Large Genus Bounds for the Distribution of Triangulated Surfaces in Moduli Space

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Abstract. Triangulated surfaces are compact Riemann surfaces equipped with a conformal triangulation by equilateral triangles. In 2004, Brooks and Makover asked how triangulated surfaces are distributed in the moduli space of Riemann surfaces as the genus tends to infinity. Mirzakhani raised this question in her 2010 ICM address. We show that in the large genus case, triangulated surfaces are well distributed in moduli space in a fairly strong sense. We do this by proving upper and lower bounds for the number of triangulated surfaces lying in a Teichmüller ball in moduli space. In particular, we show that the number of triangulated surfaces lying in a Teichmüller unit ball is at most exponential in the number of triangles, independent of the genus.

Contents

1. Introduction

In this paper, we consider compact Riemann surfaces obtained by gluing together equilateral triangles. We call such surfaces triangulated surfaces. We give genus independent bounds for the distribution of triangulated surfaces in the moduli space of Riemann surfaces.

Brooks and Makover started the study of triangulated surfaces in the context of hyperbolic geometry. In [5], they asked the following question.

Question 1. *What are the geometric properties of a large genus random triangulated surface?*

Here, the triangulated surface is assumed to have genus at least 2, and is equipped with the hyperbolic metric. Randomness is with respect to the counting measure on the discrete set of triangulated surfaces. Brooks and Makover studied T-triangle genus g triangulated surfaces in the range $T \sim 4g$, and showed that the systole of a random triangulated surface

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is asymptotically almost surely bounded below by a constant. The Cheeger constant and first eigenvalue of the Laplacian are also asymptotically almost surely bounded below by a constant. The diameter is asymptotically almost surely bounded above by around the logarithm of the genus. Subsequently Guth, Parlier and Young [22] studied Question 1 with respect to the canonical flat metric on triangulated surfaces and showed that a random triangulated surface asymptotically almost surely has large total pants length. Budzinski and Louf [10] also studied the flat metric on triangulated surfaces when $T \sim \theta g$ (for constants $\theta > 4$). They showed that a random point on a random triangulated surface asymptotically almost surely does not lie close to a short loop of nontrivial homotopy class.

A related question is:

Question 2. *What are the geometric properties of a large genus random hyperbolic surface?*

Here, randomness is with respect to the Weil-Petersson measure on moduli space $\mathcal{M}_g$. Mirzakhani studied Question 2 in [37] (following [34] and [35]). The systole of a surface is bounded below by a constant with positive probability in the large genus limit. The Cheeger constant and first eigenvalue of the Laplacian are asymptotically almost surely bounded below by a constant. The diameter is asymptotically almost surely bounded above by around the logarithm of the genus. Mirzakhani also proved in [37] that a random point on a random hyperbolic surface almost surely does not lie close to a short loop of nontrivial homotopy class. Guth, Parlier and Young [22] studied Question 2 and showed that a random hyperbolic surface asymptotically almost surely has large total pants length.

Further results on the geometry of random surfaces have been obtained in [8], [9], [11], [30], [39], [40], [41], [42], [48] and [49]. Similarities between answers to Question 1 and Question 2 motivate the following question.

Question 3. *How are triangulated surfaces distributed in the moduli space of Riemann surfaces, quantitatively?*

Question 3 was first asked by Brooks and Makover in [5]. Subsequently it has been raised in Mirzakhani's 2010 ICM address [36] as well as in [8], [13], [22] and [37].

We answer Question 3 by proving well distribution results for triangulated surfaces. Our answer is with respect to the Teichmüller metric (see Remark 7). One consequence of our main results (stated in Section 1.1) is the following simplified answer to Question 3.

Theorem 1.1. *In a Teichmüller* 1*-ball in* $\mathcal{M}_g$*, there are at most* C^T *number of* T*-triangle genus* g *triangulated surfaces where* C *is a constant independent of* g *and* T*.*

Henceforth, all generic universal constants will be denoted by C. In [18], Fletcher, Kahn and Markovic showed that the number of Teichmüller 1-balls needed to cover the thick part of $\mathcal{M}_g$ is around g^{2g}. For $g = O(T)$, the number of T-triangle genus g surfaces is around $g^{2g}C^T$, which was computed by Budzinski and Louf in [10]. In this range, for triangulated surfaces to be well distributed in moduli space means there are around C^T surfaces in each 1-ball. Theorem 1.1 gives such an upper bound. We also prove a lower bound (with different constants), which is stated in Section 1.1.

Note that the sphere admits a unique conformal structure, and the number of distinct T-triangle triangulations of the sphere is also around C^T. In this sense, Theorem 1.1 is genus independent. When we fix the conformal class, the higher genus case behaves just like the genus 0 case.

Triangulated surfaces are combinatorial objects, while moduli space parametrizes geometric objects. The difficulty in proving Theorem 1.1 lies in the difficulty of distinguishing if two particular triangulated surfaces are close in moduli space. Moreover, given a triangulated surface, it is difficult to explicitly determine the hyperbolic metric on it. Given two marked hyperbolic surfaces, it is difficult to determine if they are close in moduli space. Indeed, the results of Mirzakhani on random hyperbolic surfaces show that the usual geometric quantities like systole, diameter and Cheeger constant fail to distinguish most hyperbolic surfaces.

1.1. **Statement of main results.** We prove two results which describe the distribution of triangulated surfaces in $\mathcal{M}_g$. The following lower bound describes when a surface in moduli space can be approximated by a triangulated surface.

Theorem 1.2. *Let $g \geq 2$. Let $X \in \mathcal{M}_g$ equipped with the conformal hyperbolic metric. Let*

$$R = \sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \mathrm{length}(\gamma) < 2\,\mathrm{arcsinh}(1)}} \mathrm{length}(\gamma)^{-1}.$$

Then for $r \in (0,1]$, there exists a $T(r)$-triangle triangulated surface inside a Teichmüller r-ball around X, where $T(r) \leq C(R+g)r^{-2}$. Here, C is a universal constant (independent of g, T, r and R).

Remark 1. In particular, if X lies in the thick part of $\mathcal{M}_g$, meaning $\mathrm{sys}\, X \geq 2\,\mathrm{arcsinh}(1)$, then there exists a Cgr^{-2}-triangle triangulated surface within Teichmüller distance r from X.

Remark 2. The condition that there should not be too short geodesics on X is necessary, as we shall see in Lemma 2.4.

We also have the following upper bound regarding the distribution of triangulated surfaces in $\mathcal{M}_g$.

Theorem 1.3. *There exists at most C^{T+rg} number of T-triangle triangulated surfaces in a Teichmüller r-ball in $\mathcal{M}_g$. Here, C is a universal constant (independent of g, T and r).*

Remark 3. By Euler characteristic conditions, $g = O(T)$. So substituting $r = 1$ in Theorem 1.3 we obtain Theorem 1.1.

Remark 4. In a Teichmüller 1-ball in the thick part of $\mathcal{M}_g$, the lower and upper bounds for the number of triangulated surfaces given by Theorem 1.2 and Theorem 1.3 differ by a multiple of $\exp(O(T))$ for T/g sufficiently large.

Remark 5. It is useful to first ask the combinatorial question: what is the number of T-triangle genus g triangulated surfaces, as a function of T and g? The best bounds for this question in the linear range are proved in [10, Theorem 3]. This bound has a multiplicative error term of $\exp(o(T))$, in contrast to our slightly worse error of $\exp(O(T))$ as stated in Remark 4. (The constants in both error terms are independent of the genus.)

Remark 6. Theorem 1.3 is most interesting in the range $T \sim \theta g$ where $\theta \geq 4$ is a constant. In this range, we obtain that the number of triangulated surfaces in a Teichmüller 1-ball grows roughly exponentially in g as $g \to \infty$. This is similar to how integer points are distributed in high dimensional Euclidean space (which is a useful toy example). In $\mathbb{R}^n$, a radius $\sim n^{1/2}$ ball has volume ~ 1. Such a ball contains around C^n integer points. It is not possible to give a better bound for the number of integer points as a small translation of the ball can change this number by an exponential multiplicative factor.

Remark 7. There are several possible choices of metrics on moduli space. In Teichmüller theory, the Teichmüller metric is a natural choice. Since we are also interested in the hyperbolic geometry of individual surfaces in moduli space, we may also consider the bi-Lipschitz metric on moduli spaces. In this metric, the distance between two surfaces measures how far apart their hyperbolic metrics are. It turns out that the bi-Lipschitz metric is comparable to the Teichmüller metric, with genus independent constants (see Proposition 3.9). So Theorem 1.2 and Theorem 1.3 hold with respect to the bi-Lipschitz metric also. Another natural choice of metric on moduli space is the Weil-Petersson metric. Note that the Weil-Petersson volume of $\mathcal{M}_g$ is around g^{2g}, as computed by Penner in [43] and Grushevsky in [21], and subsequently improved by Mirzakhani and Zograf in [38]. Up to an exponential factor of T this number is also the approximate number of T-triangle genus g triangulated surfaces. So we may ask if an analogue of our main results hold for the Weil-Petersson metric as well. However, we do not know the answer to this question because we do not yet understand the large genus local geometry of the Weil-Petersson metric in a sufficiently detailed manner.

1.2. **Key ideas in the proof of Theorem 1.2.** Our approach is based on a characterization of the Teichmüller metric in [26] using extremal length and Jenkins-Strebel differentials, which we explain in Section 3. To show Theorem 1.2, given a surface in moduli space we construct a certain nicely behaved triangulation of it, take the associated triangulated surface. We use the characterization of the Teichmüller metric above to show that the triangulated surface is close to the original surface in moduli space. We do this in Section 4.

1.3. **Key ideas in the proof of Theorem 1.3.** There are roughly four parts to our proof:

(1) Riemann surfaces equipped with a holomorphic 1-form are called translation surfaces. We first consider the subset of triangulated surfaces that are actually translation surfaces where the associated holomorphic 1-form is compatible with the triangulation. We call such surfaces combinatorial translation surfaces, a term we define precisely in Section 2. In this situation, the 1-form gives us a cohomology class, which we then deal with using Hodge theory. Suppose two combinatorial translation surfaces are close together in moduli space and so are their cohomology classes. Then, we show that constraints coming from Hodge norms imply geometric constraints on how close or far apart vertices, edges, and faces of the two surfaces must be to each other. A combinatorial argument shows that these geometric constraints imply that the two triangulations are close except on a part of the surface with much smaller genus. So we reduce the counting problem for combinatorial translation surfaces to the counting problem for triangulated surfaces in a lower dimensional moduli space. As a result, we get bounds on combinatorial translation surfaces in terms of bounds on triangulated surfaces. We do this in Section 6.

(2) Given any triangulated surface, there exists a degree six branched cover which is a combinatorial translation surface. We enumerate the number of possibilities for such covers and study the possibilities for where the branch points lie, to get bounds on triangulated surfaces given bounds on combinatorial translation surfaces. Combining with our previous bounds described in (1), we obtain recursive upper bounds for the number of triangulated surfaces lying in a ball in moduli space. We solve these recursive bounds to show Theorem 1.3. We do this in Section 7.

(3) For the technicalities in (1) and (2) to work, we require the use of bounded degree triangulations instead of arbitrary triangulations. In Section 5, we show that any triangulated surface may be approximated by a bounded degree triangulation in a way that increases the number of triangles by at most a constant factor.

These three sections contain the key steps of the proof of Theorem 1.3. Before that, in Section 2, we introduce various definitions and prove several preliminary results. In Section 3, we prove results related to quasiconformal maps and the large genus geometry of Teichmüller space.

1.4. **Comments and references.** Non-quantitative versions of Question 3 have been studied in number theory. Belyi's theorem states that Riemann surfaces defined over the algebraic numbers $\overline{\mathbb{Q}}$ are exactly the Riemann surfaces which admit a branched cover (Belyi map) to $\mathbb{P}^1$ branched only at 0, 1 and ∞. Belyi maps give triangulations on the Belyi surface, and conversely, triangulations give Belyi maps. Note that Riemann surfaces defined over $\overline{\mathbb{Q}}$ are dense in $\mathcal{M}_g$, which is a non-quantitative answer to Question 3. Given this context, Theorem 1.2 may be interpreted as a quantitative version of this statement. It describes how well one can approximate an arbitrary surface in $\mathcal{M}_g$ by a Belyi surface with respect to the Teichmüller metric, in terms of the degree of the Belyi map. Another approach to approximating arbitrary Riemann surfaces by Belyi surfaces is described by Bishop in [3] and [4, Section 15].

The study of random triangulations is a central topic of research in probability theory. In the large genus setting, progress has been made in the study of local limits of triangulations. In the range $T \sim 4g$ (which is the probabilistically expected range, as shown by Gamburd in [19]), local limits do not exist since the expected surface has very few vertices and very high degrees of vertices. In the range $T \sim \theta g$ where $\theta > 4$, convergence does occur. Planar stochastic hyperbolic triangulations introduced by Curien in [14] are a family of random triangulations in the plane. These were conjectured by Benjamini and Curien to be local limits of high genus random triangulations. This was proved by Budzinski and Louf in [10]. However, global questions, such as scaling limits, are difficult to understand in the large genus case. See [28], [31] and [33] for some results on scaling limits of random maps in the planar setting.

In [7], Buser and Sarnak related the homological systole of a hyperbolic Riemann surface to the systole of its Jacobian. In [12], Balacheff, Parlier and Sabourau gave a way to find a minimal homology basis on a hyperbolic Riemann surface, and used that to deduce more general results about the geometry of its Jacobian. The original method to prove Theorem 1.3 involved bounding the number of ways to express a hyperbolic Riemann surface as a triangulated surface by studying the geometry of its Jacobian, using [12]. Then, it turned out that these ideas were not necessary to prove Theorem 1.3, so they do not appear in the proofs henceforth.

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2. Preliminaries

2.1. **Triangulated surfaces.** Let S_g be a genus g surface, and $S_{g,b}$ a genus g surface with b boundary components. Let S be a metrized simplicial 2-complex wherein each 2-simplex is an oriented unit equilateral triangle and gluings of faces preserve orientations. We call S a triangulated surface if it is homeomorphic to S_g for some $g \geq 0$ and a triangulated surface with boundary if it is homeomorphic to $S_{g,b}$ for some $g, b \geq 0$. If S is a triangulated surface with or without boundary, we may consider each equilateral triangle as embedded in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The complex structure on each equilateral triangle of S is preserved when edges are glued. Extending the complex structure over the interior vertices of S, we obtain a canonical complex structure on S. In this way, we consider S as a Riemann surface.

Remark 8. The construction of triangulated surfaces (without boundary) in [5] is done slightly differently. Instead of gluing together equilateral triangles, hyperbolic ideal triangles are used, and the resulting surface with cusps is compactified. One may check that these two constructions result in the same Riemann surface.

Given a triangulated surface S (with or without boundary), we denote by $V(S)$ the vertices of S, $E(S)$ the edges of S, and $F(S)$ the triangular faces of S. Given $v \in V(S)$ the degree of v is the number of edges emanating from v. We also denote by $V_{>6}(S)$, $V_{\neq 6}(S)$ and $V_{<6}(S)$ the set of vertices of S of degree strictly greater than 6, not equal to 6 and strictly less than 6, respectively.

2.2. **Space of triangulated surfaces.**

Definition 1. $\mathrm{Comb}^T(\mathcal{M}_g)$ is the set of all triangulated surfaces of genus g with T triangles, up to simplicial isomorphism.

We have a map

$$\Phi : \mathrm{Comb}^T(\mathcal{M}_g) \to \mathcal{M}_g$$

which takes a triangulated surface to its underlying Riemann surface in $\mathcal{M}_g$. We also have canonical biholomorphisms $\Phi_S : S \to \Phi(S)$ for all $S \in \mathrm{Comb}^T(\mathcal{M}_g)$.

Note that $T \geq 2$ for $\mathrm{Comb}^T(\mathcal{M}_g)$ to be nonempty; moreover, Euler characteristic conditions imply that $T \geq 4g - 4$. Together these imply $g/T \leq 1/2$. In the future we will implicitly assume this condition.

Finally, we denote by $\mathcal{M}_{\leq g}$ the union of $\mathcal{M}_{g'}$ over all $g' \leq g$. Similarly, we denote by $\mathrm{Comb}^{\leq T}(\mathcal{M}_{\leq g})$ the union of $\mathrm{Comb}^{T'}(\mathcal{M}_{g'})$ over all $g' \leq g$, $T' \leq T$. We will use this type of notation with respect to other spaces we will define later as well.

2.3. **Translation surfaces and combinatorial translation surfaces.**

Definition 2. A translation surface is a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form on X.

The metric $|\omega|$ is a flat metric on X with singularities at zeros of ω. See [47] for an introduction to translation surfaces. The following result gives an alternative definition of translation surfaces.

Proposition 2.1 ([47], Proposition 1.6 and Proposition 1.8)**.** *Any translation surface* (X, ω) *can be expressed in the following manner:* X *is the union of a collection of polygons* $P_1, ..., P_n$ *in* $\mathbb{C}$ *together with a choice of identification of parallel boundary edges of equal length on opposite sides, and* ω *is the* 1*-form* dz *on each polygon. Similarly, any collection of polygons* $P_1, ..., P_n \subset \mathbb{C}$ *with edge identifications as above defines a translation surface.*

We shall see that some triangulated surfaces are canonically translation surfaces as well. To this end, we define a combinatorial translation structure on a triangulated surface. Let S be a triangulated surface.

Definition 3. A combinatorial translation structure on S is an assignment, to each edge $e \in E(S)$ and vertex $v \in V(S)$ such that e emanates from v, a 6th root of unity $\zeta(e, v)$ (called a directional weight). Directional weights must satisfy the following two properties:

(1) if e contains the vertices v and w, then $\zeta(e, v) = -\zeta(e, w)$ and
(2) if e_1 and e_2 are two edges emanating from a vertex v that lie on a triangle of S such that e_1 lies counterclockwise from e_2 according to the orientation on S, then $\zeta(e_1, v) = e^{\pi i/3}\zeta(e_2, v)$.

Conditions 1 and 2 imply that for each triangle, there are only two possibilities for directional weights, which we label as Type A and Type B as seen in Fig. 1 and Fig. 2, respectively.

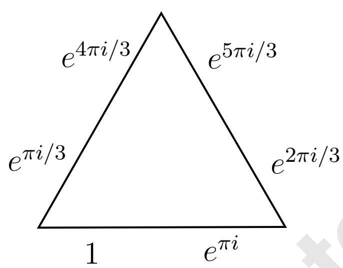

FIGURE 1. Type A triangle

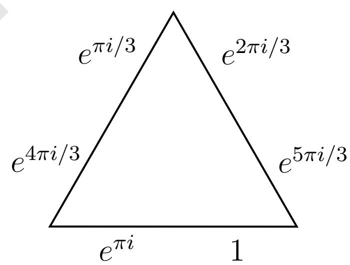

FIGURE 2. Type B triangle

Then, we have the following proposition.

Proposition 2.2. *Each triangulated surface* S *that admits a combinatorial translation structure is a translation surface wherein the associated flat metric agrees with the flat metric coming from the triangulation. Moreover, the associated holomorphic* 1*-form is canonical in the sense that it only depends on* S *and the combinatorial translation structure.*

Proof. Rotating as necessary, we identify Type A triangles with the equilateral triangle in $\mathbb{C}$ having vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and Type B triangles with the equilateral triangle in $\mathbb{C}$ having vertices at 0, 1 and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Conditions 1 and 2 in the definition of combinatorial translation surface imply that all edge identifications must be of a Type A triangle with a Type B triangle on opposite sides along parallel edges. By Proposition 2.1, S is a translation surface. Under the identification of Type A triangles and Type B triangles with triangles in $\mathbb{C}$ described above, the 1-forms dz on each triangle glue to give a 1-form ϕ on S. The 1-form ϕ only depends on S and the combinatorial translation structure. Finally $|dz|$ is simply the Euclidean metric on each triangle. $\square$

Note that given a combinatorial translation structure on S, then for $0 \leq i \leq 6$ we have another combinatorial translation structure on S obtained by multiplying each directional weight by $e^{\pi i/3}$. These six structures are the only valid combinatorial translation structures on S: once we assign directional weights to one triangle on S, there is only one choice for all other directional weights.

Definition 4. A combinatorial translation surface is a triangulated surface equipped with a combinatorial translation structure.

Definition 5. $\mathrm{Comb}^T(\mathcal{H}_g)$ is the set of combinatorial translation surfaces of genus g with T triangles.

By Proposition 2.2, any $S \in \mathrm{Comb}^T(\mathcal{H}_g)$ determines a canonical holomorphic 1-form on S that we denote by ϕ_S. We call the flat metric on S the S-metric. Its length element is $ds_S = |\phi_S|$, and area element is $|\phi_S|^2$. Distances in this metric shall be denoted by $d_S(\cdot,\cdot)$. As in the case of triangulated surfaces, we denote by

$$\Phi : \mathrm{Comb}^T(\mathcal{H}_g) \to \mathcal{T}_g$$

the map which sends a combinatorial translation surface to the underlying Riemann surface.

2.4. **Extremal length on annuli.** Let A be a Riemann surface that is topologically an annulus. By the uniformization theorem A is biholomorphic to

$$A(r) = \{z \in \mathbb{C} | 1 < |z| < r\}$$

for some $r > 1$. The modulus of A, denoted $\mathrm{mod}(A)$, is the quantity $(1/2\pi)\log r$.

Now, denote by γ a generator of $H_1(A,\mathbb{Z})$.

Definition 6. Given a Riemannian metric ρ on A, the quantity

$$\mathrm{length}_\rho(\gamma)$$

is defined to be the infimum of lengths in the ρ-metric over all curves representing γ.

Definition 7. The extremal length of γ on A is

$$\mathrm{Ext}_A(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(A)}$$

where the supremum is taken over all conformal metrics ρ on A.

The following result is proved in [1, Section 1.D].

Proposition 2.3. *We have,* $\mathrm{Ext}_A(\gamma) = \mathrm{mod}(A)^{-1}$.

2.5. **Hyperbolic metric on a triangulated surface.** In this section we prove two lemmas about the hyperbolic metric on a triangulated surface. The first lemma is about short geodesics.

Lemma 2.4. *Let S be a T-triangle triangulated surface of genus g, and $X = \Phi(S)$. Denote by ρ_X the hyperbolic metric on X. Then*

$$\sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \mathrm{length}_{\rho_X}(\gamma) < 2\operatorname{arcsinh}(1)}} \mathrm{length}_{\rho_X}(\gamma)^{-1} \leq CT$$

where C is a universal constant.

Proof. Let γ be a simple closed geodesic on X with $\operatorname{length}_{\rho_X}(\gamma) < 2\operatorname{arcsinh}(1)$. By the collar theorem [6, Theorem 4.1.1], γ has an associated hyperbolic annular collar A_γ of width

$$C \log(\operatorname{length}_{\rho_X}(\gamma)^{-1}).$$

Moreover, the collars A_γ associated to the short geodesics are all mutually disjoint. The modulus of A_γ is

$$C \operatorname{length}_{\rho_X}(\gamma)^{-1}.$$

By Proposition 2.3,

$$\frac{\operatorname{sys}_S(A_\gamma)^2}{\operatorname{area}_S(A_\gamma)} \leq C \operatorname{length}_{\rho_X}(\gamma).$$

Since $\operatorname{sys}_S(A_\gamma) \geq 1$, we have

$$\operatorname{area}_S(A_\gamma) \geq C \operatorname{length}_{\rho_X}(\gamma)^{-1}.$$

Since the A_γ are all disjoint,

$$\begin{aligned}\sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \operatorname{length}_{\rho_X}(\gamma) < 2\operatorname{arcsinh}(1)}} \operatorname{length}_{\rho_X}(\gamma)^{-1} &\leq C \sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \operatorname{length}_{\rho_X}(\gamma) < 2\operatorname{arcsinh}(1)}} \operatorname{area}_S(A_\gamma) \\ &\leq CT\end{aligned}$$

as desired. $\square$

Next, we show that the hyperbolic metric on a triangulated surface admits a nicely behaved covering by hyperbolic balls.

Lemma 2.5. *Let X be a hyperbolic surface, with metric ρ_X. Suppose*

$$\sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \operatorname{length}_{\rho_X}(\gamma) < 2\operatorname{arcsinh}(1)}} \operatorname{length}_{\rho_X}(\gamma)^{-1} \leq R.$$

Then, there exist hyperbolic disks $U_1, ..., U_N$, $V_1, ..., V_N$ and $W_1, ..., W_N$ on X such that the following conditions are satisfied:

1. *The $\{W_i\}$ together cover X.*
2. $$\operatorname{center}(U_i) = \operatorname{center}(V_i) = \operatorname{center}(W_i)$$
   and
   $$\operatorname{radius}(U_i) = 2\operatorname{radius}(V_i) = 4\operatorname{radius}(W_i) \leq \operatorname{arcsinh}(1)/2.$$
3. *If U_i nontrivially intersects U_j, then* $\operatorname{radius}(U_i) \leq C \operatorname{radius}(U_j)$.
4. *Any point $x \in X$ is contained in at most C of the U_i.*
5. *Any U_i nontrivially intersects at most C of the U_j.*
6. *$N \leq C(R + g)$.*

Here, C is a universal constant.

Remark 9. Once the U_i and W_i are constructed, the existence of the V_i is clear. Lemma 2.5 will be used to prove Theorem 1.2 in Section 5, and in that proof the V_i will become relevant. Lemma 2.5 will also be used in Section 7.5.

Proof. We divide X into the thick part and thin part. The thick part has injectivity radius at least arcsinh(1). The thin part is the union of disjoint annuli around each short geodesic.

The thick part has area at most Cg. On this part, we take a maximal arcsinh(1)/16-separated set. This set has around Cg points. Let the W_i be radius arcsinh(1)/8 disks around the points in the separated set, the V_i radius arcsinh(1)/4 disks around these points and the U_i radius arcsinh(1)/2 disks around these points.

Next, we construct the disks on the thin part. Let γ be a geodesic on X with length less than $2\operatorname{arcsinh}(1)$. Let A_γ be the annulus associated to γ, which is a connected component of the thin part. By the collar theorem, each annulus A_γ has width approximately

$$C\log(\mathrm{length}_{\rho_X}(\gamma)^{-1}).$$

We have coordinates (r,θ) on A_j where $r=0$ on γ and $r\in[w(A_\gamma)/2, w(A_\gamma)/2]$ such that the hyperbolic metric on A_γ is

$$d\rho_X^2 = dr^2+\cosh^2 r d\theta^2.$$

The injectivity radius of A_γ at any point (r,θ) (denoted $\mathrm{injrad}_{(r,\theta)}(A_\gamma)$) is around

$$C\,\mathrm{length}_{\rho_X}(\gamma)\cosh r.$$

Now, for $s\in\mathbb{N}\cap[-w(A_\gamma)/2, w(A_\gamma)/2]$, let

$$A_\gamma^s = A_\gamma\cap\{r\in[s-1,s+1]\}.$$

Denote by $\mathrm{injrad}(A_\gamma^s)$ the injectivity radius of A_γ^s. Choose a $\mathrm{injrad}(A_\gamma^s)/16$-separated set on A_γ^s to be the centers of W_i, so that the W_i are radius $\mathrm{injrad}(A_\gamma^s)/8$ disks around these centers, the V_i are radius $\mathrm{injrad}(A_\gamma^s)/4$ disks, and the U_i are radius $\mathrm{injrad}(A_\gamma^s)/2$ disks. In each A_γ^s there are at most

$$(C\,\mathrm{length}_{\rho_X}(\gamma)\cosh s)^{-1}$$

centers. In total, there are at most

$$\begin{aligned} C\sum_{s=1}^{\lceil w(A_\gamma)\rceil}(\mathrm{length}_{\rho_X}(\gamma)\cosh s)^{-1} &\le C\,\mathrm{length}_{\rho_X}(\gamma)^{-1}\int_{s=0}^{\infty}(\cosh s)^{-1}\\ &\le C\,\mathrm{length}_{\rho_X}(\gamma)^{-1}\end{aligned}$$

centers in A_γ.

We simply take all the U_i, V_i and W_i constructed on the thick part and thin part. By construction, conditions 1, 2, 3, 4 and 5 are satisfied. The total number of U_i (or V_i or W_i) is at most

$$C\left(g+\sum_{\substack{\gamma \text{ simple closed geodesic on } X\\ \mathrm{length}_{\rho_X}(\gamma)<2\operatorname{arcsinh}(1)}}\mathrm{length}_{\rho_X}(\gamma)^{-1}\right)\le C(R+g)$$

which shows condition 6. $\square$

2.6. **Conformal doubles and triangulated surfaces.** Denote by $S_{g,b}$ a surface of genus g with b boundary components. Let X be a Riemann surface of genus g with b boundary components, meaning that X is homeomorphic to $S_{g,b}$.

Definition 8. The conformal mirror X^{-1} of X is the Riemann surface whose local coordinates are obtained by composing each local coordinate z on X with the anti-holomorphic map $z \to \overline{z}$.

We may construct another Riemann surface, X^d, by gluing X and X^{-1} along their boundaries ∂X and ∂X^{-1} which are canonically identified. The complex structure on X^d in a neighborhood U of a point $p \in \partial X$ is given as follows. If z is a local coordinate on X under which $U \cap X$ is conformally a half-disk around p, then the local coordinate on U which is z on $U \cap X$ and $\overline{z}$ on $U \cap X^{-1}$ identifies U with a disk. This gives a complex structure on U which can be checked to be independent of the choice of local coordinate z. In this way, we obtain a complex structure on X^d.

Definition 9. The conformal double of X is the surface X^d constructed above.

Then X^d is a compact genus $2g+b-1$ surface which has an anti-holomorphic involution fixing $\partial X = \partial X^{-1} \subset X^d$. We have the following statement about conformal doubles of triangulated surfaces.

Proposition 2.6. *Suppose S is a triangulated surface with boundary. Then S^d is canonically a triangulated surface without boundary.*

Proof. Denote by S^{-1} the triangulated surface wherein each triangle of S is equipped with the opposite orientation. Since ∂S and ∂S^{-1} are naturally identified, gluing S and S^{-1} under this identification produces a triangulated surface S^d which can be checked to be the conformal double of S. $\square$

2.7. **Triangulated surfaces with boundedness properties.** In this section we define several subsets of $\mathrm{Comb}^T(\mathcal{M}_g)$ that shall be useful in the proof of Theorem 1.3.

Let E be an equilateral triangle in $\mathbb{C}$ of side length ℓ.

Definition 10. A k-subdivision of E is the unique triangulation by k^2 equilateral triangles of side length ℓ/k.

Let T be a triangulation of a surface (possibly with boundary).

Definition 11. The k-subdivision of T is the new triangulation constructed by replacing each triangle in T with its k-subdivision.

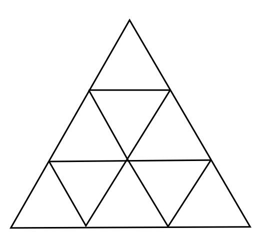

FIGURE 3. A 3-subdivision of an equilateral triangle.

Definition 12. A locally bounded triangulated surface is a triangulated surface S which satisfies the following two properties.

(1) The maximum degree of any vertex of S is 7 and
(2) There exists a triangulation S_{lb} of the surface S into equilateral triangles of side length 5 such that the triangulation S is a 5-subdivision S_{lb}.

The set of locally bounded triangulated surfaces with T triangles and genus g (resp. $\leq T$ triangles and genus $\leq g$) is denoted $\mathrm{Comb}_{\mathrm{lb}}^{T}(\mathcal{M}_g)$ (resp. $\mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{M}_{\leq g})$).

Similarly, we also define locally bounded combinatorial translation surfaces. First, we have a preliminary lemma that motivates the definition.

Lemma 2.7. *Let S be a combinatorial translation surface. Let $V_{>6}(S)$ be the set of vertices of S with degree greater than 6. If γ is an arc representing an element of $H_1(S, V_{>6}(S), \mathbb{Z})$, then*

$$\int_\gamma \phi_S \in \mathbb{Z} + e^{\pi i/3}\mathbb{Z}.$$

Proof. Note that any arc γ on S with $\partial\gamma \subset V_{>6}(S)$ is homotopic to a piecewise smooth arc wherein each piece is an edge of S. Since dz integrates to an element of $\mathbb{Z} + e^{\pi i/3}\mathbb{Z}$ along sides of the equilateral triangle with vertices $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ in $\mathbb{C}$,

$$\int_\gamma \phi_S \in \mathbb{Z} + e^{\pi i/3}\mathbb{Z},$$

as desired. □

Definition 13. A locally bounded combinatorial translation surface is a combinatorial translation surface S that satisfies the following two properties.

(1) The maximum degree of any vertex of S is 42 and
(2) Given any two vertices $x, y \in V_{>6}(S)$ (not necessarily distinct) and γ any arc from x to y,
$$\int_\gamma \phi_S \in 5\mathbb{Z} + 5e^{\pi i/3}\mathbb{Z}.$$

The set of locally bounded combinatorial translation surfaces with T triangles and genus g (resp. $\leq T$ triangles and genus $\leq g$) is denoted $\mathrm{Comb}_{\mathrm{lb}}^{T}(\mathcal{H}_g)$ (resp. $\mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{H}_{\leq g})$).

Remark 10. Given Lemma 2.7, one would expect condition 2 in Definition 13 to be the translation surface version of condition 2 in Definition 12. More rigorously, we shall see in Section 7.4 that triangulated surfaces have a canonical branched 6-cover which is a combinatorial translation surface. Locally bounded combinatorial translation surfaces are defined so that the canonical branched 6-covers of locally bounded triangulated surfaces are locally bounded combinatorial translation surfaces. See Proposition 7.2.

It is also useful to consider triangulated surfaces for which the number of vertices of degree other than 6 is bounded. (Note that combinatorial translation surfaces of genus g automatically satisfy the property that the number of vertices of degree other than 6 is bounded by Cg.)

Definition 14. The set $\mathrm{Comb}^{\leq T, \leq m}(\mathcal{M}_{\leq g})$ consists of $S \in \mathrm{Comb}^{\leq T}(\mathcal{M}_{\leq g})$ which satisfy the property that $|V_{\neq 6}(S)| \leq m$.

Definition 15. The set $\mathrm{Comb}_{\mathrm{lb}}^{\leq T, \leq m}(\mathcal{M}_{\leq g})$ consists of $S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{M}_{\leq g})$ which satisfy the property that $|V_{\neq 6}(S)| \leq m$.

Euler characteristic conditions imply that $|V(S)| \leq T/2$ when $g \geq 1$, so in the future, we will implicitly assume the condition $m/T \leq 1/2$ when $g \geq 1$.

2.8. **Counting functions for the image of Φ and roadmap to prove Theorem 1.3.**

Definition 16. Let

$$N^{\mathcal{M}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}^{\leq T}(\mathcal{M}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\},$$
$$N_{\mathrm{lb}}^{\mathcal{M}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{M}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\},$$
$$N^{\mathcal{M}}(T, g, m, r) = \sup\{\#(\{S \in \mathrm{Comb}^{\leq T, \leq m}(\mathcal{M}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\},$$
$$N_{\mathrm{lb}}^{\mathcal{M}}(T, g, m, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T, \leq m}(\mathcal{M}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\}$$

and

$$N_{\mathrm{lb}}^{\mathcal{H}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{H}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\}.$$

Remark 11. Here, d_T denotes the Teichmüller metric on moduli space $\mathcal{M}_g$. The Teichmüller metric is only defined for $g \geq 2$. For the purpose of notation, in our counting functions when we count genus g triangulated surfaces for $g = 0, 1$ we simply count all the surfaces and omit the radius variable.

To show Theorem 1.3, it is necessary to find an upper bound for $N^{\mathcal{M}}(T, g, r)$. We do by proving several bounds related to the other counting functions introduced in Definition 16. In Section 5, we bound $N^{\mathcal{M}}$ in terms of $N_{\mathrm{lb}}^{\mathcal{M}}$. In Section 6, we bound $N_{\mathrm{lb}}^{\mathcal{H}}$ in terms of $N^{\mathcal{M}}$. In Section 7, we bound $N_{\mathrm{lb}}^{\mathcal{M}}$ in terms of $N_{\mathrm{lb}}^{\mathcal{H}}$. Meanwhile, in Section 3 we prove bounds about the geometry of Teichmüller balls in $\mathcal{M}_g$ which shall be useful for the bounds in Section 6 and Section 7 as well. Finally, in Section 7, we combine all these bounds and use a recursive argument to prove Theorem 1.3.

3. Geometry of Teichmüller space

In this section we give a brief review of definitions in Teichmüller theory and describe the large genus geometry of Teichmüller space. Detailed exposition can be found in [1], [20] and [24]. We also prove some results about quasiconformal maps.

3.1. **Quasiconformal maps.** Let U and V be Riemann surfaces and $K \geq 1$.

Definition 17. An orientation preserving diffeomorphism $f : U \to V$ is K-quasiconformal if it satisfies

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right|$$

where $k = (K-1)/(K+1)$ and z is a holomorphic coordinate on U.

Given conformal metrics on U and V, this means locally there exist oriented orthonormal bases on U and V with respect to which df has singular values λ_1 and λ_2 satisfying $K^{-1} \leq \lambda_1/\lambda_2 \leq K$. The smallest such quantity K is called the quasiconformal dilatation.

Remark 12. Quasiconformal maps are generally required to be homeomorphisms only. See [24, Sections 4.1 and 4.5] for definitions in this setting. In future sections, we explicitly use only the definition we have given above, but we implicitly use the definition in the homeomorphism setting in Section 3.4 and Section 3.7.

3.2. **Hodge norm on $H^1(X,\mathbb{C})$.** Let X be a compact Riemann surface of genus $g \geq 2$ and ρ_X the hyperbolic metric on X. In this section we record certain constructions from Hodge theory on Riemann surfaces. See [15] for a detailed exposition. Let $\mathcal{E}^1(X,\mathbb{C})$ denote the space of complex valued differential 1-forms on X; also denote closed and exact 1-forms by $Z^1(X,\mathbb{C})$ and $B^1(X,\mathbb{C})$, respectively. The Hodge star

$$* : \mathcal{E}^1(X,\mathbb{C}) \to \mathcal{E}^1(X,\mathbb{C})$$

is an $\mathbb{C}$-linear map which fixes the space of harmonic 1-forms $\mathcal{H}^1(X,\mathbb{C}) \subset \mathcal{E}^1(X,\mathbb{C})$.

Definition 18. The Hodge inner product (a Hermitian inner product) on $\mathcal{E}^1(X,\mathbb{C})$ is defined as

$$\langle \omega_1, \omega_2 \rangle_X = \int_X \omega_1 \wedge *\overline{\omega_2} = \int_X \langle \omega_1, \omega_2 \rangle_{\rho_X} d\rho_X^2.$$

The Hodge norm on $\mathcal{E}^1(X,\mathbb{C})$ is defined as

$$\|\omega\|_X^2 = \int_X \omega \wedge *\overline{\omega}$$

for $\omega, \omega_1, \omega_2 \in \mathcal{E}^1(X,\mathbb{C})$.

By the Hodge decomposition theorem, the space of closed complex valued differential 1-forms on X splits as

$$\mathcal{Z}^1(X,\mathbb{C}) = \mathcal{H}^1(X,\mathbb{C}) \oplus \mathcal{B}^1(X,\mathbb{C}),$$

and the splitting is orthogonal with respect to the Hodge inner product.

Definition 19. The Hodge norm of a cohomology class

$$u \in \mathcal{Z}^1(X,\mathbb{C})/\mathcal{B}^1(X,\mathbb{C}) \simeq H^1(X,\mathbb{C}),$$

is

$$\|u\|_X = \inf_{[\omega]=u} \|\omega\|_X.$$

The infimum in Definition 19 is attained by the unique harmonic representative of u.

3.3. **Quasiconformal maps and Hodge norm.** In this section we study how the Hodge norm behaves under a quasiconformal map. All quasiconformal maps in this section are assumed to be diffeomorphisms. Let X and Y be compact Riemann surfaces of genus $g \geq 2$, with hyperbolic metrics ρ_X and ρ_Y, respectively.

Lemma 3.1. *If $f : X \to Y$ is a K-quasiconformal map and ω is a complex valued differential 1-form on Y, then*

$$(1/K)^{1/2}\|\omega\|_Y \leq \|f^*\omega\|_X \leq K^{1/2}\|\omega\|_Y.$$

Proof. We have,

$$\begin{aligned}
\|f^*\omega\|_X^2 &= \int_X \langle f^*\omega, f^*\omega \rangle_{\rho_X} \rho_X^2 \\
&\leq \int_X \|df\|^2 f^*(\langle \omega, \omega \rangle_{\rho_Y}) |\det(df)|^{-1} f^*\rho_Y^2 \\
&\leq K \int_Y \langle \omega, \omega \rangle_{\rho_Y} \rho_Y^2 \\
&= K\|\omega\|_Y^2.
\end{aligned}$$

Applying an analogous argument to f^{-1}, which is also K-quasiconformal, we obtain $\|\omega\|_Y \leq K^{1/2}\|f^*\omega\|_X$. □

As a corollary, we have:

Corollary 3.2. *Let $f : X \to Y$ be a K-quasiconformal map. Let $u \in H^1(Y, \mathbb{C})$. Then*

$$(1/K)^{1/2}\|u\|_Y \leq \|f^*u\|_X \leq K^{1/2}\|u\|_Y.$$

Proof. Let ω be the harmonic form on Y representing u. Then by Lemma 3.1,

$$\begin{aligned}\|f^*u\|_X &\leq \|f^*\omega\|_X \\ &\leq K^{1/2}\|\omega\|_Y \\ &= K^{1/2}\|u\|_Y.\end{aligned}$$

The analogous argument applied to f^{-1}, gives $\|u\|_Y \leq K^{1/2}\|f^*u\|_X$. □

Finally, we show that the pullback of a harmonic form under a quasiconformal map is close to its harmonic representative.

Lemma 3.3. *Let $f : X \to Y$ be a K-quasiconformal map. Let ω be a harmonic 1-form on Y. Denote by $(f^*\omega)^h$ the unique harmonic 1-form on X cohomologous to $f^*\omega$. Then*

$$\|f^*\omega - (f^*\omega)^h\|_X \leq ((K^2 - 1)/K)^{1/2}\|\omega\|_Y.$$

Proof. We have,

$$\begin{aligned}\|f^*\omega - (f^*\omega)^h\|_X^2 &= \langle f^*\omega - (f^*\omega)^h, f^*\omega - (f^*\omega)^h\rangle_X \\ &= \langle f^*\omega, f^*\omega\rangle_X - \langle (f^*\omega)^h, (f^*\omega)^h\rangle_X - 2\langle f^*\omega - (f^*\omega)^h, (f^*\omega)^h\rangle_X.\end{aligned}$$

By Lemma 3.1 and Corollary 3.2,

$$\langle f^*\omega, f^*\omega\rangle_X - \langle (f^*\omega)^h, (f^*\omega)^h\rangle_X \leq ((K^2 - 1)/K)\|\omega\|_Y^2.$$

Exact forms are orthogonal to harmonic forms, so since $f^*\omega - (f^*\omega)^h$ is exact and $(f^*\omega)^h$ is harmonic,

$$2\langle f^*\omega - (f^*\omega)^h, (f^*\omega)^h\rangle_X = 0.$$

The lemma follows. □

3.4. **Quasisymmetric maps and quasicircles.** In this section, we introduce some definitions related to quasiconformal maps.

Let (U, d_U) and (V, d_V) be metric spaces bi-Lipschitz to domains in $\mathbb{C}$.

Definition 20. An embedding $f : U \to V$ is called K-weakly-quasisymmetric if for all $x, y, z \in U$,

$$d_V(f(x), f(y)) \leq K d_V(f(x), f(z))$$

if

$$d_U(x, y) \leq d_U(x, z).$$

When U and V are oriented topological manifolds, we will assume f is orientation preserving. When U and V are simply domains in $\mathbb{C}$, the metrics on U and V are taken to be the restrictions of the Euclidean metric to U and V, respectively. Then weakly-quasisymmetric maps of the unit circle may be extended to quasiconformal maps of the unit disk:

Proposition 3.4 ([1], Theorem IV.B.2). *Suppose $f' : S^1 \to S^1$ is K-weakly-quasisymmetric. Then f' extends to a $C(K)$-quasiconformal map $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ where $C(K)$ only depends on K.*

Definition 21. A simple closed curve (resp. simple arc) in $\mathbb{C}$ is a K-quasicircle (resp. K-quasiarc) is the image of the unit circle (resp. unit line segment) under a K-quasiconformal homeomorphism of the plane.

Let (U, d_U) be a metric space bi-Lipschitz and homeomorphic to a domain in $\mathbb{C}$. When U is simply a domain in $\mathbb{C}$, d_U is taken to be the restriction of the Euclidean metric to U.

Definition 22. A simple arc γ in U is a K-bounded-turning curve if it satisfies the following condition: for all $x, y, z \in \gamma$ lying in order,

$$d_U(x, y) + d_U(y, z) \leq K d_U(x, z).$$

A simple closed curve γ in U is a K-bounded-turning curve if every point of γ lies in the interior of a subarc of γ that is a K-bounded-turning curve.

In the rest of this section, we note some results about quasicircles, quasiarcs, bounded-turning curves and quasisymmetric maps that shall be useful in later sections.

The following statement relates quasicircles and quasiarcs to the bounded-turning property. A proof can be found in [29, Sections 2.8.7, 2.8.8 and 2.8.9].

Proposition 3.5. *Suppose a simple closed curve (resp. simple arc) γ in $\mathbb{C}$ is a K-bounded-turning curve. Then γ is a $C(K)$-quasicircle (resp. $C(K)$-quasiarc) where $C(K)$ is a constant only depending on K. Similarly, suppose a simple closed curve (resp. simple arc) γ is a K-quasicircle (resp. K-quasiarc). Then it is a $C(K)$-bounded-turning curve.*

We have the following extension result about quasisymmetric maps.

Lemma 3.6. *Let $U_1 \subset \mathbb{C}$ and $U_2 \subset \mathbb{C}$ be simply connected open domains such that ∂U_1 is a K_1-quasicircle and ∂U_2 is a K_2-quasicircle. Suppose*

$$f' : \partial U_1 \to \partial U_2$$

is a K_3-weakly-quasisymmetric homeomorphism. Then f' extends to a map

$$f : \overline{U_1} \to \overline{U_2}$$

that is $C(K_1, K_2, K_3)$-quasiconformal. Here, $C(K_1, K_2, K_3)$ is a constant that depends only on K_1, K_2 and K_3.

Proof. Let

$$f_1 : \mathbb{C} \to \mathbb{C}$$

be a K_1-quasiconformal mapping taking $\overline{U_1}$ to $\overline{\mathbb{D}}$. Then f_1 is $C(K_1)$-weakly-quasisymmetric for some $C(K_1)$ only depending on K_1 (see [46, Theorem 18.1]). Similarly, let

$$f_2 : \mathbb{C} \to \mathbb{C}$$

be a K_2-quasiconformal mapping taking $\overline{U_2}$ to $\overline{\mathbb{D}}$. Then f_2 is $C(K_2)$-weakly-quasisymmetric. Then identifying $\partial\mathbb{D} \simeq S^1$, the map

$$f_2 \circ f' \circ f_1^{-1} : S^1 \to S^1$$

is $C(K_1, K_2, K_3)$-weakly-quasisymmetric (note that compositions and inverses of weakly-quasisymmetric maps of the plane are weakly-quasisymmetric by Theorem 10.6 and Theorem 10.19 in [23]). By Proposition 3.4,

$$f_2 \circ f' \circ f_1^{-1}$$

extends to a $C(K_1, K_2, K_3)$-quasiconformal map

$$f'' : \overline{\mathbb{D}} \to \overline{\mathbb{D}}.$$

Taking

$$f = f_2^{-1} \circ f'' \circ f_1$$

gives a $C(K_1, K_2, K_3)$-quasiconformal map from $\overline{U_1}$ to $\overline{U_2}$ that is an extension of f'. □

3.5. **Teichmüller space and Teichmüller metric.** Let $g \geq 2$ and S_g a smooth oriented genus g surface. A marked Riemann surface is a Riemann surface X along with a diffeomorphism $S_g \to X$.

Let X and Y be marked Riemann surfaces of genus g, with markings

$$f_X : S_g \to X$$

and

$$f_Y : S_g \to Y,$$

respectively. The marked surfaces X and Y are considered equivalent if there is a biholomorphism

$$f : X \to Y$$

satisfying the property that $f_Y^{-1} \circ f \circ f_X$ is isotopic to the identity.

Definition 23. Teichmüller space $\mathcal{T}_g$ is the set of equivalence classes of marked Riemann surfaces. The Teichmüller distance, denoted d_T, is given by

$$d_T(X, Y) = \inf \left\{ \frac{1}{2} \log K | f : X \to Y \text{ is } K\text{-quasiconformal} \right\}$$

where f satisfies the property that $f_Y^{-1} \circ f \circ f_X$ is isotopic to the identity.

Teichmüller's theorem asserts that the infimum in Definition 23 is attained by a homeomorphism and is unique. The unique map that attains the infimum is called a Teichmüller map.

The space $\mathcal{T}_g$ admits a natural complex structure under which it is a complex manifold of dimension $3g - 3$. The cotangent space at a point X may be naturally identified with $Q(X)$, the space of holomorphic quadratic differentials on X. Under this identification, the Teichmüller metric is the L^1-norm on $Q(X)$, given by

$$\|\phi\| = \int_X |\phi|$$

for a holomorphic quadratic differential ϕ on X.

Moduli space $\mathcal{M}_g$ can be obtained from $\mathcal{T}_g$ by quotienting by the action of the mapping class group Mod_g. The Teichmüller metric also descends to $\mathcal{M}_g$.

3.6. **Extremal length and Teichmüller metric.** The Teichmüller metric has a description in terms of extremal length, as we shall explain now. Let X be a Riemann surface of genus g. Let γ a free homotopy class of a simple closed curve on X.

Definition 24. Given a Riemannian metric ρ on X, the quantity

$$\text{length}_\rho(\gamma)$$

is defined to be the infimum of lengths in the ρ-metric over all curves representing γ.

Definition 25. The extremal length of γ on X is defined to be

$$\mathrm{Ext}_X(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(X)}$$

where the supremum is taken over all conformal metrics ρ on X.

The next theorem describes when the supremum in Definition 25 is achieved.

Theorem 3.7 (Jenkins [25], Strebel [45], see also Theorem 3.1 in [26])**.** *The supremum of*

$$\sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(X)}$$

is achieved when ρ is the flat metric $|\phi|$ associated to a holomorphic quadratic differential

$$\phi \in Q(X).$$

The Teichmüller distance has the following description due to Kerckhoff [26, Theorem 4].

Theorem 3.8. *For $X, Y \in \mathcal{T}_g$,*

$$d_T(X,Y) = \frac{1}{2}\log\sup_\gamma \frac{\mathrm{Ext}_Y(\gamma)}{\mathrm{Ext}_X(\gamma)},$$

where the supremum is taken over all free homotopy classes of a simple closed curve on X.

Remark 13. Since X and Y are marked surfaces, γ (which was initially defined as a free homotopy class on X) is also automatically a free homotopy class on Y.

3.7. **Bi-Lipschitz metric.** Let $g \geq 2$.

Definition 26. The bi-Lipschitz metric d_L on $\mathcal{M}_g$ is

$$d_L(X,Y) = \inf\left\{\log K | f : X \to Y \text{ is a K-bi-Lipschitz diffeomorphism}\right\}.$$

Here, the bi-Lipschitz constant is measured with respect to the unique hyperbolic metrics on X and Y.

The following result establishes a comparison between d_T and d_L.

Proposition 3.9. *There exists a universal constant C such that for all g,*

$$d_T \leq d_L \leq C d_T$$

on $\mathcal{M}_g$.

Proof. The inequality $d_T \leq d_L$ follows from the fact that any K-bi-Lipschitz map is automatically K^2-quasiconformal.

In the other direction, we must show that $d_L \leq C d_T$. The following proof is due to Maxime Fortier Bourque. Fix $K_0 > 0$. Let $X, Y \in \mathcal{M}_g$, such that

$$d_T(X,Y) = \log K.$$

Cut the Teichmüller geodesic from X to Y into around

$$n = \left\lceil \frac{\log K}{\log K_0} \right\rceil$$

segments of length around

$$\frac{\log K}{n}.$$

Let $Z_0 = X, Z_1, ..., Z_n = Y$ be the endpoints of these segments. Since

$$d_T(Z_i, Z_{i+1}) \leq \log K_0,$$

there is a quasiconformal map

$$f_i : Z_i \to Z_{i+1}$$

with dilatation at most K_0^2. Using the Douady-Earle extension [16, Theorem 5.2], f_i may be replaced with a map g_i that is $C(K_0^2)$-bi-Lipschitz between the hyperbolic metrics on Z_i and Z_{i+1}. Take

$$g = g_{n-1} \circ ... \circ g_0 : X \to Y.$$

Then g is $C(K_0^2)^n$-bi-Lipschitz between the hyperbolic metrics on X and Y. Therefore,

$$\begin{aligned} d_L(X, Y) &\leq C \frac{\log C(K_0^2)}{\log K_0} \log K \\ &\leq C \log K \end{aligned}$$

for a universal constant C. □

3.8. **Kobayashi metric.** Let M be a complex manifold of arbitrary dimension. Roughly speaking, the Kobayashi pseudometric is the largest pseudometric on M such that all holomorphic maps into M are distance decreasing. See [27, Section 4.1] for a rigorous construction. In general, the Kobayashi pseudometric may not be a metric (i.e. may not separate points). However, we shall see that in all cases of M relevant to us, the Kobayashi metric exists. In particular, the Kobayashi metric exists when M is a bounded domain in $\mathbb{C}^n$ [27, Corollary 4.4.6]. The Kobayashi metric satisfies the following important property.

Proposition 3.10 ([27], Proposition 4.1.1)**.** *Let M and N be complex manifolds. Suppose they admit Kobayashi metrics d_M and d_N, respectively. Let $f : M \to N$ be a holomorphic map. Then for all $x, y \in M$,*

$$d_M(x, y) \geq d_N(f(x), f(y)).$$

If f is a biholomorphism, then equality holds.

We also have the following theorem due to Royden [44, Theorem 3].

Theorem 3.11. *On $\mathcal{T}_g$, the Kobayashi metric exists and is the Teichmüller metric d_T.*

3.9. **Bers embedding of $\mathcal{T}_g$.** Let $g \geq 2$ and fix $X \in \mathcal{T}_g$. Recall from Definition 8 that X^{-1} denotes the conformal mirror of X. Let $Q^\infty(X^{-1})$ denote the space of quadratic differentials on X^{-1}, equipped with the norm

$$\|\phi\|_\infty = \sup_{x \in X^{-1}} \frac{|\phi(x)|}{|\rho_{X^{-1}}|^2}.$$

(Here, $\rho_{X^{-1}}$ is the hyperbolic metric on X^{-1}.) The Bers embedding [20, Section 5.4] is a holomorphic embedding of $\mathcal{T}_g$ into $Q^\infty(X^{-1})$ sending $X \in \mathcal{T}_g$ to the origin in $Q^\infty(X^{-1})$.

Theorem 3.12 ([20], Theorem 5.4.1)**.** *The Bers embedding*

$$\beta_X : \mathcal{T}_g \to Q^\infty(X^{-1})$$

satisfies

$$B_\infty(0, 1/2) \subset \beta_X(\mathcal{T}_g) \subset B_\infty(0, 3/2)$$

where $B_\infty(0, r)$ denotes the norm ball of radius r in the space $Q^\infty(X^{-1})$.

Remark 14. The constants in [20, Theorem 4.5.1] are different because the normalization of the Teichmüller metric is different. See [32, Theorem 2.2] for normalization and constants that agree with ours.

3.10. **Asymptotic geometry of Teichmüller space.** In this section, we use the Bers embedding to obtain bounds on the geometry of d_T on $\mathcal{T}_g$. This technique has been used to prove [32, Theorem 8.2] and [18, Theorem 1.5]. Our bounds are variations of the bounds in the latter.

Fix $X \in \mathcal{T}_g$. Denote by $Q = B_\infty(0,1)$ the open unit norm ball in $Q^\infty(X^{-1})$. Since $Q^\infty(X^{-1})$ is $3g-3$-dimensional, Q is an open $3g-3$ complex manifold. Let d_Q denote the Kobayashi metric on Q. Putting together Proposition 3.10, Theorem 3.11 and Theorem 3.12, we have the following two metric comparison results.

Lemma 3.13. *Let $Y, Z \in \mathcal{T}_g$. Then*

$$\|\beta_X(Y)\|_\infty \leq 3/2$$

and

$$\|\beta_X(Z)\|_\infty \leq 3/2.$$

Moreover,

$$d_Q((2/3)\beta_X(Y), (2/3)\beta_X(Z)) \leq d_T(Y, X).$$

Lemma 3.14. *There exists a holomorphic map*

$$\beta_X^{-1} : B_\infty(0, 1/2) \to \mathcal{T}_g$$

such that

$$\beta_X \circ \beta_X^{-1} = \mathrm{id}$$

on $B_\infty(0,1/2)$. Moreover, for $\phi_1, \phi_2 \in B_\infty(0,1/2)$,

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq d_Q(2\phi_1, 2\phi_2).$$

The following lemma gives us an approximation for d_Q.

Lemma 3.15. *For $\phi_1, \phi_2 \in B_\infty(0,1/8)$,*

$$C_1\|\phi_1 - \phi_2\|_\infty \leq d_Q(\phi_1, \phi_2) \leq C_2\|\phi_1 - \phi_2\|_\infty$$

for universal constants C_1 and C_2.

Proof. First, we show that

$$d_Q(\phi_1, \phi_2) \geq C_1\|\phi_1 - \phi_2\|_\infty.$$

To see this, let $x \in X^{-1}$ (the conformal mirror of X), and let v be a unit tangent vector at x with respect to the hyperbolic metric $\rho_{X^{-1}}$ on X^{-1}. We define a map

$$I_x : B_\infty(0,1) \to \mathbb{D}$$

that sends ϕ to the evaluation of ϕ at $v \otimes v$. Note that for all $\phi \in B_\infty(0,1)$,

$$\begin{aligned} |I_x(\phi)| &= \frac{|\phi|}{|\rho_{X^{-1}}^2|}(x) \\ &\leq 1. \end{aligned}$$

Also, for all $\phi \in B_\infty(0,1/2)$,

$$|I_x(\phi)| \leq 1/2.$$

Furthermore, I_x is affine, thus holomorphic. Therefore, by Proposition 3.10, I_x is distance decreasing with respect to d_Q on $B_\infty(0,1)$ and the Kobayashi metric on $\mathbb{D}$ which is the Poincare metric. So for $\phi_1, \phi_2 \in B_\infty(0,1/2)$,

$$\begin{aligned} d_Q(\phi_1,\phi_2) &\geq \rho_{\mathbb{D}}(I_x(\phi_1), I_x(\phi_2)) \\ &\geq C_1 |I_x(\phi_1) - I_x(\phi_2)| \\ &= C_1 \frac{|\phi_1 - \phi_2|}{|\rho_X|^2}(x). \end{aligned}$$

Since this is true for all $x \in X^{-1}$,

$$d_Q(\phi_1,\phi_2) \geq C_1 \|\phi - \phi_2\|_\infty$$

as desired.

Next, we show that for $\phi_1, \phi_2 \in B_\infty(0,1/8)$,

$$d_Q(\phi_1,\phi_2) \leq C_2 \|\phi_1 - \phi_2\|_\infty.$$

To do this, consider the map $I : \mathbb{D} \to B_\infty(0,1)$ sending $\zeta \in \mathbb{D}$ to

$$\phi_1 + \zeta \frac{\phi_2 - \phi_1}{2\|\phi_2 - \phi_1\|_\infty}.$$

The image of I is contained in $B_\infty(0,1)$ because

$$\begin{aligned} \left\| \phi_1 + \zeta \frac{\phi_2 - \phi_1}{2\|\phi_2 - \phi_1\|_\infty} \right\|_\infty &\leq \|\phi_1\|_\infty + \left| \frac{\zeta}{2} \right| \\ &\leq 1. \end{aligned}$$

Note that I is affine and therefore holomorphic. Moreover, $I(0) = \phi_1$ and

$$I(2\|\phi_1 - \phi_2\|_\infty) = \phi_2.$$

By Proposition 3.10, I must be distance decreasing with respect to the Poincare metric on $\mathbb{D}$. Since $\phi_1, \phi_2 \in B_\infty(0,1/8)$,

$$2\|\phi_1 - \phi_2\|_\infty \leq 1/2.$$

Therefore,

$$\begin{aligned} d_Q(\phi_1,\phi_2) &\leq d_{\rho_{\mathbb{D}}}(0, 2|\phi_1 - \phi_2\|_\infty) \\ &\leq C_2 \|\phi_1 - \phi_2\|_\infty, \end{aligned}$$

as desired. □

Lemma 3.13, Lemma 3.14 and Lemma 3.15 together imply:

Lemma 3.16. *Let*

$$\phi_1, \phi_2 \in B_\infty(0, 1/16).$$

Then

$$C_3 \|\phi_1 - \phi_2\|_\infty \leq d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq C_4 \|\phi_1 - \phi_2\|_\infty.$$

Proof. By Lemma 3.14,

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq d_Q(2\phi_1, 2\phi_2).$$

Since

$$2\phi_1, 2\phi_2 \in B_\infty(0, 1/8),$$

by Lemma 3.15,

$$d_Q(2\phi_1, 2\phi_2) \leq C_4\|2\phi_1 - 2\phi_2\|_\infty \leq C_4\|\phi_1 - \phi_2\|_\infty.$$

This gives one direction of the comparison in the lemma statement. To obtain the other direction, by Lemma 3.13,

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \geq d_Q((2/3)\phi_1, (2/3)\phi_2).$$

Since

$$(2/3)\phi_1, (2/3)\phi_2 \in B_\infty(0, 1/8),$$

Lemma 3.15 implies

$$d_Q((2/3)\phi_1, (2/3)\phi_2) \geq C_3\|(2/3)\phi_1 - (2/3)\phi_2\|_\infty \geq C_3\|\phi_1 - \phi_2\|_\infty,$$

as desired. □

Finally, we have:

Lemma 3.17. *There exist universal constants C_3 and C_4 such that for*

$$Y, Z \in B_{d_T}(X, C_3/20),$$

$$C_3\|\beta_X(Y) - \beta_X(Z)\|_\infty \leq d_T(Y, Z) \leq C_4\|\beta_X(Y) - \beta_X(Z)\|_\infty.$$

Proof. Consider the map

$$\beta_X^{-1} : B_\infty(0, 1/20) \to \mathcal{T}_g.$$

This is a biholomorphism onto its image. Note that

$$C_3\|\phi_1 - \phi_2\|_\infty \leq d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2))$$

for

$$\phi_1, \phi_2 \in B_\infty(0, 1/16)$$

by Lemma 3.16. In particular,

$$C_3\|\phi\|_\infty \leq d_T(X, \beta_X^{-1}(\phi))$$

for

$$\phi \in B_\infty(0, 1/16).$$

Thus $\beta_X^{-1}(B_\infty(0, 1/20))$ contains the point $X \in \mathcal{T}_g$, but $\partial\beta_X^{-1}(B_\infty(0, 1/20))$ does not intersect $B_{d_T}(X, C_3/20)$. This means $\beta_X^{-1}(B_\infty(0, 1/20))$ contains $B_{d_T}(X, C_3/20)$. Now Lemma 3.16 gives the desired inequalities. □

Lemma 3.18. *Let C_3 be as in Lemma 3.17 and suppose $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 \leq C_3/20$ and $r_1 \leq r_2$. Let $X_1, ..., X_N \in B_{d_T}(X, r_2) \subset \mathcal{T}_g$ such that*

$$d_T(X_i, X_j) \geq r_1$$

for all $i, j \in \{1, ..., N\}$ distinct. If $r_2 \leq C_3/20$, then

$$N \leq (C/r_1)^{6g-6}.$$

For all $r_2 \geq C_3/20$,

$$N \leq (C/r_1)^{Cr_2(6g-6)}.$$

Here, C is a universal constant.

Proof. First we treat the case where $r_2 \leq C_3/20$. For all $i, j \in \{1, ..., N\}$ distinct,

$$r_1/C_4 \leq \|\beta_X(X_i) - \beta_X(X_j)\|_\infty$$

by Lemma 3.17. Therefore norm balls of radius $r_1/(2C_4)$ around the $\beta_X(X_i)$ are disjoint. Again by Lemma 3.17,

$$\|\beta_X(X_i)\|_\infty \leq r_2/C_3$$

for all $i \in \{1, ..., N\}$. Now, $Q^\infty(X^{-1}) \simeq \mathbb{C}^{3g-3}$ as a vector space. Hence it admits a Euclidean volume. Denote by $V(r)$ the Euclidean volume of a norm ball of radius r in $Q^\infty(X^{-1})$. Then

$$V(r_1/(2C_4)) \geq ((4C_4r_2)/(C_3r_1))^{-(6g-6)}V(2r_2/C_3)$$

since a ball of radius $2r_2/C_3$ is simply a scaled copy of a ball of radius $r_1/(2C_4)$. Note that

$$B_\infty(\beta_X(X_i), r_1/(2C_4)) \subset B_\infty(0, 2r_2/C_3)$$

because $\beta_X(X_i) \in \overline{B_\infty(0, r_2/C_3)}$ for all $i \in \{1, ..., N\}$ and $r_1 \leq r_2$, $C_3 \leq C_4$. The former balls are also mutually disjoint, so

$$V(2r_2/C_3) \geq N \cdot V(r_1/(2C_4)).$$

Therefore

$$N \leq (C/r_1)^{6g-6}.$$

This proves the first part of the lemma.

Now, to prove the second part, let us assume that $r_2 \geq C_3/20$. We have that at most $(C/r_1)^{6g-6}$ of the X_i are contained in $B_{d_T}(X, C_3/20)$. Also, at most $(C/r_1)^{6g-6}$ of the X_j are contained in $B_{d_T}(X_i, C_3/20)$ for all X_i. By induction we obtain the desired result. □

As a corollary, we have:

Corollary 3.19. *Let C_3 be as in Lemma 3.17 and suppose $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 \leq C_3/20$ and $r_1 \leq r_2$. If*

$$r_2 \leq C_3/20,$$

then any r_2-radius ball in $\mathcal{M}_g$ in the Teichmüller metric can be covered by

$$(C/r_1)^{6g-6}$$

number of r_1-radius balls. If

$$r_2 \geq C_3/20,$$

then any r_2-radius ball in $\mathcal{M}_g$ can be covered by

$$(C/r_1)^{Cr_2(6g-6)}$$

number of r_1-radius balls. Here, C is a universal constant.

Proof. By Lemma 3.18, these statements are true on $\mathcal{T}_g$. Thus they are true on $\mathcal{M}_g$ also. □

4. Lower bounds

In this section, we prove Theorem 1.2.

Let $X \in \mathcal{M}_g$. Let ρ_X be the conformal hyperbolic metric on X. Assume

$$\sum_{\substack{\gamma \text{ simple closed geodesic on } X \\ \text{length}_{\rho_X}(\gamma) < 2\,\text{arcsinh}(1)}} \text{length}_{\rho_X}(\gamma)^{-1} \leq R.$$

Choose $U_1, ..., U_N$, $V_1, ..., V_N$ and $W_1, ..., W_N$ as in Lemma 2.5. Here, $N \leq C(R+g)$. Recall that the $\{W_i\}$ each cover X by condition 1 of Lemma 2.5. The circles ∂W_i divide X into M connected components that we label $P_1, ..., P_M$. By construction, each P_j is contained in some W_i. Moreover, each ∂P_j is a union of hyperbolic circular arcs. There are at most C such arcs in ∂P_j: P_j is contained in some W_i, which in turn is contained in U_i, which intersects at most C of the other $U_1, ..., U_N$ (by condition 5 of Lemma 2.5). For the same reason, each W_i is a union of at most C of the $P_1, ..., P_M$.

Let

$$\mathbb{D}_{\tanh(\text{radius}(W_i)/2)} = \{z \in \mathbb{D} \,|\, |z| < \tanh(\text{radius}(W_i)/2)\}.$$

Let

$$f_i : W_i \to \mathbb{D}_{\tanh(\text{radius}(W_i)/2)}$$

be a conformal map (unique up to rotation) that sends center(W_i) to 0, that is C-bi-Lipschitz between W_i with ρ_X and $\mathbb{D}_{\tanh(\text{radius}(W_i)/2)}$ with the Euclidean metric.

Now, we construct triangulations T_r of X by triangulating each P_j as follows. Fix $r > 0$ sufficiently small.

(1) For each P_j, choose an arbitrary $W_{s(j)}$ such that $P_j \subset W_{s(j)}$.

(2) Divide $f_{s(j)}(P_j)$ into
$$P_j^B = B_{\text{Euc}}(\partial(f_{s(j)}(P_j)), r \cdot \text{radius}(W_{s(j)}))$$
and
$$P_j^I = f_{s(j)}(P_j) \setminus P_j^B.$$
We triangulate each P_j separately.

(3) Triangulate a subset of $f_{s(j)}(P_j)$ by the standard hexagonal triangulation with side length
$$r \cdot \text{radius}(W_{s(j)})$$
such that P_j^I is covered by the triangulation.

(4) Pullback by $f_{s(j)}$ to obtain a partial triangulation of a subset $X_E \subset X$.

(5) In the triangulation of X_E, replace all border edges between two vertices by the shortest hyperbolic geodesic connecting these two vertices.

(6) Choose a set of vertices on $X \setminus X_E$ that is a $C_1 r \cdot \text{radius}(W_i)$-separated set and a $C_2 r \cdot \text{radius}(W_i)$-net on each W_i. Using these chosen vertices, complete the triangulation of $X \setminus X_E$ by hyperbolic triangles such that each triangle is contained in some V_i, and within V_i is contained in a $C_3 r \cdot \text{radius}(W_i)$-radius hyperbolic ball.

We have the following lemma regarding Step 5 of the triangulation process described.

Lemma 4.1. *There exists a universal constant r_0 such that for all $r < r_0$, the following statements hold. Let $x, y \in W_i$ such that $f_i^* d_{\text{Euc}}(x, y) \leq r \cdot \text{radius}(W_i)$. Let γ_{Euc} be the geodesic in W_i connecting x and y in the metric $f_i^* d_{\text{Euc}}$, and γ_X be the geodesic in W_i connecting x and y in the metric ρ_X. The angles between $f_i(\gamma_X)$ and $f_i(\gamma_{\text{Euc}})$ are at most $\pi/7$. Moreover,*

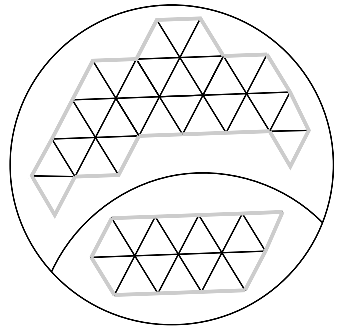

FIGURE 4. Two of the P_j contained in the same W_i, with the partial triangulation constructed in Step 3. The gray edges are replaced in Step 5.

the curvatures of $f_i(\gamma_X)$ and $f_i(\gamma_{\mathrm{Euc}})$ are bounded above by a universal constant.

Proof. The map f_i maps W_i conformally to the disk $\mathbb{D}_{\tanh(\mathrm{radius}(W_i)/2)}$ (sending center(W_i) to 0) such that the restriction of the Poincare metric $\rho_{\mathbb{D}}$ on $\mathbb{D}$ is ρ_X on W_i. Geodesics with respect to d_{Euc} correspond to straight lines in $\mathbb{D}_{\tanh(\mathrm{radius}(W_i)/2)}$, whereas geodesics with respect to $\rho_{\mathbb{D}}$ are circular arcs which meet $\partial\mathbb{D}$ at a right angle. Since radius$(W_i) \leq \mathrm{arcsinh}(1)/8$ (see condition 2 of Lemma 2.5), $\tanh(\mathrm{radius}(W_i)/2)$ is bounded away from 1. So a geodesic of $\rho_{\mathbb{D}}$ passing through $\mathbb{D}_{\tanh(\mathrm{radius}(W_i)/2)}$ must have curvature bounded above by a universal constant C. If two points in $\mathbb{D}$ are sufficiently close, then the line between them and constant curvature arc (of curvature at most C) between them meet at an angle of at most $\pi/7$. The lemma follows. $\square$

Thus we obtain, for each $0 < r < r_0$, a triangulation of X that we label T_r. There are two types of edges in T_r: those constructed in Step 3 and left unchanged (which we call flat edges), and those constructed in Step 5 or 6 (which we call hyperbolic edges). There are three types of triangles: those constructed in Step 3 and left unchanged (which we call equilateral triangles), those constructed in Step 5 (which we call semi-equilateral triangles), and those constructed in Step 6 (which we call hyperbolic triangles). We denote by T_E the union of the equilateral triangles of T_r, T_{SE} the union of the semi-equilateral triangles of T_r, and T_H the union of the hyperbolic triangles of T_r.

Denote by T_r^1 the 1-skeleton of the triangulation T_r. We define a metric $d_{T_r^1}$ on T_r^1 as follows. On a hyperbolic edge, we define $d_{T_r^1}$ to the restriction of d_{ρ_X} (the hyperbolic distance on X) to the hyperbolic edge. On a flat edge that was constructed by triangulating $f_{s(j)}(P_j)$ in Step 3, we define $d_{T_r^1}$ to be the restriction of the pullback under $f_{s(j)}$ of the Euclidean distance d_{Euc} on $f_{s(j)}(P_j) \subset \mathbb{D}$ to the flat edge.

We let S_r be the unique triangulated surface with the same underlying triangulation as T_r. Note that T_r has at most $Cr^{-2}(R+g)$ triangles, since the number of equilateral and semi-equilateral triangles is bounded by CNr^{-2} and the number of hyperbolic triangles is bounded by

$$Cr^{-1}\sum_i \mathrm{length}_{\rho_X}(L_i)\,\mathrm{radius}(W_i)^{-1} \leq Nr^{-1}.$$

Hence

$$S_r \in \mathrm{Comb}^{\leq Cr^{-2}(R+g)}(\mathcal{M}_g).$$

Let

$$X_r = \Phi(S_r) \in \mathcal{M}_g.$$

We will now show:

Theorem 4.2. *For $0 < r < r_0$, $d_T(X, X_r) < Cr$. Here, C is a universal constant.*

This directly implies Theorem 1.2. First, we have a preliminary lemma.

Lemma 4.3. *Let Q be a hyperbolic triangle contained in a hyperbolic ball of radius $r < 1/4$. Then there exists a $Cr^2 \operatorname{area}(Q)^{-1}$-quasiconformal map from Q (equipped with the hyperbolic metric) to the unit equilateral triangle T (equipped with the Euclidean metric) which sends vertices of Q to vertices of T and is a scaling map on each of the three boundary edges.*

Proof. The hyperbolic triangle Q admits a C-quasiconformal map to some flat triangle Q_1, as follows. Consider Q as contained in $\mathbb{D}$ (equipped with the Poincare metric $\rho_{\mathbb{D}}$) with one of the vertices the origin.

Let

$$H = \{x, y, z \in \mathbb{R} | x^2 + y^2 + (z - 1/2)^2 = 1/4, z \leq 1/2\}$$

be a hemisphere in $\mathbb{R}^3$. View $\mathbb{D}$ as a unit disk in the x, y-plane in $\mathbb{R}^3$, centered at the origin. Taking the inverse stereographic projection of Q to H, then the orthographic projection back to the x, y-plane gives a map from Q to a Euclidean triangle Q_1.

Away from $\partial\mathbb{D}$ and ∂H, the stereographic projection is C-bi-Lipschitz between the spherical metric on H and the hyperbolic metric on $\mathbb{D}$. The orthographic projection is C-bi-Lipschitz between the spherical metric on H and the Euclidean metric on its image. Since Q is contained in a hyperbolic ball of radius $r < 1/4$, we have a C-bi-Lipschitz map F_1 from Q (with the hyperbolic metric) to Q_1 (with the Euclidean metric). Let Q_2 be Q_1 scaled by r^{-2}; then we have a map $F_2 : Q_1 \to Q_2$ that is conformal. The map F_2 satisfies the property that F_2 is r^{-2}-Lipschitz and F_2^{-1} is r^2-Lipschitz. Finally, note that Q_2 is contained in a C-radius ball in the Euclidean metric. There exists an $\operatorname{area}(Q_2)^{-1}$-quasiconformal affine map F_3 from Q_2 to T, which sends vertices of Q_2 to vertices of vertices of T and is scaling on each boundary edge of ∂Q_2. Now, since $\operatorname{area}(Q_2)$ is around $Cr^{-2} \operatorname{area}(Q)$,

$$F_3 \circ F_2 \circ F_1 : Q \to T$$

is $Cr^2 \operatorname{area}(Q)^{-1}$-quasiconformal. Let

$$\mathrm{sc} : \partial Q \to \partial T$$

be a map that sends vertices to vertices and is scaling on each boundary edge, with respect to the hyperbolic metric on ∂Q and the Euclidean metric on ∂T. Then by construction,

$$\mathrm{sc} \circ (F_3 \circ F_2 \circ F_1)^{-1} : \partial T \to \partial T$$

is C-bi-Lipschitz with respect to the Euclidean metric, therefore C-weakly-quasisymmetric. By Lemma 3.6, it extends to a C-quasiconformal map from T to T. Precomposing this map with $F_3 \circ F_2 \circ F_1$ gives the map desired in the lemma statement. □

Proof of Theorem 4.2. We construct a map $F_r : X \to X_r$ triangle-by-triangle as follows. On vertices, F_r is naturally defined. On edges, we define F_r to be scaling (with respect to the metrics $d_{T_r^1}$ on T_r^1 and d_{S_r}, the canonical flat metric given by S_r, on X_r). Then F_r conformally extends to all equilateral triangles (triangles in T_E) of T_r. To define F_r on T_{SE}, note that by construction each semi-equilateral triangle Q is contained in V_i for some i. By Lemma 4.1, Step 6 of the triangulation construction, and conformal property of f_i, the triangular region $f_i(Q)$ has angles bounded below by $\pi/21$. By Lemma 4.1, Step 6 of the triangulation construction and the C-bi-Lipschitz property of f_i, the edges of the triangular region $f_i(Q)$ have lengths bounded above and below by $Cr\,\mathrm{radius}(W_i)$ and curvature bounded above by C. Therefore $f_i(\partial Q)$ is a C-quasicircle. Let

$$\mathrm{sc} : \partial Q \to \partial T$$

denote a scaling map, with respect to the metric $d_{T_r^1}$ on ∂Q and Euclidean metric on ∂T, that sends vertices of ∂Q to vertices of ∂T. Since

$$\mathrm{sc} \circ f_i^{-1} : f_i(\partial Q) \to \partial T$$

is $Cr^{-1}\,\mathrm{radius}(W_i)^{-1}$-Lipschitz with a $Cr\,\mathrm{radius}(W_i)$-Lipschitz inverse (with respect to the Euclidean metrics), it is C-weakly-quasisymmetric. Hence by Lemma 3.6, $\mathrm{sc} \circ f_i^{-1}$ extends to a C-quasiconformal map from $f_{s(i)}(Q)$ to T which means F_r extends to a C-quasiconformal map on Q. Finally, by Lemma 4.3, F_r extends to a $Cr^2\,\mathrm{area}(Q)^{-1}$-quasiconformal map on each hyperbolic triangle Q in T_H. In this way, we construct a map $F_r : X \to X_r$.

We use the characterization of the Teichmüller metric in terms of extremal length from Theorem 3.8. We have on $\mathcal{M}_g$,

$$d_T(X, X_r) = \frac{1}{2} \log \inf_f \sup_\gamma \frac{\mathrm{Ext}_X(\gamma)}{\mathrm{Ext}_{X_r}(f(\gamma))}$$

where the infimum runs through all quasiconformal maps $f : X \to X_r$ and the supremum runs through all free homotopy classes of simple closed curves γ on X. To show this quantity is bounded by Cr, it suffices to show that for all free homotopy classes γ of a simple closed curve on X,

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) \geq (1 - Cr)\,\mathrm{Ext}_X(\gamma).$$

Recall that

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) = \sup_{\rho_r} \frac{\mathrm{length}_{\rho_r}(F_r(\gamma))^2}{\mathrm{area}_{\rho_r}(X_r)}$$

where the supremum ranges over all conformal metrics ρ_r on X_r. Similarly,

$$\mathrm{Ext}_X(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(X)}$$

where the supremum ranges over all conformal metrics ρ on X. By Theorem 3.7, the ρ which achieves this supremum is given by $|\phi|^{1/2}$ for a holomorphic quadratic differential ϕ on X. To show

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) \geq (1 - Cr)\,\mathrm{Ext}_X(\gamma),$$

it suffices to exhibit a conformal metric ρ_r on X_r such that

$$\frac{\mathrm{length}_{\rho_r}(F_r(\gamma))^2}{\mathrm{area}_{\rho_r}(X_r)} \geq (1 - Cr)\frac{\mathrm{length}_\phi(\gamma)^2}{\mathrm{area}_\phi(X)}.$$

To do this, we consider the metric $(F_r)_*|\phi|^{1/2}$ (which is not a conformal metric on X_r), and let ρ_r be the smallest conformal metric on X_r such that $\rho_r \geq (F_r)_*|\phi|^{1/2}$. Then

$$\mathrm{length}_{\rho_r}(F_r(\gamma))^2 \geq \mathrm{length}_\phi(\gamma)^2$$

by construction. It remains to see that

$$\mathrm{area}_{\rho_r}(X_r) \leq (1 + Cr)\,\mathrm{area}_\phi(X).$$

To see this, for each $1 \leq i \leq N$ let

$$m_i = \sup_{x \in V_i} \frac{|\phi|}{\rho_X^2}(x).$$

By the mean value property,

$$\int_{U_i} |\phi| \geq C \operatorname{radius}(U_i)^2 m_i.$$

Therefore, by condition 4 of Lemma 2.5,

$$\operatorname{area}_\phi(X) \geq C \sum_{i=1}^{N} \operatorname{radius}(U_i)^2 m_i. \tag{1}$$

Now, for any equilateral triangle Q that is a face of T_r, $\operatorname{area}_{\rho_r}(F_r(Q)) = \operatorname{area}_\phi(Q)$ because F_r is conformal on Q. Therefore,

$$\operatorname{area}_{\rho_r}(F_r(T_E)) \leq \operatorname{area}_\phi(X) \tag{2}$$

Any triangle Q in T_{SE} is contained in W_i for some $1 \leq i \leq N$. By C-quasiconformality of F_r on Q,

$$\begin{aligned} \operatorname{area}_{\rho_r}(F_r(Q)) &\leq C \operatorname{area}_\phi(Q) \\ &\leq C m_i \operatorname{area}_{\rho_X}(Q). \end{aligned} \tag{3}$$

Now, by Step 5 in the triangulation construction semi-equilateral triangles contained in V_i are actually contained in a $Cr \operatorname{radius}(U_i)$ hyperbolic neighborhood of $\left(\cup_{i=1}^N L_i\right) \cap V_i$, and the latter has length at most $C \operatorname{radius}(U_i)$ in the hyperbolic metric by condition 3 and condition 5 of Lemma 2.5. Summing Eq. (3) over all semi-equilateral triangles Q we obtain

$$\begin{aligned} \operatorname{area}_{\rho_r}(F_r(T_{SE})) &\leq Cr \sum_{i=1}^{N} \operatorname{radius}(U_i)^2 m_i \\ &\leq Cr \operatorname{area}_\phi(X) \end{aligned} \tag{4}$$

by Eq. (1).

Finally, any triangle Q in T_H is contained in V_i for some $1 \leq i \leq N$. By $Cr^2 \operatorname{area}_{\rho_X}(Q)^{-1}$-quasiconformality of F_r on Q, we have

$$\begin{aligned} \operatorname{area}_{\rho_r}(F_r(Q)) &\leq Cr^2 \operatorname{area}_{\rho_X}(Q)^{-1} \operatorname{area}_\phi(Q) \\ &\leq Cr^2 m_i. \end{aligned} \tag{5}$$

Since by Step 6 of the triangulation construction at most $C \operatorname{radius}(U_i)^2 r^{-1}$ hyperbolic triangles Q are contained in any V_i, summing Eq. (5) over all hyperbolic triangles Q we obtain

$$\begin{aligned} \operatorname{area}_{\rho_r}(F_r(T_H)) &\leq Cr \sum_{i=1}^{N} \operatorname{radius}(U_i)^2 m_i \\ &\leq Cr \operatorname{area}_\phi(X) \end{aligned} \tag{6}$$

by Eq. (1).

Eq. (2), Eq. (4) and Eq. (6) together give

$$\begin{aligned} \operatorname{area}_{\rho_r}(X_r) &= \operatorname{area}_{\rho_r}(F_r(T_E)) + \operatorname{area}_{\rho_r}(F_r(T_{SE})) + \operatorname{area}_{\rho_r}(F_r(T_H)) \\ &\leq (1 + Cr) \operatorname{area}_\phi(X) \end{aligned}$$

which completes the proof. □

5. Approximating triangulated surfaces with bounded degree triangulations

5.1. Approximation theorem.

The goal of this section is to show the following theorem:

Theorem 5.1. *There exists a map $B : \mathrm{Comb}^T(\mathcal{M}_g) \to \mathrm{Comb}^{\leq \sigma T}(\mathcal{M}_g)$, and universal constants $C, \sigma, \mu > 0$, such that:*

(1) There is a C-quasiconformal map $f_S : S \to B(S)$,

(2) The maximum degree of any vertex of $B(S)$ is 7,

(3) Given any two vertices $x, y \in V_{\neq 6}(B(S))$ (not necessarily distinct), and γ any arc from x to y that is homotopically nontrivial in $B(S) \setminus \{x, y\}$, we have

$$\int_\gamma |\psi_{B(S)}|^{1/6} \geq 3,$$

(4)

$$|V_{\neq 6}(B(S))| \leq \mu(|V_{\neq 6}(S)| + g),$$

and

(5) The fiber of B over $B(S)$ has cardinality at most $C^{|V_{\neq 6}(B(S))|}$.

Here, C, σ and μ are universal constants.

To show Theorem 5.1, we first construct a local replacement for high degree vertices as follows.

For $d \in \mathbb{N}$, define the triangulated disk TD_d to be the following triangulation of a topological disk by unit equilateral triangles: d unit equilateral triangles are glued together to form a topological disk with one interior vertex of degree d. The boundary of TD_d, denoted ∂TD_d, consists of d edges. We thus obtain a triangulated surface with boundary that we also call TD_d, which comes with a flat metric that may have a singularity at the interior vertex.

Lemma 5.2. *There is a conformal map $f : TD_d \to \overline{\mathbb{D}}$ such that $f : \partial TD_d \to S^1$ is a scaling map with respect to the restriction of the flat metric on TD_d to ∂TD_d, and the Euclidean metric on S^1.*

Remark 15. Scaling map here means all distances get scaled by a constant factor which is

$$\frac{\mathrm{length}(\partial TD_d)}{\mathrm{length}(S^1)}.$$

Proof. Denote by $W_{\pi/3}$ the closed sector of the unit circle with angle $\pi/3$, and denote by T the unit equilateral triangle. By Lemma 3.6, there exists a C-quasiconformal map $W_{\pi/3} \to T$ that sends boundary vertices of $W_{\pi/3}$ to boundary vertices of T and is scaling on each boundary edge. Gluing, we obtain a conformal map from TD_d to the unit cone of angle $d\pi/3$, which is scaling on the boundary. The unit cone of angle $d\pi/3$ admits a conformal map to $\overline{\mathbb{D}}$ which is scaling on the boundary. Composing these two maps, we obtain the statement in the lemma. □

For $d \geq 8$, we now construct the triangulated hyperbolic disk TH_d to be another topological disk formed by gluing unit equilateral triangles satisfying $\partial TH_d \simeq \partial TD_d$ (that is, TH_d also has d boundary edges). However, TH_d will have more interior points, and therefore more triangles. The number of triangles in TH_d will be bounded above by Cd. We construct TH_d inductively by constructing annular layers starting from its boundary.

Label the boundary points of $\partial TH_d \simeq \partial TD_d$ by $x_{0,0}, ..., x_{0,d-1}$. If $d \geq 8$, construct TA_1, a triangulated annulus with outer and inner boundaries ∂TA_1^+ and ∂TA_1^-, respectively, such that

(1) $\partial TA_1^+ = \partial TH_d$,
(2) $\deg_{TA_1}(x_{0,j}) = 3$ if j is even and 4 if j is odd,
(3) TA_1 has no interior vertices.

(Recall that the degree of a vertex of a triangulated surface with boundary is the number of edges emanating from the vertex.) These conditions determine TA_1 uniquely. That is, ∂TA_1^- consists of d_1 edges and vertices where $d_1 = \lfloor d/2 \rfloor$. The vertices of ∂TA_1^- may be labelled $x_{1,0}, ..., x_{1,d_1-1}$ (in cyclic order), such that the following holds.

When d is even, in TA_1, $x_{0,j}$ is connected by an edge to $x_{1,j/2}$ if j is even, and $x_{0,j}$ is connected by an edge to $x_{1,(j-1)/2}$ and $x_{1,(j+1)/2}$ if $j < d-1$ is odd. Finally, $x_{0,d-1}$ is connected by an edge to $x_{1,(d-2)/2}$ and $x_{1,0}$.

When d is odd, $x_{0,j}$ is connected by an edge to $x_{1,j/2}$ if $j < d-1$ is even, and $x_{0,j}$ is connected by an edge to $x_{1,(j-1)/2}$ and $x_{1,(j+1)/2}$ if $j < d-2$ is odd. Moreover, $x_{0,d-1}$ is connected by an edge to $x_{1,0}$, while $x_{0,d-2}$ is connected by an edge to $x_{1,(d-3)/2}$ and $x_{1,0}$.

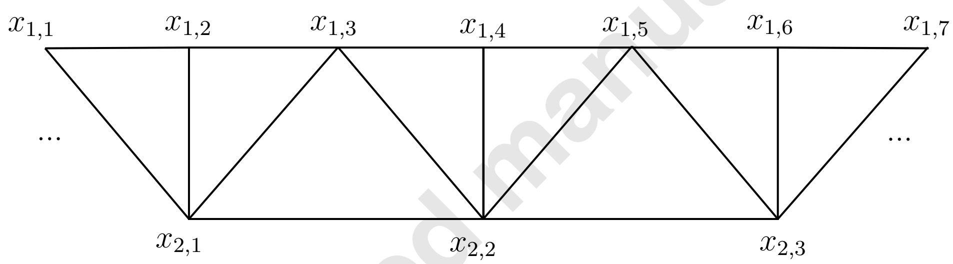

FIGURE 5. A piece of TA_1, with the triangles not to scale.

We now describe the second inductive step, constructing another annulus TA_2 with outer and inner boundaries ∂TA_2^+ and ∂TA_2^-, respectively, such that

(1) $\partial TA_2^+ = \partial TA_1^-$,
(2) $\deg_{TA_2}(x_{1,j}) = 3$ if j is even and 4 if j is odd,
(3) TA_2 has no interior vertices.

We continue this inductive process. At the ith inductive step, wherein the vertices of TA_{i-1}^- are denoted $x_{i-1,0}, ..., x_{i-1,d_{i-1}-1}$ in cyclic order (where $d_{i-1} \sim d/2^{i-1}$), we construct annulus TA_i with outer and inner boundaries ∂TA_i^+ and ∂TA_i^- such that

(1) $\partial TA_i^+ = \partial TA_{i-1}^-$,
(2) $\deg_{TA_i}(x_{i,j}) = 3$ if j is even and 4 if j is odd,
(3) A_i has no interior vertices.

Here, ∂TA_i^- consists of d_i edges and vertices where $d_i = \lfloor d_{i-1}/2 \rfloor$. The vertices of ∂TA_i^- may by labelled $x_{i,0}, ..., x_{i,d_i-1}$ (in cyclic order) such that the following holds.

When d_{i-1} is even, in TA_i, $x_{i-1,j}$ is connected by an edge to $x_{i,j/2}$ if j is even, and $x_{i-1,j}$ is connected by an edge to $x_{i,(j-1)/2}$ and $x_{i,(j+1)/2}$ if $j < d_{i-1} - 1$ is odd. Finally, $x_{i-1,d_{i-1}-1}$ is connected by an edge to $x_{i,(d_{i-1}-2)/2}$ and $x_{i,0}$.

When d_{i-1} is odd, $x_{i-1,j}$ is connected by an edge to $x_{i,j/2}$ if $j < d_{i-1} - 1$ is even, and $x_{i-1,j}$ is connected by an edge to $x_{i,(j-1)/2}$ and $x_{i,(j+1)/2}$ if $j < d_{i-1} - 2$ is odd. Moreover, $x_{i,d_{i-1}-1}$ is connected by an edge to $x_{i,0}$, while $x_{i-1,d_{i-1}-2}$ is connected by an edge to $x_{i,(d_{i-1}-3)/2}$ and $x_{i,0}$.

We continue until we reach the kth step wherein $d_{k-1} \leq 7$. We let $TA_k = TD_{d_{k-1}}$ and glue TA_k to TA_{k-1} along the boundary $T\partial A_{k-1}^-$. Our construction of the $TA_1, ..., TA_k$ naturally identifies ∂TA_{i-1}^- with ∂TA_i^+. We define TH_d to be the union of the $TA_1, ..., TA_k$. We call the unique interior vertex of TA_k the center of TH_d. From the construction described above, the following lemma is evident.

Lemma 5.3. *Let v be the center of TH_d. For $2 \leq i \leq k$, the closed star of the union of $TA_i, ..., TA_k$ is the union of $TA_{i-1}, ..., TA_k$, and TA_k is the closed star of v. In particular, the kth successive closed star of v is TH_d.*

We also have the following lemma.

Lemma 5.4. *There are at most Cd vertices of TH_d. All interior vertices of TH_d have degree at most 7 and all boundary vertices have degree at most 4. Moreover, each TA_i for $2 \leq i \leq k$ contains at least one outer boundary vertex that has degree 7 in TH_d.*

Proof. The total number of vertices of TA_i is at most $Cd/2^i$, so the number of vertices of TH_d is at most Cd. By construction, outer boundary vertices of TA_i have degree at most 4 (in particular, $x_{i-1,0}$ has degree 3). Inner boundary vertices of TA_i have degree at most 5, except for vertex $x_{i,0}$ which has degree 6. From this we obtain the second claim in the lemma. Finally, for $2 \leq i \leq k$, $x_{i-1,0}$ has degree 7 in TH_d. $\square$

Our next goal is to show the following lemma.

Lemma 5.5. *There exists a C-quasiconformal map*

$$f : TH_d \to TD_d$$

which on the boundaries agrees with an identification $\partial TH_d \simeq \partial TD_d$.

To do this, we have the following preliminary lemma.

Lemma 5.6. *Each triangulated annulus TA_i admits a C-quasiconformal map to a closed annulus*

$$A(r, 1) = \{x \in \mathbb{C} | r \leq |x| \leq 1\}$$

whose restriction $\partial TA_i \to \partial A(r, 1)$ is a scaling map. Here, C is a universal constant independent of TA_i.

Proof. We will construct a quasiconformal map to

$$A(\alpha, \beta) = \{x \in \mathbb{C} | \alpha \leq |x| \leq \beta\}.$$

Then, composing with a scaling map gives the lemma statement. We will choose α and β later. We first partition $A(\alpha, \beta)$ into triangular regions. Each triangular region consists of three boundary components, two of which are straight lines and one which is a circular arc. Let (r, θ) be polar coordinates on $\mathbb{C}$ centered at 0. Denote by

$$C_- = \{(r, \theta) | r = \alpha\}$$

and

$$C_+ = \{(r, \theta) | r = \beta\}$$

the two boundary circles of $A(\alpha,\beta)$ of radius α and β respectively. Recall that the vertices of ∂TA_i^+ are $x_{i-1,0},...,x_{i,d_{i-1}-1}$ and the vertices of ∂TA_i^- are $x_{i,0},...,x_{i,d_i-1}$, where $d_i = \lfloor d_{i-1}/2 \rfloor$. Now, let $y_{i-1,j}$ denote the vertex

$$(r,\theta) = (\beta, 2\pi j/d_{i-1})$$

(for $0 \le j \le d_{i-1}-1$). Let $y_{i,j}$ denote the vertex

$$(r,\theta) = (\alpha, 2\pi j/d_i)$$

(for $0 \le j \le d_i - 1$). We construct a triangulation of $A(\alpha,\beta)$ as follows. We let the vertices of the triangulation be the $y_{i-1,j}$ and $y_{i,j}$. Two vertices y_{i-1,j_1} and y_{i,j_2} are connected by an edge if x_{i-1,j_1} and x_{i,j_2} in TA_i are connected by an edge. In this case, the edge between y_{i-1,j_1} and y_{i,j_2} is a straight line.

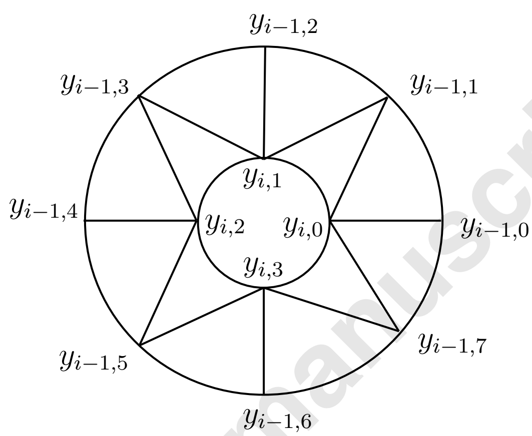

FIGURE 6. Triangulation of $A(\alpha,\beta)$ when $d_{i-1}=8$, which is combinatorially equivalent to TA_i when $d_{i-1}=8$.

For d_i sufficiently large, we take $\alpha = d_{i-1}-1$ and $\beta = d_{i-1}$, and this decomposition of $A(\alpha,\beta)$ into vertices and edges gives a triangulation of $A(\alpha,\beta)$. This is true since by the construction of TA_i, two vertices y_{i-1,j_1} and y_{i,j_2} are connected by an edge only if the distance between them is at most C. For small d_i but still greater than 7, we take $\alpha = 1/10$ and $\beta = 1$ to obtain a triangulation where the angles of each triangular region are nonzero. We claim that when d_i is large, the angles of each triangular region are bounded below by a universal constant. We also claim that when d_i is large, the lengths of the triangular sides are bounded below and above by universal constants. To see this, note that each triangular region Q consists of two straight lines along with a circular arc. Associated to Q is a genuine flat triangle Q' which may be obtained by replacing the circular arc with a straight line, which we call the base of Q'. By construction, the circular arc has length bounded below and above by universal constants, and since d_i is assumed to be sufficiently large, the straight line approximates the circular arc. So the length of the base of the triangle is bounded below and above by universal constants. Since the two circles C_α and C_β are distance 1 apart, the height of each triangle is also bounded below and above by universal constants. Hence, ∂Q has three sides, two of which are straight lines and the third a circular arc, each with lengths bounded below and above by universal constants. Moreover the angles of ∂Q are bounded below by a universal constant because the angles of $\partial Q'$ are bounded below by a universal constant. Therefore for all d_i, each triangular region Q is a C-quasicircle, where

C is independent of d_i. The map from ∂Q to ∂T (the boundary of an equilateral triangle) which sends vertices of to vertices and is a scaling map on each of the three sides is C-bi-Lipschitz with respect to the Euclidean metric, therefore also C-weakly-quasisymmetric. Using Lemma 3.6 on each triangular region, we obtain a map to the unit equilateral triangle that is scaling on each boundary component. Gluing the inverses of these maps together, we obtain a map from TA_i to $A(\alpha, \beta)$ which is a scaling map on the boundary. $\square$

Proof of Lemma 5.5. We already have shown that there exists a C-quasiconformal map

$$TD_d \to \overline{\mathbb{D}}$$

which is a scaling map on the boundary. To show Lemma 5.5, it suffices to construct a C-quasiconformal map

$$TH_d \to \overline{\mathbb{D}}$$

that is a scaling map on the boundary. We do this inductively as follows. First, by Lemma 5.6, TA_1 admits a C-quasiconformal map to the annulus $A(r_1, 1)$ which is a scaling map on the boundary. Then TA_2 admits a C-quasiconformal map to the annulus $A(r_1 r_2, r_1)$ which is a scaling map on the boundary. In general, for $i \in \{1, ..., k-1\}$, TA_i admits a C-quasiconformal map to the annulus $A(r_1...r_i, r_1...r_{i-1})$ which is scaling on the boundary. Finally, by Lemma 5.2 TA_k (which is a triangulated disk) admits a C-quasiconformal map to the Euclidean disk of radius $r_1...r_{k-1}$ which is a scaling map on the boundary. These maps (after possible rotations) glue together to give the desired map $TH_d \to \overline{\mathbb{D}}$. $\square$

Proof of Theorem 5.1. First, we replace S with a surface S_1 in $\mathrm{Comb}^{16T}(\mathcal{M}_g)$ which is the 4-subdivision of S rescaled. Then we may replace the closed star around every vertex in S_1 of degree d greater than 7 (which is a copy of TD_d) by TH_d to obtain a triangulated surface S_2. (These closed stars are all disjoint.) Then, take the rescaled 5-subdivision of S_2 to obtain $B(S)$, a genus g triangulated surface with at most σT triangles, for a constant σ independent of T and g. By construction, conditions 2 and 3 in the statement of Theorem 5.1 are satisfied. Applying Lemma 5.5 for each triangulated disk replacement and gluing, we have a C-quasiconformal map $f : S \to S_2$, and S_2 is naturally conformally equivalent to $B(S)$. This shows condition 1 in the statement of Theorem 5.1.

Now,

$$|V_{\neq 6}(S)| = |V_{\neq 6}(S_1)|$$

and

$$|V_{\neq 6}(B(S))| = |V_{\neq 6}(S_2)|.$$

Since S_2 is formed by replacing disjoint copies of TD_d in S_1 with TH_d, by Lemma 5.4, we have

$$|V_{\neq 6}(S_2)| \leq |V_{<6}(S_1)| + C \sum_{x \in V_{>6}(S_1)} \deg x.$$

By Euler characteristic considerations,

$$\sum_{x \in V_{>6}(S_1)} \deg x \leq C|V_{<6}(S_1)| + Cg.$$

So

$$\begin{aligned} |V_{\neq 6}(S_2)| &\leq C|V_{<6}(S_1)| + Cg \\ &\leq C|V_{\neq 6}(S_1)| + Cg. \end{aligned}$$

Hence

$$|V_{\neq 6}(B(S))| \leq C(|V_{\neq 6}(S)| + g),$$

showing condition 4 in the statement of Theorem 5.1.

It remains to bound the cardinality of the fibers of B. Given $B(S)$, we may recover S_2 in the following way. We start by choosing a vertex of $B(S)$ with degree strictly greater than 6 (which must exist by Euler characteristic considerations). This vertex must also be a vertex of S_2, and from this vertex we may inductively reconstruct S_2. Recall that S_1 may be constructed from S_2 by replacing certain (disjoint) copies of TH_d by copies of TD_d. Each copy TD_d contains its center, which is a vertex of degree $d > 6$, and these centers must be distinct. There are therefore at most $C^{|V_{\neq 6}(B(S))|}$ total choices for the set of centers, and the set of centers has cardinality at most $|V_{\neq 6}(B(S))|$.

Lemma 5.7. *Suppose $v \in V(S_2)$ is the center of a copy of TH_d. Given S_2 and v, there are at most two possible choices for d.*

Proof. Suppose that TH_d, $TH_{d'}$ and $TH_{d''}$, with $d < d' < d''$ are all contained in S_2 and centered at v. By construction of S_2, all the boundary vertices of the closed star of TH_d have degree 6 in S_2. (This is true because S_1 is the rescaled 4-subdivision of S, and to construct S_2 from S_1 we only replace stars of vertices of degree $d \geq 8$ by copies of TH_d.) Since $TH_{d'}$ and $TH_{d''}$ are also subsets of S_2 centered at v and $d < d' < d''$, by Lemma 5.3 we must have that the closed star of TH_d in S_2 is contained in the interior of $TH_{d''}$. In other words, boundary vertices of the closed start of TH_d are interior vertices of $TH_{d''}$. This contradicts Lemma 5.4. □

We continue with the proof of Theorem 5.1. Given a set of centers, by Lemma 5.7 there are at most $C^{|V_{\neq 6}(B(S))|}$ possibilities for the choice of triangulated hyperbolic disks TH_d centered at these centers. Once these disks are chosen, there is a unique choice of replacement (TD_d) for each triangulated hyperbolic disk, hence S_1 may be reconstructed. Finally, given S_1, S can be recovered by choosing a vertex of S_1 with degree strictly greater than 6, which must also be a vertex of S, and inductively reconstructing S starting from this vertex. Thus, given $B(S)$, there are at most $C^{|V_{\neq 6}(B(S))|}$ possibilities for S. This shows condition 5 in the statement of Theorem 5.1. □

5.2. **Upper bounds for triangulated surfaces in terms of locally bounded surfaces.**

We have the following corollary of Theorem 5.1.

Corollary 5.8. *There exists a universal constant C such that*

$$N^{\mathcal{M}}(T, g, m, r) \leq C^{m+g} N^{\mathcal{M}}_{\mathrm{lb}}(\sigma T, g, \mu(m+g), r + C)$$

for $g \geq 2$. Here, σ and μ are the constants defined in the statement of Theorem 5.1.

Proof. Let $X \in \mathcal{T}_g$. Suppose $S \in \mathrm{Comb}^{\leq T, \leq m}(\mathcal{M}_g)$ such that $\Phi(S) \in B_{d_T}(X, r)$. By Theorem 5.1, there exists a triangulated surface $B(S)$ in $\mathrm{Comb}^{\leq \sigma T, \leq \mu(m+g)}_{\mathrm{lb}}(\mathcal{M}_g)$ such that

$$d_T(\Phi(B(S)), \Phi(S)) \leq C.$$

Here, B is the map defined in statement of Theorem 5.1. Since

$$\begin{aligned} |V_{\neq 6}(B(S))| &\leq \mu(|V_{\neq 6}(S)| + g) \\ &\leq \mu(m+g), \end{aligned}$$

the fibers of B have cardinality at most C^{m+g}. Summing over all $g' \le g$, we have

$$N^{\mathcal{M}}(T,g,m,r) \le C^{m+g} N^{\mathcal{M}}_{\mathrm{lb}}(\sigma T, g, \mu(m+g), r+C),$$

as desired. □

6. Upper bounds for combinatorial translation surfaces via triangulated surfaces

In this section, we shall prove the following result.

Lemma 6.1. *There exists a universal constant C such that*

$$N^{\mathcal{H}}_{\mathrm{lb}}(T,g,r) \le (T/g)^{C(1+r)g} \sum_{\substack{n \le (1/100)g \\ T_1+\ldots+T_n \le 2T \\ g_1+\ldots+g_n \le \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \ge 2 \\ m_1+\ldots+m_n \le \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{M}}(T_i, g_i, m_i, r+C)$$

for $g \ge 2$. Here, μ is the constant defined in the statement of Theorem 5.1.

6.1. **Hodge norms and roadmap to prove Lemma 6.1.** Let $X \in \mathcal{M}_g$. We will first compute

$$|\{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X,r)\}|.$$

To do this, we will first compute

$$|\{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X,(g/T)^{\kappa_0})\}|$$

for an appropriate integer $\kappa_0 \in \mathbb{N}$ to be chosen later. Then we will use Corollary 3.19 to compute

$$|\{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X,r)\}|.$$

For any

$$Y \in B_{d_T}(X,(g/T)^{\kappa_0}),$$

there exists a diffeomorphism surfaces

$$f : X \to Y$$

such that f is $1+8(g/T)^{\kappa_0}$-quasiconformal.

We define a map

$$H_X : \{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X,(g/T)^{\kappa_0})\} \to H^1(X,\mathbb{C})$$

that sends S_Y to the cohomology class represented by $f^*\phi_{S_Y}$. We count the quantity

$$|\{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X,(g/T)^{\kappa_0})\}|$$

in two steps. First, we compute the number of $S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ such that

$$\Phi(S_Y) \in B_{d_T}(X,(g/T)^{\kappa_0})$$

and $H_X(S_Y)$ is close in the Hodge metric to a fixed cohomology class in $H^1(X,\mathbb{C})$. Then we bound the number of cohomology classes, quantitatively.

Lemma 6.2. *Let $X \in \mathcal{M}_g$ and suppose $S_X \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ such that $\Phi(S_X) = X$. Then there are at most*

$$(T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{M}}(2T_i, g_i, m_i, r + C)$$

number of $S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ that satisfy the following properties:

(1)
$$Y = \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_0})$$
and

(2)
$$\|\phi_{S_X} - f^*\phi_{S_Y}\|_X \leq \alpha_1 (g/T)^{\kappa_1} g^{1/2}.$$

Here, α_1 is a sufficiently small universal constant and κ_1 is a sufficiently large universal constant. We choose these constants in Section 6.9.

As a corollary:

Corollary 6.3. *We have,*

$$\begin{aligned}&|\{S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_2})\}| \\ &\qquad \leq (T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{M}}(2T_i, g_i, m_i, r + C)\end{aligned}$$

where κ_2 is a sufficiently large universal constant chosen in Section 6.9, satisfying $\kappa_2 \geq \kappa_0$, $(\kappa_2 - 1)/2 \geq \kappa_1$ and

$$100\alpha_1^{-1}(1/2)^{(\kappa_2-1)/2-\kappa_1} \leq 1.$$

First, a preliminary lemma.

Lemma 6.4. *The Hodge norm squared of ϕ_S is*

$$\int_X \phi_S \wedge *\overline{\phi_S} = (\sqrt{3}/4)T.$$

Proof. On the equilateral triangle with vertices $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ in $\mathbb{C}$, the Hodge norm squared of dz is $\frac{\sqrt{3}}{4}$. Since there are T triangles in S, the lemma follows. □

Proof of Corollary 6.3. The key idea is to reduce Corollary 6.3 to Lemma 6.2 by using Lemma 3.3. Lemma 3.3 allows us to deduce, from a condition about cohomology classes being close (condition 2 of Lemma 6.2), a much stronger condition about the individual forms being close averaged over the surface.

Let $S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ satisfying

$$\Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_2}).$$

By Lemma 6.4 and Corollary 3.2,

$$\|H_X(S_Y)\| \le CT^{1/2}.$$

Now, cover the $CT^{1/2}$ radius Hodge norm ball in $H^1(X,\mathbb{C})$ centered at 0 by $(T/g)^{Cg}$ number of $(g/T)^{\kappa_3} g^{1/2}$ radius balls $B_1, ..., B_{(T/g)^{Cg}}$. Here, κ_3 is a constant to be chosen, and we assume that $\kappa_3 \ge (\kappa_2 - 1)/2$. Given

$$S_Y, S_Z \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$$

satisfying

$$\Phi(S_Y), \Phi(S_Z) \in B_{d_T}(X, (g/T)^{\kappa_2})$$

and

$$H_X(S_Y), H_X(S_Z) \in B_k,$$

let $f : X \to Y$ and $g : Y \to Z$ be $1 + 8(g/T)^{\kappa_2}$-quasiconformal maps. Let

$$(f^*\phi_{S_Y})^h$$

be the harmonic form on X representing $H_X(S_Y)$, the cohomology class of $f^*\phi_{S_Y}$. Let

$$((g \circ f)^*\phi_{S_Z})^h$$

be the harmonic form on X representing $H_X(S_Z)$, the cohomology class of $(g \circ f)^*\phi_{S_Z}$.

By Lemma 3.3 and Lemma 6.4,

$$\|f^*\phi_{S_Y} - (f^*\phi_{S_Y})^h\|_X \le 4(\sqrt{3}/4)^{1/2}(g/T)^{\kappa_2/2}T^{1/2}$$

and

$$\|(g \circ f)^*\phi_{S_Z} - ((g \circ f)^*\phi_{S_Z})^h\|_X \le 8(\sqrt{3}/4)^{1/2}(g/T)^{\kappa_2/2}T^{1/2}.$$

Since $H_X(S_Y), H_X(S_Z) \in B_k$,

$$\|(f^*\phi_{S_Y})^h - (f^*\phi_{S_Z})^h\| \le 2(g/T)^{\kappa_3} g^{1/2}.$$

Summing, we obtain

$$\begin{aligned}\|f^*\phi_{S_Y} - (g \circ f)^*\phi_{S_Z}\| &\le (20(g/T)^{(\kappa_2-1)/2} + 2(g/T)^{\kappa_3})g^{1/2} \\ &\le 40(g/T)^{(\kappa_2-1)/2} g^{1/2}\end{aligned}$$

assuming $\kappa_3 \ge (\kappa_2 - 1)/2$. Pulling back to Y under f^{-1}, by Lemma 3.1 we obtain

$$\|\phi_{S_Y} - g^*\phi_{S_Z}\| \le 100(g/T)^{(\kappa_2-1)/2} g^{1/2}.$$

Since $g/T \le 1/2$, the assumptions $\kappa_1 \le (\kappa_2 - 1)/2$ and

$$100\alpha_1^{-1}(1/2)^{(\kappa_2-1)/2-\kappa_1} \le 1$$

imply condition 2 of Lemma 6.2 is satisfied. Assuming $\kappa_2 \ge \kappa_0$, condition 1 of Lemma 6.2 is satisfied. Applying Lemma 6.2, we obtain

$$|(H_X)^{-1}(B_k)| \le (T/g)^{Cg} \sum_{\substack{n \le (1/100)g \\ T_1+...+T_n \le T \\ g_1+...+g_n \le \mu^{-1}(1/100)g \\ g_1,...,g_n \ge 2 \\ m_1+...+m_n \le \mu^{-1}(1/100)g}} \prod_{i=1}^n N^{\mathcal{M}}(2T_i, g_i, m_i, r + C)$$

for all $k \in \{1, ..., (T/g)^{Cg}\}$. The lemma statement follows. □

Corollary 6.3 now implies Lemma 6.1:

Proof of Lemma 6.1. By Corollary 3.19, we may cover $B_{d_T}(X, r)$ with $(T/g)^{C\kappa_2(1+r)g}$ many $B_{d_T}(\cdot, (g/T)^{\kappa_2})$ balls. Applying Corollary 6.3 to each ball, then summing over all $T' \leq T$ and $g' \leq g$ gives the desired bound on $N^{\mathcal{H}}(T, g, r)$. □

We now turn to the proof of Lemma 6.2, which will take the rest of Section 6. Condition 2 in the statement of Lemma 6.2 can be written as

$$\int_X |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} |\phi_{S_X}|^2 \leq \alpha_2 (g/T)^{\kappa_4} g. \tag{7}$$

Here, $\alpha_2 = \alpha_1^2$ and $\kappa_4 = 2\kappa_1$. Also, $|\cdot|_{S_X}$ here denotes the operator norm of a 1-form at a particular point of X with respect to the S_X-metric.

Remark 16. Roughly speaking, Eq. (7) gives a measure of how Lipschitz the map f is with respect to the metrics d_{S_X} and d_{S_Y}, averaged over the entire surface X. However, Eq. (7) also implies that on most of X, the integral of ϕ_X on a curve is close to the integral of the 1-form ϕ_Y on the image of the curve.

The first step to prove Lemma 6.2 is to show that Eq. (7) implies most vertices of S_X and S_Y of degree strictly greater than 6 must be close to each other under f. Then we show that Eq. (7) implies many edges of S_X and S_Y must be close to each other under f. Finally we show that many faces of S_X and S_Y must be close to each other under f. We do this precisely in Section 6.2, Section 6.3 and Section 6.4 below.

The second step to prove Lemma 6.2 is to show that the geometric conditions about many vertices, edges and faces being close together under f imply that f is close to a simplicial isomorphism on all of S_X except for a part that has much lower genus. To do this, we decompose S_X into around g parallelograms of length and width at most around T/g. We use the geometric conditions to say that f is close to a simplicial isomorphism on most of the parallelograms. The remaining parallelograms form a surface of much smaller genus. This allows us to reduce the problem of counting combinatorial translation surfaces in moduli space to the problem of counting triangulated surfaces in a lower dimensional moduli space. We do this in Section 6.5, Section 6.6 and Section 6.7. Finally in Section 6.8 we prove Lemma 6.2.

6.2. **Vertices.** In this section, we show that most vertices of S_X of degree strictly greater than 6 are close under f to vertices of S_Y of degree strictly greater than 6.

Let $0 < \varepsilon_0 < 1/2$, to be chosen in Section 6.9. Assume that

$$8 \cdot (1/2)^{\kappa_0} \leq (1/10)^{10}.$$

Let V_X be the set of vertices $x \in V(S_X)$ such that there exists $y \in V(S_Y)$ satisfying $d_{S_Y}(y, f(x)) < \varepsilon_0 \cdot (g/T)$. Note that there is at most one choice of such y, since vertices of S_Y are at least distance 1 apart in the S_Y-metric.

Lemma 6.5. *We have,* $|V_{>6}(S_X) \setminus V_X| \leq \alpha_3 g$ *where* $\alpha_3 = 10^{10}\alpha_2(g/T)^{\kappa_4 - 10}\varepsilon_0^{-10}$.

Note that α_3 is not a universal constant, since it depends on T/g. To show Lemma 6.5, we first show the following lemma.

Lemma 6.6. *Suppose* $x \in V_{>6}(S_X)$ *and*

$$\int_{B_{S_X}(x,1/2)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} |\phi_{S_X}|^2 \leq \beta.$$

Then there exists $y \in V_{>6}(S_Y)$ *satisfying*

$$d_{S_Y}(y, f(x)) < \eta(\beta).$$

Here,

$$\eta(\beta) = 10\beta^{1/10} < 1/2.$$

Proof. In the following argument we write $\eta = \eta(\beta)$. Suppose the contrary, that $f(x)$ is not within a η-neighborhood of any vertex in $V_{>6}(S_Y)$. Let $a = \deg x/6$. Since S_X is a locally bounded combinatorial translation surface, $a \le 7$. Let (r_X, θ_X) (where $0 \le \theta_X < 2\pi a$) be polar coordinates on $B_{S_X}(x, \eta)$ around x such that

$$\phi_{S_X} = e^{i\theta_X} dr_X + i r_X e^{i\theta_X} d\theta_X.$$

Let (r_Y, θ_Y) (where $0 \le \theta < 2\pi$) be polar coordinates on $B_{S_Y}(f(x), \eta)$ around y such that

$$\phi_{S_Y} = e^{i\theta_Y} dr_Y + i r_Y e^{i\theta_Y} d\theta_Y.$$

Let

$$C_r = \{(r_X, \theta_X) | r_X = r\}.$$

By assumption,

$$\int_{r=\eta/200}^{\eta/100} \int_{C_r} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} r_X dr_X d\theta_X \le \beta.$$

Therefore for some $u \in [\eta/200, \eta/100]$, we have

$$\int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_X \le 40000\beta\eta^{-2}.$$

By Cauchy-Schwarz,

$$\begin{aligned} \int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X &\le \left(\int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_X \right)^{1/2} (2\pi a\eta)^{1/2} \\ &\le 400\eta^{-1/2}(\pi a\beta)^{1/2}. \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned} &\left| \int_{C_u} \left| f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) \right| d\theta_X - \int_{C_u} \left| \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) \right| d\theta_X \right| \\ &\qquad \le \int_{C_u} \left| f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) \right| d\theta_X \\ &\qquad \le \int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X \\ &\qquad \le 400\eta^{-1/2}(\pi a\beta)^{1/2}. \end{aligned}$$

Now,

$$\text{length}_{S_X}(C_u) = u \int_{C_u} \left| \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) \right| d\theta_X$$

and

$$\text{length}_{S_Y}(f(C_u)) = u \int_{C_u} \left| f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) \right| d\theta_X.$$

Thus

$$2\pi a u - 400\eta^{-1/2}(\pi a\beta)^{1/2} \leq \mathrm{length}_{S_Y}(f(C_u)) \leq 2\pi a u + 400\eta^{-1/2}(\pi a\beta)^{1/2}.$$

Since $u \in [\eta/200, \eta/100]$, $a \leq 7$ and $\beta \leq \eta^{10}/10^{10}$, $f(C_u)$ is contained in $B_{S_Y}(f(x), \eta)$.

For $v \in [0, 2a\pi)$, let x_v be a point on C_u wherein $r_X = u$ and $\theta_X = v$. Letting $C_u(x_0, x_v)$ be an arc of C_u from x_0 to x_v, we have

$$\begin{aligned}
&\left|(ue^{iv} - u) - \int_{C_u(x_0,x_v)} f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| \\
&\qquad = \left|\int_{C_u(x_0,x_v)} f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \int_{C_u(x_0,x_v)} \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| \\
&\qquad \leq \int_{C_u(x_0,x_v)} \left|f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| \\
&\qquad \leq \int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X \\
&\qquad \leq 400\eta^{-1/2}(\pi a\beta)^{1/2}.
\end{aligned}$$

by Eq. (8). Hence,

$$|f(x_v) - f(x_0) - (ue^{iv} - u)|_{S_Y} \leq 400\eta^{-1/2}(\pi a\beta)^{1/2}.$$

This means $f(C_u)$ lies in a $400\eta^{-1/2}(\pi a\beta)^{1/2}$-neighborhood of a radius u-ball passing through $f(x_0)$ (in the S_Y-metric). Therefore,

$$\mathrm{area}_{S_Y}(f(B_{S_X}(x,u))) \leq \pi(u + 400\eta^{-1/2}(\pi a\beta)^{1/2})^2. \tag{9}$$

Finally, by $1 + 8(g/T)^{\kappa_0}$-quasiconformality of f,

$$\begin{aligned}
&\mathrm{area}_{S_X}(B_{S_X}(x,u))^{1/2} - (1 + 8(g/T)^{\kappa_0})^{1/2}\,\mathrm{area}_{S_Y}(f(B_{S_X}(x,u)))^{1/2} \\
&\qquad \leq \left(\int_{B_{S_X}(x,u)} |\phi_{S_X}|^2\right)^{1/2} - \left(\int_{B_{S_X}(x,u)} |f^*\phi_{S_Y}|^2_{S_X}|\phi_{S_X}|^2\right)^{1/2} \\
&\qquad \leq \left(\int_{B_{S_X}(x,u)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X}|\phi_{S_X}|^2\right)^{1/2} \\
&\qquad \leq \beta^{1/2}.
\end{aligned}$$

Therefore

$$\mathrm{area}_{S_Y}(f(B_{S_X}(x,u)))^{1/2} \geq (1 - 8(g/T)^{\kappa_0})^{1/2}((a\pi)^{1/2}u - \beta^{1/2}).$$

Since $a \geq 2$, $g/T \leq 1/2$, $\beta \leq \eta^{10}/10^{10}$ and $u \in [\eta/200, \eta/100]$, assuming $8\cdot(1/2)^{\kappa_0} \leq (1/10)^{10}$ we have a contradiction with Eq. (9). □

Proof of Lemma 6.5. Combining Eq. (7) and Lemma 6.6, we obtain the desired result. □

6.3. **Edges.** In this section, we show that many edges of S_X and S_Y are close to each other under f.

Lemma 6.7. *Let $0 < \varepsilon \leq 1/1000$. Suppose $x_0 \in V(S_X)$ and $f(x_0) \in B_{S_Y}(y_0, \varepsilon)$ for a vertex $y_0 \in V(S_Y)$. Suppose $e(x_0, x_1)$ is an edge in S_X from x_0 to x_1. Suppose further that*

$$\int_{B_{S_X}(e(x_0,x_1),1/2)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} |\phi_{S_X}|^2 \leq \beta, \tag{10}$$

where β satisfies

$$\tau(\varepsilon, \beta) = 100\beta^{1/4}\varepsilon^{-1} + \varepsilon \leq 1/1000.$$

Then

$$f(x_1) \in B_{S_Y}(y_1, \tau(\varepsilon, \beta))$$

for some vertex y_1 in S_Y. Also, y_1 is connected to y_0 by an edge $e(y_0, y_1)$ such that

$$f(e(x_0, x_1)) \subset B_{S_Y}(e(y_0, y_1), 10\tau(\varepsilon, \beta)).$$

Proof. Let T be a triangle in S_X containing edge $e(x_0, x_1)$, and let T' be the other triangle containing edge $e(x_0, x_1)$. As shown in Fig. 7, identify T with the triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, with x_0 and x_1 identified with 0 and 1. Then T' is naturally identified with the triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Let (r_0, θ_0) be polar coordinates on $T \cup T'$ centered at 0, and (r_1, θ_1) be polar coordinates on $T \cup T'$ centered at 1. Note that the form ϕ_{S_X} may be written as

$$\zeta e^{i\theta_0} dr_0 + i\zeta r_0 e^{i\theta_0} d\theta_0$$

in the (r_0, θ_0) coordinates and

$$-\zeta e^{-i\theta_0} dr_0 + i\zeta r_0 e^{-i\theta_0} d\theta_0$$

in the (r_1, θ_1)-coordinates, for some 6th root of unity ζ. Let $\upsilon < 1/4$ be sufficiently small, to be chosen later in this proof. For $\theta \in [-\pi/3, \pi/3]$, define

$$L_\theta = \{(r_0, \theta_0) \in T | \theta_0 = \theta, \upsilon \leq r_0 \leq (1/2)(\cos\theta)^{-1}\},$$

and

$$R_\theta = \{(r_1, \theta_1) \in T | \theta_1 = \pi - \theta, \upsilon \leq r_1 \leq (1/2)(\cos\theta)^{-1}\}.$$

We assume that L_θ (resp. R_θ) is oriented so that r_0 (resp. r_1) is increasing.

By elementary trigonometry, we see that for $\theta \in [-\varepsilon, \varepsilon]$, $L_\theta \cup R_\theta \subset B_{S_X}(e(x_0, x_1), \varepsilon)$. By the co-area formula along with Eq. (10),

$$\int_{\theta=0}^{\varepsilon} \int_{L_\theta} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0 d\theta_0 + \int_{\theta=0}^{\varepsilon} \int_{R_\theta} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1 d\theta_1 \leq \beta.$$

This means for some $u \in [0, \varepsilon]$,

$$\int_{L_u} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0 + \int_{R_u} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1 \leq \beta\varepsilon^{-1}.$$

By Cauchy-Schwarz,

$$\int_{L_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_0 + \int_{R_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_1 \tag{11}$$

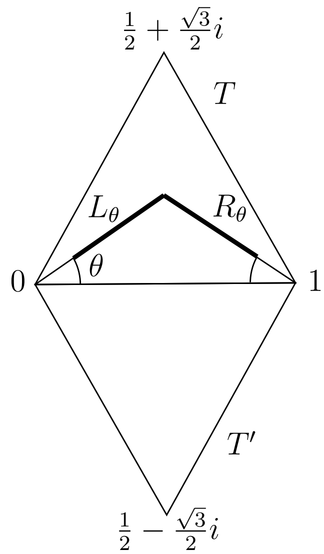

FIGURE 7. A diagram of T and T' with L_θ and R_θ in bold.

$$\leq \left(\int_{L_u} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0\right)^{1/2} \left(\int_{r_0=\upsilon}^{(1/2)(\cos\theta)^{-1}} r_0^{-1}\right)^{1/2} + \left(\int_{R_u} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1\right)^{1/2} \left(\int_{r_1=\upsilon}^{(1/2)(\cos\theta)^{-1}} r_1^{-1}\right)^{1/2}$$

thus

$$\int_{L_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_0 + \int_{R_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_1 \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \tag{12}$$

Since

$$\left|f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_0}\right)\right| \leq |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}$$

on L_u and

$$\left|f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_1}\right)\right| \leq |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}$$

on R_u, combining with Eq. (12) we obtain

$$\int_{L_u} \left|f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_0}\right)\right| dr_0 + \int_{R_u} \left|f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_1}\right)\right| dr_1 \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \tag{13}$$

For an appropriate 6th root of unity ζ,

$$\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) = e^{iu}\zeta$$

on L_u and

$$\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) = -e^{-iu}\zeta$$

on R_u. Plugging into Eq. (13) we have

$$\int_{L_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - e^{iu}\zeta \right| dr_0 + \int_{R_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) + e^{-iu}\zeta \right| dr_1 \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}.$$

This means

$$\left| \int_{L_u \cup R_u} f^*\phi_{S_Y} - \int_{L_u} e^{iu}\zeta dr_0 - \int_{R_u} e^{-iu}\zeta dr_1 \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \tag{14}$$

Note that when integrating over $L_u \cup R_u$, the orientation of R_u switches, hence there is a sign change. By construction of L_u and R_u,

$$\int_{L_u} e^{iu}\zeta dr_0 = ((1/2)(\cos u)^{-1} - \upsilon)\zeta e^{iu}$$

and

$$\int_{R_u} e^{-iu}\zeta dr_1 = ((1/2)(\cos u)^{-1} - \upsilon)\zeta e^{-iu},$$

hence combining with Eq. (14) we obtain

$$\left| \zeta - \int_{L_u \cup R_u} f^*\phi_{S_Y} \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{15}$$

Pushing forward to Y we have

$$\left| \zeta - \int_{f(L_u \cup R_u)} \phi_{S_Y} \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{16}$$

From Eq. (15) we also have

$$\int_{L_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) \right| dr_0 + \int_{R_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) \right| dr_1 \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon.$$

The left-hand-side is the length of $f(L_u \cup R_u)$ in the S_Y-metric. Hence,

$$\text{length}_{S_Y}(f(L_u \cup R_u)) \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{17}$$

Now, let $C_{0,r}$ be the circle of radius r around x_0, with respect to the S_X-metric. Similarly, let $C_{1,r}$ be the circle of radius r around x_1, with respect to the S_X-metric. By assumption, for $\upsilon < 1/4$ we have

$$\int_{r=\upsilon}^{2\upsilon} \int_{C_{0,r}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}^2 r_0 dr_0 d\theta_0 \leq \beta.$$

Therefore for some $w \in [\upsilon, 2\upsilon]$, we have

$$\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}^2 d\theta_0 \leq \beta\upsilon^{-2}.$$

By Cauchy-Schwarz,

$$\begin{aligned}\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_0 &\leq \left(\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}^2 d\theta_0 \right)^{1/2} (2\pi\upsilon)^{1/2} \\ &\leq (2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\int_{C_{0,w}}\left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right|d\theta_0-\int_{C_{0,w}}\left|\phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right|d\theta_0\right|&\\ &\leq\int_{C_{0,w}}\left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)-\phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right|d\theta_0\\ &\leq\int_{C_{0,w}}|f^*\phi_{S_Y}-\phi_{S_X}|_{S_X}d\theta_0\\ &\leq(2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}.\end{aligned}$$

Now,

$$\mathrm{length}_{S_X}(C_{0,w})=w\int_{C_{0,w}}\left|\phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right|d\theta_0$$

and

$$\mathrm{length}_{S_Y}(f(C_{0,w}))=w\int_{C_{0,w}}\left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right|d\theta_0.$$

Thus

$$2\pi w-(2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}\leq\mathrm{length}_{S_Y}(f(C_{0,w}))\leq 2\pi w+(2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}.$$

Since $w\in[\upsilon,2\upsilon]$ and assuming $\beta\leq\upsilon^4$, $f(C_{0,w})$ is contained in $B_{S_Y}(f(x_1),20\upsilon)$. Since $f(C_{0,\upsilon})$ is enclosed by $f(C_{0,w})$, $f(C_{0,\upsilon})$ is also contained in $B_{S_Y}(f(x_0),20\upsilon)$. Analogously, we may show that $f(C_{1,\upsilon})$ is contained in $B_{S_Y}(f(x_1),20\upsilon)$.

The arc $L_u\cup R_u$ has two endpoints, one wherein $r_0=\upsilon$ and $\theta_0=u$, which we label $x_{0,u}$, and the other wherein $r_1=\upsilon$ and $\theta_1=\pi-u$, which we label $x_{1,u}$. Since $f(x_{0,u})$ lies on $f(C_{0,\upsilon})$ and $f(x_{1,u})$ lies on $f(C_{1,\upsilon})$,

$$f(x_{0,u})\in B_{S_Y}(f(x_0,20\upsilon))$$

and

$$f(x_{1,u})\in B_{S_Y}(f(x_1,20\upsilon)).$$

By Eq. (16) we have,

$$d_{S_Y}(f(x_0),f(x_1))\leq 1+4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}+50\upsilon,$$

so

$$\begin{aligned}d_{S_Y}(y_0,f(x_1))&\leq d_{S_Y}(y_0,f(x_0))+d_{S_Y}(f(x_0),f(x_1))\\ &\leq 1+4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}+50\upsilon+\varepsilon.\end{aligned}$$

Let y_1 be a vertex of S_Y nearest to $f(x_1)$. Since $f(x_1)$ lies on a triangle in S_Y,

$$d_{S_Y}(y_1,f(x_1))\leq\sqrt{3}^{-1}. \tag{18}$$

Therefore

$$\begin{aligned}d_{S_Y}(y_0,y_1)&\leq d_{S_Y}(y_0,f(x_1))+d_{S_Y}(y_0,f(x_1))\\ &\leq 1+4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}+50\upsilon+\varepsilon+\sqrt{3}^{-1}.\end{aligned}$$

Choose υ later such that

$$4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}+50\upsilon+\varepsilon<1/1000.$$

We must have that either

$$d_{S_Y}(y_0,y_1)=0$$

or

$$d_{S_Y}(y_0, y_1) = 1.$$

The former possibility is ruled out by Eq. (16), therefore the latter equation is true. This necessarily means that y_0 and y_1 are adjacent vertices on S_Y.

We must show that

$$d(x_1, f(y_1)) \leq \tau(\varepsilon, \beta)$$

and also find an edge of S_Y connecting y_0 and y_1. To do this, letting q be a shortest path from $f(x_1)$ to y_1, we have

$$\begin{aligned} d_{S_Y}(f(x_1), y_1) &= \left|\int_q \phi_Y\right| \\ &\leq \sqrt{3}^{-1} \end{aligned} \tag{19}$$

by Eq. (18). Let p be a shortest path from y_0 to $f(x_0)$ in S_Y. Then

$$\left|\int_p \phi_{S_Y}\right| \leq \varepsilon. \tag{20}$$

Let s_0 be a shortest path from $f(x_{0,u})$ to $f(x_0)$ and s_1 be a shortest path from $f(x_{1,u})$ to $f(x_1)$. Finally, let t be a shortest path from y_0 to y_1 that is homotopic to the path

$$p \cup s_0 \cup f(L_u \cup R_u) \cup s_1 \cup q$$

(which also has the endpoints y_0 and y_1). By construction q is homotopic to

$$t^{-1} \cup p \cup s_0 \cup f(L_u \cup R_u) \cup s_1,$$

where here t^{-1} denotes the path t with the opposite orientation. So to compute

$$\int_q \phi_Y,$$

it suffices to compute

$$\int_{t^{-1}\cup p\cup s_0\cup f(L_u\cup R_u)\cup s_1} \phi_Y = \int_p \phi_Y + \int_{s_0} \phi_Y + \int_{f(L_u\cup R_u)} \phi_Y + \int_t \phi_Y + \int_{s_1} \phi_Y.$$

Since $s_0 \subset B(f(x_1), 20\upsilon)$,

$$\left|\int_{s_0} \phi_Y\right| \leq 20\upsilon. \tag{21}$$

Similarly,

$$\left|\int_{s_1} \phi_Y\right| \leq 20\upsilon. \tag{22}$$

From Eq. (16), Eq. (19), Eq. (20), Eq. (21) and Eq. (22), we have

$$\left|\zeta - \int_t \phi_Y\right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon + \sqrt{3}^{-1}.$$

Since

$$\int_t \phi_Y \subset \mathbb{Z} + e^{\pi i/3}\mathbb{Z},$$

we must necessarily have

$$\int_t \phi_Y = \zeta.$$

Combining with Eq. (16), Eq. (20), Eq. (21) and Eq. (22) we obtain

$$\begin{aligned} d(y_1, f(x_1)) &= \left| \int_q \phi_Y \right| \\ &\leq \left| \int_{t^{-1} \cup p \cup s_0 \cup f(L_u \cup R_u) \cup s_1} \phi_Y \right| \\ &\leq 4\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon, \end{aligned}$$

as desired. Finally, since t is a shortest path from y_0 to y_1 and

$$\int_t \phi_Y = \zeta,$$

t must be an edge which we denote $e(y_0, y_1)$.

It remains to show that

$$f(e(x_0, x_1)) \subset B_{S_Y}(e(y_0, y_1), 10\tau(\varepsilon, \beta)).$$

To do this, we have shown by Eq. (17) the existence of $u \in [0, \varepsilon]$ such that

$$\text{length}_{S_Y}(f(L_u \cup R_u)) \leq 1 + 4\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 2\upsilon.$$

Analogously, there exists $u' \in [-\varepsilon, 0]$ such that

$$\text{length}_{S_Y}(f(L_{u'} \cup R_{u'})) \leq 1 + 4\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 2\upsilon.$$

Now, the endpoints of $f(L_u \cup R_u)$ and $f(L_{u'} \cup R_{u'})$ are contained in a 20υ-ball around $f(x_0)$ or $f(x_1)$ in the S_Y-metric. Also,

$$f(x_0) \in B_{S_Y}(y_0, \varepsilon)$$

while

$$f(x_1) \in B_{S_Y}(y_1, 4\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon).$$

Thus both $f(L_u \cup R_u)$ and $f(L_{u'} \cup R_{u'})$ must be contained in

$$B_{S_Y}(e(y_0, y_1), 40\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 500\upsilon + 10\varepsilon).$$

The $L_u \cup R_u$ and $L_{u'} \cup R_{u'}$, along with arcs of $C_{0,\upsilon}$ and $C_{1,\upsilon}$ bound a closed simply connected region R in S_X such that

$$e(x_0, x_1) \subset B_{S_X}(x_0, \upsilon) \cup R \cup B_{S_X}(x_1, \upsilon).$$

Hence,

$$\begin{aligned} f(e(x_0, x_1)) &\subset f(B_{S_X}(x_0, \upsilon)) \cup f(R) \cup f(B_{S_X}(x_1, \upsilon)) \\ &\subset B_{S_Y}(f(x_0), 20\upsilon) \cup f(R) \cup B_{S_Y}(f(x_1), 20\upsilon)). \end{aligned}$$

Therefore,

$$f(e(x_0, x_1)) \subset B_{S_Y}(e(y_0, y_1), 40\beta^{1/2} \varepsilon^{-1/2} (\log(1/\upsilon))^{1/2} + 500\upsilon + 10\varepsilon)$$

as well. Finally, choosing

$$\upsilon = \beta^{1/4}$$

satisfies

$$\beta \leq \upsilon^4,$$

$$\begin{aligned} 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon &\leq 100\beta^{1/4}\varepsilon^{-1} + \varepsilon \\ &\leq 1/1000, \end{aligned}$$

and

$$\upsilon \leq 1/4,$$

which we assumed in the proof. □

6.4. **Faces.** In this section, we note that if vertices and edges of S_X are close to vertices and edges of S_Y under f, then so are the faces they bound.

Lemma 6.8. *Let $0 < \varepsilon \leq 1/4$. Suppose $x_1, x_2, x_3 \in V(S_X)$ and $y_1, y_2, y_3 \in V(S_Y)$ such that*

$$f(x_1) \in B_{S_Y}(y_1, \varepsilon),$$

$$f(x_2) \in B_{S_Y}(y_2, \varepsilon),$$

and

$$f(x_3) \in B_{S_Y}(y_3, \varepsilon).$$

Suppose x_1, x_2, x_3 are the vertices of a triangle in S_X. Let $e(x_i, x_j)$ be the edge of the triangle that connects vertices x_i and x_j. Suppose additionally that for each $i, j \in \{1, 2, 3\}$ satisfying $i \neq j$, there is an edge $e(y_i, y_j)$ in S_Y such that

$$f(e(x_i, x_j)) \subset B_{S_Y}(e(y_i, y_j), \varepsilon).$$

Then the y_1, y_2, y_3 along with the $e(y_i, y_j)$ bound a triangle in S_Y.

Proof. Consider the space

$$P = B_{S_Y}(e(y_1, y_2), \varepsilon) \cup B_{S_Y}(e(y_2, y_3), \varepsilon) \cup B_{S_Y}(e(y_3, y_1), \varepsilon).$$

Note that P is homotopy equivalent to S^1. In particular, $\pi_1(P) \simeq \mathbb{Z}$. The groupoid version of Van Kampen's theorem, along with our hypotheses, implies that

$$f(e(x_1, x_2)) \cup f(e(x_2, x_3)) \cup f(e(x_3, x_1))$$

is a generator of $\pi_1(P)$. Note that

$$e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$$

is also a generator of $\pi_1(P)$. So

$$f(e(x_1, x_2)) \cup f(e(x_2, x_3)) \cup f(e(x_3, x_1))$$

and

$$e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$$

are freely homotopic in P. Since $P \subset S_Y$, they are freely homotopic in S_Y as well. The former loop is contractible in S_Y since

$$e(x_1, x_2) \cup e(x_2, x_3) \cup e(x_3, x_1)$$

is contractible in S_X. Therefore

$$e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$$

is contractible in S_Y. Gauss-Bonnet along with the fact that S_Y is a combinatorial translation surface implies that

$$e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$$

bounds a triangle in S_Y with vertices y_1, y_2 and y_3. □

6.5. **Parallelogram decomposition.** In this section, given $S \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$, we decompose S into parallelograms wherein each parallelogram contains a vertex of degree greater than 6. This decomposition applied to S_X will allow us to use Lemma 6.5 and Lemma 6.7 repeatedly to show (in Section 6.6) that most edges of S_X are close to edges of S_Y under f.

The translation surface structure of S determines foliations (with singularities) on S whose leaves are constant trajectories for ϕ_S. In particular, the horizontal foliation is the foliation obtained in this way satisfying $\phi_S = \pm 1$ on each leaf. The skew vertical foliation is the foliation obtained in this way satisfying $\phi_S = \pm e^{\pi i/3}$ on each leaf. Both these foliations have singularities at the zeros of ϕ_S, but are otherwise nonsingular.

Lemma 6.9. *All leaves of the horizontal and skew vertical foliations of S are closed.*

Proof. Let

$$T = \mathbb{C}/(\mathbb{Z} + e^{\pi i/3}\mathbb{Z}).$$

The combinatorial translation structure of S is equivalent to a branched cover $S \to T$ branched only over 0. The form ϕ_S is simply the pullback of dz on T under the branched covering map. Since the leaves of the horizontal and skew vertical foliations of T are closed, so are the leaves of horizontal and skew vertical foliations of S. □

The singular leaves of the horizontal foliation are the leaves that travel through a vertex of degree strictly greater than 6. Label them $L_1, ..., L_k$. Note that each L_i is the union of some edges of the triangulation S.

Lemma 6.10. *The surface $S \setminus \bigcup_{i=1}^k L_i$ is a disjoint union of annuli. Moreover, the S-metric restricted to each annulus is flat (with no singularities).*

Proof. Let A be a connected component of $S \setminus \bigcup_{i=1}^k L_i$. The geodesic curvature of a boundary component of A with respect to the S-metric on A is 0. The curvature at any point in the interior of A is also 0. By Gauss-Bonnet, $\chi(A) = 0$. Therefore, A is a torus or an annulus. However, by construction, A must have at least one boundary component, so A cannot be a torus. The lemma follows. □

Denote by $\mathcal{A}$ the set of annular components of $S \setminus \bigcup_{i=1}^k L_i$. Let $A \in \mathcal{A}$.

Lemma 6.11. *Each component of ∂A contains at least one vertex in $V_{>6}(S)$.*

Proof. Suppose the contrary. Let L be a boundary component of A which does not contain any vertices in $V_{>6}(S)$. Then L is a closed leaf of the horizontal foliation on S. But L does not contain any singularities, which is a contradiction. □

We now further decompose S into parallelograms. Let $A \in \mathcal{A}$. Consider the skew vertical foliation on S restricted to A. Let $M_{A,1}, ..., M_{A,j_A}$ be the leaves of the restricted foliation that pass through $\partial A \cap V_{>6}(S)$. Note that the $M_{i,A}$ are the union of some edges of triangulation S. By construction and Lemma 6.11:

Lemma 6.12. *Each component of $A \setminus \bigcup_{i=1}^{j_A} M_{A,i}$ is a parallelogram.*

Such a parallelogram has four corner vertices and four edges. Two parallel edges are segments of leaves of the horizontal foliation. We call these horizontal edges. The other two parallel edges are segments of leaves of the skew vertical foliation. We call these skew vertical edges. By construction, we have:

Lemma 6.13. *Let R be parallelogram in $A \setminus \bigcup_{i=1}^{j_A} M_{A,i}$. Each skew vertical edge of a parallelogram contains a vertex in $V_{>6}(S)$ as one of its endpoints. In particular, R contains at least two corner vertices in $V_{>6}(S)$.*

We have decomposed S into annuli, and further decomposed each annulus into parallelograms. The union of the boundaries of these parallelograms is naturally a 1-complex on S which we will denote B_1.

That is,

$$B_1 = \bigcup_{i=1}^{k} L_i \cup \bigcup_{A\in\mathcal{A}} \bigcup_{i=1}^{j_A} M_{A,i}.$$

The 1-complex B_1 naturally decomposes into a horizontal component

$$B_1^{\text{hor}} = \bigcup_{i=1}^{k} L_i$$

and a skew vertical component

$$B_1^{\text{skv}} = \bigcup_{A\in\mathcal{A}} \bigcup_{i=1}^{j_A} M_{A,i}.$$

By construction:

Lemma 6.14. *B_1^{hor} is the union of some leaves of the horizontal foliation.*

Let B be the 2-polytope homeomorphic to S whose 1-skeleton is B_1. That is, the vertices of B (denoted $V(B)$) are the intersection points $B_1^{\text{hor}} \cap B_1^{\text{skv}}$. (Note that this set contains $V_{>6}(S)$.) Away from these points, B_1 is locally a leaf of either the horizontal or skew vertical foliation. The edges of B (denoted $E(B)$) are the connected components of $B_1 \setminus V(B)$. Each such edge is contained in either B_1^{hor} or B^{skv}. The faces of B (denoted $F(B)$) are the connected components of $S \setminus B_1$. By construction, B is a polytope. Note that the faces of B are geometrically parallelograms, but need not be quadrilaterals as faces of a polytope.

Lemma 6.15. *Let γ be an edge of the polytope B. Suppose $\gamma \subset B_1^{\text{skv}}$. Then γ is the edge of a rectangle $R \in F(B)$.*

Proof. Since $\gamma \in E(B)$, γ must not intersect B_1^{hor} except at its endpoints. Therefore, γ is contained in some annulus A in the annular decomposition. By construction, the lemma follows. $\square$

We have some additional combinatorial properties of B. Recall the construction of directional weights from Definition 3.

Lemma 6.16. *The polytope B satisfies the following properties.*

(1) *If $e \in E(S)$ is contained in B_1 and $x \in V(S)$ is a vertex which is an endpoint of e, then $\zeta(e,x) = 1$, -1, $e^{\pi i/3}$ or $-e^{\pi i/3}$.*
(2) *If $x \in V_{>6}(S)$ and $e \in E(S)$ is an edge emanating from x satisfying $\zeta(e,x) = 1$, -1, $e^{\pi i/3}$ or $-e^{\pi i/3}$, then $e \subset B_1$.*

(3) If $x \in V(B)$ and $e \in E(S)$ is an edge emanating from x satisfying $\zeta(e,x) = 1$ or -1, then $e \subset B_1$.

(4) Given two vertices $x, y \in V(B)$ and a path γ from x to y on S,

$$\int_\gamma \phi_S \in 5\mathbb{Z} + 5e^{\pi i/3}\mathbb{Z}.$$

Proof. Property 1 is true since each edge of B is a segment of a leaf of the horizontal or skew vertical foliation. Property 2 is true by construction.

To see property 3: if $x \in V(B)$, then $x \in B_1^{\mathrm{hor}}$, so property 3 follows from Lemma 6.14.

Finally, we show property 4. By condition 2 of Definition 13, it suffices to show that for all $x \in V(B)$, there exists $y \in V_{>6}(S)$ and a path γ from x to y, such that

$$\int_\gamma \phi_S \in 5\mathbb{Z} + 5e^{\pi i/3}\mathbb{Z}.$$

Suppose $x \in V(B)$. Then there is a segment in B_1^{skv} emanating from x, which by Section 6.5 is a skew vertical edge γ of a parallelogram R. By Lemma 6.13, the other endpoint y of γ lies in $V_{>6}(S)$. The parallelogram R is contained in an annulus $A \in \mathcal{A}$. The annulus A has two boundary components, one of which contains x and the other y. The boundary component which contains x must contain another vertex $x' \in V_{>6}(S)$, by Lemma 6.11. Let p be a path from x to x' in A along the boundary component of A. Since p lies on a boundary component of A, it lies in B_1^{hor}, so

$$\int_p \phi_S \in \mathbb{Z}.$$

Since γ is a skew vertical edge of R, it lies in B_1^{skv}, so

$$\int_\gamma \phi_S \in e^{\pi i/3}\mathbb{Z}.$$

However,

$$\int_{p\cup\gamma} \phi_S = \int_p \phi_S + \int_\gamma \phi_S \in 5\mathbb{Z} + 5e^{\pi i/3}\mathbb{Z}.$$

Therefore

$$\int_\gamma \phi_S \in 5\mathbb{Z} + 5e^{\pi i/3}\mathbb{Z}$$

which completes the proof of property 4. ∎

Given a parallelogram $R \in F(B)$, let $\ell(R)$ denote the length of its horizontal edges. Let $w(R)$ denote the length of its skew vertical edges. Property 4 of Lemma 6.16 implies:

Lemma 6.17. *We have, $\ell(R), w(R) \geq 5$.*

Finally, we bound the number of parallelograms in our decomposition.

Lemma 6.18. *We have, $|F(B)| \leq 12(g-1)$.*

Proof of Lemma 6.18. Since S is a combinatorial translation surface, all vertices of S have degree a multiple of 6, so

$$\begin{aligned}\sum_{x\in V_{>6}(S)} \deg x &\leq -12|V(S)| + 4|E(S)| \\ &= -12\chi(S) \\ &\leq 24(g-1).\end{aligned} \tag{23}$$

By Lemma 6.13, the lemma follows. ☐

6.6. **Lower genus triangulated surface from parallelogram decomposition.** In Section 6.5, we constructed a parallelogram decomposition for locally bounded combinatorial translation surfaces. In this section, we apply this decomposition to S_X. We show that f is close to a simplicial isomorphism on most of the parallelograms. As a consequence, we show that the part of S_X on which f is not close to a simplicial isomorphism is a lower genus surface.

Decompose S_X into parallelograms as in Section 6.5 and let B_X denote the corresponding 2-polytope. Enumerate the parallelograms in $F(B_X)$ by $R_1, ..., R_N$. Recall that V_X is the set of vertices $x \in V(S_X)$ such that there exists $y \in V(S_Y)$ satisfying $d_{S_Y}(y, f(x)) < \varepsilon_0 \cdot (g/T)$. Let E_0 be the set of R_i which contain a vertex in V_X.

Lemma 6.19. *We have, $N - |E_0| \leq 42\alpha_3 g$.*

Proof. By Lemma 6.5 we have that

$$|V_{>6}(S_X) \setminus V_X| \leq \alpha_3 g.$$

Now, if $R_i \notin E_0$, then by Lemma 6.13, R_i contains a vertex in $V_{>6}(S_X) \setminus V_X$, which means R_i also contains a triangle with a vertex belonging to $V_{>6}(S_X) \setminus V_X$. Since S_X is locally bounded, the number of such triangles is at most $42\alpha_3 g$. Since the interiors of the R_i are all disjoint, at most $42\alpha_3 g$ number of R_i can contain such a triangle. The lemma follows. ☐

Let $\delta \in \mathbb{N}$, to be chosen later. Let $E_1 \subset E_0$ be the subset of R_i in E_0 which satisfy the additional property that

$$\ell(R_i), w(R_i) \leq \delta(T/g).$$

Lemma 6.20. *We have, $|E_0| - |E_1| \leq \delta^{-1} g/2$.*

Proof. If either $\ell(R_i)$ or $w(R_i)$ is greater than $\delta(T/g)$, then

$$\mathrm{area}_{S_X}(R_i) \geq (\sqrt{3}/2)\delta(T/g).$$

Since

$$\mathrm{area}_{S_X}(S_X) = (\sqrt{3}/4)T,$$

there can be at most $\delta^{-1}g/2$ such R_i. ☐

Let $0 < \varepsilon_1 < 1/1000$ also satisfy $\varepsilon_0 \leq \varepsilon_1\delta^{-1}/40$, to be chosen in Section 6.9. Let $E_2 \subset E_1$ be the subset of R_i in E_1 which satisfies

$$\int_{B_{S_X}(R_i, 3+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 \leq (\varepsilon_1\delta^{-1}/80)^8 (g/T)^{36}.$$

Lemma 6.21. *If $R_i \in E_2$, then R_i satisfies the following two properties:*

(1) If

$$x \in V(S_X) \cap \overline{B_{S_X}(R_i, 3)},$$

then there exists a unique vertex $y \in V(S_Y)$ such that

$$f(x) \in B_{S_Y}(y, \varepsilon_1).$$

This vertex will henceforth be denoted $\mathfrak{f}(x)$.

(2) Let

$$x_1, x_2 \in V(S_X) \cap \overline{B_{S_X}(R_i, 3)}$$

be vertices and

$$e(x_1, x_2) \in E(S_X) \cap R_i$$

an edge connecting x_1 and x_2. Then there exists a unique edge

$$e(\mathfrak{f}(x_1), \mathfrak{f}(x_2)) \in E(S_Y)$$

connecting the vertices $\mathfrak{f}(x_1)$ and $\mathfrak{f}(x_2)$ and satisfying,

$$f(e(x_1, x_2)) \subset B_{S_Y}(\mathfrak{f}(e(x_1, x_2)), \varepsilon_1).$$

This unique edge will henceforth be denoted $\mathfrak{f}(e(x_1, x_2))$.

Remark 17. The uniqueness condition in property 1 automatically follows from the existence condition in property 1 since any two distinct vertices in S_Y are at least distance 1 apart in the S_Y-metric. Because S_Y is a locally bounded combinatorial translation surface, two vertices of S_Y cannot have two distinct edges between them. Thus the uniqueness condition in property 2 automatically follows from property 1 and the existence condition in property 2.

Proof. Suppose $R_i \in E_2$ and does not satisfy one of the properties in the lemma statement. Let

$$\xi = \varepsilon_1 \delta^{-1}/80.$$

Since $\delta \geq 1$ and $g/T \leq 1/2$ by construction,

$$(g/T)(8 + 2\delta(T/g))\xi \leq \varepsilon_1/10. \tag{24}$$

Define

$$\xi_k = (g/T)(1 + k)\xi$$

and

$$\beta = \xi^8 (g/T)^{36}.$$

It is easy to check that

$$\tau(\xi_k, \beta) \leq \xi_{k+1}, \tag{25}$$

where τ is as defined in Lemma 6.7. Now, since $R_i \in E_1$, there exists a vertex $x \in V_X \cap \overline{R_i}$. For all

$$x' \in V(S_X) \cap \overline{B_{S_X}(R_i, 3)},$$

denote by

$$d_G(x', x)$$

the graph distance between x and x' with respect to the 1-skeleton of S_X restricted to $\overline{B_{S_X}(R_i, 3)}$. Note that for all

$$x' \in V(S_X) \cap \overline{B_{S_X}(R_i, 3)},$$

$$d_G(x,x') \le 2\delta(T/g) + 6 \tag{26}$$

since $R_i \in E_0$. Eq. (24) and Eq. (26) imply that for all

$$x' \in (S_X) \cap \overline{B_{S_X}(R_i,3)},$$

$$\xi_{d_G(x,x')+1} \le \varepsilon_1/10. \tag{27}$$

We will show a stronger condition than the lemma statement. Namely, we will show that:

Lemma 6.22. *(1) If*

$$x_1 \in V(S_X) \cap \overline{B_{S_X}(R_i,3)},$$

then there exists a unique vertex $\mathfrak{f}(x_1) \in V(S_Y)$ such that

$$f(x_1) \in B_{S_Y}(\mathfrak{f}(x_1), \xi_{d_G(x,x_1)+1}).$$

(2) Let

$$x_1, x_2 \in V(S_X) \cap \overline{B_{S_X}(R_i,3)}$$

be vertices and

$$e(x_1,x_2) \in E(S_X) \cap R_i$$

an edge connecting x_1 and x_2. Then there exists a unique edge $\mathfrak{f}(e(x_1,x_2)) \in E(S_Y)$ connecting the vertices $\mathfrak{f}(x_1)$ and $\mathfrak{f}(x_2)$ and satisfying,

$$f(e(x_1,x_2)) \subset B_{S_Y}(\mathfrak{f}(e(x_1,x_2)), 10\xi_{d_G(x,x_1)+1}).$$

By Eq. (27), Lemma 6.22 implies Lemma 6.21. So it suffices to prove Lemma 6.22.

Proof of Lemma 6.22. First, because of our assumption that

$$\varepsilon_0 \le \varepsilon_1 \delta^{-1}/40,$$

we have

$$\varepsilon_0 \cdot (g/T) \le \xi_1.$$

Since $x \in V_X$, there exists vertex $\mathfrak{f}(x) \in V(S_Y)$ satisfying

$$f(x) \in B_{S_Y}(\mathfrak{f}(x), \varepsilon_0 \cdot (g/T)) \subset B_{S_Y}(\mathfrak{f}(x), \xi_1).$$

Now, assume that the lemma statement is false. Then there is a vertex closest to x with respect to the graph distance at which either property 1 or property 2 fails. More precisely, there exist adjacent

$$x_1, x_2 \in V(S_X) \cap \overline{B_{S_X}(R_i,3)},$$

with edge $e(x_1,x_2)$ between them in R_i, with a unique vertex $y_1 \in V(S_Y)$ such that

$$f(x_1) \in B_{S_Y}(y_1, \xi_{d_G(x_1,x)})$$

and satisfying:

(1) there does not exist a vertex $y_2 \in V(S_Y)$ such that

$$f(x_2) \in B_{S_Y}(y_2, \xi_{d_G(x_1,x)+1})$$

or

(2) condition 1 holds but there does not exist an edge $e(y_1,y_2) \in E(S_Y)$ connecting y_1 and y_2 such that

$$f(e(x_1,x_2)) \subset B_{S_Y}(e(y_1,y_2), 10\xi_{d_G(x_1,x)+1}).$$

Then by Lemma 6.7

$$\int_{B_{S_X}(e(x_1,x_2),\xi_{d_G(x_1,x)})} |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 > \beta. \tag{28}$$

(Note that Lemma 6.7 requires β to be sufficiently small compared to $\xi_{d_G(x_1,x)}$ which holds in this case because of Eq. (24), Eq. (25) and Eq. (26) combined.) Eq. (28) implies that

$$\begin{aligned}\int_{B_{S_X}(R_i,3+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 &> \beta \\ &= \xi^8 (g/T)^{36} \\ &= (\varepsilon_1 \delta^{-1}/80)^8 (g/T)^{36}.\end{aligned}$$

Therefore $R_i \notin E_2$, which is a contradiction. □

This completes the proof of Lemma 6.21 as well. □

Lemma 6.23. *We have,*

$$|E_1| - |E_2| \leq 10^5 (\varepsilon_1 \delta^{-1}/80)^{-8} \alpha_2 (g/T)^{\kappa_4 - 36} g.$$

Proof. By Eq. (7),

$$\int_X |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 \leq \alpha_2 (g/T)^{\kappa_4} g.$$

By definition, if $R_i \in E_1 \setminus E_2$,

$$\int_{B_{S_X}(R_i,3+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 \geq (\varepsilon_1 \delta^{-1}/80)^8 (g/T)^{36}.$$

Since $S_X \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g)$, degrees of vertices in S_X are bounded above by 42 and distances between distinct vertices of degree strictly greater than 6 are bounded below by 5. Therefore, any radius $3+\varepsilon_1$-ball in the flat metric on S_X centered at a vertex of S_X nontrivially intersects at most 10^5 vertices, edges and triangles. Combined with the fact that the R_is are disjoint, this means any point in S_X is contained in at most 10^5 of the $B_{S_X}(R_i, 3+\varepsilon_1)$. The lemma follows. □

Given a subcomplex $Q \subset S_X$, let $ST(Q)$ denote its closed star. Let

$$G_X = \bigcup_{R_i \in E_2} ST(ST(ST(R_i))).$$

Lemma 6.24. *There exists a 2-subcomplex G_Y of S_Y along with a simplicial map*

$$\mathfrak{f} : G_X \to G_Y$$

such that

(1) $\mathfrak{f}$ is an isomorphism,

(2) for every vertex $x \in V(G_X)$,

$$f(x) \in B_{S_Y}(\mathfrak{f}(x), \varepsilon_1),$$

(3) for every edge $e \in E(G_X)$,

$$f(e) \subset B_{S_Y}(\mathfrak{f}(e), \varepsilon_1)$$

and

(4) for every triangular face $t \in F(G_X)$,

$$f(t) \subset B_{S_Y}(\mathfrak{f}(t), \varepsilon_1).$$

Proof. In Lemma 6.21 we already constructed $\mathfrak{f}$ on vertices and edges of G_X. By Lemma 6.8, $\mathfrak{f}$ extends to faces of G_X as well, is a simplicial map, and satisfies conditions 2, 3 and 4 in the statement of the lemma. We show now that $\mathfrak{f}$ is injective onto its image. Suppose the contrary. Then there are distinct triangular faces $t_{X,1}, t_{X,2} \in F(G_X)$ such that

$$\mathfrak{f}(t_{X,1}) = \mathfrak{f}(t_{X,2}) = t_Y \in F(S_Y).$$

Conditions 2 and 3 imply that

$$f(\partial t_{X,1}), f(\partial t_{X_2}) \subset B_{S_Y}(\partial t_Y, \varepsilon_1)$$

while condition 4 implies that

$$f(t_{X,1}), f(t_{X,2}) \subset B_{S_Y}(t_Y, \varepsilon_1).$$

As a result the intersection

$$f(\text{interior}\,(t_{X,1})) \cap f(\text{interior}\,(t_{X,2}))$$

must be nonempty. However, this implies f is not injective, which is a contradiction. Therefore $\mathfrak{f}$ is injective onto its image. Let $G_Y = \mathfrak{f}(G_X)$. The lemma follows. $\square$

Let

$$V_{\text{cor}}(F(B_X) \setminus E_2)$$

the set of corner vertices of R_i over all $R_i \in F(B_X) \setminus E_2$. Let

$$G'_X = \overline{\left(\bigcup_{R_i \in E_2} R_i\right) \setminus \left(\bigcup_{x \in V_{\text{cor}}(F(B_X)\setminus E_2) \cap \left(\partial \cup_{R_i \in E_2} R_i\right)} ST(x)\right)}.$$

(By Lemma 6.17, these stars are all disjoint.) Let

$$J_X = \overline{S_X \setminus G'_X}.$$

Also, let

$$G'_Y = \mathfrak{f}(G'_X),$$

and

$$J_Y = \overline{S_Y \setminus G'_Y}.$$

Lemma 6.25. *(1) Each connected component of the simplicial complexes G'_X, G'_Y, J_X and J_Y is a triangulated surface with boundary.*

(2) Let x be a vertex of G'_X contained in $\partial G'_X$. Then

$$ST(ST(ST(x))) \subset G_X.$$

Here, the closed stars are taken in S_X.

(3) Let x be a vertex of G'_X contained in $\partial G'_X$. Then

$$ST(ST(ST(\mathfrak{f}(x)))) \subset G_Y.$$

Here, the closed stars are taken in S_Y.

(4) The simplicial complex

$$\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$$

is homotopy equivalent to J_X.

Proof. First, we show that each connected component of G'_X is a triangulated surface with boundary. To do this, it suffices to show that each vertex in $V(B_X)$ lying on $\partial G'_X$ has a neighborhood in $\partial G'_X$ homeomorphic to a half-disk. (Neighborhoods of the other boundary points are automatically homeomorphic to half-disks by construction.) Suppose the contrary. Then $\partial G'_X$ contains a vertex $v \in V(B_X)$, whose 1/4-neighborhood in G'_X is not homeomorphic to a half-disk. By property 4 of Lemma 6.16, v is not contained in the closed star of any other vertex in $V(B_X)$. Therefore, locally near v, G'_X looks like $\cup_{R_i \in E_2} R_i$. This means there are edges e_1, e_2, e_3 and e_4 emanating from v, such that

$$e_1, e_2, e_3, e_4 \subset \partial \bigcup_{R_i \in E_2} R_i = \partial \bigcup_{R_i \in F(B_X) \setminus E_2} R_i.$$

Now, we divide into two cases. In the first case, $v \in V_{=6}(S_X)$. Let

$$R_j \in F(B_X) \setminus E_2$$

be a parallelogram containing the vertex v (which exists since v lies on the boundary of $\cup_{R_i \in E_2} R_i$). Suppose v is not a corner vertex of R_j. Then there exists $e \in E(S_X)$ emanating from v such that $\zeta(e, v) = \pm 1$ or $\pm e^{\pi i/3}$, $e \subset R_j$ but e does not lie on ∂R_j. By property 1 of Lemma 6.16, e must be one of the e_1, e_2, e_3 or e_4. Since

$$e \subset \partial \bigcup_{R_i \in F(B_X) \setminus E_2} R_i,$$

$e \subset \partial R_j$, which is a contradiction. For the second case, suppose $v \in V_{>6}(S_X)$. Then by properties 1 and 2 of Lemma 6.16, v is a corner vertex of all parallelograms that contain it, which means that

$$v \in V_{\mathrm{cor}}(F(B_X) \setminus E_2).$$

This is also a contradiction. Thus, connected components of G'_X are triangulated surfaces with boundary.

Since $J_X = \overline{S_X \setminus G'_X}$, connected components of J_X are also triangulated surfaces with boundary. Since $G'_Y \simeq G'_X$ and $J_Y = \overline{G'_Y \setminus J_Y}$, connected components of G'_Y and J_Y are triangulated surfaces with boundary as well.

Next, since

$$G_X = \bigcup_{R_i \in E_2} ST(ST(ST(R_i)))$$

and

$$G'_X \subset \bigcup_{R_i \in E_2} R_i,$$

$$ST(ST(ST(G'_X))) \subset G_X.$$

Since

$$\mathfrak{f} : G_X \to G_Y$$

is an isomorphism,

$$ST(ST(ST(\mathfrak{f}(G'_X)))) \subset G_Y.$$

Finally, we show the last statement in the lemma. The subcomplex J_X is obtained by adding the closed star of some boundary vertices of $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$ to it. These additions are all disjoint and deformation retract onto subsets of $\partial \overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, hence J_X deformation retracts onto $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$. $\square$

Now, suppose J_X has n connected components. Let J_X^i be the ith connected component, a surface of genus h_i with b_i number of boundary components. By Lemma 6.24, since $\partial J_X^i \subset G_X$,

$$f(J_X^i) \subset S_Y$$

is isotopic to a connected component of J_Y that we label J_Y^i. Also by Lemma 6.24, all connected components of J_Y arise in this manner. Similarly, $f(G'_X)$ is isotopic to G'_Y. Therefore and G'_X and G'_Y are homeomorphic, and J_X and J_Y are homeomorphic.

Lemma 6.26. *The Euler characteristic of J_X satisfies*

$$\chi(J_X) \geq -\alpha_4 g$$

where

$$\alpha_4 = 96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36}).$$

Similarly,

$$\chi(J_Y) \geq -\alpha_4 g.$$

Moreover, J_X and J_Y each have at most $\alpha_4 g$ connected components.

Proof. By Lemma 6.19, Lemma 6.20 and Lemma 6.23,

$$\begin{aligned} |F(B_X) \setminus E_2| &= N - |E_2| \\ &\leq (42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})g. \end{aligned} \tag{29}$$

Now, $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$ is naturally a subpolytope of B_X. We denote this subpolytope by B'_X. The vertices of B'_X (denoted $V(B'_X)$) are the vertices of B_X in $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, the edges of B'_X (denoted $E(B'_X)$) are the edges of B_X in $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, and the faces of B'_X are the $R_i \in F(B_X) \setminus E_2$. As in the case of B_X, the faces of B'_X are not necessarily quadrilaterals as polytope faces since not every vertex in $V(B'_X)$ is a corner vertex of every parallelogram that contains it.

Given a vertex $v \in V(B'_X)$, let the degree of v in B'_X denote the number of edges of B'_X emanating from it (with loops contributing 2 to the degree). Denote by $V_{=2}(B'_X)$ the set of vertices of B'_X that have degree 2 in B'_X, and $V_{>2}(B'_X)$ the set of vertices of B'_X that have degree strictly greater than 2 in B'_X.

We claim that

$$V_{>2}(B'_X) \subset V_{\mathrm{cor}}(F(B_X) \setminus E_2).$$

To see this, note that if $v \in V_{>2}(B'_X)$ and $v \in V_{>6}(S_X)$, then by properties 1 and 2 of Lemma 6.16, v is a corner vertex of every R_i in $F(B_X) \setminus E_2$ containing v. Now suppose $v \in V_{>2}(B'_X)$ and $v \in V_{=6}(S_X)$. Because $v \in V_{>2}(B'_X)$, there are at least two distinct R_i and R_j parallelograms in $F(B_X) \setminus E_2$ containing v. If v is a corner vertex of neither, then $R_i \cup R_j$ contains a neighborhood of v, which would be a contradiction to the fact that that degree of v strictly greater than 2. So v is a corner of one of the parallelograms.

Since each R_i has at most 4 corner vertices, Eq. (29) implies

$$|V_{>2}(B'_X)| \leq 4(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})g.$$

The total number of edges of B'_X is half the sum of the degrees of all the vertices in $V(B'_X)$. Since S_X is locally bounded, the sum of the degrees of all the vertices in $V(B'_X)$ is at most

$$2|V_{=2}(B'_X)| + 42|V_{>2}(B'_X)|.$$

(Note that since B'_X is a polytope, it does not have any vertices of degree 1.) Thus,

$$\begin{aligned}\chi(B'_X) &\geq |V(B'_X)| - |E(B'_X)| \\ &\geq -24|V_{>2}(B'_X)| \\ &\geq -96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})g \\ &= -\alpha_4 g.\end{aligned}$$

By Lemma 6.25, B'_X and J_X are homotopy equivalent, so

$$\chi(J_X) \geq -\alpha_4 g.$$

Since J_Y is homeomorphic to J_X, we have

$$\chi(J_X) \geq -\alpha_4 g$$

as well. The number of connected components of J_X is at most $|F(B_X) \setminus E_2|$ since each face in $F(B_X) \setminus E_2$ belongs to at most one connected component. Hence this number, along with the number of connected components of J_Y, is at most $\alpha_4 g$. □

Let

$$K_X^i = (J_X^i)^d,$$

the conformal double of J_X^i as in Definition 9. Since J_X^i is a triangulated surface of genus h_i with b_i boundary components, its conformal double is a triangulated surface (without boundary) of genus $g_i = 2h_i + b_i - 1$. By construction, ∂J_X^i is identified with $\partial(J_X^i)^{-1}$ which in turn is identified with $\partial(J_Y^i)^{-1}$. Let

$$K_Y^i = J_Y^i \cup (J_X^i)^{-1},$$

glued along this identification. Then K_Y^i is also a triangulated surfaces of genus $g_i = 2h_i + b_i - 1$.

Recall that n is the number of connected components of J_X (and therefore also J_Y, K_X and K_Y).

Lemma 6.27. *We have,*

$$\sum_{i=1}^{n} g_i \leq \alpha_5 g$$

where g_i denotes the genus of K_X^i which is equal to the genus of K_Y^i. Here, $\alpha_5 = 2\alpha_4$.

Proof. By Lemma 6.26,

$$\chi(K_X) = 2\chi(J_X) \geq -2\alpha_4 g.$$

Now, $\chi(K_X^i) = 2 - 2g_i$, so

$$\begin{aligned}2n - 2\sum_{i=1}^{n} g_i &= \chi(K_X) \\ &\geq -2\alpha_4 g.\end{aligned}$$

Since $n \leq \alpha_4 g$ by Lemma 6.26, we have

$$\sum_{i=1}^{n} g_i \leq 2\alpha_4 g$$

as desired. □

Next, we bound the number of vertices of K_X and K_Y that have degree not equal to 6.

Lemma 6.28. *We have,*

$$|V_{\neq 6}(K_X)| \leq \alpha_6 g$$

and

$$|V_{\neq 6}(K_Y)| \leq \alpha_6 g$$

where $\alpha_6 = 500\alpha_4$.

Proof. First, we compute $|V_{<6}(K_X)|$. Since K_X is the double of J_X and J_X is a subset of S_X (which, being in $\mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$, does not have any vertices of degree strictly less than 6), all vertices of K_X having degree strictly less than 6 must lie on $\partial J_X \subset K_X$. Such a vertex will have degree strictly less than 4 considered as a vertex of J_X, so it suffices to compute the number of vertices in ∂J_X that have degree strictly less than 4 in J_X. Denote by $V_{<4}(\partial J_X)$ the set of vertices in ∂J_X that have degree strictly less than 4 in J_X. (Recall that degree for a vertex in a triangulated surface with boundary is defined as the number of edges emanating from the vertex. For a vertex on the boundary, this quantity is different from the number of faces containing the vertex.)

Since $J_X = \overline{S_X \setminus G'_X}$ and G'_X are created by removing stars of boundary vertices of $\cup_{R_i \in E_2} R_i$ that are also in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$, any vertex in $V_{<4}(\partial J_X)$ is either in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$ or adjacent to a vertex in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$. Since by Eq. (29),

$$\begin{aligned} |V_{\mathrm{cor}}(F(B_X) \setminus E_2)| &\leq 4|F(B_X) \setminus E_2| \\ &\leq \alpha_4 g \end{aligned}$$

and each vertex in J_X has degree at most 42, $|V_{<4}(\partial J_X)| \leq 43\alpha_4 g$. Therefore

$$|V_{<6}(K_X)| \leq 43\alpha_4 g \tag{30}$$

as well. Note that

$$\begin{aligned} \chi(K_X) &= |V(K_X)| - |E(K_X)| + |F(K_X)| \\ &= |V(K_X)| - (1/3)|E(K_X)|. \end{aligned}$$

Since there are at least

$$3|V_{=6}(K_X)| + (7/2)|V_{>6}(K_X)|$$

edges in K_X,

$$\chi(K_X) \leq |V_{<6}(K_X)| - (1/6)|V_{>6}(K_X)|$$

which means

$$|V_{>6}(K_X)| \leq -6\chi(K_X) + 6|V_{<6}(K_X)|.$$

From Lemma 6.27 and Eq. (30), we obtain

$$\begin{aligned} |V_{\neq 6}(K_X)| &\leq |V_{<6}(K_X)| + |V_{>6}(K_X)| \\ &\leq 500\alpha_4 g. \end{aligned}$$

Since ∂J_X is identified with ∂J_Y, a similar argument shows that $|V_{<6}(K_Y)| \leq 43\alpha_4 g$. Since $\chi(K_X) = \chi(K_Y)$, we obtain

$$|V_{\neq 6}(K_Y)| \leq 500\alpha_4 g$$

too. ∎

6.7. **Quasiconformal extension on a triangulated surface.** The goal of this section is to show:

Lemma 6.29. *There exists a universal constant C such that*

$$d_T(\Phi(K_X^i), \Phi(K_Y^i)) \leq C$$

for all i satisfying $g_i \geq 2$.

We will do this by using the description of the Teichmüller metric via extremal length.

First, we consider $\Phi(K_X^i)$ and $\Phi(K_Y^i)$ as points in $\mathcal{T}_g$. To do this, it suffices to fix a marking of $\Phi(K_X^i)$ and construct a diffeomorphism $K_X^i \to K_Y^i$. To do this, we first construct a diffeomorphism

$$F : J_X^i \to J_Y^i.$$

On

$$J_X^i \setminus ST(\partial J_X^i),$$

we let

$$F = f.$$

On ∂J_X^i, we let

$$F = \mathfrak{f}.$$

By Lemma 6.24 and Lemma 6.25, we may extend F to $ST(\partial J_X^i)$ so that

$$F(x) \in B_{J_Y^i}(\mathfrak{f}(x), 1).$$

Gluing F with the identity map $(J_X^i)^{-1} \to (J_X^i)^{-1}$, we obtain the desired diffeomorphism $K_X^i \to K_Y^i$.

Let γ be a free homotopy class on K_X^i. Since we have constructed a diffeomorphism $K_X^i \to K_Y^i$, γ also corresponds to a free homotopy class on K_Y^i that we will also denote by γ. By Theorem 3.8, to show Lemma 6.29, we need to show,

$$\frac{\mathrm{Ext}_{K_Y^i}(\gamma)}{\mathrm{Ext}_{K_X^i}(\gamma)} \leq C.$$

Now,

$$\mathrm{Ext}_{K_Y^i}(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(K_Y^i)},$$

where ρ ranges over conformal metrics on K_Y^i. By Theorem 3.7, the supremum is achieved by the flat metric $|\phi|^{1/2}$ associated to a holomorphic quadratic differential ϕ on K_Y^i.

We have,

$$\mathrm{Ext}_{K_X^i}(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(K_X^i)}.$$

We will show that

$$\mathrm{Ext}_{K_X^i}(\gamma) \geq C\,\mathrm{Ext}_{K_Y^i}(\gamma)$$

by constructing a conformal metric ρ on K_X^i for which

$$\mathrm{length}_\rho(\gamma) \geq C\,\mathrm{length}_\phi(\gamma)$$

and

$$\mathrm{area}_\rho(K_X^i) \leq C\,\mathrm{area}_\phi(K_Y^i).$$

Let

$$A^i_X = J^i_X \setminus ST(\partial J^i_X)$$

and

$$A^i_Y = J^i_X \setminus ST(\partial J^i_Y).$$

Let

$$A^i_{X,1/4} = B_{A^i_X}(\partial A^i_X, 1/4)$$

and

$$A^i_{Y,1/4} = B_{A^i_Y}(\partial A^i_Y, 1/4).$$

By construction and Lemma 6.17, $A^i_{X,1/4}$ and $A^i_{Y,1/4}$ are both the union of disjoint annuli.

Now, we construct a conformal metric on K^i_X as follows. On A^i_X, let $\rho_{\phi'}$ be the smallest conformal metric such that $\rho_{\phi'} \geq f^*(|\phi|^{1/2})$. This is well defined since

$$f(A^i_X) \subset B_{S_Y}(A^i_Y, \varepsilon_1) \subset J^i_Y \subset K^i_Y$$

by Lemma 6.24 and Lemma 6.25. Extend $\rho_{\phi'}$ to the rest of K^i_X (for instance, multiplying by a partition of unity) so that the area increases infinitesimally.

Define

$$C^i_X = J^i_X \cap B_{K^i_X}(ST(ST(\partial J^i_X)), \varepsilon_1)$$

and

$$C^i_Y = J^i_Y \cap B_{K^i_Y}(ST(ST(\partial J^i_Y)), \varepsilon_1).$$

By Lemma 6.17, C^i_X and C^i_Y are the disjoint union of annuli. By Lemma 6.24 and Lemma 6.25

$$\mathfrak{f} : C^i_X \to C^i_Y$$

is an isometry onto its image, and agrees with the identity map on ∂J^i_X. Hence, there is an isometry

$$\mathrm{id} \cup \mathfrak{f} : (J^i_X)^{-1} \cup C^i_X \to (J^i_Y)^{-1} \cup C^i_Y.$$

On $(J^i_X)^{-1} \cup C^i_X$, we let ρ_ϕ be the pullback of $|\phi|^{1/2}$ defined on $(J^i_X)^{-1} \cup C^i_Y$. As before, we may extend ρ_ϕ to the rest of K^i_X so that the area increases infinitesimally.

Cover C^i_X by balls $\{U_k\}$ of radius 1/4 in the S_X-metric such that every 1/16-radius ball on C^i_X is contained in some U_k. Because S_X is a locally bounded combinatorial translation surface, we may choose the cover so that each ball U_k is either flat, or a cone with center(U_k) the cone point. We may also assume that every 2-ball in B_X nontrivially intersects at most C of the U_ks. For each U_k, there is a holomorphic map $g_k : \mathbb{D} \to U_k$ sending 0 to center(U_k). Take

$$m_k = \max_{U_j \cap B_{S_Y}(U_k, 2) \neq 0} \frac{|(\mathfrak{f} \circ g_j)^*(\phi)|}{|dz|^2}. \tag{31}$$

By the mean value property,

$$m_k \leq \max_{U_j \cap B_{S_Y}(U_k, 2) \neq 0} C \int_{U_j} |\phi|. \tag{32}$$

Let

$$\rho_k = (g_k)^*(m_k \cdot \rho_{\mathrm{Euc}})$$

on U_k, and extend it to the rest of K^i_X as before.

Finally, we let

$$\rho = \rho_{\phi'} + \rho_\phi + \sum_k \rho_k.$$

By construction,

$$\begin{aligned}
\operatorname{area}_\rho(K_X^i) &\leq \operatorname{area}_{\rho_{\phi'}}(K_X^i) + \operatorname{area}_{\rho_\phi}(K_X^i) + \sum_k \operatorname{area}_{\rho_k}(K_X^i) \\
&\leq C \operatorname{area}_\phi(K_Y^i) + \sum_k \operatorname{area}_{\rho_k}(K_X^i) && \text{quasiconformality of } f \\
&\leq C \operatorname{area}_\phi(K_Y^i) + C \sum_k \int_{U_k} |\phi| && \text{Eq. (32)} \\
&\leq C \operatorname{area}_\phi(K_Y^i).
\end{aligned}$$

It remains to show that

$$\operatorname{length}_\rho(\gamma) \geq C \operatorname{length}_\phi(\gamma).$$

Let α be a length minimizing curve on K_X^i the free homotopy class γ. We divide α into disjoint pieces in the following way.

Now, K_X^i may be decompose as

$$\left((J_X^i)^{-1} \cup ST(\partial J_X^i)\right) \bigcup A_{X,1/4}^i \bigcup A_X^i \setminus A_{X,1/4}^i.$$

Decompose α into arcs $\alpha_1 \cup ... \cup \alpha_\ell$ such that each α_j is contained in one of $A_{X,1/4}^i$, $(J_X^i)^{-1} \cup ST(\partial J_X^i)$ or $A_X^i \setminus A_{X,1/4}^i$ with endpoints on its boundary. Label the arcs so that α_j and α_{j+1} share an endpoint, and α_ℓ and α_1 share an endpoint.

We now construct a curve β on K_Y^i. We do this by constructing arcs $\beta_1, ..., \beta_\ell$, where each arc β_j will be constructed from the arc α_j on K_X^i. We will construct the β_j so that it shares an endpoint with β_{j+1}, and β_1 and β_ℓ also share an endpoint. Thus, the β_j will glue together to form β.

If $\alpha_j \subset A_X^i \setminus A_{X,1/4}^i$, we let

$$\beta_j = f(\alpha_j).$$

If $\alpha_j \subset (J_X^i)^{-1} \cup ST(\partial J_X^i)$, we let

$$\beta_j = (\mathrm{id} \cup \mathfrak{f})(\alpha_j).$$

By construction, in both these cases,

$$\operatorname{length}_\phi(\beta_j) \leq \operatorname{length}_\rho(\alpha_j).$$

Next, we treat the case wherein $\alpha_j \subset A_{X,1/4}^i$. The region $A_{X,1/4}^i$ has two types of boundary components. Let $\partial_0 A_{X,1/4}^i$ be union of the boundary components of $A_{X,1/4}^i$ which are also boundary components of $(J_X^i)^{-1} \cup ST(\partial J_X^i)$. Let $\partial_{1/4} A_{X,1/4}^i$ be the union of the boundary components which are also boundary components of $A_X^i \setminus A_{X,1/4}^i$. If α_i has both endpoints on $\partial_0 A_{X,1/4}^i$, we let

$$\beta_j = \mathfrak{f}(\alpha_j).$$

If α_j has both endpoints on $\partial_{1/4} A_{X,1/4}^i$, we let

$$\beta_j = f(\alpha_j).$$

In these situations,

$$\operatorname{length}_\phi(\beta_j) \leq \operatorname{length}_\rho(\alpha_j)$$

as before.

It remains to treat the case wherein α_j has one endpoint x on $\partial_{1/4}A^i_{X,1/4}$ and the other endpoint y on $\partial_0 A^i_{X,1/4}$. Given a point z on α_j, let α_j^x denote the arc of α_j from x to z, and α_j^y denote the arc of α_j from z to y. By construction,

$$\alpha_j = \alpha_j^x \cup \alpha_j^y.$$

Now, there exists

$$z \in \alpha_j \cap B_{K^i_X}(x, 1/16)$$

such that

$$\mathrm{length}_{S_X}(\alpha_j^x) \geq 1/16.$$

By construction of the U_ks, there is some U_k containing x and z. Since U_k is a radius 1/4-ball and $x \in \partial_{1/4}A^i_{X,1/4}$, U_k is flat. Hence, $g_k : \mathbb{D} \to U_k$ is an isometry from the Euclidean metric on $\mathbb{D}$ to the S_X-metric on U_k. Thus,

$$\mathrm{length}_{\mathrm{Euc}}(g_k^{-1}(\alpha_j^x)) \geq 1/16.$$

This means

$$\mathrm{length}_\rho(\alpha_j^x) \geq C \cdot m_k.$$

Since U_k is a radius 1/4-ball in the S_X-metric containing x, $U_k \subset C^i_X$.

By construction, $\mathfrak{f}(z) \in C^i_Y$. Note that $f(x) \in C^i_Y$ by Lemma 6.24 and Lemma 6.25. Again by Lemma 6.24 and Lemma 6.25,

$$d_{S_Y}(f(x), \mathfrak{f}(z)) \leq 1.$$

Let

$$\beta_j^x \subset B_{S_Y}(f(x), 1) \cap C^i_Y$$

be an arc from $f(x)$ to $\mathfrak{f}(z)$. We may construct β_j^x so that it passes through at most C of the $\mathfrak{f}(U_h)$. For each $\mathfrak{f}(U_h)$ through which β_j^x passes, we may assume

$$\mathrm{length}_{\mathrm{Euc}}((\mathfrak{f} \circ g_h)^{-1}(\beta_j^x \cap \mathfrak{f}(U_h))) \leq 1/2$$

by replacing with an another arc with the same endpoints if necessary. By Eq. (31), on each such $\mathfrak{f}(U_h)$,

$$\mathrm{length}_\phi(\beta_j^x \cap \mathfrak{f}(U_h)) \leq m_k/2.$$

Therefore,

$$\mathrm{length}_\phi(\beta_j^x) \leq C \cdot m_k.$$

Now, define

$$\beta_j^y = \mathfrak{f}(\alpha_j^y)$$

and

$$\beta_j = \beta_j^x \cup \beta_j^y.$$

By construction, β_j is well defined with endpoints $f(x)$ and $\mathfrak{f}(y)$. Also,

$$\mathrm{length}_\phi(\beta_j) \leq C\,\mathrm{length}_\rho(\alpha_j).$$

Let $\beta = \beta_1 \cup ... \cup \beta_\ell$. By construction, under the identification of free homotopy classes coming from the diffeomorphism $K^i_X \to K^i_Y$, α and β belong to the same free homotopy class. Finally, we have

$$\mathrm{length}_\phi(\beta) \leq C\,\mathrm{length}_\rho(\alpha)$$

which implies

$$\mathrm{length}_\rho(\gamma) \geq C\,\mathrm{length}_\phi(\gamma)$$

as desired. This completes the proof of Lemma 6.29.

6.8. **Proof of Lemma 6.2.** We must compute the number of possible S_Y. To do this, it suffices to compute the number of possible G'_Y, the number of possible J_Y and the number of possible gluings of J_Y with G'_Y. Now, G'_Y admits a simplicial isomorphism to G'_X, which itself is a subcomplex of S_X that is uniquely determined by the subset $E_2 \subset F(B_X)$. (Note that E_2 depends on S_Y, but $F(B_X)$ does not.) So the number of possibilities of G'_Y is the number of possibilities of G'_X, and this quantity is bounded above by the number of possibilities for E_2. Since

$$|F(B_X)| \leq Cg$$

by Lemma 6.18, the number of possibilities for E_2 is bounded above by C^g.

To compute the number of possibilities for J_Y, we first compute the number of possibilities for K_Y. We start by treating the case of the connected components which have genus 0 or 1. Relabelling as necessary, we may assume that

$$g_1, ..., g_{n'} \in \{0, 1\}$$

and

$$g_{n'+1}, ..., g_n \geq 2.$$

Here, g_i denotes the genus of K_X^i and K_Y^i.

Lemma 6.30. *There are at most $(T/g)^{Cg}$ total choices for the K_Y^1,..., $K_Y^{n'}$.*

Proof. To do this, we first construct a graph A^i on each K_Y^i for $i \in \{1, ..., n'\}$. We construct a graph A_Y^i on J_Y^i and a graph A_X^i on $(J_X^i)^{-1}$, then glue them together.

First, we construct A_Y^i on J_Y^i. Let

$$V_{\text{notflat}}(J_Y^i) = \left(\text{interior}(J_Y^i) \cap V_{\neq 6}(J_Y^i)\right) \cup \left(\partial J_Y^i \cap V_{\neq 4}(J_Y^i)\right).$$

Recall from Section 6.5 that S_Y has a horizontal foliation. Restrict this foliation to J_Y^i. Let $\mathcal{L}$ be the set of leaves of this restricted foliation which pass through a vertex in $V_{\text{notflat}}(J_Y^i)$. Define

$$A_Y^i = \partial J_Y^i \bigcup_{L \in \mathcal{L}}.$$

Now, A_Y^i is naturally a graph on J_Y^i. The vertices of A_Y^i in interior(J_Y^i) are

$$\text{interior}(J_Y^i) \cap V_{\neq 6}(J_Y^i).$$

The vertices of A_Y^i in ∂J_Y^i are

$$\partial J_Y^i \cap \bigcup_{L \in \mathcal{L}}.$$

Let

$$V(A_Y^i) = \left(\text{interior}(J_Y^i) \cap V_{\neq 6}(J_Y^i)\right) \cup \left(\partial J_Y^i \cap \bigcup_{L \in \mathcal{L}}\right).$$

The edges of A_Y^i are the connected components of $A_Y^i \setminus V(A_Y^i)$. Each edge of A_Y^i is contained in either

$$\partial J_Y^i$$

or

$$\bigcup_{L \in \mathcal{L}}.$$

If an edge of A_Y^i is contained in

$$\partial J_Y^i,$$

we call it a boundary edge; otherwise, an interior edge. Each edge of A_Y^i is the union of some parallel edges of J_Y^i, and therefore has both a length and a directional weight associated to it. By construction,

$$|V(A_Y^i)| \leq C \cdot V_{\text{notflat}}(J_Y^i).$$

Moreover, since S_Y is locally bounded, the degrees of vertices of A_Y^i is bounded by C. Therefore the number of edges is bounded by $C|V_{\text{notflat}}(J_Y^i)|$.

We claim that each face of F of $J_Y^i \setminus A_Y^i$ is a flat cylinder, or embeds isometrically in the plane. To see this, first note that by construction, the angles of ∂F are at most π, and the interior of F is flat. Hence, by Gauss-Bonnet, $\chi(F) \geq 0$ which means $\chi(F) = 0$ or $\chi(F) = 1$. In the case $\chi(F) = 0$, ∂F must not contain any angles less than π, hence F is a flat cylinder. In the case $\chi(F) = 1$, F is a flat topological disk with piecewise geodesic boundary such that ∂F does not contain any angles strictly greater than π. So F is geodesically convex. Therefore the exponential map gives an isometry between a planar domain and F.

Analogously, we may construct a graph A_X^i on $(J_X^i)^{-1}$ which satisfies the same vertex and edge bounds as above. Taking the union of A_X^i and A_Y^i, we obtain a graph A^i on K_X^i. Since

$$ST(\partial J_X^i) \simeq ST(\partial J_Y^i),$$

we have

$$V_{\text{notflat}}((J_X^i)^{-1}) \cup V_{\text{notflat}}(J_Y^i) = V_{\neq 6}(K_Y^i).$$

Thus, A^i has at most $|V_{\neq 6}(K_Y^i)|$ edges.

The number of topological possibilities for A^i as a graph embedded in a torus or sphere is at most $C^{|V_{\neq 6}(K_Y^i)|}$. This is true because A^i is contained in a rooted map on the sphere or torus with at most $C \cdot |V_{\neq 6}(K_Y^i)|$ edges. There are at most $C^{|V_{\neq 6}(K_Y^i)|}$ possibilities for such a rooted map (see [2], in which a bound for the number of rooted maps with prescribed number of edges on a surface is given). Now, A^i may be obtained from the rooted map by removing some edges and vertices. There are at most $C^{|V_{\neq 6}(K_Y^i)|}$ ways to do this. Therefore, there are at most $C^{|V_{\neq 6}(K_Y^i)|}$ topological possibilities for A^i. This means, there are at most $C^{|V_{\neq 6}(K_Y^i)|}$ topological possibilities for A_Y^i. By Lemma 6.28, there are at most C^g possibilities for all the A_Y^i for $i \in \{1, ..., n'\}$. To reconstruct J_Y^i, it suffices to assign directional weights and lengths to each edge of each A_Y^i. There are at most $(T/g)^{Cg}$ choices to do that. Hence there are at most $(T/g)^{Cg}$ choices for the J_Y^i for $i \in \{1, ..., n'\}$. There are at most $(T/g)^{Cg}$ ways to glue all the J_Y^i to the $(J_X^i)^{-1}$. The lemma follows. $\square$

By Lemma 6.27, Lemma 6.29, Lemma 6.28 and Lemma 6.30 there are at most

$$(T/g)^{Cg} \sum_{\substack{n \leq \alpha_4 g \\ T_1+...+T_n \leq 2T \\ g_1+...+g_n \leq \alpha_5 g \\ g_1,...,g_n \geq 2 \\ m_1+...+m_n \leq \alpha_6 g}} \prod_{i=1}^{n} N^{\mathcal{M}}(T_i, g_i, m_i, r+C)$$

choices of K_Y.

We must compute how many choices for J_Y there are given K_Y. Since K_Y is formed by gluing J_Y and $(J_X)^{-1}$, it suffices to count the number of embeddings from $(J_X^i)^{-1}$ to K_Y^i for $i \in \{1, ..., n\}$. Such an embedding is determined by its values on one triangle. Thus there are at most $(T/g)^{Cg}$ such embeddings.

The gluing of J_Y and G'_Y is determined by the decomposition of S_X as $G'_X \cup J_X$, the isomorphism G'_Y to G'_X, and the embedding of $(J_X)^{-1}$ into K_Y.

Finally, as long as $\alpha_4 \leq 1/100$ and $\alpha_5, \alpha_6 \leq \mu^{-1}(1/100)$ (see Section 6.9), the number of choices for S_Y is at most

$$(T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1+\ldots+T_n \leq 2T \\ g_1+\ldots+g_n \leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{M}}(T_i, g_i, m_i, r+C).$$

This completes the proof of Lemma 6.2.

6.9. **Constants.** In this section, we show that the constants α_1, δ, ε_0, ε_1, κ_0, κ_1, κ_2 and κ_3 can be appropriately chosen. We list the relationships between these constants below:

(1) $\mu \in \mathbb{N}$ is a universal constant, introduced in Theorem 5.1
(2) $\kappa_0 \in \mathbb{N}$, to be chosen, introduced in the beginning of Section 6.1
(3) $\kappa_1 \in \mathbb{N}$, to be chosen, introduced in statement of Lemma 6.2
(4) $\kappa_2 \in \mathbb{N}$, to be chosen, introduced in statement of Corollary 6.3
(5) $\kappa_3 \in \mathbb{N}$, to be chosen, introduced in proof of Corollary 6.3
(6) $\alpha_1 > 0$, to be chosen, introduced in statement of Lemma 6.2
(7) $\alpha_2 = \alpha_1^2$, introduced in Eq. (7)
(8) $\kappa_4 = 2\kappa_1$, introduced in Eq. (7)
(9) $0 < \varepsilon_0 < 1/2$, to be chosen, introduced in the beginning of Section 6.2
(10) $\alpha_3 = 10^{10}\alpha_2(g/T)^{\kappa_4-10}\varepsilon_0^{-10}$, introduced in statement of Lemma 6.5
(11) $0 < \varepsilon_1 < 1/1000$, to be chosen, introduced before Lemma 6.21
(12) $\delta \in \mathbb{N}$, to be chosen, introduced in the beginning of Section 6.6
(13) $\alpha_4 = 96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})$, introduced in statement of Lemma 6.26
(14) $\alpha_5 = 2\alpha_4$, introduced in statement of Lemma 6.27
(15) $\alpha_6 = 500\alpha_4$, introduced in statement of Lemma 6.28
(16) $\varepsilon_0 \leq \varepsilon_1\delta^{-1}/40$, assumed in Section 6.6 before Lemma 6.21
(17) $\alpha_4 \leq 1/100$, assumed in proof of Lemma 6.2 in Section 6.8
(18) $\alpha_5 \leq (1/100)\mu^{-1}$, assumed in proof of Lemma 6.2 in Section 6.8
(19) $\alpha_6 \leq (1/100)\mu^{-1}$, assumed in proof of Lemma 6.2 in Section 6.8
(20) $\kappa_1 \leq (\kappa_2 - 1)/2$, assumed in statement of Corollary 6.3
(21) $100\alpha_1^{-1}(1/2)^{(\kappa_2-1)/2-\kappa_1} \leq 1$, assumed in statement of Corollary 6.3
(22) $\kappa_3 \geq (\kappa_2 - 1)/2$, assumed in proof of Corollary 6.3
(23) $\kappa_2 \geq \kappa_0$, assumed in statement of Corollary 6.3
(24) $8 \cdot (1/2)^{\kappa_0} \leq (1/10)^{10}$, assumed in the beginning of Section 6.2

Noting that $g/T \leq 1/2$, it is clear that κ_0, κ_1, κ_2, κ_3, α_1, ε_0, ε_1 and δ can be chosen to satisfy conditions 16 through 24, in the following way. First, choose ε_1 sufficiently small. Then, choose δ sufficiently large. Then, choose ε_0 sufficiently small so that condition 16 is satisfied. Finally, choose α_1 sufficiently small and κ_1 sufficiently large so that $\kappa_4 > 36$, α_3, α_4, α_5 and α_6 are sufficiently small. Then choose κ_0 sufficiently large such that condition 24 is satisfied. Finally, choose κ_2 and κ_3 sufficiently large so that conditions 20 through 23 are satisfied.

7. Upper bounds for triangulated surfaces via combinatorial translation surfaces

The first goal of this section is to show:

Lemma 7.1. *For $g \geq 2$, we have,*

$$N_{\text{lb}}^{\mathcal{M}}(T, g, m, r) \leq C^{(1+r)g}(T/m)^{Cm} N_{\text{lb}}^{\mathcal{H}}(6T_i, 6g + 5m, r + C)$$

for a universal constant C.

Note that if $m = 0$, then $g = 1$. See also Remark 11.

To show Lemma 7.1, in Section 7.1 and Section 7.2 we associate to each triangulated surface a branched 6-cover which is a combinatorial translation surface. In Section 7.3 we enumerate the number of combinatorial possibilities for the branched 6-cover given the branch points. In Section 7.4, Section 7.5, Section 7.6 and Section 7.7, we study where the branch points lie. In Section 7.8, we prove Lemma 7.1 by showing that the connected components of the branched 6-cover of a triangulated surface lie in the union of a relatively small number of balls of radius around a constant in a higher dimensional moduli space.

The second goal of this section is to complete the proof of Theorem 1.3, which follows from Corollary 5.8, Lemma 6.1 and Lemma 7.1. We do this in Section 7.9.

7.1. **Triangulated surfaces and 6-differentials.** Given a Riemann surface X, a meromorphic 6-differential is a global section of the sixth tensor power of the sheaf of meromorphic differentials on X. Locally, in a neighborhood in X with holomorphic coordinate z, a 6-differential behaves like $f(z)dz^6$ where f is a holomorphic function. Although generally triangulated surfaces may not admit a combinatorial translation structure (and therefore do not admit a canonical holomorphic 1-form), they do admit a canonical meromorphic 6-differential. Given a triangulated surface S, we may associate a meromorphic 6-differential ψ_S as follows: we identify each triangle of S with the unit equilateral triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and define

$$\psi_S = dz^6$$

under this identification. Gluings of triangles must preserve the form dz^6, and ψ_S extends holomorphically over vertices of S, therefore ψ_S is a globally defined 6-differential on S. It has a zero/pole of order $(\deg x) - 6$ at a point $x \in V_{\neq 6}(S)$. The flat metric on S (equivalently denoted S-metric) coming from the Euclidean metric on each individual unit equilateral triangle has length element given by

$$ds_S = |\psi_S|^{1/6},$$

and area element

$$|\psi_S|^{1/3}.$$

Distances in this metric shall be denoted by $d_S(\cdot, \cdot)$.

7.2. **Canonical cover.** A triangulated surface S has an associated canonical holomorphic differential on an appropriate 6-degree branched cover that we construct as follows. We cover $S \setminus V_{\neq 6}(S)$ (which is also $S \setminus \{\text{zeros and poles of } \psi_S\}$) by open balls $\{U_i\}$ with transition functions $f_{i,i'}$ on $U_i \cap U_{i'}$ whenever the latter is nonempty. Let $\zeta = e^{\pi i/3}$. To each U_i, we associate $U_{i,1}, ..., U_{i,6}$, each consisting of the pair $(U_i, \omega_{i,j})$ where

$$\omega_{i,j} = \zeta^j \psi_S^{1/6}$$

for some branch of $\psi_S^{1/6}$ on U_i. We glue $U_{i,j}$ and $U_{i',j'}$ if

$$U_i \cap U_{i'} \neq \emptyset$$

and

$$f_{i,i'}^* \omega_{i,j} = \omega_{i',j'}$$

on $U_i \cap U_{i'}$. Compactifying, we obtain a (possibly disconnected) degree 6 branched cover

$$F : \widetilde{S} \to S$$

such that

$$F^*(\psi_S) = \widetilde{\phi}_S^6$$

for some holomorphic 1-form $\widetilde{\phi}_S$ on $\widetilde{S}$. Furthermore, the surface $\widetilde{S}$ is equipped with a canonical degree 6 automorphism

$$A : \widetilde{S} \to \widetilde{S}$$

defined by

$$A : U_{i,j} \to U_{i,j'}$$

is the canonical identification if

$$\omega_{i,j} = \zeta \omega_{i,j'}.$$

The surface S may be recovered from the cover by quotienting by A, i.e.

$$S = \widetilde{S}/A.$$

Proposition 7.2. *Let S be a locally bounded $\leq T$-triangle triangulated surface of genus at most g with*

$$|V_{\neq 6}(S)| \leq m.$$

Then $\widetilde{S}$ has at most six connected components that we label $\widetilde{S}^1, ..., \widetilde{S}^6$ (where some components may be empty). For each $i \in \{1, ..., 6\}$, $\widetilde{S}^i$ is a locally bounded $\leq 6T$-triangle combinatorial translation surface of genus at most $6g + 5m$.

Proof. Since $\widetilde{S}$ is a degree 6 branched cover of S, $\widetilde{S}$ has at most six connected components. If $\widetilde{S}^i$ is a connected component, then $F|_{\widetilde{S}^i}$ is a degree d branched cover for some $d \leq 6$. Denote by

$$c_1, ..., c_{n'}$$

the critical points on $\widetilde{S}^i$ of this covering map, and

$$p_1, ..., p_n$$

the branch points on S. Since branch points must be vertices of S of degree not equal to 6,

$$n \leq m.$$

Since each branch point has a preimage which is a critical point,

$$n' \geq n.$$

By the Riemann-Hurwitz formula,

$$2 \operatorname{genus}(\widetilde{S}^i) - 2 + n' = d(2g - 2) + dn.$$

Thus $\widetilde{S}^i$ has genus at most $6g+5m$. Since all branch points of F are vertices of S, the pullback of the triangulation S under F gives a triangulation of $\widetilde{S}^i$ with corresponding 1-form $\widetilde{\phi}_S$ by construction. Hence $\widetilde{S}^i$ is a combinatorial translation surface. Since

$$F : \widetilde{S} \to S$$

is a degree 6 branched cover, $\widetilde{S}^i$ has at most $6T$ triangles.

Finally, we claim that $\widetilde{S}^i$ is locally bounded. To see this, note that if x is a vertex of the triangulation of $\widetilde{S}^i$, then $F(x)$ is a vertex of S. If $F(x)$ is not a branch point, then

$$\deg x = \deg(F(x)) \leq 7$$

as S is locally bounded. If $F(x)$ is a branch point, then the ramification index of x, denoted $e(x)$, is at most 6, so

$$\deg x = e(x)\deg(F(x)) \leq 42.$$

Thus $\widetilde{S}^i$ satisfies condition 1 in Definition 12 definition of local boundedness for combinatorial translation surfaces. Next we show condition 2 in Definition 12. Since S is locally bounded, we also have a triangulation S_{lb} of S by equilateral triangles of side length 5 in the flat metric, such that S is a 5-subdivision of S_{lb}. The pullback of the triangulation S_{lb} under F gives a triangulation $\widetilde{S}^i_{\text{lb}}$ of the surface $\widetilde{S}^i$ wherein each triangle has side length 5, such that the triangulation $\widetilde{S}^i$ is the 5-subdivision of the triangulation $\widetilde{S}^i_{\text{lb}}$. Thus $\widetilde{S}^i$ is a locally bounded combinatorial translation surface. □

Given such a canonical cover, we can reconstruct the original surface up to a factor.

Lemma 7.3. *Let $\widetilde{S}'$ be a locally bounded $\leq 6T$-triangle combinatorial translation surface of genus at most $6g+m$. There are at most CT number of triangulated surfaces S for which $\widetilde{S}'$ is a component of the canonical cover of S. Here, C is a universal constant.*

Proof. The automorphism on the canonical cover of S determines a degree 1, 2, 3 or 6 automorphism

$$A' : \widetilde{S}' \to \widetilde{S}'.$$

Since A' preserves $\psi_{\widetilde{S}'}$, A' is a simplicial isomorphism. Furthermore, A' is determined by its value on one triangle. Thus there are at most CT number of choices for A'. Finally,

$$S \simeq \widetilde{S}'/A',$$

so there are at most CT number of choices for S. □

7.3. **Combinatorics of branched 6-covers.** In this section, we enumerate the number of degree 6 branched covers of a fixed surface with fixed branched points.

Lemma 7.4. *Let S_g be a topological surface of genus g and $P \subset S$ a nonempty set of n marked points on S_g. There are at most C^{g+n} choices of branched 6-covers $\widetilde{S}_g \to S_g$ (up to isomorphism of branched covers), such that the branch points are contained in the set P.*

Proof. Cut S along elements of $H_1(S_g, P, \mathbb{Z})$ until we obtain S_g', a simply connected surface with boundary such that $P \subset \partial S_g'$. The boundary $\partial S_g'$ may be decomposed into $4g+2n-2$ curves which come in pairs

$$\gamma^{+,1}, \gamma^{-,1}, ..., \gamma^{+,2g+n-1}, \gamma^{-,2g+n-1}$$

that are glued together to form S'_g. Here, each $\gamma^{\pm,i}$ represents one element of $H_1(S_g, P, \mathbb{Z})$. Any degree 6 branched cover $\widetilde{S}_g$ with branched points in the set P necessarily admits a decomposition into isomorphic copies of S'_g denoted by $S'^1_g, ..., S'^6_g$. Boundary curve $\gamma_j^{+,i}$ (for $j \in \{1, ...6\}$ must necessarily be glued to $\gamma_{j'}^{-,i}$ for some $j' \in \{1, ..., 6\}$) under this decomposition of $\widetilde{S}_g$. Each decomposition corresponds to an isomorphism class of branched 6-covers $\widetilde{S}_g \to S_g$. For a fixed i, there are C choices to glue all the $\gamma_j^{+,i}$ and $\gamma_{j'}^{-,i}$, and i ranges from 1 to $2g + n - 1$. Hence, the total number of gluing choices (and total number of branched covers) is bounded by C^{g+n}. □

For $n > 1$, we denote by $\mathcal{F}_{g,n}$ the set of topological branched covers enumerated above.

7.4. **Mean value property and 6-differentials.** In this section, we show a rough mean value property for 6-differentials arising from triangulated surfaces, which will be useful in Section 7.5.

Lemma 7.5. *Let S be a triangulated surface and let $X = \Phi(S)$. Suppose $v, w \in V(S)$, $v \neq w$ and let $\mathbb{D} \subset X$ a conformal identification of $\mathbb{D}$ with a subset of X containing v and w such that 0 is identified with v. Suppose further that*

$$|w| \leq r < 3/4$$

in $\mathbb{D}$. Then

$$\int_{\mathbb{D}} |\psi_S|^{1/3} \geq Cr^{-2}.$$

Proof. Denote by C_α the circle of radius α around 0 in $\mathbb{D}$. Note that for $r \leq \alpha \leq 1$ the length of C_α in the d_S metric must be at least 1. Thus, writing

$$\psi = g(z)dz^6$$

for a meromorphic function g on $\mathbb{D}$, we have that

$$|g(z_0)|^{1/6} \geq Cr^{-1}$$

for some $z_0 \in C_r$. Let

$$a = g(z_0)^{1/6}.$$

Define

$$\psi' = \frac{g(z)}{(z - z_0)^6} dz^6,$$

a meromorphic 6-differential on $\mathbb{D}$. On a small neighborhood around z_0,

$$\psi' = \phi'^6$$

for a 1-form ϕ'. Locally,

$$\phi' = \frac{g(z)^{1/6}}{(z - z_0)} dz$$

(note that $g(z)^{1/6}$ makes sense on a small neighborhood of z_0). Therefore the residue of ϕ' at z_0 is a. Let $f : \widetilde{X} \to X$ be the canonical 6-cover associated to ψ (constructed in Section 7.1). This means that

$$f^*(\psi') = \widetilde{\phi}'^6$$

for a meromorphic 1-form ϕ defined on $f^{-1}(\mathbb{D})$. The 1-form $\widetilde{\phi}'$ is holomorphic except for a pole at $f^{-1}(z_0)$. On a neighborhood of $f^{-1}(z_0)$,

$$f^{-1}(\phi') = \widetilde{\phi}'$$

so the residue of $\widetilde{\phi}'$ at $f^{-1}(z_0)$ is a. Since $r < 3/4$, for $7/8 \leq \alpha \leq 1$ we have

$$\int_{f^{-1}(C_\alpha)} \widetilde{\phi}' = 2\pi i a.$$

So

$$\int_{f^{-1}(C_\alpha)} |\widetilde{\phi}'| \geq Ca.$$

Pushing forward this integral to $\mathbb{D}$ we obtain

$$\begin{aligned}\int_{C_\alpha} |g(z)|^{1/6}|dz| &\geq C \int_{C_\alpha} \frac{|g(z)|^{1/6}}{|z-z_0|}|dz| \\ &\geq Ca,\end{aligned}$$

since for $z \in C_\alpha$, $|z - z_0| \geq 1/8$. By Jensen's inequality, we have

$$\begin{aligned}\int_{C_\alpha} |g(z)|^{1/3}|dz| &\geq C \left(\int_{C_\alpha} |g(z)|^{1/6}|dz|\right)^2 \\ &\geq Ca^2.\end{aligned}$$

Then,

$$\begin{aligned}\int_{\mathbb{D}} |g(z)|^{1/3}|dz|^2 &\geq \int_{\alpha=7/8}^{1} \int_{C_\alpha} |g(z)|^{1/3}|dz|d\alpha \\ &\geq C \int_{\alpha=7/8}^{1} a^2 d\alpha \\ &\geq Ca^2 \\ &\geq Cr^{-2}\end{aligned}$$

as desired. □

We also have a quantitative version:

Lemma 7.6. *Let S be a triangulated surface and let $X = \Phi(S)$. Denote by ρ_X the hyperbolic metric on X, and $\rho_{\mathbb{D}}$ the Poincare metric on $\mathbb{D}$. Suppose*

$$v, w \in V(S)$$

with $v \neq w$. Let $U \subset X$ be a region containing v and w and

$$f : U \to B_{\rho_{\mathbb{D}}}(0, s)$$

a K-bi-Lipschitz diffeomorphism (with respect to metrics ρ_X on U and $\rho_{\mathbb{D}}$ on $B_{\rho_{\mathbb{D}}}(0, s)$) such that $f(v) = 0$. Let $r < 3/4$ and suppose that

$$d_{\rho_X}(v, w) \leq (1/100)rs/K.$$

Then

$$\int_U |\psi_S|^{1/3} \geq Cr^{-2}$$

where C is a universal constant.

Proof. Since

$$f : U \to B_{\rho_{\mathbb{D}}}(0, s)$$

is a K-bi-Lipschitz map, U contains

$$B_{\rho_X}(v, s/K),$$

the ball of radius s/K around v. Note that

$$w \in B_{\rho_X}(v, s/K)$$

too. This ball may be isometrically identified with

$$\mathbb{D}_{\tanh(s/2K)} = \{z \in \mathbb{D} | |z| \leq \tanh(s/2K)\}$$

(with v identified with 0) equipped with the restriction of the Poincare metric $\rho_{\mathbb{D}}$. Now,

$$\begin{aligned} d_{\rho_X}(v, w) &\leq (1/100) rs/K \\ &\leq 2\tanh^{-1}(r \tanh(s/(2K))) \end{aligned}$$

since $s \leq 1$, $r < 3/4$ and $K \geq 1$. So in $\mathbb{D}_{\tanh(s/(2K))}$ (under our isometric identification),

$$|w| \leq r \tanh(s/(2K)).$$

By Lemma 7.5,

$$\int_{B_{\rho_X}(v,s/K)} |\psi|^{1/3} \geq Cr^{-2}.$$

Since $B_{\rho_X}(v, s/K) \subset U$, the lemma statement follows. □

7.5. **Location of branch points.** Fix an arbitrary constant $r_0 > 0$. Let $X \in \mathcal{T}_g$. In this section, to each locally bounded triangulated surface whose conformal class lies in $B_{d_T}(X, r_0)$, we associate combinatorial data that is a discrete measure of where the vertices of degree not equal to 6 are located.

Denote by ρ_X the hyperbolic metric on X. We take hyperbolic disks $U_1, ..., U_N$, $V_1, ..., V_N$ and $W_1, ..., W_N$ as in Lemma 2.5. By Lemma 2.4 and Lemma 2.5, we may assume $N \leq CT$.

Definition 27. For each $i \in \{1, ..., N\}$ and $M \geq 1$, define

$$\{W^M_{i,j}\}|_{j\in\{1,...,M\}}$$

to be a collection of hyperbolic balls on X satisfying the following properties:

(1) For all $j, j' \in \{1, ..., M\}$,

$$\mathrm{radius}(W^M_{i,j}) = \mathrm{radius}(W^M_{i,j'}) \leq CM^{-1/2}\,\mathrm{radius}(W_i)$$

where C is a universal constant,

(2)

$$W^M_{i,j} \subset V_i$$

and

$$\mathrm{center}(W^M_{i,j}) \in W_i,$$

(3) the

$$\{B_{\rho_X}(\mathrm{center}(W^M_{i,j}), \mathrm{radius}(W^M_{i,j})/2)\}_{j\in\{1,...,M\}}$$

cover W_i and

(4) each $x \in U_i$ is contained in at most C of the $W^M_{i,j}$ for a universal constant C.

Such a collection may be constructed by taking $\sim M^{-1/2}\,\mathrm{radius}(W_i)$ radius balls around a maximal $\sim (1/8)M^{-1/2}\,\mathrm{radius}(W_i)$-separated set on W_i.

Definition 28. Define $\mathcal{D}$ to be the set of all values of the following data of (I, L, W_L):

(1) subset
$$I \subset \{1, ..., N\}$$
with
$$|I| \leq \alpha m$$
(2) function
$$L : I \to \mathbb{N}$$
such that
$$\sum_{i \in I} L(i) \leq \alpha T$$
and
(3) subset
$$W_L \subset \bigcup_i \{W_{i,j}^{\kappa L(i)}\}|_{j \in \{1,...,\kappa L(i)\}}$$
with
$$|W_L| \leq \alpha m.$$

Here, $\alpha, \kappa > 1$ are sufficiently large universal constants which will be chosen in the proofs of Lemma 7.7 and Lemma 7.8 later in this section.

Lemma 7.7. *We have*
$$|\mathcal{D}| \leq (\kappa T/m)^{C\alpha m}.$$

Proof. The number of subsets I is bounded above by $(T/m)^{C\alpha m}$. Given I, the number of functions L is bounded above by $(T/m)^{C\alpha m}$. Given I and L,
$$\left| \bigcup_{i \in I} \{W_{i,j}^{L(i)}\}|_{j \in \{1,...,\kappa L(i)\}} \right| \leq \alpha \kappa T.$$
Trivially, $m \leq 3T$. Thus for $\kappa \geq 10$, the number of subsets of
$$\bigcup_i \{W_{i,j}^{L(i)}\}|_{j \in \{1,...,\kappa L(i)\}}$$
with cardinality at most αm is bounded above by $(\kappa T/m)^{C\alpha m}$. Therefore
$$|\mathcal{D}| \leq (\kappa T/m)^{C\alpha m},$$
as desired. □

For all $Y \in B_{d_T}(X, r_0)$, by Proposition 3.9, we choose a $e^{\xi r_0}$-quasiconformal map
$$f : X \to Y$$
that is $e^{\xi r_0}$-bi-Lipschitz with respect to the hyperbolic metrics ρ_X on X and ρ_Y on Y. Here, ξ is a universal constant from Proposition 3.9.

Lemma 7.8. *Let*

$$S \in \mathrm{Comb}^{T,m}(\mathcal{M}_g)$$

and suppose

$$Y = \Phi(S) \in d_T(X, r_0).$$

Then there exist an element

$$(I, L, W_L) \in \mathcal{D}$$

associated to S satisfying

(1) *$i \in I$ if and only if W_i contains a vertex in $f^{-1}(V_{\neq 6}(S))$,*
(2) *each ball*
$$B \in W_L$$
contains a unique vertex in $f^{-1}(V_{\neq 6}(S))$, and this vertex lies in
$$B_{\rho_X}(\mathrm{center}(B), \mathrm{radius}(B)/2),$$
(3) *each vertex in $f^{-1}(V_{\neq 6}(S))$ is contained in a unique $B \in W_L$ and*
(4) *each point $x \in X$ is contained in at most C of the $W \in W_L$.*

Proof of Lemma 7.8. We let I be the subset of $i \in \{1, ..., N\}$ for which $f(W_i)$ contains an element of $V_{\neq 6}(S)$. Condition 1 in the statement of Lemma 7.8 is automatically satisfied. We also let

$$L(i) = \left\lceil \int_{f(U_i)} |\psi_S|^{1/3} \right\rceil.$$

Note that

$$|I| \leq |V_{\neq 6}(S)| \leq m.$$

Now,

$$\int_Y |\psi_S|^{1/3} \leq CT.$$

Also, each point of Y is contained in at most C of the $f(U_i)$, by condition 4 of Lemma 2.5. Therefore,

$$\sum_{i \in I} L(i) \leq \alpha T$$

for a sufficiently large constant α.

Finally, we let W_L be the subset of B in

$$\bigcup_{i \in I} \{W_{i,j}^{L(i)}\}_{j \in 1, \dots, \kappa(L(i))}$$

which satisfy the property that

$$B_{\rho_X}(\mathrm{center}(B), \mathrm{radius}(B)/2)$$

contains a vertex in $f^{-1}(V_{\neq 6}(S))$.

We now show the uniqueness part of condition 2 in the statement of Lemma 7.8. Suppose the contrary; that there exists $B \in W_L$ containing $f^{-1}(x)$ and $f^{-1}(y)$ for vertices $x, y \in V_{\neq 6}(S)$. Then

$$f^{-1}(x), f^{-1}(y) \in V_i$$

for some $i \in \{1, ..., N\}$. For sufficiently large κ,

$$\mathrm{radius}(B) \leq \mathrm{radius}(W_i),$$

implying
$$f^{-1}(y) \in B_{\rho_X}(f^{-1}(x), 2\,\mathrm{radius}(W_i)) \subset U_i.$$
Now, Lemma 7.6 applied to
$$f^{-1} : f(B_{\rho_X}(f^{-1}(x), 2\,\mathrm{radius}(W_i))) \to B_{\rho_X}(f^{-1}(x), 2\,\mathrm{radius}(W_i))$$
implies
$$d_{\rho_Y}(x, y) \geq CL(i)^{-1/2}\,\mathrm{radius}(U_i)/e^{\xi r_0}.$$
Thus
$$\begin{aligned} d_{\rho_X}(f^{-1}(x), f^{-1}(y)) &\geq CL(i)^{-1/2}\,\mathrm{radius}(U_i)/e^{2\xi r_0} \\ &\geq CL(i)^{-1/2}\kappa^{-1/2}\,\mathrm{radius}(U_i) \\ &\geq CL(i)^{-1/2}\kappa^{-1/2}\,\mathrm{radius}(W_i) \\ &\geq 2\,\mathrm{radius}(B) \qquad \text{condition 2 in Definition 27} \end{aligned}$$
for κ sufficiently large. Hence we have a contradiction to the assumption that
$$f^{-1}(x), f^{-1}(y) \in B.$$
This completes the proof of condition 2 in the statement of Lemma 7.8.

Now, condition 4 in the statement of Lemma 7.8 follows from condition 4 of Lemma 2.5 along with condition 4 in Definition 27. Condition 3 in the statement of Lemma 7.8 can be ensured by deleting unnecessary elements of W_L. □

For
$$S \in \mathrm{Comb}^{T,m}_{\mathrm{lb}}(\mathcal{M}_g)$$
such that
$$Y = \Phi(S) \in B_{d_T}(X, r_0),$$
we choose an element
$$(I, L, W_L) \in \mathcal{D}$$
as in Lemma 7.8. We label this particular choice of element in $\mathcal{D}$ by
$$(I_S, L_S, W_{L_S}).$$

7.6. **Quasiconformal map between surfaces with marked points.** In this section, we associate to triples in $\mathcal{D}$ a set of marked points on X. Then, given a triangulated surface S whose conformal class lies near X, we construct a quasiconformal map from X to S which take the marked points on X to $V_{\neq 6}(S)$.

Let $(I, L, W_L) \in \mathcal{D}$ and suppose it arises as the triple associated to a triangulated surface. Enumerate the elements $B_1, \ldots., B_N$ in W_L. For each $i \in \{1, \ldots, N\}$, choose
$$x_i \in B_{\rho_X}(\mathrm{center}(B_i), \mathrm{radius}(B_i)/2)$$
so that no element of W_L contains x_i. Such a choice can be made because of Lemma 7.8 and our assumption that (I, L, W_L) arises as the triple associated to a triangulated surface. Let
$$O = \{x_i\}|_{i \in \{1, \ldots, N\}}.$$

Suppose
$$S \in \mathrm{Comb}^{T,m}(\mathcal{M}_g)$$
and
$$\Phi(S) \in B_T(X, r_0).$$

Let

$$f : X \to S$$

be the associated $e^{\xi r_0}$-quasiconformal, $e^{\xi r_0}$-bi-Lipschitz map. Suppose

$$(I_S, L_S, W_{L_S}) = (I, L, W_L).$$

Let

$$P = V_{\neq 6}(S).$$

There is a natural identification

$$\theta : O \to P$$

which sends $x_i \in O$ to the unique $p_i \in V_{\neq 6}(S)$ satisfying $q_i = f^{-1}(p_i) \in B_i$ (unique by Lemma 7.8).

Lemma 7.9. *There is a C-quasiconformal map*

$$F : X \to S$$

isotopic to f such that

$$F(O) = P$$

and

$$F|_O = \theta.$$

Here, C is a universal constant.

First, we prove an elementary lemma.

Lemma 7.10. *For all $z_1, z_2 \in \mathbb{D}$ with*

$$|z_1|, |z_2| \leq 3/4,$$

there exists a diffeomorphism

$$f_{z_1,z_2} : \mathbb{D} \to \mathbb{D}$$

satisfying the following conditions:

(1) $f_{z_1,z_2}|_{\partial\mathbb{D}}$ is the identity map,
(2) $Df_{z_1,z_2}|_{\partial\mathbb{D}}$ is the identity map,
(3) $f_{z_1,z_2}(z_1) = z_2$ and
(4) f_{z_1,z_2} is C-quasiconformal.

Here, C is a universal constant.

Proof. We first treat the case where $z_1 = 0$. Denote by $f_{0,3/4}$ any chosen diffeomorphism $\mathbb{D} \to \mathbb{D}$ satisfying conditions (1), (2) and (3) above for $z = 3/4$. Let C be the quasiconformal dilatation of $f_{0,3/4}$. Note that C is finite since f is defined on a compact set. Now, suppose $z \in \mathbb{R}$ such that $z \leq 3/4$. The function

$$f_{0,z}(w) = \begin{cases} (4/3)z f_{0,3/4}((3/4)z^{-1}w) & \text{if } |w| \leq (4/3)z \\ w & \text{otherwise} \end{cases}$$

satisfies the conditions in the lemma statement. Suppose next that $z \in \mathbb{D}$ with $|z| \leq (3/4)$. We have already constructed $f_{0,|z|}$, since $|z| \in \mathbb{R}$. We take

$$f_{0,z}(w) = \frac{z}{|z|} f_{0,|z|}\left(\left(\frac{z}{|z|}\right)^{-1} w\right).$$

Now removing the assumption that $z_1 = 0$, we simply take

$$f_{z_1,z_2} = f_{0,z_2} \circ f_{0,z_1}^{-1},$$

and note that f_{z_1,z_2} satisfies the desired properties. $\square$

Proof of Lemma 7.9. It suffices to construct a C-quasiconformal diffeomorphism

$$F' : X \to X$$

isotopic to the identity such that

$$F'(x_i) = q_i.$$

Then, taking

$$F = f \circ F'$$

gives the desired map in the lemma statement.

To do this, recall that $B_1, ..., B_N$ are the elements of W. By Lemma 2.5 and condition 2 in Definition 27, the B_i are hyperbolic disks of maximal radius arcsinh(1). Recall that each B_i contains exactly one element of O (which is x_i), and exactly one element of $f^{-1}(P)$ (which is q_i). By condition 2 in Lemma 7.8 along with the upper bound on radius(B_i), under a conformal identification of B_i with $\mathbb{D}$ sending its center to 0, both x_i and q_i lie inside

$$\mathbb{D}_{3/4} = \{z \in \mathbb{D} | |z| < 3/4\}.$$

Thus there exists f_{x_i,q_i} satisfying the conditions in the statement of Lemma 7.10. The map f_{x_i,q_i} extends to a C-quasiconformal map

$$f^i : X \to X$$

that is the identity outside B_i. Composing all such maps f^i (for $i \in \{1, ..., N\}$) in an arbitrary order we obtain a map

$$F' : X \to X$$

such that

$$F'(x_i) = q_i.$$

Now, by condition 3 of Lemma 7.8, the set $B_1, ..., B_N$ satisfies the property that any $x \in X$ is contained in at most C of the B_i. Therefore F' is C-quasiconformal. $\square$

7.7. **Distance between covers in higher dimensional moduli space.** In this section, we show that if a triangulated surface S has conformal class close to X, then a component of its canonical cover is close to a branched cover of X in a higher dimensional moduli space.

Let

$$(I, L, W_L) \in \mathcal{D}$$

and assume that (I, L, W_L) arises as the triple associated to a triangulated surface. Recall that

$$O \subset X$$

is a subset, depending on the triple (I, L, W_L), generated in Section 7.6.

The set $\mathcal{F}_{g,m}$ enumerates the combinatorial 6-covers over X whose branch points are a subset of O. That is, associated to $\mathfrak{f} \in \mathcal{F}_{g,m}$ is a topological branched cover $\widetilde{X}_{\mathfrak{f}} \to X$. The holomorphic structure of X determines a holomorphic structure on $\widetilde{X}_{\mathfrak{f}}$. Note that $\widetilde{X}_{\mathfrak{f}}$ may be disconnected. Choose a connected component and label it $\widetilde{X}'_{\mathfrak{f}}$.

Suppose

$$S \in \mathrm{Comb}^{T,m}(\mathcal{M}_g)$$

and

$$\Phi(S) \in B_T(X, r_0).$$

Let $f : X \to S$ be the associated $e^{\xi r_0}$-quasiconformal, $e^{\xi r_0}$-bi-Lipschitz map. Suppose

$$(I_S, L_S, W_{L_S}) = (I, L, W_L).$$

The composition

$$\widetilde{S} \to S \xrightarrow{F^{-1}} X$$

is a topological branched cover over X having O as the set of branch points. Hence, there is an element $\mathfrak{f} \in \mathcal{F}_{g,m}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X}_{\mathfrak{f}} & \longrightarrow & \widetilde{S} \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & S \end{array}$$

commutes and the top vertical map (henceforth denoted $\widetilde{F}$) is a homeomorphism. By Lemma 7.9, F is C-quasiconformal. Also, the vertical maps are conformal. Therefore $\widetilde{F}$ is C-quasiconformal. Let $\widetilde{S}'$ be the connected component of $\widetilde{S}$ determined by $F(\widetilde{X}'_{\mathfrak{f}})$.

Lemma 7.11. *We have,*

$$d_T(\widetilde{X}'_{\mathfrak{f}}, \Phi(\widetilde{S}')) \leq C.$$

7.8. **Proof of Lemma 7.1.** Let $m > 0$ and $X \in \mathcal{M}_g$. We must count the number of locally bounded triangulated surfaces lying in $B_{d_T}(X, r_0)$ that have at most m vertices of degree not equal to 6. Any such triangulated surface has associated to it a triple (I, L, W) satisfying the conditions in Lemma 7.8. By Lemma 7.7, there are at most $(T/m)^{Cm}$ such triples.

Fixing a triple (I, L, W), it suffices to count triangulated surfaces with this particular triple as its associated triple. By Proposition 7.2, any such triangulated surface has associated to it a topological branched cover in $\mathcal{F}_{g,\leq m}$. By Lemma 7.4, there are at most C^{g+m} such topological branched covers.

It suffices to count triangulated surfaces with a fixed underlying topological branched cover. In this situation, by Lemma 7.11, a connected component of the canonical cover must be closed to a fixed branched cover of X in a higher dimensional moduli space. By Proposition 7.2, this connected component is a combinatorial translation surface. By Lemma 7.3 it suffices to count combinatorial translation surfaces lying in a ball in the higher genus moduli space. Thus we obtain

$$N^{\mathcal{M}}_{\text{lb}}(T, g, m, r_0) \leq C^g (T/m)^{Cm} N^{\mathcal{H}}_{\text{lb}}(6T_i, 6g + 5m, C).$$

Combining with Corollary 3.19 gives Lemma 7.1.

7.9. **Proof of Theorem 1.3.** Recall that for $g \geq 2$,

$$N^{\mathcal{M}}(T, g, m, r) \leq C^{m+g} N^{\mathcal{M}}_{\text{lb}}(\sigma T, g, \mu(m + g), r + C)$$

by Corollary 5.8,

$$N^{\mathcal{M}}_{\text{lb}}(T, g, m, r) \leq C^{(1+r)g} (T/m)^{Cm} N^{\mathcal{H}}_{\text{lb}}(6T, 6g + 5m, r + C)$$

by Lemma 7.1 and

$$N^{\mathcal{H}}_{\mathrm{lb}}(T,g,r) \leq (T/g)^{C(1+r)g} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 2T \\ g_1+\ldots+g_n\leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n\geq 2 \\ m_1+\ldots+m_n\leq \mu^{-1}(1/100)g}} \prod_{i=1}^n N^{\mathcal{M}}(T_i,g_i,m_i,r+C)$$

by Lemma 6.1.

Choosing r_1 sufficiently small (independent of T and g), we may apply Corollary 3.19 to obtain

$$N^{\mathcal{M}}(T,g,m,r_1) \leq C^{m+g} N^{\mathcal{M}}_{\mathrm{lb}}(\sigma T, g, \mu(m+g), r_1), \tag{33}$$

$$N^{\mathcal{M}}_{\mathrm{lb}}(T,g,m,r_1) \leq C^g (T/m)^{Cm} N^{\mathcal{H}}_{\mathrm{lb}}(6T, 6g+5m, r_1) \tag{34}$$

and

$$N^{\mathcal{H}}_{\mathrm{lb}}(T,g,r_1) \leq (T/g)^{Cg} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 2T \\ g_1+\ldots+g_n\leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n\geq 2 \\ m_1+\ldots+m_n\leq \mu^{-1}(1/100)g}} \prod_{i=1}^n N^{\mathcal{M}}(T_i,g_i,m_i,r_1).$$

Together, these three bounds give

$$\begin{aligned} (35)\quad & N^{\mathcal{H}}_{\mathrm{lb}}(T,g,r_1) \\ &\leq (T/g)^{Cg} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 2T \\ g_1+\ldots+g_n\leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n\geq 2 \\ m_1+\ldots+m_n\leq \mu^{-1}(1/100)g}} \prod_{i=1}^n N^{\mathcal{M}}(T_i,g_i,m_i,r_1) \\ &\leq (T/g)^{Cg} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 2T \\ g_1+\ldots+g_n\leq (1/100)g \\ g_1,\ldots,g_n\geq 2 \\ m_1+\ldots+m_n\leq (1/50)g}} \prod_{i=1}^n C^{m_i+g_i} N^{\mathcal{M}}_{\mathrm{lb}}(\sigma T_i,g_i,m_i,r_1) \\ &\leq (T/g)^{Cg} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 2T \\ g_1+\ldots+g_n\leq (1/100)g \\ g_1,\ldots,g_n\geq 2, \\ m_1+\ldots+m_n\leq (1/50)g \\ m_1,\ldots,m_n\geq 1}} \prod_{i=1}^n (T_i/m_i)^{Cm_i} N^{\mathcal{H}}_{\mathrm{lb}}(6\sigma T_i, 6g_i+5m_i, r_1) \\ &\leq (T/g)^{Cg} \sum_{\substack{n\leq (1/100)g \\ T_1+\ldots+T_n\leq 12\sigma T \\ g_1+\ldots+g_n\leq g/4}} \prod_{i=1}^n N^{\mathcal{H}}_{\mathrm{lb}}(T_i,g_i,r_1). \end{aligned}$$

where the last inequality follows from renaming variables along with the arithmetic mean-geometric mean inequality.

We claim that

$$N^{\mathcal{H}}_{\mathrm{lb}}(T, g, r_1) \le \Theta^g (T/g)^{\Omega g}$$

for universal constants Ω and Θ that we choose later. We prove this by induction. The quantity $N^{\mathcal{H}}_{\mathrm{lb}}(T, 4, r_1)$ is bounded by a polynomial in T. (See [17] for more precise results.) To show the induction hypothesis, assume

$$N^{\mathcal{H}}_{\mathrm{lb}}(T, g', r_1) \le \Theta^{g'} (T/g')^{\Omega g'}$$

for all $g' \le g$. We have,

$$\begin{aligned}
N^{\mathcal{H}}_{\mathrm{lb}}(T, g, r_1) &\le (T/g)^{Cg} \sum_{\substack{n \le (1/100)g \\ T_1+\ldots+T_n \le 12\sigma T \\ g_1+\ldots+g_n \le g/4}} \prod_{i=1}^{n} N^{\mathcal{H}}_{\mathrm{lb}}(T_i, g_i, r_1) && \text{Eq. (35)} \\
&\le (T/g)^{Cg} \sum_{\substack{n \le (1/100)g \\ T_1+\ldots+T_n \le 12\sigma T \\ g_1+\ldots+g_n \le g/4}} \prod_{i=1}^{n} \Theta^{g_i} (T_i/g_i)^{\Omega g_i} \\
&\le (T/g)^{Cg} \sum_{\substack{n \le (1/100)g \\ T_1+\ldots+T_n \le 12\sigma T \\ g_1+\ldots+g_n \le g/4}} \Theta^{g/4} (48\sigma T/g)^{\Omega g/4} && \text{arithmetic mean-geometric mean} \\
&\le (T/g)^{Cg} \Theta^{g/4} (48\sigma T/g)^{\Omega g/4}
\end{aligned}$$

where the last inequality follows since the sum contains at most $(T/g)^{Cg}$ terms. Noting that $T/g \ge 2$, there exists a choice of constants Θ and Ω such that

$$(T/g)^{Cg} \Theta^{g/4} (48\sigma T/g)^{\Omega g/4} \le \Theta^g (T/g)^{\Omega g}.$$

This completes the induction step. Therefore, we have

$$\begin{aligned}
N^{\mathcal{H}}_{\mathrm{lb}}(T, g, r_1) &\le \Theta^g (T/g)^{\Omega g} \\
&\le C^T.
\end{aligned} \tag{36}$$

Finally,

$$\begin{aligned}
N^{\mathcal{M}}(T, g, r_1) &\le N^{\mathcal{M}}(T, g, 3T, r_1) \\
&\le C^T N^{\mathcal{M}}_{\mathrm{lb}}(\sigma T, g, 3\mu T, r_1) && \text{Eq. (33)} \\
&\le C^T N^{\mathcal{H}}_{\mathrm{lb}}(6\sigma T, 6g + 15\mu T, r_1) && \text{Eq. (34)} \\
&\le C^T && \text{Eq. (36)}
\end{aligned}$$

where the second inequality follows from the fact that a T-triangle triangulated surface has at most $3T$ vertices. Combining with Corollary 3.19 gives

$$N^{\mathcal{M}}(T, g, r) \le C^{T + rg},$$

which completes the proof of Theorem 1.3.

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