THE ESTIMATION OF NATURAL FREQUENCIES AND DAMPING RATIOS OF OFFSHORE STRUCTURES

by

ROBERT BRADLEE CAMPBELL

B.S.M.E., University of Miami (1973)

M.S., University of Miami (1975)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

OCEAN ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

December 1979

(i.e. February, 1980)

Massachusetts Institute of Technology 1979

Signature of Author

Department of Ocean Engineering December 17, 1979

Certified by

J. Kim Vandiver

Thesis Supervisor

Accepted by

Chairman, Department Committee
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Submitted to the Department of Ocean Engineering on December 17, 1979 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Ocean Engineering

ABSTRACT

This research focusses on the estimation of natural frequencies and modal damping ratios from measured response spectra, with particular emphasis on the dynamic response of offshore structures to wind and wave excitation. At present, estimates of natural frequencies and damping ratios are computed from the location and half-power bandwidths of resonant peaks in a structure's ambient response power spectrum. While reliable natural frequency estimates are typically obtained in this manner, half-power bandwidth damping estimates are found to be highly sensitive to the method employed in estimating the response spectrum. The lack of confidence bounds on natural frequency and damping estimates further restricts the utility of the estimates. An alternative method is developed based on a powerful method of spectral estimation known as the Maximum Entropy Method (MEM). The resulting technique yields estimates of natural frequencies and modal damping ratios as well as approximate statistics on the reliability of the estimates. Performance of this new method is explored through extensive Monte Carlo simulation of one and two degree-of-freedom systems. Conventional estimates are also simulated for comparison with the MEM parameter estimator. The use of the MEM parameter estimator is further illustrated with ambient response data from Shell Oil's South Pass 62C platform. The MEM parameter estimates show excellent agreement with natural frequency and damping estimates obtained during recent tests conducted using forced excitation.

Thesis Supervisor: Dr. J. Kim Vandiver

Title: Associate Professor of Ocean Engineering
ACKNOWLEDGEMENTS

This research was made possible through the support of the Branch of Marine Oil and Gas Operations of the United States Geological Survey, Department of the Interior. In particular, I would like to thank the program manager, John Gregory, for the faith and patience he demonstrated during the past years.

I would also like to thank the engineers at Union Oil Company and Shell Development Company for their contribution of field data which greatly aided in the practical evaluation of the method developed in this thesis.

For their contributions during the course of this research, I would like to express my appreciation to the members of my thesis committee, Erik H. Vanmarcke and Arthur B. Baggeroer.

A very special thanks must be given to my advisor, my thesis committee chairman, my friend, J. Kim Vandiver, whose ever-present encouragement and enthusiasm made my work at M.I.T. a joy.

I would also like to express my thanks to Bruce Dunwoody, whose insight and probing questions aided immeasurably in the development of my thesis.

Finally, an expression of thanks or gratitude could never adequately convey to my wife Kerry how indebted I am to her. Her limitless energy and unfaltering love were always there when I needed it. This thesis truly belongs to her.
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CHAPTER 1

Introduction

1.1 Motivation

To date, considerable effort has been expended in attempts to measure dynamic response properties of offshore structures. Structural natural frequencies and modal damping ratios are the parameters which have received the greatest attention. The necessity for accurate measurement of natural frequencies has been motivated to a large extent by widespread interest in the development of techniques for detection of structural damage, commonly referred to as integrity monitoring. The integrity monitoring efforts have not been altogether successful. In part this has been due to the frequent inability of investigators to obtain reproducible estimates of natural frequencies above the fundamentals in bending and torsion. It is also true that some types of damage do not result in significant shifts in the natural frequencies of the lower structural modes.

The interest in accurate estimates of modal damping ratios has been stimulated by more fundamental issues regarding the dynamic amplification of structural response to wave excitation. For deep-water structures the lowest natural frequencies in flexure and torsion correspond to quite energetic regions of normal wave amplitude spectra. Fatigue life estimates for such structures are sensitive to the dynamic amplification, and hence in the modal damping ratios used in making the
estimates. Unfortunately, reproducible measures of damping from ambient response records have been difficult to achieve [1]. Until reliable measures from existing structures are obtained, it is difficult to select appropriate values to apply as criteria in the design of future structures.
1.2 Scope of This Work

This research focusses on the estimation of natural frequencies and damping ratios from measured response spectra, with particular emphasis on the dynamic response of offshore structures to wind and wave excitation. The state of the art of this particular facet of vibration testing is reviewed in the remainder of this chapter. The underlying assumptions of the various methods are delineated and prerequisite background material is provided.

Chapter 2 investigates the nature of the errors introduced into so-called maximum response natural frequency estimates and half-power bandwidth damping estimates as a result of violating certain fundamental assumptions on which the estimators are based. The results of this chapter lead to qualitative measures of uncertainty in estimated parameters.

Chapter 3 describes the development of a new method of estimating natural frequencies and damping ratios. This method combines the peak response natural frequency estimator and the half-power bandwidth damping estimator with an extremely powerful method of spectral estimation known as the Maximum Entropy Method (MEM). The resulting technique yields estimates of natural frequencies and damping ratios as well as statistics on the reliability of the estimated parameters.

Performance of this new method is explored in Chapter 4 through extensive Monte Carlo simulation of one and two degree of freedom systems. Conventional estimates of the parameters are also simulated
for comparison with the MEM parameter estimates.

The use of conventional and MEM parameter estimators is further illustrated in Chapter 5 with ambient response data from Shell Oil's South Pass 62C platform. The results are compared with estimates of natural frequencies and damping ratios recently obtained from tests conducted with forced excitation.
1.3 Background

The estimation of natural frequencies and damping ratios of dynamic structural systems can be viewed as the task of matching the theoretical and measured response of a system. In this context, it is assumed that a model for the system exists and that this model is parametric in a set of natural frequencies and damping ratios. By exciting the model and the system with identical forces, one can adjust the parameters of the model until the theoretical response predicted by the model matches the measured response of the system in some optimum manner. As one might expect, the parameter estimation problem stated in these terms admits a variety of solution strategies. In most cases, methods for estimating natural frequencies and damping ratios differ in the nature of the excitation, the complexity of the structural model, the rigor of the matching scheme and the domain of analysis (i.e. frequency or time). The reader is refered to an excellent survey article by Schiff [2] which described the most popular methods used for large structures and details some of the most frequently encountered problems.

Vibration testing of offshore platforms has, to great extent, been limited to the use of estimation procedures which extract the desired parameters from power spectra computed from recordings of a structure's response to wind and wave excitation. These techniques have found special favor as a result of their low cost and the simplicity of data collection compared to alternative schemes
which required the use of massive shakers. However, these methods do present some special problems since the exact nature of the excitation is unknown.

The estimation of natural frequencies and damping ratios from ambient response spectra can be divided into three basic topics: estimation of power spectra from ambient response measurements, evaluation of the system's theoretical response, and definition and application of the parameter estimators. Each of these subjects will now be considered in some detail in an attempt to provide the reader with a cohesive overview of the problem while reviewing techniques and defining terms which will be encountered throughout this thesis.
1.3.1 The Conventional Spectral Estimate

Consider for the moment the task of estimating a power spectrum. In general this involves a finite duration observation of an ongoing random process and a transformation of that data into an estimate of the power spectrum of the underlying process. This transformation may take on many different forms, including both analog and digital formats. At present, the vast majority of spectral estimation is accomplished with a general class of estimators which, for lack of a better name, will be referred to as conventional estimators. These estimators are based on numerical approximations of theoretical definitions of the power spectrum. One such definition, for example, is the Fourier Transform relationship between the autocorrelation function $R(\tau)$ and the power spectrum $S(f)$ expressed by equation (1.1). While this definition is in terms of another unknown function, $R(\tau)$,

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi ft} d\tau$$

(1.1)

and hence not directly useful for spectral estimation, the autocorrelation function can be obtained from a sample function of an ergodic random process $x(t)$ as shown in equation (1.2).

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t + \tau) \, dt$$

(1.2)
These equations form the basis of what is commonly referred to as the Blackman-Tukey spectral estimator [3]. In this method the numerical approximation of equations (1.1) and (1.2) is made using $N$ samples of the observed time history $x_0(t = n\Delta) \triangleq x_0(n)$ where $\Delta$ is the time interval between samples. Consequently, an estimate of the autocorrelation function at discrete lags, $\hat{R}(\tau = m\Delta) \triangleq \hat{R}(m)$, is given by equation (1.3).

$$R(m) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x(n)x(n + m)$$

(1.3)

where: $x(n) = \begin{cases} x_0(n) & 0 \leq n \leq N-1 \\ 0 & n > N-1 \end{cases}$

Upon careful examination of equation (1.3) it is found that estimates of the autocorrelation function for $m > N$ are identically zero. This fact will prove to be a major impediment in the performance of this estimator. As the final step in this algorithm, the spectral estimate is obtained as the discrete Fourier Transform of the first $M$ lags of $\hat{R}(m)$.

$$\hat{S}(f) = \Delta \sum_{m=-\infty}^{\infty} \hat{R}(m)e^{-j2\pi fm}; \ |f| \leq \frac{1}{2\Delta}$$

(1.4)

where: $\hat{R}(m) = \begin{cases} \hat{R}(m) & \text{for } |m| \leq M-1 \\ 0 & \text{for } |m| > M-1 \end{cases}$
While specifying the estimation algorithm, equations (1.3) and (1.4) also reveal that the estimates $\hat{R}(m)$ and $\hat{S}(f)$ are functions of the random variables $x(0), \ldots, x(N-1)$ and thus random variables themselves. As a result, it is convenient to characterize the performance of this and other spectral estimators in terms of their expected value and variance. For example, it is easily shown [4] that the expected value of the Blackman-Tukey spectrum, expressed by equation (1.5), is the convolution

$$E[\hat{S}(f)] = \frac{1}{2\Delta} \int_{-1/2\Delta}^{1/2\Delta} S(g)W(f-g)dg \quad (1.5)$$

where:

$$W(f) = \frac{\sin[\pi f(2M-1)\Delta]}{\sin[\pi f\Delta]}$$

of the true spectrum $S(f)$ with the spectral window function $W(f)$. The form of this result indicates that the estimated spectrum will be a smeared version of the true spectrum. As the lag, $L = M\Delta$, is increased, the width of the spectral window decreases thereby increasing the amount of detail that can be seen in the estimated spectrum. A classical measure of the fidelity of the spectral estimate (or resolution as it is frequently called) is the frequency separation $\Delta f$, that a bi-harmonic spectrum may possess and still display two discernable peaks in the estimated spectrum. It can be shown [4] that the resolution is approximately equal to the width.
of the main lobe of the spectral window function. The width is frequently measured as the interval between the first positive and negative frequencies for which the window is zero. Accordingly, the spectral estimator defined by equation (1.4) has a resolution, \( \Delta f = 1/L \). This fundamental limit in resolution is a direct consequence of extending the estimated autocorrelation function with zeros for lags \( m \geq M \).

The relationship between resolution and lag would not, in itself, be a major impediment. Unfortunately, the improved spectral fidelity realized by increasing the lag occurs at the expense of statistical reliability of the estimated spectrum. This is demonstrated by the expression for the variance of the spectral estimate shown in equation (1.6). Consequently, for a fixed record length, \( T = N \Delta \), a

\[
\text{VAR}[\hat{S}(f)] = \frac{2M}{N} S^2(f) \tag{1.6}
\]

tradeoff must be made between resolution and variance in order to find an "optimum" spectral estimate.

While the estimator defined by equation (1.4) serves as introduction to the Blackman-Tukey spectral estimate, it is actually one case of more general estimator defined by

\[
\hat{S}(f) = \Delta \sum_{m=-\infty}^{\infty} R(m)w(m)e^{-j2\pi fm \Delta} ; |f| \leq \frac{1}{2\Delta} \tag{1.7}
\]

where the lag window, \( w(m) \), has the following properties:
w(0) = 1 \quad (1.8a)

w(m) = w(-m) \quad (1.8b)

w(m) = 0 ; |m| > M-1 \quad (1.8c)

One of the benefits of this more general formation can be seen in terms of the expected value of the spectral estimate given by

\[ E[\hat{S}(f)] = \frac{1}{2\Delta} \int_{-1/2\Delta}^{1/2\Delta} S(g)W(f-g)dg \quad (1.9) \]

where: \[ W(f) = \sum_{m=-\infty}^{\infty} w(m)e^{-j2\pi fm\Delta}; |f| < \frac{1}{2\Delta} \]

This expression shows that, as before, the characteristics of the spectral estimate are governed by the shape of the spectral window. Thus, by using an appropriate lag window one may obtain smooth spectral estimates with fewer of the spurious peaks frequently obtained from spectral window functions with large secondary lobes. Furthermore, the smoothing introduced by these windows is accompanied by a reduction in variance as expressed by equation (1.10). Three of the most

\[ \text{VAR} [\hat{S}(f)] = R \cdot S^2(f) \quad (1.10) \]

where: \[ R = \frac{1}{N} \sum_{m=-M+1}^{M-1} w^2(m) \]
commonly used lag windows are described in Table 1.1 along with their characteristics resolution and variance.

This recognizedly brief treatment of spectral estimation has presented the basic concepts required to understand the methods used in this thesis. For a much more detailed treatment of conventional spectral estimation, the reader is referred to the excellent works by Jenkins and Watts [5], Blackman and Tukey [3], and Bendat and Piersol [6] to list just a few.
Table 1.1  Lag Windows and Their Characteristics

<table>
<thead>
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<th>Window Name</th>
<th>Description</th>
<th>Resolution $\Delta f$</th>
<th>Variance Ratio $R$</th>
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<tr>
<td>Rectangular</td>
<td>$w_R(m) = \begin{cases} 1 &amp; \text{if }</td>
<td>m</td>
<td>\leq M-1 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$w_B(m) = \begin{cases} 1 -</td>
<td>m</td>
<td>/M &amp; \text{if }</td>
</tr>
<tr>
<td>Hanning</td>
<td>$w_H(m) = \begin{cases} 1/2 (1 + \cos[\pi m/(M-1)]) &amp; \text{if }</td>
<td>m</td>
<td>\leq M-1 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>

* Note: Valid for $M$ small compared to $N$
1.3.2 The Theoretical Ambient Response Spectrum

In modeling the structural dynamics of an offshore platform one generally encounters a wide variety of formulations, each boasting the inclusion of various special features. While this diversity exists in the basic equations of motion, the method of normal modes provides the unifying link between these different models. This method provides a transformation from the linearized equations of motion to a model which describes the motion of the structure in terms of its "N" free vibration mode shapes. Accordingly, the method of normal modes expresses the motion of a point on the structure in terms of the modal coefficients, $c_k$, and the generalized modal coordinates $n_k(t)$ as shown in equation (1.11). The modal coefficients quantify the participation of each mode in the description of the total response, $y(t)$. The generalized modal coordinates represent the time varying amplitudes of the free-vibration mode shapes. It can be shown [7] that the time history of the $k$th modal coordinate is given by the response of the single degree-of-freedom system defined by equation (1.12). Thus the response of the $k$th mode is characterized

$$y(t) = \sum_{k=1}^{N} c_k n_k(t) \quad (1.11)$$

$$\dddot{n}_k(t) + 2\xi_k \omega_k \dot{n}_k(t) + \omega_k^2 n_k(t) = P_k(t) \quad (1.12)$$
by the natural frequency, $\omega_k$, the modal damping ratio, $\xi_k$, and the modal force $P_k(t)$. For further details on the method of normal modes, the reader is referred to Clough and Penzein [7], Biggs [8], Meirovitch [9] or virtually any other text on vibrations or structural dynamics.

In order to find a spectral representation of the response, it is necessary to assume that the exciting forces used in the equations of motion are stationary random processes. Under this assumption it can be shown [7] that the power spectral density function (henceforth abbreviated to spectrum) for the response, $S_{yy}(f)$, is given by

$$S_{yy}(f) = \sum_{k=1}^{N} \sum_{\ell=1}^{N} c_k c_{\ell} S_{k\ell}(f)$$

(1.13)

where $S_{k\ell}(f)$ is the cross-spectrum of the modal response $\eta_k(t)$ and $\eta_\ell(t)$. From the theory of random vibration it can also be shown that the cross-spectrum of the modal responses can be expressed in terms of the cross-spectrum of the modal forces $P_k(t)$ and $P_\ell(t)$ as shown in equation (1.14).

$$S_{\eta_k\eta_\ell}(f) = H_k(f)H^*_\ell(f)S_{P_kP_\ell}(f)$$

(1.14)

where:

$$H_k(f) = \frac{(1/2\pi)^2}{f_k^2 - f^2 + j2\xi_k f_k f}$$
\[
f_k = \frac{\omega_k}{2\pi}
\]

\[
H_k^*(f) = \text{complex conjugate of } H_k(f)
\]

For structures with light damping and widely separated natural frequencies it is generally reasonable to assume that the cross-terms of equation (1.13) are nearly equal to zero. In this case equations (1.13) and (1.14) combine to give:

\[
S_{S(f)} = \sum_{k=1}^{N} c_k^2 |H_k(f)|^2 S_{P_k P_k}(f) \quad (1.15)
\]

As the final step in modeling the response of a structure to wind and wave excitation, it is necessary to assume a reasonable form of modal force spectra. One of the most useful, the uniform white noise modal force spectrum can be written as

\[
S_{P_k P_k}(f) = S_k \quad \text{for all } f \quad (1.16)
\]

After substituting equation (1.16) into equation (1.15) the response spectrum can be written as

\[
S_{YY}(f) \approx \sum_{k=1}^{N} c_k^2 S_k |H_k(f)|^2 \quad (1.17)
\]
An alternative model for the modal force spectrum is a less presumptuous form of white noise. In this model, the modal force spectrum is assumed to be constant in a band around the natural frequency and unspecified for all other frequencies. This "locally white" modal force spectrum is extremely useful when one is only interested in the response spectrum for frequencies in the immediate vicinity of a natural frequency. In this case, equation (1.15) can be rewritten as

\[
S_{yy}(f) = c^2_{m} S_{m} | H_{m}(f) |^2 + \sum_{k=1}^{N} c^2_k |H_k(f)|^2 S_{p_k p_k}(f) ; \quad B_1 \leq |f-f_m| \leq B_2 \quad (1.18)
\]

where \(B_1\) and \(B_2\) prescribe the interval over which the modal force spectrum is uniform. Given the previous assumption of light damping and widely separated modes, the summation term of equation (1.18) can be approximated as constant if the modal force spectrum is relatively smooth in the band \(B_1 \leq |f-f_m| \leq B_2\). As a result, the response spectrum in the immediate vicinity of the \(m\)th natural frequency can be approximated by

\[
S_{yy}(f) \approx c^2_{m} S_{m} | H_{m}(f) |^2 + C ; \quad B_1 \leq |f-f_m| \leq B_2 \quad (1.19)
\]
This very important result indicates that under the specified restrictions, the response spectrum in the vicinity of a resonance has the same shape as the magnitude-squared of that "modal resonator's" transfer function. Consequently, that region of the response spectrum can be treated as the response spectrum of a single degree-of-freedom system excited by white noise.
1.3.3 Definition and Application of Natural Frequency and Damping Estimators

The previous two sections have described the model for the theoretical response and indicated the methods which may be used to measure the response spectrum. In order to estimate the natural frequency and damping, one must develop a scheme for fitting the theoretical response spectrum to the measured response spectrum. In general, power spectrum based natural frequency and damping estimators fall into two categories: those which fit the model of the spectrum to measured spectrum and those which fit geometric characteristics of the model to the same characteristics of the measured spectrum. An example of this first class of estimators is the method developed by Schiff and Fiel [10]. This scheme obtains estimates of a natural frequency and damping by fitting equation (1.19) to a peak in the measured spectrum which has been identified with a natural frequency. The fit is accomplished using the weighted error term given by equation (1.20) evaluated over a set of frequencies, \( f_i \), which define the peak. The mean-square error, \( E \), over the frequency band is

\[
\epsilon_i = \frac{|H(f_i)|^2 - \hat{S}(f_i)}{|H(f_i)|^2} \tag{1.20}
\]

where: \( D \) = a scale factor

\( \hat{S}(f_i) \) = the estimated spectrum

\( @f = f_i \)
\(|H_m(f_i)|^2 = \) the squared magnitude of the transfer function of the modal resonator

computed as shown in equation (1.21) and the parameters \(f_n, \xi, \) and \(D\) are determined as the values which minimize the mean-square error with respect to each of the parameters.

\[
E = \sum_{i=1}^N \varepsilon_i^2
\]

(1.21)

where: \(N = \# \) of discrete frequencies, \(f_i\)

Evaluation of \(f_n, \xi, \) and \(D\) is accomplished using a Newton-Raphson root finding solution for equations (1.22). In practice, the

\[
\frac{\partial E}{\partial f_n} = 0
\]

(1.22a)

\[
\frac{\partial E}{\partial \xi} = 0
\]

(1.22b)

\[
\frac{\partial E}{\partial D} = 0
\]

(1.22c)

application of this parameter estimator requires the analyst to choose \(N, \) the number of frequencies to be included in the analysis and the resolution (and consequently, the variance given a fixed record length)
of the measured spectrum. The selection of these values is a non-trivial matter since they govern the amount of bias and variance in the parameter estimates. It should also be noted that this formulation neglects the presence of all additive noise (i.e. \( C \) in equation (1.19) and other measurement noise). It will be shown in Chapter 2 that this assumption can introduce substantial bias into damping estimates.

Another method belonging to the first category of estimators was described by Gersch et al [11, 12, 13]. This technique simultaneously estimates all natural frequencies and dampings observed in the measured spectrum. In actuality, this method is usually considered as a time-domain analysis since it is based on a discrete-time model of equation (1.12). However, the spectral interpretation is equally valid and provides a convenient viewpoint. In practice, the application of this method requires the analyst to specify the number of modes observed in the measured spectrum. However, extreme care must be exercised when using this and other similar methods since the parameter estimates become biased by the inclusion or exclusion of various features in the measured spectrum. That is, if a mode is omitted from consideration or a peak is not readily fitted by the modal resonator, then parameter estimates (especially damping) of properly identified modes will be biased. While this method is unique in its ability to furnish the approximate bias and variance of the parameter estimates, the bias due to improper modeling is indeterminate.

The second group of estimators, those using geometric characteristics of the spectrum, are in general more easily implemented and consequently
more widely used. By far, the best known of these estimators are the 'maximum response' natural frequency estimator and the 'half-power bandwidth' damping estimator. The maximum response natural frequency estimator is based on the observation that the spectral model given by equation (1.19) attains a relative maxima at the damped natural frequency. It is easily shown [7] that when the damping is less than 10%, the damped natural frequency and the natural frequency differ by less than 1%. Thus, the maximum response natural frequency estimate is simply the frequency corresponding to the relative maxima of a spectral peak which has been identified with a natural frequency. This estimator, when applied to a conventional spectral estimate, reduces to the problem of selecting the natural frequency from a finite set of points plotted in the spectrum. Estimates of the natural frequency determined in this manner are limited in accuracy by the spacing of the points in the spectrum along the frequency axis. Moreover, highly resolved spectral estimates often exhibit several local maxima in a single spectral peak. In this case, estimation of the natural frequency becomes ambiguous.

The half-power bandwidth damping estimator is based on the theoretical relationship between damping and the width of the spectral peak. In particular, using the spectral model of equation (1.19) it can be shown [7] that, B, the width of the spectral peak measured 3dB down from the resonant peak is related to the damping by

\[
\xi = \frac{B}{2f_n} \quad (1.23)
\]
This approximate equality is valid when the damping is small (<10%), the additive noise is negligible and equation (1.19) is valid. The quantity B is the so-called half-power bandwidth. This damping estimator is applied to conventional spectral estimates after first estimating the natural frequency by the maximum response method. Having located the resonant peak, the bandwidth 3dB down from the estimated resonant peak is measured using linear interpolation when necessary. This method is very sensitive to errors in locating the resonant peak, bias in the spectral estimate and to a lesser extent, the linear interpolation used in measuring the bandwidth. Further discussion of the half-power damping estimator and the maximum response natural frequency estimator will be postponed until Chapter 2 where the validity of these estimators will be examined.

Another method belonging to the second category of estimators is VanMarcke's method of moments [14, 15]. This estimator is based on the relationship between partial moments of the power spectrum \( \lambda_i \), defined by equation (1.24) and a natural frequency and damping. For example, it can be shown that when the spectrum is modeled by equation (1.19) and

\[
\lambda_i = \int_{f_A}^{f_B} f^i S(f) \, df
\]

(1.24)

the additive noise is negligible, then the natural frequency and damping estimates are defined by
\[ f_n = \left( \frac{\lambda_2}{\lambda_0} \right)^{1/2} \quad \text{(1.25a)} \]

\[ \xi = \left( \frac{1 + \Omega_A}{1 - \Omega_A} \right) \cdot \frac{\pi}{4} \left( 1 - \frac{\lambda_1}{\lambda_0 \lambda_2} \right) \quad \text{(1.25b)} \]

where: \( \Omega_A = \frac{f_A}{f_n} = \frac{f_n}{f_B} \)

In practice this technique suffers from the same difficulty in specifying resolution and analysis interval (i.e., \( f_A \) & \( f_B \)) as did the parametric curve fitting method of Schiff and Feil.

As a postscript to this brief review of spectral based natural frequency and damping estimators, it is noted that with the exception of Gersch's method, none of the currently available estimators contain measures of one's confidence in the parameter estimates. Since the estimates themselves are random variables and can therefore take on any value allowed by their probability distributions, one must specify the probable estimation error if the estimate is to have any meaning at all. This is conveniently accomplished in terms of the expected value and variance of the parameter estimator. These measures describe the results that would be obtained if one were to compute parameter estimates from a large collection of time histories obtained by independent repetitions of an experiment. As the number of experiments increase one would find that the average value and variance of the parameter estimates would converge to the expected value and variance of the parameter estimator. Furthermore, if the probability
distribution of the parameter estimator is known, then one can compute the probability that the true value of the parameter is contained in an interval determined by a computed estimate. For example, suppose that it is known that an estimator for the parameter $\theta$, has a Gaussian distribution with expected value $\theta$ and variance $\sigma^2$. If an estimate $\hat{\theta}$ is then obtained, it can be shown that there is a 95% probability that the parameter $\theta$ is contained in the interval $[\hat{\theta} - 2\sigma, \hat{\theta} + 2\sigma]$. Probabilistic statements of this nature should be the ultimate goal of all parameter estimation algorithms.
CHAPTER 2

An Error Analysis of the Maximum Response Natural Frequency Estimator and the Half-Power Bandwidth Damping Estimator

The preceding section described the use of half-power bandwidths and the frequencies of peak response to obtain estimates of dampings and natural frequencies. These estimators, when applied to ambient response spectra require the implicit use of assumptions concerning the form of the excitation spectra, the separation of modes, and the noise content of the measured response. However, these assumptions can never be completely satisfied in practice. One must recognize that the parameter estimates will be biased and that the amount of this "systematic" bias is directly governed by the degree to which the assumptions are valid. Since the adherence to the model is, in most cases, not demonstrable, one can obtain a qualitative measure of the systematic bias by investigating the sensitivity of the estimators to violation of certain assumptions. In this way, insight is gained into the conditions under which the parameter estimates may be substantially biased.

Each of the sections to follow will deal with a single assumption and investigate the impact of its violation on natural frequency and damping estimates. In general, this will be accomplished by formulating a model for the response spectrum which explicitly includes a more realistic form of the assumption. By applying the standard estimator to this spectral model, one can directly observe the effect on
the parameter estimates. It should also be noted that in each of the following sections (with the exception of Section 2.1) the analysis will be based exclusively on the acceleration response spectrum. The analysis of displacement and velocity response spectra is omitted as a consequence of the similarity between parameter estimates made with each type of response. The relative infrequency of displacement and velocity measurements on offshore platforms provides further motivation for the omission of their analysis.
2.1 The Half-Power Bandwidth Damping Estimator - Its Definition and Validity

The half-power bandwidth damping estimator was defined in Chapter 1 as

\[
\xi_{\text{HP}} = \frac{B}{2f_n}
\]  

(2.1)

where \(f_n\) is a natural frequency and \(B\) is the so-called half-power bandwidth. This estimator is based on the response characteristics of a single degree-of-freedom system and involves a number of approximations which restrict the use of this estimator to lightly damped systems. The basis of this estimator and the nature of the approximations can be demonstrated by considering the response of a single degree-of-freedom system excited by a sinusoidally varying force. The half-power bandwidth is calculated from equation (2.2) where \(f_+\) and \(f_-\) are the pair of frequencies for which the average power output is half the average power output at resonance. The mean square output, \(\langle x^2 \rangle\), of any linear, time invariant system excited by a sinusoidal input with unit amplitude and frequency, \(f\), is given by

\[
\langle x^2 \rangle = \frac{1}{2} |H(f)|^2
\]  

(2.3)
where $H(f)$ is the complex frequency response (transfer function) of the system. When response of the single degree-of-freedom is a displacement, the squared-magnitude of the transfer function is given by

$$|H_{F-D}(f)|^2 = \frac{(1/4\pi^2M)^2}{(f_n^2 - f^2)^2 + (2\xi f_n f)^2}$$  \hspace{1cm} (2.4)

where: $M =$ mass of the system

Expressing the definition of the half-power frequencies in terms of equation (2.3) yields an expression, equation (2.5), whose roots are the half-power frequencies $f_+$ and $f_-$. After substituting equation (2.4) into equation (2.5)

$$\frac{1}{2} |H(f)|^2 = \frac{1}{4} |H(f_n)|^2$$  \hspace{1cm} (2.5)

and solving it is found that

$$f_{+/-}^2 = \xi n^2 [ (1 - \xi^2)^2 + 2\xi^2 \sqrt{1 + \xi^2} ]$$  \hspace{1cm} (2.6)

To arrive at the desired result, the approximations shown in equation (2.7a and b) must be used. Subtracting the simplified expressions for

$$\sqrt{1 + \xi^2} = 1$$  \hspace{1cm} (2.7a)
(1 + x)^n = 1 + nx \hspace{1cm} (2.7b)

the half-power frequencies, one finds that the damping is approximately given by

$$\xi \approx \frac{f_+ - f_-}{2f_n}$$ \hspace{1cm} (2.8)

Since equation (2.8) is an approximation, it is important to investigate the conditions under which the expression provides a useful estimate of the damping. This can be accomplished by evaluating equation (2.1) with the quantities that would actually be measured. That is, in evaluating equation (2.1) from measured response, one actually uses the frequency of maximum response in place of the natural frequency and the bandwidth 3dB down from the maximum response instead of the bandwidth measured relative to the resonant response. To determine the half-power frequencies that would be measured, one need only re-solve equation (2.5) using the damped natural frequency, \(f_d = f_n \sqrt{1 - 2\xi^2}\) in place of the natural frequency. After some straightforward manipulation, it is found that the estimated half-power frequencies \(\hat{f}_{+/-}\), are given exactly by

$$\hat{f}_{+/-} = f_n \sqrt{1 - 2\xi^2 + 2\xi \sqrt{1 - \xi^2}}$$ \hspace{1cm} (2.9)

Consequently, the quantity that one would measure from an ideal transfer
function in hopes of estimating the true damping is expressed by equation (2.10). The nature of this result is illustrated in Figure 2.1 where the error in the damping estimate (ie. \( \frac{\hat{\xi}_{HP} - \xi}{\xi} \)) is plotted as a function of the true damping. This figure reveals that the approximation used in the half-power damping estimator inflates the damping estimate in virtually all cases. Moreover, when a system is characterized by the transfer function shown in equation (2.4) and exhibits damping less than 10%, then the half-power bandwidth damping estimator contains less than 2% error. The origin of the half-power bandwidth damping estimator has been demonstrated in terms of a single degree-of-freedom system whose input and output are force and displacement respectively. While it will not be proven here, it is a simple matter to show that the half-power bandwidth damping estimator is also valid when the observed response to an input force is velocity or acceleration. The force to velocity or acceleration transfer functions can be expressed as shown in equations (2.11a and 2.11b). Since equations (2.11a and 2.11b) achieve a maximum

\[
\begin{align*}
|H_{F-V}(f)|^2 &= f^2|H_{F-D}(f)|^2 \quad \text{(2.11a)} \\
|H_{F-A}(f)|^2 &= f^4|H_{F-D}(f)|^2 \quad \text{(2.11b)}
\end{align*}
\]
FIG. 2.1 ERROR IN THE "HALF-POWER" DAMPING ESTIMATE CAUSED BY SMALL DAMPING APPROXIMATION
at $f_n$ and $f_n(1 - \xi^2)^{-1/2}$ respectively, the dampings estimated from the ideal frequency response will contain different errors. As previously discussed, one can compute the actual quantities that would be measured from the ideal frequency response and consequently quantify the size of the inherent bias. From the velocity or acceleration response transfer functions, (2.11a and b), it can be shown that the ideal half-power bandwidth damping estimates are given by equations (2.12a and b) respectively. The error in the damping estimate defined by equation (2.12b) is plotted

$$\hat{\xi}_{HP} = \frac{1}{2} \left( \frac{\sqrt{1+2\xi^2 + 2\xi\sqrt{1+\xi^2}}}{\sqrt{1+2\xi^2 - 2\xi\sqrt{1+\xi^2}}} \right)$$ (2.12a)

$$\hat{\xi}_{HP} = \frac{1}{2} \sqrt{\frac{1-2\xi^2}{1-8\xi^2 (1-\xi^2)}} \left( \sqrt{1-2\xi^2 + 2\xi\sqrt{1-\xi^2}} - \sqrt{1-2\xi^2 - 2\xi\sqrt{1-\xi^2}} \right)$$ (2.12b)

in Figure (2.1) as a dashed line while the error in estimates obtained from equation (2.12a) is so small that it is indistinguishable from zero when plotted in Figure (2.1). Estimates of damping obtained from acceleration response are found to exceed the true value of damping by less than 5% when the system has less than 10% damping. These results indicate that errors incurred from the small damping approximation are within the tolerance of normal engineering accuracy when the true damping is less than 10%.
2.2 The Effect of Additive Noise on the Parameter Estimators

Thus far the half-power bandwidth damping estimator has been established in terms of an ideal single degree-of-freedom system. In practice, however, one must contend with the presence of noise in the response measurements. This effect is commonly modeled as shown in Figure 2.2. It is assumed that \( n(t) \) is a zero-mean, white noise process that is uncorrelated with \( x(t) \). Consequently, the output spectrum of the system shown in Figure 2.2 is given by

\[
S_z(f) = |H(f)|^2 S_x(f) + N_0
\]  

(2.13)

where: \( N_0 = \text{intensity of the noise spectrum, } S_n(f) \)

If it is further assumed that the input is white noise with intensity \( S_o \), then equation (2.13) simplifies to
Before proceeding with the analysis of this model, it is useful to note the similarity between equations (2.14) and (1.19). The correspondence between these two forms indicates that neighboring resonances have the same effect on the shape of a resonant peak as the contamination of the measured response with an equivalent amount of additive noise. Consequently, the analysis to follow accounts for measurement noise as well as the effect of surrounding resonant peaks.

The bias in half-power damping estimates that results from the presence of additive noise can be evaluated by considering a single degree-of-freedom system with an input and output of force and acceleration respectively. Substituting the transfer function for the system, given in equation (2.11b), into equation (2.14) yields

\[
S_z(f) = |H(f)|^2 S_0 + N_0
\]  

(2.14)

Observe that

\[
\frac{dS_z(f)}{df} = \frac{dS_y(f)}{df}
\]  

(2.16)

reveals that the maximum response natural frequency estimate is unaffected
by the presence of additive noise. Consequently, the half-power bandwidth damping estimate can be computed in the usual manner from the relation

\[ S_z(f) = \frac{1}{2} S_z(\hat{f}_n) \]  

(2.17)

where \( \hat{f}_n \) is the frequency of maximum response. Substituting equation (2.15) into equation (2.17) and simplifying gives

\[
\left( f^4 + 2(2\xi^2 - 1)f_n^2 f^2 + f_n^4 \right) \left( 1 - \frac{4\xi^2(1-\xi^2)N_0}{C_1S_0} \right) - 8\xi^2(1-\xi^2)f^4 = 0 
\]  

(2.18)

where: \( C_1 = \left(\frac{1}{4\pi^2M}\right)^2 \)

A convenient measure of the noise content in the observed response \( z(t) \) is the signal to noise ratio defined by equation (2.19). Evaluating the signal to noise ratio for the current system and substituting into equation (2.18) produces the expression shown in equation (2.20).

\[
\left( f_n^4 + 2(2\xi^2 - 1)f_n^2 f^2 + f_n^4 \right) \left( \text{SNR} - 2/\text{SNR} - 1 \right) - 8\xi^2(1-\xi^2)f_n^4 = 0
\]

(2.20)

After determining the roots of equation (2.20) and evaluating the half-power bandwidth damping estimate, the final result is given by equation (2.21).
where: $\beta = \text{SNR} - 2/\text{SNR} - 1$

This expression is an exact solution for the half-power bandwidth damping estimates that would be obtained in the presence of additive noise. The implications of this result are demonstrated in Figure 2.3 where the damping estimation error, $((\hat{\xi}_{HP} - \xi)/\xi)$, is plotted against the signal-to-noise ratio expressed in decibels (SNR$_{\text{dB}} = 10 \log(\text{SNR})$). By varying the amount of damping in the system, one finds that the damping estimates are generally inflated by the presence of additive noise. Figure 2.3 also reveals that signal-to-noise ratios below 15 dB will generally produce estimation errors in excess of 5% in lightly damped systems. Conversely, one also finds that damping estimates are insensitive to noise contamination as long as the signal-to-noise ratio is large ($\geq 30$ dB). This assertion is demonstrated in Figure 2.3 by convergence of estimation errors to the values predicted for use of the small damping approximation.

These observations point out the necessity for caution when estimating damping from resonant peaks which lie close to the noise floor. In general, the response of an offshore structure in the fundamental flexure-torsion mode group exhibits such a large signal-to-noise ratio that one need not be concerned with these effects. However, the response of the
FIG. 2.3 ERROR IN THE HALF-POWER BANDWIDTH DAMPING ESTIMATE CAUSED BY ADDITIVE NOISE
higher order mode groups generally exhibit substantially lower signal-to-noise ratios. Consequently, one must recognize the possibility for large bias in the damping estimates for these modes.
2.3 The Effect of a Linearly Varying Excitation Spectrum on the Parameter Estimators

In the development of methods for estimating natural frequencies and dampings from ambient response spectra, it was necessary to assume that the excitation was locally white. This means that the modal force spectra must be uniform over the bandwidth utilized in the estimation algorithm. In all but contrived cases, local whiteness is an approximation and as such introduces some error into the estimates. To examine and quantify this source of error, a more reasonable form for the force spectrum will be used to determine the response spectrum of a single degree-of-freedom system. Thus when the standard estimators are applied to the more realistic response spectrum, the error introduced by a non-white force spectrum will be established.

In selecting the force spectrum to be used in this analysis, it was recognized that in most cases a locally non-white spectrum could be adequately approximated by a sloping straight line. For example, Figure 2.4 shows a typical Pierson-Moskowitz sea spectrum superposed on the squared-magnitude of a typical lightly damped transfer function. This figure demonstrates that a linearly varying spectrum provides a reasonable model for the behavior of the excitation spectrum in the vicinity of a resonant peak. In view of the uncertainty in actual excitation spectra, it was decided that a more complicated model would be of little value. The force spectrum, \( S_x(f) \), used in this analysis is given by equation (2.22). This form was chosen such that the resonant response amplitude is
FIG. 2.4 TYPICAL PIETSON-MOSKOWITZ SEA SPECTRUM AND STRUCTURAL TRANSFER FUNCTION
\[ S_x(f) = m(f - f_n) + S_0 \] \hspace{1cm} (2.22)

independent of the spectrum slope, \( m \), when \( f = f_n \) and the input spectrum degenerates to white noise with intensity, \( S_0 \), when the slope equals zero. As in the proceeding section, the output of the single degree-of-freedom system is acceleration. Consequently, the response spectrum can be written as

\[ S_y(\Omega) \propto \frac{[\gamma(\Omega - 1) + 1] \Omega^4}{(\Omega^2 - 1)^2 + 4\xi^2\Omega^2} \] \hspace{1cm} (2.23)

where: \( \Omega = f/f_n \)
\[ \gamma = mf_n/S_0 \]

As usual, the natural frequency is estimated from the response spectrum as the frequency corresponding to the maximum spectral ordinate. Thus the estimate of the natural frequency is defined as a solution of equation (2.24). Substituting equation (2.23) into equation (2.24) and simplifying

\[ \frac{dS_y(\Omega)}{d\Omega} = 0 \] \hspace{1cm} (2.24)

the result of the specified differentiation yields equation (2.25).

\[ M_e\dot{\Omega}^2 + 6M_e(2\xi^2 - 1)\Omega^3 + 4(2\xi^2 - 1)(2\xi - M_e)\dot{\Omega}^2 + 5M_e\dot{\Omega} + 4(2\xi - M_e) = 0 \] \hspace{1cm} (2.25)
where: \( \Omega = f/f_n \)

\[
M_e = 2m\xi f_n \quad = \frac{mB}{S_0} \quad = \frac{\Delta S}{S_0}
\]

The real solutions of this fifth-order polynomial identify frequencies which correspond to extrema of the response spectrum. Similarly, given \( \Omega_0 \), the frequency of maximum response, one can obtain the corresponding half-power frequencies, \( \Omega_{+/-} \), as the solutions of

\[
S_y(\Omega) = \frac{1}{2} S_y(\Omega_0)
\]  \hspace{1cm} (2.26)

After substituting equation (2.23) into equation (2.26) and simplifying, it is found that the half-power frequencies are determined from the real solutions of equation (2.27). The form of equations (2.25) and (2.27)

\[
M_e \Omega^5 + [(2 - S_y(\Omega_0) \xi - M_e) \Omega^4 - 2 \xi S_y(\Omega_0)(2 \xi^2 - 1) \Omega^3 - \xi S_y(\Omega_0) = 0 \]  \hspace{1cm} (2.27)

reveals that the natural frequency and half-power frequency estimates are a function of the natural frequency, \( f_n \), the damping ratio, \( \xi \), and effective slope, \( M_e \). An interpretation of the effective slope can be obtained with the aid of Figure 2.5. By plotting the excitation spectrum on top of the squared magnitude of the transfer function, it is found that \( M_e \) represents the ratio of the change in the excitation spectrum across the half-power band, \( \Delta S \), to the level of excitation at resonance, \( S_0 \). Although the effective slope can take on any value in the range \( \pm \infty \), the preceding interpretation leads one to expect realistic values to fall in
FIG. 2.5 GEOMETRIC INTERPRETATION OF EFFECTIVE SLOPE

\[ S_x(t) = m(f-f_0) + S_0 \]
Estimates of the natural frequencies and the half-power frequencies were obtained from equations (2.25) and (2.27) respectively, using a standard computer program to evaluate roots of a polynomial. Solutions were obtained for values of $M_0$ between $\pm 0.5$ and damping ratios of 1%, 3%, and 5%. The results are presented in Figures 2.6a and b. These figures reveal the functional relationship between the parameter estimation errors (defined as in previous sections) and the effective slope of the excitation spectrum. Perhaps the most important result of this analysis is the somewhat surprising sensitivity of damping estimates to variation in the excitation spectrum. In general, one is usually content with the assumption of a "locally white excitation" when it is felt that the excitation spectrum is "reasonably smooth". However, the results of Figure (2.6b) point out that damping estimates may be biased in excess of 5% without warning even if the spectrum is smooth. To incur a bias of this magnitude the analysis indicates that the effective slope of the excitation spectrum must be large. This may occur when either the actual slope of the input spectrum is large, giving rise to a large change, $\Delta S$, over the half-power band, or when the magnitude of the spectrum, $S_0$, at the natural frequency, $f_n$, is very low. However, compared to the other sources of damping estimation error, the effect of a non-white input spectrum is reasonably small.
FIG. 2.6  ERROR IN PARAMETER ESTIMATES CAUSED BY LINEARLY VARYING EXCITATION SPECTRUM
2.4 The Effect of Closely Spaced Natural Frequencies on the Parameter Estimators

The preceding analyses have investigated biases in parameter estimates made from the response of a single degree-of-freedom system. These cases, while providing analytical tractability, do not reflect the complex spectral features of any "interesting" structural response. This section will investigate biasing of natural frequency and half-power bandwidth damping estimates by the proximity of two resonant peaks. To examine and quantify this effect, the acceleration response of a two degree-of-freedom system is analyzed for the case when the forcing functions are mutually uncorrelated, white noise. This assumption greatly simplifies the problem and permits the study of bias in the damping estimate independent of cross-spectral effects. Utilizing the results of section 1.2.2, it is easily shown that the acceleration response spectrum for the two degree-of-freedom system can be written as

\[ S_y(f) = \frac{f^4}{(f^2 - f_i^2)^2 + (2\xi_if_i^2)^2} + \frac{cf^4}{(f^2 - f_2^2)^2 + (2\xi_2f_2^2)^2} \]  

(2.28)

where:

- \( f_i \) = natural frequency of the ith mode
- \( \xi_i \) = damping ratio of the ith mode
- \( c \) = scale factor

This simplified form of the response spectrum can be utilized without loss of generality since the natural frequency and damping estimators only
use the shape of the spectrum to obtain estimates. Estimates of the two natural frequencies are obtained as solutions of

\[
\frac{dS_y(f)}{df} = 0 \tag{2.29}
\]

After substituting equation (2.28) into equation (2.29), differentiating and simplifying, one obtains the following result:

\[
a_5 \beta^5 + a_4 \beta^4 + a_3 \beta^3 + a_2 \beta^2 + a_1 \beta + a_0 = 0 \tag{2.30}
\]

where: \( \beta = \frac{f^2}{f_1^2} \)

\[
a_5 = d_1 + cd_2
\]
\[
a_4 = 4d_1 d_2 r^2 (c+1) + cr^4 + 1
\]
\[
a_3 = 2r^2 [(2d_2^2 + 1 + 2c)d_1 r^2 + (2cd_1^2 + c + 2)d_2]
\]
\[
a_2 = 2r^2 [2d_1 d_2 (r^4 + c) + (2d_2^2 + 1)r^2 + (2d_1^2 + 1)cr^2]
\]
\[
a_1 = r^2 [(r^4 + 4c)d_1 r^2 + (4r^4 + c)d_2]
\]
\[
a_0 = r^4 (r^4 + c)
\]
\[
d_1 = 2\xi_1^2 - 1
\]
\[
d = 2\xi_2^2 - 1
\]
\[
r = \frac{f^2}{f_2^2}
\]

The real roots of this fifth-order polynomial are frequencies which correspond to extrema of the spectrum. Consequently, one need only solve
equation (2.30) with standard numerical methods in order to obtain the frequencies one would measure in estimating the natural frequencies.

Having obtained the frequencies of maximum response, \( \hat{f}_1 \) and \( \hat{f}_2 \), one can compute the half-power damping estimates from the half-power frequencies defined by

\[
S_y(f) = \frac{1}{2} S_y(\hat{f}_i) ; \quad i = 1, 2
\]  

(2.31)

Substituting equation (2.28) into equation (2.31) and simplifying yields the final result shown in equation (2.32). The real solutions of this fourth-order polynomial contain the half-power frequencies of the resonant peak located at the ith natural frequency. Reviewing the variables in equations (2.30) and (2.32) it is found that the natural frequency and

\[
a_4 \beta^4 + a_3 \beta^3 + a_2 \beta^2 + a_1 \beta + a_0 = 0
\]  

(2.32)

where:

\[
a_4 = \frac{1}{2} S_y(\hat{f}_4) - c - 1
\]
\[
a_3 = 2[ d \left( \frac{1}{2} S_y(\hat{f}_3) - c \right) + d_2 \left( \frac{1}{2} S_y(\hat{f}_2) - 1 \right) r^2 ]
\]
\[
a_2 = \frac{1}{2} S_y(\hat{f}_2) [r^4 + 4d_1 d_2 r^2 + 1] - c - r^4
\]
\[
a_1 = S_y(\hat{f}_1) r^2 [d_1 r^2 + d_2]
\]
\[
a_0 = \frac{1}{2} S_y(\hat{f}_1) r^4
\]
\[
d_1 = 2\xi_1^2 - 1
\]
\[
d_2 = 2\xi_2^2 - 1
\]
\[
r = \frac{f_2}{\hat{f}_1}
\]
\[
\beta = \frac{f_2}{\hat{f}_1^2}
\]
damping estimates are dependent upon the true damping ratios, the ratio of the natural frequencies, $r$, and the scale factor, $c$, for the response of the second mode. To investigate the role each variable plays in biasing the parameter estimates, two special cases are studied. The first case delineates the importance of modal separation when the spectrum contains two equal amplitude peaks. This analysis is accomplished by computing parameter estimates for different values of $r$, while adjusting $c$ to maintain equal amplitude resonant peaks (i.e. $S(f_1) = S(f_2)$). To simplify the interpretation of the results, modal damping ratios are set equal to each other. The results of this analysis are presented in Figures 2.7 and 2.8 in the form of curves which describe the relationship between the estimation errors and the modal separation for different amounts of damping. The minimum natural frequency ratio used with each curve is determined by the natural frequency ratio for which the half-power band of the particular mode is not obscured. Noting that each curve represents a clearly defined parameter estimate, it is concluded that the existence of measurable half-power bands is not a guarantee of unbiased natural frequency and damping estimates. Measurable damping estimates may easily be high by as much as 25% for true damping of 3% or less. Natural frequency estimates, however, seem to suffer less from the interaction with an adjacent mode and exhibit errors of less than 1%. One should also observe that in each case considered, the modal interaction consistently produces damping estimates which exceed the true value. In all cases, the estimation error dropped significantly as the separation between modes increased. It is interesting to note that the asymptotic values of the estimation
FIG. 2.7  ERROR IN THE PARAMETER ESTIMATES FOR 1ST MODE
AS A FUNCTION OF MODAL SEPARATION
FIG. 2.8 ERROR IN THE PARAMETER ESTIMATES FOR 2ND MODE AS A FUNCTION OF MODAL SEPARATION
errors tend toward the theoretical minimum for the first mode while the estimates for the second mode converge to a somewhat higher value. This behavior is understood when it is recalled that the first term of equation (2.28) becomes a constant at frequencies much larger than the natural frequency. Thus the response spectrum in the vicinity of the second natural frequency is equivalent to that of a single degree-of-freedom system with additive noise. Consequently, asymptotic parameter estimation errors will be governed by the effective signal-to-noise ratio as discussed in Section 2.2.

The second and final case considered in this section examines the effect of unequal resonant responses on the parameter estimates. In this example, the relative contribution of the two modes is characterized by the modal contribution ratio (MCR) defined in decibels as

\[
MCR = 10 \log \frac{|H_1(f_1)|^2}{c|H_2(f_2)|^2}
\]  

(2.33)

where: \( H_i(f) \) = transfer function for the i\( \text{th} \) mode

The analysis is conducted with both of the modal damping ratios set equal to 3%. The results of this study are presented in Figures 2.9 and 2.10 where each curve describes the relationship between the estimation error and the modal contribution ratio for a given modal separation. The smallest value of the MCR which allows an estimate of the first damping ratio corresponds to the appearance of a half-power band which is not obscured by the resonant peak of the second mode. Similarly, the maximum value of the MCR
used in estimating the natural frequency and damping of the second mode corresponds to the disappearance of a measurable spectral peak and half-power band respectively. For the case where $r$ equals 1.05, the figures reveal that damping estimation for the first mode commences with the disappearance of the second spectral peak. Thus one finds that extremely large biases can occur in damping estimates for the first mode as a result of interaction with an unseen second mode. Furthermore, the damping estimation error for this case does not reach tolerable levels until the modal contribution ratio exceeds 15 dB. One finds, however, that as the modal separation increases, the second resonant peak remains visible over the range of significant bias thereby providing warning of possible error. When the natural frequency ratio increases to 1.2, it becomes possible to estimate damping ratios for the second mode. In this case, the shoulder of the first resonant peak becomes the noise floor for the second resonant peak. Thus the sizeable estimation errors observed in Figure 2.10b result from a marked decrease in the effective signal-to-noise ratio that occurs with increasing MCR.

The results of this analysis demonstrate the potential difficulties that arise in making damping estimates from the response spectra of a structure with two nearly coincident natural frequencies. Such is the case for the lowest broadside and end-on flexural modes of offshore platforms. To avoid eventual estimation errors, great care must be taken to align accelerometers on the structure so as to obtain records which reflect nearly pure response of only one mode. The results presented in Figure 2.9b indicate that the potential for very large damping estimation errors exists even for small deviations from a unimodal response measurement.
FIG. 2.9 ERROR IN THE PARAMETER ESTIMATES FOR 1ST MODE AS A FUNCTION OF MODAL CONTRIBUTION
FIG. 2.10 ERROR IN THE PARAMETER ESTIMATES FOR 2ND MODE AS A FUNCTION OF MODAL CONTRIBUTION RATIO

\[ \tilde{\xi}_1 = \tilde{\xi}_2 = 3 \% \]

\[ r = 1.2 \]

\[ r = 1.05 \]

\[ r = 1.1 \]
CHAPTER 3

The MEM Parameter Estimators

3.1 Introduction

This chapter introduces a new method of estimating natural frequencies and damping ratios based on the combination of the conventional natural frequency and damping estimators discussed in the previous chapters, with the maximum entropy method of spectral estimation (MEM). This technique, while capitalizing on MEM's inherent ability to provide smooth, highly resolved response spectra from short-time histories, also furnishes approximations for the bias and variance of its estimates. Prior to the development of these estimators in Sections 3.3 and 3.4, an extensive introduction to MEM will be provided in view of its relative newness to the field of vibration testing. In an attempt to obviate an extensive background in time series analysis and digital signal processing, the development will favor intuitive arguments over mathematical vigor. For completeness, however, an extensive bibliography is included so that the interested reader can draw from the rich background material available to build a working knowledge of the method.
3.2 The Maximum Entropy Method of Spectral Estimation

3.2.1 The Concept

To introduce the philosophy behind the maximum entropy method, it is convenient to describe MEM in terms of its similarity with the Blackman-Tukey spectral estimator. It is recalled that the Blackman-Tukey method computes spectral estimates as the Fourier Transform of autocorrelation functions estimated from finite duration time histories. Spectra computed with the Blackman-Tukey method were shown in Section 1.2 to be inherently limited in resolution as a result of the required truncation of the estimated autocorrelation function and its implicit extension of the truncated function to infinite lag with zeros. The Maximum entropy method, on the other hand, seeks to improve resolution by analytically extending the autocorrelation function from its truncation point to infinite lag in a manner consistent with the observed random process. It is this extension of the autocorrelation function which forms the basis of the maximum entropy spectral estimate.

The extension of the autocorrelation function which MEM achieves can be viewed as the result of fitting a special model to a finite portion of the known autocorrelation function. Correspondingly, MEM provides an analytic means of extrapolating from "p + 1" samples of a known autocorrelation function, R(k), k=0, 1, ... p, to obtain the remaining values, R(k), k = p + 1, ... \(\infty\), while requiring that the Fourier Transform of the resulting infinite duration autocorrelation
function always be positive. This last condition guarantees that the function created by the MEM extension is acceptable as an autocorrelation function in that it corresponds to a non-negative power spectrum. While one can imagine an uncountably large number of candidate functions that are capable of satisfying the modelling requirements, MEM chooses from among these, the function that introduces the least number of artificial features into the power spectrum. In this way the MEM spectrum attains superior resolution over conventional methods while preserving the character of the underlying random process.

In a superb mathematical development, Burg [16] demonstrates that the maximum entropy extension of the autocorrelation function is given by the following recursion equation:

\[ R(k) = -A_1 R(k-1) - A_2 R(k-2) - \ldots - A_p R(k-p) \quad ; \quad k > 0 \quad (3.1) \]

The parameters \{A\} are obtained as the solution of equation (3.2) which is commonly known as the Yule-Walker equations. Thus given \( p+1 \) lags of

\[
\begin{bmatrix}
R(0) & R(1) & \cdots & \cdots & R(p-1) \\
R(1) & R(0) & \cdots & & \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
R(p-1) & R(1) & R(0) & \cdots & \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_p \\
\end{bmatrix}
= 
\begin{bmatrix}
R(1) \\
R(2) \\
\vdots \\
R(p) \\
\end{bmatrix}
\quad (3.2)
\]
an autocorrelation function, one can uniquely define the pth order MEM coefficients $A_1, \ldots, A_p$ and subsequently generate the extended autocorrelation function. While the infinite duration autocorrelation function determined by equations (3.1) and (3.2) provides an appealing conceptual framework for MEM, one need never actually compute the extended function in the course of determining the power spectrum. Instead, it can be shown that the pth order MEM spectrum is uniquely defined as

$$S(f) = \frac{\sigma_p^2 \Delta}{|1 + \sum_{k=1}^{p} A_k e^{-j2\pi kf\Delta}|^2}$$

(3.3)

where: $A_k$ are the pth order MEM coefficients

$$\sigma_p^2 = R(0) - \sum_{k=1}^{p} A_k R(k)$$

$\Delta$ is the sampling increment

$$j = \sqrt{-1}$$

An alternative interpretation of the maximum entropy spectral estimation is provided by the equivalence noted by van der Bos [17] between MEM and autoregressive (AR) modelling of a random process. Accordingly, a random process is modelled as the output of the system defined by

$$x(n) = -\sum_{k=1}^{p} A_k x(n-k) + w(n)$$

(3.4)
where: \( x(n) \) is the model's representation of the process
\( A_k \) are the coefficients of the AR model
\( w(n) \) is discrete-time white noise
\( p \) is the order of the AR model

The correspondence between MEM and AR modelling becomes apparent when one discovers that the Yule-Walker equations (equation (3.2)), are obtained in the course of estimating the AR coefficients by minimizing the mean-square error between the observed random process and that predicted by the AR model. Further, equation (3.1) which defines the MEM extension of an autocorrelation function, is identical to the relation governing the theoretical autocorrelation function of a pth order AR process. Consequently, the spectra obtained from a pth order AR process and a pth MEM extension of an autocorrelation function are one and the same. It is for this reason that MEM is frequently referred to as autoregressive spectral estimation.

One of the most frequently encountered uses of the AR model is as a system for the removal of correlation from an arbitrary time history, commonly known as a whitening filter (also called a prediction error filter). The manner in which the AR formulation accomplishes this task is demonstrated by recasting equation (3.4) in the form

\[
e(n) = x(n) + \sum_{k=1}^{p} A_k x(n-k) \quad (3.5)
\]

In this expression, \( x(n) \) is the current observation of the random process,
the summation term represents the optimum prediction of the current observation based on a linear combination of the past "p" values of the random process and e(n) is the error between the predicted and observed value. It can be shown that as the order of the AR model is increased, the mean-square of the prediction error decreases to an approximately asymptotic value, at which time the output of the system, e(n) is approximately white (i.e., uncorrelated). When this operation is viewed in the frequency domain, it is found that the squared-magnitude of this system's transfer function must be an exact inverse of the input power spectrum in order to output white noise. Consequently, once the optimum whitening filter is found, its transfer function provides an estimate of the desired power spectrum. This interpretation of MEM in terms of a whitening filter has played an important role in the advancement of MEM spectral estimation and is frequently encountered in discussions concerning the selection of "p", the order of the MEM spectrum.
3.2.2 Implementation of the Spectral Estimator

The description of MEM given thus far has tacitly assumed availability of the exact autocorrelation function for the underlying random process. In practice, one only has a finite duration observation of the random process with which to obtain an approximation of the exact MEM coefficients. Consequently, certain estimation algorithms must be employed to obtain estimates of the MEM coefficients, thereby specifying the spectral estimate.

One of the most widely used methods for estimating the MEM spectrum is known as the correlation method. This scheme obtains its estimates of the MEM coefficients as the solution of the Yule-Walker equation (equation (3.2)) using estimates of the autocorrelation function in place of the true values. The estimate of the autocorrelation function used for this purpose is obtained with the lagged-product formula, equation (1.3) discussed in connection with the Blackman-Tukey spectral estimate. While a "brute-force" solution of the Yule-Walker equations can be accomplished by general numerical techniques (e.g. Gaussian elimination), another more efficient method exists which takes advantage of the special diagonal symmetry of the autocorrelation matrix. This technique, developed by Levinson [18] and simplified by Durbin [19] makes use of the so-called Toeplitz form to express the (p+1)th order MEM coefficients in terms of the pth order coefficients. In this way, the desired solution is obtained by recursively generating MEM coefficients of increasing order until arriving at the specified order. Durbin's recursive scheme is summarized
as follows:

\[
\sigma_0^2 = \hat{R}(0) \tag{3.6a}
\]

\[
k_{p+1} = -\left[\hat{R}(p+1) - \sum_{\lambda=1}^{p} A_\lambda^{(p)} \hat{R}(p+1-\lambda)\right]/\sigma_p^2 \tag{3.6b}
\]

\[
\begin{bmatrix}
A_1^{(p+1)} \\
A_2^{(p+1)} \\
\vdots \\
A_p^{(p+1)}
\end{bmatrix}
= \begin{bmatrix}
A_1^{(p)} \\
A_2^{(p)} \\
\vdots \\
A_p^{(p)}
\end{bmatrix}
+ k_{p+1}
\begin{bmatrix}
A_1^{(p)} \\
A_2^{(p)} \\
\vdots \\
A_p^{(p)}
\end{bmatrix} \tag{3.6c}
\]

\[
\sigma_{p+1}^2 = (1 - k_{p+1}^2) \sigma_p^2 \tag{3.6d}
\]

It can be shown that the recursive estimation of pth order MEM coefficients requires approximately \(p^2\) operations (multiplications and divisions) and 2\(p + 1\) memory locations while solution by Gaussian elimination would require approximately \(p^3\) operations and \(p^2\) memory locations. It should be noted, however, that in those cases where the number of data points greatly exceeds the desired order of the MEM spectrum, the major computational load comes from the \(n \cdot p\) operations required to estimate the
autocorrelation function rather than determination of the MEM coefficients.

Another prominent estimation technique in use today was developed by Burg [16] in an effort to estimate MEM coefficients in a manner consistent with the philosophy of MEM. In Burg's thesis, it is pointed out that use of the correlation method requires the implicit extension of the observed data with zeros in order to estimate the autocorrelation function. This observation is demonstrated by equation (1.3). In view of the fact that MEM attempts to rectify problems encountered in appending zeros to an autocorrelation function, it would seem inconsistent to implement MEM using the similarly poor practice of appending zeros to the data. To remedy this situation, Burg suggested a "Levinson-type", recursive algorithm which operates directly on the observed time series without implicit assumption regarding the unobserved data. The key to this algorithm is the clever use of the predictive-error filter in the determination of the so-called reflection coefficients, \( k_i \), \( i=1, p \), encountered in the Levinson algorithm, equation (3.6). A review of equation (3.6) reveals that once determined, the reflection coefficients \( k_1, \ldots, k_p \), completely define the MEM coefficients of all orders up to and including \( p \), without the use of the autocorrelation function (except \( \hat{R}(0) \)).

As yet, no decisive evidence has been presented in the literature to indicate the superiority of one method over the other. However, experience indicates that the enhanced resolution obtained with the Burg method is accompanied by the tendency for tone splitting (the representation of one peak as two). It has also been noted that
differences between these two methods (as well as other methods) disappear as the data length increases relative to the order of the MEM spectrum.

The final step in the implementation of MEM is the actual evaluation of the spectral estimate given by equation (3.3) utilizing the estimated MEM coefficients. This can be accomplished efficiently by noting the similarity between the denominator of equation (3.3) and the discrete Fourier Transform which can be written as shown in equation (3.7).

\[
\text{DFT} \{x(k)\} = \frac{1}{M} \left[ x(0) + \sum_{k=1}^{M-1} x(k)e^{-j2\pi km/M} \right]; \quad m=0, M-1 \tag{3.7}
\]

where: \( x(k) \) is a time series of length \( M \)

Accordingly, the denominator of equation (3.3) can be evaluated at the frequencies \( f = m/p\Delta; \quad m=0, \ldots, p-1 \) with a single discrete Fourier Transform (accomplished with an FFT) of the sequence \( \{1, A_1, \ldots, A_p\} \). However, the frequency increment, \( 1/p\Delta \), will in general be too large to be of any value. This easily remedied by appending \( M-p \) zeros to the end of the sequence \( \{1, A_1, \ldots, A_p\} \) as shown in equation (3.8), thereby reducing the frequency increment to \( 1/M\Delta \).

\[
\text{DFT} \{a_k\} = \left[ 1 + \sum_{k=1}^{p} a_k e^{-j2\pi df\Delta} \right], f=\frac{m}{M\Delta}, \quad m=0, \ldots, M-1 \tag{3.8}
\]
where:  \[ a_k = \begin{cases} 
1, & k=0 \\
A_k, & k=1, \ldots, p \\
0, & k=p+1, \ldots, M-1
\end{cases} \]

It is important to exercise care in choosing the frequency increment since too small a value can result in the omission of important features in the plot of the spectrum. The author has found \( M = 4096 \) more than adequate for most work in vibration measurement of offshore platforms.
3.2.3 Selection of the Optimum Order

An issue which must be dealt with in the course of computing MEM spectral estimates is the selection of the order which best matches the MEM spectrum to the true spectrum. In theory, the problem is trivial since the fidelity of an MEM spectrum increases until, in the limit as the order approaches infinity, the MEM spectrum is identically equal to the true spectrum. In practice, a finite portion of the random process is used to estimate the MEM coefficients thereby introducing statistical uncertainty into the computed spectrum. Consequently, as the order of the spectral estimate is increased one observes a classical decrease in statistical stability (i.e., increased variance) accompanied by an increase in resolution. This trade-off between resolution and variance, while similar in nature to that of conventional spectral estimates, is made more difficult by the lack of simple "rules of thumb" to aid in the decision-making process. Instead, the optimum order is chosen with the aid of various order selection criteria, tempered by insight into the characteristics of the true spectrum.

One type of selection criteria involves hypothesis tests based on theoretical properties of AR processes. Included in the criteria of this type are prediction-error and reflection coefficient tests. Simply stated, these tests utilize the theoretical result that; given that the observed data is from a $p_0$ th order AR process, then the mean-square prediction error, $\sigma_p^2$ and the reflection coefficients satisfy the following relations:
\[
\sigma_p^2 = \sigma_{p_0}^2 \quad \text{for } p \geq p_0 \quad (3.9a)
\]
\[
k_p = 0 \quad \text{for } p > p_0 \quad (3.9b)
\]

In theory, the observance of these conditions in the computed mean-square prediction error and reflection coefficients indicates the proper order to be used in the spectral estimate. In the use of these relations with real data, one must account for the statistical nature of the estimated mean-square prediction error and reflection coefficients in the formulation of the test. For instance, it can be shown that if an AR process is of order \( p_0 \) then the reflection coefficients for \( p > p_0 \) are zero mean, normal random variables with variance approximately equal to the inverse of the number of data points, \( N \). This suggests the use of a threshold test of the form

\[
|k_p| < \frac{2}{\sqrt{N}} \quad (3.10)
\]

This test is performed by a search for the smallest order after which virtually all reflection coefficients stay within the two standard deviation band defined by equation (3.10). Similarly, the prediction error test is adjusted for use with real data by searching for the first 'significant plateau' in the mean-square vs. order curve. This suggests the use of the threshold test shown in equation (3.11) where \( \varepsilon \) is a
small number. This test must be passed for several consecutive orders

\[
1 - \frac{\sigma^2}{\sigma^2_P + 1} < \varepsilon
\]  

(3.11)

before declaring that the plateau has been located. Note that this test is easily "tuned" by the selection of \( \varepsilon \) and the number of consecutive tests to be passed. Thus as the analyst gains more experience with a specific type of data, the test criteria can be updated. The author has used \( \varepsilon = 0.001 \) and 5 consecutive tests with varying degrees of success.

A second type of order selection criteria involves the minimization of a function which formally expresses a particular trade-off between resolution and variance of the spectral estimate. Two of the most widely used criteria are Akaike's Final Prediction Error (FPE) and Information Theoretic Criterion (AIC) shown in equations (3.12a and b) respectively.

\[
FPE(p) = \ln\left(\frac{N + 1 + p}{N - 1 - p}\right) \frac{\sigma^2}{p} 
\]  

(3.12a)

\[
AIC(p) = \ln(\frac{\sigma^2}{p}) + 2 \frac{p}{N}
\]  

(3.12b)

In practice these "cost functions" are evaluated at all orders up to the maximum order of interest and the optimum order is selected as the order corresponding to the absolute minimum of the function. Akaike has suggested
that the maximum order to be used in these criteria should be less than $3\sqrt{N}$. In view of the fact that these criteria have been used with varying degrees of success, the orders specified by these functions should be viewed more as guides than absolute indicators. It is suggested that spectra should be computed at a number of different orders bracketing a proposed order thereby revealing trends in convergence. This method of order selection is encountered in conventional spectral estimation and has proved to be satisfactory.
3.2.4 Performance of MEM with Theoretical Autocorrelations

To gain insight into the properties of the MEM spectral estimator, it is often useful to obtain spectral estimates based on exact theoretical values of the autocorrelation function rather than estimates from simulated data. In this way, the spectral estimates can be compared with theoretical spectra without concern over sampling properties.

The first example, which was reported by Lacoss [20], involves the spectrum of two sinusoids in a background of white noise. In this case the spectral estimates were obtained from eleven samples of the prescribed autocorrelation function given by equation (3.13). The eleven samples were taken at intervals of \( \Delta = 1 \) second resulting in a total lag of ten seconds and a conventionally expected resolution of 0.1 Hz. Spectral estimates were obtained with the Blackman-Tukey method (using a Bartlett window) and MEM. The resulting spectra shown in Figure 3.1 have been normalized to their peak values for the purpose of comparison. This figure indicates that while the Blackman-Tukey method is unable to resolve the two spectral lines, MEM clearly reveals two peaks in the estimated spectrum. It is also important to note that both methods are

\[
R(k) = 5.33 \cos(0.3\pi k) + 10.66 \cos(0.4\pi k) + \delta_k
\]  \hspace{1cm} (3.13)

where: \( \delta_k \) is the Kronecker delta function
FIG. 3.1 RESOLVING CAPABILITY OF SPECTRAL ESTIMATORS
subject to the spurious ripple known as sidelobes. Another point that should be mentioned is that MEM actually calculates the power spectral density. Thus in order to judge the relative power in the two peaks it is necessary to integrate the spectrum under the peaks. With conventional methods the power is directly reflected in each point of the spectrum, and a summation is all that is required to obtain the total power.

As a second example, the theoretical autocorrelation function for the response of a single degree-of-freedom system excited by white noise was evaluated by the same spectral estimators. The damping was selected to be 1% of critical, the natural frequency was 10Hz, and a signal-to-noise ratio of 30dB was adopted. As before, eleven lags were used with a one second sampling interval. The resulting spectra are shown in Figure 3.2 along with the true spectrum. These results clearly indicate the potential of MEM in estimating natural frequencies and dampings from response spectra. While the MEM estimate is virtually indistinguishable from the true response in the region of resonance, the Blackman-Tukey estimate exhibits an erroneously high damping as a result of the previously described smearing phenomenon. It may be argued that this example is based on such a small lag that it does not reflect the true power of conventional estimates with more realistic lags. While this observation may be justified, this example is intended to demonstrate the accuracy with which MEM fits the true spectrum, even with a very short total lag.
FIG. 3.2 SPECTRAL ESTIMATE OF SINGLE DOF RESPONSE TO WHITE NOISE
3.3 The MEM Natural Frequency Estimator

3.3.1 Derivation of the Estimator

The estimation of natural frequencies was described in Chapter 1 as the selection of frequencies corresponding to relative maxima of a response spectrum which have been identified as modes of vibration. When implemented with conventional spectral estimates, the natural frequency estimator must be interpreted graphically. That is, since conventional spectral estimates can only be evaluated at a finite set of frequencies, natural frequency estimation reduces to the problem of choosing spectral peaks and their corresponding frequencies from a graph (or tabular listing) of the spectrum. MEM, on the other hand, allows the natural frequency estimator to be applied analytically. To understand how this is possible, one need only recall the form of the pth order MEM spectrum shown in equation (3.3) and repeated here for convenience.

\[
S(f) = \frac{\sigma^2 P^2}{\left(1 + \sum_{k=1}^{P} A_k e^{-j2\pi k f P}\right)^2}
\]  

(3.3)

Assuming for the moment that the parameters \(\{A_k\}\) and \(\sigma^2\) are known, it is found that equation (3.3) is actually a closed-form analytic equation for the response spectrum. Consequently, the search for relative maxima of the spectrum can be accomplished with the aid of a classical result from
calculus which states the derivative of a function is zero at an extrema. Since the spectrum is available in functional form, one can locate any relative extrema of a power spectrum (i.e. relative maxima or minima) by solving

\[ \frac{dS(f)}{df} = 0 \]  

(3.14)

All frequencies which form the solution set of equation (3.14) must correspond to extrema of the spectrum. Thus, this set of frequencies, which will be known as the set of critical frequencies, contains the natural frequency estimates.

Before substituting the equation for the MEM spectrum into equation (3.14), it is convenient to recast equation (3.3) into a form more easily manipulated. The following identity is used for that purpose.

\[ \sum_{k=0}^{p} A_k e^{-j2\pi kf\Delta} = \rho_0 + 2 \sum_{k=1}^{p} \rho_k \cos(2\pi kf\Delta) \]  

(3.15)

where: \( A_0 = 1 \)

\[ \rho_k = \sum_{i=0}^{p-k} A_i A_{i+k} \]
The veracity of this identity can be demonstrated by a straightforward manipulation involving an interchange of summation order. Alternatively, if the coefficients $A_k$ are viewed as a time series, then $\rho_k$ can be interpreted as an autocorrelation function of the $A_k$'s. Thus equation (3.15) is simply a restatement of the well known relationship between the Fourier Transform of a time series (ie. $A_0, A_1, ..., A_\rho$) and the Fourier Transform of its autocorrelation function (ie, $\rho_k$). Utilizing the identity of equation (3.15), the $p$th order MEM spectrum can be rewritten as

$$S(f) = \frac{\sigma_p^2 \Delta}{\rho_0 + 2 \sum_{k=1}^p \rho_k \cos(2\pi kf\Delta)}$$

(3.16)

After substituting equation (3.16) into equation (3.14) and evaluating the derivative, the following result is obtained

$$\frac{\sum_{k=1}^p k \rho_k \sin(2\pi kf\Delta)}{\rho_0 + 2 \sum_{k=1}^p \rho_k \cos(2\pi kf\Delta)} = 0$$

(3.17)

One set of solutions for equation (3.17) is obtained by finding all frequencies that cause the denominator to become infinite. However, comparing the denominators of equations (3.17) and (3.3), it is found that any solution, $f_0$, obtained from the denominator of equation (3.17)
becoming infinite, corresponds to a zero in the spectrum (i.e. \( S(f_0) = 0 \)). Since this condition is of no value in estimating peaks of the spectrum, the numerator of equation (3.17) must include all solutions which correspond to maxima of the spectrum. Thus, any frequency which corresponds to a relative maximum of the spectrum must be a solution of equation (3.18).

\[
\sum_{k=1}^{p} k_p \sin(2\pi k f_0) = 0
\]  

(3.18)

Conversely, any frequency which is a solution of equation (3.18) must correspond to an extrema of the spectrum.

While the solution set of equation (3.18) includes both maxima and minima (excluding zeros of the spectrum) it is a relatively simple task to determine which type of extrema a given solution corresponds to. One way of checking the solution type is to simply plot the spectrum and verify the extrema graphically. Since spectral plots are generally desired anyway, this method of validation requires little or no extra effort. Another method of determining solution types uses the second derivative test for extrema. This test identifies maxima and minima of a function with positive and negative second derivatives of the function respectively. Accordingly, the second derivative of a pth order MEM spectrum evaluated at \( f_0 \), a solution of equation (3.18), is given by
\[
\frac{d^2 S(f)}{df^2} \bigg|_{f=f_0} = \frac{4\pi^2 \sigma^2 \Delta^2}{p} \sum_{k=1}^{p} k^2 \rho_k \cos(2\pikf_0 \Delta) \\
\left[ \rho_0 + 2 \sum_{k=1}^{p} \rho_k \cos(2\pikf_0 \Delta) \right]^2
\]

Since the denominator of equation (3.19) is always positive, the sign of the second derivative is decided by the numerator. Consequently, the second derivative test can be written as

\[
\sum_{k=1}^{p} k^2 \rho_k \cos(2\pikf_0 \Delta) \begin{cases} 
< 0 \rightarrow \text{relative maxima \ @ \ } f_0 \\
= 0 \rightarrow \text{indeterminate with this test} \\
> 0 \rightarrow \text{relative minima \ @ \ } f_0
\end{cases}
\]

This test has proved very useful in the estimation of natural frequencies in that solutions of equation (3.18) can be quickly and easily verified as a maxima without the use of graphics or interaction with the analyst.
3.3.2 Statistics of the MEM Natural Frequency Estimator

The preceding section described the development of an equation whose solutions are critical frequencies of an MEM power spectrum. In theory, it is known that the MEM spectrum is capable of modelling any spectrum with any desired goodness of fit and in the limit, as the order approaches infinity, the MEM spectrum will identically equal the true spectrum of the process. Consequently, critical frequencies as defined by equation (3.18) can be made arbitrarily close to the critical frequencies of the true spectrum by judicious choice of order. In practice, however, the MEM spectral estimate is limited in goodness of fit by the inherent bias and variance of the spectral estimate introduced as a result of using estimates of the MEM coefficients. In an attempt to characterize the impact of these corrupted spectral estimates on natural frequency estimates and aid in the selection of an optimum order, a first-order approximation of the expected value and variance of the MEM natural frequency estimator will be presented.

In considering the statistics of the MEM natural frequency estimator, it is useful to recast equation (3.18) in terms of the pth order MEM coefficients. This is accomplished by substituting the equation defining the coefficients \( \rho_k \) (see equation (3.15)) into equation (3.18). After interchanging the order of summations and simplifying, one obtains the following equivalent expression for the MEM natural frequency estimator

\[
\sum_{k=0}^{P} \sum_{l=0}^{P} kA_k A_l \sin[2\pi(l-k)f\Delta] = 0; \quad A_0 = 1
\] (3.21)
When utilizing equation (3.21) to obtain critical frequencies, $f_0$, one does not know the true values of \( A_k \) which correspond to the pth order MEM model of the random process. Instead, estimates of the MEM coefficients \( \hat{A}_k \) are used to determine estimates of the critical frequencies, $\hat{f}_0$. Consequently, the relationship between the estimated pth order MEM coefficients and the estimated critical frequencies of the spectrum defined by those coefficients is given by

$$
\sum_{k=0}^{p} \sum_{\ell=0}^{p} k\hat{A}_k \hat{A}_\ell \sin[2\pi(\ell-k)f_0 \Delta] = 0 \tag{3.22}
$$

To understand the properties of the critical frequency estimate, one can expand equation (3.22) about the true critical frequency of the pth order spectrum. This is accomplished by writing the estimated critical frequency as

$$
\hat{f}_0 = f_0 + \delta f_0 \tag{3.23}
$$

where $\delta f_0$ accounts for the variation of the estimate about the true value. Substituting equation (3.23) into equation (3.22) gives

$$
\sum_{k=0}^{p} \sum_{\ell=0}^{p} k\hat{A}_k \hat{A}_\ell \sin[2\pi(\ell-k)(f_0 + \delta f_0) \Delta] = 0 \tag{3.24}
$$
Rewriting equation (3.24) using the trigonometric identity for the sine of the sum of two angles yields

\[ \sum_{k=0}^{P} \sum_{\ell=0}^{P} k \hat{A}_k \hat{A}_\ell \{ \sin[2\pi(\ell-k)\delta f_0 \Delta] \cos[2\pi(\ell-k)\delta f_0 \Delta] \\ + \sin[2\pi(\ell-k)\delta f_0 \Delta] \cos[2\pi(\ell-k)\delta f_0 \Delta] \} = 0 \] (3.25)

If it is assumed that the error in the critical frequency estimate is small then the small angle approximations \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \) can be used to give the following result:

\[ \delta f_0 = -\frac{\sum_{k=0}^{P} \sum_{\ell=0}^{P} k \hat{A}_k \hat{A}_\ell \sin[2\pi(\ell-k)\delta f_0 \Delta]}{2\pi \Delta \sum_{k=0}^{P} \sum_{\ell=0}^{P} k(\ell-k) \hat{A}_k \hat{A}_\ell \cos[2\pi(\ell-k)\delta f_0 \Delta]} \] (3.26)

If one examines equation (3.26) together with equation (3.23) it is found that the estimate of the critical frequency has been expressed, to first-order, as a non-linear function of the estimates of the \( p \)th order MEM coefficients and the true critical frequency. It is this result which will permit the evaluation of the expected value and variance of the natural frequency estimator.
Expected Value of the Natural Frequency Estimator

The expected value of the natural frequency estimator, or more generally the critical frequency estimator, is evaluated with the aid of a first-order approximation for the expected value of a non-linear function of random variables. The approximation, which is derived in detail by Jenkins and Watts [5], states that the expected value of a function, \( f(X_1, X_2, ..., X_n) \) of the random variables \( X_1, X_2, ..., X_n \) which have means \( \mu_i \), can be evaluated to first-order as

\[
E[f(X_1, X_2, ..., X_n)] \approx f(\mu_1, \mu_2, ..., \mu_n)
\]

(3.27)

In the case of the critical frequency estimator, the expected value, found by taking the expectation of equation (3.23), is given by

\[
E[\hat{f}_0] = f_0 + E[\delta f_0]
\]

(3.28)

However, \( \delta f_0 \) is a non-linear function of the p-random variables, \( \{A_k\} \), which can be shown to have expected values which asymptotically equal to the true values \( \{A_k\} \), [21]. Consequently, by applying equation (3.27) to equation (3.26) it is found that

\[
\lim_{N \to \infty} E[\delta f_0] = - \frac{\sum_{k=0}^{p} \sum_{\ell=0}^{p} k A_k A_\ell \sin[2\pi(k-\ell)\hat{f} \Delta]}{2\pi \Delta \sum_{k=0}^{p} \sum_{\ell=0}^{p} k (\ell-k) A_k A_\ell \cos[2\pi(k-\ell)\hat{f} \Delta]}
\]

(3.29)
Comparing the numerator of equation (3.29) with equation (3.21) reveals that

\[ \lim_{N \to \infty} E[\delta f_0] = 0 \quad (3.30) \]

and therefore

\[ \lim_{N \to \infty} E[\hat{f}_0] = f_0 \quad (3.31) \]

Thus equation (3.31) reveals that the critical frequency estimator yields asymptotically unbiased estimates of the critical frequencies of a pth order MEM spectrum. In terms of natural frequencies, equation (3.31) means that the estimator can provide asymptotically unbiased estimates of natural frequencies provided that the exact pth order MEM spectrum matches the maximum of the true spectrum at the desired natural frequency. This point will be advanced further after the derivation of the variance of the natural frequency estimator to follow.

**Variance of the Natural Frequency Estimator**

An approximation to the variance of the critical frequency estimator can be determined by evaluating the variance of equation (3.23), as shown in equation (3.32). To evaluate the variance of \( \delta f_0 \) one must again
Utilize a first-order approximation for the variance of a non-linear function of random variables. Specifically, it is shown in Jenkins and Watts [5] that the variance of a function, \( f(X_1, X_2, \ldots, X_n) \) of the random variables \( X_1, X_2, \ldots, X_n \) with means \( \mu_i \), is given to first-order by

\[
\text{Var}[f(X_1, X_2, \ldots, X_n)] \approx \sum_{k=1}^{n} \sum_{l=1}^{n} \left( \frac{\partial f}{\partial X_k} \right) \left( \frac{\partial f}{\partial X_l} \right) \mu \text{Cov}[X_k, X_l] \quad (3.33)
\]

where \( \text{Cov}[X_k, X_l] \) represents the covariance between \( X_k \) and \( X_l \) and \( \left( \frac{\partial f}{\partial X_k} \right) \) represents the partial derivative with respect to \( X_k \) evaluated at the point \( (X_1, X_2, \ldots, X_n) = (\mu_1, \mu_2, \ldots, \mu_n) \). Applying this formula to equation (3.26) yields the following result,

\[
\lim_{N \to \infty} \text{Var}[\hat{f}_0] = \sum_{k=1}^{p} \sum_{l=1}^{p} \left( \frac{\partial \delta f_0}{\partial \hat{A}_k} \right) \left( \frac{\partial \delta f_0}{\partial \hat{A}_l} \right) \text{Cov}[\hat{A}_k, \hat{A}_l] \quad (3.34)
\]

where use has been made of the asymptotic properties of the MEM coefficients. The partial derivatives required for the evaluation of equation (3.34) can be shown, by straightforward differentiation of equation (3.36), to be given by
\[
\left( \frac{\partial \delta f_0}{\partial A_i} \right)_A = \sum_{k=0}^{P} \frac{(k-i) A_k \sin[2\pi(k-i)f_0 \Delta]}{2\pi \Delta \sum_{k=0}^{P} \sum_{\ell=0}^{P} k(\ell-k) A_k A_{\ell} \cos[2\pi(\ell-k)f_0 \Delta]}
\] (3.35)

Finally, the expression for the variance of the critical frequency estimator becomes complete upon substitution of an approximation for the covariance matrix given by Box and Jenkins [21] and shown in equation (3.36)

\[
\text{Cov}[\hat{A}] = \frac{\sigma^2}{N} \begin{bmatrix}
R(0) & R(1) & \cdots & R(p-1) \\
R(1) & R(0) & & \\
\vdots & \vdots & \ddots & \vdots \\
R(p-1) & & & R(1) & R(0)
\end{bmatrix}^{-1}
\] (3.36)

where \( \text{Cov}[\hat{A}] \) represents the covariance matrix whose elements, \((i, j)\) are \( \text{Cov}[\hat{A}_i, \hat{A}_j] \). One important characteristic of the critical frequency estimator is revealed in equation (3.34) when it is noted that the expression for the variance of the estimator is independent of the number of data points, \( N \), except as explicitly shown in equation (3.36). Consequently, the critical frequency estimator is, to first order, asymptotically consistent since its bias and variance decrease to zero as
the number of data points approaches infinity. This implies estimates of critical frequencies can be obtained to any desired precision by choosing an appropriately large record length.

To place the results of this section in perspective, it is important to understand exactly what one obtains in using equation (3.18). Toward that end, it is recalled that if the first \( p+1 \) lags of an exact (ie. theoretical) autocorrelation function are known, then a \( p \)th order MEM spectrum is uniquely defined. This spectrum, which may or may not match the true spectrum defined by the infinite duration, theoretical autocorrelation function, exhibits a number of extrema which are themselves uniquely defined for this \( p \)th order spectrum. It is the critical frequencies of these extrema which are obtained as solutions of equation (3.18). While critical frequencies are of little value when determined from MEM spectra which poorly models the true spectrum, one is assured by theory that critical frequencies will converge to the critical frequencies of the true spectrum as the order is increased. Consequently, when the MEM spectrum matches an extrema of a true spectrum, then the critical frequency determined by equation (3.18) takes on the physical significance of that particular frequency of the true spectrum. For example, referring to the tenth order MEM spectrum shown in Figure 3.2, there should be little doubt that the natural frequency would be estimated extremely well by the critical frequency of the MEM spectrum.

When one does not know the exact autocorrelation function of a process, then use of an estimated (ie, approximate) autocorrelation function introduces error into the corresponding MEM coefficients and
consequently corrupts the MEM spectrum. This estimated spectrum will, for a given set of coefficients, have its own characteristic extrema whose critical frequencies are exactly obtainable from equation (3.18) and the given coefficients. The error between an estimate of a critical frequency and the value obtained from the exact MEM spectrum is statistically described by the expected value and variance shown in equations (3.31) and (3.34). Thus these measures characterize the scatter in estimates of a critical frequency that result from using estimates of the MEM coefficients instead of the exact values. As before, theory guarantees that as the number of data points increases, estimates of the critical frequencies will converge in the mean to critical frequencies of the true spectrum. However, in this case record length will govern the variance and hence the usefulness of the estimate. Furthermore, an extrema of an exact pth order MEM spectrum matches an extrema of the true spectrum then the expected value and variance of estimates of that critical frequency can be interpreted as the expected value and variance of the physical parameter characterizing that frequency.
3.4 The MEM Damping Estimator

3.4.1 Derivation of the Estimator

The preceding section has described a new method of estimating natural frequencies from an ambient response spectrum computed by MEM. Based on the observation that the MEM spectral estimate is an analytic equation for the spectrum, an equation was developed whose solutions include the natural frequencies of the spectrum. Having developed this ability to precisely locate the maxima of the spectrum, it is now possible to extract half-power damping estimates from the MEM spectra in a similar manner. Recalling the form of the half-power damping estimator, repeated here for convenience, it is observed that one need only obtain an estimator

\[ \xi_{HP} = \frac{B}{2f_n} = \frac{f_+ - f_-}{2f_n} \]  

(2.1)

for the half-power frequencies, \( f_+ \), \( f_- \), in order to complete the tools required to estimate damping.

Half-power frequencies are defined, in the present context as the pair of frequencies bracketing the natural frequency which corresponds to spectral ordinates 3dB down from the resonant peak. In more concise terms, the half-power frequencies are solutions of

\[ S(f) = \frac{1}{2} S(f_n) \]  

(3.37)
Utilizing the MEM representation of the spectrum given by equation (3.16) permits equation (3.39) to be expressed as

\[
\frac{\sigma_p^2 \Delta}{\rho_o + 2 \sum_{k=1}^{p} \rho_k \cos(2\pi kf\Delta)} = \frac{1}{2} S(f_n) \tag{3.38}
\]

Rearranging the terms of equation (3.38) one arrives at the final result given by equation (3.39). This equation provides the analytic capability of determining the half-power frequencies corresponding to any given maxima of a pth order MEM spectrum. The pair of frequencies determined by equation (3.39) together with the previously determined natural frequency completely define the MEM damping estimate when substituted into equation (2.1).

While the development of equation (3.39) was posed in terms of damping and natural frequencies, this result applies to the more general problem of finding half-power frequencies of a specific maxima of pth order MEM spectrum. As with the critical frequency estimator given by equation (3.18), attachment of physical meaning to the estimate is done by the analyst and is not implicit in the method. Accordingly, damping
estimates are only available after a maxima is judged to be a mode of vibration.

It will prove convenient in the sections to follow to have the half-power frequency expressed in terms of the pth order MEM coefficients instead of $S(f_n)$ and the coefficients $\rho_k$ as shown in equation (3.39). This alternate form is easily obtained by substituting the definition for $S(f)$ and $\rho_k$ given by equations (3.3) and (3.15) respectively and simplifying. Accordingly, it can be shown that the half-power frequency estimator can be written as

$$\sum_{k=0}^{P} \sum_{\ell=0}^{P} A_k A_\ell \{ \cos[2\pi(k-\ell)f_{\Delta}] - 2 \cos[2\pi(k-\ell)f_n \Delta]\} = 0 \quad (3.40)$$

where: $A_0 = 1$
3.4.2 Statistics of the MEM Damping Estimator

The MEM half-power frequency estimator has been defined in equation (3.40) in terms of the exact MEM coefficients, \( \{A_k\} \), and the natural frequency (i.e. the critical frequency). In practice, however, one only has estimates of these quantities and hence can only obtain estimates of the half-power frequencies \( f_+ \) and \( f_- \). Substituting estimates of the MEM parameters into equation (3.40) produces equation (3.41), an expression for the half-power frequencies which correspond to the specified parameters.

In order to describe the statistics of these half-power estimates, an expression must be obtained of the form

\[
\sum_{k=0}^{P} \sum_{\ell=0}^{P} \hat{A}_k \hat{A}_\ell \{\cos[2\pi(k-\ell)\hat{f}_+\Delta] - 2 \cos[2\pi(k-\ell)\hat{f}_-\Delta]\} = 0 \tag{3.41}
\]

where: \( \hat{f}_+ = \hat{f}_+ \) and/or \( \hat{f}_- \)

Following the derivation of the natural frequency estimator, one can write the half-power frequency estimate as a perturbation about the true value using

\[
\hat{f}_* = f_* + \delta f_* \tag{3.43}
\]
Upon substituting equation (3.43) into equation (3.41) and invoking the small angle approximations, the following result is obtained

$$
\delta f = \frac{\sum_{k=0}^{p} \sum_{\ell=0}^{p} \hat{A}_k \hat{A}_\ell \{ \cos[2\pi(k-\ell)f_* \Delta] - 2\cos[2\pi(k-\ell)\hat{f}_n \Delta] \}}{2\pi \Delta \sum_{k=0}^{p} \sum_{\ell=0}^{p} (k-\ell) \hat{A}_k \hat{A}_\ell \sin[2\pi(k-\ell)f_* \Delta]}
$$

(3.44)

while equation (3.44) is only a first-order approximation to the desired result, it does permit an otherwise impossible investigation of the expected value and variance of the estimator.

**Expected Value of the Damping Estimator**

The expected value of the half-power frequency estimator can be determined by computing the expected value of equation (3.43) as shown in equation (3.45). However, since $\delta f_*$ is a non-linear function of the random variables, $\hat{A}_k$ and $\hat{f}_n$ a first-order approximation to the expected value can be obtained using the method described in Section (3.3.2) and summarized by equation (3.27). Accordingly, it can be shown that

$$
E[\hat{f}_*] = f_* + E[\delta f_*]
$$

(3.45)
where the following asymptotic means have been used.

\[
\lim_{N \to \infty} E[\hat{A}_k] = A_k
\]  
(3.47a)

\[
\lim_{N \to \infty} E[\hat{f}_n] = f_n
\]  
(3.47b)

Comparing the numerators of equations (3.46) and (3.40), it is found that

\[
\lim_{N \to \infty} E[\delta f_*] = 0
\]  
(3.48)

and therefore

\[
\lim_{N \to \infty} E[\hat{f}_*] = f_*
\]  
(3.49)

In order to evaluate the expected value of the damping estimate obtained from the estimates of the half-power frequencies and the natural frequencies
one must again draw upon a first-order approximation, since damping is computed as a non-linear function of three random variables. Thus, utilizing equation (3.27), it can be shown that

\[
\lim_{N \to \infty} E[\hat{\xi}_{HP}] = \frac{E[\hat{f}_+] - E[\hat{f}_-]}{2E[\hat{f}_n]} = \xi_{HP} \quad (3.50)
\]

This result indicates that asymptotically unbiased estimates of the half-power damping ratio are attainable with the MEM damping estimator. By specifying that convergence is to the half-power damping ratio, the reader is reminded that the MEM damping estimator is still a half-power damping estimator. As such, estimates obtained by this method are still subject to the limitations discussed in Chapter 2. However under those conditions when the half-power damping estimate is accurate, then the MEM damping estimator provides accurate estimates of the true damping.

**Variance of the Damping Estimator**

Following the derivation of the variance of the natural frequency estimator, one can compute a first-order approximation for the variance of the half-power frequency estimator. By a straightforward application of the general formula for computing the approximate variance shown in equation (3.33), it can be shown that
\[ \lim_{N \to \infty} \text{Var}[\hat{f}_n] = \sum_{k=1}^{P} \sum_{\ell=1}^{P} \left( \frac{\partial \delta f_n}{\partial \lambda_k} \right) \left( \frac{\partial \delta f_n}{\partial \lambda_{\ell}} \right) \text{Cov}[\lambda_k, \lambda_{\ell}] \]  

(3.51)

where:

\[ \left( \frac{\partial \delta f_n}{\partial \lambda_k} \right) = \frac{\sum_{k=0}^{P} \hat{\lambda}_k \{ \cos[2\pi(k-\ell)f_n\Delta] - 2 \cos[2\pi(k-\ell)f_n\Delta] \} }{2\pi \Delta \sum_{k=0}^{P} \sum_{\ell=0}^{P} \hat{\lambda}_k \hat{\lambda}_{\ell} \sin[2\pi(k-\ell)f_n\Delta]} \]

\[ + 2 \frac{\sum_{k=0}^{P} \sum_{\ell=0}^{P} \hat{\lambda}_k \hat{\lambda}_{\ell} \sin[2\pi(k-\ell)f_n\Delta]}{\sum_{k=0}^{P} \sum_{\ell=0}^{P} \hat{\lambda}_k \hat{\lambda}_{\ell} \sin[2\pi(k-\ell)f_n\Delta]} \cdot \left( \frac{\partial \delta f_n}{\partial \lambda_k} \right) \]

\[ \left( \frac{\partial \delta f_n}{\partial \lambda_k} \right) = \text{equation (3.35)} \]

\[ \text{Cov}[\hat{\lambda}_k, \hat{\lambda}_{\ell}] = \text{equation (3.36)} \]

Before proceeding with the calculation of the variance of the MEM damping estimator, it will prove useful to determine the variance of the half-power bandwidth. This quantity, defined as the difference between the
half-power frequencies, can be expressed in perturbation form as shown in equation (3.52). The desired variance is obtained through the

\[ \hat{B} = B + \delta f_+ - \delta f_- \]  
(3.52)

application of the first-order approximation for the variance to equation (3.52). The result of this operation is given by equation 3.53:

\[ \lim_{N \to \infty} \text{Var}[B] = \sum_{k=1}^{P} \sum_{i=1}^{P} \left( \frac{\partial (\delta f_+ - \delta f_-)}{\partial A_k} \right)_{A} \left( \frac{\partial (\delta f_+ - \delta f_-)}{\partial A_i} \right)_{A} \text{Cov}[\hat{A}_k, \hat{A}_i] \]  
(3.53)

where:

\[ \left( \frac{\partial (\delta f_+ - \delta f_-)}{\partial A_i} \right)_{A} = \left( \frac{\partial \delta f_+}{\partial A_i} \right)_{A} - \left( \frac{\partial \delta f_-}{\partial A_i} \right)_{A} \]

\[ \left( \frac{\partial \delta f_+}{\partial A_i} \right)_{A} = \text{equation (3.51)} \]

While expressions have been derived for the variance of the natural frequency estimator and the half-power bandwidth estimator, as yet, the variance of the damping estimator remains unspecified.
As in previous derivations, one must resort to the first-order approximation formula in order to compute the variance of the damping estimator. This approximation is applied to the damping estimator written in the form

\[
\hat{\xi}_{HP} = \frac{\hat{B}}{2\hat{f}_n} \quad (3.54)
\]

Then according to equation (3.33)

\[
\lim_{N \to \infty} \text{Var}[\hat{\xi}_{HP}] = \sum_{k=1}^{P} \sum_{l=1}^{P} \left( \frac{\partial}{\partial A_k} \left[ \frac{\hat{B}}{2\hat{f}_n} \right] \right)_A \left( \frac{\partial}{\partial A_l} \left[ \frac{\hat{B}}{2\hat{f}_n} \right] \right)_A \text{Cov}[\hat{A}_k, \hat{A}_l] \quad (3.55)
\]

Expanding the partial derivatives produces the final result expressed by equation (3.56).

\[
\lim_{N \to \infty} \frac{\text{Var}[\hat{\xi}_{HP}]}{\hat{\xi}^2_{HP}} = \frac{\text{Var}[\hat{B}]}{B^2} + \frac{\text{Var}[\hat{f}_n]}{f^2_n} - \frac{2\text{Cov}[\hat{f}_n, \hat{B}]}{B} \quad (3.56)
\]

where: \( \text{Var}[\hat{B}] = \) equation (3.33)

\( \text{Var}[\hat{f}_n] = \) equation (3.34)
The expression for the variance of the MEM damping estimator provides little insight into the statistical characteristics of the damping estimator. However, from the form of equation (3.55), it can be noted that the variance of the MEM damping estimator is inversely proportional to the number of data points. Consequently, the MEM damping estimator is, to first-order, asymptotically consistent. That is, the variance and expected value of the estimator go to zero as the number of data points becomes infinite.

It is important to remember when using the damping and natural frequency estimators that the expected value and variance derived for each of these estimators reflect the statistical scatter about the natural frequency and damping that would be calculated with the exact value of the MEM coefficients. This quantity can be considerably different from the natural frequency and damping of the "true" spectrum.
However, the theory of MEM guarantees that increasing the order of the spectrum, brings the MEM parameter estimates closer, in the mean, to the true value. While convergence in the mean is obtained at the cost of increasing the variance of the estimate, this variance is given by the theory and can be used to establish one's confidence in the estimates.
3.5 Implementation of the Estimators

The final topic to be dealt with in this chapter concerns the actual utilization of the estimators developed in the two preceding sections. As one might expect, the complexity of the estimators limit the evaluation of closed form solutions to the simplest cases. Consequently, numerical techniques must be used to solve the estimation equations. Reviewing the form, as expressed by equations (3.18) and (3.39), it is found that estimates are obtained as zeros of trigonometric equations. Of the plethora of numerical methods applicable to this type of problem, Newton's method is perhaps the best known [22]. This technique iteratively solves for a zero of a function $f(x)$ using the formula given by equation (3.57)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (3.57)$$

where: $x_i$ = $i$th estimate of the zero
$x_{i+1}$ = updated estimate of the zero
$f'(x_i) = \frac{df}{dx} \bigg|_{x=x_i}$

Iteration is continued until the difference between two consecutive estimates of the zero yields the desired degree of accuracy. It can be shown that this method will always converge quadratically to the correct answer if the initial guess, $x_1$, is close enough to the zero.
Applying equation (3.57) to the natural frequency and half-power frequency estimators results in the iteration formula given by equations (3.58) and (3.59) respectively.

\[ f_{i+1} = f_i - \frac{\sum_{k=1}^{p} k \rho_k \sin(2\pi kf_i \Delta)}{2\pi \Delta \sum_{k=1}^{p} k^2 \rho_k \cos(2\pi kf_i \Delta)} \]  

(3.58)

\[ f_{*,i+1} = f_{*,i} - \frac{\sum_{k=1}^{p} \rho_k \cos(2\pi kf_{*,i} \Delta) - \frac{\sigma f_{*,i} \Delta}{S(f_{*,i})} + \frac{1}{2} \rho_{i+1}}{2\pi \Delta \sum_{k=1}^{p} k \rho_k \sin(2\pi kf_{*,i} \Delta)} \]  

(3.59)

In practice, the initial value of the natural frequency is obtained by guessing the value from a cursory scan of the plotted spectrum. After iterating with equation (3.59) and arriving at a final estimate of the natural frequency \( \hat{f}_o \), initial values for the half-power frequencies can be obtained using

\[ f_{+/-} = (1 \pm \xi) \hat{f}_0 \]  

(3.60)
Experience has shown that in most cases a guess of $\xi = 0.05$ is close enough to ensure convergence. In a very few cases, a poor initial guess caused convergence to an incorrect value. Fortunately, in each case, the value obtained was grossly in error and therefore immediately obvious. An improved guess always provided the correct value.

As a final note, it is observed that the equations defining the variance for each of the estimator have been written in terms of the MEM coefficients and consequently required double summations to be computed. However, each of these can be transformed into a more computational efficient format using the following identities:

\[
\sum_{k=0}^{p} \sum_{l=0}^{p} A_k A_l \sin[2\pi(\ell-k)f\Delta] = -\sum_{k=1}^{p} kA_k \sin(2\pi k f\Delta) \quad (3.61a)
\]

\[
\sum_{k=0}^{p} \sum_{l=0}^{p} A_k A_l \cos[2\pi(\ell-k)f\Delta] = -\sum_{k=1}^{p} k^2 \rho_k \cos(2\pi k f\Delta) \quad (3.61b)
\]

where: $\rho_k = \sum_{i=0}^{p} A_i A_{i+k}$

The use of equations (3.61) significantly reduces the number of multiplications and the complexity of the required program.
CHAPTER 4

Performance of Natural Frequency and Damping Estimators with Simulated Data

The preceding chapter introduced a new method of estimating natural frequencies and damping ratios which is based on the maximum entropy method of spectral analysis. An approximation for the expected value and variance of the estimators was presented and their expected values were shown to approach the true values for long record lengths (ie. they are asymptotically unbiased). In this chapter, Monte Carlo simulation techniques are used to study the behavior of these estimators and validate the theoretically predicted statistics. The results of this study will be used in Chapter 5 to define a concise methodology for estimating natural frequencies and damping ratios from response measurements.

The expected value and variance of natural frequencies and damping ratios obtained through the use of Blackman-Tukey spectral estimation techniques are compared to the results obtained with MEM. The desired statistics are evaluated by constructing an ensemble of finite-duration, Gaussian random processes where each of the realizations are mutually independent and governed by the same set of parameters (ie, $f_n$, $\xi$, SNR, $\Delta$, and $R(0)$). By applying a parameter estimator to each of the realizations, an ensemble of parameter estimates is constructed. This ensemble of estimates is subsequently used to compute approximations to
the desired statistics. These results provide the information necessary to evaluate the performance of each of the estimation algorithms. In this way, a fair comparison can be made between different methods.
4.1 Simulation of the Response of Linear Structural Systems to Stationary Random Excitation

Performance of the natural frequency and damping estimators is evaluated in this chapter using the response of an n degree-of-freedom system described by a second order, coupled, stochastic differential equation:

\[
[M] \{\ddot{y}\} + [C] \{\dot{y}\} + [K] \{y\} = \{F\}
\] (4.1)

where: \( \{y\} = n \)-component system response
\[= \{y_1(t), y_2(t), \ldots, y_n(t)\} \]
\( \{F\} = n \)-component white noise excitation
with \( E[\{F(t)\}^T \{F(s)\}] = [D]\delta(t-s) \)

The response of the system defined by equation (4.1) is simulated using the discrete-time model shown in equation (4.2). The basis of this

\[
\sum_{k=0}^{2n} \alpha_k y_{t-k} = \sum_{k=0}^{2n-1} \beta_k x_{t-k} ; \quad \delta_0 = \beta_0 = 1
\] (4.2)

where: \( n \) = number of degree-of-freedom
\( \{\alpha_k\} \) and \( \{\beta_k\} \) = model parameters
\( x_t \) = zero mean, uncorrelated, discrete-time process with \( E[x_t x_s] = \sigma_x^2 \delta_{t,s} \)
model as a simulator has been demonstrated by Gersch [23] in terms of the equivalence between the response of the discrete-time model and the sampled response of the continuous system. When the observed response of the continuous time model is contaminated by a zero mean, uncorrelated additive noise process, \( n(t) \), as illustrated in Figure 2.2, then it can be shown [23] that the equivalent discrete-time series is of the form

\[
\sum_{k=0}^{2n} \alpha_k z_{t-k} = \sum_{k=0}^{2n} \beta_k x_{t-k}
\]  

(4.3)

where: \( z_t = y_t + n_t \)

The models defined by equations (4.2) and (4.3) are commonly known as autoregressive-moving average models (ARMA). Aside from the obvious simplicity of this formulation, it can be shown that the ARMA model duplicates the autocovariance function of the regularly sampled multivariate input - scalar output of the system defined by equation (4.1). Consequently, the output spectrum of the ARMA model, \( S_z(f) \), is identical to the response spectrum of the sampled continuous process \( y_i(t) \) (providing certain constraints on sampling frequency, etc. are met). This property is important in the simulation of time histories since it assures the equivalence of the second order statistics for the continuous and discrete time histories. The ability of an ARMA model to match the
response spectrum of the continuous system makes it the ideal choice for studying the properties of natural frequency and damping estimators.

The actual algorithm used in generating simulated response measurements is revealed when equation (4.3) is written in the form

$$z_t = - \sum_{k=1}^{2n} \alpha_k z_{t-k} + \sum_{k=0}^{2n} \beta_k x_{t-k}; \quad \beta_0 = 1 \quad (4.4)$$

The value of the simulated time history at time, $t$, is recursively computed from the current input, $x_t$, and the preceding $2n$-values of the response and the input. Simulation is started by setting the initial values of the response to zero and proceeding with the recursion. The resultant starting transients are accounted for by simply discarding the model output until steady-state is achieved. The time-series used as input to the ARMA model is obtained by generating zero mean, uncorrelated random variables with a digital computer. In this research, the input time series was a sequence of uncorrelated, normal random variables generated using the Box-Mueller algorithm [24]. Whiteness of the random variates was verified using tests described by Jenkins and Watts [5]. The ARMA model coefficients used in this thesis were computed using Gersch's two-stage least-squares procedure [23]. For further details on this simulation method, the reader is referred to the comprehensive treatment of the topic by Gersch [23].
4.2 Performance of the Estimators with One Degree-of-Freedom Simulations

In the two Sections to follow, the bias and variance of the conventional and MEM natural frequency and damping estimators are evaluated using the simulated displacement response of a single degree-of-freedom system. The simulation model was constructed with a natural frequency of 1 Hz, sampling frequency of 8 Hz and signal-to-noise ratio (defined in Section 2.2) of 30 dB. Damping ratios of 1%, 2%, and 3% were used to demonstrate the variation in estimator statistics with the true damping. An ensemble of 50 independent realizations was constructed for each of three models. Each realization consisted of 10,240 samples, making a maximum record length of 21.3 minutes. Partial records of 2048, 4096, and 10240 samples were used to estimate autocorrelation functions for each of the realizations. All of the partial records started at the first sample of the realization. Thus, combinations of the three record lengths and three models results in nine complete sets of statistics for each of the parameter estimators tested.
4.2.1 Evaluation of the Maximum Response Natural Frequency Estimator and Half-power Bandwidth Damping Estimator with Blackman-Tukey Spectral Estimates

Natural frequency and damping estimates were obtained from Blackman-Tukey spectral estimates computed for each of the three autocorrelation functions available from each realization. The spectral estimates were computed with a Bartlett window and fifteen different lag lengths ranging from 5 seconds to 64 seconds. In each case, the spectrum was evaluated at a frequency increment of 0.004 Hz by appending zeros to the end of each autocorrelation function. Accordingly, natural frequencies and half-power frequencies were determined with a precision of ± 0.002 Hz. Expected values and variances were computed as the sample average and variance using the 50 realizations for each parameter estimate. The results of this simulation study are presented in Figures (4.1 - 4.18). Each page contains the natural frequency or damping estimator statistics for a specific simulation model and record length. The top figure on each page is a plot of the average parameter estimate as a function of the ratio of the maximum lag used in the spectral estimate to the total record length. Since the record length is fixed in each figure, the abscissa can be equivalently interpreted as indicating lag or resolution. The dashed line in the top figure shows the true value of the parameter. The bracket included with each average is the 95% confidence interval associated with estimating a mean from 50 samples. The bottom figure on each page is a plot of the coefficient of variation (ie. standard
deviation/true value) for the estimated parameters as a function of the lag to record length ratio. The brackets in these figures are the 95% confidence interval associated with the computation of a coefficient of variation from 50 samples. The dashed line in the bottom figures is the Cramér-Rao lower bound on the coefficient of variation for the estimation of the parameter. This quantity represents the smallest coefficient of variation that any unbiased estimator of the parameter can have for the specified record length and system characteristics. Any unbiased estimator which achieves this bound is called an efficient estimator and can be shown to possess numerous desirable properties [25]. The Cramér-Rao bound shown in these figures was calculated using a technique described by Gersch [26].

The results of this simulation study reveal that the maximum response natural frequency estimator is an asymptotically (i.e. increasingly valid for large records) efficient estimator of the maximum response frequency. This statement is demonstrated in the simulation results by the convergence of the average natural frequency estimate to the maximum response frequency (which is slightly less than the natural frequency) while achieving the Cramér-Rao bound. One should note that efficient estimates are only obtained for a small range in the lag to record length ratio. The departure from efficiency is of little consequence in these natural frequency estimates since the coefficient of variation remains well within the tolerance of engineering accuracy.

The results of this study also reveal that the half-power bandwidth
The damping estimator is not an efficient estimator when applied to the Blackman-Tukey spectral estimator. This is demonstrated in the simulation results by a large bias which is present at the ratio of lag to record length corresponding to attainment of the Cramér-Rao bound. As the lag increases, the departure of the average damping estimate from the true value decreases while the coefficient of variation increases to a level which is substantially larger than the lower bound. The use of such a non-efficient damping estimator may severely limit the utility of the estimated damping ratios. For example, in Figure 4.18, the average damping estimate equals the true value and the coefficient of variation is 0.25 at a lag to record length ratio of 5%. Putting this information in the form of a confidence bound for an estimate $\hat{\xi}$, one can state with 95% confidence (assuming a Gaussian distribution) that the true damping ratio lies in the interval $(\hat{\xi} - 1.5, \hat{\xi} + 1.5)$. However, an efficient estimator would exhibit a confidence bound of $(\hat{\xi} - 0.5, \hat{\xi} + 0.5)$.

The results further indicate that the average damping estimate can be less than the true value. Consequently, one cannot simply estimate the damping with increasingly finer resolution and be assured of convergence to the true value. In practice, one does not know at what lag the estimator becomes unbiased. Furthermore, since there is no measure of the coefficient of variation, one cannot know how close the estimate is to the true value. The impact of this rather distressing observation was made clear when a "blindfold test" was conducted using a 20 minute realization of single degree-of-freedom system with the true damping unrevealed. Analysis of this time series showed that while the
natural frequency was easily determined, the estimate of the damping ratio varied from 5% to 4% when the true damping was 2%.
FIG. 4.1 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.2 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.3 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.4 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.5 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.6 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.7 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.8  CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.9 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.10 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
**FIG. 4.11** CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF

- $f_n = 1.0$ Hz
- $\xi = 2.0$%
- $f_s = 8.0$ Hz
- SNR = 30 dB
- $T = 21.3$ Min
FIG. 4.12 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.13 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.14 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.15  CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
FIG. 4.16 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION - 1 DOF
FIG. 4.17 CONVENTIONAL NATURAL FREQUENCY ESTIMATION - 1 DOF
Fig. 4.18 Conventional "Half-Power" Damping Estimation - 1 DOF
4.2.2 Evaluation of the MEM Parameter Estimators

MEM natural frequency and damping estimates were obtained from each of the autocorrelation functions using the MEM coefficients \( \{ A_k \} \) evaluated at 24 different orders ranging from 4 to 50 (i.e., lags from 5 to 51). In each case, the parameter estimates were computed by solving equations (3.18) and (3.40) using Newton's iteration. The test for convergence in this iterative solution scheme was set at 0.01%. The statistics of the MEM parameter estimators were determined as outlined in the preceding section and are presented in Figures (4.19 - 4.36). While the format of these figures is unchanged, an additional line has been added to the coefficient of variation computed for each estimate. This line is the coefficient of variation predicted by the first order approximation for the variance of the MEM estimator. Predicted values were obtained by computing the "exact" MEM coefficients from the theoretical autocorrelation function and evaluating the coefficients of variation at the average values of the parameters shown in the figures.

The results of this simulation study demonstrate that the MEM estimate of the maximum response frequency is asymptotically efficient. As in the previous section, it is found that the estimator is efficient over a finite range of orders. Unlike estimates made from conventional Blackman-Tukey spectra, the MEM estimates include an approximate coefficient of variation which can be used as a guide in selecting the appropriate order. Comparison of the theoretical coefficient of variation with the simulation results reveal that the first-order
approximation provides an excellent match to the data.

The simulation results also reveal that the MEM damping estimator is not an asymptotically efficient estimator due to the presence of a small bias (less than 5%) at orders corresponding to the attainment of the Cramér-Rao bound. The results do reveal that the average damping converges to consistently high values, whose magnitudes decrease with increasing true damping. It is also noted that the order required for the "best" estimate of damping or natural frequency is independent of the record length. Further, it is found that the best estimates of natural frequency and damping occur at approximately the same order. While these observations demonstrate the quality of the obtainable estimates, the greatest strength of the MEM estimator lies in the ability to compute the approximate coefficients of variation for the damping. In each of the test cases shown, it is found that the first-order approximation for the coefficient of variation for the damping estimator provides an excellent match to the simulation results. In a repetition of the previously described "blind-fold" test, the natural frequency and damping of the simulated time history was easily found with the aid of the approximate coefficient of variation.
FIG. 4.19 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR – 1 DOF
FIG. 4.20 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.21 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.22 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.23 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.24 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF

- TRUE VALUE
- PREDICTED
- C-R BOUND
FIG. 4.25  PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.26 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.27 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.28 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.29 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.30 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
Fig. 4.31 Performance of MEM Nat. Frequency Estimator - 1 DOF
FIG. 4.32 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.33 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR - 1 DOF
FIG. 4.34 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
FIG. 4.35 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR – 1 DOF
FIG. 4.36 PERFORMANCE OF MEM DAMPING ESTIMATOR - 1 DOF
4.3 Performance of the Estimators with Two Degree-of-Freedom Simulations

The results of the preceding Section has demonstrated the superiority of the MEM parameter estimators when applied to the response of a single degree-of-freedom system. While these results are enlightening, the simplicity of the simulation model prohibits generalization of the observed characteristics. To furnish the required evidence, simulation studies were conducted with two degree-of-freedom systems. Two simulation models were utilized in this work and the squared-magnitude of the system transfer functions are shown in Figure (4.37a and b). Both models contain equal modal damping ratios of 2% and equal amplitude resonant responses. Figure (4.37a and b) also show that the noise floor was set at 30 dB below the resonant response amplitude. The natural frequencies were 0.5 Hz and 1 Hz for one model and 0.75 Hz and 1 Hz for the other. The sampling frequency was 8 Hz in both models. As in the one degree-of-freedom simulations, 50 realizations with 10,240 samples were generated. In this study, partial records of 4096 and 10,240 samples were constructed and autocorrelation functions were estimated for each record.
FIG. 4.37a & b  TWO DOF SIMULATION MODELS
4.3.1 Evaluation of the Maximum Response Natural Frequency Estimator and the Half-Power Bandwidth Damping Estimator with Blackman-Tukey Spectral Estimates

Natural frequency and damping estimates were computed for both response peaks from Blackman-Tukey spectral estimates. The spectra used in this analysis were determined in accordance with the procedure used in Section 4.2.1. The parameter estimation statistics were computed for both response peaks and the results are presented in Figures (4.38 - 4.53) where the first 16 figures correspond to estimates for the first mode and the remainder for the second mode. The format of these plots is unchanged from the preceding Sections.

In scanning the results contained in the 16 figures, one finds that the presence of a second peak in the response spectrum has virtually no effect on the behavior of the parameter estimators. A careful comparison of the averages and coefficients of variation for the second mode of the two degree-of-freedom with the equivalent one degree-of-freedom results reveals that the only measurable difference between the two cases is slightly higher bias in the natural frequency estimates for the two degree-of-freedom system. The equivalence of the results is not surprising when one considers the minimal interaction that takes place between the frequency bands of conventional spectral estimates. Accordingly, the characteristics delineated in Section 4.2.1 can be generalized to a multi-moded response when the natural frequencies are well separated.
FIG. 4.38 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 1ST MODE
FIG. 4.39 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 1ST MODE
FIG. 4.40 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 1ST MODE
FIG. 4.41 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 1ST MODE
FIG. 4.42 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 1ST MODE
\( f_n: 0.75 \text{ Hz and } 1.0 \text{ Hz} \)
\( \xi: 2.0 \% \text{ and } 2.0 \% \)
\( \text{SNR}: 30 \text{ dB and } 30 \text{ dB} \)
\( f_s = 8.0 \text{ Hz} \)
\( T = 8.5 \text{ Min} \)

**Fig. 4.43** CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 1ST MODE
FIG. 4.44 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 1ST MODE
**FIG. 4.45 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION**

2 DOF - 1ST MODE
FIG. 4.46 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 2ND MODE
FIG. 4.47 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 2ND MODE
FIG. 4.48 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 2ND MODE
**FIG. 4.49**  CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 2ND MODE
FIG. 4.50 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 2ND MODE
FIG. 4.51 CONVENTIONAL "HALF-POWER" DAMPING ESTIMATION
2 DOF - 2ND MODE
FIG. 4.52 CONVENTIONAL NATURAL FREQUENCY ESTIMATION
2 DOF - 2ND MODE
**Figure 4.53** Conventional 'Half-Power' Damping Estimation 2 DOF - 2nd Mode
4.3.2 Evaluation of MEM Parameter Estimators

Evaluation of the MEM parameter estimators using the simulated response of a two degree-of-freedom system followed the same procedures outlined in Section 4.2.1 with a few minor exceptions. As before, parameter estimates were evaluated at 24 different orders; however the minimum and maximum orders were set at 10 and 56 respectively in order to assure resolution of both spectral peaks. Natural frequency and damping estimates were computed for both modes of response and statistics of the estimates were computed as previously described. The results of the two degree-of-freedom simulation study are presented in Figures (4.54 - 4.69). Averages and coefficients of variation for estimates of the first and second mode are contained in Figures (4.54 - 4.61) and Figures (4.62 - 4.69) respectively. The format of these plots is unchanged from the preceding sections. A quick glance through these figures reveals that the presence of a second spectral peak has significantly changed the relationship between the expected value of the estimated parameters and lag (i.e. order of the MEM spectrum). The monotonic convergence of the average value observed with one degree-of-freedom systems has changed to a cyclical variation which decays with increasing order. Examining the natural frequency estimates more closely, it is found that the average natural frequency estimate cycles about a slightly biased estimate of the frequency of maximum response. The amount of bias in these natural frequency estimates is found to increase with decreasing modal separation and decrease with record length. However, when a sufficiently
large order is used in estimating natural frequencies, the inherent bias is found to be less than $\frac{1}{4}$. While the bias in the natural frequency estimate is substantially altered by the presence of a second peak, a comparison between estimates for one and two degree-of-freedom systems reveals that the coefficient of variation is essentially unchanged. An example of this observation is the equivalence between the coefficients of variation shown in Figures 4.30 and 4.69. It is concluded that with the proper selection of order, the MEM natural frequency estimator is, to a close approximation, an asymptotically efficient estimator. MEM damping estimates from the simulations reveal that as the order increases, the average damping exhibits large fluctuations which quickly decay to small cycles about a biased estimate. When the natural frequencies are well separated, one finds that the amount of bias in the second damping ratio is virtually identical to that observed with single degree-of-freedom systems while the damping in the first mode appears unbiased. However, when natural frequencies are more closely spaced, one finds that the average damping estimate for both modes tends toward slightly higher values. In each case, the coefficient of variation appears to be unaffected by the presence of a second mode. In summarizing these observations, it can be said that the MEM damping estimator contains little bias (5% or less, depending on the true damping) and a coefficient of variation very near the Cramér-Rao bound when the record length is long and the natural frequencies are well separated. As modal separation diminished, bias in the damping estimator appears to increase slightly. While the exact behavior of the MEM damping
estimator is unknown for very small modal separation, it is felt that the large systematic biases discussed in Section 2.4 will dominate the damping estimation error.

Figures (4.54 - 4.69) also illustrate the validity of the final order approximation for the coefficients of variation. Unlike the results presented for the single degree-of-freedom simulations, it is found that the theoretical coefficient of variation is substantially different from the measured results for orders corresponding to biased estimates. However, the first order approximation demonstrates good agreement with the simulation results when transients in the averages have decayed. In all cases where the theory departed from the measured values, it was found that the predicted value of the coefficient of variation exceeded the observed value. Consequently, one finds a characteristic "cusp" shape in most of the theoretical coefficients of variation. This trend in the coefficient of variation will prove to be useful in selecting the optimum parameter estimates in the analysis of the field data.
**FIG. 4.54** PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.55 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.56 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.57 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.58 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.59 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 1ST MODE
FIG. 4.60 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 1ST MODE
Figure 4.61: Performance of MEM Damping Estimator
2 DOF - 1st Mode

- Parameter values: $f_n = 0.75 \text{ Hz and 1.0 Hz}$
  - $\xi = 2.0\%$ and $2.0\%$
  - SNR: $30 \text{ dB and 30 dB}$
  - $f_s = 8.0 \text{ Hz}$
  - $T = 21.3 \text{ Min}$
Fig. 4.62 Performance of MEM nat. frequency estimator
2 DOF – 2nd mode
\[ f_n: 0.50 \text{ Hz and } 1.0 \text{ Hz} \]
\[ \xi: 2.0\% \text{ and } 2.0\% \]
\[ \text{SNR: } 30 \text{ dB and } 30 \text{ dB} \]
\[ f_s = 8.0 \text{ Hz} \]
\[ T = 8.5 \text{ Min} \]

FIG. 4.63 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 2ND MODE
FIG. 4.64 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 2ND MODE
Fig. 4.65 Performance of MEM damping estimator

2 DOF - 2nd mode
FIG. 4.66 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 2ND MODE
FIG. 4.67 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 2ND MODE
FIG. 4.68 PERFORMANCE OF MEM NAT. FREQUENCY ESTIMATOR
2 DOF - 2ND MODE
FIG. 4.69 PERFORMANCE OF MEM DAMPING ESTIMATOR
2 DOF - 2ND MODE
CHAPTER 5

Performance of the Estimators with Field Data

The MEM natural frequency and damping estimators have thus far been studied in an ideal setting. Theoretical properties of the estimators were developed in Chapter 3 and characteristics with simulated data were delineated in Chapter 4. The information obtained from these analyses are combined in this chapter into a procedure for the estimation of natural frequencies and dampings from experimentally measured response data.

The guidelines for practical parameter estimation are readily illustrated by means of an example analysis. The data used for this purpose was obtained from ambient response recordings from Shell Oil's South Pass 62C platform. This platform is an eight-leg diagonally braced jacket structure which stands in 327 feet of water. The estimation of this structure's natural frequencies and damping ratios has been the subject of numerous experimental and analytical studies which include a recently conducted full-scale forced vibration test [1, 27, 28]. The platform provides an ideal test for the MEM estimators.

Ambient response measurements were obtained using two accelerometers positioned as shown in Figure 5.1. Acceleration response was
recorded for 32 minutes using an FM tape recorder. The signal was low-pass filtered at 3 Hz prior to recording to remove high frequency machine noise. Records were digitized with a sampling frequency of 6.4 Hz and autocorrelation functions were computed for each of the records with a maximum lag of 160 seconds. MEM spectral estimates computed for both response measurements are shown in Figures (5.2a and b). The fundamental end-on, broadside and torsion modes are clearly revealed in the autospectra. The peak located at 0.5 Hz in end-on response spectrum is attributed to machinery induced vibration and consequently excluded from further consideration. MEM natural frequency and damping estimates and their corresponding standard deviations were determined for each of the fundamental modes. Estimates were evaluated at 44 different orders ranging from 12 to 100 (ie. lag = 2.03 - 15.8 seconds). The resulting estimates are shown in Figures (5.3 - 5.8). The top
FIG. 5.2a & b  RESPONSE SPECTRA FOR SOUTH PASS 62C
figure on each page shows the parameter estimates obtained at different lag to record length ratios (ie. orders of the MEM spectra). The standard deviation \( \hat{\sigma} \), computed with each of the parameter estimates is plotted in the bottom figure. The standard deviation is also used to compute \( + 2\hat{\sigma} \) confidence bounds which are shown in the top figures as dashed lines. These figures, together with the tabular listings of the estimates in Tables (5.1 - 5.3), completes the information necessary to select the "best" estimates of the natural frequencies and dampings.

The essential task in selecting an optimum estimate is to find the value which the estimates converge to as the lag ratio (ie. the lag to total record length ratio) increases. The use of this value as the estimate is supported by results of the simulation studies. In each of the cases described in Chapter 4, the average parameter estimates consistently converged to the true value (or a reasonable approximation of it) as the lag ratio increased. Consequently, selection of an estimate from the "region of convergence" provides reasonable assurance of a minimally biased estimate.

Identification of convergence is in most cases a reasonably straightforward process. Examining Figures 5.3 - 5.3, it is found that the behavior of the parameter estimates as a function of lag ratio is characterized by a period of erratic fluctuations which degenerate to small cyclical variations in the estimates. Similar patterns observed in two degree-of-freedom simulations identify these small variations as characteristic of converging estimates. Simulation results also suggest that additional evidence of convergence is supplied by the
FIG. 5.3 MEM NATURAL FREQUENCY ESTIMATES
1ST BROADSIDE FLEXURE MODE
FIG. 5.4 MEM DAMPING RATIO ESTIMATES
1ST BROADSIDE FLEXURE MODE
FIG. 5.5 MEM NATURAL FREQUENCY ESTIMATES
1ST TORSION MODE
FIG. 5.6  MEM DAMPING RATIO ESTIMATES
1ST TORSION MODE
FIG. 5.7 MEM NATURAL FREQUENCY ESTIMATES
1ST END-ON FLEXURE MODE
FIG. 5.8 MEM DAMPING ESTIMATES
1ST END-ON FLEXURE MODE
Table 5.1
MEM Estimates for First Broadside Flexure Mode

<table>
<thead>
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<th>LAG</th>
<th>NAT. FREQUENCY</th>
<th>DAMPING RATIO</th>
<th>EST</th>
<th>STD[ ]</th>
<th>EST</th>
<th>STD[ ]</th>
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<td>0.002</td>
<td>5.006</td>
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<td></td>
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<td>0.122</td>
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<td>0.002</td>
<td>2.505</td>
<td>0.204</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.641</td>
<td>0.001</td>
<td>2.072</td>
<td>0.181</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.640</td>
<td>0.001</td>
<td>2.375</td>
<td>0.205</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.171</td>
<td>0.643</td>
<td>0.002</td>
<td>3.529</td>
<td>0.311</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.187</td>
<td>0.651</td>
<td>0.001</td>
<td>2.610</td>
<td>0.242</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.203</td>
<td>0.650</td>
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STD[ ] = STANDARD DEVIATION
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<td>0.001</td>
<td>1.923</td>
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<td>0.002</td>
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<td>0.002</td>
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<tr>
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<td>0.633</td>
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<td>0.007</td>
<td>3.673</td>
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<td>2.720</td>
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<td>0.005</td>
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<tr>
<td>0.822</td>
<td>0.637</td>
<td>0.004</td>
<td>3.105</td>
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</tbody>
</table>

**NOTES:**

EST = ESTIMATED VALUE  
STD[ ] = STANDARD DEVIATION
presence of a gradual increase in the standard deviation across the region of convergence. While the standard deviations shown for the field data exhibit fairly strong fluctuations, the prescribed trend is clearly visible.

Once convergence of the parameter estimate has been identified, the "best" estimate can be read from the plot or calculated as the numerical average of the estimates in the region of convergence. For example, the region of convergence in Figure 5.4 can be defined by the lag ratios 0.33% - 0.82%. Accordingly, the best estimate of the damping, determined as the average of the corresponding entries in Table 5.2, is 2.0%. The standard deviation which quantifies the uncertainty in the best estimate is given by the approximate value observed at the onset of convergence. For the preceding example, the standard deviation is approximately 0.3. In view of the judgement required to locate the region of convergence and the approximate nature of the standard deviation calculation, estimates should be limited to three significant figures for natural frequencies, two for damping ratios, and one for standard deviations.

This selection procedure was applied Figures 5.3 - 5.8 and the resulting natural frequency and damping estimates are shown in Table 5.4 with their corresponding ±2σ uncertainties. Natural frequencies and dampings were also computed from Blackman-Tukey spectral estimates using a maximum lag of 160 seconds. These results are included in Tables 5.4 and 5.5 along with estimates obtained by J.A. Ruhl [28] using forced, vibration. The slight disparity between the MEM natural frequency estimates
Table 5.4
South Pass 62C Natural Frequency Estimates for First-Order Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Blackman-Tukey</th>
<th>MEM</th>
<th>Frequency Sweep</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broadside</td>
<td>0.651</td>
<td>0.643 ± .004</td>
<td>0.648</td>
</tr>
<tr>
<td>End-on</td>
<td>0.638</td>
<td>0.639 ± .004</td>
<td>0.642</td>
</tr>
<tr>
<td>Torsion</td>
<td>0.955</td>
<td>0.955 ± .004</td>
<td>0.960</td>
</tr>
</tbody>
</table>

All natural frequencies measured in Hz

Table 5.5
South Pass 62C Damping Ratio Estimates for First-Order Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Blackman-Tukey</th>
<th>MEM</th>
<th>0-4</th>
<th>5-9</th>
<th>0-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broadside</td>
<td>1.14</td>
<td>2.0 ± 0.6</td>
<td>1.65</td>
<td>0.86</td>
<td>1.38</td>
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<tr>
<td>End-on</td>
<td>0.45</td>
<td>2.1 ± 0.6</td>
<td>1.72</td>
<td>1.29</td>
<td>1.53</td>
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<tr>
<td>Torsion</td>
<td>0.27</td>
<td>1.3 ± 0.4</td>
<td>1.20</td>
<td>1.03</td>
<td>1.11</td>
</tr>
</tbody>
</table>

All damping ratios measured in %
and Ruhl's estimates is the result of using two different tape recorders for recording and play-back of the response measurements. A 2 Hz calibration signal was used but its frequency was not known with the required precision. The MEM damping estimates show very good agreement with Ruhl's transient decay tests. In each case, Ruhl's estimates computed over cycles 0-4 and 0-10 are included in the confidence interval for the MEM estimates. Conversely, damping estimates compute from Blackman-Tukey spectra severely underpredict the damping. However, the use of slightly different resolution (i.e. alternate windows and/or maximum lags) produced damping estimates which significantly overpredicted the damping. In these cases, there was no apparent means of choosing the appropriate estimate.
CHAPTER 6

Summary and Recommendations

The primary result of this work is the development of a new method for estimating natural frequencies and damping ratios from measurements of a structure's response to ambient excitations. The estimation algorithm was obtained by analytically combining the maximum response natural frequency estimator and the half-power bandwidth damping estimator with a powerful method of spectral analysis known as the Maximum Entropy Method (MEM). The resulting scheme was shown to yield estimates of natural frequencies and damping ratios using the numerical solutions of two "estimation equations." The analytic form of the estimators also made possible the derivation of a first-order approximation for the bias and variance of the parameter estimators.

Monte Carlo simulations were used to demonstrate the utility of the MEM estimators and their approximate statistics. It was found that when the order of the MEM spectrum was correctly chosen the parameter estimators exhibited little bias (excluding systematic bias) and variances approximately equaled the Cramér-Rao lower bound. The results of simulation studies of one and two degree-of-freedom systems were also used to formulate guidelines for the selection of "optimum" order associated with each of the parameters.

Simulation studies were also conducted for maximum response natural frequency estimates and half-power bandwidth damping estimates computed from Blackman-Tukey spectral estimates. The results demonstrated a classical tradeoff which must be accomplished between the bias and variance associated with the parameter estimator. Unfortunately, no guidelines are
available to aid the analyst in computing the optimum estimates. It was found that the bias and variance of typical natural frequency estimates is so small that virtually any reasonable resolution yields adequate results. These simulations also revealed that the variance of "optimum" damping estimates made using 30 min records is so large that estimates are virtually useless for engineering purposes. This problem can be remedied by increasing the record length but violation of the stationary assumption may introduce undesirable effects.

While this research has provided an extensive introduction to the MEM parameter estimators, several possibilities exist for additional research. One topic which as yet remains unexplored concerns the probability distribution of the estimators. The approximate efficiency of the estimators suggests a tendency toward a normal distribution but an indepth analysis is required to support this hypothesis.

Another issue involves parameter estimation from a record containing a deterministic component. This problem was first encountered in chapter 5 when machinery induced vibration contaminated the broadside response of the platform. In computing the optimum natural frequency and damping estimates of the first broadside mode it was found that the parameter estimates did not strictly adhere to the previously specified selection guidelines -- an effect thought to be caused by the deterministic signal. In this instance difficulties were circumvented by a judicious choice of the "region of convergence." However, this anomaly suggests the need for further simulation studies.

As a final note it is observed that this thesis has only used the single-channel capabilities of MEM spectral analysis. The complete multi-
channel analysis tools currently available for MEM have yet to be implemented in vibration testing. It is felt that these methods will prove extremely valuable especially in the area of mode shape identification.
REFERENCES


ADDITIONAL REFERENCES ON MEM


