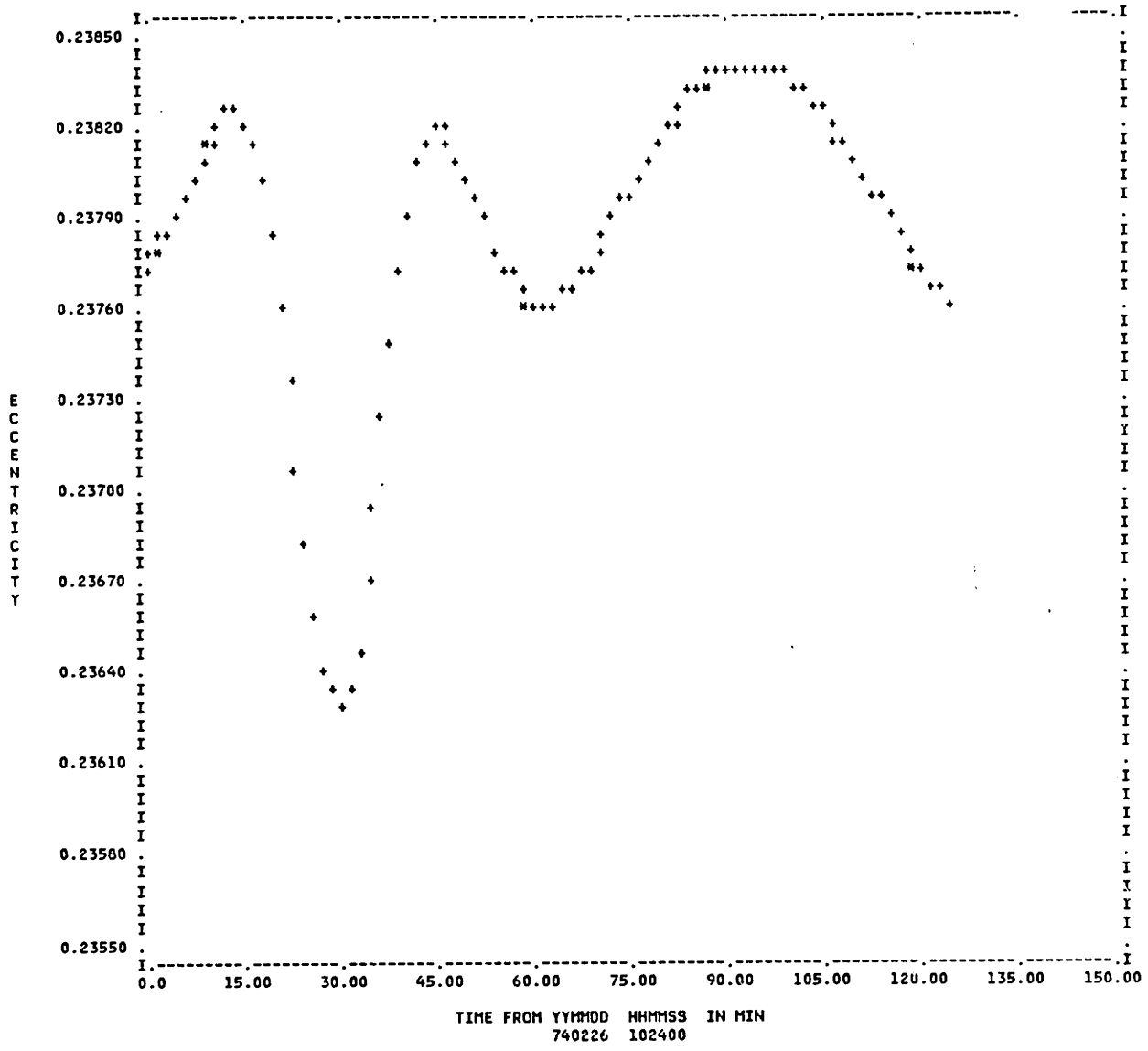
Initial conditions and perturbations as in Table 2-9.

Figure 2-37. Osculating Semimajor Axis Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



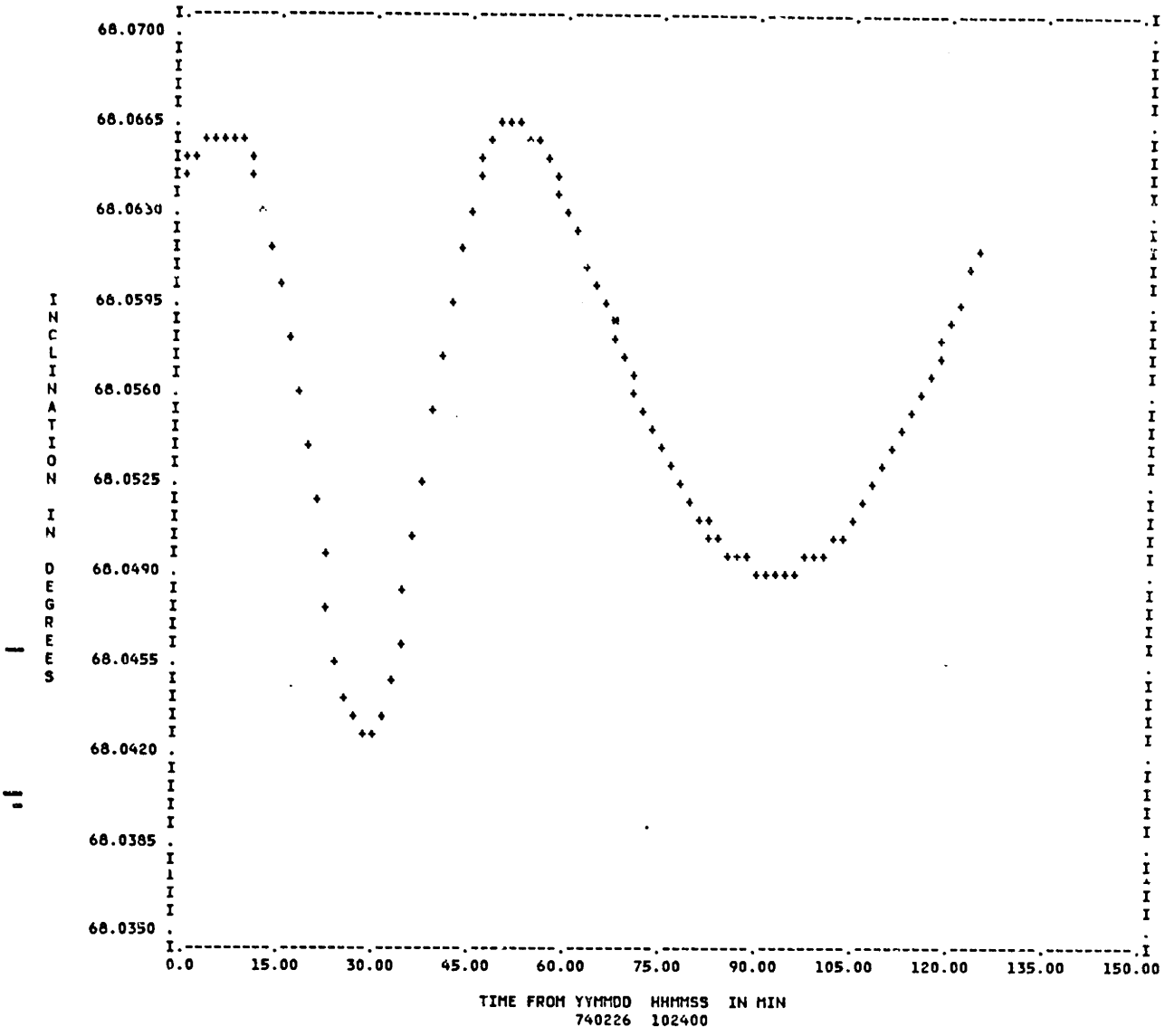
Initial conditions and perturbations as in Table 2-9.

Figure 2-38. Osculating Eccentricity Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



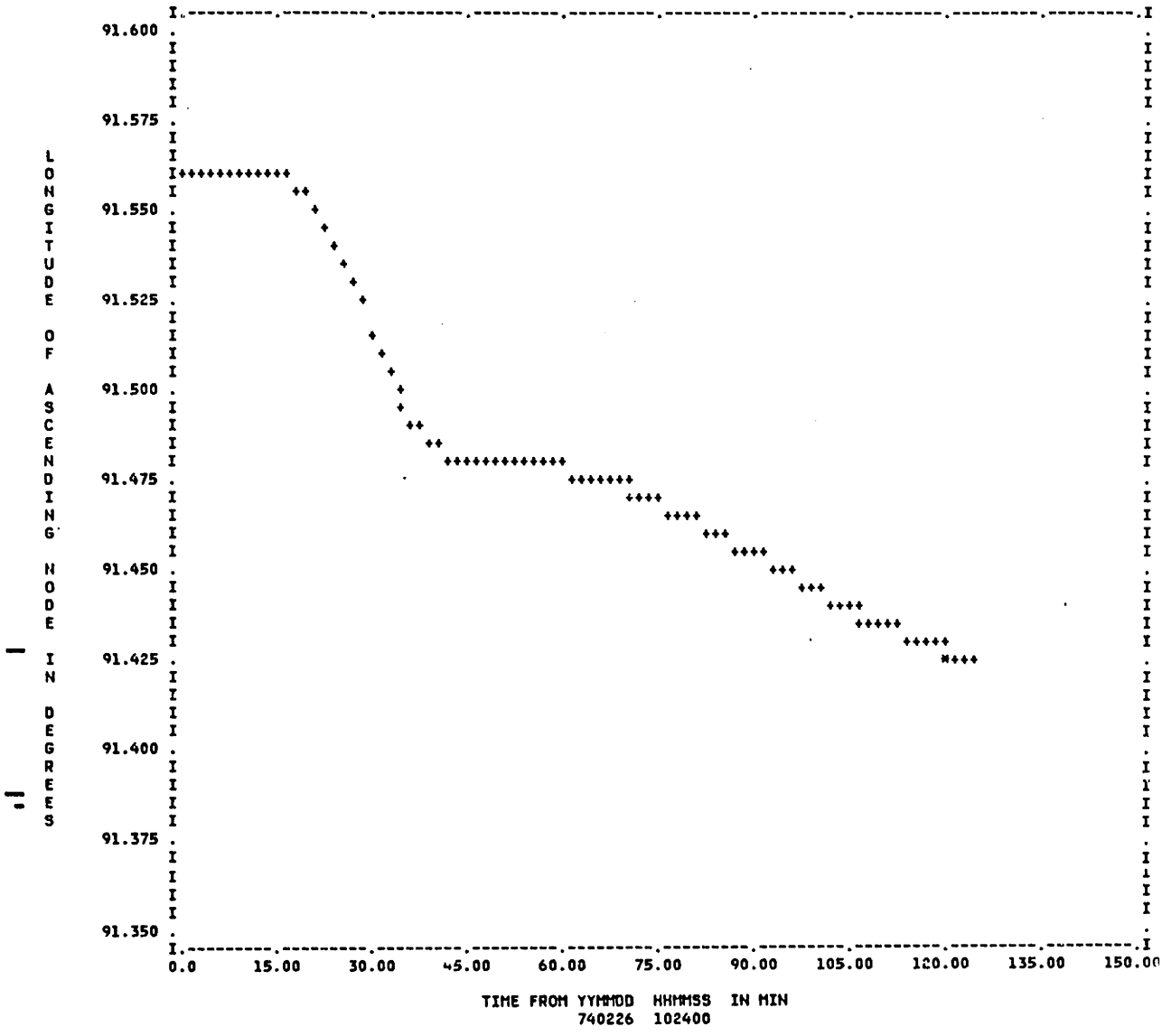
Initial conditions and perturbations as in Table 2-9.

Figure 2-39. Osculating Inclination Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



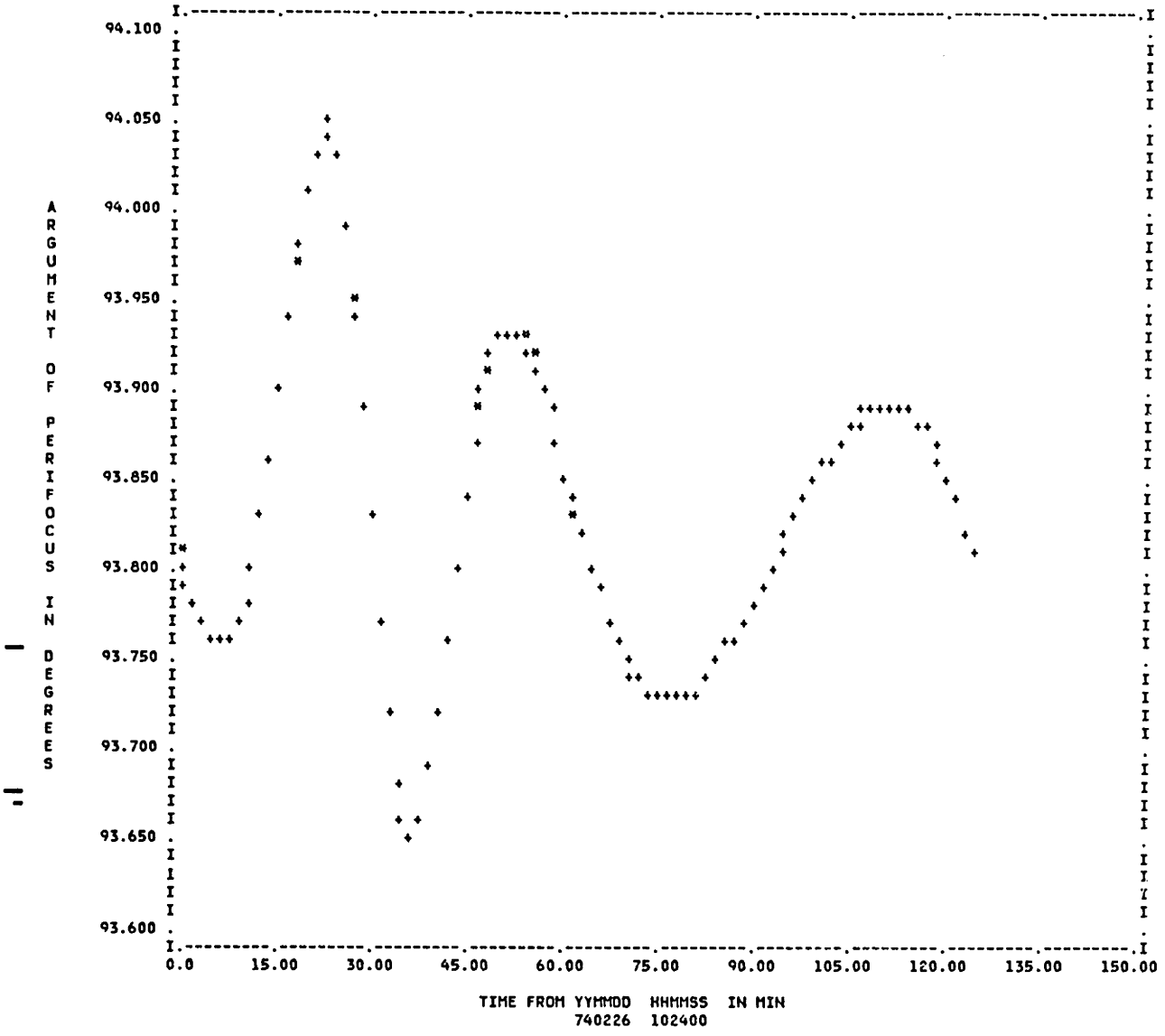
Initial conditions and perturbations as in Table 2-9.

Figure 2-40. Osculating Ω Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



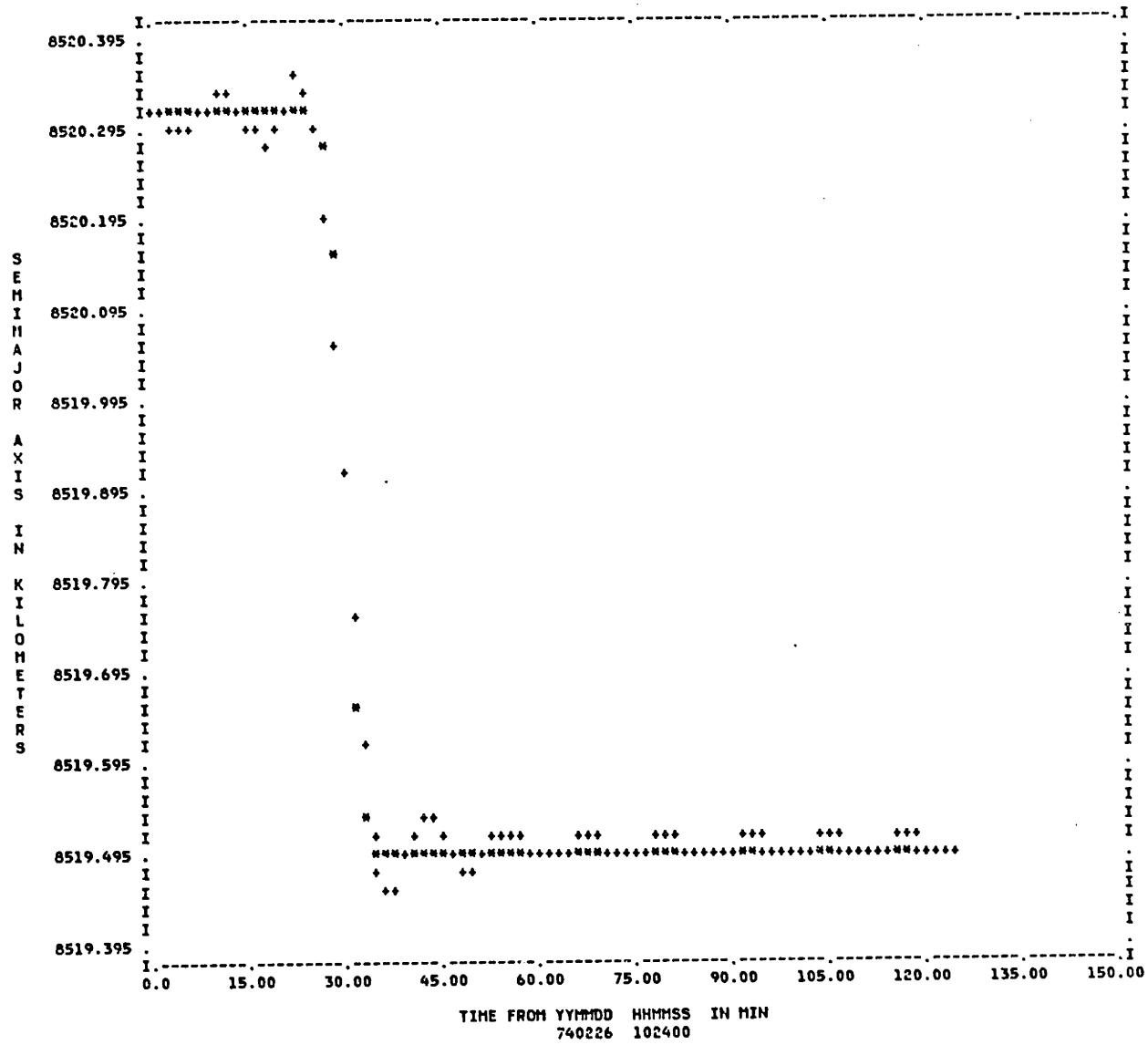
Initial conditions and perturbations as in Table 2-9.

Figure 2-41. Osculating ω Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



Initial conditions and perturbations as in Table 2-9.

Figure 2-42. Osculating Semimajor Axis Comparison/Semianalytical versus Cowell for the Low Altitude Eccentric Test Case



Perturbations

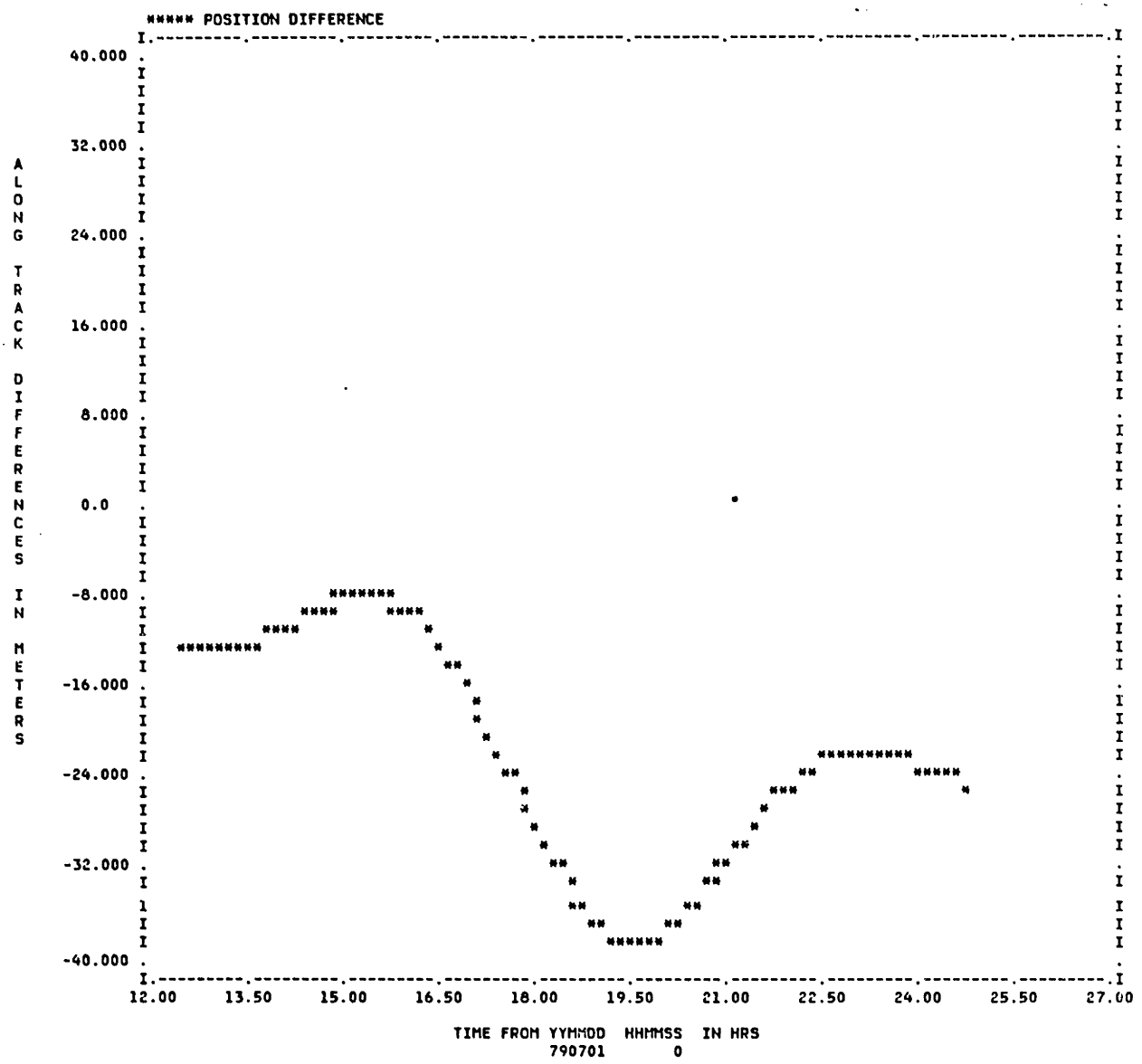
Cowell: drag only (10 sec numerical integration time step)

AOG: drag only, option 1 (48 pt quadrature)
1 day numerical integration time step

SPG: 1st order drag only (10/48), time-independent formulation

Initial conditions: NOM

Figure 2-43. Along Track Difference after 12 hours from Epoch/Semianalytical* minus Cowell for the Medium Altitude Circular Test Case



* Weak Time-Dependent corrections were not used.

Perturbations:

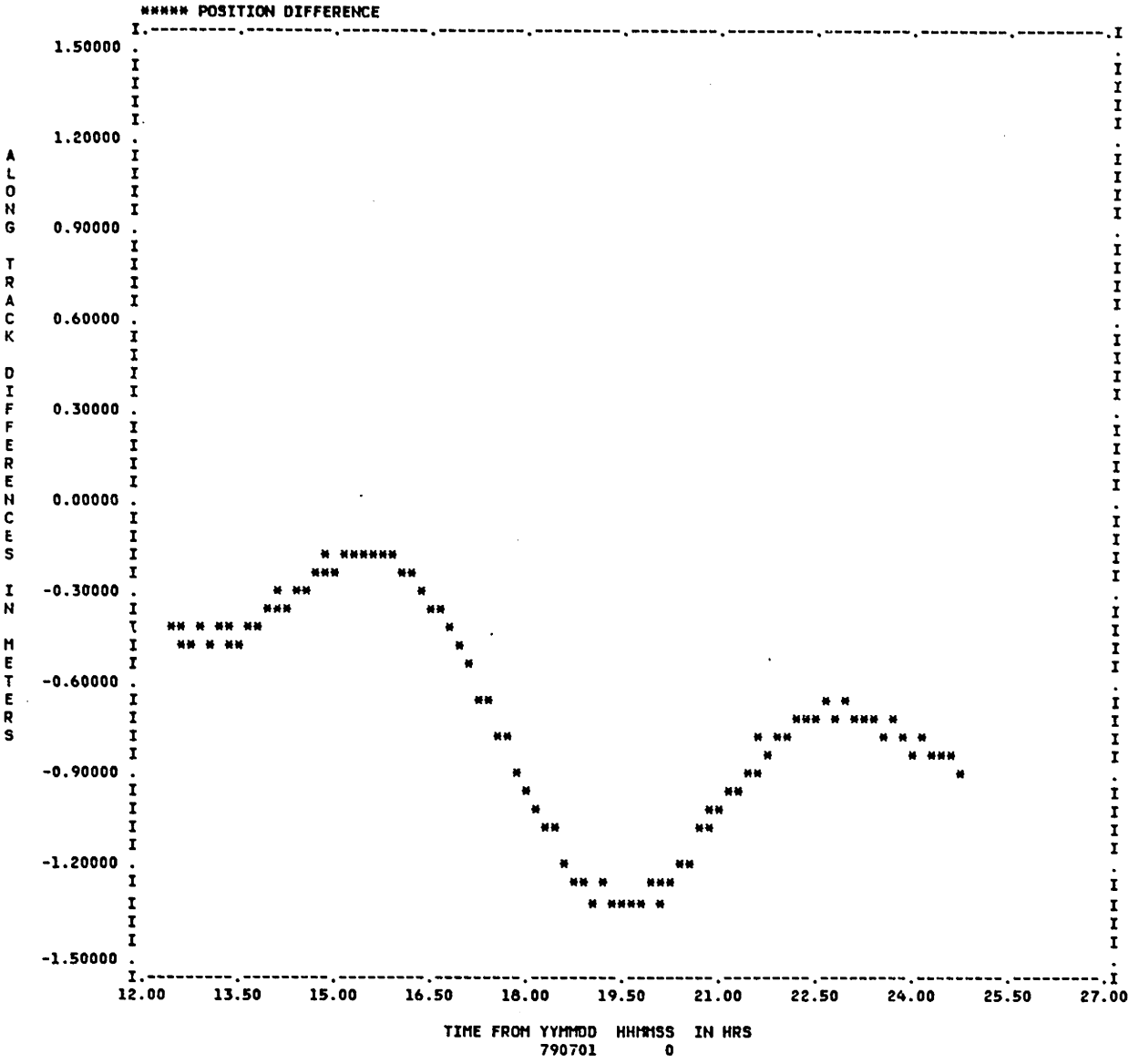
Cowell: 6x0 gravitational field, lunar-solar, solar radiation pressure: 300 sec integration time step

AOG: 1st order analytical expressions for 6x0 gravitational field and lunar-solar point mass effects, Zeis' J_2^2 expressions, numerical solar radiation pressure effects (48 pt quadrature): 1 day integration time steps

SPG: 1st order time-independent model for 6x0 gravitational field (7/48), lunar-solar effects (7/48), solar radiation pressure effects (4/48), Zeis' J_2^2 expressions.

Initial Conditions: $[EPC]^{-1}$ (see Table 2-11)

Figure 2-44. Along Track Difference after 12 hours from Epoch/Semianalytical minus Cowell for the Medium Altitude Circular Test Case



Perturbations:

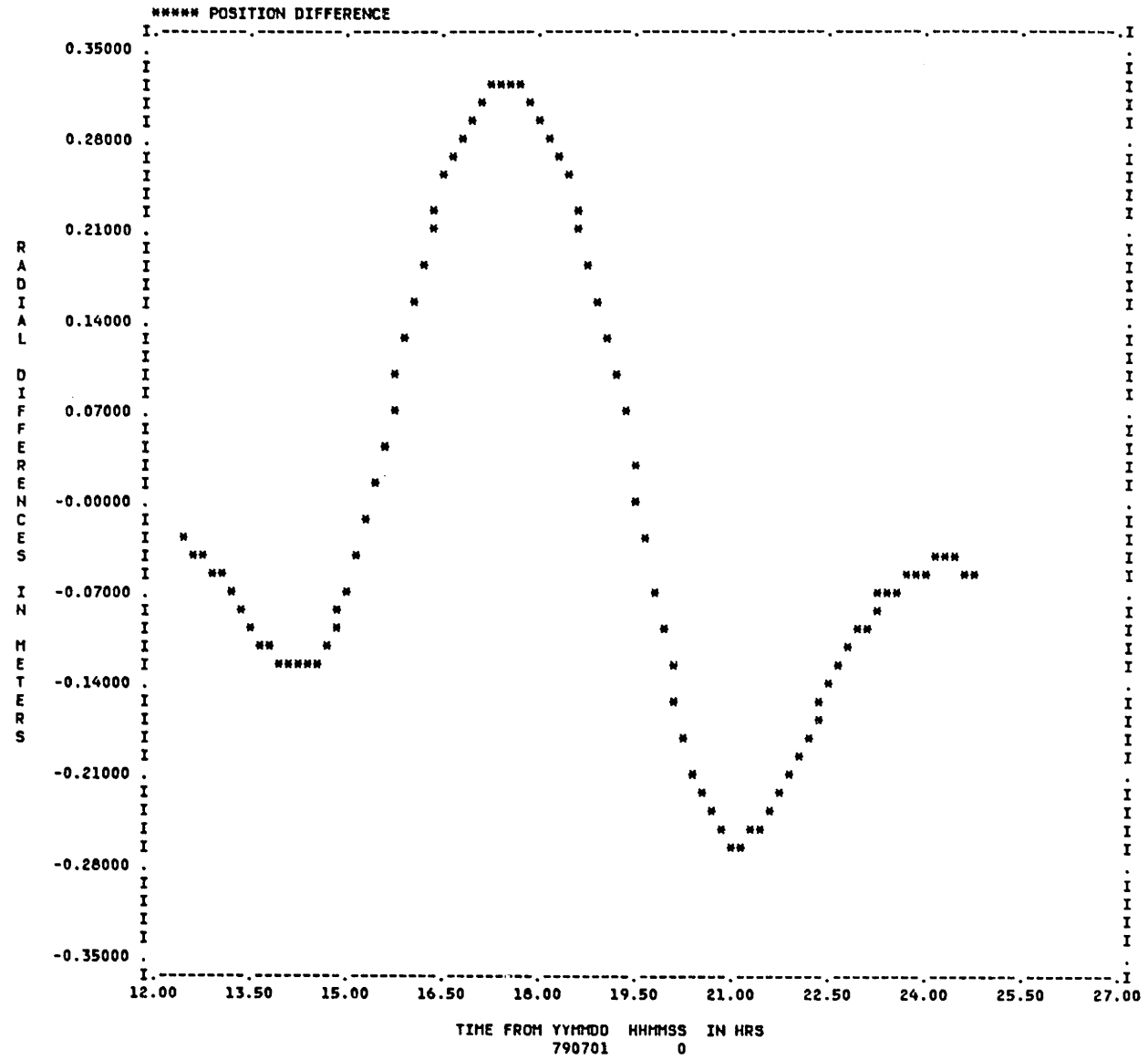
Cowell: 6x0 gravitational field, lunar-solar, solar radiation pressure: 300 sec. integration time step

AOG: 1st order analytical expressions for 6x0 gravitational field and lunar-solar point mass effects, Zeis' J_2^2 expressions, numerical solar radiation pressure effects (48 pt quadrature order): 1 day integration time step

SPG: 1st order weak time-dependent model for 6x0 gravitational field (7/48), lunar-solar effects (7/48), solar radiation pressure effects (4/48), Zeis' J_2^2 expressions.

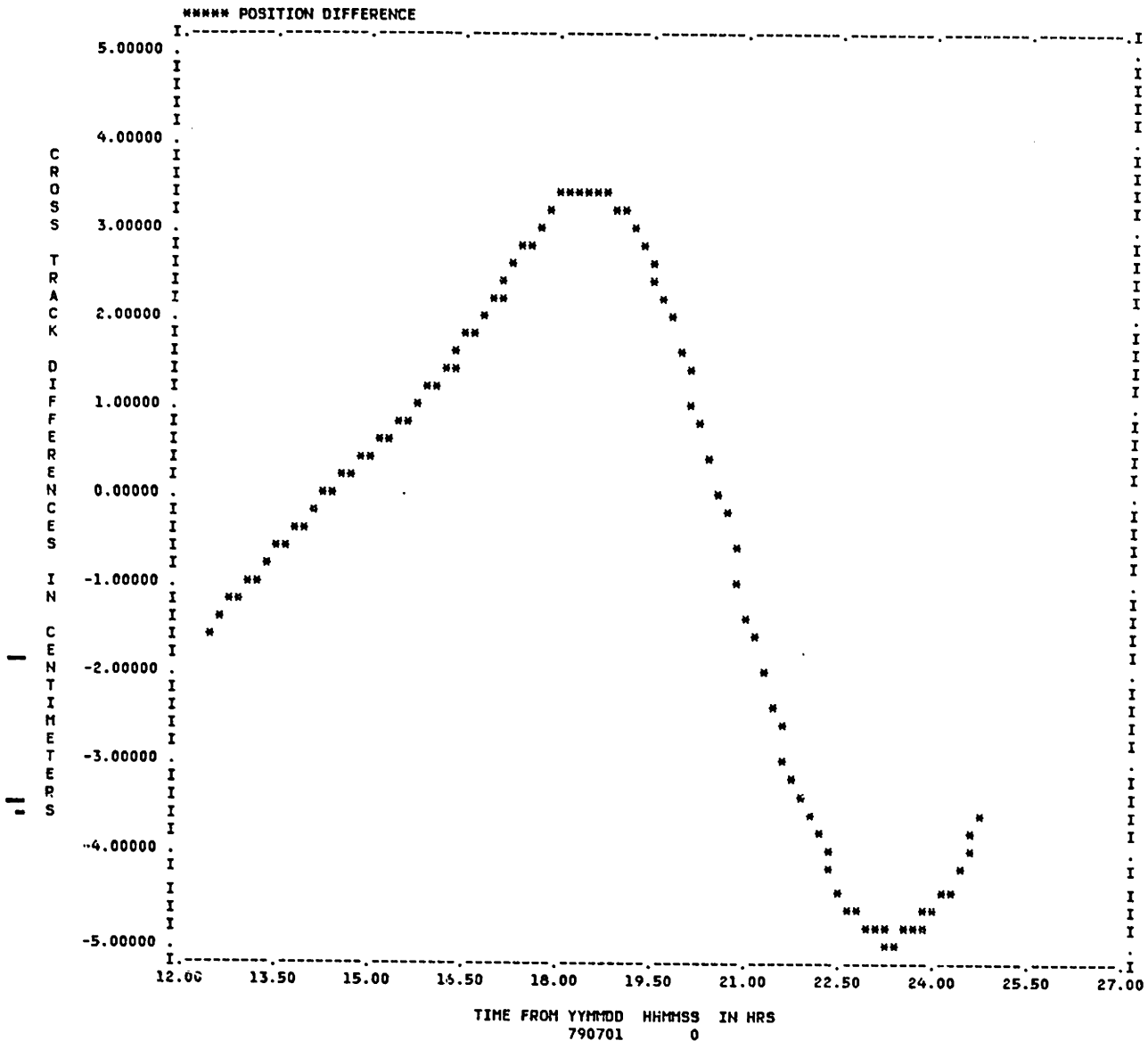
Initial Conditions: $[EPC]^{-1}$ (see Table 2-11)

Figure 2-45. Radial Difference after 12 hours from Epoch/Semianalytical minus Cowell for the Medium Altitude Circular Test Case



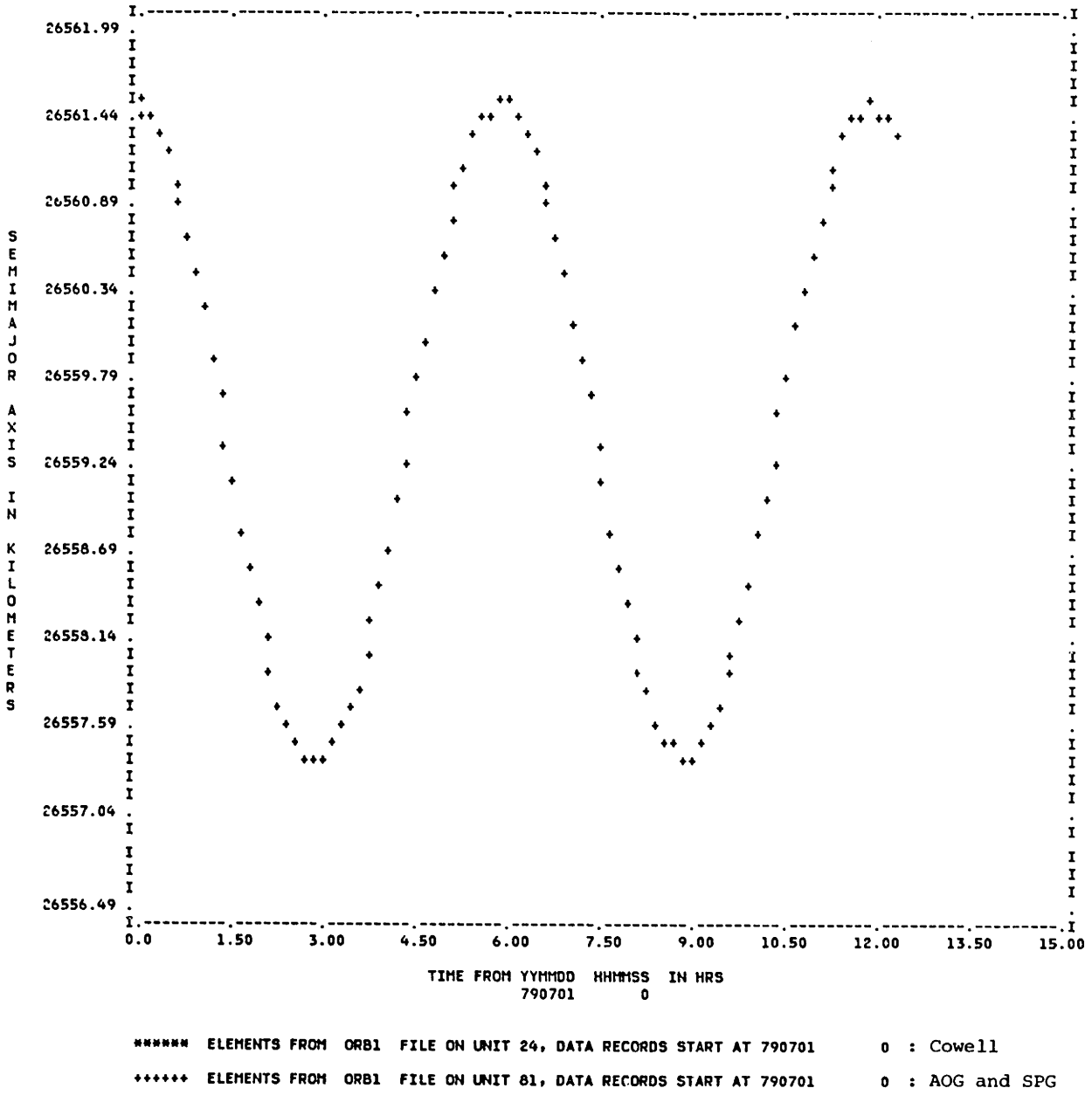
Initial conditions and perturbations as in Figure 2-44.

Figure 2-46. Cross Track Difference after 12 hours from Epoch/Semianalytical minus Cowell for the Medium Altitude Circular Test Case



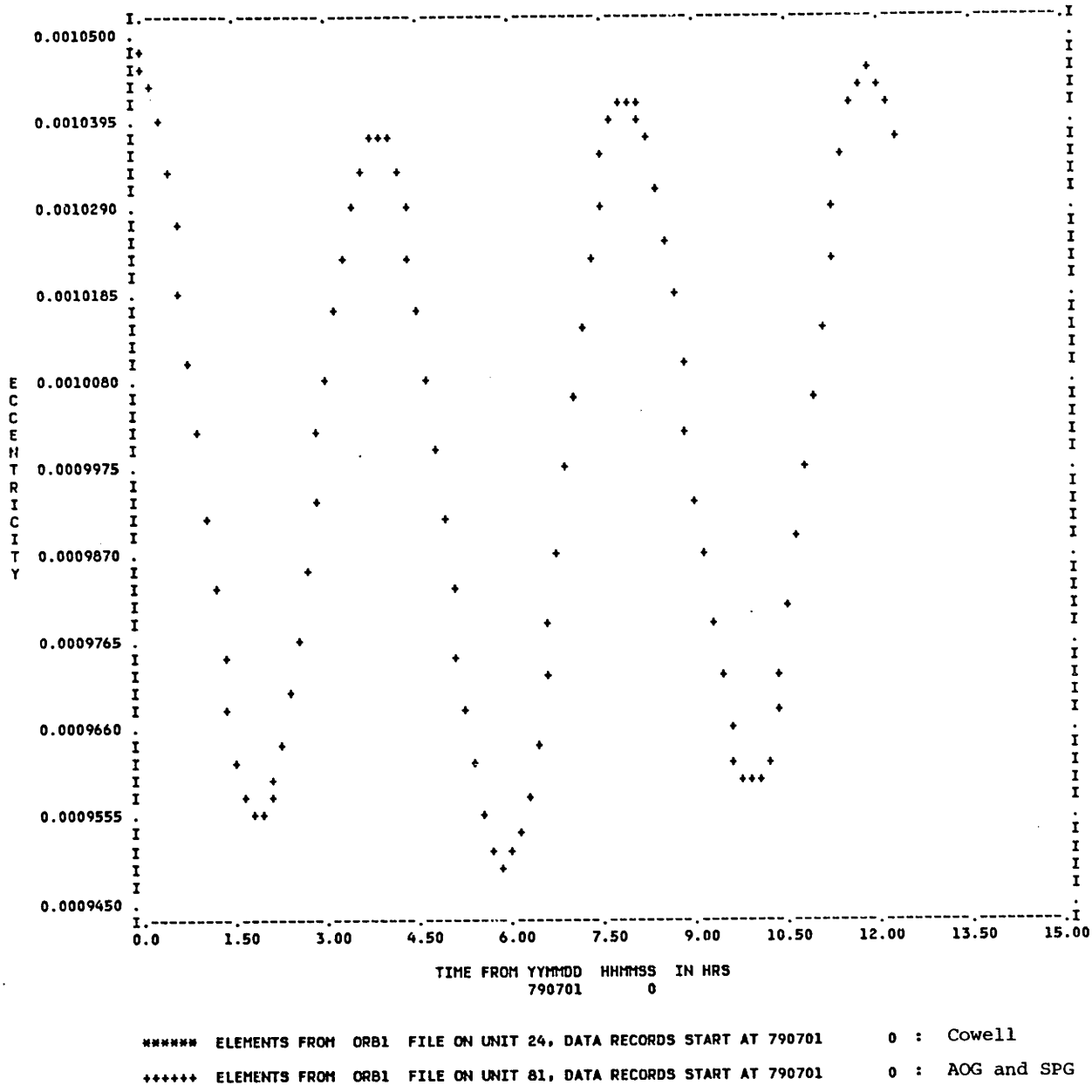
Initial conditions and perturbations as in Figure 2-44.

Figure 2-47. Osculating Semimajor Axis Comparison/Semianalytical versus Cowell for the Medium Altitude Circular Test Case.



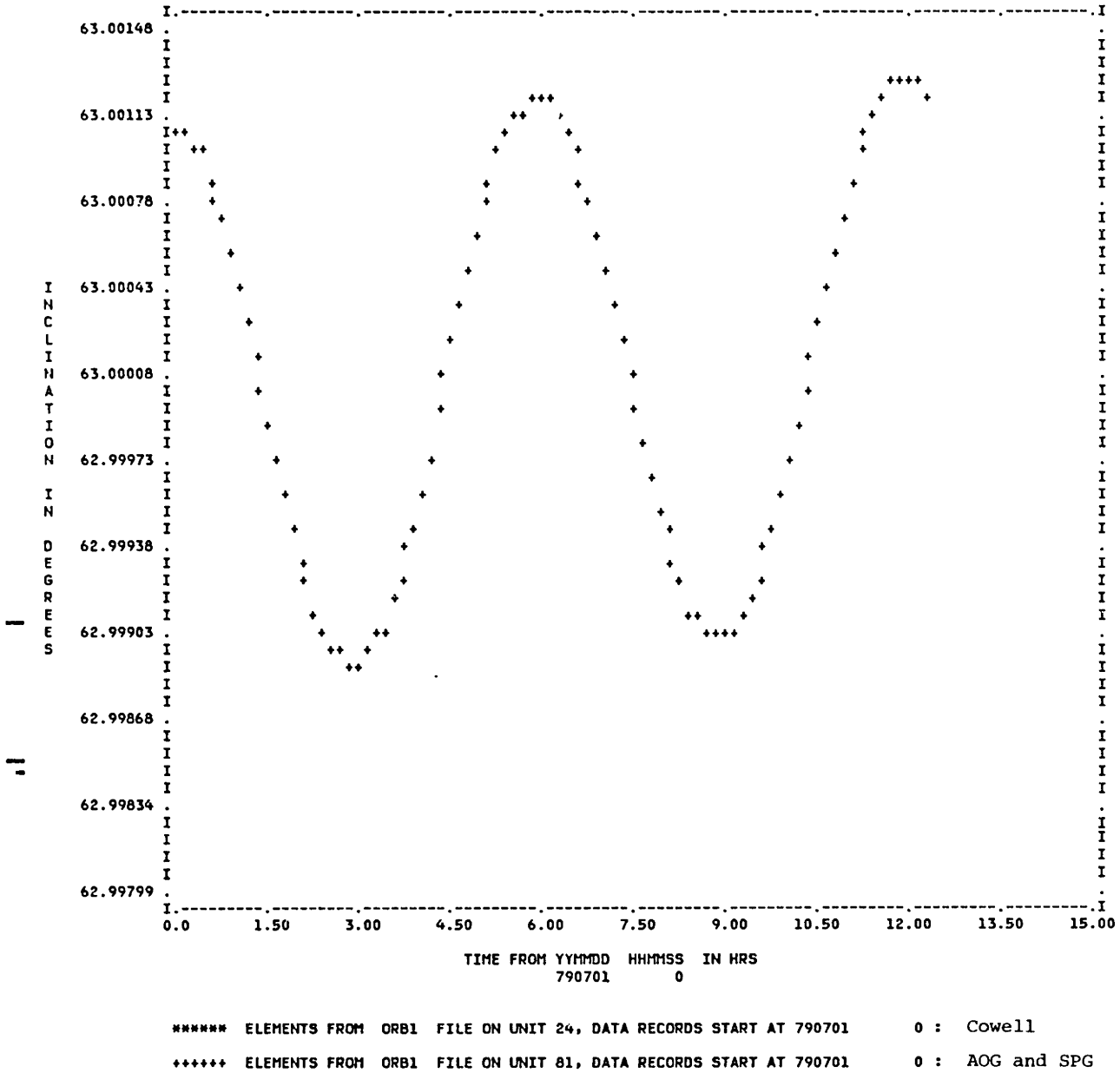
Same initial conditions and perturbations as in Figure 2-44.

Figure 2-48. Osculating Eccentricity Comparison/Semianalytical versus Cowell for the Medium Altitude Circular Test Case



Same initial conditions and perturbations as in Figure 2-44.

Figure 2-49. Osculating Inclination Comparison/Semianalytical versus Cowell for the Medium Altitude Circular Test Case



Same initial conditions and perturbations as in Figure 2-44.

Figure 2-50. One Orbit Time History of the Solar Radiation Pressure Semimajor Axis First Order Short Periodic Variations for the Medium Altitude Circular Test Case

Same initial conditions and perturbations as in Figure 2-44.

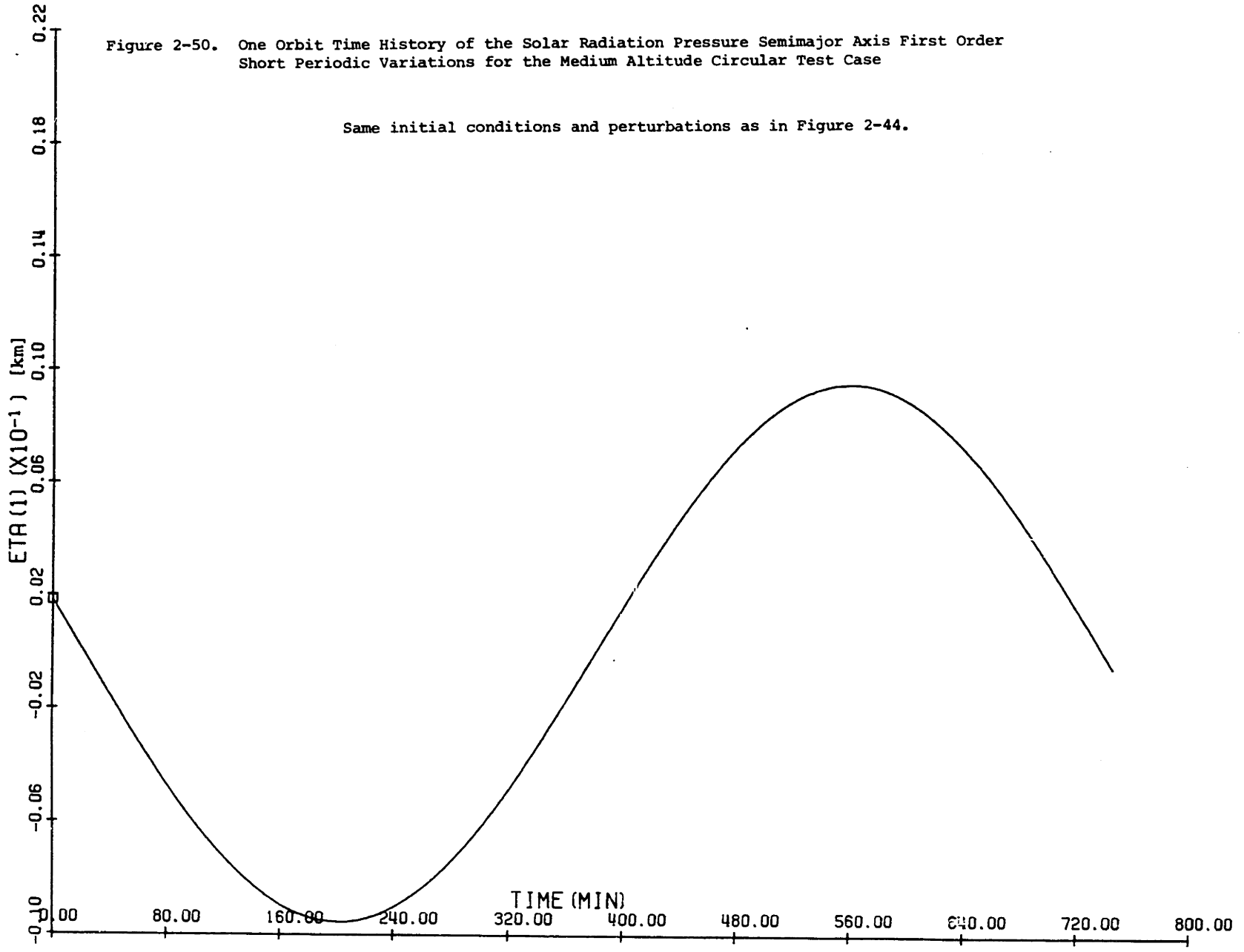
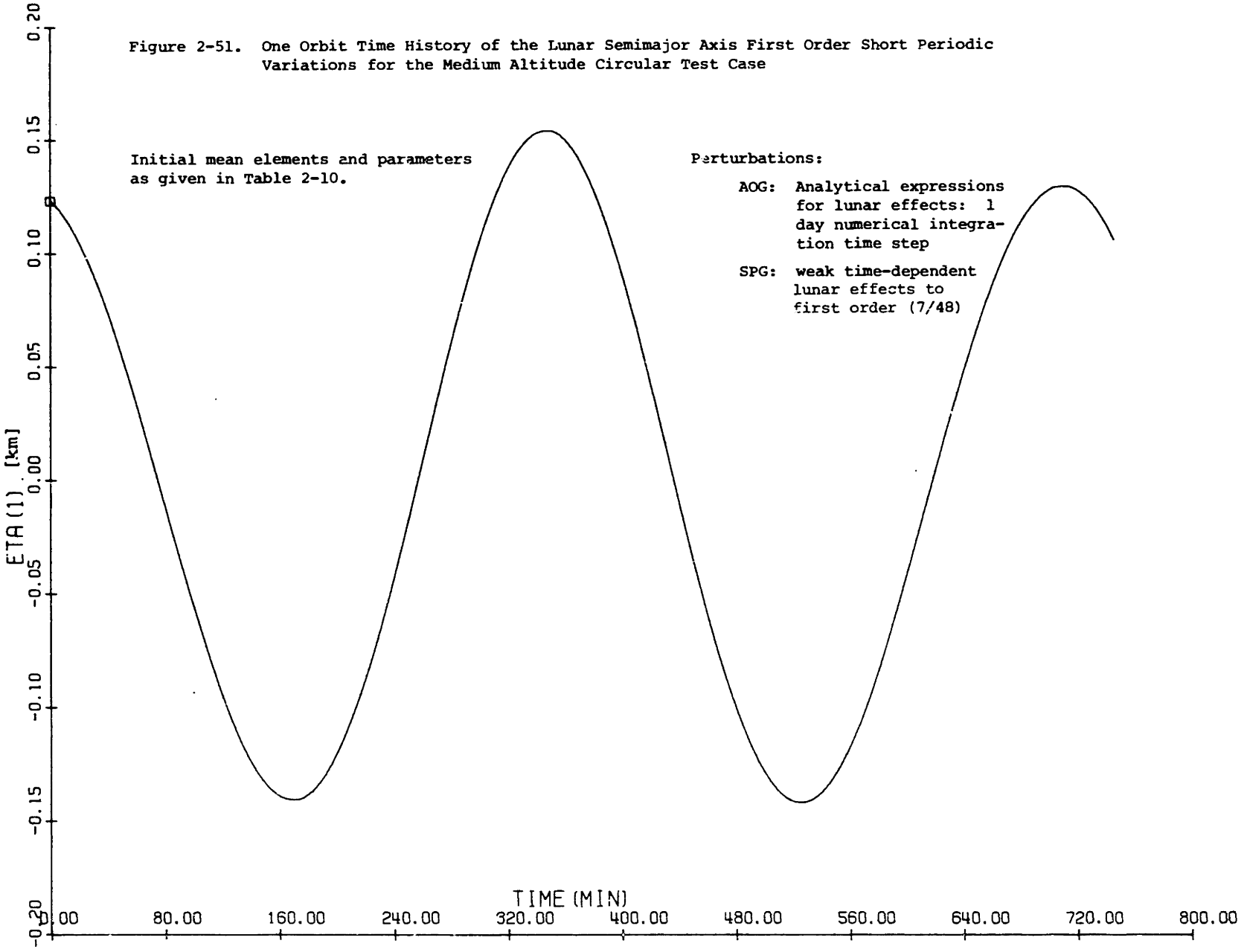


Figure 2-51. One Orbit Time History of the Lunar Semimajor Axis First Order Short Periodic Variations for the Medium Altitude Circular Test Case



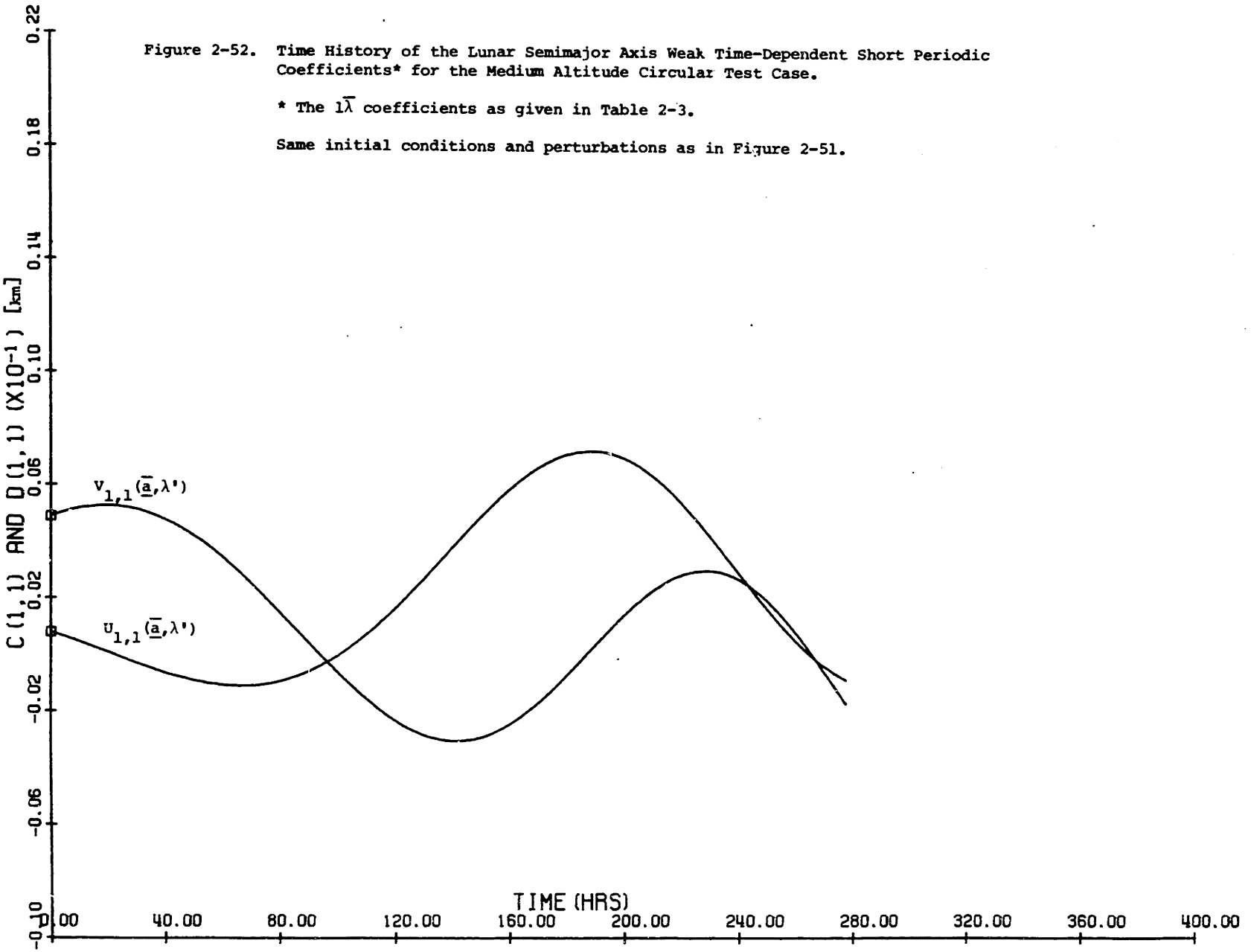
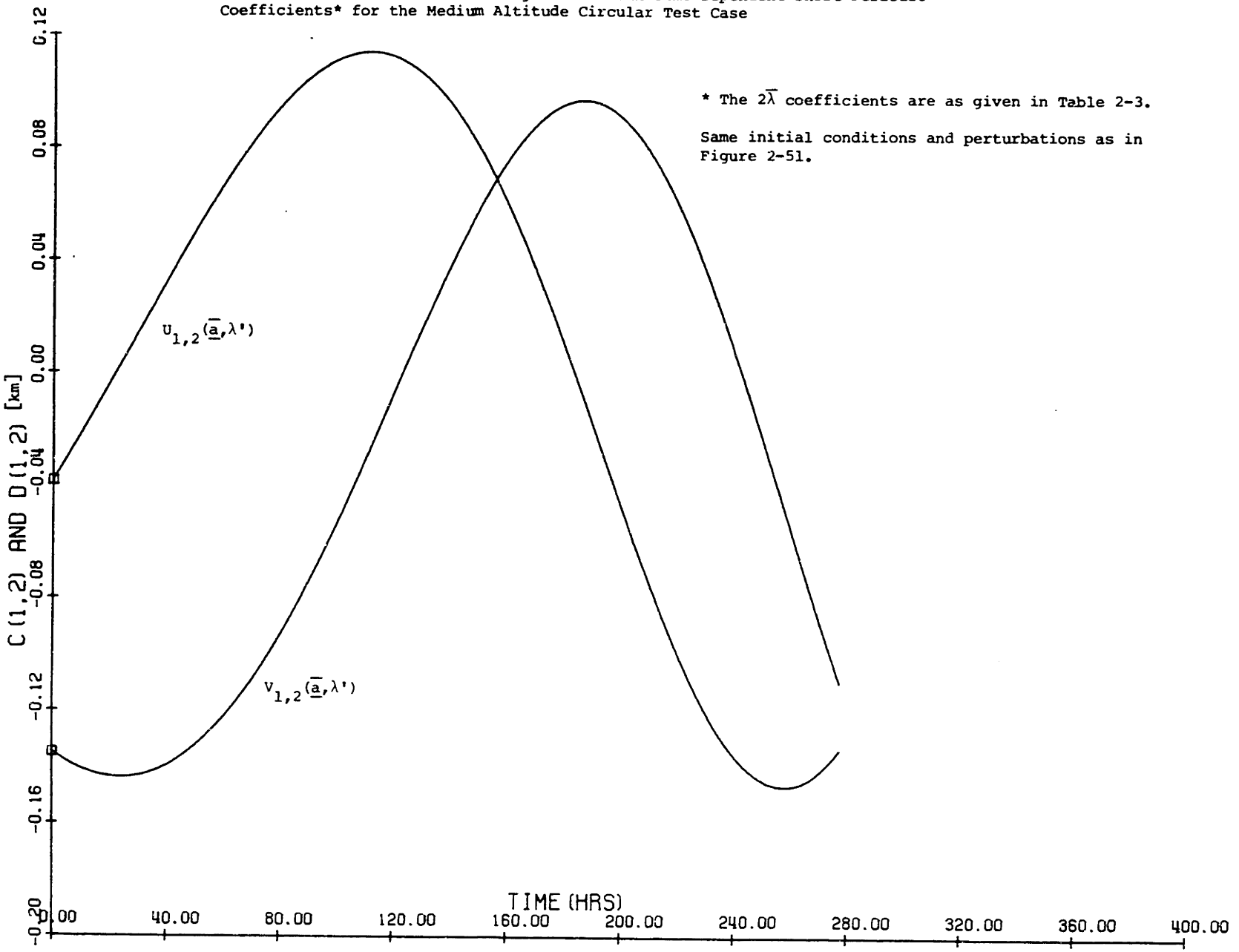


Figure 2-52. Time History of the Lunar Semimajor Axis Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case.

* The $1\bar{\lambda}$ coefficients as given in Table 2-3.

Same initial conditions and perturbations as in Figure 2-51.

Figure 2-53. Time History of the Lunar Semimajor Axis Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case



* The $2\bar{\lambda}$ coefficients are as given in Table 2-3.

Same initial conditions and perturbations as in Figure 2-51.

Figure 2-54. Time History of the Lunar Semimajor Axis Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case

* The $4\bar{\lambda}$ coefficients as given in Table 2-3.

Same initial conditions and perturbations as given in Figure 2-51.

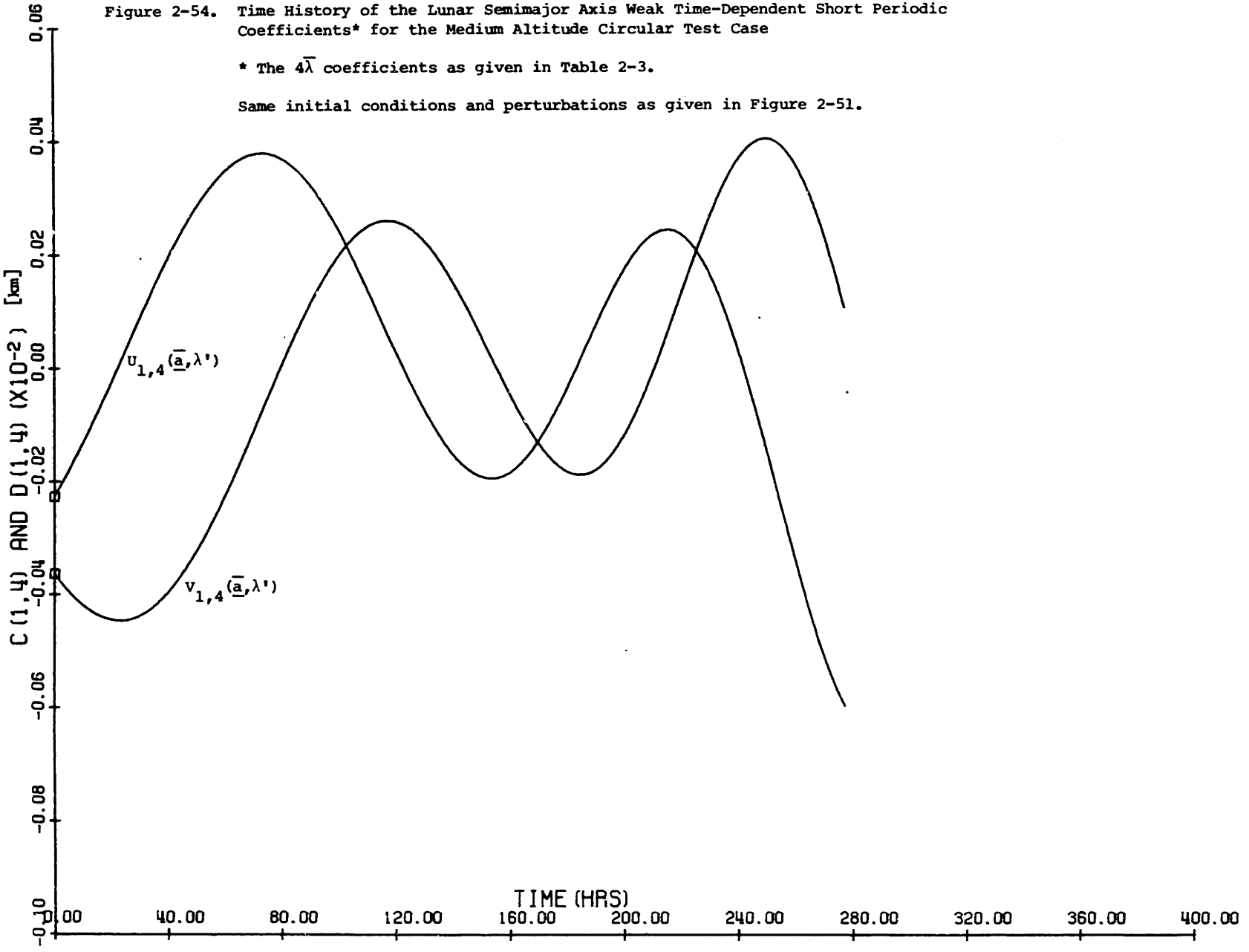


Figure 2-55. Time History of the Lunar h Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case

* The $3\bar{\lambda}$ coefficients as given in Table 2-3.

Same initial conditions and perturbations as given in Figure 2-51.

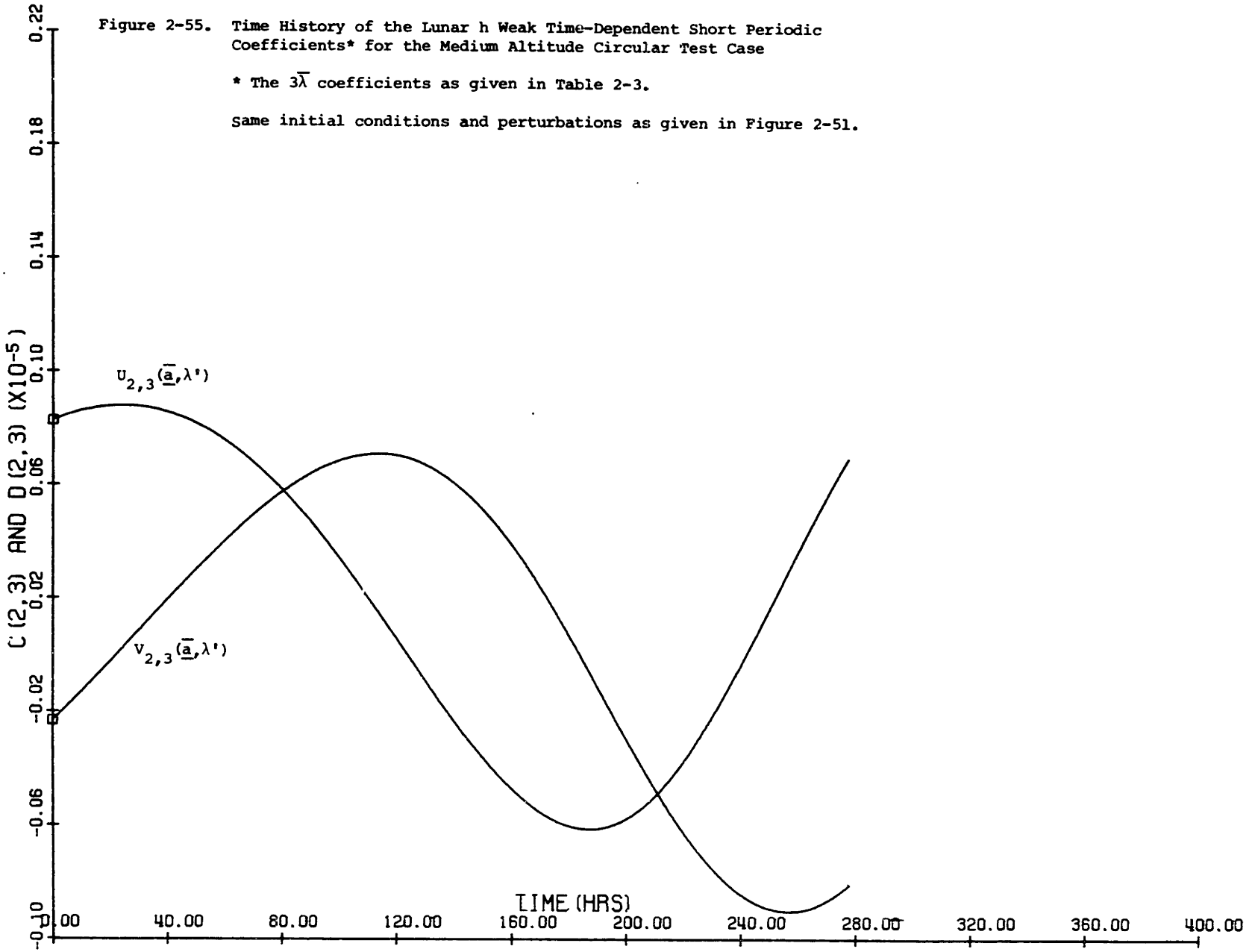


Figure 2-56. Time History of the Lunar k Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case

* The $2\bar{\lambda}$ coefficients as given in Table 2-3.

Same initial conditions and perturbations as given in Figure 2-51.

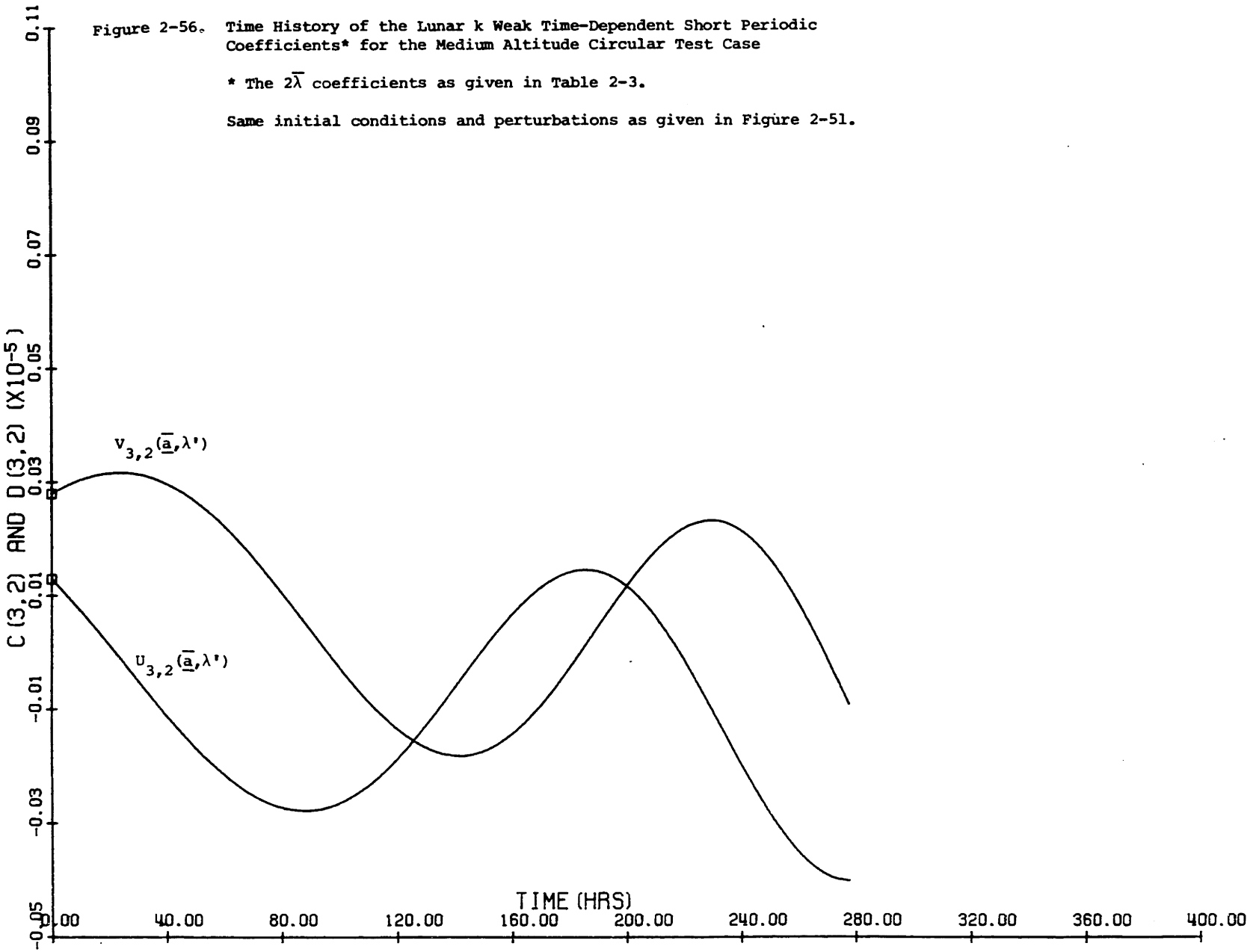
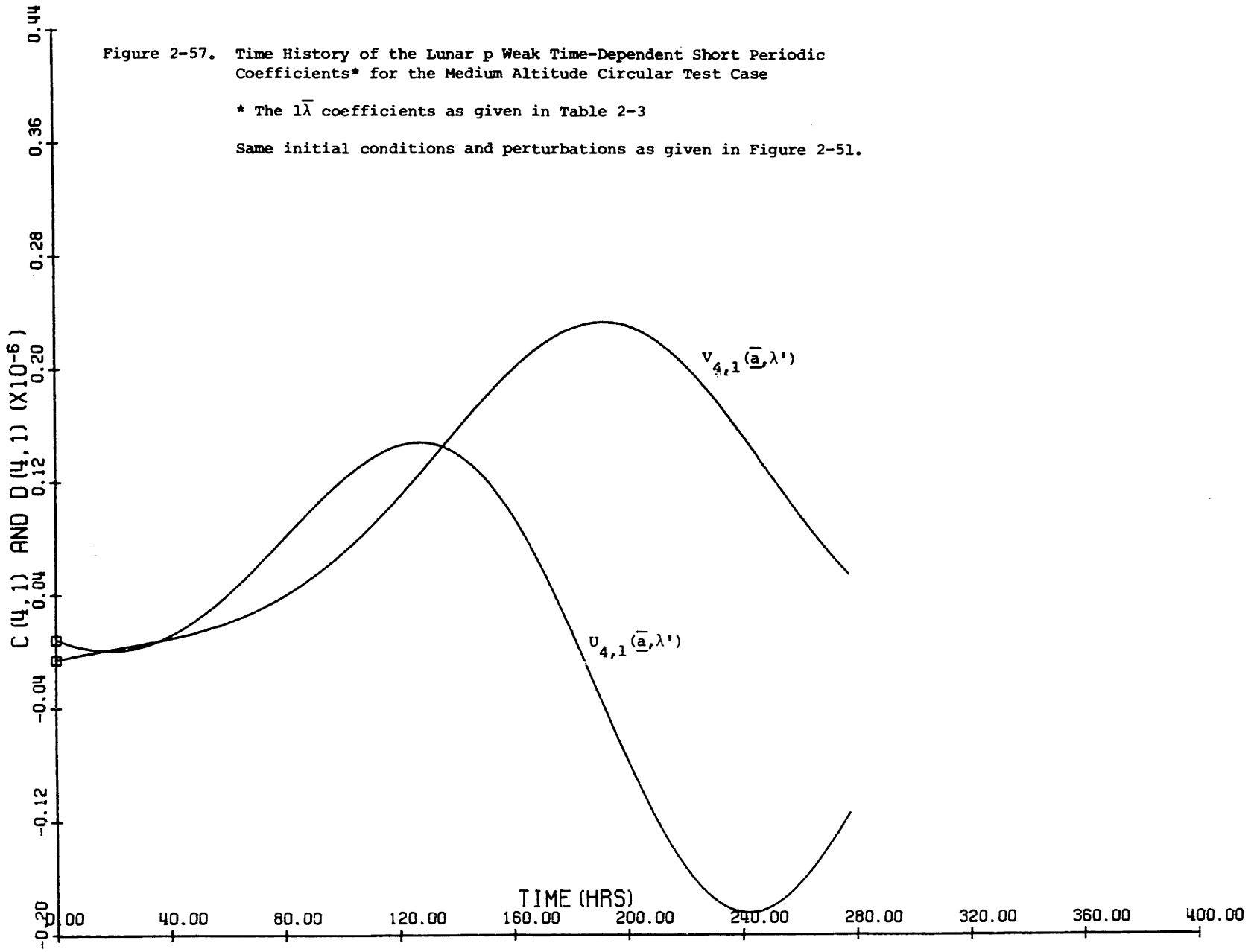


Figure 2-57. Time History of the Lunar p Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients as given in Table 2-3

Same initial conditions and perturbations as given in Figure 2-51.



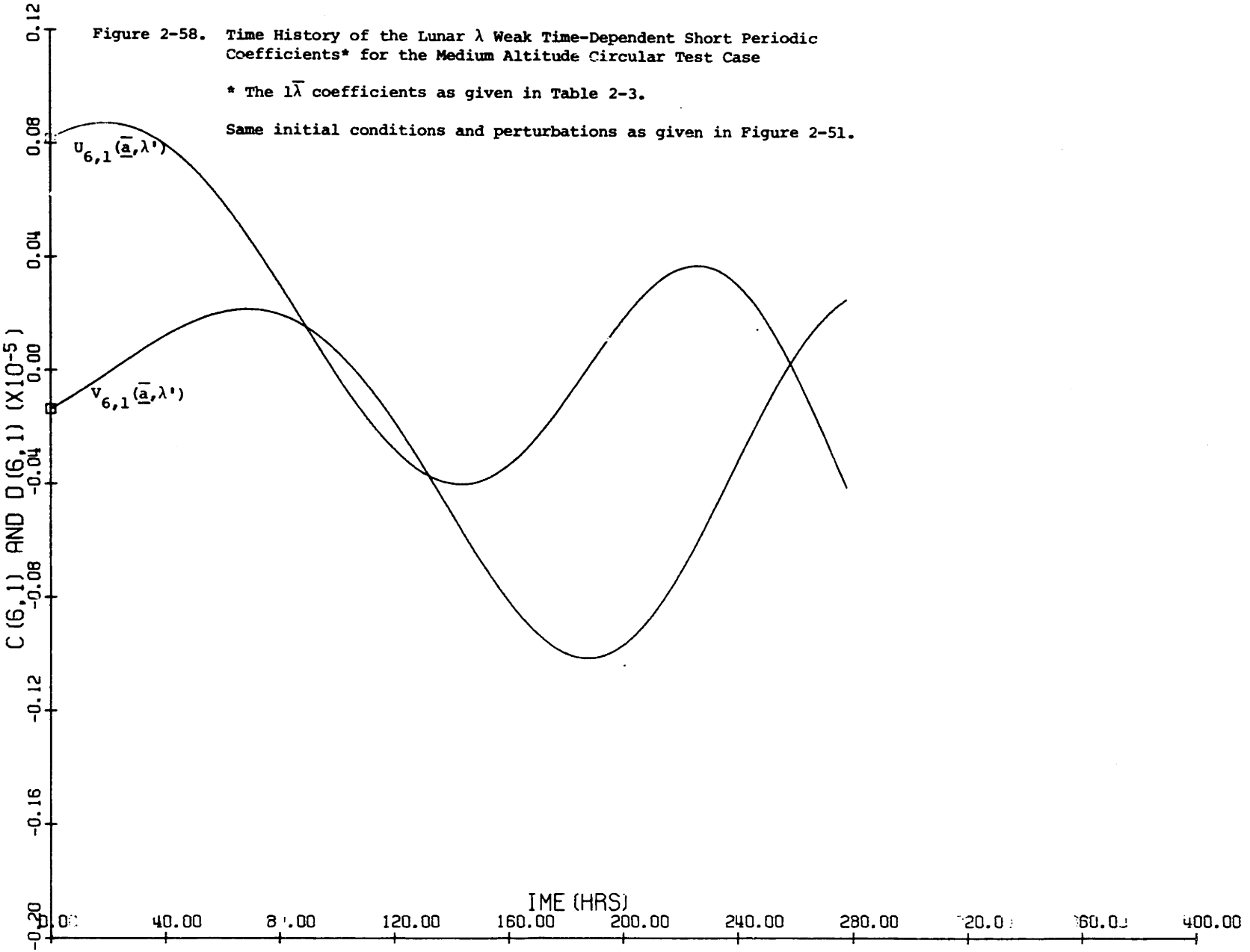


Figure 2-58. Time History of the Lunar λ Weak Time-Dependent Short Periodic Coefficients* for the Medium Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients as given in Table 2-3.

Same initial conditions and perturbations as given in Figure 2-51.

CHAPTER 3

ORBIT DETERMINATION WITH A SEMIANALYTICAL BATCH FILTER

If it were possible to model the disturbing forces experienced by an artificial satellite exactly, there would be no need to use observation data for the purpose of improving the estimate of the satellite's state. Obviously this is not the case and estimation procedures become a very important part of an OD system. This is especially true for near-Earth satellites due to the presence of atmospheric drag and the uncertainties associated with this perturbation.

An estimation procedure which is used extensively by most OD systems is the batch filter. The batch filter is a least squares differential correction algorithm which takes all the tracking data over an observation span and determines the best estimate of the satellite's state at a reference time (usually epoch). An adaptive batch filter solves for uncertain parameters within the force model in addition to the satellite's state (position and velocity or an orbital element set). A major portion of the batch filter, or for that matter any estimation process, is the satellite theory used for orbit generation. The previous chapter demonstrated that the semianalytical satellite theory is a viable orbit generator. The coupling of the semianalytical satellite theory with an adaptive batch filter may significantly increase the performance of OD systems.

It is the purpose of this chapter to present an adaptive semianalytical batch filter. Drag perturbed satellites are of primary concern in this chapter. A means of adaptively estimating modeled parameters within an atmospheric density model is presented. Observational data is simulated and is processed by the semianalytical batch filter.

This chapter is divided into three sections. Section 3.1 presents the basic algorithmic structure of the semianalytical batch filter. The semianalytical partial derivatives of perturbed motion, which are required for a semianalytical batch filter, are developed in Section 3.2. Numerical results of the adaptive semianalytical batch filter are presented in Section 3.3.

3.1 A Semianalytical Batch Filter

This section develops the basic functional formulation of the semianalytical batch filter. In this research the term batch filter will imply a weighted least squares estimator. First a brief heuristic review of the weighted least squares algorithm is presented. Next the weighted least squares algorithm is formulated in terms of the semianalytical satellite theory.

A Review of the Batch Filter

Assume that the system dynamics can be expressed as

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t)] + \underline{w}(t) \qquad (3\text{-}1)$$

where $\underline{x}(t)$ is the actual or true state vector of the system and $\underline{w}(t)$ is the modeling/system noise. The best estimate of the actual state will be denoted as $\hat{\underline{x}}(t)$. Associated with $\hat{\underline{x}}(t)$ is the covariance matrix $P(t)$. The covariance matrix is defined as

$$P(t) = E\{[\hat{\underline{x}}(t) - \underline{x}(t)][\hat{\underline{x}}(t) - \underline{x}(t)]^T\} \qquad (3\text{-}2)$$

where $E\{\ \}$ denotes the expected or mean value of the quantity within the brackets (see Ref. 72). It is noted that

$$\text{Trace}[P(t)] = E\{[\hat{\underline{x}}(t) - \underline{x}(t)]^T[\hat{\underline{x}}(t) - \underline{x}(t)]\} \qquad (3\text{-}3)$$

where Trace $[P(t)]$ denotes the operation of adding together the terms on the main diagonal of $P(t)$. It can be seen from Equation (3-3) that as the Trace$[P(t)]$ decreases, the magnitude of the expected error between $\hat{\underline{x}}(t)$ and $\underline{x}(t)$ decreases. Therefore, the covariance matrix can be thought of as an indicator of the confidence in $\hat{\underline{x}}(t)$.

Obviously, the state can be propagated forward (or backward) in time by the following differential equation

$$\dot{\hat{\underline{x}}}(t) = \underline{f}[\hat{\underline{x}}(t)] \tag{3-4a}$$

with the initial conditions

$$\hat{\underline{x}}(t_o) = \tilde{\underline{x}}_o \tag{3-4b}$$

$$P(t_o) = P_o \tag{3-4c}$$

The symbol $\sim$ denotes that this is an a priori estimate of $\underline{x}(t_o)$. P_o is the covariance matrix (confidence) associated with $\tilde{\underline{x}}_o$. From Equation (3-4a) it is seen that $\hat{\underline{x}}(t)$ will depend upon $\hat{\underline{x}}(t_o)$. In other words, given a set of initial conditions, a unique system trajectory, $\hat{\underline{x}}(t)$, is defined through Equation (3-4a).

In this development the solve-for parameters for the least squares estimate (batch filter) will be $\hat{\underline{x}}(t_o)$. The notation $\hat{\underline{x}}(t_o-)$ will denote the value of $\hat{\underline{x}}(t_o)$ prior to the estimation process. Similarly, $\hat{\underline{x}}(t_o+)$ will indicate the value of $\hat{\underline{x}}(t_o)$ after the estimation process. The notation $\hat{\underline{x}}(t+)$ or $\hat{\underline{x}}(t-)$ will denote the estimated system trajectory [analytical or numerical solution to Equation (3-4a)] associated with the initial conditions $\hat{\underline{x}}(t_o+)$ or $\hat{\underline{x}}(t_o-)$ respectively.

Defining $\underline{O}_a$ to be a vector of all the actual observations and $\underline{O}_c[\underline{x}(t)]$ as the model of the observations (the computed observations), it is seen that

$$\underline{O}_a = \underline{O}_c[\underline{x}(t)] + \underline{v}(t) \tag{3-5}$$

where $\underline{v}(t)$ is the noise associated with the observation process. The measurement noise is assumed to have a normal distribution with zero mean and a covariance of R. Naturally, the computed observations will depend upon the actual system trajectory, $\underline{x}(t)$.

It is desired that after the weighted least squares estimation process, the following be true

$$\hat{\underline{x}}(t+) = \underline{x}(t) \tag{3-6}$$

Equation (3-6) indicates that the estimation process is required to yield an updated system trajectory which is equal to the actual (true) system trajectory. Let $\delta\underline{b}$ denote the observation residual vector prior to the estimation process and $\delta\hat{\underline{x}}(t_o)$ denotes the resulting update for the solve-for vector. Mathematically these quantities can be expressed as

$$\delta\underline{b} \equiv \underline{O}_a - \underline{O}_c[\hat{\underline{x}}(t-)] \tag{3-7}$$

$$\delta\hat{\underline{x}}(t_o) \equiv \hat{\underline{x}}(t_o+) - \hat{\underline{x}}(t_o-) \tag{3-8}$$

Expanding Equation (3-5) in a Taylor series about $\hat{\underline{x}}(t_o-)$ and noting that $\hat{\underline{x}}(t-)$ is the system trajectory corresponding to $\hat{\underline{x}}(t_o-)$, the following expression results

$$\underline{O}_a = \underline{O}_c[\hat{\underline{x}}(t-)] + F\delta\hat{\underline{x}}(t_o) + \underline{E} \tag{3-9a}$$

where $\underline{E}$ is the error vector associated with such a formulation and where

$$F = \left[\frac{\partial \underline{O}_c[\underline{x}(t)]}{\partial \underline{x}(t_o)}\right]_{\hat{\underline{x}}(t_o-)} \tag{3-9b}$$

Equation (3-9a) can be written as

$$\delta\underline{b} = F\delta\hat{\underline{x}}(t_o) + \underline{E} \tag{3-10}$$

The basic concept of a weighted least squares estimation procedure is to pick $\delta\hat{\underline{x}}(t_o)$ such that the weighted square of $\underline{E}$ is minimized. The cost function for such a concept can be expressed as

$$J = \underline{E}^T W \underline{E} \tag{3-11}$$

where W is the weighting matrix. The weighting matrix is usually taken to be the inverse of the covariance of the measurement/observation noise vector (Ref. 67). Assuming the

measurements to be uncorrelated and the errors to have a zero mean value, the weighting matrix can be expressed as

$$W = R^{-1} = \begin{bmatrix} \sigma_{m1}^{-2} & & & 0 \\ & \sigma_{m2}^{-2} & & \\ & & \ddots & \\ 0 & & & \sigma_{mp}^{-2} \end{bmatrix} \tag{3-12}$$

where σ_{mi} is the standard deviation of the ith measurement.

Substitution of Equation (3-10) into Equation (3-11) results in

$$J = [\delta\underline{b} - F\ \delta\hat{\underline{x}}(t_o)]^T\ W[\delta\underline{b} - F\ \delta\hat{\underline{x}}(t_o)]$$

which reduces to

$$J = \delta\underline{b}^T\ W\ \delta\underline{b} - 2\delta\underline{b}^T\ W\ F\ \delta\hat{\underline{x}}(t_o) + \delta\hat{\underline{x}}^T(t_o)\ F^T\ W\ F\ \delta\hat{\underline{x}}(t_o) \tag{3-13}$$

As was previously indicated, the weighted least squares estimator determines $\delta\hat{\underline{x}}(t_o)$ such that Equation (3-13) is minimized. Setting $\frac{\partial J}{\partial [\delta\hat{x}(t_o)]}$ equal to zero results in

$$0 = -2\delta\underline{b}^T\ W\ F + 2\delta\hat{\underline{x}}^T(t_o)\ F^T\ W\ F \tag{3-14}$$

which implies that

$$\delta\hat{\underline{x}}(t_o) = (F^T WF)^{-1} F^T W \delta\underline{b} \qquad (3\text{-}15)$$

Taking $\frac{\partial^2 J}{\partial [\delta\hat{\underline{x}}(t_o)]^2}$ results in $2F^T WF$ which is positive definite and therefore Equation (3-15) is the value of $\delta\hat{\underline{x}}(t_o)$ which minimizes J (Ref. 72).

Equation (3-15) is the classical weighted least squares estimator. The residual, $\delta\underline{b}$, and the matrix F are computed* and substituted into Equation (3-15) in order to determine $\delta\hat{\underline{x}}(t_o)$. The updated estimate of the epoch conditions, $\hat{\underline{x}}(t_o+)$, is determined from Equation (3-8). Knowing $\hat{\underline{x}}(t_o+)$, the system trajectory, $\hat{\underline{x}}(t+)$, can be obtained by solving Equation (3-4a).

Due to linearization errors and other errors associated with the formulation, the classical weighted least squares estimator usually has to iterate on Equation (3-15) until the minimum is reached to within a specified tolerance. Letting k denote the iteration number it is easily shown that

$$\delta\hat{\underline{x}}_k(t_o) = (F^T WF)^{-1} F^T W \delta\underline{b}_k \qquad (3\text{-}16a)$$

$$\delta\underline{b}_k = \underline{O}_a - \underline{O}_c[\hat{\underline{x}}_k(t-)] \qquad (3\text{-}16b)$$

---

* The actual algorithmic structure is explained in Chapter 8 of Reference 67.

$$F = \left[\frac{\partial\, O_c[\underline{x}(t)]}{\partial\, \underline{x}(t_o)}\right]_{\hat{\underline{x}}_k(t_o-)} \tag{3-16c}$$

$$\dot{\hat{\underline{x}}}(t) = f[\hat{\underline{x}}(t)] \text{ with } \hat{\underline{x}}(t_o) = \hat{\underline{x}}_k(t_o) \tag{3-16d}$$

where

$$\hat{\underline{x}}_k(t_o+) = \hat{\underline{x}}_k(t_o-) + \delta\hat{\underline{x}}_k(t_o) \tag{3-16e}$$

The above development did not take into account any information about the a priori knowledge of the initial conditions. For example, if a component of the initial state is known exactly, the batch filter should be constrained such that it does not adjust that component during the update. As the confidence in the a priori initial conditions, $\tilde{\underline{x}}_o$, decreases, the estimation process should have less constraints in choosing the update. This information can be conveyed to the batch filter by adding a term to the cost function in Equation (3-13) which takes into account the value of $\hat{\underline{x}}(t_o)$ to within the limits of its uncertainty (Ref. 67). The new cost function, on the kth iteration, can be written as

$$J = \delta\underline{b}_k^T W\, \delta\underline{b}_k - 2\delta\underline{b}_k^T WF\delta\hat{\underline{x}}_k(t_o) + \delta\hat{\underline{x}}_k^T(t_o)\, F^T WF\delta\hat{\underline{x}}_k(t_o) + [\hat{\underline{x}}_k(t_o+) - \tilde{\underline{x}}_o]^T P_o^{-1}[\hat{\underline{x}}(t_o+) - \tilde{\underline{x}}_o] \tag{3-17}$$

By defining $\delta\tilde{\underline{x}}_k(t_o-)$ as follows

$$\delta\tilde{\underline{x}}_k(t_o-) = \tilde{\underline{x}}_o - \hat{\underline{x}}_k(t_o-) \tag{3-18}$$

it is seen that

$$[\hat{\underline{x}}(t_o+) - \tilde{\underline{x}}_o] = \delta\hat{\underline{x}}_k(t_o) - \delta\tilde{\underline{x}}_k(t_o-) \tag{3-19}$$

From Equation (3-18) it is noted that on the first iteration, $\delta\tilde{\underline{x}}_1(t_o-) = 0$ since $\hat{\underline{x}}_1(t_o-) = \tilde{\underline{x}}_o$. Substitution of Equation (3-19) into Equation (3-17) and expanding yields

$$\begin{aligned} J &= \delta\underline{b}_k^T W \delta\underline{b}_k - 2\delta\underline{b}_k^T W F \delta\hat{\underline{x}}_k(t_o) \\ &+ \delta\hat{\underline{x}}^T(t_o) F^T W F^T \delta\hat{\underline{x}}(t_o) + \delta\hat{\underline{x}}_k^T(t_o) P_o^{-1} \delta\hat{\underline{x}}_k^T(t_o) \\ &- 2\delta\hat{\underline{x}}_k^T(t_o) P_o^{-1} \delta\tilde{\underline{x}}_k(t_o) + \delta\tilde{\underline{x}}_k(t_o-) P_o^{-1} \delta\tilde{\underline{x}}_k(t_o-) \end{aligned} \tag{3-20}$$

Taking $\dfrac{\partial J}{\partial [\partial \hat{\underline{x}}_k(t_o)]}$, setting equal to zero and rearranging yields (Ref. 67)

$$\delta\hat{\underline{x}}_k(t_o) = (F^T W F + P_o^{-1})^{-1} [F^T W \delta\underline{b}_k + P_o^{-1} \delta\tilde{\underline{x}}_k(t_o-)] \tag{3-21}$$

Equation (3-21) is the batch filter used in GTDS-RD (Refs. 67 and 73). This is summarized in Table 3-1.

The covariance matrix associated with the converged estimate of the initial conditions is the gain matrix in Equation (3-21). This is expressed as

$$P_c(t_o) = (F^T W F + P_o^{-1})^{-1} \tag{3-22}$$

where $P_c(t_o)$ indicates that this is for the converged solution. Reference 67 gives a detailed derivation of Equation (3-22).

Semianalytical Satellite Theory and the Batch Filter

The solve-for vector in the adaptive semianalytical batch filter is defined as

$$\underline{x}(t_o) = \begin{bmatrix} \underline{\bar{a}}^*(t_o) \\ \underline{c} \end{bmatrix} = \begin{bmatrix} \underline{\bar{a}}^*_o \\ \underline{c} \end{bmatrix} \tag{3-23}$$

where

$$\underline{\bar{a}}^*_o = \begin{bmatrix} \underline{\bar{a}}_o \\ \bar{\lambda}_o \end{bmatrix} \tag{3-24a}$$

$$\underline{\bar{a}}_o = [\bar{a}_o, \bar{h}_o, \bar{k}_o, \bar{p}_o, \bar{q}_o]^T \tag{3-24b}$$

Table 3-1

The Batch Filter

$$\delta\hat{\underline{x}}_k(t_o) = [F^T WF + P_o^{-1}]^{-1}[F^T W\delta\underline{b}_k + P_o^{-1}\,\delta\tilde{\underline{x}}_k(t_o-)]$$

$$\delta\hat{\underline{x}}_k(t_o) = \hat{\underline{x}}_k(t_o+) - \hat{\underline{x}}_k(t_o-)$$

$$\delta\tilde{\underline{x}}_k(t_o) = \tilde{\underline{x}}_o - \hat{\underline{x}}_k(t_o-)$$

$$\delta\underline{b}_k = \underline{O}_a - \underline{O}_c\,[\hat{\underline{x}}_k(t-)]$$

$$F = \left[\frac{\partial \underline{O}_c[\underline{x}(t)]}{\partial \underline{x}(t_o)}\right]_{\hat{\underline{x}}_k(t_o-)}$$

$$\dot{\hat{\underline{x}}}(t) = f[\hat{\underline{x}}(t)] \text{ with } \hat{\underline{x}}(t_o) = \hat{\underline{x}}_k(t_o)$$

$\hat{\underline{x}}_k(t_o-)$ ≡ best estimate of the initial state prior to kth iteration of the batch filter

$\hat{\underline{x}}_k(t_o+)$ ≡ best estimate of the initial state after the kth iteration of the batch filter

$\tilde{\underline{x}}_o$ ≡ a priori initial state

P_o ≡ a priori covariance matrix associated with $\tilde{\underline{x}}_o$

$\underline{O}_a$ = actual observations

$\underline{O}_c[\hat{\underline{x}}_k(t-)]$ = computed observations on kth iteration of the batch filter

and where $\underline{c}$ represents the modeled solve-for parameters. Obviously $\bar{\underline{a}}^*_o$ denotes the epoch conditions for the six mean equinoctial elements as given in Chapter 2. The modeled parameters considered in this research deal only with the drag formulation. The drag solve-for parameters include C_D [see Equation (1-1)] or the a_1 through a_5 parameters in the Adaptive Modified Harris-Priester density model (see Appendix C). The actual estimation options are described in Section 3.3.

From Table 3-1 it is seen that in order to couple the semianalytical satellite theory with the batch filter, the observational residual vector, $\delta\underline{b}_k$, and the F matrix must be determined. In other words, the adaptive semianalytical batch filter is the algorithm presented in Table 3-1 with the solve-for vector defined in Equation (3-23) and the observation residual vector and F matrix computed in terms of the semianalytical satellite theory.

The computation of the observation residual vector, $\delta\underline{b}_k$, in terms of the semianalytical satellite theory is rather simple and straightforward. Given the best estimate for the epoch mean elements, the AOG, as described in Chapter 2, is used to determine the mean elements throughout the observation span. Again it is noted that the AOG uses the large numerical integration time step associated with the semianaly-

tical satellite theory (typically 1 day). The mean elements are determined at the observation times by use of an interpolator. Next the short periodic variations at the observation times are computed by use of the SPG as described in Chapter 2. It is noted that the concept of determining the short periodic coefficients on the integration grid and interpolating to obtain them at the output/observation times should greatly reduce the computational costs of the SPG. Given the mean elements and short periodic functions at an observation time, the precision osculating equinoctial elements can then be determined by use of the near-identity transformation. Knowing the osculating equinoctial elements, the position and velocity in the local tangent coordinate frame (frame the observations are determined in) can be obtained [see Equations (2-9) and (2-18)]. Given the observation model, the computed observation is then determined (see Ref. 67 or Appendix G). The observation residual vector can be obtained by subtracting the computed observations from the actual observations. For example, suppose a radar tracking station determined a range observation at time t [denote the actual observation as $\rho_a(t_i)$]. Also suppose that this range observation is modeled as

$$\rho_c(t_i) = \sqrt{\underline{r}_{LT}^T(t_i) \cdot \underline{r}_{LT}(t_i)} \qquad (3\text{-}25)$$

Obviously Equation (3-25) assumes a perfect measurement and does not take into account signal propagation characteristics,

biases, atmospheric refraction or any other effects (see Chapter 7 of Ref. 67). After determining $\underline{r}_{LT}(t_i)$ from Equation (2-18), $\rho_c(t_i)$ can be determined from Equation (3-25). The residual observation at time t_i is therefore given as

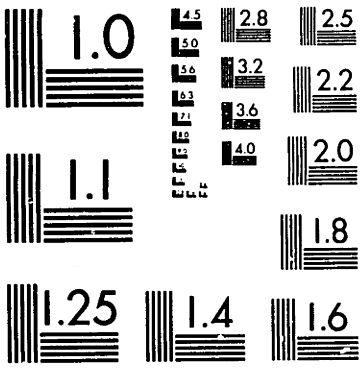
$$\delta b(t_i) = \rho_a(t_i) - \rho_c(t_i) \tag{3-26}$$

The F matrix is the partial derivative of the observations with respect to the solve-for parameters. Since the observation models are usually defined in terms of the local tangent coordinate frame, the F matrix can be written as

$$F = \left[\frac{\partial \underline{O}_c[\underline{r}_{LT}, \dot{\underline{r}}_{LT}]}{\partial \underline{x}(t_o)}\right]_{\hat{\underline{x}}(t_o)} \tag{3-27}$$

By use of the chain rule of differentiation and assuming that there are p measurements and ℓ solve-for parameters, it is seen that

$$F_{p\times\ell} = \left\{\left[\left\{\frac{\partial \underline{O}_c[\underline{r}_{LT}, \dot{\underline{r}}_{LT}]}{\partial \underline{r}_{LT}(t)}\right\} \middle| \left\{\frac{\partial \underline{O}_c[\underline{r}_{LT}, \dot{\underline{r}}_{LT}]}{\partial \dot{\underline{r}}_{LT}(t)}\right\}\right]_{p\times 6} \cdot \left[\begin{array}{c|c} \left\{\frac{\partial \underline{r}_{LT}(t)}{\partial \underline{R}(t)}\right\} & \left\{\frac{\partial \underline{r}_{LT}(t)}{\partial \dot{\underline{R}}(t)}\right\} \\ \hline \left\{\frac{\partial \dot{\underline{r}}_{LT}(t)}{\partial \underline{R}(t)}\right\} & \left\{\frac{\partial \dot{\underline{r}}_{LT}(t)}{\partial \dot{\underline{R}}(t)}\right\} \end{array}\right]_{6\times 6} \cdot \left[\begin{array}{c} \left\{\frac{\partial \underline{R}(t)}{\partial \underline{a}^*(t)}\right\} \\ \hline \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \underline{a}^*(t)}\right\} \end{array}\right]_{6\times 6} \cdot \left[\frac{\partial \underline{a}^*(t)}{\partial \underline{x}(t_o)}\right]_{6\times\ell} \Bigg\}_{\hat{\underline{x}}(t_o)} \tag{3-28}$$



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where

$\underline{r}_{LT}, \dot{\underline{r}}_{LT}$ ≡ position and velocity vector with respect to the local tangent frame

$\underline{R}(t), \dot{\underline{R}}(t)$ ≡ position and velocity vector with respect to an inertial reference frame [mean of 1950 or true of date]

$\underline{a}^*(t)$ ≡ osculating equinoctial elements

The first matrix in Equation (3-28) is the partial derivatives of the observations with respect to the position and velocity vectors in the local tangent coordinate frame. For example, if the perfect range measurement model of Equation (3-25) is assumed, the following would be true

$$\left\{\frac{\partial O_c[\underline{r}_{LT}, \dot{\underline{r}}_{LT}]}{\partial \underline{r}_{LT}(t)}\right\} = \frac{\underline{r}_{LT}^T(t)}{\rho_c(t)} \qquad (3\text{-}29a)$$

$$\left\{\frac{\partial O_c[\underline{r}_{LT}, \dot{\underline{r}}_{LT}]}{\partial \dot{\underline{r}}_{LT}(t)}\right\} = 0 \qquad (3\text{-}29b)$$

Reference 67 gives a detailed discussion of the above partial derivatives for realistic observation models (see Appendix G also) and this is presently implemented into GTDS-RD.

The second matrix in Equation (3-28) can be obtained from the transformation equations between the local tangent frame and the inertial frame (see Section 2.1.1). Assuming

that the inertial coordinate frame is the mean of 1950 coordinate frame, the following can be obtained from Equation (2-15)

$$\left\{\frac{\partial \underline{r}_{LT}(t)}{\partial \underline{R}(t)}\right\} = M_{LT}\, B(t)\, C \qquad (3\text{-}30a)$$

$$\left\{\frac{\partial \underline{r}_{LT}(t)}{\partial \dot{\underline{R}}(t)}\right\} = 0 \qquad (3\text{-}30b)$$

$$\left\{\frac{\partial \dot{\underline{r}}_{LT}(t)}{\partial \underline{R}(t)}\right\} = M_{LT}\, \dot{B}(t)\, C \qquad (3\text{-}30c)$$

$$\left\{\frac{\partial \dot{\underline{r}}_{LT}(t)}{\partial \dot{\underline{R}}(t)}\right\} = M_{LT}\, B(t)\, C \qquad (3\text{-}30d)$$

where M_{LT}, B(t) and C are coordinate transformations described in Section 2.1.1 and Reference 67. If the true of date coordinate frame is used as the inertial frame, the above expressions will still apply as long as the C matrix is replaced with the identity matrix.

The third matrix in Equation (3-28) can be obtained directly from the two-body relations between the inertial position and velocity and the equinoctial orbital elements [see Equations (2-2) and (2-9)]. Reference 74 gives a complete list of these derivatives.

The last matrix in Equation (3-28) is defined as

$$G = \left[\left\{ \frac{\partial \underline{a}^*(t)}{\partial \underline{\bar{a}}_o} \right\} \middle| \left\{ \frac{\partial \underline{a}^*(t)}{\partial \underline{c}} \right\} \right] \qquad (3\text{-}31)$$

The elements of the G matrix are the partial derivatives of perturbed motion. This is discussed in detail in the next section.

3.2 A Semianalytical Satellite Theory for the Partial Derivatives of Perturbed Motion

This section develops the semianalytical partial derivatives of perturbed motion. These partial derivatives are essential to the development and implementation of the semianalytical batch filter. A general formulation for the semianalytical partial derivatives is presented. The generation of the semianalytical partial derivatives is done by an Averaged Partial Generator (APG) and a Short Periodic Partial Generator (SPPG). The APG integrates a matrix differential equation for the mean semianalytical partial derivatives. The APG uses the large numerical integration time step associated with the semianalytical satellite theory. The SPPG determines, on the output/observation grid, the short periodic portion of the semianalytical partial derivatives. Many of the advantages that result from using the AOG and SPG in orbit generation are also applicable to the use of the APG and SPPG in partial derivative generation.

3.2.1 A General Formulation for the Semianalytical Partial Derivatives

The averaged equations of motion will be assumed to be of the following vector form

$$\dot{\underline{\bar{a}}}^{*} = \bar{n}\,\underline{e}_6 + \varepsilon \underline{A}_1(\underline{\bar{a}}) \tag{3-32}$$

where

$$\underline{\bar{a}}^{*} = [\bar{a},\bar{h},\bar{k},\bar{p},\bar{q},\bar{\lambda}]^T$$

$$\bar{n} = \text{mean-mean motion}$$

$$\underline{e}_6 = [0,0,0,0,0,1]^T$$

$$\varepsilon \underline{A}_1(\underline{\bar{a}}) = [\varepsilon A_{1,1}(\underline{\bar{a}}), \varepsilon A_{2,1}(\underline{\bar{a}}), \varepsilon A_{3,1}(\underline{\bar{a}}), \varepsilon A_{4,1}(\underline{\bar{a}}), \varepsilon A_{5,1}(\underline{\bar{a}}), \varepsilon A_{6,1}(\underline{\bar{a}})]^T$$

The vector form of the near-identity transformation is assumed to be

$$\underline{a}^{*} = \underline{\bar{a}}^{*} + \varepsilon \underline{\eta}_1(\underline{\bar{a}}^{*}) \tag{3-33}$$

where

$$\varepsilon \underline{\eta}_1(\underline{\bar{a}}) = \sum_{\sigma=1}^{\infty} [\varepsilon \underline{C}_\sigma \sin(\sigma\bar{\lambda}) - \varepsilon \underline{D}_\sigma \cos(\sigma\bar{\lambda})] \tag{3-34}$$

and where

$$\varepsilon\underline{C}_\sigma = [\varepsilon C_{1\sigma}, \varepsilon C_{2\sigma}, \varepsilon C_{3\sigma}, \varepsilon C_{4\sigma}, \varepsilon C_{5\sigma}, \varepsilon C_{6\sigma}]^T$$

$$\varepsilon\underline{D}_\sigma = [\varepsilon D_{1\sigma}, \varepsilon D_{2\sigma}, \varepsilon D_{3\sigma}, \varepsilon D_{4\sigma}, \varepsilon D_{5\sigma}, \varepsilon D_{6\sigma}]^T \quad (3\text{-}35)$$

Although the development in this chapter will only consider the case of a single time-independent disturbing force to first order in the small parameter in both the AOG and SPG, the extension to a more general case is straightforward.

Equation (3-31) can be written as

$$G = [G_2^0 \mid G_2^1] \quad (3\text{-}36)$$

where

$$G_2^0 = \left[\frac{\partial \underline{a}^*(t)}{\partial \bar{\underline{a}}^*(t)}\right] \left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right] \quad (3\text{-}37a)$$

and

$$G_2^1 = \left[\frac{\partial \underline{a}^*(t)}{\partial \bar{\underline{a}}^*(t)}\right] \left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right] + \left[\frac{\partial \underline{a}^*(t)}{\partial \underline{c}}\right]_{\text{explicit}} \quad (3\text{-}37b)$$

Equation (3-37b) indicates that the osculating elements have both an implicit and an explicit dependence upon the modeled parameters. The implicit dependence results because a change in the modeled parameters change the trajectory of the mean elements. The explicit dependence is due to the fact that the near-identity transformation is dependent upon the force model. Since the near-identity transformation does not explicitly depend upon the initial mean elements, Equation (3-37a) shows only an implicit dependence.

Substitution of Equations (3-37a) and (3-37b) into Equation (3-36) results in the following after some rearranging

$$G = \left[\frac{\partial \underline{a}^*(t)}{\partial \bar{\underline{a}}^*(t)}\right] \cdot \left[\left\{\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right\} \vdots \left\{\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right\}\right]$$

$$+ \left[\{0\} \vdots \left\{\frac{\partial \underline{a}^*(t)}{\partial \underline{c}}\right\}_{\text{explicit}}\right] \qquad (3\text{-}38)$$

From Equation (3-33) it is seen that

$$\left[\frac{\partial \underline{a}^*(t)}{\partial \bar{\underline{a}}^*(t)}\right] = \left[I + \left\{\frac{\partial\ \varepsilon \underline{\eta}_1(\bar{\underline{a}}^*)}{\partial \bar{\underline{a}}^*(t)}\right\}\right] \qquad (3\text{-}39)$$

and

$$\left[\frac{\partial \underline{a}^*(t)}{\partial \underline{c}}\right]_{\text{explicit}} = \left[\frac{\partial\ \varepsilon\underline{\eta}_1(\bar{\underline{a}}^*)}{\partial \underline{c}}\right] \tag{3-40}$$

Equation (3-38) can therefore be written as

$$G = [I + B_1][B_2 \mid B_3] + [0 \mid B_4] \tag{3-41}$$

where

$$B_2 = \left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right]_{6x6} \tag{3-42}$$

$$B_3 = \left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right]_{6x(\ell-6)} \tag{3-43}$$

$$B_1 = \left[\frac{\partial\ \varepsilon\underline{\eta}_1(\bar{\underline{a}}^*)}{\partial \bar{\underline{a}}^*(t)}\right]_{6x6} \tag{3-44}$$

$$B_4 = \left[\frac{\partial\ \varepsilon\underline{\eta}_1(\bar{\underline{a}}^*)}{\partial \underline{c}}\right]_{6x(\ell-6)} \tag{3-45}$$

The matrices B_1 and B_4 represent the short periodic portion of the semianalytical partial derivatives and will be called the Short Periodic Partial Generator (SPPG). The B_2 and B_3 matrices are the partial derivatives of the mean elements with respect to the solve-for parameters and will be called the

Averaged Partial Generator (APG). The similarities between the APG/SPPG and the AOG/SPG are evident.

The differential equations for the APG can be obtained from the averaged equations of motion. Taking the partial derivatives of Equation (3-32) with respect to the solve-for vector, it is seen that

$$\left[\frac{\partial \dot{\bar{\underline{a}}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right] = \left[\frac{\partial \{\bar{n}\ \underline{e}_6 + \varepsilon \underline{A}_1(\bar{\underline{a}})\}}{\partial \bar{\underline{a}}^*(t)}\right]\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right] \tag{3-46a}$$

$$\left[\frac{\partial \dot{\bar{\underline{a}}}^*(t)}{\partial \underline{c}}\right] = \left[\frac{\partial \{\bar{n}\ \underline{e}_6 + \varepsilon \underline{A}_1(\bar{\underline{a}})\}}{\partial \bar{\underline{a}}^*(t)}\right]\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right] + \left[\frac{\partial \{\bar{n}\ \underline{e}_6 + \varepsilon \underline{A}_1(\bar{\underline{a}})\}}{\partial \underline{c}}\right]_{\text{explicit}} \tag{3-46b}$$

Equation (3-46b) indicates that the partial derivatives of the mean element rates with respect to the modeled parameters have both an implicit and explicit dependence. Assuming that the mean elements are "well-behaved," it is possible to interchange the order of differentiation on the left hand side of Equations (3-46a) and (3-46b). This results in

$$\frac{d}{dt}\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}_o}\right] = \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \bar{\underline{a}}^*(t)}\right]\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*_o}\right] \tag{3-47a}$$

$$\frac{d}{dt}\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right] = \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \bar{\underline{a}}^*(t)}\right]\left[\frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}}\right] + \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \underline{c}}\right] \tag{3-47b}$$

where it is understood that $[\dot{\bar{\underline{a}}}^*]$ stands for the right hand side of the averaged equations of motion. It is also noted that at the epoch time, $\left[\frac{\partial \bar{\underline{a}}^*(t_o)}{\partial \bar{\underline{a}}^*_o}\right]$ is equal to the identity matrix and $\left[\frac{\partial \bar{\underline{a}}^*(t_o)}{\partial \underline{c}}\right]$ is equal to the null matrix. This last fact is true because a change in the modeled parameters at epoch will not affect the value of the mean elements at epoch. Therefore the APG can be expressed as

$$\dot{B}_2 = A\,B_2 \qquad \text{with } B_2|_{t_o} = I \tag{3-48}$$

and

$$\dot{B}_3 = A\,B_3 + D \text{ with } B_3|_{t_o} = 0 \tag{3-49}$$

where

$$A = \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \bar{\underline{a}}^*(t)}\right]_{6x6} \tag{3-50}$$

and

$$D = \left[\frac{\partial [\dot{\underline{\bar{a}}}^*]}{\partial \underline{c}}\right]_{6x(\ell-6)} \tag{3-51}$$

Since the mean element rates do not contain the fast variable ($\bar{\lambda}$), the sixth column of the A matrix will contain all zeroes. The A and D matrices are functions of only the five slowly varying mean elements and therefore the integration of the APG can be done with the same numerical integration time step associated with the AOG (1 day). The determination of the B_2 and B_3 matrices at the output/observation times can be done with an interpolator.

The SPPG (B_1 and B_4 matrices) can be obtained by a direct application of Equations (3-44) and (3-45). If the short periodic functions are expressed in terms of a Fourier series representation [see Equations (3-34) and (3-35)] and if it is noted that the short periodic coefficients ($\varepsilon\underline{C}_\sigma$ and $\varepsilon\underline{D}_\sigma$) are functions of the five slowly varying elements only, it is seen that the B_1 and B_4 matrices can be written

$$B_1 = \left\{\left[\sum_{\sigma=1}^{\infty}\left\{M_\sigma[\sin(\sigma\bar{\lambda})] - N_\sigma[\cos(\sigma\bar{\lambda})]\right\}\right]_{6x5}\left[\sum_{\sigma=1}^{\infty}\left\{\varepsilon\underline{C}_\sigma\,\sigma\,\cos(\sigma\bar{\lambda}) + \varepsilon\underline{D}_\sigma\,\sigma\,\sin(\sigma\bar{\lambda})\right\}\right]_{6x1}\right\} \tag{3-52}$$

$$B_4 = \left\{ \sum_{\sigma=1}^{\infty} \left[K_\sigma [\sin(\sigma\bar{\lambda})] - L_\sigma [\cos(\sigma\bar{\lambda})] \right] \right\}_{6x(\ell-6)} \tag{3-53}$$

where

$$M_\sigma = \left[\frac{\partial \ \varepsilon \underline{C}_\sigma}{\partial \ \underline{\bar{a}}(t)} \right]_{6x5} \tag{3-54a}$$

$$N_\sigma = \left[\frac{\partial \ \varepsilon \underline{D}_\sigma}{\partial \ \underline{\bar{a}}(t)} \right]_{6x5} \tag{3-54b}$$

$$K_\sigma = \left[\frac{\partial \ \varepsilon \underline{C}_\sigma}{\partial \underline{c}} \right]_{6x(\ell-6)} \tag{3-54c}$$

$$L_\sigma = \left[\frac{\partial \ \varepsilon \underline{D}_\sigma}{\partial \underline{c}} \right]_{6x(\ell-6)} \tag{3-54d}$$

and where $[\sin(\sigma\bar{\lambda})]$ and $[\cos(\sigma\bar{\lambda})]$ are square matrices [5x5 or $(\ell-6)x(\ell-6)$] with $\sin(\sigma\bar{\lambda})$ or $\cos(\sigma\bar{\lambda})$, respectively, for each of the main diagonal elements and zero for all the off-diagonal elements. The M_σ, N_σ, K_σ and L_σ matrices are functions of only the five slowly varying mean elements ($\underline{\bar{a}}$). It should therefore be possible to determine these matrices on the large integration grid and interpolate to obtain them at the output/

observation times. This should greatly reduce the computational costs associated with the SPPG.

The APG and SPPG are summarized in Table 3-2. An obvious advantage of the semianalytical partial derivatives is that the APG and SPPG can be considered independent of each other. In other words, the force models of the APG and SPPG could be completely different. It would even be possible to completely neglect the SPPG.

The A, D, B_1 and B_4 matrices must be determined in order to obtain the semianalytical partial derivatives of perturbed motion. These matrices can be obtained by analytical, quadrature or finite-differencing approaches.

Analytical expressions for the A, D, B_1 and B_4 matrices are obtained by taking the indicated partial derivatives of the analytical expressions for the mean element rates and short periodic functions. For example, the A and B_1 matrices, corresponding to J_2 effects, could be obtained by taking the required partial derivatives of Zeis' or Kaniecki's expressions for the J_2 mean element rates or short periodic functions* (Refs. 6 or 69). The M_σ and N_σ matrices could be obtained by taking the indicated partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ as

* Recent development efforts at the Draper Laboratory have implemented into GTDS-RD the analytical expressions for the A and B_1 matrices due to J_2 effects.

Table 3-2

Semianalytical Partial Derivative Generator

$$G = \left[\left\{\frac{\partial \underline{a}^*(t)}{\partial \underline{\bar{a}}_o}\right\} \vdots \left\{\frac{\partial \underline{a}^*(t)}{\partial \underline{c}}\right\}\right]$$

$$G = [I + B_1][B_2 \vdots B_3] + [0 \vdots B_4]$$

THE APG

$$\dot{B}_2 = AB_2 \qquad B_2|_{t=t_o} = I$$

$$\dot{B}_3 = AB_3 + D \qquad B_3|_{t=t_o} = 0$$

$$A = \left[\frac{\partial [\dot{\underline{a}}^*]}{\partial \underline{\bar{a}}^*(t)}\right] \qquad D = \left[\frac{\partial [\dot{\underline{a}}^*]}{\partial \underline{c}}\right]$$

THE SPPG

$$B_1 = \left[\frac{\partial\ \varepsilon \underline{\eta}_1(\underline{\bar{a}}^*)}{\partial \underline{\bar{a}}^*(t)}\right] \qquad B_4 = \left[\frac{\partial\ \varepsilon \underline{\eta}_1(\underline{\bar{a}}^*)}{\partial \underline{c}}\right]$$

$$B_1 = \left\{\left[\sum_{\sigma=1}^{\infty} \{M_\sigma[\sin(\sigma\bar{\lambda})] - N_\sigma[\cos(\sigma\bar{\lambda})]\}\right] \vdots \left[\sum_{\sigma=1}^{\infty} \{\varepsilon \underline{C}_\sigma \sigma \cos(\sigma\bar{\lambda}) + \varepsilon \underline{D}_\sigma \sigma \sin(\sigma\bar{\lambda})\}\right]\right\}$$

$$B_4 = \left\{\sum_{\sigma=1}^{\infty} \left[K_\sigma[\sin(\sigma\bar{\lambda})] - L_\sigma[\cos(\sigma\bar{\lambda})]\right]\right\}$$

$$M_\sigma = \left[\frac{\partial \varepsilon \underline{C}_\sigma}{\partial \underline{\bar{a}}(t)}\right] \qquad N_\sigma = \left[\frac{\partial \varepsilon \underline{D}_\sigma}{\partial \underline{\bar{a}}(t)}\right] \qquad K_\sigma = \left[\frac{\partial \varepsilon \underline{C}_\sigma}{\partial \underline{c}}\right] \qquad L_\sigma = \left[\frac{\partial \varepsilon \underline{D}_\sigma}{\partial \underline{c}}\right]$$

given in Table 2-2. The analytical approach yields expressions which are generally more accurate and computationally more efficient than the quadrature or finite-differencing approaches.

The required matrices can also be obtained by a quadrature approach. From Equations (2-22) and (2-28) it is seen that

$$\dot{\bar{a}}_i = \bar{n}\,\delta_{i6} + \frac{1}{2\pi}\int_0^{2\pi} \varepsilon F_i(\underline{\bar{a}},\bar{\lambda})\, d\bar{\lambda} \qquad (3\text{-}55a)$$

where $\varepsilon F_i(\underline{\bar{a}},\bar{\lambda})$ is the disturbing function for the ith element, $\bar{n}$ is the mean-mean motion and δ_{i6} is the Kroenecker Delta function defined in Equation (2-24). Define the vector $\underline{x}$ as the state vector at time t, i.e.,

$$\underline{x} = \underline{x}(t) = \begin{bmatrix} \underline{\bar{a}}^*(t) \\ \\ \underline{c} \end{bmatrix} \qquad (3\text{-}55b)$$

It is noted that $\underline{x}(t_o)$ is the solve-for vector [see Equation (3-23)]. Taking the required partial derivatives of Equation (3-55a) and interchanging the order of integration and differentiation, it is seen that

$$[A \,\vdots\, D]_{ij} = \left[\frac{\partial\,[\dot{\bar{a}}_i]}{\partial x_j}\right] = -\left(\frac{3\bar{n}}{2\bar{a}_1}\right)\delta_{i6}\,\delta_{1j} + \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial\ \varepsilon F_i(\underline{\bar{a}},\gamma)}{\partial x_j}\right] d\gamma \qquad (3\text{-}56)$$

where γ is used as the dummy variable in the integration operation, x_j represents the jth element of the vector $\underline{x}$ and where $[A \mid D]_{ij}$ represents the element in the ith row and jth column of the 6xℓ matrix $[A \mid D]$. The direct application of a quadrature algorithm to Equation (3-56) allows the determination of the A and D matrices. It should be noted that the sixth column of the $[A \mid D]$ matrix will be zero since $\left[\frac{\partial \varepsilon F(\bar{\underline{a}},\gamma)}{\partial \bar{\lambda}}\right] = 0$.

Similarly, the short periodic coefficient matrices M_σ, N_σ, K_σ and L_σ can be obtained by taking the respective partial derivatives of $\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$ as given in Table 2-1. The resulting expressions can be written as

$$[M_\sigma \mid 0 \mid K_\sigma]_{ij} = \left[\frac{\partial \varepsilon C_{i\sigma}}{\partial x_j}\right] = \frac{1}{\sigma \bar{n} \pi} \int_0^{2\pi} \left[\frac{\partial \varepsilon F_i(\bar{\underline{a}},\gamma)}{\partial x_j}\right] \cos(\sigma\gamma)\, d\gamma$$

$$+ \left[\frac{3\varepsilon C_{i\sigma}}{2\bar{a}_1} - \frac{9\varepsilon D_{1\sigma}\ \delta_{i6}}{4\sigma(\bar{\underline{a}}_1)^2}\right] \delta_{1j}$$

$$+ \left[\frac{3[N_\sigma \mid 0 \mid L_\sigma]_{1j}}{2\sigma\bar{a}_1} - \frac{3\varepsilon D_{1\sigma}\ \delta_{1j}}{2\sigma(\bar{a}_1)^2}\right] \delta_{i6} \qquad (3\text{-}57a)$$

$$[N_\sigma \mid 0 \mid L_\sigma]_{ij} = \left[\frac{\partial \varepsilon D_{i\sigma}}{\partial x_j}\right] = \frac{1}{\sigma \bar{n} \pi} \int_0^{2\pi} \left[\frac{\partial \varepsilon F_i(\bar{\underline{a}}, \gamma)}{\partial x_j}\right] \sin(\sigma\gamma)\, d\gamma$$

$$+ \left[\frac{3\varepsilon D_{i\sigma}}{2\bar{a}_1} - \frac{9\varepsilon C_{1\sigma}\, \delta_{i6}}{4\sigma(\bar{\underline{a}}_1)^2}\right] \delta_{1j}$$

$$- \left[\frac{3[M_\sigma \mid 0 \mid K_\sigma]_{1j}}{2\sigma\bar{a}_1} - \frac{3\varepsilon C_{1\sigma}\, \delta_{1j}}{2\sigma(\bar{a}_1)^2}\right] \delta_{i6} \qquad (3\text{-}57b)$$

It is noted that the major portion of Equations (3-56), (3-57a) and (3-57b) is the evaluation of the partial derivative of the disturbing function, i.e., $\left[\frac{\partial \varepsilon F_i(\bar{\underline{a}}, \gamma)}{\partial x_j}\right]$. By saving the integrand of Equation (3-56) when the A and D matrices are determined and multiplying by $\sin(\sigma\gamma)$ or $\cos(\sigma\gamma)$ and then applying the same quadrature algorithm, the M_σ, N_σ, K_σ and L_σ can be obtained at minimum computational costs. Knowing the M_σ, N_σ, K_σ and L_σ matrices, the B_1 and B_4 can be obtained by applying Equations (3-52) and (3-53).

A finite-differencing approach for the determination of the A, D, M_σ, N_σ, K_σ and L_σ matrices is discussed in the next section.

3.2.2 Determination of the Semianalytical Partial Derivatives by Finite-Differencing of Slowly Varying Quantities

As was previously indicated, the determination of the semianalytical partial derivatives requires the determination of the A, D, M_σ, N_σ, K_σ and L_σ matrices. A finite-differencing approach can be used to obtain these matrices. It is noted that these matrices are functions of the five slowly varying mean elements and therefore are slowly varying themselves.

An obvious advantage of a finite-differencing approach is that the force models and much of the software of the AOG and SPG can be used. This permits minimum software development in implementing the semianalytical partial derivatives. This also makes it possible to easily change the perturbation models that are included in the semianalytical partial derivatives.

In this development, a double-sided finite-differencing approach (see Appendix B) was used to determine the A, D, M_σ, N_σ, K_σ and L_σ matrices. Functionally the finite-differencing approach used can be written as

$$[A \mid D]_{ij} \sim \frac{\dot{\bar{a}}_i(x_j + \Delta x_j) - \dot{\bar{a}}_i(x_j - \Delta x_j)}{2\Delta x_j} \qquad (3\text{-}58a)$$

$$[M_\sigma \mid 0 \mid K_\sigma]_{ij} \sim \frac{\varepsilon C_{i\sigma}(x_j + \Delta x_j) - \varepsilon C_{i\sigma}(x_j - \Delta x_j)}{2\Delta x_j} \tag{3-58b}$$

$$[M_\sigma \mid 0 \mid L_\sigma]_{ij} \sim \frac{\varepsilon D_{i\sigma}(x_j + \Delta x_j) - \varepsilon D_{i\sigma}(x_j - \Delta x_j)}{2\Delta x_j} \tag{3-58c}$$

where x_j is the jth element of the vector $\underline{x}$ as defined in Equation (3-55b) and where $[\ ]_{ij}$ denotes the element in the ith row and jth column of the matrix as indicated in [ ]. In this development it was found that good results were obtained when Δx_j was 10^{-5} of x_j, i.e.,

$$\Delta x_j = (10^{-5})x_j \tag{3-59}$$

The A and D matrices were computed at the request times of the integrator in subroutine 'VARDIF' (see Appendix F). The short periodic coefficient matrices, M_σ, N_σ, K_σ and L_σ were computed at the output/observation times in subroutine 'VRSPFD' (see Appendix F). The concept of determining the short periodic coefficient matrices on the integration grid and using an interpolator to obtain them at the output/observation times was not explored in this research.

The actual implementation into GTDS-RD uses the initial mean position and mean velocity vectors as the state solve-for vector. Therefore, the actual solve-for vector in GTDS-RD can be written as

$$\underline{x}(t_o) = \begin{bmatrix} \underline{\bar{R}}(t_o) \\ \underline{\dot{\bar{R}}}(t_o) \\ \underline{c} \end{bmatrix} \tag{3-60}$$

where $\underline{\bar{R}}(t_o)$ and $\underline{\dot{\bar{R}}}(t_o)$ are the mean position and velocity vectors associated with the mean elements $\underline{\bar{a}}^*_o$. The only matrix which is different than previously defined is the B_2 matrix. The B_2 matrix which is implemented into GTDS-RD is defined as follows

$$B_2 = \left[\left\{ \frac{\partial \underline{\bar{a}}^*(t)}{\partial \underline{\bar{R}}(t_o)} \right\} \middle| \left\{ \frac{\partial \underline{\bar{a}}^*(t)}{\partial \underline{\dot{\bar{R}}}(t_o)} \right\} \right]_{6x6} \tag{3-61}$$

where

$$\dot{B}_2 = AB_2 \tag{3-62a}$$

with the initial conditions

$$B_2|_{t_o} = \left[\left\{ \frac{\partial \underline{\bar{a}}^*(t_o)}{\partial \underline{\bar{R}}(t_o)} \right\} \middle| \left\{ \frac{\partial \underline{\bar{a}}^*(t_o)}{\partial \underline{\dot{\bar{R}}}(t_o)} \right\} \right] \tag{3-62b}$$

Equation (3-62b) comes from the two-body relationships between the position and velocity and the equinoctial orbital elements (see Ref. 74 for a listing of these derivatives). The A matrix is as defined in Equation (3-50).

The remainder of this section presents some numerical results of the semianalytical partial derivatives. First the semianalytical partial derivatives are compared to the high precision Cowell partial derivatives. Next some results with changing the perturbation model of the APG and SPPG are presented.

Comparison with Cowell Partial Derivatives

The Cowell partial derivatives are defined as follows

$$Z(t) = \left[\begin{array}{c|c} \left\{\frac{\partial \underline{R}(t)}{\partial \underline{R}(t_o)}\right\} & \left\{\frac{\partial \underline{R}(t)}{\partial \dot{\underline{R}}(t_o)}\right\} \\ \hline \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \underline{R}(t_o)}\right\} & \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \dot{\underline{R}}(t_o)}\right\} \end{array}\right] \qquad (3\text{-}63)$$

The semianalytical partial derivatives are defined as

$$Y(t) = \left[\begin{array}{c} \left\{\frac{\partial \underline{R}(t)}{\partial \underline{a}^*(t)}\right\} \\ \hline \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \underline{a}^*(t)}\right\} \end{array}\right] G \qquad (3\text{-}64a)$$

where the G matrix is defined in Equation (3-41) with the B_2 matrix as given in Equation (3-61). This can be rewritten as

$$Y(t) = \left[\begin{array}{c|c} \left\{\frac{\partial \underline{R}(t)}{\partial \underline{R}(t_o)}\right\} & \left\{\frac{\partial \underline{R}(t)}{\partial \dot{\underline{R}}(t_o)}\right\} \\ \hline \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \underline{R}(t_o)}\right\} & \left\{\frac{\partial \dot{\underline{R}}(t)}{\partial \dot{\underline{R}}(t_o)}\right\} \end{array}\right] \tag{3-64b}$$

The Cowell partial derivatives and the semianalytical partial derivatives can be compared at time t by use of the following relation

$$Z(t) = Y(t)\ [Y(t_o)]^{-1} \tag{3-65}$$

This can be rewritten as

$$Z(t)\ Y(t_o) = Y(t) \tag{3-66}$$

Equation (3-66) allows a comparison of the semianalytical partial derivatives to the Cowell partial derivatives. A matrix multiplication of the Cowell partial derivatives at time t and the epoch semianalytical partial derivatives should yield the semianalytical partial derivatives at time t.

The above comparison was done for a low altitude satellite (similar to the Low Altitude Circular Test Case of Section 2.4.3) with the initial conditions and parameters as given in Table 3-3. The perturbation models used in the partial derivative test are given in Table 3-4. In the SPG the notation (7/48) indicates that 7 short periodic coefficients were used with a 48 point numerical quadrature. Likewise, in the SPPG the notation (5/48) indicates that 5 short periodic coefficient matrices were used in the determination of the B_1 matrix [see Equation (3-52)] and a 48 point quadrature algorithm was used in the determination of the short periodic coefficients. The comparison was done at a time of 2 days into the orbit generation. The computer printout of the partial derivatives in GTDS-RD shows only four (4) significant digits. The above comparison [Equation (3-66)] agreed to the four significant digits shown. Comparisons at other times between 1 and 2 days gave similar results. Some discrepancies were noted in a few of the elements in the 4th significant digit. These discrepancies were probably due to round-off errors.

It is felt that the above tests verify the theoretical developments of Section 3.2.1 and the finite-differencing approach presented in this section.

Table 3-3

Initial Conditions and Parameters for the Partial Derivative Test

Initial Conditions: $[EPC]^{-1}$

Mean Elements		Osculating Elements	
$\bar{a}$ =	6644.586 km	a =	6652.796017 km
$\bar{e}$ =	.01	e =	.009397030651
$\bar{i}$ =	67.98538419°	i =	67.99983930°
$\bar{\Omega}$ =	91.99738418°	Ω =	91.99985809°
$\bar{\omega}$ =	200.6741688°	ω =	201.1973744°
$\bar{M}$ =	164.3173126°	M =	163.8032616°

Parameters:

Epoch	=	10 hrs. 24 min. 21 Oct 1974
Reference Frame	=	True of Date
C_D	=	2.0
Area	=	1.86 m^2
Mass	=	677 kg
Density	=	Modified Harris-Priester with $\bar{F}_{10.7}$ = 150 10^{-22} watts m^{-2} Hz^{-1}

Table 3-4

Perturbation Models for the Partial Derivative Test

Cowell: 6x0 gravitational field and drag effects with a 30 second integration time step for both the equations of motion and the variational equations (partial derivatives).

Semianalytical:

AOG and APG

6x0 gravitational field (analytical), Zeis' J_2^2 expressions, drag option 7 (4/48): 1 day numerical integration time step.

SPG

6x0 gravitational field and drag to first order with a time-independent formation (7/48), Zeis' corrected analytical J_2^2 expressions

SPPG

6x0 gravitational field and drag to first order with time-independent formulation (5/48)

Initial Conditions: $[EPC]^{-1}$, see Table 3-3

Comparison with Different Perturbation Models

A comparison of the semianalytical partial derivatives obtained with different perturbation models is presented in Table 3-5. The low altitude test case of Table 3-3 was used. All comparisons were made to the semianalytical partial derivatives obtained with the perturbations as given in Table 3-4. The comparisons in Table 3-5 were done at 2 days from epoch. It is noted that at this time the baseline partial derivatives were shown to equal the Cowell partial derivatives to at least 4 significant digits.

The number of significant digits indicates that all the partial derivatives agreed to the baseline partial derivatives for that many significant digits. For example, two (2) significant digits in Table 3-5 indicates that all the partial derivatives had the first two digits agreeing with the baseline. Some of the elements may agree to more than 2 significant digits but all had at least two significant digits. The notation 2^- indicates that some of partial derivatives (one to three) did not agree to two significant digits. Table 3-5 should be viewed as a very coarse measurement of the effects of different perturbations on the semianalytical partial derivatives. It is also noted again that the computer printout showed only four significant digits and that round-off in the fourth digit could be causing some of the apparent agreements or disagreements.

Table 3-5

Comparison* of the Semianalytical Partial Derivatives with Different Perturbations

Run #	Perturbation Model**		# Significant Digits at 2 days from Epoch
	APG†	SPPG††	
A	J_2	---	2
B	J_2	J_2	3
C	J_2 and drag option 1	---	2
D	J_2 and drag option 2	---	3
E	J_2 and drag option 2	J_2	4
F	J_2, J_2^2 and drag option 2	J_2	4
G	J_2-J_6, J_2^2 and drag option 2	---	3
H	J_2-J_6, J_2^2 and drag option 2	J_2-J_6	4

* The baseline was the semianalytical partial derivatives obtained with the initial conditions, parameters and perturbations of Tables 3-3 and 3-4.

** Initial conditions and parameters as given in Table 3-3. The AOG and SPG had perturbations as given in Table 3-4.

† Analytical expressions were used for the element rates due to the gravitational field, drag options (see Section 2.4.1) used a 48 point numerical quadrature.

†† First order time-independent formulation, (5/48).

From Table 3-5 it can be concluded that drag perturbations do not need to be included in the SPPG (at least for the case considered). It also appears that for precision semianalytical partial derivatives, J_2 effects need to be considered in the SPPG*. For the case considered, the APG must include at least J_2 effects and drag effects with Izsak's height correction (see Section 2.4.1). It is interesting to note that the batch filter results presented in the next section used perturbation models equivalent to Run G of Table 3-5. The analytical partial derivatives that are currently in GTDS-RD were also determined for this test case and only one (1) significant digit of agreement with the baseline of Table 3-5 was obtained.

3.3 Numerical Results of the Semianalytical Batch Filter

This section presents some of the numerical results obtained with the semianalytical batch filter. It is felt that the results presented in this section demonstrate the semianalytical batch filter and the concept of adaptive parameter estimation of the drag formulation for low altitude satellites. Again it should be noted that this research was primarily concerned with demonstrating the accuracy and feasibility of the semianalytical batch filter. Development efforts now in progress are concerned with improving the computational efficiency as well as the accuracy of the semianalytical batch filter.

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* Recent development efforts at the CSDL have included analytical expressions for the B_1 matrix.

An overview of the software used in the current investigation is presented first in this section. Next, the test philosophy used is explained. Three separate test cases are presented and discussed.

3.3.1 Software Overview

The semianalytical batch filter, as described in Sections 3.1 and 3.2, has been implemented into a specially modified version of GTDS-RD that operates on an Amdahl 470 V6 computer at the Charles Stark Draper Laboratory in Cambridge, MA. A detailed description of the software used to implement the semianalytical batch filter is presented in Appendix F.

The batch filter of GTDS-RD is given in Table 3-1. The solve-for vector for the semianalytical batch filter is the initial mean position vector, the initial mean velocity vector and a vector of modeled parameters [see Equation (3-60)]. The semianalytical partial derivatives, as described in Section 3.2, were coupled to the batch filter of Table 3-1. The coordinate transformations, observation models and partial derivatives of the observation models that are presently in GTDS-RD were not modified [see Equation (3-28) and Appendix G].

A Cowell batch filter will refer to a batch filter which uses a Cowell orbit generator (SP theory). The term

states will refer to the initial conditions (the first six elements of the solve-for vector) of the equations of motion. For a Cowell batch filter the states are the initial osculating position and velocity vectors. For a semianalytical batch filter the states are the initial mean position and velocity vectors. It should be noted that the orbital elements (osculating or mean) corresponding to the initial state can be obtained by the two-body relations (see Section 2.1.1). The results presented in this section are usually expressed in terms of the osculating and/or mean orbital elements.

The Cowell batch filter in GTDS-RD currently has two estimation options for the drag formulation (see Table 3-6). The first option is to solve for the states only. In other words, no modeled parameters are solved for in this option. The second option is to solve for the states and the variation in the drag coefficient (ρ_1). The drag coefficient is written as (Ref. 67)

$$C_D = C_{DO}(1 + \rho_1) \tag{3-67}$$

The a priori guess of ρ_1 is usually taken as zero.

The semianalytical batch filter, when the Adaptive Modified Harris-Priester density model (see Appendix C) is used, has 6 estimation options. These options are listed in Table 3-6.

It should be noted that setting the adaptive parameters in the Adaptive Modified Harris-Priester density model to the following

$$\begin{aligned} a_1 &= a_2 = 1.0 \\ a_3 &= 6.0 \\ a_4 &= a_5 = 0.0 \end{aligned} \tag{3-68}$$

yields the standard Modified Harris-Priester atmosphere of GTDS-RD. Equation (3-68) is usually the a priori guess for the adaptive parameters in the Adaptive Modified Harris-Priester atmosphere.

The semianalytical batch filter and Cowell batch filter allow the inclusion of an a priori covariance matrix associated with the a priori solve-for vector (see Table 3-1 and/or Section 3.1). The a priori covariance matrix (P_o) is assumed diagonal with the diagonal elements equal to the respective variance ($\sigma^2_{x_i}$) of the a priori solve-for parameters. If the variances of the a priori solve-for parameters are not given, the P_o matrix is neglected in the implementation of Table 3-1. It is also possible to include some of the a priori variances and neglect others. For example, if in the states and C_D estimation option, only the variance of the a priori value of C_D is included, the P_0^{-1} matrix of Table 3-1 would be written as

$$P_0^{-1} = \left\{ \begin{array}{c|c} [0]_{6x6} & 0 \\ \hline 0 & \dfrac{1}{\sigma_{C_D}^2} \end{array} \right\} \qquad (3\text{-}69)$$

It will be assumed in the remainder of this chapter that if the variance of an a priori solve-for parameter is not given, its respective element on the main diagonal of P_o^{-1} will be zero. Likewise, if the variance of an a priori solve-for parameter is given, its corresponding element on the main diagonal of P_o^{-1} will be $\dfrac{1}{\sigma_{x_i}^2}$.

The covergence criteria in GTDS-RD is based on iterative reduction of the square root of the mean square of the observation residuals (Ref. 67). This is expressed as

$$RMS = \left\{ \frac{1}{p} \left[\delta \underline{b}_k^T \, W \, \delta \underline{b}_k + \delta \tilde{\underline{x}}_k(t_o-) P_0^{-1} \delta \underline{x}_k(t_o-) \right] \right\}^{1/2} \qquad (3\text{-}70)$$

where $\delta \underline{b}_k$, P_o, W and $\delta \tilde{\underline{x}}_k(t_o-)$ are defined in Table 3-1 and where p is the number of observations. The predicted RMS is computed by the following formula (Ref. 67)

$$RMSP = \left\{ \frac{1}{p} \left[[\delta \underline{b}_k - F \, \delta \hat{\underline{x}}_k(t_o)]^T W [\delta \underline{b}_k - F \, \delta \hat{\underline{x}}_k(t_o)]^T + [\delta \hat{\underline{x}}_k(t_o) - \delta \tilde{\underline{x}}_k(t_o-)]^T P_0^{-1} [\delta \hat{\underline{x}}_k(t_o) - \delta \underline{x}_k(t_o-)] \right] \right\}^{1/2} \qquad (3\text{-}71)$$

where F and $\delta\hat{\underline{x}}_k(t_o)$ are defined in Table 3-1. If the value of the RMS [Equation (3-70)] decreases during two consecutive iterations, the solution is converging. The batch filter is considered converged when the following is true (Ref. 67)

$$\left|\frac{RMSB - RMSP}{RMSB}\right| < \zeta \tag{3-72}$$

where

RMSB ≡ the smallest RMS achieved

ζ ≡ tolerance criteria

Table 3-6
Estimation Options

Cowell Batch Filter	Semianalytical Batch Filter*
1. States only $(\underline{R}_o, \dot{\underline{R}}_o)$	1. States only $(\bar{\underline{R}}_o, \dot{\bar{\underline{R}}}_o)$
2. States and ρ_1 $[C_D = C_{DO}(1 + \rho_1)]$	2. States and C_D
	3. States, a_1, a_2, a_3, a_4 and a_5
	4. States, a_1, a_2 and a_3
	5. States, a_1 and a_2
	6. States, C_D and a_4

* see Adaptive Modified Harris-Priester density model in Appendix C.

GTDS-RD contains a Data Simulation (Datasim) Program which computes simulated observations for a given set of tracking stations and observation intervals (Ref. 73). The Datasim Program requires a satellite ephemeris. The simulated observations are based upon the observation models in GTDS-RD (see Ref. 67 or Appendix G) and are computed at a specified frequency. Random and bias errors can be included in the observations. No modifications were made to the Datasim Program of GTDS-RD in this research.

3.3.2 Test Philosophy

The test philosophy used in this research is presented in Table 3-7. The truth ephemeris is generated with the Cowell orbit generator. The density model used in the truth ephemeris is generally the Analytical Jacchia Roberts density (the first test case used the Modified Harris-Priester atmosphere).

Table 3-7

Orbit Determination Test Philosophy

1. Generate truth ephemeris
2. Simulate observations using the truth ephemeris in the Datasim Program.
3. Process the observations with operational force models in the Batch Filter.
4. Predict the orbit using the operational force models and solve-for vector obtained from the Batch Filter.
5. Compare the predicted ephemeris to the truth ephemeris.

An observation net is assumed and the Datasim Program of GTDS-RD is used to generate simulated observations over an assumed observation time. The simulated observations are processed in a semianalytical or Cowell batch filter to give the solve-for vector associated with the various estimation options. The batch filter uses force models which are generally different than the force models used to generate the truth ephemeris. For the cases considered in this research, the density models were different. A predicted ephemeris, corresponding to the solve-for vector, is then determined by using the same orbit generator and force models that were associated with the solve-for vector in the data processing step. The predicted ephemeris is then compared to the truth ephemeris.

3.3.3 Test Case One

This test case was used to demonstrate that the implementation of the semianalytical batch filter was correct and that modeling errors could be sensed by the filter.

Tables 3-8 and 3-9 give the truth ephemeris and data simulation for Test Case One. It is noted that the truth ephemeris used the standard form of the Modified Harris-Priester density model.

Table 3-8

Truth Ephemeris for Test Case One

Initial Osculating Elements and Parameters

a	= 6644.586 km	Epoch	= 1024 hrs. 21 Oct 1974
e	= .01	Reference Frame	= True of Date
i	= 67.98538419°	C_D	= 2.0
Ω	= 91.99738418°	Area	= 1.86 m^2
ω	= 200.6741688°	Mass	= 677 kg
M	= 164.3173126°		

Perturbations

6x0 gravitational field with drag. The Modified Harris-Priester Atmosphere was used ($a_1 = a_2 = 1$, $a_3 = 6$, $a_4 = a_5 = 0$). $\bar{F}_{10.7}$ was 150 10^{-22} watts m^{-2} Hz^{-1}

Orbit Generator

Cowell orbit generator with a 30 second numerical integration time step was used (12th order Adams-Bashforth Predictor/Adams-Moulton Corrector integrator).

Table 3-9

Data Simulation for Test Case One

Tracking Stations:	Net A (see Table 3-10)
Observation Type:	VHF range (ρ) and range rate ($\dot{\rho}$)
	σ_ρ = 5 meters $\sigma_{\dot{\rho}}$ = .55 cm/sec
Observations:	Generated every 60 seconds while in sight (15° minimum elevation). Maximum time of observation interval per satellite pass is 300 seconds. No random noise included.
Observation Span:	From Epoch to 1000 hrs 22 Oct. 1974
Truth Ephemeris:	see Table 3-8

Table 3-10

Tracking Net A

Station Name	Symbol	Latitude (deg.)	Longitude (deg.)	Height (m.)
Canberra	CANV	-35.5833900	148.9777300	1149
Carnarvon	CARV	-24.9066444	113.7253249	8
Guam	UAMV	13.3104666	144.7374610	76
Dodaira	DODV	36.0055300	139.1919900	879
Kokee	KOKV	22.1261200	200.3348900	1123
Fairbanks	FAIV	64.9719499	212.486686	340
Goldstone	GOLV	35.3415110	243.1261749	920
Guaymas	GUAV	27.9629777	249.2787249	-28
Rosman	OSSV	35.1960110	277.1238332	818
Merritt Is.	MERV	28.5080944	279.3064194	-45
Grand Bahama	GRAV	26.6326638	281.7621972	-46
Santiago	SANV	-33.1497000	289.3315700	710
Ascension	ASCV	- 7.9553600	345.6724400	555
Winkfield	WINV	51.4462200	359.3022000	90
Greece	GREV	38.0789900	23.9327800	490
Shiraz	SHIV	29.6371600	52.5197900	1564
Nainital	NAIV	29.3592500	79.4575200	1856
Addis Ababa	ADDV	8.7474200	38.9591600	1901
Johannesburg	JOHV	-25.8838300	27.7072800	1541
Tananarive	TANV	-19.0110800	47.4247200	1367

Geodetic Coordinates, taken from Table 5.1 of Reference 17

Semianalytical Batch Filter - States and C_D as the Solve-for Vector

A semianalytical batch filter with states and C_D as the solve-for vector was used to process the simulated observations (see Table 3-9). The perturbation model of the semianalytical batch filter is given in Table 3-11. Table 3-12 gives the a priori and final values of the solve-for vector. It is noted that the only difference between the perturbation model that generated the truth ephemeris and the perturbation model used in the semianalytical batch filter is that the a priori C_D in the batch filter is about 15% larger.

From Table 3-12 it is seen that the semianalytical batch filter converged (7 iterations) to a value of C_D that varies from the true C_D by less than 0.0035%. A comparison of the final solve-for vector in Table 3-12 to the initial PCE elements of the Low Altitude Circular Test Case of Section 2.4.3 (see Table 2-7), reveals that the initial mean semi-major axis differs by about 2 centimeters (note that drag option 6 was used in generating Table 2-7).

Semianalytical Batch Filter - States, a_1 and a_2 as the Solve-for Vector

A semianalytical batch filter with states, a_1 and a_2 as the solve-for vector was also used to process the simulated

Table 3-11
Perturbation Model for the
Semianalytical Batch Filter*

AOG**: First order analytical expressions for the 6x0 gravitational field, Zeis' expressions for the J_2^2 effects, drag option 7 (4/48) for drag effects: .5 day numerical integration time step

SPG: 6x0 gravitational field (7/48) and drag (7/48) both to first order (time-independent), Zeis' J_2^2 expressions

APG**: Analytical expressions for the mean element rates due to 6x0 gravitational field, drag option 2 for the drag element rates (48 pt quadrature)

SPPG: neglected

* An Adaptive Modified Harris-Priester Density model was used,

Test Case One: $\bar{F}_{10.7} = 150\ 10^{-22}$ watts m^{-2} Hz^{-1}

Test Cases Two and Three: $\bar{F}_{10.7} = 75\ 10^{-22}$ watts m^{-2} Hz^{-1}

** A 4th order Runge-Kutta integrator was used (see Ref. 75)

Table 3-12

Results of the Semianalytical Batch Filter with States and C_D as the Solve-For Vector

A Priori*		Final**	
$\bar{a}$ =	6636.45790974 km	$\bar{a}$ =	6636.37973177 km
$\bar{e}$ =	.010600946793	$\bar{e}$ =	.010604560838
$\bar{i}$ =	67.9708552769°	$\bar{i}$ =	67.9709267987°
$\bar{\Omega}$ =	91.9949096046°	$\bar{\Omega}$ =	91.9949014161°
$\bar{\omega}$ =	200.213582516°	$\bar{\omega}$ =	200.210968206°
$\bar{M}$ =	164.768801765°	$\bar{M}$ =	164.771253150°
C_D =	2.30, $\sigma_{C_D}^2$ = .333	C_D =	2.00006917685

* Initial state elements ($\bar{\underline{a}}^*_o$) were obtained by a NOM procedure applied to the initial osculating elements of Table 3-8 with $a_1 = a_2 = 1$, $a_3 = 6$, $a_4 = a_5 = 0$ and $C_D = 2.3$.

** Took 7 iterations to converge

observations of Table 3-9. The perturbation model of the filter is given in Table 3-11. Table 3-13 gives the a priori and final values of the solve-for vector. Different a priori values for a_1 and a_2 are the only differences between the perturbation models of the semianalytical batch filter and the truth ephemeris.

From Table 3-13 it is seen that the semianalytical batch filter required 15 iterations to converge. The converged values of a_1 and a_2 were within .4% of the true values of a_1 and a_2. A comparison of the final solve-for vector in Table 3-13 to the PCE elements of Table 2-7 shows that the semimajor axis differs by about 3.5 millimeters.

It is felt that the above tests demonstrate that the adaptive semianalytical batch filter is sensitive to and can correct for modeling errors in the drag formulation. The above tests give confidence that the software implementation into GTDS-RD is correct.

3.3.4 Test Case Two

This test case is basically the test case presented by Dowd in Reference 17. The truth ephemeris, observation net and observations are similar. Dowd used an Extended Kalman filter (coupled to an SP orbit generator) and adaptively estimated parameters in an Asymmetric Modified Harris Priester density

Table 3-13

Results of the Semianalytical Batch Filter with States, a_1 and a_2 as the Solve-For Vector

A Priori*	Final**
$\bar{a}$ = 6636.458 km	$\bar{a}$ = 6636.37971418
$\bar{e}$ = .01060095	$\bar{e}$ = .010604645740
$\bar{i}$ = 67.97086°	$\bar{i}$ = 67.9709289907°
$\bar{\Omega}$ = 91.99491°	$\bar{\Omega}$ = 91.9949029114°
$\bar{\omega}$ = 200.2137°	$\bar{\omega}$ = 200.210673276°
$\bar{M}$ = 164.7687°	$\bar{M}$ = 164.771556441°
a_1 = 1.20, $\sigma^2_{a_1}$ = .1	a_1 = .99526846
a_2 = 1.10, $\sigma^2_{a_2}$ = .1	a_2 = 1.00335613

* Initial states elements ($\bar{\underline{a}}^*_o$) were obtained by a NOM procedure applied to initial osculating elements of Table 3-8 with $a_1 = 1.2$, $a_2 = 1.1$, $a_3 = 6.0$, $a_4 = a_5 = 0.0$ and $C_D = 2.0$.

** Took 15 iterations to converge.

model which basically correspond to the a_1, a_2 and a_3 parameters of the Adaptive Modified Harris-Priester atmosphere (see Appendix C).

Table 3-14 gives the inputs for the truth ephemeris for Test Case Two. This test case has an initial perigee height of about 300 km and an initial apogee height of about 309 km. It is noted that the density model used to generate the truth ephemeris was the Analytical Jacchia-Roberts density model. The simulation of the observational data is explained in Table 3-15 and the GTDS station pass report is given in Table 3-16. The simulated observations did not include random noise and were generated at a rate of one set of observations (range and range rate) every 10 seconds while in the field of view of the tracking station. The maximum time that a tracking station was allowed to observe the satellite during a single pass was 5 minutes. A total of 331 range observations and 331 range-rate observations were generated.

Cowell Batch Filter - States Only as the Solve-For Vector

A Cowell batch filter with states only as the solve-for vector was used to process the simulated data. The perturbation model and a priori values used in the Cowell Batch filter are given in Tables 3-17 and 3-18. The results of the Cowell batch filter are presented in Table 3-19. The filter took

Table 3-14

Truth Ephemeris for Test Case Two

Initial Osculating Elements and Parameters

a	=	6682.47355 km	Epoch	=	2 hrs. 47.93 min. 16 Dec. 1973
e	=	.000646254	Reference Frame	=	True of Date
i	=	67.991163°	C_D	=	2.0
Ω	=	254.26°	Area	=	1.86 m^2
ω	=	0°	Mass	=	677 kg
M	=	0°			

Perturbations

6x0 gravitational field with drag. The Analytical Jacchia-Roberts density model was used.

Orbit Generator

Cowell with a 30 second numerical integration time step

Table 3-15

Data Simulation for Test Case Two

Tracking Station:	Net A (see Table 3-10)
Observation Type:	VHF range (ρ) and range rate ($\dot{\rho}$) σ_ρ = 5 meters $\sigma_{\dot{\rho}}$ = .55 cm/sec
Observations:	Generated every 10 seconds while in sight (15° minimum elevation). Maximum time of observation interval per satellite pass is 300 seconds. No random noise included.
Observation Span:	From Epoch to 1630 hrs 16 Dec 1973
Truth Ephemeris:	See Table 3-14

Table 3-16. Station Pass Report for Test Case Two

PEV NUM	0	1	2	3	4	5	6	7
START	7312161 24008	7312161 41645	7312161 54921	7312161 71957	7312161 85034	7312161 102110	7312161 115147	7312161 132223
C AOS								7312161
A								141150
N LOS								7312161
V								142040
ELMAX								69
AOSL								7312171
								7451
C AOS	7312161							
A	41100							
P LOS	7312161							
V	41450							
ELMAX	0							
AOSL	7312161							
	114554							
U AOS	7312161							
A	25000							
M LOS	7312161							
V	25600							
ELMAX	0							
ACSL	7312161							
	120357							
D AOS	7312161	7312161						
O	25310	42750						
D LOS	7312161	7312161						
V	30150	43320						
ELMAX	49	0						
AOSL	7312161	7312161						
	120956	134436						
K AOS				7312161				
O				75510				
K LOS				7312161				
V				75610				
ELMAX				0				
AOSL				7312161				
				211630				
F AOS	7312161	7312161	7312161					
A	30610	43910	61150					
I LOS	7312161	7312161	7312161					
V	31500	44300	62030					
ELMAX	40	66	31					
AOSL	7312161	7312191	7312161					
	171607	184907	202147					
G AOS		7312161						
O		44850						
L LOS		7312161						
V		45540						
ELMAX		0						
AOSL		7312161						
		210120						

PEV NUM	0	1	2	3	4	5	6	7
START	7312161 24008	7312161 41645	7312161 54921	7312161 71957	7312161 85034	7312161 102110	7312161 115147	7312161 132223
G AOS		7312161						
U		45110						
A LOS		7312161						
V		45310						
ELMAX		0						
AOSL		7312161						
		212917						
O AOS	7312161							
S	31850							
S LOS	7312161							
V	32720							
ELMAX	34							
AOSL	7312161							
	214720							
M AOS	7312161							
E	32040							
P LOS	7312161							
V	32890							
ELMAX	25							
AOSL	7312161							
	215754							
G AOS	7312161							
R	30110							
A LOS	7312161							
V	32950							
ELMAX	33							
ACSL	7312161							
	220313							
S AOS	7312161	7312161						
A	33910	50040						
N LOS	7312161	7312161						
V	34040	51720						
ELMAX	0	48						
AOSL	7312161	7312171						
	225630	25601						
A AOS						7312161		
S						114410		
C LOS						7312161		
V						115250		
ELMAX						53		
AOSL						7312171		
						104651		
W AOS							7312161	
I							120020	
N LOS							7312161	
V							120740	
ELMAX							0	
AOSL							7312171	
							115733	

PEV NUM	0	1	2	3	4	5	6	7
START	7312161 24008	7312161 41645	7312161 54921	7312161 71957	7312161 85034	7312161 102110	7312161 115147	7312161 132223
G AOS						7312161		
P						102610		
E LOS						7312161		
V						103440		
ELMAX						35		
AOSL						7312161		
						120154		
S AOS				7312161				
H				72400				
I LOS				7312161				
V				72830				
ELMAX				0				
AOSL				7312161				
				105405				
N AOS			7312161					
A			55240					
I LOS			7312161					
V			55920					
ELMAX			0					
AOSL			7312161					
			111030					
A AOS				7312161				
D				84800				
D LOS				7312161				
V				85540				
ELMAX				63				
AOSL				7312161				
				112350				
J AOS				7312161				
O				85540				
H LOS				7312161				
V				84720				
ELMAX				71				
AOSL				7312161				
				102930				
T AOS			7312161					
A			70940					
N LOS			7312161					
V			71600					
ELMAX			29					
AOSL			7312161					
			101922					

AOS = Acquisition of Signal
LOS = Loss of Signal

ELMAX = Maximum Elevation Angle this Pass
AOSL = Local Time of AOS

Table 3-17

Perturbation Model for the Cowell Orbit Generator/Batch Filter

6x0 gravitational field with drag. A Modified Harris-Priester Atmosphere was used with $\bar{F}_{10.7} = 75\ 10^{-22}$ watts m^{-2} Hz^{-1}. The Cowell orbit generator used a 12th order Adams-Bashforth Predictor/Adams-Moulton Corrector integrator with a 30 second numerical integration time step.

Table 3-18

A Priori Values for Test Case Two

Semianalytical Batch Filter*	Cowell Batch Filter**
$\bar{a}$ = 6674.04345478	a = 6682.47355 km
$\bar{e}$ = .000016055664	e = .000646254
$\bar{i}$ = 67.9763516649°	i = 67.991163°
$\bar{\Omega}$ = 254.260152633°	Ω = 254.26°
$\bar{\omega}$ = 359.449849018°	ω = 0.0°
$\bar{M}$ = .549789997042°	M = 0.0°
C_D = 2.0, $\sigma^2_{C_D}$ = .333	C_{DO} = 2.0
	$C_D = C_{DO}(1+\rho_1)$
$a_1 = a_2$ = 1.0, $\sigma^2_{a_1} = \sigma^2_{a_2}$ = .1	ρ_1 = 0.0
a_3 = 6.0, $\sigma^2_{a_3} = 10^{-4}$	
$a_4 = a_5$ = 0.0, $\sigma^2_{a_4} = \sigma^2_{a_5} = 10^{-12}$	

* Initial state elements were obtained by a NOM procedure applied to initial osculating elements of Table 3-14 with $a_1 = a_2 = 1$, $a_3 = 6$, $a_4 = a_5 = 0$ and $C_D = 2.0$.

** Cowell batch filter could not use the Adaptive Modified Harris-Priester Atmosphere.

Table 3-19

Final* Results of the Cowell Batch Filter with States Only as the Solve-For Vector

$a = 6682.463898$ km

$e = .0006465268$

$i = 67.99106851°$

$\Omega = 254.2604117°$

$\omega = .1711699991°$

$M = 359.8277233°$

* Took 3 iterations to converge, RMS = 73.96.

3 iterations to converge. This is not surprising since the a priori guess of the state (Table 3-18) was the actual initial state of the truth ephemeris (Table 3-14). A comparison of Tables 3-19 and 3-18 reveals that the Cowell batch filter decreased the semimajor axis by about 10 meters from the a priori starting value.

A predicted ephemeris corresponding to the solve-for vector of Table 3-19 was generated and compared to the truth ephemeris over a 72 hour time span from epoch. The along track error was dominant and is shown in Figure 3-1.

Semianalytical Batch Filter - States Only as the Solve-For Vector

The results of a semianalytical batch filter with states only as the solve-for vector are shown in Table 3-20. The semianalytical batch filter used the perturbation model of Table 3-11 and the a priori solve-for vector of Table 3-18. The along

Table 3-20

Final* Results of the Semianalytical Batch Filter with States Only as the Solve-For Vector

$\bar{a}$	=	6673.954887 km
$\bar{e}$	=	,0000149613
$\bar{i}$	=	67.97686382°
$\bar{\Omega}$	=	254.2604506°
$\bar{\omega}$	=	21.28302544°
$\bar{M}$	=	338.7155903°

* Values came from the 13th iteration. The batch filter was converging but had not converged to a final value, RMS = 34.02.

track difference between the predicted ephemeris corresponding to the solve-for vector of Table 3-20 and the truth ephemeris is presented in Figure 3-2. The semianalytical orbit generator used to obtain the predicted ephemeris is given in Table 3-21.

Table 3-21

Perturbation Model of the Semianalytical Orbit Generator Used in Generating the Predicted Ephemerides

AOG:	First order analytical expressions for the 6x0 gravitational field, Zeis' expressions for the J_2^2 effects, drag* option 7 (4/48): 1 day numerical integration time step.
SPG:	6x0 gravitational field (7/48) and drag* (7/48), both to first order (time-independent), Zeis' J_2^2 expressions.
Integrator:	For Test Case Two - 6th order Runge-Kutta For Test Case Three - 12th order Adams-Bashforth Predictor/Adams-Moulton Corrector

* An Adaptive Modified Harris-Priester Atmosphere with the respective adaptive parameters and with $\overline{F}_{10.7} = 75\ 10^{-22}$ watts m^{-2} Hz^{-1}.

Comparison of Figures 3-2 and 3-1 shows that the semianalytical batch filter yields similar results to the Cowell batch filter.

Cowell Batch Filter - States and ρ_1 as the Solve-For Vector

Table 3-22 shows the final results of processing the simulated data of Test Case Two with a Cowell batch filter which solves for the variation in C_D (ρ_1) in addition to the initial states. Tables 3-17 and 3-18 give the perturbations and a priori values of the adaptive Cowell batch filter. Figure 3-3 shows the along track error between the predicted ephemeris associated with the solve-for vector of Table 3-22 and the truth ephemeris of Test Case Two. A comparison of Figures 3-1 and 3-3 indicates that solving for states and C_D (or ρ_1) gives better prediction results over the 72 hours predict span than solving for states only.

Table 3-22
Final* Results of the Cowell Batch Filter
with States and ρ_1 as the Solve-For Vector

a =	6682.474163 km	ω =	.0054429055°
e =	.0006465726	M =	359.9945420°
i =	67.99115226°	ρ_1 =	.2335674365
Ω =	254.2599985°		

* Values came from the 15th iteration. The batch filter was converging but had not converged to a final value, RMS = .8966.

Semianalytical Batch Filter - States and C_D as the Solve-For Vector

A semianalytical batch filter with states and C_D as the solve-for vector was also used to process the simulated data of Test Case Two. Tables 3-11 and 3-18 give the perturbation model and the a priori solve-for vector used in the batch filter. Table 3-23 gives the final results of the semianalytical batch filter for this test case. The final RMS value in Table 3-23 is slightly larger than the RMS in Table 3-22 [see Equation (3-70) for the definition of the RMS]. The larger value of the RMS in Table 3-23 could be due partly to the fact that the a priori value of C_D had a variance of .333 and therefore P_0^{-1} was not zero in Equation (3-70). It is also interesting to note that the value of C_D in Table 3-23 varies by less than .025% from the final value of C_D obtained with the Cowell batch filter.

Table 3-23

Final* Results of the Semianalytical Batch Filter with States and C_D as the Solve-For Vector

$\bar{a}$ =	6673.966073 km	$\bar{\omega}$ =	356.8969264°
$\bar{e}$ =	.0000128192	$\bar{M}$ =	3.102968644°
$\bar{I}$ =	67.97637954°	C_D =	2.466603290
$\bar{\Omega}$ =	254.2601548°		

* Values came from the 15th iteration. The batch filter was converging but had not converged to a final value, RMS = 1.331.

Figure 3-4 shows the along track difference between the predicted ephemeris corresponding to Table 3-23. A comparison of Figures 3-3 and 3-4 shows that the semianalytical batch filter gives results very similar to the Cowell Batch filter.

Semianalytical Batch Filter - States, a_1 and a_2 as the Solve-For Vector

Table 3-24 gives the final results of the semianalytical batch filter with states, a_1 and a_2 as the solve-for vector. The perturbation model and a priori values are given in Tables 3-11 and 3-18. The final value for the epoch mean semimajor axis is about .5 meters more than the epoch mean semimajor axis as given in Table 3-23. It is also noted that the RMS value is considerably less than any of the previously obtained RMS values [see Equation (3-70) for definition of RMS].

Table 3-24

Final* Results of the Semianalytical Batch Filter with States, a_1 and a_2 as the Solve-For Vector

$\bar{a}$ =	6673.966564 km	$\bar{\omega}$ =	356.7534244°
$\bar{e}$ =	.0000122015	$\bar{M}$ =	3.246535807°
$\bar{i}$ =	67.97638780°	a_1 =	1.453672222
$\bar{\Omega}$ =	254.2601561°	a_2 =	.9847238034

* Values came from the 14th iteration. The batch filter was converging but had not converged to a final value, RMS = .5882.

Figure 3-5 gives the along track difference between the predicted ephemeris associated with this estimation option and the truth ephemeris of this test case. Comparing Figures 3-2 and 3-5 indicates that the states plus a_1 and a_2 estimation option gives better results than a states only estimation option. Comparing Figures 3-4 and 3-5 indicates that for this test case the states plus C_D estimation option is superior to the states plus a_1 and a_2 estimation option over the prediction span.

Summary for Test Case Two

Table 3-25 gives a 72 hour comparison between the predicted ephemerides of this test case and the truth ephemeris. This table has to be viewed with a certain amount of caution since the sampling rate of once per 45 minutes is approximately equal to one half the orbital period.

Table 3-25 shows that for states only and states plus C_D as the solve-for parameters, the semianalytical and Cowell batch filters give nearly equivalent results for the orbit prediction problem. Table 3-25 also indicates that over the three day predict span, the states plus C_D estimation option gave the best results. This demonstrates that adaptive estimation of the drag formulation does significantly improve the orbit determination and prediction processes for low altitude satellites.

Table 3-25 also indicates that the states plus a_1 and a_2 estimation option in the semianalytical batch filter is not as good as the states plus C_D estimation option over the two day predict span. Comparing the final RMS values of the batch filter in Tables 3-19, 3-20, 3-22, 3-23 and 3-24 indicates that the states plus a_1 and a_2 may be slightly superior to the other estimation options over the observation span. This point will be examined in Test Case Three.

The other semianalytical estimation options as given in Table 3-6 were also used to process the simulated data of this test case. It was found that the semianalytical batch filter with these estimation options (states plus a_1, a_2 and a_3, states plus a_1, a_2, a_3, a_4 and a_5 and states plus C_D and a_4) would not converge unless the variances on a_3, a_4 and a_5 were set to the values given in Table 3-18. This is basically "telling" the batch filter that a_3, a_4 and a_5 are known extremely well. The parameters a_3, a_4 and a_5 of the Adaptive Modified Harris-Priester density model are, for all practical purposes, unobservable to the semianalytical batch filter. The states, a_1, a_2 and a_3 estimation option (using $\sigma^2_{a_3} = 10^{-4}$) gives results that are nearly equivalent to the results of the states plus a_1 and a_2 estimation option. Likewise, the states, C_D and a_4 estimation option (using $\sigma^2_{a_4} = 10^{-12}$) resulted in a predicted ephemeris which was not as good as the predicted ephemeris of the states and C_D estimation option.

Table 3-25

72 Hour Comparison of the Predicted Ephemerides with the Truth Ephemeris of Test Case Two

Batch Filter	Solve-For Vector	Maximum Sampled* Position Error (km)				Sampled* Position RMS (km)			
		Radial	Cross Track	Along Track	$\|\Delta r\|$	Radial	Cross Track	Along Track	$\|\Delta r\|$
Cowell	states only	.1343	.0445	18.31	18.31	.0581	.0267	7.696	7.696
Cowell	states and ρ_1	.0273	.0025	4.334	4.334	.0136	.0018	2.079	2.079
Semianalytical	states only	.1598	.0916	17.98	17.98	.0628	.0749	7.519	7.52
Semianalytical	states and C_D	.0275	.0290	4.363	4.363	.0137	.0135	2.090	2.090
Semianalytical	states, a_1 and a_2	.0290	.0300	6.397	6.397	.0163	.0141	2.935	2.935

* Sampling rate was one point every 45 minutes.

3.3.5 Test Case Three

This test case is representative of a real-world operational OD system. The inputs for the truth ephemeris used in this test case are presented in Table 3-26. Note that the perigee height (200 km) is approximately 100 km lower than for the test case of Section 3.3.4. As can be seen, the truth ephemeris is the Low Altitude Test Case presented in Section 2.4.3 with an Analytical Jacchia-Roberts atmosphere. The simulation of the observational data is explained in Tables 3-27 and 3-28. The simulated data did include random noise corresponding to the standard deviation of the measurements as given in Table 3-27. A unified S-band tracking net was used. A total of 51 range observations, 30 range rate observations and 60 gimbal angle observations (27 X_{30}, 27 Y_{30}, 3 X_{85} and 3 Y_{30} measurements, see Appendix G or Ref. 67) were generated.

Cowell Batch Filter - States Only as the Solve-For Vector

A Cowell batch filter with states only as the solve-for vector was used to process the simulated data of Test Case Three. The perturbation model used in the batch filter is given in Table 3-17. The a priori values of the solve-for parameters are given in Table 3-30. The results of the Cowell batch filter are given in Table 3-31. Figures 3-6 and 3-7 give 12 hour and 72 hour comparisons, respectively, of the along track difference between the predicted ephemeris corresponding to the solve-for vector of Table 3-31 and the truth ephemeris of this test case.

Table 3-26

Truth Ephemeris for Test Case Three

Initial Osculating Elements and Parameters

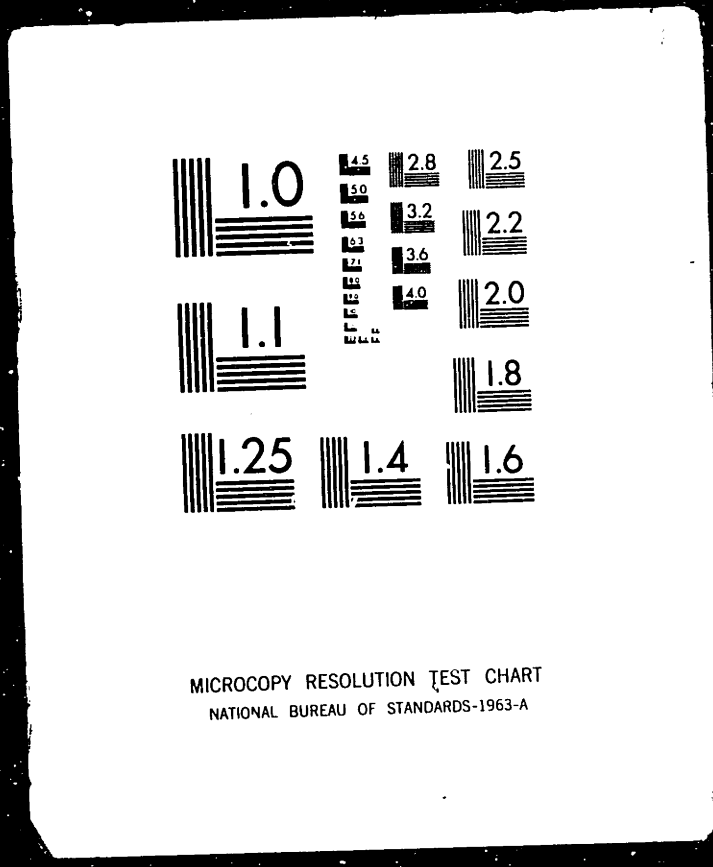
a	= 6644.586 km	Epoch	= 10 hrs. 24 min. 21 Oct. 74
e	= .01	Reference Frame	= True of Date
i	= 67.98538419°	C_D	= 2.0
Ω	= 91.99738418°	Area	= 1.86 m^2
ω	= 200.6741688°	Mass	= 677 kg
M	= 164.3173126°		

Perturbations

6x0 gravitational field with drag. The Analytical Jacchia-Roberts density model was used.

Orbit Generator

Cowell with a 30 second numerical integration time step.



20X

Table 3-27

Data Simulation for Test Case Three

Tracking Stations:	Net B (see Table 3-29)
Observation Type;	Unified S-band range (ρ), range rate ($\dot{\rho}$) and gimbal angles (X_{30}, Y_{30}, X_{85} or Y_{85})
	σ_ρ = 15 meters, $\sigma_{\dot{\rho}}$ = 2 cm/sec
	$\sigma_{X_{30}} = \sigma_{X_{85}} = \sigma_{Y_{30}} = \sigma_{Y_{85}} = 3506.5017$ sec-arc
Observations:	Generated every 60 seconds while in sight (20° minimum elevation). Maximum time of observation interval per satellite pass is 300 seconds. Random noise corresponding to observation type was included.
Observation Span:	From Epoch to 0600 hrs 22 Oct. 1974
Truth Ephemeris:	see Table 3-26

Table 3-28. Station Pass Report for Test Case Three

REV NUM	0	1	2	3	4	5	6	7	8	9	10	11	12	13
START	741021 102252	741021 115242	741021 132233	741021 145223	741021 162213	741021 175204	741021 192154	741021 205144	741021 222135	741021 235125	741022 12115	741022 25105	741022 42056	741022 55046
AOS					741021							741022		
					170600							41300		
LOS					741021							741022		
					171200							42000		
ELMAX					35							22		
AOSL					741022							741023		
					160840							31542		
AOS								741021						741022
								214200						70500
LOS								741021						741022
								214700						71300
ELMAX								42						24
AOSL								741022						741023
								165920						22220
AOS	741021													
	103000													
LOS	741021													
	103500													
ELMAX	0													
AOSL	741022													
	61122													
AOS	741021	741021				741021								
	103000	120300				182200								
LOS	741021	741021				741021								
	103800	120800				182400								
ELMAX	57	0				0								
AOSL	741022	741022				741022								
	52238	65538				131438								
AOS		741021	741021											
		115800	132900											
LOS		741021	741021											
		120300	133700											
ELMAX		0	24											
AOSL		741022	741022											
		41030	54130											
AOS					741021									
					175100									
LOS					741021									
					175600									
ELMAX					0									
AOSL					741022									
					32957									
AOS				741021	741021									
				145300	162700									
LOS				741021	741021									
				150100	163200									
ELMAX				24	0									
AOSL				741022	741022									
				41420	54820									
AOS			741021	741021										
			135200	152300										
LOS			741021	741721										
			135700	153000										
ELMAX			0	23										
AOSL			741022	741022										
			133520	150620										
AOS	741021						741021							
	103000						195500							
LOS	741021						741021							
	103500						200000							
ELMAX	55						0							
AOSL	741022						741022							
	50714						143214							
AOS				741021										
				160700										
LOS				741021										
				161400										
ELMAX				0										
AOSL				741022										
				20250										
AOS				741021	741021	741021	741021	741021						
				150600	163600	180700	193900	211200						
LOS				741021	741021	741021	741021	741021						
				151100	164300	181500	194700	211900						
ELMAX				0	28	56	35	51						
AOSL				741022	741022	741022	741022	741022						
				51557	64557	81657	94857	112157						

AOS = ACQUISITION OF SIGNAL
LOS = LOSS OF SIGNAL
ELMAX = MAXIMUM ELEVATION ANGLE THIS PASS
AOSL = LOCAL TIME OF AOS

Table 3-29

Tracking Net B

Station Name	Symbol	Latitude (deg.)	Longitude (deg.)	Height (m)
Ascension	ACN3W	- 7.954825	345.672936	528.0
Santiago	AGO3W	-33.150994	289.333633	706.18
Bermuda	BDA3W	32.3503	295.342206	34.0
Greenbelt	ETC3W	38.998567	283.157286	- 1.0
Goldstone	GDS3W	35.342205	243.126545	912.71
Guam	GWM3W	13.310625	144.736814	116.0
Kauai	HAW3W	22.126239	200.334842	1139.0
Madrid	MAD8W	40.455464	355.831553	808.0
Merritt Is.	MIL3W	28.50875	279.306625	-55.0
Orroral	ORR3W	-35.627969	148.957039	929.0
Fairbanks	ULA3W	64.972	212.487053	338.51

Geodetic Coordinates, taken from GTDS-RD User's Guide (Ref. 73)

Table 3-30

A Priori Values for Test Case Three

Semianalytical Batch Filter*	Cowell Batch Filter**
$\bar{a}$ = 6636.45593986 km	a = 6644.586 km
$\bar{e}$ = .010600675342	e = .01
$\bar{i}$ = 67.9708551529°	i = 67.98538419°
$\bar{\Omega}$ = 91.9949093860°	Ω = 91.99738418°
$\bar{\omega}$ = 200.209177391°	ω = 200.6741688°
$\bar{M}$ = 164.773135897°	M = 164.3173126°
C_D = 2.0, $\sigma^2_{C_D}$ = .333	C_{DO} = 2.0
$a_1 = a_2$ = 1.0, $\sigma^2_{a_1} = \sigma^2_{a_2}$ = .1	ρ_1 = 0.0 $\quad C_D = C_{DO}(1+\rho_1)$
a_3 = 6.0, $\sigma^2_{a_3} = 10^{-4}$	
$a_4 = a_5$ = 0.0, $\sigma^2_{a_4} = \sigma^2_{a_5} = 10^{-12}$	

* Initial state elements were obtained by a NOM procedure applied to the initial osculating elements of Table 3-26.

** Cowell batch filter could not use the Adaptive Modified Harris-Priester Atmosphere

Table 3-31

Final* Results of the Cowell Batch Filter with States Only as the Solve-For Vector

a	=	6644.37884315 km
e	=	.010294090436
i	=	67.9743471306°
Ω	=	91.9960440160°
ω	=	201.305628665°
M	=	163.656474108

* took 12 iterations to converge, RMS = 59.87

Semianalytical Batch Filter - States Only as the Solve-For Vector

The results of processing the observational data of Test Case Three with a semianalytical batch filter with states only as the solve-for vector are shown in Table 3-32. The semianalytical batch filter used the perturbation model of Table 3-11 and the a priori solve-for vector of Table 3-30. Figures 3-8 and 3-9 show the along track difference between the predicted ephemeris corresponding to the solve-for vector of Table 3-32 and the truth ephemeris of this test case over a 12 hour and a 72 hour time span. It is noted that the semianalytical batch filter with the states only estimation option gives results equivalent to the Cowell batch filter with the same estimation option.

Table 3-32

Final* Results of the Semianalytical Batch Filter with States Only as the Solve-For Vector

$\bar{a}$ = 6636.17653458 km

$\bar{e}$ = .010897761885

$\bar{i}$ = 67.9598842035°

$\bar{\Omega}$ = 91.9936086178°

$\bar{\omega}$ = 200.816521950°

$\bar{M}$ = 164.136382471°

* Took 12 iterations to converge, RMS = 59.89

Cowell Batch Filter - States and ρ_1 as the Solve-For Vector

Using the perturbation models of Table 3-17 and the a priori values of Table 3-30, a Cowell batch filter with states and ρ_1 as the solve-for vector gave the final results of Table 3-33. The along track error between the predicted ephemeris associated with Table 3-33 and the truth ephemeris is shown in Figures 3-10 and 3-11 for 12 and 72 hour comparisons, respectively. It is interesting to note that, as in Test Case Two, the states plus C_D (or ρ_1) estimation option gives significantly better 72 hour prediction results than the states only estimation option. The states plus C_D estimation option also significantly improves the prediction error during the observation span (Figure 3-10).

Table 3-33

Final* Results of the Cowell Batch Filter with States and ρ_1 as the Solve-For Vector

a	=	6644.56026507 km
e	=	.009996692517
i	=	67.9863983353°
Ω	=	91.9971347242°
ω	=	200.664730933°
M	=	164.326691616°
ρ_1	=	.269831999344

* Took 13 iterations to converge, RMS = 3.412

Semianalytical Batch Filter - States and C_D as the Solve-For Vector

Table 3-34 gives the final results of processing the observational data of Test Case Three with a semianalytical batch filter with states and C_D as the solve-for parameters. The perturbation models and a priori values used are given in Tables 3-11 and 3-30. Figures 3-12 and 3-13 are plots of the along track difference between the predicted ephemeris with the initial conditions of Table 3-34 and the truth ephemeris of this test case. As expected, the semianalytical batch filter with states plus C_D as the estimation option is equivalent to the Cowell batch filter with states and ρ_1 (C_D) as the solve-for vector.

Table 3-34

Final* Results of the Semianalytical Batch Filter with States and C_D as the Solve-For Vector

$\bar{a}$ = 6636.35295242 km

$\bar{e}$ = .010601078192

$\bar{i}$ = 67.9720308217°

$\bar{\Omega}$ = 91.9946124323°

$\bar{\omega}$ = 200.200474381°

$\bar{M}$ = 164.781700625°

C_D = 2.54063409983

* Took 13 iterations to converge, RMS = 3.317

Semianalytical Batch Filter - States, a_1 and a_2 as the Solve-for Vector

The semianalytical batch filter with states, a_1 and a_2 as the solve-for vector was also used to process the simulated data of Test Case Three. The perturbation model and a priori values used are given in Tables 3-11 and 3-30. Table 3-35 gives the final results of the semianalytical batch filter with states plus a_1 and a_2 as the solve-for parameters. Figures 3-14 and 3-15 are plots of the along track difference between the predicted ephemeris corresponding to Table 3-35 and the truth ephemeris of this test case for 12 and 72 hour comparisons, respectively. Figures 3-12 and 3-14 indicate that the

Table 3-35

Final* Results of the Semianalytical Batch Filter with States, a_1 and a_2 as the Solve-For Vector

$\bar{a}$ = 6636.35451813 km

$\bar{e}$ = .010595641200

$\bar{i}$ = 67.9716923016°

$\bar{\Omega}$ = 91.9945682623°

$\bar{\omega}$ = 200.239053854°

$\bar{M}$ = 164.742323444°

a_1 = 1.54777983011

a_2 = 1.12096690690

* Took 13 iterations to converge, RMS = 3.102

states plus a_1 and a_2 estimation option gives slightly better results over the observation span than the states plus C_D estimation option. Over the prediction span the states plus a_1 and a_2 estimation option is not as good as the states plus C_D estimation option.

Summary for Test Case Three

Tables 3-36 and 3-37 give a 12 hour and a 72 hour comparison, respectively, between the predicted ephemerides of this test case and the truth ephemeris. Table 3-36 represents the observation span (from Table 3-28 it is seen that the

Table 3-36

12 Hour Comparison of the Predicted Ephemerides with the Truth Ephemeris of Test Case Three

Batch Filter	Solve-For Vector	Maximum Sampled* Position Error (km)				Sampled* Position RMS (km)			
		Radial	Cross Track	Along Track	$\|\Delta\underline{r}\|$	Radial	Cross Track	Along Track	$\|\Delta\underline{r}\|$
Cowell	states only	2.264	1.295	8.347	8.422	1.509	.9081	4.119	4.480
Cowell	states and ρ_1	.0502	.1209	.3760	.3811	.0231	.0845	.1967	.2153
Semianalytical	states only	2.268	1.296	8.358	8.433	1.511	.9074	4.125	4.486
Semianalytical	states and C_D	.0503	.1348	.3733	.3809	.0231	.0928	.1948	.2170
Semianalytical	states, a_1 and a_2	.0914	.0999	.3344	.3472	.0458	.0676	.1646	.1837

* Sampling rate was one point every 8 minutes

Table 3-37

72 Hour Comparison of the Predicted Ephemerides with the Truth Ephemeris of Test Case Three

Batch Filter	Solve-For Vector	Maximum Sampled* Position Error (km)				Sampled Position RMS (km)			
		Radial	Cross Track	Along Track	$\|\Delta\underline{r}\|$	Radial	Cross Track	Along Track	$\|\Delta\underline{r}\|$
Cowell	states only	6.509	.9775	167.9	168.1	2.729	.5183	68.22	68.28
Cowell	states and ρ_1	.1511	.0954	16.60	16.60	.0755	.0684	8.910	8.910
Semianalytical	states only	6.523	1.012	168.2	168.4	2.783	.5370	68.34	68.40
Semianalytical	states and C_D	.1523	.0719	16.55	16.55	.0758	.0609	8.878	8.878
Semianalytical	states, a_1 and a_2	.2628	.1528	23.64	23.65	.1049	.0488	11.40	11.40

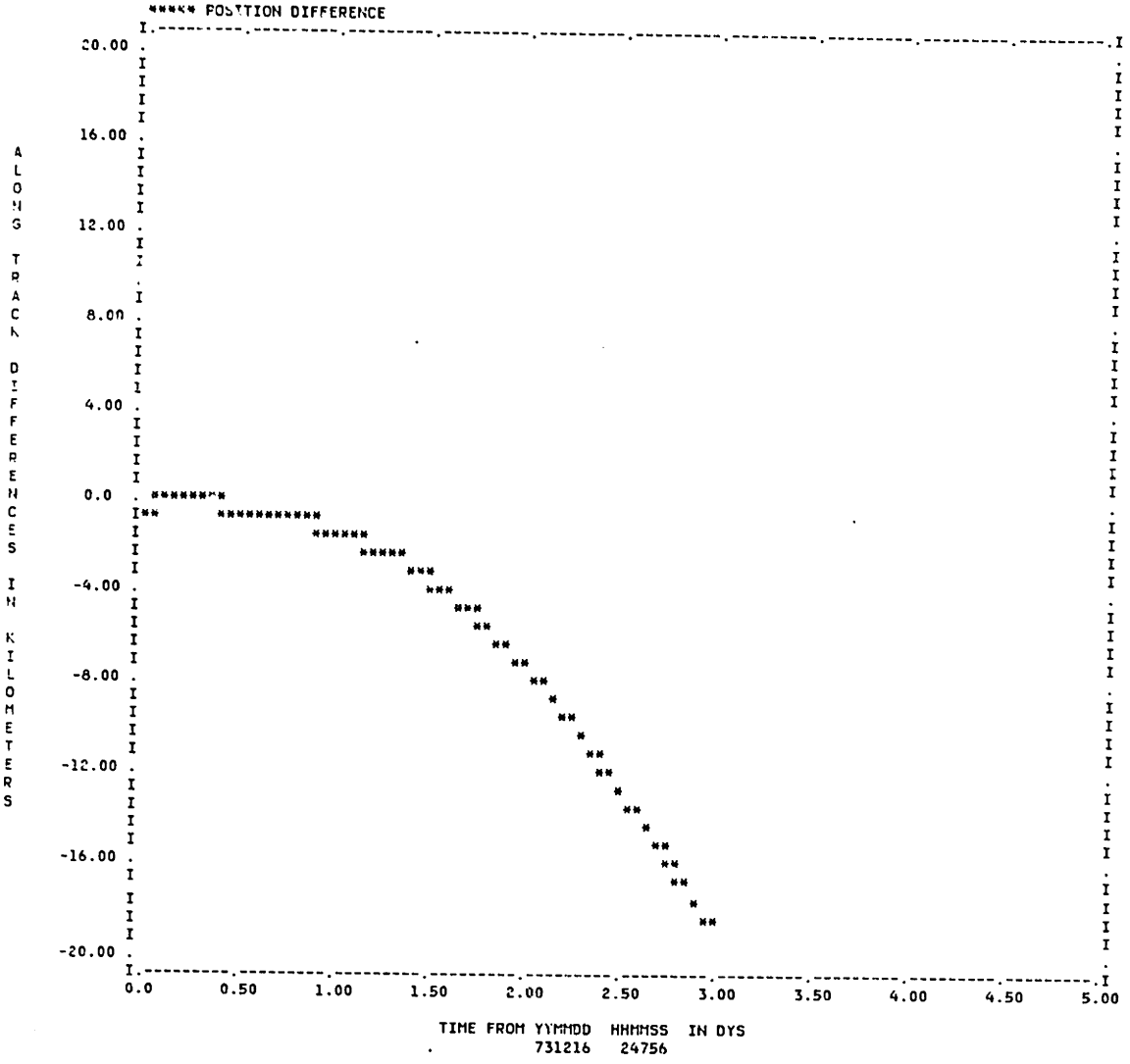
* Sampling rate was one point every 45 minutes.

effective observation span is 10.5 hours). Table 3-37 represents the prediction span. The sampling rate in Table 3-36 is one point every 8 minutes whereas the sampling rate in Table 3-37 is one point every 45 minutes.

From Tables 3-36 and 3-37 it is seen that for the states only and the states plus C_D (or ρ_1) estimation options, the semianalytical batch filter yields results that are equivalent to the Cowell batch filter. From Table 3-36 it is seen that over the observation span the states plus a_1 and a_2 estimation option gives results which are about 30 meters (total position error) better than the states plus C_D estimation option. Over the prediction span, the states plus C_D estimation option gives the best results.

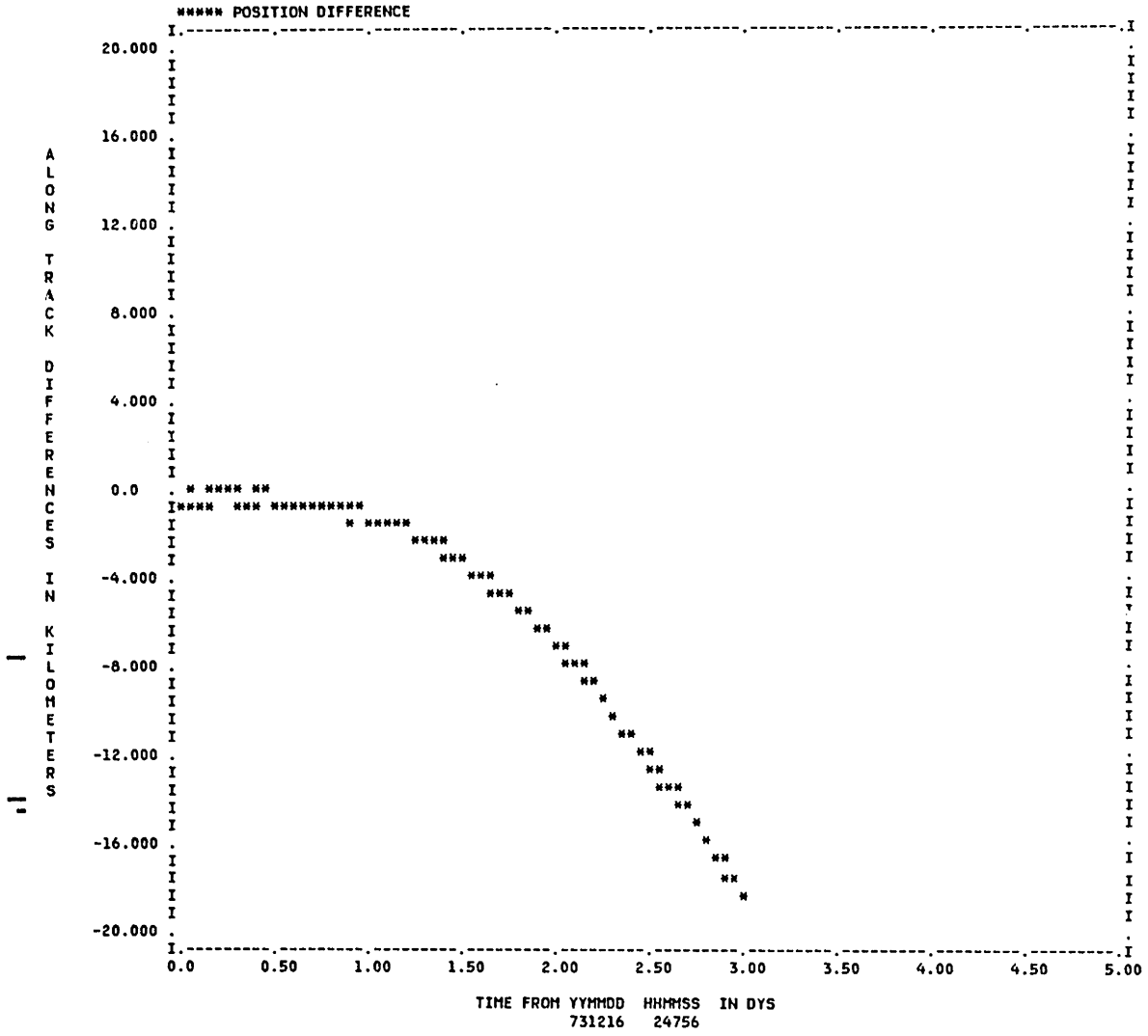
As was done in Test Case Two, the other estimation options, as given in Table 3-6, were used to process the simulated observational data of this test case. The semianalytical batch filter would not converge unless the variances on a_3, a_4 and a_5 were set to the values given in Table 3-30. It was also found that the results obtained with these estimation options were not as good as the states plus a_1 and a_2 estimation option and/or states plus C_D estimation option.

Figure 3-1. Along Track Difference/Cowell* minus the Truth Ephemeris of Test Case Two



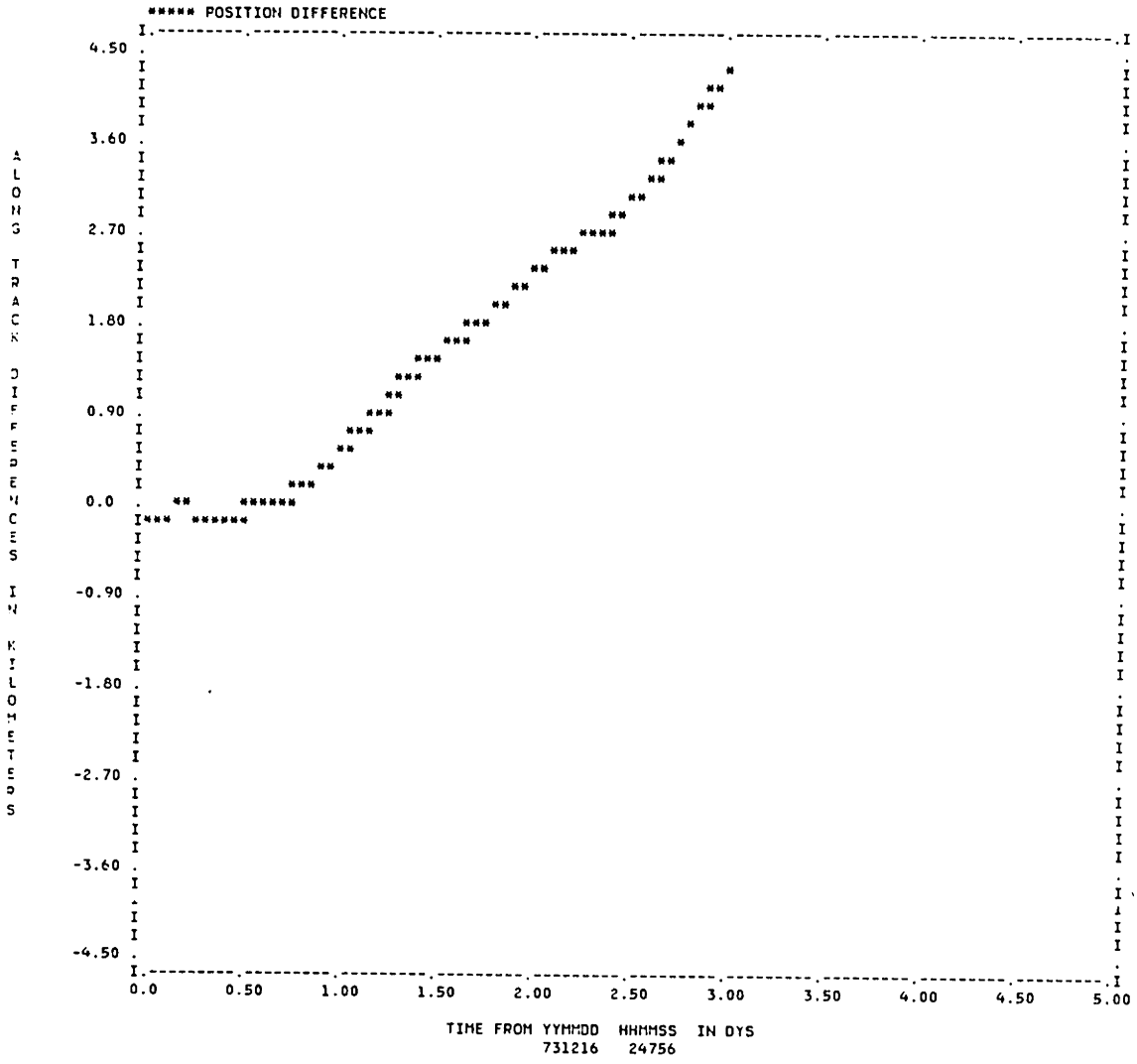
* Initial conditions from the Cowell batch filter with states only as solve-for vector (see Table 3-19). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-2. Along Track Difference/Semianalytical* minus the Truth Ephemeris of Test Case Two



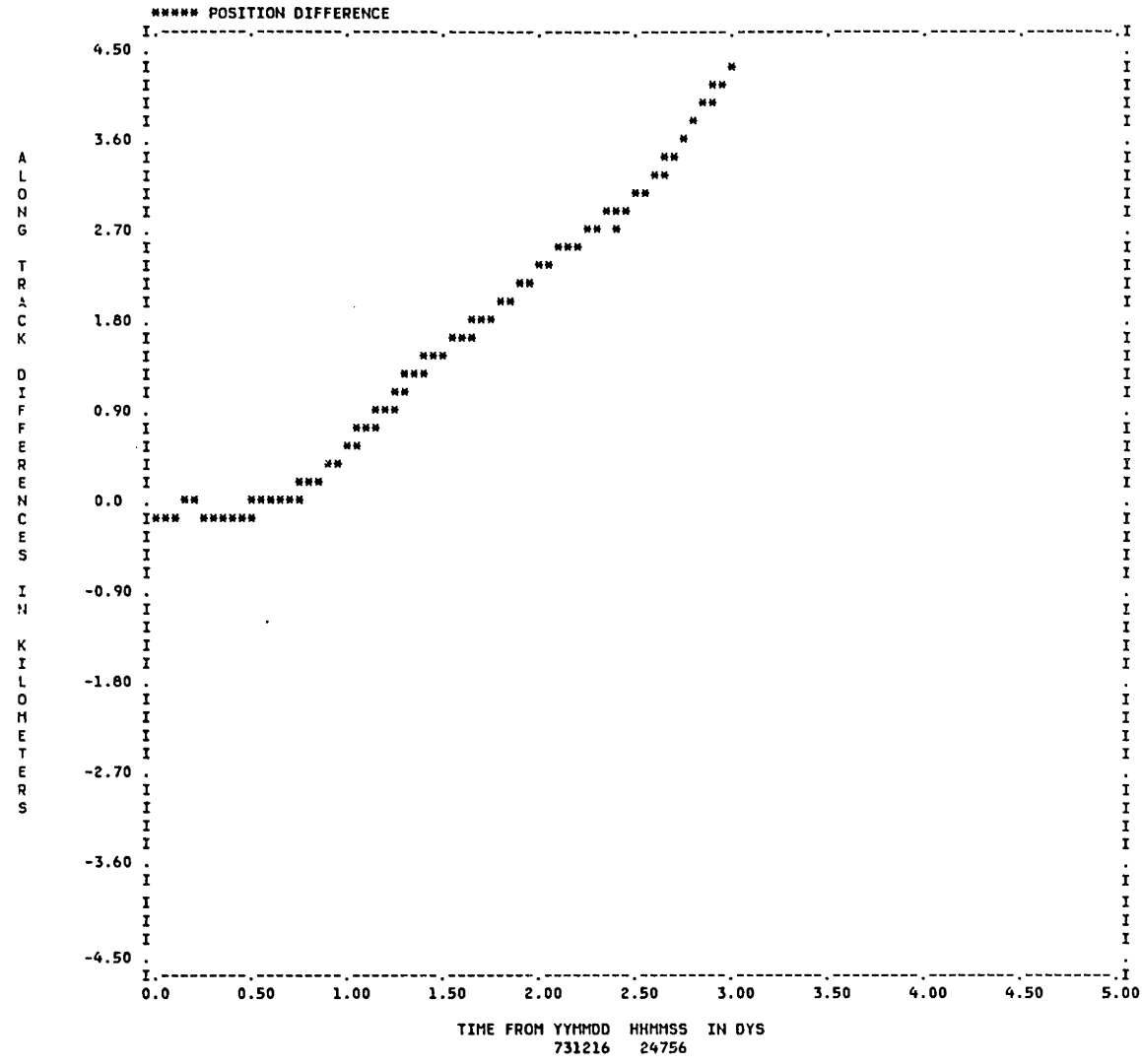
* Initial conditions from the semianalytical batch filter with states only as solve-for vector (see Table 3-20). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-3. Along Track Difference/Cowell* minus the Truth Ephemeris of Test Case Two



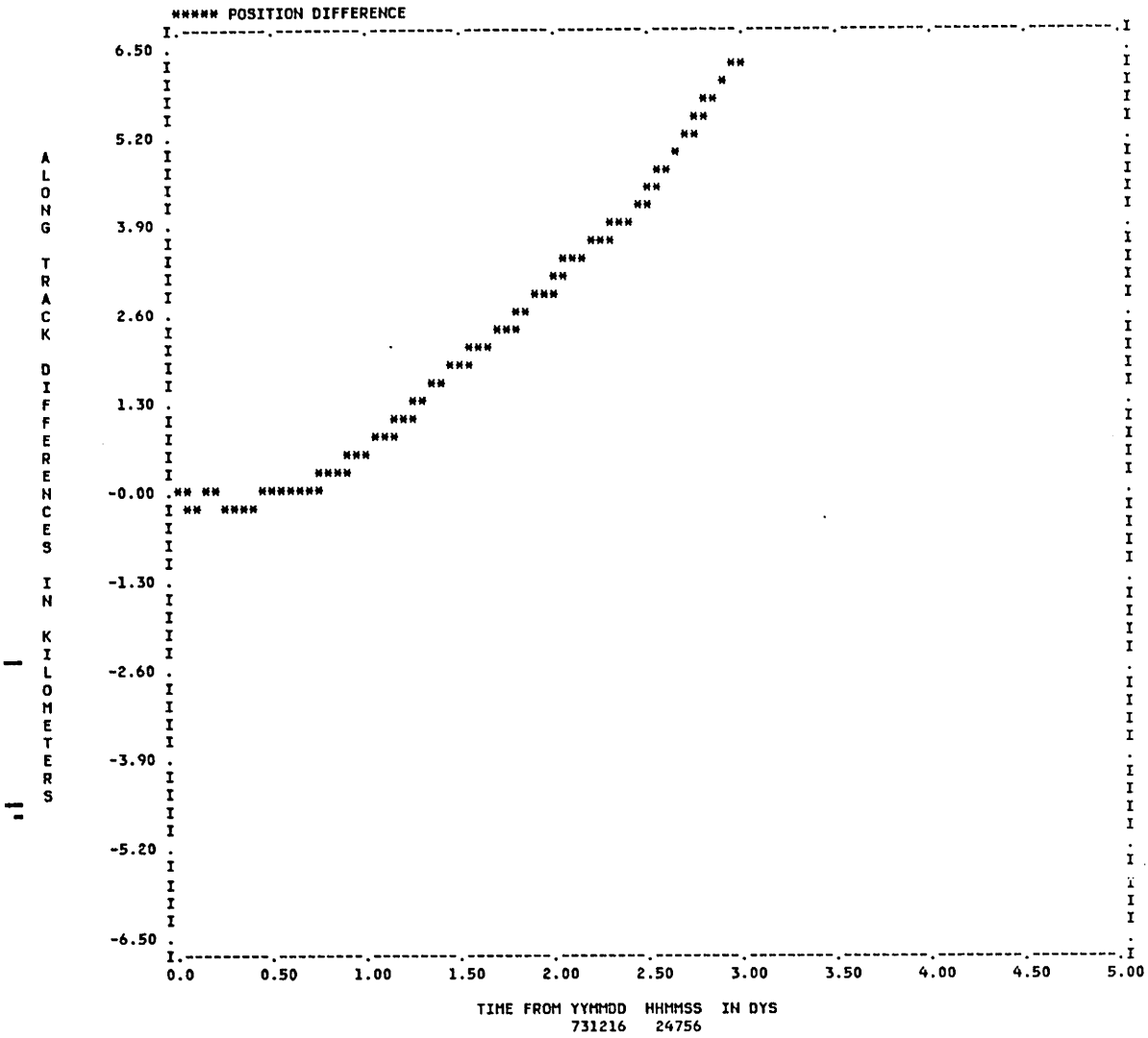
* Initial conditions from the Cowell batch filter with states and ρ_1 as solve-for vector (see Table 3-22). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-4. Along Track Difference/Semianalytical* minus the Truth Ephemeris of Test Case Two



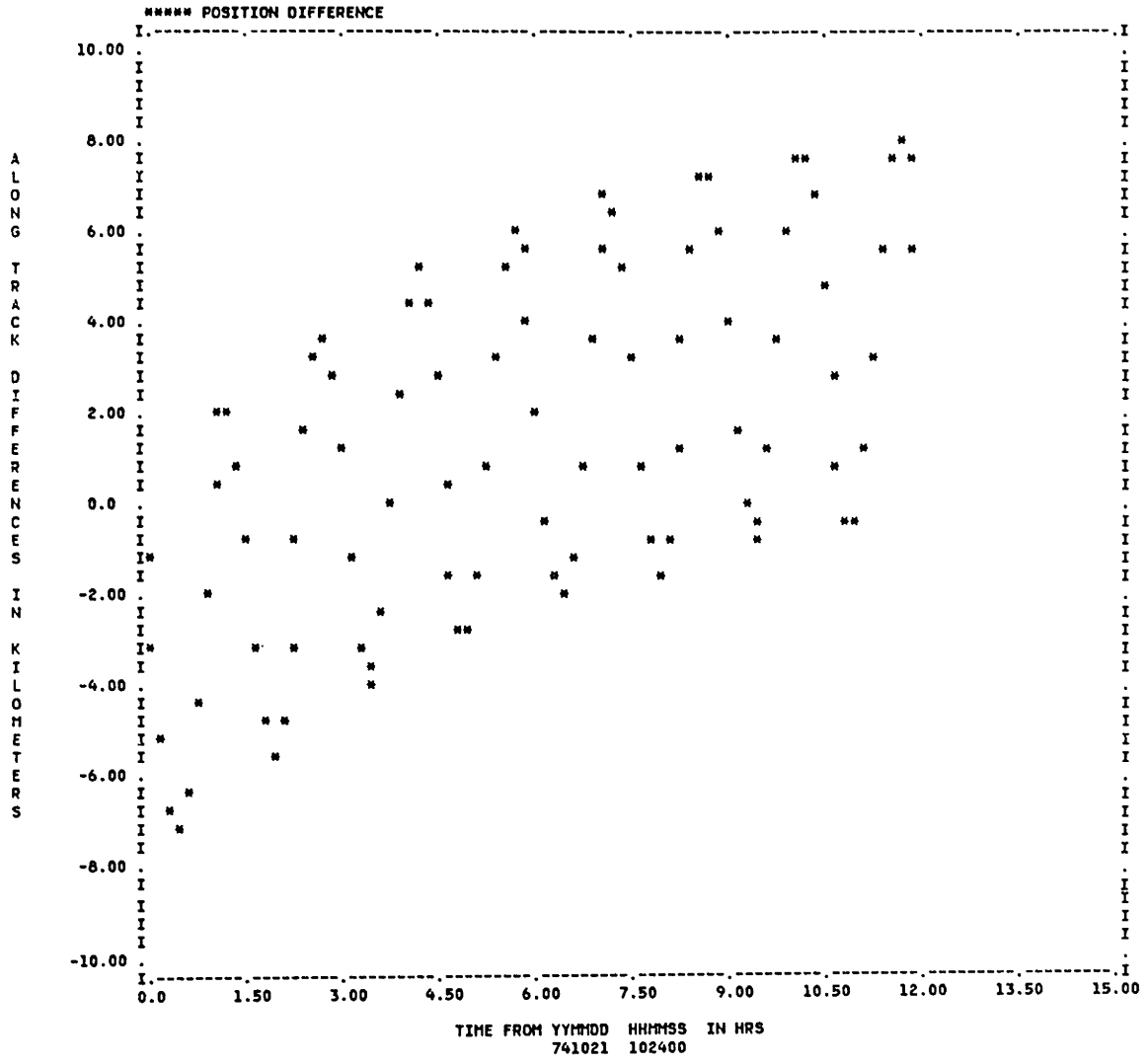
* Initial conditions from the semianalytical batch filter with states and C_D as solve-for vector (see Table 3-23). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-5. Along Track Difference/Semianalytical* minus the Truth Ephemeris of Test Case Two



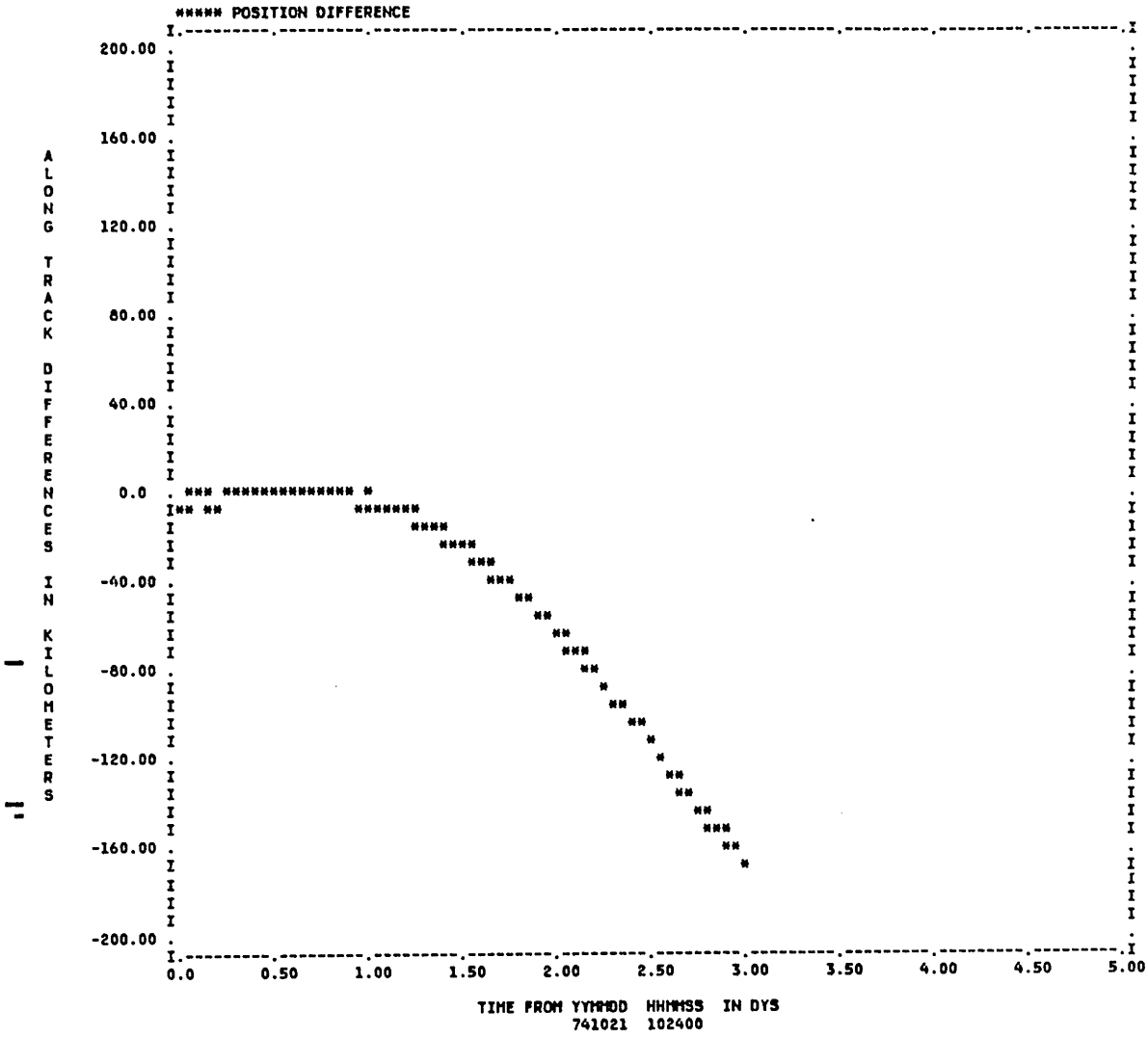
* Initial conditions from the semianalytical batch filter with states, a_1 and a_2 as solve-for vector (see Table 3-24). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-6. Along Track Difference over 12 Hours/Cowell* minus the Truth Ephemeris of Test Case Three



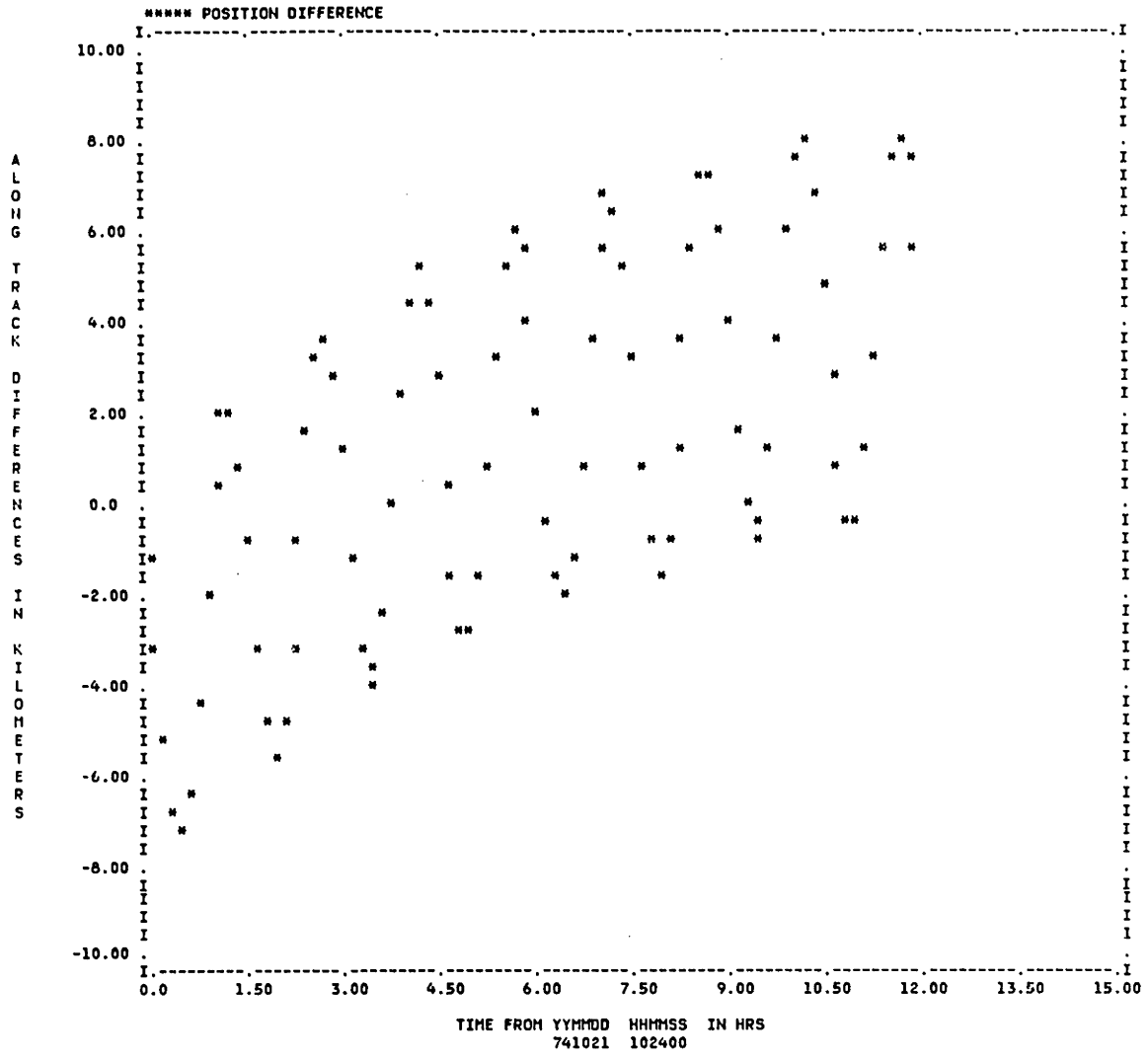
* Initial conditions from the Cowell batch filter with states only as the solve-for vector (see Table 3-31). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-7. Along Track Difference over 72 Hours/Cowell* minus the Truth Ephemeris of Test Case Three



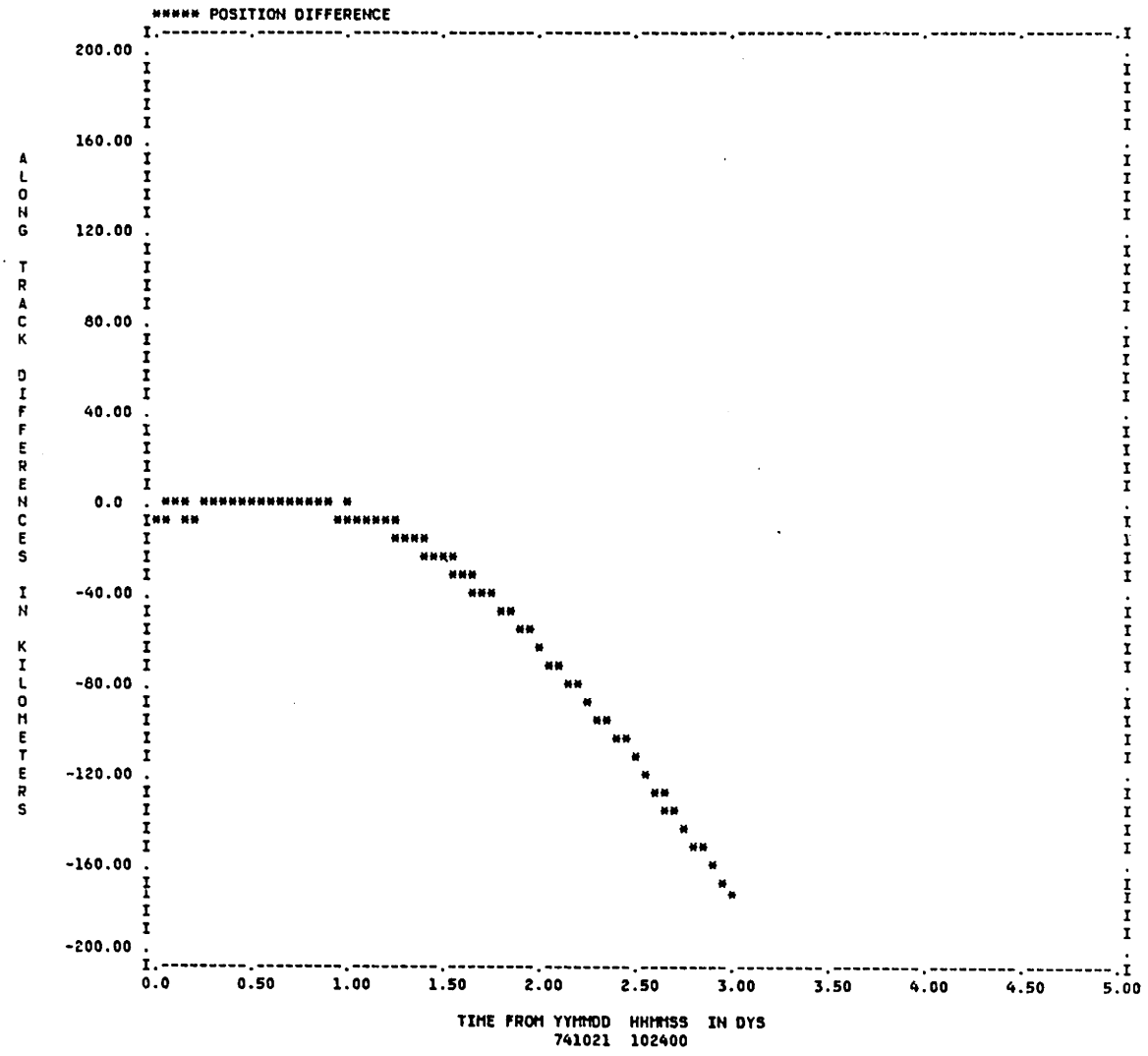
* Initial conditions from the Cowell batch filter with states only as the solve-for vector (see Table 3-31). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-8. Along Track Difference over 12 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



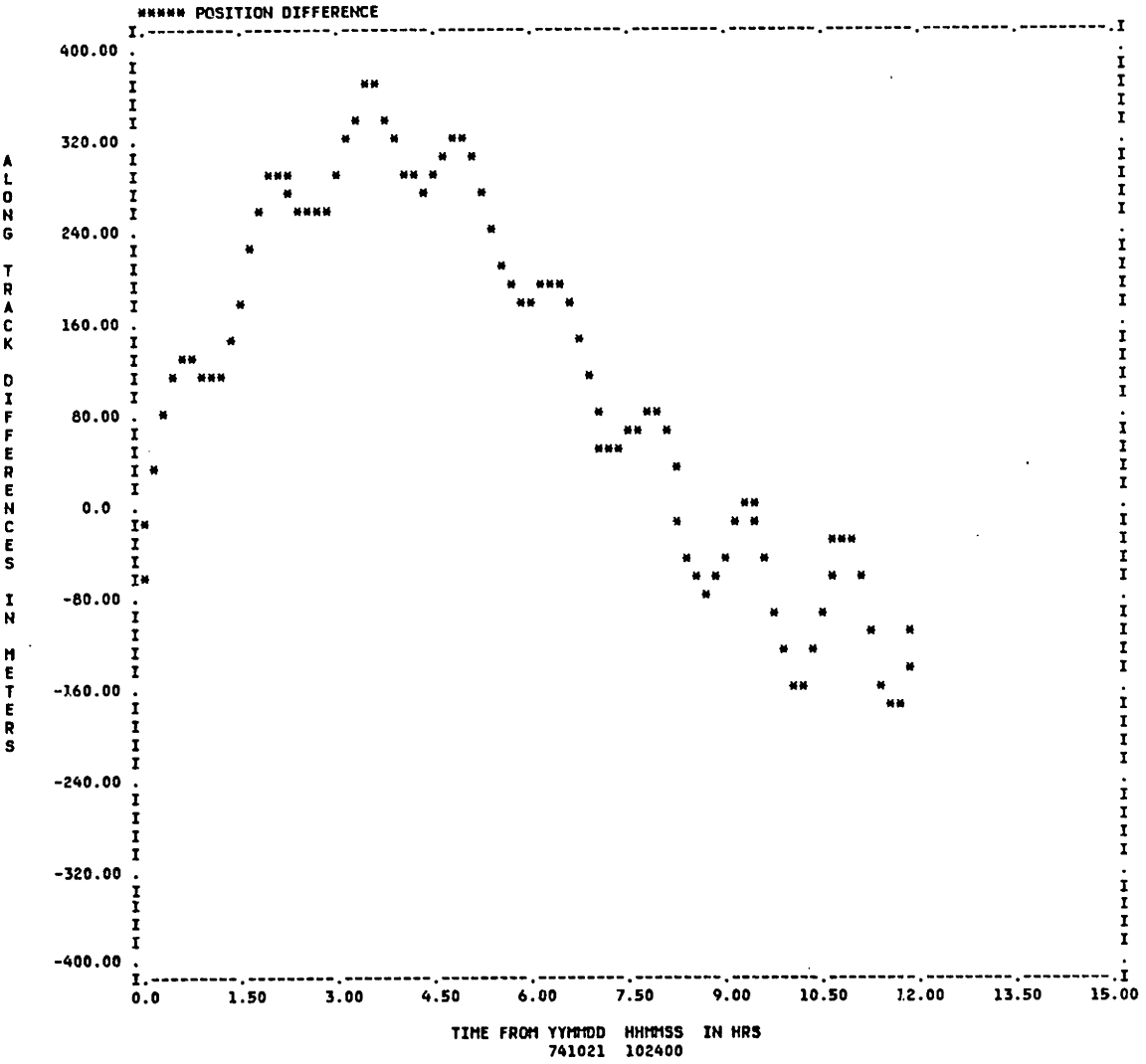
* Initial conditions from the semianalytical batch filter with states only as the solve-for vector (see Table 3-32). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-9. Along Track Difference over 72 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



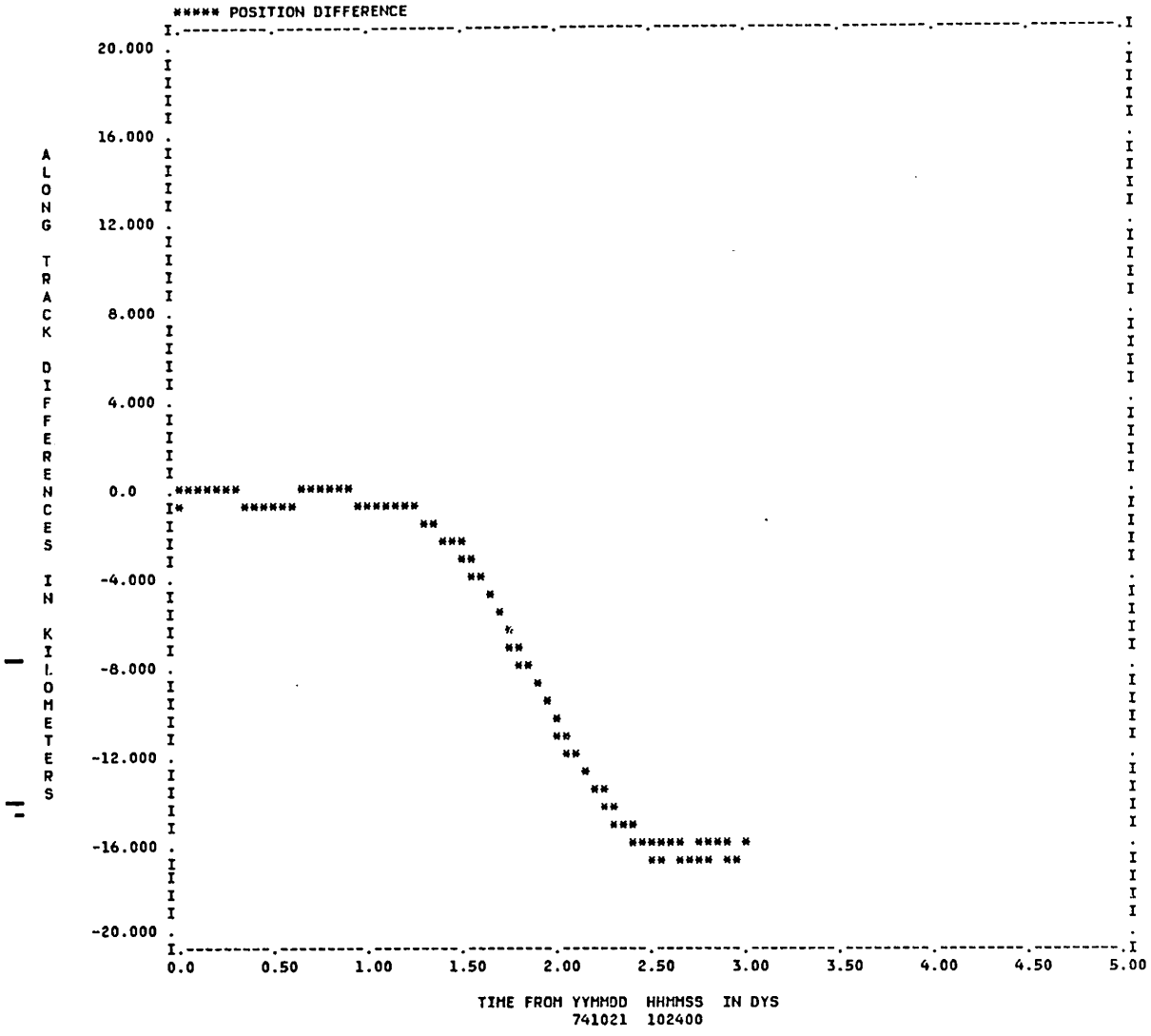
* Initial conditions from the semianalytical batch filter with states only as the solve-for vector (see Table 3-32). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-10. Along Track Difference over 12 Hours/Cowell* minus the Truth Ephemeris of Test Case Three



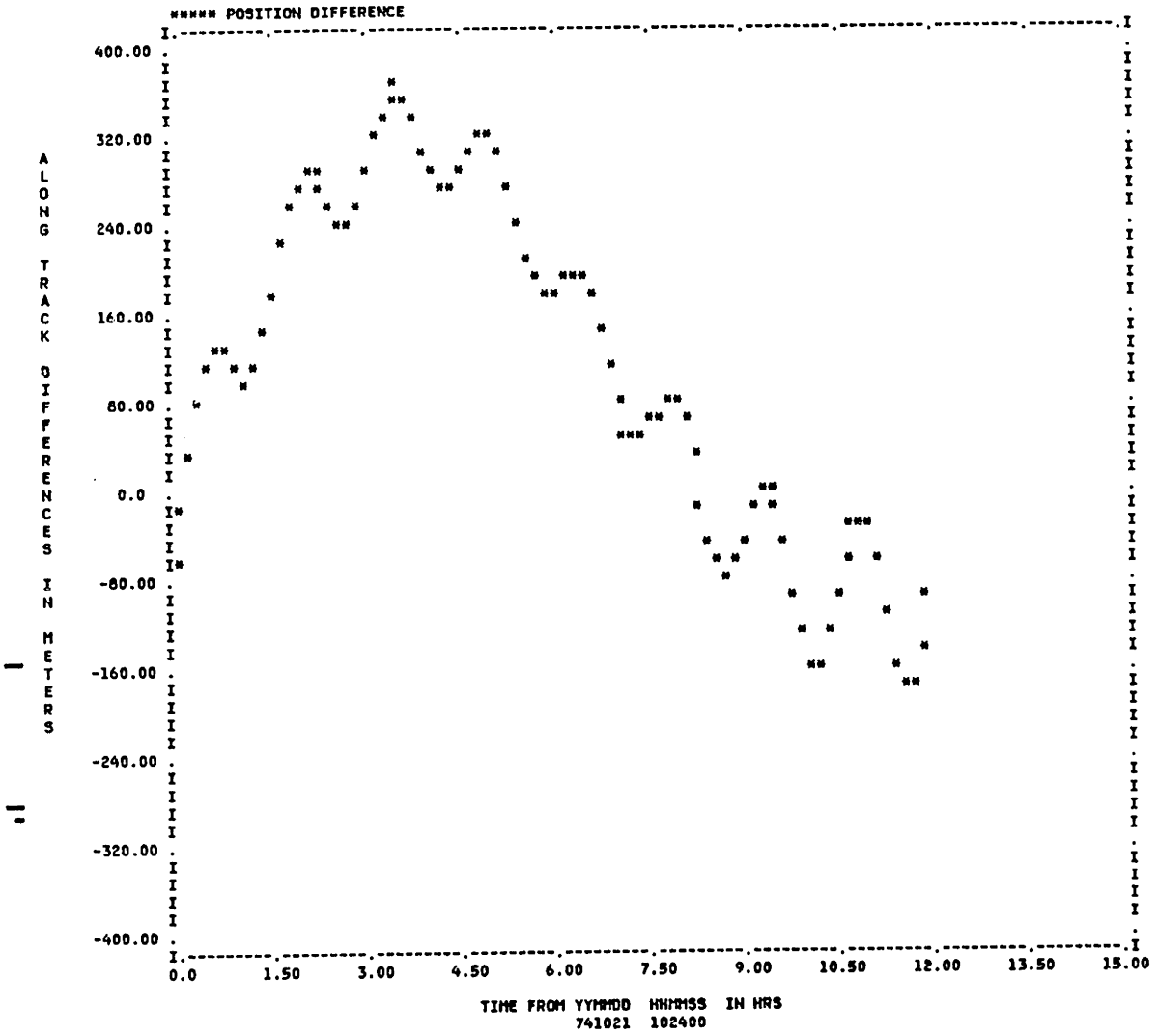
* Initial conditions from the Cowell batch filter with states and ρ_1 as the solve-for vector (see Table 3-33). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-11. Along Track Difference over 72 Hours/Cowell* minus the Truth Ephemeris of Test Case Three



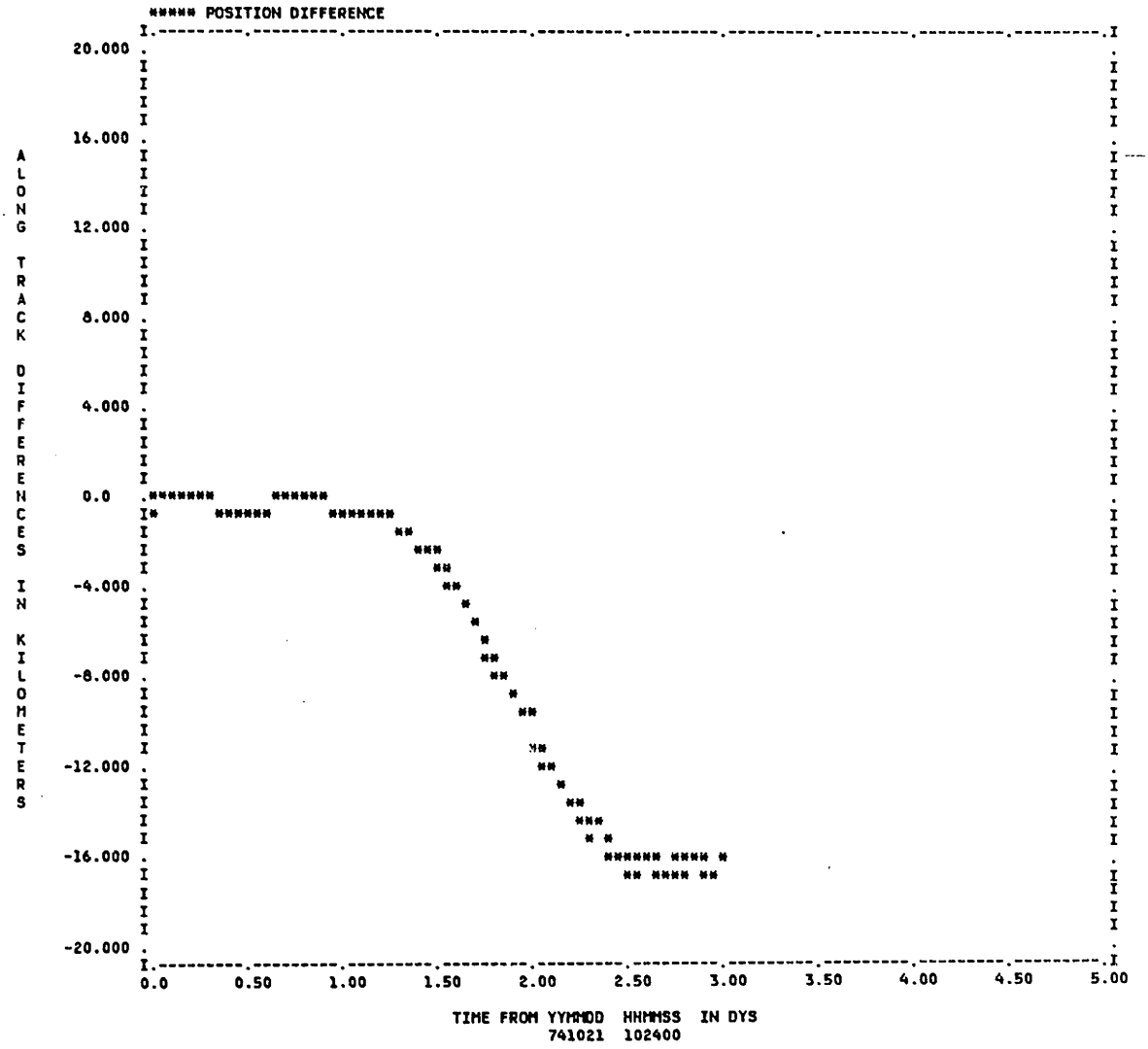
* Initial conditions from the Cowell batch filter with states and ρ_1 as the solve-for vector (see Table 3-33). The perturbation model for the Cowell orbit generator is given in Table 3-17.

Figure 3-12. Along Track Difference over 12 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



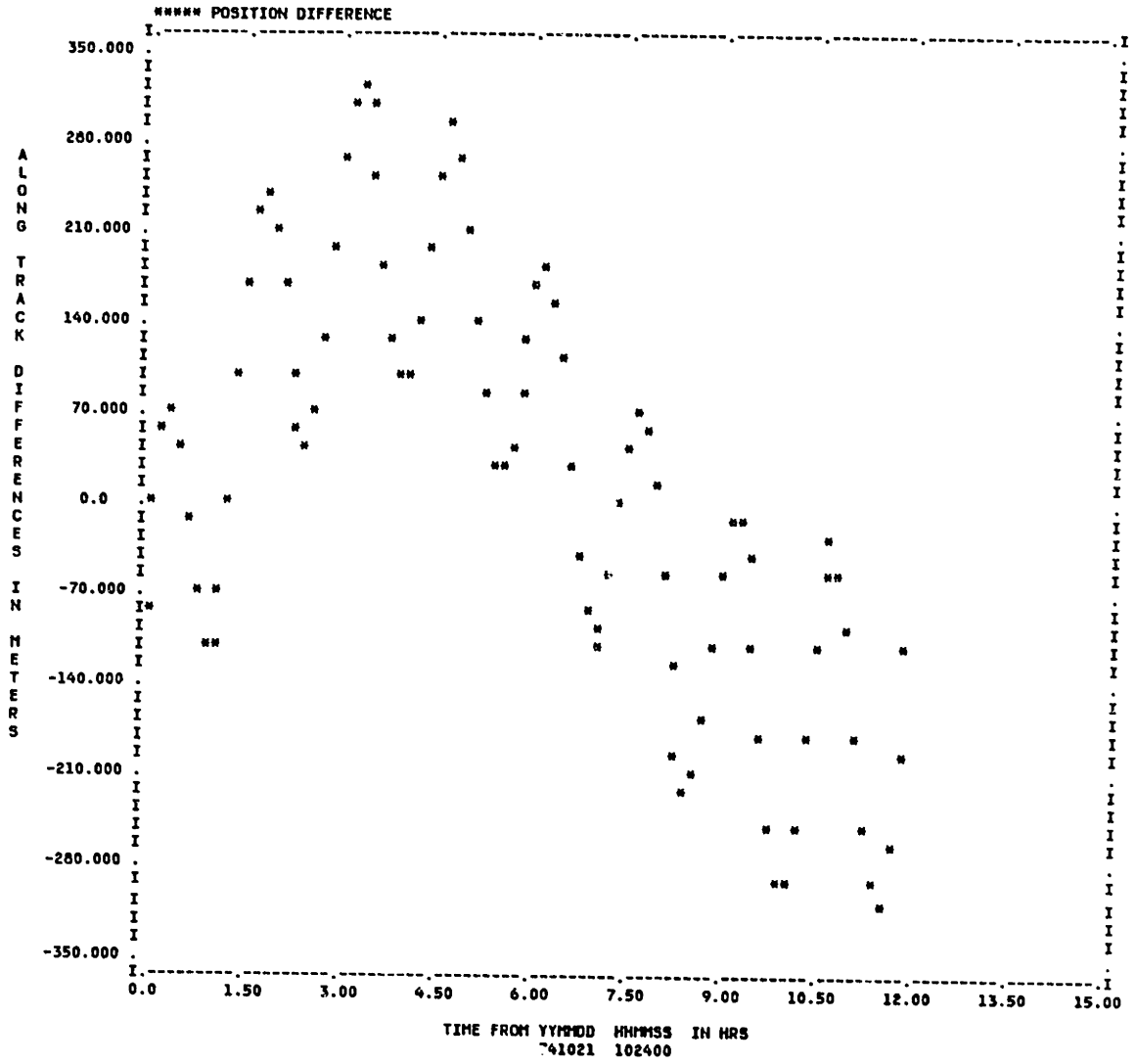
* Initial conditions from the semianalytical batch filter with states and C_D as the solve-for vector (see Table 3-34). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-13. Along Track Difference over 72 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



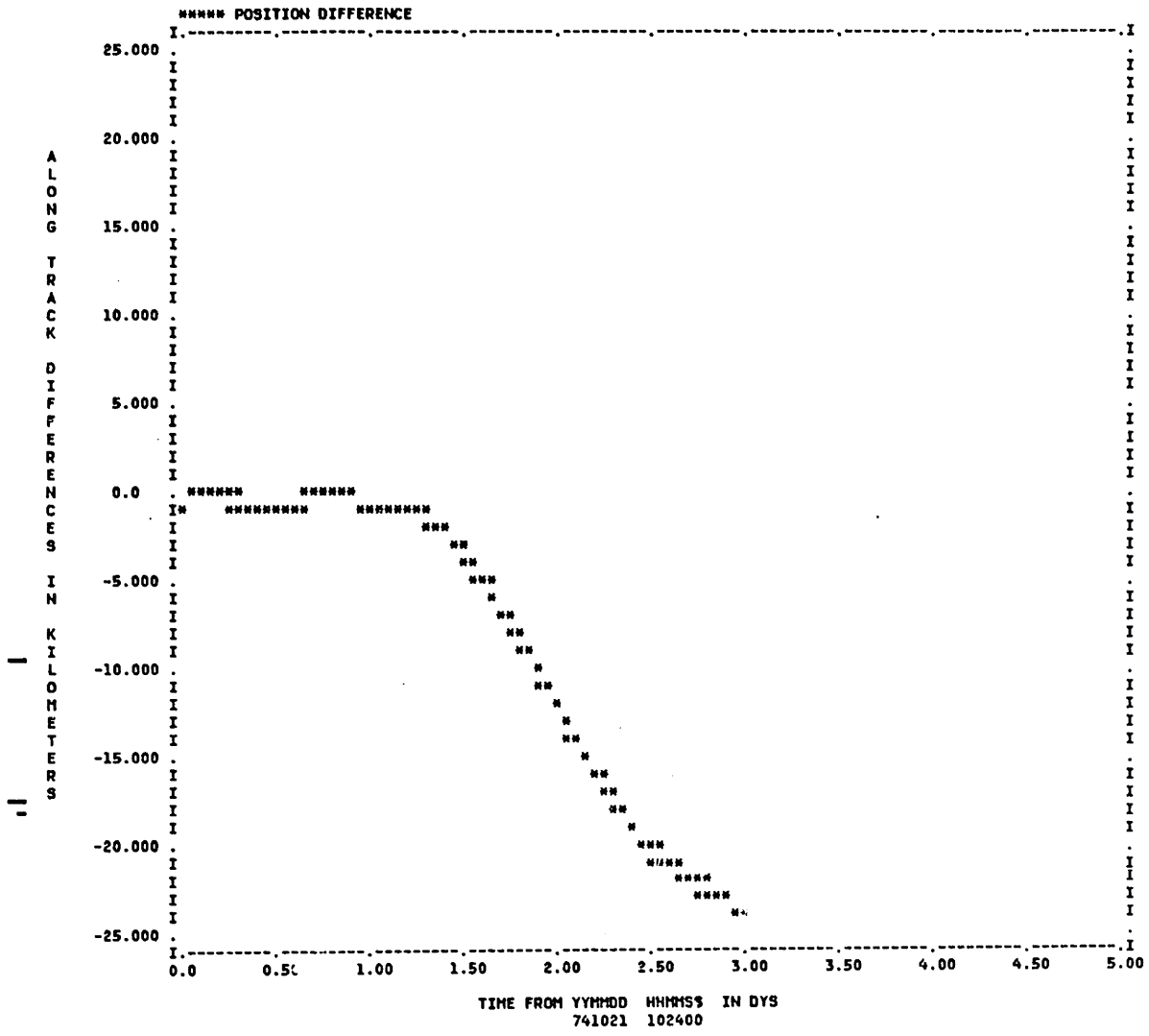
* Initial conditions from the semianalytical batch filter with states and C_D as the solve-for vector (see Table 3-34). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-14. Along Track Difference over 12 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



* Initial conditions from the semianalytical batch filter with states, a_1 and a_2 as the solve-for vector (see Table 3-35). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Figure 3-15. Along Track Difference over 72 Hours/Semianalytical* minus the Truth Ephemeris of Test Case Three



* Initial conditions from the semianalytical batch filter with states, a_1 and a_2 as the solve-for vector (see Table 3-35). The perturbation model for the semianalytical orbit generator is given in Table 3-21.

Chapter 4

SEMIANALYTICAL SATELLITE THEORY AND THE KALMAN FILTER

The previous chapter demonstrated that the semianalytical satellite theory can be coupled with a batch filter. A batch filter uses all the measurements over an observation span to determine the best estimate of the epoch (or any reference time) conditions. It sometimes becomes desirable to use an estimation procedure which processes the observations sequentially and computes the current estimate of a satellite's state. This is especially true for a space surveillance system which is required to operate in a real-time environment.

A sequential filter is an estimation procedure which does not need to use past measurements for the computation of the current estimate. For a linear system the optimal sequential filter is the Kalman filter (Ref. 72). The processing of observational data by a Kalman filter is done presently in only a few operational OD systems due to real-time requirements and/or the sequential nature of this estimation procedure. If the semianalytical satellite theory is to be applicable to all aspects of an OD system, it is necessary to couple the semianalytical satellite theory with the Kalman filter. Combining the semianalytical satellite theory with the Kalman filter may improve the accuracy and efficiency of current OD systems.

This chapter will present the algorithmic structure for coupling the semianalytical satellite theory with the Kalman filter. First a brief, heuristic review of the Kalman filter is presented. A Semianalytical Linearized Kalman filter is developed and discussed next.

4.1 Review of the Kalman Filter

Since this research is concerned with using the Kalman filter with an orbit determination process, it will be assumed that the system dynamics are continuous and that the observational data is received in a discrete fashion. In this section, the Kalman filter for a linear system with discrete measurements will be discussed first. Next the Linearized Kalman filter and Extended Kalman filter will be briefly reviewed.

For a linear system, the system dynamics are assumed to be of the form

$$\dot{\underline{x}} = A\underline{x} + \underline{\omega}(t) \qquad (4\text{-}1a)$$

where $\underline{x}$ is the actual state, $\underline{\omega}(t)$ is the noise vector and A is the system matrix. The noise vector is assumed to have a normal distribution with zero mean and a covariance of $Q(t)$. This is represented as

$$\underline{\omega}(t) \sim N(O,Q(t)) \tag{4-1b}$$

The measurement model is assumed to be of the form

$$\underline{O}_a(t_k) = H_k\underline{x}_k + \underline{v}_k \tag{4-2a}$$

$$\underline{v}_k \sim N(O,R_k) \tag{4-2b}$$

The basic concept of a Kalman filter is that given an a priori estimate of the state at t_k, $\hat{\underline{x}}(t_k-)$, an updated estimate, $\hat{\underline{x}}(t_k+)$, can be obtained based only on $\hat{\underline{x}}(t_k-)$ and $\underline{O}_a(t_k)$. This can be written as

$$\hat{\underline{x}}(t_k+) = K'_k\,\hat{\underline{x}}(t_k-) + K_k\,\underline{O}_a(t_k) \tag{4-3}$$

where K'_k and K_k are time-varying weighting matrices that must be determined. In order to determine K'_k and K_k, the following conditions are placed upon the filter

$$E[\delta\hat{\underline{x}}(t_k+)] = 0 \tag{4-4a}$$

$$\text{minimize } E[\delta\hat{\underline{x}}^T(t_k+)\,\delta\hat{\underline{x}}(t_k+)] \tag{4-4b}$$

where

$$\delta\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k+) - \underline{x}(t_k) \tag{4-5}$$

and where $\hat{\underline{x}}(t_k+)$ is the updated estimate, $\underline{x}(t_k)$ is the actual state and E[d] denotes the expected or mean value of d. Equation (4-4a) implies that the expected value of $\hat{\underline{x}}(t_k+)$ equals the expected value of $\underline{x}(t_k)$. Equation (4-4b) implies that the filter will give an estimate that makes the length of the error vector, $\hat{\underline{x}}(t_k+) - \underline{x}(t_k)$, minimum. The covariance matrix of $\delta\hat{\underline{x}}(t_k)$ is defined as

$$P \equiv E[\delta\hat{\underline{x}}(t_k)\ \delta\underline{x}^T(t_k)] \tag{4-6}$$

It is noted that P is a symmetric matrix and that the trace of P is called the variance of $\delta\hat{\underline{x}}(t_k)$. It is also noted that

$$\text{Trace}(P) = E[\delta\hat{\underline{x}}^T(t_k)\,\delta\hat{\underline{x}}(t_k)] \tag{4-7}$$

It becomes obvious that the desired Kalman filter [Equations (4-3), (4-4a) and (4-4b)] is unbiased and minimum variance.

From Equation (4-5) it is seen that

$$\begin{aligned} \hat{\underline{x}}(t_k+) &= \delta\hat{\underline{x}}(t_k+) + \underline{x}(t_k) \\ \hat{\underline{x}}(t_k-) &= \delta\hat{\underline{x}}(t_k-) + \underline{x}(t_k) \end{aligned} \tag{4-8}$$

Substitution of Equation (4-8) into Equation (4-3) yields after some rearranging

$$\delta\hat{\underline{x}}(t_k+) = [K'_k + K_k H_k - I]\ \underline{x}(t_k)$$

$$+ K'_k\ \delta\hat{\underline{x}}(t_k-) + K_k \underline{v}_k$$

Taking the expectation of the above expression and noting that

$$E[\delta\hat{\underline{x}}(t_k+)] = E[\delta\hat{\underline{x}}(t_k-)] = E[\underline{v}_k] = 0$$

yields

$$0 = [K'_k + K_k H_k - I]\ E[\underline{x}(t_k)]$$

Since $E[\underline{x}(t_k)]$ does not necessarily equal zero, it follows that

$$K'_k = I - K_k H_k \tag{4-9}$$

Equation (4-5) can be rewritten as

$$\delta\hat{\underline{x}}(t_k+) = [I - K_k H_k]\hat{\underline{x}}(t_k-) + K_k[H_k\ \hat{\underline{x}}(t_k+) + \underline{v}_k]$$

which reduces to

$$\delta\hat{\underline{x}}(t_k+) = [I - K_k H_k]\ \delta\hat{\underline{x}}(t_k-) + K_k \underline{v}_k \tag{4-10}$$

From Equations (4-6) and (4-10) it follows that

$$P(t_k+) = (I - K_kH_k)\, E[\delta\underline{\hat{x}}(t_k-)\delta\underline{\hat{x}}^T(t_k-)]\,(I - K_kH_k)^T$$

$$+ K_k\, E[\underline{v}_k\underline{v}_k^T]\, K_k^T + (I - K_kH_k)\, E[\delta\underline{\hat{x}}(t_k-)\, \underline{v}_k^T]\, K_k^T$$

$$+ K_k\, E[\underline{v}_k\, \delta\underline{\hat{x}}^T(t_k-)]\, (I - K_kH_k)^T$$

Assuming that the measurement errors are uncorrelated with $\delta\underline{\hat{x}}(t_k-)$ and that $E[\underline{v}_k\underline{v}_k^T] = R_k$, the above expression reduces to

$$P(t_k+) = (I - K_kH_k)\, P(t_k-)\, (I - K_kH_k)^T + K_kR_kK_k^T \qquad (4\text{-}11a)$$

which yields

$$P(t_k+) = P(t_k-) - K_kH_kP(t_k-) - P(t_k-)\, H_k^TK_k^T + K_kH_kP_k(t_k-)\, H_k^TK_k^T + K_kR_kK_k^T \qquad (4\text{-}11b)$$

It can be seen from Equations (4-4b) and (4-7) that the second filter criterion can be expressed as minimizing the trace of $P(t_k+)$. The following relations with regard to the operation of the trace of a matrix are noted

$$\text{Trace}[A+B] = \text{Trace}[A] + \text{Trace}[B] \qquad (4\text{-}12a)$$

$$\text{Trace}[A^T] = \text{Trace}[A] \qquad (4\text{-}12b)$$

$$\frac{\partial [\text{Trace}(ABA^T)]}{\partial A} = 2AB \quad (B \text{ is symmetric}) \qquad (4\text{-}12c)$$

$$\frac{\partial [\text{Trace}(BA^T)]}{\partial A} = B \qquad (4\text{-}12d)$$

Applying the filter conditions as given in Equation (4-4b) results in

$$\frac{\partial \{\text{Trace}[P(t_k+)]\}}{\partial K_k} = 0 \qquad (4\text{-}13)$$

By use of Equations (4-11b) and (4-12), it is seen that

$$K_k = P(t_k-)H_k^T [H_k P(t_k-)H_k^T + R_k]^{-1} \qquad (4\text{-}14)$$

Substitution of Equation (4-14) into Equation (4-11) gives the updated covariance matrix as

$$P(t_k+) = [I - K_k H_k] P(t_k-) \qquad (4\text{-}15)$$

The general form of the Kalman filter is obtained by substituting Equation (4-9) into Equation (4-3). This results in

$$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K_k[\underline{O}_a(t_k) - H_k\hat{\underline{x}}(t_k-)] \qquad (4\text{-}16)$$

The next problem that must be solved is the propagation of the updated state and covariance matrix to the next observation

time. Taking the expectation of each side of Equation (4-1a) and noting that $E[\dot{\underline{x}}] = \frac{d}{dt}[E[\underline{x}]] = \dot{\hat{\underline{x}}}$ and that $E[\underline{\omega}(t)] = 0$, it is seen that

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} \tag{4-17}$$

Equation (4-17) can be used to propagate the state forward by a numerical integration technique. Equation (4-17) can also be solved by use of the state transition matrix, i.e.,

$$\hat{\underline{x}}(t) = \Phi(t,t_o)\,\hat{\underline{x}}(t_o) \tag{4-18a}$$

where

$$\dot{\Phi}(t,t_o) = A\,\Phi(t,t_o) \tag{4-18b}$$

$$\Phi(t_o,t_o) = I \tag{4-18c}$$

It is noted that the state transition matrix has the following important characteristics (Ref. 72)

$$\Phi(t_2,t_o) = \Phi(t_2,t_1)\,\Phi(t_1,t_o) \tag{4-18d}$$

$$\Phi^{-1}(t_k,t_o) = \Phi(t_o,t_k) \tag{4-18e}$$

$$\Phi(t,t) = I \tag{4-18f}$$

The error covariance matrix is written as [Equation (4-6)]

$$P(t) = E[\delta\hat{\underline{x}}(t)\ \delta\hat{\underline{x}}^T(t)] \tag{4-19}$$

From Equation (4-5) it is seen that

$$\delta\hat{\underline{x}}(t) = \hat{\underline{x}}(t) - \underline{x}(t)$$

Therefore

$$\dot{\delta\hat{\underline{x}}}(t) = \dot{\hat{\underline{x}}}(t) - \dot{\underline{x}}(t)$$

and

$$\dot{\delta\hat{\underline{x}}} = A\ \delta\hat{\underline{x}} + \underline{\omega}(t) \tag{4-20}$$

Taking the derivative of Equation (4-19), noting that $\frac{d}{dt} = E[a] = E[\frac{da}{dt}]$ and using Equation (4-20), the following expression results

$$\begin{aligned}
\dot{P}(t) &= E[A\ \delta\hat{\underline{x}}\ \delta\hat{\underline{x}}^T] + E[\underline{\omega}(t)\delta\hat{\underline{x}}^T] \\
&\quad + E[\delta\hat{\underline{x}}\ \delta\hat{\underline{x}}^T A^T] + E[\delta\hat{\underline{x}}\ \underline{\omega}^T(t)] \\
\dot{P}(t) &= A\ P(t) + P(t)A^T + E[\underline{\omega}(t)\delta\hat{\underline{x}}^T] \\
&\quad + E[\delta\hat{\underline{x}}\ \underline{\omega}^T(t)]
\end{aligned} \tag{4-21}$$

Equation (4-20) can be solved by the variation of constant formula (Ref. 72)

$$\delta\hat{\underline{x}}(t) = \Phi(t,t_o)\delta\hat{\underline{x}}(t_o) + \int_{t_o}^{t} \Phi(t,\tau)\underline{\omega}(\tau)d\tau \qquad (4\text{-}22)$$

Post-multiplying Equation (4-22) by $\underline{\omega}^T(t)$ and taking the expectation of the resulting expression yields the following

$$E[\delta\hat{\underline{x}}(t)\ \underline{\omega}^T(t)] = \Phi(t,t_o)\ E[\delta\hat{\underline{x}}(t_o)\underline{\omega}^T(t)] + \int_{t_o}^{t} \Phi(t,\tau)\ E[\underline{\omega}(\tau)\underline{\omega}^T(t)]d\tau$$

Assuming that the system noise and initial state deviation, $\delta\hat{\underline{x}}(t_o)$, are independent implies that

$$E[\delta\hat{\underline{x}}(t)\ \underline{\omega}^T(t)] = \int_{t_o}^{t} \Phi(t,\tau)\ Q(\tau)\delta(t-\tau)d\tau$$

where $\delta(t-\tau)$ is the Dirac-Delta function. Noting that (Ref. 72)

$$\int_a^b f(x)\ \delta(b - x)dx = \frac{1}{2} f(b) \qquad (4\text{-}23)$$

results in

$$E[\delta\hat{\underline{x}}(t)\ \underline{\omega}^T(t)] = \frac{1}{2}\Phi(t,t)\ Q(t) = \frac{1}{2} Q(t) \qquad (4\text{-}24a)$$

In a similar manner it can be shown that

$$E[\underline{\omega}(t)\ \delta\hat{\underline{x}}^T(t)] = \frac{1}{2} Q(t) \tag{4-24b}$$

Therefore Equation (4-21) becomes

$$\dot{P}(t) = AP(t) + P(t)A^T + Q(t) \tag{4-25}$$

The error covariance matrix can be propagated by a numerical integration of Equation (4-25). The homogeneous solution of Equation (4-25) is

$$P(t) = \Phi(t,t_o)\ P(t_o)\ \Phi^T(t,t_o) \tag{4-26}$$

where $\Phi(t,t_o)$ is defined in Equations (4-18b) through (4-18f). Equation (4-26) can be shown to be correct by taking the time derivative and reducing this to the homogeneous part of Equation (4-25), i.e.,

$$\begin{aligned}
\dot{P}(t) &= \dot{\Phi}(t,t_o)\ P(t_o)\ \Phi^T(t,t_o) \\
&\quad + \Phi(t,t_o)\ P(t_o)\ \dot{\Phi}^T(t,t_o)
\end{aligned}$$

$$\begin{aligned}
\dot{P}(t) &= A\ \Phi(t,t_o)\ P(t_o)\ \Phi^T(t,t_o) \\
&\quad + \Phi(t,t_o)\ P(t_o)\ \Phi^T(t,t_o)\ A^T
\end{aligned}$$

$$\dot{P}(t) = A\ P(t) + P(t)\ A^T$$

By use of the variation of constants formula it can be shown that the solution to Equation (4-25) is (Ref. 72)

$$P(t) = \Phi(t,t_o)\, P(t_o)\, \Phi^T(t,t_o) + \int_{t_o}^{t} \Phi(t,\tau)\, Q(\tau)\, \Phi^T(t,\tau)\, d\tau \qquad (4\text{-}27)$$

A summary of the Kalman filter for a linear system is given in Table 4-1 (Ref. 72). The Kalman filter as given in Table 4-1 must be modified for non-linear systems. Two such modifications are the Extended Kalman filter (EKF) and the Linearized Kalman filter (LKF).

The Linearized Kalman filter results when the system dynamics and measurement models are linearized about a nominal trajectory, $\underline{x}_N(t)$. The nominal trajectory can be obtained from a reduced dynamic model or from a numerical integration of the equations of motion. The system model can be expressed functionally as

$$\dot{\underline{x}} = f(\underline{x},t) + \underline{\omega}(t) \qquad (4\text{-}28)$$

The estimated dynamics are assumed to be

$$\dot{\hat{\underline{x}}} = \underline{f}(\hat{\underline{x}},t) \qquad (4\text{-}29)$$

Table 4-1

Kalman Filter for a Linear System with Discrete Measurements

System Model:	$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{\omega}(t) \qquad \underline{\omega}(t) \sim N(O,Q(t))$
Measurement Model:	$\underline{O}_a(t_k) = H_k\,\underline{x}(t_k) + \underline{v}_k \qquad \underline{v}_k \sim N(O,R_k)$
Initial Conditions:	$\underline{x}(0) \sim N(\hat{\underline{x}}_o, P_o)$
Assumptions:	$E[\underline{v}_k\ \delta\hat{\underline{x}}^T(t_k-)] = 0 \qquad \delta\hat{\underline{x}}(t_k) = \hat{\underline{x}}(t_k) - \underline{x}(t_k)$
	$E[\underline{\omega}(t)\ \delta\hat{\underline{x}}^T(t_o)] = 0 \qquad E[\underline{\omega}(t)\ \underline{v}_k^T] = 0$
State Estimate Propagation:	$\dot{\hat{\underline{x}}} = A\hat{\underline{x}}$ $\hat{\underline{x}}(t) = \Phi(t,t_o)\ \hat{\underline{x}}(t_o)$
Error Covariance Propagation:	$\dot{P}(t) = AP(t) + P(t)A^T + Q(t)$ $P(t) = \Phi(t,t_o)\ P(t_o)\ \Phi^T(t,t_o) + \int_{t_o}^{t} \Phi(t,\tau)\ Q(\tau)\ \Phi^T(t,\tau)\,d\tau$
State Estimate Update:	$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K_k[\underline{O}_a(t_k) - H_k\ \hat{\underline{x}}(t_k-)]$
Error Covariance Update:	$P(t_k+) = [I - K_kH_k]\ P(t_k-)$
Gain Matrix:	$K_k = P(t_k-)\ H_k^T[H_k\ P(t_k-)H_k^T + R_k]^{-1}$
State Transition Matrix:	$\dot{\Phi}(t,t_o) = A\ \Phi(t,t_o)$ $\Phi(t_o,t_o) = I$

Now linearizing about the nominal trajectory yields

$$\dot{\hat{\underline{x}}} = \underline{f}(\underline{x}_N, t) + \left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]_{\underline{x}_N} (\hat{\underline{x}} - \underline{x}_N) + O[(\hat{\underline{x}} - \underline{x}_N)^2] \tag{4-30}$$

The following quantities are defined

$$F \equiv \left[\frac{\partial \underline{f}(\underline{x}, t)}{\partial \underline{x}}\right]_{\underline{x}_N} \tag{4-31a}$$

$$\Delta\hat{\underline{x}} = \hat{\underline{x}} - \underline{x}_N \tag{4-31b}$$

Equation (4-30) can therefore be written to first order in $\Delta\hat{\underline{x}}$ as

$$\dot{\Delta\hat{\underline{x}}} = F\, \Delta\hat{\underline{x}} \tag{4-32}$$

The actual measurement model can be expressed as

$$\underline{O}_a(t_k) = \underline{h}_k[\underline{x}(t_k)] + \underline{v}_k \tag{4-33a}$$

The best estimate of the measurement model is

$$\underline{O}_c(t_k) = \underline{h}_k[\hat{\underline{x}}(t_k)] \tag{4-33b}$$

where $\underline{O}_c(t_k)$ represents the computed observations. Linearizing about the nominal trajectory yields

$$\underline{Z}(t_k) = H_k \, \Delta\hat{\underline{x}} \tag{4-34a}$$

where

$$H_k \equiv \left[\frac{\partial \underline{h}_k[\underline{x}(t_k)]}{\partial \underline{x}}\right]_{\underline{x}_N} \tag{4-34b}$$

$$\underline{Z}(t_k) \equiv \underline{O}_a(t_k) - \underline{h}_k[\underline{x}_N(t_k)] \tag{4-34c}$$

Equations (4-32) and (4-34a) are in linear form so the Kalman filter as given in Table 4-1 applies. A summary of the Linearized Kalman filter is given in Table 4-2 (Ref. 72). The main advantage of the LKF is that once the nominal trajectory has been obtained, the error covariance matrix and Kalman gain matrix are defined (assuming that the observation model is known for each observation point in advance). This means that P(t) and K_k can be obtained off-line, i.e., prior to when the actual estimation process is required. A major disadvantage of the LKF is that over long observation spans, the nominal trajectory can vary greatly from the actual trajectory causing erroneous estimates.

Table 4-2

Linearized Kalman Filter for a Non-Linear System with Discrete Measurements

System Model: $\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t),t] + \underline{\omega}(t) \quad \underline{\omega}(t) \sim N(0,Q(t))$

Measurement model: $\underline{O}_a(t_k) = \underline{h}_k[\underline{x}(t_k)] + \underline{v}_k \quad \underline{v}_k \sim N(0,R_k)$

Initial Conditions: $\underline{x}(0) \sim N(\hat{\underline{x}}_o, P_o)$

Assumptions: $E[\underline{\omega}(t)\underline{v}_k^T] = 0,\ E[\underline{v}_k(\Delta\hat{\underline{x}}(t_k-) - \Delta\underline{x}(t_k-))^T] = 0$

$E[\underline{\omega}(t)(\Delta\hat{\underline{x}}(t_o) - \Delta\underline{x}(t_o))^T] = 0,$

$\Delta\hat{\underline{x}}(t) \equiv \hat{\underline{x}}(t) - \underline{x}_N(t)$

State Estimate Propagation: $\Delta\dot{\hat{\underline{x}}} = F\Delta\hat{\underline{x}}\ ,\ \Delta\hat{\underline{x}}(t) = \Phi(t,t_o)\ \Delta\hat{\underline{x}}(t_o)$

$\dot{\hat{\underline{x}}} = \underline{f}(\underline{x}_N,t) + F[\hat{\underline{x}}(t) - \underline{x}_N(t)]$

Error Covariance Propagation: $P(t) = \Phi(t,t_o)\ P(t_o)\ \Phi^T(t,t_o) + \int_{t_o}^{t} \Phi(t,\tau)Q(\tau)\ \Phi^T(t,\tau)d\tau$

$\dot{P}(t) = F\ P(t) + P(t)\ F^T + Q(t)$

State Estimate Update: $\Delta\hat{\underline{x}}(t_k+) = \Delta\hat{\underline{x}}(t_k-) + K_k[\underline{O}_a(t_k) - \underline{h}_k[\hat{\underline{x}}(t_k-)]]$

$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K_k[\underline{O}_a(t_k) - \underline{h}_k[\hat{\underline{x}}(t_k-)]]$

$\underline{h}_k[\hat{\underline{x}}(t_k-)] = \underline{h}_k[\underline{x}_N(t_k)] + H_k\ \Delta\hat{\underline{x}}(t_k-)$

Table 4-2
(cont.)

Error Covariance Update:

$$P(t_k^+) = [I - K_k H_k]\, P(t_k^-)$$

Gain Matrix:

$$K_k = P(t_k^-)\, H_k^T [H_k\, P(t_k^-)\, H_k^T + R_k]^{-1}$$

Matrix Definitions:

$$F = \left[\frac{\partial \underline{f}[\underline{x}(t),t]}{\partial \underline{x}}\right]_{\underline{x}_N(t)}$$

$$H_k = \left[\frac{\partial \underline{h}_k[\underline{x}(t)]}{\partial \underline{x}}\right]_{\underline{x}_N(t)}$$

State Transition Matrix:

$$\dot{\Phi}(t,t_o) = F\, \Phi(t,t_o)$$

$$\Phi(t_o,t_o) = I$$

Nominal Trajectory:

$$\dot{\underline{x}}_N(t) = \underline{f}[\underline{x}_N(t),t], \quad \underline{x}_N(t_o) = \hat{\underline{x}}_o$$

The Extended Kalman filter develops a nominal trajectory between observations based upon the best estimate of the state at the last observation time. Linearization about the nominal trajectory is done only for the propagation of the error covariance matrix and in the determination of the Kalman gain matrix. The propagation of the state is done by a numerical/analytical integration of Equation (4-29) with the updated value of the state at the last observation time as the initial condition. This usually results in the nominal trajectory being closer to the actual state and therefore reduces the linearization errors associated with the LKF. A summary of the EKF is shown in Table 4-3. A major disadvantage of the EKF is that the state and error covariance matrix must be integrated (usually numerically) between observation times with new initial conditions after each observation. An advantage of the EKF is that since the nominal trajectory is being improved, longer observation spans are usually possible.

It is also possible to iterate in order to improve the state estimate. Using the best estimate of state at the end of the observation span, the state is integrated back to the epoch conditions and then the estimation process is continued over the observation span until the sum of the residuals is minimum.

Table 4-3
Extended Kalman Filter for a Non-Linear
System with Discrete Measurements

System Model:	$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t),t] + \underline{\omega}(t) \quad \underline{\omega}(t) \sim N(O,Q(t))$
Measurement Model:	$\underline{O}_a(t_k) = \underline{h}_k[\underline{x}(t_k)] + \underline{v}_k \quad \underline{v}_k \sim N(O,R_k)$
Initial Conditions:	$\underline{x}(0) \sim N(\hat{\underline{x}}_o, P_o)$
Assumptions:	$E[\underline{\omega}(t)\underline{v}_k^T] = 0, \; E[\underline{v}_k(\Delta\hat{\underline{x}}(t_k-) - \Delta\underline{x}(t_k-))^T] = 0$
	$E[\underline{\omega}(t)(\Delta\hat{\underline{x}}(t_o) - \Delta\underline{x}(t_o-))^T] = 0$
	$\Delta\hat{\underline{x}}(t) \equiv \hat{\underline{x}}(t) - \underline{x}_N(t)$
State Estimate Propagation:	$\dot{\hat{\underline{x}}} = \underline{f}[\hat{\underline{x}}(t),t]$
Error Covariance Propagation:	$\dot{P}(t) = F\,P(t) + P(t)\,F^T + Q(t)$
State Estimate Update:	$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K_k[\underline{O}_a(t_k) - \underline{h}_k[\hat{\underline{x}}(t_k-)]]$
Error Covariance Update:	$P(t_k+) = [I - K_kH_k]\,P(t_k-)$
Gain Matrix:	$K_k = P(t_k-)\,H_k^T[H_k\,P(t_k-)H_k^T + R_k]^{-1}$

Table 4-3
(cont.)

Matrix Definitions:

$$F = \left[\frac{\partial \underline{f}[\underline{x}(t),t]}{\partial \underline{x}}\right]_{\underline{x}_N(t)}$$

$$H_k = \left[\frac{\partial \underline{h}_k[\underline{x}(t)]}{\partial \underline{x}}\right]_{\underline{x}_N(t_k)}$$

Nominal Trajectory:

$$\dot{\underline{x}}_N(t) = \underline{f}[\underline{x}_N(t),t], \quad \underline{x}_N(t_k) = \hat{\underline{x}}(t_k+)$$

$$t_k \leq t \leq t_{k+1}$$

A problem with both the LKF and the EKF is that filter divergence can develop (erroneous estimates). Filter divergence generally occurs due to modeling errors. Because of the linearization of the system model, the error covariance matrix tends to decrease as time increases. As the error covariance matrix decreases, the filter believes it is following the actual trajectory and therefore disregards new measurements. Because of the modeling errors the actual and estimated trajectories diverge. It should also be noted that computational procedures (round-off errors) can also cause filter divergence.

4.2 The Semianalytical Linearized Kalman Filter

In this section the semianalytical satellite theory is coupled to the Linearized Kalman filter. It is felt that the coupling of these two concepts may increase the computational speed and accuracy of current OD systems. The computational speed will be increased due to the computational efficiency of the semianalytical orbit generator. Accuracy improvements may result because the linearization assumptions used in the LKF may be better satisfied due to the near-linear behavior of the mean elements over an integration step (typically one day). This development will be called a Semianalytical Linearized Kalman Filter (SLKF).

The system vector, $\underline{x}(t)$, is defined as

$$\underline{x}(t) = \begin{bmatrix} \bar{\underline{a}}^* \\ \underline{c} \end{bmatrix} \qquad (4\text{-}35a)$$

where

$$\underline{\bar{a}}^* = \begin{bmatrix} \underline{\bar{a}} \\ \bar{\lambda} \end{bmatrix} \tag{4-35b}$$

with

$$\underline{\bar{a}} = [\bar{a}, \bar{h}, \bar{k}, \bar{p}, \bar{q}]^T \tag{4-35c}$$

and where $\underline{c}$ represents the modeled solve-for parameters. The system dynamics can be represented as

$$\dot{\underline{x}}(t) = \begin{bmatrix} \bar{n}\ \underline{e}_6 + \varepsilon \underline{A}_1(\underline{\bar{a}}) \\ \hline 0 \end{bmatrix} + \underline{\omega} \tag{4-36a}$$

where $\underline{\omega}$ is the system noise associated with the formulation and where $\bar{n}$, $\underline{e}_6$ and $\underline{A}_1(\underline{\bar{a}})$ were previously defined in Equation (3-32). The system noise is assumed to have a normal distribution with zero mean and a covariance of $Q(t)$. This is represented as

$$\underline{\omega} \sim N(O, Q(t)) \tag{4-36b}$$

The vector form of the near-identity transformation is assumed to be of the following form

$$\underline{a}^* = \underline{\bar{a}}^* + \varepsilon\, \underline{\eta}_1(\underline{\bar{a}}^*) \tag{4-37}$$

where

$$\varepsilon \underline{\eta}_1(\underline{\bar{a}}^*) = \sum_{\sigma=1}^{\infty} [\varepsilon \underline{C}_\sigma \sin(\sigma\bar{\lambda}) - \varepsilon \underline{D}_\sigma \cos(\sigma\bar{\lambda})] \tag{4-38}$$

and where $\varepsilon \underline{C}_\sigma$ and $\varepsilon \underline{D}_\sigma$ are defined in Equation (3-35). It is noted that this development will only consider a single time-independent disturbing force to first order in the small parameter in both the AOG and SPG. The extension to a more general case is straightforward.

A nominal trajectory for the system vector, $\underline{x}$, is formed over the integration grid. The system vector is composed of the six mean elements and the modeled parameters, $\underline{c}$. The AOG (see Chapter 2) is used to generate a nominal trajectory for $\underline{\bar{a}}^*$ over an integration step of the semianalytical satellite theory (typically 1 day). The modeled parameters are assumed constant over the integration step for the nominal trajectory. The nominal trajectory is denoted as $\underline{x}_n$ which can be written as

$$\underline{x}_n = \begin{bmatrix} \underline{\bar{a}}^*_n \\ \underline{c} \end{bmatrix} \tag{4-39}$$

As was discussed in Chapter 2, an interpolator is used to determine the mean elements, $\underline{\bar{a}}^*_n$, at points off the integration grid.

From Table 4-2 it is seen that the linearized system matrix can be represented as

$$F = \left[\frac{\partial [\dot{\underline{x}}(t)]}{\partial \underline{x}(t)}\right]_{\underline{x}_n} \tag{4-40a}$$

which can be written as

$$F = \left[\begin{array}{c|c} \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \bar{\underline{a}}^*}\right] & \left[\frac{\partial [\dot{\bar{\underline{a}}}^*]}{\partial \underline{c}}\right] \\ \hline 0 & 0 \end{array}\right]_{\underline{x}_n} \tag{4-40b}$$

where it is understood that $[\dot{\bar{\underline{a}}}^*]$ stands for the right hand side of the averaged equations of motion. From Equations (3-50) and (3-51) it is seen that the F matrix is composed of the A and D matrices of the semianalytical partial derivatives (see Table 3-2). Equation (4-40b) can therefore be written as

$$F = \left[\begin{array}{c|c} A & D \\ \hline 0 & 0 \end{array}\right] \tag{4-41}$$

where the A and D matrix are defined in Section 3.2.1. The determination of the A and D matrices can be done as discussed in Sections 3.2.1 and 3.2.2. Much of the software that is used to compute the A and D matrices in the semianalytical batch filter can be used to calculate the F matrix. It is noted that the

F matrix is a function of the slowly varying mean elements. It should therefore be possible to evaluate the F matrix on the integration grid and use an interpolator in order to obtain the F matrix on the output/observation grid. This should greatly reduce the computational costs associated with the determination of the F matrix.

The state transition matrix is expressed as

$$\dot{\Phi}(t,t_o) = F\,\Phi(t,t_o) \tag{4-42a}$$

with

$$\Phi(t_o,t_o) = I \tag{4-42b}$$

Since the F matrix is a function of only the five slowly varying mean elements, the state transition matrix can be integrated with the large numerical integration time step associated with the semianalytical satellite theory. It should be noted that because of the interpolator for the F matrix, the integration algorithm will not require a full re-evaluation of the F matrix. In other words, at the request points of the integration algorithm for the state transition matrix, the interpolator for the F matrix can be used. In this development such an integration scheme will be called an integrator-interpolator algorithm. It is therefore expected that using an integrator-interpolator

algorithm with a large numerical integration time step should allow the determination of the state transition matrix at minimum computational cost. Due to the slowly varying nature of the state transition matrix, an interpolation algorithm can also be used to determine the state transition matrix at the output/observation times. By using Equation (4-18d) and the interpolator for the state transition matrix over the integration grid, the state transition matrix between observations can be obtained. For example, the state transition matrix between observations at t_k and t_{k+1} can be expressed as

$$\Phi(t_{k+1},t_k) = \Phi(t_{k+1},t_o)\ \Phi^{-1}(t_k,t_o) \qquad (4-43)$$

where $\Phi(t_{k+1},t_o)$ and $\Phi(t_k,t_o)$ can be determined directly from the interpolator for the state transition matrix.

The observation model is assumed to be of the form

$$\underline{O}_a(t_k) = \underline{h}_k(\underline{r}_{LT},\dot{\underline{r}}_{LT}) + \underline{v}_k \qquad (4-44a)$$

where $\underline{v}_k$ is the noise vector associated with the measurement at t_k and where $\underline{r}_{LT}$ and $\dot{\underline{r}}_{LT}$ are the position and velocity vector in the local tangent frame (see Section 2.1.1). The noise is assumed to have a normal distribution with a zero mean and a covariance of R_k. This is denoted as

$$\underline{v}_k \sim N(0, R_k) \tag{4-44b}$$

Obviously the computed observations would be expressed as

$$\underline{O}_c(t_k) = \underline{h}_k(\hat{\underline{r}}_{LT}, \dot{\hat{\underline{r}}}_{LT}) \tag{4-45}$$

From Table 4-2 it is seen that the linearized measurement matrix can be written as follows

$$H_k = \left\{ \left[\frac{\partial [\underline{O}_c(t_k)]}{\partial \bar{\underline{a}}^*} \right]_{px6} \Big| \left[\frac{\partial [\underline{O}_c(t_k)]}{\partial \underline{c}} \right]_{px(\ell-6)} \right\}_{\underline{x}_n} \tag{4-46}$$

where p is the number of observations at t_k and ℓ is the number of solve-for parameters. With the use of Equation (4-45) it is seen that

$$H_k = \left\{ \left[\frac{\partial \underline{h}_k(\underline{r}_{LT}, \dot{\underline{r}}_{LT})}{\partial \underline{a}^*} \right] \left[\left\{ \frac{\partial \underline{a}^*}{\partial \bar{\underline{a}}^*} \right\} \Big| \left\{ \frac{\partial \underline{a}^*}{\partial \underline{c}} \right\} \right] \right\}_{\underline{x}_n} \tag{4-47}$$

where $\underline{a}^*$ represents the six osculating orbital elements. By taking the partial derivatives of the near-identity transformation with respect to $\bar{\underline{a}}^*$ and $\underline{c}$, it is seen that Equation (4-47) can be written as

$$H_k = \left[\frac{\partial \underline{h}_k(\underline{r}_{LT}, \dot{\underline{r}}_{LT})}{\partial \underline{a}^*}\right]\left[I + B_1 \;\vdots\; B_4\right] \qquad (4\text{-}48)$$

where the B_1 and B_4 matrices are defined in Equation (3-44) and (3-45) [see Table 3-2 also]. When the short periodic functions are assumed to be of the form given in Table 2-1 [or Equation (4-38)], the B_1 and B_4 matrices are given in Equations (3-52) and (3-53). From Equations (3-28) and (3-30), it can be shown that

$$H_k = \left[\left\{\frac{\partial \underline{h}_k(\underline{r}_{LT}, \dot{\underline{r}}_{LT})}{\partial \underline{r}_{LT}}\right\} \vdots \left\{\frac{\partial \underline{h}_k(\underline{r}_{LT}, \dot{\underline{r}}_{LT})}{\partial \dot{\underline{r}}_{LT}}\right\}\right] \cdot$$

$$\cdot \left[\begin{array}{c|c} M_{LT}B(t)C & 0 \\ \hline M_{LT}\dot{B}(t)C & M_{LT}B(t)C \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{c} \left\{\dfrac{\partial \underline{R}}{\partial \underline{a}^*}\right\} \\ \hline \left\{\dfrac{\partial \dot{\underline{R}}}{\partial \underline{a}^*}\right\} \end{array}\right] \cdot \left[I + B_1 \;\vdots\; B_4\right] \qquad (4\text{-}49)$$

where $\underline{r}_{LT}$, $\dot{\underline{r}}_{LT}$, $\underline{R}$, $\dot{\underline{R}}$, $\underline{a}^*$, M_{LT}, B(t) and C are defined in Section 2.1.1. Sections 3.1 and 3.2 explain the determination of each of the matrices in Equation (4-49). Again it is noted that much of the same software used in the semianalytical batch filter can be used in the SLKF.

Linearization about the nominal trajectory yields the following for the variation of the system vector from the nominal trajectory (see Section 4.1)

$$\dot{\hat{\underline{x}}}(t) = F\,\hat{\Delta\underline{x}}(t) \tag{4-50}$$

where

$$\hat{\Delta\underline{x}}(t) = \hat{\underline{x}}(t) - \underline{x}_N \tag{4-51}$$

The initial condition for Equation (4-50) is zero at the beginning of the integration step [$\hat{\Delta\underline{x}}(t_o) = 0$] and equal to $\hat{\Delta\underline{x}}(t_k+)$ after the processing of the observations at t_k [see the state estimate update algorithm in Table 4-2]. The propagation of the system variation vector, $\hat{\Delta\underline{x}}(t)$, can be done by the use of an integrator-interpolator algorithm applied to Equation (4-50). Equivalently, the state transition matrix could be used to propagate $\hat{\Delta\underline{x}}(t)$ between observations. For example, between t_k and t_{k+1}, the system variation vector would be propagated by

$$\hat{\Delta\underline{x}}(t_{k+1}-) = \Phi(t_{k+1},t_k)\,\hat{\Delta\underline{x}}(t_k+) \tag{4-52}$$

where $\Phi(t_{k+1},t_k)$ is given in Equation (4-43).

In order to process the observations at t_k, it is necessary to add the short periodic variations to $\bar{\underline{a}}_N$ such that the nominal osculating orbital elements can be determined. As was demonstrated in Chapter 2, the short periodic coefficients are slowly varying and can be determined on the integration grid and an interpolator can be used to determine them at the output/observation times. The nominal values of the short periodic coefficients will be denoted as $[\varepsilon\underline{C}_\sigma]_n$ and $[\varepsilon\underline{D}_\sigma]_n$. Obviously, as the estimate of the system vector varies from the nominal trajectory, the estimate of the short periodic coefficients also varies from their nominal values. One way of obtaining the updated short periodic coefficients is to re-evaluate the short periodic coefficients (analytically or numerically) using the updated value of the system vector, $\hat{\underline{x}}(t_k)$. This could be computationally expensive if done many times. Another way of obtaining the updated short periodic coefficients is by performing a linearization about the nominal trajectory. To first order in $\Delta\hat{\underline{x}}(t)$ it is seen that

$$\hat{\varepsilon\underline{C}}_\sigma = [\varepsilon\underline{C}_\sigma]_n + \left[\frac{\partial \varepsilon\underline{C}_\sigma}{\partial \underline{x}}\right]_{\underline{x}_N} \Delta\hat{\underline{x}}(t)$$

$$\hat{\varepsilon\underline{D}}_\sigma = [\varepsilon\underline{D}_\sigma]_n + \left[\frac{\partial \varepsilon\underline{D}_\sigma}{\partial \underline{x}}\right]_{\underline{x}_N} \Delta\hat{\underline{x}}(t) \qquad (4\text{-}53)$$

where $\varepsilon\hat{\underline{C}}_\sigma$ and $\varepsilon\hat{\underline{D}}_\sigma$ represent the updated short periodic coefficients. From Table 3-2 and/or Equation (3-57) it is seen that the change in the short periodic coefficients can be expressed as

$$\begin{aligned} \Delta\varepsilon\underline{C}_\sigma &= [M_\sigma \mid 0 \mid K_\sigma]_{6x\ell}\ \Delta\hat{\underline{x}}(t) \\ \Delta\varepsilon\underline{D}_\sigma &= [N_\sigma \mid 0 \mid L_\sigma]_{6x\ell}\ \Delta\hat{\underline{x}}(t) \end{aligned} \tag{4-54a}$$

where

$$M_\sigma = \left\{\left[\frac{\partial\,\varepsilon\underline{C}_\sigma}{\partial\,\underline{\bar{a}}}\right]_{6x5}\right\}_{\underline{x}_n} \tag{4-54b}$$

$$N_\sigma = \left\{\left[\frac{\partial\,\varepsilon\underline{D}_\sigma}{\partial\,\underline{\bar{a}}}\right]_{6x5}\right\}_{\underline{x}_n} \tag{4-54c}$$

$$K_\sigma = \left\{\left[\frac{\partial\,\varepsilon\underline{C}_\sigma}{\partial\,\underline{c}}\right]_{6x(\ell-6)}\right\}_{\underline{x}_n} \tag{4-54d}$$

$$L_\sigma = \left\{\left[\frac{\partial\,\varepsilon\underline{D}_\sigma}{\partial\,\underline{c}}\right]_{6x(\ell-6)}\right\}_{\underline{x}_n} \tag{4-54e}$$

and where ℓ is the total number of solve-for parameters (including the six orbital elements). From Equation (4-53) it is seen that the updated short periodic coefficients can be expressed as

$$\hat{\varepsilon \underline{C}}_\sigma = [\varepsilon \underline{C}_\sigma]_n + \Delta\, \varepsilon \underline{C}_\sigma$$

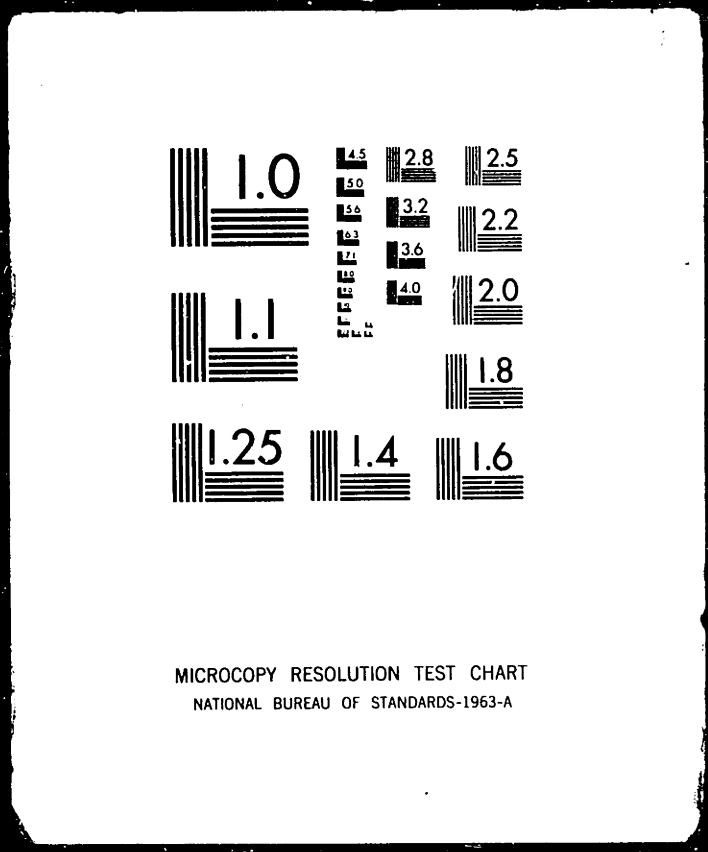
$$\hat{\varepsilon \underline{D}}_\sigma = [\varepsilon \underline{D}_\sigma]_n + \Delta\, \varepsilon \underline{D}_\sigma \qquad (4\text{-}55)$$

Since the B_1 and B_4 matrices, and therefore the M_σ, N_σ, K_σ and L_σ matrices [see Equations (3-52) and (3-53)], have to be determined in the computation of the linearized observation matrix, (H_k), the use of Equations (4-54) and (4-55) to update the short periodic coefficients may be advantageous. The determination of the A, D, M_σ, N_σ, K_σ and L_σ matrices was described in Section 3.2. The use of the quadrature approach as given in Equations (3-56) and (3-57) is interesting in that it allows the calculation of these matrices with minimum evaluations of the partial derivatives of the disturbing functions.

The calculation of the nominal osculating elements at the observation times is determined by the following

$$\underline{a}^*_n = \bar{\underline{a}}^*_n + \sum_{\sigma=1}^{\infty} \{[\varepsilon \underline{C}_\sigma]_n \sin(\sigma\bar{\lambda}) - [\varepsilon \underline{D}_\sigma]_n \cos(\sigma\bar{\lambda})\} \qquad (4\text{-}56)$$

The updated osculating elements can be written as



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$$\hat{\underline{a}}^{*} = \hat{\underline{\bar{a}}}^{*}_{N} + \Delta\underline{\bar{a}}^{*}(t) + \sum_{\sigma=1}^{\infty} \{[[\varepsilon \underline{C}_{\sigma}]_{n} + \Delta\varepsilon \underline{C}_{\sigma}] \sin(\sigma\bar{\lambda}) - [[\varepsilon \underline{D}_{\sigma}]_{n} + \Delta\varepsilon \underline{D}_{\sigma}] \cos(\sigma\bar{\lambda})\} \tag{4-57}$$

The computed observations and linearized measurement matrix are calculated using the nominal values of the orbital elements (see Table 4-2). Linearization errors associated with the measurement model may be reduced if the updated osculating orbital elements [Equation (4-57)] are used instead of the nominal osculating elements in the calculation of $\underline{h}_k(\underline{r}_{LT}, \dot{\underline{r}}_{LT})$ and H_k.

From Table 4-2 it is seen that the differential equation for the error covariance matrix can be written as

$$\dot{P}(t) = F\,P(t) + P(t)\,F^{T} + Q(t) \tag{4-58}$$

where F is the linearized system matrix and Q(t) is the covariance matrix associated with the system noise. The initial conditions for Equation (4-58) is the a priori value of the error covariance matrix at t_o [$P(t_o)$] and equal to $P(t_k+)$ after the estimation update at t_k [see the error covariance update in Table 4-2]. It is assumed that the system noise is slowly varying in the same manner as the mean elements, i.e., Q(t)

contains only secular and long periodic terms. This should not be a bad assumption since the mean elements and modeled parameters contain only secular and long periodic terms. Modeling errors in the short periodic variations will result in transformation errors and will generally not affect the system dynamics (at least to first order). This means that Q(t) should not contain short periodic terms.

Since the F matrix is slowly varying and can be obtained by an interpolator, and since Q(t) is assumed to be slowly varying, the error covariance matrix can be propagated between observations by using an integrator-interpolator algorithm applied to Equation (4-58) with the large numerical integration time step. The state transition matrix can also be used to propagate P(t). Between t_k and t_{k+1}, the error covariance matrix would be propagated by the following (see Table 4-2)

$$P(t_{k+1}^-) = \Phi(t_{k+1},t_k)\ P(t_k^+)\ \Phi^T(t_{k+1},t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\ Q(\tau)\ \Phi^T(t_{k+1},\tau)\,d\tau \qquad (4\text{-}59)$$

The matrix $\Lambda(t,t_o)$ is defined as follows

$$\Lambda(t,t_o) \equiv \int_{t_o}^{t} \Phi(t,\tau)\ Q(\tau)\ \Phi^T(t,\tau)\,d\tau \qquad (4\text{-}60)$$

Using Equation (4-60), it is seen that Equation (4-59) can be written as

$$P(t_{k+1}-) = \Phi(t_{k+1},t_k)\ P(t_k+)\ \Phi^T(t_{k+1},t_k) + \Lambda(t_{k+1},t_k) \tag{4-61}$$

$P(t_k-)$ comes from the error covariance update (see Table 4-2) and $\Phi(t_{k+1},t_k)$ is determined in Equation (4-43). The matrix $\Lambda(t_{k+1},t_k)$ could be obtained by an integrator-interpolator algorithm applied between each observation. For many observations this could be computationally expensive. It is possible to determine $\Lambda(t_{k+1},t_k)$ in what may be a more efficient manner.

Since the state transition matrix and $Q(t)$ are slowly varying matrices, $\Lambda(t,t_o)$ can be determined with an integrator-interpolator algorithm over the entire integration grid. The large numerical integration time step associated with the semi-analytical satellite theory can be used in determining $\Lambda(t,t_o)$. An interpolator can be used to determine $\Lambda(t,t_o)$ at the output/observation times. It is also known that

$$\int_{t_o}^{t_{k+1}} \Phi(t_{k+1},\tau)\ Q(\tau)\ \Phi^T(t_{k+1},\tau)d\tau = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)Q(\tau)\Phi^T(t_{k+1},\tau)d\tau$$

$$+ \int_{t_o}^{t_k} \Phi(t_{k+1},\tau)\ Q(\tau)\ \Phi^T(t_{k+1},\tau)d\tau$$

From Equation (4-60) it is seen that the above expression can be written as

$$\Lambda(t_{k+1},t_o) = \Lambda(t_{k+1},t_k) + \int_{t_o}^{t_k} \Phi(t_{k+1},\tau)\, Q(\tau)\, \Phi^T(t_{k+1},\tau)\, d\tau \tag{4-62}$$

From Equation (4-18d) it is seen that

$$\Phi(t_{k+1},\tau) = \Phi(t_{k+1},t_k)\, \Phi(t_k,\tau) \tag{4-63}$$

Substitution of Equation (4-63) into Equation (4-62) and using the definition of $\Lambda(t_k,t_o)$ [see Equation (4-60)] it is seen that

$$\Lambda(t_{k+1},t_o) = \Lambda(t_{k+1},t_k) + \Phi(t_{k+1},t_k)\, \Lambda(t_k,t_o)\, \Phi^T(t_{k+1},t_k)$$

which can be written as

$$\Lambda(t_{k+1},t_k) = \Lambda(t_{k+1},t_o) - \Phi(t_{k+1},t_k)\, \Lambda(t_k,t_o)\, \Phi^T(t_{k+1},t_k) \tag{4-64}$$

Equation (4-64) is useful in that it allows the calculation of $\Lambda(t_{k+1},t_k)$ in terms of readily obtainable quantities. $\Lambda(t_{k+1},t_o)$ and $\Lambda(t_k,t_o)$ are obtained from the interpolator for $\Lambda(t,t_o)$ over the integration grid. $\Phi(t_{k+1},t_k)$ is given in Equation (4-43). Therefore, with the use of Equations (4-61) and (4-64), the error covariance matrix can be propagated between observations.

Iterations on the integration grid are possible with the SLKF. After processing all the observational data over the integration grid of the semianalytical satellite theory, the system vector is integrated backwards to the beginning of the integration step and the estimation process is repeated. The iteration procedure would be continued until the sum of the residuals is minimum and/or no longer changes to within a specified amount. It should also be noted that the SLKF also allows iterations over the entire observation span (global iteration).

The SLKF also permits rectification on the integration grid. After all observational data is processed within an integration grid of the semianalytical satellite theory (to include integration grid iterations), a new nominal trajectory is formed for the next integration grid based upon the best estimate of the system vector at the end of the last integration grid. In other words, the best estimate of $\hat{\underline{x}}(t_f)$ and $P(t_f)$ at the end of an integration grid (t_f), is used as the initial conditions

for the nominal trajectory for the next integration grid. The short periodic coefficients, linearized system matrix, state transition matrix, linearized measurement matrix, $\Lambda(t,t_o)$ and short periodic coefficient matrices are all re-initialized and determined based upon the new nominal trajectory. This should greatly reduce the linearization errors and allow the SLKF to process long observation spans without experiencing filter divergence.

The SLKF is best explained by giving the sequence of events which would be followed in processing observational data. The following is a suggested sequence of events for the SLKF when the state transition matrix is used in the propagation of $\Delta\hat{\underline{x}}(t)$ and $P(t)$ and when the short periodic functions are formulated as given in Equation (4-38).

Integration Grid Calculations

1. Form the nominal trajectory for $\bar{\underline{a}}^*$ over an integration time step (modeled parameters are assumed constant).
   a. Integrate the mean equations of motion (AOG) with the best estimate of the mean elements and modeled parameters at the beginning of the integration grid.
   b. Form an interpolator for $\bar{\underline{a}}^*_n$ over the integration grid.

2. Determine the short periodic coefficients on the integration grid and form the interpolators for them over the integration grid.

3. Determine the F matrix and short periodic coefficient matrices (M_σ, N_σ, K_σ and L_σ) on the integration grid and form the interpolators for them over the integration grid. Note that for those perturbations that the quadrature approach is used, these matrices can be determined jointly [see Equations (3-56) and (3-57)].

4. Determine the state transition matrix on the integration grid.
   a. Use integrator-interpolator on Equation (4-42) with the large integration time step.
   b. Form interpolator over the integration grid.

5. Determine $\Lambda(t,t_o)$ on the integration grid.
   a. Use integrator-interpolator on Equation (4-60) with the large integration time step.
   b. Form interpolator over the integration grid.

Observation Processing at t_{k+1}

6. Determine by the use of the interpolators, $\underline{\bar{a}}^*_N$, $[\varepsilon \underline{C}_\sigma]_n$, $[\varepsilon \underline{D}_\sigma]_n$, M_σ, N_σ, K_σ and L_σ at the observation time, t_{k+1}.

7. Calculate $\underline{a}^*_n$ [see Equation (4-56)].

8. Determine $\Phi(t_{k+1},t_k)$ and $\Lambda(t_{k+1},t_k)$ [see Equations (4-43) and (4-64)].

9. Determine the system variation vector at the observation time, $\Delta\hat{\underline{x}}(t_{k+1}-)$ [see Equation (4-52)].

10. Determine the error covariance matrix at the observation time, $P(t_{k+1}-)$ [see Equation (4-61)].

11. Determine the computed observation, $\underline{O}_c(t_{k+1}-)$, the linearized measurement matrix, H_{k+1}, and the Kalman gain matrix, K_{k+1} [see Equations (4-45), (4-49) and (4-14)].

12. Update the system vector, $\Delta\hat{\underline{x}}(t_{k+1}+)$ [see Table 4-2].

13. Update the error covariance matrix, $P(t_{k+1}+)$ [see Table 4-2].

14. Determine updated short periodic coefficients if output and/or updated osculating orbital elements are required [see Equation (4-57)].

15. Move to next observation time and repeat steps 6 through 14.

16. After the last observation in the integration grid is processed, propagate the best estimate of the system vector and error covariance matrix to the end of the integration step (t_f).

17. If integration grid iteration is done, integrate the system vector back to the start time and repeat steps 1 through 16 until the sum of the residuals is minimized (or acceptable).

Rectification at the End of the Integration Step (t_f)

18. Determine $\hat{\underline{x}}(t_f-)$ and set this equal to $\underline{x}_N(t_o)$.

19. Determine $P(t_f-)$ and set this equal to $P(t_o)$.

20. Set $\Phi(t_f,t_f) = \Phi(t_o,t_o) = I$.

21. The short periodic coefficients at t_f are calculated using $\hat{\underline{x}}(t_f-)$ and set equal to $[\varepsilon \underline{C}_\sigma(t_o)]_n$ and $[\varepsilon \underline{D}_\sigma(t_o)]_n$.

22. Repeat steps 1 through 21 for next integration grid.

The above procedure is continued until the end of the observation span is reached. If a global iteration procedure is used, the best estimate of the system vector at the end of the observation span is integrated back to the epoch time and steps 1 through 22 are repeated. This is continued until the sum of the residuals is minimized (or acceptable).

If the short periodic formulation as given in Equation (4-38) is not used and/or if analytical expressions for the B_1 and B_4 matrices are used, the calculation of the M_σ, N_σ, K_σ and L_σ matrices would not be necessary. The B_1 and B_4 matrices would be calculated at each observation time when the linearized measurement matrix is determined (step 11 in the above procedure). It should also be possible to use reduced perturbation models in the generation of the F, B_1 and B_4 matrices. For example, it may be possible to consider J_2 perturbations only in the F matrix and neglect the B_1 and B_4 matrices.

It is felt that the above development has coupled the semianalytical satellite theory with the Kalman filter. The

performance of current OD systems may be increased with the use of a Semianalytical Linearized Kalman filter.

Chapter 5

CONCLUSIONS AND FUTURE WORK

This research has been concerned with adding new capabilities to the semianalytical satellite theory and combining the theory with current estimation procedures such that orbit determination and prediction processes for low altitude satellites are improved. Enhancements to the semianalytical satellite theory include a short periodic development which is applicable to all types of perturbations (conservative and non-conservative) and a second order drag theory which allows a complete treatment of all second order drag effects. The semianalytical satellite theory has been combined with a batch filter (least squares differential correction algorithm) such that the epoch mean orbital elements and various parameters within the drag formulation can be determined. The semianalytical batch filter necessitated the development and implementation of a semianalytical theory for the partial derivatives of perturbed motion. The solve-for parameters within the drag formulation include the drag coefficient and adaptive parameters within the Adaptive Modified Harris-Priester density model. An algorithmic structure for coupling the semianalytical satellite theory with the Kalman filter has also been developed.

5.1 Conclusions

It is concluded that the semianalytical satellite theory can be used in an orbit determination (OD) system to accurately determine and predict satellite ephemerides. It is further concluded that a semianalytical OD system can accurately and efficiently handle drag perturbed satellites. The accuracy of the semianalytical orbit generator and semianalytical batch filter is comparable to special perturbation techniques, i.e., Cowell orbit generator and Cowell batch filter.

The first order time-independent short periodic formulation, as given in Section 2.2.1, is a valid means of determining the short periodic variations due to zonal terms of the central body gravitational field, third body effects for low altitude satellites, atmospheric drag and solar radiation pressure. The coupling in the short periodic formulation (see Table 2-1) between the semimajor axis and the satellite's fast variable is a major first order effect and must be considered for accurate orbit generation. The time-independent short periodic formulation permits both analytical and numerical implementations. The numerical approach has been implemented into the Research and Development version of the Goddard Trajectory Determination System (GTDS-RD). The numerical approach for drag and solar radiation pressure perturbations allows the consideration of complicated density models and/or spacecraft solar radiation pressure models.

The time-independent short periodic coefficients ($\varepsilon C_{i\sigma}$ and $\varepsilon D_{i\sigma}$, see Table 2-1) are functions of the five slowly varying mean elements and are therefore slowly varying themselves. This fact makes it possible to determine the short periodic coefficients on the integration grid of the semianalytical satellite theory and use an interpolator to determine them at the output/observation time within the integration grid. This should significantly reduce the computational costs associated with the short periodic generator.

For the low altitude near-circular test case considered in Section 2.4.3, the drag short periodic variation was a 35 meter effect in the semimajor axis and can cause up to a 50 meter effect in the along track error. It is concluded, at least for the test cases examined in this research, that the drag short periodic variation must be considered for precision orbit generation.

The weak time-dependent short periodic formulation, as given in Section 2.2.2, is required in order to achieve high accuracy for third body perturbations on medium altitude satellites (such as GPS). The weak time-dependent short periodic variations are obtained by including corrections to the time-independent short periodic coefficients. Both analytical and numerical methods are possible for the calculation of the weak time-dependent short periodic coefficients. For the test case

considered in Section 2.4.5, it appears that the lunar weak time-dependent short periodic coefficient can also be determined on the integration grid of the semianalytical satellite theory and an interpolator can be used to determine them at the output times.

The dependence of the drag and solar radiation pressure short periodic variations on the solar motion was shown to be very small for the test cases considered. It appears that these perturbations can be considered adequately by a time-independent short periodic formulation.

For the osculating to mean conversion, the Precision Conversion of Elements (PCE) initialization procedure coupled with an accurate Short Periodic Generator (SPG) is superior to the Numerical Osculating to Mean (NOM) initialization procedure (see Sections 2.4.2 and 2.4.3). It also appears that the Epoch Point Conversion (EPC) with an accurate SPG will be an accurate and efficient means of converting initial osculating elements to initial mean elements. It is felt that the EPC procedure will be superior to the NOM procedure.

The generalized method of averaging permits a systematic approach to the second order averaged equations of motion based upon the first order short periodic functions. This is particularly useful in the investigation of the second order effects

of oblateness and drag. This research has shown that the second order oblateness/drag effect can be broken into four parts: J_2-drag, drag-J_2, J_2^2 and drag-squared effects. Analytical expressions good to zeroth order in the eccentricity and for direct orbits only have been developed for the drag-J_2 effect. These results are given in Appendix D. The second order oblateness/drag effects, including both numerical averaging and analytical drag-J_2 formulations, have been implemented into GTDS-RD. The implementation into GTDS-RD is completely general with respect to the density model.

This second order treatment of the oblateness/drag effects is comparable, in purpose, to the canonical approaches of Brouwer-Hori (Ref. 48), Lane (Ref. 50), Sherrill (Ref. 53), Willey and Pisacane (Ref. 51), Watson, Mistretta and Bonavito (Ref. 54), and Scheifele, Mueller and Starke (Refs. 55, 76). The present work also extends the treatment of the drag perturbed orbits given by Lutsky and Uphoff (Ref. 20) and Long and McClain (Ref. 12).

For the low altitude near-circular case considered, the J_2-drag effect is the dominant second order oblateness/drag effect. Izsak's J_2 height (radial) correction, applied to the density calculations, gives a good approximation to the J_2-drag effect. It has also been shown that the drag-J_2 effect must be considered for high accuracy orbit generation. For a near

circular satellite at a height of 200 kilometers, the drag-squared effect is small and can be neglected over a short arc. The present numerical testing with the Modified Harris-Priester atmosphere demonstrates the capability to include complex density variations with height, oblate atmosphere, diurnal, atmospheric rotation, and $\overline{F}_{10.7}$ variations.

The net result of this research effort is that the semianalytical satellite theory as implemented can generate an ephemeris for the low altitude circular test case of Section 2.4.3 that will vary from a precision Cowell generated ephemeris by less than 15 meters at the end of a one day time span (500 meters at the end of a 5 day time span).

A semianalytical theory for the partial derivatives of perturbed motion has been developed and implemented into GTDS-RD. The semianalytical partial derivatives are composed of an Average Partial Generator (APG) and Short Periodic Partial Generator (SPPG). The APG integrates, with the large numerical integration time step, a matrix differential equation of the mean semianalytical partial derivatives. The SPPG determines the semianalytical partial derivatives of the short periodic variations at the output/observation times. The perturbation models of the APG and SPPG can be different from each other. The semianalytical partial derivatives can be determined by analytical and/or numerical methods. The semianalytical partial

derivatives, as implemented in this research, have been shown to be equivalent to high precision Cowell generated partial derivatives with the same perturbation models.

For the test case considered in Section 3.2.2, it appears that for accurate semianalytical partial derivatives, J_2 effects must be considered in the SPPG and the APG must include at least J_2 effects and drag effects with Izsak's height correction applied to the density calculations (approximation to the J_2-drag effects). Drag perturbations do not need to be considered in the SPPG.

A semianalytical batch filter has been developed and implemented into GTDS-RD. The semianalytical batch filter can solve for the six mean epoch orbital elements and parameters within the drag formulation. The drag parameters that can be solved for are the coefficient of drag (C_D) and adaptive parameters (a_1 through a_5) within the Adaptive Modified Harris-Priester Atmosphere (see Appendix C). For the same estimation option, the semianalytical batch filter, as implemented in this research, has been shown to be equivalent to a Cowell batch filter.

For the test cases considered in Section 3.3, the states plus C_D estimation option was superior to the states only estimation option over both the observation span and the prediction

span. The states plus a_1 and a_2 estimation option was better than the states plus C_D estimation option over the observation span. Over the prediction span used in Section 3.3, the states plus C_D estimation option gave the best results of all the estimation options considered. The other adaptive density estimation options (see Table 3-6) were not as good as the states plus a_1 and a_2 estimation option for the test cases considered.

It is felt that this research has shown that adaptive estimation of the drag formulation is a means of improving the performance of OD systems designed for near-Earth satellites. Adaptive estimation of the drag coefficient, which is presently used by most operational OD systems, does significantly improve the orbit determination and prediction of low altitude satellites. Adaptive estimation of the density model also appears to be a feasible and viable means of accounting for uncertainties in the drag formulation. Adaptive estimation of different density models and using longer observation spans may show dramatic improvements over the results obtained by this research.

The algorithmic structure for developing the Semianalytical Linearized Kalman filter (SLKF) has been presented. The system equations of motion, short periodic coefficients, linearized system matrix, state transition matrix and covariance matrix are functions of only the five slowly varying mean elements and therefore need only to be evaluated on the large

integration grid of the semianalytical satellite theory. Interpolators can be used to obtain these quantities at the output/observation times within the integration grid. The SLKF demonstrates that the semianalytical satellite theory can be gracefully interfaced with sequential as well as batch estimation procedures.

5.2 Future Work

This research has investigated the concept of developing a complete operational OD system based upon the semianalytical satellite theory. It is felt that this research has demonstrated this concept and has also indicated areas in which future work needs to be done in order to improve the performance of the semianalytical OD system. These areas of future work are discussed below.

Additional work that should be performed in the short periodic generator includes the implementation of analytical short periodic variations of the central body, including m-dailies, and third-body gravitational effects based upon Reference 5. Zeis' J_2^2 short periodic variations should either be extended to a higher order in the eccentricity or be replaced by closed-form expressions. Eventually, the J_2^2 formulation should be extended to the retrograde case. The determination of the short periodic coefficients should be put on the integration grid of the semianalytical satellite theory to improve

computational speed. The analytical short periodic formulation, as given in Reference 5, should be reformulated in terms of short periodic coefficients that are functions of the five slowly varying elements. The short periodic variation may need to be reformulated in terms of another geometric variable or orthogonal function such that the drag short periodic variation associated with a higher eccentricity orbit can be better modeled (see Figure 2-42). Related investigations of the solar radiation pressure shadow function may be helpful in this area. Further testing of the numerical short periodic generator should be performed so that its accuracy can be determined for much lower (more strongly drag perturbed) and much higher orbits (where lunar-solar and solar radiation pressure short-periodic variations are important). It should be noted that some of the above mentioned areas of future work are presently being done in the MIT/CSDL community.

Future work in the average orbit generator should involve improvements and refinements to the present models. Zeis' expressions for the J_2^2 mean element rates should be extended to at least the J_2^2 e terms or be replaced with closed form expressions. The inclusion of closed form analytical J_2 short periodic variations should also improve the accuracy of the higher order AOG. The drag-J_2 analytical expressions should be extended to the retrograde case and also to higher order in the eccentricity.

Most of the improvements of the AOG will involve increasing the accuracy of higher eccentricity orbits.

Future research also needs to be done on the density models used in OD systems. One way of improving density determination is by developing improved density models. Research in this area should continue but due to the unpredictable nature of the upper atmosphere it is expected that this approach will not yield dramatic increases in accuracy for drag perturbed satellites. It appears that adaptive estimation of the density model is a means of improving the accuracy of orbit determination and prediction on near-Earth satellites and should be considered in future research. Adaptive estimation of other than the Adaptive Modified Harris-Priester density model should be attempted. A density model that should readily permit adaptive density estimation is the USSR density model (see Ref. 17 for a good description of this model). Since the major portion of the drag effect occurs around perigee (see Figures 2-13 and 2-42), and since the Sun's position and the location of perigee, with respect to inertial space, changes slowly, it may be advantageous to develop perigee density models. A perigee density model tries to accurately predict the density around perigee but may not be accurate at other points away from perigee. The diurnal-oblate exponential density model [see Equation (1-4)], with the perigee point as the reference point, may be a good perigee density model.

Another aspect that must be considered with density models used in the semianalytical satellite theory is how to handle the discontinuous nature of the $F_{10.7}$, $\bar{F}_{10.7}$ and K_p over the integration time step. One way of handling the discrete nature of these quantities is to fit them with polynominal interpolators over the integration grid of the semianalytical satellite theory. This would require that analytical continuous density models be used.

Additional work also needs to be done in the semianalytical batch filter. The analytical and quadrature approaches for determining the semianalytical partial derivatives need to be implemented. More testing of the semianalytical batch filter also needs to be done. The effects of longer and shorter observation spans, different perturbation models (to include different density models), different observation types and increased measurement errors needs to be examined. The semianalytical batch filter also needs to be tested in different flight regimes such that its accuracy for much lower and much higher orbits can be determined. The semianalytical batch filter should also be tested with real data. The test protocol used in this research was to generate simulated data from a Cowell generated truth orbit and compare the predicted ephemeris to the truth orbit. The actual performance of the semianalytical batch filter should be determined in an operational environment.

The semianalytical satellite theory should be coupled to sequential estimation procedures.* The SLKF appears to be a means of accomplishing this. Extensive testing and evaluation of this concept should be done.

The storing of satellite ephemerides by using the mean elements and the short periodic formulation developed in Chapter 2 should also be examined. It appears that the mean elements and short periodic coefficients can be represented over a one day time span by a low order interpolator. It should therefore be possible to store the ephemeris of a satellite over a long time span with a relatively small amount of data points. For example, suppose that the time, six mean elements and 120 short periodic coefficients (assuming upper limit of summation in Table 2-1 is 10) are stored at one day intervals, a 5th order Lagrangian interpolator could be used to determine the six mean elements and short periodic coefficients at any time within the five day time span. Knowing the mean elements and short periodic coefficients, the six osculating orbital elements can be determined by the use of the near-identity transformation. This may have application to the on-board computation and storage of satellite ephemerides.

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* Currently being done in a MIT Masters thesis at the CSDL by 2Lt S. Taylor, ENG, U.S. Army, Hertz Fellow.

Appendix A

VARIATION OF PARAMETERS EQUATIONS OF MOTION

The Lagrangian and Gaussian form of the Variation of Parameters (VOP) equations of motion are presented. The development is brief and taken from References 15, 21, 74, 78, 79 and 80.

Lagrangian VOP Equations of Motion

The Lagrangian form of the VOP equations of motion is applicable only for conservative forces, i.e., forces derivable from a potential. The equation of motion can be written in Cartesian coordinates as

$$\ddot{\underline{x}} = - \underline{\nabla} V \tag{A-1}$$

where $\underline{x}$ is the position vector, $\underline{\nabla}$ is the gradient operation and V is the potential function. This can be written in scalar form as

$$\frac{dx_i}{dt} = \dot{x}_i$$

$$\frac{d\dot{x}_i}{dt} = - \frac{\partial V_i}{\partial x_i} \tag{A-2}$$

$$(i = 1,2,3)$$

where x_i are the 3 components of the position vector and $\dot{x}_i$ are the 3 components of the velocity vector. With a set of six osculating orbital elements (a_k) which describe the motion, Equation (A-2) can be written as

$$\sum_{k=1}^{6} \frac{\partial x_i}{\partial a_k} \dot{a}_k = \dot{x}_i \qquad (i = 1,2,3) \tag{A-3a}$$

$$\sum_{k=1}^{6} \frac{\partial \dot{x}_i}{\partial a_k} \dot{a}_k = - \frac{\partial V}{\partial x_i} \tag{A-3b}$$

Multiplying Equation (A-3a) by $- \frac{\partial \dot{x}_i}{\partial a_\ell}$ and Equation (A-3b) by $+ \frac{\partial x_i}{\partial a_\ell}$, then adding both expressions together and summing over i, yields the following after some rearranging

$$\sum_{k=1}^{6} \sum_{i=1}^{3} \left[\frac{\partial x_i}{\partial a_\ell} \frac{\partial \dot{x}_i}{\partial a_k} - \frac{\partial \dot{x}_i}{\partial a_\ell} \frac{\partial x_i}{\partial a_k} \right] \dot{a}_k = - \sum_{i=3}^{3} \left[\frac{\partial \dot{x}_i}{\partial a_\ell} \dot{x}_i + \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial a_\ell} \right] \tag{A-4}$$

Define the Lagrange bracket as follows

$$[a_\ell, a_k] \equiv \sum_{i=1}^{3} \left[\frac{\partial x_i}{\partial a_\ell} \frac{\partial \dot{x}_i}{\partial a_k} - \frac{\partial \dot{x}_i}{\partial a_\ell} \frac{\partial x_i}{\partial a_k} \right] \tag{A-5}$$

From the definition of the Lagrange bracket it is seen that

$$[a_\ell, a_\ell] = 0 \tag{A-6a}$$

$$[a_\ell, a_k] = -[a_k, a_\ell] \tag{A-6b}$$

It can also be shown that

$$\frac{\partial}{\partial t} [a_\ell, a_k] = 0 \tag{A-6c}$$

The matrix of Lagrange brackets is defined as [LM], viz.

$$[LM] = \begin{bmatrix} [a_1, a_1] & [a_1, a_2] & \cdots & [a_1, a_6] \\ [a_2, a_1] & & & \\ \vdots & & \ddots & \vdots \\ [a_6, a_1] & \cdots & & [a_6, a_6] \end{bmatrix} \tag{A-7}$$

The matrix of Lagrange brackets can be written as

$$[LM] = \Gamma^{-T} J \Gamma^{-1} \tag{A-8}$$

where

$$\Gamma = \left\{ \left[\frac{\partial \underline{a}^*}{\partial \underline{x}} \right] \Big| \left[\frac{\partial \underline{a}^*}{\partial \dot{\underline{x}}} \right] \right\}_{6x6} \tag{A-9}$$

$$J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right]_{6x6} \tag{A-10}$$

and where $\underline{a}^*$ represents the vector of six orbital elements. It is noted that Γ is the matrizant of the two-body problem and naturally its inverse exists and can be written as

$$\Gamma^{-1} = \left\{ \begin{array}{c} \left[\frac{\partial \underline{x}}{\partial \underline{a}^*} \right] \\ \hline \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*} \right] \end{array} \right\}_{6x6} \tag{A-11}$$

It is also noted that

$$J^{-1} = -J \tag{A-12a}$$

and

$$\Gamma^{-T} = [\Gamma^T]^{-1} = [\Gamma^{-1}]^T \tag{A-12b}$$

Substitution of Equation(A-5) into Equation (A-4) yields

$$\sum_{k=1}^{6} [a_\ell, a_k]\dot{a}_k = -\sum_{i=1}^{3}\left[\frac{\partial \dot{x}_i}{\partial a_\ell}\dot{x}_i + \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial a_\ell}\right] \qquad \text{(A-13)}$$

It is noted that

$$\sum_{i=1}^{3}\left[\frac{\partial \dot{x}_i}{\partial a_\ell}\dot{x}_i\right] = \frac{\partial}{\partial a_\ell}\left[\sum_{i=1}^{3}\left\{\frac{1}{2}(\dot{x}_i)^2\right\}\right] = \frac{\partial T}{\partial a_\ell} \qquad \text{(A-14)}$$

where T is the kinetic energy of the system.

It is also noted that

$$\sum_{i=1}^{3}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial a_\ell}\right] = \frac{\partial V}{\partial a_\ell} \qquad \text{(A-15)}$$

Substitution of Equations (A-14) and (A-15) into Equation (A-13) yields after some rearranging

$$\sum_{k=1}^{6} [a_\ell, a_k]\dot{a}_k = \frac{\partial F}{\partial a_\ell} \qquad \text{(A-16)}$$

where F is the Delaunay function which is equal to the negative of the Hamiltonian of the system. This is expressed as

$$F = -H = -(T + V) \qquad \text{(A-17)}$$

It is noted that the potential can be written as

$$V = -\frac{\mu}{r} + R \tag{A-18}$$

where $-\frac{\mu}{r}$ is the two-body part of the potential and where R is the disturbing potential. A two-body relation between kinetic energy and potential energy is

$$T - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{A-19}$$

Substitution of Equations (A-18) and (A-19) into Equation (A-17) yields

$$F = \frac{\mu}{2a} - R \tag{A-20}$$

Equation (A-16) can be written in vector form with the use of the matrix of the Lagrange brackets as

$$[LM]\dot{\underline{a}}^* = \left[\frac{\partial F}{\partial \underline{a}^*}\right]^T \tag{A-21}$$

The element rates can therefore be written as

$$\dot{\underline{a}}^* = [LM]^{-1}\left[\frac{\partial F}{\partial \underline{a}^*}\right]^T \tag{A-22}$$

From Equation (A-8) it is seen that

$$[LM]^{-1} = -\Gamma\, J\, \Gamma^T \tag{A-23}$$

The matrix of Poisson Brackets is defined as

$$[PM] = \Gamma\ J\ \Gamma^T \tag{A-24}$$

which denotes the following matrix

$$[PM] = \begin{bmatrix} (a_1,a_1) & (a_1,a_2) & \cdots & (a_1,a_6) \\ (a_2,a_1) & & & \\ \vdots & \ddots & & \\ (a_6,a_1) & \cdots & & (a_6,a_6) \end{bmatrix} \tag{A-25}$$

From Equation (A-24) it can be shown that a scalar Poisson bracket is defined as

$$(a_\ell,a_k) = \sum_{i=1}^{3} \left[\frac{\partial a_\ell}{\partial x_i} \frac{\partial a_k}{\partial \dot{x}_i} - \frac{\partial a_\ell}{\partial \dot{x}_i} \frac{\partial a_k}{\partial x_i} \right] \tag{A-26}$$

It can be easily shown that the Poisson brackets have the following characteristics

$$(a_\ell,a_\ell) = 0 \tag{A-27a}$$

$$(a_\ell,a_k) = -(a_k,a_\ell) \tag{A-27b}$$

$$\frac{\partial}{\partial t}(a_\ell,a_k) = 0 \tag{A-27c}$$

Obviously, the relation between the matrix of Lagrange brackets and the matrix of Poisson brackets is

$$[LM]^{-1} = -[PM] \tag{A-27}$$

Substitution of Equation (A-27) into Equation (A-22) yields

$$\dot{\underline{a}}^* = -[PM] \left[\frac{\partial F}{\partial \underline{a}^*}\right]^T \tag{A-28}$$

Substitution of Equation (A-20) into Equation (A-28) yields after some rearranging

$$\dot{\underline{a}}^* = -[PM] \begin{bmatrix} -\dfrac{\mu}{2(a_1)^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - [PM] \left[\frac{\partial R}{\partial \underline{a}^*}\right]^T$$

which can be written as

$$\dot{\underline{a}}^* = \begin{bmatrix} (a_1,a_1) \\ (a_2,a_1) \\ (a_3,a_1) \\ (a_4,a_1) \\ (a_5,a_1) \\ (a_6,a_1) \end{bmatrix} \left[\frac{\mu}{2(a_1)^2}\right] - [PM] \left[\frac{\partial R}{\partial \underline{a}^*}\right]^T \tag{A-29}$$

It has been shown in the above mentioned references that

$$(a_j, a_1) = 0 \qquad \text{for } j = 1,2,3,4,5$$

$$(a_6, a_1) = \frac{2}{na_1} \tag{A-30}$$

where n is the mean motion and is equal to $\mu^{1/2}(a_1)^{-3/2}$. Substitution of Equation (A-30) into Equation (A-29) yields the vector form of the Lagrangian VOP equations of motion, viz.

$$\dot{\underline{a}}^* = \underline{e}_6 n - [PM]\left[\frac{\partial R}{\partial \underline{a}^*}\right]^T \tag{A-31a}$$

where

$$\underline{e}_6 = [0\ 0\ 0\ 0\ 0\ 1]^T \tag{A-31b}$$

In scalar form the Lagrangian VOP equations of motion can be written as

$$\dot{a}_i = n\delta_{i6} - \sum_{\ell=1}^{6} (a_i, a_\ell)\frac{\partial R}{\partial a_\ell} \tag{A-32}$$

$$(i = 1,2,3,4,5,6)$$

where δ_{ij} is the Kroenecker Delta function defined in Equation (2-24) and where R is the disturbing potential.

Gaussian VOP Equations of Motion

The Gaussian form of the VOP equations of motion is applicable for conservative or non-conservative forces. The general form of the equation of motion in Cartesian coordinates can be expressed as

$$\ddot{\underline{r}} + \frac{\mu \underline{r}}{r^3} = \underline{Q} \qquad \text{(A-33)}$$

where $\underline{r}$ is the position vector in the inertial Cartesian coordinate frame, $\ddot{\underline{r}}$ is the acceleration vector, μ is the gravitational constant and $\underline{Q}$ is the total perturbing acceleration. The unperturbed or two-body problem can be written as

$$\ddot{\underline{x}} + \frac{\mu \underline{x}}{x^3} = 0 \qquad \text{(A-34)}$$

where $\underline{x}$ denotes unperturbed or two-body position vector. The solution to Equation (A-34) is the two-body solution and is denoted functionally as

$$\underline{x} = \underline{x}(\underline{a}^*) \qquad \text{(A-35)}$$

where $\underline{a}^*$ denotes the vector of orbital elements which describe the two-body motion. It is noted that the orbital element set is such that the matrizant of the two-body problem (equivalently the Jacobian of the transformation) and its inverse exist. In other words, Γ and Γ^{-1}, as given in Equations (A-9) and (A-11),

exist and are well defined. The two-body velocity vector can be obtained by taking the time derivative of Equation (A-35), i.e.,

$$\dot{\underline{x}} = \left[\frac{\partial \underline{x}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^* \qquad \text{(A-36)}$$

In the case of the equinoctial orbital element set it is seen that

$$\underline{a}^* = [a \; h \; k \; p \; q \; \lambda]^T \qquad \text{(A-37a)}$$

and

$$\dot{\underline{a}}^* = [0 \; 0 \; 0 \; 0 \; 0 \; n]^T \qquad \text{(A-37b)}$$

where n is the mean motion.

The solution to the perturbed case is assumed to have the following functional form

$$\underline{r} = \underline{x}[\underline{a}^*(t)] \qquad \text{(A-38)}$$

where $\underline{a}^*(t)$ is the orbital element set that contains both the unperturbed and perturbed motion. This is represented as

$$\underline{a}^*(t) = \underline{a}^* + \underline{a}^*_p(t) \qquad \text{(A-39)}$$

where $\underline{a}^*$ represents the two-body motion of the orbital element set and $\underline{a}^*_p(t)$ represents the perturbed motion of the orbital element set. Equation (A-38) is requiring that the perturbed position vector and the two-body position vector be the same at each instant of time (the concept of the osculating ellipse). Taking the time derivative of Equation(A-38) yields the following for the perturbed velocity vector

$$\dot{\underline{r}} = \left[\frac{\partial \underline{x}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*(t) \tag{A-40}$$

From Equation (A-39) it is seen that

$$\dot{\underline{a}}^*(t) = \dot{\underline{a}}^* + \dot{\underline{a}}^*_p(t) \tag{A-41}$$

With the use of Equations (A-41) and (A-36), Equation (A-40) can be written as

$$\dot{\underline{r}} = \dot{\underline{x}} + \left[\frac{\partial \underline{x}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t) \tag{A-42}$$

The concept of the osculating ellipse requires that the perturbed velocity vector equal the two-body velocity vector at each instant of time. In other words, it is required that

$$\left[\frac{\partial \underline{x}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t) = 0 \tag{A-43}$$

Equation (A-42) can therefore be expressed as

$$\dot{\underline{r}} = \dot{\underline{x}} \tag{A-44}$$

Taking the time derivative of Equation (A-44) yields

$$\ddot{\underline{r}} = \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*(t)$$

which is

$$\ddot{\underline{r}} = \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^* + \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t) \tag{A-45}$$

From Equation (A-36) and Equation (A-37b), it is seen that the two-body velocity vector can be expressed functionally in terms of the vector of orbital elements. In other words

$$\dot{\underline{x}} = \dot{\underline{x}}(\underline{a}^*) \tag{A-46}$$

Taking the time derivative of Equation (A-46) yields

$$\ddot{\underline{x}} = \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^* \tag{A-47}$$

With the use of Equations (A-33) and (A-47) it is seen that Equation (A-45) can be rewritten as

$$\underline{Q} - \frac{\mu \underline{r}}{r^3} = \ddot{\underline{x}} + \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t)$$

Using Equation (A-38) it is seen that

$$\underline{Q} = \ddot{\underline{x}} + \frac{\mu \underline{x}}{x^3} + \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t)$$

The two-body equations of motion [Equation (A-34)] requires that

$$\underline{Q} = \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right] \dot{\underline{a}}^*_p(t) \qquad \text{(A-48)}$$

Therefore, the concept of the osculating ellipse requires that Equations (A-43) and (A-48) be satisfied. This can be written as

$$\left\{\begin{array}{c}\left[\frac{\partial \underline{x}}{\partial \underline{a}^*}\right] \\ \hline \left[\frac{\partial \dot{\underline{x}}}{\partial \underline{a}^*}\right]\end{array}\right\} \dot{\underline{a}}^*_p(t) = \left[\begin{array}{c}0 \\ \hline \underline{Q}\end{array}\right] \qquad \text{(A-49)}$$

With the use of the two-body matrizant, as given in Equations (A-9) and (A-11), Equation (A-49) is equivalently expressed as

$$\Gamma^{-1} \dot{\underline{a}}^*_p(t) = \left[\begin{array}{c}0 \\ \hline \underline{Q}\end{array}\right] \qquad \text{(A-50)}$$

Inverting the two-body matrizant in the above expression yields

$$\dot{\underline{a}}^*_p(t) = \Gamma \begin{bmatrix} 0 \\ --- \\ \underline{Q} \end{bmatrix} \tag{A-51}$$

which reduces to

$$\dot{\underline{a}}^*_p(t) = \left[\frac{\partial \underline{a}^*}{\partial \dot{\underline{x}}}\right] \cdot \underline{Q} \tag{A-52}$$

Substitution of Equation (A-52) into Equation (A-41) and noting that the two-body element rate is given in Equation (A-37b), yields the vector form of the Gaussian VOP equations of motion, viz.

$$\underline{a}^*(t) = \underline{e}_6 \, n + \left[\frac{\partial \underline{a}^*}{\partial \dot{\underline{x}}}\right] \cdot \underline{Q} \tag{A-53}$$

where $\underline{e}_6$ is defined in Equation (A-31b). In scalar form the Gaussian VOP equations of motion can be written as

$$\dot{a}_i = n \, \delta_{i6} + \sum_{k=1}^{3} \left\{\left[\frac{\partial a_i}{\partial x_k}\right] Q_k\right\} \tag{A-54}$$

where δ_{i6} is the Kroenecker Delta function defined in Equation (2-24) and where Q_k denotes the kth component of the disturbing acceleration. The matrizant for the two-body problem is given in the above mentioned references.

Appendix B

FINITE-DIFFERENCING

Single-sided and double-sided finite-differencing techniques were used in this research to obtain the partial derivatives of various quantities. In this appendix both techniques will be developed in terms of a scalar function of x, i.e., f(x).

The Double-Sided Finite-Differencing Technique

By use of a Taylor series it is seen that

$$f(x_o + \Delta x) = f(x_o) + \Delta x \left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} + \frac{(\Delta x)^2}{2}\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_o} + \frac{(\Delta x)^3}{6}\left[\frac{\partial^3 f(x)}{\partial x^3}\right]_{x=x_o} + \ldots \quad \text{(B-1)}$$

It is also seen that

$$f(x_o - \Delta x) = f(x_o) - \Delta x \left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} + \frac{(\Delta x)^2}{2}\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_o} - \frac{(\Delta x)^3}{6}\left[\frac{\partial^3 f(x)}{\partial x^3}\right]_{x=x_o} + \ldots \quad \text{(B-2)}$$

Subtracting $f(x_o - \Delta x)$ from $f(x_o + \Delta x)$ yields

$$f(x_o + \Delta x) - f(x_o - \Delta x) = 2(\Delta x)\left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} + \frac{(\Delta x)^3}{6}\left[\frac{\partial^3 f(x)}{\partial x^3}\right]_{x=x_o} + \cdots \tag{B-3}$$

Obviously the partial derivative of $\left[\frac{\partial f(x)}{\partial x}\right]_{x_o}$ can be approximated as

$$\left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} \sim \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2(\Delta x)} \tag{B-4}$$

Equation (B-4) is the form of the double-sided finite-differencing technique. There are two sources of error for this approach: neglect of higher order terms and round-off. The error due to the neglect of higher order terms will be less than or equal to the following

$$\text{Error} \leq \left\{\frac{(\Delta x)^2}{6}\left[\max\left|\frac{\partial^3 f(x)}{\partial x^3}\right|\right]\right\} \tag{B-5}$$

where max $\left|\frac{\partial^3 f(x)}{\partial x^3}\right|$ denotes the maximum of this quantity over the range of interest. Round-off errors are due to the finite word length of the digital computer that this technique is implemented on. Varying the stepsize (Δx) can significantly affect the round-off error. With double precision on the Amdahl 470 V6 (16 digits), it was found that good results were obtained when Δx was 10^{-5} of x, i.e.

$$\Delta x = (10^{-5})x \tag{B-6}$$

With single precision (8 digits), a good stepsize is

$$\Delta x = (10^{-3})x \tag{B-7}$$

The Single-Sided Finite-Differencing Technique

Subtracting $f(x_o)$ from $f(x_o + \Delta x)$ as given in Equation (B-1) yields

$$f(x_o + \Delta x) - f(x_o) = \Delta x \left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} + \frac{(\Delta x)^2}{2}\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_o} + \cdots \tag{B-8}$$

The single-sided finite-difference approximation is therefore

$$\left[\frac{\partial f(x)}{\partial x}\right]_{x=x_o} \sim \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} \qquad \text{(B-9)}$$

Again it is noted that the errors in this technique are the neglect of higher order terms and round-off. The error due to the neglect of higher order terms will be less than or equal to the following

$$\text{Error} \leq \left\{\frac{\Delta x}{2}\left[\max\left|\frac{\partial^2 f(x)}{\partial x^2}\right|\right]\right\} \qquad \text{(B-10)}$$

Again it is noted that the stepsize can significantly affect the round-off error. It was found that a stepsize as given in Equations (B-6) and (B-7) also gave good results for the single-sided finite-difference technique. The single-sided finite-differencing technique is not recommended for the semianalytical partial derivatives of Chapter 3.

Appendix C

THE ADAPTIVE MODIFIED HARRIS-PRIESTER ATMOSPHERE

The Adaptive Modified Harris-Priester Atmosphere is defined as

$$\rho = \exp[a_4 h + a_5/h] \{a_1 \rho_{min}(h) + [a_2 \rho_{max}(h) - a_1 \rho_{min}(h)] \cos^{a_3}(\psi/2)\} \quad \text{(C-1)}$$

where

ρ = density at point in question

a_1, a_2, a_3, a_4, a_5 = adaptive density parameters

h = height above the reference ellipsoid ($h_1 \leq h \leq h_2$)

ψ = angle between the diurnal bulge and satellite

$$\rho_{min}(h) = \rho_{min_1} \exp\left[\frac{h_1 - h}{H_{min}}\right] \quad \text{(C-2)}$$

$$\rho_{max}(h) = \rho_{max_1} \exp\left[\frac{h_1 - h}{H_{max}}\right] \quad \text{(C-3)}$$

$$H_{min} = \frac{h_1 - h_2}{\ell n[\rho_{min_2}/\rho_{min_1}]} \tag{C-4}$$

$$H_{max} = \frac{h_1 - h_2}{\ell n[\rho_{max_2}/\rho_{max_1}]} \tag{C-5}$$

and where h_1, h_2, ρ_{min_1}, ρ_{min_2}, ρ_{max_1} and ρ_{max_2} come from the Modified Harris-Priester density table corresponding to the $\bar{F}_{10.7}$ of interest. It is noted that the standard form of the Modified Harris-Priester density model results when the adaptive density parameters are

$$\begin{aligned} a_1 &= a_2 = 1.0 \\ a_3 &= 6.0 \\ a_4 &= a_5 = 0.0 \end{aligned} \tag{C-6}$$

The Modified Harris-Priester density model is discussed in detail in References 17 and 67.

Appendix D

DRAG-J_2 ANALYTICAL EXPRESSIONS

From Equation (2-101b) it can be seen that the drag-J_2 term of the mean element rates can be written as

$$\varepsilon\nu B_{i,DRJ2}(\bar{\underline{a}}) = ADRJ2i + \left(\frac{15\bar{n}\ \delta_{i6}}{16\bar{a}_1^{\,2}}\right) \sum_{\sigma=1}^{\infty} [\varepsilon C_{1\sigma}\ \nu G_{1\sigma} + \varepsilon D_{1\sigma}\ \nu H_{1\sigma}] \tag{D-1}$$

where

$$ADRJ2i = \frac{1}{2\pi} \int_0^{2\pi} \sum_{\ell=1}^{6} \nu\psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda}) \left[\frac{\partial\, \varepsilon F_i(\bar{\underline{a}},\bar{\lambda})}{\partial\, \bar{a}_\ell}\right] d\bar{\lambda} \tag{D-2}$$

and where

$\nu\psi_{\ell,1}(\bar{\underline{a}},\bar{\lambda})$ = drag short periodic variations [see Equation (2-99)]

$\varepsilon F_i(\bar{\underline{a}},\bar{\lambda})$ = VOP disturbing function due to J_2 effects

$\varepsilon C_{i\sigma}, \varepsilon D_{i\sigma}$ = J_2 short periodic coefficients

$\nu G_{i\sigma}, \nu H_{i\sigma}$ = drag short periodic coefficients

Equation (D-2) was solved using MACSYMA (Ref. 22). MACSYMA is a large computer programming system used for performing symbolic as well as numerical mathematical manipulations. MACSYMA is written in LISP and is being developed by the Mathlab Group at the MIT Laboratory for Computer Science. The VOP disturbing function due to J_2 effects, $\varepsilon F_i(\bar{\underline{a}},\bar{\lambda})$, was obtained by the MACSYMA orbital mechanics package of Zeis as explained in Section 2.4 of Reference 6. Zeis' package is only good for direct orbits. Due to the present limitations of the MACSYMA system hardware, $\varepsilon F_i(\bar{\underline{a}},\bar{\lambda})$ could only be obtained to first order in powers of the eccentricity. Taking the required partial derivatives of $\varepsilon F_i(\bar{\underline{a}},\bar{\lambda})$ as indicated in Equation (D-2), made the results obtained for ADRJ2i good only to zeroth order in the eccentricity. Performing the indicated operations of Equation (D-2) on the MACSYMA system gave the analytical drag-J_2 expressions. These expressions are given below. It should be noted that due to the limitations of the present MACSYMA computer system hardware, Equation (D-2) had to be broken into smaller component parts and the final results added together. The following notation was used:

Cx(i,σ),Dx(i,σ)	≡	drag short periodic coefficients $\nu G_{i\sigma}$, $\nu H_{i\sigma}$
MU	≡	μ = gravitational constant
RE	≡	r_e = equatorial radius of the Earth
XN	≡	n = mean motion
C	≡	$P^2 + Q^2$
A,H,K,P,Q,L	≡	equinoctial elements = a,h,k,p,q,λ

ADRJ21;

$$\frac{12\,J^2\,DX(5,2)\,MU\,P\,(4\,Q^2 - C - 1)\,RE^2}{A^4\,(C+1)^3\,XN}$$

$$-\frac{12\,J^2\,CX(5,2)\,MU\,(4\,P^2 - C + 1)\,Q\,RE^2}{A^4\,(C+1)^3\,XN}$$

$$+\frac{12\,J^2\,DX(4,2)\,MU\,(4\,P^2 - C - 1)\,Q\,RE^2}{A^4\,(C+1)^3\,XN} - \frac{24\,J^2\,CX(6,2)\,MU\,P\,Q\,RE^2}{A^4\,(C+1)^2\,XN}$$

$$-\frac{63\,J^2\,DX(3,3)\,MU\,P\,Q\,RE^2}{A^4\,(C+1)^2\,XN} + \frac{3\,J^2\,DX(3,1)\,MU\,P\,Q\,RE^2}{A^4\,(C+1)^2\,XN}$$

$$+\frac{63\,J^2\,CX(2,3)\,MU\,P\,Q\,RE^2}{A^4\,(C+1)^2\,XN} + \frac{3\,J^2\,CX(2,1)\,MU\,P\,Q\,RE^2}{A^4\,(C+1)^2\,XN}$$

$$+\frac{30\,DX(1,2)\,J^2\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN} - \frac{12\,J^2\,CX(4,2)\,MU\,P\,(4\,P^2 - 3\,C - 1)\,RE^2}{A^4\,(C+1)^3\,XN}$$

$$-\frac{3\,J^2\,CX(3,1)\,MU\,(2\,P^2 + C^2 - 5\,C + 1)\,RE^2}{2\,A^4\,(C+1)^2\,XN}$$

$$+\frac{3\,J^2\,DX(2,1)\,MU\,(2\,P^2 - C^2 + 3\,C - 1)\,RE^2}{2\,A^4\,(C+1)^2\,XN}$$

$$
\begin{aligned}
&- \frac{12\, J_2^2\, DX(6,2)\, MU\, (2P^2 - C)\, RE^2}{A^4\, (C+1)^2\, XN} + \frac{63\, J_2^2\, CX(3,3)\, MU\, (2P^2 - C)\, RE^2}{2\, A^4\, (C+1)^2\, XN} \\
&+ \frac{63\, J_2^2\, DX(2,3)\, MU\, (2P^2 - C)\, RE^2}{2\, A^4\, (C+1)^2\, XN} - \frac{15\, CX(1,2)\, J_2^2\, MU\, (2P^2 - C)\, RE^2}{A^5\, (C+1)^2\, XN}
\end{aligned}
$$

ADRJ22:

$$-\frac{21\,J^2\,CX(5,3)\,MU\,P\,(4\,Q^2-C-1)\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J^2\,CX(5,1)\,MU\,P\,(4\,Q^2-C-1)\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$-\frac{21\,J^2\,DX(5,3)\,MU\,(4\,P^2-C+1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$-\frac{21\,J^2\,CX(4,3)\,MU\,(4\,P^2-C-1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J^2\,CX(4,1)\,MU\,(4\,P^2-C-1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J^2\,DX(5,1)\,MU\,(4\,P^2-7\,C+7)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}-\frac{63\,J^2\,DX(6,3)\,MU\,P\,Q\,RE^2}{2\,A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J^2\,DX(6,1)\,MU\,P\,Q\,RE^2}{2\,A^5\,(C+1)^2\,XN}+\frac{51\,J^2\,CX(3,4)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}$$

$$
\begin{aligned}
&\frac{51\ J_2^2\ DX(2,4)\ MU\ P\ 0\ RE^2}{A^5\ (C+1)^2\ XN} + \frac{3\ J_2^2\ DX(2,2)\ MU\ P\ 0\ RE^2}{A^5\ (C+1)^2\ XN} \\
&- \frac{147\ CX(1,3)\ J_2^2\ MU\ P\ 0\ RE^2}{4\ A^6\ (C+1)^2\ XN} + \frac{21\ CX(1,1)\ J_2^2\ MU\ P\ 0\ RE^2}{4\ A^6\ (C+1)^2\ XN} \\
&- \frac{3\ J_2^2\ CX(3,2)\ (C+4)\ MU\ P\ 0\ RE^2}{A^5\ (C+1)^2\ XN} \\
&- \frac{3\ J_2^2\ DX(3,2)\ MU\ (4\ (C+4)\ P^2 + C^2 - 20\ C + 3)\ RE^2}{4\ A^5\ (C+1)^2\ XN} \\
&- \frac{3\ J_2^2\ CX(2,2)\ MU\ (4\ P^2 - (C-3)\ (3\ C-1))\ RE^2}{4\ A^5\ (C+1)^2\ XN} \\
&- \frac{21\ J_2^2\ DX(4,3)\ MU\ P\ (4\ P^2 - 3\ C - 1)\ RE^2}{2\ A^5\ (C+1)^3\ XN} \\
&+ \frac{3\ J_2^2\ DX(4,1)\ MU\ P\ (4\ P^2 - 9\ C + 5)\ RE^2}{2\ A^5\ (C+1)^3\ XN}
\end{aligned}
$$

$$- \frac{3\,J_2\,CX(6,1)\,MU\,(2\,P^2 + C^2 - 5\,C + 1)\,RE^2}{4\,A^5\,(C+1)^2\,XN}$$

$$+ \frac{21\,DX(1,1)\,J_2\,MU\,(2\,P^2 + C^2 - 5\,C + 1)\,RE^2}{8\,A^6\,(C+1)^2\,XN}$$

$$+ \frac{63\,J_2\,CX(6,3)\,MU\,(2\,P^2 - C)\,RE^2}{4\,A^5\,(C+1)^2\,XN} + \frac{51\,J_2\,DX(3,4)\,MU\,(2\,P^2 - C)\,RE^2}{2\,A^5\,(C+1)^2\,XN}$$

$$- \frac{51\,J_2\,CX(2,4)\,MU\,(2\,P^2 - C)\,RE^2}{2\,A^5\,(C+1)^2\,XN} - \frac{147\,DX(1,3)\,J_2\,MU\,(2\,P^2 - C)\,RE^2}{8\,A^6\,(C+1)^2\,XN}$$

ADRJ23;

$$\frac{21\,J_2\,DX(5,3)\,MU\,P\,(4\,Q^2 - C - 1)\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J_2\,DX(5,1)\,MU\,P\,(4\,Q^2 - C - 1)\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$-\frac{3\,J_2\,CX(5,1)\,MU\,(4\,P^2 + 5\,C - 5)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$-\frac{21\,J_2\,CX(5,3)\,MU\,(4\,P^2 - C + 1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{21\,J_2\,DX(4,3)\,MU\,(4\,P^2 - C - 1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J_2\,DX(4,1)\,MU\,(4\,P^2 - C - 1)\,Q\,RE^2}{2\,A^5\,(C+1)^3\,XN} - \frac{63\,J_2\,CX(6,3)\,MU\,P\,Q\,RE^2}{2\,A^5\,(C+1)^2\,XN}$$

$$-\frac{3\,J_2\,CX(6,1)\,MU\,P\,Q\,RE^2}{2\,A^5\,(C+1)^2\,XN} - \frac{51\,J_2\,DX(3,4)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}$$

$$
\frac{3\,J_2\,DX(3,2)\,MU\,P\,0\,RE^2}{A^5\,(C+1)^2\,XN} + \frac{51\,J_2\,CX(2,4)\,MU\,P\,0\,RE^2}{A^5\,(C+1)^2\,XN}
$$

$$
\frac{147\,DX(1,3)\,J_2\,MU\,P\,0\,RE^2}{4\,A^6\,(C+1)^2\,XN} + \frac{21\,DX(1,1)\,J_2\,MU\,P\,0\,RE^2}{4\,A^6\,(C+1)^2\,XN}
$$

$$
+ \frac{3\,J_2\,CX(2,2)\,(C+4)\,MU\,P\,0\,RE^2}{A^5\,(C+1)^2\,XN}
$$

$$
\frac{3\,J_2\,DX(2,2)\,MU\,(4\,(C+4)\,P^2 - 5\,C^2 + 4\,C - 3)\,RE^2}{4\,A^5\,(C+1)^2\,XN}
$$

$$
- \frac{3\,J_2\,CX(3,2)\,MU\,(4\,P^2 + 3\,C^2 - 14\,C + 3)\,RE^2}{4\,A^5\,(C+1)^2\,XN}
$$

$$
- \frac{3\,J_2\,CX(4,1)\,MU\,P\,(4\,P^2 + 3\,C - 7)\,RE^2}{2\,A^5\,(C+1)^3\,XN}
$$

$$
- \frac{21\,J_2\,CX(4,3)\,MU\,P\,(4\,P^2 - 3\,C - 1)\,RE^2}{2\,A^5\,(C+1)^3\,XN}
$$

$$
- \frac{3\, J_2\, DX(6,1)\, MU\, (2P^2 - C^2 + 3C - 1)\, RE^2}{4\, A^5\, (C+1)^2\, XN}
$$

$$
- \frac{21\, CX(1,1)\, J_2\, MU\, (2P^2 - C^2 + 3C - 1)\, RE^2}{8\, A^6\, (C+1)^2\, XN}
$$

$$
- \frac{63\, J_2\, DX(6,3)\, MU\, (2P^2 - C)\, RE^2}{4\, A^5\, (C+1)^2\, XN} + \frac{51\, J_2\, CX(3,4)\, MU\, (2P^2 - C)\, RE^2}{2\, A^5\, (C+1)^2\, XN}
$$

$$
+ \frac{51\, J_2\, DX(2,4)\, MU\, (2P^2 - C)\, RE^2}{2\, A^5\, (C+1)^2\, XN} - \frac{147\, CX(1,3)\, J_2\, MU\, (2P^2 - C)\, RE^2}{8\, A^6\, (C+1)^2\, XN}
$$

ADRJ241

$$\frac{3\,J_2\,DX(5,2)\,MU\,(4\,Q^2+C^2-1)\,RE^2}{4\,A^5\,(C+1)^2\,XN}-\frac{3\,J_2\,CX(5,2)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,DX(4,2)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}+\frac{3\,J_2\,CX(6,2)\,(C-1)\,MU\,Q\,RE^2}{2\,A^5\,(C+1)\,XN}$$

$$+\frac{21\,J_2\,DX(3,3)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}-\frac{21\,J_2\,DX(3,1)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$-\frac{21\,J_2\,CX(2,3)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}+\frac{15\,J_2\,CX(2,1)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$-\frac{21\,DX(1,2)\,J_2\,(C-1)\,MU\,Q\,RE^2}{8\,A^6\,(C+1)\,XN}-\frac{3\,J_2\,CX(4,2)\,MU\,(4\,P^2+C^2-1)\,RE^2}{4\,A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,DX(6,2)\,(C-1)\,MU\,P\,RE^2}{2\,A^5\,(C+1)\,XN}-\frac{21\,J_2\,CX(3,3)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$+\frac{3\,J_2\,CX(3,1)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}-\frac{21\,J_2\,DX(2,3)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$-\frac{3\,J_2\,DX(2,1)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}+\frac{21\,CX(1,2)\,J_2\,(C-1)\,MU\,P\,RE^2}{8\,A^6\,(C+1)\,XN}$$

ADPJ251

$$\frac{3\,J_2\,CX(5,2)\,MU\,(4\,Q^2+C^2-1)\,RE^2}{4\,A^5\,(C+1)^2\,XN}+\frac{3\,J_2\,DX(5,2)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,CX(4,2)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}-\frac{3\,J_2\,DX(6,2)\,(C-1)\,MU\,Q\,RE^2}{2\,A^5\,(C+1)\,XN}$$

$$+\frac{21\,J_2\,CX(3,3)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}-\frac{3\,J_2\,CX(3,1)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$+\frac{21\,J_2\,DX(2,3)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}+\frac{3\,J_2\,DX(2,1)\,(C-1)\,MU\,Q\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$-\frac{21\,CX(1,2)\,J_2\,(C-1)\,MU\,Q\,RE^2}{8\,A^6\,(C+1)\,XN}+\frac{3\,J_2\,DX(4,2)\,MU\,(4\,P^2+C^2-1)\,RE^2}{4\,A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,CX(6,2)\,(C-1)\,MU\,P\,RE^2}{2\,A^5\,(C+1)\,XN}+\frac{21\,J_2\,DX(3,3)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$+\frac{15\,J_2\,DX(3,1)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}-\frac{21\,J_2\,CX(2,3)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}$$

$$-\frac{21\,J_2\,CX(2,1)\,(C-1)\,MU\,P\,RE^2}{8\,A^5\,(C+1)\,XN}-\frac{21\,DX(1,2)\,J_2\,(C-1)\,MU\,P\,RE^2}{8\,A^6\,(C+1)\,XN}$$

ADRJ261

$$\frac{3\,J_2\,CX(5,2)\,MU\,P\,(2\,(C-15)\,Q^2-(C-7)\,(C+1))\,RE^2}{A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J_2\,CX(4,2)\,MU\,(2\,(C-15)\,P^2-(C-7)\,(C+1))\,Q\,RE^2}{A^5\,(C+1)^3\,XN}$$

$$+\frac{3\,J_2\,DX(5,2)\,MU\,(2\,(C-15)\,P^2+9\,C-7)\,Q\,RE^2}{A^5\,(C+1)^3\,XN}$$

$$+\frac{6\,J_2\,DX(6,2)\,(C-7)\,MU\,P\,Q\,RE^2}{A^5\,(C+1)^2\,XN}+\frac{21\,CX(1,2)\,J_2\,(C-7)\,MU\,P\,Q\,RE^2}{2\,A^6\,(C+1)^2\,XN}$$

$$-\frac{21\,J_2\,CX(3,3)\,(2\,C-15)\,MU\,P\,Q\,RE^2}{4\,A^5\,(C+1)^2\,XN}+\frac{3\,J_2\,CX(3,1)\,(2\,C-15)\,MU\,P\,Q\,RE^2}{4\,A^5\,(C+1)^2\,XN}$$

$$-\frac{21\,J_2\,DX(2,3)\,(2\,C-15)\,MU\,P\,Q\,RE^2}{4\,A^5\,(C+1)^2\,XN}-\frac{3\,J_2\,DX(2,1)\,(2\,C-15)\,MU\,P\,Q\,RE^2}{4\,A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,CX(2,1)\,MU\,(2\,(2\,C-15)\,P^2+23\,C^2-49\,C+13)\,RE^2}{8\,A^5\,(C+1)^2\,XN}$$

$$+\frac{3\,J_2\,DX(3,1)\,MU\,(2\,(2\,C-15)\,P^2-27\,C^2+79\,C-13)\,RE^2}{8\,A^5\,(C+1)^2\,XN}$$

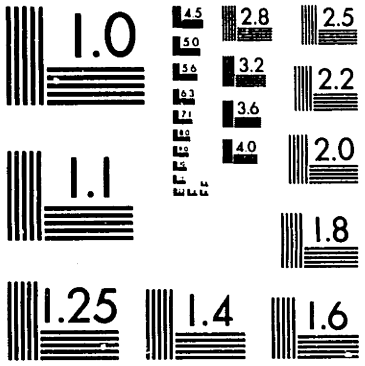
$$+\frac{3\,J_2\,DX(4,2)\,MU\,P\,(2\,(C-15)\,P^2-2\,C^2+21\,C+7)\,RE^2}{A^5\,(C+1)^3\,XN}$$

$$-\frac{3\,J_2\,CX(6,2)\,(C-7)\,MU\,(2\,P^2-C)\,RE^2}{A^5\,(C+1)^2\,XN}$$

$$+\frac{21\,DX(1,2)\,J_2\,(C-7)\,MU\,(2\,P^2-C)\,RE^2}{4\,A^6\,(C+1)^2\,XN}$$

$$-\frac{21\,J_2\,DX(3,3)\,(2\,C-15)\,MU\,(2\,P^2-C)\,RE^2}{8\,A^5\,(C+1)^2\,XN}$$

$$+\frac{21\,J_2\,CX(2,3)\,(2\,C-15)\,MU\,(2\,P^2-C)\,RE^2}{8\,A^5\,(C+1)^2\,XN}$$



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

20X

Appendix E

CORRECTIONS TO ZEIS' J_2^2 SHORT PERIODIC EXPRESSIONS

The second order J_2 short periodic variation for the fifth* (Δq) and the sixth ($\Delta\lambda$) variables as given in Reference 6 were rederived during the present development effort. The results obtained were different than Zeis' expressions. The new expressions obtained are

$$\begin{aligned} J22ETA52 = \frac{-3J_2^2(\bar{c}-1)R_e^4}{16\bar{a}^4(\bar{c}+1)^3}\Big[&\bar{q}(72\bar{p}^2-12\bar{c})\cos(2\bar{\lambda}) \\ &+\bar{p}(72\bar{p}^2-48\bar{c})\sin(2\bar{\lambda})+\bar{q}[(12\bar{c}+20)\bar{p}^2-3\bar{c}^2 \\ &-5\bar{c}]\cos(4\bar{\lambda})+\bar{p}[(12\bar{c}+20)\bar{p}^2-9\bar{c}^2-15\bar{c}]\sin(4\bar{\lambda})\Big] \end{aligned} \tag{E-1}$$

and

$$\begin{aligned} J22ETA62 = \frac{3J_2^2R_e^4}{16\bar{a}^4(\bar{c}+1)^4}\Big[&16\bar{p}\,\bar{q}(13\bar{c}^2-13\bar{c}+7)\cos(2\bar{\lambda}) \\ &-8(\bar{q}^2-\bar{p}^2)(13\bar{c}^2-13\bar{c}+7)\sin(2\bar{\lambda}) \\ &-4\,\bar{p}\,\bar{q}(\bar{q}^2-\bar{p}^2)(3\bar{c}^2+10\bar{c}-45)\cos(4\bar{\lambda}) \\ &+(\bar{c}^2-8\bar{p}^2\,\bar{q}^2)(3\bar{c}^2+10\bar{c}-45)\sin(4\bar{\lambda})\Big] \end{aligned} \tag{E-2}$$

---

* J. P. Kaniecki discovered the error in the Δq in August 1979.

where

$\bar{a},\bar{h},\bar{k},\bar{p},\bar{q}$, and $\bar{\lambda}$ = the mean equinoctial elements

$\bar{c} = \bar{p}^2 + \bar{q}^2$

$R_e \equiv$ equatorial radius of Earth

J22ETA52 $\equiv \varepsilon^2 \eta_{5,2}(\underline{\bar{a}},\bar{\lambda})$

J22ETA62 $\equiv \varepsilon^2 \eta_{6,2}(\underline{\bar{a}},\bar{\lambda})$

Appendix F

SOFTWARE IMPLEMENTATION

The short periodic development, second order drag theory, semianalytical partial derivatives and semianalytical batch filter were implemented into GTDS-RD. The new subroutines developed are presented in this appendix. Subroutines 'ORBITV' and 'AVRAGE' which are standard subroutines of GTDS-RD are also presented due to the necessary interfaces that were required in these subroutines.

The necessary switches and flags used in order to exercise different options were set in subroutines 'HWIRE' and 'ESTSET.' Subroutine 'HWIRE' set the necessary switches and flags used in the short periodic development and in the implementation of the second order drag theory. Subroutine 'HWIRE' is called from the GTDS-RD subroutine 'AVRINT.' Subroutine 'ESTSET' sets the necessary switches and flags used in the implementation of the semianalytical partial derivatives and the semianalytical batch filter. Subroutine 'ESTSET' is called from the GTDS-RD subroutine 'INTOGN.' These subroutines and the other new subroutines were linked with the GTDS-RD load module. Eventually these switches and flags will be standard input options and/or initialization procedures in GTDS-RD.

The software implementation of this research can be divided into the following areas: SPG, second order drag effects in the AOG, APG, SPPG and semianalytical batch filter. Each of these areas will be discussed separately followed by a listing of new subroutines in alphabetical order.

SPG

The present SPG is on the output grid. A flow diagram for the SPG is given in Table F-1. The numerical time-independent short periodic formulation was implemented. Weak time-dependent short periodic corrections, using a single-sided finite-differencing technique, were also implemented. Zeis' J_2^2 short periodic expressions (with the corrections as indicated in Appendix E) were implemented by Leo W. Early of CSDL. Present development efforts are implementing the analytical short periodic expressions as given in Reference 5. The implementation of the SPG also required changes in the GTDS-RD subroutines 'ANAVIN' and 'AVRINT.'

Second Order Drag Effects in the AOG

A flow chart for the implementation of the second order drag effects into the AOG is presented in Table F-2. This implementation also required changes in the GTDS-RD subroutines 'ANAVIN,' 'AVRINT,' 'EQINT' and 'EQUNPV.'

Table F-1

Flow Chart for the SPG

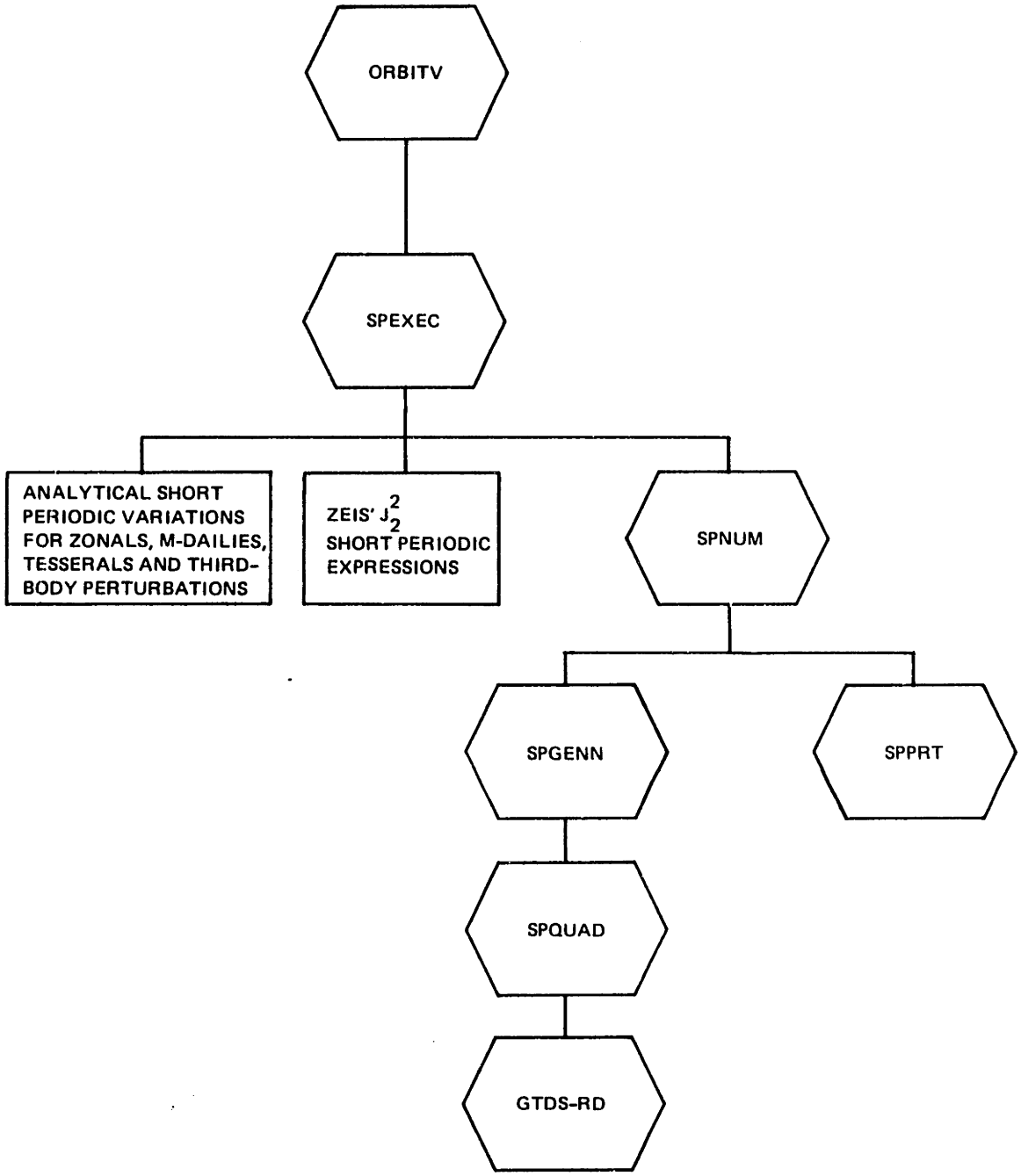
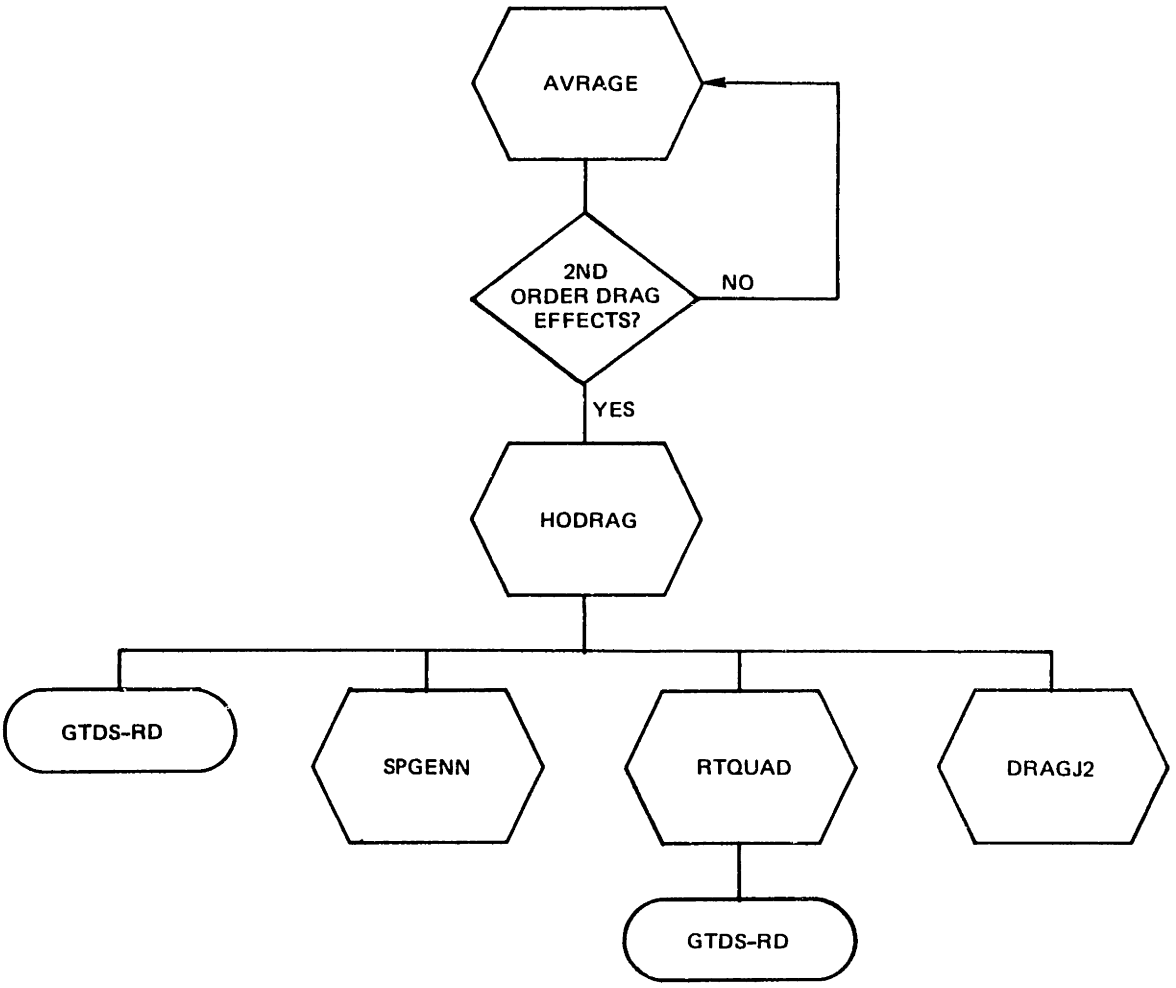


Table F-2

Flow Chart for Second Order Drag Effects in the AOG



APG

The APG is on the integration grid of the semianalytical satellite theory. A flow chart for the APG is presented in Table F-3. The determination of the A and D matrices (see Chapter 3) was done by a double-sided finite-differencing technique. The integration subroutines are 'CSTEPX' and 'RKINTG.' Subroutine 'CSTEPX' is the predictor-corrector integrator of GTDS-RD. Subroutine 'RKINTG' is a Runge-Kutta integrator and was recently implemented into GTDS-RD by A. Bobick of CSDL. The initialization of the APG in the GTDS-RD subroutine 'RESINV' required the use of subroutine 'EPART.' Subroutine 'EPART' was recently implemented into GTDS-RD by R. Proulx of CSDL. Subroutine 'EPART' calculates the matrizant of the two-body problem. Subroutine 'EPART' is also used on the output grid to form the required partial derivatives as given by Equation (3-28). The implementation of the APG also required changes in the GTDS-RD subroutines 'ATMOS,' 'CSTEPX,' 'GVCVL,' 'INTOGN,' 'MULSTP,' 'OUTPAR,' 'RESINV' and 'SOLTAB.' Subroutine 'ATMOS' had to be changed in order to implement the Adaptive Modified Harris-Priester density model.

SPPG

The SPPG is on the output grid. Table F-4 shows the flow chart for the SPPG. A double-sided finite-differencing technique

Table F-3

Flow Chart for the APG

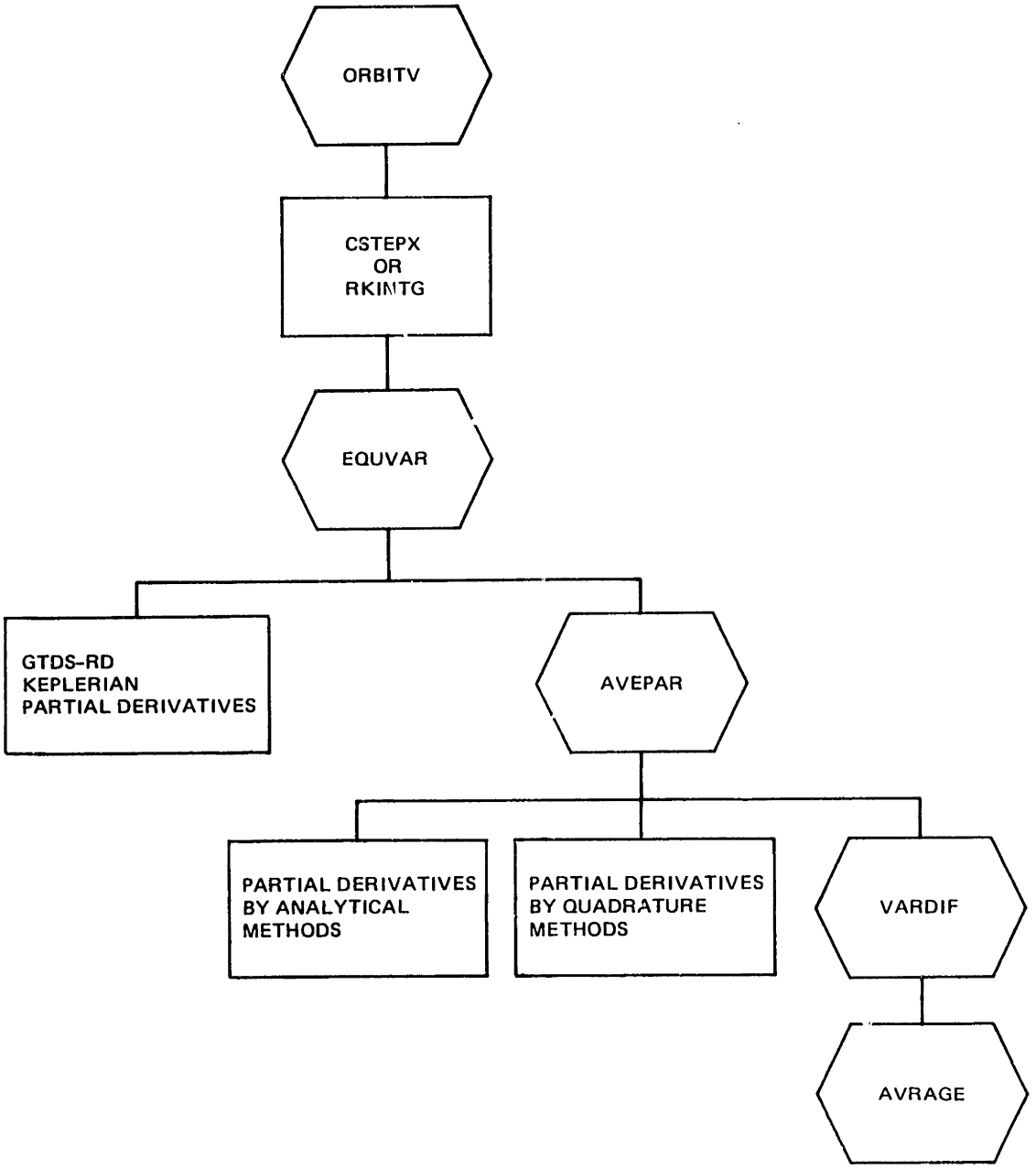
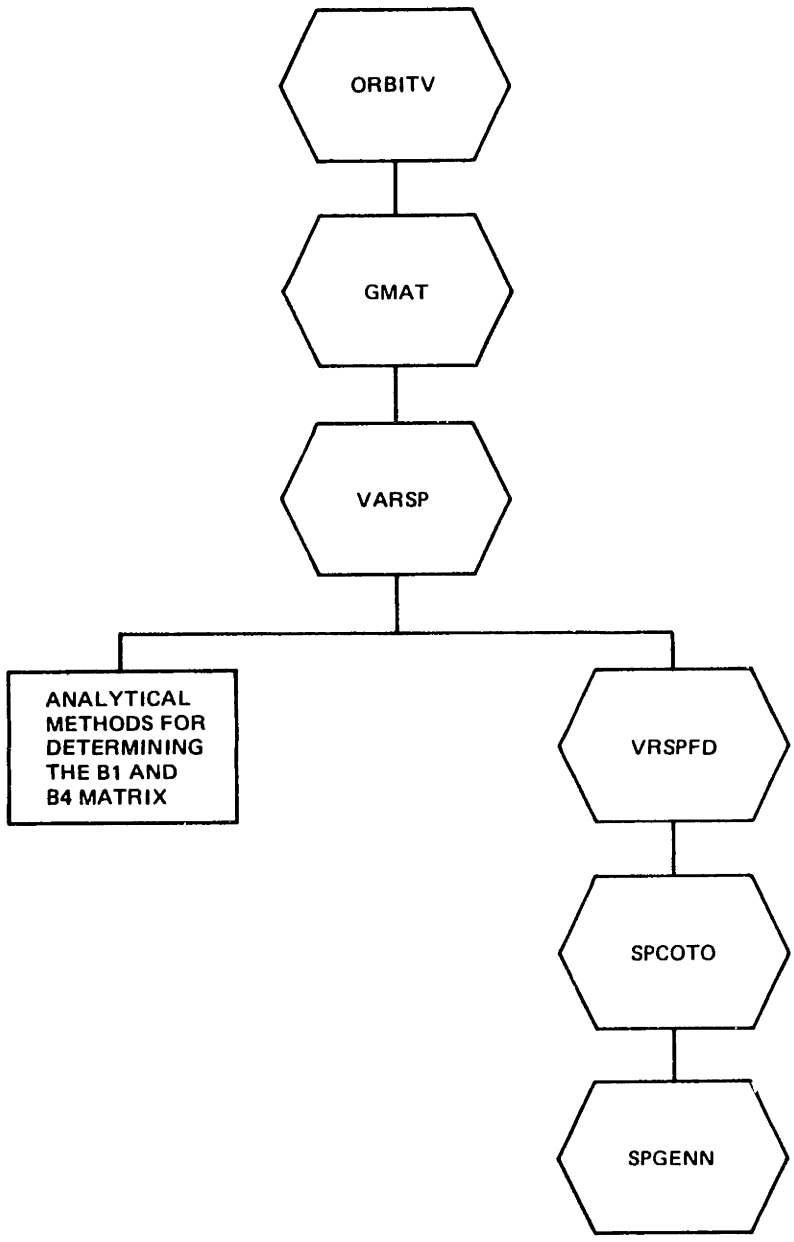


Table F-4

Flow Chart for the SPPG



was used to obtain the M_σ, N_σ, K_σ and L_σ matrices. The implementation of the SPPG also required changes in the GTDS-RD subroutine 'INTOGN.'

Semianalytical Batch Filter

The implementation of the semianalytical batch filter required interfacing the semianalytical orbit generator (including the generation of semianalytical partial derivatives) to the batch filter of GTDS-RD. The development of the semianalytical batch filter involved changes in the following GTDS-RD subroutines: 'GVCVL,' 'INTOGN,' 'PSET,' 'RESINV,' and 'SOLTAB.'

AVEPAR AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE AVEPAR                                                 00000010
                                                                        00000020
C
C     VERSION OF 20 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6   00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C        EXECUTIVE PROGRAM FOR THE DETERMINATION OF THE A AND D MATRIX  00000060
C        IN THE DIFFERENTIAL EQUATIONS FOR THE PERTURBED PARTIALS.      00000070
C                                                                       00000080
C     PARAMETERS                                                        00000090
C                                                                       00000100
C       /ESTFLG/                                                        00000110
C                                                                       00000120
C                                                                       00000130
C       IANAL      I     CONTROL SWITCH FOR ANALYTICAL APPROACH IN THE  00000140
C                        AVERAGED PARTIAL GENERATOR.*                   00000150
C       IDIFF      I     CONTROL SWITCH FOR FINITE-DIFFERENCING APPROACH00000160
C                        IN THE AVERAGED PARTIAL GENERATOR.             00000170
C                        0=APPROACH NOT USED                            00000180
C                        1=A MATRIX ONLY                                00000190
C                        2=D MATRIX ONLY                                00000200
C                        3=BOTH A AND D MATRICES                        00000210
C       IQDRT      I     CONTROL SWITCH FOR QUADRATURE APPROACH IN THE  00000220
C                        AVERAGED PARTIAL GENERATOR                     00000230
C                        0=APPROACH NOT USED                            00000240
C                        1=A MATRIX ONLY                                00000250
C                        2=D MATRIX ONLY                                00000260
C                        3=BOTH A AND D MATRICES                        00000270
C       ICBVAR     I     CENTRAL GRAVITATIONAL BODY CONTROL SWITCH FOR   00000280
C                        THE AVERAGED PARTIAL GENERATOR.                00000290
C                        0=NONE                                         00000300
C                        1=ANALYTICALLY                                 00000310
C                        2=FINITE-DIFFERENCING WITH THE AOG CENTRAL      00000320
C                          BODY FIELD                                   00000330
C                        3=FINITE-DIFFERENCING WITH A NXO FIELD         00000340
C       J22VAR     I     J2 SQUARED CONTROL SWITCH IN THE AVERAGED      00000350
C                        PARTIAL GENERATOR.                             00000360
C                        0=OFF                                          00000370
C                        1=FINITE-DIFFERENCING                          00000380
C       ITBVAR     I     THIRD-BODY CONTROL SWITCH IN THE AVERAGED      00000390
C                        PARTIAL GENERATOR.                             00000400
C                        0=OFF                                          00000410
C                        1=FINITE-DIFFERENCING WITH THE AOG THIRD-BODY  00000420
C                          MODEL                                        00000430
C       IDRVAR     I     DRAG CONTROL SWITCH IN THE AVERAGED PARTIAL    00000440
C                        GENERATOR.                                     00000450
C                        0=OFF                                          00000460
C                        1=FINITE-DIFFERENCING                          00000470
C                        2=QUADRATURE APPROACH                          00000480
C       ISRVAR     I     SOLAR RADIATION PRESSURE CONTROL SWITCH IN THE 00000490
C                        AVERAGED PARTIAL GENERATOR.                    00000500
C                        0=OFF                                          00000510
C                        1=FINITE-DIFFERENCING                          00000520
C                        2=QUADRATURE APPROACH                          00000530
C       NVAR       I     MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITATIONAL00000540
C                        FIELD TO BE CONSIDERED IN THE AVERAGED PARTIAL 00000550
C                        GENERATOR WHEN ICBVAR=3.                       00000560
C       LEVAR      I     MAXIMUM POWER OF E IN THE POWER SERIES EXPANSION00000570
```

```
C                                FOR THE ANALYTICAL HARMONIC POTENTIAL CONSIDERED00000580
C                                IN THE AVERAGED PARTIAL GENERATOR WHEN ICBVAR=3 00000590
C                                (AT MOST CAN BE NVAR-2).                        00000600
C       KDRFLG      I            CONTROL SWITCH FOR CONSIDERING 2ND ORDER DRAG   00000610
C                                EFFECTS IN THE AVERAGED PARTIAL GENERATOR.      00000620
C                                0=NO 2ND ORDER DRAG EFFECTS                     00000630
C                                1=J2-DRAG EFFECTS                               00000640
C                                2=J2-DRAG AND DRAG-DRAG EFFECTS                 00000650
C                                3=J2-DRAG,DRAG-J2 AND DRAG-DRAG EFFECTS         00000660
C                                  DONE NUMERICALLY                              00000670
C                                4=J2-DRAG AND DRAG-DRAG EFFECTS DONE            00000680
C                                  NUMERICALLY, DRAG-J2 DONE ANALYTICALLY        00000690
C                                5=ISZAK'S J2 HEIGHT CORRECTION AND ANALYTIC     00000700
C                                  DRAG-J2 EFFECTS                               00000710
C                                6=ISZAK'S J2 HEIGHT CORRECTION ONLY             00000720
C                                                                                00000730
C       /FRC/                                                                    00000740
C                                                                                00000750
C       NMAX        0            MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITAIONAL 00000760
C                                FIELD.                                          00000770
C       MMAX        0            MAXIMUM ORDER OF THE CENTRAL BODY GRAVITATIONAL 00000780
C                                FIELD.                                          00000790
C                                                                                00000800
C       /ANAVIN/                                                                 00000810
C                                                                                00000820
C       ITESS       0            LOW-ORDER CENTRAL BODY SPHERICAL HARMONIC       00000830
C                                AVERAGING OPTION.                               00000840
C                                0 = NO LOW-ORDER SPHERICAL HARMONICS.           00000850
C                                1 = TIME-DEPENDENT NUMERICAL AVERAGING.         00000860
C                                2 = TIME-INDEPENDENT NUMERICAL AVERAGING.       00000870
C                                3 = ANALYTICAL AVERAGING.                       00000880
C       NEND        0            DEGREE OF ANALYTICALLY AVERAGED NONRESONANT     00000890
C                                CENTRAL BODY SPHERICAL HARMONIC FIELD.          00000900
C       MEND        0            ORDER OF ANALYTICALLY AVERAGED NONRESONANT      00000910
C                                CENTRAL BODY SPHERICAL HARMONIC FIELD.          00000920
C       LEND        0            MAXIMUM POWER OF E IN THE POWER SERIES          00000930
C                                EXPANSION FOR THE AN.  TICALLY AVERAGED         00000940
C                                NONRESONANT CENTRAL BODY SPHERICAL HARMONIC      00000950
C                                POTENTIAL.                                      00000960
C       NCSRES      0            NUMBER OF ANALYTICALLY AVERAGED RESONANT        00000970
C                                CENTRAL BODY SPHERICAL HARMONIC TERMS.          00000980
C       ITHIRD      0            THIRD-BODY AVERAGING OPTION.                    00000990
C                                POTENTIALS OF THE NONCENTRAL BODIES.            00001000
C       JTWOSQ      0            SECOND-ORDER J2 PERTURBATION CONTRL SWITCH.     00001010
C       IDRAG       0            QUADRATURE ORDER CONTROL SWITCH FOR DRAG.       00001020
C       ISOLAR      0            QUADRATURE ORDER CONTROL SWITCH FOR SOLAR       00001030
C                                RADIATION PRESSURE.                             00001040
C       ICCRR       0            CONTROL SWITCH FOR SOLAR RADIATION PRESSURE.*   00001050
C       IDRDR       0            CONTROL SWITCH FOR 2ND ORDER DRAG EFFECTS.      00001060
C                                                                                00001070
C       /VARMAT/                                                                 00001080
C                                                                                00001090
C       AMAT        I            THE A MATRIX.                                   00001100
C       DMAT        I            THE D MATRIX.                                   00001110
C                                                                                00001120
C       /WORKER/                                                                 00001130
C                                                                                00001140
```

```
C       PVOP      O     THE TOTAL A MATRIX.                                    00001150
C       VELPAR    O     THE TOTAL D MATRIX.                                    00001160
C       NEQ       I     THE NUMBER OF VARIATIONAL EQUATIONS.                   00001170
C       NOPARM    I     THE NUMBER OF ORBITAL ELEMENTS.                        00001180
C                                                                              00001190
C       /SWITCH/                                                               00001200
C                                                                              00001210
C       IRESON    O     CENTRAL BODY FIELD CONTROL SWITCH.                     00001220
C                       2=NXM FIELD                                            00001230
C                       4=NO CENTRAL BODY FIELD                                00001240
C                                                                              00001250
C       REFERENCES                                                             00001260
C         "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW                00001270
C         ALTITUDE SATELLITES." BY A.J. GREEN                                  00001280
C                                                                              00001290
C      ANALYSIS                                                                00001300
C         ANDREW J. GREEN, CPT, U.S. ARMY                                      00001310
C                                                                              00001320
C       PROGRAMMER                                                             00001330
C         ANDREW J. GREEN, CPT, U.S. ARMY                                      00001340
C                                                                              00001350
C                                                                              00001360
      IMPLICIT REAL*8(A-H,O-Z)                                                 00001370
      DIMENSION PVOP(6,6),VELPAR(6,20)                                         00001380
C                                                                              00001390
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00001400
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00001410
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00001420
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00001430
     4                 ITHIRD      ,MENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00001440
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00001450
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00001460
     7                 KA          ,KB          ,KC          ,KGHA        ,00001470
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00001480
     9                 IGHA        ,ISTRES      ,ITERM       ,ISNAD       ,00001490
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00001500
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00001510
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00001520
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00001530
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00001540
     6                 NMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR       00001550
C                                                                              00001560
      COMMON/FRC   /   RFRC(1300)  ,IFRC(50)                                   00001570
      EQUIVALENCE      (NMAX                    ,IFRC(3)                   ),00001580
     1                 (NMAX                    ,IFRC(4)                   ) 00001590
C                                                                              00001600
      COMMON/WORKER/ DPWRK(3264),INWRK(13)                                     00001610
      EQUIVALENCE      (DPWRK(2227)             ,PVOP(1,1)                 ),00001620
     1                 (DPWRK(2263)             ,VELPAR(1,1)               ),00001630
     2                 (INWRK(3)                ,NOPARM                    ),00001640
     3                 (INWRK(5)                ,NEQ                       ) 00001650
C                                                                              00001660
      COMMON/ESTFLG/ IANAL     ,IDIFF     ,IGDRT     ,ICBVAR    ,J22VAR    , 00001670
     1               ITBVAR    ,IDRVAR    ,ISRVAR    ,NVAR      ,LEVAR     , 00001680
     2               KDRFLG    ,KFAR      ,KSTEP(20),KVRFLG     ,KCPSPP    , 00001690
     3               KCBSPP    ,KTESPP    ,KDRSPP    ,KSRSPP    ,NVARSP    , 00001700
     4               IVSPNC(4),KATMOS     ,KPRTB1    ,KPRTB2    ,KPRTB3    , 00001710
```

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     5              KPRTB4                                                  00001720
C                                                                           00001730
      COMMON/SWITCH/ IND(225)                                               00001740
      EQUIVALENCE    (IRESON                    ,IND(38)                  ) 00001750
C                                                                           00001760
      COMMON/VARMAT/ AMAT(6,6),DMAT(6,14),B1MAT(6,6),B4MAT(6,14)            00001770
C                                                                           00001780
C     INITIALIZE AND SAVE THE VARIOUS SWITCHES.                             00001790
C                                                                           00001800
      N=NOPARM+1                                                            00001810
      DO 30 I=1,6                                                           00001820
      DO 10 J=1,6                                                           00001830
      PVOP(I,J)=0.D0                                                        00001840
   10 CONTINUE                                                              00001850
      DO 20 L=1,NEQ                                                         00001860
      VELPAR(I,L)=0.D0                                                      00001870
   20 CONTINUE                                                              00001880
   30 CONTINUE                                                              00001890
C                                                                           00001900
      KNMAX=NMAX                                                            00001910
      KMMAX=MMAX                                                            00001920
      KNEND=NEND                                                            00001930
      KMEND=MEND                                                            00001940
      KLEND=LEND                                                            00001950
      KTHIRD=ITHIRD                                                         00001960
      KJ2SQ=JTWOSQ                                                          00001970
      KDRAG=IDRAG                                                           00001980
      KDRDR=IDRDR                                                           00001990
      KCORR=ICORR                                                           00002000
      KSOLAR=ISOLAR                                                         00002010
      KTESS=ITESS                                                           00002020
      KNCRES=NCSRES                                                         00002030
      KRES=IRESON                                                           00002040
C                                                                           00002050
C     DETERMINATION OF THE A MATRIX ANALYTICALLY.                           00002060
C                                                                           00002070
      IF(IANAL.NE.1) GO TO 100                                              00002080
      CALL VARANL                                                           00002090
C                                                                           00002100
C     ACCUMULATE THE A MATRIX.                                              00002110
C                                                                           00002120
      DO 50 I=1,6                                                           00002130
      DO 40 J=1,6                                                           00002140
      PVOP(I,J)=AMAT(I,J)                                                   00002150
   40 CONTINUE                                                              00002160
   50 CONTINUE                                                              00002170
C                                                                           00002180
C     DETERMINATION OF THE A AND/OR D MATRIX BY FINITE-DIFFERENCING.        00002190
C                                                                           00002200
  100 IF(IDIFF.EQ.0) GO TO 200                                              00002210
C                                                                           00002220
C     CENTRAL BODY GRAVITATIONAL EFFECTS.                                   00002230
C                                                                           00002240
      IF(ICBVAR.LE.1) GO TO 110                                             00002250
      IF(ICBVAR.EQ.2) GO TO 120                                             00002260
      NMAX=NVAR                                                             00002270
      MMAX=0                                                                00002280
```

```
      IRESON=2                                                          00002290
      NEND=NVAR                                                         00002300
      MEND=0                                                            00002310
      NCSRES=0                                                          00002320
      LEND=LEVAR                                                        00002330
      IF(NVAR.GE.2)GO TO 120                                            00002340
  110 NMAX=0                                                            00002350
      MMAX=0                                                            00002360
      IRESON=4                                                          00002370
      ITESS=0                                                           00002380
      NCSRES=0                                                          00002390
C                                                                       00002400
C     J2 SQUARED EFFECTS.                                               00002410
C                                                                       00002420
  120 IF(J22VAR.EQ.1) GO TO 130                                         00002430
      JTWOSQ=2                                                          00002440
C                                                                       00002450
C     THIRD-BODY EFFECTS.                                               00002460
C                                                                       00002470
  130 IF(ITBVAR.EQ.1) GO TO 140                                         00002480
      ITHIRD=0                                                          00002490
C                                                                       00002500
C     DRAG EFFECTS.                                                     00002510
C                                                                       00002520
  140 IF(IDRVAR.EQ.1) GO TO 145                                         00002530
      IDRAG=0                                                           00002540
      GO TO 150                                                         00002550
  145 IDRDR=KDRFLG                                                      00002560
      ICORR=2                                                           00002570
      IF(IDRDR.LT.5)GO TO 150                                           00002580
      ICORR=1                                                           00002590
      IF(IDRDR.EQ.6) IDRDR=0                                            00002600
C                                                                       00002610
C     SOLAR RADIATION PRESSURE EFFECTS.                                 00002620
C                                                                       00002630
  150 IF(ISRVAR.EQ.1) GO TO 160                                         00002640
      ISOLAR=0                                                          00002650
C                                                                       00002660
C     COMPUTE THE MATRICES.                                             00002670
C                                                                       00002680
  160 CALL VARDIF                                                       00002690
C                                                                       00002700
C     ACCUMULATE THE A AND D MATRICES.                                  00002710
C                                                                       00002720
      DO 190 I=1,6                                                      00002730
      DO 170 J=1,6                                                      00002740
      PVOP(I,J)=PVOP(I,J)+AMAT(I,J)                                     00002750
  170 CONTINUE                                                          00002760
      DO 180 L=N,NEQ                                                    00002770
      II=L-NOPARM                                                       00002780
      VELPAR(I,L)=DMAT(I,II)                                            00002790
  180 CONTINUE                                                          00002800
  190 CONTINUE                                                          00002810
C                                                                       00002820
C     DETERMINATION OF THE A AND/OR D MATRIX BY A QUADRATURE APPROACH.  00002830
C                                                                       00002840
  200 IF(IQDRT.EQ.0) GO TO 300                                          00002850
```

```
C                                                                       00002860
C     ONLY DRAG AND SOLAR RADIATION PRESSURE ARE DONE.                  00002870
C                                                                       00002880
      NMAX=0                                                            00002890
      MMAX=0                                                            00002900
      IRESON=4                                                          00002910
      ITESS=0                                                           00002920
      NCSRES=0                                                          00002930
      ITHIRD=0                                                          00002940
      JTWOSQ=2                                                          00002950
C                                                                       00002960
C     DRAG EFFECTS.                                                     00002970
C                                                                       00002980
      IF(IDRVAR.EQ.2) GO TO 210                                         00002990
      IDRAG=0                                                           00003000
      GO TO 220                                                         00003010
  210 IDRAG=KDRAG                                                       00003020
      IDRDR=KDRFLG                                                      00003030
      ICORR=2                                                           00003040
      IF(IDRDR.LT.5)GO TO 220                                           00003050
      ICORR=1                                                           00003060
      IF(IDRDR.EQ.6) IDRDR=0                                            00003070
C                                                                       00003080
C     SOLAR RADIATION PRESSURE EFFECTS.                                 00003090
C                                                                       00003100
  220 IF(ISRVAR.EQ.2) GO TO 230                                         00003110
      ISOLAR=0                                                          00003120
      GO TO 240                                                         00003130
  230 ISOLAR=KSOLAR                                                     00003140
C                                                                       00003150
C     COMPUTE THE MATRICES.                                             00003160
C                                                                       00003170
  240 CALL VARQDR                                                       00003180
C                                                                       00003190
C     ACCUMULATE THE A AND D MATRICES.                                  00003200
C                                                                       00003210
      DO 270 I=1,6                                                      00003220
      DO 250 J=1,6                                                      00003230
      PVOP(I,J)=PVOP(I,J)+AMAT(I,J)                                     00003240
  250 CONTINUE                                                          00003250
      DO 260 L=N,NEQ                                                    00003260
      II=L-NOPARM                                                       00003270
      VELPAR(I,L)=VELPAR(I,L)+DMAT(I,II)                                00003280
  260 CONTINUE                                                          00003290
  270 CONTINUE                                                          00003300
C                                                                       00003310
C     RESTORE THE FLAGS.                                                00003320
C                                                                       00003330
  300 NMAX=KNMAX                                                        00003340
      MMAX=KMMAX                                                        00003350
      NEND=KNEND                                                        00003360
      MEND=KMEND                                                        00003370
      LEND=KLEND                                                        00003380
      ITHIRD=KTHIRD                                                     00003390
      JTWOSQ=KJ2SQ                                                      00003400
      IDRAG=KDRAG                                                       00003410
      IDRDR=KDRDR                                                       00003420
```

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```
      ICORR=KCORR                                                    00003430
      ISOLAR=KSOLAR                                                  00003440
      ITESS=KTESS                                                    00003450
      NCSRES=KNCRES                                                  00003460
      IRESON=KRES                                                    00003470
C                                                                    00003480
      RETURN                                                         00003490
      END                                                            00003500
```

******** END OF MEMBER AVEPAR 350 RECORDS ********

```
AVRAGE              AJG1324.GTDS.UPDATE.FORT

      SUBROUTINE AVRAGE                                                  00000010
C                                                                        00000020
C     VERSION OF 19 MARCH 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6      00000030
C                                                                        00000040
C     PURPOSE                                                            00000050
C          TO CONTROL THE EVALUATION OF THE AVERAGED ELEMENT RATES FOR   00000060
C          THE AVERAGED VOP ORBIT GENERATOR                              00000070
C                                                                        00000080
C     CALLING SEQUENCE                                                   00000090
C          CALL AVRAGE                                                   00000100
C                                                                        00000110
C     INPUT                                                              00000120
C          VIA BLOCK COMMON                                              00000130
C                                                                        00000140
C          VARIABLES IN COMMON CONST                                     00000150
C                                                                        00000160
C          PI       = THE CONSTANT PI                                    00000170
C          TWOPI    = TWO TIMES PI                                       00000180
C                                                                        00000190
C          VARIABLES IN COMMON FRC                                       00000200
C                                                                        00000210
C          NB       = NUMBER OF CENTRAL AND NON-CENTRAL BODIES           00000220
C          GM       = ARRAY OF GRAVITATIONAL CONSTANTS                   00000230
C          ICENT    = CENTRAL BODY INDICATOR                             00000240
C                                                                        00000250
C          VARIABLES IN COMMON SWITCH                                    00000260
C                                                                        00000270
C          IND      = ARRAY OF INTEGRATION SWITCHES                      00000280
C                                                                        00000290
C          VARIABLES IN COMMON WORKER                                    00000300
C                                                                        00000310
C          VPELMT   = ORBITAL ELEMENT VECTOR                             00000320
C          K        = DERIVATIVE ARRAY INDICATOR OF VALUE AT T=T(K)      00000330
C          NOPARM   = NUMBER OF ORBITAL ELEMENTS                         00000340
C          T        = TIME FROM EPOCH IN SECONDS                         00000350
C                                                                        00000360
C          VARIABLES IN COMMON ANAVIN                                    00000370
C                                                                        00000380
C          JTWOSQ   = CONTROL SWITCH FOR 2ND ORDER J2 PERTURBATION       00000390
C          ITMDEP   = CONTROL SWITCH FOR NUMERICAL TIME-DEPENDENT        00000400
C                     AVERAGING                                          00000410
C          ITMIND   = CONTROL SWITCH FOR NUMERICAL TIME-INDEPENDENT      00000420
C                     AVERAGING                                          00000430
C          IDRAG    = CONTROL SWITCH FOR DRAG                           00000440
C          ISOLAR   = CONTROL SWITCH FOR SOLAR RADIATION PRESSURE       00000450
C          ITESS    = CONTROL SWITCH FOR LOW-ORDER CENTRAL BODY         00000460
C                     SPHERICAL HARMONIC AVERAGING                       00000470
C          ITRES    = CONTROL SWITCH FOR HIGH-ORDER CENTRAL BODY        00000480
C                     SPHERICAL HARMONIC AVERAGING                       00000490
C          IDRDR    = CONTROL SWITCH FOR 2ND ORDER DRAG EFFECTS          00000500
C                                                                        00000510
C     OUTPUT                                                             00000520
C                                                                        00000530
C          VARIABLES IN COMMON SWITCH                                    00000540
C                                                                        00000550
C          NBOPT    = ARRAY OF CENTRAL AND NON-CENTRAL BODY INDICATORS   00000560
C          IQUAD    = ARRAY OF QUADRATURE CONTROL SWITCHES               00000570
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AVRAGE              AJG1324.GTDS.UPDATE.FORT

C                                                                       00000580
C          VARIABLES IN COMMON WORKER                                   00000590
C                                                                       00000600
C          DPARRY  = ARRAY OF ELEMENT DERIVATIVES                       00000610
C          XMEAN   = MEAN MOTION                                        00000620
C          XMEANO  = XMEAN AT TZERO                                     00000630
C          TZERO   = TIME OF THE DERIVATIVE EVALUATION                  00000640
C          SELEMS  = VPELMT AT TZERO                                    00000650
C          XLAMDA  = VALUE OF THE MEAN LONGITUDE AT TIME TZERO          00000660
C          NPERTB  = PERTURBATION INDICATOR                             00000670
C                  = 0 FOR CONTINUOUS PERTURBATIONS                     00000680
C                  = 1 FOR DRAG ONLY                                    00000690
C                  = 2 FOR SOLAR RADIATION PRESSURE ONLY                00000700
C          NTIME   = COUNTER FOR QUADRATURE INTEGRANDS                  00000710
C          AVRATE  = VECTOR OF INTEGRALS EVALUATED IN VOQUAD            00000720
C                                                                       00000730
C     EXTERNAL REFERENCES                                               00000740
C                                                                       00000750
C          VOQUAD                                                       00000760
C          LIMITE                                                       00000770
C                                                                       00000780
C     REFERENCE                                                         00000790
C                                                                       00000800
C          GTDS SUBROUTINE DESCRIPTION AVRAGE                           00000810
C                                                                       00000820
C     METHOD                                                            00000830
C          METHOD IS OUTLINED IN REFERENCE                              00000840
C                                                                       00000850
C     ANALYSIS                                                          00000860
C          ANNE C. LONG, COMPUTER SCIENCES CORPORATION                  00000870
C                                                                       00000880
C     PROGRAMMER                                                        00000890
C          ANNE C. LONG, COMPUTER SCIENCES CORPORATION                  00000900
C                                                                       00000910
C     MODIFICATIONS                                                     00000920
C          06/15/74 BY R.PAJERSKI- ADDED NEW ELEMENT SET(KEPLERIAN)     00000930
C          01/01/75 BY J. M. MEYER TO MODULARIZE THE AVERAGING PROGRAM  00000940
C          1/1/78 BY LEO W. EARLY JR. TO MODULARIZE AVERAGING, ELIM-    00000950
C          INATE ERRORS, AND PUT CODE IN STRUCTURED FORM.               00000960
C          19/3/79 BY ANDREW J. GREEN, CPT, U.S. ARMY, CSDL, INCLUDED   00000970
C          2ND-ORDER DRAG EFFECTS                                       00000980
C          7/30/79 BY A. BOBICK, CSDL, TO COUNT FORCE EVALUATIONS       00000985
C                                                                       00000990
C*****START PROGRAM*************************************************00001000
C                                                                       00001010
C                                                                       00001020
      IMPLICIT REAL*8(A-H,O-Z)                                          00001030
      DIMENSION       DPARRY(15,10)          ,VPELMT(10) ,AVRATE(10) ,00001040
     2                GM(11)      ,POSVEL(6,8),DUMMY(3)                00001050
      DIMENSION       IBODY(9)    ,IBODYS(3)                            00001060
C                                                                       00001070
      COMMON/ANAVIN/  ISTESS      ,ITESSE     ,ITRESE     ,ISTHIR     ,00001080
     1                ITHRAR      ,ITHRE      ,ITESS      ,NEND       ,00001090
     2                MEND        ,LEND       ,ITRES      ,NRES(10)   ,00001100
     3                NRES(10)    ,JRES(10)   ,NCSRES     ,LENDRS     ,00001110
     4                ITHIRD      ,MENDTH(8)  ,MENDTH(8)  ,JTWOSQ     ,00001120
     5                IQUAD(4)    ,NQUAD(4)   ,JPOS       ,JA         ,00001130
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     6                JB          ,JC          ,JGHA        ,KPOS        ,00001140
     7                KA          ,KB          ,KC          ,KGHA        ,00001150
     8                ICORR       ,IFOS        ,IMOON       ,IMEAN       ,00001160
     9                IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00001170
     1                JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00001180
     2                J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00001190
     3                JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00001200
     4                KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00001210
     5                KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00001220
     6                MMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR       00001230
      EQUIVALENCE     (ITMDEP                  ,IQUAD(1)                 ),00001240
     2                (ITMIND                  ,IQUAD(2)                 ),00001250
     3                (IDRAG                   ,IQUAD(3)                 ),00001260
     4                (ISOLAR                  ,IQUAD(4)                 ) 00001270
C                                                                          00001280
      COMMON/CONST /  RCONST(51)                                           00001290
      EQUIVALENCE     (TWOPI                   ,RCONST(1)                ) 00001300
C                                                                          00001310
      COMMON/FRC   /  RFRC(1300)  ,IFRC(50)                                00001320
      EQUIVALENCE     (GM(1)                   ,RFRC(2)                  ),00001330
     2                (POSVEL(1,1)             ,RFRC(101)                ),00001340
     3                (NBODY                   ,IFRC(7)                  ),00001350
     4                (ISUN                    ,IFRC(14)                 ) 00001360
C                                                                          00001370
      COMMON/SWITCH/  ISWIT(225)                                           00001380
      EQUIVALENCE     (IVOP                    ,ISWIT(3)                 ),00001390
     2                (ITHOBD                  ,ISWIT(6)                 ),00001400
     3                (IRESCN                  ,ISWIT(39)                ),00001410
     4                (IBODY(1)                ,ISWIT(201)               ),00001420
     5                (ICENT                   ,IBODY(1)                 ) 00001430
C                                                                          00001440
      COMMON/TIMER/   TIM(51),     NOFC                                    00001443
C                                                                          00001446
      COMMON/WORKER/  RWORK(3264) ,IWORK(13)                               00001450
      EQUIVALENCE     (DPARRY(1,1)             ,RWORK(7)                 ),00001460
     2                (VPELMT(1)               ,RWORK(2829)              ),00001470
     3                (AVRATE(1)               ,RWORK(3068)              ),00001480
     4                (XLAMDA                  ,RWORK(2851)              ),00001490
     5                (XMEAN                   ,RWORK(2814)              ),00001500
     6                (T                       ,RWORK(3215)              ),00001510
     7                (TZERO                   ,RWORK(2850)              ) 00001520
      EQUIVALENCE     (K                       ,IWORK(2)                 ),00001530
     2                (NOPARM                  ,IWORK(3)                 ),00001540
     3                (NFERT                   ,IWORK(9)                 ) 00001550
C                                                                          00001560
C ***************** START PROGRAM ***************                         00001561
C                                                                          00001562
C  *** INCREMENT NUMBER OF FORCE EVALUATIONS                               00001563
C                                                                          00001564
      NOFC = NOFC + 1                                                      00001565
C                                                                          00001566
C     SAVE THE TIME AND FAST VARIABLE.                                     00001570
C                                                                          00001580
      TZERO = T                                                            00001590
      XLAMDA = VPELMT(NOPARM)                                              00001600
C                                                                          00001610
C     INITIALIZE THE AVERAGED ELEMENT RATES.                               00001620
```

```
C
      DO 100 I=1,NOPARM                                                 00001630
         DPARRY(K,I) = 0.D0                                             00001640
  100 CONTINUE                                                          00001650
C                                                                       00001660
C     COMPUTE THE MEAN MOTION AND TEST FOR TWO-BODY MOTION AND KEPLERIAN00001670
C     AVERAGING.                                                        00001680
C                                                                       00001690
      XMEAN = DSQRT(GM(ICENT)/(VPELMT(1)*VPELMT(1)*VPELMT(1)))          00001700
      IF (ITWOBD.EQ.1) GO TO 600                                        00001710
      IF (IVOP.NE.12) GO TO 200                                         00001720
C                                                                       00001730
C     COMPUTE AUXILIARY QUANTITIES.                                     00001740
C                                                                       00001750
      CALL AUXPAR(VPELMT)                                               00001760
C                                                                       00001770
C     COMPUTE THE ELEMENT RATES FOR ANALYTICALLY AVERAGED PERTURBATIONS.00001780
C                                                                       00001790
      CALL ANAVR                                                        00001800
C                                                                       00001810
C     ACCUMULATE THE ELEMENT RATES.                                     00001820
C                                                                       00001830
      DO 110 I=1,NOPARM                                                 00001840
         DPARRY(K,I) = AVRATE(I)                                        00001850
  110 CONTINUE                                                          00001860
C                                                                       00001870
C                 COMPUTE THE ELEMENT RATES DUE TO THE J2 SQUARED      00001880
C                 PERTURBATION.                                         00001890
C                                                                       00001900
      IF (JTWOSQ.NE.1) GO TO 200                                        00001910
      CALL J2SQR(AVRATE)                                                00001920
C                                                                       00001930
C                 ACCUMULATE THE ELEMENT RATES.                         00001940
C                                                                       00001950
      DO 120 I=1,NOPARM                                                 00001960
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)                          00001970
  120 CONTINUE                                                          00001980
C                                                                       00001990
C                 COMPUTE THE ELEMENT RATES DUE TO NUMERICALLY AVERAGED00002000
C                 TIME-DEPENDENT CONTINUOUS PERTURBATIONS.              00002010
C                                                                       00002020
  200 IF (ITMDEP.EQ.0) GO TO 300                                        00002030
C                                                                       00002040
C                 SET THE CENTRAL BODY SPHERICAL HARMONIC SWITCH.       00002050
C                                                                       00002060
      IRESNS = IRESON                                                   00002070
C                                                                       00002080
      IF (ITESS.NE.1) GO TO 202                                         00002090
      IF (ITRES.NE.1) GO TO 201                                         00002100
      IF (IRESON.NE.1) GO TO 201                                        00002110
         IRESON = 1                                                     00002120
         GO TO 210                                                      00002130
  201    IRESON = 2                                                    00002140
         GO TO 210                                                      00002150
C                                                                       00002160
  202 IF (ITRES.NE.1) GO TO 203                                         00002170
      IF (IRESON.NE.1) GO TO 203                                        00002180
                                                                        00002190
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```
      IRESON = 3                                                        00002200
      GO TO 210                                                         00002210
  203 IRESON = 4                                                        00002220
C                                                                       00002230
C                 COMPUTE THE LIMITS AND VALUE OF THE AVERAGING         00002240
C                 INTEGRAL.                                             00002250
C                                                                       00002260
  210 NPERT = 1                                                         00002270
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00002280
      CALL VOQUAD(FLOW,FHIGH)                                           00002290
C                                                                       00002300
C                 ACCUMULATE THE ELEMENT RATES.                         00002310
C                                                                       00002320
      DO 220 I=1,NOPARM                                                 00002330
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                    00002340
  220 CONTINUE                                                          00002350
C                                                                       00002360
C                 RESTORE THE CENTRAL BODY HIGH-ORDER RESONANCE SWITCH.00002370
C                                                                       00002380
      IRESON = IRESNS                                                   00002390
C                                                                       00002400
C                 RESTORE THE TIME.                                     00002410
C                                                                       00002420
      T = TZERO                                                         00002430
C                                                                       00002440
C                 COMPUTE THE ELEMENT RATES DUE TO NUMERICALLY AVERAGED00002450
C                 TIME-INDEPENDENT CONTINUOUS PERTURBATIONS.            00002460
C                                                                       00002470
  300 IF (ITMIND.EQ.0) GO TO 400                                        00002480
C                                                                       00002490
C                 SET THE CENTRAL BODY SPHERICAL HARMONIC SWITCH.       00002500
C                                                                       00002510
      IRESNS = IRESON                                                   00002520
C                                                                       00002530
      IF (ITESS.NE.2) GO TO 301                                         00002540
         IRESON = 2                                                     00002550
         GO TO 310                                                      00002560
  301    IRESON = 4                                                     00002570
C                                                                       00002580
C                 COMPUTE THE POSITIONS OF THE PERTURBING BODIES.       00002590
C                                                                       00002600
  310 CALL EVAL(TZERO,KPOS,KA,KB,KC,KGHA,IBODY,POSVEL)                  00002610
C                                                                       00002620
C                 COMPUTE THE LIMITS AND VALUE OF THE AVERAGING         00002630
C                 INTEGRAL.                                             00002640
C                                                                       00002650
      NPERT = 2                                                         00002660
      IF (ITMDEP.EQ.0) CALL LIMITS(FLOW,FHIGH,XLAMDA)                   00002670
      CALL VOQUAD(FLOW,FHIGH)                                           00002680
C                                                                       00002690
C                 ACCUMULATE THE ELEMENT RATES.                         00002700
C                                                                       00002710
      DO 320 I=1,NOPARM                                                 00002720
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                    00002730
  320 CONTINUE                                                          00002740
C                                                                       00002750
C                 RESTORE THE CENTRAL BODY HIGH-ORDER RESONANCE SWITCH.00002760
```

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C                                                                       00002770
      IRESON = IRESNS                                                   00002780
C                                                                       00002790
C                   COMPUTE THE ELEMENT RATES DUE TO DRAG.              00002800
C                                                                       00002810
  400 IF (IDRAG.EQ.0) GO TO 500                                         00002820
C                                                                       00002830
C                   COMPUTE THE HIGHER ORDER DRAG EFFECTS               00002840
C                                                                       00002850
      IF (IDRDR.EQ.0) GO TO 410                                         00002860
      CALL HODRAG                                                       00002870
      IF(IDRDR.NE.5)GO TO 500                                           00002880
C                                                                       00002890
C                   COMPUTE FIRST-ORDER EFFECTS IF NECESSARY.           00002900
C                                                                       00002910
C                   COMPUTE THE LIMITS OF THE AVERAGING INTEGRAL.       00002920
C                                                                       00002930
  410 NPERT = 3                                                         00002940
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00002950
      IF (NPERT.EQ.0) GO TO 500                                         00002960
C                                                                       00002970
C                   FIND THE POSITION OF THE SUN SO THAT THE DIURNAL    00002980
C                   BULGE IN THE ATMOSPHERE CAN BE LOCATED.             00002990
C                                                                       00003000
      IBODYS(2) = IBODY(2)                                              00003010
      IBODYS(3) = IBODY(3)                                              00003020
      NBODYS = NBODY                                                    00003030
C                                                                       00003040
      IBODY(2) = 3                                                      00003050
      IBODY(3) = 0                                                      00003060
      NBODY = 2                                                         00003070
C                                                                       00003080
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                           00003090
C                                                                       00003100
C                   COMPUTE THE VALUE OF THE AVERAGING INTEGRAL.        00003110
C                                                                       00003120
      CALL VOQUAD(FLOW,FHIGH)                                           00003130
C                                                                       00003140
C                   ACCUMULATE THE ELEMENT RATES.                       00003150
C                                                                       00003160
      DO 420 I=1,NOPARM                                                 00003170
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                    00003180
  420 CONTINUE                                                          00003190
C                                                                       00003200
C                   RESTORE THE THIRD BODIES.                           00003210
C                                                                       00003220
  440 IF (ISOLAR.NE.0) GO TO 510                                        00003230
      IBODY(2) = IBODYS(2)                                              00003240
      IBODY(3) = IBODYS(3)                                              00003250
      NBODY = NBODYS                                                    00003260
      GO TO 600                                                         00003270
C                                                                       00003280
C                   COMPUTE THE ELEMENT RATES DUE TO SOLAR RADIATION    00003290
C                   PRESSURE.                                           00003300
C                                                                       00003310
  500 IF (ISOLAR.EQ.0) GO TO 600                                        00003320
C                                                                       00003330
```

```
C                  FIND AND STORE THE POSITION OF THE SUN.                00003340
C                                                                         00003350
      IBODYS(2) = IBODY(2)                                                00003360
      IBODYS(3) = IBODY(3)                                                00003370
      NBODYS = NBODY                                                      00003380
C                                                                         00003390
      IBODY(2) = 3                                                        00003400
      IBODY(3) = 0                                                        00003410
      NBODY = 2                                                           00003420
C                                                                         00003430
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                             00003440
C                                                                         00003450
  510 ISUNS = ISUN                                                        00003460
      ISUN = - 1                                                          00003470
      CALL PMASS(IBODY,DUMMY,DUMMY,IERR)                                  00003480
      ISUN = 1                                                            00003490
C                                                                         00003500
C                  COMPUTE THE LIMITS AND VALUE OF THE AVERAGING          00003510
C                  INTEGRAL.                                              00003520
C                                                                         00003530
      NPERT = 4                                                           00003540
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                      00003550
      CALL VOQUAD(FLOW,FHIGH)                                             00003560
C                                                                         00003570
C                  ACCUMULATE THE ELEMENT RATES.                          00003580
C                                                                         00003590
      DO 520 I=1,NOPARM                                                   00003600
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                      00003610
  520 CONTINUE                                                            00003620
C                                                                         00003630
C                  RESTORE THE THIRD BODIES.                              00003640
C                                                                         00003650
      IBODY(2) = IBODYS(2)                                                00003660
      IBODY(3) = IBODYS(3)                                                00003670
      NBODY = NBODYS                                                      00003680
      ISUN = ISUNS                                                        00003690
C                                                                         00003700
C                  RESTORE THE FAST VARIABLE AND ACCUMULATE THE TWO-      00003710
C                  BODY ELEMENT RATES.                                    00003720
C                                                                         00003730
  600 VPELMT(NOPARM) = XLAMDA                                             00003740
      DPARRY(K,NOPARM) = DPARRY(K,NOPARM) + XMEAN                         00003750
      RETURN                                                              00003760
C                                                                         00003770
      END                                                                 00003780
```

******** END OF MEMBER AVRAGE 387 RECORDS ********

```
      SUBROUTINE DRAGJ2                                                  00000010
                                                                         00000020
C                                                                        00000030
C     VERSION OF 11 MAY 1979 IMPLEMENTED AT CSDL ON ANDAHL 470 V6        00000040
C                                                                        00000050
C     PURPOSE                                                            00000060
C        ANALYTICALLY DETERMINE THE DRAG-J2 COUPLING EFFECT, GOOD TO     00000070
C        ZERO ORDER IN THE ECCENTRICITY AND FOR DIRECT ORBITS ONLY.       00000080
C                                                                        00000090
C     PARAMETERS                                                         00000100
C                                                                        00000110
C       /WORKER/                                                         00000120
C                                                                        00000130
C        A       I   MEAN A                                             00000140
C        XH      I   MEAN H                                             00000150
C        XK      I   MEAN K                                             00000160
C        P       I   MEAN P                                             00000170
C        Q       I   MEAN Q                                             00000180
C        XL      I   MEAN LAMDA                                         00000190
C        XN      I   MEAN MEAN MOTION                                   00000200
C        AVRATE  I   ARRAY OF THE MEAN ELEMENT RATES                    00000210
C                                                                        00000220
C       /FRC/                                                            00000230
C                                                                        00000240
C        AB      I   ARRAY OF EQUATORIAL RADIUS FOR THE VARIOUS BODIES  00000250
C        GM      I   ARRAY OF THE GRAVITAIONAL CONSTANTS(I.E. MU)       00000260
C        CS      I   ARRAY OF THE SPHERICAL HARMONIC COEFFICIENTS       00000270
C                                                                        00000280
C     REFERENCES                                                         00000290
C                                                                        00000300
C       "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW ALTITUDE   00000310
C       SATELLITES."  BY ANDREW J. GREEN                                 00000320
C                                                                        00000330
C     ANALYSIS                                                           00000340
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                            00000350
C                                                                        00000360
C     PROGRAMMER                                                         00000370
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                            00000380
C                                                                        00000390
C*****START PROGRAM*************************************************** 00000400
C                                                                        00000410
C                                                                        00000420
      IMPLICIT REAL*8(A-H,O-Z)                                           00000430
      DIMENSION       CS(504),AVRATE(10) ,GM(11),AB(11)                  00000440
      DIMENSION       IBODY(9)                                           00000450
C                                                                        00000460
      COMMON/FRC   /  RFRC(1300)  ,IFRC(50)                             00000470
      EQUIVALENCE     (GM(1)                  ,RFRC(2)                 ),00000480
     1                (AB(1)                  ,RFRC(24)                ),00000490
     2                (CS(1)                  ,RFRC(355)               ) 00000500
C                                                                        00000510
      COMMON/SWITCH/  ISWIT(225)                                         00000520
      EQUIVALENCE     (IBODY(1)               ,ISWIT(201)              ),00000530
     1                (ICENT                  ,IBODY(1)                ) 00000540
C                                                                        00000550
      COMMON/WORKER/  RWORK(3264) ,IWORK(13)                             00000560
      EQUIVALENCE     (A                      ,RWORK(2829)             ),00000570
     2                (XH                     ,RWORK(2830)             ),00000580
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     3                 (XK                    ,RWORK(2831)          ),00000580
     4                 (P                     ,RWORK(2832)          ),00000590
     5                 (Q                     ,RWORK(2833)          ),00000600
     6                 (XL                    ,RWORK(2834)          ),00000610
     7                 (AVRATE(1)             ,RWORK(3068)          ),00000620
     8                 (XLAMDA                ,RWORK(2851)          ),00000630
     9                 (XN                    ,RWORK(2814)          ) 00000640
      COMMON/SPCOEF/   CX(10,10),DX(10,10)                             00000650
C                                                                      00000660
C     SET THE COMMON FACTORS.                                          00000670
C                                                                      00000680
      P2=P*P                                                           00000690
      Q2=Q*Q                                                           00000700
      C=P2+Q2                                                          00000710
      C2=C*C                                                           00000720
      A4=A*A*A*A                                                       00000730
      CP1=C+1                                                          00000740
      CP12=CP1*CP1                                                     00000750
      RE=AB(ICENT)                                                     00000760
      FACT42=(-CS(2)*GM(ICENT)*RE*RE)/(A4*CP12*XN)                     00000770
      FACT43=FACT42/CP1                                                00000780
      FACT52=FACT42/A                                                  00000790
      FACT53=FACT52/CP1                                                00000800
      FACT62=FACT52/A                                                  00000810
      FACT51=FACT52*CP1                                                00000820
      FACT61=FACT62*CP1                                                00000830
C                                                                      00000840
C     DETERMINE A-DOT.                                                 00000850
C                                                                      00000860
      ADOT42=FACT42*P*Q*(-24.D0*CX(6,2)-63.D0*DX(3,3)+3.D0*DX(3,1)+    00000870
     *63.D0*CX(2,3)+3.D0*CX(2,1))+                                     00000880
     *FACT42*(-3.D0*CX(3,1)*(2.D0*P2+C2-5.D0*C+1.D0)/2.D0+             00000890
     *3.D0*DX(2,1)*(2.D0*P2-C2+3.D0*C-1.D0)/2.D0)+                     00000900
     *FACT42*(2.D0*P2-C)*(-12.D0*DX(6,2)+63.D0*CX(3,3)/2.D0+           00000910
     *63.D0*DX(2,3)/2.D0)                                              00000920
      ADOT43=FACT43*(12.D0*DX(5,2)*P*(4.D0*Q2-C-1.D0)-                 00000930
     *12.D0*CX(5,2)*Q*(4.D0*P2-C+1.D0)+12.D0*DX(4,2)*Q*(4.D0*P2-C-    00000940
     *1.D0)-12.D0*CX(4,2)*P*(4.D0*P2-3.D0*C-1.D0))                     00000950
      ADOT52=FACT52*(30.D0*DX(1,2)*P*Q-15.D0*CX(1,2)*(2.D0*P2-C))      00000960
C                                                                      00000970
      AVRATE(1)=ADOT42+ADOT43+ADOT52                                   00000980
C                                                                      00000990
C     DETERMINE H-DOT                                                  00001000
C                                                                      00001010
      DOTH52=FACT52*P*Q*(-63.D0*DX(6,3)/2.D0+3.D0*DX(6,1)/2.D0+        00001020
     *51.D0*CX(3,4)+51.D0*DX(2,4)+3.D0*DX(2,2)-3.D0*CX(3,2)*(C+4.D0))- 00001030
     *FACT52*(3.D0*DX(3,2)*(4.D0*(C+4.D0)*P2+C2-20.D0*C+3.D0)+         00001040
     *3.D0*CX(2,2)*(4.D0*P2-(C-3.D0)*(3.D0*C-1.D0)))/4.D0+             00001050
     *FACT52*(2.D0*P2-C)*(63.D0*CX(6,3)/4.D0+51.D0*DX(3,4)/2.D0-       00001060
     *51.D0*CX(2,4)/2.D0)-                                             00001070
     *FACT52*3.D0*CX(6,1)*(2.D0*P2+C2-5.D0*C+1.D0)/4.D0                00001080
      DOTH53=FACT53*(4.D0*Q2-C-1.D0)*P*(-21.D0*CX(5,3)/2.D0+           00001090
     *3.D0*CX(5,1)/2.D0)+                                              00001100
     *FACT53*(4.D0*P2-C-1.D0)*Q*(-21.D0*CX(4,3)/2.D0+                  00001110
     *3.D0*CX(4,1)/2.D0)+                                              00001120
     *FACT53*(-21.D0*DX(5,3)*(4.D0*P2-C+1.D0)*Q/2.D0+                  00001130
     *3.D0*DX(5,1)*(4.D0*P2-7.D0*C+7.D0)*Q/2.D0)+                      00001140
```

```
     *FACT53*(-21.D0*DX(4,3)*P*(4.D0*P2-3.D0*C-1.D0)+                   00001150
     *3.D0*DX(4,1)*P*(4.D0*P2-9.D0*C+5.D0))/2.D0                        00001160
      DOTH62=FACT62*P*Q*(-147.D0*CX(1,3)+21.D0*CX(1,1))/4.D0+           00001170
     *FACT62*(21.D0*DX(1,1)*(2.D0*P2+C2-5.D0*C+1.D0)-                   00001180
     *147.D0*DX(1,3)*(2.D0*P2-C))/8.D0                                  00001190
C                                                                       00001200
      AVRATE(2)=DOTH52+DOTH53+DOTH62                                    00001210
C                                                                       00001220
C     DETERMINE K-DOT.                                                  00001230
C                                                                       00001240
      DOTK52=FACT52*P*Q*(-63.D0*CX(6,3)/2.D0-3.D0*CX(6,1)/2.D0-         00001250
     *51.D0*DX(3,4)+51.D0*CX(2,4)+3.D0*DX(3,2)+3.D0*CX(2,2)*(C+4.D0))+  00001260
     *FACT52*(3.D0*DX(2,2)*(4.D0*(C+4.D0)*P2-5.D0*C2+4.D0*C-3.D0)-      00001270
     *3.D0*CX(3,2)*(4.D0*P2+3.D0*C2-14.D0*C+3.D0))/4.D0+                00001280
     *FACT52*(2.D0*P2-C)*(-63.D0*DX(6,3)/4.D0+51.D0*CX(3,4)/2.D0+       00001290
     *51.D0*DX(2,4)/2.D0)-                                              00001300
     *FACT52*3.D0*DX(6,1)*(2.D0*P2-C2+3.D0*C-1.D0)/4.D0                 00001310
      DOTK53=FACT53*(4.D0*Q2-C-1.D0)*P*(21.D0*DX(5,3)/2.D0+             00001320
     *3.D0*DX(5,1)/2.D0)+                                               00001330
     *FACT53*(4.D0*P2-C-1.D0)*Q*(21.D0*DX(4,3)/2.D0+                    00001340
     *3.D0*DX(4,1)/2.D0)+                                               00001350
     *FACT53*(-21.D0*CX(5,3)*(4.D0*P2-C+1.D0)*Q/2.D0-                   00001360
     *3.D0*CX(5,1)*(4.D0*P2+5.D0*C-5.D0)*Q/2.D0)-                       00001370
     *FACT53*(21.D0*CX(4,3)*P*(4.D0*P2-3.D0*C-1.D0)+                    00001380
     *3.D0*CX(4,1)*P*(4.D0*P2+3.D0*C-7.D0))/2.D0                        00001390
      DOTK62=FACT62*P*Q*(147.D0*DX(1,3)+21.D0*DX(1,1))/4.D0-            00001400
     *FACT62*(21.D0*CX(1,1)*(2.D0*P2-C2+3.D0*C-1.D0)+                   00001410
     *147.D0*CX(1,3)*(2.D0*P2-C))/8.D0                                  00001420
C                                                                       00001430
      AVRATE(3)=DOTK52+DOTK53+DOTK62                                    00001440
C                                                                       00001450
C     DETERMINE P-DOT.                                                  00001460
C                                                                       00001470
      PDOT51=FACT51*(C-1.D0)*Q*(3.D0*CX(6,2)/2.D0+(21.D0*DX(3,3)-       00001480
     *21.D0*DX(3,1)-21.D0*CX(2,3)+15.D0*CX(2,1))/8.D0)+                 00001490
     *FACT51*(C-1.D0)*P*(3.D0*DX(6,2)/2.D0+(-21.D0*CX(3,3)+3.D0*CX(3,1)-00001500
     *21.D0*DX(2,3)-3.D0*DX(2,1))/8.D0)                                 00001510
      PDOT52=FACT52*(3.D0*DX(5,2)*(4.D0*Q2+C2-1.D0)/4.D0+               00001520
     *3.D0*P*Q*(-CX(5,2)+DX(4,2))-3.D0*CX(4,2)*(4.D0*P2+C2-1.D0)/4.D0)  00001530
      PDOT61=FACT61*21.D0*(C-1.D0)*(-Q*DX(1,2)+P*CX(1,2))/8.D0          00001540
C                                                                       00001550
      AVRATE(4)=PDOT51+PDOT52+PDOT61                                    00001560
C                                                                       00001570
C     DETERMINE Q-DOT.                                                  00001580
C                                                                       00001590
      QDOT51=FACT51*(C-1.D0)*Q*(-3.D0*DX(6,2)/2.D0+(21.D0*CX(3,3)-      00001600
     *3.D0*CX(3,1)+21.D0*DX(2,3)+3.D0*DX(2,1))/8.D0)+                   00001610
     *FACT51*(C-1.D0)*P*(3.D0*CX(6,2)/2.D0+(21.D0*DX(3,3)+15.D0*DX(3,1)-00001620
     *21.D0*CX(2,3)-21.D0*CX(2,1))/8.D0)                                00001630
      QDOT52=FACT52*(3.D0*CX(5,2)*(4.D0*Q2+C2-1.D0)/4.D0+               00001640
     *3.D0*P*Q*(DX(5,2)+CX(4,2))+3.D0*DX(4,2)*(4.D0*P2+C2-1.D0)/4.D0)   00001650
      QDOT61=FACT61*21.D0*(C-1.D0)*(-Q*CX(1,2)-P*DX(1,2))/8.D0          00001660
C                                                                       00001670
      AVRATE(5)=QDOT51+QDOT52+QDOT61                                    00001680
C                                                                       00001690
C     DETERMINE LAMDA-DOT.                                              00001700
C                                                                       00001710
```

```
DRAGJ2              AJG1324.GTDS.UPDATE.FORT

      C7=C-7.D0                                                          00001720
      C15=C-15.D0                                                        00001730
      C215=2.D0*C-15.D0                                                  00001740
      DOTL52=FACT52*P*Q*(C7*6.D0*DX(6,2)+C215*(-21.D0*CX(3,3)+           00001750
     *3.D0*CX(3,1)-21.D0*DX(2,3)-3.D0*DX(2,1))/4.D0)+                    00001760
     *FACT52*3.D0*CX(2,1)*(2.D0*P2*C215+23.D0*C2-49.D0*C+                00001770
     *13.D0)/8.D0+                                                       00001780
     *FACT52*3.D0*DX(3,1)*(2.D0*P2*C215-27.D0*C2+79.D0*C-                00001790
     *13.D0)/8.D0+                                                       00001800
     *FACT52*(2.D0*P2-C)*(-3.D0*C7*CX(6,2)+21.D0*C215*                   00001810
     *(-DX(3,3)+CX(2,3))/8.D0)                                           00001820
      DOTL53=FACT53*3.D0*CX(5,2)*P*(2.D0*C15*Q2-C7*CP1)+                 00001830
     *FACT53*3.D0*CX(4,2)*Q*(2.D0*C15*P2-C7*CP1)+                        00001840
     *FACT53*3.D0*DX(5,2)*(2.D0*C15*P2+9.D0*C-7.D0)*Q+                   00001850
     *FACT53*3.D0*DX(4,2)*P*(2.D0*C15*P2-2.D0*C2+21.D0*C+7.D0)           00001860
      DOTL62=FACT62*21.D0*C7*(CX(1,2)*P*Q/2.D0+DX(1,2)*(2.D0*P2-C)/4.D0)00001870
C                                                                        00001880
      AVRATE(6)=DOTL52+DOTL53+DOTL62                                     00001890
C                                                                        00001900
      RETURN                                                             00001910
C                                                                        00001920
      END                                                                00001930

******** END OF MEMBER DRAGJ2     193 RECORDS ********
```

EQUVAR AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE EQUVAR                                                 00000010
C                                                                       00000020
C     PURPOSE                                                           00000030
C                                                                       00000040
C        EXECUTIVE PROGRAM FOR THE EVALUATION OF THE RIGHT HAND         00000050
C        SIDE OF THE DIFFERENTIAL EQUATION FOR THE PARTIALS.            00000060
C                                                                       00000070
C     PARAMETERS                                                        00000080
C                                                                       00000090
C        /SWITCH/                                                       00000100
C                                                                       00000110
C            IND(3)   I   VOP INDICATOR.                                00000120
C                         8  = OSCULATING KEPLERIAN                     00000130
C                         12 = AVERAGED EQUINOCTIAL                     00000140
C                                                                       00000150
C     PROGAMMER                                                         00000160
C                                                                       00000170
C        ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                          00000180
C                                                                       00000190
C                                                                       00000200
      IMPLICIT REAL*8(A-H,O-Z)                                          00000210
      COMMON/SWITCH/ IND(225)                                           00000220
C                                                                       00000230
      IF(IND(3).EQ.8) GO TO 10                                          00000240
      IF(IND(3).EQ.12) GO TO 20                                         00000250
      GO TO 90                                                          00000260
   10 CALL VARFRC                                                       00000270
      CALL VARFRV                                                       00000280
      GO TO 100                                                         00000290
   20 CALL AVEPAR                                                       00000300
      GO TO 100                                                         00000310
   90 WRITE(6,1000)                                                     00000320
  100 RETURN                                                            00000330
 1000 FORMAT(/,'PARTIALS ARE ASKED FOR BUT THE VOP ORBIT TYPE IS WRONG')00000340
      END                                                               00000350
```

******** END OF MEMBER EQUVAR 35 RECORDS ********

ESTSET AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE ESTSET                                                 00000010
C                                                                       00000020
C     PURPOSE                                                           00000030
C                                                                       00000040
C       FOR THE INITIAL DESIGN, THIS SUBROUTINE SETS THE VALUES OF THE  00000050
C       SWITCHES USED IN THE AVERAGE PARTIAL GENERATOR AND THE SHORT    00000060
C       PERIODIC PARTIAL GENERATOR.                                     00000070
C                                                                       00000080
C     PROGRAMMER                                                        00000090
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                           00000100
C                                                                       00000110
      IMPLICIT REAL*8(A-H,O-Z)                                          00000120
C                                                                       00000130
      COMMON/ESTFLG/ IANAL    ,IDIFF    ,IQDRT    ,ICBVAR   ,J22VAR   , 00000140
     1               ITBVAR   ,IDRVAR   ,ISRVAR   ,NVAR     ,LEVAR    , 00000150
     2               KDRFLG   ,KPAR     ,KSTEP(20),KVRFLG   ,KCPSPP   , 00000160
     3               KCBSPP   ,KTBSPP   ,KDRSPP   ,KSRSPP   ,NVARSP   , 00000170
     4               IVSPNC(4),KATMOS   ,KPRTB1   ,KPRTB2   ,KPRTB3   , 00000180
     5               KPRTB4                                             00000190
C                                                                       00000200
      COMMON/NEWPAR/ XS(20)   ,APAR(5)  ,APARVR(5),CDRVAR               00000210
C                                                                       00000220
      COMMON/FRC   /   RFRC(1300)  ,IFRC(50)                            00000230
      EQUIVALENCE      (NMAX                    ,IFRC(3)              ),00000240
     1                 (MMAX                    ,IFRC(4)              ) 00000250
C                                                                       00000260
C   /ESTFLG/                                                            00000270
C                                                                       00000280
C       IANAL           CONTROL SWITCH FOR ANALYTICAL APPROACH IN THE   00000290
C                       AVERAGED PARTIAL GENERATOR.*                    00000300
C       IDIFF           CONTROL SWITCH FOR FINITE-DIFFERENCING APPROACH 00000310
C                       IN THE AVERAGED PARTIAL GENERATOR.              00000320
C                       0=APPROACH NOT USED                             00000330
C                       1=A MATRIX ONLY                                 00000340
C                       2=D MATRIX ONLY                                 00000350
C                       3=BOTH A AND D MATRICES                         00000360
C       IQDRT           CONTROL SWITCH FOR QUADRATURE APPROACH IN THE   00000370
C                       AVERAGED PARTIAL GENERATOR                      00000380
C                       0=APPROACH NOT USED                             00000390
C                       1=A MATRIX ONLY                                 00000400
C                       2=D MATRIX ONLY                                 00000410
C                       3=BOTH A AND D MATRICES                         00000420
C       ICBVAR          CENTRAL GRAVITATIONAL BODY CONTROL SWITCH FOR    00000430
C                       THE AVERAGED PARTIAL GENERATOR.                 00000440
C                       0=NONE                                          00000450
C                       1=ANALYTICALLY                                  00000460
C                       2=FINITE-DIFFERENCING WITH THE AOG CENTRAL       00000470
C                         BODY FIELD                                    00000480
C                       3=FINITE-DIFFERENCING WITH A NXO FIELD          00000490
C       J22VAR          J2 SQUARED CONTROL SWITCH IN THE AVERAGED       00000500
C                       PARTIAL GENERATOR.                              00000510
C                       0=OFF                                           00000520
C                       1=FINITE-DIFFERENCING                           00000530
C       ITBVAR          THIRD-BODY CONTROL SWITCH IN THE AVERAGED       00000540
C                       PARTIAL GENERATOR.                              00000550
C                       0=OFF                                           00000560
C                       1=FINITE-DIFFERENCING WITH THE AOG THIRD-BODY    00000570
```

```
C                    MODEL                                                  00000580
C       IDRVAR       DRAG CONTROL SWITCH IN THE AVERAGED PARTIAL             00000590
C                    GENERATOR.                                             00000600
C                    0=OFF                                                  00000610
C                    1=FINITE-DIFFERENCING                                  00000620
C                    2=QUADRATURE APPROACH                                  00000630
C       ISRVAR       SOLAR RADIATION PRESSURE CONTROL SWITCH IN THE          00000640
C                    AVERAGED PARTIAL GENERATOR.                            00000650
C                    0=OFF                                                  00000660
C                    1=FINITE-DIFFERENCING                                  00000670
C                    2=QUADRATURE APPROACH                                  00000680
C       NVAR         MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITATIONAL       00000690
C                    FIELD TO BE CONSIDERED IN THE AVERAGED PARTIAL         00000700
C                    GENERATOR WHEN ICBVAR=3.                               00000710
C       LEVAR        MAXIMUM POWER OF E IN THE POWER SERIES EXPANSION       00000720
C                    FOR THE ANALYTICAL HARMONIC POTENTIAL CONSIDERED       00000730
C                    IN THE AVERAGED PARTIAL GENERATOR WHEN ICBVAR=3        00000740
C                    (AT MOST CAN BE NVAR-2).                               00000750
C       KDRFLG       CONTROL SWITCH FOR CONSIDERING 2ND ORDER DRAG          00000760
C                    EFFECTS IN THE AVERAGED PARTIAL GENERATOR.             00000770
C                    0=NO 2ND ORDER DRAG EFFECTS                            00000780
C                    1=J2-DRAG EFFECTS                                      00000790
C                    2=J2-DRAG AND DRAG-DRAG EFFECTS                        00000800
C                    3=J2-DRAG,DRAG-J2 AND DRAG-DRAG EFFECTS                00000810
C                      DONE NUMERICALLY                                     00000820
C                    4=J2-DRAG AND DRAG-DRAG EFFECTS DONE                   00000830
C                      NUMERICALLY, DRAG-J2 DONE ANALYTICALLY               00000840
C                    5=ISZAK'S J2 HEIGHT CORRECTION AND ANALYTIC            00000850
C                      DRAG-J2 EFFECTS                                      00000860
C                    6=ISZAK'S J2 HEIGHT CORRECTION ONLY                    00000870
C       KPAR         PARAMETER ESTIMATION OPTION.                           00000880
C                    0=NONE                                                 00000890
C                    1=SOLVE FOR CDRAG                                      00000900
C                    2=SOLVE FOR THE 5 PARAMETERS,A1 - A5, IN THE           00000910
C                      ADAPTIVE MODIFIED HARRIS-PRIESTER ATMOSPHERE.        00000920
C                    3=SOLVE FOR THE 3 PARAMETERS,A1 - A3, IN THE           00000930
C                      ADAPTIVE MODIFIED HARRIS-PRIESTER ATMOSPHERE.        00000940
C       KSTEP        ARRAY OF CONTROL SWITCHES FOR THE PERTURBING           00000950
C                    STEPSIZE.                                              00000960
C                    1=STEPSIZE IS GIVEN IN XS                              00000970
C                    2=STEPSIZE IS XS*PARAMETER                             00000980
C       KVRFLG       CONTROL SWITCH FOR THE SHORT PERIODIC PARTIALS.         00000990
C                    0=NONE                                                 00001000
C                    1=B1 MATRIX - ANALYTICALLY                             00001010
C                    2=B1 MATRIX ONLY - FINITE DIFFERENCING                 00001020
C                    3=B4 MATRIX ONLY - FINITE DIFFERENCING                 00001030
C                    4=BOTH B1 AND B4 MATRICES - FINITE DIFFERENCING        00001040
C       KCPSPP       CONTROL SWITCH FOR CONTINUOUS TIME-INDEPENDENT         00001050
C                    PERTURBATIONS IN THE SHORT PERIODIC PARTIALS.*          00001060
C       KCBSPP       CONTROL SWITCH FOR CENTRAL BODY PERTURBATIONS           00001070
C                    IN THE SHORT PERIODIC PARTIALS.*                        00001080
C       KTBSPP       CONTROL SWITCH FOR THIRD-BODY PERTURBATIONS IN          00001090
C                    THE SHORT PERIODIC PARTIALS.*                           00001100
C       KDRSPP       CONTROL SWITCH FOR DRAG PERTURBATIONS IN THE            00001110
C                    SHORT PERIODIC PARTIALS.*                               00001120
C       KSRSPP       CONTROL SWITCH FOR SOLAR RADIATION PRESSURE             00001130
C                    PERTURBATIONS IN THE SHORT PERIODIC PARTIALS.*          00001140
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ESTSET              AJG1324.GTDS.UPDATE.FORT

C       NVARSP          MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITATIONAL 00001150
C                       FIELD TO BE CONSIDERED IN THE SHORT PERIODIC      00001160
C                       GENERATOR.                                       00001170
C       IVSPNC          ARRAY OF THE NUMBER OF SHORT PERIODIC COEF-       00001180
C                       FICIENTS FOR EACH PERTURBATIONS TO BE             00001190
C                       CONSIDERED IN THE SHORT PERIODIC PARTIALS.        00001200
C       KATMOS          CONTROL SWITCH FOR THE TYPE OF MODIFIED HARRIS-   00001210
C                       PRIESTER ATMOSPHERIC MODEL.                       00001220
C                       1=STANDARD GTDS                                   00001230
C                       2=FULL ADAPTIVE(KPAR=2)                           00001240
C                       3=PART ADAPTIVE(KPAR=3)                           00001250
C       KPRTB1          CONTROL SWITCH FOR PRINTING THE B1 MATRIX.*       00001260
C       KPRTB2          CONTROL SWITCH FOR PRINTING THE B2 MATRIX.*       00001270
C       KPRTB3          CONTROL SWITCH FOR PRINTING THE B3 MATRIX.*       00001280
C       KPRTB4          CONTROL SWITCH FOR PRINTING THE B4 MATRIX.*       00001290
C                                                                         00001300
C   /NEWPAR/                                                              00001310
C                                                                         00001320
C       XS              ARRAY OF STEPSIZES FOR THE SOLVE-FOR-PARAMETERS   00001330
C                       USED TO COMPUTE THE PARTIALS WITH FINITE-         00001340
C                       DIFFERENCING.                                     00001350
C       APAR            ARRAY OF DENSITY PARAMETERS USED IN THE ADAPTIVE  00001360
C                       HARRIS-PRIESTER ATMOSPHERE.                       00001370
C       APARVR          ARRAY OF THE VARIANCE FOR APAR.                   00001380
C       CDRVAR          VARIANCE OF THE COEFFICIENT OF DRAG.              00001390
C                                                                         00001400
C                                                                         00001410
C         * 1=YES     *=NO                                                00001420
C                                                                         00001430
C     ESTSET FLAGS                                                        00001440
C                                                                         00001450
      IANAL  = 2                                                          00001460
      IDIFF  = 3                                                          00001470
      IQDRT  = 0                                                          00001480
C                                                                         00001490
        ICBVAR = 2                                                        00001500
           NVAR = NMAX                                                    00001510
           LEVAR= NVAR-2                                                  00001520
        J22VAR = 1                                                        00001530
        ITBVAR = 0                                                        00001540
        IDRVAR = 1                                                        00001550
        ISRVAR = 0                                                        00001560
        KDRFLG = 6                                                        00001570
C                                                                         00001580
      KPAR   = 1                                                          00001590
      KATMOS = 1                                                          00001600
C                                                                         00001610
      DO 10 I=1,20                                                        00001620
      KSTEP(I)=2                                                          00001630
      XS(I) =0.00001D0                                                    00001640
   10 CONTINUE                                                            00001650
C                                                                         00001660
C                                                                         00001670
      KVRFLG = 0                                                          00001680
      KCPSPP = 2                                                          00001690
        KCBSPP = 2                                                        00001700
           NVARSP = NMAX                                                  00001710
```

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ESTSET              AJG1324.GTDS.UPDATE.FORT

        KTBSPP =2                                                       00001720
      KDRSPP =2                                                         00001730
      KSRSPP =2                                                         00001740
C                                                                       00001750
      IVSPNC(1) = 5                                                     00001760
      IVSPNC(2) = 5                                                     00001770
      IVSPNC(3) = 5                                                     00001780
      IVSPNC(4) = 5                                                     00001790
C                                                                       00001800
      KPRTB1 = 2                                                        00001810
      KPRTB2 = 2                                                        00001820
      KPRTB3 = 2                                                        00001830
      KPRTB4 = 2                                                        00001840
C                                                                       00001850
C                                                                       00001860
      APAR(1) = 1.D0                                                    00001870
      APAR(2) = 1.D0                                                    00001880
      APAR(3) = 6.D0                                                    00001890
      APAR(4) = 0.D0                                                    00001900
      APAR(5) = 0.D0                                                    00001910
C                                                                       00001920
      APARVR(1) = .333D0                                                00001930
      APARVR(2) = .333D0                                                00001940
      APARVR(2) = .333D0                                                00001950
      APARVR(2) = .333D0                                                00001960
      APARVR(2) = .333D0                                                00001970
C                                                                       00001980
      CORVAR = .333D0                                                   00001990
C                                                                       00002000
C                                                                       00002010
      RETURN                                                            00002020
      END                                                               00002030

******** END OF MEMBER ESTSET     203 RECORDS ********
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```
      SUBROUTINE GMAT(TEMPPK)                                                   00000010
                                                                                00000020
C                                                                               
C     VERSION OF 20 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6.          00000030
C                                                                               00000040
C     PURPOSE                                                                   00000050
C        EXECUTIVE PROGRAM FOR THE DETERMINATION OF THE G MATRIX -              00000060
C        PARTIAL OF THE OSCULATING EQUINOCTIAL ELEMENTS WITH RESPECT            00000070
C        TO THE SOLVE-FOR-PARAMETERS.                                           00000080
C                                                                               00000090
C     PARAMETERS                                                                00000100
C                                                                               00000110
C        TEMPPK    I    ARRAY OF THE PARTIAL OF THE MEAN EQUINOCTIAL           00000120
C                       ELEMENTS WRT THE SOLVE-FOR-PARAMETERS.                  00000130
C                  O    ARRAY OF THE PARTIAL OF THE OSCULATING EQUI-           00000140
C                       NOCTIAL ELEMENTS WRT THE SOLVE-FOR-PARAMETERS.          00000150
C                                                                               00000160
C        /WORKER/                                                               00000170
C                                                                               00000180
C        XLAMDA    I    MEAN FAST VARIABLE.                                    00000190
C        VPELMT    I    ARRAY OF MEAN ELEMENTS.                                00000200
C                                                                               00000210
C        /VARMAT/                                                               00000220
C                                                                               00000230
C        B1MAT     O    ARRAY OF THE PARTIAL OF THE SHORT PERIODICS            00000240
C                       WRT THE MEAN EQUINOCTIAL ELEMENTS.                      00000250
C        B4MAT     O    ARRAY OF THE PARTIAL OF THE SHORT PERIODICS            00000260
C                       WRT THE SOLVED-FOR-MODELLED PARAMETERS.                 00000270
C                                                                               00000280
C        /ESTFLG/                                                               00000290
C                                                                               00000300
C        KVRFLG    I    CONTROL SWITCH FOR THE SHORT PERIODIC PARTIAL          00000310
C                       GENERATOR.                                              00000320
C        KPRTB1    I    CONTROL SWITCH FOR PRINTING THE B1 MATRIX.             00000330
C        KPRTB2    I    CONTROL SWITCH FOR PRINTING THE B2 MATRIX.             00000340
C        KPRTB3    I    CONTROL SWITCH FOR PRINTING THE B3 MATRIX.             00000350
C        KPRTB4    I    CONTROL SWITCH FOR PRINTING THE B4 MATRIX.             00000360
C                                                                               00000370
C     REFERENCES                                                                00000380
C                                                                               00000390
C        "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW-                 00000400
C        ALTITUDE SATELLITES." BY A.J. GREEN                                    00000410
C                                                                               00000420
C     ANALYSIS                                                                  00000430
C        ANDREW J. GREEN, CPT, U.S. ARMY                                        00000440
C                                                                               00000450
C     PROGRAMMER                                                                00000460
C        ANDREW J. GREEN, CPT, U.S. ARMY                                        00000470
C                                                                               00000480
C                                                                               00000490
      IMPLICIT REAL*8(A-H,O-Z)                                                  00000500
      DIMENSION TEMPPK(6,20),TMAT(6,20),VPELMT(10)                              00000510
C                                                                               00000520
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                                   00000530
      EQUIVALENCE      (VPELMT(1)              ,RWORK(2829)                   ),00000540
     2                 (XLAMDA                 ,RWORK(2851)                   ),00000550
     3                 (XMEAN                  ,RWORK(2814)                   ) 00000560
      EQUIVALENCE      (NOPARM                 ,IWORK(3)                      ),00000570
```

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     2                (NEQ                    ,IWORK(5)              ),00000580
     3                (NPERT                  ,IWORK(9)              ) 00000590
C                                                                        00000600
      COMMON/DCINT/    DPDCI(432), INDCI(26)                             00000610
      EQUIVALENCE      (TTO                    ,DPDCI(127)            ) 00000620
C                                                                        00000630
      COMMON/ESTFLG/ IANAL     ,IDIFF     ,IQDRT     ,ICBVAR    ,J22VAR   , 00000640
     1               ITBVAR    ,IDRVAR    ,ISRVAR    ,NVAR      ,LEVAR    , 00000650
     2               KDRFLG    ,KPAR      ,KSTEP(20),KVRFLG     ,KCPSPP   , 00000660
     3               KCBSPP    ,KTBSPP    ,KDRSPP    ,KSRSPP    ,NVARSP   , 00000670
     4               IVSFNC(4),KATMOS     ,KPRTB1    ,KPRTB2    ,KPRTB3   , 00000680
     5               KPRTB4                                                 00000690
C                                                                        00000700
      COMMON/VARMAT/ AMAT(6,6),DMAT(6,14),B1MAT(6,6),B4MAT(6,14)         00000710
C                                                                        00000720
C     SAVE THE FAST VARIABLE                                             00000730
C                                                                        00000740
      XLAMDA=VPELMT(NOPARM)                                              00000750
C                                                                        00000760
C     PRINT THE B2 AND B3 MATRICIES                                      00000770
C                                                                        00000780
      TIMOUT=TTO/86400.D0                                                00000790
      IF(KPRTB2.NE.1) GO TO 3                                            00000800
      WRITE(6,1000) TIMOUT                                               00000810
      DO 2 I=1,6                                                         00000820
      WRITE(6,1001) (TEMPPK(I,J),J=1,6)                                  00000830
      WRITE(6,1003)                                                      00000840
    2 CONTINUE                                                           00000850
    3 IF(KPRTB3.NE.1) GO TO 5                                            00000860
      WRITE(6,1002) TIMOUT                                               00000870
      DO 4 I=1,6                                                         00000880
      WRITE(6,1001) (TEMPPK(I,J),J=7,NEQ)                                00000890
      WRITE(6,1003)                                                      00000900
    4 CONTINUE                                                           00000910
C                                                                        00000920
C     CALCULATE THE B1 AND B4 MATRICES.                                  00000930
C                                                                        00000940
    5 CALL VARSP                                                         00000950
C                                                                        00000960
C     PRINT THE B1 AND B4 MATRICES.                                      00000970
C                                                                        00000980
      IF(KPRTB1.NE.1) GO TO 7                                            00000990
      WRITE(6,1004) TIMOUT                                               00001000
      DO 6 I=1,6                                                         00001010
      WRITE(6,1001) (B1MAT(I,J),J=1,6)                                   00001020
      WRITE(6,1003)                                                      00001030
    6 CONTINUE                                                           00001040
    7 IF(KPRTB4.NE.1) GO TO 9                                            00001050
      WRITE(6,1006) TIMOUT                                               00001060
      NL=NEQ-6                                                           00001070
      DO 8 I=1,6                                                         00001080
      WRITE(6,1001) (B4MAT(I,J),J=1,NL)                                  00001090
      WRITE(6,1003)                                                      00001100
    8 CONTINUE                                                           00001110
C                                                                        00001120
C     FORM THE I+B1 MATRIX.                                              00001130
C                                                                        00001140
```

```
     9 DO 10 I=1,6                                                          00001150
       B1MAT(I,I)=B1MAT(I,I)+1.D0                                           00001160
    10 CONTINUE                                                             00001170
C                                                                           00001180
C      FORM THE (I+B1)(TEMPPK) MATRIX.                                      00001190
C                                                                           00001200
       CALL MATMUL(B1MAT,TEMPPK,TMAT,6,6,NEQ)                               00001210
C                                                                           00001220
C      ADD THE B4 MATRIX.                                                   00001230
C                                                                           00001240
       DO 30 I=1,6                                                          00001250
       DO 20 J=7,NEQ                                                        00001260
       N=J-6                                                                00001270
       TMAT(I,J)=TMAT(I,J)+B4MAT(I,N)                                       00001280
    20 CONTINUE                                                             00001290
    30 CONTINUE                                                             00001300
C                                                                           00001310
C      FORM THE OUTPUT MATRIX.                                              00001320
C                                                                           00001330
       NN=NEQ+1                                                             00001340
       DO 60 I=1,6                                                          00001350
       DO 40 J=1,NEQ                                                        00001360
       TEMPPK(I,J)=TMAT(I,J)                                                00001370
    40 CONTINUE                                                             00001380
       DO 50 L=NN,20                                                        00001390
       TEMPPK(I,L)=0.D0                                                     00001400
    50 CONTINUE                                                             00001410
    60 CONTINUE                                                             00001420
C                                                                           00001430
       RETURN                                                               00001440
  1000 FORMAT(////,41X,'THE B2 MATRIX AT TIME(DAYS) =',1P1D20.10,//)        00001450
  1001 FORMAT(1P6D22.12)                                                    00001460
  1002 FORMAT(////,41X,'THE B3 MATRIX AT TIME(DAYS) =',1P1D20.10,//)        00001470
  1003 FORMAT(/)                                                            00001480
  1004 FORMAT(////,41X,'THE B1 MATRIX AT TIME(DAYS) =',1P1D20.10,//)        00001490
  1006 FORMAT(////,41X,'THE B4 MATRIX AT TIME(DAYS) =',1P1D20.10,//)        00001500
       END                                                                  00001510
```

******** END OF MEMBER GMAT 151 RECORDS ********

HODRAG AJG1324.GTDS.UPDATE.FORT

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      SUBROUTINE HODRAG                                                 00000010
C                                                                       00000020
C     VERSION OF 19 MARCH 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6     00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C          TO CONTROL THE EVALUATION OF THE AVERAGED ELEMENT RATES       00000060
C          DUE TO DRAG, J2-DRAG, DRAG-J2 AND DRAG-DRAG EFFECTS.  ALL    00000070
C          SHORT PERIODICS ARE EXPRESSED IN TERMS OF A FOURIER SERIES   00000080
C          IN THE MEAN-MEAN LONGITUDE.                                  00000090
C                                                                       00000100
C     PARAMETERS                                                        00000110
C                                                                       00000120
C     /ANAVIN/                                                          00000130
C                                                                       00000140
C       ITHIRD    I   THIRD-BODY AVERAGING OPTION.                      00000150
C                     0=NO THIRD-BODY                                   00000160
C                     1=TIME-DEPENDENT NUMERICAL AVERAGING              00000170
C                     2=TIME-INDEPENDENT NUMERICAL AVERAGING            00000180
C                     3=ANALYTICAL AVERAGING                            00000190
C       NQUAD     I   QUADRATURE ORDER(UNTIL THE AUTOMATIC QUADRATURE   00000200
C                     ORDER SELECTION OPTION IS IMPLEMENTED IN 'SPGENN',00000210
C                     THIS SUBROUTINE ASSUMES THAT THE QUADRATURE ORDER 00000220
C                     HAS BEEN DETERMINED BY A KEYWORD CARD OR BY A     00000230
C                     PREVIOUS CALL TO 'VOQUAD'                         00000240
C       ICORR     I   CONTROL SWITCH FOR ISZAK'S J2 HEIGHT CORRECTION   00000250
C                     IN THE DRAG QUADRATURE.                           00000260
C       IDRDR     I   CONTROL SWITCH FOR CONSIDERING 2ND ORDER          00000270
C                     DRAG EFFECTS.                                     00000280
C                     0=NO 2ND ORDER DRAG EFFECTS                       00000290
C                     1=J2-DRAG EFFECTS                                 00000300
C                     2=J2-DRAG AND DRAG-DRAG EFFECTS                   00000310
C                     3=J2-DRAG,DRAG-J2 AND DRAG-DRAG EFFECTS           00000320
C                       DONE NUMERICALLY                                00000330
C                     4=J2-DRAG AND DRAG-DRAG EFFECT DONE               00000340
C                       NUMERICALLY, DRAG-J2 EFFECTS DONE ANALYTICALLY  00000350
C                     5=ISZAK'S J2 HEIGHT CORRECTION AND ANALYTIC       00000360
C                       DRAG-J2 EFFECTS.                                00000370
C       NCHOJ2    I   NUMBER OF J2 SHORT PERIODIC COEFFICIENTS TO BE    00000380
C                     CONSIDERED IN 'HODRAG'.                           00000390
C       NCHODR    I   NUMBER OF DRAG SHORT PERIODIC COEFFICIENTS TO BE  00000400
C                     CONSIDERED IN 'HODRAG'.                           00000410
C                                                                       00000420
C     /FRC/                                                             00000430
C                                                                       00000440
C       NMAX      O   MAXIMUM DEGREE FOR THE EARTH OR MOON FIELDS.      00000450
C       MMAX      O   MAXIMUM ORDER FOR THE EARTH OR MOON FIELDS.       00000460
C                                                                       00000470
C     /SWITCH/                                                          00000480
C                                                                       00000490
C       IBURN     O   CONTROL SWITCH FOR POLYNOMIAL BURN MODEL. *       00000500
C       IRESON    O   CENTRAL BODY FIELD SWITCH.                        00000510
C                     1=NXM CENTRAL BODY FIELD PLUS HIGH ORDER          00000520
C                       RESONANCE                                       00000530
C                     2=NXM CENTRAL BODY FIELD ONLY                     00000540
C                     3=HIGH ORDER RESONANCE ONLY                       00000550
C                     4=NO CENTRAL BODY FIELD                           00000560
C       IBODY     O   CENTRAL AND NON-CENTRAL BODY INDICATORS.          00000570
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C                                                                       00000580
C     /WORKER/                                                          00000590
C                                                                       00000600
C       OSCELM    O   ARRAY OF OSCULATING ELEMENTS.                     00000610
C       ETA       O   ARRAY OF SHORT PERIODICS.                         00000620
C       VPELMT    I   ARRAY OF MEAN ELEMENTS.                           00000630
C       XLAMBA    I   MEAN FAST VARIABLE(MEAN MEAN LONGITUDE).          00000640
C       XMEAN     O   MEAN MOTION.                                      00000650
C       T         I   TIME FROM EPOCH.                                  00000660
C       NOPARM    I   NUMBER OF PARAMETERS.                             00000670
C       NPERT     O   PERTURBATION INDICATOR.                           00000680
C                     1=CONTINUOUS TIME-DEPENDENT                       00000690
C                     2=CONTINUOUS TIME-INDEPENDENT                     00000700
C                     3=DRAG                                            00000710
C                     4=SOLAR RADIATION PRESSURE                        00000720
C                                                                       00000730
C       *     1=YES         2=NO                                        00000740
C                                                                       00000750
C                                                                       00000760
C     REFERENCES                                                        00000770
C                                                                       00000780
C       "A FOURIER SERIES FORMULATION OF THE SHORT PERIODIC VARIATIONS   00000790
C       IN TERMS OF EQUINOCTIAL ELEMENTS." BY A.J. GREEN AND P.J. CEFOLA00000800
C                                                                       00000810
C     ANALYSIS                                                          00000820
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                           00000830
C                                                                       00000840
C     PROGRAMMER                                                        00000850
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                           00000860
C                                                                       00000870
C*****START PROGRAM**************************************************00000880
C                                                                       00000890
C                                                                       00000900
C                                                                       00000910
      IMPLICIT REAL*8(A-H,O-Z)                                          00000920
      DIMENSION        DPARRY(15,10)           ,VPELMT(10) ,AVRATE(10) ,00000930
     2                 GM(11)       ,POSVEL(6,8),DUMMY(3)               00000940
      DIMENSION        TEMPC(10,10),TEMPD(10,10)                        00000950
      DIMENSION        IBODY(9)     ,IBODYS(3)                          00000960
C                                                                       00000970
      COMMON/ANAVIN/   ISTESS       ,ITESSE      ,ITRESE      ,ISTHIR  ,00000980
     1                 ITHRAR       ,ITHRE       ,ITESS       ,NEND    ,00000990
     2                 MEND         ,LEND        ,ITRES       ,NRES(10) ,00001000
     3                 MRES(10)     ,JRES(10)    ,NCSRES      ,LENDRS  ,00001010
     4                 ITHIRD       ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ  ,00001020
     5                 IQUAD(4)     ,NQUAD(4)    ,JPOS        ,JA      ,00001030
     6                 JB           ,JC          ,JGHA        ,KPOS    ,00001040
     7                 KA           ,KB          ,KC          ,KGHA    ,00001050
     8                 ICORR        ,IPOS        ,INOON       ,IMEAN   ,00001060
     9                 IGHA         ,ISTRES      ,ITERM       ,ISHAD   ,00001070
     1                 JTMDEP       ,JTMIND      ,JDRAG       ,JSOLAR  ,00001080
     2                 J2SQSP       ,JSPCGB      ,JSPTHR      ,JSPPOS  ,00001090
     3                 JSPA         ,JSPB        ,JSPC        ,JSPGHA  ,00001100
     4                 KSPPOS       ,KSPA        ,KSPB        ,KSPC    ,00001110
     5                 KSPGHA       ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP  ,00001120
     6                 MMAXSP       ,IDRDR       ,NCHOJ2      ,NCHODR   00001130
      EQUIVALENCE      (ITMDEP                   ,IQUAD(1)              ),00001140
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```
     2                (ITMIND                  ,IQUAD(2)            ),00001150
     3                (IDRAG                   ,IQUAD(3)            ),00001160
     4                (ISOLAR                  ,IQUAD(4)            ) 00001170
C                                                                      00001180
      COMMON/CONST /  RCONST(51)                                       00001190
      EQUIVALENCE     (TWOPI                   ,RCONST(1)           ) 00001200
C                                                                      00001210
      COMMON/FRC   /  RFRC(1300)  ,IFRC(50)                            00001220
      EQUIVALENCE     (GM(1)                   ,RFRC(2)             ),00001230
     2                (POSVEL(1,1)             ,RFRC(101)           ),00001240
     3                (NBODY                   ,IFRC(7)             ),00001250
     4                (ISUN                    ,IFRC(14)            ) 00001260
C                                                                      00001270
      COMMON/SWITCH/  ISWIT(225)                                       00001280
      EQUIVALENCE     (IVOP                    ,ISWIT(3)            ),00001290
     2                (ITWOBD                  ,ISWIT(6)            ),00001300
     3                (IBURN                   ,ISWIT(12)           ),00001310
     4                (IRESON                  ,ISWIT(38)           ),00001320
     5                (IBODY(1)                ,ISWIT(201)          ),00001330
     6                (ICENT                   ,IBODY(1)            ) 00001340
C                                                                      00001350
      COMMON/WORKER/  RWORK(3264) ,IWORK(13)                           00001360
      EQUIVALENCE     (DPARRY(1,1)             ,RWORK(7)            ),00001370
     2                (TXMEAN                  ,RWORK(2671)         ),00001380
     3                (VPELMT(1)               ,RWORK(2829)         ),00001390
     4                (AVRATE(1)               ,RWORK(3068)         ),00001400
     5                (XLAMDA                  ,RWORK(2851)         ),00001410
     6                (XMEAN                   ,RWORK(2814)         ),00001420
     7                (T                       ,RWORK(3215)         ),00001430
     8                (TZERO                   ,RWORK(2850)         ) 00001440
      EQUIVALENCE     (K                       ,IWORK(2)            ),00001450
     2                (NOPARM                  ,IWORK(3)            ),00001460
     3                (NPERT                   ,IWORK(9)            ) 00001470
      COMMON/SPCOEF/  CQUAD(10,10),DQUAD(10,10)                        00001480
C                                                                      00001490
C     STORE THE MEAN MOTION AND SET THE VARIOUS SWITCHES.              00001500
C                                                                      00001510
      TXMEAN = XMEAN                                                   00001520
      IRESNS = IRESON                                                  00001530
      NMAXS  = NMAX                                                    00001540
      MMAXS  = MMAX                                                    00001550
      ITHRTE = ITHIRD                                                  00001560
      ICORRT = ICORR                                                   00001570
      IBURNT = IBURN                                                   00001580
      KTEMTI = KSPNC(2)                                                00001590
      KTEMDR = KSPNC(3)                                                00001600
      IRESON = 2                                                       00001610
      NMAX   = 2                                                       00001620
      MMAX   = 0                                                       00001630
      ITHIRD = 0                                                       00001640
      ICORR  = 2                                                       00001650
      IBURN  = 2                                                       00001660
      KSPNC(2) = NCHOJ2                                                00001670
      KSPNC(3) = NCHODR                                                00001680
      KMAXNC = NCHOJ2                                                  00001690
      IF(IDRDR.EQ.5) GO TO 100                                         00001700
C                                                                      00001710
```

```
C     COMPUTE THE SHORT PERIODIC COEFFICIENTS DUE TO J2 USING          00001720
C     TIME-INDEPENDENT NUMERICAL AVERAGING.                            00001730
C                                                                      00001740
      NPERT=2                                                          00001750
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                   00001760
C                                                                      00001770
C     STORE THE J2 LIMITS.                                             00001780
C                                                                      00001790
      TFLOW=FLOW                                                       00001800
      TFHIGH=FHIGH                                                     00001810
C                                                                      00001820
      CALL EVAL(TZERO,2,KA,KB,KC,2,IBODY,POSVEL)                       00001830
      CALL SPGENN(FLOW,FHIGH)                                          00001840
      IF(IDRDR.EQ.1) GO TO 100                                         00001850
C                                                                      00001860
C     STORE THE J2 COEFFICIENTS.                                       00001870
C                                                                      00001880
      KMAXNC=MAX0(NCHOJ2,NCHODR)                                       00001890
      DO 40 I = 1,NOPARM                                               00001900
      DO 30 J = 1,KMAXNC                                               00001910
      TEMPC(I,J) = CQUAD(I,J)                                          00001920
      TEMPD(I,J) = DQUAD(I,J)                                          00001930
   30 CONTINUE                                                         00001940
   40 CONTINUE                                                         00001950
C                                                                      00001960
C     COMPUTE THE ELEMENT RATES DUE TO DRAG, J2-DRAG AND               00001970
C     DRAG-DRAG EFFECTS.                                               00001980
C                                                                      00001990
  100 NPERT=3                                                          00002000
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                   00002010
      IF(NPERT.EQ.0) GO TO 600                                         00002020
C                                                                      00002030
C                 FIND THE POSITION OF THE SUN SO THAT THE DIURNAL     00002040
C                 BULGE IN THE ATMOSPHERE CAN BE LOCATED.              00002050
C                                                                      00002060
      IBODYS(2) = IBODY(2)                                             00002070
      IBODYS(3) = IBODY(3)                                             00002080
      NBODYS = NBODY                                                   00002090
C                                                                      00002100
      IBODY(2) = 3                                                     00002110
      IBODY(3) = 0                                                     00002120
      NBODY = 2                                                        00002130
C                                                                      00002140
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                          00002150
      IF(IDRDR.EQ.1) GO TO 120                                         00002160
C                                                                      00002170
C     COMPUTE THE SHORT PERIODIC COEFFICIENTS DUE TO DRAG.             00002180
C                                                                      00002190
      CALL SPGENN(FLOW,FHIGH)                                          00002200
      IF(IDRDR.EQ.5) GO TO 160                                         00002210
C                                                                      00002220
C     ACCUMULATE THE SHORT PERIODIC COEFFICIENTS AND DETERMINE THE     00002230
C     CROSS COUPLING TERM IN THE FAST VARIABLE RATE.                   00002240
C                                                                      00002250
      TK=2.D0                                                          00002260
      IF (IDRDR.EQ.2) TK=1.D0                                          00002270
      DELLAM=0.D0                                                      00002280
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```
      XFACT=(15.D0*XMEAN)/(16.D0*VPELMT(1)*VPELMT(1))                   00002290
      DO 115 I=1,NOPARM                                                 00002300
      DO 110 J=1,KMAXNC                                                 00002310
      TEM1=CQUAD(I,J)*CQUAD(I,J)+DQUAD(I,J)*DQUAD(I,J)                  00002320
      TEM2=CQUAD(I,J)*TEMPC(I,J)+DQUAD(I,J)*TEMPD(I,J)                  00002330
      DELLAM=DELLAM+XFACT*(TEM1+TK*TEM2)                                00002340
      CQUAD(I,J)=CQUAD(I,J)+TEMPC(I,J)                                  00002350
      DQUAD(I,J)=DQUAD(I,J)+TEMPD(I,J)                                  00002360
  110 CONTINUE                                                          00002370
  115 CONTINUE                                                          00002380
C                                                                       00002390
C     ADD THE CROSS COUPLING TERM TO THE FAST VARIABLE RATE.            00002400
C                                                                       00002410
      DPARRY(K,NOPARM) = DPARRY(K,NOPARM) + DELLAM                      00002420
C                                                                       00002430
C     COMPUTE THE VALUE OF THE ELEMENT RATES.                           00002440
C                                                                       00002450
  120 CALL RTQUAD(AVRATE,FLOW,FHIGH,NQUAD(NPERT),KMAXNC)                00002460
C                                                                       00002470
C                   ACCUMULATE THE ELEMENT RATES.                       00002480
C                                                                       00002490
      DO 125 I=1,NOPARM                                                 00002500
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                    00002510
  125 CONTINUE                                                          00002520
C                                                                       00002530
C                   RESTORE THE THIRD BODIES.                           00002540
C                                                                       00002550
      IBODY(2) = IBODYS(2)                                              00002560
      IBODY(3) = IBODYS(3)                                              00002570
      NBODY = NBODYS                                                    00002580
      IF(IDRDR.LE.2) GO TO 600                                          00002590
C                                                                       00002600
C     COMPUTE THE VALUE OF THE ELEMENT RATES DUE TO DRAG-J2 EFFECTS.    00002610
C                                                                       00002620
C                                                                       00002630
C     THE DRAG SHORT PERIODIC COEFFICIENTS HAVE ALREADY BEEN DETERMINED.00002640
C                                                                       00002650
      KMAXNC=NCHODR                                                     00002660
      DO 135 I=1,NOPARM                                                 00002670
      DO 130 J=1,KMAXNC                                                 00002680
      CQUAD(I,J)=CQUAD(I,J) - TEMPC(I,J)                                00002690
      DQUAD(I,J)=DQUAD(I,J) - TEMPD(I,J)                                00002700
  130 CONTINUE                                                          00002710
  135 CONTINUE                                                          00002720
      IF (IDRDR.EQ.4) GO TO 160                                         00002730
C                                                                       00002740
C     DETERMINE THE ELEMENT RATES NUMERICALLY.                          00002750
C                                                                       00002760
      NPERT=2                                                           00002770
      CALL EVAL(TZERO,2,KA,KB,KC,2,IBODY,POSVEL)                        00002780
  140 CALL RTQUAD(AVRATE,TFLOW,TFHIGH,NQUAD(NPERT),KMAXNC)              00002790
C                                                                       00002800
C                   ACCUMULATE THE ELEMENT RATES.                       00002810
C                                                                       00002820
      DO 145 I=1,NOPARM                                                 00002830
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)/TWOPI                    00002840
  145 CONTINUE                                                          00002850
```

```
C                                                                       00002860
C     SINCE THE J2 EFFECT HAS ALREADY BEEN INCLUDED PREVIOUSLY,         00002870
C     IT MUST BE NOW BE REMOVED.                                        00002880
C                                                                       00002890
      XMEAN = TXMEAN                                                    00002900
      CALL AUXPAR(VPELMT)                                               00002910
      CALL VOQUAD(TFLOW,TFHIGH)                                         00002920
      DO 150 I=1,NOPARM                                                 00002930
         DPARRY(K,I) = DPARRY(K,I) - AVRATE(I)/TWOPI                    00002940
  150 CONTINUE                                                          00002950
      GO TO 601                                                         00002960
C                                                                       00002970
C     DETERMINE THE ELEMENT RATES ANALYTICALLY.                         00002980
C                                                                       00002990
  160 XMEAN=TXMEAN                                                      00003000
      CALL AUXPAR(VPELMT)                                               00003010
      CALL DRAGJ2                                                       00003020
C                                                                       00003030
C     ACCUMULATE THE ELEMENT RATES.                                     00003040
C                                                                       00003050
      DO 165 I=1,NOPARM                                                 00003060
         DPARRY(K,I) = DPARRY(K,I) + AVRATE(I)                          00003070
  165 CONTINUE                                                          00003080
      GO TO 601                                                         00003090
C                                                                       00003100
C     RESTORE THE SWITCHES.                                             00003110
C                                                                       00003120
  600 XMEAN = TXMEAN                                                    00003130
      CALL AUXPAR(VPELMT)                                               00003140
  601 IRESON = IRESNS                                                   00003150
      NMAX   = NMAXS                                                    00003160
      MMAX   = MMAXS                                                    00003170
      ITHIRD = ITHRTE                                                   00003180
      ICORR  = ICORRT                                                   00003190
      IBURN  = IBURNT                                                   00003200
      KSPNC(2) = KTEMTI                                                 00003210
      KSPNC(3) = KTEMDR                                                 00003220
C                                                                       00003230
      RETURN                                                            00003240
      END                                                               00003250
```

******** END OF MEMBER HODRAG 325 RECORDS ********

HWIRE AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE HWIRE                                                      00000010
C                                                                           00000020
C  PURPOSE                                                                  00000030
C     FOR THE INITIAL DESIGN THIS SUBROUTINE SETS THE VALUES OF             00000040
C     THE SHORT PERIODIC SWITCHES.                                          00000050
C                                                                           00000060
      IMPLICIT REAL*8(A-H,O-Z)                                              00000070
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00000080
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00000090
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00000100
     3                 NRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00000110
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00000120
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00000130
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00000140
     7                 KA          ,KB          ,KC          ,KGHA        ,00000150
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00000160
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00000170
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00000180
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00000190
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00000200
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00000210
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00000220
     6                 MMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR       00000230
C                                                                           00000240
      COMMON/FRC    /  RFRC(1300)  ,IFRC(50)                               00000250
      EQUIVALENCE      (NMAX                    ,IFRC(3)                  ),00000260
     1                 (MMAX                    ,IFRC(4)                  ) 00000270
C                                                                           00000280
C       JTMDEP         CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE     00000290
C                      TO TIME-DEPENDENT CONTINUOUS PERTURBATIONS.    *     00000300
C       JTMIND         CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE     00000310
C                      TO TIME-INDEPENDENT CONTINUOUS PERTURBATIONS.  *     00000320
C       JSPCGB         CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE     00000330
C                      TO THE CENTRAL GRAVITATIONAL BODY.                   00000340
C                      0=NO CENTRAL BODY FIELD                              00000350
C                      1=TIME-DEPENDENT NUMERICAL SHORT PERIODICS           00000360
C                      2=TIME=INDEPENDENT NUMERICAL SHORT PERIODICS         00000370
C                      3=ANALYTICAL SHORT PERIODICS                         00000380
C       JSPTHR         CONTROL SWITCH FOR THIRD-BODY NUMERICAL SHORT        00000390
C                      PERIODICS.                                           00000400
C                      0=NO THIRD BODY                                      00000410
C                      1=TIME-DEPENDENT NUMERICAL SHORT PERIODICS           00000420
C                      2=TIME-INDEPENDENT NUMERICAL SHORT PERIODICS         00000430
C                      3=ANALYTICAL SHORT PERIODICS                         00000440
C       JDRAG          CONTROL SWITCH FOR DRAG SHORT PERIODICS. *           00000450
C       JSOLAR         CONTROL SWITCH FOR SOLAR RADIATION PRESSURE          00000460
C                      SHORT PERIODICS. *                                   00000470
C       J2SQSP         CONTROL SWITCH FOR J2 SQUARED SHORT PERIODICS.       00000480
C       NMAXSP         MAXIMUM DEGREE FOR THE CENTRAL BODY FIELD THAT       00000490
C                      WILL BE USED IN THE DETERMINATION OF THE SHORT       00000500
C                      PERIODICS.                                           00000510
C       MMAXSP         MAXIMUM ORDER FOR THE CENTRAL BODY FIELD THAT        00000520
C                      WILL BE USED IN THE DETERMINATION OF THE SHORT       00000530
C                      PERIODICS.                                           00000540
C       KSPNC(NPERT)   NUMBER OF SHORT PERIODIC COEFFICIENTS.               00000550
C       KSPPRT(NPERT)  CONTROL SWITCH FOR SHORT PERIODIC PRINT OPTION.*     00000560
C       NPERT          PERTURBATION INDICATOR                               00000570
```

```
C                    1=TIME-DEPENDENT CONTINUOUS PERTURBATIONS          00000580
C                    2=TIME-INDEPENDENT CONTINUOUS PERTURBATIONS        00000590
C                    3=DRAG PERTURBATIONS                              00000600
C                    4=SOLAR RADIATION PRESSURE PERTURBATIONS           00000610
C       IDRDR     I  CONTROL SWITCH FOR CONSIDERING 2ND ORDER          00000620
C                    DRAG EFFECTS.                                     00000630
C                    0=NO 2ND ORDER DRAG EFFECTS                       00000640
C                    1=J2-DRAG EFFECTS                                 00000650
C                    2=J2-DRAG AND DRAG-DRAG EFFECTS                   00000660
C                    3=J2-DRAG,DRAG-J2 AND DRAG-DRAG EFFECTS           00000670
C                      DONE NUMERICALLY                                00000680
C                    4=J2-DRAG AND DRAG-DRAG EFFECTS DONE              00000690
C                      NUMERICALLY, DRAG-J2 DONE ANALYTICALLY          00000700
C                    5=ISZAK'S J2 HEIGHT CORRECTION AND ANALYTIC       00000710
C                      DRAG-J2 EFFECTS.                                00000720
C       NCHOJ2    I  NUMBER OF J2 SHORT PERIODIC COEFFICIENTS TO BE    00000730
C                    CONSIDERED IN 'HODRAG'.                           00000740
C       NCHODR    I  NUMBER OF DRAG SHORT PERIODIC COEFFICIENTS TO BE  00000750
C                    CONSIDERED IN 'HODRAG'.                           00000760
C                                                                      00000770
C       *       1=YES          2=NO                                    00000780
C     HWIRE FLAGS                                                      00000790
C                                                                      00000800
      JTNDEP   = 2                                                     00000810
      JTNIND   = 1                                                     00000820
      JDRAG    = 1                                                     00000830
      JSOLAR   = 2                                                     00000840
      J2SQSP   = 1                                                     00000850
C                                                                      00000860
      JSPCGB   = 2                                                     00000870
      NMAXSP   = NMAX                                                  00000880
      MMAXSP   = MMAX                                                  00000890
C                                                                      00000900
      JSPTHR   = 0                                                     00000910
C                                                                      00000920
      JSPPOS   = 2                                                     00000930
      IF(JSPTHR.EQ.1) JSPPOS=1                                         00000940
      JSPA     = 2                                                     00000950
      JSPB     = 2                                                     00000960
      JSPC     = 2                                                     00000970
      JSPGHA   = 2                                                     00000980
C                                                                      00000990
      KSPPOS   = 2                                                     00001000
      IF(JSPTHR.EQ.2) KSPPOS=1                                         00001010
      KSPA     = 2                                                     00001020
      KSPB     = 2                                                     00001030
      KSPC     = 2                                                     00001040
      KSPGHA   = 2                                                     00001050
C                                                                      00001060
      KSPNC(1) = 7                                                     00001070
      KSPNC(2) = 7                                                     00001080
      KSPNC(3) = 7                                                     00001090
      KSPNC(4) = 7                                                     00001100
C                                                                      00001110
      KSPPRT(1)= 2                                                     00001120
      KSPPRT(2)= 2                                                     00001130
      KSPPRT(3)= 2                                                     00001140
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HWIRE                AJG1324.GTDS.UPDATE.FORT

      KSPPRT(4)= 2                                                    00001150
C                                                                     00001160
      IDRDR    = 5                                                    00001170
      NCHOJ2   = 4                                                    00001180
      NCHODR   = 4                                                    00001190
C                                                                     00001200
      RETURN                                                          00001210
      END                                                             00001220

******** END OF MEMBER HWIRE      122 RECORDS ********
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```
ORBITV              AJG1324.GTDS.UPDATE.FORT

      SUBROUTINE ORBITV(INTFLG,*)                                          00000100
C                                                                          00000200
C     VERSION 2.2                                                          00000300
C                                                                          00000400
C     PURPOSE                                                              00000500
C           TO INTEGRATE FOR POSITION AND VELOCITY AT THE REQUEST TIME     00000600
C           USING THE VARIATION-OF-PARAMETERS ORBIT GENERATOR              00000700
C                                                                          00000800
C     CALLING SEQUENCE                                                     00000900
C           CALL ORBITV(INTFLG,*)                                          00001000
C               INTFLG=1 RETURN                                            00001100
C               INTFLG=2 PERFORM VOP INTEGRATION                           00001200
C                                                                          00001300
C     COMMON BLOCK VARIABLES                                               00001400
C           INPUT                                                          00001500
C               H           STEPSIZE                                       00001600
C               IND         ARRAY OF CONTROL SWITCHES                      00001700
C               NPART       NUMBER OF PARTIALS REQUIRED                    00001800
C               NSTR        NUMBER OF BACK POINTS RECOPIED                 00001900
C               NOPARM      NUMBER OF ORBITAL ELEMENTS INTEGRATED          00002000
C               NORDER      NUMBER OF BACK POINTS USED IN INTEGRATOR       00002100
C               IORDER(1)   ORDER OF INTERPOLATION OF EQS. OF MOTION       00002200
C               IORDER(2)   ORDER OF INTERPOLATION OF VARIATIONAL EQS.     00002300
C               STREG       VALUE OF INDEPENDENT VARIABLE AT T=T(K)        00002400
C               I2NDOR      2ND ORDER AVERAGING SWITCH FOR DRAG WHICH      00002410
C                             IS USED IN 'RTQUAD'                          00002420
C                                                                          00002500
C                                                                          00002600
C           INPUT-OUTPUT                                                   00002700
C               NCONDT      SECTIONING CONDITION FLAG                      00002800
C               XXSUM       FIRST SUM VECTOR OF DERIVATIVES                00002900
C               TTO         OUTPUT REQUEST TIME                            00003000
C               DPARRY      DERIVATIVE ARRAY                               00003100
C               PARARY      ELEMENT VECTOR ARRAY                           00003200
C               VEPARY      PARTIAL DERIVATIVE ARRAY                       00003300
C               VEVOPD      PARTIAL DERIVATIVE RATES ARRAY                 00003400
C                                                                          00003500
C           OUTPUT                                                         00003600
C               K           DERIVATIVE ARRAY INDICATOR OF VALUE AT T=T(K)  00003700
C               PXTXO       VARIATIONAL PARTIAL DERIVATIVES                00003800
C               T           TIME FROM EPOCH OF K TH  DERIVATIVE            00003900
C               XDTO        S/C VELOCITY VECTOR AT TIME TTO                00004000
C               XTO         S/C POSITION VECTOR AT TIME TTO                00004100
C                                                                          00004200
C                                                                          00004300
C     EXTERNAL REFERENCES                                                  00004400
C           CSTEPX                                                         00004500
C           SECHEK                                                         00004600
C           CROSSV                                                         00004700
C           HEMITR                                                         00004800
C           HERMIT                                                         00004900
C           INTPAR                                                         00005000
C           DALLPV                                                         00005100
C           EQUNPV                                                         00005200
C           IDLPV                                                          00005300
C           DPEL2X                                                         00005400
C           ANPART                                                         00005500
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ORBITV              AJG1324.GTDS.UPDATE.FORT

C           CELEM                                                          00005600
C           RKINTG                                                         00005610
C           SPEXEC                                                         00005620
C                                                                          00005700
C     PROGRAMMER                                                           00005800
C          R. W. NELSON(PROGRAMMED ORIGINAL ORBIT SUBROUTINE)              00005900
C     PROGRAM MODIFICATIONS                                                00006000
C          4/1/72 ORIGINAL SUBROUTINE ORBITV BY A. C. LONG                 00006100
C          7/15/73 MODIFIED BY A. C. LONG TO INCLUDE MORE COMMENTS         00006200
C          6/15/74 BY R.PAJERSKI- ADDED NEW ELEMENT SET(KEPLERIAN)         00006300
C          1/10/75 BY R.PAJERSKI- DELETED ABS.VALUE TEST                   00006400
C          7/20/79 BY A.J. GREEN,CSDL - INCLUDE THE DETERMINATION          00006410
C                  OF THE PARTIALS FOR IVOP=12,(APG AND SPFG)              00006420
C          7/30/79 BY A. BOBICK ,C.S.DRAPER LAB - TO INCLUDE R-K INTEG.    00006500
C                       AND HERMIT INTERPOLATION OPTIONS                   00006600
C                                                                          00006700
      IMPLICIT REAL*8(A-H,O-Z)                                             00006800
      COMMON          TEMPPK(6,20),     PKPC(6,6)                        00006900
      COMMON/DCINT/   DPDCI(432),       INDCI(26)                        00007000
      COMMON/ FRC /   DPFRC(1300),      INTFRC(47)                       00007100
      COMMON/INTEG/   DPINT(149),       INT(9)                           00007200
      COMMON/SECTN/   DPSEC(675),       INSEC(325)                       00007300
      COMMON/SWITCH/ INSWI(200)                                          00007400
      COMMON/TIMREG/ DTRG(94)                                            00007500
      COMMON/VEPART/ VEPAR(1800)                                         00007600
      COMMON/WORKER/ DPWOR(3264),       INWOR(13)                        00007700
C                                                                        00007800
      DIMENSION       VPELMT(10),       DPARRY(15,10),       XXSUM(10),  00007900
     1                IND(100),         XTO(3),              XDTO(3),    00008000
     2                PXTXO(6,20),      IORDER(2),      VEPARY(15,6,20), 00008100
     3                OSCELM(10)                                          00008110
      DIMENSION       TREG(40),         U(4),                US(4)       00008200
      DIMENSION       GM(11)                                             00008300
C                                                                        00008400
      EQUIVALENCE     (DPDCI(1),XTO(1))      ,     (DPDCI(4),XDTO(1)),   00008500
     1                (DPDCI(7),PXTXO(1,1)) ,     (DPDCI(127),TTO),     00008600
     2                (INDCI(2),NPART)                                    00008700
      EQUIVALENCE     (DPWOR(7),DPARRY(1,1))  , (DPWOR(2819),XXSUM(1)),  00008800
     1                (DPWOR(3215),T)         , (DPWOR(2653),OSCELM(1)), 00008900
     2                (DPWOR(3216),H)         , (INWOR(2),K),            00009000
     3                (INWOR(3),NOPARM)       , (INWOR(13),NSTR),        00009100
     4                (INWOR(10), I2NDOR)     , (DPWOR(2829),VPELMT(1))  00009110
      EQUIVALENCE     (VEPAR(1),VEPARY(1,1,1))                           00009200
      EQUIVALENCE     (INT(1),NORDER)         , (INT(8),IORDER(1))       00009300
      EQUIVALENCE     (INSWI(1),IND(1))                                  00009400
      EQUIVALENCE     (INSEC(313),NCONDT)                                00009500
      EQUIVALENCE     (STREG,DTRG(92))        , (TREG(1),DTRG(1))        00009600
C                                                                        00009700
      DIMENSION        PARARY(15,10)                                     00009800
      EQUIVALENCE     (DPWOR(157),PARARY(1,1))                           00009900
      EQUIVALENCE     (DPFRC(2),GM(1))        , (INTFRC(1),ICENT)        00010000
C                                                                        00010100
      DIMENSION       VEVOP(6,20)          ,VEVOPD(15,6,20)    ,         00010200
     *                XVSUM(6,20)                                         00010300
      EQUIVALENCE     (DPWOR(307)          ,VEVOP(1,1))        ,         00010400
     *                (DPWOR(427)          ,VEVOPD(1,1,1))     ,         00010500
     *                (DPWOR(2383)         ,XVSUM(1,1))        ,         00010600
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ORBITV              AJG1324.GTDS.UPDATE.FORT

     *               (INWOR(5)           ,NEQ)                                  00010700
      EQUIVALENCE    (INSWI(54),NSTATE)  ,(INSEC(315)        ,MANDON)           00010800
                                                                                 00010900
C                                                                                00011000
C                                                                                00011100
      NX=NORDER+1                                                                00011200
      STIME=TTO                                                                  00011300
      IND3=IND(3)                                                                00011400
      IF(INTFLG.EQ.1)GO TO 180                                                   00011500
C                                                                                00011600
C     TEST FOR TERMINAL SECTION                                                  00011700
C                                                                                00011800
   20 IF(NCONDT.GT.100) GO TO 999                                                00011900
C                                                                                00012000
C     TEST FOR SECTIONING WITH INTERVENING PRINT                                 00012100
C                                                                                00012200
      IF(NCONDT.LT.0) GO TO 150                                                  00012300
C                                                                                00012400
C     IS REQUEST TIME LESS THEN CURRENT TIME, IF SO INTERPOLATE                  00012500
C                                                                                00012600
      IF(IND3.EQ.7)   T=TREG(K-2)                                                00012700
   40 IF(DABS(TTO).LE.DABS(T)) GO TO 168                                         00012800
C                                                                                00012900
C     SET PARAMETERS TO INTEGRATE ANOTHER STEP                                   00013000
C                                                                                00013100
C *** UPDATE INDEPENDENT VARIABLE                                                00013200
C                                                                                00013300
      STREG = STREG + H                                                          00013400
C                                                                                00013500
C *** IF DOING R-K INTEGRATION  SET K TO 15 AND INTEGRATE                        00013600
C                                                                                00013700
      IF (IND(2) .NE. 4) GO TO 50                                                00013800
      K = 15                                                                     00013900
      CALL RKINTG(IERR)                                                          00014000
      GO TO 150                                                                  00014100
C                                                                                00014200
C *** CONTINUE HERE FOR STANDARD ADAMS METHOD (CSTEPX)                           00014300
C                                                                                00014400
   50 K=K+1                                                                      00014500
C                                                                                00014600
C     IS THE DERIVATIVE ARRAY FULL, IF SO RECOPY THE LAST NSTR ROWS              00014700
C                                                                                00014800
      IF(K.LE.15) GO TO 110                                                      00014900
      DO 100 I=1,NSTR                                                            00015000
      NBCK = 15 - NSTR + I                                                       00015100
      IF(IND3.NE.6.AND.IND3.NE.7)   GO TO 90                                     00015200
      TREG(I)=TREG(NBCK)                                                         00015300
   90 CONTINUE                                                                   00015400
      DO 100 J=1,NOPARM                                                          00015500
      DPARRY(I,J)=DPARRY(NBCK,J)                                                 00015600
      IF(IND3.EQ.7.AND.IND(26).NE.1)  PARARY(I,J)=PARARY(NBCK,J)                 00015700
C                                                                                00015800
      IF (NEQ.LE.0) GO TO 100                                                    00015900
      DO 95 L=1,NEQ                                                              00016000
      VEVOPD(I,J,L) = VEVOPD(15-NSTR+I,J,L)                                      00016100
   95 CONTINUE                                                                   00016200
  100 CONTINUE                                                                   00016300
      K=NSTR+1
```

```
C                                                                       00016400
C     INTEGRATE TO NEXT STEP                                            00016500
C                                                                       00016600
  110 CALL CSTEPX(IERR)                                                 00016700
C                                                                       00016800
C     CHECK FOR SECTIONING                                              00016900
C                                                                       00017000
  150 CALL SECHEK(&160,&168,&160)                                       00017100
      GO TO 40                                                          00017200
C                                                                       00017300
C     SECTIONING HAS OCCURRED                                           00017400
C                                                                       00017500
  160 CALL CROSSV(MSGER,&165)                                           00017600
      GO TO 20                                                          00017700
C                                                                       00017800
C     ERROR RETURN IF MAXIMUM CORRECTOR ITERATIONS WERE VIOLATED OR     00017900
C     ERROR FROM CROSSV                                                 00018000
C                                                                       00018100
  165 RETURN1                                                           00018200
C                                                                       00018300
C *** SKIP IF NOT USING R-K INTEGRATION AND HERMITE INTERPOLATION        00018400
C                                                                       00018500
  168 IF (IND(2) .NE. 4) GO TO 169                                      00018600
      CALL HERMIT(VPELMT,PARARY,DPARRY,NOPARM,STIME,H,STREG,IORDER(1))  00018700
      GO TO 170                                                         00018800
C                                                                       00018900
C     IS THIS A KS RUN, IF SO CALL THE TIME INTERPOLATOR                00019000
C                                                                       00019100
  169 IF(IND3.EQ.6) CALL HEMITR(STIME,TTO,K,H,STREG,TREG)               00019200
C                                                                       00019300
C      IS THIS A DS RUN,IF SO CALL THE INTERPOLATOR                     00019400
C                                                                       00019500
      IF(IND3.EQ.7)  CALL HEMIDS(STIME,TTO,K,H,STREG,TREG,NX,XTO,XDTO)  00019600
      IF(IND3.EQ.7)  GO TO 180                                          00019700
C                                                                       00019800
C     INTERPOLATE FOR ORBITAL ELEMENTS AT THE REQUEST TIME              00019900
C                                                                       00020000
      CALL INTPAR(VPELMT,DPARRY,XXSUM,STREG,STIME,K,.FALSE.,NX,NOPARM,H)00020100
C                                                                       00020200
C     BRANCH TO THE APPROPRIATE SUBROUTINE FOR COMPUTATION OF POSITION  00020300
C     AND VELOCITY FROM THE ORBITAL ELEMENTS                            00020400
C                                                                       00020500
  170 IF(IND3.GT.10) IND3=IND3-10                                       00020600
      GO TO (171,172,172,173,173,174,171,176),IND3                      00020700
  171 CALL DALLPV(VPELMT,XTO,XDTO,JERROR)                               00020800
      GO TO 180                                                         00020900
C                                                                       00020910
C     DETERMINE THE OSCULATING EQUINOCTIAL ORBITAL ELEMENTSPRIOR        00020920
C     TO DETERMINING POSITION AND VELOCITY                              00020930
C                                                                       00020940
  172 CALL SPEXEC                                                       00020950
      CALL EQUNPV(OSCELM,XTO,XDTO,JERROR,IND3)                          00021000
      GO TO 180                                                         00021100
  173 CALL IDLPV(VPELMT,XTO,XDTO,IND3)                                  00021200
      GO TO 180                                                         00021300
  174 CALL DPEL2X(VPELMT,STIME,U,US,XTO,XDTO,SE2,CE2)                   00021400
      GO TO 180                                                         00021500
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ORBITV              AJG1324.GTDS.UPDATE.FORT

  176 CALL CELEM(VPELMT,GM(1CENT),XTO,XDTO)                          00021600
C                                                                    00021700
C     ARE ANALYTIC PARTIALS REQUIRED                                 00021800
C                                                                    00021900
  180 IF(NPART.NE.0) GO TO 999                                       00022000
      IF (IND(4).EQ.1) GO TO 195                                     00022100
      IF (NSTATE.EQ.0) GO TO 190                                     00022200
C                                                                    00022300
C     CALCULATE ANALYTIC STATE PARTIALS                              00022400
C                                                                    00022500
      CALL ANPART(XTO,XDTO,TTO,2,PXTXO)                              00022600
C                                                                    00022700
  188 IF (NEQ.LE.0) GO TO 999                                        00022800
C                                                                    00022900
C     SET UP TO COMPUTE DYNAMIC PARTIALS FOR NON-STATE PARAMETERS    00023000
C     WHEN ANALYTIC OPTION IS ON.                                    00023100
C                                                                    00023200
  190 N = NSTATE                                                     00023300
      GO TO 245                                                      00023400
C                                                                    00023500
C     TEST FOR MANEUVERS                                             00023600
C                                                                    00023700
  195 N = 0                                                          00023800
      IF (MANDON.EQ.0) GO TO 245                                     00023900
C                                                                    00024000
C     SET UP PARTIAL OUTPUT FOR NON-MANEUVER CASE                    00024100
C                                                                    00024200
  245 DO 250 I=1,NEQ                                                 00024300
C  *** IF DOING R-K CALL HERMIT INTERFOLATOR OTHERWISE INTPAR        00024400
           IF (IND(2) .NE. 4) GO TO 247                              00024500
           CALL HERMIT(TEMPPK(1,N+I),VEPARY(1,1,I),VEVOPD(1,1,1),    00024600
     *                 NOPARM,STIME,H,STREG,IORDER(2))              00024700
           GO TO 250                                                 00024800
  247      CALL INTPAR (TEMPPK(1,N+I),VEVOPD(1,1,I),XVSUM(1,I),STREG, 00024900
     *                  STIME,K,.TRUE.,NX,NOPARM,H)                  00025000
  250 CONTINUE                                                       00025100
C                                                                    00025200
C     COMPUTE CARTESIAN VARIATIONAL PARTIALS FROM KEPLERIAN          00025300
C     VOP VARIATIONSL PARTIALS.                                      00025400
C                                                                    00025500
      IF (IND(3) .EQ. 8) GO TO 300                                   00025510
      IF (IND(3) .NE. 12) GO TO 999                                  00025520
      CALL GMAT(TEMPK)                                               00025530
      CALL EPART(XTO,XDTO,PKPC)                                      00025540
      GO TO 400                                                      00025550
  300 CALL KPART (XTO,PKPC)                                          00025600
      CALL INV2 (6)                                                  00025700
  400 CALL MATMUL (PKPC,TEMPPK(1,N+1),PXTXO(1,N+1),6,6,20-N)         00025800
C                                                                    00025900
C                                                                    00026000
  999 RETURN                                                         00026100
      END                                                            00026200

******** END OF MEMBER ORBITV     280 RECORDS ********
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      SUBROUTINE RTQUAD(QUAD,AA,BB,NQUAD,ISPNC)                          00000010
C                                                                        00000020
C     VERSION OF 19 MARCH 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6      00000030
C                                                                        00000040
C     PURPOSE                                                            00000050
C       EVALUATES A DEFINITE INTEGRAL WITH A GUASSIAN NUMERICAL QUAD-    00000060
C       RATURE FORMULA WHEN FIRST ORDER SHORT PERIODICS ARE INCLUDED     00000070
C       IN THE ORBITAL ELEMENTS.                                         00000080
C                                                                        00000090
C                                                                        00000100
C     PARAMETERS                                                         00000110
C       QUAD     O    VALUE OF THE QUADRATURE.                         00000120
C       AA       I    LOWER LIMIT OF INTEGRATION.                      00000130
C       BB       I    UPPER LIMIT OF INTEGRATION.                      00000140
C       NQUAD    I    QUADRATURE ORDER PARAMETER.                      00000150
C       ISPNC    I    NUMBER OF COEFFICIENTS                           00000160
C                                                                        00000170
C     /WORKER/                                                           00000180
C       DPARDT   I    PRECISION ELEMENT RATES COMPUTED IN *GQFUN*.     00000190
C       NOPARM   I    NUMBER OF ORBITAL ELEMENTS.                      00000200
C       VPELMT   I    MEAN ORBITAL ELEMENT VECTOR.                     00000210
C       I2NDOR   I    2ND ORDER SWITCH WHICH IS USED IN EQINT.         00000220
C                     1=2ND ORDER EFFECTS ARE BEING CONSIDERED IN      00000230
C                       SUBROUTINE 'RTQUAD'.                           00000240
C                     2=2ND ORDER EFFECTS ARE NOT BEING CONSIDERED     00000250
C       ROA      O    THE MEAN R/A.                                    00000260
C       XML      O    THE MEAN-MEAN LONGITUDE.                         00000270
C                                                                        00000280
C     /SPCOEF/                                                           00000290
C       CQUAD    I    VALUE OF C COEFFICIENT                           00000300
C       DQUAD    I    VALUE OF D COEFFICIENT                           00000310
C                                                                        00000320
C                                                                        00000330
C     EXTERNAL REFERENCES                                                00000340
C       GQFUN         COMPUTES PRECISION ORBITAL ELEMENT RATES.         00000350
C                                                                        00000360
C     REFERENCES                                                         00000370
C       GTDS SUBROUTINE  QUADRT                                          00000380
C                                                                        00000390
C       "A FOURIER SERIES FORMULATION OF THE SHORT PERIODIC VARIATIONS    00000400
C       IN TERMS OF EQUINOCTIAL ELEMENTS." BY A.J. GREEN AND P.J. CEFOLA 00000410
C                                                                        00000420
C     ANALYSIS                                                           00000430
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                            00000440
C                                                                        00000450
C     PROGRAMMER                                                         00000460
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                            00000470
C                                                                        00000480
C*****START PROGRAM*************************************************** 00000490
C                                                                        00000500
C                                                                        00000510
      IMPLICIT REAL*8(A-H,O-Z)                                           00000520
      COMMON/WORKER/ RWORK(3264)          ,IWORK(12)                     00000530
      DIMENSION      QUAD(10)  ,DPARDT(10),VPELMT(10),TEMELM(10)         00000540
      DIMENSION      IXWBEG(7) ,IXWEND(7)                                00000550
      EQUIVALENCE    (ROA                ,RWORK(2670)       ),          00000560
     *               (XML                ,RWORK(2672)       ),          00000570
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     *               (VPELMT(1)          ,RWORK(2829)       ),          00000580
     *               (DPARDT(1)          ,RWORK(2840)       ),          00000590
     *               (NOPARM             ,IWORK(3)          ),          00000600
     *               (I2NDOR             ,IWORK(10)         )           00000610
      COMMON/SPCOEF/ CQUAD(10,10),DQUAD(10,10)                           00000620
C                                                                        00000630
      DIMENSION      XTAB(96)  ,WTAB(96)                                 00000640
      DIMENSION      XTB1  ( 9)          ,XTB10 (10)         ,          00000650
     *               XTB20 (10)          ,XTB30 (10)         ,          00000660
     *               XTB40 (10)          ,XTB50 (10)         ,          00000670
     *               XTB60 (10)          ,XTB70 (10)         ,          00000680
     *               XTB80 (10)          ,XTB90 ( 7)                    00000690
      DIMENSION      WTB1  ( 9)          ,WTB10 (10)         ,          00000700
     *               WTB20 (10)          ,WTB30 (10)         ,          00000710
     *               WTB40 (10)          ,WTB50 (10)         ,          00000720
     *               WTB60 (10)          ,WTB70 (10)         ,          00000730
     *               WTB80 (10)          ,WTB90 ( 7)                    00000740
C                                                                        00000750
      EQUIVALENCE    (XTB1  (1)          ,XTAB(  1)         ),          00000760
     *               (XTB10 (1)          ,XTAB( 10)         ),          00000770
     *               (XTB20 (1)          ,XTAB( 20)         ),          00000780
     *               (XTB30 (1)          ,XTAB( 30)         ),          00000790
     *               (XTB40 (1)          ,XTAB( 40)         ),          00000800
     *               (XTB50 (1)          ,XTAB( 50)         ),          00000810
     *               (XTB60 (1)          ,XTAB( 60)         ),          00000820
     *               (XTB70 (1)          ,XTAB( 70)         ),          00000830
     *               (XTB80 (1)          ,XTAB( 80)         ),          00000840
     *               (XTB90 (1)          ,XTAB( 90)         )           00000850
      EQUIVALENCE    (WTB1  (1)          ,WTAB(  1)         ),          00000860
     *               (WTB10 (1)          ,WTAB( 10)         ),          00000870
     *               (WTB20 (1)          ,WTAB( 20)         ),          00000880
     *               (WTB30 (1)          ,WTAB( 30)         ),          00000890
     *               (WTB40 (1)          ,WTAB( 40)         ),          00000900
     *               (WTB50 (1)          ,WTAB( 50)         ),          00000910
     *               (WTB60 (1)          ,WTAB( 60)         ),          00000920
     *               (WTB70 (1)          ,WTAB( 70)         ),          00000930
     *               (WTB80 (1)          ,WTAB( 80)         ),          00000940
     *               (WTB90 (1)          ,WTAB( 90)         )           00000950
C                                                                        00000960
C                   ABSCISSAE FOR ORDER  12 IN SEQUENCE  12.             00000970
C                                                                        00000980
      DATA XTB  1   /.1252334085114689D+00          ,                    00000990
     2               .3678314989981802D+00          ,                    00001000
     3               .5873179542866174D+00          ,                    00001010
     4               .7699026741943047D+00          ,                    00001020
     5               .9041172563704749D+00          ,                    00001030
     6               .9815606342467192D+00          ,                    00001040
C                                                                        00001050
C                   ABSCISSAE FOR ORDER  16 IN SEQUENCE  16.             00001060
C                                                                        00001070
     7                .9501250983763743D-01         ,                    00001080
     8                .2816035507792589D+00         ,                    00001090
     9                .4580167776572274D+00         /                    00001100
      DATA XTB 10   /.6178762444026437D+00          ,                    00001110
     1               .7554044083550030D+00          ,                    00001120
     2               .8656312023878317D+00          ,                    00001130
     3               .9445750230732326D+00          ,                    00001140
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     4              .9894009349916499D+00         ,                  00001150
C                                                                      00001160
C                  ABSCISSAE FOR ORDER  20 IN SEQUENCE  20.          00001170
C                                                                      00001180
     5              .7652652113349732D-01         ,                  00001190
     6              .2277858511416451D+00         ,                  00001200
     7              .3737060887154195D+00         ,                  00001210
     8              .5108670019508271D+00         ,                  00001220
     9              .6360536807265150D+00         /                  00001230
      DATA XTB 20  /.7463319064601508D+00         ,                  00001240
     1              .8391169718222188D+00         ,                  00001250
     2              .9122344282513259D+00         ,                  00001260
     3              .9639719272779138D+00         ,                  00001270
     4              .9931285991850949D+00         ,                  00001280
C                                                                      00001290
C                  ABSCISSAE FOR ORDER  24 IN SEQUENCE  24.          00001300
C                                                                      00001310
     5              .6405689286260562D-01         ,                  00001320
     6              .1911188674736163D+00         ,                  00001330
     7              .3150426796961634D+00         ,                  00001340
     8              .4337935076260451D+00         ,                  00001350
     9              .5454214713888395D+00         /                  00001360
      DATA XTB 30  /.6480936519369756D+00         ,                  00001370
     1              .7401241915785544D+00         ,                  00001380
     2              .8200019859739029D+00         ,                  00001390
     3              .8864155270044010D+00         ,                  00001400
     4              .9382745520027328D+00         ,                  00001410
     5              .9747285559713095D+00         ,                  00001420
     6              .9951872199970214D+00         ,                  00001430
C                                                                      00001440
C                  ABSCISSAE FOR ORDER  32 IN SEQUENCE  32.          00001450
C                                                                      00001460
     7              .4830766568773832D-01         ,                  00001470
     8              .1444719615827965D+00         ,                  00001480
     9              .2392873622521371D+00         /                  00001490
      DATA XTB 40  /.3318686022821276D+00         ,                  00001500
     1              .4213512761306353D+00         ,                  00001510
     2              .5068999089322294D+00         ,                  00001520
     3              .5877157572407623D+00         ,                  00001530
     4              .6630442669302152D+00         ,                  00001540
     5              .7321821187402897D+00         ,                  00001550
     6              .7944837959679424D+00         ,                  00001560
     7              .8493676137325700D+00         ,                  00001570
     8              .8963211557660521D+00         ,                  00001580
     9              .9349060759377397D+00         /                  00001590
      DATA XTB 50  /.9647622555875064D+00         ,                  00001600
     1              .9856115115452683D+00         ,                  00001610
     2              .9972638618494816D+00         ,                  00001620
C                                                                      00001630
C                  ABSCISSAE FOR ORDER  40 IN SEQUENCE  40.          00001640
C                                                                      00001650
     3              .3877241750605082D-01         ,                  00001660
     4              .1160840706752552D+00         ,                  00001670
     5              .1926975807013711D+00         ,                  00001680
     6              .2681521850072537D+00         ,                  00001690
     7              .3419940908257585D+00         ,                  00001700
     8              .4137792043716050D+00         ,                  00001710
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     9                .4830758016861787D+00         /                00001720
      DATA XTB 60    /.5494671250951282D+00         ,                00001730
     1                .6125538896679802D+00         ,                00001740
     2                .6719566846141795D+00         ,                00001750
     3                .7273182551899271D+00         ,                00001760
     4                .7783056514265194D+00         ,                00001770
     5                .8246122308333117D+00         ,                00001780
     6                .8659595032122595D+00         ,                00001790
     7                .9020988069688743D+00         ,                00001800
     8                .9328128082786765D+00         ,                00001810
     9                .9579168192137917D+00         /                00001820
      DATA XTB 70    /.9772599499837743D+00         ,                00001830
     1                .9907262386994570D+00         ,                00001840
     2                .9982377097105592D+00         ,                00001850
C                                                                      00001860
C                   ABSCISSAE FOR ORDER  48 IN SEQUENCE  48.         00001870
C                                                                      00001880
     3                .3238017096286936D-01         ,                00001890
     4                .9700469920946270D-01         ,                00001900
     5                .1612223560688917D+00         ,                00001910
     6                .2247637903946891D+00         ,                00001920
     7                .2873624873554556D+00         ,                00001930
     8                .3487558862921607D+00         ,                00001940
     9                .4086864819907167D+00         /                00001950
      DATA XTB 80    /.4669029047509584D+00         ,                00001960
     1                .5231609747222330D+00         ,                00001970
     2                .5772247260839727D+00         ,                00001980
     3                .6288673967765136D+00         ,                00001990
     4                .6778723796326639D+00         ,                00002000
     5                .7240341309238146D+00         ,                00002010
     6                .7671590325157403D+00         ,                00002020
     7                .8070662040294426D+00         ,                00002030
     8                .8435882616243935D+00         ,                00002040
     9                .8765720202742479D+00         /                00002050
      DATA XTB 90    /.9058791367155697D+00         ,                00002060
     1                .9313866907065543D+00         ,                00002070
     2                .9529877031604309D+00         ,                00002080
     3                .9705915925462472D+00         ,                00002090
     4                .9841245837228269D+00         ,                00002100
     5                .9935301722663508D+00         ,                00002110
     6                .9987710072524261D+00         /                00002120
C                                                                      00002130
C                   WEIGHTS FOR ORDER  12 IN SEQUENCE  12.           00002140
C                                                                      00002150
      DATA WTB  1    /.2491470458134028D+00         ,                00002160
     2                .2334925365383548D+00         ,                00002170
     3                .2031674267230659D+00         ,                00002180
     4                .1600783285433462D+00         ,                00002190
     5                .1069393259953184D+00         ,                00002200
     6                .4717533638651183D-01         ,                00002210
C                                                                      00002220
C                   WEIGHTS FOR ORDER  16 IN SEQUENCE  16.           00002230
C                                                                      00002240
     7                .1894506104550685D+00         ,                00002250
     8                .1826034150449236D+00         ,                00002260
     9                .1691565193950025D+00         /                00002270
      DATA WTB 10    /.1495959888165767D+00         ,                00002280
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     1              .1246289712555339D+00          ,                00002290
     2              .9515851168249277D-01          ,                00002300
     3              .6225352393864789D-01          ,                00002310
     4              .2715245941175409D-01          ,                00002320
C                                                                    00002330
C                 WEIGHTS FOR ORDER  20 IN SEQUENCE  20.             00002340
C                                                                    00002350
     5              .1527533871307258D+00          ,                00002360
     6              .1491729864726037D+00          ,                00002370
     7              .1420961093183820D+00          ,                00002380
     8              .1316886384491766D+00          ,                00002390
     9              .1181945319615184D+00          /                00002400
      DATA WTB 20   /.1019301198172404D+00          ,                00002410
     1              .8327674157670474D-01          ,                00002420
     2              .6267204833410905D-01          ,                00002430
     3              .4060142980036694D-01          ,                00002440
     4              .1761400713915212D-01          ,                00002450
C                                                                    00002460
C                 WEIGHTS FOR ORDER  24 IN SEQUENCE  24.             00002470
C                                                                    00002480
     5              .1279381953467521D+00          ,                00002490
     6              .1258374563466283D+00          ,                00002500
     7              .1216704729278034D+00          ,                00002510
     8              .1155056805372560+00           ,                00002520
     9              .1074442701159656D+00          /                00002530
      DATA WTB 30   /.9761865210411388D-01          ,                00002540
     1              .8619016153195327D-01          ,                00002550
     2              .7334648141108030D-01          ,                00002560
     3              .5929858491543678D-01          ,                00002570
     4              .4427743881741981D-01          ,                00002580
     5              .2853138662693366D-01          ,                00002590
     6              .1234122979998720D-01          ,                00002600
C                                                                    00002610
C                 WEIGHTS FOR ORDER  32 IN SEQUENCE  32.             00002620
C                                                                    00002630
     7              .9654008851472780D-01          ,                00002640
     8              .9563872007927485D-01          ,                00002650
     9              .9384439908080455D-01          /                00002660
      DATA WTB 40   /.9117367869576388D-01          ,                00002670
     1              .8765209300440391D-01          ,                00002680
     2              .8331192422694675D-01          ,                00002690
     3              .7819389578707030D-01          ,                00002700
     4              .7234579410884850D-01          ,                00002710
     5              .6582222277636184D-01          ,                00002720
     6              .5868409347853555D-01          ,                00002730
     7              .5099805926237618D-01          ,                00002740
     8              .4283589802222668D-01          ,                00002750
     9              .3427386291302143D-01          /                00002760
      DATA WTB 50   /.2539206530926206D-01          ,                00002770
     1              .1627439473090567D-01          ,                00002780
     2              .7018610009470096D-02          ,                00002790
C                                                                    00002800
C                 WEIGHTS FOR ORDER  40 IN SEQUENCE  40.             00002810
C                                                                    00002820
     3              .7750594797842481D-01          ,                00002830
     4              .7703981816424796D-01          ,                00002840
     5              .7611036190062624D-01          ,                00002850
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     6              .7472316905796826D-01       ,
     7              .7288658239580405D-01       ,
     8              .7061164739128677D-01       ,
     9              .6791204581523390D-01       /
      DATA WTB 60  /.6480401345660103D-01       ,
     1              .6130624249292894D-01       ,
     2              .5743976909939155D-01       ,
     3              .5322784698393682D-01       ,
     4              .4869580763507223D-01       ,
     5              .4387090818567327D-01       ,
     6              .3878216797447202D-01       ,
     7              .3346019528254785D-01       ,
     8              .2793700693002340D-01       ,
     9              .2224584919416696D-01       /
      DATA WTB 70  /.1642105838190789D-01       ,
     1              .1049828453115281D-01       ,
     2              .4521277098533191D-02       ,
C
C                WEIGHTS FOR ORDER  48 IN SEQUENCE  48.
C
     3              .6473769681268392D-01       ,
     4              .6446616443595007D-01       ,
     5              .6392423858464819D-01       ,
     6              .6311419228625402D-01       ,
     7              .6203942315989266D-01       ,
     8              .6070443916589388D-01       ,
     9              .5911483969839564D-01       /
      DATA WTB 80  /.5727729210040321D-01       ,
     1              .5519950369998416D-01       ,
     2              .5289018948519367D-01       ,
     3              .5035903555385447D-01       ,
     4              .4761665849249047D-01       ,
     5              .4467456085669428D-01       ,
     6              .4154508294346475D-01       ,
     7              .3824135106583071D-01       ,
     8              .3477722256477044D-01       ,
     9              .3116722783279809D-01       /
      DATA WTB 90  /.2742650970835695D-01       ,
     1              .2357076083932438D-01       ,
     2              .1961616045735553D-01       ,
     3              .1557931572294385D-01       ,
     4              .1147723457923454D-01       ,
     5              .7327553901276262D-02       ,
     6              .3153346052305839D-02       /
C
      DATA IXWBEG  /1         ,7         ,15        ,25        ,37        ,
     *              53        ,73        /
      DATA IXWEND  /6         ,14        ,24        ,36        ,52        ,
     *              72        ,96        /
C
C                INITIALIZATION
C
      I2NDOR=1
      NOPAR=NOPARM-1
      XX=(BB+AA)/2.D0
      DD=(BB-AA)/2.D0
      DO 1 I=1,NOPARM
```

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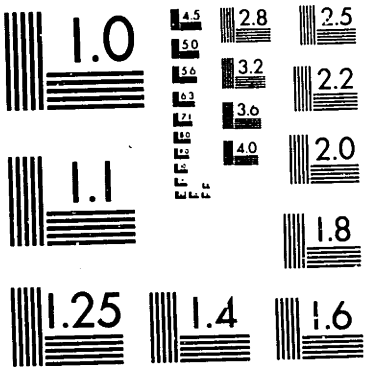
      QUAD(I)=0.D0                                                      00003430
      TEMELM(I)=VPELMT(I)                                               00003440
    1 CONTINUE                                                          00003450
C                                                                       00003460
      IBEG=IXWBEG(NQUAD)                                                00003470
      IEND=IXWEND(NQUAD)                                                00003480
      INDEX=IEND                                                        00003490
C                                                                       00003500
C               ACCUMULATE THE QUADRATURE SUM OVER THE FIRST HALF OF    00003510
C               THE INTERVAL.                                           00003520
    3 CONTINUE                                                          00003530
C                                                                       00003540
C     DETERMINE THE OSCULATING ELEMENTS TO 1ST ORDER.                   00003550
C                                                                       00003560
C     DETERMINE MEAN ECCENTRIC LONGITUDE.                               00003570
C                                                                       00003580
      EL=XX-DD*XTAB(INDEX)                                              00003590
      XML=EL-TEMELM(3)*DSIN(EL)+TEMELM(2)*DCOS(EL)                      00003600
C                                                                       00003610
C     INITIALIZE THE OSCULATING ELEMENTS.                               00003620
C                                                                       00003630
      DO 4 I=1,NOPAR                                                    00003640
      VPELMT(I)=TEMELM(I)                                               00003650
    4 CONTINUE                                                          00003660
      VPELMT(NOPARM)=XML                                                00003670
C                                                                       00003680
      DO 6 J=1,ISPNC                                                    00003690
         TT=J                                                           00003700
         TTML= TT*XML                                                   00003710
         DSJ = DSIN(TTML)                                               00003720
         DCJ = DCOS(TTML)                                               00003730
      DO 5 I=1,NOPARM                                                   00003740
         VPELMT(I) = VPELMT(I) + CQUAD(I,J)*DSJ-DQUAD(I,J)*DCJ          00003750
    5 CONTINUE                                                          00003760
    6 CONTINUE                                                          00003770
C                                                                       00003780
C     COMPUTE THE R/A FACTOR.                                           00003790
C                                                                       00003800
      ROA=1.D0-TEMELM(3)*DCOS(EL)-TEMELM(2)*DSIN(EL)                    00003810
C                                                                       00003820
         CALL GQFUN(EL)                                                 00003830
         WGHT=WTAB(INDEX)                                               00003840
         DO 7 I=1,NOPARM                                                00003850
            QUAD(I)=QUAD(I)+WGHT*DPARDT(I)                              00003860
    7    CONTINUE                                                       00003870
         INDEX=INDEX-1                                                  00003880
      IF (INDEX.GE.IBEG) GO TO 3                                        00003890
C                                                                       00003900
      IF (XTAB(IBEG).NE.0.D0) GO TO 8                                   00003910
      INDEX=IBEG                                                        00003920
C                                                                       00003930
C               ACCUMULATE THE QUADRATURE SUM OVER THE SECOND HALF OF   00003940
C               THE INTERVAL.                                           00003950
C                                                                       00003960
    8 CONTINUE                                                          00003970
         INDEX=INDEX+1                                                  00003980
C                                                                       00003990
```

RTQUAD AJG1324.GTDS.UPDATE.FORT

```
C     DETERMINE THE OSCULATING ELEMENTS TO 1ST ORDER.                    00004000
C                                                                        00004010
C     DETERMINE MEAN ECCENTRIC LONGITUDE.                                00004020
C                                                                        00004030
      EL=XX+DD*XTAB(INDEX)                                               00004040
      XML=EL-TEMELM(3)*DSIN(EL)+TEMELM(2)*DCOS(EL)                       00004050
C                                                                        00004060
C     INITIALIZE THE OSCULATING ELEMENTS.                                00004070
C                                                                        00004080
      DO 9 I=1,NOPAR                                                     00004090
      VPELMT(I)=TEMELM(I)                                                00004100
    9 CONTINUE                                                           00004110
      VPELMT(NOPARM)=XML                                                 00004120
C                                                                        00004130
      DO 11 J=1,ISPNC                                                    00004140
         TT=J                                                            00004150
         TTML= TT*XML                                                    00004160
         DSJ = DSIN(TTML)                                                00004170
         DCJ = DCOS(TTML)                                                00004180
      DO 10 I=1,NOPARM                                                   00004190
         VPELMT(I) = VPELMT(I) + CQUAD(I,J)*DSJ-DQUAD(I,J)*DCJ           00004200
   10 CONTINUE                                                           00004210
   11 CONTINUE                                                           00004220
C                                                                        00004230
C     COMPUTE THE R/A FACTOR.                                            00004240
C                                                                        00004250
      ROA=1.D0-TEMELM(3)*DCOS(EL)-TEMELM(2)*DSIN(EL)                     00004260
C                                                                        00004270
         CALL GQFUN(EL)                                                  00004280
         WGHT=WTAB(INDEX)                                                00004290
         DO 12 I=1,NOPARM                                                00004300
            QUAD(I)=QUAD(I)+WGHT*DPARDT(I)                               00004310
   12    CONTINUE                                                        00004320
      IF (INDEX.LT.IEND) GO TO 8                                         00004330
C                                                                        00004340
C     RETURN THE VALUE OF THE QUADRATURE AND MEAN ELEMENTS.              00004350
C                                                                        00004360
      DO 13 I=1,NOPARM                                                   00004370
         QUAD(I)=DD*QUAD(I)                                              00004380
         VPELMT(I)=TEMELM(I)                                             00004390
   13 CONTINUE                                                           00004400
      IENDCR=2                                                           00004410
      RETURN                                                             00004420
C                                                                        00004430
      END                                                                00004440
```

******** END OF MEMBER RTQUAD 444 RECORDS ********

```
      SUBROUTINE SPCOTO                                                 00000010
C                                                                       00000020
C     VERSION OF 20 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6.  00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C                                                                       00000060
C        TO DETERMINE THE TOTAL SHORT PERIODIC COEFFICIENTS.            00000070
C                                                                       00000080
C     PARAMETERS                                                        00000090
C                                                                       00000100
C       /ESTFLG/                                                        00000110
C                                                                       00000120
C       KCPSPP    I    CONTROL SWITCH FOR CONTINUOUS TIME-INDEPENDENT   00000130
C                      PERTURBATIONS IN THE SHORT PERIODIC PARTIALS.*    00000140
C       KCBSPP    I    CONTROL SWITCH FOR CENTRAL BODY PERTURBATIONS     00000150
C                      IN THE SHORT PERIODIC PARTIALS.*                 00000160
C       KTBSPP    I    CONTROL SWITCH FOR THIRD-BODY PERTURBATIONS IN    00000170
C                      THE SHORT PERIODIC PARTIALS.*                    00000180
C       KDRSPP    I    CONTROL SWITCH FOR DRAG PERTURBATIONS IN THE      00000190
C                      SHORT PERIODIC PARTIALS.*                        00000200
C       KSRSPP    I    CONTROL SWITCH FOR SOLAR RADIATION PRESSURE       00000210
C                      PERTURBATIONS IN THE SHORT PERIODIC PARTIALS.*    00000220
C       NVARSP    I    MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITATIONAL  00000230
C                      FIELD TO BE CONSIDERED IN THE SHORT PERIODIC      00000240
C                      GENERATOR.                                       00000250
C                                                                       00000260
C       /SPCOEF/                                                        00000270
C                                                                       00000280
C       CQUAD     O    ARRAY OF C SHORT PERIODIC COEFFICIENT.           00000290
C       DQUAD     O    ARRAY OF D SHORT PERIODIC COEFFICIENT.           00000300
C                                                                       00000310
C       /FRC/                                                           00000320
C                                                                       00000330
C       NBODY     I     NUMBER OF SOLAR SYSTEM BODIES CONSIDERED.       00000340
C       ISUN      I     SOLAR VECTOR INDICATOR.                         00000350
C       GM        I     GRAVITATIONAL CONSTANT TIMES THE MASS OF THE    00000360
C                       SOLAR SYSTEM BODY.                              00000370
C       NMAX      O     MAXIMUM DEGREE OF THE CENTRAL BODY GRAVITAIONAL 00000380
C                       FIELD.                                          00000390
C       MMAX      O     MAXIMUM ORDER OF THE CENTRAL BODY GRAVITATIONAL 00000400
C                       FIELD.                                          00000410
C                                                                       00000420
C       /ANAVIN/                                                        00000430
C                                                                       00000440
C       ITHIRD    O     THIRD-BODY AVERAGING OPTION.                    00000450
C                       POTENTIALS OF THE NONCENTRAL BODIES.            00000460
C       JTWOSQ    O     SECOND-ORDER J2 PERTURBATION CONTRL SWITCH.     00000470
C       IDRAG     O     QUADRATURE ORDER CONTROL SWITCH FOR DRAG.       00000480
C       ISOLAR    O     QUADRATURE ORDER CONTROL SWITCH FOR SOLAR       00000490
C                       RADIATION PRESSURE.                             00000500
C       ICORR     O     CONTROL SWITCH FOR SOLAR RADIATION PRESSURE.*    00000510
C       IDRDR     O     CONTROL SWITCH FOR 2ND ORDER DRAG EFFECTS.       00000520
C       KSPNC     I     ARRAY OF THE NUMBER OF SHORT PERIODIC            00000530
C                       COFFICIENTS.                                    00000540
C                                                                       00000550
C       /SWITCH/                                                        00000560
C                                                                       00000570
```



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

20X

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SFCOTO              AJG1324.GTDS.UPDATE.FORT

C       IRESON    O     CENTRAL BODY FIELD CONTROL SWITCH.                  00000580
C                       2=NXM FIELD                                        00000590
C                       4=NO CENTRAL BODY FIELD                            00000600
C                                                                          00000610
C       /WORKER/                                                           00000620
C                                                                          00000630
C       VPELMT    I/O   ARRAY OF THE MEAN ELEMENTS.                       00000640
C       XMEAN     O     MEAN MEAN MOTION.                                  00000650
C       XLAMDA    I     MEAN FAST VARIABLE.                                00000660
C       NOPARM    I     NUMBER OF PARAMETERS.                              00000670
C       NEQ       I     NUMBER OF VARIATIONAL EQUATIONS.                   00000680
C       NPERT     O     PERTURBATION INDICATOR.                            00000690
C                       1=CONTINUOUS TIME-DEPENDENT                        00000700
C                       2=CONTINUOUS TIME-INDEPENDENT                      00000710
C                       3=DRAG                                             00000720
C                       4=SOLAR RADIATION PRESSURE                         00000730
C                                                                          00000740
C     REFERENCE                                                            00000750
C                                                                          00000760
C        "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW             00000770
C        ALTITUDE SATELLITES." BY A.J. GREEN                               00000780
C                                                                          00000790
C     ANALYSIS                                                             00000800
C        ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                             00000810
C                                                                          00000820
C     PROGRAMMER                                                           00000830
C        ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                             00000840
C                                                                          00000850
C                                                                          00000860
      IMPLICIT REAL*8(A-H,O-Z)                                             00000870
      DIMENSION       CTEMP(10,10),STEMP(10,10)                            00000880
      DIMENSION       ETA(10)     ,OSCELM(10) ,VPELMT(10) ,                00000890
     2                GM(11)      ,POSVEL(6,8),DUMMY(3)                    00000900
      DIMENSION       IBODY(9)    ,IBODYS(3)                               00000910
C                                                                          00000920
      COMMON/ANAVIN/  ISTESS      ,ITESSE     ,ITRESE     ,ISTHIR      ,00000930
     1                ITHRAR      ,ITHRE      ,ITESS      ,NEND        ,00000940
     2                MEND        ,LEND       ,ITRES      ,NRES(10)    ,00000950
     3                MRES(10)    ,JRES(10)   ,NCSRES     ,LENDRS      ,00000960
     4                ITHIRD      ,NENDTH(8)  ,MENDTH(8)  ,JTWOSQ      ,00000970
     5                IQUAD(4)    ,NQUAD(4)   ,JPOS       ,JA          ,00000980
     6                JB          ,JC         ,JGHA       ,KPOS        ,00000990
     7                KA          ,KB         ,KC         ,KGHA        ,00001000
     8                ICORR       ,IPOS       ,IMOON      ,IMEAN       ,00001010
     9                IGHA        ,ISTRES     ,ITERM      ,ISHAD       ,00001020
     1                JTMDEP      ,JTMIND     ,JDRAG      ,JSOLAR      ,00001030
     2                J2SQSP      ,JSPCGB     ,JSPTHR     ,JSPPOS      ,00001040
     3                JSPA        ,JSPB       ,JSPC       ,JSPGHA      ,00001050
     4                KSPPOS      ,KSPA       ,KSPB       ,KSPC        ,00001060
     5                KSPGHA      ,KSPNC(4)   ,KSPPRT(4)  ,NMAXSP      ,00001070
     6                NMAXSP      ,IDRDR                                00001080
C                                                                          00001090
      COMMON/FRC    / RFRC(1300)  ,IFRC(50)                              00001100
      EQUIVALENCE     (GM(1)                  ,RFRC(2)                ),00001110
     2                (POSVEL(1,1)            ,RFRC(101)              ),00001120
     3                (NMAX                   ,IFRC(3)                ),00001130
     4                (NMAX                   ,IFRC(4)                ),00001140
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SPCOTO              AJG1324.GTDS.UPDATE.FORT

     5                (NBODY                    ,IFRC(7)              ),00001150
     6                (ISUN                     ,IFRC(14)             ) 00001160
C                                                                        00001170
      COMMON/SWITCH/  ISWIT(225)                                         00001180
      EQUIVALENCE     (IVOP                     ,ISWIT(3)             ),00001190
     2                (ITWOBD                   ,ISWIT(6)             ),00001200
     3                (IBURN                    ,ISWIT(12)            ),00001210
     4                (IRESON                   ,ISWIT(38)            ),00001220
     5                (IBODY(1)                 ,ISWIT(201)           ),00001230
     6                (ICENT                    ,IBODY(1)             ) 00001240
C                                                                        00001250
      COMMON/WORKER/  RWORK(3264) ,IWORK(13)                             00001260
      EQUIVALENCE     (OSCELM(1)                ,RWORK(2653)          ),00001270
     2                (ETA(1)                   ,RWORK(2673)          ),00001280
     3                (VPELMT(1)                ,RWORK(2829)          ),00001290
     4                (XLAMDA                   ,RWORK(2851)          ),00001300
     5                (XMEAN                    ,RWORK(2814)          ),00001310
     6                (T                        ,RWORK(3215)          ),00001320
     7                (TZERO                    ,RWORK(2850)          ) 00001330
      EQUIVALENCE     (NOPARM                   ,IWORK(3)             ),00001340
     2                (NPERT                    ,IWORK(9)             ) 00001350
C                                                                        00001360
      COMMON/DCINT/   DPDCI(432), INDCI(26)                              00001370
      EQUIVALENCE     (TTO                      ,DPDCI(127)           ) 00001380
C                                                                        00001390
      COMMON/SPCOEF/ CQUAD(10,10),DQUAD(10,10)                           00001400
C                                                                        00001410
      COMMON/ESTFLG/ IANAL    ,IDIFF    ,IQDRT    ,ICBVAR   ,J22VAR    , 00001420
     1               ITBVAR   ,IDRVAR   ,ISRVAR   ,NVAR     ,LEVAR     , 00001430
     2               KDRFLG   ,KPAR     ,KSTEP(20),KVRFLG   ,KCPSPP    , 00001440
     3               KCBSPP   ,KTBSPP   ,KDRSPP   ,KSRSPP   ,NVARSP    , 00001450
     4               IVSPNC(4),KATMOS   ,KPRTB1   ,KPRTB2   ,KPRTB3    , 00001460
     5               KPRTB4                                              00001470
C                                                                        00001480
C     SAVE THE TIME AND INITIALIZE.                                      00001490
C                                                                        00001500
      TZERO=TTO                                                          00001510
      DO 20 I=1,NOPARM                                                   00001520
      DO 10 J=1,10                                                       00001530
      CTEMP(I,J)=0.D0                                                    00001540
      DTEMP(I,J)=0.D0                                                    00001550
   10 CONTINUE                                                           00001560
   20 CONTINUE                                                           00001570
C                                                                        00001580
C     COMPUTE THE MEAN MOTION AND AUXILARY QUANTITIES.                   00001590
C                                                                        00001600
      XMEAN = DSQRT(GM(ICENT)/(VPELMT(1)*VPELMT(1)*VPELMT(1)))          00001610
      CALL AUXPAR(VPELMT)                                                00001620
C                                                                        00001630
C     COMPUTE THE SHORT PERIODIC COEFFICIENTS DUE TO NUMERICAL           00001640
C     TIME-INDEPENDENT CONTINUOUS PERTURBATIONS.                         00001650
C                                                                        00001660
  200 IF(KCPSPP.EQ.2) GO TO 300                                          00001670
C                                                                        00001680
C     SET THE CENTRAL BODY SPHERICAL HARMONIC SWITCH.                    00001690
C                                                                        00001700
      IRESNS=IRESON                                                      00001710
```

```
      NMAXS=NMAX                                                        00001720
      MMAXS=MMAX                                                        00001730
      IF(KCBSPP.NE.1) GO TO 210                                         00001740
      NMAX=NVARSP                                                       00001750
      MMAX=0                                                            00001760
      IRESON=2                                                          00001770
      GO TO 220                                                         00001780
  210 NMAX=0                                                            00001790
      MMAX=0                                                            00001800
      IRESON=4                                                          00001810
C                                                                       00001820
C     SET THE THIRD-BODY AND BURN SWITCH.                               00001830
C                                                                       00001840
  220 ITHRTE=ITHIRD                                                     00001850
      IF(KTBSPP.NE.1) GO TO 225                                         00001860
      ITHIRD=2                                                          00001870
      GO TO 226                                                         00001880
  225 ITHIRD=0                                                          00001890
  226 IBURNT=IBURN                                                      00001900
      IBURN=2                                                           00001910
C                                                                       00001920
C     COMPUTE THE POSITIONS OF THE PERTURBING BODIES.                   00001930
C                                                                       00001940
      CALL EVAL(TZERO,KSPPOS,KSPA,KSPB,KSPC,KSPGHA,IBODY,POSVEL)       00001950
C                                                                       00001960
C     COMPUTE THE LIMITS AND VALUE OF THE SHORT PERIODIC COEFFICIENTS.  00001970
C                                                                       00001980
      NPERT = 2                                                         00001990
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00002000
      CALL SPGENN(FLOW,FHIGH)                                           00002010
C                                                                       00002020
C     ACCUMULATE THE SHORT PERIODIC COEFFICIENTS.                       00002030
C                                                                       00002040
      KNC=KSFNC(NPERT)                                                  00002050
      DO 240 I=1,NOPARM                                                 00002060
      DO 230 J=1,KNC                                                    00002070
      CTEMP(I,J)=CTEMP(I,J)+CQUAD(I,J)                                  00002080
      DTEMP(I,J)=DTEMP(I,J)+DQUAD(I,J)                                  00002090
  230 CONTINUE                                                          00002100
  240 CONTINUE                                                          00002110
C                                                                       00002120
C     RESTORE THE SWITCHES.                                             00002130
C                                                                       00002140
      IRESON = IRESNS                                                   00002150
      ITHIRD = ITHRTE                                                   00002160
      IBURN  = IBURNT                                                   00002170
      NMAX   = NMAXS                                                    00002180
      MMAX   = MMAXS                                                    00002190
C                                                                       00002200
C     COMPUTE THE SHORT PERIODIC COEFFICIENTS DUE TO DRAG PERTURBATIONS.00002210
C                                                                       00002220
  300 IF(KDRSPP.NE.1) GO TO 400                                         00002230
C                                                                       00002240
C     COMPUTE THE LIMITS OF THE AVERAGING INTEGRAL.                     00002250
C                                                                       00002260
      NPERT = 3                                                         00002270
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00002280
```

```
      IF (NPERT.EQ.0) GO TO 400                                          00002290
C                                                                        00002300
C     FIND THE POSITION OF THE SUN SO THAT THE DIURNAL BULGE IN THE      00002310
C     ATMOSPHERE CAN BE LOCATED.                                         00002320
C                                                                        00002330
      IBODYS(2) = IBODY(2)                                               00002340
      IBODYS(3) = IBODY(3)                                               00002350
      NBODYS = NBODY                                                     00002360
C                                                                        00002370
      IBODY(2) = 3                                                       00002380
      IBODY(3) = 0                                                       00002390
      NBODY = 2                                                          00002400
C                                                                        00002410
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                            00002420
C                                                                        00002430
C     COMPUTE THE VALUE OF THE SHORT PERIODIC COEFFICIENTS.              00002440
C                                                                        00002450
      CALL SPGENN(FLOW,FHIGH)                                            00002460
C                                                                        00002470
C     ACCUMULATE THE SHORT PERIODIC COEFFICIENTS.                        00002480
C                                                                        00002490
      KNC=KSPNC(NPERT)                                                   00002500
      DO 340 I=1,NOPARM                                                  00002510
      DO 330 J=1,KNC                                                     00002520
      CTEMP(I,J)=CTEMP(I,J)+CQUAD(I,J)                                   00002530
      DTEMP(I,J)=DTEMP(I,J)+DQUAD(I,J)                                   00002540
  330 CONTINUE                                                           00002550
  340 CONTINUE                                                           00002560
C                                                                        00002570
C     RESTORE THE THIRD BODIES.                                          00002580
C                                                                        00002590
      IF (KSRSPP.EQ.1) GO TO 410                                         00002600
      IBODY(2) = IBODYS(2)                                               00002610
      IBODY(3) = IBODYS(3)                                               00002620
      NBODY = NBODYS                                                     00002630
      GO TO 500                                                          00002640
C                                                                        00002650
C     COMPUTE THE SHORT PERIODICS DUE TO SOLAR RADIATION PRESSURE.       00002660
C                                                                        00002670
  400 IF (KSRSPP.NE.1) GO TO 500                                         00002680
C                                                                        00002690
C     FIND AND STORE THE POSITION OF THE SUN.                            00002700
C                                                                        00002710
      IBODYS(2) = IBODY(2)                                               00002720
      IBODYS(3) = IBODY(3)                                               00002730
      NBODYS = NBODY                                                     00002740
C                                                                        00002750
      IBODY(2) = 3                                                       00002760
      IBODY(3) = 0                                                       00002770
      NBODY = 2                                                          00002780
C                                                                        00002790
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                            00002800
C                                                                        00002810
  410 ISUNS = ISUN                                                       00002820
      ISUN = - 1                                                         00002830
      CALL PMASS(IBODY,DUMMY,DUMMY,IERR)                                 00002840
      ISUN = 1                                                           00002850
```

SPCOTO AJG1324.GTDS.UPDATE.FORT

```
C                                                                       00002860
C     COMPUTE THE LIMITS AND VALUE OF THE SHORT PERIODIC COEFFICIENTS.  00002870
C                                                                       00002880
      NPERT = 4                                                         00002890
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00002900
      CALL SPGENN(FLOW,FHIGH)                                           00002910
C                                                                       00002920
C     ACCUMULATE THE SHORT PERIODIC COEFFICIENTS.                       00002930
C                                                                       00002940
      KNC=KSPNC(NPERT)                                                  00002950
      DO 440 I=1,NOPARM                                                 00002960
      DO 430 J=1,KNC                                                    00002970
      CTEMP(I,J)=CTEMP(I,J)+CQUAD(I,J)                                  00002980
      DTEMP(I,J)=DTEMP(I,J)+DQUAD(I,J)                                  00002990
  430 CONTINUE                                                          00003000
  440 CONTINUE                                                          00003010
C                                                                       00003020
C     RESTORE THE THIRD BODIES.                                         00003030
C                                                                       00003040
      IBODY(2) = IBODYS(2)                                              00003050
      IBODY(3) = IBODYS(3)                                              00003060
      NBODY = NBODYS                                                    00003070
      ISUN = ISUNS                                                      00003080
C                                                                       00003090
C     RESTORE THE FAST VARIABLE AND RETURN THE SHORT PERIODIC           00003100
C     COEFFICIENTS.                                                     00003110
C                                                                       00003120
  500 VPELMT(NOPARM)=XLAMDA                                             00003130
      DO 520 I=1,NOPARM                                                 00003140
      DO 510 J=1,10                                                     00003150
      CQUAD(I,J)=CTEMP(I,J)                                             00003160
      DQUAD(I,J)=DTEMP(I,J)                                             00003170
  510 CONTINUE                                                          00003180
  520 CONTINUE                                                          00003190
C                                                                       00003200
      RETURN                                                            00003210
      END                                                               00003220
```

******** END OF MEMBER SPCOTO 322 RECORDS ********

SPEXEC AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE SPEXEC                                                 00000010
C                                                                       00000020
C     VERSION OF 22 FEB 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6.      00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C          TO CONTROL THE EVALUATION OF THE OSCULATING ELEMENTS FOR      00000060
C          THE AVERAGED VOP ORBIT GENERATOR.                            00000070
C                                                                       00000080
C     PARAMETERS                                                        00000090
C                                                                       00000100
C     /ANAVIN/                                                          00000110
C                                                                       00000120
C       J2SPSQ    I   CONTROL SWITCH FOR J2 SHORT PERIODICS. *        00000130
C       JTMDEP    I   CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE 00000140
C                     TO TIME-DEPENDENT CONTINUOUS PERTURBATIONS.  *  00000150
C       JTMIND    I   CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE 00000160
C                     TO TIME-INDEPENDENT CONTINUOUS PERTURBATIONS. * 00000170
C       JSPCGB    I   CONTROL SWITCH FOR NUMERICAL SHORT PERIODICS DUE 00000180
C                     TO THE CENTRAL GRAVITATIONAL BODY.              00000190
C                     0=NO CENTRAL BODY FIELD                         00000200
C                     1=TIME-DEPENDENT NUMERICAL SHORT PERIODICS      00000210
C                     2=TIME=INDEPENDENT NUMERICAL SHORT PERIODICS    00000220
C                     3=ANALYTICAL SHORT PERIODICS                    00000230
C       ITHIRD    0   THIRD-BODY AVERAGING OPTION.                    00000240
C                     0=NO THIRD-BODY                                 00000250
C                     1=TIME-DEPENDENT NUMERICAL AVERAGING            00000260
C                     2=TIME-INDEPENDENT NUMERICAL AVERAGING          00000270
C                     3=ANALYTICAL AVERAGING                          00000280
C       JSPTHR    I   CONTROL SWITCH FOR THIRD-BODY NUMERICAL SHORT   00000290
C                     PERIODICS.                                      00000300
C                     0=NO THIRD BODY                                 00000310
C                     1=TIME-DEPENDENT NUMERICAL SHORT PERIODICS      00000320
C                     2=TIME-INDEPENDENT NUMERICAL SHORT PERIODICS    00000330
C                     3=ANALYTICAL SHORT PERIODICS                    00000340
C       JDRAG     I   CONTROL SWITCH FOR DRAG SHORT PERIODICS. *      00000350
C       JSOLAR    I   CONTROL SWITCH FOR SOLAR RADIATION PRESSURE     00000360
C                     SHORT PERIODICS. *                              00000370
C       NMAXSP    I   MAXIMUM DEGREE FOR THE CENTRAL BODY FIELD THAT  00000380
C                     WILL BE USED IN THE DETERMINATION OF THE SHORT  00000390
C                     PERIODICS.                                      00000400
C       MMAXSP    I   MAXIMUM ORDER FOR THE CENTRAL BODY FIELD THAT   00000410
C                     WILL BE USED IN THE DETERMINATION OF THE SHORT  00000420
C                     PERIODICS.                                      00000430
C                                                                     00000440
C     /FRC/                                                           00000450
C                                                                     00000460
C       NBODY     O   NUMBER OF SOLAR SYSTEM BODIES CONSIDERED.       00000470
C       ISUN      O   SOLAR VECTOR INDICATOR.                         00000480
C       GM        I   GRAVITATIONAL CONSTANT TIMES THE MASS OF THE    00000490
C                     SOLAR SYSTEM BODY.                              00000500
C       NMAX      O   MAXIMUM DEGREE FOR THE EARTH OR MOON FIELDS.    00000510
C       MMAX      O   MAXIMUM ORDER FOR THE EARTH OR MOON FIELDS.     00000520
C                                                                     00000530
C     /SWITCH/                                                        00000540
C                                                                     00000550
C       IVOP      I   VOP TYPE.                                       00000560
C                     12=AVERAGED EQUINOCTIAL ELEMENTS                00000570
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SPEXEC              AJG1324.GTDS.UPDATE.FORT

C       ITWOBD    I   TWO-BODY MOTION. *                                    00000580
C       IBURN     O   CONTROL SWITCH FOR POLYNOMIAL BURN MODEL. *           00000590
C       IRESON    O   CENTRAL BODY FIELD SWITCH.                            00000600
C                     1=NXM CENTRAL BODY FIELD PLUS HIGH ORDER              00000610
C                       RESONANCE                                           00000620
C                     2=NXM CENTRAL BODY FIELD ONLY                         00000630
C                     3=HIGH ORDER RESONANCE ONLY                           00000640
C                     4=NO CENTRAL BODY FIELD                               00000650
C       IBODY     O   CENTRAL AND NON-CENTRAL BODY INDICATORS.              00000660
C                                                                           00000670
C     /WORKER/                                                              00000680
C                                                                           00000690
C       OSCELM    O   ARRAY OF OSCULATING ELEMENTS.                         00000700
C       ETA       O   ARRAY OF SHORT PERIODICS.                             00000710
C       VPELMT    I   ARRAY OF MEAN ELEMENTS.                               00000720
C       XLAMBA    I   MEAN FAST VARIABLE(MEAN MEAN LONGITUDE).              00000730
C       XMEAN     O   MEAN MOTION.                                          00000740
C       T         I   TIME FROM EPOCH OF KTH DERIVATIVE.                    00000750
C       NOPARM    I   NUMBER OF PARAMETERS.                                 00000760
C       NPERT     O   PERTURBATION INDICATOR.                               00000770
C                     1=CONTINUOUS TIME-DEPENDENT                           00000780
C                     2=CONTINUOUS TIME-INDEPENDENT                         00000790
C                     3=DRAG                                                00000800
C                     4=SOLAR RADIATION PRESSURE                            00000810
C                                                                           00000820
C     /DCINT/                                                               00000830
C                                                                           00000840
C       TTO       I   OUTPUT TIME.                                          00000850
C                                                                           00000860
C                                                                           00000870
C       *    1=YES        2=NO                                             00000880
C                                                                           00000890
C                                                                           00000900
C     REFERENCES                                                            00000910
C                                                                           00000920
C       "A FOURIER SERIES FORMULATION OF THE SHORT PERIODIC VARIATIONS       00000930
C       IN TERMS OF EQUINOCTIAL ELEMENTS." BY A.J. GREEN AND P.J. CEFOLA    00000940
C                                                                           00000950
C     ANALYSIS                                                              00000960
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                               00000970
C                                                                           00000980
C     PROGRAMMER                                                            00000990
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                               00001000
C                                                                           00001010
C*****START PROGRAM*************************************************************00001020
C                                                                           00001030
C                                                                           00001040
      IMPLICIT REAL*8(A-H,O-Z)                                              00001050
      DIMENSION        ETA(10)     ,OSCELM(10) ,VPELMT(10) ,                00001060
     2                 GM(11)      ,POSVEL(6,8),DUMMY(3)                    00001070
      DIMENSION        IBODY(9)    ,IBODYS(3)                               00001080
C                                                                           00001090
      COMMON/ANAVIN/   ISTESS      ,ITESSE     ,ITRESE     ,ISTHIR     ,00001100
     1                 ITHRAR      ,ITHRE      ,ITESS      ,NEND       ,00001110
     2                 MEND        ,LEND       ,ITRES      ,NRES(10)   ,00001120
     3                 MRES(10)    ,JRES(10)   ,NCSRES     ,LENDRS     ,00001130
     4                 ITHIRD      ,NENDTH(8)  ,MENDTH(8)  ,JTWOSQ     ,00001140
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     5              IQUAD(4)     ,NQUAD(4)     ,JPOS        ,JA          ,00001150
     6              JB           ,JC           ,JGHA        ,KPOS        ,00001160
     7              KA           ,KB           ,KC          ,KGHA        ,00001170
     8              ICORR        ,IPOS         ,IMOON       ,IMEAN       ,00001180
     9              IGHA         ,ISTRES       ,ITERM       ,ISHAD       ,00001190
     1              JTMDEP       ,JTMIND       ,JDRAG       ,JSOLAR      ,00001200
     2              J2SQSP       ,JSPCGB       ,JSPTHR      ,JSPPOS      ,00001210
     3              JSPA         ,JSPB         ,JSPC        ,JSPGHA      ,00001220
     4              KSPPOS       ,KSPA         ,KSPB        ,KSPC        ,00001230
     5              KSPGHA       ,KSPNC(4)     ,KSPPRT(4)   ,NMAXSP      ,00001240
     6              MMAXSP       ,IDRDR                                   00001250
C
      COMMON/FRC   / RFRC(1300)   ,IFRC(50)                                00001260
      EQUIVALENCE    (GM(1)                     ,RFRC(2)                  ),00001270
     2               (POSVEL(1,1)               ,RFRC(101)                ),00001280
     3               (NMAX                      ,IFRC(3)                  ),00001290
     4               (MMAX                      ,IFRC(4)                  ),00001300
     5               (NBODY                     ,IFRC(7)                  ),00001310
     6               (ICUN                      ,IFRC(14)                 ) 00001320
C                                                                           00001330
      COMMON/SWITCH/ ISWIT(225)                                             00001340
      EQUIVALENCE    (IVOP                      ,ISWIT(3)                 ),00001350
     2               (ITWOBD                    ,ISWIT(6)                 ),00001360
     3               (IBURN                     ,ISWIT(12)                ),00001370
     4               (IRESCN                    ,ISWIT(38)                ),00001380
     5               (IBODY(1)                  ,ISWIT(201)               ),00001390
     6               (ICENT                     ,IBODY(1)                 ),00001400
C                                                                           00001410
      COMMON/WORKER/ RWORK(3264) ,IWORK(13)                                 00001420
      EQUIVALENCE    (OSCELM(1)                 ,RWORK(2653)              ),00001430
     2               (ETA(1)                    ,RWORK(2673)              ),00001440
     3               (VPELMT(1)                 ,RWORK(2829)              ),00001450
     4               (XLAMDA                    ,RWORK(2851)              ),00001460
     5               (XMEAN                     ,RWORK(2814)              ),00001470
     6               (T                         ,RWORK(3215)              ),00001480
     7               (TZERO                     ,RWORK(2850)              ),00001490
      EQUIVALENCE    (NOPARM                    ,IWORK(3)                 ),00001500
     2               (NPERT                     ,IWORK(9)                 ),00001510
C                                                                           00001520
      COMMON/DCINT/  DPDCI(432), INDCI(26)                                  00001530
      EQUIVALENCE    (TTO                       ,DPDCI(127)               ),00001540
C                                                                           00001550
C     SAVE THE TIME AND FAST VARIABLE.                                      00001560
C                                                                           00001570
      TZERO = TTO                                                           00001580
      XLAMDA = VPELMT(NOPARM)                                               00001590
C                                                                           00001600
C     INITIALIZE THE OSCULATING ELEMENTS.                                   00001610
C                                                                           00001620
      DO 10 I=1,NOPARM                                                      00001630
         OSCELM(I) = VPELMT(I)                                              00001640
   10 CONTINUE                                                              00001650
C                                                                           00001660
C     COMPUTE THE MEAN MOTION AND TEST FOR TWO-BODY MOTION AND              00001670
C     AVERAGED EQUINOCTIAL ELEMENTS.                                        00001680
C                                                                           00001690
      XMEAN = DSQRT(GM(ICENT)/(VPELMT(1)*VPELMT(1)*VPELMT(1)))              00001700
                                                                            00001710
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      IF (ITWOBD.EQ.1) GO TO 500                                        00001720
      IF (IVOP.NE.12) GO TO 500                                         00001730
C                                                                       00001740
C     COMPUTE AUXILIARY QUANTITIES.                                     00001750
C                                                                       00001760
      CALL AUXPAR(VPELMT)                                               00001770
C                                                                       00001780
C     COMPUTE THE ANALYTICAL SHORT PERIODICS.                           00001790
C                                                                       00001800
      CALL SPANAL                                                       00001810
C                                                                       00001820
C     ACCUMULATE THE OSCULATING ELEMENTS.                               00001830
C                                                                       00001840
      DO 11 I=1,NOPARM                                                  00001850
         OSCELM(I) = OSCELM(I) + ETA(I)                                 00001860
   11 CONTINUE                                                          00001870
C                                                                       00001880
C     COMPUTE THE SHORT PERIODICS DUE TO THE J2 SQUARED PERTURBATIONS.   00001890
C                                                                       00001900
      IF (J2SQSP.NE.1) GO TO 100                                        00001910
      CALL SPJ2SQ                                                       00001920
C                                                                       00001930
C      ACCUMULATE THE OSCULATING ELEMENTS.                              00001940
C                                                                       00001950
      DO 12 I=1,NOPARM                                                  00001960
         OSCELM(I) = OSCELM(I) + ETA(I)                                 00001970
   12 CONTINUE                                                          00001980
C                                                                       00001990
C     COMPUTE THE SHORT PERIODICS DUE TO NUMERICAL TIME-DEPENDENT       00002000
C     CONTINUOUS PERTURBATIONS. REMEMBER THAT THIS FORMULATION IS       00002010
C     NOT RIGOROUS.                                                     00002020
C                                                                       00002030
  100 IF (JTMDEP.NE.1) GO TO 200                                        00002040
C                                                                       00002050
C     SET THE CENTRAL BODY SPHERICAL HARMONIC SWITCHES AND THE SWITCHES 00002060
C     TO BE USED IN SUBROUTINE 'EVAL'.                                  00002070
C                                                                       00002080
      IRESNS = IRESON                                                   00002090
      NMAXS = NMAX                                                      00002100
      MMAXS = MMAX                                                      00002110
      JPOST = JPOS                                                      00002120
      JAT   = JA                                                        00002130
      JBT   = JB                                                        00002140
      JCT   = JC                                                        00002150
      JGHAT = JGHA                                                      00002160
      TSAVE = T                                                         00002170
C                                                                       00002180
      NMAX = NMAXSP                                                     00002190
      MMAX = MMAXSP                                                     00002200
      JPOS = JSPPOS                                                     00002210
      JA   = JSPA                                                       00002220
      JB   = JSPB                                                       00002230
      JC   = JSPC                                                       00002240
      JGHA = JSPGHA                                                     00002250
C                                                                       00002260
      IF (JSPCGB.EQ.1) GO TO 110                                        00002270
      IRESON = 4                                                        00002280
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      GO TO 120                                                          00002290
  110 IRESON = 2                                                         00002300
C                                                                        00002310
C     SET THE THIRD-BODY AND BURN SWITCH.                                00002320
C                                                                        00002330
  120 ITHRTE = ITHIRD                                                    00002340
      IF(JSPTHR.NE.1) GO TO 125                                          00002350
      ITHIRD = 1                                                         00002360
      GO TO 126                                                          00002370
  125 ITHIRD = 0                                                         00002380
  126 IBURNT = IBURN                                                     00002390
      IBURN  = 2                                                         00002400
C                                                                        00002410
C     COMPUTE THE LIMITS AND VALUE OF THE SHORT PERIODICS.               00002420
C                                                                        00002430
      NPERT = 1                                                          00002440
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                     00002450
      CALL SPNUM(FLOW,FHIGH)                                             00002460
C                                                                        00002470
C     ACCUMULATE THE OSCULATING ELEMENTS.                                00002480
C                                                                        00002490
      DO 130 I=1,NOPARM                                                  00002500
         OSCELM(I) = OSCELM(I) + ETA(I)                                  00002510
  130 CONTINUE                                                           00002520
C                                                                        00002530
C     RESTORE THE SWITCHES.                                              00002540
C                                                                        00002550
      IRESON = IRESNS                                                    00002560
      ITHIRD = ITHRTE                                                    00002570
      IBURN  = IBURNT                                                    00002580
      NMAX   = NMAXS                                                     00002590
      MMAX   = MMAXS                                                     00002600
      JPOS   = JPOST                                                     00002610
      JA     = JAT                                                       00002620
      JB     = JBT                                                       00002630
      JC     = JCT                                                       00002640
      JGHA   = JGHAT                                                     00002650
      T = TSAVE                                                          00002660
C                                                                        00002670
C     COMPUTE THE SHORT PERIODICS DUE TO NUMERICAL TIME-INDEPENDENT      00002680
C     CONTINUOUS PERTURBATIONS.                                          00002690
C                                                                        00002700
  200 IF (JTMIND.NE.1) GO TO 300                                         00002710
C                                                                        00002720
C     SET THE CENTRAL BODY SPHERICAL HARMONIC SWITCH.  FOR               00002730
C     SHORT PERIODICS M = 0.                                             00002740
C                                                                        00002750
      IRESNS = IRESON                                                    00002760
      NMAXS = NMAX                                                       00002770
      MMAXS = MMAX                                                       00002780
      NMAX = NMAXSP                                                      00002790
      MMAX = 0                                                           00002800
C                                                                        00002810
      IF (JSPCGB.EQ.2) GO TO 210                                         00002820
      IRESON = 4                                                         00002830
      GO TO 220                                                          00002840
  210 IRESON = 2                                                         00002850
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C                                                                       00002860
C     SET THE THIRD-BODY AND BURN SWITCH.                               00002870
C                                                                       00002880
  220 ITHRTE = ITHIRD                                                   00002890
      IF(JSPTHR.NE.2) GO TO 225                                         00002900
      ITHIRD = 2                                                        00002910
      GO TO 226                                                         00002920
  225 ITHIRD = 0                                                        00002930
  226 IBURNT = IBURN                                                    00002940
      IBURN  = 2                                                        00002950
C                                                                       00002960
C     COMPUTE THE POSITIONS OF THE PERTURBING BODIES.                   00002970
C                                                                       00002980
      CALL EVAL(TZERO,KSPPOS,KSPA,KSPB,KSPC,KSPGHA,IBODY,POSVEL)        00002990
C                                                                       00003000
C     COMPUTE THE LIMITS AND VALUE OF THE SHORT PERIODICS.              00003010
C                                                                       00003020
      NPERT = 2                                                         00003030
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00003040
      CALL SPNUM(FLOW,FHIGH)                                            00003050
C                                                                       00003060
C     ACCUMULATE THE OSCULATING ELEMENTS.                               00003070
C                                                                       00003080
      DO 230 I=1,NOPARM                                                 00003090
         OSCELM(I) = OSCELM(I) + ETA(I)                                 00003100
  230 CONTINUE                                                          00003110
C                                                                       00003120
C     RESTORE THE SWITCHES.                                             00003130
C                                                                       00003140
      IRESON = IRESNS                                                   00003150
      ITHIRD = ITHRTE                                                   00003160
      IBURN  = IBURNT                                                   00003170
      NMAX   = NMAXS                                                    00003180
      MMAX   = MMAXS                                                    00003190
C                                                                       00003200
C     COMPUTE THE SHORT PERIODICS DUE TO DRAG.                          00003210
C                                                                       00003220
  300 IF (JDRAG.NE.1) GO TO 400                                         00003230
C                                                                       00003240
C     COMPUTE THE LIMITS OF THE AVERAGING INTEGRAL.                     00003250
C                                                                       00003260
      NPERT = 3                                                         00003270
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00003280
      IF (NPERT.EQ.0) GO TO 400                                         00003290
C                                                                       00003300
C     FIND THE POSITION OF THE SUN SO THAT THE DIURNAL BULGE IN THE     00003310
C     ATMOSPHERE CAN BE LOCATED.                                        00003320
C                                                                       00003330
      IBODYS(2) = IBODY(2)                                              00003340
      IBODYS(3) = IBODY(3)                                              00003350
      NBODYS = NBODY                                                    00003360
C                                                                       00003370
      IBODY(2) = 3                                                      00003380
      IBODY(3) = 0                                                      00003390
      NBODY = 2                                                         00003400
C                                                                       00003410
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                           00003420
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C                                                                       00003430
C     COMPUTE THE VALUE OF THE SHORT PERIODICS.                         00003440
C                                                                       00003450
      CALL SPNUM(FLOW,FHIGH)                                            00003460
C                                                                       00003470
C     ACCUMULATE THE OSCULATING ELEMENTS.                               00003480
C                                                                       00003490
      DO 310 I=1,NOPARM                                                 00003500
         OSCELM(I) = OSCELM(I) + ETA(I)                                 00003510
  310 CONTINUE                                                          00003520
C                                                                       00003530
C     RESTORE THE THIRD BODIES.                                         00003540
C                                                                       00003550
      IF (JSOLAR.EQ.1) GO TO 410                                        00003560
      IBODY(2) = IBODYS(2)                                              00003570
      IBODY(3) = IBODYS(3)                                              00003580
      NBODY = NBODYS                                                    00003590
      GO TO 500                                                         00003600
C                                                                       00003610
C     COMPUTE THE SHORT PERIODICS DUE TO SOLAR RADIATION PRESSURE.      00003620
C                                                                       00003630
  400 IF (JSOLAR.NE.1) GO TO 500                                        00003640
C                                                                       00003650
C     FIND AND STORE THE POSITION OF THE SUN.                           00003660
C                                                                       00003670
      IBODYS(2) = IBODY(2)                                              00003680
      IBODYS(3) = IBODY(3)                                              00003690
      NBODYS = NBODY                                                    00003700
C                                                                       00003710
      IBODY(2) = 3                                                      00003720
      IBODY(3) = 0                                                      00003730
      NBODY = 2                                                         00003740
C                                                                       00003750
      CALL EVAL(TZERO,1,2,2,2,2,IBODY,POSVEL)                           00003760
C                                                                       00003770
  410 ISUNS = ISUN                                                      00003780
      ISUN = - 1                                                        00003790
      CALL PMASS(IBODY,DUMMY,DUMMY,IERR)                                00003800
      ISUN = 1                                                          00003810
C                                                                       00003820
C     COMPUTE THE LIMITS AND VALUE OF THE SHORT PERIODICS.              00003830
C                                                                       00003840
      NPLT = 4                                                          00003850
      CALL LIMITS(FLOW,FHIGH,XLAMDA)                                    00003860
      CALL SPNUM(FLOW,FHIGH)                                            00003870
C                                                                       00003880
C     ACCUMULATE THE OSCULATING ELEMENTS.                               00003890
C                                                                       00003900
      DO 420 I=1,NOPARM                                                 00003910
         OSCELM(I) = OSCELM(I) + ETA(I)                                 00003920
  420 CONTINUE                                                          00003930
C                                                                       00003940
C     RESTORE THE THIRD BODIES.                                         00003950
C                                                                       00003960
      IBODY(2) = IBODYS(2)                                              00003970
      IBODY(3) = IBODYS(3)                                              00003980
      NBODY = NBODYS                                                    00003990
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SPEXEC              AJG1324.GTDS.UPDATE.FORT

      ISUN = ISUNS                                                    00004000
C                                                                     00004010
C     RESTORE THE FAST VARIABLE.                                      00004020
C                                                                     00004030
  500 VPELMT(NOPARM) = XLAMDA                                         00004040
      RETURN                                                          00004050
C                                                                     00004060
      END                                                             00004070

******** END OF MEMBER SPEXEC     407 RECORDS ********
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SPGENN AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE SPGENN(FLOW,FHIGH)                                          00000010
C                                                                            00000020
C     VERSION OF 22 FEB 79 IMPLEMENTED AT CSDL ON AMDAHL 470 V6.             00000030
C                                                                            00000040
C     PURPOSE                                                                00000050
C         CONTROLS THE NUMERICAL PROCEDURE FOR DETERMINING THE SHORT         00000060
C         PERIODIC COEFFICIENTS.                                             00000070
C                                                                            00000080
C     PROGRAMMER                                                             00000090
C         ANDREW J. GREEN, CPT,  U.S. ARMY, CSDL                             00000100
C                                                                            00000110
C*****START PROGRAM*************************************************************00000120
C                                                                            00000130
C                                                                            00000140
      IMPLICIT REAL*8(A-H,O-Z)                                               00000150
      DIMENSION   OSCELM(10),VPELMT(10),ETA(10)                              00000160
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00000170
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00000180
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00000190
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00000200
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00000210
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00000220
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00000230
     7                 KA          ,KB          ,KC          ,KGHA        ,00000240
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00000250
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00000260
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00000270
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00000280
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00000290
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00000300
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00000310
     6                 MMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR       00000320
C                                                                            00000330
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                                00000340
      EQUIVALENCE      (OSCELM(1)                ,RWORK(2653)              ),00000350
     2                 (ETA(1)                   ,RWORK(2673)              ),00000360
     3                 (VPELMT(1)                ,RWORK(2829)              ),00000370
     4                 (XLAMDA                   ,RWORK(2851)              ),00000380
     5                 (XMEAN                    ,RWORK(2814)              ),00000390
     6                 (T                        ,RWORK(3215)              ),00000400
     7                 (TZERO                    ,RWORK(2850)              ) 00000410
      EQUIVALENCE      (NOPARM                   ,IWORK(3)                 ),00000420
     2                 (NPERT                    ,IWORK(9)                 ) 00000430
C                                                                            00000440
C     IF THE NUMBER OF SHORT PERIODIC COEFFICIENTS IS GIVEN, SIMPLY          00000450
C     CALCULATE THE COEFFICIENTS. THE QUADRATURE ORDER FOR THE               00000460
C     DETERMINATION OF THE COEFFICIENTS IS THE SAME AS USED IN THE           00000470
C     DETERMINATION OF THE MEAN ELMENT RATES.                                00000480
C                                                                            00000490
      IF(KSPNC(NPERT).LE.0) GO TO 100                                        00000500
      CALL SPQUAD(FLOW,FHIGH,NQUAD(NPERT),KSPNC(NPERT))                      00000510
      RETURN                                                                 00000520
C                                                                            00000530
C     DETERMINE THE NUMBER OF COEFFICIENTS TO BE USED. THE CRITERION         00000540
C     WILL BE DETERMINED. FOR THE PRESENT TIME THE MAXIMUM ALLOWABLE         00000550
C     NUMBER OF COEFFICIENTS WILL BE SET.                                    00000560
C                                                                            00000570
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SPGENN              AJG1324.GTDS.UPDATE.FORT

  100 KSPNC(NPERT) = 10                                                 00000580
      RETURN                                                            00000590
C                                                                       00000600
      END                                                               00000610

******** END OF MEMBER SPGENN      61 RECORDS ********
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SPNUM AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE SPNUM(FLOW,FHIGH)                                          00000010
C                                                                           00000020
C     VERSION OF FEB 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6.             00000030
C                                                                           00000040
C     PURPOSE                                                               00000050
C          TO COMPUTE THE NUMERICAL SHORT PERIODICS.                        00000060
C                                                                           00000070
C     PARAMETERS                                                            00000080
C                                                                           00000090
C          ETA      O    ARRAY OF SHORT PERIODICS                         00000100
C          CQUAD    O    ARRAY OF C COEFFICIENTS                          00000110
C          DQUAD    O    ARRAY OF D COEFFICIENTS                          00000120
C          KSPNC    I    NUMBER OF NUMERICAL SHORT PERIODIC               00000130
C                         COEFFICIENTS. MAX ALLOWABLE IS 10.               00000140
C                         (.LE.0)=PROGRAM WILL DETERMINE NUMBER            00000150
C                         (.GT.0)=PROGRAM WILL USE THE NUMBER GIVEN        00000160
C          KSPPRT   I    CONTROL SWITCH FOR THE PRINT OPTION.             00000170
C                         1=YES                                            00000180
C                         2=NO                                             00000190
C          FLOW     I    LOWER LIMIT OF INTEGRATION.                      00000200
C          FHIGH    I    UPPER LIMIT OF INTEGRATION.                      00000210
C                                                                           00000220
C          SEE SUBROUTINE SPEXEC FOR DEFINITION OF OTHER PARAMETERS         00000230
C                                                                           00000240
C     ANALYSIS                                                              00000250
C          ANDREW J. GREEN, CPT,  U.S. ARMY, CSDL                           00000260
C                                                                           00000270
C     PROGRAMMER                                                            00000280
C          ANDREW J. GREEN, CPT,  U.S. ARMY, CSDL                           00000290
C                                                                           00000300
C*****START PROGRAM*************************************************************00000310
C                                                                           00000320
C                                                                           00000330
      IMPLICIT REAL*8(A-H,O-Z)                                              00000340
      DIMENSION OSCELM(10),ETA(10),VPELMT(10)                               00000350
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00000360
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00000370
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00000380
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00000390
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00000400
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00000410
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00000420
     7                 KA          ,KB          ,KC          ,KGHA        ,00000430
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00000440
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00000450
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00000460
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00000470
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00000480
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00000490
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00000500
     6                 MMAXSP      ,IDRDR                                  00000510
C                                                                           00000520
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                               00000530
      EQUIVALENCE      (OSCELM(1)                ,RWORK(2653)              ),00000540
     2                 (ETA(1)                   ,RWORK(2673)              ),00000550
     3                 (VPELMT(1)                ,RWORK(2829)              ),00000560
     4                 (XLAMDA                   ,RWORK(2851)              ),00000570
```

SPNUM AJG1324.GTDS.UPDATE.FORT

```
     5                (XMEAN                    ,RWORK(2814)          ),00000580
     6                (T                        ,RWORK(3215)          ),00000590
     7                (TZERO                    ,RWORK(2850)          ) 00000600
      EQUIVALENCE     (NOPARM                   ,IWORK(3)             ),00000610
     2                (NPERT                    ,IWORK(9)             ) 00000620
C                                                                         00000630
      COMMON/SPCOEF/   CQUAD(10,10), DQUAD(10,10)                        00000640
C                                                                         00000650
C     INITIALIZE THE SHORT PERIODIC FUNCTIONS.                            00000660
C                                                                         00000670
      DO 100 I=1,NOPARM                                                   00000680
         ETA(I) = 0.D0                                                    00000690
  100 CONTINUE                                                            00000700
C                                                                         00000710
C     COMPUTE THE SHORT PERIODIC COEFFICIENTS.                            00000720
C                                                                         00000730
      CALL SPGENN(FLOW,FHIGH)                                             00000740
C                                                                         00000750
C     COMPUTE THE SHORT PERIODIC FUNCTIONS.                               00000760
C                                                                         00000770
      KTI = KSPNC(NPERT)                                                  00000780
      DO 110 J=1,KTI                                                      00000790
         TT=J                                                             00000800
         TLJ = TT*XLAMDA                                                  00000810
         DSJ = DSIN(TLJ)                                                  00000820
         DCJ = DCOS(TLJ)                                                  00000830
      DO 105 I=1,NOPARM                                                   00000840
         TEMPER = CQUAD(I,J)*DSJ-DQUAD(I,J)*DCJ                           00000850
         ETA(I) = ETA(I) + TEMPER                                         00000860
  105 CONTINUE                                                            00000870
  110 CONTINUE                                                            00000880
C                                                                         00000890
C     PRINT THE SHORT PERIODIC COEFFICIENTS.                              00000900
C                                                                         00000910
      IF(KSPPRT(NPERT).EQ.1) CALL SPPRT                                   00000920
      RETURN                                                              00000930
C                                                                         00000940
      END                                                                 00000950
```

******** END OF MEMBER SPNUM 95 RECORDS ********

SPPRT AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE SPPRT                                                          00000010
C                                                                               00000020
C     VERSION OF FEB 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6.                 00000030
C                                                                               00000040
C     PURPOSE                                                                   00000050
C      TO PRINT THE NUMERICAL SHORT PERIODIC FUNCTIONS AND COEFFICIENTS.        00000060
C                                                                               00000070
C     PARAMETERS                                                                00000080
C                                                                               00000090
C        /DCINT/                                                                00000100
C                                                                               00000110
C           TTO      I   OUTPUT REQUEST TIME IN SECONDS.                       00000120
C                                                                               00000130
C        SEE SUBROUTINES SPEXEC AND SPNUM FOR ADDITIONAL DEFINITIONS.           00000140
C                                                                               00000150
C     PROGRAMMER                                                                00000160
C        ANDREW J. GREEN, CPT,  U.S. ARMY, CSDL                                 00000170
C                                                                               00000180
C*****START PROGRAM*************************************************************00000190
C                                                                               00000200
C                                                                               00000210
      IMPLICIT REAL*8(A-H,O-Z)                                                  00000220
      DIMENSION   OSCELM(10),VPELMT(10),ETA(10)                                 00000230
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00000240
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00000250
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00000260
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00000270
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00000280
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00000290
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00000300
     7                 KA          ,KB          ,KC          ,KGHA        ,00000310
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00000320
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00000330
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00000340
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00000350
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00000360
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00000370
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00000380
     6                 MMAXSP      ,IDRDR                                   00000390
C                                                                             00000400
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                                 00000410
      EQUIVALENCE      (OSCELM(1)                ,RWORK(2653)               ),00000420
     2                 (ETA(1)                   ,RWORK(2673)               ),00000430
     3                 (VPELMT(1)                ,RWORK(2829)               ),00000440
     4                 (XLAMDA                   ,RWORK(2851)               ),00000450
     5                 (XMEAN                    ,RWORK(2814)               ),00000460
     6                 (T                        ,RWORK(3215)               ),00000470
     7                 (TZERO                    ,RWORK(2850)               ) 00000480
      EQUIVALENCE      (NOPARM                   ,IWORK(3)                  ),00000490
     2                 (NPERT                    ,IWORK(9)                  ) 00000500
C                                                                             00000510
      COMMON/SPCOEF/   CQUAD(10,10), DQUAD(10,10)                             00000520
C                                                                             00000530
      COMMON/DCINT/    DPDCI(432), INDCI(26)                                  00000540
      EQUIVALENCE      (TTO                      ,DPDCI(127)                ) 00000550
C                                                                             00000560
      KTI = KSPNC(NPERT)                                                      00000570
```

SPPRT AJG1324.GTDS.UPDATE.FORT

```
      TIMOUT=TTO/86400.0D0                                                  00000580
      WRITE(6,1000)                                                         00000590
      WRITE(6,1001)NPERT,NQUAD(NPERT),TIMOUT                                00000600
      WRITE(6,1002)                                                         00000610
      WRITE(6,1006)(VPELMT(I),I=1,5),XLAMDA                                 00000620
      WRITE(6,1003)                                                         00000630
      WRITE(6,1006)(ETA(I),I=1,NOPARM)                                      00000640
      WRITE(6,1004)                                                         00000650
      DO 10 K=1,KTI                                                         00000660
      WRITE(6,1006)(CQUAD(I,K),I=1,NOPARM)                                  00000670
   10 CONTINUE                                                              00000680
      WRITE(6,1005)                                                         00000690
      DO 20 K=1,KTI                                                         00000700
      WRITE(6,1006)(DQUAD(I,K),I=1,NOPARM)                                  00000710
   20 CONTINUE                                                              00000720
      RETURN                                                                00000730
 1000 FORMAT(' ++++++++++++++++++++++++++++++++++++++++++++++++SHORT PERI00000740
     *CDICS++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++00000750
     *++++++++')                                                            00000760
 1001 FORMAT(/,' NPERT = ',I4,6X,'QUADRATURE PARAMETER = ',I4,              00000770
     *6X,'TIME(DAYS) = ',1P1D20.10)                                         00000780
 1002 FORMAT(/,9X,'VPELMT(1)',13X,'VPELMT(2)',13X,'VPELMT(3)',13X,          00000790
     *'VPELMT(4)',13X,'VPELMT(5)',13X,'VPELMT(6)')                          00000800
 1003 FORMAT(/,10X,'ETA(1)',16X,'ETA(2)',16X,'ETA(3)',16X,'ETA(4)',         00000810
     *16X,'ETA(5)',16X,'ETA(6)')                                            00000820
 1004 FORMAT(/,8X,'CQUAD(1,K)',12X,'CQUAD(2,K)',12X,'CQUAD(3,K)',12X,       00000830
     *'CQUAD(4,K)',12X,'CQUAD(5,K)',12X,'CQUAD(6,K)')                       00000840
 1005 FORMAT(/,8X,'DQUAD(1,K)',12X,'DQUAD(2,K)',12X,'DQUAD(3,K)',12X,       00000850
     *'DQUAD(4,K)',12X,'DQUAD(5,K)',12X,'DQUAD(6,K)')                       00000860
 1006 FORMAT(1P6D22.12)                                                     00000870
      END                                                                   00000880
```

******** END OF MEMBER SPPRT 88 RECORDS ********

```
SPQUAD              AJG1324.GTDS.UPDATE.FORT

      SUBROUTINE SPQUAD(AA,BB,ISPQO,ISPNC)                              00000010
C                                                                       00000020
C     VERSION OF FEB 1979 IMPLEMENTED AT CSDL ON AMDAHL 470 V6.         00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C          EVALUATES THE COEFFICIENTS OF THE SHORT PERIODICS WITH A      00000060
C          GAUSSIAN NUMERICAL QUADRATURE FORMULA.                       00000070
C                                                                       00000080
C     PARAMETERS                                                        00000090
C          AA        I    LOWER LIMIT OF INTEGRATION.                  00000100
C          BB        I    UPPER LIMIT OF INTEGRATION.                  00000110
C          ISPQO     I    QUADRATURE ORDER PARAMETER.                  00000120
C          ISPNC     I    NUMBER OF COEFFICIENTS.                      00000130
C                                                                       00000140
C     /WORKER/                                                          00000150
C          DPARDT    I    PRECISION ELEMENT RATES COMPUTED IN *GQFUN*.  00000160
C          NOPARM    I    NUMBER OF ORBITAL ELEMENTS.                  00000170
C          VPELMT    I    ORBITAL ELEMENT VECTOR.                      00000180
C                                                                       00000190
C     /SPCOEF/                                                          00000200
C          CQUAD     O    VALUE OF THE C COEFFICIENTS QUADRATURE.      00000210
C          DQUAD     O    VALUE OF THE D COEFFICIENTS QUADRATURE.      00000220
C                                                                       00000230
C     EXTERNAL REFERENCES                                               00000240
C          GQFUN          COMPUTES PRECISION ORBITAL ELEMENT RATES.    00000250
C                                                                       00000260
C     REFERENCES                                                        00000270
C          SUBROUTINE QUADRT                                            00000280
C                                                                       00000290
C     ANALYSIS                                                          00000300
C          ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                        00000310
C                                                                       00000320
C     PROGRAMMER                                                        00000330
C          ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                        00000340
C                                                                       00000350
      IMPLICIT REAL*8(A-H,O-Z)                                          00000360
      COMMON/WORKER/ RWORK(3264)          ,IWORK(12)                    00000370
      DIMENSION      VPELMT(10),DPARDT(10)                              00000380
      DIMENSION      IXWBEG(7) ,IXWEND(7)                               00000390
      EQUIVALENCE    (XMEAN              ,RWORK(2814)       ),         00000400
     *               (VPELMT(1)          ,RWORK(2829)       ),         00000410
     *               (DPARDT(1)          ,RWORK(2840)       ),         00000420
     *               (NOPARM             ,IWORK(3)          )          00000430
      COMMON/CONST/  TWOPI,PI                                           00000440
      COMMON/SPCOEF/ CQUAD(10,10),DQUAD(10,10)                          00000450
C                                                                       00000460
      DIMENSION      XTAB(96)  ,WTAB(96)                                00000470
      DIMENSION      XTB1  ( 9)          ,XTB10 (10)        ,          00000480
     *               XTB20 (10)          ,XTB30 (10)        ,          00000490
     *               XTB40 (10)          ,XTB50 (10)        ,          00000500
     *               XTB60 (10)          ,XTB70 (10)        ,          00000510
     *               XTB80 (10)          ,XTB90 ( 7)                   00000520
      DIMENSION      WTB1  ( 9)          ,WTB10 (10)        ,          00000530
     *               WTB20 (10)          ,WTB30 (10)        ,          00000540
     *               WTB40 (10)          ,WTB50 (10)        ,          00000550
     *               WTB60 (10)          ,WTB70 (10)        ,          00000560
     *               WTB80 (10)          ,WTB90 ( 7)                   00000570
```

```
C
      EQUIVALENCE    (XTB1  (1)          ,XTAB(  1)        ),
     *               (XTB10 (1)          ,XTAB( 10)        ),
     *               (XTB20 (1)          ,XTAB( 20)        ),
     *               (XTB30 (1)          ,XTAB( 30)        ),
     *               (XTB40 (1)          ,XTAB( 40)        ),
     *               (XTB50 (1)          ,XTAB( 50)        ),
     *               (XTB60 (1)          ,XTAB( 60)        ),
     *               (XTB70 (1)          ,XTAB( 70)        ),
     *               (XTB80 (1)          ,XTAB( 80)        ),
     *               (XTB90 (1)          ,XTAB( 90)        )
      EQUIVALENCE    (WTB1  (1)          ,WTAB(  1)        ),
     *               (WTB10 (1)          ,WTAB( 10)        ),
     *               (WTB20 (1)          ,WTAB( 20)        ),
     *               (WTB30 (1)          ,WTAB( 30)        ),
     *               (WTB40 (1)          ,WTAB( 40)        ),
     *               (WTB50 (1)          ,WTAB( 50)        ),
     *               (WTB60 (1)          ,WTAB( 60)        ),
     *               (WTB70 (1)          ,WTAB( 70)        ),
     *               (WTB80 (1)          ,WTAB( 80)        ),
     *               (WTB90 (1)          ,WTAB( 90)        )
C
C                 ABSCISSAE FOR ORDER  12 IN SEQUENCE  12.
C
      DATA XTB  1   /.12523340851146890D+00          ,
     2               .36783149899818020D+00          ,
     3               .58731795428661740D+00          ,
     4               .76990267419430470D+00          ,
     5               .90411725637047490D+00          ,
     6               .98156063424671920D+00          ,
C
C                 ABSCISSAE FOR ORDER  16 IN SEQUENCE  16.
C
     7               .95012509837637430D-01          ,
     8               .28160355077925890D+00          ,
     9               .45801677765722740D+00          /
      DATA XTB 10   /.61787624440264370D+00          ,
     1               .75540440835500300D+00          ,
     2               .86563120238783170D+00          ,
     3               .94457502307323260D+00          ,
     4               .98940093499164990D+00          ,
C
C                 ABSCISSAE FOR ORDER  20 IN SEQUENCE  20.
C
     5               .76526521133497320D-01          ,
     6               .22778585114164510D+00          ,
     7               .37370608871541950D+00          ,
     8               .51086700195082710D+00          ,
     9               .63605368072651500D+00          /
      DATA XTB 20   /.74633190646015080D+00          ,
     1               .83911697182221880D+00          ,
     2               .91223442825132590D+00          ,
     3               .96397192727791380D+00          ,
     4               .99312859918509490D+00          ,
C
C                 ABSCISSAE FOR ORDER  24 IN SEQUENCE  24.
C
```

```
SPQUAD              AJG1324.GTDS.UPDATE.FORT

     5                .6405689286260562D-01          ,                    00001150
     6                .1911188674736163D+00          ,                    00001160
     7                .3150426796961634D+00          ,                    00001170
     8                .4337935076260451D+00          ,                    00001180
     9                .5454214713888395D+00          /                    00001190
      DATA XTB 30    /.6480936519369756D+00          ,                    00001200
     1                .7401241915785544D+00          ,                    00001210
     2                .8200019859739029D+00          ,                    00001220
     3                .8864155270044010D+00          ,                    00001230
     4                .9382745520027328D+00          ,                    00001240
     5                .9747285559713095D+00          ,                    00001250
     6                .9951872199970214D+00          ,                    00001260
C                                                                          00001270
C                   ABSCISSAE FOR ORDER  32 IN SEQUENCE  32.             00001280
C                                                                          00001290
     7                .4830766568773832D-01          ,                    00001300
     8                .1444719615827965D+00          ,                    00001310
     9                .2392873622521371D+00          /                    00001320
      DATA XTB 40    /.3318686022821276D+00          ,                    00001330
     1                .4213512761306353D+00          ,                    00001340
     2                .5068999089322294D+00          ,                    00001350
     3                .5877157572407623D+00          ,                    00001360
     4                .6630442669302152D+00          ,                    00001370
     5                .7321821187402897D+00          ,                    00001380
     6                .7944837959679424D+00          ,                    00001390
     7                .8493676137325700D+00          ,                    00001400
     8                .8963211557660521D+00          ,                    00001410
     9                .9349060759377397D+00          /                    00001420
      DATA XTB 50    /.9647622555875064D+00          ,                    00001430
     1                .9856115115452683D+00          ,                    00001440
     2                .9972638618494816D+00          ,                    00001450
C                                                                          00001460
C                   ABSCISSAE FOR ORDER  40 IN SEQUENCE  40.             00001470
C                                                                          00001480
     3                .3877241750605082D-01          ,                    00001490
     4                .1160840706752552D+00          ,                    00001500
     5                .1926975807013711D+00          ,                    00001510
     6                .2681521850072537D+00          ,                    00001520
     7                .3419940908257585D+00          ,                    00001530
     8                .4137792043716050D+00          ,                    00001540
     9                .4830759016861787D+00          /                    00001550
      DATA XTB 60    /.5494671250951282D+00          ,                    00001560
     1                .6125538896679802D+00          ,                    00001570
     2                .6719566846141795D+00          ,                    00001580
     3                .7273182551899271D+00          ,                    00001590
     4                .7783056514265194D+00          ,                    00001600
     5                .8246122308333117D+00          ,                    00001610
     6                .8659595032122595D+00          ,                    00001620
     7                .9020988069688743D+00          ,                    00001630
     8                .9328128082786765D+00          ,                    00001640
     9                .9579168192137917D+00          /                    00001650
      DATA XTB 70    /.9772599499837743D+00          ,                    00001660
     1                .9907262386994570D+00          ,                    00001670
     2                .9982377097105592D+00          ,                    00001680
C                                                                          00001690
C                   ABSCISSAE FOR ORDER  48 IN SEQUENCE  48.             00001700
C                                                                          00001710
```

```
     3             .3238017096286936D-01          ,
     4             .9700469920946270D-01          ,
     5             .1612223560688917D+00          ,
     6             .2247637903946891D+00          ,
     7             .2873624873554556D+00          ,
     8             .3487558862921607D+00          ,
     9             .4086864819907167D+00          /
      DATA XTB 80 /.4669029047509584D+00          ,
     1             .5231609747222330D+00          ,
     2             .5772247260839727D+00          ,
     3             .6288673967765136D+00          ,
     4             .6778723796326639D+00          ,
     5             .7240341309238146D+00          ,
     6             .7671590325157403D+00          ,
     7             .8070662040294426D+00          ,
     8             .8435882616243935D+00          ,
     9             .8765720202742479D+00          /
      DATA XTB 90 /.9058791367155697D+00          ,
     1             .9313866907065543D+00          ,
     2             .9529877031604309D+00          ,
     3             .9705915925462472D+00          ,
     4             .9841245837228269D+C0          ,
     5             .9935301722663508D+00          ,
     6             .9987710072524261D+00          /
C
C                 WEIGHTS FOR ORDER  12 IN SEQUENCE  12.
C
      DATA WTB  1 /.2491470458134028D+00          ,
     2             .2334925365383548D+00          ,
     3             .2031674267230659D+00          ,
     4             .1600783285433462D+00          ,
     5             .1069393259953184D+00          ,
     6             .4717533638651183D-01          ,
C
C                 WEIGHTS FOR ORDER  16 IN SEQUENCE  16.
C
     7             .1894506104550685D+00          ,
     8             .1826034150449236D+00          ,
     9             .1691565193950025D+00          /
      DATA WTB 10 /.1495959888165767D+00          ,
     1             .1246289712555339D+00          ,
     2             .9515851168249277D-01          ,
     3             .6225352393864789D-01          ,
     4             .2715245941175409D-01          ,
C
C                 WEIGHTS FOR ORDER  20 IN SEQUENCE  20.
C
     5             .1527533871307258D+00          ,
     6             .1491729864726037D+00          ,
     7             .1420961093183820D+00          ,
     8             .1316886384491766D+00          ,
     9             .1181945319615184D+00          /
      DATA WTB 20 /.1019301198172404D+00          ,
     1             .8327674157670474D-01          ,
     2             .6267204833410905D-01          ,
     3             .4060142980038694D-01          ,
     4             .1761400713915212D-01          ,
```

```
C                                                                       00002290
C                  WEIGHTS FOR ORDER  24 IN SEQUENCE  24.               00002300
C                                                                       00002310
     5               .1279381953467521D+00          ,                  00002320
     6               .1258374563468283D+00          ,                  00002330
     7               .1216704729278034D+00          ,                  00002340
     8               .1155056680537256D+00          ,                  00002350
     9               .1074442701159656D+00          /                  00002360
      DATA WTB 30   /.9761865210411388D-01          ,                  00002370
     1                .8619016153195327D-01          ,                  00002380
     2                .7334648141108030D-01          ,                  00002390
     3                .5929855491543678D-01          ,                  00002400
     4                .4427743881741981D-01          ,                  00002410
     5                .2853138862893366D-01          ,                  00002420
     6                .1234122979998720D-01          ,                  00002430
C                                                                       00002440
C                  WEIGHTS FOR ORDER  32 IN SEQUENCE  32.               00002450
C                                                                       00002460
     7               .9654008851472780D-01          ,                  00002470
     8               .9563872007927485D-01          ,                  00002480
     9               .9384439908080455D-01          /                  00002490
      DATA WTB 40   /.9117387869576388D-01          ,                  00002500
     1                .8765209300440381D-01          ,                  00002510
     2                .8331192422694675D-01          ,                  00002520
     3                .7819389578707030D-01          ,                  00002530
     4                .7234579410884850D-01          ,                  00002540
     5                .6582222277636184D-01          ,                  00002550
     6                .5868409347853555D-01          ,                  00002560
     7                .5099805926237618D-01          ,                  00002570
     8                .4283589802222669D-01          ,                  00002580
     9                .3427386291302143D-01          /                  00002590
      DATA WTB 50   /.2539206530926206D-01          ,                  00002600
     1                .1627439473090567D-01          ,                  00002610
     2                .7018610009470096D-02          ,                  00002620
C                                                                       00002630
C                  WEIGHTS FOR ORDER  40 IN SEQUENCE  40.               00002640
C                                                                       00002650
     3               .7750594797842481D-01          ,                  00002660
     4               .7703981816424796D-01          ,                  00002670
     5               .7611036190062624D-01          ,                  00002680
     6               .7472316905798826D-01          ,                  00002690
     7               .7288658239580405D-01          ,                  00002700
     8               .7061164739128677D-01          ,                  00002710
     9               .6791204581523390D-01          /                  00002720
      DATA WTB 60   /.6480401345660103D-01          ,                  00002730
     1                .6130624249292894D-01          ,                  00002740
     2                .5743976909939155D-01          ,                  00002750
     3                .5322784698393682D-01          ,                  00002760
     4                .4869580763507223D-01          ,                  00002770
     5                .4387090818567327D-01          ,                  00002780
     6                .3878216797447202D-01          ,                  00002790
     7                .3346019528254785D-01          ,                  00002800
     8                .2793700698002340D-01          ,                  00002810
     9                .2224584919416696D-01          /                  00002820
      DATA WTB 70   /.1642105838190789D-01          ,                  00002830
     1                .1049828453115281D-01          ,                  00002840
     2                .4521277098533191D-02          ,                  00002850
```

```
SPQUAD              AJG1324.GTDS.UPDATE.FORT

C                                                                       00002860
C                  WEIGHTS FOR ORDER  48 IN SEQUENCE  48.               00002870
C                                                                       00002880
     3               .6473769681268392D-01          ,                  00002890
     4               .6446616443595007D-01          ,                  00002900
     5               .6392423858464819D-01          ,                  00002910
     6               .6311419228625402D-01          ,                  00002920
     7               .6203942315969266D-01          ,                  00002930
     8               .6070443916589388D-01          ,                  00002940
     9               .5911483969839564D-01          /                  00002950
      DATA WTB 80   /.5727729210040321D-01          ,                  00002960
     1               .5519950369998416D-01          ,                  00002970
     2               .5289018948519367D-01          ,                  00002980
     3               .5035903555385447D-01          ,                  00002990
     4               .4761665849249047D-01          ,                  00003000
     5               .4467456085669428D-01          ,                  00003010
     6               .4154508294346475D-01          ,                  00003020
     7               .3824135106583071D-01          ,                  00003030
     8               .3477722256477044D-01          ,                  00003040
     9               .3116722783279809D-01          /                  00003050
      DATA WTB 90   /.2742650970335695D-01          ,                  00003060
     1               .2357076083932438D-01          ,                  00003070
     2               .1961616045735553D-01          ,                  00003080
     3               .1557931572294365D-01          ,                  00003090
     4               .1147723457923454D-01          ,                  00003100
     5               .7327553901276262D-02          ,                  00003110
     6               .3153346052305839D-02          /                  00003120
C                                                                       00003130
      DATA IXWBEG   /1         ,7        ,15       ,25       ,37      ,00003140
     *               53        ,73       /                            00003150
      DATA IXWEND   /6         ,14       ,24       ,36       ,52      ,00003160
     *               72        ,96       /                            00003170
C                                                                       00003180
C                  INITIALIZATION                                       00003190
C                                                                       00003200
      XX=(BB+AA)/2.D0                                                   00003210
      DD=(BB-AA)/2.D0                                                   00003220
      DO 2 K=1,10                                                       00003230
      DO 1 I=1,NOPARM                                                   00003240
         CQUAD(I,K)=0.D0                                                00003250
         DQUAD(I,K)=0.D0                                                00003260
    1 CONTINUE                                                          00003270
    2 CONTINUE                                                          00003280
C                                                                       00003290
      IBEG=IXWBEG(ISPQO)                                                00003300
      IEND=IXWEND(ISPQO)                                                00003310
      INDEX=IEND                                                        00003320
C                                                                       00003330
C ACCUMULATE THE QUADRATURE SUM OVER THE FIRST HALF OF THE INTERVAL.    00003340
C                                                                       00003350
    3 CONTINUE                                                          00003360
C                                                                       00003370
C        COMPUTE THE ELEMENT RATES.                                     00003380
C                                                                       00003390
         EL=XX-DD*XTAB(INDEX)                                           00003400
         CALL GQFUN(EL)                                                 00003410
         WGHT=WTAB(INDEX)                                               00003420
```

```
C                                                                       00003430
C       COMPUTE THE MEAN MEAN LONGITUDE.                                00003440
C                                                                       00003450
        XML=EL-VPELMT(3)*DSIN(EL)+VPELMT(2)*DCOS(EL)                    00003460
C                                                                       00003470
C       COMPUTE THE SHORT PERIODIC COEFFICIENTS OVER THE FIRST HALF.    00003480
C                                                                       00003490
        DO 5 K=1,ISPNC                                                  00003500
          TT=K                                                          00003510
          DCK = DCOS(TT*XML)                                            00003520
          DSK = DSIN(TT*XML)                                            00003530
        DO 4 I=1,NOPARM                                                 00003540
          WGDP = WGHT*DPARDT(I)                                         00003550
          CQUAD(I,K)=CQUAD(I,K)+WGDP*DCK                                00003560
          DQUAD(I,K)=DQUAD(I,K)+WGDP*DSK                                00003570
    4   CONTINUE                                                        00003580
    5   CONTINUE                                                        00003590
        INDEX=INDEX-1                                                   00003600
      IF (INDEX.GE.IBEG) GO TO 3                                        00003610
C                                                                       00003620
      IF (XTAB(IBEG).NE.0.D0) GO TO 6                                   00003630
      INDEX=IBEG                                                        00003640
C                                                                       00003650
C ACCUMULATE THE QUADRATURE SUM OVER THE SECOND HALF OF THE INTERVAL.   00003660
C                                                                       00003670
C                                                                       00003680
    6 CONTINUE                                                          00003690
C                                                                       00003700
C       COMPUTE THE ELEMENT RATES.                                      00003710
C                                                                       00003720
        INDEX=INDEX+1                                                   00003730
        EL = XX+DD*XTAB(INDEX)                                          00003740
        CALL GQFUN(EL)                                                  00003750
        WGHT=WTAB(INDEX)                                                00003760
C                                                                       00003770
C       COMPUTE THE MEAN MEAN LONGITUDE.                                00003780
C                                                                       00003790
        XML=EL-VPELMT(3)*DSIN(EL)+VPELMT(2)*DCOS(EL)                    00003800
C                                                                       00003810
C       COMPUTE THE SHORT PERIODIC COEFFICIENTS OVER THE SECOND HALF.   00003820
C                                                                       00003830
        DO 8 K=1,ISPNC                                                  00003840
          TT=K                                                          00003850
          DCK=DCOS(TT*XML)                                              00003860
          DSK=DSIN(TT*XML)                                              00003870
        DO 7 I=1,NOPARM                                                 00003880
          WGDP=WGHT*DPARDT(I)                                           00003890
          CQUAD(I,K)=CQUAD(I,K)+WGDP*DCK                                00003900
          DQUAD(I,K)=DQUAD(I,K)+WGDP*DSK                                00003910
    7   CONTINUE                                                        00003920
    8   CONTINUE                                                        00003930
      IF (INDEX.LT.IEND) GO TO 6                                        00003940
C                                                                       00003950
C RETURN THE VALUE OF THE COEFFICIENTS AND DIVIDE BY THE MEAN           00003960
C MOTION AND THE COEFFICIENT NUMBER.                                    00003970
C                                                                       00003980
      DDPI = DD/(PI*XMEAN)                                              00003990
```

SPQUAD AJG1324.GTDS.UPDATE.FORT

```
      DO 10 K=1,ISPNC                                                   00004000
         TT=K                                                           00004010
      DO 9 I=1,NOFARM                                                   00004020
         CQUAD(I,K)=(DDPI*CQUAD(I,K))/TT                                00004030
         DQUAD(I,K)=(DDPI*DQUAD(I,K))/TT                                00004040
    9 CONTINUE                                                          00004050
   10 CONTINUE                                                          00004060
C                                                                       00004070
C        ACCOUNT FOR THE COUPLING BETWEEN THE FIRST AND SIXTH MEAN      00004080
C        ELEMENTS IN THE SHORT PERIODICS.                               00004090
C                                                                       00004100
      A15=1.5D0/VPELMT(1)                                               00004110
      DO 11 K=1,ISPNC                                                   00004120
         TT=K                                                           00004130
         AK15=A15/TT                                                    00004140
         CQUAD(6,K)=CQUAD(6,K)+AK15*DQUAD(1,K)                          00004150
         DQUAD(6,K)=DQUAD(6,K)-AK15*CQUAD(1,K)                          00004160
   11 CONTINUE                                                          00004170
C                                                                       00004180
      RETURN                                                            00004190
C                                                                       00004200
      END                                                               00004210
```

******** END OF MEMBER SPQUAD 421 RECORDS ********

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VARDIF              AJG1324.GTDS.UPDATE.FORT

      SUBROUTINE VARDIF                                                 00000010
C                                                                       00000020
C     VERSION OF 20 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6.  00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C       TO COMPUTE THE A AND/OR D MATRICES WITH A FINITE-DIFFERENCING   00000060
C       TECHNIQUE.                                                      00000070
C                                                                       00000080
C     PARAMETERS                                                        00000090
C                                                                       00000100
C       /FRC/                                                           00000110
C                                                                       00000120
C       CDRAG     I/O   DRAG COEFFICIENT.                             00000130
C                                                                       00000140
C       /NEWPAR/                                                        00000150
C                                                                       00000160
C       APAR      I/O   ARRAY OF DENSITY PARAMETERS.                  00000170
C       XS        I     ARRAY OF STEPSIZE USED TO COMPUTE PARTIALS.   00000180
C                                                                       00000190
C       /ESTFLG/                                                        00000200
C                                                                       00000210
C       IDIFF     I     CONTROL SWITCH FOR FINITE-DIFFERENCING.       00000220
C                       0=APPROACH NOT USED                           00000230
C                       1=A MATRIX ONLY                               00000240
C                       2=D MATRIX ONLY                               00000250
C                       3=BOTH A AND D MATRICES                       00000260
C       KPAR      I     PARAMETER ESTIMATION OPTION.                  00000270
C                       0=NONE                                        00000280
C                       1=SOLVE FOR CDRAG ONLY                        00000290
C                       2=SOLVE FOR THE 5 PARAMETERS IN THE ADAPTIVE  00000300
C                         HARRIS-PRIESTER ATMOSPHERE.                 00000310
C                       3=SOLVE FOR THE 3 PARAMETERS IN THE ADAPTIVE  00000320
C                         HARRIS-PRIESTER ATMOSPHERE.                 00000330
C       KSTEP     I     ARRAY OF CONTROL SWITCHES FOR THE PERTURBING  00000340
C                       STEPSIZE.                                     00000350
C                       1=STEPSIZE IS GIVEN IN XS                     00000360
C                       2=STEPSIZE IS XS*PARAMETER                    00000370
C                                                                       00000380
C       /VARMAT/                                                        00000390
C                                                                       00000400
C       AMAT      O     THE A MATRIX.                                 00000410
C       DMAT      O     THE D MATRIX.                                 00000420
C                                                                       00000430
C       /WORKER/                                                        00000440
C                                                                       00000450
C       VPELMT    I     ARRAY OF MEAN ELEMENTS.                       00000460
C       NOPARM    I     NUMBER OF PARAMETERS.                         00000470
C       NPERT     O     PERTURBATION INDICATOR.                       00000480
C                       1=CONTINUOUS TIME-DEPENDENT                   00000490
C                       2=CONTINUOUS TIME-INDEPENDENT                 00000500
C                       3=DRAG                                        00000510
C                       4=SOLAR RADIATION PRESSURE                    00000520
C       DPARRY    I     ARRAY OF ELEMENT DERIVATIVES.                 00000530
C                                                                       00000540
C     REFERENCES                                                        00000550
C       "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW-ALTITUDE 00000560
C       SATELLITES." BY A.J. GREEN                                      00000570
```

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VARDIF              AJG1324.GTDS.UPDATE.FORT

C                                                                       00000580
C     ANALYSIS                                                          00000590
C       ANDREW J. GREEN, CPT, U.S. ARMY                                 00000600
C                                                                       00000610
C     PROGRAMMER                                                        00000620
C       ANDREW J. GREEN, CPT, U.S. ARMY                                 00000630
C                                                                       00000640
C                                                                       00000650
C                                                                       00000660
      IMPLICIT REAL*8(A-H,O-Z)                                          00000670
      DIMENSION TEMSAV(10),TEMRAT(10),DPARRY(15,10),VPELMT(10),         00000680
     2          SAE(3),TSAE(3)                                          00000690
C                                                                       00000700
      COMMON/THRUST/  DPTHR(150),  INTHR(50)                            00000710
      EQUIVALENCE    (DPTHR(61)                ,SAE(1)                 )00000720
C                                                                       00000730
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR    ,00000740
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND      ,00000750
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)  ,00000760
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS    ,00000770
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ    ,00000780
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA        ,00000790
     6                 JB          ,JC          ,JGHA        ,KPOS      ,00000800
     7                 KA          ,KB          ,KC          ,KGHA      ,00000810
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN     ,00000820
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD     ,00000830
     1                 JTMDEP      ,JTMIND      ,JDRAG       ,JSOLAR    ,00000840
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS    ,00000850
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA    ,00000860
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC      ,00000870
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP    ,00000880
     6                 MMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR     00000890
C                                                                       00000900
      COMMON/FRC    /   RFRC(1300)  ,IFRC(50)                           00000910
      EQUIVALENCE       (CDRAG                  ,RFRC(154)            ),00000920
     1                  (NMAX                   ,IFRC(3)              ),00000930
     2                  (MMAX                   ,IFRC(4)              ) 00000940
C                                                                       00000950
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                           00000960
      EQUIVALENCE      (DPARRY(1,1)             ,RWORK(7)             ),00000970
     2                 (VPELMT(1)               ,RWORK(2829)          ) 00000980
      EQUIVALENCE      (K                       ,IWORK(2)             ),00000990
     2                 (NOPARM                  ,IWORK(3)             ),00001000
     3                 (NEQ                     ,IWORK(5)             ),00001010
     4                 (NPERT                   ,IWORK(9)             ) 00001020
C                                                                       00001030
      COMMON/ESTFLG/ IANAL    ,IDIFF    ,IQDRT    ,ICBVAR   ,J22VAR   , 00001040
     1               ITBVAR   ,IDRVAR   ,ISRVAR   ,NVAR     ,LEVAR    , 00001050
     2               KDRFLG   ,KPAR     ,KSTEP(20),KVRFLG   ,KCPSPP   , 00001060
     3               KCBSPP   ,KTBSPP   ,KDRSPP   ,KSRSPP   ,NVARSP   , 00001070
     4               IVSPNC(4),KATMOS   ,KPRTB1   ,KPRTB2   ,KPRTB3   , 00001080
     5               KPRTB4                                             00001090
C                                                                       00001100
      COMMON/VARMAT/ AMAT(6,6),DMAT(6,14),B1MAT(6,6),B4MAT(6,14)        00001110
C                                                                       00001120
      COMMON/NEWPAR/ XS(20)   ,APAR(5)  ,APARVR(5),CDRVAR               00001130
C                                                                       00001140
```

VARDIF AJG1324.GTDS.UPDATE.FORT

```
C     SAVE THE DERIVATIVE ARRAY AND INITIALIZE THE A AND D MATRICES.     00001150
C                                                                        00001160
      DO 10 I=1,NOPARM                                                   00001170
      TEMSAV(I)=DPARRY(K,I)                                              00001180
   10 CONTINUE                                                           00001190
      DO 40 I=1,6                                                        00001200
      DO 20 J=1,6                                                        00001210
      AMAT(I,J)=0.D0                                                     00001220
   20 CONTINUE                                                           00001230
      DO 30 L=1,14                                                       00001240
      DMAT(I,L)=0.D0                                                     00001250
   30 CONTINUE                                                           00001260
   40 CONTINUE                                                           00001270
C                                                                        00001280
C     COMPUTE THE A MATRIX.                                              00001290
C                                                                        00001300
      IF(IDIFF.EQ.2) GO TO 300                                           00001310
      DO 100 J=1,5                                                       00001320
      TEMP=VPELMT(J)                                                     00001330
      STEP=XS(J)*DABS(VPELMT(J))                                         00001340
      IF((DABS(VPELMT(J)).LT.1.D-10).OR.(KSTEP(J).EQ.1)) STEP=XS(J)      00001350
      VPELMT(J)=VPELMT(J)+STEP                                           00001360
      CALL AVRAGE                                                        00001370
      DO 50 I=1,6                                                        00001380
      TEMRAT(I)=DPARRY(K,I)                                              00001390
   50 CONTINUE                                                           00001400
      VPELMT(J)=VPELMT(J)-2.D0*STEP                                      00001410
      CALL AVRAGE                                                        00001420
      DO 75 I=1,6                                                        00001430
      AMAT(I,J)=(TEMRAT(I)-DPARRY(K,I))/(2.D0*STEP)                      00001440
   75 CONTINUE                                                           00001450
      VPELMT(J)=TEMP                                                     00001460
  100 CONTINUE                                                           00001470
C                                                                        00001480
C     COMPUTE THE D MATRIX.                                              00001490
C                                                                        00001500
  300 IF((IDIFF.EQ.1).OR.(KPAR.EQ.0)) GO TO 900                          00001510
C                                                                        00001520
C     CHECK FOR THE PARAMETER ESTIMATION OPTION. FOR OPTIONS 1,2, AND 3  00001530
C     ONLY THE DRAG FORCE MODEL NEEDS TO BE CONSIDERED.                  00001540
C                                                                        00001550
      IF(KPAR.GT.3) GO TO 500                                            00001560
C                                                                        00001570
      KNMAX=NMAX                                                         00001580
      KMMAX=MMAX                                                         00001590
      KRES=IRESON                                                        00001600
      KTESS=ITESS                                                        00001610
      KNCRES=NCSRES                                                      00001620
      KTHIRD=ITHIRD                                                      00001630
      KTWOSQ=JTWOSQ                                                      00001640
      KTMDEP=ITMDEP                                                      00001650
      KTMIND=ITMIND                                                      00001660
      KSOLAR=ISOLAR                                                      00001670
C                                                                        00001680
      NMAX=0                                                             00001690
      MMAX=0                                                             00001700
      IRESON=4                                                           00001710
```

```
      ITESS=0                                                          00001720
      NCSRES=0                                                         00001730
      ITHIRD=0                                                         00001740
      JTWOSQ=2                                                         00001750
      ITMDEP=2                                                         00001760
      ITMIND=2                                                         00001770
      ISOLAR=2                                                         00001780
C                                                                      00001790
      KKPAR=KPAR                                                       00001800
      GO TO (310,400,401), KKPAR                                       00001810
C                                                                      00001820
C     DETERMINE THE D MATRIX WHEN CDRAG IS A SOLVE-FOR PARAMETER.      00001830
C                                                                      00001840
  310 TEMP=CDRAG                                                       00001850
      DO 320 I=1,3                                                     00001860
      TSAE(I)=SAE(I)                                                   00001870
  320 CONTINUE                                                         00001880
      STEP=XS(7)*CDRAG                                                 00001890
      IF((CDRAG.LT.1.D-10).OR.(KSTEP(7).EQ.1)) STEP=XS(7)              00001900
      CDRAG=CDRAG+STEP                                                 00001910
      RHO=CDRAG/TEMP                                                   00001920
      DO 330 I=1,3                                                     00001930
      SAE(I)=TSAE(I)*RHO                                               00001940
  330 CONTINUE                                                         00001950
      CALL AVRAGE                                                      00001960
      DO 350 I=1,6                                                     00001970
      TEMRAT(I)=DPARRY(K,I)                                            00001980
  350 CONTINUE                                                         00001990
      CDRAG=CDRAG-2.D0*STEP                                            00002000
      RHO=CDRAG/TEMP                                                   00002010
      DO 360 I=1,3                                                     00002020
      SAE(I)=TSAE(I)*RHO                                               00002030
  360 CONTINUE                                                         00002040
      CALL AVRAGE                                                      00002050
      DO 375 I=1,6                                                     00002060
      DMAT(I,1)=(TEMRAT(I)-DPARRY(K,I))/(2.D0*STEP)                    00002070
  375 CONTINUE                                                         00002080
      CDRAG=TEMP                                                       00002090
      DO 380 I=1,3                                                     00002100
      SAE(I)=TSAE(I)                                                   00002110
  380 CONTINUE                                                         00002120
      GO TO 490                                                        00002130
C                                                                      00002140
C     DETERMINE THE D MATRIX WHEN PARAMETERS WITHIN THE ADAPTIVE       00002150
C     MODIFIED HARRIS-PRIESTER ATMOSPHERE ARE THE SOLVE-FOR-PARAMETERS. 00002160
C                                                                      00002170
  400 N=5                                                              00002180
      GO TO 403                                                        00002190
  401 N=3                                                              00002200
  403 DO 430 J=1,N                                                     00002210
      L=6+J                                                            00002220
      TEMP=APAR(J)                                                     00002230
      STEP=XS(L)*DABS(APAR(J )                                         00002240
      IF((DABS(APAR(J)).LT.1.D-10).OR.(KSTEP(L).EQ.1)) STEP=XS(L)      00002250
      APAR(J)=APAR(J)+STEP                                             00002260
      CALL AVRAGE                                                      00002270
      DO 410 I=1,6                                                     00002280
```

VARDIF AJG1324.GTDS.UPDATE.FORT

```
      TEMRAT(I)=DPARRY(K,I)                                          00002290
  410 CONTINUE                                                       00002300
      APAR(J)=APAR(J)-2.D0*STEP                                      00002310
      CALL AVRAGE                                                    00002320
      DO 420 I=1,6                                                   00002330
      DMAT(I,J)=(TEMRAT(I)-DPARRY(K,I))/(2.D0*STEP)                  00002340
  420 CONTINUE                                                       00002350
      APAR(J)=TEMP                                                   00002360
  430 CONTINUE                                                       00002370
C                                                                    00002380
C     RESET THE SWITCHES.                                            00002390
C                                                                    00002400
  490 NMAX=KNMAX                                                     00002410
      MMAX=KMMAX                                                     00002420
      ITESS=KTESS                                                    00002430
      NCSRES=KNCRES                                                  00002440
      ITHIRD=KTHIRD                                                  00002450
      JTWOSQ=KTWOSQ                                                  00002460
      ITMDEP=KTMDEP                                                  00002470
      ITMIND=KTMIND                                                  00002480
      ISOLAR=KSOLAR                                                  00002490
      GO TO 900                                                      00002500
C                                                                    00002510
C     ADDITIONAL ESTIMATION OPTIONS WOULD BE INCLUDED HERE.          00002520
C                                                                    00002530
  500 CONTINUE                                                       00002540
C                                                                    00002550
C     RESET THE DERIVATIVE ARRAY.                                    00002560
C                                                                    00002570
  900 DO 910 I=1,6                                                   00002580
      DPARRY(K,I)=TEMSAV(I)                                          00002590
  910 CONTINUE                                                       00002600
C                                                                    00002610
      RETURN                                                         00002620
      END                                                            00002630
```

******** END OF MEMBER VARDIF 263 RECORDS ********

VARSP AJG1324.GTDS.UPDATE.FORT

```
      SUBROUTINE VARSP                                                      00000010
C                                                                           00000020
C     VERSION OF 13 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6.      00000030
C                                                                           00000040
C     PURPOSE                                                               00000050
C                                                                           00000060
C        EXECUTIVE PROGRAM FOR THE DETERMINATION OF THE B1 AND B4           00000070
C        MATRICES.                                                          00000080
C                                                                           00000090
C     PARAMETERS                                                            00000100
C                                                                           00000110
C     /ESTFLG/                                                              00000120
C                                                                           00000130
C     KVTFLG     I    CONTROL SWITCH FOR THE SHORT PERIODIC PARTIALS        00000140
C                     0=NEGLECT THE B1 AND B4 MATRICES                      00000150
C                     1=B1 MATRIX ONLY,DONE ANALYTICALLY                    00000160
C                     2=B1 MATRIX ONLY,DONE BY FINITE-DIFFERENCING          00000170
C                     3=B4 MATRIX ONLY,DONE BY FINITE-DIFFERENCING          00000180
C                     4=BOTH B1 AND B4 MATRICES,DONE BY FINITE-             00000190
C                       DIFFERENCING                                        00000200
C                                                                           00000210
C     REFERENCE                                                             00000220
C                                                                           00000230
C        "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW-             00000240
C        ALTITUDE SATELLITES." BY A.J. GREEN                                00000250
C                                                                           00000260
C     ANALYSIS                                                              00000270
C        ANDREW J. GREEN, CPT,U.S. ARMY, CSDL                               00000280
C                                                                           00000290
C     PROGRAMMER                                                            00000300
C       ANDREW J. GREEN, CPT, U.S. ARMY, CSDL                               00000310
C                                                                           00000320
      IMPLICIT REAL*8(A-H,O-Z)                                              00000330
C                                                                           00000340
      COMMON/ESTFLG/ IANAL    ,IDIFF    ,IQDRT    ,ICBVAR   ,J22VAR   ,   00000350
     1               ITBVAR   ,IDRVAR   ,ISRVAR   ,NVAR     ,LEVAR    ,   00000360
     2               KDRFLG   ,KPAR     ,KSTEP(20),KVRFLG   ,KCPSPP   ,   00000370
     3               KCBSPP   ,KTBSPP   ,KDRSPP   ,KSRSPP   ,NVARSP   ,   00000380
     4               IVSFNC(4),KATMOS   ,KPRTB1   ,KPRTB2   ,KPRTB3   ,   00000390
     5               KPRTB4                                               00000400
C                                                                           00000410
      COMMON/VARMAT/ AMAT(6,6),DMAT(6,14),B1MAT(6,6),B4MAT(6,14)            00000420
C                                                                           00000430
C                                                                           00000440
C     CHECK FOR TYPE OF SHORT PERIODIC PARTIALS.                            00000450
C                                                                           00000460
      IF(KVRFLG.EQ.0) GO TO 300                                             00000470
      IF(KVRFLG.NE.1) GO TO 200                                             00000480
C                                                                           00000490
C     ANALYTICAL SHORT PERIODIC PARTIALS.                                   00000500
C                                                                           00000510
  100 CALL VRSPAN                                                           00000520
      GO TO 400                                                             00000530
C                                                                           00000540
C     SHORT PERIODIC PARTIALS VIA FINITE-DIFFERENCEING.                     00000550
C                                                                           00000560
  200 CALL VRSPFD                                                           00000570
```

VARSP AJG1324.GTDS.UPDATE.FORT

```
      GO TO 400                                                          00000580
C                                                                        00000590
C     INITIALIZE THE B1 AND B4 MATRICES AND RETURN IF KVRFLG=0.          00000600
C                                                                        00000610
  300 DO 330 I=1,6                                                       00000620
      DO 310 J=1,6                                                       00000630
      B1MAT(I,J)=0.D0                                                    00000640
  310 CONTINUE                                                           00000650
      DO 320 L=1,14                                                      00000660
      B4MAT(I,L)=0.D0                                                    00000670
  320 CONTINUE                                                           00000680
  330 CONTINUE                                                           00000690
C                                                                        00000700
  400 RETURN                                                             00000710
      END                                                                00000720
```

******** END OF MEMBER VARSP 72 RECORDS ********

```
VRSPFD              AJG1324.GTDS.UPDATE.FORT

      SUBROUTINE VRSPFD                                                 00000010
C                                                                       00000020
C     VERSION OF 13 JULY 1979 IMPLEMENTED AT CSDL ON AN AMDAHL 470 V6.  00000030
C                                                                       00000040
C     PURPOSE                                                           00000050
C                                                                       00000060
C        TO DETERMINE THE B1 AND/OR B4 MATRICES BY A FINITE-DIFFERENCING00000070
C        TECHNIQUE.                                                     00000080
C                                                                       00000090
C     PARAMETERS                                                        00000100
C                                                                       00000110
C        /ANAVIN/                                                       00000120
C                                                                       00000130
C        ITHIRD     O    CONTROL SWITCH FOR 3RD BODIES.                 00000140
C        KSPNC      O    ARRAY OF THE NUMBER OF SHORT PERIODIC COEF-    00000150
C                        FICIENTS FOR EACH PERTURBATIONS.               00000160
C                                                                       00000170
C        /ESTFLG/                                                       00000180
C                                                                       00000190
C        KPAR       I    PARAMETER ESTIMATION OPTION.                   00000200
C                        0=NONE                                         00000210
C                        1=SOLVE FOR CDRAG                              00000220
C                        2=SOLVE FOR THE 5 PARAMETERS,A1 - A5, IN THE   00000230
C                          ADAPTIVE MODIFIED HARRIS-PRIESTER ATMOSPHERE.00000240
C                        3=SOLVE FOR THE 3 PARAMETERS,A1 - A3, IN THE   00000250
C                          ADAPTIVE MODIFIED HARRIS-PRIESTER ATMOSPHERE.00000260
C        KSTEP      I    ARRAY OF CONTROL SWITCHES FOR THE PERTURBING   00000270
C                        STEPSIZE.                                      00000280
C                        1=STEPSIZE IS GIVEN IN XS                      00000290
C                        2=STEPSIZE IS XS*PARAMETER                     00000300
C        KVRFLG     I    CONTROL SWITCH FORT HE SHORT PERIODIC PARTIALS.00000310
C                        0=NONE                                         00000320
C                        1=B1 MATRIX - ANALYTICALLY                     00000330
C                        2=B1 MATRIX ONLY - FINITE DIFFERENCING         00000340
C                        3=B4 MATRIX ONLY - FINITE DIFFERENCING         00000350
C                        4=BOTH B1 AND B4 MATRICES - FINITE DIFFERENCING00000360
C                                                                       00000370
C        /NEWPAR/                                                       00000380
C                                                                       00000390
C        APAR       I/O  ARRAY OF DENSITY PARAMETERS.                  00000400
C        XS         I    ARRAY OF STEPSIZE USED TO COMPUTE PARTIALS.   00000410
C                                                                       00000420
C        /SPCOEF/                                                       00000430
C                                                                       00000440
C        CQUAD      I    ARRAY OF THE C SHORT PERIODIC COEFFICIENT.    00000450
C        DQUAD      I    ARRAY OF THE D SHORT PERIODIC COEFFICIENT.    00000460
C                                                                       00000470
C        /WORKER/                                                       00000480
C                                                                       00000490
C        VPELMT     I/O  ARRAY OF THE MEAN ELEMENTS.                   00000500
C        XLAMDA     I    MEAN FAST VARIABLE.                           00000510
C        NOPARM     I    NUMBER OF PARAMETERS.                         00000520
C        NEQ        I    NUMBER OF VARIATIONAL EQUATIONS.              00000530
C                                                                       00000540
C     REFERENCES                                                        00000550
C                                                                       00000560
C        "ORBIT DETERMINATION AND PREDICTION PROCESSES FOR LOW-         00000570
```

```
C       ALTITUDE SATELLITES." BY A.J. GREEN                                    00000580
C                                                                              00000590
C     ANALYSIS                                                                 00000600
C        ANDREW J. GREEN, CPT, U.S. ARMY                                       00000610
C                                                                              00000620
C     PROGRAMMER                                                               00000630
C        ANDREW J. GREEN, CPT, U.S. ARMY                                       00000640
C                                                                              00000650
C                                                                              00000660
      IMPLICIT REAL*8(A-H,O-Z)                                                 00000670
      DIMENSION XMMAT(6,5),XNMAT(6,5),XCQUAD(6,10),XDQUAD(6,10),               00000680
     2          XKMAT(6,14),XLMAT(6,14),VPELMT(10),SAE(3),TSAE(3)              00000690
      DIMENSION INC(4)                                                         00000700
C                                                                              00000710
      COMMON/THRUST/  DPTHR(150),  INTHR(50)                                   00000720
      EQUIVALENCE    (DPTHR(61)                ,SAE(1)                       )00000730
C                                                                              00000740
      COMMON/ANAVIN/   ISTESS      ,ITESSE      ,ITRESE      ,ISTHIR      ,00000750
     1                 ITHRAR      ,ITHRE       ,ITESS       ,NEND        ,00000760
     2                 MEND        ,LEND        ,ITRES       ,NRES(10)    ,00000770
     3                 MRES(10)    ,JRES(10)    ,NCSRES      ,LENDRS      ,00000780
     4                 ITHIRD      ,NENDTH(8)   ,MENDTH(8)   ,JTWOSQ      ,00000790
     5                 IQUAD(4)    ,NQUAD(4)    ,JPOS        ,JA          ,00000800
     6                 JB          ,JC          ,JGHA        ,KPOS        ,00000810
     7                 KA          ,KB          ,KC          ,KGHA        ,00000820
     8                 ICORR       ,IPOS        ,IMOON       ,IMEAN       ,00000830
     9                 IGHA        ,ISTRES      ,ITERM       ,ISHAD       ,00000840
     1                 JTNDEP      ,JTMIND      ,JDRAG       ,JSOLAR      ,00000850
     2                 J2SQSP      ,JSPCGB      ,JSPTHR      ,JSPPOS      ,00000860
     3                 JSPA        ,JSPB        ,JSPC        ,JSPGHA      ,00000870
     4                 KSPPOS      ,KSPA        ,KSPB        ,KSPC        ,00000880
     5                 KSPGHA      ,KSPNC(4)    ,KSPPRT(4)   ,NMAXSP      ,00000890
     6                 MMAXSP      ,IDRDR       ,NCHOJ2      ,NCHODR       00000900
C                                                                              00000910
      COMMON/FRC   /   RFRC(1300)  ,IFRC(50)                                   00000920
      EQUIVALENCE      (CDRAG                   ,RFRC(154)                  ) 00000930
C                                                                              00000940
      COMMON/WORKER/   RWORK(3264) ,IWORK(13)                                  00000950
      EQUIVALENCE      (VPELMT(1)               ,RWORK(2829)                ),00000960
     2                 (XLAMDA                  ,RWORK(2851)                ) 00000970
      EQUIVALENCE      (NOPARM                  ,IWORK(3)                   ),00000980
     2                 (NEQ                     ,IWORK(5)                   ),00000990
     3                 (NPERT                   ,IWORK(9)                   ) 00001000
C                                                                              00001010
      COMMON/ESTFLG/ IANAL     ,IDIFF     ,IQDRT     ,ICBVAR    ,J22VAR    , 00001020
     1               ITBVAR    ,IDRVAR    ,ISRVAR    ,NVAR      ,LEVAR     , 00001030
     2               KDRFLG    ,KPAR      ,KSTEP(20),KVRFLG     ,KCRSPP    , 00001040
     3               KCBSPP    ,KTBSPP    ,KDRSPP    ,KSRSPP    ,NVARSP    , 00001050
     4               IVSPNC(4),KATMOS     ,KPRTB1    ,KPRTB2    ,KPRTB3    , 00001060
     5               KPRTB4                                                    00001070
C                                                                              00001080
      COMMON/VARMAT/ AMAT(6,6),DMAT(6,14),B1MAT(6,6),B4MAT(6,14)               00001090
C                                                                              00001100
      COMMON/NEWPAR/ XS(20)   ,APAR(5)   ,APARVR(5),CDRVAR                    00001110
C                                                                              00001120
      COMMON/SPCOEF/ CQUAD(10,10),DQUAD(10,10)                                 00001130
C                                                                              00001140
```

```
C     SET THE NUMBER OF SHORT PERIODIC COEFFICIENTS AND INITIALIZE.      00001150
C                                                                        00001160
      DO 1 I=1,4                                                         00001170
      INC(I)=KSPNC(I)                                                    00001180
      KSPNC(I)=IVSPNC(I)                                                 00001190
    1 CONTINUE                                                           00001200
      KMAX=MAX0(KSPNC(1),KSPNC(2),KSPNC(3),KSPNC(4))                     00001210
C                                                                        00001220
      DO 7 I=1,6                                                         00001230
      DO 5 J=1,6                                                         00001240
      B1MAT(I,J)=0.D0                                                    00001250
    5 CONTINUE                                                           00001260
      DO 6 L=1,14                                                        00001270
      B4MAT(I,L)=0.D0                                                    00001280
    6 CONTINUE                                                           00001290
    7 CONTINUE                                                           00001300
C                                                                        00001310
C     COMPUTE THE B1 MATRIX                                              00001320
C                                                                        00001330
      IF(KVRFLG.EQ.3) GO TO 300                                          00001340
      DO 250 J=1,5                                                       00001350
      TEMP=VPELMT(J)                                                     00001360
      STEP=XS(J)*DABS(VPELMT(J))                                         00001370
      IF((DABS(VPELMT(J)).LT.1.D-10).OR.(KSTEP(J).EQ.1)) STEP=XS(J)      00001380
      VPELMT(J)=VPELMT(J)+STEP                                           00001390
      CALL SPCOTO                                                        00001400
      DO 220 I=1,6                                                       00001410
      DO 210 L=1,KMAX                                                    00001420
      XCQUAD(I,L)=CQUAD(I,L)                                             00001430
      XDQUAD(I,L)=DQUAD(I,L)                                             00001440
  210 CONTINUE                                                           00001450
  220 CONTINUE                                                           00001460
      VPELMT(J)=VPELMT(J)-2.D0*STEP                                      00001470
      CALL SPCOTO                                                        00001480
      DO 240 L=1,KMAX                                                    00001490
      XL=L                                                               00001500
      DO 230 I=1,6                                                       00001510
      XMMAT(I,J)=(XCQUAD(I,L)-CQUAD(I,L))/(2.D0*STEP)                    00001520
      XNMAT(I,J)=(XDQUAD(I,L)-DQUAD(I,L))/(2.D0*STEP)                    00001530
      B1MAT(I,J)=B1MAT(I,J)+XMMAT(I,J)*DSIN(XL*XLAMDA)-                  00001540
     *        XNMAT(I,J)*DCOS(XL*XLAMDA)                                00001550
  230 CONTINUE                                                           00001560
  240 CONTINUE                                                           00001570
      VPELMT(J)=TEMP                                                     00001580
  250 CONTINUE                                                           00001590
C                                                                        00001600
C     DO THE 6TH COLUMN OF THE B1 MATRIX                                 00001610
C                                                                        00001620
      CALL SPCOTO                                                        00001630
      DO 270 L=1,KMAX                                                    00001640
      XL=L                                                               00001650
      DO 260 I=1,6                                                       00001660
      B1MAT(I,6)=B1MAT(I,6)+CQUAD(I,L)*XL*DCOS(XL*XLAMDA)+               00001670
     *        DQUAD(I,L)*XL*DSIN(XL*XLAMDA)                             00001680
  260 CONTINUE                                                           00001690
  270 CONTINUE                                                           00001700
C                                                                        00001710
```

```
C     COMPUTE THE B4 MATRIX.                                                    00001720
C                                                                               00001730
  300 IF((KVRFLG.LE.2).OR.(KPAR.EQ.0)) GO TO 900                                00001740
C                                                                               00001750
C     CHECK FOR THE PARAMETER ESTIMATION OPTION. FOR OPTIONS 1,2 AND 3,         00001760
C     ONLY THE DRAG FORCE MODEL NEEDS TO BE CONSIDERED.                         00001770
C                                                                               00001780
      N=NEQ-6                                                                   00001790
      IF(KPAR.GT.3) GO TO 900                                                   00001800
C                                                                               00001810
      MCPSPP=KCPSPP                                                             00001820
      MSRSPP=KSRSPP                                                             00001830
      KCPSPP=2                                                                  00001840
      KSRSPP=2                                                                  00001850
C                                                                               00001860
      KKPAR=KPAR                                                                00001870
      GO TO (303,400,400), KKPAR                                                00001880
C                                                                               00001890
C     DETERMINE THE B4 MATRIX WHEN CDRAG IS A SOLVE-FOR-PARAMETER.              00001900
C                                                                               00001910
  303 TEMP=CDRAG                                                                00001920
      DO 304 I=1,3                                                              00001930
      TSAE(I)=SAE(I)                                                            00001940
  304 CONTINUE                                                                  00001950
      STEP=XS(7)*CDRAG                                                          00001960
      IF((CDRAG.LT.1.D-10).OR.(KSTEP(7).EQ.1)) STEP=XS(7)                       00001970
      CDRAG=CDRAG+STEP                                                          00001980
      RHO=CDRAG/TEMP                                                            00001990
      DO 305 I=1,3                                                              00002000
      SAE(I)=TSAE(I)*RHO                                                        00002010
  305 CONTINUE                                                                  00002020
      CALL SPCOTO                                                               00002030
      DO 320 I=1,6                                                              00002040
      DO 310 L=1,KMAX                                                           00002050
      XCQUAD(I,L)=CQUAD(I,L)                                                    00002060
      XDQUAD(I,L)=DQUAD(I,L)                                                    00002070
  310 CONTINUE                                                                  00002080
  320 CONTINUE                                                                  00002090
      CDRAG=CDRAG-2.D0*STEP                                                     00002100
      RHO=CDRAG/TEMP                                                            00002110
      DO 325 I=1,3                                                              00002120
      SAE(I)=TSAE(I)*RHO                                                        00002130
  325 CONTINUE                                                                  00002140
      CALL SPCOTO                                                               00002150
      DO 340 L=1,KMAX                                                           00002160
      XL=L                                                                      00002170
      DO 330 I=1,6                                                              00002180
      XKMAT(I,1)=(XCQUAD(I,L)-CQUAD(I,L))/(2.D0*STEP)                           00002190
      XLMAT(I,1)=(XDQUAD(I,L)-DQUAD(I,L))/(2.D0*STEP)                           00002200
      B4MAT(I,1)=B4MAT(I,1)+XKMAT(I,1)*DSIN(XL*XLAMDA)-                         00002210
     *           XLMAT(I,1)*DCOS(XL*XLAMDA)                                     00002220
  330 CONTINUE                                                                  00002230
  340 CONTINUE                                                                  00002240
      CDRAG=TEMP                                                                00002250
      DO 345 I=1,3                                                              00002260
      SAE(I)=TSAE(I)                                                            00002270
  345 CONTINUE                                                                  00002280
```

```
      GO TO 490                                                          00002290
                                                                         00002300
C                                                                        00002310
C     DETERMINE THE B4 MATRIX WHEN THE APAR'S ARE THE SOLVE-FOR-         00002320
C     PARAMETERS.                                                        00002330
C                                                                        00002340
  400 DO 450 J=1,N                                                       00002350
      JLL=6+J                                                            00002360
      TEMP=APAR(J)                                                       00002370
      STEP=XS(JLL)*DABS(APAR(J))                                         00002380
      IF((DABS(APAR(J)).LT.1.D-10).OR.(KSTEP(JLL).EQ.1)) STEP=XS(JLL)    00002390
      APAR(J)=APAR(J)+STEP                                               00002400
      CALL SPCOTO                                                        00002410
      DO 420 I=1,6                                                       00002420
      DO 410 L=1,KMAX                                                    00002430
      XCQUAD(I,L)=CQUAD(I,L)                                             00002440
      XDQUAD(I,L)=DQUAD(I,L)                                             00002450
  410 CONTINUE                                                           00002460
  420 CONTINUE                                                           00002470
      APAR(J)=APAR(J)-2.D0*STEP                                          00002480
      CALL SPCOTO                                                        00002490
      DO 440 L=1,KMAX                                                    00002500
      XL=L                                                               00002510
      DO 430 I=1,6                                                       00002520
      XKMAT(I,J)=(XCQUAD(I,L)-CQUAD(I,L))/(2.D0*STEP)                    00002530
      XLMAT(I,J)=(XDQUAD(I,L)-DQUAD(I,L))/(2.D0*STEP)                    00002540
      B4MAT(I,J)=B4MAT(I,J)+XKMAT(I,J)*DSIN(XL*XLAMDA)-                  00002550
     *           XLMAT(I,J)*DCOS(XL*XLAMDA)                              00002560
  430 CONTINUE                                                           00002570
  440 CONTINUE                                                           00002580
      APAR(J)=TEMP                                                       00002590
  450 CONTINUE                                                           00002600
C                                                                        00002610
C     RESET THE FLAGS.                                                   00002620
C                                                                        00002630
  490 KCPSPP=MCPSPP                                                      00002640
      KSRSPP=MSRSPP                                                      00002650
C                                                                        00002660
C     ADDITIONAL ESTIMATION OPTIONS WOULD BE INCLUDED HERE.              00002670
C                                                                        00002680
  500 CONTINUE                                                           00002690
C                                                                        00002700
C     RESET THE NUMBER OF SHORT PERIODIC COEFFICIENTS.                   00002710
C                                                                        00002720
  900 DO 910 I=1,4                                                       00002730
      KSPNC(I)=INC(I)                                                    00002740
  910 CONTINUE                                                           00002750
C                                                                        00002760
      RETURN                                                             00002770
      END
```

******** END OF MEMBER VRSPFD 277 RECORDS ********

Appendix G

MEASUREMENT EQUATIONS AND PARTIAL DERIVATIVES

This appendix is taken from Section 7.2.3 of Reference 67. Range, range rate and the gimbal angle (X_{30}, Y_{30}, X_{85}, Y_{85}) measurements used in VHF and USB (Unified S-Band) tracking stations are presented in this appendix. A more detailed discussion is given in Reference 67.

Range Observations for USB

The expected value of $\rho(t_R)$ is computed from ephemeris information and station coordinates using the following equation

$$\rho(t_R) = \frac{1}{2}\{|\bar{r}_v(t_v) - \bar{r}_T(t_T)| + |\bar{r}_v(t_v) - \bar{r}_R(t_R)|\} \tag{G-1}$$

For simplicity, this equation is presented in an inertial reference frame, where

$\bar{r}_v$ ∿ spacecraft inertial position vector

$\bar{r}_T$ ∿ transmitting site inertial position vector

$\bar{r}_R$ ∿ receiving site inertial position vector

t_T ∿ time at which the measured signal left the ground transmitter

t_v ∿ time at which the measured signal was received and retransmitted by the spacecraft. The assumption of instantaneous turnaround is used; the constant bias in the observed range caused by the spacecraft electronic delay is accounted for as a measurement error elsewhere in GTDS.

t_r ∿ time tag of the reduced observed range (that is, the time at which the measured signal arrived at the ground receiver).

The algorithm used in GTDS to compute $\rho(t_R)$ proceeds as follows:

1. Calculate $\bar{r}_R(t_R)$
2. Calculate iteratively the downlink propagation distance $\rho_d(t_R)$ using the following equations:

$$\begin{aligned} &(a) \quad \rho_d(t_R) = |\bar{r}_v(t_v) - \bar{r}_R(t_R)| \\ &(b) \quad \delta_d(t_R) = \rho_d(t_R)/c \\ &(c) \quad t_v = t_R - \delta_d(t_R) \end{aligned} \tag{G-2}$$

The iteration process is initiated by assuming that $t_v = t_R$, and is terminated when successive values of $\delta_d(t_R)$ agree to within 10^{-8} seconds.

3. Calculate iteratively the uplink propagation distance $\rho_u(t_R)$ using the following equations:

$$\begin{aligned} &(a) \quad \rho_u(t_R) = |\bar{r}(t_v) - \bar{r}_T(t_T)| \\ &(b) \quad \delta_u(t_R) = \rho_u(t_R)/c \\ &(c) \quad t_T = t_v - \delta_u(t_R) \end{aligned} \tag{G-3}$$

The iteration is initiated by assuming that $\delta_u(t_R) = \delta_d(t_R)$, and is terminated when successive values of $\delta_u(t_R)$ agree to within 10^{-8} seconds.

4. The following geometrically exact equation is used to compute the expected value of the range $\rho(t_R)$ for the USB system

$$\rho(t_R) = [\rho_u(t_R) + \rho_d(t_R)]/2 \qquad \text{(G-4)}$$

The partial derivatives of the expected range [Equation (G-1)] in inertial coordinates (USB system) are

$$\frac{\partial \rho(t_R)}{\partial \bar{r}_v(t_v)} = \frac{1}{2\rho_u\rho_d}\{\rho_d[\bar{r}_v^T(t_v) - \bar{r}_T^T(t_T)] + \rho_u[\bar{r}_v^T(t_v) - \bar{r}_R^T(t_R)]\} \qquad \text{(G-5)}$$

$$\frac{\partial \rho(t_R)}{\partial \dot{\bar{r}}_v(t_v)} = 0 \qquad \text{(G-6)}$$

If it is assumed that $\rho_u = \rho_d = \rho(t_R)$, Equation (G-5) reduces to

$$\frac{\partial \rho(t_R)}{\partial \bar{r}_v(t_v)} = \frac{1}{2\rho(t_R)}\{2\bar{r}_v^T(t_v) - [\bar{r}_T^T(t_T) + \bar{r}_R^T(t_R)]\} \qquad \text{(G-7)}$$

Range Observations for VHF

Range data produced by the VHF system is less accurate than that produced by the USB tracking system; therefore, it does not warrant the application of the iterative solution described above. Instead, the following more efficient algorithm is used to determine an instantaneous approximation for $\rho(t_R)$ using VHF range data

$$\rho(t_R) = |\bar{r}_v(t_v) - \bar{r}_T(t_v)| = |\bar{r}_{lt}(t_v)|$$

$$= \sqrt{x_{lt}^2 + y_{lt}^2 + z_{lt}^2} \qquad \text{(G-8)}$$

where

$$t_v = t_R - \rho(t_R)/c$$

and $\bar{r}_{lt}$ is the spacecraft position vector in local tangent plane coordinates.

The partial derivatives of the expected range in local tangent plane coordinates are

$$\frac{\partial \rho(t_R)}{\partial \bar{r}_{lt}(t_v)} = \frac{\bar{r}_{lt}^T(t_v)}{\rho(t_R)} \qquad \text{(G-9)}$$

$$\frac{\partial \rho(t_R)}{\partial \vec{r}_{1t}(t_v)} = 0 \qquad \text{(G-10)}$$

Range Rate Observations for USB

The modeling of the expected value of the range rate which is most precise is to difference the average range at the beginning and end of the count interval as shown below

$$\dot{\rho}(t_R) = \frac{[\rho_u(t_R) + \rho_d(t_R)] - [\rho_u(t_R - \Delta t_{RR}) + \rho_d(t_R - \Delta t_{RR})]}{2\Delta t_{RR}} \qquad \text{(G-11)}$$

where

$\rho_u(t_R)$ ∿ uplink propagation path of a signal arriving at the receiver at t_R

$\rho_d(t_R)$ ∿ downlink propagation path of a signal arriving at the receiver at t_R

Δt_{RR} ∿ Doppler count time interval

The calculations for these uplink and downlink ranges are iteratively corrected for the light time delay in exactly the same manner as the USB range observations. This method is used for USB measurements where the time tag on the observed data is t_R (corresponding to the end of the count interval) and the count interval Δt_{RR} corresponds to the sample interval. This method is accurate for both two-way and three-way Doppler measurements

using the USB system. Two-way Doppler measurements are obtained when the transmitting and receiving antennas are the same, while three-way Doppler measurements are obtained when the transmitting and receiving antennas are different.

The range-rate partial derivatives with respect to the epoch state elements $\bar{R}$ and $\dot{\bar{R}}$ are computed most efficiently by using the algorithms for the range partial derivatives

$$\frac{\partial \dot{\rho}(t_R)}{\partial \bar{R}} = \frac{\frac{\partial \rho(t_R)}{\partial \bar{R}} - \frac{\partial \rho(t_R - \Delta t_{RR})}{\partial \bar{R}}}{\Delta t_{RR}} \qquad \text{(G-12)}$$

$$\frac{\partial \dot{\rho}(t_R)}{\partial \dot{\bar{R}}} = \frac{\frac{\partial \rho(t_R)}{\partial \dot{\bar{R}}} - \frac{\partial \rho(t_R - \Delta t_{RR})}{\partial \dot{\bar{R}}}}{\Delta t_{RR}} \qquad \text{(G-13)}$$

Range Rate Observations for VHF

The VHF range rate, which is the least accurate but most efficient, calculates the instantaneous range rate at the mid-point of the Doppler count interval as seen at the spacecraft. This value is used to approximate the average range rate over the uplink and downlink paths, and is therefore denoted $\dot{\rho}_{avg}$. It is computed as

$$\dot{\rho}_{avg} = \frac{\bar{r}_{lt}(t_v) \cdot \dot{\bar{r}}_{lt}(t_v)}{|\bar{r}_{lt}(t_v)|} \qquad \text{(G-14)}$$

The position and velocity vectors are expressed in station-centered local tangent plane coordinates evaluated at the vehicle turnaround time t_v. When this method is used the preprocessor modifies the time tag on the VHF data according to the relationship

$$t_v = t_R + \frac{\Delta t_{RR}}{2} - \frac{|\bar{r}_{lt}(t_v)|}{c} \qquad \text{(G-15)}$$

The partial derivatives of $\dot{\rho}_{avg}$ with respect to local tangent plane coordinates are

$$\frac{\partial \dot{\rho}_{avg}}{\partial \bar{r}_{lt}} = \frac{1}{\rho(t_R)} \left[\dot{\bar{r}}_{lt}^{T} - \frac{\dot{\rho}_{avg}}{\rho(t_R)} \bar{r}_{lt}^{T} \right] \qquad \text{(G-16)}$$

$$\frac{\partial \dot{\rho}_{avg}}{\partial \dot{\bar{r}}_{lt}} = \frac{\bar{r}_{lt}^{T}}{\rho(t_R)} \qquad \text{(G-17)}$$

Gimbal Angles X_{30} and Y_{30}

The gimbal angles for the 30-foot antennas in the VHF and USB systems are denoted X_{30} and Y_{30}. The X_{30}-axis is aligned north-south in the local horizon (tangent) plane at the tracking

station. The reference plane for the angular measurements is the vertical plane which is aligned east-west and includes the tracking station zenith. The angle X_{30} is measured from the vertical axis (zenith) to the projection of the station-to-spacecraft vector onto the reference plane. This angle is positive when the spacecraft is east of the station, i.e.

$$X_{30} = \tan^{-1}\left(\frac{x_{1t}}{z_{1t}}\right) \qquad -\frac{\pi}{2} \leq X_{30} \leq \frac{\pi}{2} \tag{G-18}$$

The angle Y_{30} is measured from the projection of the station-to-spacecraft vector onto the reference plane to the vector itself. This angle is positive when the spacecraft is north of the station, i.e.

$$Y_{30} = \tan^{-1}\left(\frac{y_{1t}}{\sqrt{x_{1t}^2 + z_{1t}^2}}\right) \quad -\frac{\pi}{2} \leq Y_{30} \leq \frac{\pi}{2} \tag{G-19}$$

The partial derivatives of X_{30} and Y_{30} with respect to the local tangent plane coordinates are

$$\frac{\partial X_{30}}{\partial \bar{r}_{1t}} = \frac{1}{(x_{1t}^2 + z_{1t}^2)} \left[z_{1t}, 0, -x_{1t}\right] \tag{G-20}$$

$$\frac{\partial X_{30}}{\partial \dot{\bar{r}}_{1t}} = 0$$

and

$$\frac{\partial Y_{30}}{\partial \bar{r}_{1t}} = \frac{1}{\rho^2}\left[\frac{-x_{1t}y_{1t}}{\sqrt{x_{1t}^2 + z_{1t}^2}}, \sqrt{x_{1t}^2 + z_{1t}^2}, \frac{-y_{1t}z_{1t}}{\sqrt{x_{1t}^2 + z_{1t}^2}}\right] \tag{G-21}$$

$$\frac{\partial Y_{30}}{\partial \dot{\bar{r}}_{1t}} = 0$$

where

$$\rho = \sqrt{x_{1t}^2 + y_{1t}^2 + z_{1t}^2}$$

Gimbal Angles X_{85} and Y_{85}

The gimbal angles associated with the USB 85-foot antennas are denoted X_{85} and Y_{85}. The X_{85}-axis is aligned east-west in the local horizon (tangent) plane at the tracking station. The reference plane for the angular measurements is the vertical plane which is aligned north-south and includes the tracking station zenith. The angle X_{85} is measured from the vertical axis (zenith) to the projection of the station-to-spacecraft vector onto the reference plane. This angle is positive when the spacecraft is south of the station, i.e.

$$X_{85} = \tan^{-1}\left(-\frac{y_{lt}}{z_{lt}}\right) \qquad -\frac{\pi}{2} \leq X_{85} \leq \frac{\pi}{2} \tag{G-22}$$

The angle Y_{85} is measured from the projection of the station-to-spacecraft vector onto the reference plane to the vector itself. This angle is positive when the spacecraft is east of the station, i.e.,

$$Y_{85} = \tan^{-1}\left(\frac{x_{LT}}{\sqrt{y_{lt}^2 + z_{lt}^2}}\right) \qquad -\frac{\pi}{2} \leq Y_{85} \leq \frac{\pi}{2} \tag{G-23}$$

The partial derivatives of X_{85} and Y_{85} with respect to the local tangent plane coordinates are

$$\frac{\partial X_{85}}{\partial \bar{r}_{lt}} = \frac{1}{(y_{lt}^2 + z_{lt}^2)} \left[0, -z_{lt}, y_{lt}\right] \tag{G-24}$$

$$\frac{\partial X_{85}}{\partial \dot{\bar{r}}_{lt}} = 0$$

and

$$\frac{\partial Y_{85}}{\partial \bar{r}_{lt}} = \frac{1}{\rho^2}\left[\sqrt{y_{lt}^2 + z_{lt}^2}\,,\ \frac{-x_{lt}y_{lt}}{\sqrt{y_{lt}^2 + z_{lt}^2}}\,,\ \frac{-x_{lt}z_{lt}}{\sqrt{y_{lt}^2 + z_{lt}^2}}\right] \tag{G-25}$$

$$\frac{\partial Y_{85}}{\partial \dot{\bar{r}}_{lt}} = 0$$

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BIOGRAPHY

Andrew Joseph Green was born on 2 February 1950 in Utica, New York, the son of Kathleen and Hugh Green. He attended Waterville Central High School in Waterville, New York from 1964 to 1968. In May 1970 he received an Applied Science degree from Mohawk Valley Community College in Utica, New York. From 1970 to 1974 he attended the United States Military Academy. Upon graduation from the Military Academy, he was commissioned a Second Lieutenant in the Armor Branch of the United States Army. He married Theresa Belfield on 5 June 1974. He has successfully completed the following Army schools: Ranger, Airborne, Armor Officer Basic, Motor Officer. He served 17 months with the 3rd Armor Cavalry Regiment in Fort Bliss, Texas. In September 1976 he entered Massachusetts Institute of Technology. He received a Master of Science from MIT in September 1977. He is a Fannie and John Hertz Foundation Fellow and is currently a Captain in the U. S. Army. He has two children, Christopher and Andrea.