

ON THE DIFFRACTION OF FREE SURFACE WAVES

BY A SLENDER SHIP

by

Paul Sclavounos

Diploma, National Technical University of Athens
Department of Naval Architecture and Marine Engineering
(1977)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 1981

© Massachusetts Institute of Technology 1981

Signature of Author Signature redacted
Department of Ocean Engineering
January 1981

Certified by Signature redacted Signature redacted
Ronald W. Yeung J. N. Newman
Supervisor Supervisor

Accepted by Signature redacted
ARCHIVES
Chairman, Departmental Committee on Graduate Students
OF TECHNOLOGY

APR 21 1981

ON THE DIFFRACTION OF FREE SURFACE WAVES

BY A SLENDER SHIP

by

Paul Sclavounos

Submitted to the Department of Ocean Engineering
in February, 1981, in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

ABSTRACT

A linear theory is presented for the analysis of the interaction of a slender ship fixed at its mean advancing position with a small amplitude monochromatic and unidirectional incident wave train. The length of the incident waves and the angle of incidence are unrestricted and the ship forward speed is assumed constant. The "unified theory" is applied, appropriately modified for the present problem. The smallness of the ship beam to length ratio is exploited to apply the method of matched asymptotic expansions for the solution of the resulting boundary value problem. Predicted values of the pressure distribution, sectional forces and the total exciting force and moment in head waves are compared with existing theories and experiments for two bodies of revolution.

Supervisor: Ronald W. Yeung
Title: Associate Professor of Naval Architecture

Supervisor: J. N. Newman
Title: Professor of Naval Architecture

AKNOWLEDGEMENTS

I wish to express my sincere thanks to my advisor Professor J. Nicholas Newman for his advice, encouragement and stimulation during my graduate studies. I also direct my appreciation to Professor Ronald W. Yeung for his help and computational assistance, as well as to Professors Chiang C. Mei, Odd M. Faltinsen and David J. Benney for many valuable discussions.

Financial support, while in M.I.T., was provided by the Office of Naval Research, National Science Foundation, and the Naval Sea System Command's General Hydromechanics Research Program administered by the David W. Taylor Naval Ship Research and Development Center, and it is gratefully acknowledged.

For her proficiency and patience in typing this manuscript, many thanks I owe to Ms. Joanne Sullivan.

	<u>PAGE</u>
ABSTRACT	2
ACKNOWLEDGEMENTS	3
TABLE OF CONTENTS	4
LIST OF TABLES	6
LIST OF FIGURES	6
I. INTRODUCTION	8
II. THE BOUNDARY VALUE PROBLEM	22
III. THE OUTER PROBLEM	31
A. THE SHORT-WAVELENGTH APPROXIMATION FOR $U=0$	45
B. THE "LARGE τ " APPROXIMATION	51
C. THE SMALL τ APPROXIMATION	54
IV. THE INNER PROBLEM	56
V. THE MATCHING	69
VI. THE PRESSURE FORCE	75
VII. NUMERICAL RESULTS AND COMPARISONS WITH EXPERIMENTS	80
VIII. CONCLUSIONS	120
REFERENCES	124
APPENDIX 1. The inner expansion of the outer solution	127
APPENDIX 2. Reduction of the kernel	133
APPENDIX 3. The short-wavelength approximation for $U=0$	141

	<u>PAGE</u>
APPENDIX 4. The "large τ " approximation	148
APPENDIX 5. Evaluation of m_3 for a prolate spheroid	153

LIST OF TABLES

		<u>PAGE</u>
Table 1	Source strength, sectional force and velocity potential of the two-dimensional homogeneous problem for a circular boundary, versus va	90

LIST OF FIGURES

Figure 1	Coordinate system	22
Figure 2a	Regular part (f_R) of the kernel of (3.35) as a function of the longitudinal coordinate $ x /L$ for $\tau=0.2$ and $Fr=0.2$	91
Figure 2b	Same as in Figure 2a with $\tau=0.7$	92
Figure 3a	Longitudinal distribution of the normalized pressure amplitude on the model used by Maruo and Sasaki, for $\theta=0^\circ$, $\lambda/L=0.621$ and $U=0$.	93
Figure 3b	Same as in Figure 3a with $\theta=40^\circ$	94
Figure 3c	Same as in Figure 3a with $\theta=70^\circ$	95
Figure 4a	Normalized pressure amplitude at the section $x=0.3L$ on the model used by Maruo and Sasaki, at $\theta=40^\circ$ and $U=0$ versus λ/L .	96
Figure 4b	Same as in Figure 4a with $x=0$.	97
Figure 4c	Same as in Figure 4a with $x=-0.3L$	98
Figure 5a	Longitudinal distribution of the normalized sectional force amplitude on a prolate spheroid ($\epsilon=1/6$) in head waves at $Fr=0.123$ and $\lambda/L=0.5$.	99
Figure 5b	Same as in Figure 5a with $\lambda/L=1.0$.	100

	<u>PAGE</u>
Figure 5c	Same as in Figure 5a with $\lambda/L=1.5$. 101
Figure 6a	Same as in Figure 5a with $Fr=0.205$ 102
Figure 6b	Same as in Figure 6a with $\lambda/L=1.0$. 103
Figure 6c	Same as in Figure 6a with $\lambda/L=1.5$. 104
Figure 7a	Same as in Figure 5a with $Fr=0.246$ 105
Figure 7b	Same as in Figure 7a with $\lambda/L=1.0$. 106
Figure 7c	Same as in figure 7a with $\lambda/L=1.5$. 107
Figure 8a	Normalized amplitude of heave exciting force on a prolate spheroid ($\epsilon=1/6$) in head waves at $Fr=0.123$, versus λ/L . 108
Figure 8b	Phase angle of heave exciting force (leading incident wave elevation at $x=0$) at $Fr=0.123$, versus λ/L 109
Figure 9a	Normalized amplitude of pitch exciting moment on a prolate spheroid ($\epsilon=1/6$) in head waves at $Fr=0.123$, versus λ/L . 110
Figure 9b	Phase angle of pitch exciting moment (leading incident wave elevation at $x=0$) at $Fr=0.123$, versus λ/L . 111
Figure 10a	Same as in Figure 8a with $Fr=0.205$. 112
Figure 10b	Same as in Figure 8b with $Fr=0.205$. 113
Figure 11a	Same as in Figure 9a with $Fr=0.205$. 114
Figure 11b	Same as in Figure 9b with $Fr=0.205$. 115
Figure 12a	Same as in Figure 8a with $Fr=0.246$. 116
Figure 12b	Same as in Figure 8b with $Fr=0.246$. 117
Figure 13a	Same as in Figure 9a with $Fr=0.246$. 118
Figure 13b	Same as in Figure 9b with $Fr=0.246$. 119
Figure A2-1	Mapping from the z to the k plane. 135

INTRODUCTION

Seagoing ships often encounter unfavorable weather conditions and the accurate prediction of the ship motions, structural loads and drift forces is of unquestionable importance for a safe and economical operation.

Using the techniques of spectral analysis, an ambient seaway is traditionally represented as the superposition of small amplitude linear unidirectional and monochromatic wave components. The hydrodynamic interaction of a ship advancing at a constant forward speed with such a wave component can be decomposed, through a systematic perturbation procedure, into the *radiation problem* where the ship undergoes a predetermined oscillatory motion on an otherwise calm free surface, and the *diffraction problem* where the ship is kept fixed at its mean advancing position and is otherwise free to interact with the incident wave train. In the present study the latter problem is analysed for a ship advancing at a constant forward speed among waves of unrestricted wavelength and angle of incidence.

The first hydrodynamic analysis known of the ship motion problem is connected with the names of Froude (1861) and Krylov (1896). Froude and Krylov derived differential equations for the motion of the ship in waves by including the inertial and hydrostatic forces for the radiation

problem, without attempting to further analyse the hydrodynamic disturbance associated with the ship oscillatory motion. The exciting force was evaluated using the undisturbed pressure field of the incident wave, and subsequently has been known as the Froude-Krylov exciting force.

Korvin-Kroukovsky (1955), in a pioneering effort, used the concepts of the slender body theory of aerodynamics together with substantial physical insight to derive a three-dimensional strip theory for the approximation of the ship motions in head waves. In order to evaluate the exciting forces, the relative motion hypothesis was used, where the ship was assumed to undergo a snakelike vertical oscillatory motion in the absence of the incident wave, with a local velocity opposite to that of the undisturbed wave profile. Hydrodynamic interactions were neglected between adjacent ship sections, and the exciting force was determined from the integration of the sectional forces obtained from the solution of the two-dimensional heaving problems. Refinements were subsequently provided by Korvin-Kroukovsky and Jacobs (1957).

A modification of the Korvin-Kroukovsky strip theory was presented by Gerritsma and Beukelman (1967) who introduced a mean effective depth for each ship section

where the diffraction pressure was assumed to act. This is equivalent to the assumption that the velocity of each ship section in the relative motion hypothesis is opposite to the vertical fluid velocity at the mean effective depth due to the incident wave.

A comprehensive derivation of a strip theory, valid for short waves, was carried out by Salvesen, Tuck and Faltinsen (1970) in the context of the prediction of the ship motions and loads in waves.

A rational derivation of a short-wavelength slender body theory for the radiation problem was carried out by Ogilvie and Tuck (1969), where a strip theory approximation was shown to be an adequate representation of the flow field adjacent to the ship hull.

The complementary regime of wavelengths comparable to the ship length was analysed by Newman (1964) and Newman and Tuck (1964) in the context of a long-wavelength *ordinary* slender body theory of ship motions. Both the radiation and diffraction problems were analysed, use of an outer and an inner solution was made and Green's theorem was used to establish the compatibility of the flow field representations in the inner and outer regions. The resulting three-dimensional interactions appeared as an additive function of x in the two-dimensional inner solution,

which, due to the long-wavelength assumption, was assumed to satisfy a zero flux free surface boundary condition.

Ursell (1968a) analysed the interaction of a regular, wave train with an infinitely long semisubmerged circular cylinder, and proved that the solution for the diffraction potential can be expanded in a convergent multipole series; the Green function used for the oblique incidence case was required to satisfy outgoing waves at *infinity* and turned out to be singular for head waves. In this limit the expansion theorem was modified by using a Green function that grows linearly at infinity, but is otherwise bounded at finite transverse distances. This property was used by Ursell (1968b) to show that head waves cannot propagate along an infinitely long circular cylinder without deformation. The interaction of a semi-infinite circular cylinder with head waves was analysed by Ursell (1975) where the deformation of the incident waves was predicted.

A detailed analysis of the head-wave diffraction problem for a slender ship was carried out by Faltinsen (1971) for the case where the incident wavelength is of the order of the ship beam. When formulating the inner problem, Faltinsen assumed that to leading order the diffracted wave cancels out the incident wave, allowed three-dimensional interaction effects only in the outer

problem and obtained a solution possessing a square root singularity at the bow. The analysis was carried out to next order, three-dimensional interaction effects were introduced in the inner problem, but the square root singularity persisted.

Maruo and Sasaki (1974) presented a modified approach for the zero speed short-wavelength head-sea diffraction problem which removes the bow singularity. The Maruo and Sasaki solution, appropriately corrected, is derived as the short-wavelength limit of the present approach.

Choo (1975) and Troesh (1976) analysed the oblique incidence diffraction problem for zero and forward speed respectively and wavelengths of the order of the ship beam. No three-dimensional interactions were found between the adjacent ship sections, and a strip theory representation was derived for both the symmetric and antisymmetric parts of the diffraction potential. The solution is singular in the head-sea limit, indicating the failure of strip theory to model the head incidence diffraction problem.

Skjrdal and Faltinsen (1980) generalized the Maruo and Sasaki theory to account for forward speed effects, in the context of deriving a linear theory of springing, and Liapis and Faltinsen (1980) extended the same theory to oblique waves using a composite approach.

Mei and Tuck (1980) extended the parabolic approximation to the diffraction of short head waves by thin obstacles where, due to the body geometry or the small water depth, the two-dimensional Helmholtz equation is satisfied in the fluid domain. Haren and Mei (1981) applied the same method to the analysis of the head-wave diffraction by an articulated raft of small draft in the context of wave energy absorption. The parabolic approximation leads to the solution of a Volterra integral equation of the second kind for the scattered wave amplitude, similar to the Maruo and Sasaki integral equation and in agreement with the short-wavelength limit of the present approach.

For the radiation problem, the restriction of short waves was removed by Newman (1978). A *unified* slender body theory was derived under the single assumption of a small beam to length ratio. This theory covered the gap between the ordinary slender body theory and the strip theory, and both theories were recovered in the limit of long and short wavelengths respectively. The velocity potential in the inner region included a particular solution similar to that of strip theory, plus a homogeneous component to account for the interaction effects, analogous to the additive function of x in the ordinary slender body theory. The inner solution was subsequently matched to a complementary outer solution for the quantitative evaluation of the interaction effects. A similar approach was attempted for the diffraction problem where the particular component of the inner solution canceled out the normal velocity of the incident wave on the body boundary, and the homogeneous solution again accounted for the longitudinal interaction effects. Both the particular and the homogeneous solutions satisfied the Helmholtz equation in the fluid domain (due to the factorization of the longitudinal wave component), and were constructed using the two-dimensional Green function which turns out to be singular for head-incidence. The same singularity was inevitably

shared by the approximation of the outer solution in the overlap region.

In the present study the application of the unified theory to the diffraction problem is completed and modified where necessary in order to preserve the regularity of the solution for head incidence. In what follows, the flow pattern is discussed from the physical standpoint, differences and similarities to the radiation problem are pointed out and the framework is created for the mathematical analysis of the subsequent chapters.

Conventional ship hulls are slender in the sense that the geometry variation in the longitudinal (x-axis) direction is small compared to the corresponding variation in the transverse (y and z axes) directions.

At radial distances of the order of the ship length or greater (*outer* region), the flow is fully three-dimensional and depends mainly on the elongated nature of the ship hull, being relatively insensitive to its local geometrical details. Hence, the oscillatory disturbance resulting in the outer region from the diffraction of the incident regular waves, satisfies the three-dimensional Laplace equation, the linearized free surface condition, with or without forward speed dependence, and the condition of outgoing waves at infinity. Since at distances of the

order of the ship length disturbances associated with comparable length scales dominate, we may approximate the outer solution by a distribution of free surface wave sources and dipoles along the ship centerline that satisfy the three-dimensional flow equations and the radiation condition at infinity. The precise quantitative information about the mechanism of diffraction is determined only after the flow close to the ship hull is analysed. Thus, the source strength and dipole moment distributions are at the present stage unknown and the representation of the outer solution is so far primarily of a qualitative importance.

At transverse distances of the order of the ship beam (*inner* region), the flow gradients, after the x-component of the incident wave is factored out, are dictated by the respective order of the geometry gradients. Consequently, the wave disturbance generated in the inner region from the successive reflections of the incident wave train along the ship hull, is required to satisfy a set of two-dimensional equations and boundary conditions, and is thus insensitive to the nature of the three-dimensional environment that the outwards propagating waves will encounter. This qualitative information is provided from the previous representation of the outer solution, appropriately

approximated in an *overlap* region. The mathematical analysis of the present study is largely devoted to the derivation of this approximation, which, in the jargon of the method of matched asymptotic expansions is called inner expansion of the outer solution. This approximation has to be compatible with the corresponding outer expansion of the inner solution in the overlap region, for the possibility of exchanging information.

Similar arguments apply to the radiation problem, as far as the outer region is concerned. The reason is that both wave disturbances, generated either by the diffraction of an incident regular wave train or by a prescribed hull normal velocity, will eventually propagate into the same three-dimensional environment. The fundamental difference between the radiation and the diffraction problems lies in the inner region and is associated with the process of the wave generation.

When a ship performs a forced oscillatory motion on an otherwise calm free surface, wave disturbances generated along the ship hull, will include three dimensional interactions due to the finite ship length, the longitudinal geometry variation and the possible x -dependence of the mode of oscillation. The strength of such interactions decays as the generated wavelength to ship length ratio

decreases. In the limiting case of wavelengths comparable to the ship transverse dimensions, the strip theory is applicable.

The situation is different when diffraction phenomena are involved. Head waves are considered first and the conclusions drawn are subsequently generalized for oblique waves. An incident wave impinging on the ship bow gradually deforms as it propagates downstream, and the energy it carries is progressively reflected away. The resulting total wave elevation in the stern region is reduced in amplitude and shifted in phase, compared to the undisturbed incident train. Furthermore, the ratio of the total wave elevation in the stern region to the incident wave amplitude decreases with decreasing incident wavelength, due to the increased number of collisions that a short incident wave undergoes with the ship hull. Hence, as opposed to the radiation problem, the strength of the longitudinal flow interactions increases with decreasing incident wavelength to ship length ratio, and a strip theory representation is ruled out.

In oblique waves the resulting three-dimensional interactions along the ship hull are weaker. This is especially apparent in the short-wavelength limit where, in contrast to the head-incidence case, the interactions are absent and strip theory is valid. The previous arguments do not apply for angles of incidence β close to 180° ($\beta=180^\circ$ for head waves) where the longitudinal interactions are still strong.

From the technical standpoint, it is more convenient to consider the total velocity potential as our unknown in the inner region. After factorizing the longitudinal component of the incident wave, the sum of the incident and diffraction velocity potentials satisfies the two-dimensional Helmholtz equation in the fluid domain, the wave free surface condition and a zero normal velocity on the ship section of interest. No radiation condition is necessary in the inner problem, since it has been already satisfied by the diffraction potential in the outer problem. The general solution of the previous homogeneous boundary value problem at a plane of constant x-coordinate can be written as the product of a normalized two-dimensional homogeneous solution, times an *interaction coefficient*, function of x, which represents the wave amplitude attenuation along the ship and does not violate the two-dimensional flow equations.

The regularity of the homogeneous solution for head incidence is achieved by using a two-dimensional Green function that satisfies standing waves at infinity, which for head waves becomes identical to the modified Green function used by Ursell (1968a). An appropriate modification of the inner expansion of the outer solution leads to the compatibility of the inner and the outer problems and to a solution regular for all angles of incidence.

The interaction coefficients of the symmetric and antisymmetric inner solutions, together with the unknown source and dipole distributions of the outer solution, are determined from solving a system of four equations with four unknowns, resulting from the compatibility requirements of both problems in the overlap region. As expected, it turns out that the interaction coefficient of the symmetric disturbance is equal to unity at the ship bow and decreases toward the stern, the total decrease being larger for shorter waves. On the other hand, the corresponding coefficient of the antisymmetric disturbance is constant, indicating the absence of longitudinal interactions.

In Chapter II, the exact linearized boundary value problem for the unknown diffraction potential is formulated, and the fundamental assumptions of the unified theory are

explicitly stated. The method of matched asymptotic expansions is used for the solution of the problem. The analysis of the outer solution is carried out in Chapter III, where the "inner expansion" is derived together with its short-wavelength approximations for $U=0$ and $U>0$. The inner problem is formulated in Chapter IV and the compatibility conditions (or matching) are described in Chapter V. Chapter VI deals with the determination of the pressure on the hull and the form of the exciting forces and moments. The numerical solution of the two-dimensional homogeneous problem and of the equations resulting from the matching are described in Chapter VII. In the same Chapter the predicted pressure and sectional force distributions and exciting forces and moments are compared with existing theories and experimental measurements for two bodies of revolution interacting with head waves for $U=0$ and $U>0$. Finally, in Chapter VIII conclusions are drawn and recommendations are made for further research.

II. THE BOUNDARY-VALUE PROBLEM

The ship is assumed to be fixed at its mean position and advancing at a constant forward velocity U . It is convenient to introduce two Cartesian coordinate systems, one fixed in space with $\vec{x}_0 = (x_0, y_0, z_0)$ and one with $\vec{x} = (x, y, z)$ fixed with respect to the ship and defined by

$$\vec{x} = \vec{x}_0 - \vec{U}t \quad (2.1)$$

The undisturbed free surface is taken at $z_0 = 0$ and the ship centerplane at $y_0 = 0$, the positive x_0 axis pointing in the direction of the ship forward velocity and the z_0 axis being positive upwards. The fluid is assumed ideal and incompressible with constant density ρ and the fluid motion irrotational. Surface tension effects are neglected.

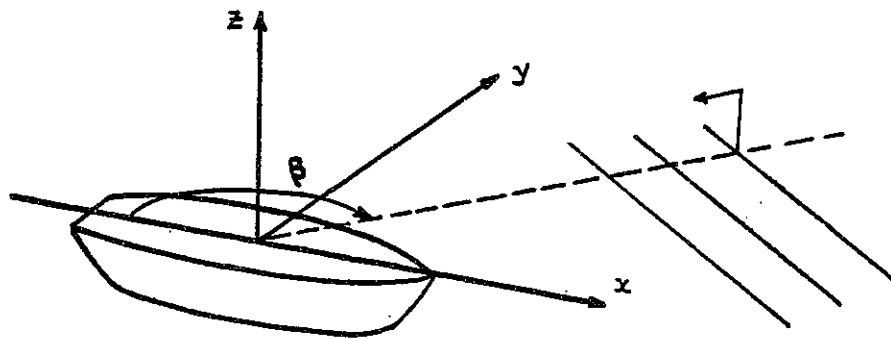


Figure 1 - Coordinate system

The fluid velocity vector $\vec{v}(x_0)$ is defined as the gradient of the velocity potential $\phi(\vec{x}_0, t)$ satisfying the Laplace equation

$$\nabla^2 \phi(\vec{x}_0, t) = 0 \quad (2.2)$$

in the fluid domain. The normal velocity of the ship submerged surface S is equal to the adjacent fluid normal velocity,

$$\vec{v} \cdot \vec{n} = \vec{U} \cdot \vec{n} \quad \text{on } S \quad (2.3)$$

where \vec{n} is the unit normal vector pointing out of the fluid domain.

On the free surface, defined by its elevation $z_0 = \zeta(x_0, y_0, t)$, the kinematic boundary condition

$$\frac{D}{Dt}(\zeta - z_0) = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) (\zeta - z_0) = 0 \quad \text{on } z_0 = \zeta \quad (2.4)$$

and the dynamic boundary condition of zero pressure

$$-p/\rho = \phi_t + \frac{1}{2} \vec{v}^2 + g z_0 = 0 \quad \text{on } z_0 = \zeta \quad (2.5)$$

lead to the exact non-linear free surface condition

$$\phi_{tt} + 2\nabla\phi \cdot \nabla\phi_t + \frac{1}{2} \nabla\phi \cdot \nabla(\nabla\phi \cdot \nabla\phi) + g\phi_{z_0} = 0 \quad \text{on } z_0 = \zeta \quad (2.6)$$

with

$$\zeta = -\frac{1}{g}(\phi_t + \frac{1}{2}v^2) \quad z_0 = \zeta \quad (2.7)$$

The fluid velocity vanishes at $z_0 = -\infty$ and a radiation condition has to be imposed on the ship generated disturbance potential.

The set of equations (2.1) - (2.7) formulate the exact boundary value problem of a floating or submerged three dimensional body fixed at its mean position, advancing at a constant forward speed U and interacting with an arbitrary wave system.

The purpose of the present study is to analyse the unsteady interaction of a slender ship with free surface plane progressive waves of small amplitude A , sinusoidal profile and direction of propagation forming an angle β with the positive x semi-axis. For deep water, their velocity potential is

$$\phi_I = \text{Re} \left\{ \frac{igA}{\omega_0} \exp[\nu(z_0 - ix_0 \cos\beta - iy_0 \sin\beta) + i\omega_0 t] \right\} \quad (2.8)$$

where ω_0 is the radian frequency in the space fixed coordinate system, $\nu = \omega_0^2/g = 2\pi/\lambda$ is the wavenumber, λ the wavelength and g the acceleration of gravity. Re , stands for the real part of the quantity involved.

The potential ϕ_I satisfies Laplace's equation (2.2) and the linearized free surface condition that results from (2.6) and (2.7) after the non-linear terms have been dropped out

$$\phi_{I \text{ tt}} + g \phi_{I \text{ z}_0} = 0 \quad \text{on } z_0 = 0 \quad (2.9)$$

In the moving coordinate system we define

$$\phi_I(x + Ut, y, z, t) = \tilde{\phi}_I(x, y, z, t) \quad (2.10)$$

hence

$$\phi_I(x_0, y_0, z_0, t) = \tilde{\phi}_I(x_0 - Ut, y, z, t) \quad (2.11)$$

The application of the operator $\frac{\partial}{\partial t}$ on both sides of (2.11) leads to

$$\phi_{I \text{ t}} = \frac{\partial}{\partial t} \tilde{\phi}_I(x_0 - Ut, y, z, t) = \left(\frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) \tilde{\phi}_I(x, y, z, t) \quad (2.12)$$

In the translating coordinate system the free surface condition (2.9) takes the form

$$\left(\frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right)^2 \tilde{\phi}_I + g \tilde{\phi}_{I \text{ z}} = 0 \quad \text{on } z=0 \quad (2.13)$$

Substituting (2.8) in (2.10) we obtain

$$\tilde{\phi}_I = \text{Re} \left\{ \phi_I(\vec{x}) e^{i\omega t} \right\} \quad (2.14)$$

with

$$\phi_I = \frac{igA}{\omega_0} \exp [v(z-ix\cos\beta - iysin\beta)] \quad (2.15)$$

where ω is the frequency of encounter

$$\omega = \omega_0 - Uv\cos\beta \quad (2.16)$$

Since the amplitude of the incident waves is small we hereafter assume that products of oscillatory quantities are of higher order and are thus neglected. On the other hand the steady disturbance associated with the ship's forward motion is not necessarily small compared to the unsteady one unless geometrical restrictions are imposed. Under the previous assumptions it is legitimate to decompose the total velocity potential into a steady and an unsteady part

$$\phi(\vec{x}_0, t) = \bar{\phi}(\vec{x}_0, t) + \phi_I(\vec{x}_0, t) + \phi_7(\vec{x}_0, t) \quad (2.17)$$

where $\bar{\phi}$ is the velocity potential due to the steady translation and ϕ_7 is the unsteady potential of the diffracted free surface waves.

The free surface condition satisfied by ϕ_7 is obtained from substitution of (2.17) in (2.6) and use of (2.9). Neglecting second order terms and subtracting the steady part one obtains an equation linear in ϕ_7 with varying coefficients, functions of $\bar{\phi}$ and its derivatives, and a non-homogeneous right hand side involving products of ϕ_I and $\bar{\phi}$.

In the present work we neglect any interactions between the steady and unsteady disturbances. This assumption is justified for a slender ship at moderate speeds. A more detailed derivation and discussion of the boundary value problem and the interaction effects is presented in Newman (1978).

The resulting free surface boundary condition for ϕ_7 is (2.9) and (2.13) in the fixed and moving coordinate systems respectively. In accordance with the notation introduced in (2.14)

$$\tilde{\phi}_7 = \text{Re} \left\{ \phi_7(\vec{x}) e^{i\omega t} \right\} \quad (2.18)$$

Because of the linearity of the mathematical problem, the operator $\text{Re}(\cdot e^{i\omega t})$ is omitted and understood hereafter.

Equation (2.13) takes the form

$$(i\omega - U \frac{\partial}{\partial x})^2 \phi_{I,7} + g \frac{\partial}{\partial z} \phi_{I,7} = 0 \quad \text{on } z=0 \quad (2.19)$$

and the hull boundary condition (2.3)

$$\frac{\partial}{\partial n} (\phi_I + \phi_7) = 0 \quad (2.20)$$

on the undisturbed wetted surface S_0 .

In summary, the linear boundary value problem satisfied by $\phi_7(\vec{x})$ in the moving reference frame is

$$\nabla^2 \phi_7 = 0 \quad \text{in the fluid domain} \quad (2.21)$$

$$(i\omega - U \frac{\partial}{\partial x})^2 \phi_7 + g \phi_{7z} = 0 \quad \text{on } z=0 \quad (2.22)$$

$$\frac{\partial}{\partial n} \phi_7 = - \frac{\partial}{\partial n} \phi_I \quad \text{on } S_0 \quad (2.23)$$

$$\nabla \phi_7 \rightarrow 0 \quad \text{at } z=-\infty \quad (2.24)$$

supplemented by the requirement that ϕ_7 represents outgoing waves at infinity.

John (1950) proved the existence and uniqueness of the solution of (2.21) - (2.24) for zero speed and certain smoothness conditions at the free surface hull intersection. No such proof exists, to the author's knowledge, for finite forward speed. For an arbitrary geometry the solution can be obtained using Green's theorem. An integral equation for the unknown potential on the wetted surface and/or the free surface is derived. The solution involves in general the inversion of a big matrix and lengthy computations for the forward speed case.

Since conventional ship hulls are slender, our effort will be concentrated in exploiting the small beam and draft to length ratio ϵ by applying the method of matched asymptotic expansions.

Before any mathematical manipulation we state our basic assumptions. The ship forward speed U is restricted only to the extent that interactions between the steady and unsteady disturbances remain relatively unimportant. We set

$$U \leq 0 \quad (1) \quad (2.25)$$

with the interpretation that U assumes any value from zero to any quantity of order one, independently of how small ϵ is.

The heading angle β is unrestricted. The limiting cases $\beta=0^\circ, 180^\circ$ merit special attention primarily from the mathematical standpoint.

The wavelength of the incident waves varies independently of the ship dimensions. Nevertheless it will be of interest to investigate the behaviour and accuracy of the solution for long and short waves. It is usual when applying the method of matched asymptotic expansions to introduce the parametrization

$$\omega_0 = 0(\epsilon^{-\gamma}) \quad (2.26)$$

For $\gamma=0$ the wavelength $\lambda=2\pi g/\omega_0^2$ is of the order of the ship length and for $\gamma=1/2$ of the order of the beam.

In the next section we analyse the outer problem first for reasons that will be readily apparent.

III. THE OUTER SOLUTION

The outer solution defined for $R = (x^2 + y^2 + z^2)^{1/2} \gg 0(1)$, satisfies the following set of equations

$$\nabla^2 \phi_7 = 0 \quad \text{in the fluid domain} \quad (3.1)$$

$$\left[i\omega - U \frac{\partial}{\partial x} \right]^2 \phi_7 + g \frac{\partial}{\partial z} \phi_7 = 0 \quad \text{on } z=0 \quad (3.2)$$

$$\text{Outgoing waves at infinity} \quad (3.3)$$

As the slenderness ratio ϵ tends to zero the ship shrinks to a line segment. It is thus justified to approximate the far field disturbance by a superposition of singularities over the ship centerline that satisfy the set of equations (3.1)-(3.3). The symmetric and anti-symmetric parts of ϕ_7 are represented by source and transverse dipole distributions respectively.

$$\phi_7(\vec{x}) = \int_L d\xi \left\{ q_7(\xi) + \mu_7(\xi) \frac{\partial}{\partial y} \right\} G(x-\xi, y, z) \quad (3.4)$$

where $G(x, y, z)$ is the velocity potential of a pulsating source of unit strength located at $\vec{x}=0$ in the presence of a uniform stream $-U$, defined as follows

$$G(x, y, z) = - \frac{1}{8\pi^2} \lim_{\mu \rightarrow 0^+} \int_0^\infty dk k \int_0^{2\pi} d\theta \times \frac{\exp[kz + ik(x\cos\theta + y\sin\theta)]}{k - (\omega - i\mu - Uk\cos\theta)^2/g} \quad (3.5)$$

The details of the derivation can be found in Wehausen and Laitone (1960) and Lighthill (1967).

The source and dipole strengths $q_7(x)$ and $\mu_7(x)$ are unknown quantities and will be determined from the matching with the inner solution. A more accurate representation of the outer flow field will involve higher order singularities but to the present order of approximation this is unnecessary.

It is preferable to work in the Fourier domain where the convolution integral (3.4) reduces to the product of the Fourier transforms of the functions involved. We define

$$f^*(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x) \quad (3.6)$$

Taking the Fourier transform of both sides of (3.4) we obtain

$$\phi_7^*(y, z; k) = \{q_7^*(k) + \mu_7^*(k) \frac{\partial}{\nu \partial y}\} G^*(y, z; k) \quad (3.7)$$

where

$$G^*(y, z; k) = -\frac{1}{4\pi} \lim_{\mu \rightarrow 0^+} \int_{-\infty}^{\infty} du \frac{\exp[z(k^2 + u^2)^{1/2} + iyu]}{(k^2 + u^2)^{1/2} - (\omega - i\mu + Uk)^{2/g}} \quad (3.8)$$

A proper behaviour of $q_7(x)$ and $\mu_7(x)$ is assumed so that their F.T. exist and the convolution theorem is applicable in (3.4). Further discussion is postponed to the appropriate section of this chapter.

The expressions (3.5) and (3.8) were derived under the assumption that the frequency of encounter ω is positive. Definition (2.16) suggests that for $\cos\beta > 0$, a negative ω is possible if $U\cos\beta$ is greater than the wave phase velocity g/ω_0 . In order to preserve the radiation condition of outgoing waves at infinity for negative ω , we continue (3.5) and (3.8) as follows

$$G_-(x,y,z,\omega) = \overline{G_+}(x,y,z,\omega) , \omega < 0 \quad (3.9)$$

where G_- and G_+ are the definitions of the Green function (3.5) for negative or positive ω respectively and the overbar stands for the complex conjugate of the expression involved. The corresponding continuation formula for the Fourier transform of G_- can be derived by combining (3.9) with (3.6)

$$G_-^*(y,z;k,\omega) = \overline{G_+^*}(y,z;-k,\omega) , \omega < 0 \quad (3.10)$$

where G_+^* is defined in (3.8) for positive ω . Further use of (3.8) gives

$$\bar{G}_+^*(y, z; -k, \omega) = G_+^*(y, z; k, |\omega|) , \omega < 0 \quad (3.11)$$

or

$$G_-^*(y, z; k, \omega) = G_+^*(y, z; k, -\omega) , \omega < 0 \quad (3.12)$$

hence, the Fourier transform of the Green function is an even function of ω and we hereafter take $\omega = |\omega_0 - Uv \cos \beta|$.

It is convenient to introduce the notation

$$l = -v \cos \beta \quad (3.13)$$

$$\phi_7(x, y, z) = \frac{igA}{\omega_0} e^{ilx} \psi_7(x, y, z) \quad (3.14)$$

$$q_7(x) = \frac{igA}{\omega_0} e^{ilx} Q_7(x) \quad (3.15)$$

$$\mu_7(x) = \frac{igA}{\omega_0} e^{ilx} M_7(x) \quad (3.16)$$

It is straightforward that

$$\phi_7^*(k-l) = \frac{igA}{\omega_0} \psi_7^*(k) \quad (3.17)$$

$$q_7^*(k-l) = \frac{igA}{\omega_0} Q_7^*(k) \quad (3.18)$$

$$\mu_7^*(k-l) = \frac{igA}{\omega_0} M_7^*(k) \quad (3.19)$$

Combining (3.7) and (3.17)-(3.19) we obtain

$$\psi_7^*(y, z; k) = \{Q_7^*(k) + M_7^*(k) \frac{\partial}{v \partial y}\} G^*(y, z; k-l) \quad (3.20)$$

The expansion of G^* for small values of νr and finite k is derived in Appendix 1 in the form

$$G^*(y, z; k-\ell) = H_{2D}(y, z) + \frac{1}{2\pi} (1+\nu z) h^*(k) \quad (3.21)$$

with an error factor $E = 1+O((\kappa-\nu)z, \nu^2 r^2, |k-\nu|^2 r^2)$ and

$$H_{2D}(y, z) = -\frac{1}{2\pi} \int_0^\infty dw \frac{\cosh w}{\cosh w - |\sec\beta|} \exp(z|\ell| \cosh w) \cos(|\ell| y \sinh w) \quad (3.22)$$

$$h^*(k) = \left\{ \begin{array}{l} \pi \operatorname{isgn}(\omega_0 + Uk) + \cosh^{-1} \left(\frac{\kappa}{|k-\ell|} \right) \\ -\pi + \cos^{-1} \left(\frac{\kappa}{|k-\ell|} \right) \end{array} \right\} \left| 1 - \frac{(k-\ell)^2}{\kappa^2} \right|^{-1/2} - \csc\beta \cosh^{-1}(|\sec\beta|) + \ln \left| \frac{k-\ell}{\ell} \right| \quad (3.23)$$

where the upper or lower expression is applicable according as $\kappa/|k-\ell| \gtrless 1$ and $\kappa = (\omega_0 + Uk)^2/g$.

The inner expansion of the outer solution is obtained from (3.20) using the approximation (3.21) for the symmetric and $G^*(y, z; -\ell)$ for the antisymmetric mode respectively. It follows that

$$\begin{aligned} \psi_7^*(y, z; k) &= Q_7^*(k) \{ H_{2D}(y, z) + \frac{1}{2\pi} (1+\nu z) h^*(k) \} \\ &+ M_7^*(k) \frac{1}{\nu} \frac{\partial}{\partial y} G_{2D}(y, z) \end{aligned} \quad (3.24)$$

where $G_{2D}(y, z) = G^*(y, z; -\ell)$ and the same error factor as in (3.21) is understood.

Expression (3.22) defines a two-dimensional source potential satisfying Helmholtz's equation $\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \ell^2 \right]$

$H_{2D} = 0$ and the free surface condition $\left(\frac{\partial}{\partial z} - v \right) H_{2D} = 0$

on $z=0$.

The limiting cases of (3.22) for $\beta=0^\circ, 180^\circ$ can be obtained if the series expansion of H_{2D} for small vr is first derived (Ursell (1962))

$$H_{2D}(y, z) = \frac{1}{2\pi} \left\{ \csc\beta \cosh^{-1}(|\sec\beta|) \cdot \exp(vz) \cdot \cos(vy \sin\beta) \right. \\ \left. - K_0(|\ell|r) - 2 \sum_{m=1}^{\infty} (-)^{m-1} \frac{\partial}{\partial \lambda} [I_\lambda(|\ell|r) \cos\lambda\theta]_{\lambda=m} \cdot \sinh m\eta \cdot \coth\eta \right\} \quad (3.25)$$

where

$$\ell = -v \cos\beta, \quad \cosh\eta = |\sec\beta|, \quad y = r \sin\theta, \quad z = -r \cos\theta$$

and $K_0(z)$, $I_\lambda(z)$ are the modified Bessel functions defined in Abramowitz and Stegun (1964). Using the limiting values

$$\csc\beta \cdot \cosh^{-1}(|\sec\beta|) \rightarrow 1 \quad \text{as } \beta \rightarrow 0^\circ \text{ or } 180^\circ \quad (3.26)$$

$$\sinh m\eta \cdot \coth\eta \rightarrow m \quad \text{as } \beta \rightarrow 0^\circ \text{ or } 180^\circ \quad (3.27)$$

we easily obtain the correct interpretation of expression (3.22) for $\beta=0^\circ, 180^\circ$.

In the limit $\beta=90^\circ$, H_{2D} reduces to the two dimensional source potential that satisfies Laplace's equation and standing waves as $|vy| \rightarrow \infty$, defined by

$$H_{2D}(y, z) \Big|_{\beta=90^\circ} = -\frac{1}{2\pi} \int_0^\infty dk \frac{e^{kz} \cos(ky)}{k-v} \quad (3.28)$$

The second term in the inner expansion of G^* defined in (3.23) is also regular at $\beta=0^\circ, 180^\circ$ in view of (3.26). For $\beta=90^\circ$, $\ell \rightarrow 0$ and the logarithmic infinities present in $\cosh^{-1}(|\sec\beta|)$ and $\ln \left| \frac{k-\ell}{\ell} \right|$ cancel each other out. The antisymmetric part of (3.24) is well behaved at all limits.

The approximation (3.24) is valid in a region where the error factor $E \rightarrow 1$ as $\epsilon \rightarrow 0$. Let

$$r = O(\epsilon^\alpha), \quad \alpha < 1 \quad (3.29)$$

It may be seen that

$$(\kappa-v)z = O(U\omega_0 z) = O(\epsilon^{\alpha-\gamma}) \quad (3.30)$$

$$v^2 r^2, |k-v|^2 r^2 = O(\epsilon^{2\alpha-4\gamma}) \quad (3.31)$$

Hence, it is legitimate to expand G^* in the manner indicated by (3.24) only if $\alpha-\gamma > 0$ and $\alpha-2\gamma > 0$, thus

$$\alpha > 2\gamma \quad (3.32)$$

Hence, in the matching region we have

$$2\gamma < \alpha < 1 \quad (3.33)$$

The main result so far is the inner approximation of the outer solution defined in (3.24). Transforming back in the physical space we obtain

$$\psi_7(x, y, z) = Q_7(x) H_{2D}(y, z) + \frac{1}{2\pi} (1 + \nu z) \int_L Q_7(\xi) e^{i\nu(x-\xi)\cos\beta} \cdot f(x-\xi) d\xi + M_7(x) \frac{1}{\nu} \frac{\partial}{\partial y} G_{2D}(y, z) \quad (3.34)$$

with the same error factor and $f(x)$ defined as the inverse Fourier transform of $h^*(k - \nu \cos\beta)$. This approximation will be matched with a corresponding expression of the inner solution to result in a system of equations for the unknowns involved in both problems. Before proceeding to the formulation of the inner problem it is of interest to further analyse the convolution integral in (3.34) which represents the three-dimensional information coming from the outer problem. We define

$$I(x) = \int_L Q_7(\xi) e^{i\nu(x-\xi)\cos\beta} f(x-\xi) d\xi \quad (3.35)$$

where $f(x)$ is the inverse Fourier transform of $f^*(k)$ defined as follows

$$\begin{aligned}
 \bar{f}^*(k) = & \left\{ \begin{array}{l} \pi i \operatorname{sgn}(\omega + Uk) \\ -\pi \end{array} \right\} |1 - k^2/\tilde{\kappa}^2|^{-1/2} \\
 & + \left\{ \begin{array}{l} \cosh^{-1}(\tilde{\kappa}/|k|) \\ \cos^{-1}(\tilde{\kappa}/|k|) \end{array} \right\} |1 - k^2/\tilde{\kappa}^2|^{-1/2} - \operatorname{csc}\beta \cosh^{-1}(|\sec\beta|) \\
 & + \ln|k/v\cos\beta|, \quad \tilde{\kappa}/|k| \geq 1 \qquad (3.36)
 \end{aligned}$$

where

$$\tilde{\kappa} = (\omega + Uk)^2/g \qquad (3.37)$$

with the frequency of encounter ω hereafter assumed positive. The function $f(x-\xi)$ represents the value of the velocity potential along the x axis due to a source of unit strength, located at $x=\xi$ and pulsating at a frequency ω in the presence of uniform stream $-U$. If in addition we assume a strength distribution $Q_7(\xi)$ along the ship axis, the resulting value, modulated by the wave factor $e^{i\nu(x-\xi)\cos\beta}$, is given by $I(x)$. The corresponding result for the radiation problem is obtained for $\cos\beta=0$.

The $U=0$ case will be analysed first due to the relative simplicity in comparison to the forward speed problem. Following a similar mathematical procedure as in Ursell (1962) and Mays (1978), it can be shown that

$$\begin{aligned}
 I(x) = & -Q_7(x) [\gamma + \csc\beta \cosh^{-1}(|\sec\beta|) - \ln(2|\sec\beta|)] \\
 & - \frac{1}{2} \int_L \left[\frac{d}{d\xi} - i\nu \cos\beta \right] Q_7(\xi) e^{i\nu(x-\xi)\cos\beta} \operatorname{sgn}(x-\xi) \log(2\nu|x-\xi|) d\xi \\
 & + \frac{\pi\nu}{4} \int_L Q_7(\xi) e^{i\nu(x-\xi)\cos\beta} K[\nu(x-\xi)] d\xi \quad (3.38)
 \end{aligned}$$

where $\gamma = 0.577 \dots$ is Euler's constant and

$$K(x) = Y_0(|x|) + 2i J_0(|x|) + iH_0(|x|) \quad (3.39)$$

where Y_0 , J_0 , and H_0 are the Bessel and Struve functions of zeroth order defined in Abramowitz and Stegun (1964).

The derivation of the corresponding result for $U > 0$ is more difficult but the underlying physical interpretations are similar. The wave generating part of $f(x)$ is not an even function of x as the corresponding part $K(\nu|x|)$ in the zero speed case and this is due to the directionality of the flow introduced by the forward speed. Any wave-systems present along the x -axis are associated with the square-root singularities of the F.T. of $f(x)$. The first term in (3.36) has four square root singularities in the complex k -plane defined as

$$(\omega + Uk_i)^4/g^2 = k_i^2, \quad i = 1, \dots, 4 \quad (3.40)$$

where

$$k_{1,2} = -\frac{g}{2U^2} \{1 + 2\tau \pm \sqrt{1+4\tau}\} \leq 0, \quad \tau \geq 0 \quad (3.41)$$

$$k_{3,4} = \frac{g}{2U^2} \left\{ \begin{array}{l} 1 - 2\tau \mp \sqrt{1-4\tau}, \quad 0 \leq \tau < 1/4 \\ 1 - 2\tau \mp i\sqrt{4\tau-1}, \quad \tau > 1/4 \end{array} \right\} \quad (3.42)$$

and $\tau = \omega U/g$.

As $U \rightarrow 0$, $k_{1,4} \rightarrow \mp \infty$ and $k_{2,3} \rightarrow \mp v$. Similarly, as $\omega \rightarrow 0$, $k_{1,4} \rightarrow \mp g/U^2$ and $k_{2,3} \rightarrow 0$. As $k \rightarrow \pm \infty$ the first term in (3.36) behaves like $\pm \pi i$, being finite at $k=0$.

The remaining part of $f^*(k)$, say $f_2^*(k)$, is free from any square root singularities since k_i are also zeros of the inverse functions in brackets. Furthermore $f_2^*(k)$ is finite on the finite real k axis with a logarithmic behaviour as $|k| \rightarrow \infty$. Exploiting the analytical properties of $f_2^*(k)$ in the complex k plane, we may end up with a considerable simplification in the final form of $I(x)$ defined in (3.35). Combining (3.35)-(3.36) we may write $I(x)$ into the form

$$I(x) = Q_7(x) \{ \pi i + \ln(2|\sec\beta|) - \csc\beta \cosh^{-1}(|\sec\beta|) - \ln \frac{v}{K} \} \\ - \int_L \left[\frac{d}{d\xi} - iv \cos\beta \right] Q_7(\xi) e^{iv(x-\xi)\cos\beta} F(x-\xi) d\xi \quad (3.43)$$

where

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \frac{g^*(k)}{(-ik)} dk \quad (3.44)$$

with

$$g^*(k) = \ln \frac{2K}{|k|} + \pi i - \left\{ \begin{array}{l} \pi i \operatorname{sgn}(\omega + Uk) + \cosh^{-1}(\tilde{\kappa}/|k|) \\ -\pi + \cos^{-1}(\tilde{\kappa}/|k|) \end{array} \right\} \\ \times |1 - k^2/\tilde{\kappa}^2|^{-1/2}, \quad \tilde{\kappa}/|k| \geq 1 \quad (3.45)$$

where $K = \omega^2/g$. The function $F(x)$ was also used in Newman and Sclavounos (1980) in the context of the radiation problem.

In the process of deriving (3.43) the existence of the first derivative of $Q_7(x)$ has been assumed, defined as follows

$$Q_7'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-ik) e^{-ikx} Q_7^*(k) dk \quad (3.46)$$

for all x , including $x=0$. If we ask that $k Q_7^*(k)$ is an absolutely integrable function over the real k axis we conclude, using the L^1 Fourier transform theory (Stein and Weiss (1975)), that $Q_7'(x)$ must be uniformly continuous. Combining this property with the vanishing of $Q_7(x)$ at the ship ends we obtain

$$Q_7(\pm L/2) = Q_7'(\pm L/2) = 0 \quad (3.47)$$

Assuming a similar behaviour for $M_7(x)$ and using (3.8) and (3.18)-(3.19), it can be seen that the transition from (3.4) to (3.7) is legitimate.

Leaving the mathematical derivation for Appendix 2, we may reduce $F(x)$ in the form

$$F(x) = \begin{cases} \bar{F}_1(x) + F_2(x) & , x < 0 \\ F_2(x) & , x > 0 \end{cases} \quad (3.48)$$

where

$$F_1(x) = \left[\int_{-\infty}^{k_1} + \int_{k_2}^0 \right] e^{-ikx} [(1 - k^2/\tilde{\kappa}^2)^{-1/2} - 1] dk/k + E_1(i|k_1x|) + E_1(i|k_2x|) \quad (3.49)$$

$$F_2(x) = \frac{1}{2} \left[\int_0^{k_3} + \int_{k_4}^{\infty} \right] e^{-ikx} [(1 - k^2/\tilde{\kappa}^2)^{-1/2} - 1] dk/k + \frac{1}{2} \int_{k_3}^{k_4} e^{-ikx} [i(k^2/\tilde{\kappa}^2 - 1)^{-1/2} - 1] dk/k, \tau < 1/4 \quad (3.50)$$

$$F_2(x) = \frac{1}{2} \int_0^{\infty} e^{-ikx} [(1 - k^2/\tilde{\kappa}^2)^{-1/2} - 1] dk/k, \tau > 1/4 \quad (3.51)$$

where $\tilde{\kappa}$ is defined in (3.37), k_i in (3.41)-(3.42) and $E_1(z)$ is the exponential integral defined in Abramowitz and Stegun (1964).

The previous definition indicates that $F(x)$ is regular at $x=0^+$ and logarithmically singular at $x=0^-$ because of the presence of the exponential integrals, the other terms being finite for all x . The large $|x|$ leading order asymptotic behaviour of $F_1(x)$ and $F_2(x)$ show for $x>0$ an exponentially decaying disturbance if $\tau>1/4$ and a wave system with a wavelength $2\pi/k_3$ if $\tau<1/4$. For $x<0$ there are always two wave systems associated with the roots $k_{1,2}$ with wavelengths $2\pi/k_1$ and $2\pi/k_2$ respectively and an additional one if $\tau<1/4$ with wavelength $2\pi/k_4$. In the zero speed limit $k_{1,4} = \bar{\tau} \infty$ and $k_{2,3} = \bar{\tau} v$ with a resulting symmetric disturbance in x . A zero frequency of encounter occurs either for $\omega_0=0$ and $\cos\beta<0$ or for $\omega_0=g/U\cos\beta$ and $\cos\beta>0$. At this limit, only one wave system of wavelength $2\pi U^2/g$ survives for $x<0$, and this is the transverse wave-system associated with a source of constant strength advancing at a speed U , the diverging wave system not being present along the x -axis.

In Figure 2 the regular part of $f(x)$ defined in (3.35)-(3.36) is shown as a function of $|x|/L$ for a Froude number 0.2 and two values of τ , 0.2 and 0.7 less and greater than 1/4 respectively.

A. THE SHORT-WAVELENGTH APPROXIMATION FOR U=0

The short-wavelength approximation of $I(x)$ presents a particular interest because of the simplicity of the final expressions. Furthermore, previous evidence from the high-frequency theory of Maruo and Sasaki (1974) for $U=0$ and $\beta=180^\circ$ suggests that the resulting predictions may be in a quite good agreement with the exact theory and experimental measurements.

An obvious way to proceed is to formally expand the integrals in (3.38) for large v . However, we may reduce the amount of work required if we begin with definition (3.35) of $I(x)$ and apply the convolution theorem to obtain

$$I(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} Q^*(k) f^*(k+v\cos\beta) dk \quad (3.52)$$

where $f^*(k+v\cos\beta) = h^*(k)$ according to the original definition of $f^*(k)$. In the present form of $I(x)$, the only frequency dependent term in (3.52) is $h^*(k)$ which can be written as the sum of $h_1^*(k)$ and $h_2^*(k)$ defined by

$$h_1^*(k) = \left\{ \begin{array}{c} \pi i \\ -\pi \end{array} \right\} |1 - (k+v\cos\beta)^2/v^2|^{-1/2} \quad (3.53)$$

and

$$h_2^*(k) = \left\{ \begin{array}{l} \cosh^{-1}(v/|k+v\cos\beta|) \\ \cos^{-1}(v/|k+v\cos\beta|) \end{array} \right\} |1 - (k+v\cos\beta)^2/v^2|^{-1/2} \\ - \csc\beta \cosh^{-1}(|\sec\beta|) + \ln(|k+v\cos\beta|/|v\cos\beta|) \quad (3.54)$$

where the upper or lower term in brackets is applicable according as $v/|k+v\cos\beta| \gtrless 1$. We observe that $h_2^*(k)$ is finite along the finite k -axis with a logarithmic behaviour as $|k| \rightarrow \infty$. Furthermore for $v \gg 1$ and k finite, $|k+v\cos\beta|/v = |\cos\beta| + O(k/v)$ and $\ln(|k+v\cos\beta|/|v\cos\beta|) = O(k/|v\cos\beta|)$, hence for $\cos\beta \neq 0$

$$h_2^*(k) = O(k/|v\cos^2\beta|) \quad (3.55)$$

where for $v \gg 1$ the upper term in brackets is applicable and the limiting behaviour

$$|1 - (k+v\cos\beta)^2/v^2|^{-1/2} \cdot \cosh^{-1}(v/|k+v\cos\beta|) \\ = \csc\beta \cosh^{-1}(|\sec\beta|) + O(k/|v\cos^2\beta|) \quad (3.56)$$

was used to obtain (3.55). For beam waves, h_2^* is written in the alternative form

$$h_2^*(k) = \cosh^{-1}(v/|k|) \{1 + O(k^2/v^2)\} + \ln|k|/v \\ \cos\beta \rightarrow 0 \\ - \csc\beta \cosh^{-1}(|\sec\beta|) - \ln|\cos\beta| \quad (3.57)$$

For large arguments z , $\cosh^{-1}(z) = \ln 2z + o(z^{-2})$ and the resulting logarithmic singularities in the first two terms cancel each other out with a remaining $\ln 2$ term. A similar cancellation occurs in the last two terms as $\cos\beta \rightarrow 0$ with an opposite remainder. Hence for $\cos\beta = 0$ and $v \gg 1$

$$h_2^*(k) = o(k^2/v^2) \quad (3.58)$$

Consequently, the contribution from h_2^* to $I(x)$ is of $o(1)$. The estimates (3.55) and (3.58) hold for any k such that $k/v \ll 1$ for $v \gg 1$. For larger k the logarithmic behaviour of $h_2^*(k)$ is offset by the algebraic decay of $Q_7^*(k)$ which is required for the absolute integrability of $kQ_7^*(k)$.

Similar arguments do not apply for h_1^* defined in (3.53) which has at least two square root singularities on the real k axis. Proceeding with the zero speed case first, we substitute the corresponding expression for h_1^* in (3.52) to obtain

$$I(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} Q_7(k) \begin{cases} \pi i \\ -\pi \end{cases} |1 - (k + v \cos\beta)^2 / v^2|^{-1/2} \quad (3.59)$$

where the upper or lower term is applicable according as $v/|k + v \cos\beta| \gtrless 1$.

Applying successively the coordinate transformation $k' = k + v \cos \beta$, the convolution theorem and using the definition the Hankel function $H_0^{(2)}(z) = J_0(z) - iY_0(z)$, we obtain:

$$I(x) = \frac{\pi v i}{2} \int_L Q_7(\xi) e^{i v (x-\xi) \cos \beta} H_0^{(2)}(v|x-\xi|) d\xi \quad (3.60)$$

The substitution of the leading order behaviour of $H_0^{(2)}(v|x-\xi|)$ for large v in (3.60) is correct only if the remainder is shown to be of higher order. Even though this is not the case, we will pursue the analysis since in the $\beta=180^\circ$ case the theory of Maruo and Sasaki(1974) is recovered. It follows from (3.60) that

$$I(x) = -\frac{1}{2}(1-i) \sqrt{\pi v} \int_L Q_7(\xi) \frac{\exp i v [(x-\xi) \cos \beta - |x-\xi|]}{|x-\xi|^{1/2}} d\xi \quad (3.61)$$

The $\beta=180^\circ$ limit can be easily obtained from (3.61) in the form

$$I(x) = -\frac{1}{2}(1-i) \sqrt{\pi v} \left\{ \int_x^{L/2} \frac{Q_7(\xi)}{(\xi-x)^{1/2}} d\xi + \int_{-L/2}^x \frac{Q_7(\xi)}{(x-\xi)^{1/2}} e^{-2i v (x-\xi)} d\xi \right\} \quad (3.62)$$

The second term in (3.62) can be further simplified for large ν as follows

$$\int_{-L/2}^x \frac{Q_7(\xi)}{(x-\xi)^{1/2}} e^{-2i\nu(x-\xi)} d\xi = \sqrt{\frac{\pi}{2\nu}} e^{-\pi i/4} Q_7(x) + o(1/\nu)$$

(3.63)

with the final result

$$I(x) = -\frac{1}{2}(1-i) \sqrt{\pi\nu} \int_x^{L/2} \frac{Q_7(\xi)}{(\xi-x)^{1/2}} d\xi + \frac{\pi i}{2} Q_7(x) + o(\nu^{-1/2})$$

(3.64)

This form of $I(x)$ is identical to the corresponding term in Maruo and Sasaki's theory except for the $1/2$ factor in the first term of (3.64) that does not appear in their theory. If we approximate $I(x)$ in a similar manner for oblique angles of incidence using (3.63), we obtain to leading order

$$I(x) = \frac{\sqrt{2}}{2} \pi i \frac{(1+\cos\beta)^{1/2} + (1-\cos\beta)^{1/2}}{\sin\beta} Q_7(x)$$

(3.65)

For beam waves it follows that

$$I(x) = \sqrt{2} \pi i Q_7(x)$$

(3.66)

The $\sqrt{2}$ factor in (3.66) is not correct, as can be seen if we set $\cos\beta=0$ in (3.53) and then let $\nu \rightarrow \infty$. In this case $h_1^*(k) \rightarrow \pi i$ and (3.52) suggests that the correct result

should be $\pi i Q_7(x)$. A consistent large ν expansion of $I(x)$ is obtained in Appendix 3 and since we analyse the $U=0$ case we may assume without loss of generality that $-1 < \cos\beta < 0$. The two-term short-wavelength expansion of (3.60) takes the form

$$I(x) = -\frac{1}{2} (1-i) \sqrt{\pi\nu} \int_x^{L/2} Q_7(\xi) \frac{\exp[-i\nu(\xi-x)(1+\cos\beta)]}{(\xi-x)^{1/2}} d\xi \\ + \pi i \left[\csc\beta - \frac{1}{2} \sqrt{\frac{2}{1+\cos\beta}} \right] Q_7(x) + o(\nu^{-1/2}) \quad (3.67)$$

In the head sea limit, the second term in (3.67) vanishes with a net result

$$I(x) = -\frac{1}{2} (1-i) \sqrt{\pi\nu} \int_x^{L/2} \frac{Q_7(\xi)}{(\xi-x)^{1/2}} d\xi + o(\nu^{-1/2}) \quad (3.68)$$

Hence, the last term in (3.64) and in the theory of Maruo and Sasaki is not present in a consistent short-wavelength expansion of (3.60). Finally for oblique waves we may approximate the integral in (3.67) as follows

$$\int_x^{L/2} Q_7(\xi) \frac{\exp[-i\nu(\xi-x)(1+\cos\beta)]}{(\xi-x)^{1/2}} d\xi \\ = \sqrt{\frac{\pi}{\nu(1+\cos\beta)}} e^{-\pi i/4} Q_7(x) + o(1/\nu) \quad (3.69)$$

The substitution of (3.69) in (3.67) results into a cancellation of the first and last terms, thus

$$I(x) = \pi i \csc\beta Q_7(x) + o(v^{-1/2}) \quad (3.70)$$

which leads to the correct result $\pi i Q_7(x)$ for beam waves.

In summary for $U=0$ and short wavelengths $I(x)$ is given by (3.68) for head seas, and by (3.70) otherwise.

B. THE "LARGE τ " APPROXIMATION

In the forward speed case, a short wavelength does not lead necessarily to a high frequency of encounter if $\cos\beta > 0$ and the Froude number is appropriately small. A zero frequency of encounter occurs when the wave phase velocity g/ω_0 is equal to $U\cos\beta$, or in terms of non-dimensional quantities

$$Fr = (2\pi\cos^2\beta)^{-1/2} (\lambda/L)^{1/2} \quad (3.71)$$

For a slenderness ratio $\epsilon=1/6$, a wavelength equal to the ship beam and $\beta=40^\circ$, the corresponding Froude number is equal to 0.21. Consequently, a small frequency of encounter is in practice possible even for short waves, if $\cos\beta > 0$.

The situation is different if $\cos\beta < 0$. In this case even moderate wavelength-to-ship-length ratios lead to high frequencies of encounter and only trailing wave-systems for $\tau > 1/4$. For simplicity we examine the head wave case

with the understanding that the same description applies qualitatively for $-1 < \cos\beta < 0$. For $\tau > 1/4$ there are only two trailing wave-systems along the x-axis, one associated with the steady-state disturbance with wavelength to ship length ratio

$$\lambda_1/L = 4\pi Fr^2 [1+2\tau + (1+4\tau)^{1/2}]^{-1} \quad (3.72)$$

and a second associated with the oscillatory disturbance generated by the ship-wave interaction with

$$\lambda_2/L = 4\pi Fr^2 [1+2\tau - (1+4\tau)^{1/2}]^{-1} \quad (3.73)$$

At $Fr=0.2$, $\cos\beta = -1$ and $\lambda/L = 1/2$, we get $\tau = 1.21$ and $L/\lambda_1 \approx 12$ compared to its $\omega_0 = 0$ limit $L/\lambda_1 \approx 4$. These values indicate the sensitivity of L/λ_1 on ω_0 at a given Froude number. At the same time $L/\lambda_2 \approx 2$ compared to $L/\lambda_2=2$ at $U=0$, indicating the insensitivity of L/λ_2 on Froude number changes at a constant frequency ω_0 .

In the forward speed case the relevant non-dimensional parameter is not the incident wavelength-to-ship-length ratio as for $U=0$ but the nondimensional parameter τ . The $\beta = 40^\circ$ example clearly shows that, even for short waves, τ can still be close to zero for a realistic ship speed. In this case no approximation is sought for

$F(x)$ and the exact expressions (3.49)-(3.51) have to be used. For $\cos\beta < 0$ and a realistic Fr.No, τ can take values in the vicinity of one for moderate to small incident wavelength-to-ship-length ratios. In this case L/λ_1 is quite large and in fact always larger than its steady state limit. On the contrary, L/λ_2 is less sensitive to the forward speed effects staying close to its $U=0$ limit.

The previous discussion suggests that the terms associated with k_1 in (3.49) can be approximated even for moderate values of τ (always $> 1/4$), the same being true for the k_2 -related terms for relatively shorter incident waves. It is shown in Appendix 4 that the leading order asymptotic behaviour of $F(x)$ for large L/λ_1 and L/λ_2 is

$$F(x) = \left[\frac{\pi}{2|x/\delta|} \right]^{1/2} (1+4\tau)^{-1/4} e^{-\pi i/4} x$$

$$\left(e^{-i\rho_1 x/\delta} |\rho_1|^{-1/2} + e^{-i\rho_2 x/\delta} |\rho_2|^{-1/2} \right) + O\left[(|\rho_{1,2} x/\delta|)^{-1} \right]$$

(3.74)

where $\rho_{1,2} = -[1+2\tau \pm (1+4\tau)^{1/2}]/2$, $\tau = \omega U/g$ and $\delta = U^2/g$.

The approximation (3.74) is not uniformly valid as $|x| \rightarrow 0$. It is a typical situation when deriving high-frequency approximations to end up with a square-root singularity at $x=0$ instead of a logarithmic one. When the ratio of the

body length to the generated wavelength is large compared to one, the far field of the wave-systems involved, where (3.74) is valid, appears within the body-length scale. Consequently, the elementary square root behaviour in (3.74) is not able at the same time to describe the logarithmic behaviour of the disturbance in the very close vicinity at $x=0$. This can be achieved only by the use of functions like the Bessel functions.

At this point only numerical computations will show the range over which the "large τ " approximation is in satisfactory agreement with the exact theory and experiments.

C. THE SMALL τ APPROXIMATION

In parallel with the "large" values, small values of τ occur in practice, as we have already mentioned, for $\cos\beta > 0$ and $U\cos\beta$ close to the wave phase velocity, but also for small ship-length to incident-wavelength ratios and low speeds. Consequently a small τ approximation can be quite useful for the stability analysis of the ship motions in following waves or for maneuvering and dynamic positioning considerations of a ship interacting with long waves. The small τ approximation of $F(x)$ is obtained in Newman and Sclavounos (1970, eq. 43) in the form

$$\begin{aligned}
 F(x) = & \pm \frac{1}{2} [\ln(2K|x|) + \gamma + \pi i] \\
 & \mp \frac{\pi}{4} \int_0^{K|x|} \{Y_0(\tau) + 2i J_0(\tau) + H_0(\tau)\} d\tau \\
 & - \frac{\pi}{4} [(2\mp 1) Y_0(g|x|/U^2) - H_0(g|x|/U^2)], \quad x \geq 0 \quad (3.75)
 \end{aligned}$$

where Y_0 , J_0 and H_0 are the Bessel and Sturve functions of order zero. For $U=0$ the last line in (3.75) vanishes and the zero speed result is recovered. As $\omega \rightarrow 0$ the second line vanishes and the steady state result is obtained in agreement with Tuck (1963). The $\ln(K)$ singularity present in the first term of (3.75) is cancelled out by the $\ln(K)$ singularity present in the first term of (3.43). If $\cos\beta < 0$, a small K implies a small v and the $\ln(v)$ singularity in the first term of (3.43) is cancelled out by an opposite singularity present in $H_{2D}(y,z)$.

The small τ approximation concludes the analysis of the outer problem. In the next chapter the inner problem will be analysed, where the form of the approximation (3.34) will be of decisive importance for the correct formulation of the inner solution.

IV. THE INNER PROBLEM

The inner problem is defined in an appropriately restricted region close to the ship hull where, taking advantage of the ship slenderness we may reduce the three-dimensional boundary value problem (2.21)-(2.24) into a two-dimensional one. The resulting relative errors must be small in the limit $\epsilon \rightarrow 0$.

The basic assumption in the inner region is that the flow gradients in the x direction, after the $e^{-ivxcos\beta}$ wave component has been factored out, are small compared to the transverse ones. Assuming that the y and z coordinates are both of $O(r)$, a coordinate stretching suggests that

$$\frac{\partial}{\partial x} = O(1), \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} = O(r^{-1}) \quad (4.1)$$

In accordance with the formulation of the outer problem we set

$$\phi_7(x, y, z) = \frac{igA}{\omega_0} e^{-ivxcos\beta} \psi_7(x, y, z) \quad (4.2)$$

The flow equations (2.21)-(2.24) satisfied by ϕ_7 will be subsequently expressed in terms of ψ_7 and approximated by using (4.1).

The Laplace equation takes the form

$$\frac{\partial^2 \psi_7}{\partial y^2} + \frac{\partial^2 \psi_7}{\partial z^2} - v^2 \cos^2 \beta \psi_7 - 2iv \cos \beta \frac{\partial \psi_7}{\partial x} + \frac{\partial^2 \psi_7}{\partial x^2} = 0 \quad (4.3)$$

where $v = \omega_0^2 / g = O(\epsilon^{-2\gamma})$. For $\gamma=0$ or wavelengths of the order of the ship length, the first two terms in (4.3) are the dominant ones. As γ grows, the third term becomes increasingly important being of order ϵ^{-2} for $\gamma=1$, or wavelengths of the order of the ship beam. Hence we approximate (4.3) as follows

$$\frac{\partial^2 \psi_7}{\partial y^2} + \frac{\partial^2 \psi_7}{\partial z^2} - v^2 \cos^2 \beta \psi_7 = 0 \quad (4.4)$$

with an error factor $1 + O\left\{v \frac{\partial}{\partial x} / \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right\} = 1 + O(vr^2)$.

Consequently, $\psi_7(y, z; x)$ satisfies the two dimensional Helmholtz equation in the (y, z) plane.

The body boundary condition (2.23) can be written in the form

$$(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}) (\phi_I + \phi_7) = 0 \quad \text{on } S_0 \quad (4.5)$$

where (n_1, n_2, n_3) are the (x, y, z) components of the unit vector \vec{n} . Since the beam and draft to ship length ratio is of $O(\epsilon)$ we assume

$$n_1 = O(\epsilon), \quad n_2, n_3 = O(1) \quad (4.6)$$

Thus, the body boundary condition reduces to

$$(n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}) (\phi_I + \phi_7) = \frac{\partial}{\partial N} (\psi_I + \psi_7) = 0 \text{ on } S_0 \quad (4.7)$$

where

$$\psi_I(y, z) = e^{vz - ivy \sin \beta} \text{ and } \vec{N} = (n_2, n_3) \quad (4.8)$$

with an error factor $1 + O(n_1 \partial / \partial x / n_2 \partial / \partial y, n_3 \partial / \partial z) = 1 + O(\epsilon v r)$, where $\frac{\partial}{\partial x} = O(v)$ when applied on $\phi_{I,7}$.

The free surface boundary condition is purposely left last because of the particular interest it presents. By substituting (4.2) in (2.2) and using (2.16) we obtain

$$g \frac{\partial \psi_7}{\partial z} - \omega_0^2 \psi_7 - 2i\omega_0 U \frac{\partial \psi_7}{\partial x} + U^2 \frac{\partial^2 \psi_7}{\partial x^2} = 0 \text{ on } z=0 \quad (4.9)$$

Only the first two terms in (4.9) are kept uniformly in the wavelength range, thus

$$\frac{\partial \psi_7}{\partial z} - v \psi_7 = 0, \text{ on } z=0 \quad (4.10)$$

with a resulting error factor $1 + O(U\omega_0 \partial / \partial x / \partial / \partial z) = 1 + O(\omega_0 r)$

Two points are of interest in (4.9) and (4.10). First, the boundary condition (4.10) is independent of the forward speed U . This fact combined with (4.4) and (4.7) renders the inner problem free from any explicit U dependence.

A U -dependent interaction coefficient $C(x)$ will be

subsequently introduced in the representation of the inner solution and its form will be determined only after the matching with the outer solution is performed. The dependence of the inner problem on ω_0 instead of ω is further discussed in Newman (1977).

The second point is related to the possible improvement on (4.10) by including the $\partial\psi_7/\partial x$ term in the free surface condition. This term becomes relatively more important for wavelengths of the order of the ship beam or $\omega_0 = O(\epsilon^{-1/2})$. In this particular case the error factor that multiplies (4.10) is $1+O(\epsilon^{1/2})$ with $r=O(\epsilon)$. It is shown in Ogilvie and Tuck (1969), in the context of the radiation problem and in Newman (1978), that there are terms resulting from the interaction of the steady and unsteady disturbance potentials also of order $\epsilon^{1/2}$ in the same wavelength range. Consequently, a consistent perturbation scheme requires that if the third term in (4.9) was to be kept in the free surface condition in a higher order theory, then all interaction terms of order $\epsilon^{1/2}$ should be also included. The corresponding error factors in the field equation (4.4) and the body boundary condition (4.7) are $1+O(\epsilon)$ in the short-wavelength limit and can be consistently neglected.

The preceding discussion applies only to the inner region. In the outer region, the steady state disturbance potential satisfies the linear free surface condition (2.9) and no interactions are possible with the oscillatory disturbance. The unified theory derived in Newman (1978) and applied by Newman and Sclavounos (1980) for the radiation problem, as well as the present study are consistently neglecting steady-unsteady interactions in the inner region with an error factor at most equal to $1 + O(\epsilon^{1/2})$.

In summary, the boundary value problem satisfied by the inner solution takes the form

$$\frac{\partial^2 \psi_7}{\partial y^2} + \frac{\partial^2 \psi_7}{\partial z^2} - v^2 \cos^2 \beta \psi_7 = 0 \quad \text{in the fluid domain}$$

$$\times 1 + O(vr^2) \quad (4.11)$$

$$\frac{\partial \psi_7}{\partial z} - v \psi_7 = 0 \quad \text{on } z=0$$

$$\times 1 + O(\omega_0 r) \quad (4.12)$$

$$\frac{\partial \psi_7}{\partial N} = - \frac{\partial \psi_I}{\partial N} \quad \text{on } S_0$$

$$\times 1 + O(\epsilon vr) \quad (4.13)$$

Assuming that the ship geometry is symmetric with respect to the $y=0$ plane, we may decompose the inner problem into an even and odd part. The resulting symmetric and antisymmetric two-dimensional problems have been analysed by Bolton and Ursell (1973), Choo (1975), Troesh (1976) and Liapis and Faltinsen (1980), for an oblique angle of incidence and under the assumption that the solutions of both problems represent outgoing waves as $|vy| \rightarrow \infty$.

As $\sin\beta \rightarrow 0$, the antisymmetric disturbance vanishes and the symmetric solution blows up like $\csc\beta$, unless the condition of outgoing waves is removed. Since the radiation condition at infinity has been already satisfied in the outer problem, the inner problem may be relaxed from such a constraint, provided that the inner solution is compatible with the inner expansion of the outer solution (3.34) in an appropriate overlap region.

Following Newman (1978), we may write the inner velocity potential in the form

$$\psi_7 = \psi_7^P + C_S(x) \psi_{7S}^H + C_A(x) \psi_{7A}^H \quad (4.14)$$

where ψ_7^P satisfies the non-homogeneous B.V.P. (4.11)-(4.13) and ψ_{7S}^H , ψ_{7A}^H are symmetric and antisymmetric homogeneous solutions multiplied by two interaction functions $C_S(x)$

and $C_A(x)$ respectively, which represent arbitrary constants for each two-dimensional problem and their form will be determined from the matching. One particular solution is simply $-\psi_I$, thus

$$\psi_7^P = -e^{vz} - ivy \sin \beta \quad (4.15)$$

The $\sin \beta = 0$ value of ψ_7^P is the leading order inner solution in the theory of Faltinsen (1970).

From the physical standpoint, the homogeneous disturbance potentials ψ_{7S}^H and ψ_{7A}^H can be identified from the interaction of two waves incident from opposite sides on a ship cross section, with equal or opposite phases for the symmetric and antisymmetric problems respectively. From the mathematical standpoint, Ursell (1968a) proved that for a two-dimensional velocity potential satisfying the Helmholtz equation (4.11), the wave free surface condition (4.12) and an arbitrary body boundary condition, the solution outside a circle of radius greater than the body-boundary maximum radius can be expanded in a convergent series of the form

$$\begin{aligned} \chi(y, z) = & C_1 \left[v^{-1} \frac{\partial}{\partial y} \right] G_{2D}(y, z) + C_2 e^{vz} \begin{bmatrix} \cos \\ \sin \end{bmatrix} (v y \sin \beta) \\ & + \sum_{m=1}^{\infty} \begin{bmatrix} s_{2m} S_{2m}(r, \theta) \\ a_{2m} A_{2m}(r, \theta) \end{bmatrix} \end{aligned} \quad (4.16)$$

where the upper or lower terms in brackets are applicable for the symmetric and antisymmetric problem respectively, and

$$G_{2D}(y, z) = \frac{1}{2} \operatorname{icsc} \beta e^{vz} \cos(v y \sin \beta) + H_{2D}(y, z) \quad (4.17)$$

with $H_{2D}(y, z)$ defined in (3.25) and

$$\begin{aligned} S_{2m}(r, \theta) = & K_{2m-2}(|\ell| r) \cos(2m-2)\theta + K_{2m}(|\ell| r) \cos 2m\theta \\ & + 2|\sec \beta| K_{2m-1}(|\ell| r) \cos(2m-1)\theta \end{aligned} \quad (4.18)$$

$$\begin{aligned} A_{2m}(r, \theta) = & K_{2m-1}(|\ell| r) \sin(2m-1)\theta + K_{2m+1}(|\ell| r) \sin(2m+1)\theta \\ & + 2|\sec \beta| K_{2m}(|\ell| r) \sin 2m\theta \end{aligned} \quad (4.19)$$

where $\ell = -v \cos \beta$, $y = r \sin \theta$, $z = -r \cos \theta$ and $K_n(z)$, $n=1, 2, \dots$ are the modified Bessel functions of real argument and integer order defined in Abramowitz and Stegun (1964).

The source function, G_{2D} also defined in (A1.7) is required to have a cut along the positive z axis across which it is continuous with its y -derivative discontinuous. The requirement of outgoing waves as $|vy| \rightarrow \infty$ doesn't appear in the proof. In fact while deriving $G_{2D}(y,z)$, it is stated that the pole of the integrand in (A1.7) may be avoided from below instead from above with resulting incoming waves from infinity. If we just take the average, we end up with $H_{2D}(y,z)$ defined in (3.22) and (A1.9) and which has the required properties along the positive z axis. Hence it is possible to modify the expansion theorem with $G_{2D}(y,z)$ replaced by $H_{2D}(y,z)$ which is regular as $\sin\beta \rightarrow 0$.

The final form of the symmetric part of the inner problem can be obtained from (4.14), (4.15) and the modified form of (4.16)

$$\begin{aligned} \psi_{7S}(y,z;x) = & -e^{\nu z} \cos(\nu y \sin\beta) + C_S(x) \{ \sigma_S(x) H_{2D}(y,z) \\ & + e^{\nu z} \cos(\nu y \sin\beta) + S(r,\theta) \} \quad (4.20) \end{aligned}$$

or

$$\psi_{7S}(y,z;x) = \left[C_S(x) - 1 \right] e^{vz} \cos(vy \sin \beta) + C_S(x) \left[\sigma_A(x) H_{2D}(y,z) + S(r,\theta) \right] \quad (4.21)$$

where in (4.20) the function in brackets satisfies the homogeneous boundary value problem (4.11)-(4.13) and has been normalized by taking $C_2 = 1$ in (4.16). The unknown functions $\sigma_S(x)$ and $S(r,\theta)$ are determined from the numerical solution of the homogeneous problem. The corresponding form of the antisymmetric problem is

$$\psi_{7A}(y,z;x) = i e^{vz} \sin(vy \sin \beta) + C_A(x) \left\{ \sigma_A(x) \frac{\partial}{\partial y} G_{2D}(y,z) + e^{vz} \sin(vy \sin \beta) + A(r,\theta) \right\} \quad (4.22)$$

or

$$\psi_{7A}(y,z;x) = \left[i + C_A(x) \right] e^{vz} \sin(vy \sin \beta) + C_A(x) \left[\sigma_A(x) \frac{\partial}{\partial y} G_{2D}(y,z) + A(r,\theta) \right] \quad (4.23)$$

where $G_{2D}(y,z)$ is defined in (3.24) and a corresponding definition for $\sigma_A(x)$ and $A(r,\theta)$ is understood. The solution of the symmetric and antisymmetric homogeneous problems for a circular geometry is described in the chapter dealing with the numerical results.

Before we proceed to the expansion of the inner solution in the matching region we need estimates of the order of magnitude of $S(r, \theta)$ and $A(r, \theta)$ in the limit $\epsilon \rightarrow 0$. The multipole expansion representation will be used outside a circle of radius equal to the maximum body boundary radius. We only consider $S(r, \theta)$, the same order of magnitude holding for $A(r, \theta)$.

The r -derivative of $e^{vz} \cos(vy \sin \beta)$ on a circle of radius $r = O(\epsilon)$ is of $O(v)$ as $\epsilon \rightarrow 0$. This normal velocity must be of the same order as the sum of the normal velocities of $\sigma_s(x) H_{2D}(y, z)$ and $S(r, \theta)$. Furthermore, since the functional dependence of these two terms on the transverse coordinates y and z is different, we may assume that for a general v they are of the same order of magnitude, hence

$$\frac{\partial}{\partial r} S(r, \theta) |_{r=O(\epsilon)} = O(v) \quad (4.24)$$

The leading order term in $S(r, \theta)$ is $s_2 K_0(|\ell|/r)$, its r -derivative being of order $O(s_2 v/r)$. Consequently

$$O(s_2 v/r) = O(v) \quad (4.25)$$

or

$$\bar{s}_2 = O(r) = O(\epsilon) \quad (4.26)$$

For $m > 1$, a similar argument proves that s_{2m} are of higher order. The same estimates hold for s_2 and a_{2m} , $m > 1$, hence

$$S(r, \theta), A(r, \theta) = O(\epsilon \ln vr), \quad r = O(\epsilon) \quad (4.27)$$

Since $C_S(x)$ and $C_A(x)$ are of $O(1)$, the error factor involved if we ignore the multipole terms is $1 + O(\epsilon)$. Logarithmic terms in the order of magnitude considerations are taken to be of $O(1)$ unless their display is essential.

The symmetric part of the inner solution can be approximated in the form

$$\psi_{7S}(y, z; x) = [C_S(x) - 1] (1 + vz) + C_S(x) \sigma_S(x) H_{2D}(y, z) \quad (4.28)$$

with an error factor $1 + O(vr^2, \omega_0 r, \epsilon vr, v^2 r^2, \epsilon)$.

The first three terms in the error factor come from the boundary value problem (4.11)-(4.13), the $v^2 r^2$ term from the approximation of the particular solution and the last term from neglecting $S(r, \theta)$.

Similarly, the antisymmetric problem is approximated by

$$\begin{aligned} \psi_{7A}(y, z; x) = [1 + C_A(x)] e^{vz} \sin(vy \sin \beta) \\ + C_A(x) \sigma_A(x) \frac{\partial}{v \partial y} G_{2D}(y, z) \end{aligned} \quad (4.29)$$

with an error factor $1 + O(vr^2, \omega_0 r, \epsilon vr, \epsilon)$. The approxi-

mations (4.28) and (4.29) are valid in a region where the errors involved are small in the limit $\epsilon \rightarrow 0$. This region has in addition to overlap with the region where the corresponding approximation of the outer solution is valid.

If we again introduce the notation $r=0(\epsilon^\alpha)$ and $\omega_0 = 0(\epsilon^{-\gamma})$, the former requirements are simultaneously met only if

$$2\gamma < \alpha \leq 1 \tag{4.30}$$

in the limit $\epsilon \rightarrow 0$.

It is now possible to correlate the terms in (3.34) with the corresponding terms in (4.28) and (4.29). In the next chapter the set of equations that $C_S(x)$, $Q_7(x)$ and $C_A(x)$, $M_7(x)$ satisfy will be derived and the solutions will be discussed.

V. THE MATCHING

In the third and fourth chapters the outer and inner problems were analysed respectively and the corresponding solutions (3.4) and (4.20)-(4.22) were expanded in the forms (3.34) and (4.28)-(4.29) respectively in an intermediate overlap region where they have to be compatible. The errors involved in (3.34) and (4.28)-(4.29) must be small in the limit $\epsilon \rightarrow 0$. This requirement is met if both (3.33) and (4.30) hold or

$$2\gamma < \alpha \leq 1 \quad (5.1)$$

Hence the matching is possible only if

$$\gamma \leq 1/2 \quad (5.2)$$

Condition (5.2) implies wavelengths of order equal or lower than the ship transverse dimensions as $\epsilon \rightarrow 0$.

The matching conditions are easily derived by comparing (3.34) with the sum of (4.28) and (4.29). Equating the coefficients of the linearly independent functions $1+vz$ and $H_{2D}(y,z)$, we obtain for the symmetric mode

$$Q_7(x) = C_s(x) \sigma_s(x) \quad (5.3)$$

$$C_s(x) = 1 + \frac{1}{2\pi} \int_L Q_7(\xi) e^{i\nu(x-\xi) \cos \beta_f(x-\xi)} d\xi \quad (5.4)$$

For the antisymmetric mode it follows that

$$M_7(x) = C_A(x) \sigma_A(x) \quad (5.5)$$

$$C_A(x) + i = 0 \quad (5.6)$$

The elimination of $C_S(x)$ from (5.3) and (5.4) results in an integral equation for the unknown source strength $Q_7(x)$

$$Q_7(x) = \sigma_S(x) + \frac{\sigma_S(x)}{2\pi} \int_L Q_7(\xi) e^{i\nu(x-\xi)\cos\beta} f(x-\xi) d\xi \quad (5.7)$$

with $C_S(x)$ determined subsequently from (5.3) or (5.4).

For the antisymmetric problem we simply have

$$C_A(x) = -i \quad (5.8)$$

In the general wavelength case, (5.7) takes the form

$$\begin{aligned} L(x) Q_7(x) + \frac{\sigma_S(x)}{2\pi} \int_L d\xi F(x-\xi) e^{i\nu(x-\xi)\cos\beta} \left[\frac{d}{d\xi} - i\nu\cos\beta \right] Q_7(\xi) \\ = \sigma_S(x) \end{aligned} \quad (5.9)$$

where

$$\begin{aligned} L(x) = 1 + \frac{\sigma_S(x)}{2\pi} [\ln(\nu/K) - \pi i + \csc\beta \cosh^{-1}(|\sec\beta|)] \\ - \ln 2 |\sec\beta| \end{aligned} \quad (5.10)$$

and where $F(x)$ is defined in (3.49)-(3.51). The small τ approximation leads to the same equation with $F(x)$ defined

in (3.75). The $U=0$ limit of (5.9)-(5.10) is obtained using either (3.38) or (3.75)

$$L_0(x) Q_7(x) + \frac{\sigma_s(x)}{2\pi} \left\{ \frac{1}{2} \int_L d\xi \operatorname{sgn}(x-\xi) \ln 2v|x-\xi| e^{iv(x-\xi)\cos\beta} \left[\frac{d}{d\xi} - iv\cos\beta \right] Q_7(\xi) - \frac{\pi v}{4} \int_L Q_7(\xi) e^{iv(x-\xi)\cos\beta} K[v(x-\xi)] d\xi \right\} = \sigma_s(x) \quad (5.11a)$$

where

$$L_0(x) = 1 + \frac{\sigma_s(x)}{2\pi} [\gamma + \operatorname{csc}\beta \cosh^{-1}(|\sec\beta|) - \ln 2|\sec\beta|] \quad (5.11b)$$

and

$$K(x) = Y_0(|x|) + 2i J_0(|x|) + \mathbb{H}_0(|x|) \quad (5.11c)$$

The difference between the $U=0$ limit of $L(x)$ and $L_0(x)$ is a $\pi i + \gamma$ term which in addition with the logarithmic singularity at $x=0^-$ is built in the exponential integrals of definition (3.49). The rest of (5.11) is obtained as a regular limit of (5.9) as $U \rightarrow 0$.

The short-wavelength approximations of (5.9) and (5.11) are obtained by using the corresponding expressions for the convolution integral in (5.7). Using (3.74), it

follows for $U > 0$

$$R(x) Q_7(x) + \frac{1}{4\pi} (1-i) [\pi (1+4\tau)^{-1/2}]^{1/2} \times$$

$$\left\{ \int_x^{L/2} [|\rho_1|^{-1/2} e^{i(\xi-x)\rho_1/\delta} + |\rho_2|^{-1/2} e^{i(\xi-x)\rho_2/\delta}] \right.$$

$$\frac{e^{-iv(\xi-x)\cos\beta}}{(\xi-x)^{1/2}} \cdot$$

$$\left. \left[\frac{d}{d\xi} - iv\cos\beta \right] Q_7(\xi) d\xi \right\} = 1 \quad (5.12)$$

where $R(x) = L(x)/\sigma_s(x)$ with $L(x)$ defined in (5.10), and

$$\rho_{1,2} = -\frac{1}{2} [1+2\tau \pm (1+4\tau)^{1/2}] \quad (5.13)$$

For $U=0$, we use (3.67) to obtain

$$P(x) Q_7(x) + \frac{1}{4\pi} (1-i) (\pi v)^{1/2} \int_x^{L/2} Q_7(\xi) \frac{e^{-iv(\xi-x)(1+\cos\beta)}}{(\xi-x)^{1/2}} d\xi$$

$$= 1 \quad (5.14)$$

where

$$P(x) = \frac{1}{\sigma_s(x)} - \frac{1}{2} i \left[\csc\beta - \frac{1}{2} \left(\frac{2}{1+\cos\beta} \right)^{1/2} \right] \quad (5.15)$$

In the limit of head waves (5.14) reduces to

$$Q_7(x)/\sigma_s(x) + \frac{1}{4\pi} (1-i) \sqrt{\pi v} \int_x^{L/2} \frac{Q_7(\xi)}{(\xi-x)^{1/2}} d\xi = 1 \quad (5.16)$$

Maruo and Sasaki derived a similar integral equation which includes an additional $\pi i Q_7(x)/2$ term that should not be present, and they miss a $1/2$ factor in the second term. For an oblique incidence, (5.14) can be further approximated using (3.72), with the solution

$$Q_7(x) = \sigma_s(x) \left(1 + \frac{1}{2} icsc\beta\right) \quad (5.17)$$

The solution of the integrodifferential equations (5.9) and (5.11) can be obtained using an iterative scheme, the first iteration being the two dimensional source strength $\sigma_s(x)$. Questions related to the existence and uniqueness of the solution requires a further mathematical analysis which is beyond the scope of the present thesis. A discussion on the numerical solution of (5.9)-(5.14) is postponed to a later chapter dealing with the numerical computations.

As a final comment we note that the solution $Q_7(x)$ of the integral equations has to fall into the class of the uniformly continuous functions. Assuming that this requirement is satisfied we conclude from (5.3) and (5.4) that $\sigma_s(x)$ has the same property, hence

$$\sigma_s(\pm L/2) = \sigma_s'(\pm L/2) = 0 \quad (5.18)$$

The condition (5.18) combined with the fact that σ_s tends to zero linearly for small nondimensional frequencies (Newman

(1978)), implies that from the mathematical standpoint the ship waterline at the ends has to have a zero slope.

VI. THE PRESSURE FORCE

The pressure on the body surface is determined by using Bernoulli's equation

$$p(\vec{x}_0) = -\rho \left\{ \phi_t + \frac{1}{2} (\nabla\phi)^2 + gz \right\} \quad (6.1)$$

where ϕ is the total velocity potential with respect to the coordinate system fixed in space. Keeping terms linear in ϕ , transforming (6.1) back to the (x, y, z) coordinate system and omitting the $\exp(i\omega t)$ factor, we obtain

$$p(\vec{x}) = -\rho \left\{ (i\omega - U \frac{\partial}{\partial x}) \phi + U \nabla\bar{\phi}\nabla\phi + gz \right\} \quad (6.2)$$

where $\phi(\vec{x}) = \phi_I + \phi_7$ and $\phi(\vec{x})$ are the oscillatory and the steady state velocity potentials respectively. The hydrostatic pressure term is hereafter neglected, assuming that the ship weight balances the buoyancy force and that the pressure integration is performed over the mean hull wetted surface.

While formulating the inner boundary value problem for ϕ_7 all interactions with the steady state disturbance were neglected, with an error factor at most $1+O(\epsilon^{1/2})$. A consistent perturbation scheme in ϵ requires that the second term in (6.2) must be present. This can be seen

as follows; to leading order we may approximate $\bar{\phi}(\vec{x})$ by the double body velocity potential which satisfies the hull boundary condition

$$\vec{n} \cdot \nabla \phi = n_1 \quad \text{on } S_0 \quad (6.3)$$

and the free surface condition

$$\partial \bar{\phi} / \partial z = 0 \quad \text{on } z=0 \quad (6.4)$$

Using (4.1) and (4.6) in (6.3) we obtain

$$\bar{\phi} = O(\epsilon^2) \quad (6.5)$$

The correct order of magnitude is $\epsilon^2 \ln \epsilon$, but we have already assumed that $\ln \epsilon = O(1)$. A more detailed discussion can be found in Ogilvie and Tuck (1969), Ogilvie (1977) and Newman (1978). The use of (6.5) in (6.2) indicates that the second term is of $O(\phi)$ and cannot be neglected.

The forces and moments acting on the ship are defined by

$$X_i = \iint_{S_0} p(\vec{x}) n_i ds, \quad i = \begin{matrix} 3,5 \\ 2,6 \end{matrix} \quad (6.6)$$

where $X_{3,5}$ and $X_{2,6}$ are the vertical and horizontal exciting forces and moments respectively, and $n_5 = -x n_3$, $n_6 = x n_2$. The substitution of (6.2) in (6.6) yields

$$X_i = -\rho \iint_{S_0} \{i\omega\phi + UV(\bar{\phi} - x)\nabla\phi\} n_i ds \quad (6.7)$$

Applying Tuck's theorem [Ogilvie and Tuck(1969)], and assuming that the slope of the ship hull at the inter-section with the free surface is vertical, we get

$$X_i = -\rho \iint_{S_0} (i\omega\phi n_i - Um_i)\phi ds \quad (6.8)$$

where

$$m_3 = -(\vec{n} \cdot \nabla) \bar{\phi}_z \quad (6.9)$$

$$m_5 = -xm_3 + n_3 \quad (6.10)$$

$$m_2 = -(\vec{n} \cdot \nabla) \bar{\phi}_y \quad (6.11)$$

$$m_6 = xm_2 - n_2 \quad (6.12)$$

In the inner region, the velocity potential $\phi = \phi_I + \phi_7$ takes the form

$$\phi = \frac{igA}{\omega_0} e^{-ivxcos\beta} [e^{vz-ivycos\beta} + \{C_S(x)-1\} e^{vz-ivycos\beta} + C_S(x)\psi_S - i\psi_A] \quad (6.13)$$

where

$$\psi_S(x, y, z) = \sigma_S(x) H_{2D}(y, z) + S(r, \theta) \quad (6.14)$$

$$\psi_A(x, y, z) = \sigma_A(x) \frac{\partial}{\partial y} G_{2D}(y, z) + A(r, \theta) \quad (6.15)$$

are the even and odd strip-theory solutions that satisfy the even and odd parts of the boundary value problem (4.11) - (4.13) respectively. The interaction coefficient $C_S(x)$ is determined from (5.4) after solving the integral equations (5.9) - (5.14).

It is shown in Appendix 5 that the exact expression of m_3 for a prolate spheroid has the form

$$m_3(\phi, \theta, \epsilon) = \frac{2}{L} \frac{1-\epsilon^2}{D(\epsilon)} \cos\phi \sin\phi \cos\theta [1+2\epsilon^2/(\sin^2\phi + \epsilon^2 \cos^2\phi)] / (\sin^2\phi + \epsilon^2 \cos^2\phi)^{3/2} \quad (6.16)$$

where L is the length of the spheroid, $\epsilon=B/L$ is the slenderness ratio, with B being the maximum beam, and

$$\cos\phi = 2x/L, \quad y = r\sin\theta, \quad z = -r\cos\theta \quad (6.17)$$

$$D(\epsilon) = 1 - \frac{1}{2} \epsilon^2 (1-\epsilon^2)^{-1/2} \ln\{[1+(1-\epsilon^2)^{1/2}]/[1-(1-\epsilon^2)^{1/2}]\} \quad (6.18)$$

It is easy to see that $m_3(x)$ is an odd function of x , vanishing at $x = \pm L/2$.

The pressure on the body boundary can be simplified if we write

$$\nabla\bar{\phi} \cdot \nabla\phi = \frac{\partial\bar{\phi}}{\partial x} \frac{\partial\phi}{\partial x} + \frac{\partial\bar{\phi}}{\partial N} \frac{\partial\phi}{\partial N} + \frac{\partial\bar{\phi}}{\partial M} \frac{\partial\phi}{\partial M} \quad (6.19)$$

where the y and z components of the velocities associated with $\bar{\phi}$ and ϕ are expressed with respect to a local right handed orthogonal coordinate system with the N and M axes normal and tangent to the body surface respectively and the positive N direction pointing inside the hull.

The second term in (6.19) is identically zero since the oscillatory velocity potential satisfies a zero normal velocity on the body boundary. Furthermore (6.5) suggests that the first term in (6.19) is of $O(\epsilon^2\phi)$ and can be consistently neglected if the pressure is required at a point sufficiently away from the body ends. Consequently, only the product of the tangential velocity components gives a finite contribution consistent with the perturbation scheme. Proceeding even further, we note that for a geometry symmetric port and starboard, $\partial\bar{\phi}/\partial M$ vanishes along $y=0$ and by definition along $z=0$. For a body of revolution it vanishes on the whole body boundary. Thus, the magnitude of $\partial\bar{\phi}/\partial M$ depends on how much the body surface deviates from a body of revolution. The integral over the body wetted surface of the pressure approximated in the previous manner does not lead to (6.8) due to the failure of assumption (6.5) at the body ends. Hence, the use of Tuck's theorem is essential in avoiding errors coming from the ends when evaluating the exciting forces.

VII. NUMERICAL RESULTS AND COMPARISON WITH EXPERIMENTS

As a first check, two bodies of revolution are analysed for which experiments exist in head waves and both zero and finite forward speed.

The numerical effort involved in the determination of the homogeneous solution of the two-dimensional inner problem is greatly reduced when the hull cross-sections are circular. As the form of the body geometry depends only on the local radius $a(x)$, the normalized two-dimensional solution is needed only for a set of values of va that cover the range of practical interest. Furthermore, the solution can be obtained using Ursell's multipole method [Ursell(1968a)], with the Green function modified as indicated in Chapter IV. For head waves, the normalized homogeneous solution takes the form

$$\begin{aligned} \psi_{7H}(vr, va, \theta) = & \sigma_S(va) H_{2D}(vr, \theta) + e^{-vrcos\theta} \\ & + \sum_{m=1}^{\infty} s_{2m}(va) S_{2m}(vr, \theta) \end{aligned} \quad (7.1)$$

where $H_{2D}(vr)$ and S_{2m} are defined in (3.25) and (4.18) respectively, with $\beta=180^\circ$. Thirty-seven terms are kept in (7.1) and a zero normal velocity is satisfied at an equal number of equally spaced angles θ along the body boundary.

The source strength $\sigma_S(va)$ and the coefficients $a_{2m}(va)$ are subsequently determined by a matrix inversion. The values of $\sigma_S(va)$, the sectional force $a_3(va)$, defined by

$$a_3(va) = \int_0^{\pi/2} \psi_{7H}(va, \theta) \cos\theta d\theta \quad (7.2)$$

and of the velocity potential ψ_{7H} at $\theta=0^\circ-80^\circ(\text{mod } 20^\circ)$, are listed in Table 1 versus va . Interpolating these data, the two-dimensional information required for a body of revolution can be easily obtained. The computation time required to complete Table 1 is 10 minutes on a PDP11-34 minicomputer or 10 seconds on an IBM370 computer.

The next step involves the solution for the center-line source strength distribution $Q_7(x)$. For a given distribution of the body sectional radius $a(x)$, the two-dimensional source strength distribution $\sigma_S[va(x)] = \sigma_S(x)$ is easily determined by interpolating the second column of Table 1.

For $U=0$, the relevant integro-differential equation is given in (5.11 a,b,c) with $\beta=180^\circ$. Its short-wavelength approximation is given in (5.16). The evaluation of $K(x)$ in (5.11c) is straightforward since it involves only Bessel functions. An iteration scheme is used for the solution of (5.11c), the first iteration of $Q_7(x)$ being $\sigma_S(x)$. The iterations converge for all tested values of

λ/L , even for $\lambda/L=\epsilon$. No attempt has been made to determine the range of λ/L for which the iteration scheme converges but the presence of the factor $\exp[i\nu(x-\xi)\cos\beta]$ suggests convergence for all values of λ/L (except for beam waves). This conjecture is strengthened if we examine the short-wavelength approximation of (5.11a) for head waves.

Equation (5.16) is a Volterra integral equation of the second kind and the iteration procedure is known to converge for any value of νL . At most 15 iterations are required for the determination of $Q_7(x)$ from both (5.11a) and (5.16), with a relative error 10^{-4} . On the PDP11-34 minicomputer, 15 seconds are required for the solution of (5.11a) and 10 seconds for (5.16) or 0.25 and 0.16 seconds on the IBM370 respectively.

For $U>0$, the corresponding integro-differential equation is given in (5.9)-(5.10) and its high-frequency approximation in (5.12)-(5.13). The kernel of (5.9) is defined in (3.49)-(3.51) and is evaluated by numerical quadratures. Similar conclusions are reached as far as the range of λ/L for which the iterations converge is concerned. The rate of convergence is slower however, with at most 50 iterations being necessary for (5.9) and 70 for (5.12), for a relative error 10^{-4} . The computation times for the solution of (5.9) and (5.12) are 1 minute and 30 seconds on the PDP11-34 minicomputer, and 1 and 0.5 seconds on the

IBM370 computer, respectively.

The evaluation of $Q_7(x)$ essentially completes the numerical work required for the solution of the problem. The interaction coefficient $C_S(x)$ is obtained using (5.3) with the velocity potential of the inner solution, the pressure and the exciting force and moment subsequently determined from (4.20), (6.2) and (6.8) respectively.

Maruo and Sasaki (1974) performed experimental measurements of the pressure distribution on a body of revolution in head waves at $U=0$. The radius variation along the body length is given by

$$a(x) = \begin{cases} L/20, & |x| \leq 3L/10 \\ L/20 - 5L(|x| - 3L/10)^2/4, & 3L/10 \leq |x| \leq L/2 \end{cases} \quad (7.3)$$

where the length of the model used is $L=2m$. The pressure was measured at four sections, located at distances $0.3L, 0.15L, 0, -0.15L$ and $-0.3L$ from midships. For each section, the pressure gages were located at the angles $\theta=0^\circ, 40^\circ$ and 70° , the keel being at $\theta=0^\circ$. The model was kept at the centerline of a tank 3.6m wide, the wavelength to model length ratio was taken to vary between 0.347 and 1.22 and the wave amplitude was kept constant at 20mm.

In Figures 3a,b,c the measured pressure amplitude along

the model length for $\lambda/L = 0.621$ and $\theta = 0^\circ, 40^\circ, 70^\circ$ is compared with the numerical results. The strip theory shown is based on the two-dimensional Laplace equation and the relative motion hypothesis. In particular each section is assumed to oscillate vertically on an otherwise calm free surface with a velocity $V_R(x)$ opposite to the fluid velocity due to the incident wave at the mean effective depth, defined by

$$V_R(x) = -iA\omega_0 e^{i\nu x} \int_0^{\pi/2} e^{-\nu a(x) \cos\theta} \cos\theta \, d\theta \quad (7.4)$$

Results are also shown of the normalized two-dimensional solution which is based on the Helmholtz equation, with the velocity potential given by (7.1). The product of this solution with the interaction coefficient $C_S(x)$ results in the unified theory solution, their ratio being a measure of the three-dimensional interactions that occur along the body length.

The numerical results presented in Figures 3a,b,c suggest that the original Maruo and Sasaki theory, the unified theory and its short-wavelength approximation are all very close and compare well with experimental data. It must be mentioned however that the short-wavelength approximation is essentially the Maruo and Sasaki theory corrected as noted in Chapters III and V. The theory of

Faltinsen is in reasonable agreement with experiments on the afterbody but is singular at the bow. The strip theory solution is free from three-dimensional interactions and compares well with experiments only in a region close to the bow where the memory effects due to the flow upstream are not yet strong. The growing importance of these effects downstream is displayed by the experimental measurements which indicate that the pressure amplitude along the parallel middle body decreases towards the stern, compared to the constant value predicted by strip theory.

The ratio of the unified theory prediction to the two-dimensional Helmholtz results at a given position along the body length is independent of the angle θ at which the pressure is measured and is equal to the interaction coefficient $C_S(x)$. The magnitude of $C_S(x)$ decreases towards the stern and in particular $C_S(-L/2) \approx 1/2$ compared to $C_S(L/2) = 1$. This indicates that the total wave amplitude at the stern is reduced to about half of the incident wave amplitude when $\lambda/L = 0.621$.

In Figures 4a,b,c the pressure amplitude measured at $\theta=40^\circ$ and at the positions $x=0.3L, 0$ and $-0.3L$ along the body is compared to the theoretical predictions for different values of λ/L . These results show clearly the better agreement of strip theory with the experiments at $x=0.3L$ compared to the agreement downstream. It is also

interesting to notice that the rate of decrease of the wave disturbance along the body length increases with decreasing λ/L , suggesting stronger interactions for shorter waves.

Lee (1964) performed experimental measurements of the sectional force distribution and the total exciting force and moment on a prolate spheroid advancing in head waves. The model length was 1m with a beam to length ratio 1/6. The towing tests were performed at Froude numbers ranging from 0.082 to 0.328 and λ/L ratios ranging from 0.5 to 2.0. Pressure measurements were taken at six stations located symmetrically with respect to $x=0$, and the measured sectional forces were subsequently integrated to yield the exciting force and moment. Comparisons were made with direct force and moment measurements.

In Figures 5a,b,c the measured sectional force amplitude is compared to the predicted values for $Fr=0.123$ and $\lambda/L=0.5, 1.0$ and 1.5 respectively. Similar comparisons at $Fr=0.205$ and 0.246 are shown in Figures 6a,b,c and 7a,b,c respectively.

The unified theory and its short-wavelength approximation are again close to each other and are both in good agreement with experiments. The decay of the sectional force towards the stern is again apparent. The strip theory prediction, is symmetric with respect to $x=0$

and its agreement with experiments is less satisfactory when compared to the unified theory predictions. Both the unified theory and the short-wavelength predictions of the normalized sectional force at the bow is unity, but due to end effects this value is not approached in a smooth manner.

A more careful analysis of the results indicates that the normalized sectional force amplitude along the model length increases with increasing λ/L and this trend is predicted by both strip theory and unified theory. At a given λ/L ratio, both the measured sectional force amplitude and the unified theory prediction tend to increase with increasing Froude number, the rate of increase being larger between $Fr=0.123$ and 0.205 and more pronounced for short waves. The strip theory prediction is independent of forward speed effects, being equal to the $U=0$ values. Combining the pronounced rate of increase of the normalized sectional force amplitude at low Froude numbers with the independence of the strip theory predictions on speed changes, it is possible to explain the better agreement of strip theory with the experiments for U finite compared to the zero speed case.

Taking again the ratio of the two-dimensional Helmholtz to the unified theory values as a measure of the three-dimensional interaction effects, we conclude that

their strength decreases with increasing λ/L at a fixed Froude number, being relatively insensitive to speed changes at a given λ/L ratio.

Comparisons of the measured amplitude and phase of the heave exciting force and pitch exciting moment with the theoretical predictions are shown in Figures 8a,b-13a,b for all three Froude numbers 0.123, 0.205 and 0.246.

The amplitude of the exciting force predicted by strip theory and unified theory are close for $\lambda/L > 1.0$ for all Froude numbers, and in this range they are both in good agreement with the experiments. For $\lambda/L < 1.0$, the unified theory predictions seem to agree better with the experiments, especially for $Fr = 0.205$ and 0.246 . The scatter of the measurements, however, prevents a confident conclusion. The same conclusions hold for the amplitude of the exciting moment, with the additional comment that strip theory seems to be in better agreement relative to others with the experiments for $\lambda/L > 1.5$. The unified theory predictions of the phase angles are in better agreement with the measured values for all Froude numbers, compared to the corresponding strip theory predictions. The high-frequency approximation of the exciting force and moment amplitudes is in good agreement with the exact theory only for $\lambda/L < 0.5$, with the single exception of the pitching moment at $Fr = 0.123$. The corresponding values of the phase angles are in good agreement with the exact theory

for all λ/L ratios and Froude numbers. For all cases tested $\tau = \omega U/g$ was greater than $1/4$.

The unified theory predictions indicate a small increase of the exciting force and moment amplitude with increasing Froude number for $\lambda/L > 0.5$, while the phase angles remain relatively insensitive to speed changes. A similar insensitivity is also observed in the experimental measurements for $\lambda/L > 0.5$, no conclusion being possible for the measured exciting force and moment amplitudes due to the experimental scatter.

v_a	σ_s	a_3	$\theta=0^\circ$	$\theta=20^\circ$	$\theta=40^\circ$	$\theta=60^\circ$	$\theta=80^\circ$
0.0	0.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	0.30978	0.85569	0.82440	0.83340	0.85906	0.89731	0.94071
0.2	0.52271	0.79566	0.73302	0.75075	0.80182	0.87931	0.96936
0.3	0.67278	0.74997	0.65589	0.68201	0.75817	0.87625	1.01747
0.4	0.77212	0.70610	0.58228	0.61590	0.71524	0.87316	1.06838
0.5	0.82972	0.66068	0.51064	0.55037	0.66954	0.86422	1.11377
0.6	0.85441	0.61374	0.44219	0.48640	0.62111	0.84772	1.14959
0.7	0.85474	0.56655	0.37865	0.42566	0.57136	0.82415	1.17462
0.8	0.83832	0.52052	0.32131	0.36959	0.52199	0.79500	1.18938
0.9	0.81131	0.47682	0.27075	0.31906	0.47439	0.76203	1.19527
1.0	0.77833	0.43616	0.22699	0.27433	0.42952	0.72684	1.19400
1.1	0.74263	0.39890	0.18964	0.23527	0.38793	0.69072	1.18719
1.2	0.70635	0.36510	0.15807	0.20150	0.34983	0.65464	1.17626
1.3	0.67084	0.33464	0.13157	0.17247	0.31520	0.61927	1.16235
1.4	0.63688	0.30729	0.10943	0.14762	0.28391	0.58502	1.14636
1.5	0.60485	0.28278	0.09098	0.12641	0.25572	0.55216	1.12895
1.6	0.57492	0.26082	0.07565	0.10832	0.23038	0.52083	1.11061
1.7	0.54711	0.24113	0.06291	0.09290	0.20763	0.49109	1.09170
1.8	0.52133	0.22345	0.05234	0.07975	0.18720	0.46294	1.07250
1.9	0.49748	0.20755	0.04357	0.06853	0.16886	0.43634	1.05318
2.0	0.47541	0.19321	0.03628	0.05894	0.15239	0.41125	1.03389
2.1	0.45499	0.18026	0.03023	0.05075	0.13759	0.38760	1.01471
2.2	0.43605	0.16852	0.02520	0.04374	0.12428	0.36533	0.99572
2.3	0.41848	0.15787	0.02102	0.03773	0.11231	0.34435	0.97696
2.4	0.40215	0.14816	0.01755	0.03258	0.10154	0.32460	0.95847
2.5	0.38695	0.13931	0.01465	0.02815	0.09183	0.30601	0.94027
2.6	0.37276	0.13121	0.01224	0.02435	0.08309	0.28851	0.92237
2.7	0.35950	0.12379	0.01023	0.02108	0.07520	0.27204	0.90478
2.8	0.34709	0.11697	0.00856	0.01825	0.06808	0.25653	0.88752
2.9	0.33545	0.11069	0.00716	0.01582	0.06165	0.24192	0.87057

Table 1 - Source strength, sectional force and velocity potential at $\theta=0^\circ-80^\circ$ (mod 20°) of the two-dimensional homogeneous problem (2-D Helmholtz problem) for a circular boundary, versus v_a .

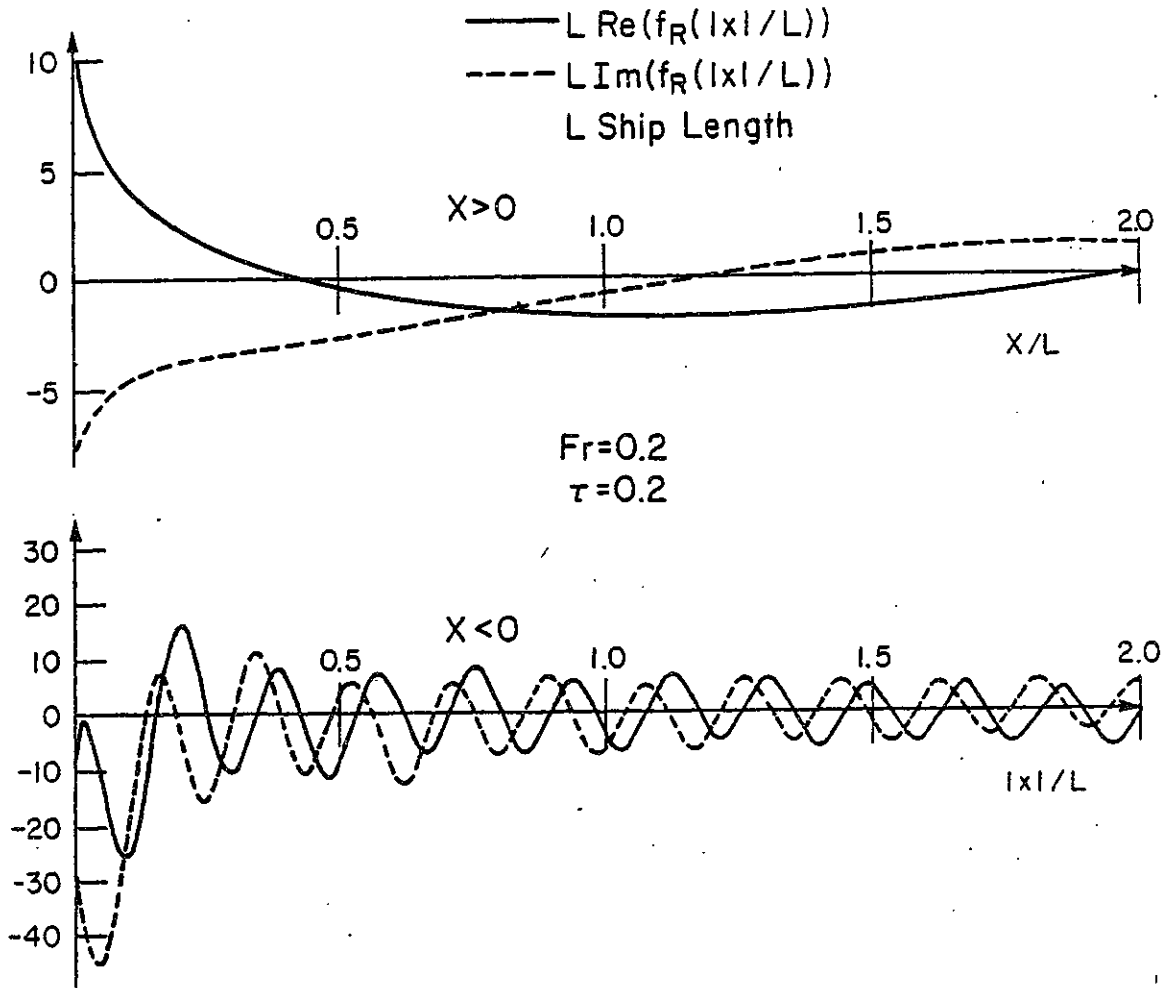


Figure 2a- Regular part (f_R) of the kernel of (3.35) as a function of the longitudinal coordinate $|x|/L$ for $\tau = \omega U/g$ equal to 0.2 and 0.7 (Figures 2a and 2b respectively). Waves are present upstream only in the first case associated with the root k_3 in (3.42) and with the wavelength to ship length ratio $2\pi/k_3 L \approx 3.3$. For $\tau = 0.7$ (Figure 2b) no waves exist upstream. Downstream of the disturbance the most obvious wave motion is associated with the largest root k_1 , and with the wavelength to ship length ratio 0.19 ($\tau = 0.2$) and 0.12 ($\tau = 0.7$). Longer wavelengths also exist downstream, associated with the roots k_2 and k_4 for $\tau < 1/4$ and with k_2 alone for $\tau > 1/4$. Their superposition upon the shorter wave system is more apparent in Figure 2b. (Reprinted from Newman and Sclavounos (1980)).

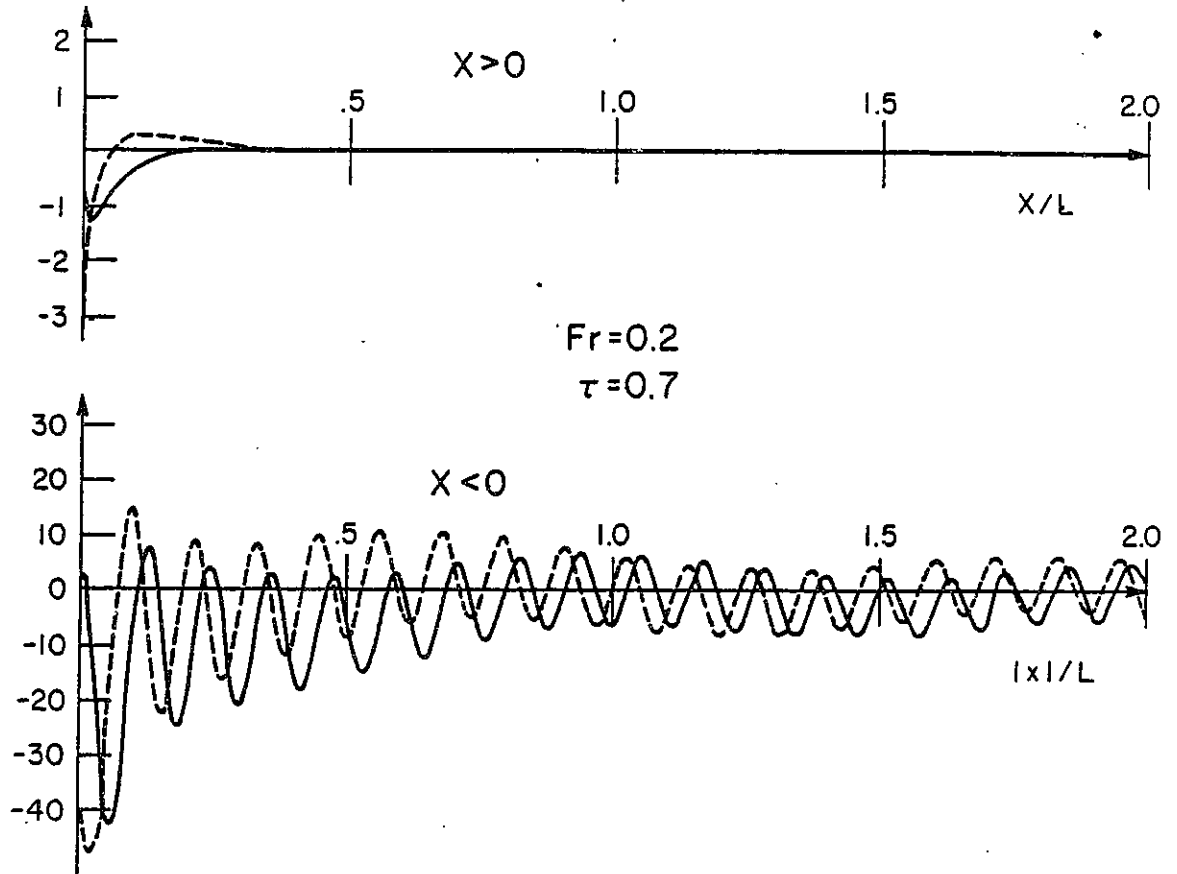


Figure 2b - see Figure 2a

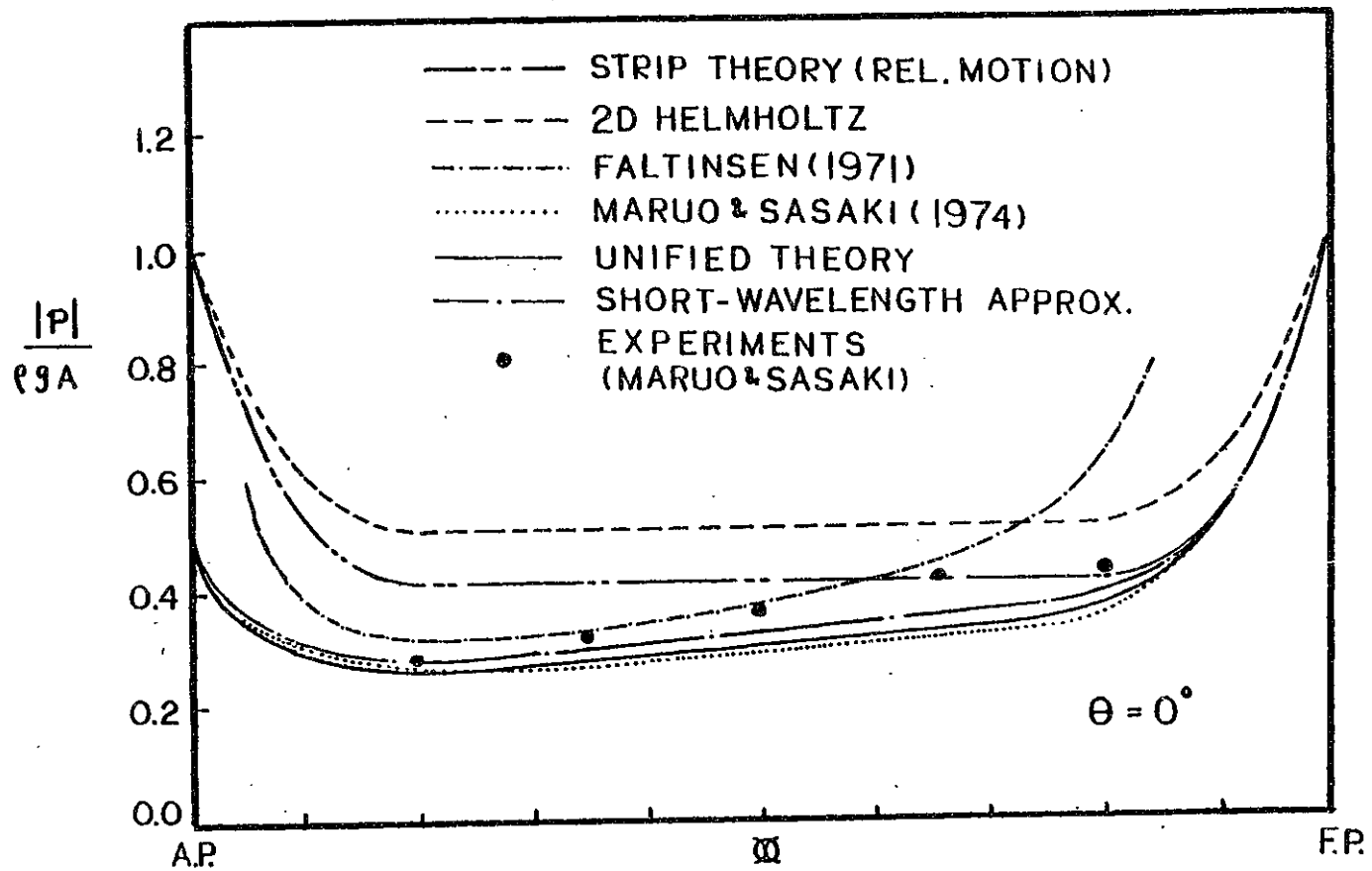


Figure 3a - Longitudinal distribution of the normalized pressure amplitude on the model used by Maruo and Sasaki (1974), for $\theta=0^\circ$, 40° and 70° (Figures 3a, 3b and 3c respectively), $\lambda/L=0.621$ and $U=0$

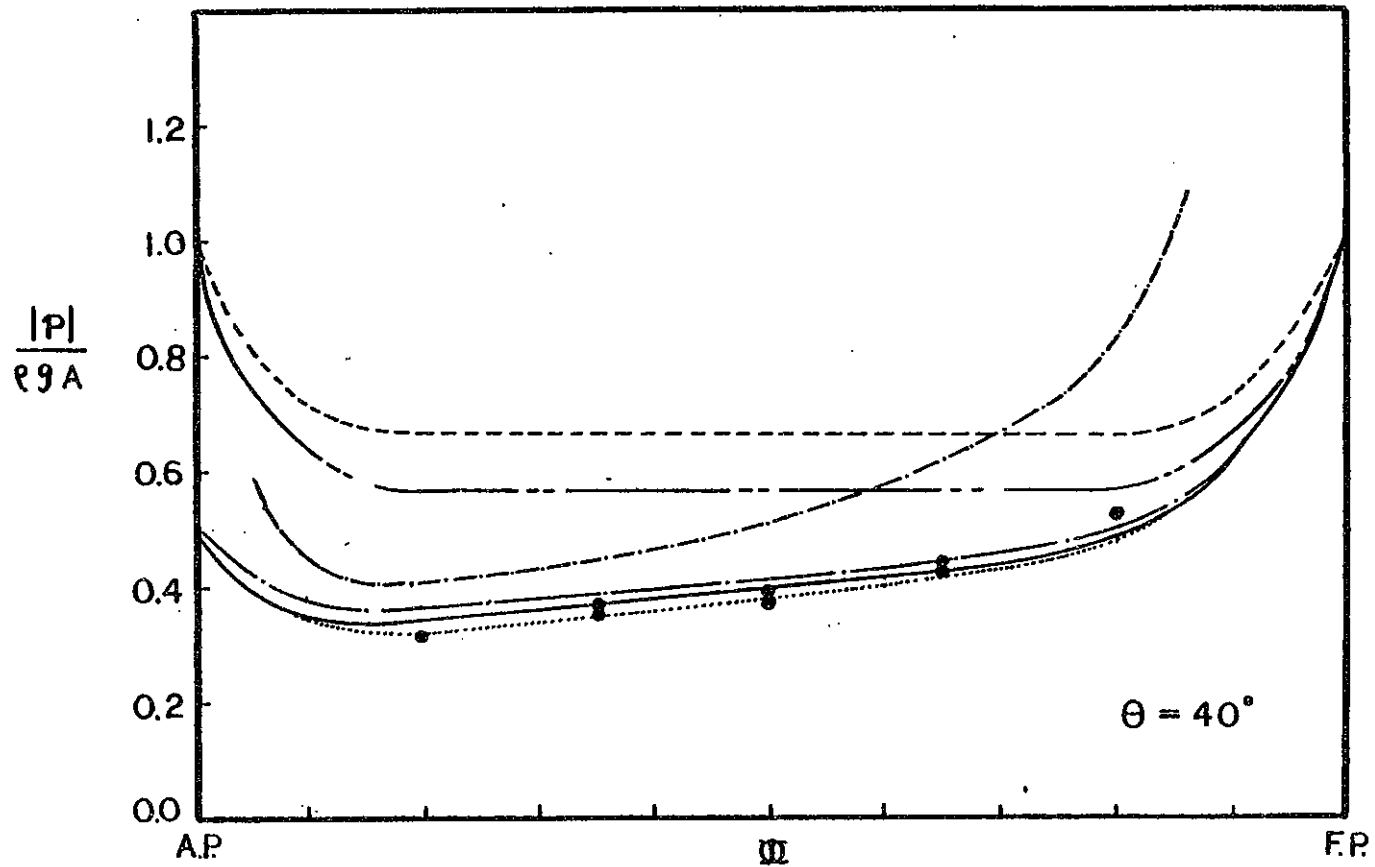


Figure 3b - See Figure 3a

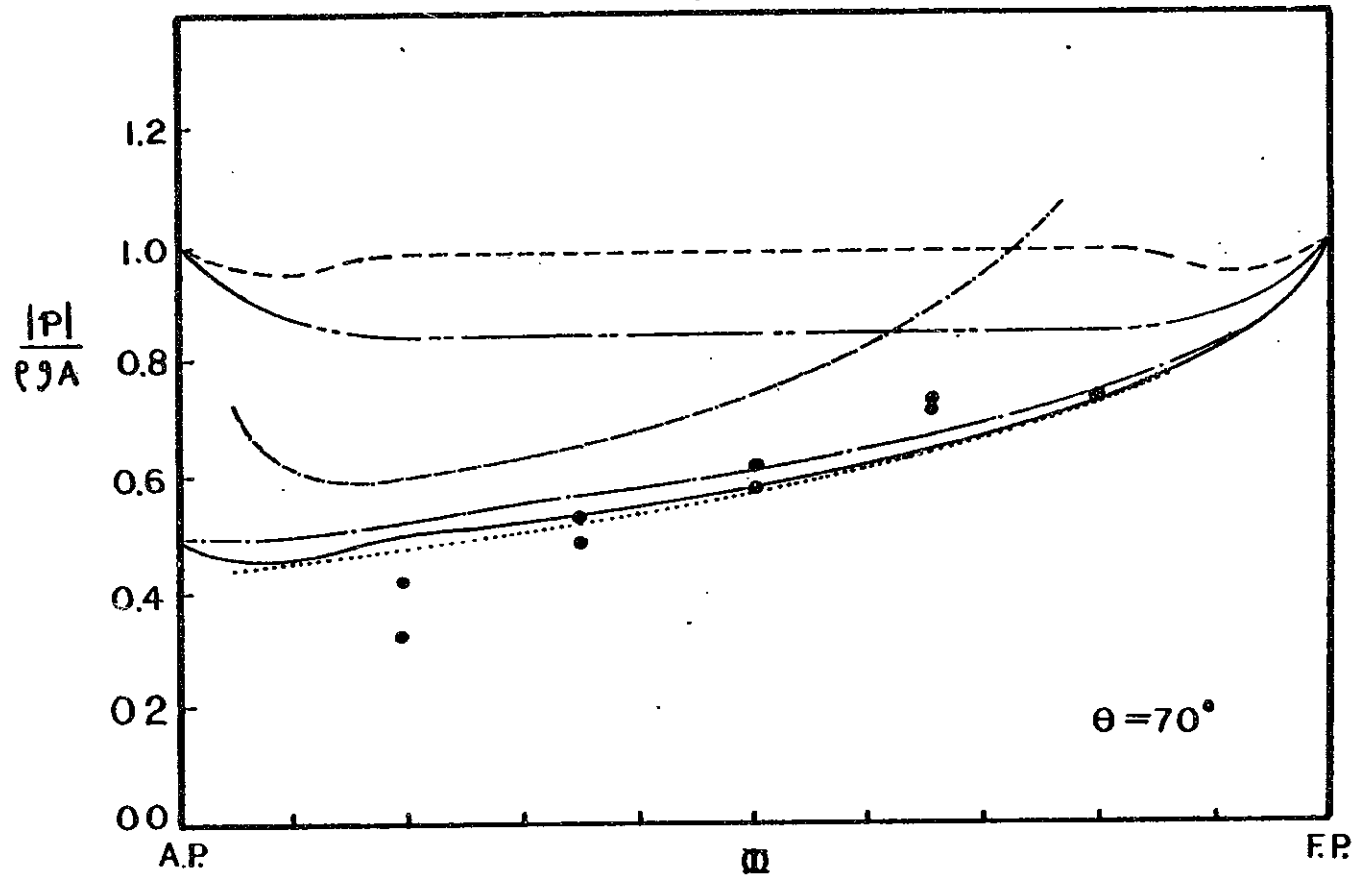


Figure 3c - See Figure 3a

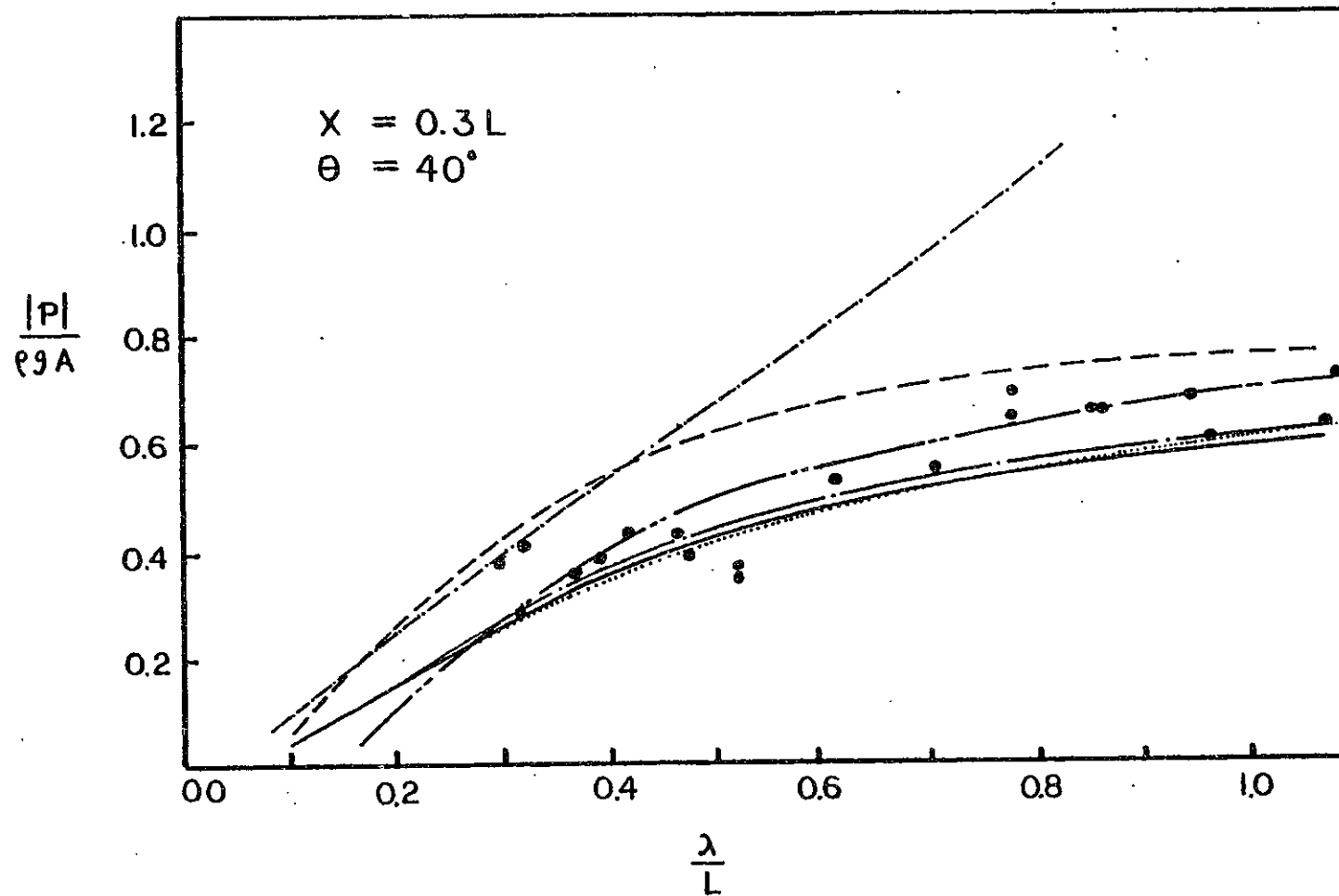


Figure 4a - Normalized pressure amplitude at the sections $x=0.3L$, 0 , $-0.3L$ (Figures 4a, 4b and 4c respectively) of the model used by Maruo and Sasaki (1974), vs λ/L at $\theta=40^\circ$ and $U=0$

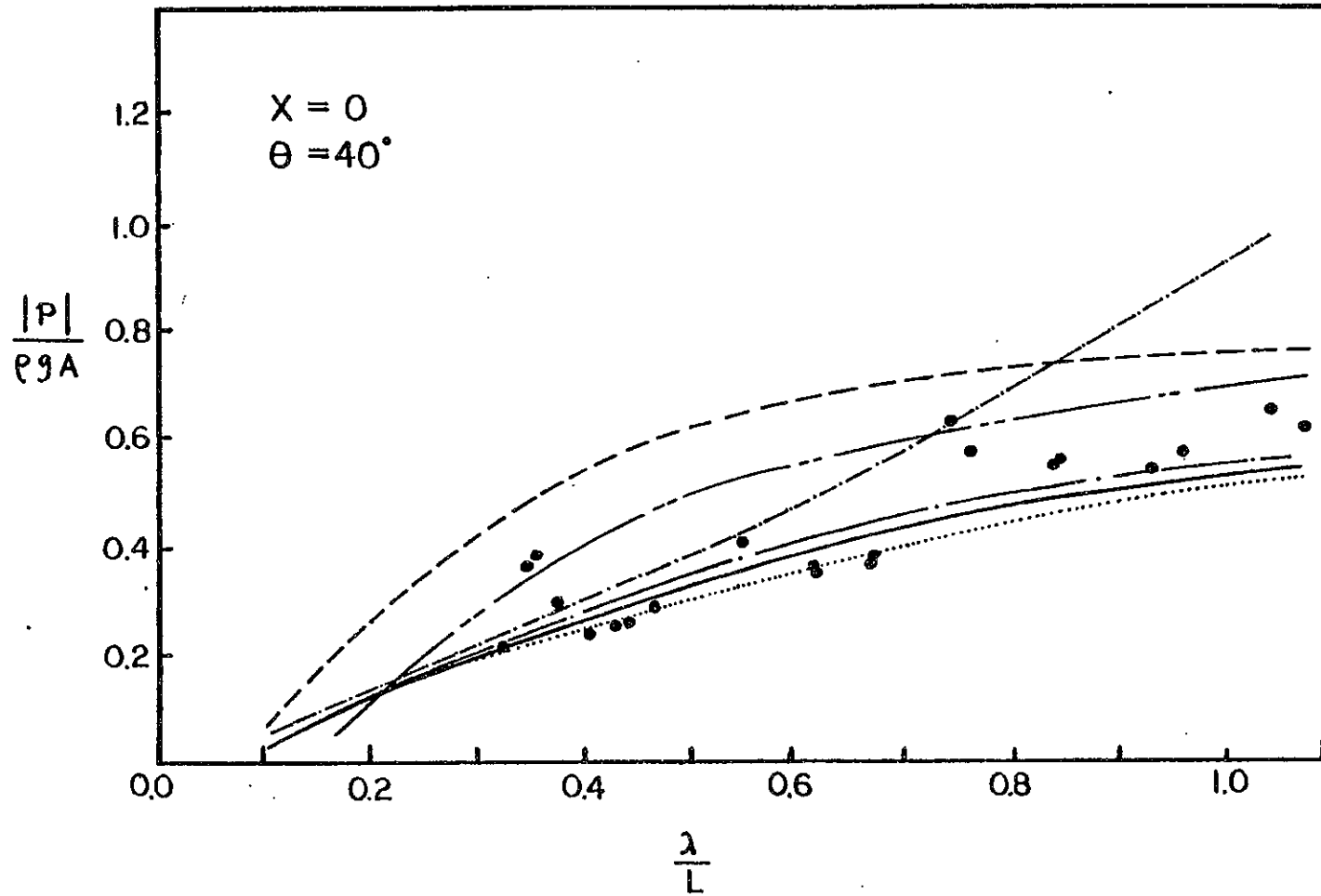


Figure 4b - See Figure 4a

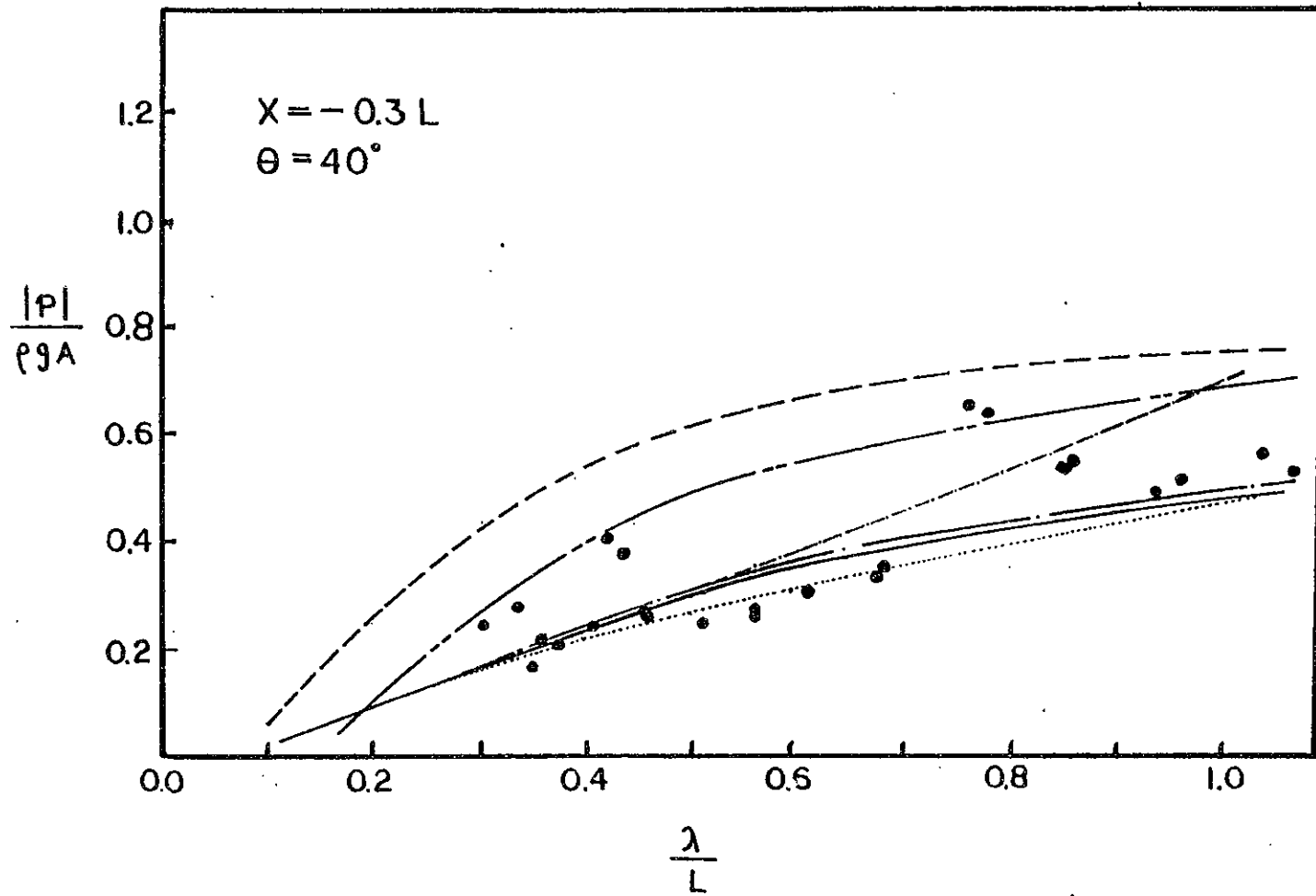


Figure 4c - See Figure 4a

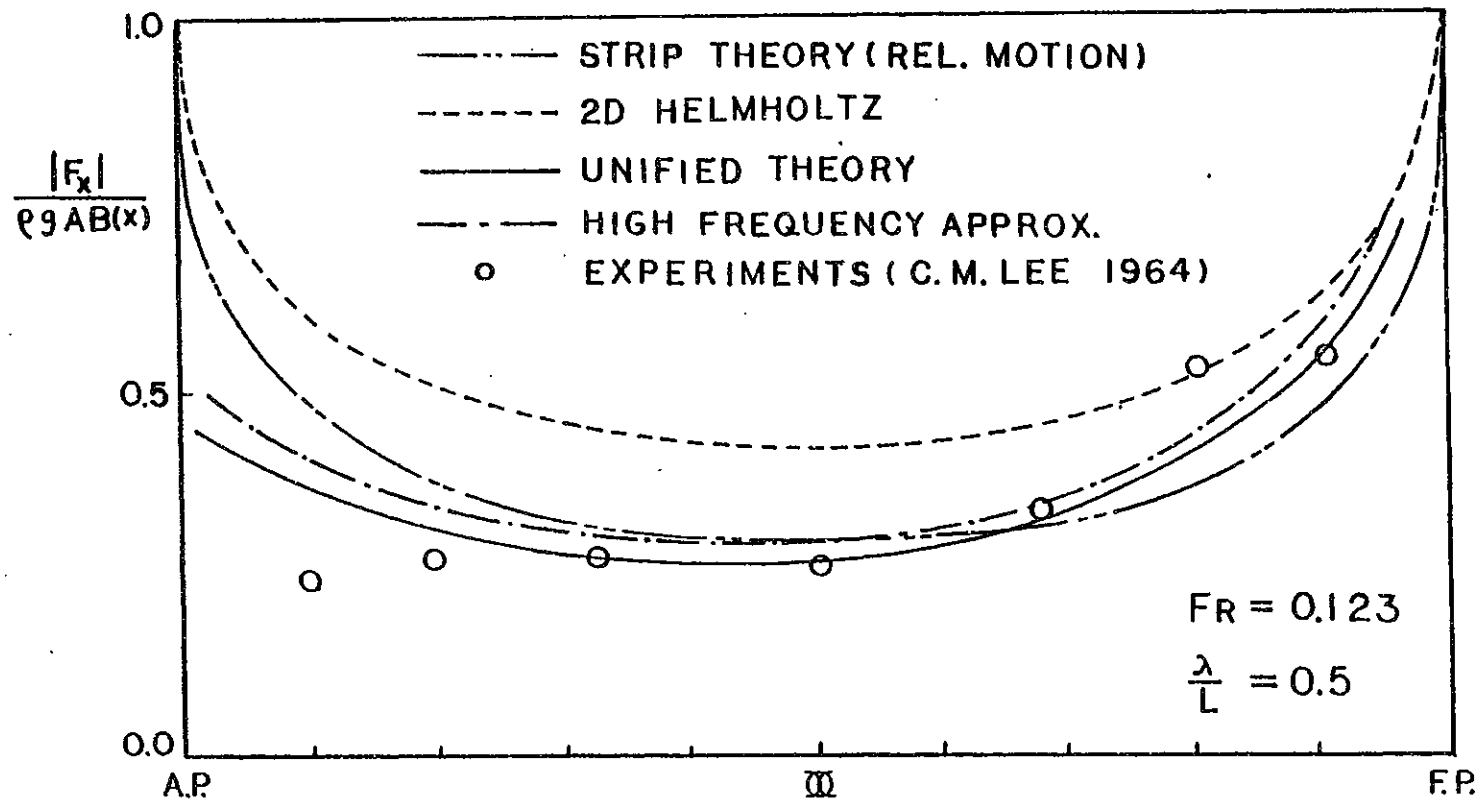


Figure 5a - Longitudinal distribution of normalized sectional force amplitude for a prolate spheroid ($\epsilon=1/6$) in head waves with $\lambda/L=0.5, 1.0$ and 1.5 (Figures 5a, 5b and 5c respectively) at $Fr=0.123$

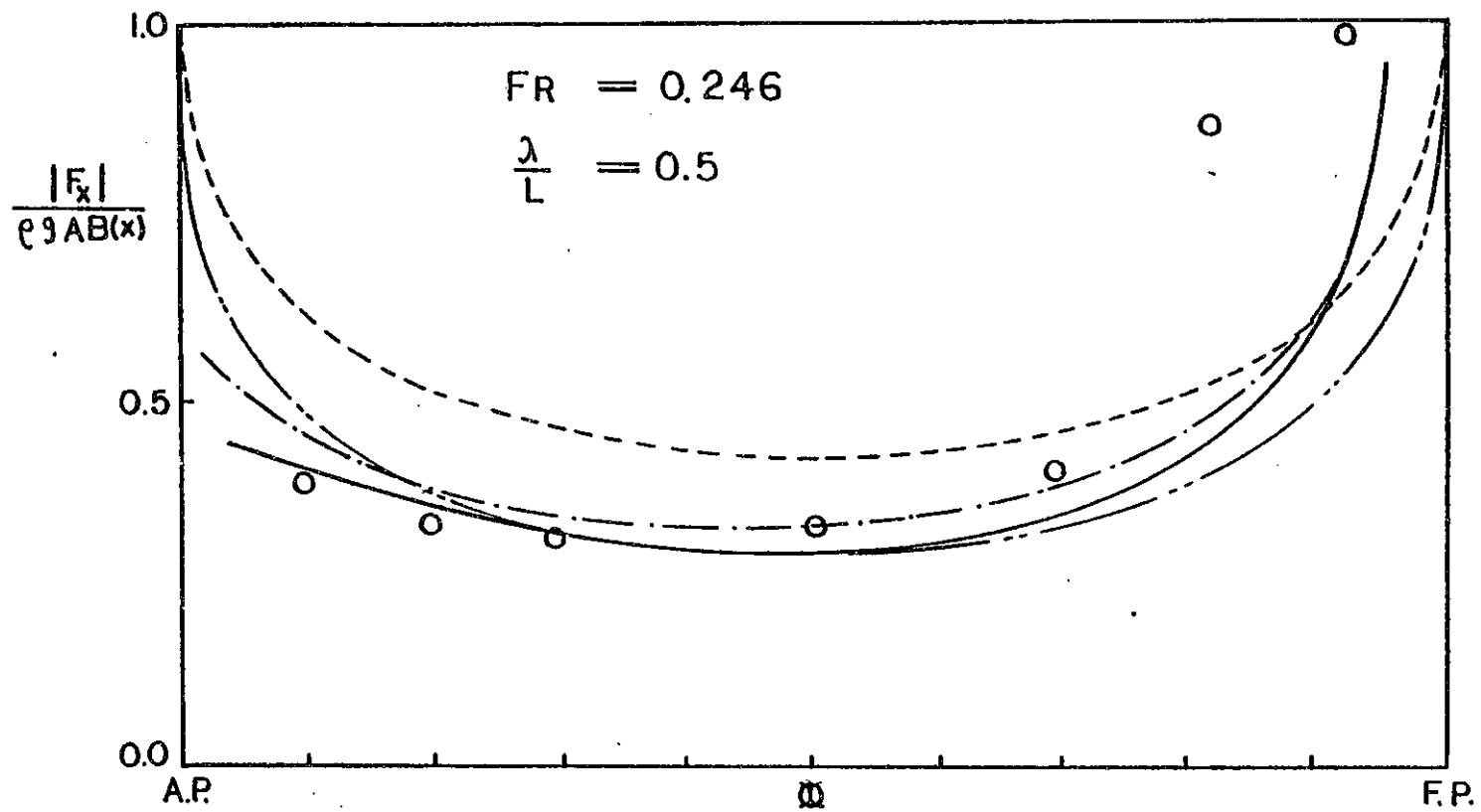


Figure 7a - Same as in Figure 5a with Fr=0.246

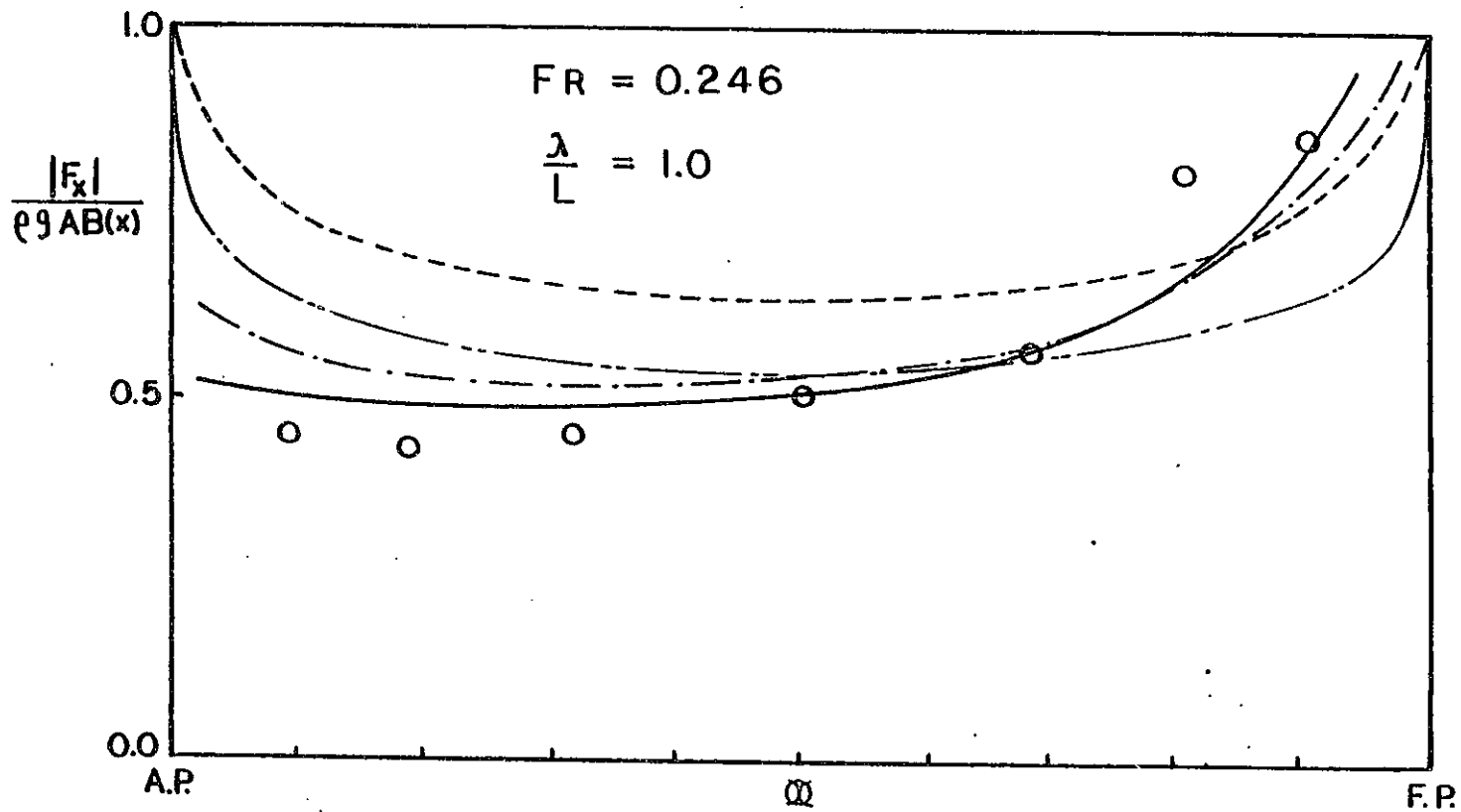


Figure 7b - See Figure 7a

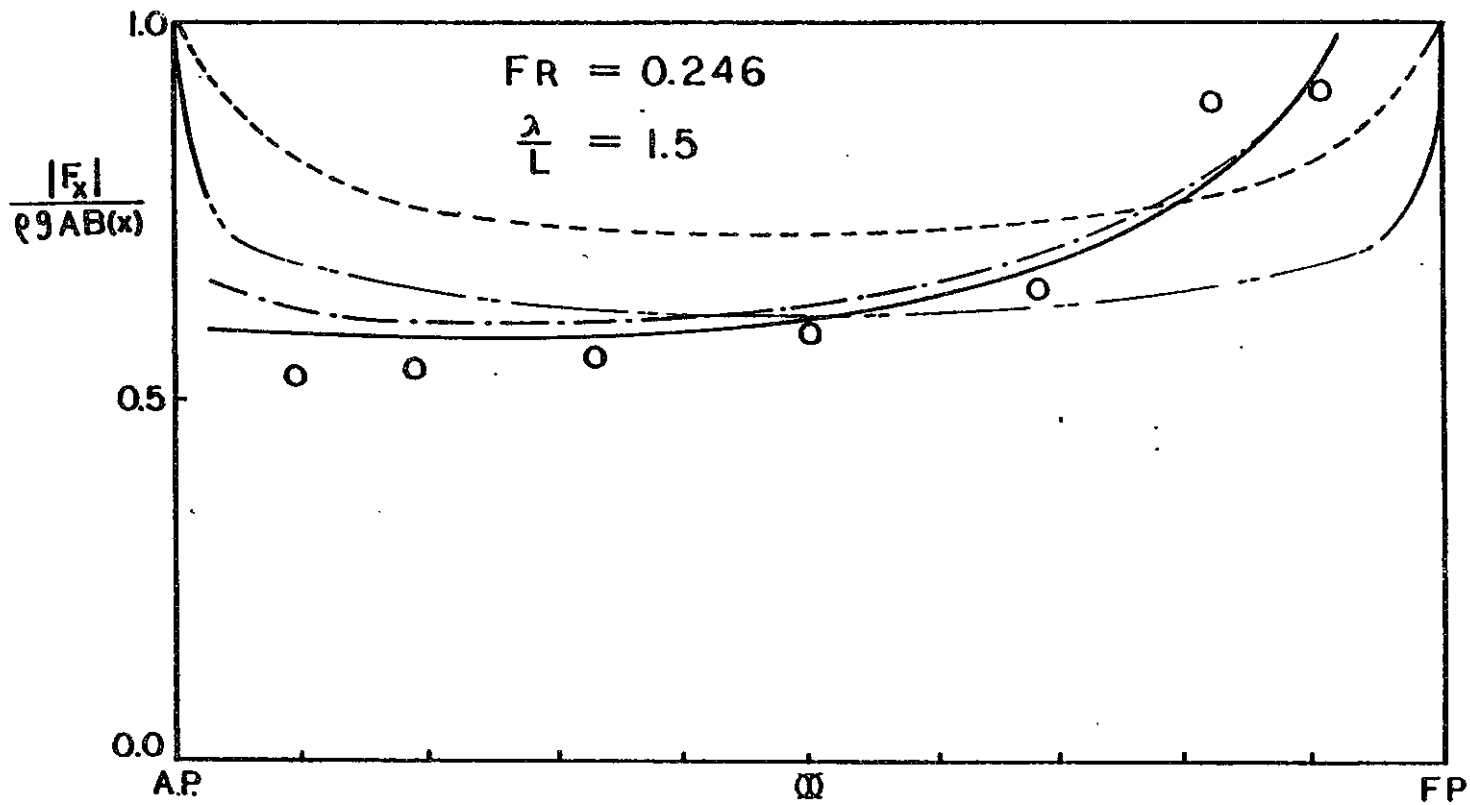


Figure 7c - See Figure 7a

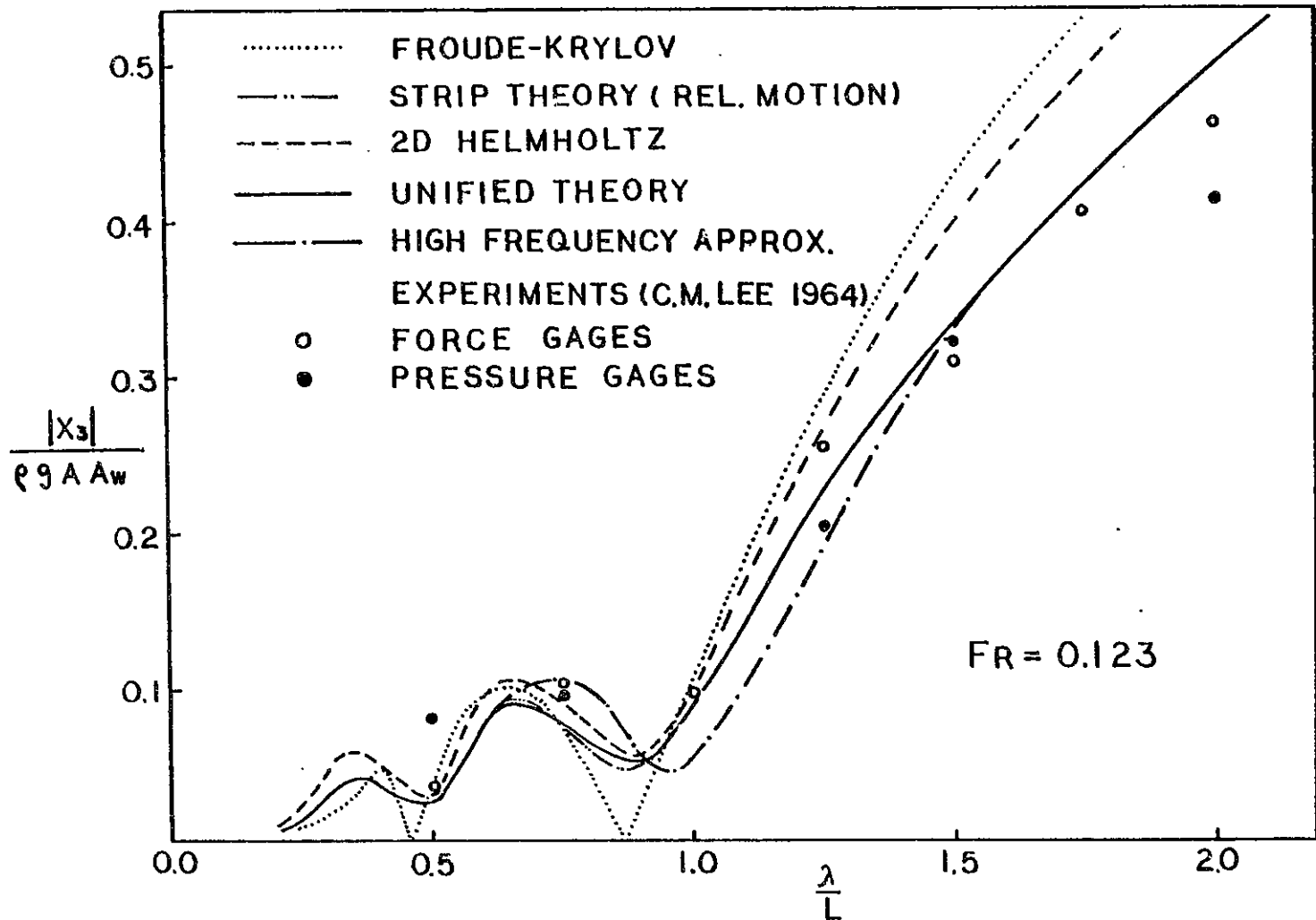


Figure 8a - Normalized amplitude of heave exciting force on a prolate spheroid ($\epsilon=1/6$) in head waves vs λ/L at $Fr=0.123$

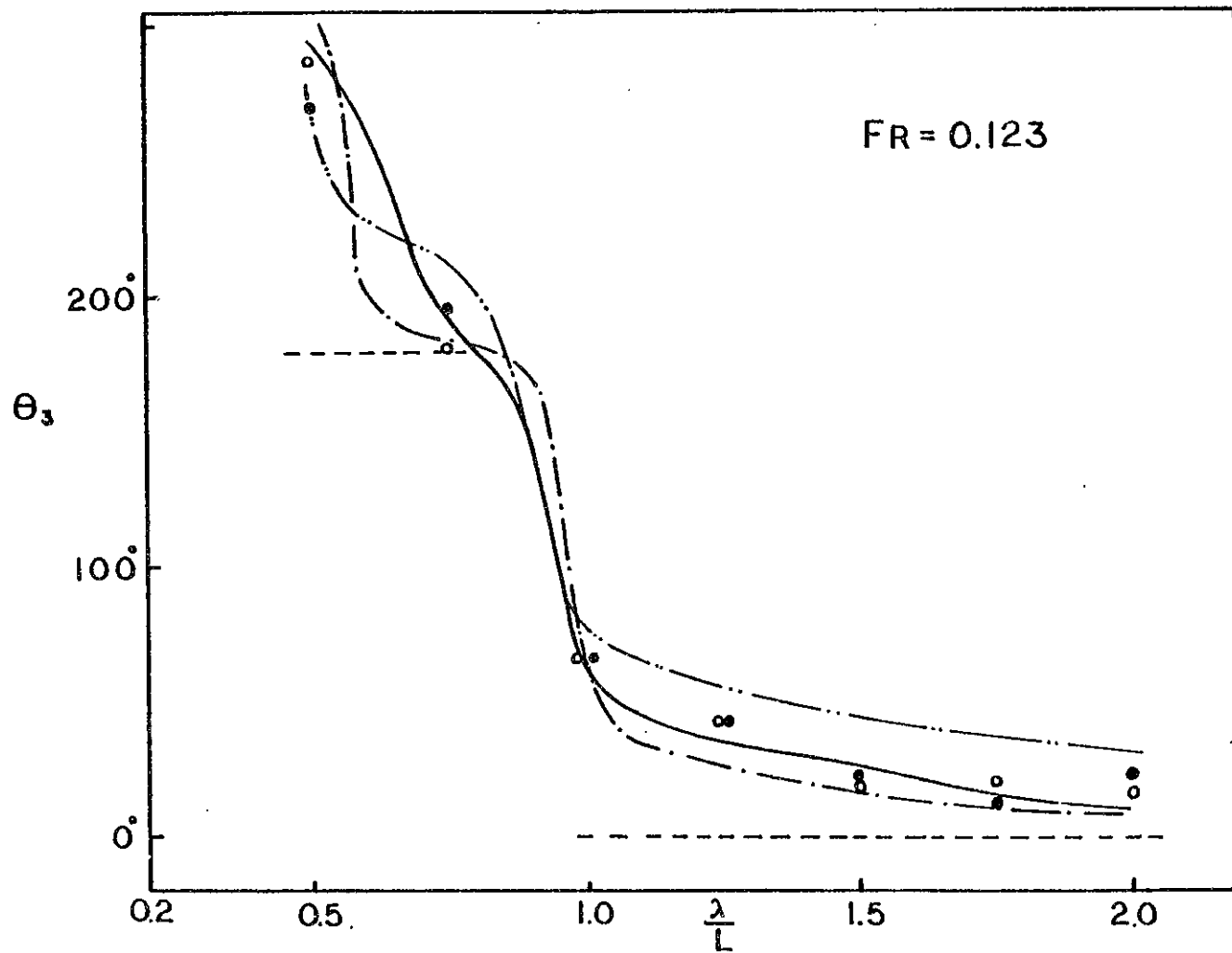


Figure 8b - Phase angle by which the heave exciting force leads the incident wave elevation at $x=0$ vs λ/L at $Fr=0.123$

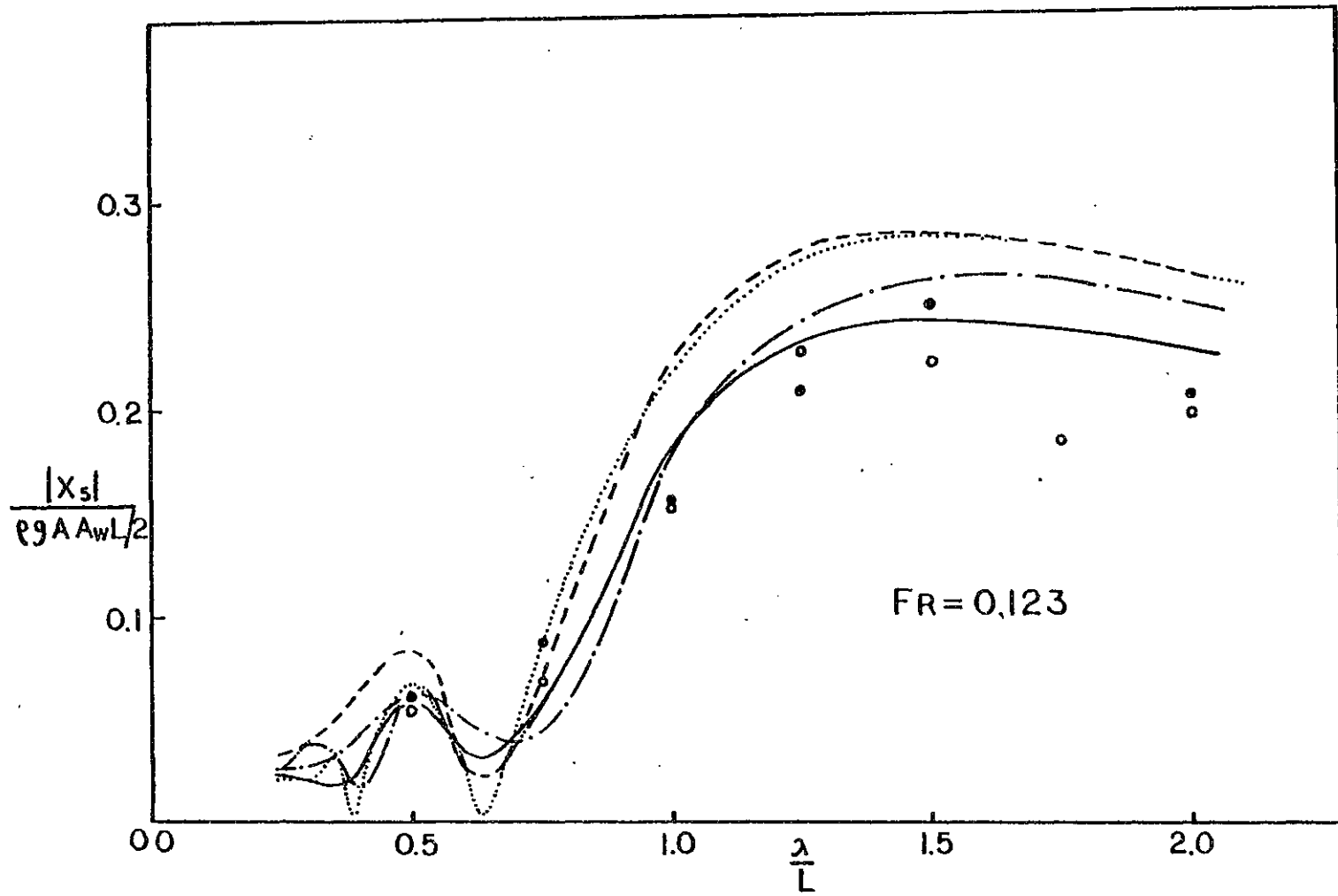


Figure 9a - Normalized amplitude of pitch exciting moment on a prolate spheroid ($\epsilon=1/6$) in head waves vs λ/L at $Fr=0.123$

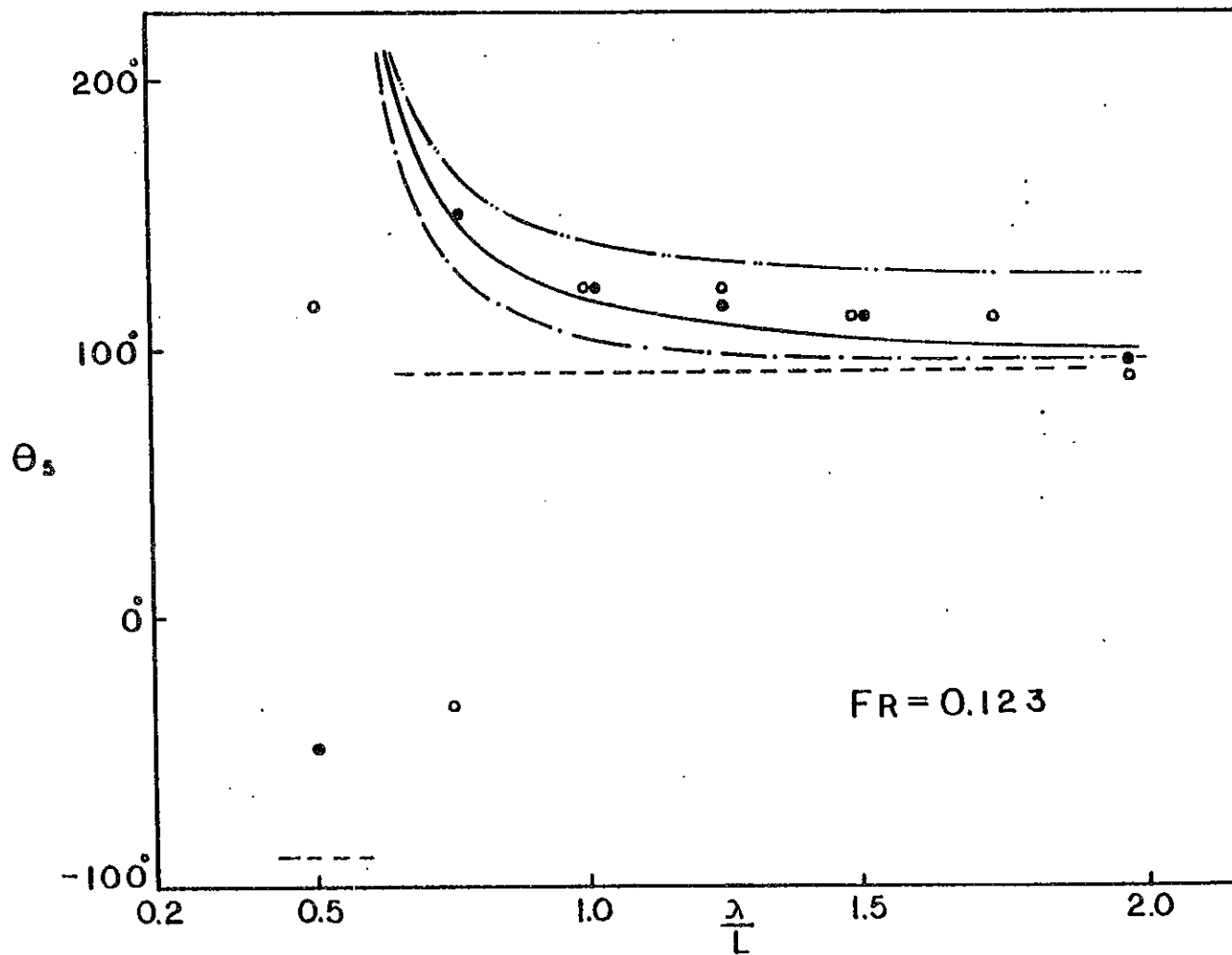


Figure 9b - Phase angle by which the pitch exciting moment leads the incident wave elevation at $x=0$ vs λ/L at $Fr=0.123$

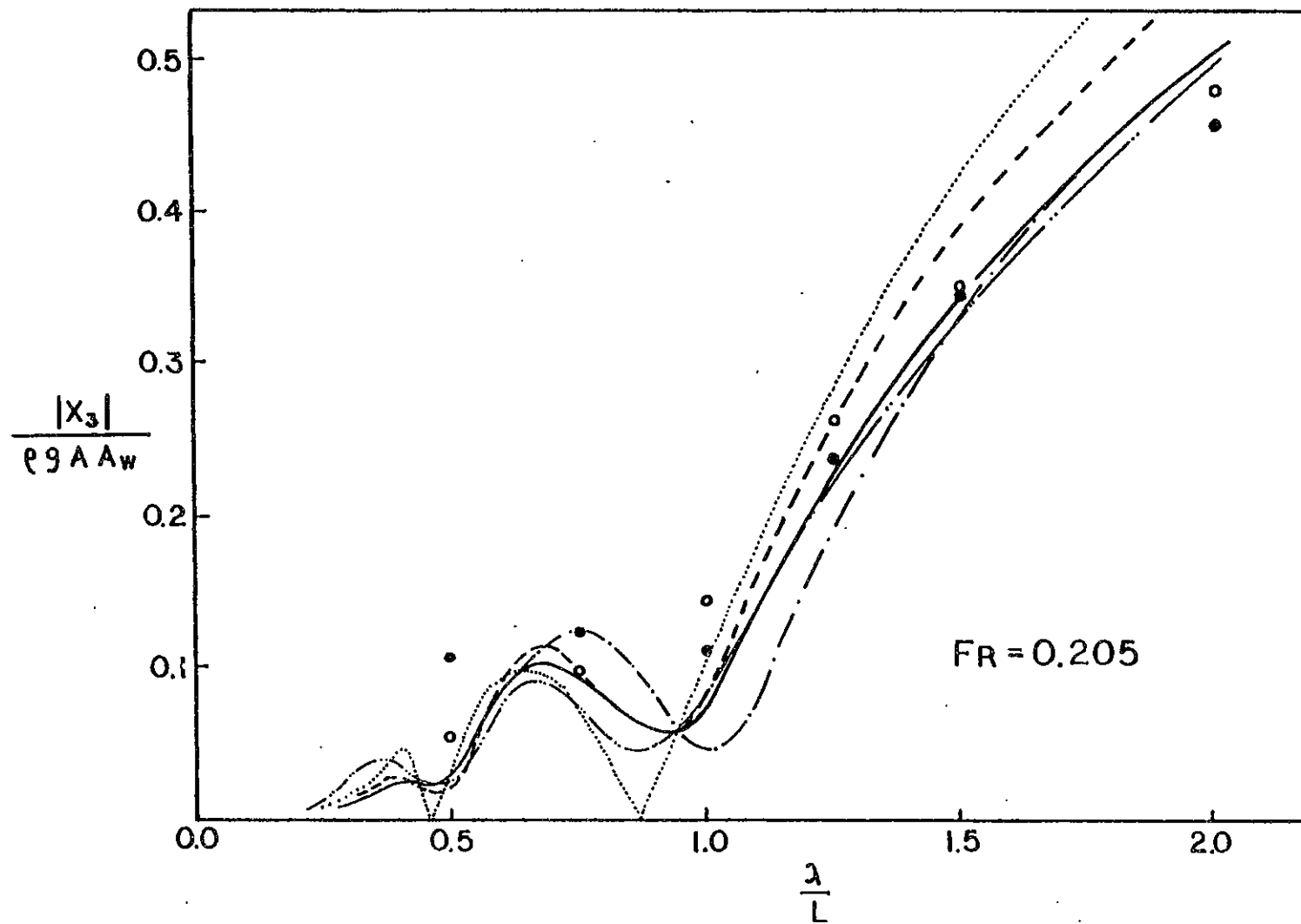


Figure 10a- Same as in Figure 8a with Fr=0.205

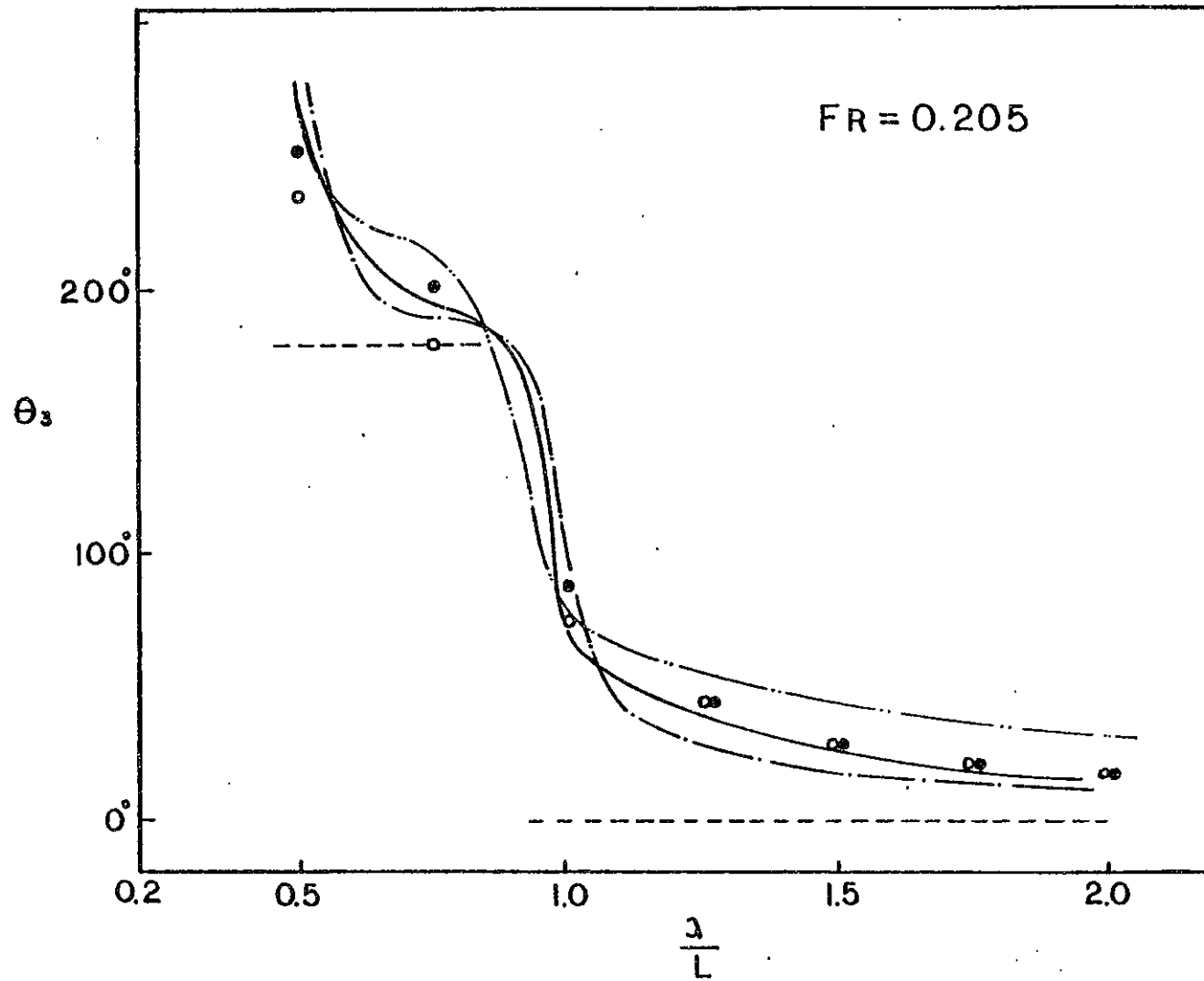


Figure 10b- Same as in Figure 8b with Fr=0.205

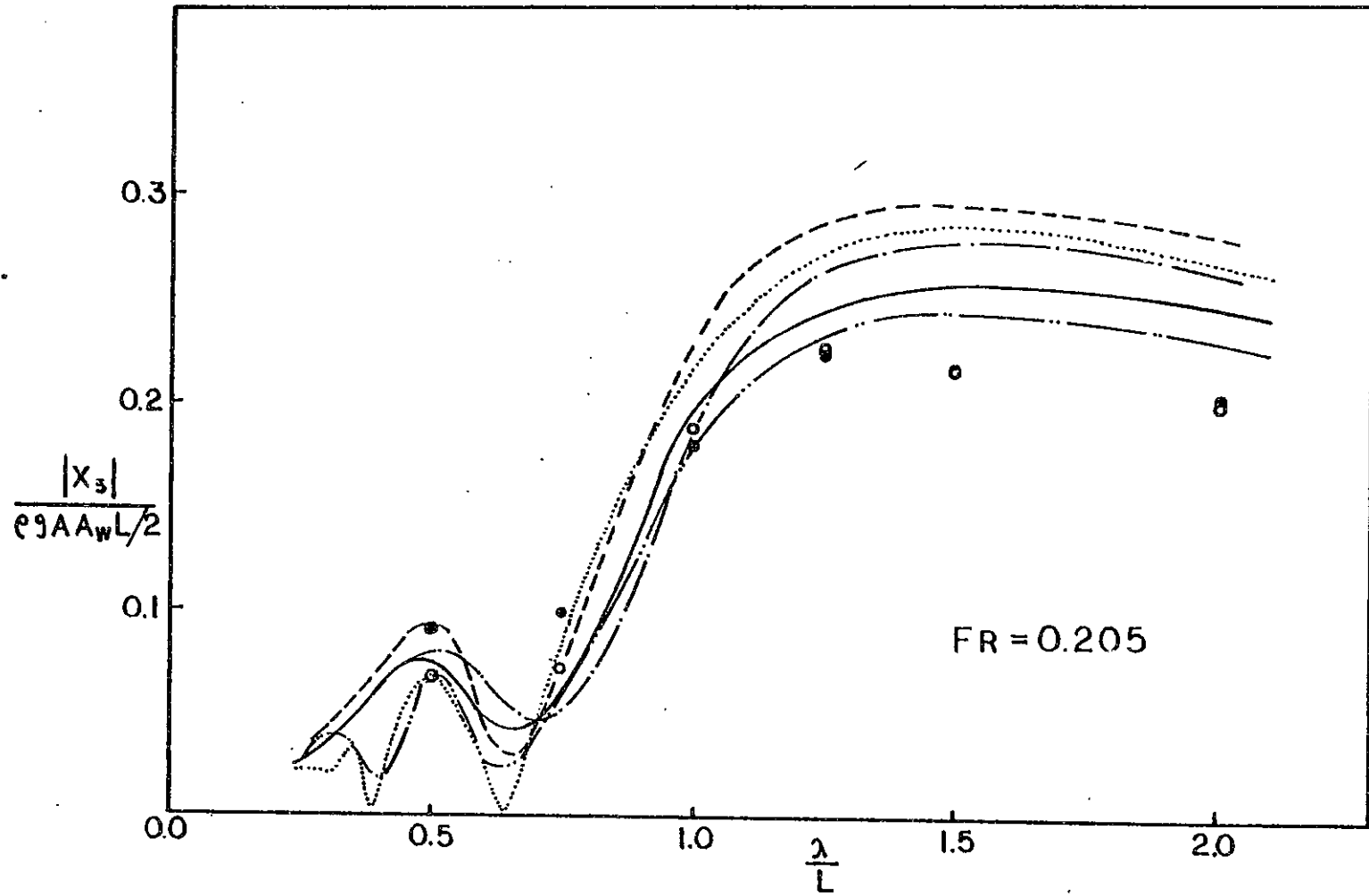


Figure 11a- Same as in Figure 9a' with Fr=0.205

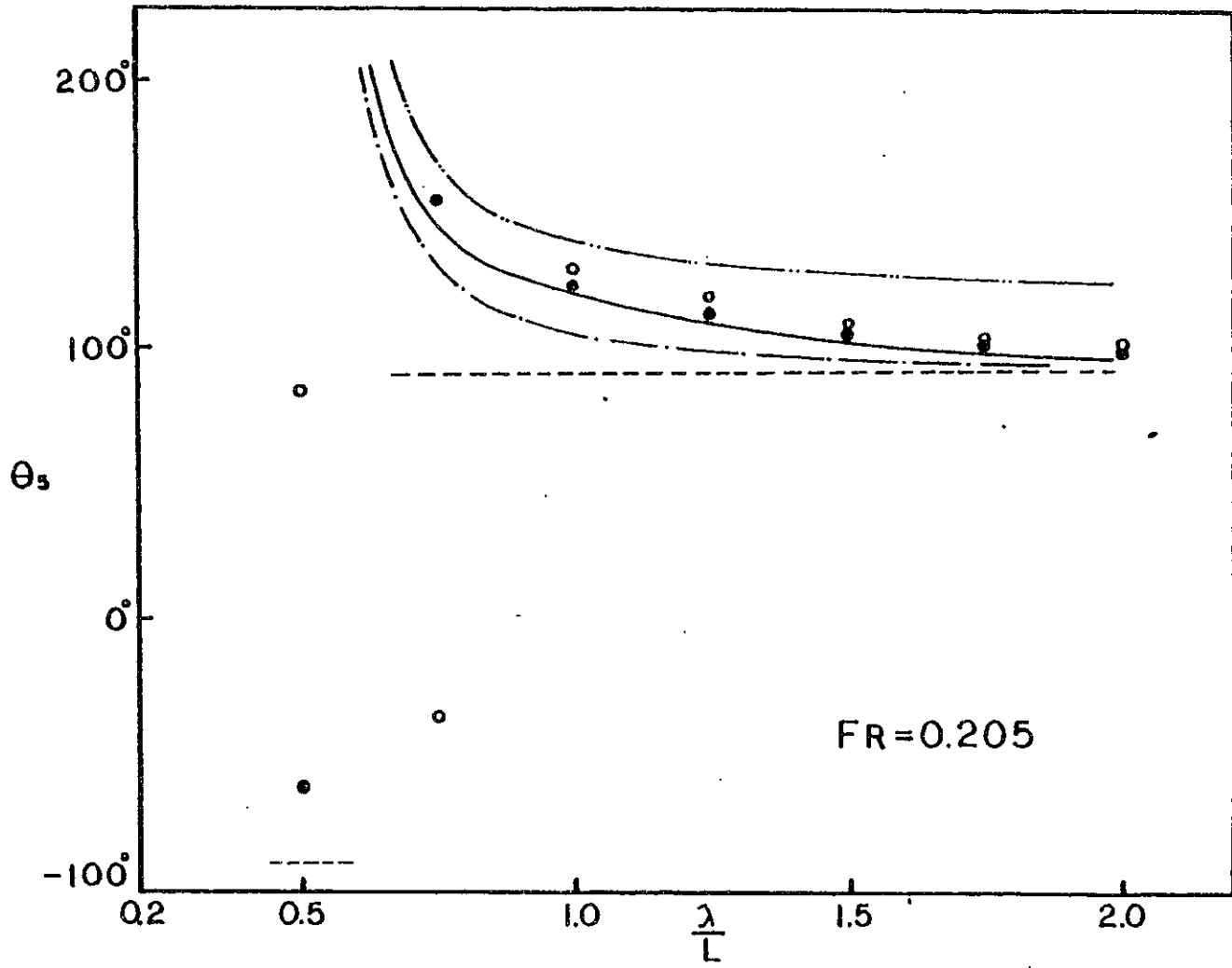


Figure 11b- Same as in Figure 9b with Fr=0.205

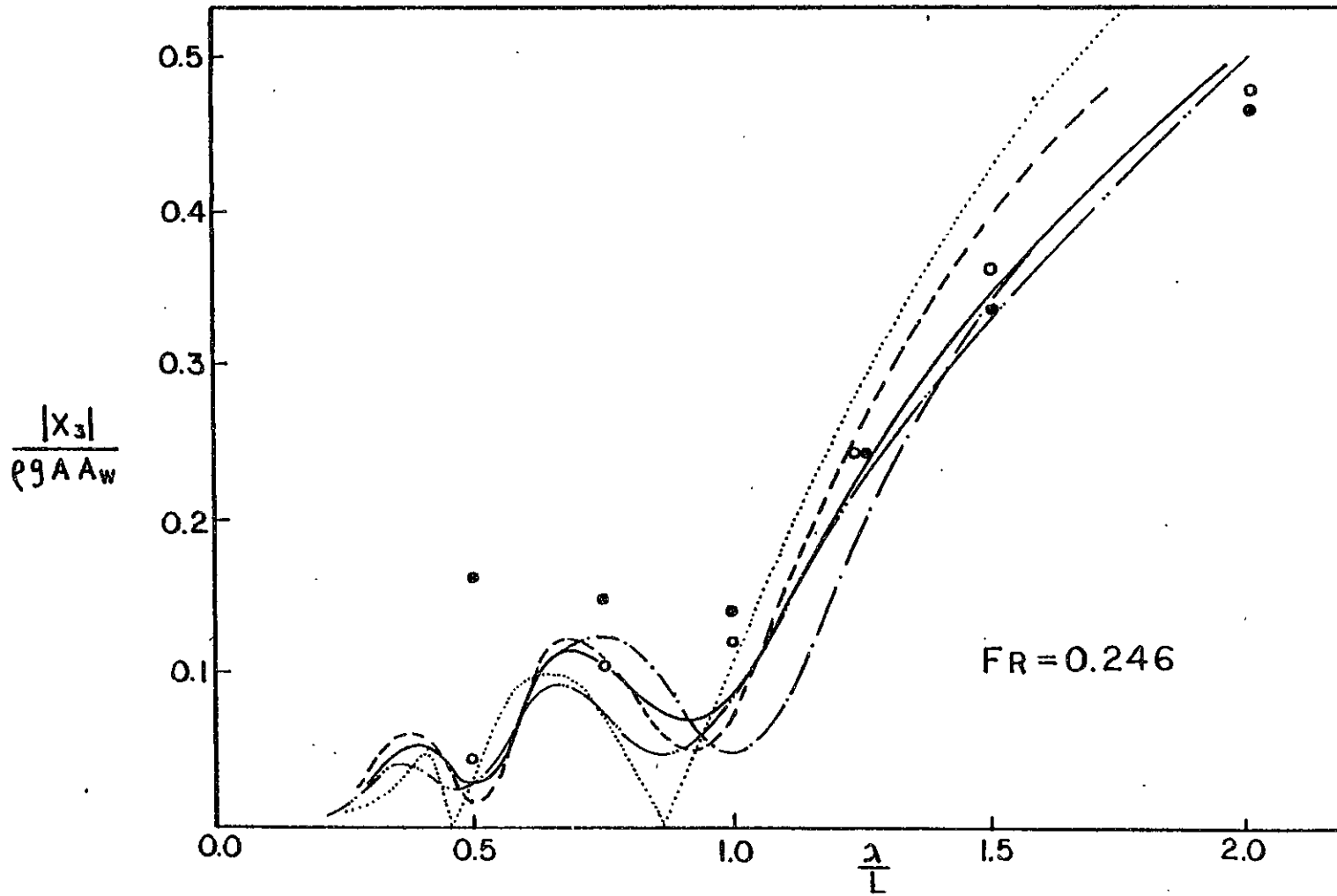


Figure 12a- Same as in Figure 8a with Fr=0.246

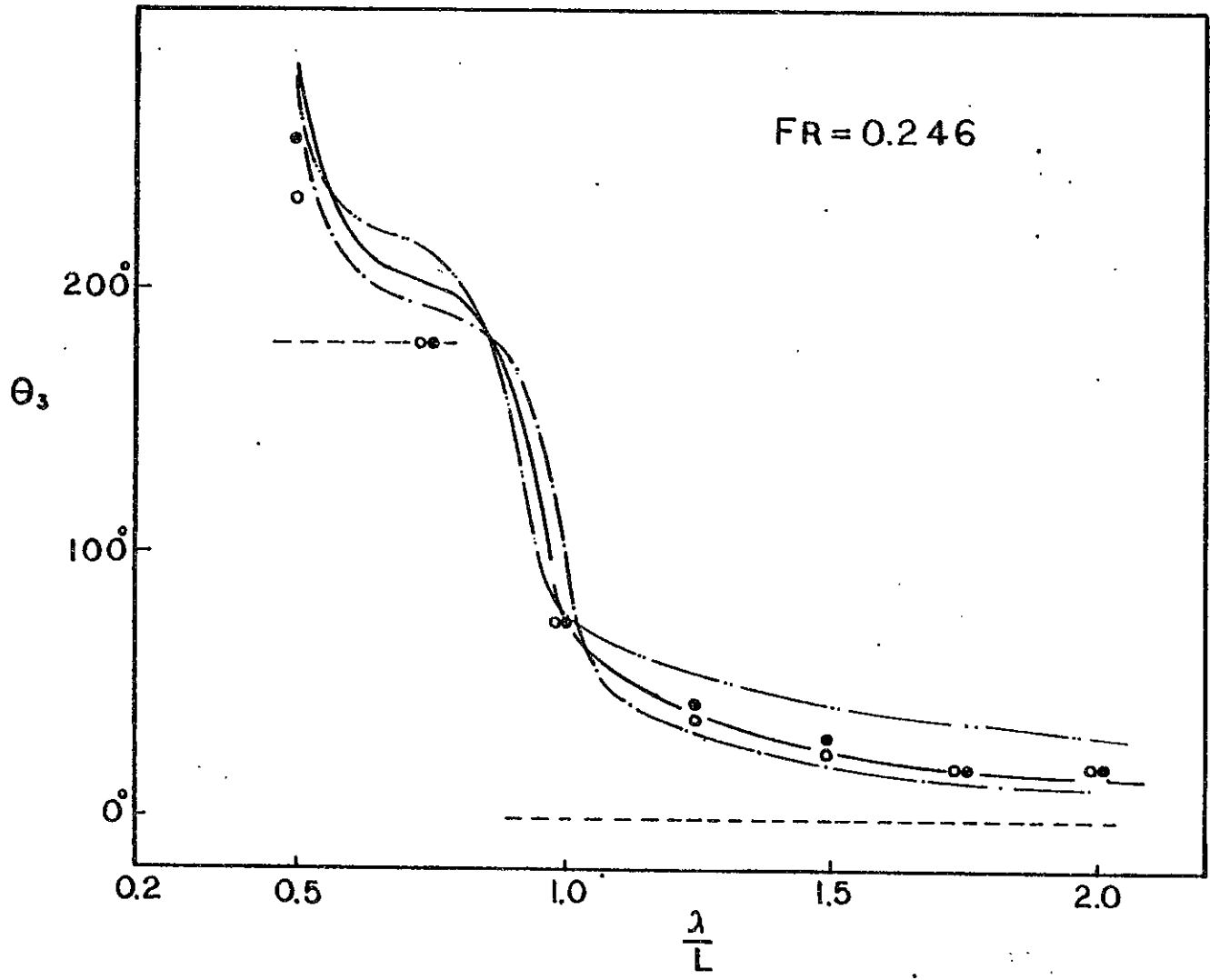


Figure 12b- Same as in Figure 8b with Fr=0.246

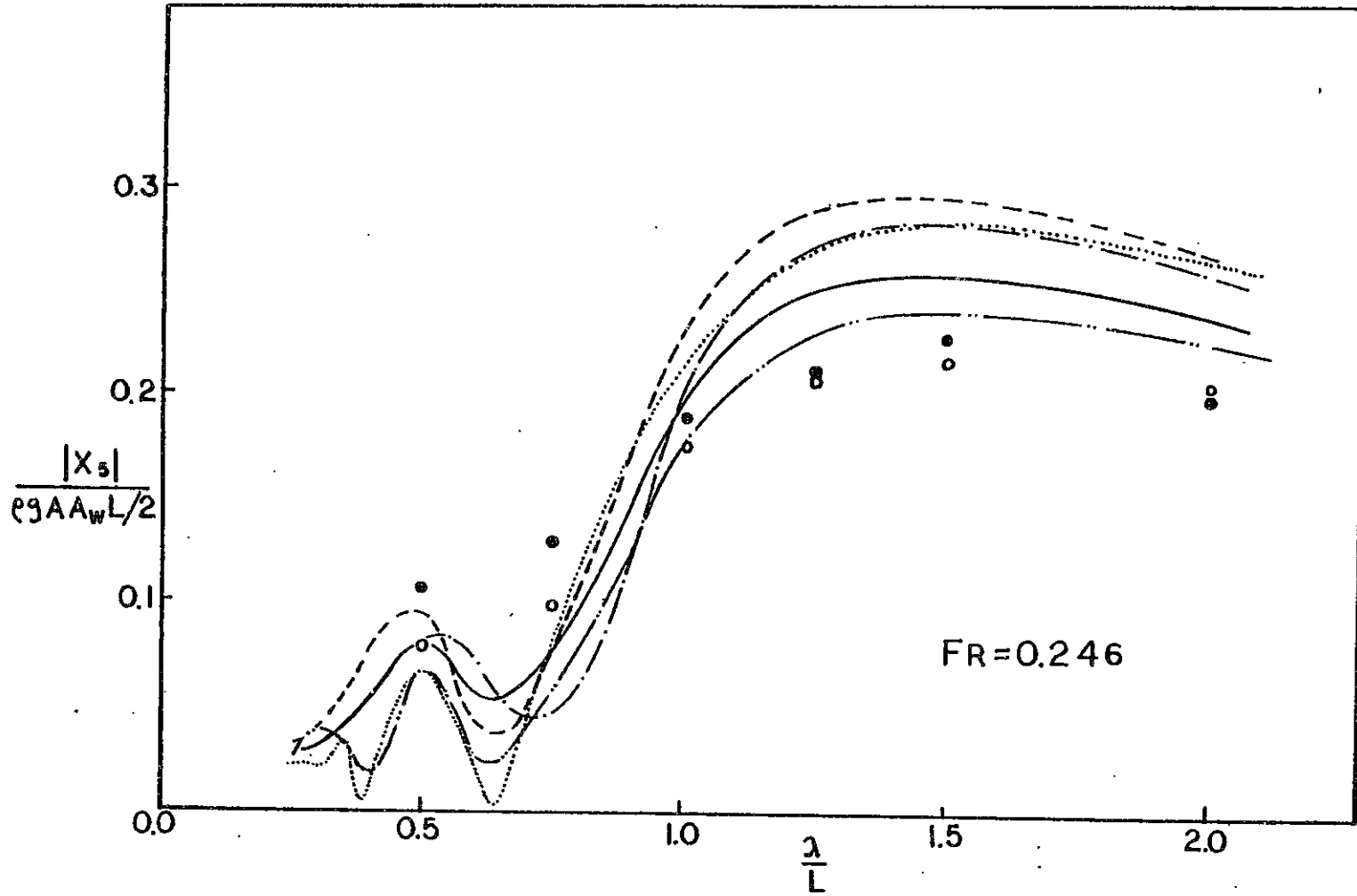


Figure 13a- Same as in Figure 9a with $Fr=0.246$

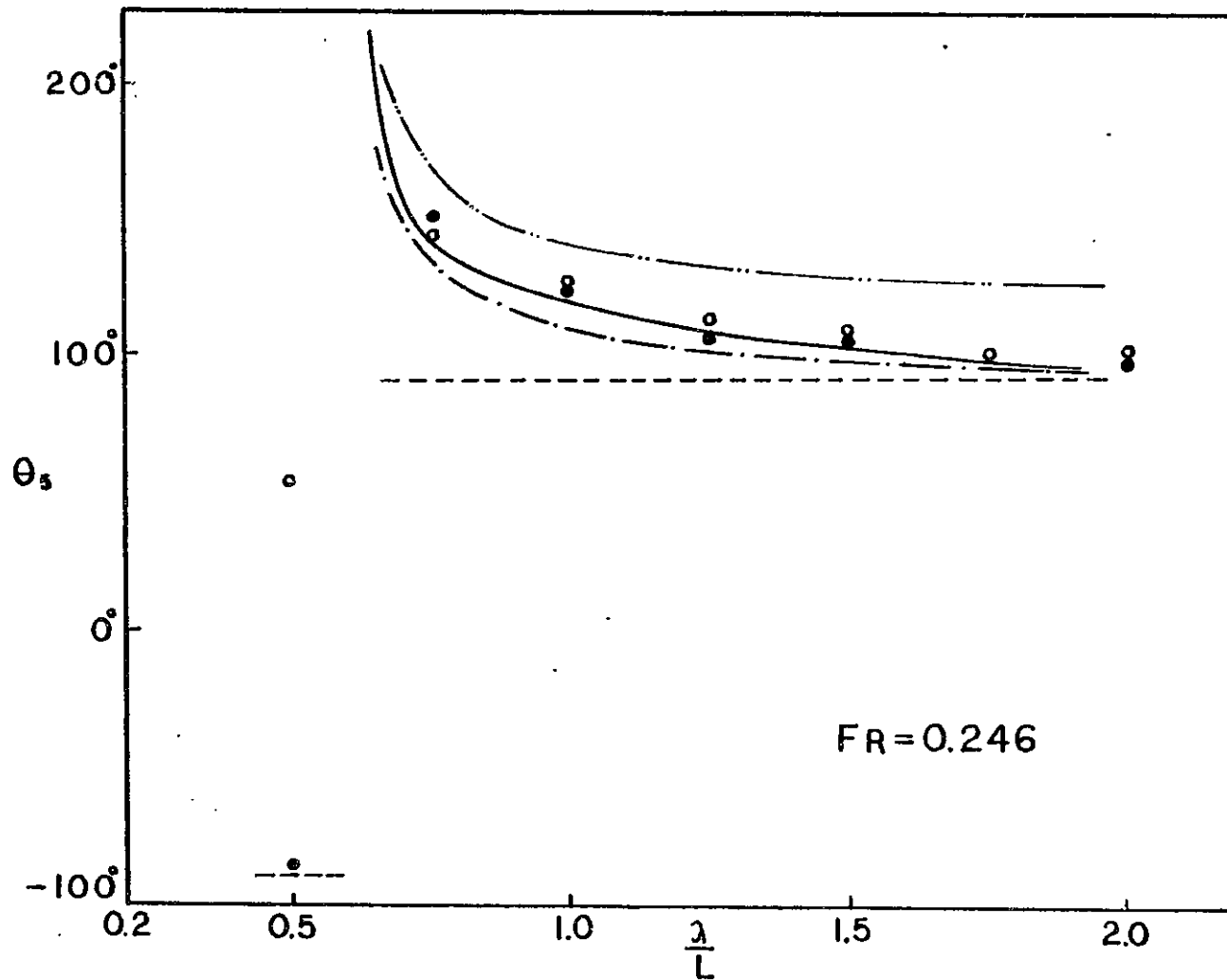


Figure 13b- Same as in Figure 9b with Fr=0.246

VIII. CONCLUSIONS

A linear theory has been presented for the interaction of a regular free surface wave train of arbitrary wavelength and angle of incidence with a slender ship fixed at its mean floating position and advancing at a constant forward speed.

The unified theory derived by Newman(1978) is appropriately modified to preserve the regularity of the solution for head (or following) waves. The flow interactions between adjacent ship sections are found to be important for head waves and of increasing strength for decreasing wavelength to ship length ratio.

As the incidence varies from head to oblique, the strength of the longitudinal interactions gradually diminishes. This decrease is reflected in the short-wavelength limit where the strip theory solution of Choo (1975) and Troesh(1976) is recovered. This solution, however, is singular for head waves indicating the presence of a nonuniformity when the short-wavelength approximation is derived first and then the angle of incidence is let to approach 0° or 180° .

Comparisons have been made with experimental measurements on two bodies of revolution interacting with head waves both for zero and finite forward speed. The observed decay of the wave amplitude along the body length

is predicted by the present approach. Calculations of the exciting force, moment and pressure distribution are in good agreement with the experiments and present an improvement over existing predictions.

Numerical results and comparisons with experiments for the added-mass and damping coefficients for realistic ship hulls are presented in Newman and Sclavounos (1980). A straightforward superposition of the radiation and the diffraction problems will yield the ship motions in waves. The refinements of unified theory are expected to be especially apparent in the prediction of the structural loads since they result from the difference of the pressure forces along the hull and not the sum.

An alternative way of evaluating the ship exciting forces in waves is the application of the Haskind relations. Further research is required in that direction in order to establish a rational connection between the Haskind relations, which involve information from the radiation problem, and the pressure integration method resulting from the diffraction problem.

The simplified representation of the flow field at large distances from the ship obtained by unified theory, is particularly convenient when applying energy and momentum conservation theorems to evaluate the drift forces and moments on a ship in waves. Comparison of this approach

to the alternative pressure integration method, will provide an indication of the relative accuracy of the two methods based on the outer and inner solutions respectively.

Of a more fundamental interest is the further investigation of the interactions occurring between the steady-state and the oscillatory disturbance. The analysis of such interactions involves a free-surface condition with variable coefficients, associated with a wave propagation in a non-uniform medium. Additional research in that direction will certainly clarify the importance of end effects but the possibility of practical implications remains questionable.

The decay of the wave amplitude along the ship hull for head waves has an analogue in the wave resistance problem. The downstream interaction of the bow-generated waves with the ship hull, results in a similar amplitude decay, known as the "sheltering effect". Although the slender body theory has not yet been successful in modeling the wave resistance problem, research based on the coupling of the unified theory with the investigations of Tuck(1963), Ogilvie(1972) and Reed(1975), may lead to a useful wave resistance theory.

Linear acoustics bear a lot of similarities with linear free-surface hydrodynamics and the application of unified theory to the interaction of acoustic waves with slender bodies seems straightforward. In that context Newman(1976) analysed the wave radiation by a slender body oscillating in an acoustic medium. The treatment of the corresponding diffraction problem follows from the present study. What makes linear acoustics particularly attractive is the possibility of checking the validity and limitations of the unified theory against exact results for spheroids and ellipsoids.

Interaction effects between the steady state and the oscillatory disturbances are also present for large speeds of advance of a slender body in acoustic medium. Apart from the interest it presents by itself, an investigation in that direction will provide guidelines for similar interactions in free surface flows.

Finally, the application of geometrical acoustics in the very short-wavelength limit, will trace the borderlines between unified theory and ray theory and, possibly, suggest ways of treatment of very high-frequency problems in ship hydrodynamics, like the springing problem.

REFERENCES

- Abramowitz, M., and Stegun, I., eds. (1964). "Handbook of Mathematical Functions." U.S. Gov. Print. Off., Washington, D.C.
- Bolton, W. E., and Ursell, F. (1973). The Wave Force on an Infinitely Long Circular Cylinder in an Oblique Sea. *J. Fluid Mech.* 57, 241-256.
- Copson, E. T. (1965). "Asymptotic Expansions." Cambridge Univ. Press, Cambridge.
- Choo, K. Y. (1975). Exciting Forces and Pressure Distribution on a Ship in Oblique Waves. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge Massachusetts.
- Faltinsen, O. (1971). Wave Forces on a Restrained Ship in Head-sea Waves. Ph.D. Thesis, University of Michigan, Ann Arbor
- Froude, W. (1861). On the Rolling of Ships. *Inst. Nav. Archit., Trans.* 2, 180-229.
- Gerritsma, J. and Beukelman, W. (1967). Analysis of the Modified Strip Theory for the Calculation of Ship Motions and Wave Bending Moments. *Int. Shipbuilding Prog.* 14, 319-337.
- Haren, G. P. and Mei, C. C. (1981). Head Sea Diffraction by a Slender Raft - with Application to Wave Power Absorption. *J. Fluid Mech.* (In press).
- John, F. (1950). On the Motion of Floating Bodies. II. *Comm. Pure Appl. Math.* 3, 45-101.
- Korvin-Kroukovsky, B. V. (1955). Investigation of Ship Motions in Regular Waves. *Soc. Nav. Archit. Mar. Eng., Trans.* 63, 386-485.
- Korvin-Kroukovsky, B. V. and Jacobs, W. R. (1957). Pitching and Heaving Motions of a Ship in Regular Waves. *Soc. Nav. Archit. Mar. Eng., Trans.* 65, 590-632.
- Kriloff, A. (1896). A New Theory of the Pitching Motion of Ships on Waves, and of the Stresses Produced by this Motion. *Inst. Nav. Archit., Trans.* 37, 326-368.

- Lamb, H. (1932). "Hydrodynamics." 6th Edition. Cambridge Univ. Press, Cambridge.
- Lee, C. M. (1964). Heaving Forces and Pitching Moments on a Semi-submerged and Restrained Prolate Spheroid Proceeding in Regular Head Waves. Rep. No NA-64-2, Dept. of Naval Architecture, Univ. of California, Berkeley.
- Liapis, N. and Faltinsen, O. M. (1980). Diffraction of Waves Around a Ship. *J. Ship Res.* 24(3), 147-155.
- Lighthill, M. J. (1967). On Waves Generated in Dispersive Systems by Travelling Forcing Effects, with Applications to the Dynamics of Rotating Fluids. *J. Fluid Mech.* 27, 725-752.
- Maruo, H. and Sasaki, N. (1974). On the Wave Pressure Acting on the Surface of an Elongated Body Fixed in Head Waves. *J. Soc. Nav. Archit. Jpn.* 136, 34-42.
- Mays, J. H. (1978). Wave Radiation and Diffraction by a Floating Slender Body. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts.
- Mei, C. C. and Tuck, E. O. (1980). "Forward Scattering by Long Thin Bodies." *Siam. J. Appl. Math.* 39, 178-191.
- Newman, J. N. (1964). A Slender Body Theory for Ship Oscillations in Waves. *J. Fluid Mech.* 18, 602-618.
- Newman, J. N. (1978). Wave Radiation from Slender Bodies. *Proc. Symp. Appl. Math. Dedicated to the Late Prof. Dr. R. Timman*, pp. 101-115. Sijthoff & Nordhoff, Groningen.
- Newman, J. N. (1977). "Marine Hydrodynamics". MIT Press., Cambridge, Massachusetts.
- Newman, J. N. (1976). The Theory of Ship Motions. *Adv. Appl. Mech.* 18, 221-283.
- Newman, J. N. and Tuck, E. O. (1964). Current Progress in the Slender Body Theory of Ship Motions. *Proc. Symp. Nav. Hydrodyn., 5th ACR-112*, pp. 129-167. Off. Nav. Res., Washington, D.C.
- Newman, J. N. and Sclavounos, P. (1980). The Unified Theory of Ship Motions. *Proc. Symp. Nav. Hydrodyn., 13th*, Tokyo, (in press).
- Ogilvie, T. F. (1972). The Wave Generated By a Fine Ship Bow. *Proc. Symp. Nav. Hydrodyn., 9th*, Paris.

- Ogilvie, T. F. (1977). Singular Perturbation Problems in Ship Hydrodynamics. *Adv. Appl. Mech.* 17, 91-188.
- Ogilvie, T. F. and Tuck, E. O. (1969). "A Rational Strip Theory for Ship Motions", Part 1. Rep. No. 013 Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor.
- Reed, A. M. (1975). Wave Making: A Low-speed Slender-body Theory. Rep. No. 169. Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor.
- Salvesen, N., Tuck, E. O. and Faltinsen, O. (1970). Ship Motions and Sea Loads. *Soc. Nav. Archit. Mar. Eng., Trans.* 78, 250-287.
- Skjrdal, S. O. and Faltinsen, O. M. (1980). A Linear Theory of Springing. *J. Ship Res.* 24(2), 74-84.
- Stein, E. M. and Weiss, G. (1975). "Introduction to Fourier Analysis on Euclidean Spaces." Princeton Univ. Press, Princeton.
- Tuck, E. O. (1963). On the Steady Motion of a Slender Ship. Ph.D. Thesis, University of Cambridge.
- Troesh, A. W. (1976). The Diffraction Potential for a Slender Ship Moving Through Oblique Waves. Ph.D. Thesis, University of Michigan, Ann Arbor.
- Ursell, F. (1962). Slender Oscillating Ships at Zero Forward Speed. *J. Fluid Mech.* 19, 496-516.
- Ursell, F. (1968a). The Expansion of Water-wave Potentials at Great Distances. *Proc. Cambridge Philos. Soc.* 64, 811-826.
- Ursell, F. (1968b). On Head Seas Travelling Along a Horizontal Cylinder. *J. Inst. Math. Its Appl.* 4, 414-427.
- Ursell, F. (1975). The Refraction of Head Seas by a Long Ship. *J. Fluid Mech.* 67, 689-703.
- Wehausen, J. V. and Laitone, E. V. (1960). Surface Waves. In "Handbuch der Physik" (S. Flugge, ed.), Vol. 9, pp. 446-778. Springer-Verlag, Berlin and New York.

APPENDIX 1: The Inner Expansion of the Outer Solution

We start with the Fourier transform of the Green Function defined in (3.8) with $k-\ell$ in place of k

$$G^*(y, z; k-\ell) = -\frac{1}{4\pi} \lim_{\mu \rightarrow 0^+} \int_{-\infty}^{\infty} du \frac{e^{iuy+z[u^2+(k-\ell)^2]^{1/2}}}{[u^2+(k-\ell)^2]^{1/2} - (\omega_0 - i\mu + Uk)^{2/g}} \quad (A1.1)$$

where $\ell = -v \cos \beta$. Applying the coordinate transformation

$$u = |k-\ell| \sinh w \quad (A1.2)$$

we may reduce (A1.1) to the form

$$G^*(y, z; k-\ell) = -\frac{1}{4\pi} \lim_{\mu \rightarrow 0^+} \int_{-\infty}^{\infty} dw \frac{\cosh w}{\cosh w - (\omega_0 - i\mu + Uk)^{2/g} |k-\ell|} \cdot \exp(z|k-\ell| \cosh w) \cos(y|k-\ell| \sinh w) \quad (A1.3)$$

Ursell (1962) derived a series representation of (A1.3) for $U=0$. This result, modified for $U>0$, takes the form

$$2\pi G^*(y, z; s) = \left[I_0(sr) + 2 \sum_{m=1}^{\infty} (-1)^m I_m(sr) \cos m\theta \begin{bmatrix} \cosh m\alpha \\ \cos m\alpha^* \end{bmatrix} \right] \times \left\{ \begin{array}{l} [\pi i \operatorname{sgn}(\omega_0 + Uk) + \alpha] \coth \alpha \\ [-\pi + \alpha^*] \cot \alpha^* \end{array} \right\} - K_0(sr) - 2 \sum_{m=1}^{\infty} (-1)^{m-1} \left[\frac{\partial}{\partial v} (I_v(sr) \cos v\theta) \right]_{v=m} \begin{bmatrix} \sinh m\alpha \coth \alpha \\ \sin m\alpha^* \cot \alpha^* \end{bmatrix} \quad (A1.4)$$

where I_n , K_0 are the modified Bessel functions defined in Abramowitz and Stegun (1964), $y = r \sin \theta$, $z = -r \cos \theta$ and the upper and lower terms in brackets are applicable according as

$$\left\{ \begin{array}{l} \cosh \alpha \\ \cos \alpha^* \end{array} \right\} = \kappa/s \gtrless 1 \quad (\text{A1.5})$$

where $\kappa = (\omega_0 + Uk)^2/g$ and $s = |k-l|$.

We are interested in the expansion of (A1.4) for small values of sr . Using the ascending series expansions for I_n and K_0 and neglecting terms of $O(s^2 r^2)$ we obtain

$$\begin{aligned} G^*(y, z; s) = & \frac{1}{2\pi} \left\{ (1+vz) \times \left[\begin{array}{l} \pi i \operatorname{sgn}(\omega_0 + Uk) + \cosh^{-1}(\kappa/s) \\ -\pi + \cos^{-1}(\kappa/s) \end{array} \right] \right. \\ & \times \left| 1 - s^2/\kappa^2 \right|^{-1/2} + [\ln(\frac{1}{2} sr) + \gamma] \\ & \left. + v[z[\ln(\frac{1}{2} sr) + \gamma - 1] - y\theta \sin \theta] \right\} \quad (\text{A1.6}) \end{aligned}$$

with an error factor $1 + O((\kappa - v)z, v^2 r^2, s^2 r^2)$. The first term in the error factor resulted from replacing κ by $v = \omega_0^2/g$ in all products of κ with y and z .

The three-dimensional characteristics of the expansion (A1.6) are associated with the presence of the Fourier parameter k . It is possible to factor them out if we compare

(A1.6) with its $k=0$ limit. If we set $k=0$ in (A1.1) we obtain, as expected, a two-dimensional function $G_{2D}(y, z)$, defined as

$$G_{2D}(y, z) = - \frac{1}{2\pi} \int_0^{\infty} du \frac{e^{z(u^2 + \ell^2)^{1/2}} \cos uy}{(u^2 + \ell^2)^{1/2} - v} \quad (A1.7)$$

which is the two dimensional Green function satisfying the Helmholtz equation, the wave free-surface condition and a radiation condition of outgoing waves as $|vy| \rightarrow \infty$. It's $\ell=0$ limit coincides with the corresponding Green function that satisfies the Laplace equation. The expansion of $G_{2D}(y, z)$ for small $|\ell|r$ is obtained from (A1.6) by keeping the upper term in brackets since $\kappa/s = |\sec\beta| \geq 1$,

$$G_{2D}(y, z) = \frac{1}{2\pi} \{ (1+vz) [\pi i + \cosh^{-1}(|\sec\beta|)] / \sin\beta + [\ln(\frac{1}{2}|\ell|r) + \gamma] + v[z[\ln(\frac{1}{2}|\ell|r) + \gamma - 1] - y\theta \sin\theta] \} \quad (A1.8)$$

with an error factor $1+O(v^2 r^2, s^2 r^2)$.

At this particular point special care is required for the $\beta=180^\circ$ ($\beta=0^\circ$) case. The limits of (A1.3) and (A1.6) as $|\cos\beta| \rightarrow 1$ exist for an arbitrary k , with two or four square-root singularities on the real k -axis one of which is located at $k=0$ for $|\cos\beta|=1$. Since all Fourier transformed quantities will be inverted in the x -space, any integrable singularities

cause no trouble in general. Expression (A1.7), however, was defined as the $k=0$ limit of $G^*(y,z;|k-\ell|)$ and is therefore singular at $|\cos\beta|=1$. A closer look at (A1.8) indicates that the singularity is associated with the pure imaginary term, since $\cosh^{-1}(|\sec\beta|)/\sin\beta \rightarrow 1$ as $\sin\beta \rightarrow 0$. We define

$$\begin{aligned} H_{2D}(y,z) &= \frac{1}{2}[G_{2D} + \bar{G}_{2D}] \\ &= -\frac{1}{2\pi} \int_0^\infty du \frac{e^{z(u^2+\ell^2)^{1/2}} \cos uy}{(u^2+\ell^2)^{1/2} - v} \end{aligned} \quad (A1.9)$$

where \bar{G}_{2D} is the complex conjugate of G_{2D} and the second term is to be interpreted as a Cauchy principal value integral. The function $H_{2D}(y,z)$ is real, satisfies the Helmholtz equation the wave free-surface condition and represents standing waves as $|vy| \rightarrow \infty$. It's expansion for small $|\ell|r$ takes the form

$$\begin{aligned} H_{2D}(y,z) &= \frac{1}{2\pi} \{ (1+vz) \csc\beta \cosh^{-1}(|\sec\beta|) + [\ln(\frac{1}{2}|\ell|r) + \gamma] \\ &\quad + v[z[\ln(\frac{1}{2}|\ell|r) + \gamma - 1] - y\theta \sin\theta] \} \end{aligned} \quad (A1.10)$$

with the same error factor as in (A1.8).

The three-dimensional effects can be factored out by adding $H_{2D}(y,z)$ to (A1.6) and subtracting its expansion (A1.10), thus

$$G^*(y, z; |k-\ell|) = H_{2D}(y, z) + \frac{1}{2\pi} (1+vz) \times$$

$$\left\{ \begin{aligned} & \left[\begin{aligned} & \pi i \operatorname{sgn}(\omega_0 + Uk) + \cosh^{-1}(\kappa/|k-\ell|) \\ & - \pi + \cos^{-1}(\kappa/|k-\ell|) \end{aligned} \right] |1 - (k-\ell)^2/\kappa^2|^{-1/2} \\ & - \csc\beta \cosh^{-1}(|\sec\beta|) + \ln(|k-\ell|/|\ell|) \end{aligned} \right\} \quad (\text{A1.11})$$

with an error factor $1+O((\kappa-v)z, v^2 r^2, s^2 r^2)$. The expansion (A1.11) is to be used for the symmetric mode. For the antisymmetric mode we can instead use $(v^{-1} \partial/\partial y) G^*(y, z; |\ell|)$ which is regular for all β , with the same error factor.

The previous procedure of factoring out the three-dimensional effects was used by Newman (1978) in the context of the radiation problem where $\beta=90^\circ$ and no singularity of the type encountered here as $\beta \rightarrow 0^\circ, 180^\circ$ exists. If we let $\cos\beta=0$ in (A1.11) the logarithmic singularities present in the last two terms cancel out with a net result $\ln(|k|/2v)$. The πi term in Newman (1978, eq. 4.13) cancels out with the imaginary part of the two-dimensional Green function defined in Newman's equation (4.3).

It is also of interest to know the large vy asymptotic behaviour of $G_{2D}(y, z)$. If we use definition (A1.7) with $\cos u = (e^{iuy} + e^{-iuy})/2$ and close the contour of integration in the upper and lower u -plane according as $y \geq 0$ in $\exp(iuy)$ or $y < 0$ in $\exp(-iuy)$, we obtain

$$G_{2D}(y,z) = \frac{1}{2} \operatorname{icsc}\beta e^{vz - i\nu|y|\sin\beta} + o(e^{-\nu|y|}) \text{ as } \nu|y| \rightarrow \infty$$

(A1.12)

The corresponding behaviour for $H_{2D}(y,z)$ can be easily obtained from (A1.9) and (A1.12)

$$H_{2D}(y,z) = \frac{1}{2} \operatorname{csc}\beta e^{vz} \sin(\nu|y|\sin\beta) + o(e^{-\nu|y|}) \text{ as } \nu|y| \rightarrow \infty$$

(A1.13)

with a regular limit $e^{vz}\nu|y|/2$ as $\sin\beta \rightarrow 0$.

APPENDIX 2: Reduction of the Kernel

The kernel of the integral equation (5.9) is defined in (3.43)-(3.45). Applying the coordinate transformation $k' = k\delta$, $\delta = \bar{U}^2/g$, we obtain

$$F(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-ikx/\delta} D(k) dk/k \quad (A2.1)$$

where

$$D(k) = \ln(2\tau^2/|k|) + \pi i - \left[\begin{array}{l} \pi i \operatorname{sgn}(k+\tau) + \cosh^{-1}[d(k)] \\ -\pi + \cos^{-1}[d(k)] \end{array} \right] \\ \times |1 - k^2/(k+\tau)^4|^{-1/2} \quad (A2.2)$$

where the upper or lower term in brackets is applicable according as

$$d(k) = (k + \tau)^2/|k| \gtrless 1 \quad (A2.3)$$

We use the definition $\cosh^{-1}(x) = \ln[x + (x^2-1)^{1/2}]$, $x \geq 1$ and start by analysing the function

$$f(z) = (1-z^{-2})^{-1/2} \ln[z+(z^2-1)^{1/2}] \quad (A2.4)$$

on the complex z plane, where $\ln z$ and $z^{1/2}$ are said to be the principal branches of the corresponding multivalued functions with $-\pi < \arg(z) \leq \pi$. The function $f(z)$ is analytic on the finite complex z plane with a branch cut

along $(-\infty, -1]$. The values of $f(z)$ along the x -axis are given by

$$f(z) = \begin{cases} |x| (x^2-1)^{-1/2} [\cosh^{-1}(|x|) \pm \pi i], & z=x \pm i0, \quad x \in (-\infty, -1) \\ |x| (1-x^2)^{-1/2} [\cos^{-1}(|x|) - \pi], & z=x \pm i0, \quad x \in (-1, 0) \\ x (1-x^2)^{-1/2} \cos^{-1}(x) & , z=x \pm i0, \quad x \in (0, 1) \\ x (x^2-1)^{-1/2} \cosh^{-1}(x) & , z=x \pm i0, \quad x \in (1, \infty) \end{cases}$$

(A2.5)

where $0 < \cos^{-1}(|x|) < \pi/2$ for $0 < |x| < 1$ and the previous definition of $\cosh^{-1}(x)$ is understood when $x \geq 1$.

The analytical structure of the function $f[(\tau+k)^2/k]$ on the complex k -plane can be determined if we first consider the mapping

$$z = x + iy = (\tau+k)^2/k, \quad k = u + iv, \quad \tau > 0 \quad (A2.6)$$

The branch point at $z=-1$ corresponds to the roots of the equation $(\tau+k)^2/k = -1$, defined by

$$\rho_{1,2} = -\frac{1}{2} [1+2\tau \pm (1+4\tau)^{1/2}] \quad (A2.7)$$

which are real and negative for all τ . The corresponding roots for $z=1$ are real and positive or complex conjugate,

according as $4\tau \lesseqgtr 1$, and are defined by

$$\rho_{3,4} = \begin{cases} \frac{1}{2}[1-2\tau \mp (1-4\tau)^{1/2}] & , \quad 4\tau < 1 \\ \frac{1}{2}[1-2\tau \mp i(4\tau-1)^{1/2}] & , \quad 4\tau > 1 \end{cases} \quad (\text{A2.8})$$

It is easy to show that the mapping function $z(k)$ is analytic, and does not take any value more than once (schlicht or simple) in the domains \mathcal{D}_{kU} and \mathcal{D}_{kL} of the lower k -plane. Thus the mapping to the corresponding domains \mathcal{D}_{zU} and \mathcal{D}_{zL} is one-to-one and we may construct the inverse mapping $k(z)$ which is also one-to-one. The previous considerations are schematically shown in the next figure where corresponding points are shown on both planes.

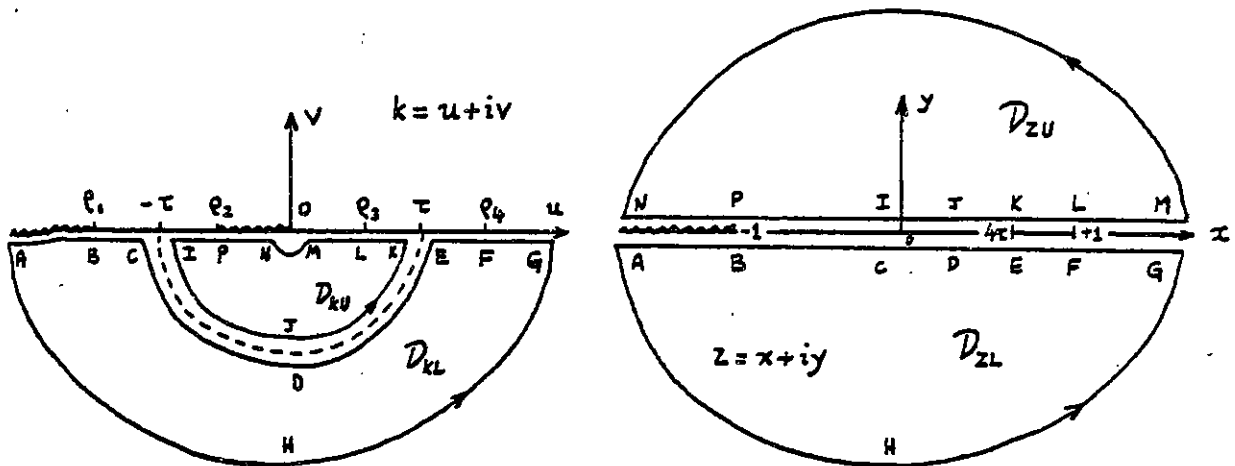


Figure A2-1: Mapping from the z to the k plane

In order to display the presence of all the roots it has been assumed that $4\tau < 1$. As 4τ grows, the pairs of points K&E and L&F get closer, coinciding at one point for $4\tau = 1$ or $\rho_3 = \rho_4 = 1/4$. For larger values of 4τ , L and F are located along the dotted semicircle across each other, and at the position of one of the two complex conjugate roots in the lower k plane. We define

$$R(k) = [1 - k^2 / (k + \tau)^4]^{-1/2} \ln \{ (k + \tau)^2 / k + [(k + \tau)^4 / k^2 - 1]^{1/2} \} \quad (\text{A2.9})$$

The function $R(k)$ is analytic in each of the domains \mathcal{D}_{kU} and \mathcal{D}_{kL} , its values being determined from the corresponding values of $f(z)$ in \mathcal{D}_{zU} and \mathcal{D}_{zL} respectively. We observe that the values of $R(k)$ along the curves CDE and IJK correspond, in the limit, to the values of $f(z)$ along the segment $(0, 4\tau)$ on the x -axis where $f(z)$ is analytic and the mapping one valued. Thus $R(k)$ can be analytically continued across the dotted semicircle from the domain \mathcal{D}_{kU} to \mathcal{D}_{kL} and vice versa, using definition (A2.9) uniformly in the lower half k -plane. Furthermore, $R(k)$ takes real values along the segments BC, IP, MK, and EG of the real k -axis and it can also be analytically continued for $\text{Im}k \geq 0^+$ using Swartz's reflection principle

$$\overline{R(k)} = R(\overline{k}) \quad (\text{A2.10})$$

where the overbar stands for the complex conjugate of the expression involved.

Consequently $R(k)$ is analytic in the finite k -plane apart from two branch cuts, one located along $(-\infty, \rho_1]$ and another along $[\rho_2, 0]$, with $-\infty, 0$ and ρ_1, ρ_2 being logarithmic and square-root branch points respectively. As $k \rightarrow 0$, $z(k) = \tau^2/k + o(1)$, and from (A2.4)

$$R(k) = \ln(2\tau^2/k) + o(k^2 \ln|k|), \quad k \rightarrow 0 \quad (\text{A2.11})$$

As $k \rightarrow \infty$, $z(k) = k + o(1)$, thus

$$R(k) = \ln 2k + o(\ln|k|/k^2), \quad k \rightarrow \infty \quad (\text{A2.12})$$

The presence of the k^{-1} term in (A2.1) urges us to establish the integrability of the functions involved at $k=0$. Thus, we define

$$W(k) = [\ln(2\tau^2/k) - R(k)]/k \quad (\text{A2.13})$$

Using (A2.11) and (A2.12), it can be seen that as $k \rightarrow 0$, $W(k) = o(k \ln|k|)$ and as $k \rightarrow \infty$, $W(k) = o(\ln|k|/k)$. The function $\ln(2\tau^2/k)$ is analytic on the finite k -plane with a branch cut along the negative k -axis. It is now possible to define the values of $D(k)/k$ along the real k axis in terms of the values of $W(k \pm i0)$, where the upper or lower sign is applicable according as $x \gtrless 0$ in (A2.1). Using

(A2.5), Fig. A2.1, (A2.9) and (A2.13) we obtain

$$D(k)/k = W(k \pm i0) + [\pi i \pm \pi i H(-k)]/k \\ + (|1 - k^2/(k + \tau)^4|^{-1/2}) g_{\pm}(k)/k \quad (\text{A2.14})$$

where $H(-k)$ is the Heaviside unit function and

$$g_{+}(k) = 2\pi i \quad , \quad -\infty < k < \rho_1 \quad (\text{A2.15a})$$

$$g_{+}(k) = 0 \quad , \quad \rho_1 < k < \rho_2 \quad (\text{A2.15b})$$

$$g_{+}(k) = -2\pi i \quad , \quad \rho_2 < k < 0 \quad (\text{A2.15c})$$

$$g_{-}(k) = 0 \quad , \quad -\infty < k < 0 \quad (\text{A2.15d})$$

$$g_{\pm}(k) = -\pi i \quad , \quad 0 < k < \rho_3 \quad , \quad \tau < 1/4 \quad (\text{A2.15e})$$

$$g_{\pm}(k) = \pi \quad , \quad \rho_3 < k < \rho_4 \quad , \quad \tau < 1/4 \quad (\text{A2.15f})$$

$$g_{\pm}(k) = -\pi i \quad , \quad \rho_4 < k < \infty \quad , \quad \tau < 1/4 \quad (\text{A2.15g})$$

$$g_{\pm}(k) = -\pi i \quad , \quad 0 < k < \infty \quad , \quad \tau > 1/4 \quad (\text{A2.15h})$$

It can be seen from (A2.14) and the appropriate definitions in (A2.15) that the singular behaviour of the second and third terms as $k \rightarrow 0$ cancel each other out with an integrable result. As $|k| \rightarrow \infty$ their sum multiplied by e^{-ikx} is integrable for all x except for $x=0^-$ where $F(x)$ has a logarithmic singularity. Applying Jordan's lemma, we obtain

$$\int_{-\infty}^{\infty} W(k \pm i0) e^{-ikx} dk/k = 0, \quad x \lesseqgtr 0 \quad (\text{A2.16})$$

Combining (A2.1), (A2.14) and (A2.15) we obtain

$$F(x) = \begin{cases} F_1(x) + F_2(x), & x < 0 \\ F_2(x), & x > 0 \end{cases} \quad (\text{A2.17})$$

where

$$\begin{aligned} F_1(x) = & - \int_{-\infty}^{\rho_1} e^{-ikx/\delta} \{1 + [1 - k^2/(k+\tau)^4]^{-1/2}\} dk/k \\ & - \int_{\rho_1}^{\rho_2} e^{-ikx/\delta} dk/k \\ & - \int_{\rho_2}^{\theta} e^{-ikx/\delta} \{1 - [1 - k^2/(k+\tau)^4]^{-1/2}\} dk/k \quad (\text{A2.18}) \end{aligned}$$

and

$$\begin{aligned} F_2(x) = & - \frac{1}{2} \left(\int_0^{\rho_3} + \int_{\rho_4}^{\infty} \right) e^{-ikx/\delta} \{1 - [1 - k^2/(k+\tau)^4]^{-1/2}\} dk/k \\ & - \frac{1}{2} \int_{\rho_3}^{\rho_4} e^{-ikx/\delta} \{1 - i[k^2/(k+\tau)^4 - 1]^{-1/2}\} dk/k, \end{aligned}$$

$$\tau < 1/4 \quad (\text{A2.19})$$

and

$$F_2(x) = -\frac{1}{2} \int_0^{\infty} e^{-ikx/\delta} \{1 - [1 - k^2/(k+\tau)^4]^{-1/2}\} dk/k, \quad \tau > 1/4 \quad (A2.20)$$

where $\delta = U^2/g$ and applying the transformation $k'=k/\delta$ in $F_2(x)$, we recover (3.50)-(3.51). Using the properties of the sine, cosine and exponential integrals it is easy to show that $F_1(x)$ reduces to

$$F_1(x) = \left[-\int_{-\infty}^{\rho_1} + \int_{\rho_2}^0 \right] e^{-ikx/\delta} \{ [1 - k^2/(k+\tau)^4]^{-1/2} - 1 \} dk/k + E_1(i|\rho_1 x/\delta|) + E_1(i|\rho_2 x/\delta|) \quad (A2.21)$$

where $E_1(z)$ is the exponential integral. Equation (3.49) can be easily obtained from (A2.21) by applying the coordinate transformation $k'=k/\delta$. An alternative expression for $F_1(x)$, where the logarithmic singularity is explicitly displayed, is given in Newman and Sclavounos (1980, eq. 42).

APPENDIX 3: The Short-Wavelength Approximation for $U=0$

The purpose of the present Appendix is to derive a consistent short-wavelength approximation of $I(x)$ defined in (3.63). Since $U=0$, we may assume without loss of generality that $-1 \leq \cos\beta \leq 0$. Equation (3.63) can be written in the form

$$I(x) = \frac{\pi v i}{2} [I_1(x) + I_2(x)] \quad (\text{A3.1})$$

where

$$\begin{aligned} I_1(x) &= \int_{-L/2}^x d\xi e^{i\nu(x-\xi)\cos\beta} Q_7(\xi) H_0^{(2)}(\nu|x-\xi|) \\ &= \int_0^\infty dt e^{i\nu t \cos\beta} Q_7(x-t) H_0^{(2)}(\nu t) dt \end{aligned} \quad (\text{A3.2})$$

where the fact that $Q_7(x)=0$ for $|x| > L/2$ was used. Similarly

$$\begin{aligned} I_2(x) &= \int_x^{L/2} d\xi e^{i\nu(x-\xi)\cos\beta} Q_7(\xi) H_0^{(2)}(\nu|x-\xi|) \\ &= \int_0^\infty dt e^{-i\nu t \cos\beta} Q_7(x+t) H_0^{(2)}(\nu t) \end{aligned} \quad (\text{A3.3})$$

For large ν , we get to leading order

$$\exp(\pm i\nu t \cos\beta) H_0^{(2)}(\nu t) \sim (2/\pi\nu t)^{-1/2} \exp[-i\nu t(l \mp \cos\beta) + \pi i/4] \quad (\text{A3.4})$$

consequently, the integrand of $I_1(x)$ oscillates at a non-zero frequency $\nu(1-\cos\beta)$ which varies from ν , if $\cos\beta=0$, to 2ν if $\cos\beta=-1$. The opposite occurs for $I_2(x)$ where the corresponding frequency $\nu(1+\cos\beta)$ varies from zero, if $\cos\beta=-1$, to ν if $\cos\beta=0$. Thus a different approach is needed for $I_2(x)$ for the derivation of a uniformly valid expansion in β for large ν . Starting with $I_1(x)$, we integrate once by parts to obtain

$$I_1(x) = \frac{1}{\nu} Q_7(x) \int_0^\infty e^{it\cos\beta} H_0^{(2)}(t) dt + R_1(x) \quad (A3.5)$$

where the vanishing of $Q_7(x)$ at $x=\pm L/2$ was used and

$$R_1(x) = \frac{1}{\nu} \int_0^\infty dt Q_7'(x-t) \int_{\nu t}^\infty du e^{iu\cos\beta} H_0^{(2)}(u) du \quad (A3.6)$$

In order to prove that $R_1(x)$ is $o(\nu^{-1})$ we add and subtract the leading order behaviour of $H_0^{(2)}(u)$ in (A3.6) to obtain

$$R_1(x) = \frac{1}{\nu} (2/\pi)^{1/2} e^{\pi i/4} \int_0^\infty dt Q_7'(x-t) \int_{\nu t}^\infty du e^{iu(\cos\beta-1)} u^{-1/2} \\ + \frac{1}{\nu} \int_0^\infty du [H_0^{(2)}(u) - (2/\pi u)^{1/2} e^{-iu+\pi i/4}] e^{iu\cos\beta} \int_0^{u/\nu} dt Q_7'(x+t) \quad (A3.7)$$

where the second term of (A3.7) was obtained by interchanging the order of integration. It is easy to prove, integrating by parts once, that

$$\int_{vt}^{\infty} du e^{iu(\cos\beta-1)} u^{-1/2} = \frac{i e^{ivt(\cos\beta-1)}}{(\cos\beta-1)(vt)^{1/2}} + o(v^{-3/2}), \quad (\text{A3.8})$$

Since $|\cos\beta-1| > 1$, there exists a finite M_1 such that

$$\left| \int_{vt}^{\infty} du e^{iu(\cos\beta-1)} u^{-1/2} \right| \leq M_1 (vt)^{-1/2} \quad (\text{A3.9})$$

Denoting the first term in (A3.7) by $R_{11}(x)$ and using (A3.9), we obtain

$$\left| R_{11}(x) \right| \leq v^{-3/2} (2/\pi)^{1/2} M_1 \left| \int_0^{\infty} dt t^{-1/2} Q_7'(x-t) \right| \quad (\text{A3.10})$$

The integral in (A3.10) is bounded since $t^{-1/2}$ is integrable at the lower limit and $Q_7'(x-t)$ is uniformly continuous and identically vanishing for $t > x+L/2$. Thus, $R_{11}(x) = o(v^{-1})$.

Proceeding to the second term, $R_{12}(x)$, of (A3.7) we first note that since $Q_7'(x+t)$ is uniformly continuous. Let

$$P(x,u) = \int_0^{u/v} dt Q_7'(x+t) = Q_7(x+u/v) - Q_7(x) \quad (\text{A3.11})$$

Using Rolle's theorem and the uniform continuity of $Q'(x)$, we

may find a $u_0(u, x)$ such that

$$P(x, u) = \frac{u}{v} Q_7'(x + u_0/v), \quad 0 \leq u_0 \leq u \quad (\text{A3.12})$$

The substitution of (A3.12) into $R_{12}(x)$ gives

$$R_{12}(x) = \frac{1}{v^2} \int_0^\infty du Q_7'(x + u_0/v) u [H_0^{(2)}(u) - (2/\pi u)^{1/2} e^{-iu + \pi i/4}] e^{iucos\beta} \quad (\text{A3.13})$$

The large u leading order asymptotic behaviour of the term in brackets is simply the second term in the corresponding expansion of $H_0^{(2)}$ and behaves like $e^{-iu} u^{-3/2}$ as $u \rightarrow +\infty$. Consequently, its product with u is an integrable function, since Q_7' is bounded and $\cos\beta \leq 0$. Thus, $R_{12}(x) = o(v^{-2})$ uniformly in x and the proof that $R_1(x)$ is $o(v^{-1})$ is concluded.

The integral in (A3.5) can be explicitly evaluated with a result

$$I_1(x) = \frac{1}{v} \csc\beta [1 - \frac{2}{\pi} \sin^{-1}(|\cos\beta|)] Q_7(x) + o(v^{-1}) \quad (\text{A3.14})$$

Proceeding in a similar manner we may obtain higher order terms which form a descending series in v^{-m} .

Integration by parts, applied to $I_2(x)$, doesn't work because the corresponding integral in (A3.5), with $-\cos\beta$ instead of $\cos\beta$, doesn't exist for $\cos\beta = -1$ due to the cancelling oscillatory behaviours suggested by (A3.4). We

instead start by subtracting and adding to the integrand the leading order behaviour (A3.4), thus

$$\begin{aligned}
 I_2(x) = & (2/\pi v)^{1/2} e^{\pi i/4} \int_0^\infty dt t^{-1/2} Q_7(x+t) e^{-i v t (1+\cos\beta)} \\
 & + \int_0^\infty dt Q_7(x+t) e^{-i v t \cos\beta} [H_0^{(2)}(vt) - (2/\pi vt)^{-1/2} e^{-i v t + \pi i/4}]
 \end{aligned}
 \tag{A3.15}$$

The first term in (A3.15) is at most of $O(v^{-1/2})$ and this occurs when $\cos\beta = -1$. The second term, hereafter denoted by $I_{22}(x)$, is in a form which can be integrated by parts once with a result

$$I_{22}(x) = \frac{1}{v} Q_7(x) \int_0^\infty e^{-i t \cos\beta} [H_0^{(2)}(t) - (2/\pi t)^{1/2} e^{-i t + \pi i/4}] + R_2
 \tag{A3.16}$$

where

$$\begin{aligned}
 R_2(x) = & \frac{1}{v} \int_0^\infty dt Q_7(x+t) \int_{vt}^\infty du e^{-i u \cos\beta} [H_0^{(2)}(u) \\
 & - (2/\pi u)^{1/2} e^{-i u + \pi i/4}]
 \end{aligned}
 \tag{A3.17}$$

Proceeding in a similar manner as for $I_1(x)$ we subtract and add the leading order behaviour of the term in brackets

$$\begin{aligned}
 R_2(x) &= \frac{1}{v} (i/8) (2/\pi)^{1/2} e^{\pi i/4} \int_0^\infty dt Q_7'(x+t) \int_{vt}^\infty du e^{-iu(\cos\beta+1)} u^{-3/2} \\
 &+ \frac{1}{v} \int_0^\infty du [H_0^{(2)}(u) - (2/\pi u)^{1/2} e^{-iu+\pi i/4} (1 + \frac{i}{8u})] e^{-iucos\beta} \int_0^{u/v} Q_7'(x+t) dt
 \end{aligned}
 \tag{A3.18}$$

The first term can be bounded as follows

$$\left| \int_{vt}^\infty du e^{-iu(\cos\beta+1)} u^{-3/2} \right| \leq \int_{vt}^\infty du u^{-3/2} = 2(vt)^{-1/2}
 \tag{A3.19}$$

and using the same argument as for $I_1(x)$, we conclude that the first term in (A3.18) is of $O(v^{-3/2})$.

We proceed in a similar manner for the second part of $R_2(x)$. The leading order behaviour of the term in brackets for large u , multiplied with the exponential term, is proportional to

$$e^{-iu(\cos\beta+1)} u^{-5/2}
 \tag{A3.20}$$

which, multiplied with the u factor coming from applying Rolle's theorem, gives an integrable behaviour $u^{-3/2}$, even if $\cos\beta+1 = 0$. The corresponding leading order behaviour as $u \rightarrow 0$ is $u^{-3/2}$ which, multiplied with the same u factor, gives an integrable singular behaviour $u^{-1/2}$. Thus, $R_{22}(x) = O(v^{-2})$ and we may consistently consider the first terms in (A3.14) and (A3.15) as the leading order behaviour of $I_2(x)$. The integral in (A3.16) can be explicitly evaluated with the

final result

$$\begin{aligned}
 I_{32}(x) &= (2/\pi v)^{1/2} e^{\pi i/4} \int_0^{\infty} dt t^{-1/2} Q_7(x+t) e^{-i v t (1+\cos\beta)} \\
 &+ \frac{1}{v} Q_7(x) \left\{ \left[1 + \frac{2}{\pi} \sin^{-1}(|\cos\beta|) \right] \csc\beta - \sqrt{\frac{2}{1+\cos\beta}} \right\} + o(v^{-3/2})
 \end{aligned}
 \tag{A3.21}$$

We may proceed in a similar manner to obtain a descending series in half powers of v in contrast to the integer powers for $I_1(x)$. It is clear now that the half powers of v are present because of the existence of a non-uniformity as $\cos\beta \rightarrow -1$.

Combining (A3.1), (A3.14) and (A3.21) we finally obtain

$$\begin{aligned}
 I(x) &= -\frac{1}{2}(1-i) \sqrt{\pi v} \int_x^{L/2} Q_7(\xi) e^{\frac{-i v (\xi-x)(1+\cos\beta)}{(\xi-x)^{1/2}}} d\xi \\
 &+ \pi i \left[\csc\beta - \frac{1}{2} \left(\frac{2}{1+\cos\beta} \right)^{1/2} \right] Q_7(x) + o(v^{-1/2})
 \end{aligned}$$

(A3.22)

APPENDIX 4: The "Large τ " Approximation

Starting with definition (A2.21) of $F_1(x)$ and denoting the terms associated with the root ρ_1 by $F_{11}(x)$, we get

$$F_{11}(x) = E_1(i|\rho_1 x/\delta|) - \int_{-\infty}^{\rho_1} e^{-ikx/\delta} \{[1-e^2(k)]^{-1/2} - 1\} dk/k \quad (A4.1)$$

where $e(k) = k^2/(k+\tau)^4$; $\delta = U^2/g$ and $x < 0$. Applying the coordinate transformation $k' = k/\rho_1$ and using the fact that $\rho_1 < 0$, we observe that the exponential integral and the second term in curly brackets cancel each other out, thus

$$F_{11}(x) = \int_1^{\infty} e^{-ik\rho_1 x/\delta} [1-e^2(\rho_1 k)]^{-1/2} dk/k \quad (A4.2)$$

The formal asymptotic expansion of $F_{11}(x)$ for large $\rho_1 x/\delta$ is obtained if we factor out the square root singularity at $k=1$ in the integrand of (A4.2) and then integrate by parts once, as described by Copson (1976, p. 23), to obtain

$$F_{11}(x) = \left[\frac{\pi}{2|\rho_1 x/\delta|} \right]^{1/2} (1+4\tau)^{-1/4} e^{-i\rho_1 x/\delta - \pi i/4} + R_{N1} \quad (A4.3)$$

where

$$|R_{N1}| \leq |\rho_1 x/\delta|^{-1} \int_1^{\infty} (k-1)^{-1/2} \left| \frac{d}{dk} \phi_1(k) \right| dk \quad (A4.4)$$

where

$$\phi_1(k) = (k-1)^{1/2} [1-e^2(\rho_1 k)]^{-1/2} / k \quad (\text{A4.5})$$

The $(k-1)^{1/2}$ factor in (A4.5) cancels out with the opposite behaviour of the term in brackets with a finite derivative of the remainder at $k=1$. Furthermore, $\phi_1(k) \sim k^{-1/2}$ as $k \rightarrow \infty$. Consequently, the integral in (A4.4) exists and $R_{N1} = O[|\rho_1 x/\delta|^{-1}]$. Proceeding in a similar manner for the rest of $F_1(x)$, we define

$$F_{12}(x) = E_1(i|\rho_2 x/\delta|) + \int_{\rho_2}^0 e^{-ikx/\delta} \{ [1-e^2(k)]^{-1/2} - 1 \} dk/k \quad (\text{A4.6})$$

Applying the coordinate transformation $k' = k/\rho_2$ in (A4.6), we obtain

$$F_{12}(x) = E_1(i|\rho_2 x/\delta|) - \int_0^1 e^{-ik\rho_2 x/\delta} (1-k)^{-1/2} \phi_2(k) dk \quad (\text{A4.7})$$

where

$$\phi_2(k) = (1-k)^{1/2} \{ [1-e^2(\rho_2 k)]^{-1/2} - 1 \} / k \quad (\text{A4.8})$$

The integrand in (A4.7) is not as strongly oscillatory as the corresponding integrand in (A4.2), as suggested by (3.72) and (3.73). This combined with the finite range of integration reduces the numerical effort for the evaluation of (A4.7).

For short waves, however, the leading order asymptotic approximation is useful both from the qualitative and the quantitative standpoint. Denoting the second term in (A4.7) by $J_{12}(x)$, and integrating once by parts, we encounter some difficulty in satisfying the inequality that corresponds to (A4.4). This is so because the derivative of $\phi_2(k)$ is square-root singular at $k=1$. We deal with this anomaly by dividing the range of integration as follows

$$J_{12}(x) = - \left[\int_0^A + \int_A^1 \right] e^{-ik\rho_2 x/\delta} \{ [1 - e^{2(\rho_2 k)}]^{-1/2} - 1 \} dk/k \quad (A4.9)$$

where $0 < A < 1$. The integrand in (A4.9) is infinitely differentiable at $k=A$, square-root singular at $k=1$ and behaves like $k/2\tau^4$ as $k \rightarrow 0$. Integrating once by parts the first term, and denoting the integrand by $r(k)$, we obtain

$$-\int_0^A e^{-ik\rho_2 x/\delta} r(k) dk = -i(\rho_2 x/\delta)^{-1} r(A) + i(\rho_2 x/\delta)^{-1} \int_0^A e^{-ik\rho_2 x/\delta} r'(k) dk \quad (A4.10)$$

The integral in the right-hand-side of (A4.10) exists, and the contribution from the first term of (A4.9) turns out

to be of $O[(\rho_2 x/\delta)^{-1}]$. The second integral in (A4.9) is broken into the following two terms

$$\int_A^1 e^{-ik\rho_2 x/\delta} dk/k \quad \text{and} \quad - \int_A^1 e^{-ik\rho_2 x/\delta} [1-e^{-2(\rho_2 k)}]^{-1/2} dk/k$$

Integrating the first term once by parts, we get a contribution of $O[(\rho_2 x/\delta)^{-1}]$. The second term is in the appropriate form to apply Copson's theorem. Proceeding in exactly the same way as for $F_{11}(x)$, we obtain

$$J_{12}(x) = \left[\frac{\pi}{2|\rho_2 x/\delta|} \right]^{1/2} (1+4\tau)^{-1/4} e^{-i\rho_2 x/\delta - \pi i/4} + R_{N2} \quad (\text{A4.11})$$

where $R_{N2} = O[(\rho_2 x/\delta)^{-1}]$, the exponential integral in (A4.7) being of the same order for large arguments. Hence, the leading order approximation of $F_1(x)$ is

$$F_1(x) = \left[\frac{\pi}{2|x/\delta|} \right]^{1/2} (1+4\tau)^{-1/4} e^{-\pi i/4} \times \\ (e^{-i\rho_1 x/\delta} |\rho_1|^{-1/2} + e^{-i\rho_2 x/\delta} |\rho_2|^{-1/2}) + O[(\rho_{1,2} x/\delta)^{-1}] \quad (\text{A4.12})$$

The corresponding approximation for $F_2(x)$ can be obtained if we observe in definition (A2.20) that the integrand

$$\ell(k) = \{ [1-k^2/(k+\tau)^4]^{-1/2} - 1 \} / k \quad (\text{A4.13})$$

is infinitely differentiable for all $k \in [0, \infty)$. Integrating by parts an infinite number of times, we conclude that the order of $F_2(x)$ is higher than any power of $|x/\delta|^{-1}$, indicating that $F_2(x)$ dies exponentially fast as $|x| \rightarrow \infty$. The form of the exponential factor can be obtained using physical arguments. For $\tau < 1/4$ there is only one wave system present upstream of the form $e^{-i\rho_3 x/\lambda} (\rho_3 x/\delta)^{-1/2}$ as $x \rightarrow +\infty$, and one additional downstream associated with $F_2(x)$ of the form $e^{+i\rho_4 x/\lambda} (\rho_4 |x|/\delta)^{-1/2}$ as $x \rightarrow -\infty$. For $\tau > 1/4$, the amplitude of both wave-systems shifts from the square-root to a combination of an exponential and square-root decay as $|x| \rightarrow \infty$, since $\rho_{3,4}$ become complex conjugate. The exponential decay factor takes the form

$$\exp[-|Jm\rho_{3,4}| |x|/\delta] \tag{A4.14}$$

where $|Jm\rho_{3,4}| = (4\tau-1)^{1/2}$, which is of the same order as $|\rho_{1,2}|$ for "large τ ". Thus, we may consistently neglect the contribution from $F_2(x)$. The precise form of the expansion of $F_2(x)$ for large $|x/\delta|$ can be obtained by diverting the contour of integration [in the definition of $F_2(x)$] in the upper or lower k -plane according as $x \lesseqgtr 0$, and picking up the contribution from the square root singularities ρ_3 and ρ_4 .

APPENDIX 5: Evaluation of m_3 for a Prolate Spheroid

The x-axis is chosen to be the axis of revolution and let a be the semi-length of this axis with c being the radius at the equatorial plane. The foci are located at $(\pm k, 0, 0)$ where

$$k = (a^2 - c^2)^{1/2} \quad (A5.1)$$

Following Lamb (1935, p. 139) we define

$$\begin{aligned} x &= k \cos \phi \cosh \eta = k \mu \zeta, \quad y = r \cos \omega, \quad z = r \sin \omega \\ r &= k \sin \phi \sinh \eta = k(1-\mu^2)^{1/2} (\zeta^2 - 1)^{1/2} \end{aligned} \quad (A5.2)$$

where $|\mu| < 1$, $1 < \zeta < \infty$, $|\omega| < \pi$ and the angle ω being equal to $\theta - \pi/2$ in our notation. The coordinates μ, ζ, ω form an orthogonal system with surfaces of constant ζ being confocal spheroids. The unit vectors in the positive μ, ζ and ω directions \vec{U}_μ , \vec{U}_ζ and \vec{U}_ω respectively are related to the \vec{i} , \vec{j} and \vec{k} unit vectors as follows

$$\vec{U}_\mu = \zeta h_\mu \vec{i} - \mu h_\zeta \cos \omega \vec{j} - \mu h_\zeta \sin \omega \vec{k} \quad (A5.3)$$

$$\vec{U}_\zeta = \mu h_\zeta \vec{i} + \zeta h_\mu \cos \omega \vec{j} + \zeta h_\mu \sin \omega \vec{k} \quad (A5.4)$$

$$\vec{U}_\omega = -\sin \omega \vec{j} + \cos \omega \vec{k} \quad (A5.5)$$

where

$$h_\mu = (1-\mu^2)^{1/2} (\zeta^2-\mu^2)^{-1/2}, \quad h_\zeta = (\zeta^2-1)^{1/2} (\zeta^2-\mu^2)^{-1/2} \quad (\text{A5.6})$$

The velocity potential associated with a translation of the spheroid in the positive x-direction, normalized for a unit speed is given by

$$\bar{\phi} = A\mu \left\{ \frac{1}{2}\zeta \ln \left[\frac{(\zeta+1)}{(\zeta-1)} \right] - 1 \right\} \quad (\text{A5.7})$$

where

$$A = -k \left\{ \zeta_0 / (\zeta_0^2 - 1) - \frac{1}{2} \ln \left[\frac{(\zeta_0 + 1)}{(\zeta_0 - 1)} \right] \right\} \quad (\text{A5.8})$$

$$\zeta_0 = a(a^2 - c^2)^{-1/2} \quad (\text{A5.9})$$

The $\vec{m} = (m_1, m_2, m_3)$ vector is defined as

$$\vec{m} = -(\vec{n} \cdot \nabla) \nabla \bar{\phi} \quad (\text{A5.10})$$

In the spheroidal coordinates

$$\nabla = \vec{U}_\mu \frac{1}{k} h_\mu \partial/\partial\mu + \vec{U}_\zeta \frac{1}{k} h_\zeta \partial/\partial\zeta \quad (\text{A5.11})$$

and since $\vec{n} = -\vec{U}_\zeta$, using the orthogonality of \vec{U}_μ and \vec{U}_ζ we obtain

$$\vec{m} = \frac{1}{k^2} h_\zeta \partial/\partial\zeta \left(\vec{U}_\mu h_\mu \partial\bar{\phi}/\partial\mu + \vec{U}_\zeta h_\zeta \partial\bar{\phi}/\partial\zeta \right) \quad (\text{A5.12})$$

Multiplying both sides of (A5.12) by the unit vector \vec{k} and using the definitions (A5.3) - (A5.8), we obtain after a somewhat lengthy algebra

$$m_3 = \frac{A}{k^2} \mu \zeta h_\mu h_\zeta^2 \sin \omega (\zeta^2 - 1)^{-1} \{ (\zeta^2 - 1)^{-1} + 2(\zeta^2 - \mu^2)^{-1} \} \quad (\text{A5.13})$$

If we let $c/a = \epsilon$, it is easy to show using (A5.6) and (A5.9) that on the spheroid surface

$$h_\mu = (1 - \epsilon^2)^{1/2} \sin \phi (\sin^2 \phi + \epsilon^2 \cos^2 \phi)^{-1/2} \quad (\text{A5.14})$$

$$h_\zeta = \epsilon (\sin^2 \phi + \epsilon^2 \cos^2 \phi)^{-1/2} \quad (\text{A5.15})$$

$$A = - \frac{a\epsilon^2}{D(\epsilon)}, \quad D(\epsilon) = 1 - \frac{\epsilon^2}{2(1-\epsilon^2)^{1/2}} \ln \frac{1+(1-\epsilon^2)^{1/2}}{1-(1-\epsilon^2)^{1/2}} \quad (\text{A5.16})$$

Substitution of (A5.14) - (A5.16) into (A5.13) gives

$$m_3 = - \frac{1}{a} \frac{1-\epsilon^2}{D(\epsilon)} \frac{\cos \phi \sin \phi \sin \omega}{(\sin^2 \phi + \epsilon^2 \cos^2 \phi)^{3/2}} \left(1 + \frac{2\epsilon^2}{\sin^2 \phi + \epsilon^2 \cos^2 \phi} \right) \quad (\text{A5.17})$$

If we set $a=L/2$ and $\sin \omega = -\cos \theta$ we obtain (6.15).