AN INFORMATION THEORETIC MODEL OF THE DECISION MAKER

by

KEVIN LLOYD BOETTCHER

B.S.M.E., Valparaiso University
(1979)

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1981

Signature of Author ______________________
Department of Electrical Engineering and Computer Science
June 15, 1981

Certified by ______________________
Alexander H. Levis
Thesis Supervisor

Accepted by ______________________
Chairman, Departmental Committee on Graduate Students

AN INFORMATION THEORETIC MODEL
OF THE DECISION MAKER

by

Kevin Lloyd Boettcher

Submitted to the Department of Electrical Engineering and Computer Science on June 15, 1981 in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering.

ABSTRACT

An analytic characterization of the process of executing a well-defined decision-making task by a human decision maker is presented. A basic two-stage model of this process is introduced in which external situations are first assessed and then responses are selected. An information theoretic framework is used in which total internal activity is described in terms of internal coordination and internal decision-making, as well as throughput and blockage. A constraint on the rate of total activity, i.e., on the rate of internal processing, is suggested as a model of bounded rationality. The correspondence between restriction on internal decision-making and bounded rationality is explored.

The model is extended to include basic interactions in an organizational context: direct and indirect control. The former is modeled as a restriction on internal decision-making by external commands and the latter is incorporated through an auxiliary situation assessment input received from the organization.

Thesis Supervisor: Alexander H. Levis
Title: Senior Research Scientist

ACKNOWLEDGEMENT

The author wishes to thank Dr. Alexander H. Levis for his guidance, support, and patience. In the course of this work, several insightful suggestions were made by David Castanon, and the author has benefitted by participation in discussions with Joe Wohl. Elizabeth Ducot is gratefully acknowledged for her help with the computations required for the example presented in Chapter 5, and David Rossi deserves special recognition for serving as a sympathetic listener as this work was in progress. Finally, the Laboratory for Information and Decision Systems has provided a positive environment in which to do the research work presented.

Laura Washburn is recognized for her efforts at typing the manuscript and Art Giordani of LIDS is thanked for drafting the figures.

The author deeply appreciates the support and encouragement of his family, to whom this thesis is dedicated: parents, sisters, and Gran Pa.

This research was carried out at the MIT Laboratory for Information and Decision Systems with partial support provided by the Air Force Office of Scientific Research under Contracts AFOSR-79-0116 (MIT OSP No. 88155) and AFOSR-80-0229 (MIT OSP No. 89620), and with partial support provided by the National Science Foundation and the Tau Beta Pi Association under graduate fellowships.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND AND POINT OF VIEW

This thesis is concerned with the analytic characterization of the process of performing a well-defined decision-making task by a human decision maker. The research presented has been motivated by the desire to design and evaluate alternative military organizational structures, particularly structures which support decision-making in a tactical environment [1]; there presently exists no unified analytic methodology for such an assessment. Central to the tactical decision-making process is the role of the human decision maker, the commander, and characteristic to the tactical environment is the problem of coordination among commanders who must make compatible decisions about overlapping areas of responsibility using varying amounts of information.

Under the assumption that the commanders in a tactical military environment make their decisions based on organization-wide objectives, the problem outlined above is of the class of problems in decentralized control known as team-decision theoretic problems. Characteristic of these problems are the five following elements [2]:

a) a set of random variables which represent the "states of nature"
b) an information structure determined by the set of observations on nature for each decision maker of the team
c) a set of decision responses for each decision maker
d) strategies (decision rules, control laws) for each decision maker which map possible observations into decision responses
e) cost criterion for the organization (team)

In order to specify eventually the appropriate form of an organization for a particular task, it is necessary to evaluate how particular information structures (b) affect the strategies of decision makers (d). Correspondingly, in this thesis the information structure and the decision-making task of each organization member are specified and well-defined, and consideration will be focused on the performance of that task by the decision maker.

In previous considerations of team-theoretic decision problems [2], [3], [4], it has been tacitly assumed that the decision maker is perfectly rational, that is, each decision maker possesses a given set of alternatives, has knowledge (in a probabilistic sense) of the consequences of choosing a particular alternative, and has a cost ordering on the consequences [5]. The result is that optimal decision strategies are obtained.

An alternative hypothesis, however, is that due to limitations in information processing ability or problem-solving ability, the decision maker is unable to construct and consider all alternatives in a given decision situation, nor can he assign perfectly a value to the consequences of choosing a particular alternative that he does consider [6]. To the extent that this is the case, the rationality of the decision maker cannot be perfect no matter how "intendedly rational" he is [7], i.e., he exhibits bounded rationality. March and Simon suggest that the boundedly rational decision maker seeks to find an alternative which meets minimally satisfactory criteria [8], that is, which satisfices. Because of the bounded rationality of the decision maker, they claim that "most human decision-making, whether individual or organizational, is concerned with the discovery and selection of satisfactory alternatives." It is believed that the notion of bounded rationality is appropriate in modeling the organization member considered in the present context, i.e., the commander in a tactical decision-making situation. The goal of this thesis, therefore, has been to represent analytically and explicitly the decision maker's boundedness and to show how it may affect the determination of strategies as solutions to the team-theoretic decision problem.

Representations of bounded rationality have been made previously [9], [10], [11]. Characteristic of these representations is that they are based on input-output models of the decision maker. The work presented in this thesis is most closely related to that of Drenick [12]. In the Drenick model, input symbols (observations) arrive and are processed into output symbols (decision responses). Each particular association of input symbol to output symbol requires a fixed amount of processing time, and the bounded rationality of the decision maker is represented by a maximum mean processing time. The problem is to determine the input-output strategy which maximizes a payoff function, subject to the mean processing time limitation. The optimal association of inputs to outputs is deterministic (pure strategy) in the unconstrained case; because of the constraint imposed, however, the strategies are in general non-deterministic (mixed).

1.2 METHOD OF APPROACH AND REVIEW OF RESULTS

The fundamental departure in this thesis from previous models of bounded rationality is to seek a characterization of the internal processing which is accomplished to determine output from input. Such a characterization of the decision-making process is achieved by synthesis of qualitative notions of decision-making with the analytic framework of information theory. The working model is expressed as a function of internal choices made, i.e., an internal decision strategy determines the input-output mapping, and the characterization is such that

1) an analytic representation of the total activity required to accomplish the internal processing is given as a function of the internal decision strategy,
2) the bounded rationality of a decision maker is readily represented as a constraint on the rate of total activity, and
3) the notions of indirect and direct control through interactions with other organization members are readily apparent in the model.

The third item is especially significant because it indicates that the model developed is adaptable to consideration of team-theoretic decision problems (static or dynamic) where each decision maker is boundedly rational according to (1) and (2). Indeed, because of the characteristic of the model as a process, an additional richness in the consideration of interactions among team members is possible since various types of inputs can enter at different points in the process.

The model of the decision-making process developed in this thesis is shown to give pure internal decision strategies as solutions in the normative context for the case of an unconstrained decision maker, which corresponds to the pure decision rules obtained as solutions in the usual team-theoretic problem (the precise correspondence is with person-by-person optimal solutions). However, for the case of the boundedly rational decision maker, the model gives mixed strategies as solutions. Similarly, in the descriptive (satisficing) context, it is shown that it is possible that solutions may be only mixed strategies.

Finally, it is interesting to note an aspect of the model which arises due to the information theoretic framework: the greater the uncertainty in the input to the decision maker, the greater is the total activity required in the decision-making process. This property is in close correspondence in spirit to recent work by Galbraith in the context of (qualitative) organization design [13]. He observes that "the greater the task uncertainty, the greater the amount of information that must be processed among decision makers during task execution in order to achieve a given level of performance." This applies to the individual tasks of decision makers as well. Though the correspondence is a loose one, it is satisfying in that the model developed in this thesis is in harmony with developments in related disciplines.

1.3 THE THESIS IN OUTLINE

The thesis is organized as follows. Chapter 2 discusses the qualitative basis for the decision-making model developed, while Chapter 3 reviews the aspects of information theory which are used as the analytic framework. In Chapter 4, the basic model of the decision-making process is developed and analyzed. Chapter 5 presents an example which illustrates many of the properties of the model. In Chapter 6 the basic model is extended to include two types of interaction with the rest of the organization: processed information inputs and command inputs. Suggestions for future work and the thesis conclusion are presented in Chapter 7.

CHAPTER 2

QUALITATIVE MODEL

In this chapter, the qualitative basis is discussed for the decision-making process model developed in subsequent chapters.

2.1 PREVIOUS QUALITATIVE PROCESS MODELS

March and Simon [8] have hypothesized that the decision-making process of the satisficing decision maker is a two-stage process of "discovery and selection." The first stage is that of determining the situation of the environment, while the second addresses the question of what action to take in a particular situation. Selection in the first stage takes the form of choosing the degree and type of the "discovery" which the decision maker wishes to make regarding his environment, while discovery in the second stage pertains to generating possible courses of action for consideration. Clearly, the stages are coupled in that the type of alternatives sought depend on the situation perceived. Together they constitute the construction of the decision situation, which, according to the notion of satisficing in decision-making, all but completes the decision process, since if the process has been accomplished adequately, a satisfactory alternative is generated. Recent work by Wise [14] has substantiated this viewpoint. He writes that a common experience has been that "once a decision task is 'well structured,' it is painfully obvious to all involved what the appropriate course of action should be," as the "bulk of thoughtful and difficult work has resided in the actual structuring of the decision situation." Wise calls this phenomenon an "emergence" of the decision.

Wohl [1] has suggested a similar two-stage model of the decision process through an extension of the classical stimulus-response model of psychology. When a stimulus is received, the initial reaction of the decision maker is to hypothesize about its origin. This is followed by the generation and evaluation of options, among which one response is selected. Wohl applies this Stimulus - Hypothesis - Option - Response (SHOR) model in a military context to the tactical decision process with favorable results.

2.2 BASIC PROCESS MODEL

Based on the above discussion, the following two-stage model is assumed, and is illustrated in Figure 2.1. The decision maker receives an input x from his environment and uses it in the situation assessment (SA) stage of processing to "hypothesize about its origin." This results

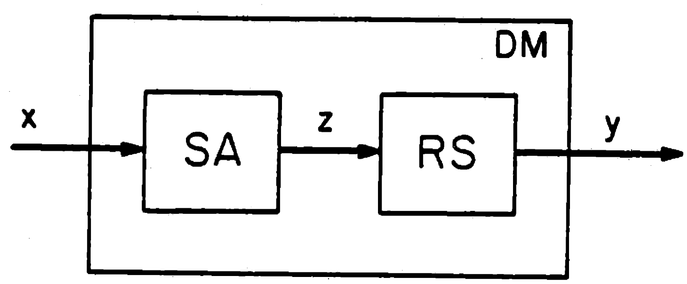


Figure 2.1. Qualitative Model

in the selection of a particular value of z. Possible alternatives of action are then evaluated in the second stage of processing, the response selection (RS) stage. The outcome of this process is the choice of action or decision response y. Many classes of decisions can be represented by the process of Figure 2.1. Consideration in this thesis will be restricted to decision-making tasks which are well-defined and which are performed in the steady-state, that is, the decision maker is assigned a particular task which he performs again and again for successively arriving inputs. For example, consider the task of monitoring a radar. A target appears on the radar screen and corresponds to an input stimulus. Initial processing is necessary to determine the type and/or status (friend or foe) of the aircraft which the target represents. Once this is decided, an appropriate response must be selected, perhaps to notify a superior (if foe) or to do nothing (if friend). This process of mapping input stimuli into output responses occurs repeatedly as the task is performed. The model developed in this thesis characterizes analytically the internal processing present in the performance of such a task using the framework of information theory. Accordingly, the relevant aspects of that theory are reviewed in the next chapter.

CHAPTER 3

ANALYTIC FRAMEWORK

The mathematics of information theory form the analytic framework for the model of the decision-making process developed in this thesis. Accordingly, in this chapter the two fundamental definitions in information theory are reviewed, as are their extension to multiple dimensions. In addition, the so-called Partition Law of Information [15] is presented and discussed; it is the information theoretic expression upon which much of the thesis work is based. Throughout this thesis, consideration will be restricted to the discrete case.

3.1 DEFINITIONS

Entropy, Conditional Entropy

The definitions of entropy and conditional entropy are due to Shannon [16]. The entropy of a random variable x which takes values according to the probability distribution p(x) is given by

$$H(x) = -\sum_{i} p(x = x^{i}) \log p(x = x^{i}) \tag{3.1}$$

$$= -\sum_{x} p(x) \log p(x) \tag{3.2}$$

Note that "x" has been used to denote both a random variable and the values that it takes. This ambiguity will be maintained where possible in order to simplify notation. The logarithm base will be taken as 2, giving H(x) the units of bits. The following definition will also be made:

$$0 \log 0 = 0 \tag{3.3}$$

H(x) is interpreted as the uncertainty in which value the random variable x will take [17]. If x takes a particular value with probability one, then

H(x) is equal to zero, i.e., x is deterministic and has no associated uncertainty. On the other hand, if x is equally likely to take each of its possible values, then H(x) attains a maximum. Finally, note that H(x) can be viewed as the expected value of the random variable log p(x). It is appropriate, therefore, to regard uncertainty as an average quantity, an interpretation which will be of use in the sequel.

The definition of conditional entropy is similar to that of marginal entropy. If two variables x and y have the conditional probability distribution $p(y|x)$, then the entropy of y conditioned on knowledge of the value of x is given by

$$H_x(y) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x) \tag{3.4}$$

Note that the above represents a double averaging, in that for a given x the log of the distribution $p(y|x)$ is averaged, and then the resulting quantities are averaged over all x. The interpretation of $H_x(y)$ is that of conditional uncertainty, that is, the uncertainty remaining in y when x is known. Two extreme cases are of particular interest. If y is independent of x, i.e., $p(y|x) = p(y)$, then the conditional uncertainty in y given x reduces to the marginal uncertainty in y:

$$H_x(y) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x) = -\sum_x p(x) \sum_y p(y) \log p(y) = H(y) \tag{3.5}$$

Alternatively, knowing x resolves none of the uncertainty in y. At the other extreme, if y is a deterministic function of x, that is, for each $x = x_i$

$$p(y|x_i) = \begin{cases} 1 & \text{for some } y = y_j \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

then $H_x(y) = 0$, since

$$p(y_j|x_i) \log p(y_j|x_i) = 0 \qquad \forall\ i,j \tag{3.7}$$

and no uncertainty remains in y when x is known.

Joint Entropy

The definitions of entropy and conditional entropy for single variables extend readily to multiple variables [16]. The joint entropy, or joint uncertainty, of two variables x and y can be computed from the joint probability distribution p(x,y), and is given by

$$H(x,y) = - \sum_{(x,y)} p(x,y) \log p(x,y) \tag{3.8}$$

Joint uncertainty can also be computed using an information theoretic identity (see Appendix A):

$$H(x,y) = H(x) + H_x(y) \tag{3.9}$$

From Eq. (3.9) and Eq. (3.5), it is apparent that if x and y are independent, their joint uncertainty is given by the sum of their respective marginal uncertainties. Similarly, if y is a deterministic function of x, then the joint uncertainty of x and y reduces to the uncertainty present in x. In general, the joint uncertainty of a set of random variables is computable from the corresponding joint probability distribution; however, it is often possible to take advantage of dependencies which may exist among variables to simplify the evaluation.

Mutual Information

The definition of the so-called mutual information, or transmission, of variables x and y follows directly from the marginal and conditional entropy definitions, and is given by

$$T(x{:}y) = H(x) + H(y) - H(x,y) \tag{3.10}$$

$$T(x{:}y) = H(y) - H_x(y) = H(x) - H_y(x) \tag{3.11}$$

Mutual information measures the relatedness of variables. If x and y are independent (not related), their mutual information is zero. If y is a deterministic function of x, then their relatedness is given by H(y); similarly, if x is a deterministic function of y, the mutual information T(x:y) is equal to H(x). Indeed, the two latter cases can be shown to be extremes, which gives

$$T(x:y) \in [0, \min\{H(x), H(x)\}] \tag{3.12}$$

It should be noted that the phrase "mutual information" has been chosen deliberately, as opposed to using only "information." The latter is used to describe several quantities which appear in the information theory literature, but the viewpoint in this development will be that "information" is only appropriate in describing the relationship of one variable to another; hence, the phrase mutual information will be used as a reminder of this interpretation [17], [18]. T(x:y) can then be considered as the information <u>in</u> y <u>about</u> x, or vice versa, since from Eq. (3.11) it is evident that mutual information is a symmetric quantity.

N-dimensional Mutual Information

McGill [19] has extended the concept of mutual information to n variables. Given n random variables x_i, their respective marginal probability distributions $p(x_i)$, and the n-dimensional joint probability distribution $p(x_1,\ldots,x_n)$, the n-dimensional mutual information of these variables is defined as

$$T(x_1:x_2:\ldots:x_n) = \sum_{i=1}^{n} H(x_i) - H(x_1,x_2,\ldots,x_n) \tag{3.13}$$

$T(x_1:x_2:\ldots:x_n)$ is zero if the n variables are mutually independent, which reflects the unrelatedness of the variables. At the other extreme, suppose knowledge of x_1 is sufficient to determine completely $x_2,\ldots,x_n$. Then

$$H(x_1,\ldots,x_n) = H(x_1) + H_{x_1}(x_2,\ldots,x_n) = H(x_1) \tag{3.14}$$

and the n-dimensional mutual information becomes (by substitution)

$$T(x_1:x_2:\ldots:x_n) = \sum_{i=2}^{n} H(x_i) \tag{3.15}$$

which gives a (relative) maximum.

In general, the greater the interrelatedness of variables, the greater is their mutual information. Furthermore, while n-dimensional mutual information is a measure of the interrelatedness of n variables considered as a group, it is certainly possible to view this global interaction as multiple interactions within and among subsets of the n variables. This approach is especially advantageous when the n variables can be partitioned into groups which are mutually independent. For example, suppose $n = 4$. The global mutual information in this case is given by

$$T(x_1:x_2:x_3:x_4) = \sum_{i=1}^{4} H(x_i) - H(x_1,x_2,x_3,x_4) \tag{3.16}$$

Considering $\{x_1,x_2,x_3,x_4\}$ to be $\{x_1,x_2\} \cup \{x_3,x_4\}$, it is possible to evaluate the 4-dimensional mutual information as the sum of interactions within $\{x_1,x_2\}$ and $\{x_3,x_4\}$ plus the interaction between the two subsets [20], [21], that is, (see Appendix A)

$$T(x_1:x_2:x_3:x_4) = T(x_1:x_2) + T(x_3:x_4) + T(x_1,x_2:x_3,x_4). \tag{3.17}$$

If $\{x_1,x_2\}$ is independent of $\{x_3,x_4\}$, then the last term on the right hand side above is zero and the 4-dimensional mutual information reduces to the sum of two binary mutual information quantities. This decomposition property will be used extensively in the sequel.

To facilitate the manipulation of information theoretic expressions involving sets of variables, it is convenient to use an aggregate representation [15], [21]. In the above example, if the various sets are denoted as

$$S^1 = \{x_1, x_2\}$$

$$S^2 = \{x_3, x_4\}$$

$$S = S^1 \cup S^2 = \{x_1, x_2, x_3, x_4\},$$

then the mutual information of S^1 and S^2 is given by

$$T(S^1:S^2) = H(S^2) + H(S^2) - H(S) \tag{3.18}$$

Eq. (3.18) expresses a more fundamental relationship given by

$$T(S^1:S^2) = T(x_1, x_2 : x_3, x_4) = H(x_1, x_2) + H(x_3, x_4) - H(x_1, x_2, x_3, x_4) \tag{3.19}$$

In general, it is possible to consider mutual information between/among arbitrary combinations of variables and sets of variables, and to evaluate an arbitrary mutual information expression provided the relevant probability distributions are known.

Finally, conditional mutual information is a useful quantity. The mutual information among n variables x_i conditioned on knowledge of another variable y is given by

$$T_y(x_1 : x_2 : \ldots : x_n) = \sum_{i=1}^{n} H_y(x_i) - H_y(x_1, \ldots, x_n) \tag{3.20}$$

and is evaluated using the appropriate (conditional) probability distributions. As an interesting example of how conditioning can influence the mutual information of a set of variables, recall the earlier example where knowledge of x_1 was sufficient to determine completely $x_2, \ldots, x_n$. The n-dimensional mutual information in this case was given by Eq. (3.15):

$$T(x_1 : x_2 : \ldots : x_n) = \sum_{i=2}^{n} H(x_i)$$

and was said to be at a (relative) maximum. Now condition this mutual information on the variable x_1. This gives

$$T_{x_1}(x_1:x_2:\ldots:x_n) = \sum_{i=2}^{n} H_{x_1}(x_i) \tag{3.21}$$

Knowledge of x_1 determines the remaining x_i, however, and the right hand side of Eq. (3.21) is equal to zero. In words, there is no interrelatedness of the n variables x_i except that due to x_1. Such conditional mutual information quantities will appear frequently as the decision-making process model is developed.

3.2 PARTITION LAWS

The application of n-dimensional information theory to systems has been made by Conant [15], and the so-called Partition Law of Information forms the basis for much of the work described in this thesis. Its derivation appears in Appendix B; the resultant equality is presented and discussed below.

Partition Law of Information

Suppose a system S is given and is described by n variables, of which one variable is the output y and the balance are the n-1 "internal variables" $w_1, w_2, \ldots, w_{n-1}$. Let $W = \{w_1, w_2, \ldots, w_{n-1}\}$ and denote the input to the system by x. For an arbitrary interconnection of the variables of the system (Figure 3.1), the Partition Law of Information (PLI) states that for $w_n \equiv y$,

$$\sum_{i=1}^{n} H(w_i) = T(x:y) + T_y(x:W) + H_x(W,y) + T(w_1:w_2:\ldots:w_{n-1}:y) \tag{3.22}$$

where $\sum_{i=1}^{n} H(w_i)$ is the sum of the marginal uncertainties of the n variables of the system. As is evident from the derivation in Appendix B, the PLI is essentially a tautological information theoretic expression. It is useful only to the extent that meaning can be attached to the individual quantities in the partition. Accordingly, the interpretation of each quantity is given in the following paragraphs.

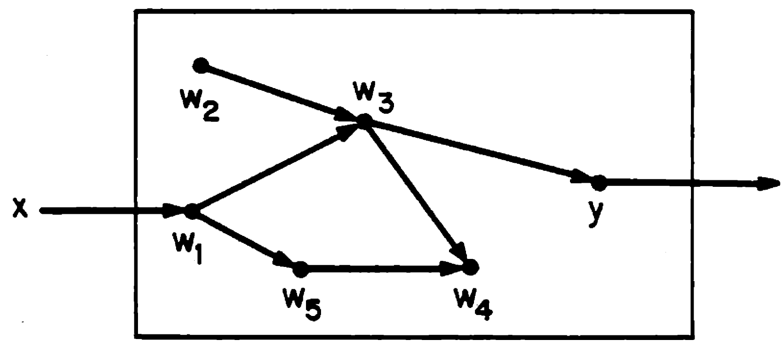


Figure 3.1. A Six-Variable System (n = 6)

Throughput, Blockage, Rejection

The first term on the right hand side of Eq. (3.22) is by definition the mutual information between input and output. Mathematically, it is exactly the same quantity which appears in the communications context as a measure of a channel's capacity. In the present context it has a similar interpretation, though it differs somewhat in that it is not necessarily a quantity to be maximized as is often the case in communications. Rather, it is one of several quantities which characterize a system, namely, that which measures the amount by which the input and output are related. Such a relationship will be designated the throughput and will be denoted by G_t.

If throughput is a measure of the extent to which an input to a system is reflected at the output, one might expect the extent to which the input is not reflected at the output to be a complementary quantity of interest. Indeed, such is the interpretation given to the second term on the right hand side of Eq. (3.22). $T_y(x:W)$ is the relatedness of the input x to the set of internal variables W, conditioned on knowledge of the output y, and expresses the blockage, denoted G_b, of the input by the system. This can be more readily seen by using the definition of mutual information to write

$$T_y(x:W) = H_y(x) - H_{y,W}(x) \tag{3.23}$$

$$T_y(x:W) = H_y(x) - H_{y,W}(x) + H(x) - H(x) \tag{3.24}$$

$$T_y(x:W) = [H(x) - H_{y,W}(x)] - [H(x) - H_y(x)] \quad (3.25)$$

$$T_y(x:W) = T(x:y,W) - T(x:y) \quad (3.26)$$

The second term on the right hand side of Eq. (3.26) is recognized as the throughput, while the first is the relationship between input and entire system. Their difference measures that part of the input which remained within the system, i.e., was blocked.

Throughput and blockage represent two possible dispositions of the input to a particular system. A third possibility is that the input may not be recognized by the system, that is, may not even cross the boundary into the system. Such a phenomenon is termed <u>rejection</u>, denoted G_r, and can be expressed as

$$G_r = H_{W,y}(x) \quad (3.27)$$

Rejection is the uncertainty which remains about the system's input when there is complete knowledge of system variables. Note that it is a passive form of blockage from the system's point of view, in that the equivalent of blockage is accomplished without adding to the total activity of the system, i.e., G_r does not enter into the PLI. G_r does appear in an auxiliary equation to the PLI which expresses the fact that there are three possible paths for an input:

$$H(x) = G_t + G_b + G_r \quad (3.28)$$

The combination of blockage and rejection represents the uncertainty about the input when the output is known. This can be shown by substitution of the definition of throughput into Eq. (3.28) and rearrangement of terms:

$$T(x:y) = H(x) - [G_b + G_r] \quad (3.29)$$

$$H(x) - H_y(x) = H(x) - [G_b + G_r] \quad (3.30)$$

$$H_y(x) = G_b + G_r \quad (3.31)$$

$H_y(x)$ is known in the communications context as equivocation [16]. The useful distinction between active equivocation [blockage] and passive equivocation [rejection] will be retained in the sequel, however.

Noise, Coordination, Total Activity

The third term on the right hand side of Eq. (3.22) is by definition the uncertainty in the system when the input is known, and reflects the extent to which the system is not a deterministic function of its input. In previous applications of the PLI [15], such uncertainty has been interpreted to be undesirable noise, denoted by G_n. In the present context the same notation will be used, although a fundamentally different interpretation will be made.

The n-dimensional mutual information appears as the last term in the PLI. As discussed previously, it measures the total interaction of the n system variables. Considering the system to be a processor of input into output, the interpretation of the mutual information of the system is that of the coordination required among the system variables to accomplish this processing. In general, however, the system need not be "dedicated" to processing input into output, and the coordination, denoted G_c, will reflect all system variable interactions. Note that it may be possible to use the decomposition property discussed earlier to partition the total system into "subsystems" and thereby evaluate the total coordination as a sum of coordinations within and among subsystems.

Having described the four quantities which partition the system, it should be noted that the partition has not been one of the variables per se, that is, a particular variable is not associated to any of the partitions. Rather, the partitions refer to a division of the activity of the system as a whole, and indeed, such is the interpretation given to the left hand side summation of the PLI. The total activity of a system, then, denoted G, can be described as the sum of the throughput, blockage, noise, and coordination present in that system:

$$G = G_t + G_b + G_n + G_c \tag{3.32}$$

In addition, the auxiliary equation (3.28) relates throughput, blockage, and rejection.

3.3 EXAMPLE

To illustrate the application of the PLI, consider the comparison tree shown in Figure 3.2. The input variable x is mapped into the output variable y, where y can take the values 1, 3, 5, 7. The two-stage comparison process represents the rounding of x into the nearest value of y. Associated with each comparison is a binary variable, w_i, which takes its value according to the outcome of the comparison. The tree can be represented as shown in Figure 3.3, where $w_1 \equiv x$.

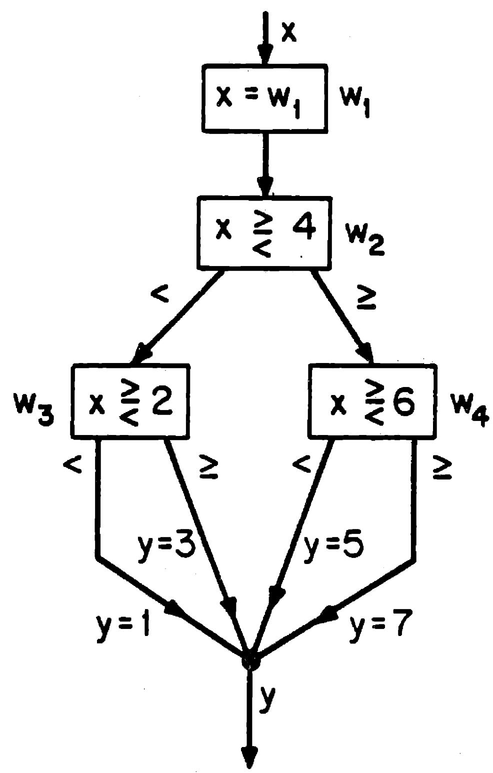


Figure 3.2. Comparison Tree

The PLI for the system of Figure 3.3 is given by ($w_5 \equiv y$)

$$\sum_{i=1}^{5} H(w_i) = T(x:y) + T_y(x:w_1,w_2,w_3,w_4) + H_x(w_1,w_2,w_3,w_4,y)$$

$$+ T(w_1:w_2:w_3:w_4:y)$$

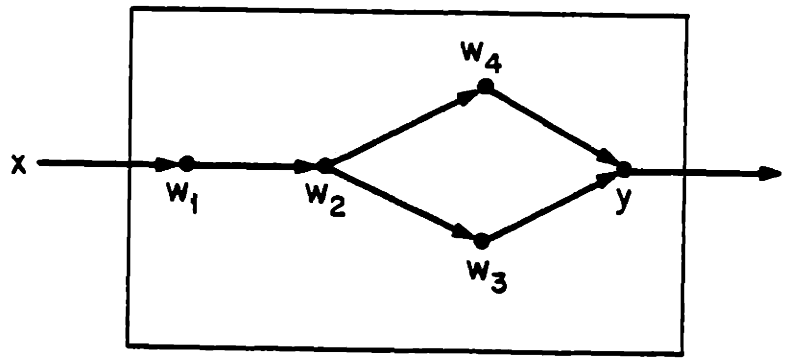


Figure 3.3. System Representation of Comparison Tree

Since the system is a deterministic function of its input, no noise is present ($G_n = 0$). In addition, the throughput can be simplified using the definition of mutual information:

$$G_t = T(x:y)$$

$$G_t = H(y) - H_x(y) = H(y)$$

Let x take the values 1, 3, 5, 7, all with equal probability. H(x), the uncertainty in the input, is therefore equal to 2 bits. Furthermore, in this case the variables w_2, w_3, w_4 each have an uncertainty of one bit. This follows from the fact that comparisons made at each node are equally likely to fall on either side of the threshold. Evaluating the total activity of the system gives

$$G = \sum_{i=1}^{5} H(w_i) = 2 + 1 + 1 + 1 + 2 = 7 \text{ bits}$$

Throughput is easily obtained:

$$G_t = H(y) = 2 \text{ bits}$$

Blockage can be evaluated using the auxiliary equation. Because $w_1 \equiv x$, there is no rejection by the system; that is,

$$G_r = H_{w_1,w_2,w_3,w_4,w_5}(x) = 0$$

This gives

$$G_b = H(x) - G_t$$

$$G_b = 2 - 2 = 0$$

Finally, the coordination is evaluated:

$$G_c = \sum_{i=1}^{5} H(w_i) - H(w_1,w_2,w_3,w_4,w_5)$$

$$G_c = \sum_{i=1}^{5} H(w_i) - H(w_1) - H_{w_1}(w_2,w_3,w_4,w_5)$$

The last term in the above equation is zero since the system is deterministic, and the second term is combined with the first to give

$$G_c = \sum_{i=2}^{5} H(w_i) = 1 + 1 + 1 + 2 = 5 \text{ bits}$$

In summary,

$$\begin{cases} G = G_t + G_b + G_n + G_c \\ 7 = 2 + 0 + 0 + 5 \text{ bits} \end{cases}$$

$$H(x) = 2 \text{ bits}$$

$$G_r = 0 \text{ bits}$$

For purposes of comparison, two other cases are considered:

Case II: $x = \begin{cases} 1,5 & \text{each with prob. } 3/8 \\ 3,7 & \text{each with prob. } 1/8 \end{cases}$

Case III: $x = \{0.5, 1.5, 6.5, 7.5\}$ with uniform probability.

The resulting PLI quantities are given in Table 3.1 together with those from Case I.

Table 3.1 PLI QUANTITIES FOR EXAMPLE CASES; UNITS ARE BITS

CASE	x-ALPHABET	H(x)	G_r	G_t	G_b	G_n	G_c	G
I	1, 3, 5, 7	2.0	0	2.0	0	0	5.0	7.0
II	1, 3, 5, 7	1.8	0	1.8	0	0	4.4	6.2
III	0.5, 1.5, 6.5, 7.5	2.0	0	1.0	1.0	0	3.0	4.0

From the above example, it is clear first of all that the PLI quantities can be evaluated for a particular system if its structure and the nature of its input is known. Secondly, it should be emphasized that the activity within a system depends not only on the structure of the system (internal variables) but also on the alphabet and uncertainty of the input, as is evident from Table 3.1. This fact will have interesting implications in the context of the model of the decision-making process.

Finally, suppose that successive inputs arrive for processing every τ seconds. The system is then required to accomplish on the average seven bits (Case I) of activity every τ seconds in order to keep up with the clock. Successive processing of inputs introduces a dynamic element into the PLI, and the corresponding PLI rates which result will be of particular interest in the sequel.

3.4 SUMMARY

Basic results from information theory have been reviewed. In particular, the Partition Law of Information was introduced and discussed. It is a general information theoretic expression, and characterizes a system according to the equation

$$G = G_t + G_b + G_n + G_c$$

$$\text{Total Activity} = \text{Throughput} + \text{Blockage} + \text{Noise} + \text{Coordination}$$

In succeeding chapters, the PLI will be used to obtain analytic expressions which describe a specific model of the decision-making process. This synthesis of the mathematics of information theory with qualitative models of decision-making begins in the next chapter with the consideration of a basic two-stage model of the decision maker.

CHAPTER 4
THE TWO-STAGE MODEL

In this chapter, a simple two-stage model is introduced to represent the internal decision process by men or machines in performing well-defined tasks in the steady state. The Partition Law of Information is used to obtain an analytical description of internal information processing and decision making. The model is then used to investigate choices of decision strategies both in the normative and in the descriptive (satisficing) contexts. The effect on the choices of the decision maker's bounded rationality is analyzed.

4.1 STRUCTURE OF TWO-STAGE MODEL

As discussed in Chapter 2, the decision-making process can be regarded as occurring in two stages, where the first is that of situation assessment and the second is that of output selection based on the assessed situation. This decision-making model is a general one; in order to obtain expressions which describe analytically internal information processing and decision-making, it is necessary to add more structure to the conceptual model. In particular, it will be assumed that the decision maker is an expert at his decision-making task. As such, each stage of the process can be considered as containing two sets of well-defined procedures, or algorithms. An algorithm belonging to the first set maps input stimuli into assessed situations and an algorithm in the second set takes assessed situations and maps them into output decision responses. The algorithms differ in quality and in the amount of resources required; no correlation between these two properties is assumed. It is assumed that the algorithms do not change as the process takes place, i.e., no learning is possible and the process is therefore assumed to be in steady state. Given this structure, the decision-making process becomes one of selecting an algorithm in each of the two stages.

To be more precise, assume that the state of the decision maker's environment is given by $\underline{x}'$, an r-dimensional vector which takes values from a finite alphabet. The decision maker receives as input $\underline{x}$, however, which is a noisy measurement of $\underline{x}'$. The vector $\underline{x}$ is also r-dimensional and takes known values from a finite alphabet according to $p(\underline{x})$. Furthermore, let y denote the outcome of the decision-making process, or the output. Assume that there are a fixed number of outcomes. The task of the decision maker, then, is to determine an output (y) appropriate to the state of the environment ($\underline{x}'$) based on a noisy measurement of that state ($\underline{x}$).

The decision maker accomplishes his task by first assessing the situation using one of the U algorithms he possesses for that purpose. Each algorithm maps measurements $\underline{x}$ into assessed situations $\underline{z}$, where $\underline{z}$ is an s-dimensional vector taking M values, with $s \leq r$. In the extreme case, the situation assessment would involve an estimation of the entire state $\underline{x}'$; however, it is more likely that in order to choose an appropriate output or decision response it is necessary to consider only some "sufficient statistic" determined from the measurement $\underline{x}$. Thus, $\underline{z}$ represents a possible aggregation of input data. The input-output mappings of the algorithms used to determine $\underline{z}$ from $\underline{x}$ are denoted by $f_i(\underline{x})$, where $i = 1,2,...,U$.

Once the situation $\underline{z}$ has been determined, another choice among algorithms is made in order to map $\underline{z}$ into an appropriate response y. The decision maker possesses V such response selection algorithms for this purpose, and their respective input-output mappings are denoted by $h_j(\underline{z})$, where $j = 1,2,...,V$.

The complete decision-making process, then, is given by the equation

$$y = h_j(f_i(\underline{x})) \tag{4.1}$$

For a given $\underline{x}$, the output response is determined by the realization of variables u and v, with each one taking values from the sets $i = 1,2,...,U$ and $j = 1,2,...,V$, respectively. These variables are <u>internal choices</u> in the decision-making process; indeed, according to the model defined above, they represent the "real" decisions made in accomplishing the overall task. The process defined above can be represented as shown in Figure 4.1, where

$\underline{q}$ is the noise source in the measurement of $\underline{x}'$, and $\underline{x} = \underline{x}' + \underline{q}$. The internal choices have been represented as switches which take positions according to the realizations of u and v. The research work described in this chapter

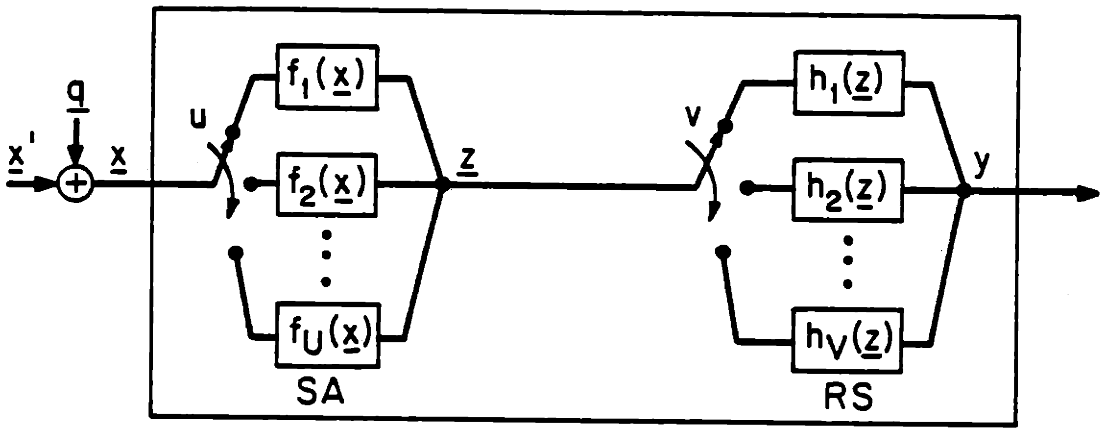


Figure 4.1. Basic Decision-Making Process Model

is based on the application of the Partition Law of Information to the class of systems shown in Figure 4.1.

While time does not appear explicitly in the development, it is implicit as defined below. The inputs to the decision maker are considered to be symbols generated by a source according to $p(\underline{x})$. A memoryless source is assumed, i.e., each symbol is generated independently. The characterization of such a source by

$$H(\underline{x}) = - \sum_{\underline{x}} p(\underline{x}) \log p(\underline{x})$$

is defined to be the entropy of the source per symbol generated [16]. If, in addition, the source is such that an input symbol is generated every τ seconds on the average, the entropy rate of the source is given by

$$\frac{H(\underline{x})}{\tau},$$

which is measured in bits per second, if 2 is the base of the logarithms in the definition of the entropy.

Furthermore, because it has been assumed that no learning takes place during the performance of a sequence of tasks, the successive values taken by the variables of the model are uncorrelated, i.e., the model is memory-less. Hence, all information theoretic expressions that are written in the development will be on a per symbol basis (per symbol of the input); rates are determined by dividing the information theoretic expressions by the mean symbol interarrival time.

A key assumption in the following sections is that the algorithms are deterministic, that is, the mappings $f_i(\underline{x})$ and $h_j(\underline{z})$ are deterministic. This assumption is made for conceptual clarity; the general form of the equations remains the same for the non-deterministic case, and the implications of the latter are considered in Chapter 7.

4.2 FIRST STAGE: SITUATION ASSESSMENT

In applying the PLI to the two-stage model, it is convenient to consider the system shown in Figure 4.1 as being composed of two structurally identical subsystems. Because of the decomposition property of n-dimensional mutual information, it is possible to study each subsystem individually and then connect them into a single system. Accordingly, in this section consideration will be given only to the situation assessment subsystem, defined as shown in Figure 4.2.

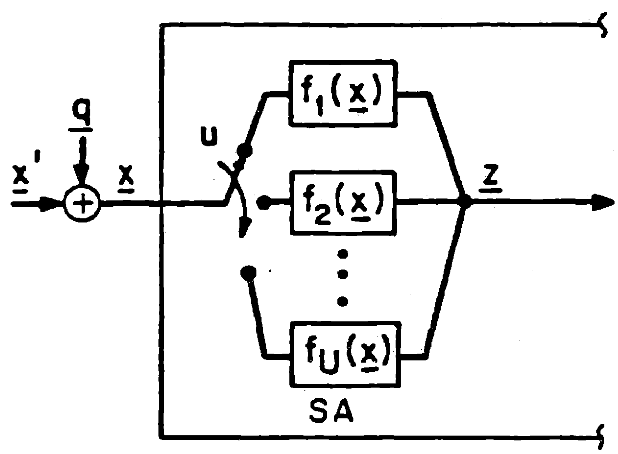


Figure 4.2. Situation Assessment Stage

4.2.1 Algorithm Variables

As discussed earlier, the mappings $f_i(\underline{x})$ represent the decision maker's situation assessment algorithms. To effect a mapping from $\underline{x}$ to $\underline{z}$ each algorithm consists, in general, of a series of steps, such as intermediate computations or comparisons. These steps determine the variables of the algorithm, which are its fundamental quantities. In other words, each algorithm can be considered as a system unto itself, with input $\underline{x}$ and output $\underline{z}$, and internal variables whose interconnection determines the steps taken in mapping the input into the output. Suppose algorithm i contains α_i variables. Denote the set of variables of algorithm i by

$$W^i = \{w_1^i, w_2^i, \ldots, w_{\alpha_i}^i\} \tag{4.2}$$

Furthermore, assume that the algorithms have no variables in common, i.e.,

$$W^i \cap W^j = \emptyset \qquad i \neq j;\ \forall\ i,j \in \{1,2,\ldots,U\} \tag{4.3}$$

Finally, assume that no part of the input $\underline{x}$ is reflected at the decision maker's input boundary, i.e., that the rejection of each algorithm is zero.

In summary, the model of the situation assessment stage consists of a system of variables, denoted S^I, where

$$S^I = \{u, W^1, W^2, \ldots, W^U, \underline{z}\} \tag{4.4}$$

The interconnection of these variables is determined by the algorithmic interconnections within sets W^i, as well as the interconnection among algorithms determined by the variable u.

4.2.2 Derivation of Analytic Expressions

To the system of variables given in Eq. (4.4) is applied the Partition Law of Information. Each term in the partition will be considered individually, beginning with the so-called noise.

Noise

By definition, the noise present in S^I is given by

$$G_n^I = H_{\underline{x}}(u, W^1, W^2, \ldots, W^U, \underline{z}) \tag{4.5}$$

Eq. (4.5) can be written as (see Appendix A)

$$G_n^I = H_{\underline{x}}(u) + H_{\underline{x},u}(W^1, W^2, \ldots, W^U, \underline{z}) \tag{4.6}$$

Because the algorithms are deterministic, the uncertainty in the system variables is zero when the input and choice of algorithm are known. Hence, Eq. (4.6) reduces to

$$G_n^I = H_{\underline{x}}(u) \tag{4.7}$$

or

$$G_n^I = -\sum_{\underline{x}} p(\underline{x}) \sum_{u} p(u|\underline{x}) \log p(u|\underline{x}) \tag{4.8}$$

Eq. (4.7) represents the uncertainty in the choice of algorithm when the input is known. Previously [15], this uncertainty has been regarded as undesirable noise, but in the present context it has a great deal of importance because it corresponds to an internal decision. The distribution $p(\underline{x})$ is external to the decision maker and assumed fixed and known. G_n^I, the <u>amount of internal decision-making</u>, is therefore determined by $p(u|\underline{x})$, the <u>internal decision strategy</u>. Such a strategy represents the inclination of the decision maker to select a particular algorithm. For successively arriving inputs, the strategy reflects the relative frequency of a particular algorithm's use.

While the expression for G_n^I includes a dependence on the input $\underline{x}$, the system is such that u is independent of $\underline{x}$, i.e., $p(u|\underline{x}) \equiv p(u)$. This is because the value of an arriving input is not known to the decision maker; rather, it is known only that an input is present. Indeed, it is the function of the situation assessment stage to identify the input and use it to determine the appropriate value of $\underline{z}$. Conditioning the first stage internal decision strategy on the value of $\underline{x}$ implies that preliminary processing must take place in order to identify $\underline{x}$ and to associate it with the appropriate strategy $p(u|\underline{x})$. Such processing does not exist in the model under consideration and hence the internal decision strategy must be independent of $\underline{x}$. The inclusion of a pre-assessment stage to accomplish preliminary processing or pre-processing constitutes a direct extension of the model presented in Section 4.3.

With the internal decision strategy given by $p(u)$, the amount of decision-making present in the situation assessment stage becomes, simply,

$$G_n^I = H(u) \tag{4.9}$$

If a particular algorithm is used exclusively, i.e., $p(u=i)=1$ for some i, then $H(u)=0$, which indicates that no real decision is being made. On the other hand, when $p(u)$ is uniform, i.e., each algorithm is equally likely to be chosen, then $H(u)$ is at a maximum. Since the remainder of the system is fixed, the relationship between the first stage processing and the decision strategy $p(u)$ employed is one of key importance in characterization of the model.

Throughput

The throughput of the system S^I is given by definition as

$$G_t^I = T(\underline{x}:\underline{z}) \tag{4.10}$$

$$G_t^I = H(\underline{z}) - H_{\underline{x}}(\underline{z}) \tag{4.11}$$

Recall that the fundamental quantity in $H(\underline{z})$ is $p(\underline{z})$; similarly, $p(\underline{z}|\underline{x})$ and $p(\underline{x})$ are needed to evaluate $H_{\underline{x}}(\underline{z})$. By Bayes' rule, $p(\underline{z}|\underline{x})$ can be written as

$$p(\underline{z}|\underline{x}) = \sum_{u} p(\underline{z}|\underline{x}u)p(u|\underline{x}) \tag{4.12}$$

Since the algorithm selection strategy is independent of the input, Eq. (4.12) reduces to

$$p(\underline{z}|\underline{x}) = \sum_{u} p(\underline{z}|\underline{x}u)p(u) \tag{4.13}$$

Furthermore, because the mappings $f_i(\underline{x})$ are known, $p(\underline{z}|\underline{x}u)$ can be obtained by determining whether a given input $\underline{x}$ yields output $\underline{z}$ when algorithm f_i is used (for all inputs, all possible outputs and all algorithms), i.e., evaluating

$$f_i(\underline{x}) \qquad \forall\ i,\underline{x}. \tag{4.14}$$

Similarly, $p(\underline{z})$ can be obtained according to

$$p(\underline{z}) = \sum_{\underline{x}} \sum_{u} p(\underline{z}|\underline{x}u)p(u)p(\underline{x}) \tag{4.15}$$

Since $p(\underline{x})$ is known, specification of $p(u)$, the internal decision strategy, determines the throughput of S^I. In addition, the dependence of G_t^I on $p(u)$ has been made explicit, as shown by Eqs. (4.13), (4.14), and (4.15).

Blockage

The blockage within S^I is obtained by application of the equation auxiliary to the PLI, repeated here for convenience:

$$H(\underline{x}) = G_t^I + G_b^I + G_r^I \tag{4.16}$$

Each algorithm has been assumed to have zero rejection; hence, because of the structure of S^I, there is no rejection present in S^I. $H(\underline{x})$ is known through $p(\underline{x})$ and G_t^I can be computed as a function of $p(u)$ as shown above. This gives

$$G_b^I = H(\underline{x}) - G_t^I \tag{4.17}$$

which is computable as a function of p(u). Note that because there is no rejection G_b represents the equivocation of the system, as discussed in Chapter 3.

Coordination

In evaluating the coordination present in S^I, the approach taken will be to view the situation assessment algorithms as subsystems and to obtain the decomposition of the global mutual information which is expressed in terms of the coordination within and among those subsystems. The coordination of S^I is given by

$$G_c^I = T(w_1^1 : w_2^1 : \ldots : w_{\alpha_1}^1 : w_1^2 : \ldots : w_{\alpha_U}^U : u : \underline{z}) \tag{4.18}$$

$$G_c^I = \sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} H(w_j^i) + H(u) + H(\underline{z}) - H(W^1, W^2, \ldots, W^U, u, \underline{z}) \tag{4.19}$$

Consider the joint uncertainty term. It can be written as (see Appendix A, Eq. (A.1))

$$H(W^1, W^2, \ldots, W^U, u, \underline{z}) = H(u) + H_u(W^1) + H_{u,W^1}(W^2) + \ldots$$

$$+ H_{u,W^1,\ldots,W^{U-1}}(W^U) + H_{u,W^1,\ldots,W^U}(\underline{z}) \tag{4.20}$$

The last term in Eq. (4.20) is zero, since no uncertainty remains in $\underline{z}$ when all other system variables are known.

Consider the term $H_{u,W^1}(W^2)$. It is the uncertainty in the variables of the second algorithm conditioned on knowledge of u and knowledge of the first algorithm's variables. In general, the variables of a particular algorithm i can be regarded as active or inactive, depending on the realization of u. If $u = i$, then the set of variables W^i is active. If $u \neq i$, then the variables W^i are not active, and the best inference that can be made from algorithm i's perspective is only that another algorithm was chosen. This follows from the assumption that algorithm variable sets are disjoint, and from the fact that the only variable linking the algorithms is u ($\underline{z}$ is not pertinent to the present discussion). Therefore, it follows that

$$H_{u,W^1}(W^2) = H_u(W^2) \tag{4.21}$$

since knowledge of the variables W^1 can only resolve the uncertainty as to the active/inactive status of the variables W^2, and this uncertainty is also resolved by knowledge of u. The same argument is applied to other terms in Eq. (4.20) to obtain

$$\begin{aligned} H_{u,W^1,W^2}(W^3) &= H_u(W^3) \\ &\vdots \\ H_{u,W^1,\ldots,W^{U-1}}(W^U) &= H_u(W^U) \end{aligned}$$

Substitution into the expression for G_c^I yields

$$G_c^I = \sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} H(w_j^i) + H(\underline{z}) - \sum_{i=1}^{U} H_u(W^i) \tag{4.22}$$

Add and subtract

$$\sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} H_u(w_j^i)$$

to obtain

$$G_c^I = \sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} H(w_j^i) - H_u(w_j^i) + \sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} H_u(w_j^i) - \sum_{i=1}^{U} H_u(W^i) + H(\underline{z}) \tag{4.23}$$

$$G_c^I = \sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} T(u:w_j^i) + \sum_{i=1}^{U} \left[\sum_{j=1}^{\alpha_i} H_u(w_j^i) - H_u(W^i) \right] + H(\underline{z}) \tag{4.24}$$

Consider the first term in Eq. (4.24), and in particular let $i = j = 1$. The mutual information between u and w_1^1 is given by

$$T(u:w_1^1) = H(u) - H_{w_1^1}(u) \tag{4.25}$$

In order to simplify Eq. (4.25), it is necessary to make more precise the meaning of active and inactive algorithm variables.

As indicated earlier, w_1^1 is active when $u = 1$. Under this condition w_1^1 takes values according to values of the input $\underline{x}$ and also according to the nature of the algorithm. If $u \neq 1$, then w_1^1 is inactive. It will be assumed that w_1^1 takes a <u>fixed</u> value when inactive which is <u>not</u> one of the values taken when active. Thus, the probability distribution for w_1^1 has two distinct modes, which can be represented in a general manner as shown in Figure 4.3, where □ denotes the value of w_1^1 when inactive.

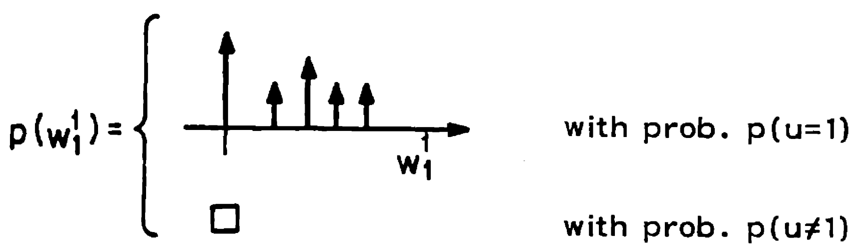


Figure 4.3. Form of Probability Distribution for Variable w_1^1

The definitions of uncertainty and conditional uncertainty can be used to write Eq. (4.25) as

$$T(u:w_1^1) = -\sum_u p(u) \log p(u) + \sum_{w_1^1} p(w_1^1) \sum_u p(u|w_1^1) \log p(u|w_1^1) \tag{4.26}$$

$$T(u:w_1^1) = -\sum_u p(u) \log p(u) + \sum_{w_1^1 \neq \Box} p(w_1^1) \sum_u p(u|w_1^1) \log p(u|w_1^1)$$

$$+ p(w_1^1 = \Box) \sum_u p(u|w_1^1 = \Box) \log p(u|w_1^1 = \Box) \tag{4.27}$$

From Figure 4.3, $p(w_1^1 = \Box) = p(u \neq 1)$. Also, for $w_1^1 \neq \Box$ it is true that

$$p(u|w_1^1 \neq \Box) = \begin{cases} 1 & u = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.28}$$

In the case where $w_1^1 = \Box$, it is known with certainty that $u \neq 1$. The relative likelihood of using the other algorithms remains the same from the perspective of algorithm 1, however. This gives the conditional distribution

$$p(u|w_1^1 = \Box) = \begin{cases} 0 & u = 1 \\ \dfrac{p(u)}{p(u\neq 1)} & u \neq 1 \end{cases} \tag{4.29}$$

where $p(u\neq 1)$ has been used to normalize the distribution. Substituting into Eq. (4.27) gives

$$T(u:w_1^1) = -\sum_u p(u) \log p(u) + \sum_{w_1^1 \neq \Box} p(w_1^1) \cdot 0 + p(u\neq 1) \sum_{u\neq 1} \frac{p(u)}{p(u\neq 1)} \log \frac{p(u)}{p(u\neq 1)} \tag{4.30}$$

$$T(u:w_1^1) = -\sum_u p(u) \log p(u) + \sum_{u\neq 1} p(u) \log p(u) - \sum_{u\neq 1} p(u) \log p(u\neq 1) \tag{4.31}$$

$$T(u:w_1^1) = - p(u=1) \log p(u=1) - p(u\neq 1) \log p(u\neq 1) \tag{4.32}$$

Eq. (4.32) is the uncertainty in a binary random variable which takes values with probability $p(u=1)$ and $1 - p(u=1)$. It is a common function encountered in the communications context and has been designated $\mathscr{H}(\cdot)$ [18]:

$$\mathscr{H}(p) = -p \log p - (1-p) \log (1-p) \tag{4.33}$$

This gives $T(u:w_1^1) = \mathscr{H}(p(u=1))$.

The above analysis applies to all variables in algorithm one. In fact, it is true in general, i.e.,

$$T(u:w_j^i) = \mathscr{H}(p(u=i)) \qquad i=1,\ldots,U \tag{4.34}$$

If specific values of the variable i are denoted by letter i, the first term of Eq. (4.24) can be written as

$$\sum_{i=1}^{U} \sum_{j=1}^{\alpha_i} T(u:w_j^i) = \sum_{i=1}^{U} \alpha_i \mathscr{H}(p_i) \tag{4.35}$$

where $p_i = p(u=i)$, $i=1,\ldots,U$. Eq. (4.24) then becomes

$$G_C^I = \sum_{i=1}^{U} \alpha_i \mathscr{H}(p_i) + \sum_{i=1}^{U} \left[\sum_{j=1}^{\alpha_i} H_u(w_j^i) - H_u(W^i) \right] + H(\underline{z}) \tag{4.36}$$

Consider the second term in Eq. (4.36) and let $i=1$. From the definition of conditional uncertainty, the resulting expression can be written as

$$\sum_{j=1}^{\alpha_1} H_u(w_j^1) - H_u(W^1) = \sum_{j=1}^{\alpha_1} \left[-\sum_u p(u) \sum_{w_j^1} p(w_j^1|u) \log p(w_j^1|u) \right] - \left[-\sum_u p(u) \sum_{W^1} p(W^1|u) \log p(W^1|u) \right] \tag{4.37}$$

Recall that for $u \neq 1$, the values of the variables w_j^1 are fixed and hence have no uncertainty. Factoring the remaining $p(u=1)$ as a pre-multiplier reduces Eq. (4.37) to

$$\sum_{j=1}^{\alpha_1} H_u(w_j^1) - H_u(W^1) = p(u=1)\left\{\sum_{j=1}^{\alpha_1}\left[-\sum_{w_j^1} p(w_j^1|u=1)\log p(w_j^1|u=1)\right]\right.$$
$$\left. -\left[-\sum_{W^1} p(W^1|u=1)\log p(W^1|u=1)\right]\right\} \tag{4.38}$$

When $u=1$, the variables w_j^1 are active and are processing the input $\underline{x}$. From the definition of n-dimensional mutual information and its interpretation as coordination, the term of Eq. (4.38) in braces is the coordination of algorithm 1 when considered as a system with input $\underline{x}$ and output $\underline{z}$. Indeed, when algorithm 1 is active, it will always be the case that it processes $\underline{x}$ into $\underline{z}$; hence the left-hand side of Eq. (4.38) reduces to the coordination of the first algorithm, denoted g_c^1, weighted by the relative frequency of its use:

$$\sum_{j=1}^{\alpha_1} H_u(w_j^1) - H_u(W^1) = p(u=1)\, g_c^1 \tag{4.39}$$

The particular form of Eq. (4.39) depends critically on the fact that the decision u is independent of the input $\underline{x}$. If an association between inputs and algorithms were possible, i.e., $p(u|\underline{x}) \not\equiv p(u)$, then algorithm 1 would process only a subset of the inputs $\underline{x}$. Since the coordination of algorithm 1, g_c^1, is determined in part by the characteristics of its input (see the example of Chapter 3), the more general form of Eq. (4.39) is

$$\sum_{j=1}^{\alpha_1} H_u(w_j^1) - H_u(W^1) = p(u=1)\, g_c^1(p(\underline{x}|u=1)) \tag{4.39a}$$

where the possible dependence of the inputs to the algorithm on the internal decision u has been shown. For the case at hand $p(\underline{x}|u=1) \equiv p(\underline{x})$ and Eq. (4.39) results when g_c^1 is evaluated using $p(\underline{x})$. A similar argument applies for the remaining algorithms, and the second term of Eq. (4.36) becomes

$$\sum_{i=1}^{U}\left[\sum_{j=1}^{\alpha_i} H_u(w_j^i) - H(W^i)\right] = \sum_{i=1}^{U} p_i g_c^i \tag{4.40}$$

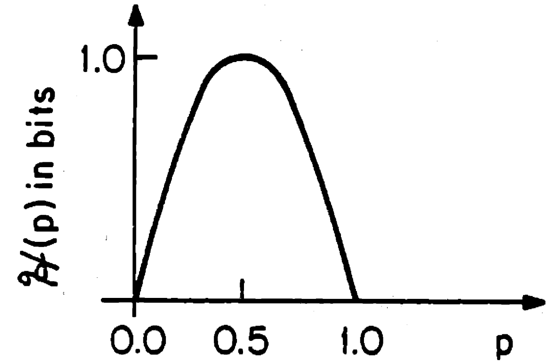
Finally, substitution of Eq. (4.40) into Eq. (4.36) yields

$$G_c^I = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) \tag{4.41}$$

4.2.3 Interpretation of the Coordination Expression

The resulting expression for the total coordination in S^I reflects the presence of a switching operation in the system. Since this switching occurs among subsystems which are disjoint and active on non-overlapping intervals of time, part of the total coordination becomes that of the coordination within each of the subsystems, weighted according to the relative frequency of each algorithm's use (internal decision strategy). Furthermore, since G_c^I measures the global coordination among all variables of S^I, and since $\underline{z}$ is the only variable within S^I which is related to all other variables, it is to be expected that G_c^I contains the term $H(\underline{z})$.

The second term of Eq. (4.41) is interpreted to be the coordination required to switch among algorithms; it can also be regarded as the effort or resource use required to initialize the variables of an algorithm prior to its use. Examination of the mathematical expression for this coordination shows that it is functionally dependent on the relative frequency of a particular algorithm's use, and furthermore, that variables of the same algorithm make an equal contribution to the total required. The latter is not unreasonable, and the former is necessary because the coordination equation represents steady-state phenomena, i.e., the coordination required to initialize algorithms is very much related to the number of times on the average such initializations must take place. Reference to Figure 4.4 shows that the nature of this relationship is such that if a particular algorithm is always used ($p = 1$ in Figure 4.4), the initialization coordination is zero as no initializations are taking place in the steady state. Similarly, if an algorithm is never used, it is never initialized ($\mathscr{H}(0) = 0$). In addition, the symmetry of $\mathscr{H}(p)$ about $p = 0.5$ is significant because it reflects the fact that frequent use of an algorithm may require on the average the same number of initializations as infrequent use. This phenomenon arises because an often used algorithm is likely to be used for successive inputs, in which case no re-initialization would take place.



$$\mathscr{H}(p) = -p \log_2 p - (1-p)\log_2(1-p)$$

Figure 4.4. Coordination Per Variable Required for Initialization

4.2.4 Summary of Situation Assessment Model

The following equations have been developed which describe the situation assessment stage of the decision making process.

- Amount of Internal Decision-Making:

$$G_n^I = H(u) \tag{4.42}$$

- Throughput:

$$G_t^I = H(\underline{z}) - H_{\underline{x}}(\underline{z}) \tag{4.43}$$

- Blockage:

$$G_b^I = H(\underline{x}) - G_t^I \tag{4.44}$$

- Coordination:

$$G_c^I = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) \tag{4.45}$$

The above equations are valid under the assumptions that the sets of algorithm variables are disjoint, that the algorithms themselves are deterministic, that no rejection occurs, and that $p(u)$ is independent of $p(\underline{x})$. In addition, if $p(\underline{x})$, $f_i(\underline{x})$, g_c^i, and α_i are known $(i = 1,\ldots,U)$, then the above equations are functions only of $p(u)$, the internal decision strategy.

In the next section, the complete two-stage process will be considered and expressions analogous to Eqs. (4.42) - (4.45) will be developed.

4.3 SITUATION ASSESSMENT AND RESPONSE SELECTION

In this section, the Partition Law of Information will be applied to the model that combines situation assessment and response selection. The development will take advantage of the similar structure of the two stages and will result in expressions which characterize the overall decision-making process as a function of two internal decision strategies.

4.3.1 System Definition

Recall the structure of the two-stage model defined in Section 4.1 and shown in Figure 4.1. In Section 4.2, additional structure was placed on the situation assessment algorithms by considering them to be composed of variables whose interconnections represented the steps taken in the algorithm. A similar structure will be assumed for the response selection algorithms, that is, associated with each algorithm is assumed to be a set of variables. The interconnections of these variables determine the input-output mapping of the algorithm. Suppose there are α_j' variables, $j = 1,2,\ldots,V$ associated with the mapping $h_j(\underline{z})$. Denote these variables by w_i^{U+j}, $i = 1,2,\ldots,\alpha_j'$ and denote the set of variables of algorithm j by

$$W^{U+j} = \left\{ w_1^{U+j}, w_2^{U+j}, \ldots, w_{\alpha_j'}^{U+j} \right\} \tag{4.46}$$

As with the situation assessment algorithms, assume that the sets W^{U+j} are mutually disjoint. The second stage of the decision-making process is then represented by the set of variables S^{II}, where

$$S^{II} = \{v, W^{U+1}, \ldots, W^{U+j}, \ldots, W^{U+V}, y\} \quad (4.47)$$

Finally, it will be assumed that each of the response selection algorithms has zero rejection.

It follows from the basic model structure that S, the set of variables of the complete model, is the union of the sets S^I and S^{II}, that is,

$$S = S^I \cup S^{II} = \{u, W^1, \ldots, W^U, \underline{z}, v, W^{U+1}, \ldots, W^{U+j}, \ldots, W^{U+V}, y\} \quad (4.48)$$

4.3.2 Analytic Expressions for Two Stage Model

The analytic expressions for system S are obtained following the procedure used for system S^I. Indeed, much of the discussion in the consideration of S^I is directly applicable and will be used to advantage where possible.

Amount of Internal Decision-Making

For system S, G_n is, by definition,

$$G_n = H_{\underline{x}}(u, W^1, \ldots, W^i, \ldots, W^U, \underline{z}, v, W^{U+1}, \ldots, W^{U+j}, \ldots, W^{U+V}, y), \quad (4.49)$$

which can be written as (see Appendix A)

$$G_n = H_{\underline{x}}(u) + H_{\underline{x},u}(v) + H_{\underline{x},u,v}(W^1, \ldots, W^U, \underline{z}, W^{U+1}, \ldots, W^{U+V}, y) \quad (4.50)$$

The last term in Eq. (4.46) is zero since the system is deterministic once $\underline{x}$, u, and v are known. Furthermore, since the internal decision strategy for the situation assessment stage has been assumed to be independent of the input $\underline{x}$, Eq. (4.46) can be written as

$$G_n = H(u) + H_{\underline{x},u}(v) \tag{4.51}$$

Consider the second term in Eq. (4.47). It is the uncertainty in the internal decision v when the previous decision u and the input $\underline{x}$ are known. By definition, $H_{\underline{x},u}(v)$ is

$$H_{\underline{x},u}(v) = -\sum_{u,\underline{x}} p(u,\underline{x}) \sum_{v} p(v|u\underline{x}) \log p(v|u\underline{x}) \tag{4.52}$$

The distribution $p(\underline{x})$ is external to the decision maker and the situation assessment strategy p(u) is independent of the input $\underline{x}$; hence $p(u,\underline{x}) = p(u)p(\underline{x})$. The remaining distribution, $p(v|u\underline{x})$, represents the internal decision strategy used to choose a response selection algorithm. Note that it depends in general on the choice of situation assessment algorithm and on the value of the input $\underline{x}$.

In contrast with the first stage decision strategies, a dependence of the second stage strategy on previous processing is essential; the appropriate conditioning is on the input to the second stage, the value of the assessed situation $\underline{z}$, which is determined by u and $\underline{x}$. Hence, it will be assumed that $p(v|u\underline{x}) \equiv p(v|\underline{z})$ in Eq. (4.52), and the amount of decision-making then reduces to

$$G_n = H(u) + H_{\underline{z}}(v) \tag{4.53}$$

Throughput

The throughput of the system is given by

$$G_t = T(\underline{x}:y) = H(y) - H_{\underline{x}}(y) \tag{4.54}$$

To evaluate Eq. (4.54) the distributions p(y), $p(y|\underline{x})$ and $p(\underline{x})$ are required. The application of Bayes' rule to p(y) and $p(y|\underline{x})$ yields

$$p(y) = \sum_{\underline{x}} \sum_{u} \sum_{\underline{z}} \sum_{v} p(y|v\underline{z})p(v|u\underline{x})p(\underline{z}|u\underline{x})p(u)p(\underline{x}) \tag{4.55}$$

$$p(y|\underline{x}) = \sum_{u}\sum_{v} p(y|\underline{x}uv)p(v|u\underline{x})p(u) \qquad (4.56)$$

The distributions $p(y|v\underline{z})$, $p(\underline{z}|u\underline{x})$, and $p(y|\underline{x}uv)$ are known through the mappings $f_i(\underline{x})$ and $h_j(\underline{z})$. The distribution $p(\underline{x})$ is assumed known. Therefore, specification of the internal decision strategies $p(u)$ and $p(v|u\underline{x}) \equiv p(v|\underline{z})$ determines $p(y)$ and $p(y|\underline{x})$, and hence determines G_t.

Blockage

Because the system has no rejection, and in view of the computability of G_t as an explicit function of the internal decision strategies, it is also possible to evaluate G_b as a function of these strategies:

$$G_b = H(\underline{x}) - G_t \qquad (4.57)$$

Coordination

To evaluate the coordination of the entire system S, the decomposition property of n-dimensional mutual information will be employed. Recall that S is composed of two disjoint subsystems S^I and S^{II}, and that, in general, the total coordination of a system of subsystems is equal to the sum of coordinations within each subsystem plus the coordination present among subsystems. The coordination of S is then given by

$$G_c = G_c^I + G_c^{II} + T(S^I:S^{II}), \qquad (4.58)$$

where G_c^I and G_c^{II} are the coordinations within S^I and S^{II}, respectively. G_c^I has been evaluated in Section 4.2:

$$G_c^I = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) \qquad (4.41)$$

The response selection stage is identical in structure to the situation assessment stage. Hence, a development similar to that for G^I gives

$$G_c^{II} = \sum_{j=1}^{V} p_j g_c^{U+j}(p(\underline{z}|v=j)) + \alpha_j' \mathcal{H}(p_j) + H(y) \tag{4.59}$$

where $p_j \stackrel{\Delta}{=} p(v=j)$, $j = 1,2,\ldots,V$. Note that because the response selection strategy is dependent on the value of $\underline{z}$, the characteristics of the input to each algorithm are determined in part by the strategy employed. This in turn influences the internal coordination of each algorithm, as represented by the functional dependence of g_c^{U+j} on $p(\underline{z}|v=j)$ shown in Eq. (4.59). The input distribution to the $j^{\underline{th}}$ algorithm can be evaluated, using Bayes rule, as

$$p(\underline{z}|v=j) = \frac{p(v=j|\underline{z})\,p(\underline{z})}{\sum_{\underline{z}} p(v=j|\underline{z})\,p(\underline{z})} \tag{4.60}$$

The final term in Eq. (4.59) is that of the coordination between the first and second stages of the model. By definition, it is given by

$$T(S^I:S^{II}) = H(S^{II}) - H_{S^I}(S^{II}) \tag{4.61}$$

Because the algorithms are deterministic and have no rejection, conditioning the uncertainty in S^{II} on knowledge of S^I is equivalent to conditioning on only the variables $\underline{x}$ and u, since together $\underline{x}$ and u determine every variable in S^I. Hence

$$T(S^I:S^{II}) = H(S^{II}) - H_{u,\underline{x}}(S^{II}) \tag{4.62}$$

Because of the symmetry of mutual information, Eq. (4.62) can also be expressed as

$$T(S^I:S^{II}) = H(u,\underline{x}) - H_{S^{II}}(u,\underline{x}) \tag{4.63}$$

Recall that the response selection algorithms have no rejection. Thus $\underline{z}$ is known within S^{II}. This fact, together with the dependence of v on u and

$\underline{x}$, leads to a simplification of the second term of Eq. (4.63):

$$H_{S^{II}}(u,\underline{x}) = H_{\underline{z},v}(u,\underline{x}) \tag{4.64}$$

The variables $\underline{z}$ and v incorporate all of the knowledge in S^{II} about the variables u and $\underline{x}$. In order to obtain a useful form of $T(S^I:S^{II})$, several information theoretic identities (see Appendix A) are applied in succession:

$$T(S^I:S^{II}) = H(u,\underline{x}) - H_{\underline{z},v}(u,\underline{x}) \tag{4.65}$$

$$= H(\underline{z},v) - H_{u,\underline{x}}(\underline{z},v) \tag{4.66}$$

$$= H(\underline{z}) + H_{\underline{z}}(v) - H_{u,\underline{x}}(\underline{z}) - H_{u,\underline{x},\underline{z}}(v) \tag{4.67}$$

$H_{u\underline{x}}(\underline{z}) = 0$ since $\underline{z}$ is a deterministic function of u and $\underline{x}$. In addition, $p(\bar{v}|u\underline{x}) \equiv p(v|\underline{z})$ implies that

$$H_{\underline{z}}(v) = H_{u,\underline{x},\underline{z}}(v) \tag{4.68}$$

and Eq. (4.67) reduces to

$$T(S^I:S^{II}) = H(\underline{z}) \tag{4.69}$$

Substitution of Eqs. (4.41), (4.59), and (4.69) into Eq. (4.58) gives

$$G_c = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) + \sum_{j=1}^{V} p_j g_c^{U+j}(p(\underline{z}|v=j)) + \alpha_j' \mathscr{H}(p_j) + H(y) + H(\underline{z}) \tag{4.70}$$

In words, the total coordination is equal to the sum of the internal coordination of each stage plus the coordination present between the two stages.

4.3.3 Summary and Discussion

The analytic expressions which describe the basic two stage model are given below.

- Amount of Internal Decision-Making:

$$G_n = H(u) + H_{\underline{z}}(v) \tag{4.71}$$

- Throughput:

$$G_t = T(\underline{x}:y) \tag{4.72}$$

- Blockage:

$$G_b = H(\underline{x}) - G_t \tag{4.73}$$

- Coordination:

$$G_c = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathcal{H}(p_i) + H(\underline{z})$$

$$+ \sum_{j=1}^{V} p_j g_c^{U+j} (p(\underline{z}|v=j)) + \alpha_j' \mathcal{H}(p_j) + H(y) + H(\underline{z}) \tag{4.74}$$

- Total Activity:

$$G = G_n + G_t + G_b + G_c \tag{4.75}$$

Eqs. (4.71) - (4.75) have been derived under the assumptions that the sets of algorithm variables are mutually disjoint, that the algorithms themselves are deterministic, and that each algorithm has no rejection. In addition, the situation assessment strategy has been assumed independent of the input, i.e., no preprocessing is assumed. The general dependencies of the response selection strategy have been retained, however. Finally, if $p(\underline{x})$, $f_i(\underline{x})$, $h_j(\underline{z})$, g_c^i, g_c^{U+j}, α_i, and α_j' are known $(i = 1,\ldots,U;$ $j = 1,\ldots,V)$, then Eqs. (4.70) - (4.75) are functions only of the internal decision strategies $p(u)$ and $p(v|\underline{z})$.

4.4 BOUNDED RATIONALITY AND PERFORMANCE EVALUATION

To complete the analytic characterization of the decision-making process model, it is necessary to consider constraints which reflect observed limitations, namely, the bounded rationality of decision makers. It is also appropriate to introduce a mechanism for the evaluation of the decision maker's performance.

4.4.1 Bounded Rationality

As discussed in Chapter 1, the notion of bounded rationality refers to the limited ability of the human being to process information. Considerable evidence has been collected in support of this hypothesis [22], [23, Chapter 6].

Since the present model is a characterization of the internal processing present in the determination of output from input, the qualitative notion of a limitation on processing ability translates readily into an analytic restriction on the total activity. The appropriate constraint is on the rate of total activity, however, because of the steady-state property of the expression for G in the present context. Denote this constraint by F, where F is expressed in bits per second. A constraint on the rate of total activity therefore requires

$$\frac{G}{\tau} \leq F, \tag{4.76}$$

where τ is the mean input symbol interarrival time. In other words, the boundedly rational decision maker is restricted from use of internal decision strategies which exceed his effective processing capacity, F.

Previous analytic representations of bounded rationality have been based on an input-output description of decision-making [9], [10], [12]. In that context, the appropriate analytic representation is of the form of an input-output capacity. While the present model includes an input-output quantity (G_t), the correspondence of the constraint of previous characterizations is with the rate of total activity constraint, F. In the present framework, if the model developed were only a throughput characterization of the process, i.e., $G_n = G_b = G_c = 0$, then the total activity rate constraint, F, would also be a rate of throughput constraint or input-output capacity and would correspond to previous representations. Additional constraints are possible [15] in the context of the present model; however, the justification of their use through empirical evidence is not possible at this time.

4.4.2 Performance Evaluation

While the PLI measures the total activity present in the decision making process, it offers little insight into the quality of performance obtained for a particular amount of total activity or resource expenditure [17], [23]. This can be illustrated by consideration of the two mappings shown in Figure 4.5. Each is a deterministic mapping of variable x into the variable y, where x takes values x_1, x_2, x_3 and y takes values y_1, y_2, y_3. The mutual information T(x:y) is given in each case by

$$T^{I}(x:y) = H(x) - H_y^{I}(x) \tag{4.77}$$

$$T^{II}(x:y) = H(x) - H_y^{II}(x) \tag{4.78}$$

Because each mapping is one-to-one, the last term in Eqs. (4.77) and (4.78) is zero, and

$$T^{I}(x:y) = T^{II}(x:y), \tag{4.79}$$

i.e., the mutual information between x and y is numerically the same for both cases. The qualitative nature of the two mappings is very different, however.

In terms of the present model, it may well be that two different decision strategies require the same amount of activity (resources), but yield

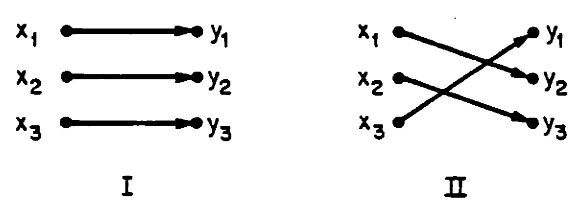


Figure 4.5. Two Deterministic Mappings

very different decision responses to the same input. In order to evaluate the quality of decision-making, then, it is necessary to compare the actual decision response with the one that should have been made. A mechanism for accomplishing this evaluation is shown in Figure 4.6. The desired decision response is represented by y' and is determined according to the function $L(x')$. Comparison of y' and y is made by the function $d(y,y')$, which assigns a value of cost e to each possible value of the difference between y' and y. The expected value of the cost can be determined for given strategies by averaging over the possible inputs, and thus a value of performance can be assigned to each pair of strategies $p(u)$ and $p(v|\underline{z})$. For example, if

$$e = d(y,y') = \begin{cases} 0 & y = y' \\ 1 & y \neq y' \end{cases} \tag{4.80}$$

the expected value of the cost is given by

$$E\{d(y,y')\} = 0 \cdot p(y = y') + 1 \cdot p(y \neq y') \tag{4.81}$$

which represents the probability of error in decision-making. The information obtained from performance evaluation can be used by the organization designer in defining and allocating tasks to the decision maker and in changing the number and contents of the situation assessment and response selection algorithms. This is achieved through training and learning; these processes, however, are outside the scope of this model, which describes a decision-making mechanism in the steady-state.

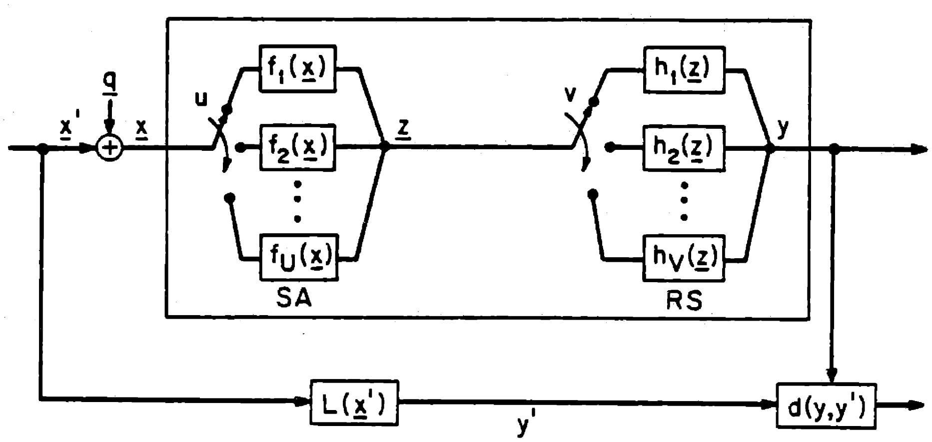


Figure 4.6. Model of Decision-Making Process With Performance Evaluation Mechanism

4.5 PROPERTIES OF MODEL

The two-stage model developed in this chapter can be used to consider several interesting questions. Two problems in particular will be stated in this section, one each in the normative and descriptive contexts: a) determination of optimal performance strategies and b) determination of those strategies which exceed a performance minimum. The investigation of the solutions to these problems will illustrate the properties of the model.

4.5.1 The Two-Stage Model

The two-stage model of the decision-making process is taken to be that shown in Figure 4.6, where the following are assumed to be specified:

- algorithms $f_i(\underline{x})$ and $h_j(\underline{z})$ with internal variables α_i and α'_j and internal coordinations g_c^i and g_c^{U+j}, respectively, which are known as a function of the characteristics of their respective inputs, $(i = 1,2,\ldots,U,\ j = 1,2,\ldots,V)$ (4.82)
- analytic characterization of the decision-making process as derived and described in Sections 4.2 - 4.3 and represented by Eqs. (4.71) - (4.75) (4.83)
- function $L(\underline{x}')$ that maps x' into y' (4.84)
- distribution $p(\underline{x}')$ from which a sample is drawn every τ seconds (4.85)
- distribution $p(\underline{q})$ from which a sample is drawn every τ seconds, simultaneously with $\underline{x}'$ (4.86)

Given the conditions (4.82) - (4.86), the additional specification of the cost function $d(y,y')$ completely characterizes the decision-making process model as a function of the internal decision strategies $p(u)$, $p(v|\underline{z})$. In particular, the expected cost can be evaluated as a function of decision strategy. Let

$$J(p(u),\ p(v|\underline{z})) = E\{d(y,y')\} \qquad (4.87)$$

The general problem of interest is then to investigate how the cost J varies according to strategy selection, where the set of possible strategies may or may not be restricted by capacity limitations, as discussed in Section 4.4.

4.5.2 Problem Statement

Two problems in particular will be considered in order to investigate the properties of the model. The cost function is chosen to be that given by Eq. (4.80), which yields

$$E\{d(y,y')\} = p(y \neq y') \qquad (4.88)$$

Eq. (4.87) then becomes

$$J(p(u),\ p(v|\underline{z})) = p(y \neq y') \qquad (4.89)$$

Finally, it will be assumed that the bounded rationality of the decision maker is represented by a total activity rate constraint, F, and

$$\frac{G}{\tau} \leq F \qquad (4.90)$$

is required. For the given conditions (4.82) - (4.86), G is determined as a function of the internal decision strategies; Eq. (4.89) can then be written equivalently as

$$G(p(u),\ p(v|\underline{z})) \leq F\tau \qquad (4.91)$$

With these additional specifications, a problem of interest in the normative context is

Problem I

Given conditions (4.82) - (4.86), (4.89), (4.91), determine p(u) and $p(v|\underline{z})$ such that

a) $J(p(u),\ p(v|\underline{z})) = p(y \neq y')$ is minimized

or

b) $J(p(u),\ p(v|\underline{z})) = p(y \neq y')$ is minimized subject to $G(p(u),\ p(v|\underline{z})) \leq F\tau$.

Problem I(a) is a minimization of the probability of error, while Problem I(b) is a determination of the minimum error strategies which do not exceed the decision maker's total activity capacity, i.e., do not overload him. Note that the solution to I(b) is in general a function of τ, and that the solution of I(a) corresponds to the case where $\tau \to \infty$.

As discussed in Chapter 2, a descriptive model of the decision maker leads to the characterization of his decision-making as satisficing rather than as optimizing. In analytic terms, satisficing can be taken to mean that a threshold cost has not been exceeded; for the model at hand, let $\bar{J}$ be that threshold. This gives the following problem of interest:

Problem II

Given conditions (4.82) - (4.86), (4.89), and (4.91), determine p(u) and $p(v|\underline{z})$ such that

a) $J(p(u),\ p(v|\underline{z})) \le \bar{J}$

or

b) $J(p(u),\ p(v|\underline{z})) \le \bar{J}$ subject to $G(p(u),\ p(v|\underline{z})) \le F\tau$.

The solutions to Problems II(a) and (b) represent strategies which satisfice. As with Problem I(b), the solutions to Problem II(b) are in general a function of τ.

The existence and characterization of solutions to Problems I and II are considered in the following sections.

4.5.3 Characterization of Solutions

Pure and Mixed Strategies

Before characterizing the solution to Problems I and II, it is necessary to consider the two classes of strategies, pure and mixed [24], which might arise as solutions. For convenience, the situation assessment decision strategy p(u) and the response selection decision strategy $p(v|\underline{z})$ will be referred to together as the decision strategy.

In general, the specification of a response selection strategy requires the specification of M distributions on v, where M is the number of values that $\underline{z}$ takes. If a response selection strategy is chosen such that there is an algorithm associated deterministically to each value of z, that is, for $m = 1, 2, \ldots, M$,

$$p(v = j | \underline{z} = \underline{z}_m) = 1 \quad \text{for some } j, \tag{4.92}$$

then a deterministic or pure response selection strategy has been employed. There exist $V \cdot M$ such strategies. Correspondingly, there are U pure situation assessment strategies, which give a total of $(U \cdot V \cdot M)$ possible pure strategies. Denote by D_K the $K^{\underline{th}}$ pure decision strategy, where $K = 1, 2, \ldots, (U \cdot V \cdot M)$.

All other possible strategies are said to be stochastic, or mixed strategies, and are represented by non-trivial distributions $p(u)$, $p(v|\underline{z})$. Every possible mixed strategy is obtained by a convex combination of pure strategies, that is, associated with each distribution $(p(u), p(v|\underline{z}))$ is a set of weights p_K such that

$$(p(u),\ p(v|\underline{z})) = \sum_{K=1}^{U \cdot V \cdot M} D_K p_K \overset{\Delta}{=} D(p_K) \tag{4.93}$$

where

$$\sum_{K=1}^{U \cdot V \cdot M} p_K = 1 \qquad p_K \geq 0 \quad \forall\ K \tag{4.94}$$

Problem I(a)

Characterization of the solutions to I(a) requires knowledge of the dependence of the probability of error on the decision strategy. This dependence can be seen by using Bayes' rule:

$$p(y \neq y') = \sum_{v, \underline{z}, u} p(y \neq y' | v\underline{z}u) p(v|\underline{z}) p(\underline{z}|u) p(u)$$

The distributions $p(y \neq y' | v\underline{z}u)$ and $p(\underline{z}|u)$ can be evaluated from known quantities in the model, and are fixed. Corresponding to each pure strategy D_K is a value of the probability of error, which can be obtained by substitution of D_K into Eq. (4.95). Denote these values by J_K. Since each possible mixed strategy is obtained by appropriate combination of pure strategies, the error associated with an arbitrary strategy is also obtained in the same fashion, i.e.,

$$J(p(u), p(v|\underline{z})) = \sum_{K=1}^{U \cdot V \cdot M} J_K p_K \qquad (4.96)$$

where particular values of p_K determine a particular strategy $(p(u), p(v|\underline{z})$. Minimization of Eq. (4.96) subject to (4.94) gives as solution a pure strategy corresponding to the minimum J_K. However, if the values of J_K are not distinct, it is possible that the minimizing pure strategy is not unique. Indeed, if there are two pure minimum error strategies, then any linear combination of these strategies is also a solution.

Problem I(b)

In order to describe the solution to I(b), it is necessary to determine the relationship between the performance J (error probability) and the total activity, G. This will be accomplished in two steps: a) the convexity of G in the decision strategy will be shown, and b) the decision strategy will be used parametrically to construct a plot of possible (J,G) pairs.

a) Convexity of G

To show the convexity of G in the decision strategy, it is necessary to show that

$$G((1-\delta)D(p_K^a) + \delta D(p_K^b)) \geq (1-\delta)G(D(p_K^a)) + \delta G(D(p_K^b)), \quad 0 \leq \delta \leq 1 \qquad (4.97)$$

where $D(p_K^a)$ and $D(p_K^b)$ represent two arbitrary decision strategies, a and b. Recall that the total activity can be evaluated as

$$G = \sum_{w} H(w) \tag{4.98}$$

where $\sum_{w} H(w)$ represents the sum of the marginal uncertainties of all system variables. Because the sum of convex functions is convex [25], in order to show the convexity of G in the decision strategy it is sufficient to show the convexity of the marginal uncertainty of each system variable.

Consider an arbitrary variable of the system w. The distribution p(w) can be written as

$$p(w) = \sum_{v,\underline{z},u} p(w|v\underline{z}u)p(v|\underline{z})p(\underline{z}|u)p(u) \tag{4.99}$$

where the quantities $p(w|v\underline{z}u)$ are fixed and known by specification of the model. Corresponding to each pure strategy is a distribution $p(w|D_K)$ which results by substitution of D_K into Eq. (4.99). The possible distributions on w which can arise by variations in decision strategy are therefore elements of the set given by

$$\Omega_w = \left\{ p(w) \,|\, p(w) = \sum_{K=1}^{U \cdot V \cdot M} p(w|D_K)p_K \right\}, \quad \text{where} \tag{4.100}$$

$$\sum_{K=1}^{U \cdot V \cdot M} p_K = 1, \; p_K \geq 0 \qquad \forall\, K$$

Ω_w represents a convex space of probability distributions. From information theory the following result holds concerning convex probability distribution spaces [17]:

If p(x) is an element of the convex space determined by

$$p(x) = (1-\delta)p_1(x) + \delta p_2(x), \qquad 0 \leq \delta \leq 1, \quad \text{then}$$

$$H(x) \geq (1-\delta)H^1(x) + \delta H^2(x)$$

where $H(x)$, $H^1(x)$, $H^2(x)$ are the marginal uncertainties obtained using $p(x)$, $p_1(x)$, and $p_2(x)$, respectively.

Let $p_a(w)$, $p_b(w)$ be distributions on w which result when strategies a and b are used, respectively; that is,

$$p_a(w) = \sum_{K=1}^{U \cdot V \cdot M} p(w|D_K)p_K^a \tag{4.101}$$

$$p_b(w) = \sum_{K=1}^{U \cdot V \cdot M} p(w|D_K)p_K^b \tag{4.102}$$

Because Ω_w is a convex space,

$$p(w) = (1-\delta)p_a(w) + \delta p_b(w) \qquad 0 \le \delta \le 1 \tag{4.103}$$

is an element of that space. Hence

$$H(w) \ge (1-\delta)H^a(w) + \delta H^b(w) \qquad 0 \le \delta \le 1 \tag{4.104}$$

for any variable w of the system, and

$$\sum_w H(w) \ge (1-\delta)\sum_w H^a(w) + \delta\sum_w H^b(w) \qquad 0 \le \delta \le 1 \tag{4.105}$$

Eq. (4.105) is equivalent to Eq. (4.97), however, and the desired result has been shown.

b) Construction of (J,G) Plot

To obtain a characterization of the possible (J,G) pairs, it is convenient to first restrict consideration to the case of strategies which are a binary variation between pure strategies. Extension of the resulting description to the case of arbitrary strategies then follows directly.

Consider the decision strategies $D(\delta)$ obtained when

$$p_K = \begin{cases} 1-\delta & K=1 \\ \delta & K=2 \\ 0 & \text{otherwise} \end{cases} \quad , \; 0 \le \delta \le 1 \tag{4.106}$$

where

$$D(\delta) \triangleq (1-\delta)D_1 + \delta D_2 \tag{4.107}$$

Denote the total activities corresponding to D_1 and D_2 by G_1 and G_2, respectively. From Eqs. (4.96) and (4.97) it is true that

$$G(D(\delta)) \ge (1-\delta)G_1 + \delta G_2 \tag{4.108}$$

and

$$J(D(\delta)) = (1-\delta)J_1 + \delta J_2 \tag{4.109}$$

Eqs. (4.108) and (4.109) are parametric in δ and can be used to describe the relationship of G and J, which is represented in Figure 4.7. The relative orientation of the points (J_1,G_1) and (J_2,G_2) is arbitrary, i.e., it is not true in general that the lowest total activity also realizes the worst performance.

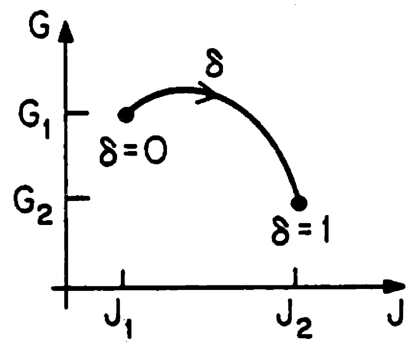


Figure 4.7. Representation of G vs. J for Binary Variation of Pure Strategies

Application of the above construction to all possible binary variations between pure strategies yields a locus of (J,G) pairs, as represented in Figure 4.8 for the case of three pure strategies. Since all possible

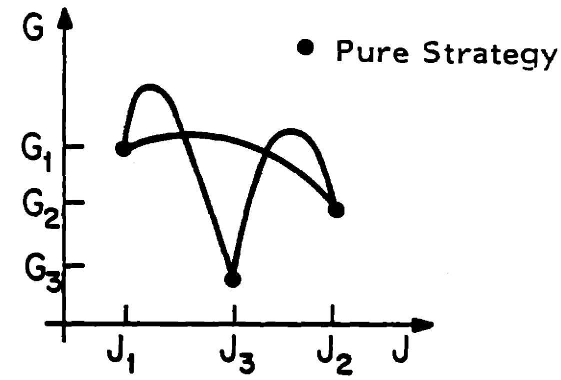


Figure 4.8. Representation of Locus of (J,G) Pairs for Binary Variation Among Three Pure Strategies

mixed strategies can be obtained by successive binary combinations of mixed strategies, the region of all (J,G) pairs is also readily obtained and is typified by the representation in Figure 4.9. Characteristic of such

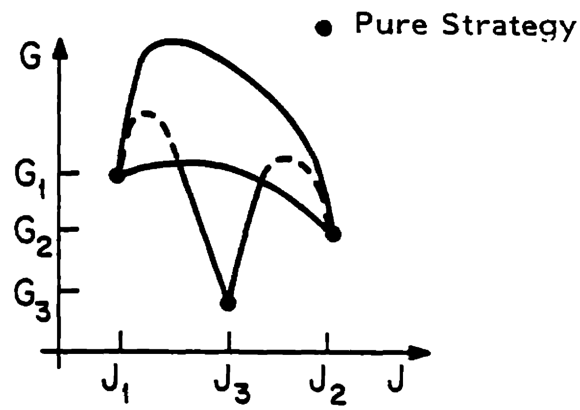


Figure 4.9. Representation of Possible (J,G) Pairs

a region is that the strategy corresponding to the minimum value of G for a particular J is either a pure strategy or a binary variation between pure strategies. Let the possible (J,G) pairs be denoted by R, i.e.,

$$R = \left\{ (J,G) \mid G = G(D(p_K)),\ J = J(D(p_K)) \right\} \tag{4.110}$$

Recall that in Problem I(b), the minimum error strategy is sought such that the decision maker's total activity capacity is not exceeded. The points (J,G) which correspond to strategies which do not exceed this constraint are given by the intersection of R and the set of points determined by

$$\{(J,G) \mid G \leq F\tau\} \tag{4.111}$$

Depending on the value of τ, two types of intersection are possible, as shown in Figure 4.10. For $\tau = \tau_1$, the minimum error is achieved by the (pure) strategy corresponding to the point (J_c, G_c), which is also the solution obtained to Problem I(a). However, as τ decreases, it may no longer be possible to use the optimal strategy, as illustrated in Figure 4.10 for $\tau = \tau_2$. In that case, the minimum error strategy is in general a mixed strategy, and for the present model is a binary variation between pure strategies.

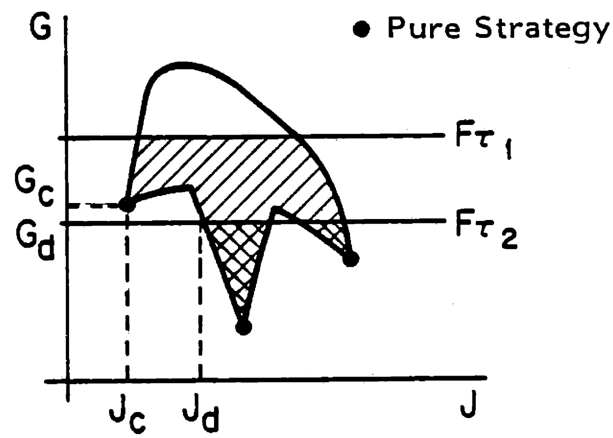


Figure 4.10. Characterization of Solutions to Problem I(b)

Problem II(a)

From the discussion of solutions to Problem I(a), the solutions to Problem II(a) can be characterized as the set of feasible solutions p_K to

$$\left.\begin{aligned} &\bar{J} \geq \sum_{K=1}^{U\cdot V\cdot M} J_K p_K \\ &\sum_{K=1}^{U\cdot V\cdot M} p_K = 1, \quad p_K \geq 0 \quad \forall\ K \end{aligned}\right\} \tag{4.112}$$

The solution set determines a partition of the region R as shown in Figure 4.11, which illustrates that while an infinite number of strategies will

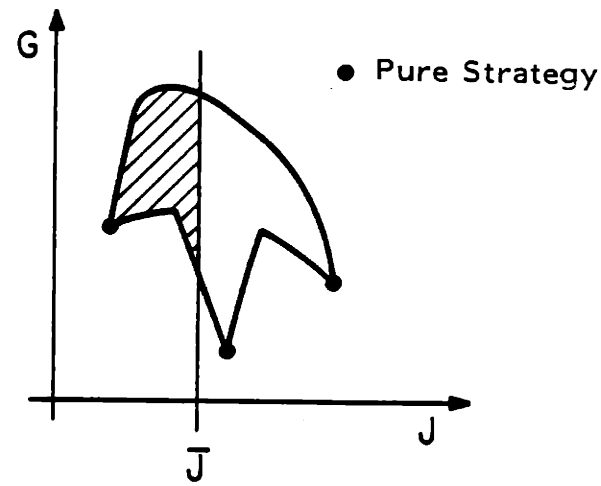


Figure 4.11. Region of Satisficing (J,G) Pairs

satisfice, the difference in total activity required for each can be significant. Note that for $\bar{J} < \min_K J_K$, no solution exists.

Problem II(b)

The solution to Problem II(b) is obtained readily as the set of strategies, represented by values of p_K, which give (J,G) pairs in the region defined by

$$R \cap \{(J,G) \mid G \leq F\tau\} \cap \{(J,G) \mid J \leq \bar{J}\} \tag{4.113}$$

Several types of intersection are possible, as illustrated in Figure 4.12. For τ sufficiently large, the set of satisficing strategies includes the minimum error strategy ($\tau = \tau_3$ in Figure 4.12). As τ decreases, however, the solution set may contain only mixed strategies ($\tau = \tau_4$), that is, satisficing performance can be achieved only by using strategies which involve non-zero amounts of internal decision-making, i.e., $G_n \neq 0$. A third type of intersection occurs when τ is decreased sufficiently so that the solution set is empty ($\tau = \tau_5$).

As τ decreases, there exists in general some value of $\tau = \tau_0$ below which the solution set is empty. A decreasing value of τ corresponds to an

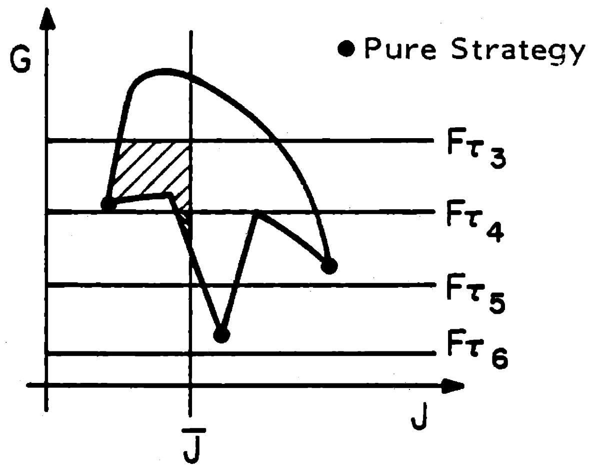


Figure 4.12. Representation of Possible Solutions to Problem II(b)

increasing rate of input arrivals, and $\tau < \tau_0$ corresponds to the case where inputs are arriving at a rate too fast for adequate processing. Such a condition represents an <u>overload</u> of the decision maker. It is possible that the decision task can still be accomplished, however, by using a strategy which realizes a higher than acceptable cost, but in general there exist values of τ for which no amount of performance compromise will avoid overload ($\tau = \tau_6$ in Figure 4.12) and the decision-making task cannot be accomplished in that event.

The above analysis has shown that the minimum probability of error is realized by a pure strategy; however, because of capacity constraints, the minimum error probability achievable by the decision maker may be realized by a mixed strategy. In the descriptive context, the set of strategies which achieves a performance minimum has been characterized, and for the case of a constrained decision maker, it was shown that such a set might contain only mixed strategies.

4.5.4 Other Properties of Model

Two other characteristics of the model are of particular interest: algorithm switching may be due entirely to external influences, and algorithm reinitialization activity may be a significant fraction of the total activity G. Both are illustrated by the following example.

Suppose that the response selection stage contains two algorithms, $h_1(\cdot)$ and $h_2(\cdot)$. Furthermore, assume that the situation assessment stage consists of a single algorithm and that it maps the inputs $\underline{x}$ into two possible values of $\underline{z}$ according to

$$p(\underline{z}) = \begin{cases} 1-\gamma & \underline{z} = \underline{z}_1 \\ \gamma & \underline{z} = \underline{z}_2 \end{cases} \qquad 0 \le \gamma \le 1 \tag{4.114}$$

Let the pure strategy D_1 be employed, where

$$D_1 = \begin{cases} p(v=1 \mid \underline{z}=\underline{z}_1) = 1 \\ p(v=2 \mid \underline{z}=\underline{z}_2) = 1 \end{cases} \tag{4.115}$$

i.e., $\underline{z}_1$ is always used in $h_1(\cdot)$ and $\underline{z}_2$ is always used in $h_2(\cdot)$. The model of the process is illustrated in Figure 4.13 and because of the strategy employed,

$$G_n = 0 \tag{4.116}$$

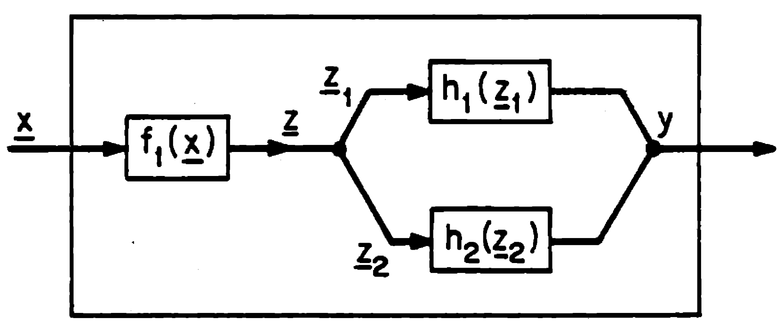


Figure 4.13. Deterministic Switching

However, it is also true that

$$p(v) = \begin{cases} 1-\gamma & v = 1 \\ \gamma & v = 2 \end{cases} \tag{4.117}$$

which illustrates that pure strategies do not necessarily imply fixed algorithm usage. The switching among second stage algorithms in this case is due entirely to variations in the value of $\underline{z}$, which derive ultimately from variations in the input $\underline{x}$; hence it is a switching which is driven totally from outside the decision maker.

A second interesting phenomenon present in the model can be illustrated by consideration of the total activity which describes the process shown in Figure 4.13. In particular, consider the coordination within the response selection stage:

$$G_c^{II} = \sum_{j=1}^{2} p_j g_c^{U+j} (p(\underline{z}|v=j)) + \alpha_j' \mathcal{H}(p_j) + H(y) \tag{4.118}$$

The input distributions to $h_1(\cdot)$ and $h_2(\cdot)$ are given by

$$p(\underline{z}|v=1) = \begin{cases} 1 & \underline{z} = \underline{z}_1 \\ 0 & \underline{z} = \underline{z}_2 \end{cases} \tag{4.119}$$

$$p(\underline{z}|v=2) = \begin{cases} 0 & \underline{z} = \underline{z}_1 \\ 1 & \underline{z} = \underline{z}_2 \end{cases} \tag{4.120}$$

i.e., they are deterministic from the point of view of the algorithms. The internal coordinations g_c^2 and g_c^3 are therefore zero and Eq. (4.118) reduces to

$$G_c^{II} = \alpha_1' \mathcal{H}(1-\gamma) + \alpha_2' \mathcal{H}(\gamma) + H(y) \tag{4.121}$$

and the activity within the second stage is due almost entirely to the reinitialization of algorithms, which activity may be a significant part of the total if the numbers of algorithm variables, α_1' and α_2', are large. This phenomenon reflects and reinforces the notion that regardless of the simplicity of a particular task or procedure, there is still required a certain amount of effort to switch attention or become oriented [23].

4.6 CHAPTER SUMMARY

In this chapter, a structure has been presented for modeling internal decision-making in the execution of a well-defined task. It consists of two stages. In the first, a measurement of the state of the decision maker's environment is taken as input and from it an assessment of the situation is made. In the second stage, a decision response is selected which is appropriate to the assessed situation. To accomplish the processing required in each stage, the decision maker, who is well-trained in the performance of his task, is modeled as possessing a set of well-defined procedures or algorithms, among which a choice is made for each input.

The Partition Law of Information has been applied to this model, with the result that analytic expressions which characterize the total activity of this process have been developed which are explicit functions of the internal choices. These expressions divide the total activity into that of throughput, blockage, coordination, and the amount of internal decision-making. In order that an evaluation of the decision maker's performance with respect to the assigned task might be made, a performance measure and a procedure for evaluating it are introduced. Finally, the limited ability of the human decision maker to process information is modeled as a constraint on the overall activity rate.

The general expressions developed were specialized in Section 4.5 to the case where the performance measure is the probability of error in decision-making, and where a possible total activity capacity constraint exists. Interpretation of the relative frequency of choice as the decision strategy was made, and two classes of decision strategies were distinguished: pure and mixed, corresponding to the cases of zero and non-zero amounts of internal decision-making, respectively. It was shown that the total activity of the process is a convex function of the decision strategy. In the normative context, it was shown that the minimum error probability is realized by a pure strategy; a capacity limitation may dictate that the minimizing strategy be mixed, however. Similarly, in the descriptive context, a characterization of those strategies which realized a minimum acceptable level of performance (satisficing) was obtained, and it was seen that the additional consideration of a capacity constraint may restrict the set of satisficing strategies such that only mixed strategies are included. It was observed by example that pure strategies do not necessarily fix the choice of algorithm, rather, a switching can occur due to variations in input. Finally, it was shown that the activity required for reinitialization of algorithms can be a significant fraction of the total.

Subsequent theoretical development in this thesis (Chapter 6) will extend the model to include possible interactions which might occur in an organization. In the next chapter, an example is presented that illustrates many of the features of the analysis presented so far.

CHAPTER 5
ILLUSTRATION OF MODEL

5.1 INTRODUCTION

An example which illustrates several of the properties of the two-stage model is presented. In particular, construction of a representative (J,G) region is made and the pure and mixed strategies as solutions to Problems I and II of Chapter 4 are illustrated. A clear indication is given that total activity, as described by the present model, rather than throughput alone, is a more appropriate quantity by which to describe bounded rationality. In addition, the significant influence on the total of the activity required to switch among options is illustrated.

The example presented is not intended as an application of the model to an actual decision-making task. Actual application requires additional considerations in task definition which are beyond the scope of this thesis. The intent is to demonstrate how simple input-output task description can be viewed as a process, and how internal choices made in that process can influence the performance and total activity required.

5.2 INPUT-OUTPUT TASK DEFINITION

To define the task of the decision maker, it is necessary to define (see Figure 4.6) the characteristics of the input $\underline{x}'$, to specify the possible output decision responses y', and finally to define the desired mapping between input and output $L(\underline{x}')$. Once this characterization is complete, a particular process of mapping input to output by the decision maker can be specified and the performance of the task by that process can be evaluated. In this section, the input characteristics to the decision maker are defined and the desired mapping $L(\underline{x}')$ of inputs to outputs is also specified.

5.2.1 General Description

The environment of the decision maker is such that coplanar line segments are generated independently every τ seconds. These segments are completely characterized by their endpoints, which are taken as the state of the decision maker's environment $\underline{x}'$. Three possible decision responses are assumed and the desired association of each possible line segment to the possible decision responses is made based on the orientation and normalized length of the line segment, both of which can be determined from its endpoints. For simplicity, it is assumed that all lines which contain the generated segments pass through a common point, which determines the origin of a coordinate system.

5.2.2 Analytic Description

Input From Environment

The line segments are defined by their endpoints (λ_1',μ_1'), (λ_2',μ_2') which are generated according to

$$\left.\begin{aligned} \lambda_1' &= m_1 t_1 \qquad & \mu_1' &= m_2 t_1 \\ \lambda_2' &= m_1 t_2 \qquad & \mu_2' &= m_2 t_2 \end{aligned}\right\} \tag{5.1}$$

where t_1 and t_2 are fixed parameters $(t_2 > t_1 > 0)$ and m_1, m_2 are independent, positive random variables which take a finite number of values with uniform probability. All possible line segments therefore occur with equal probability. The input vector $\underline{x}'$ is a random vector defined to be

$$\underline{x}' \triangleq \begin{bmatrix} \lambda_1' \\ \mu_1' \\ \lambda_2' \\ \mu_2' \end{bmatrix} \tag{5.2}$$

It is assumed that no m_1,m_2 combination gives a slope greater than 1.

Desired Input-Output Mapping

Three possible decision responses are assumed: $y' \in \{a,b,c\}$. The particular response desired for each line segment is based on the orientation of the line segment and its normalized length (ν), which are determined as shown in Figure 5.1. The analytic description of the mapping $L(\underline{x}')$ is given below, where the quantities c_1, c_2, and d_1 are fixed parameters chosen to give a decision response mapping such as that shown in Figure 5.2. (The upper boundary at 45° is by assumption; no m_1,m_2 pair gives slope greater than 1.)

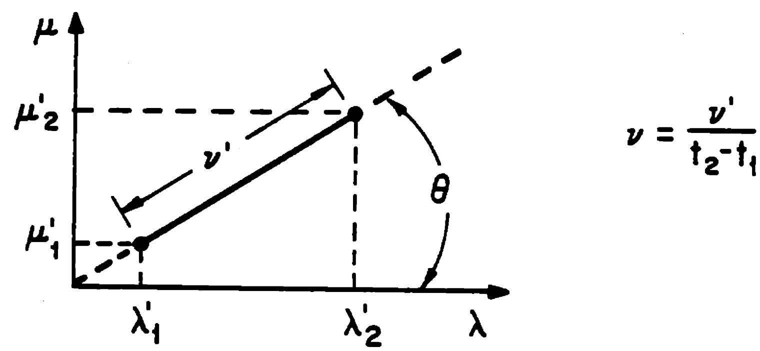


Figure 5.1. Determination of Orientation (θ) and Normalized Length (ν)

$$L(\underline{x}') : y' = \begin{cases} a & \nu \geq c_1(1-d_1\theta) \\ b & \nu \leq c_2(1-d_1\theta) \\ c & \text{otherwise} \end{cases} \tag{5.3}$$

Actual Input to Decision Maker

Recall that the decision maker does not receive $\underline{x}'$ directly, however, because of the corruption by noise represented by the variable $\underline{q}$. The actual input is $\underline{x}$, which is obtained by

$$\underline{x} = \underline{x}' + \underline{q} \tag{5.4}$$

where $\underline{q}$ is a 4-dimensional zero-mean, random vector with 4 independent elements. In analytic terms, the task of the decision maker is to select a response $y \in \{a,b,c\}$, based on the noisy measurement $\underline{x}$, which matches

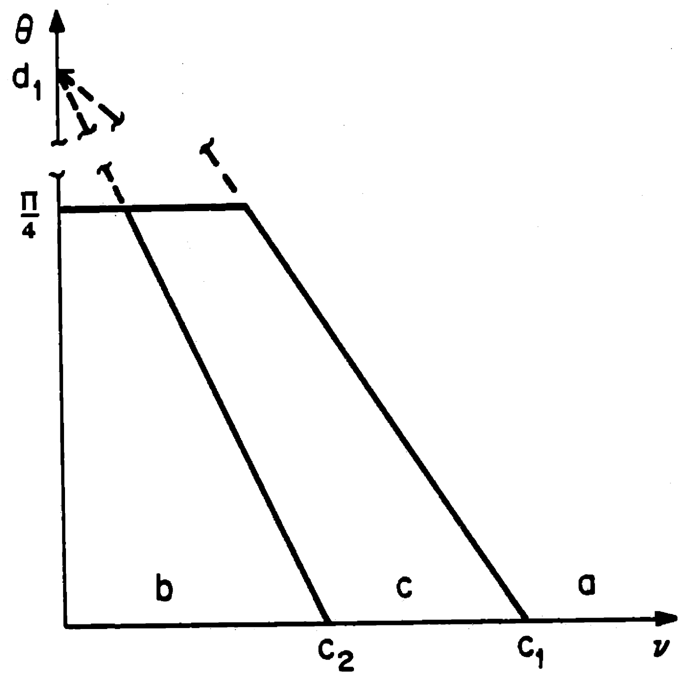


Figure 5.2. Desired Decision Mapping $L(\underline{x}')$

the desired response y' associated with the actual state $\underline{x}'$. Values chosen for the alphabets of the random variables m_1, m_2, and $\underline{q}$, as well as the values of the appropriate parameters, are given in Appendix C.

5.3 PROCESS DEFINITION

5.3.1 General Algorithm Characteristics

The decision-making process which maps inputs $\underline{x}$ into decision responses y is assumed to consist of a single situation assessment algorithm and three response selection algorithms, among which a choice is made according to the variable v as shown in Figure 5.3. For the example at hand, the appropriate variables which characterize the situation, i.e., contain all the information necessary to determine the decision response, are the variables θ and ν. Correspondingly, the situation assessment algorithm maps inputs $\underline{x}$ into estimates of θ and ν, which are then forwarded

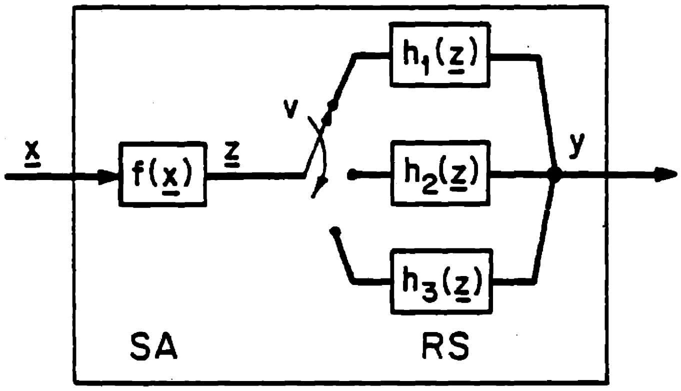


Figure 5.3. Example Decision-Making Process

to the response selection stage, that is,

$$\underline{z} \triangleq \begin{bmatrix} \hat{\theta} \\ \hat{\nu} \end{bmatrix} = f(\underline{x}) \tag{5.5}$$

The estimates $\hat{\nu}$ and $\hat{\theta}$ are obtained by the situation assessment algorithm using standard sum of squares and least squares techniques, respectively.

The three response selection algorithms are constructed to be of varying quality in terms of performance. It also happens that in this case better performance requires greater total activity (G), although this need not be true in general. The best algorithm, $h_3(\cdot)$, uses the actual decision mapping regions defined by (5.3), except that the estimates $\hat{\nu}$ and $\hat{\theta}$ are used instead of θ and ν. The worst algorithm of the three in performance, $h_1(\cdot)$, determines a decision response according to the estimate $\hat{\theta}$ only:

$$h_1(\cdot) \; : \; y = \begin{cases} a & \hat{\theta} > \beta \\ c & \text{otherwise} \end{cases} , \tag{5.5}$$

that is, the decision response is determined by the comparison of $\hat{\theta}$ with a fixed threshold β. The remaining algorithm, $h_2(\cdot)$, corresponds to a table lookup and is accomplished as a two-dimensional search. The possible values of (ν,θ) are partitioned and values of y assigned as shown in Figure 5.4, where the quantities c_i and d_j are fixed thresholds.

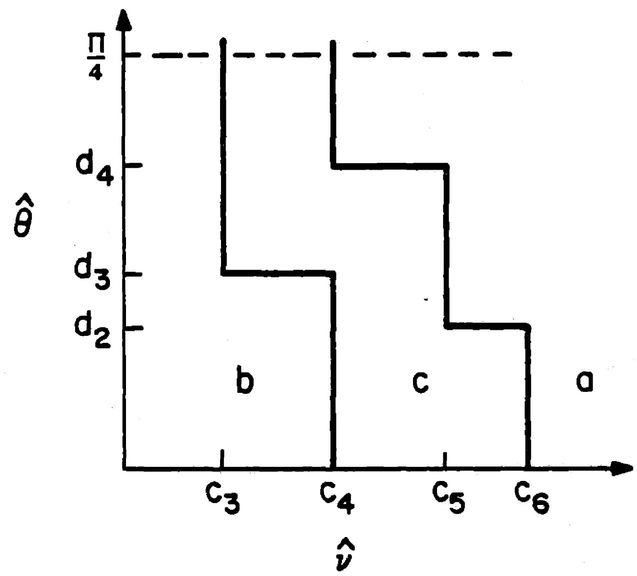


Figure 5.4. Algorithm $h_2(\cdot)$: Look-up Table

The relative characteristics of the respective response selection algorithms for the parameters chosen are shown in Figure 5.5 together with the possible (ν,θ) pairs which were generated as inputs. Numerical values of the parameters used are given in Appendix C.

5.3.2 Variable Definition

In this section, the variables of algorithm $h_3(\cdot)$ are defined. Variable definition for the other algorithms proceeds similarly to that for $h_3(\cdot)$; the algorithm variables for $f(\cdot)$, $h_1(\cdot)$ and $h_2(\cdot)$ are presented in Appendix C.

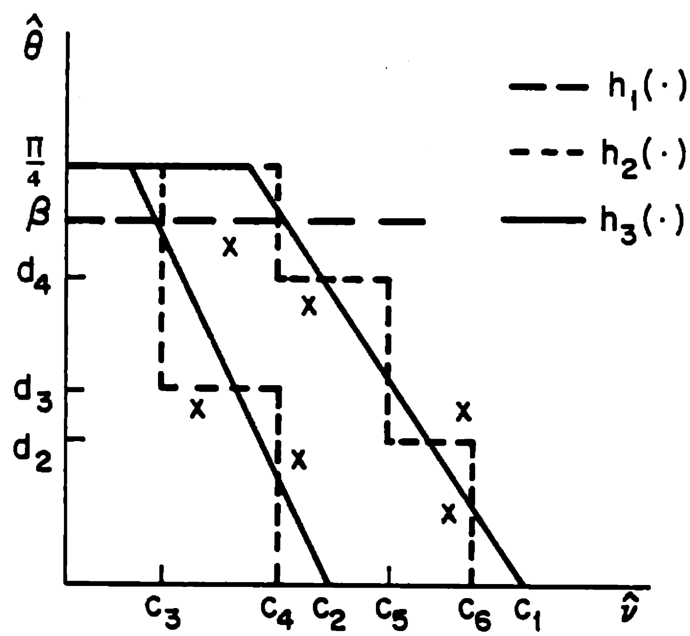


Figure 5.5. Relative Performance Characteristics of Response Selection Algorithms

Algorithm $h_3(\cdot)$ receives values of $\hat{\theta}$ and $\hat{\nu}$ as contained in the vector $\underline{z}$ and determines the decision response according to (5.3). The processing accomplished by algorithm $h_3(\cdot)$ is determined according to the internal variables of the set $W^{U+3} = W^4$ $(U = 1$ in this example) and the interconnection of these variables. The internal variables are defined according to the binary operations present: addition, multiplication, and comparison. They are given in Figure 5.6; note that $\alpha_1' = 13$. Several features of the variable definition are of particular interest. Since the characteristics of the input to each response selection algorithm may vary according to the particular strategy used, the input to algorithm $h_3(\cdot)$ is denoted by the superscript 3 (variables $\hat{\theta}^3$ and $\hat{\nu}^3$). Secondly, the constants c_1, c_2, d_1 and 1 are considered as variables of the algorithm since any reinitialization of the algorithm must also switch their status from inactive to active. Finally, the algorithm includes a two-level comparison tree, which defines additional variables (see the example of Chapter 3).

$w_1^4 = \hat{\theta}^3$ $\quad w_4^4 = c_1$ $\quad w_7^4 = 1$ $\quad w_{10}^4 = c_2(1-d_1 \cdot \hat{\theta}^3)$

$w_2^4 = \hat{\nu}^3$ $\quad w_5^4 = c_2$ $\quad w_8^4 = 1-d_1 \cdot \hat{\theta}^3$ $\quad w_{13}^4 = y^3$

$w_3^4 = d_1$ $\quad w_6^4 = d_1 \cdot \hat{\theta}^3$ $\quad w_9^4 = c_1(1-d_1 \cdot \hat{\theta}^3)$

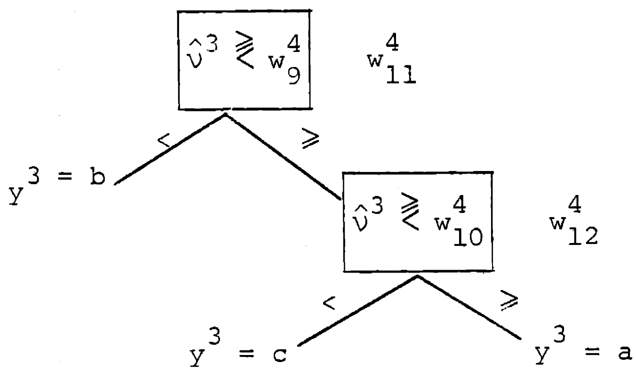


Figure 5.6. Internal Variables of Algorithm $h_3(\cdot)$

5.3.3 Computability

In order to evaluate the various quantities in the PLI, it is necessary to determine the probability distributions for each algorithm variable. Given the internal variable definition for a particular algorithm and the implied interconnection of variables which is determined by that definition, this is indeed possible if the input characteristics to the algorithm are known. To illustrate this, consider the internal variables of algorithm $h_3(\cdot)$ as defined in Figure 5.6. The variables w_3^4, w_4^4, w_5^4, and w_7^4 are constant; hence their distributions are known. Because of this fact, the variables w_6^4, w_8^4, w_9^4, and w_{10}^4 have the same distribution as $\hat{\theta}^3$, with appropriate re-labeling of values according to respective variable alphabets. Finally, the distributions on w_{11}^4 and w_{12}^4 are known if those on $\hat{\theta}^3$ and $\hat{\nu}^3$ are known, and the same is true for $p(y^3)$. Therefore, all internal variable probability distributions can be computed once the input distribution $p(\hat{\theta}^3, \hat{\nu}^3)$ is specified. The input to the response selection algorithms is determined by the output of the situation assessment stage and the strategy $p(v|\underline{z})$. The latter is specified, and the former is determined from internal variable definition of the situation assessment algorithm and the characteristics of its input $p(\underline{x})$. These quantities are all well-defined and hence the model is well-defined in that the probability distribution on each internal variable is computable.

5.4 DEMONSTRATION OF PROPERTIES

5.4.1 Admissible Strategies and Performance Measure

To demonstrate the properties of the model developed in Chapter 4, it is sufficient to consider only three pure strategies and their possible combinations. Accordingly, the admissible set of strategies is chosen as $p(v|\underline{z}) \equiv p(v)$, which includes the three pure strategies in which a single algorithm is used exclusively.

In addition, the cost function is chosen to be that of (4.80), i.e., the expected cost is the probability of error in decision-making.

5.4.2 Pure Strategies

Denote the pure strategies by D_K, where $K = 1,2,3$ and let D_K correspond to $h_K(\cdot)$, that is, D_K is the pure strategy corresponding to exclusive use of the $K^{\underline{th}}$ response selection algorithm. The analytic expressions which characterize the model for use of D_3 are given by (units are bits)

- Amount of Internal Decision-Making

$$G_n = H(v)$$

$$G_n(D_3) = 0$$

- Throughput

$$G_t = T(\underline{x}:y)$$

$$G_t = H(y) - H_{\underline{x}}(y)$$

$$G_t(D_3) = 1.4 - 0 = 1.4$$

- Blockage

$$G_b = H(\underline{x}) - G_t$$

$$G_b(D_3) = 8.9 - 1.4 = 7.5$$

- Coordination

$$G_c = g_c^1 + H(\underline{z}) + \sum_{j=1}^{3} p_j g_c^{1+j}(p(\underline{z}|v=j)) + \alpha_j^! \mathscr{H}(p_j) + H(y) + H(\underline{z})$$

$$G_c(D_3) = g_c^1 + H(\underline{z}) + g_c^4(p(\underline{z}|v=3)) + 13 \cdot \mathscr{H}(1) + H(y) + H(\underline{z})$$

$$G_c(D_3) = 91.6 + 4.5 + 14.1 + 0 + 1.4 + 4.5$$

$$G_c(D_3) = 114.1$$

The use of strategies D_1 and D_2 yields similar evaluations of the respective quantities:

$$G_n(D_1) = 0$$

$$G_t(D_1) = 0.3 - 0 = 0.3$$

$$G_b(D_1) = 8.9 - 0.3 = 8.6$$

$$G_c(D_1) = g_c^1 + H(\underline{z}) + g_c^2(p(\underline{z}|v=1)) + 5 \cdot \mathscr{H}(1) + H(y) + H(\underline{z})$$

$$= 91.6 + 4.5 + 1.6 + 0 + 0.3 + 4.5$$

$$= 102.5$$

$$G_n(D_2) = 0$$

$$G_t(D_2) = 1.2 - 0 = 1.2$$

$$G_b(D_2) = 8.9 - 1.2 = 7.7$$

$$\begin{aligned} G_c(D_2) &= g_c^1 + H(\underline{z}) + g_c^3(p(\underline{z}|v=2)) + 19 \cdot \mathscr{H}(1) + H(y) + H(\underline{z}) \\ &= 91.6 + 4.5 + 8.1 \quad + \quad 0 \quad + \quad 1.2 + 4.5 \\ &= 109.9 \end{aligned}$$

In the expressions above, it is interesting to note that variations in G_c are due primarily to the differences in the internal coordinations of the response selection algorithms. Indeed, in each case the contribution of the first stage is the same (100.6), and the remaining contribution is due almost entirely to g_c^{U+j}. As indicated in this example, the internal coordinations of the algorithm tend to dominate the contributions made by the $H(\underline{z})$ and $H(y)$ terms. The results given above are summarized in Table 5.1 together with the performance achieved.

Table 5.1 CHARACTERIZATION OF MODEL USING PURE STRATEGIES

STRATEGY	G_n	G_t	G_b	G_c	G	J
D_1	0	0.3	8.6	102.5	111.4	0.38
D_2	0	1.2	7.7	109.9	118.8	0.25
D_3	0	1.4	7.5	114.1	125.0	0.23

Two features of the model are evident in the table. The first is that the coordination activity dominates the total, and the second is that the throughput activity is an insignificant fraction of the total. The interpretation of this phenomenon is based on the following observation. Many decisions made are of the yes-no type, which corresponds to a (maximum) throughput of 1 bit. This characterization does not necessarily reflect the fact that considerable effort may have been expended in reaching the final decision (yes or no), however, and indicates that G_t is perhaps an incomplete characterization [15], [23]. The presence of the coordination term seems to account for the actual effort which takes place internally; in this example, even though only three final outcomes were possible, a great deal of computation was required to select a particular outcome as reflected in G_c.

5.4.3 Mixed Strategies

The use of mixed strategies which are binary variations of pure strategies yields the (J,G) pairs plotted in Figure 5.7. Denote by $D(\delta)$ the mixed strategy which is a binary variation between D_1 and D_2, i.e.,

$$D(\delta) = (1-\delta)D_1 + \delta D_2 \qquad 0 \le \delta \le 1 \tag{5.6}$$

It can be shown (see Appendix C) that the amount of additional total

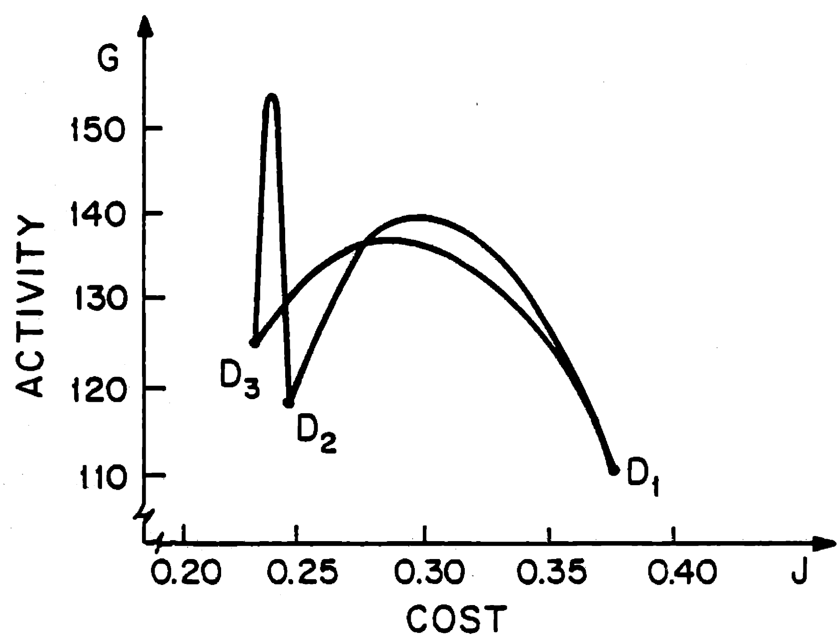


Figure 5.7. (J,G) Pairs for Mixed Strategies Which are Binary Variations of Pure Strategies

activity required for the mixed strategy $D(\delta)$ beyond that expected by a simple weighted average of G_1 and G_2 is due almost entirely to the activity required for re-initialization of algorithms, i.e.,

$$G(\delta) - [(1-\delta)G_1 + \delta G_2] \approx (13 + 19)\mathcal{H}(\delta) \qquad (5.7)$$

For binary variations between D_1 and D_2, $\delta = 0.5$ represents nearly a 25% increase in total activity in this example. This follows from the analysis in Chapter 4 and demonstrates that switching among algorithms can account for a significant fraction of the total activity.

The entire (J,G) region can be determined by all possible binary variations between pure and mixed strategies and in the present case results in the region shown in Figure 5.8.

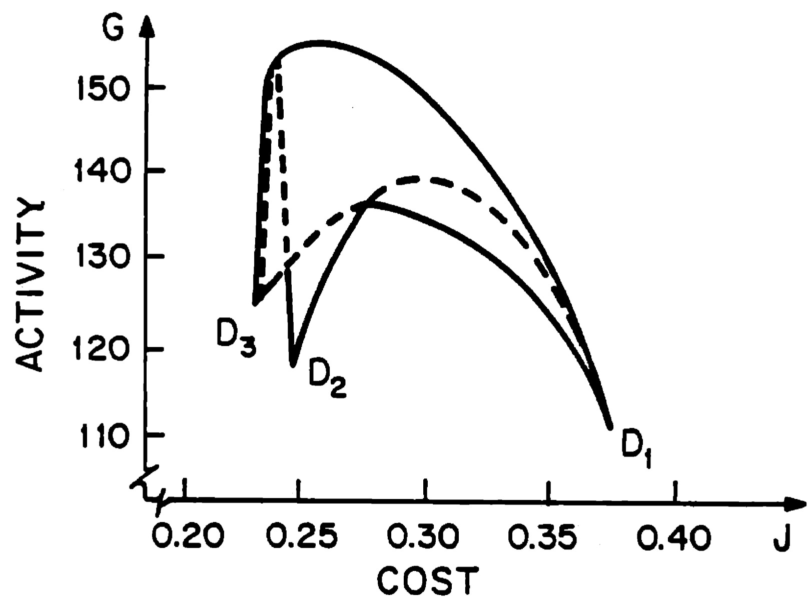


Figure 5.8. Region of Possible (J,G) Pairs

Recall Problems I and II of Chapter 4. Let $\bar{J}$, the threshold of satisficing performance, and F, the rate of total activity constraint, be 0.26 and 125 (bits per second), respectively. The total activity constraint for several values of τ is shown in Figure 5.9. The minimum possible error probability is given by the (pure) strategy D_1 (Problem I(a)), and for

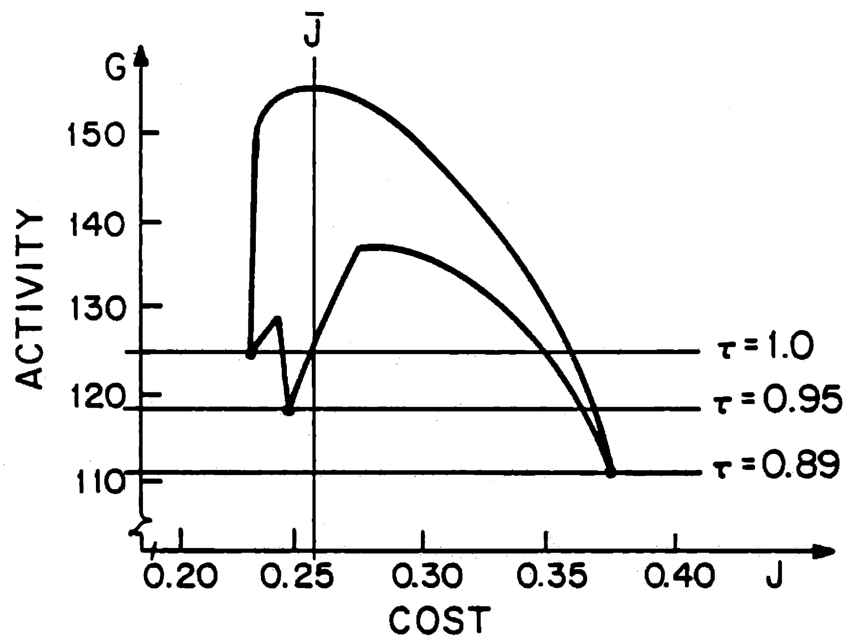


Figure 5.9. (J,G) Region With Bounded Rationality Constraints

$\tau \geq 1$ the boundedly rational decision maker can realize this level of performance. For $\tau < 1$, however, the solution to Problem I(b) becomes in general a mixed strategy. It is interesting to note that as τ decreases to less than 0.95, the best realizable performance changes significantly. This represents a general characteristic of the model, i.e., as the decision maker becomes more tightly constrained, the alterations in performance are not necessarily "smooth." Indeed, as seen in this example, performance may alter drastically for a slight decrease in mean input interarrival time, which corresponds to one type of human overload response [23].

For $\bar{J} = 0.26$, solutions to Problem II(a) exist. If $\tau \geq 0.95$ solutions exist to Problem II(b), also. If $\tau < 0.95$, however, the decision maker becomes overloaded, i.e., there are no satisficing strategies, although the task can still be accomplished for $0.89 \leq \tau < 0.95$ at much-reduced performance levels. For $\tau < 0.89$, the task cannot be accomplished.

5.5 SUMMARY

In this chapter an example was presented which verified many of the properties of the model shown in Chapter 4. The significance of the coordination activity G_c, as a part of the total was indicated, especially in comparison to the throughput. In addition, the activity required to switch among algorithms was seen to be significant. Finally, the strategies which realized minimum error and satisficing performances were considered for representative values of $\bar{J}$ and F.

CHAPTER 6

INTERACTION WITH ORGANIZATION

The two-stage model presented in Chapter 4 represents a first step toward an analytic description of a decision maker as he performs his task. In order to evaluate eventually alternative military command structures, however, it is necessary to extend the basic model to account for various types of interaction among decision makers. Two such interactions will be considered in this chapter: inputs which represent a supplement to the decision maker's situation assessment, and inputs which represent a command issued to the decision maker. A model which incorporates both is suggested, and expressions which describe the model are developed and analyzed. It is shown that the expressions are ready extensions of those for the basic model and that the characterization of the strategies as solutions to problems in the normative and descriptive context do not change in their qualitative aspects. Finally, the implications of the interactions as control exercised by the organization are considered.

6.1 INFORMATION STRUCTURES

The possible types of interaction between an organization and one of its members are determined in part by the information structure of the organization. Such a structure can be determined by partitioning the overall input to the organization, as shown in Figure 6.1 [26]. The vector X'

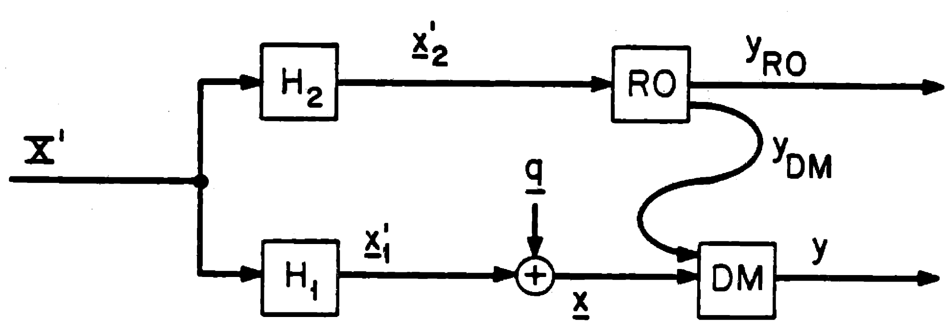


Figure 6.1. Organization Information Structure

represents the organization's input; it is partitioned according to H_1 and H_2 into the vectors $\underline{x}_1'$ and $\underline{x}_2'$, which are the inputs to the decision maker (DM) and to the rest of the organization (RO), respectively. Consideration of DM-RO interactions in this chapter will be restricted to the class of those which are additional inputs to the decision maker from the rest of the organization, as shown in Figure 6.1. In addition, it is assumed that a noisy measurement of $\underline{x}_1'$ is received by the DM, as discussed in Chapter 4.

There are three classes of information partitions which can arise for the model shown in Figure 6.1. One class consists of all partitions such that the vector $\underline{x}_1'$ is completely contained in $\underline{x}_2'$, which represents the case where all of the information received by the decision maker is also received by the rest of the organization. It is not a particularly interesting case since the decision maker does not receive any unique inputs, and the separation of the DM and RO is hence not very meaningful.

A second class of partitions contains those in which $\underline{x}_1'$ and $\underline{x}_2'$ are disjoint, that is, the decision maker receives an input which is entirely different from that of the rest of the organization. The possible DM-RO interactions are therefore such that no additional information about $\underline{x}_1'$ can be passed to the DM. Investigation of such interactions is most meaningful when the decision maker's performance is considered only in terms of the organization's overall performance, as shown in Figure 6.2.

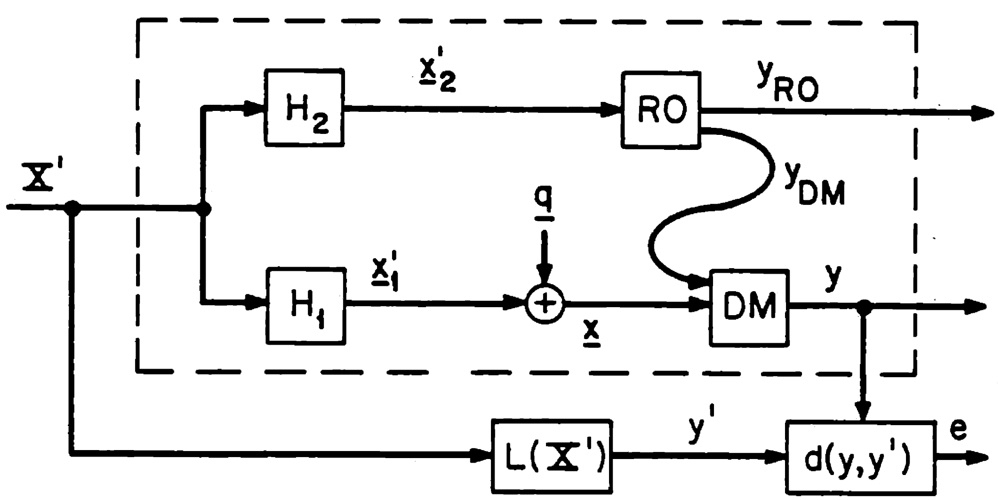


Figure 6.2. Performance Evaluation of DM With Respect to Organization

The third class of information structure arises when $\underline{x}_1'$ and $\underline{x}_2'$ are overlapping, that is, have common elements of X', but are such that $\underline{x}_1'$ is not contained in $\underline{x}_2'$. This type of structure is similar to that of disjoint inputs, but differs in that the input to the DM from RO can be related to the DM's own input. It is therefore possible to consider the effect of DM-RO interactions on the decision maker's performance of his own task, as shown in Figure 6.3. This consideration is consistent with the objective of this thesis, i.e., that of describing the decision maker as he performs his task. Accordingly, it will be assumed that the partitions H_1 and H_2 have been determined such that $\underline{x}_2'$ and $\underline{x}_1'$ have common elements, but $\underline{x}_1'$ is not contained in $\underline{x}_2'$. The decision maker's task is to determine a decision response y appropriate to an input $\underline{x}$, and the interest is to characterize the decision maker when additional inputs are available from the rest of the organization. Two such inputs are considered, as described in the next section.

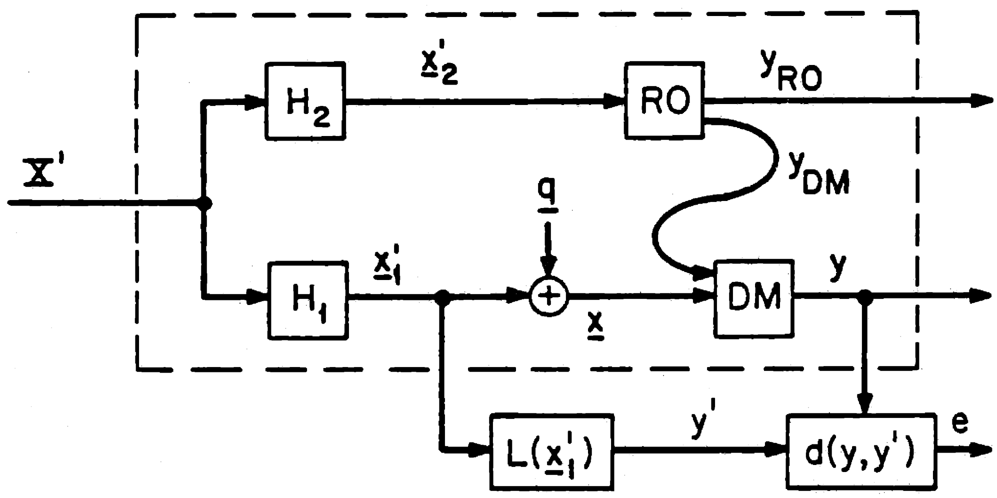


Figure 6.3. Performance Evaluation of DM With Respect to Task Performance

6.2 SITUATION INPUTS AND COMMAND INPUTS FROM THE ORGANIZATION

6.2.1 Model Definition

Because of the information structure assumed, it is possible that the rest of the organization can perform a partial assessment of the decision maker's situation. As a supplementary input, such an assessment could be used instead of or in addition to the decision maker's own assessment. A second type of interaction which could arise is that of an external command, which is of particular interest for military organizations. In terms of the model, an external command is an input which restricts the decision maker's possible options. The two types of inputs described above are incorporated into the basic model of the decision making process as shown in Figure 6.4. The decision maker's own situation assessment, $\underline{z}$,

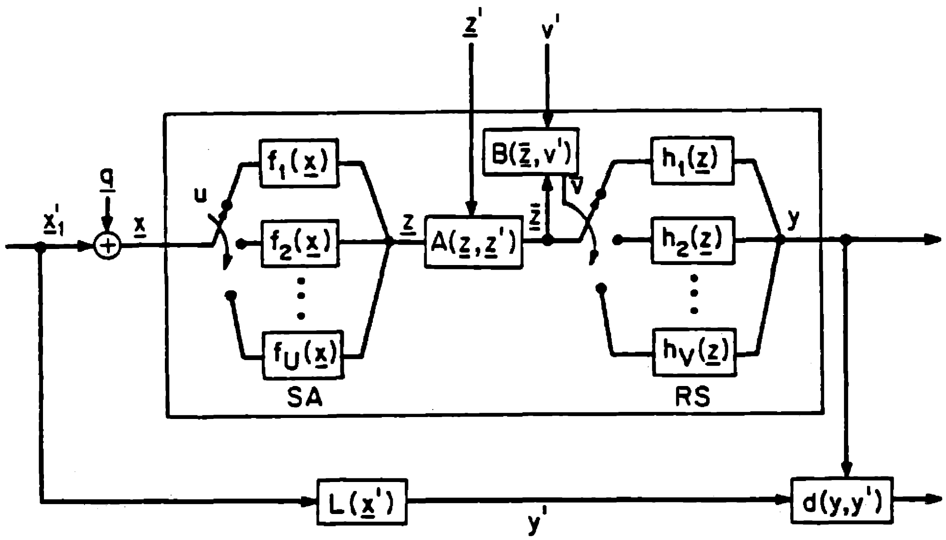


Figure 6.4. Decision-Making Process With Situation and Command Inputs

is combined with the supplementary situation input $\underline{z}'$ to obtain a final assessment $\bar{\underline{z}}$. The variables $\underline{z}$ and $\bar{\underline{z}}$ are assumed to be of the same dimension and to take values from the same alphabet. The variable $\underline{z}'$ is such that its elements combine with some subset of the elements of $\underline{z}$. The combination of $\underline{z}'$ and $\underline{z}$ is accomplished by algorithm A, which is deterministic, has no rejection, and is composed of a set W^A of α_A interconnected variables, where

$$W^A = \{w_1^A, \ldots, w_{\alpha_A}^A\} \tag{6.1}$$

To complete the definition of $\underline{z}'$ from the point of view of the decision maker, it is assumed that when an input $\underline{z}'$ is <u>not</u> received it takes a fixed (inactive) value. Correspondingly, the variables of algorithm A do not become active when $\underline{z}'$ is not received, except for the output variable $\bar{\underline{z}}$, which in that case is identically equal to $\underline{z}$.

The mechanism by which the command input v' restricts the decision maker's options is as follows. Recall that in the two-stage model, an internal decision strategy $p(v|\underline{z})$ was specified and used. In the present model, such a strategy is also specified, except that the appropriate conditioning is on $\bar{\underline{z}}$; since the alphabets of $\underline{z}$ and $\bar{\underline{z}}$ are the same, the specification is equivalent. However, the distribution $p(v|\bar{\underline{z}})$ does not determine the algorithm choice; rather, the decision maker's choice v is modified according to the input command v' to determine a final choice $\bar{v}$, where v' is a scalar and takes a finite number of values. This modification is assumed to be deterministic and is represented by

$$b\ :\ \bar{v} = b(v,v') \qquad \bar{v} = 1,2,\ldots,V \tag{6.2}$$

The overall process of mapping the assessed situation $\bar{\underline{z}}$ and the command input v' into the final choice $\bar{v}$ is represented by algorithm B in Figure 6.4, and the result of this process is a deterministic modification of the strategy $p(v|\bar{\underline{z}})$ into the effective strategy $p(\bar{v}|\bar{\underline{z}}v')$, as given by

$$p(\bar{v}|\bar{\underline{z}}v') = \sum_{v} p(\bar{v}|vv')p(v|\bar{\underline{z}}) \tag{6.3}$$

where the quantities $p(\bar{v}|vv')$ are 0 or 1. Note that algorithm B is nondeterministic; this property is due only to the dependence on $p(v|\bar{\underline{z}})$, however.

The specification of b(v,v') represents the specification of a protocol according to which the command is used, i.e., the values of $\bar{v}$ determined by b(v,v') reflect the degree of option restriction effected by the command. For example, if a particular command, say $v' = v_1'$, is completely restrictive, then

$$\bar{v} = b(v,v_1') = j \qquad \forall\ v \tag{6.4}$$

for some $j = 1,2,\ldots,V$. On the other hand, it is possible that several response selection algorithms can be consistent with the command; the decision maker's internal strategy is then partially implemented. For example, suppose the command $v' = v_2'$ combines with $v = 1,2,3$ to give

$$\bar{v} = b(v,v_2') = \begin{cases} 1 & v = 1 \\ 2 & v = 2 \\ 2 & v = 3 \end{cases} \tag{6.5}$$

In this case it is seen that only if the decision maker chooses algorithm $h_3(\cdot)$ is his choice modified by the command input. This restriction of options is not complete, i.e., the decision maker's own choice is still implemented to some degree.

It is assumed that algorithm B has no rejection and contains α_B internal variables given by

$$W^B = \{w_1^B,\ldots,w_{\alpha_B}^B\} \tag{6.6}$$

As with the situation input, if a command input is not received, the decision maker assumes v' to be fixed at its inactive value. The variables in algorithm B do not become active in that event, except for the variable $\bar{v}$ which becomes identically equal to the decision maker's algorithm choice v.

6.2.2 Analytic Expressions

The application of the PLI to the extended model is straightforward and is similar to the development in Chapter 4. The resulting expressions are summarized below while their derivation is presented in Appendix D. A key feature of the derivation is the definition of four subsystems within the model:

$$S^I = \{u, W^1, \ldots, W^U, \underline{z}\} \quad (6.7)$$

$$S^A = W^A \quad (6.8)$$

$$S^B = W^B \quad (6.9)$$

$$S^{II} = \{\bar{v}, W^{U+1}, \ldots, W^{U+V}, y\} \quad (6.10)$$

In addition, it is shown in the derivation that the analytic expressions which describe the model are functions only of the internal decision strategies $p(u)$ and $p(v|\bar{\underline{z}})$ if the algorithms are specified and if the distributions $p(\underline{x}_1')$, $p(\underline{q})$, $p(\underline{z}'|\underline{x}_1')$ and $p(v'|\underline{z}'\underline{x}_1')$ are specified. The first two distributions represent the input $\underline{x}$ to the decision maker. The latter two distributions appear because of the information structure assumed for the organization. Because RO receives a subset of the elements of X' sent to DM, the inputs $\underline{z}'$ and v' from RO are in general related to $\underline{x}_1'$. The distributions $p(\underline{z}'|\underline{x}_1')$ and $p(v'|\underline{z}'\underline{x}_1')$ represent the fundamental expression of this relationship.

- Amount of Internal Decision-Making

$$G_n = H(u) + H_{\bar{\underline{z}}}(v) \quad (6.11)$$

- Throughput

$$G_t = T(\underline{x}, \underline{z}', v' : y) \quad (6.12)$$

- Blockage

$$G_b = H(\underline{x}, \underline{z}', v') - G_t \quad (6.13)$$

- Coordination

$$G_c = G_c^I + G_c^A + G_c^B + G_c^{II} + T(S^I : S^A : S^B : S^{II}) \quad (6.14)$$

$$G_c^I = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) \quad (6.15)$$

$$G_c^A = g_c^A(p(\underline{z})) \quad (6.16)$$

$$G_c^B = g_c^B(p(\bar{\underline{z}})) \quad (6.17)$$

$$G_c^{II} = \sum_{j=1}^{V} p_j g_c^{U+j}(p(\bar{\underline{z}}|\bar{v}=j)) + \alpha_j' \mathscr{H}(p_j) + H(y) \quad (6.18)$$

$$T(S^I:S^A:S^B:S^{II}) = H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{v},\bar{\underline{z}}) + T_{\underline{z}}(\underline{x}_1':\underline{z}') + T_{\bar{\underline{z}}}(\underline{x}_1',\underline{z}':v') \quad (6.19)$$

- Total Activity

$$G = G_n + G_t + G_b + G_c \quad (6.20)$$

Of particular interest are the expressions for G_n and G_c. Despite the fact that the response selection strategy is modified by a command input, the amount of internal decision-making depends fundamentally on $p(u)$ and $p(v|\bar{\underline{z}})$, rather than on $p(\bar{v}|\bar{\underline{z}}v')$. The decision maker specifies a strategy $p(v|\bar{\underline{z}})$, which represents his <u>internal</u> decision. This internal decision may or may not be implemented, depending on the command input received, however, because the input v' alters the strategy $p(v|\bar{\underline{z}})$ into what becomes the effective decision-making strategy $p(\bar{v}|\bar{\underline{z}}v')$.

The internal coordination is seen to be the sum of the subsystem coordinations plus the inter-subsystem coordination given by Eq. (6.19). It is useful to write Eq. (6.19) as (see Appendix A)

$$T(S^I:S^A:S^B:S^{II}) = T(S^I:S^A) + T(S^I,S^A:S^B) + T(S^I,S^A,S^B:S^{II}) \quad (6.21)$$

where

$$T(S^I:S^A) = H(\underline{z}) + T_{\underline{z}}(\underline{x}_1':\underline{z}') \quad (6.22)$$

$$T(S^I,S^A:S^B) = H(\bar{\underline{z}}) + T_{\bar{\underline{z}}}(\underline{x}_1',\underline{z}':v') \tag{6.23}$$

$$T(S^I,S^A,S^B:S^{II}) = H(\bar{\underline{z}},\bar{v}) \tag{6.24}$$

The inter-subsystem coordination effected by the first term of Eq. (6.22) is analogous to that present in the two-stage model. The second term of Eq. (6.22) arises because of the relationship of $\underline{z}'$ to $\underline{x}_1'$ through $p(\underline{z}'|\underline{x}_1')$. Because of this relationship, it is possible to effect a greater coordination between S^I and S^A than that given by $H(\underline{z})$, i.e., more information about the input $\underline{x}_1'$ can be forwarded to S^A than contained in $\underline{z}$. For example, it is possible that the RO can resolve more finely a portion of the DM's input $\underline{x}_1'$ and a partial situation assessment which is more refined in some aspects than the DM's own assessment can therefore be made. In such an instance, an additional amount of the input is passed forward to S^A and the coordination between subsystems increases. This additional activity within the DM does not increase the total in general; rather, it can reduce significantly the activity required for subsequent processing, as will be illustrated later.

A similar interpretation holds for Eq. (6.23). The form of the second term reflects the fact that the command input can be determined by the RO from the situation input. Finally, since subsystem S^{II} receives two inputs, $\bar{v}$ and $\bar{\underline{z}}$ from other subsystems, the corresponding subsystem coordination is given by Eq. (6.24) as the joint uncertainty $H(\bar{\underline{z}},\bar{v})$, which is analogous to the inter-subsystem coordination of the basic model.

6.2.3 Reduction to Basic Model

If the situation and command inputs are not present, the present model reduces in a consistent fashion to that of the basic model. Such a reduction proceeds as follows. The variables $\bar{\underline{z}}$ and v' are fixed at their respective inactive values. As such, there is no uncertainty associated with either, and

$$\bar{\underline{z}} \equiv \underline{z} \tag{6.25}$$

$$\bar{v} \equiv v \tag{6.26}$$

$$p(\bar{v}|\underline{\bar{z}}v') \equiv p(v|\underline{z}) \tag{6.27}$$

Furthermore, because subsystems S^A and S^B become throughput-only subsystems (all variables inactive except $\underline{\bar{z}}$ and $\bar{v}$, respectively) their internal coordinations are zero:

$$g_c^A(p(\underline{z})) \equiv 0 \tag{6.28}$$

$$g_c^B(p(\underline{\bar{z}})) \equiv 0 \tag{6.29}$$

Finally, because $\underline{z}'$ and v' are fixed, the last two terms in Eq. (6.18) are zero, and the throughput and blockage expressions are also reduced. The analytic description of the model then becomes

$$G_n = H(u) + H_{\underline{z}}(v) \tag{6.30}$$

$$G_t = T(\underline{x}:y) \tag{6.31}$$

$$G_b = H(\underline{x}) - G_b \tag{6.32}$$

$$G_c = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z})$$

$$+ \sum_{j=1}^{V} p_j g_c^{U+j}(p(z|v=j)) + \alpha_j' \mathscr{H}(p_j) + H(y) + H(\underline{z}) + H(\underline{\bar{z}}) + H(\underline{\bar{z}},\bar{v}) \tag{6.33}$$

Eqs. (6.30) - (6.33) are identical to those obtained for the basic model, with the exception of the additional coordinations $H(\underline{\bar{z}})$ and $H(\bar{v},\underline{\bar{z}})$. Recall that the interaction of subsystems produces coordination activity, and the existence of the subsystems S^A and S^B produces the additional coordination represented by $H(\underline{\bar{z}})$ and $H(\underline{\bar{z}},\bar{v})$ regardless of whether or not the functions performed by those subsystems are trivial.

6.3 PROPERTIES OF MODEL

In this section, the characterization of solutions to problems similar to I and II of Chapter 4 is made for the present model. In the next section, the notions of indirect and direct control are demonstrated in the model using the special cases of situation input only and command input only.

6.3.1 Problem Statement

The statement of problems in the normative and descriptive contexts for the present model, with and without the constraint of bounded rationality, is made analogously to Problems I and II of Chapter 4.

Organizationally Interactive Model

The following are assumed to be specified, and characterize the model shown in Figure 6.4:

- algorithms $f_i(\underline{x})$, $h_j(\bar{\underline{z}})$, $A(\underline{z},\underline{z}')$, $B(\bar{\underline{z}},v')$, with α_i, α_j' internal variables and internal coordinations g_c^i, g_c^{U+j}, g_c^A, g_c^B, where the internal coordinations are known as a function of the characteristics of their respective inputs $(i = 1,2,\ldots,U,\ j = 1,2,\ldots,V)$ (6.34)
- analytic characterization of the decision-making process as described by Eqs. (6.11) - (6.20) (6.35)
- function $L(\underline{x}_1')$ that maps $\underline{x}_1'$ into y' (6.36)
- distributions $p(\underline{z}'|\underline{x}_1')$, $p(v'|\underline{z}'\underline{x}_1')$ as determined from the rest of the organization (6.37)
- distribution $p(\underline{q})$ (6.38)
- samples drawn simultaneously from distributions in (6.37) - (6.38) every τ seconds on the average. (6.40)

In addition, let the cost function be the probability of error, i.e.,

$$J(p(u),\ p(v|\bar{\underline{z}})) = p(y \neq y'), \tag{6.41}$$

and assume a total activity rate constraint, F, so that

$$G(p(u),\ p(v|\bar{\underline{z}})) \leq F\tau \tag{6.42}$$

is required.

The statement of Problems I and II in the present context differs only in the conditions of the model.

Problem I'

Given conditions (6.34) - (6.40), determine p(u) and $p(v|\bar{\underline{z}})$ such that

a) $J(p(u),\ p(v|\bar{\underline{z}}))$ is minimized,

or

b) $J(p(u),\ p(v|\bar{\underline{z}}))$ is minimized subject to $G(p(u),\ p(v|\bar{\underline{z}})) \le F\tau$.

Problem II'

Given conditions (6.34) - (6.40), determine p(u) and $p(v|\bar{\underline{z}})$ such that

a) $J(p(u),\ p(v|\bar{\underline{z}})) \le \bar{J}$,

or

b) $J(p(u),\ p(v|\bar{\underline{z}})) \le \bar{J}$ subject to $G(p(u),\ p(v|\bar{\underline{z}})) \le F\tau$, where $\bar{J}$ is the threshold of satisficing performance.

6.3.2 Characterization of Solutions

It is readily shown that the total activity present in the model is a convex function of the decision strategy $(p(u),\ p(v|\bar{\underline{z}}))$, and that the probability of error for an arbitrary strategy is obtained as a convex combination of the error probabilities corresponding to pure strategies. Hence the analysis which led to the characterization of solutions to Problems I and II is also valid for Problems I' and II'. In particular, the minimum error strategy is pure if all error probabilities corresponding to pure strategies are distinct. Under the condition of bounded rationality, the solution to both Problems I' and II' may contain only mixed strategies. The condition of overload arises as τ decreases. An interesting additional feature is apparent in the solutions to Problems I' and II': depending on the characteristics of Algorithm B, many internal decision strategies may be mapped into a single effective strategy, i.e., the mapping

$$p(v|\underline{\bar{z}}) \xrightarrow{B(\underline{\bar{z}},v')} p(\bar{v}|\underline{\bar{z}}v') \tag{6.43}$$

is in general a many-to-one mapping.

The characterizations of the solutions, when they exist, to Problems I'a, I'b, II'a, and II'b, are summarized below.

Problem	Strategies
I'a	pure, mixed if more than one pure strategy exists
I'b	pure and mixed, may be mixed only
II'a	pure and mixed
II'b	pure and mixed, may be mixed only

6.4 CONTROL

An especially interesting aspect of the interaction among organization members is the exertion of control, whether indirect or direct [27]. In the following paragraphs, it is seen from the consideration of the special cases of situation input only and command input only that both are reflected in the model.

6.4.1 Indirect Control

Consider the case where the model is specialized to include only the situation input. Analytic characterization is then given by the following expressions, where superscript SI denotes situation input only and $v \equiv \bar{v}$,

$$G_n^{SI} = H(u) + H_{\underline{\bar{z}}}(v) \tag{6.44}$$

$$G_t^{SI} = T(\underline{x},\underline{z}':y) \tag{6.45}$$

$$G_b^{SI} = H(\underline{x},\underline{z}') - G_t \tag{6.46}$$

$$G_c^{SI} = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) + G_c^A(p(\bar{\underline{z}}))$$

$$+ \sum_{j=1}^{V} p_j g_c^{U+j}(p(\bar{\underline{z}}|v=j)) + \alpha_j' \mathscr{H}(p_j) + H(y)$$

$$+ H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{\underline{z}},\bar{v}) + T_{\bar{\underline{z}}}(\underline{x}_1':\underline{z}') \tag{6.47}$$

As indicated earlier, one of the benefits of the situation input for the decision maker is that an improved and refined assessment of the situation is made, which contributes to better performance, all other things being equal, i.e., it does not produce sufficiently higher activity that a change in strategy is necessary to remain within rationality bounds. The ability of RO to alter performance by generation of $\underline{z}'$ represents an indirect control on the decision maker. It is indirect because it does not alter directly the decision strategy employed, but rather alters the performance by influencing the value of the assessed situation. Such an influence need not be positive. If it is possible for RO to select $\underline{z}'$ based on $\underline{x}_1'$ such that performance is improved, it is also equally possible to construct a $\underline{z}',\underline{x}_1'$ relationship which causes lower performance. In each case, once the strategy $p(v|\bar{\underline{z}})$ has been selected, the DM is subject to a control by $\underline{z}'$.

While the control that is possible by $\underline{z}'$ over performance is easily seen, it is also true, though perhaps less clear, that the situation input can cause a significant change in either direction of the total activity present. Consider the following example, which is based on the example presented in Section 4.5. The model given was that shown in Figure 6.5, where $\underline{z}$ was assumed to take values $\underline{z}_1$ and $\underline{z}_2$ with probability $1-\gamma$ and γ, respectively. Let γ be 0.5. The coordination activity present in this model is then given by

$$G_c^{II} = g_c^1 + H(\underline{z}) + (\alpha_1' + \alpha_2')\mathscr{H}(0.5) + H(y) \tag{6.48}$$

(Recall that the internal coordinations of algorithms $h_1(\cdot)$ and $h_2(\cdot)$ are zero because their respective inputs are deterministic.) Now consider the same model, but with a situation assessment subsystem, as shown in Figure

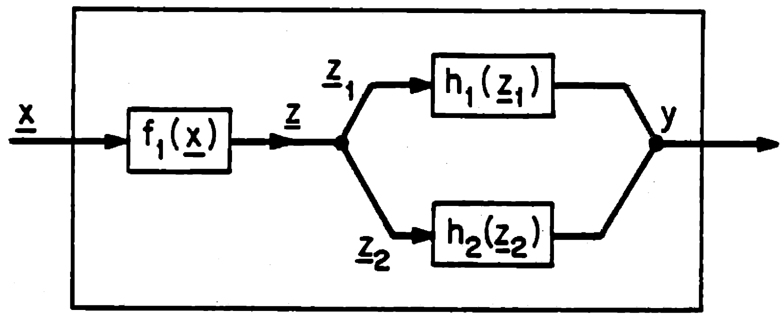


Figure 6.5. Deterministic Switching

6.6. Note that $\underline{z}$ still takes values $\underline{z}_1$, $\underline{z}_2$ with equal probability and that a correspondence $\bar{\underline{z}}_i \equiv \underline{z}_i$, $i = 1,2$, is assumed. Suppose the relationship between $\underline{x}_1'$ and $\bar{\underline{z}}'$ is such that $\underline{z}'$ is chosen so that

$$A(\underline{z},\bar{\underline{z}}') = \bar{\underline{z}}_1 \tag{6.49}$$

is always the case. Such is an extreme case, but possible within the framework of the model, and gives the result that algorithm $h_1(\bar{\underline{z}}_1)$ is always used. It is easy to show (see Appendix D, Section 5) that the difference in coordination activity between models is given by

$$G_c^{SI} - G_c^2 = [g_c^A(p(\underline{z})) + T_{\underline{z}}^{SI}(x_1':\underline{z})] - [(\alpha_1'+\alpha_2')\mathcal{H}(0.5) + H^2(y)] \tag{6.50}$$

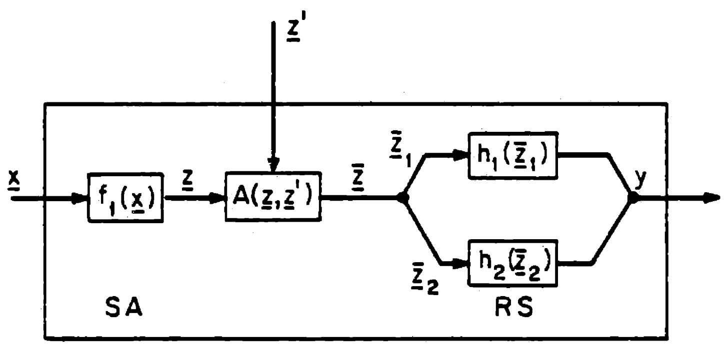


Figure 6.6. Deterministic Switching With Situation Assessment

where the superscript 2 denotes the basic two-stage model. The first bracketed term in Eq. (6.50) represents the extra coordination introduced into the model by the addition of the subsystem S^A. If the number of variables in the response selection algorithms is large (α_1' and α_2'), the coordination required in switching algorithms in the model of Figure 6.5 may exceed the amount introduced by S^A. In that case, Eq. (6.50) gives a negative result and the activity in the model of Figure 6.6 is less than that of the model of Figure 6.5 even though an additional stage of processing is present. This illustrates the significant effect that the situation input can have on the total activity of the model.

6.4.2 Direct Control

The possibility of direct control is present in the model through the command input v'. Consider the case where organization interactions are restricted to command inputs only. The expressions which characterize this model are obtained by appropriate reduction of Eqs. (6.11) - (6.20):

$$G_n^{CI} = H(u) + H_{\underline{z}}(v) \tag{6.51}$$

$$G_t^{CI} = T(\underline{x}, v':y) \tag{6.52}$$

$$G_b^{CI} = H(\underline{x}, v') - G_t \tag{6.53}$$

$$\begin{aligned} G_c^{CI} = & \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) + g_c^B(p(\bar{\underline{z}})) \\ & + \sum_{j=1}^{V} p_j g_c^{U+j}(p(\bar{\underline{z}}|\bar{v}=j)) + \alpha_j' \mathscr{H}(p_j) + H(y) \\ & + H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{\underline{z}}, \bar{v}) + T_{\bar{\underline{z}}}(\underline{x}_1':v') \end{aligned} \tag{6.54}$$

where CI denotes command input only.

The extreme case of direct control occurs when the values of v' selected as commands are such that the algorithm chosen is a deterministic function of v'. Note that this implies that a protocol function b(v,v') is possessed by the decision maker which accomplishes the proper mapping. The amount of internal decision-making in this case then becomes

$$G_n^{CI} = H(u) \tag{6.55}$$

Since the value of $\bar{v}$ is completely determined externally, an externally controlled switching is present, and considerable influence on the activity in the response selection stage is exerted. It is equally apparent that the performance will also be affected by v', and hence controlled directly, either beneficially or adversely.

Another interesting case of direct control occurs when the values of v' are such that a less than total restriction on the selection $\bar{v}$ occurs. Such a case might represent the input of commands by a commander of similar rank as the decision maker. Hence the command input would serve more to coordinate the two decision makers, i.e., eliminate incompatible choices, rather than restrict tightly the possible options.

6.4.3 Situation and Command Inputs

For the case where the decision maker receives both types of inputs, it is evident that both indirect and direct control can exist. The interaction of the two is a non-trivial occurrence in general because the characteristics of $\bar{\underline{z}}$, which are derived in part from $\underline{z}'$, determine in part the internal coordination of the $h_j(\cdot)$ algorithms, as well as influence the effective strategy determination through $B(\bar{z},v')$. The relative influence of each type of control is dependent on which stages dominate the overall performance and/or activity of the decision maker. For example, if the situation assessment stage accounts for a large fraction of the total activity and also dominates the overall performance, then it is possible that a partial assessment determined externally would reduce the total activity without compromising performance greatly. In such an instance the constrained decision maker's performance would be more robust against decreases in τ. By the same token, variation in v' would have little impact on the total process if the first stage were dominant.

6.5 CHAPTER SUMMARY

In this chapter, two interactions of the decision maker with the rest of the organization were considered: situation inputs and command inputs. Analytic expressions were derived for a model which included both interactions. Such expressions were seen to follow easily from the basic model, which demonstrates the potential usefulness of the approach in evaluation of alternative organization structures for design purposes. The concepts of indirect and direct control, which correspond to control exercised through situation inputs and control exercised by input commands, were introduced and their effects interpreted in terms of the model.

CHAPTER 7

FUTURE WORK AND CONCLUSIONS

Two extensions to the basic model are discussed in this chapter: non-zero rejection and non-deterministic algorithms. Each is seen to have implications for eventual organization structure considerations. In addition, two other issues which are of a fundamentally different nature are mentioned: task definition and modeling the acquisition of experience in the performance of a task. The chapter ends with the conclusions of the thesis.

7.1 FUTURE WORK

7.1.1 Non-Zero Rejection and Pre-Processing

The work presented in this thesis has assumed the condition of zero rejection, that is, all aspects of the input are recognized by the decision maker. Non-zero rejection corresponds to insensitivity by the decision maker to some aspects of his input. In one form this insensitivity can be an inadequate resolution of the input. For example, two elements of the input alphabet may be recognized as a single element by the decision maker. Rejection is a passive phenomenon in the present context, i.e., it does not require activity within the system. A model which incorporates algorithms with various degrees of rejection represents a useful concept, since it corresponds to the case where the total activity in the decision-making process can be reduced by choosing to be insensitive to some aspects of the input. The degree to which this can be accomplished without compromising performance requires the development of analytic expressions which incorporate rejection. This development is suggested for future work.

Rejection by the decision maker may occur in an inconsistent manner because of algorithm switching. An alternative is to incorporate a pre-processor into the model which modifies the input so that it can be processed by the decision maker without rejection [12]. Such a function incorporates elements of encoding of the input for efficient processing, and also the elimination of aspects that are not relevant to decision-making. As such it is worthy of additional consideration in the context of the model developed in this thesis.

A different form of rejection occurs when an input is completely prevented from reaching the decision maker. An interesting manifestation of this is the case where a pre-processor examines the inputs to the decision maker and eliminates those which would needlessly occupy him. Such would be the case for those inputs which clearly require no action, i.e., the decision response is to do nothing. Another case where needless activity might be eliminated occurs when the decision-making task is basically one of monitoring, and where decisions need be made only on irregularities in the received inputs. Rather than processing each arriving input, those which correspond to the "normal" environment can be filtered by the pre-processor, and only the exceptional inputs forwarded to the decision maker for determination of decision response. Selective elimination of inputs causes an effective increase in the interarrival time, from τ to τ', say. This has implications for the boundedly rational decision maker since a greater actual rate of arrivals (τ) can be processed without overload by use of the same strategy, i.e., it is possible that

$$\frac{G}{\tau'} < F < \frac{G}{\tau} \tag{7.1}$$

To be sure, one must investigate whether the pre-processor can adequately handle its own task, but the possible combination of the two represents a key structural consideration in organization design and should be investigated in the context of this model.

7.1.2 Non-Deterministic Algorithms

Deterministic algorithms have been assumed throughout this thesis. Non-deterministic procedures are an interesting notion, however, and their incorporation into the model would be a useful extension, although the conceptual clarity obtained by assuming deterministic algorithms would not be retained. For example, the quantity G_n would no longer only represent the amount of internal decision-making, but would also reflect the uncertainty present in the algorithms. The general form of the expressions derived would remain the same, however, since the fundamental difference would be a change in the form of the distributions which describe algorithm mappings rather than how the distributions are used, i.e., while the distribution $p(\underline{z}|xu)$ would appear the same in the development, it would be no longer 0 or 1.

7.1.3 Alternate Processing and Non-Deterministic Algorithms

A particularly interesting case involving non-deterministic algorithms arises when each algorithm incorporates switching among "sub-algorithms." Such a structure is shown in Figure 7.1 for the first stage of the basic two-stage model.

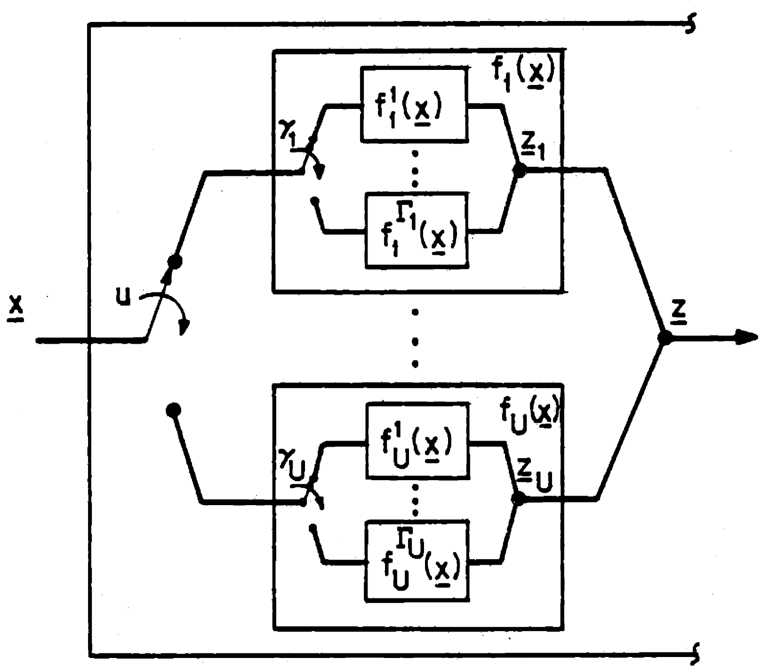


Figure 7.1. Algorithms Which Switch Among Sub-Algorithms

If all of the sub-algorithms $f_1^j(\underline{x})$, $j = 1,\ldots,\Gamma$, are deterministic, then the only uncertainty within algorithm $f_1(\underline{x})$ is the uncertainty in the choice of sub-algorithm, i.e., the realization of the variable γ_1. If the same is true for all of the sub-algorithms $f_i^j(\underline{x})$, $i = 1,2,\ldots,U$; $j = 1,2,\ldots,\Gamma_i$, then the overall model becomes one of alternate processing [12], [26] by an echelon of decision makers, where the determination of which decision maker processes a particular input is made by another decision maker according to the choice u. Such a structure preserves the property that the uncertainty that arises within the system is due only to the amount of internal decision-making. It also indicates how some organization structures, in particular hierarchical ones, might be represented using the model developed in this thesis.

7.1.4 Task Definition in Terms of Variables

In order to evaluate alternative organization structures using the model developed in this thesis, it is necessary to describe a "well-defined task" analytically. In terms of the model, a definition of the algorithms which can be used to perform the task is required. In particular, knowledge of the input-output characteristics of an algorithm, the number of variables it contains, and its internal coordination as a function of the characteristics of its input are needed in order to describe the decision-making process. The need for such a generalized characterization raises interesting and non-trivial issues in terms of variable definition and interconnection, especially in view of the complexities encountered in the consideration of the simple task used in the example in Chapter 5.

7.1.5 Learning

Recall that in this thesis a condition of no-learning has been assumed, i.e., the decision maker is not able to adapt either his choices or his algorithms according to knowledge about his performance of the task. Such an ability requires a mechanism for incorporating performance feedback into the model, i.e., generalizing the model to include dynamic effects. The phenomenon of adaptation to the task represents the acquisition of experience, and its presence in the model would give a more complete description of the commander as he performs his task.

7.2 CONCLUSION

In this thesis, qualitative notions of decision-making have been synthesized with concepts from n-dimensional information theory into a working model which represents the decision-making process of a well-trained commander in the performance of a well-defined decision-making task. In particular, a basic model has been developed in the form of a two-stage process in which the situation is first assessed and then a response is selected based on the assessed situation. The model reflects explicitly internal choices made in the decision-making process. The stochastic version of the Partition Law of Information (PLI) has been used to characterize analytically the model as a function of the choices made, i.e., the <u>decision strategy</u>. Particular strategies used determine the amount of internal decision-making G_n, which represents a key re-interpretation of that quantity in the PLI.

Analytic representation of the bounded rationality of a decision maker is made in the form of a total activity rate constraint. It has been shown that the decision strategies which realize the optimal performance (normative) or satisficing performance (descriptive), subject to the boundedness of the decision maker, may be only <u>mixed</u> strategies, i.e., the decision maker alternates among options. It was also shown, according to the model, that alternating among options requires additional activity in the form of re-initialization of variables particular to each option. This activity and the coordination activity required to execute each option once it is chosen determine a significant part of the total activity in the decision-making process. As such they represent a key consideration in the characterization of the decision maker who is bounded.

The extension of the basic model to include possible interactions in an organizational context was considered. In particular, two types of interaction, situation inputs and command inputs, have been incorporated into the model. The notion of indirect control was shown to correspond to the former, while direct control was evident in the latter. It was seen that in terms of the model such control can be exercised to affect both the performance and total activity of the decision maker either beneficially or adversely. The qualitative aspects of the decision strategies which realized optimal and satisficing performance remained unchanged, however, in both the constrained and unconstrained cases.

APPENDIX A

IDENTITIES FROM INFORMATION THEORY

1. $$H(x,y) = H(x) + H_x(y) \tag{A.1}$$

Proof:

$$H(x) + H_x(y) = -\sum_x p(x) \log p(x) - \sum_x p(x) \sum_y p(y|x) \log p(y|x)$$

$$= -\sum_x \sum_y p(x,y) \log p(x) - \sum_x \sum_y p(x,y) \log p(y|x)$$

$$= -\sum_x \sum_y p(x,y) \log p(x)\ p(y|x)$$

$$= -\sum_x \sum_y p(x,y) \log p(x,y)$$

$$= H(x,y)$$

2. $$H(x,y,z) = H(x) + H_x(y) + H_{x,y}(z) \tag{A.2}$$

Proof: Similar to A above

3. $$T(x_1:x_2:x_3:x_4) = T(x_1:x_2) + T(x_3:x_4) + T(x_1,x_2:x_3,x_4) \tag{A.3}$$

Proof:

$$T(x_1:x_2:x_3:x_4) = \sum_{i=1}^{4} H(x_i) - H(x_1,x_2,x_3,x_4)$$

$$= \sum_i H(x_i) - H(x_1,x_2) - H_{x_1,x_2}(x_3,x_4) + H(x_3,x_4)$$

$$- H(x_3,x_4)$$

$$= [H(x_1) + H(x_2) - H(x_1,x_2)] + [H(x_3) + H(x_4) - H(x_3,x_4)]$$

$$+ [H(x_3,x_4) - H_{x_1,x_2}(x_3,x_4)]$$

$$= T(x_1:x_2) + T(x_3:x_4) + T(x_1,x_2:x_3,x_4)$$

4. $$T(x_1:x_2,x_3) = T(x_1:x_2) + T_{x_2}(x_1:x_3) \tag{A.4}$$

Proof:

$$T(x_1:x_2,x_3) = H(x_1) - H_{x_2,x_3}(x_1)$$

$$= H(x_1) - H_{x_2,x_3}(x_1) + H_{x_2}(x_1) - H_{x_2}(x_1)$$

$$= [H(x_1) - H_{x_2}(x_1)] + [H_{x_2}(x_1) - H_{x_2,x_3}(x_1)]$$

$$= T(x_1:x_2) + T_{x_2}(x_1:x_3)$$

APPENDIX B

DERIVATION OF THE PARTITION LAW OF INFORMATION

Given: System S with n variables, including output y and n-1 internal variables w_j

Input x to system

Denote: $W = \{w_1, \ldots, w_{n-1}\}$

$S = \{w_1, \ldots, w_{n-1}, y\} = \{W, y\}$

Then: $$\sum_{i=1}^{n} H(\cdot) = T(x:y) + T_y(x:W) + T(w_1:w_2:\ldots:w_{n-1}:y) + H_x(W,y) \quad \text{(B.1)}$$

Proof: By definition,

$$T(x:S) = H(S) - H_x(S) \quad \text{(B.2)}$$

Application of Eq. (A.4) gives

$$T(x:S) = T(x:W,y) = T(x:y) + T_y(x:W) \quad \text{(B.3)}$$

Eqs. (B.2) and (B.3) together yield

$$H(S) - H_x(S) = T(x:y) + T_y(x:W) \quad \text{(B.4)}$$

By definition, the n-dimensional mutual information is

$$T(w_1:w_2:\ldots:w_{n-1}:y) = \sum_{i=1}^{n} H(\cdot) - H(S) \quad \text{(B.5)}$$

Substitution of Eq. (B.5) into (B.4) yields

$$\sum_{i=1}^{n} H(\bullet) - T(w_1:w_2:\ldots:w_{n-1}:y) = H_x(S) = T(x:y) + T_y(x:W) \tag{B.6}$$

and rearrangement of terms yields the final form of the result (B.1):

$$\sum_{i=1}^{n} H(\bullet) = T(x:y) + T_y(x:W) + T(w_1:w_2:\ldots:w_{n-1}:y) + H_x(W,y)$$

APPENDIX C

VARIABLE DEFINITION FOR CHAPTER 5 EXAMPLE

C.1 SUMMARY

This appendix completes the definition of the internal variables of the algorithms used in the example of Chapter 5. It also gives the values of the input alphabets and the parameters used in generating the results presented in Chapter 5.

C.2 INPUT ALPHABETS

C.2.1 Input From Environment

$m_1 = 3,4,6$ with uniform probability

$m_2 = 1,2$ with uniform probability

$t_1 = 1.\qquad t_2 = 3.$

Note that all m_1, m_2 pairs give slope less than 1.

C.2.2 Noise Corruption

$$\underline{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

where the q_i are independent, identically and uniformly distributed variables taking values

$$q_i = -\tfrac{1}{2},\ 0,\ \tfrac{1}{2}.$$

C.2.3 Parameters of $L(\underline{x}')$ and Algorithm $h_3(\cdot)$

Refer to Eq. (5.3).

$$c_1 = 7.0 \qquad c_2 = 4.6$$

C.3 SITUATION ASSESSMENT ALGORITHM

C.3.1 General Description

The situation assessment algorithm receives as input $\underline{x}$, where

$$\underline{x} = \underline{x}' + \underline{q} = \begin{bmatrix} \lambda_1' \\ \mu_1' \\ \lambda_2' \\ \mu_2' \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \triangleq \begin{bmatrix} \lambda_1 \\ \mu_1 \\ \lambda_2 \\ \mu_2 \end{bmatrix}, \tag{C.1}$$

and determines estimates of θ and ν. $\hat{\theta}$ is determined in a two-part operation. First, a least squares slope $\hat{m}$ is determined using the formula appropriate for a line known to pass through the origin:

$$\hat{m} = \frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\lambda_1^2 + \lambda_2^2} \tag{C.2}$$

Second, a series of comparisons of $\hat{m}$ with appropriate thresholds results in a value of $\hat{\theta}$ which is within 2.5° of the value of $\tan^{-1}\hat{m}$. The value of $\hat{\nu}$ is determined in a similar manner. The algorithm first computes the variable $(\hat{\nu})^2$ according to

$$(\hat{\nu})^2 = \frac{(\lambda_1 + \lambda_2)^2 + (\mu_1 + \mu_2)^2}{(t_1 + t_2)^2} \tag{C.3}$$

A series of comparisons of $(\hat{\nu})^2$ results in a value of $\hat{\nu}$ which is within

±0.175 of the value of $\sqrt{\hat{\nu}^2}$ as defined above. In order to accomplish the transformation represented by the square root operation, the comparison thresholds are determined in a quadratic fashion according to the desired resolution of $\hat{\nu}$.

C.3.2 Variable Definition

For convenience, the superscript 1 is omitted in the definition of variables below.

$$w_1 = \lambda_1 \qquad w_5 = \lambda_1 \cdot \lambda_1 \qquad w_9 = w_5 + w_6$$

$$w_2 = \lambda_2 \qquad w_6 = \lambda_2 \cdot \lambda_2 \qquad w_{10} = w_7 + w_8$$

$$w_3 = \mu_1 \qquad w_7 = \mu_1 \cdot \lambda_1 \qquad w_{11} = \frac{w_{10}}{w_9} = \hat{m}$$

$$w_4 = \mu_2 \qquad w_8 = \mu_2 \cdot \lambda_1$$

Comparison thresholds used to determine $\hat{\theta}$:

$$w_{11+i} = \tan(5° \cdot i) \qquad i = 1,2,\ldots,9$$

Comparison thresholds used to determine $\hat{\nu}$:

$$w_{20+i} = (0.35 \cdot i)^2 \qquad i = 1,2,\ldots,19$$

$$w_{40} = \lambda_1 + \lambda_2 \qquad w_{42} = \mu_1 + \mu_2 \qquad w_{44} = (t_1 + t_2)^2$$

$$w_{41} = (\lambda_1 + \lambda_2)^2 \qquad w_{43} = (\mu_1 + \mu_2)^2 \qquad w_{45} = w_{41} + w_{43}$$

$$w_{46} = w_{44} \cdot w_{45} = (\hat{\nu})^2 \qquad w_{47} = \hat{\theta} \qquad w_{48} = \hat{\nu}$$

The comparison trees used to determine $\hat{\theta}$ from $\hat{m}$ and $\hat{\nu}$ from $(\hat{\nu})^2$ are given in Figures C.1 and C.2, respectively; they define 28 additional variables in the algorithm, which gives $\alpha_1 = 76$. As discussed in Chapter 5, if the

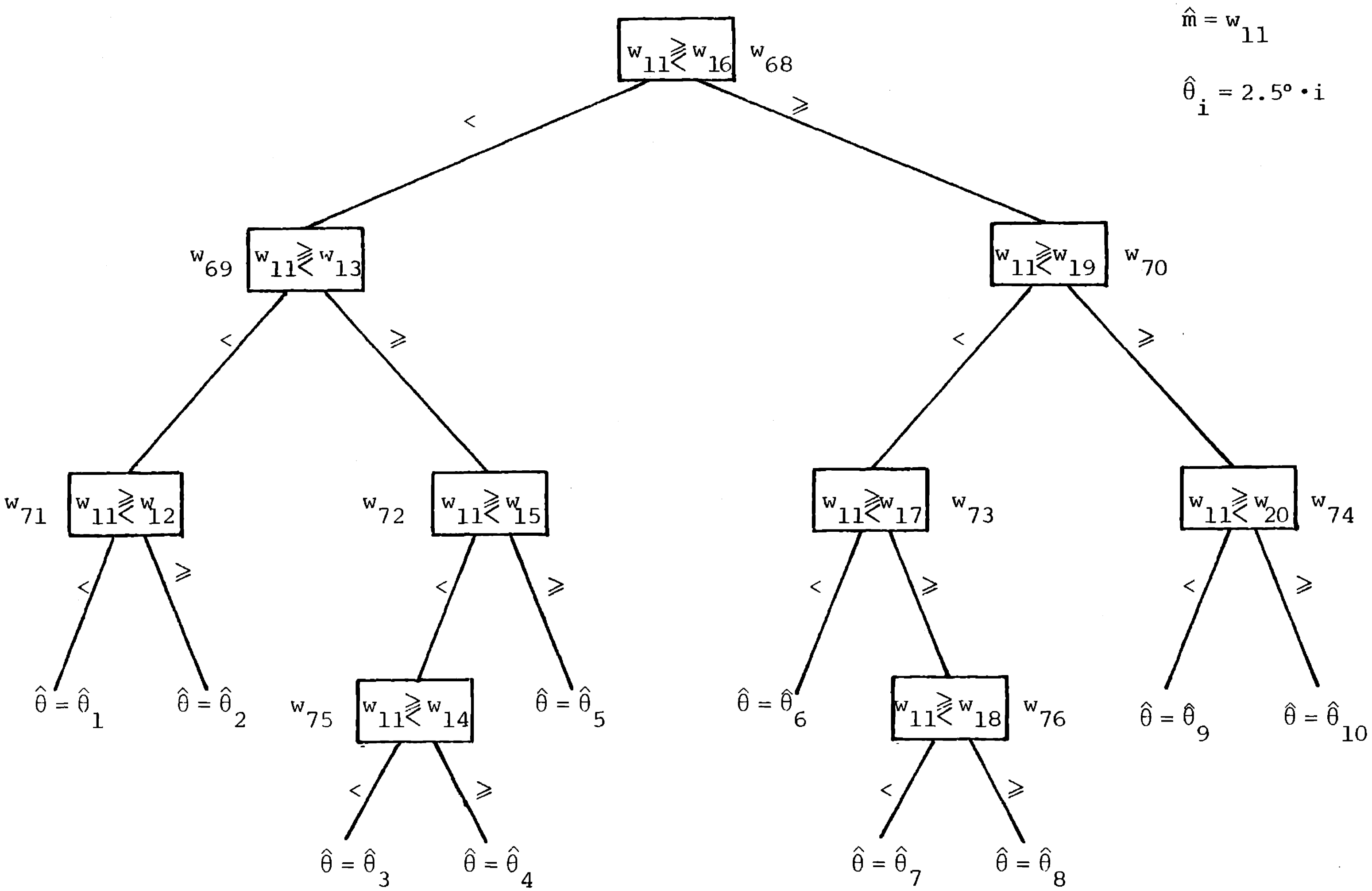


Figure C.1. Determination of $\hat{\theta}$ from $\hat{m}$ by Algorithm f(•). (Note: superscript on internal variable omitted (w_i^1).)

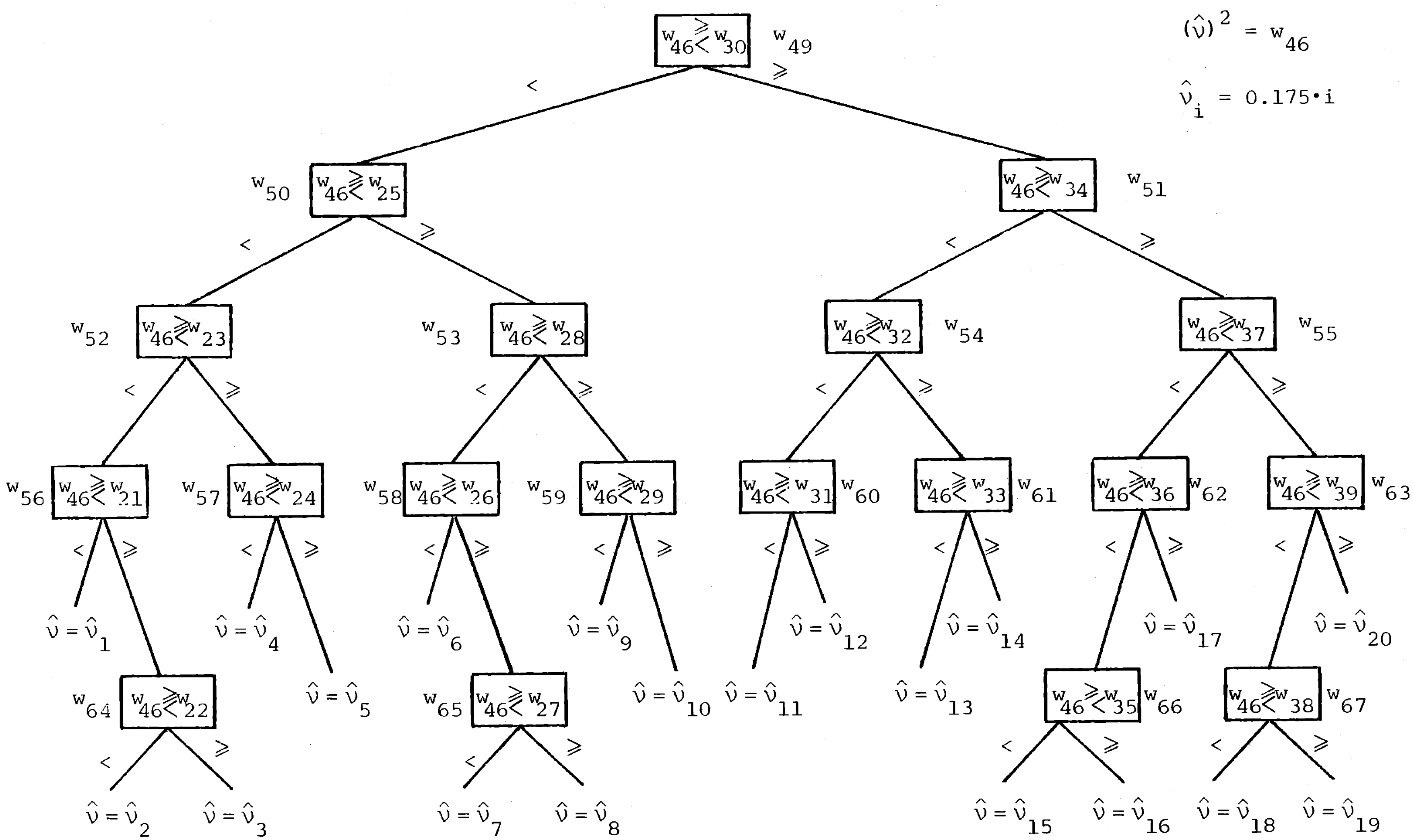


Figure C.2. Determination of $\hat{\nu}$ from $(\hat{\nu})^2$ by Algorithm f(•). (Note: superscript on internal variables omitted (w_i^1).)

characteristics of the input $\underline{x}$ are known, all variables in the algorithm defined above have known distributions and the quantities in the PLI, particularly the coordination g_c^1, can be computed. For the alphabets and distributions used,

$$g_c^1 = 91.6 \text{ bits} \tag{C.4}$$

C.4 RESPONSE SELECTION ALGORITHM $h_1(\cdot)$

C.4.1 General Description

The response selection algorithm $h_1(\cdot)$ determines the decision response based on a single comparison of $\hat{\theta}^1$ with a fixed threshold. The algorithm receives both $\hat{\theta}^1$ and $\hat{\nu}^1$, however, since it has been assumed to have no rejection. The particular value of the threshold used is

$$\beta = 35°$$

C.4.2 Variable Definition

The superscript 2 is omitted for convenience.

$$w_1 = \hat{\theta}^1 \qquad w_3 = \beta$$

$$w_2 = \hat{\nu}^1 \qquad w_4 = y^1$$

The comparison tree for determination of y^1, the output of algorithm $h_1(\cdot)$, is given in Figure C.3.

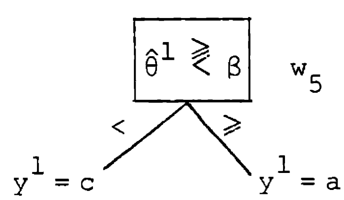


Figure C.3. Determination of y^1 From $\hat{\theta}^1$

The total number of internal variables, α_1', is equal to five. Because the admissible set of internal decision strategies is not dependent on $\underline{z}$, and because only a single situation assessment algorithm is present, $g_c^2(p(\underline{z}|v=1)) \equiv g_c^2$, i.e., the internal coordination of $h_1(\cdot)$ does not vary in this example. It is given by

$$g_c^2 = 1.6 \text{ bits} \tag{C.5}$$

C.5 RESPONSE SELECTION ALGORITHM $h_2(\cdot)$

C.5.1 General Description

Response selection algorithm $h_2(\cdot)$ receives $\hat{\theta}^2$ and $\hat{v}^2$ and executes a two-dimensional search through a "look-up table" to determine the decision response y^2. The particular thresholds of comparison used to generate the results presented are given as

$$c_3 = 2.6 \qquad c_5 = 5.2$$

$$c_4 = 4.0 \qquad c_6 = 6.3$$

$$d_2 = 14 \qquad d_3 = 21 \qquad d_4 = 30$$

C.5.2 Variable Definition

For convenience, the superscript 3 is omitted.

$$w_1 = \hat{\theta}^2 \quad w_3 = c_3 \quad w_5 = c_5 \quad w_7 = d_2 \quad w_9 = d_4$$

$$w_2 = \hat{v}^2 \quad w_4 = c_4 \quad w_6 = c_6 \quad w_8 = d_3 \quad w_{10} = y^2$$

The comparison tree used to search through the look-up table is given in Figure C.4. Nine additional internal variables are defined in order to effect the comparisons, and $\alpha_2' = 19$. Because the internal decision strategy is restricted to the class of independent distributions $p(v)$, the internal coordination of algorithm $h_2(\cdot)$ does not vary and is given in this example by

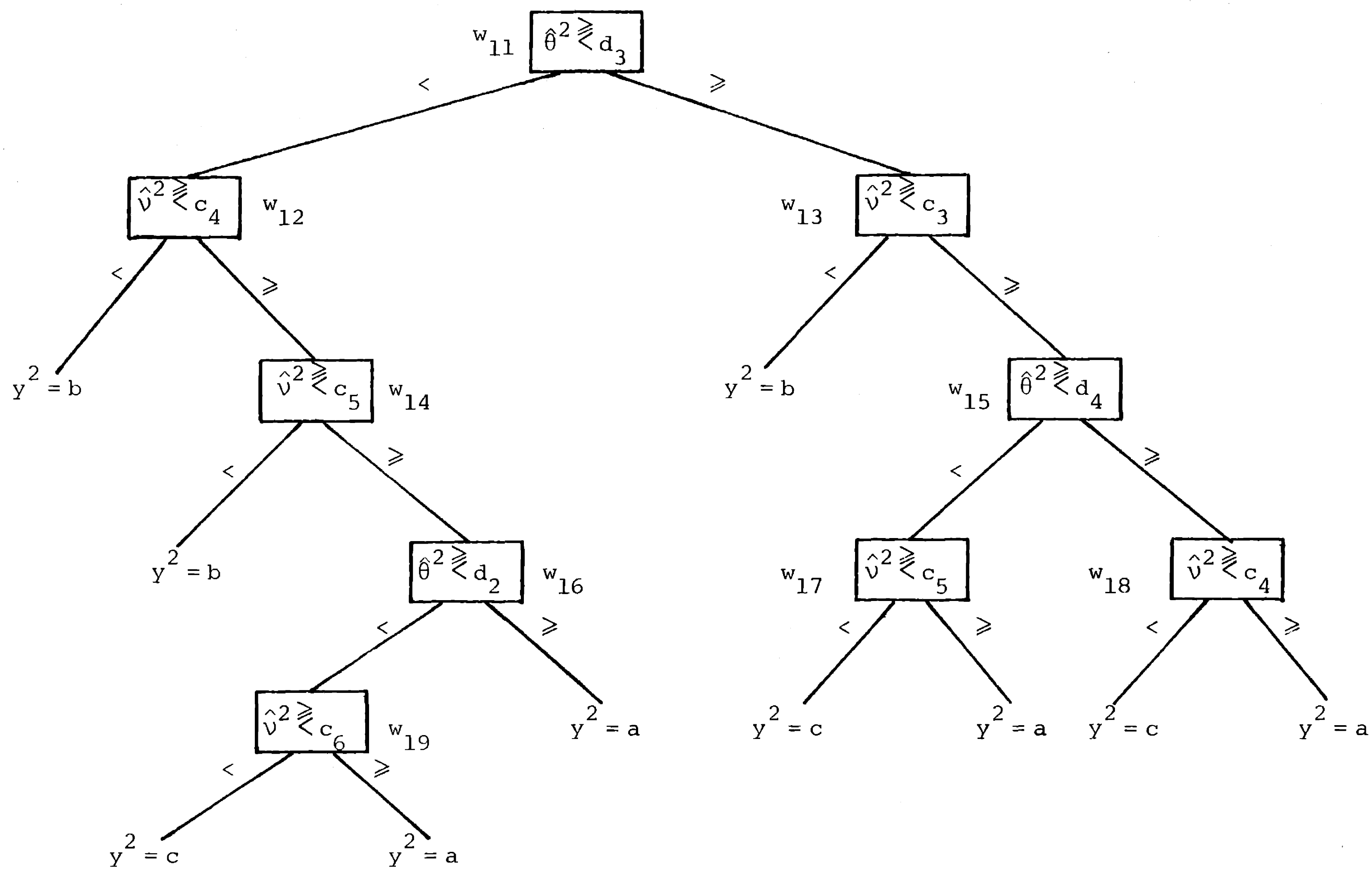


Figure C.4. Look up Table Search Tree--Algorithm $h_2(\cdot)$. (Note: superscript 3 omitted on internal variables (w_i^3).)

$$g_c^3 = 8.1 \text{ bits} \tag{C.6}$$

C.5 RESPONSE SELECTION ALGORITHM $h_3(\cdot)$

The general description and internal variable definition of algorithm $h_3(\cdot)$ are contained in Chapter 5; two facts are worthy of note here. First, the total number of variables in the algorithm, α_3', is 13. Second, the internal coordination is computed as

$$g_c^4 = 14.1 \text{ bits} \tag{C.7}$$

C.6 BINARY VARIATIONS BETWEEN D_1 AND D_2

The strategies which are binary variations of D_1 and D_2 are given by

$$D(\delta) = (1-\delta)D_1 + \delta D_2 \qquad 0 \le \delta \le 1 \tag{C.8}$$

In this section, it is shown that

$$G(D(\delta)) - [(1-\delta)G_1 + \delta G_2] \simeq (\alpha_1' + \alpha_2')\mathscr{H}(\delta) = 32\cdot\mathscr{H}(\delta), \tag{C.9}$$

where $G(D(\delta))$ is the total activity present when strategy $D(\delta)$ is used. From the analytic characterization of the model (Eqs. (4.71) - (4.75)), it is true that

$$G_1 \triangleq G(D_1) = G_t(D_1) + G_b(D_1) + g_c^1 + H(\underline{z}) + g_c^2 + H^{D_1}(y) \tag{C.10}$$

$$G_2 \triangleq G(D_2) = G_t(D_2) + G_b(D_2) + g_c^1 + H(\underline{z}) + g_c^3 + H^{D_2}(y) \tag{C.11}$$

$$G(D(\delta)) = G_n(D(\delta)) + G_t(D(\delta)) + G_b(D(\delta)) + g_c^1 + H(\underline{z}) + (1-\delta)g_c^2 + \delta g_c^3$$
$$+ (\alpha_1' + \alpha_2')\mathscr{H}(\delta) + H^{D(\delta)}(y) \tag{C.12}$$

where the superscript $D_i(D(\delta))$ denotes the use of strategy $D_i(D(\delta))$.

The sum of throughput and blockage is a constant in the model because of the auxiliary equation to the PLI, regardless of the strategy used. Eqs. (C.10) - (C.12) combine in a straightforward fashion to give

$$G(D(\delta)) - [(1-\delta)G_1 + \delta G_2] = G_n(D(\delta)) + (\alpha_1' + \alpha_2')\mathscr{H}(\delta)$$
$$+ \left\{H^{D(\delta)}(y) - [(1-\delta)H^{D_1}(y) + \delta H^{D_2}(y)]\right\} \tag{C.13}$$

Consider the term in braces in Eq. (C.13). $H^{D(\delta)}(y)$ is evaluated using the distribution $p_{D(\delta)}(y)$, where

$$p_{D(\delta)}(y) = (1-\delta)p_{D_1}(y) + \delta p_{D_2}(y) \tag{C.14}$$

and $p_{D_i}(y)$ is the distribution on y when the pure strategy D_i is used. Eq. (C.14) determines a convex distribution space and it is easy to show that

$$0 \le \left\{H^{D(\delta)}(y) - [(1-\delta)H^{D_1}(y) + \delta H^{D_1}(y)]\right\} \le \mathscr{H}(\delta) \tag{C.15}$$

Furthermore,

$$G_n(D(\delta)) = \mathscr{H}(\delta) \tag{C.16}$$

Substitution of Eqs. (C.15) and (C.16) into (C.13) yields

$$(\alpha_1' + \alpha_2' + 1)\mathscr{H}(\delta) \le G(D(\delta)) - [(1-\delta)G_1 + \delta G_2] \le (\alpha_1' + \alpha_2' + 2)\mathscr{H}(\delta) \tag{C.17}$$

Because $\alpha_1' + \alpha_2' = 32$, the following approximation is reasonable:

$$G(D(\delta)) - [(1-\delta)G_1 + \delta G_2] \simeq 32 \cdot \mathscr{H}(\delta) \tag{C.18}$$

Eq. (C.18) is Eq. (5.7).

APPENDIX D

APPLICATION OF PLI TO MODEL WITH ORGANIZATION INTERACTIONS

D.1 SUMMARY

In this appendix, the analytic expressions which characterize the model shown in Figure 6.4 are derived. The derivation proceeds by defining four subsystems in the model:

$$S^I = \{u, W^1, \ldots, W^U, \underline{z}\} \tag{D.1}$$

$$S^A = W^A \tag{D.2}$$

$$S^B = W^B \tag{D.3}$$

$$S^{II} = \{\bar{v}, W^{U+1}, \ldots, W^{U+V}, y\} \tag{D.4}$$

The resulting expressions are given by

- Amount of Internal Decision-Making

$$G_n = H(u) + H_{\underline{z}}(v) \tag{D.5}$$

- Throughput

$$G_t = T(\underline{x},\underline{z}',v':y) \tag{D.6}$$

- Blockage

$$G_b = H(\underline{x},\underline{z}',v') - G_t \tag{D.7}$$

• Coordination

$$G_c = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z})$$

$$+ g_c^A(p(\underline{z})) + g_c^B(p(\bar{\underline{z}}))$$

$$+ \sum_{j=1}^{V} p_j g_c^{U+j}(p(\bar{\underline{z}}|\bar{v}=j)) + \alpha_j' \mathscr{H}(p_j) + H(y)$$

$$+ H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{v},\bar{\underline{z}}) + T_{\underline{z}}(\underline{x}_1':\underline{z}') + T_{\bar{\underline{z}}}(\underline{x}_1',\underline{z}':v') \quad \text{(D.8)}$$

Eqs. (D.5) - (D.8) are valid if the algorithms $f_i(\underline{x})$, $h_j(\bar{\underline{z}})$, and $A(\underline{z},\underline{z}')$ are deterministic, known, and have no rejection, and if the algorithm $B(\bar{\underline{z}},v')$ is known, has no rejection, and is deterministic except for a dependence on the internal decision strategy $p(v|\bar{\underline{z}})$. In addition, if $p(\underline{x}_1')$, $p(\underline{q})$, $p(\underline{z}'|\underline{x}_1')$, and $p(v'|\underline{z}'\underline{x}_1')$ are known along with the internal coordinations g_c^i, g_c^{U+j}, g_c^A, and g_c^B as functions of their respective input characteristics and the numbers of internal variables α_i, α_j' are known, then Eqs. (D.5) - (D.8) describe the model as a function of the internal decision strategy $(p(u), p(v|\bar{\underline{z}}))$.

D.2 AMOUNT OF INTERNAL DECISION-MAKING

By definition,

$$G_n = H_{\underline{x},\underline{z}',v'}(u, W^1, \ldots, W^U, \underline{z}, W^A, W^B, \bar{v}, W^{U+1}, \ldots, W^{U+V}, y) \quad \text{(D.9)}$$

which can be written as (see Appendix A)

$$G_n = H_{\underline{x},\underline{z},v'}(u) + H_{\underline{x},\underline{z}',v',u}(W^1, \ldots, W^U, \underline{z})$$

$$+ H_{\underline{x},\underline{z}',v',u,W^1,\ldots,W^U,\underline{z}}(W^A, W^B, \bar{v})$$

$$+ H_{\underline{x},\underline{z}',v',u,W^1,\ldots,W^U,\underline{z},W^A,W^B,\bar{v}}(W^{U+1}, \ldots, W^{U+V}, y) \quad \text{(D.10)}$$

The second and last terms of Eq. (D.10) are zero because the conditioning variables determine the variables in question. As is discussed in Chapter 4, the situation assessment strategy is an independent quantity. Eq. (D.10) can then be written as

$$G_n = H(u) + H_{\underline{x},\underline{z}',v',u,W^1,\ldots,W^U,\underline{z}}(W^A)$$

$$+ H_{\underline{x},\underline{z}',v',u,W^1,\ldots,W^U,\underline{z}W^A}(W^B,\bar{v}) \tag{D.11}$$

where the identity (A.1) has been used to expand the third term in Eq. (D.10). The second term in Eq. (D.11) is zero since $\underline{z}'$ and $\underline{z}$ determine all the variables in W^A, including $\bar{\underline{z}}$. Recall that because of the definition of the operation of subsystem S^B, the variable v is an element of W^B. Let

$$W^B = \{v, \tilde{W}^B\} \tag{D.12}$$

where $\tilde{W}^B$ represents all other variables of W^B except v. The conditioning of the last term of Eq. (D.11) is equivalent to conditioning only on $\underline{z}$ and v', and G_n can then be written as

$$G_n = H(u) + H_{v'\bar{\underline{z}}}(v, \tilde{W}^A, \bar{v}) \tag{D.13}$$

or

$$G_n = H(u) + H_{v'\bar{\underline{z}}}(v) + H_{v'\bar{\underline{z}}v}(\tilde{W}^A, \bar{v}) \tag{D.14}$$

The last term in Eq. (D.14) is zero since specification of v' and v determines all variables of W^A except v. In addition, the distribution on v has been assumed to depend only on $\bar{\underline{z}}$, i.e.,

$$p(v|\bar{\underline{z}}v') \equiv p(v|\bar{\underline{z}}) \tag{D.15}$$

and the amount of internal decision-making is given in final form as

$$G_n = H(u) + H_{\underline{\bar{z}}}(v) \qquad \text{(D.16)}$$

Eq. (D.16) is computable as a function of $p(u)$ and $p(v|\underline{\bar{z}})$, if $p(\underline{\bar{z}})$ is known; the latter distribution is determined as a function of $p(u)$ from $p(\underline{x}_1')$, $p(\underline{q})$, $p(\underline{z}'|\underline{x}_1')$, and the algorithm mappings $f_i(\underline{x})$, $A(\underline{z},\underline{z}')$, $i = 1,2,\ldots,U$.

D.3 THROUGHPUT

By definition, the throughput of the system is given by

$$G_t = T(\underline{x},\underline{z}',v':y) \qquad \text{(D.17)}$$

Determination of the distributions $p(y)$, $p(y|\underline{x}\underline{z}'v')$ and $p(\underline{x}\underline{z}'v')$ is necessary in order to compute G_t. The distribution $p(y)$ is evaluated using Bayes' rule:

$$p(y) = \sum_{\bar{v},v',\underline{\bar{z}},\underline{\bar{z}}',\underline{z},u,\underline{x}_1',\underline{q}} p(y|\bar{v}v'\underline{\bar{z}}\underline{\bar{z}}'\underline{z}u\underline{x}_1'\underline{q})\,p(\bar{v}|v'\underline{\bar{z}}\underline{\bar{z}}'\underline{z}u\underline{x}_1'\underline{q})$$

$$\cdot\; p(\underline{\bar{z}}|v'\underline{z}'\underline{z}u\underline{x}_1'\underline{q})\,p(\underline{z}|v'\underline{z}'u\underline{x}_1'\underline{q})\,p(v'|\underline{z}'u\underline{x}_1'\underline{q})\,p(\underline{z}'|u\underline{x}_1'\underline{q})$$

$$\cdot\; p(u|\underline{x}_1'\underline{q})\,p(\underline{x}_1'|\underline{q})\,p(\underline{q}) \qquad \text{(D.18)}$$

Because the algorithms $f_i(\underline{x})$, $h_j(\underline{\bar{z}})$, and $A(\underline{z},\underline{z}')$ are deterministic, the following correspondence can be made:

$$p(y|\bar{v}v'\underline{\bar{z}}\underline{\bar{z}}'\underline{z}u\underline{x}_1'\underline{q}) \equiv p(y|\bar{v}\underline{\bar{z}}) \Leftrightarrow h_j(\underline{\bar{z}}) \qquad \text{(D.19)}$$

$$p(\underline{\bar{z}}|v'\underline{z}'\underline{z}u\underline{x}_1'\underline{q}) \equiv p(\underline{\bar{z}}|\underline{z}'\underline{z}) \Leftrightarrow A(\underline{z},\underline{z}') \qquad \text{(D.20)}$$

$$p(\underline{z}|v'\underline{z}'u\underline{x}_1'\underline{q}) \equiv p(\underline{z}|u\underline{x}_1'\underline{q}) \Leftrightarrow f_i(\underline{x}) \qquad \text{(D.21)}$$

In addition, $\bar{v}$ is dependent only on v' and $\underline{\bar{z}}$ through the $B(\underline{\bar{z}},v')$ algorithm, i.e.,

$$p(\bar{v}|v'\underline{\bar{z}}\underline{\bar{z}}'\underline{z}u\underline{x}_1'\underline{q}) \equiv p(\bar{v}|\underline{\bar{z}}v'), \qquad \text{(D.22)}$$

and $p(\bar{v}|\bar{\underline{z}}v')$ can be determined from $p(v|\bar{\underline{z}})$ and $b(v,v')$ as

$$p(\bar{v}|\bar{\underline{z}}v') = \sum_{v} p(\bar{v}|vv')p(v|\bar{\underline{z}}) \tag{D.23}$$

As discussed in Chapter 4,

$$p(u|\underline{x}_1'\underline{q}) \equiv p(u) \tag{D.24}$$

$$p(\underline{x}_1'|\underline{q}) \equiv p(\underline{x}_1') \tag{D.25}$$

Finally, because of the information structure of the organization

$$p(v'|\underline{z}'u\underline{x}_1'\underline{q}) \equiv p(v'|\underline{z}'\underline{x}_1') \tag{D.26}$$

$$p(\underline{z}'|u\underline{x}_1'\underline{q}) \equiv p(\underline{z}'|\underline{x}_1') \tag{D.27}$$

For known distributions $p(\underline{x}_1')$, $p(\underline{q})$, $p(\underline{z}'|\underline{x}_1')$, and $p(v'|\underline{z}'\underline{x}_1')$ and known algorithm mappings $f_i(\underline{x})$, $h_j(\bar{\underline{z}})$, $A(\bar{\underline{z}},\underline{z}')$, and $b(v,v')$, $i = 1,2,\ldots,U$, $j = 1,2,\ldots,V$ the distribution $p(y)$ is a function of the internal decision strategy $(p(u),\ p(v|\bar{\underline{z}}))$.

A similar result obtains for $p(y|\underline{x}\underline{z}'v')$, and the distribution $p(\underline{x}\underline{z}'v')$ is given by

$$p(\underline{x}\underline{z}'v') = \sum_{\underline{q},\underline{x}_1'} p(v'|\underline{z}'\underline{x}\underline{x}_1'\underline{q})p(\underline{z}'|\underline{x}\underline{x}_1'\underline{q})p(\underline{x}|\underline{x}_1'\underline{q})p(\underline{x}_1')p(\underline{q}) \tag{D.28}$$

where $\underline{x} = \underline{x}_1' + \underline{q}$. As shown by the above development, then, the throughput of the model is computable as a function of the internal decision strategy $(p(u),\ p(v|\bar{\underline{z}}))$.

D.3 BLOCKAGE

The auxiliary equation to the PLI can be used to determine G_b:

$$G_b = H(\underline{x},\underline{z}',v') - G_t \tag{D.29}$$

D.4 COORDINATION

The total coordination can be determined by use of the decomposition property of n-dimensional mutual information (see Appendix A, Eq. (A.4)):

$$G_c = G_c^I + G_c^A + G_c^B + G_c^{II} + T(S^I:S^A:S^B:S^{II}), \tag{D.30}$$

where the superscripts refer to the respective subsystems defined in Eqs. (D.1) - (D.4).

From the development in Chapter 4, the following can be written immediately:

$$G_c^I = \sum_{i=1}^{U} p_i g_c^i + \alpha_i \mathscr{H}(p_i) + H(\underline{z}) \tag{D.31}$$

$$G_c^A = g_c^A(p(\underline{z})) \tag{D.32}$$

$$G_c^B = g_c^B(p(\bar{\underline{z}})) \tag{D.33}$$

$$G_c^{II} = \sum_{j=1}^{V} p_j g_c^{U+j}(p(\bar{\underline{z}}|\bar{v}=j)) + \alpha_j' \mathscr{H}(p_j) + H(y) \tag{D.34}$$

where $p_j \triangleq p(\bar{v}=j)$ and the internal coordinations of the various algorithms have been expressed as functions of those inputs which are determined (at least in part) by strategy choice.

The last term in Eq. (D.30) represents the coordination among subsystems and can be written equivalently as (see Appendix A, Eq. (A.4)):

$$T(S^I:S^A:S^B:S^{II}) = T(S^I:S^A) + T(S^I,S^A:S^B) + T(S^I,S^A,S^B:S^{II}) \tag{D.35}$$

The first term in Eq. (D.35) is, by definition,

$$T(S^I:S^A) = H(S^A) - H_{S^I}(S^A) \tag{D.36}$$

which is equivalent to (see Appendix A)

$$T(S^I:S^A) = H(\underline{z},\underline{z}') - H_{u,\underline{x}}(\underline{z},\underline{z}') \tag{D.37}$$

$$T(S^I:S^A) = H(\underline{z}) + H_{\underline{z}}(\underline{z}') - H_{u,\underline{x}}(\underline{z}) - H_{u,\underline{x},\underline{z}}(\underline{z}') \tag{D.38}$$

The third term in Eq. (D.38) is zero since u and $\underline{x}$ determine $\underline{z}$. Eq. (D.38) then reduces to

$$T(S^I:S^A) = H(\underline{z}) + T_{\underline{z}}(u,\underline{x}:\underline{z}') \tag{D.39}$$

Consider the last term in Eq. (D.39). By Eq. (A.4) it can be written as

$$T_{\underline{z}}(u,\underline{x}:\underline{z}') = T_{\underline{z}}(u:\underline{z}') + T_{\underline{z},u}(\underline{x}:\underline{z}') \tag{D.40}$$

The variable $\underline{z}'$ is independent of the first stage algorithm choice because of the information structure of the organization; hence, the first term of Eq. (D.40) is zero. Furthermore, in the present model the relationship of $\underline{x}$ to $\underline{z}'$ is only that which is due to $\underline{x}_1'$ since $\underline{x} = \underline{x}_1' + \underline{q}$ and $\underline{q}$ is an independent quantity. Eq. (D.40) therefore reduces to

$$T_{\underline{z}}(u,\underline{x}:\underline{z}') = T_{\underline{z}}(\underline{x}_1':\underline{z}') \tag{D.41}$$

Substitution of Eq. (D.41) into Eq. (D.39) gives the desired form of the result:

$$T(S^I:S^A) = H(\underline{z}) + T_{\underline{z}}(\underline{x}_1':\underline{z}') \tag{D.42}$$

Consider the second term in Eq. (D.35). By definition, it can be written as

$$T(S^I,S^A:S^B) = H(S^B) - H_{S^I,S^A}(S^B) \tag{D.43}$$

Because of the structure of S^B, Eq. (D.43) is equivalent to

$$T(S^I,S^A:S^B) = H(\bar{\underline{z}},v,v') - H_{S^I,S^A}(\bar{\underline{z}},v,v') \quad (D.44)$$

$$= H(\bar{\underline{z}},v,v') - H_{u,\underline{x},\underline{z}',\bar{\underline{z}}}(\bar{\underline{z}},v,v') \quad (D.45)$$

where an equivalent conditioning has been used in the second term of Eq. (D.45). Successive application of Eq. (A.1) to Eq. (D.45) yields

$$T(S^I,S^A:S^B) = H(\bar{\underline{z}}) + H_{\bar{\underline{z}}}(v') + H_{\bar{\underline{z}},v'}(v) - H_{u,\underline{x},\underline{z}',\bar{\underline{z}}}(\bar{\underline{z}})$$

$$- H_{u,\underline{x},\underline{z}',\bar{\underline{z}}}(v') - H_{u,\underline{x},\underline{z}',\bar{\underline{z}},v'}(v) \quad (D.46)$$

Because v depends only on $\bar{\underline{z}}$, the third and last terms are equal in magnitude. In addition, the fourth term is zero. Eq. (D.46) is therefore equivalent to

$$T(S^I,S^A:S^B) = H(\bar{\underline{z}}) + T_{\bar{\underline{z}}}(\underline{x},u,\underline{z}':v'), \quad (D.47)$$

which can be reduced to its final form by consideration of the possible dependencies present as a result of the information structure:

$$T(S^I,S^A:S^B) = H(\bar{\underline{z}}) + T_{\bar{\underline{z}}}(\underline{x}_1',\underline{z}':v') \quad (D.48)$$

Finally, the last term in Eq. (D.35) is given by definition as

$$T(S^I,S^A,S^B:S^{II}) = H(S^I) - H_{S^I,S^A,S^B}(S^{II}) \quad (D.49)$$

which can be written equivalently as

$$T(S^I,S^A,S^B:S^{II}) = H(\bar{v},\bar{\underline{z}}) - H_{\bar{\underline{z}},\bar{v}}(\bar{v},\bar{\underline{z}}) \quad (D.50)$$

$$= H(\bar{v},\bar{\underline{z}}) \quad (D.51)$$

Substitution of Eqs. (D.42), (D.48) and (D.51) into Eq. (D.35) yields

$$T(S^I:S^A:S^B:S^{II}) = H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{v},\bar{\underline{z}}) + T_{\underline{z}}(\underline{x}_1':\underline{z}') + T_{\bar{\underline{z}}}(\underline{x}_1',\underline{z}':v') \tag{D.52}$$

and the total coordination result follows.

D.5 COMPUTATION OF COORDINATION FOR EXAMPLE IN SECTION 6.4

For the model shown in Figure 6.6., the coordination is evaluated as

$$G_c^{SI} = g_c^1 + H(\underline{z}) + g_c^A(p(\underline{z})) + \sum_{j=1}^{2} p_j g_c^{U+j}(p(\bar{z}|v=j)) + \alpha_j' \mathscr{H}(p_j) + H^{SI}(y)$$

$$+ H(\underline{z}) + H(\bar{\underline{z}}) + H(\bar{\underline{z}},v) + T_{\underline{z}}(\underline{x}_1':\underline{z}') \tag{D.53}$$

where SI denotes the situation input only model. The value of the input $\underline{z}'$ is such that $\bar{\underline{z}}$ is always $\bar{\underline{z}}_1$. Hence, no response selection algorithm switching occurs, and since the input to algorithm $h_1(\cdot)$ is deterministic, g_c^{U+1} is zero. The output takes a single value and therefore also has zero uncertainty, as does the variable $\bar{\underline{z}}$. Substitution of these facts into Eq. (D.53) yields

$$G_c^{SI} = g_c^1 + H(\underline{z}) + g_c^A(p(\underline{z})) + H(\underline{z}) + T_{\underline{z}}(x_1':\underline{z}') \tag{D.54}$$

The coordination of the basic two-stage model for this example is given by

$$G_c^2 = g_c^1 + H(\underline{z}) + (\alpha_1'+\alpha_2')\mathscr{H}(0.5) + H^2(y) + H(\underline{z}) \tag{D.55}$$

The difference between G_c^{SI} and S^2 is therefore given by

$$G_c^{SI} - G_c^2 = [g_c^A(p(z)) + T_{\underline{z}}(\underline{x}_1':\underline{z}')] - [(\alpha_1'+\alpha_2')\mathscr{H}(0.5) + H^2(y)] \tag{D.56}$$

This is Eq. (6.50).

REFERENCES

1. J. G. Wohl, "Battle Management Decisions in Air Force Tactical Command and Control," Report M79-233, MITRE, Bedford, Mass., December 1979.

2. Y. C. Ho, "Team Decision Theory and Information Structures," Proceedings of the IEEE, Vol. 68, No. 6, June 1980.

3. Y. C. Ho and K. C. Chu, "Team Decision Theory and Information Structures in Optimal Control Problems, Part I," IEEE Transactions on Automatic Control, Vol. AC-17, Feb. 1972.

4. Y. C. Ho and K. C. Chu, "Team Decision Theory and Information Structures in Optimal Control Problems, Part II," IEEE Transactions on Automatic Control, Vol. AC-17, Feb. 1972.

5. Simon, H. A., "A Behavioral Model of Rational Choice," Quarterly Journal of Economics, Vol. 69, 1956.

6. J. G. March, "Bounded Rationality, Ambiguity, and the Engineering of Choice," The Bell Journal of Economics, March, 1980.

7. H. A. Simon, Administrative Behavior, MacMillan Co., New York, 1961.

8. J. G. March and H. A. Simon, Organizations, John Wiley and Sons, New York, 1958.

9. J. Froyd and F. N. Bailey, "Performance of Capacity Constrained Decision Makers," Proceedings of the 19th IEEE Conference on Decision and Control, Albuquerque, NM, December 1980.

10. N. M. Papadopoulos, "The Garbling Decision Maker: A Model of Bounded Rationality," IEEE Transactions on Automatic Control, Vol. AC-25, No. 3, June 1980.

11. R. F. Drenick, "Organization and Control," in Directions in Large Scale Systems, Ho and Mitter, Eds., Plenum, New York, 1976.

12. R. F. Drenick and A. H. Levis, "A Mathematical Theory of Organization, Part I: Centralized Organization," Polytechnic Institute of New York, Brooklyn, NY, October, 1974.

13. J. R. Galbraith, Organization Design, Addison-Wesley, Reading, MA, 1977.

14. J. A. Wise, "Cognitive Bases for an Expanded Decision Theory," Proceedings of the 1979 IEEE International Conference on Cybernetics and Society, Denver, CO, October, 1979.

15. R. C. Conant, "Laws of Information Which Govern Systems," IEEE Transactions on Systems, Man and Cybernetics, Vol. SMC-6, No. 4, April, 1976.

16. C. E. Shannon and W. Weaver, The Mathematical Theory of Communication, University of Illinois, Urbana, 1949.

17. G. D. Forney, "Information Theory," Electrical Engineering Dept., Stanford University, 1972.

18. R. G. Gallager, Information Theory and Reliable Communication, John Wiley and Sons, New York, 1968.

19. W. J. McGill, "Multivariate Information Transmission," Psychometrika, Vol. 19, No. 2, June, 1954.

20. W. R. Ashby, "Two Tables of Identities Governing Information Flows Within Large Systems," Commun. Amer. Soc. Cybern., Vol. 1, No. 2, 1969.

21. W. R. Ashby, "Measuring the Internal Information Exchange in a System," Cybernetica, Vol. 8, 1965.

22. J. G. Miller, "Adjusting to Overloads of Information," in Organization: Systems, Control and Adaptation, Vol. II, J. A. Litterer, Ed., John Wiley and Sons, New York, 1969.

23. T. B. Sheridan and W. R. Ferrel, Man-Machine Systems, MIT Press, Cambridge, MA, 1974.

24. R. D. Luce and H. Raiffa, Games and Decisions, John Wiley and Sons, New York, 1957.

25. D. G. Luenberger, Introduction to Linear and Nonlinear Programming, Addison-Wesley Publishing Co., Reading, MA, 1973.

26. D. A. Stabile, "The Design of Information Structures: Basic Allocation Strategies for Organizations," S.M. Thesis, MIT, Cambridge, MA, June 1981.

27. R. F. Drenick, "A Mathematical Theory of Organization, Part II: Decentralized Organization," Polytechnic Institute of New York, Brooklyn, NY, December 1976.