Problem Solving in Introductory Physics: Demons and Difficulties

by

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Bachelor of Science, Massachusetts Institute of Technology (1973)

Submitted (at last!) in Partial Fulfillment
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This thesis explores problem solving in the standard introductory physics course. It documents many difficulties students encounter with the logic of and approach to problem solving (as opposed to its content), produces abstracted descriptions of these difficulties, and identifies some of the demons responsible for many of these difficulties. These difficulties are the result of student inability to employ physicist-like reasoning approaches, and teacher inability to recognize the subtlety of many of their own assumptions.

My methodology is observational and descriptive. I observe and interview in detail small numbers of students having difficulty in solving particular problems. In this way, I lose the benefits of large samples, but I preserve the details of the important aspects of the data, namely the cognitive processes used to solve the problem.

Many of the difficulties I document seem closely related to a lack of acceptance of certain values which physicists take for granted. These values include public objectivity, prediction, quantification, broad applicability, and the assumption that it is possible for a representation of X to capture the essential features of X. These values are not arbitrary or independent of each other, but they can appear that way to the student. In addition, physics uses certain non-quantitative meta-principles which some students are also reluctant to accept. These include a distinct separation of system and environment and the notion of cause and effect.

The architecture of the human problem solver also imposes some restrictions on problem solving performance. In particular, the limited capacity of immediate-access memory and the assumption that the problem solver attempts to minimize the total mental effort employed at any given moment help to explain many student difficulties.

This thesis describes two categories of problem-solving difficulties. Understanding refers to redescribing the problem in a useful manner, and interpreting the solution as it develops in light of the specific problem and the theory underlying the solution. Generation refers to the identification and formulation of relations useful in the problem's solution employing the cues given by the problem and the problem solver's background knowledge.

Difficulties with understanding can be artifactual; these difficulties include use of poor notation (e.g., use of the same symbol to denote similar but physically distinct quantities), inadequate imagery (e.g., interpretation of static image on paper as indicative of time-independent physical situation), and poor choices of coordinate systems. Alternatively, difficulties with understanding can be of a more fundamental nature; these include confusion between functional and numerical equality, ambiguities in the physicist's use of calculus (e.g., dy/dx can mean the derivative of y(x) with respect to x or the ratio of two infinitesmally small quantities dy and dx), confusion between perceptual and representational spaces, and the literal interpretation of idealized solutions.

Difficulties with generation can arise from student reluctance to accept the physicist's criteria for scientific argument, e.g., consistent, testable, deductively based on general principles. Instead, the student may employ correlational, redescriptive, or analogical argument. Alternatively, difficulties with generation can arise from student inability to work within the framework of scientific argument. The resulting problem-solving difficulties relate to the selection of useful general relations, mathematical formulation of these relations, and specification of the symbols in these relations in terms of the specific problem at hand (e.g., the inability to use coupled equations).

In addition, I discuss three other areas of difficulty which arise from the raw data themselves. These include the risk which a student takes when he attempts to solve a problem, the use of binary (rather than continuous) quantification, and the use of intuition.

The pedagogical implications of this thesis fall into two categories. One of my general suggestions is the orientation the introductory physics course more towards "socializing the student into the community of physicists" by making explicit assumptions which teachers usually take for granted, but which students do not. Some of my specific suggestions include articulating explicitly a strategy for selecting useful equations and introducing the notion of "state" much earlier in the introductory course.

Suggestions for future work focus primarily on broad-brush explorations of learning and problem-solving processes, rather than developing specific pedagogical strategies. These might include examining the development of a beginning student as he becomes an expert professional within some framework of radical conceptual or epistemological change, and documenting the the views of lay people concerning scientific ways of knowing on the assumption that beginning students are likely to be more similar to lay people than to experts.

Thesis Supervisor: Philip Morrison

Title: Institute Professor

To Scotty (now Margaret) MacVicar

who started it all

Graduate students are the adolescents of the academic world.

- H. Lin

Acknowledgments

The pursuit of this thesis has been both lonely and exciting. On one hand, I have often longed for a sense of membership in an active research group, and all the psychological support that such membership implies. Theses are lonely enterprises under the best of circumstances, and there have been many times when I could have profited from the counsel of a more advanced graduate student or a post-doc.

On the other hand, being the very first graduate student in university physics education ever to go through M.I.T. has been very exciting. I have worked on problems of my own choosing, formulated my own research approach, asked my own questions, and set foot in previously unexplored territory. I feel I have helped to develop the frontiers of an entire field and not just a narrow sub-field. On balance, I think I'm pleased with the way things have worked out.

A thesis is never a solitary effort, and this one is no exception, despite my comments above. In general, my committee kept me from going off the deep end, but pretty much allowed me to make my own mistakes and find out for myself that research is mostly dead ends or projects that cost a million dollars and twenty man-years. However, a committee is made up of individuals, and each member of my committee has contributed in his own rather special and unique ways. In particular, I will be forever grateful to the following people:

Phil Morrison, nominally chairman of my thesis committee, but in reality, one of my heroes, and the person I'd most like to be when I grow up. As advisor, he was wonderful: he left me alone when I needed to be left alone, he offered constructive criticism when I needed constructive criticism, he encouraged me when I needed encouragement. His encyclopedic knowledge gave this thesis the needed global perspective. In addition, his continuing support for such an off-beat thesis gave me hope when I thought I might be driving a cab for the rest of my life.

Ed Taylor, also from my committee, but in reality a very savvy education researcher who knows the difference between garbage and good stuff. His feedback was gentle but firm, and his sense of organization and detail played a central role in transforming a pile of disconnected tidbits into at least a semi-coherent structure.

Tony French, who asked insightful questions and forced me to clarify my more obscure scribblings. His administrative backing was also crucial.

John King, who provided the vital perspective of someone interested but uninvolved. He worried that he was little use to me, but I always found his good common sense reassuring.

Seymour Papert, who always had new ideas and new ways of looking at things. He also provided me with access to the wonderful text editing facilities of the Artificial Intelligence Laboratory at M.I.T., without which this thesis would have been much harder to do.

Andy DiSessa, whose own work in physics education made me feel a bit less isolated and alone. He also provided me with several anecdotes which I have pirated for my own use in this thesis.

Phylis Morrison, who had absolutely no responsibility to me, but who nevertheless provided encouragement and reassurance in the final days of this project.

Bob Halfman and Ned Frank, who constituted my shadow committee - informal advisors always willing to chat over lunch, offering commentary and good sounding boards for flaky ideas.

Jack Lochhead, Jof Clement, and Bob Gray (from the University of Massachusetts at Amherst), who provided me with my first membership in a group seriously concerned with education in a university physics department.

Finally, I have to acknowledge a very special debt to Dana Roberts (of Wellesley College). Originally my roommate, now a very special friend, he provided day-to-day intellectual input into my work, and vital emotional support in my times of personal crisis. More than anyone, he helped to make up for the lack of an intellectual community sympathetic to my interests.

In the beginning,......

- Genesis

Preface

I first realized that there was something quite wrong with university physics education in my sophomore year. I got stuck doing a problem, and I went to my instructor for help. He said "Well, all you have to do is this!" and all of a sudden, by magic, I was un-stuck, and I could do the problem. I then asked him "How did you know that that was what you were supposed to do?" His response: "With experience, you learn these things."

While this was of course hard to argue with, it was absolutely no help for me the next time I got stuck. Further exploration revealed that most professors couldn't offer advice that was much better. When I decided that I wanted to do research in physics education seriously, many people (including some people whose opinion I value tremendously) tried to talk me out of it, saying that the way to get taken seriously is to do first-rate research in standard areas, thereby earning the respect of your peers for your views on education. However, in talking to people who have followed this route, I have since found that people will say "He made a name for himself doing real physics; now, he's entitled to do flaky things now." People still don't listen, and don't particularly care, even if you do make your name doing straight physics.

An unnamed professor told me the following story in order to convince me that a certain physics department really did consider education an important duty:

Once, this department denied tenure to a very, very good researcher, but who couldn't teach to save his life. Apparently, he was so bad that no one ever wanted to take his courses, and students went out of their way to avoid him.

This was the sense in which this department took teaching seriously.

Several other factors make it hard to get respect for doing education research. One is the fact that there is much education research that is simply bad, and it gives the few good pieces or work a bad name by association. The second is that education is a "soft" field, especially as viewed by physicists who work within the "hard" sciences. Consequently, everyone you meet is an expert. Particularly dangerous are those people whose mild-but-greater-than-average interest in education is enough to make them think they have genuine expertise. Finally, the predominant attitude among many (perhaps even the majority of) university physics teachers is that they care mostly about the good students.

Still, I've paid my money, and I've made my choice. I am much more concerned about the loser more than I am about the winner (who will learn despite anything teachers can do to them), and I am resigned to encountering skepticism, lip-service, and even hostility to my work from many physicists. I can only hope that the future will be as kind to me as the past has been, and continue to put me in contact with the reasonable people who do exist.

Comments on this thesis are very welcome. If by some chance someone is reading this thesis as a reference, I can be located through my parents who live at 369 Edinboro Road, Staten Island, New York, New York, 10306.

The M.I.T. thesis specifications guide says that I should include some sort of biographical data, and so that follows below:

Degree History

S.B., Physics	M.I.T.	received February 1973

Thesis title: "A Qualitative Look at the Physics of Water Waves"

Thesis Supervisor: Victor Weisskopf, Institute Professor

Research Interests

(a) Problem solving strategies useful to physics students

(b) Non-curricular influences on learning in the physics classroom

(c) Similarities and differences between everyday and scientific thought

(d) Applications of the history and philosophy of physics to the design of strategies for improving conceptual understanding in physics

(e) Conceptual difficulties with advanced areas of physics

Physics and Education Related Experience

Fall 1976 to Fall 1978	Taught first-year introductory physics recitations for MIT students with only a one year interest or who had failed physics previously
Fall 1977	Designed and taught an MIT course on "How To Approach and Think About Physics Problems"
Spring 1977	Consulted for Physics Department at University of Massachusetts, Amherst, assisting in evaluation and improvement of introductory physics program
Fall 1970 to Spring 1976	Tutored for various physics courses (Newtonian mechanics, electromagnetism, statistical mechanics, and quantum mechanics) and for the MIT Experimental Study Group (an alternative first-year program, stressing individual initiative and self-directed study)
February 1973 July 1974	Worked as research associate, for Professor Victor Weisskopf, assisting in qualitative physics of nature calculations

Honors

National Science Foundation Graduate Fellowship: Honorable Mention (1974)

Danforth Fellowship for College Teaching: M.I.T. Nominee (1973)

Papers

"Newtonian Mechanics and the Human Body: Some Estimates of Performance", American Journal of Physics, Volume 46(1), January 1978, page .5

"Approaches to Clinical Research In Cognitive Process Instruction", in Cognitive Process Instruction at the College Level, edited by John Lochhead and John Clement, Franklin Institute Press, Philadelphia, 1979

"Effective Study of Physics: Tips for the Beginning Student", The Physics Teacher, Volume 17(4), April 1979, page 243

Invited Talks

"Learning Physics vs Passing Physics Courses", Physics Department Colloquium, Rensselaer Polytechnic Institute, March 7, 1979

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Chapter 1 - Introduction 30

One goal of an introductory physics course is to introduce a student to the analytic techniques which characterize the formal structure of physics. Problem solving is one accepted domain in which students come to learn these techniques. This thesis focuses on the introductory level because this level affects the largest number of students, and also because this level is farthest removed from the professional's level of expertise. This thesis documents student difficulties in problem solving (concerning primarily logic rather than content), produces abstracted descriptions of difficulties which seem to recur, and annotates these descriptions from a number of different perspectives (discussed in Chapter 4).

Chapter 2 - Previous Work 40

Piaget 41

Piaget sets forth a view of intelligence and its development that is seminal for all of developmental and cognitive psychology.

Karplus, McKinnon and Renner 47

Karplus has developed an analysis of current pedagogical materials in physics based on Piaget's theory of intellectual development. McKinnon and Renner have documented the fact that large numbers of university students do not function at Piaget's level of formal operations.

Kuhn 48

Kuhn describes the process by which professional scientists adopt new paradigms.

Kahneman 51
Kahneman summarizes research which describes the human as an information processing system limited by memory capacity and/or channel capacity.

Perry 52
Perry documents the development of a college student's epistemology from dualism to multiplicity to relativism to commitment.

McDermott, Clement 53
McDermott and Clement study naive student preconceptions about the nature of the physical universe.

Arons 55
Arons describes some of the reasoning skills expected of college students.

DiSessa 56
DiSessa has described a procedural epistemology of physics and its pedagogy.

Bloom and Broder 57
Bloom and Broder study the cognitive processes of college students solving word problems.

Polya 58
Polya has described the heuristics employed by expert mathematicians.

Schoenfeld 59
Schoenfeld has constructed procedures (executable by advanced undergraduate students) for expert-like application of Polya's heuristics.

Newell and Simon 61
Newell and Simon have conducted extensive studies on human problem-solving, emphasizing computer models of behavior.

Wickelgren 62
Wickelgren attempts to apply the heuristics of artificial intelligence and computer science to teaching people to solve problems more effectively.

Larkin 63
Larkin has extensively studied the problem-solving behavior of both beginning students and experts (though she has emphasized the modeling of expert behavior), and the role of algorithmic models in her work.

Reif and Eylon 65

Reif and Eylon study the relationship of the large-scale organization of knowledge to the performance of various tasks of physics problem solving.

This thesis differs from the works discussed above in a number of significant ways. It is a study in an area complex compared to pattern recognition or memorization of nonsense syllables. It studies problem solving behavior in a domain in which much background knowledge is required. Its goal is insight into (rather than prescription for) the problem-solving process of the weaker physics student.

Chapter 3 - Research Approach 69

Section 3.1 - Historical Perspective 69

3.1.1 - A Concern with Behavior 69

Behaviorism is an approach to psychology which denies the utility of cognitive process - it is concerned only with behavior. It was the dominant influence in American psychology for many years, and still retains strong influence in such arenas as university physics education: a student learns to solve physics problems by solving many physics problems.

3.1.2 - Process is Relevant 74

I take the view that process is relevant, since only by modifying process can we make additional improvement. Focus on reasoning (rather than on specific content) implies a certain degree of domain independence. Nevertheless, difficulties in reasoning do interact with difficulties in content.

3.1.3 - Two Concerns for Process 78

A concern for process can take two forms: deterministic (in which one tries to construct a model from which behavior can be rigorously deduced from certain assumptions) and descriptive (in which one tries to describe coherently a set of observations in a useful manner).

3.1.4 - My Choice 81

I take a descriptive approach, because I do not believe that difficulties are rule-governed, nor do I believe that deterministic models capture influences of affect and context.

Section 3.2 - Research Approach 82

I examine in detail small numbers of self-confessed "weak" students, rather than large numbers of "average" students superficially, in un-controlled (i.e., actual) teaching and learning situations with the problems with which students actually have difficulty. I attempt to map unexplored territory with a broad brush, picking out some (but not all) of the interesting features of the terrain.

Chapter 4 - Background 91

The difficulties of Part II are related to student inability to employ physicist-like reasoning approaches, and teacher inability to recognize the subtlety of many of their own assumptions.

Section 4.1 - The Values of Science 93

The physics we teach is based on certain philosophical values - public objectivity, prediction, quantification, broad applicability, and the assumption that it is possible for a representation of X to capture the essential features of X. These values contrast sharply with what is required in the everyday thinking of the lay person (and the beginning student).

Section 4.2 - Analytic Thought: On the Meta-Principles of Physics 104

Analytic thought is the individual consideration of separate and distinct aspects of a situation. The meta-principles below are elements of analytic thought as applied to physics. By contrast, non-analytic thought is diffuse and considers situations as undifferentiated wholes. Non-analytic thought is often characteristic of the beginning student.

4.2.1 - Separation of System and Environment 104
4.2.2 - Cause and Effect 105
4.2.3 - Cause and Effect vs Divine Intervention 107
4.2.4 - Discussion and Examples 108

Section 4.3 - The Relations of Physics: a Classification 112

The relations of physics divide usefully into four categories: fundamental relations (i.e., the fundamental principles of physics such as Maxwell's equations), phenomenological relations (e.g., $F = -kx$), space-time relations (which involve geometric and kinematic considerations), and definitions (which establish a common vocabulary).

Section 4.4 - On Physics Problems: States and Time Evolution 114

The most general physics problem involves an initial state evolving through time to some final state. In this section, I examine three forms of time evolution (time-independent, time-dependent, and steady-state), and two ways of specifying state (by time, and by value).

Section 4.5 - The Architecture of the Human Problem Solver 118

For certain purposes, the human problem-solver can be modeled in information processing terms. In particular, the distinction between long-term memory (with essentially unlimited capacity but slow access times) and working memory (with very limited capacity but very rapid access times) is central to many of the difficulties described later. I also make an assumption of minimal mental effort: the human problem solver attempts to minimize the number of items held in working memory at one time.

Part II - Substance 132

As a person attempts to solve a problem, he reads it and tries to understand its requirements. He generates a solution by drawing on cues given by the problem and his background knowledge. As his solution proceeds, he must understand it in light of the problem and his background knowledge, and these two phases (generation and understanding) alternate. They exhaust (though with some overlap) the activities possible in problem solving, and therefore provide good organizers for the difficulties presented in Chapters 5-8.

Chapter 5 - Understanding 140

Understanding refers generally to the problem and the solution at any instant of time; these form a data-base which grows as the solution progresses. This data-base must be continuously interpreted.

Section 5.1 - Notation 141

- 5.1.1 - Confusion of Personal Notation 142
- 5.1.2 - Lack of Mnemonic Notation 145

Section 5.2 - Imagery 147

- 5.2.1 - Literal Interpretation of Schematic Drawing 149
- 5.2.2 - Conversion of Dynamic Problem to Static Problem 155

Section 5.3 - Extraction of Information from Problem Statement 162

The text of a problem contains the problem-specific information necessary for solving the problem. A student who fails to extract all relevant information from the problem statement may be blocked from taking further action, or may supply his own (and incorrect) default information in place of the missing information.

Section 5.4 - Coordinate Systems 167
A coordinate system provides a definite way of anchoring a physical situation to the space in which is embedded. Students may be quite vague in their choice of coordinate system orientation, reference frame, or origin.

Section 5.5 - Functional and Numerical Equality 176
An equation can denote functional and/or numerical equality. Equations which express functional equality have at least six distinct interpretations: identity, meaning, causality, constraint, global definition of symbols (vs operational definition), local specification of symbols.

Section 5.6 - Calculus as Used in Physics 189

5.6.1 - Derivatives as Rates of Change and Ratios of Infinitesimals 189
A differential Δx can indicate a change in x from some initial value to some final value, or a small bit of x.

5.6.2 - Δx as Change in x and as Infinitesimal Amount of x 194
dy/dx can indicate a command to differentiate the function y(x) (e.g., a = dv/dt), or the ratio of two differentially small quantities dy and dx (e.g., i = dQ/dt).

Section 5.7 - Perceptual and Representational Spaces 198
Perceptual spaces are mapping from real-space onto real-space drawings (though perhaps distorted as a schematic might be). Representational spaces are mappings of non-spatial quantities (pressure, electric field, time) onto real-space drawings. Students can confuse these spaces.

Section 5.8 - Interpretation of Idealized Solutions 205
The manipulation of an idealized model results in idealized answers. A student who does not realize the connection between the model itself and the physical situation it represents may interpret his idealized answers without regard for the approximations which are a part of the model.

Section 5.9 - Identification of System States 210
The use of a conservation law requires the clear identification of initial and final states of a system. Students may not clearly separate initial or final state from the irrelevant intermediate states which constitute the process related to the problem.

Chapter 6 - Generation: Problem Solutions as Arguments 218

Physicists have a certain notion of what it means to "understand" a problem or a solution. This notion imposes on the student solving standard physics problems certain criteria. However, a student may not fully accept these criteria.

Section 6.1 - Criteria for Argument 220

6.1.1 - Arguments Must Use An Appropriate Approach 220
6.1.2 - Arguments Must Use Only Known Information 223
6.1.3 - Arguments Must Be Self-Consistent 224
6.1.4 - Arguments Must Be Empirically Testable 227
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Section 6.2 - What Problem-Solving Argument Is Not 236

6.2.1 - Correlative Argument 237
6.2.2 - Redescriptive Argument 237
6.2.3 - Analogical Argument 239
6.2.4 - The Place of Pseudo-Argument 241

Chapter 7 - Generating Quantitative Solutions 243

Generating a quantitative solution requires a number of different types of activity: selection of appropriate general relations, formulation of these relations in mathematical form, and defining the symbols in these relations in terms of the specific problem at hand.

Section 7.1 - Selection 244
Students have different difficulties depending on the type of relation which must be selected.

7.1.1 - Fundamental Relations 245
Students are usually capable of identifying fundamental relations which apply to a problem. They have difficulty in identifying relations which are useful in the solution of that problem.

7.1.2 - Relations of Geometric and Dynamic Constraint 251
Relations of constraint are often quite difficult for students to apply, because they are not found explicitly in mathematical form in the statement of the problem - consequently, they are easy to overlook.

7.1.3 - Definitions 253
Definitions are also easy to overlook, because their function is to provide meaning for particular terms. If a student believes he understands the meaning of a particular term, it may be so familiar to him that he will not attempt to unpack it any further.

7.1.4 - Special Case Equations 256
Direct recall of special cases of various general relations can be quite useful if these special cases occur frequently. However, their successful application requires that the student check to see that the conditions under which these special cases apply are in fact present.

7.1.5 - Arbitrary Parameter Problems 258
Independently of the type of relation, students have great difficulty in making progress on problems in which it is helpful or necessary to introduce an arbitrary parameter which ultimately drops out of the equation.

Section 7.2 - Formulation 262
The mere selection of a relation as being relevant to a problem is not a guarantee that the student can formulate it in mathematical terms.

7.2.1 - Difficulties in Translation from English to Equations 263
7.2.2 - Imprecision in Vocabulary 268

Section 7.3 - Local Symbol Specification 273
Local symbol specification is the process by which the information given in the problem is mapped onto the general relations previously identified and formulated. This step is the most difficult for students, both by their own admission, and by my observation.

7.3.1 - Physical Quantities 279
Students can confuse additive (consisting of physically distinguishable components) and non-additive (lacking separable components) quantities. Often this confusion is rooted in the lack of a clear separation of system and environment.

7.3.2 - Application of Relations to Different Aspects of Problem 287
Students can be unwilling or unable to apply same general relation to different aspects of a problem, e.g., to another system, or in a different direction.

7.3.3 - Special Cases vs General Relations 290
Students can inappropriately use a frequently occurring special case instead of a

general relation.

7.3.4 - Inability to Isolate Cause-effect Chain 294
Students may not recognize the need for construction of a causal chain.

7.3.5 - Magnitude of Quantity vs Quantity 298
Students may substitute the magnitude of a quantity for the quantity itself (which includes magnitude and sign or direction).

7.3.6 - Variables, Constants, Equations, and Ignorance 302
Students may be unwilling to defer the resolution of ignorance. This deferment is necessary for the use of coupled equations and even the use of symbolic variables.

Chapter 8 - Other Ideas 316

The ideas in this section arise directly from the data themselves.

Section 8.1 - Cognitive Risk: Fear and Courage 317
A student who attempts to solve a problem faces some risk. The manner in which he copes with risky situations shapes his actions or lack thereof. A good problem solver uses the content of the problem to guide his actions, but he actively engages in transforming the problem and his solution into more transparent forms. A poor problem solver is controlled by the form of the problem, resulting in a very conservative and passive approach to the problem.

Section 8.2 - Binary Logic 323
Physics uses a continuum of real (and imaginary) numbers to make its predictions. However, some students appear to operate using a two-valued notion of quantity. One special case of this binary logic is the distinction between quantities which are zero and quantities which are not. Students often behave very differently when confronted with a zero quantity - they abandon procedures which they use effectively in other situations involving non-zero quantities.

Section 8.3 - Intuition 337
An intuitive prediction is one which is generated without the benefit of detailed analysis. However, beginners believe that intuition is unreliable; experts put their faith in it. Intuition requires a model of a physical situation which separates essence from detail, and an organized experiential background with the phenomena in question. Frame theory (from the field of artificial intelligence) provides one way of describing this organization.

Part III - Conclusions 344

This thesis raises more questions than it answers; not surprisingly, since it is an exploratory study. Part III makes some tentative recommendations for pedagogical practice, but more importantly, outlines several paths which future work can take.

Chapter 9 - Pedagogy 346

The pedagogical implications of this thesis fall into two categories: global and specific.

Section 9.1 - Global Suggestions 348

9.1.1 - Socializing and Teaching Physics Students to be Community Members 349
Problem solutions are to the student as published papers are to the professional; both present a distorted though necessary view of physics. Socialization is an important part of the training of each.

9.1.2 - Teaching Methods of Problem-Solving Analysis Explicitly 358
9.1.3 - Teaching Micro-Skills 365
9.1.4 - Tutorials as a Complement to Lectures and Recitations 367

Section 9.2 - Specific Suggestions: Understanding (Chapter 5) 368

9.2.1 - Elements of Good Notation (Section 5.1) 368
Use subscripts rather than upper and lower case letters.

9.2.2 - Imagery: Good Diagrams and Other Tips (Section 5.2) 369
Make diagrams big and mnemonic; pay attention to angles.

9.2.3 - Extraction of All Relevant Information from Problem (Section 5.3) 370
Read the problem out loud.

9.2.4 - Choosing Good Coordinate Systems (Section 5.4) 372
Orient axes so that the acceleration need not be resolved; always include origin.

9.2.5 - Clarifying Functional and Numerical Equality (Section 5.5) 372
Use functional notation; write one line of explanation for each equation.

9.2.6 - Calculus as Used in Physics 373

Discuss explicitly different usages of differentials.

9.2.7 - Perceptual and Representational Spaces (Section 5.7) 374
Discuss explicitly connections between different vector representational spaces.

9.2.8 - Idealizations (Section 5.8) 374
Use demonstrations instead of text statements of problems.

9.2.9 - Identification of States (Section 5.9) 375
Introduce notion of state explicitly and much earlier.

Section 9.3 - Specific Suggestions: Problem Solutions as Argument 376

Section 9.4 - Specific Suggestions: Generating a Quantitative Solution 377

9.4.1 - Identification and Selection of Useful Relations (Section 7.1) 378
Articulate a managerial strategy for selection of useful equations; apply same general relation to different aspects of problem; search for different types of relation.

9.4.2 - Formulation (Section 7.2) 380
Use quick and dirty checks such as testing possibilities with numbers.

9.4.3 - Local Symbol Definition (Section 7.3) 382
Require that solutions begin with general relations; articulate correspondence rules.

9.4.4 - Determination of Sign (Section 7.3.5) 387
Insert sign on physical grounds when possible.

9.4.5 - Use of Coupled Equations (Section 7.3.6) 389
Discuss explicitly notion of "well-defined system" and "closed system of equations".

9.4.6 - Reducing Cognitive Fear and Risk (Section 8.1) 389
Employ brainstorming sessions.

9.4.7 - Zero Property and Non-Existence (Section 7.2) 390
Discuss modeling; explicitly mention that ma = 0 means that a = 0 or that m = 0.

9.4.8 - Intuition 391
Use anthropomorphic or teleological arguments when possible.

Section 9.5 - Miscellaneous Suggestions 393

9.5.1 - Periodic Summary and Simplification 394
9.5.2 - Frequent Checks 394
9.5.3 - Alternative Checks 395
Check by limiting or special cases; check sign and magnitude; check units.

9.5.4 - Grading 396
Penalize most heavily the careless and silly mistakes.

Section 9.6 - Pedagogical Suggestions: Caveats and Comments 397
This section qualifies many of the suggestions made in preceding sections, and therefore serves as a warning.

Chapter 10 - Work to Come 403

This thesis raises more questions than it answers. Here are some areas in need of further investigation.

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Section 10.8 - Hierarchical and Linear Organization and Approaches 419

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Bibliography and References 424

This chapter contains thesis references, plus a section of general information references.

Notation of This Thesis

Footnote references are set off in curly brackets like this: {1}. Footnotes are found at the end of each chapter.

Book and paper quotes are single spaced, indented, and printed in this font.

Sections are numbered as follows:

Section 3.7.2 is Chapter 3
Section 7 in Chapter 3
Sub-Section 2 in Section 7

Examples are labeled as follows:

Example 3.7.2.a is the first example of Section 3.7.2, example 3.7.2.b is the second example of Section 3.7.2, and so on.

An example is set off from the body of the text by a row of asterisks (*****) before and after the text of the example.

1. The statement of the problem or topic under discussion will be written in this font.

2. The dialogue between student and teacher (usually me) will use two fonts:

S: S(tudent) comments will appear in this font.

H: H(erb's or more generally, teacher's) comments will appear in this font.

3. Discussion or commentary concerning this (and only this) example will follow or be interspersed with the dialogue.

The longest journey begins with a single step.

- Lao-Tze

Part I - Preliminaries

Part I presents a great deal of background material necessary for understanding the substantive material presented in Part II. It (Part I) is fairly long as thesis introductions go, but this is due to the unconventional nature of this thesis: things which might normally be taken for granted must be raised explicitly.

For example, previous relevant work is scattered over a variety of disciplines: developmental psychology, philosophy of science, cognitive psychology, artificial intelligence, mathematics education, science education, cognitive science, as well as physics. Chapter 2 presents a more than indicative but less than exhaustive look at some of this work.

In addition, the methodology of this thesis (described in Chapter 3) differs from classical education research, in that this work is observationally based (rather than experimentally based). It also focuses on process rather than behavior, but does not result in a predictive model; rather, it attempts to describe the underlying processes.

Chapter 4 provides a framework from which it is easier to understand the difficulties in Part II. It explicitly discusses many assumptions concerning philosophy and value which teachers often take for granted. It also discusses certain salient characteristics of the formal structure of physics and of the problems typically assigned by teachers of introductory physics. Finally, it discusses one possible model of the human as problem solver which is useful for interpreting many of the difficulties presented in Part II.

It is important to emphasize that though Chapter 4 often appears to be cast in the spirit of cognitive psychology and the philosophy of science, its function is to provide a vocabulary useful for discussing the difficulties of Part II. It is in no sense a fundamental theory of beginners solving physics problems.

In order to get from here to there, you better know where 'here' is before you start!

- unknown

Chapter 1 - Introduction

This thesis presents a study in physics education - its purpose is not so much to contribute to the understanding of physics as to the understanding of physicists, as teachers and as students.

Most contributions to the improvement of physics education concern "getting the story straight"; these contributions include:

- elimination of unnecessary mathematical complexity
- alternative and/or simplified derivations of physical results
- relations between physics and other fields (biology, engineering) and areas of current interest (energy, environment)
- increasing the readability of the material by using more words in a more informal, colloquial manner
- introduction of new labs

Examples of the preceding would include the majority of articles in the American Journal of Physics {1}, the Berkeley Introductory Physics Series, the M.I.T. "Physics: a New Introductory Course" series, and the Feynman Lectures.

These innovations concern primarily the transmission of knowledge, by the teacher, to the student; as such, these innovations are teacher-centered. By contrast, this thesis focuses primarily on the reception of knowledge, by the student; as such, it is student-centered.

More specifically, this thesis is primarily concerned with how beginning students solve the problems typically assigned by teachers of introductory physics. My ultimate objective is, of course, to make substantial improvements to the manner in which teachers approach university physics education. A necessary first step is to analyze in detail the triad of students, teachers, and educational contexts as it actually functions. Only after completing this analysis might it be possible to suggest better ways. It may be the case that better ways do not exist; however, this is not necessarily true, and there will surely be no better way if no one attempts the analysis.

This thesis documents a wide range of difficulties which students encounter in solving introductory physics problems {2}. The demons responsible for these difficulties constitute a pseudo-theoretical framework with which insight into these difficulties can be gained.

To set the stage for this work, I will first discuss in general terms the nature of the problem under consideration. This in turn requires a discussion of goals, and the reasons behind my selection of problem-solving as an appropriate domain of inquiry. I will end with a comment concerning the audiences to which this work is directed.

What is the primary goal of introductory physics courses designed for students who will continue to more advanced work? A quite general response would be that an introductory physics course should teach the student two kinds of things. It should teach students some specific content (e.g., Newtonian mechanics or classical electricity and magnetism), and it should teach methods of analysis and a feeling for the power of physics in a broad context. Indeed, it is the latter that many physics teachers would consider the most important. For example, Halliday and Resnick (1977) write that

> The ... examples [of the chapter on Newton's laws] illustrate the method of analysis used in applying Newton's laws of motion. Each body is treated as if it were a [point] particle... Strings .... have negligible mass. Although some of the situations picked for analysis may seem simple and artificial, they are the prototypes for many interesting real situations; but, more importantly, the method of analysis - <u>which is the chief thing to understand</u> - [emphasis mine] is applicable to all the modern and sophisticated situations of classical mechanics.

Thus, it is in the introductory course that the student must begin to adopt new procedures and ways of thinking if he is to become a scientific professional. Those who fail to make the transition usually drop out by the end of the introductory course.

The specific implementation of this goal is subject to far more debate. My own position is that an introductory course should introduce the student to the kind of logical argument and abstraction that characterizes the formal structure of physics. This has the following consequences:

1. The right answer to a problem is not enough. The route to the answer must be articulated within the framework of a community-accepted formalism such as Newtonian mechanics or classical electricity and magnetism. An answer like "Current is similar to water flowing in pipes" isn't enough. Such a statement is quite useful in understanding the formalism, but I would insist on a reference to a conservation law, e.g., "What goes in must come out, or else it would accumulate."

2. Simple reference to the words of physics is insufficient. The student must be able to articulate the formalism in his own words, giving precise directions for its use, as he might program a computer. Otherwise, the formalism is incoherent and too vague to be of any concrete use.

3. Everyday intuition and common sense and gut feeling are also not sufficient: these are generally vague, fuzzy and often misleading. By contrast, students should develop a trained intuition, one that arises from full acceptance of and belief in the accepted formalisms of, say, classical physics. "Common sense thinking" should be part of an introductory physics course only to the extent that it can be formulated in terms of the formalism

under study, or that it assists the development of the formalism. By itself, it is too diffuse to provide a foundation on which to build a physics course.

4. Vague and fuzzy thinking is at the heart of most difficulties of beginning students. Therefore, a physics course should seek to replace the vague and fuzzy thinking with unambiguous articulation as soon as possible. Indeed, the essential difference between the working professional and the lay person is precisely the professional's command of a general theory, a formalism which can be applied to physical problems to draw conclusions and make predictions in which the professional community can have some confidence.

As a result of these four items, I am concerned with what a student does when he tries to function according to the rules that the world of the physicist imposes on the student's behavior. At the introductory course level, this world requires proficiency in two general areas:

a. Problems of the form "Here is a physical situation, presented with certain knowns. Something happens. Calculate something of physical interest associated with what happens or the beginning or the end states." Such a problem typically gives exactly the data needed to solve it, and is rarely soluble without the application of some (possibly quite simple) mathematical formulas. These problems are most easily and reliably solved through the application of general physical principles.

b. Explanations and discussions of general principles or specific results. For example, a student might be required to answer a question regarding the restrictions on the use of a certain equation or principle, or one which asks "Why might you have expected this result?" or "Explain this principle qualitatively in words." or "Solve this problem by using limiting cases."

These areas (characteristic of introductory physics courses) involve the general theme of theory application. Of course, the scientific professional requires more than application. The creation of theories (and experiments), the formulation of reasonable questions, and the reduction of complex and ill-defined problems and situations to tractable forms are some of the other requirements, and these are not usually addressed by the introductory course.

Nevertheless, I have chosen to take the perspective that the standard introductory physics course takes, namely that problem solving ability is essentially the same as understanding physics. I am personally not convinced that this is necessarily the case, but this assumption does allow at least some initial sharing of assumptions. [I will return to this point in Chapter 9 (Pedagogy) and argue that fewer examples of problem solving and more emphasis on "understanding" will improve even problem solving ability.] Also, there is a substantial body of work which concerns the detailed analysis of expert and novice solutions to standard physics problems. By contrast, creativity analyses exist only for seminal figures in the sciences (e.g., Holton (1973) on Einstein and others, Gruber (1974) on Darwin), and these people are hardly typical of the average struggling student.

I have directed my efforts at the difficulties encountered by students at the introductory level for two reasons:

a. It is at the introductory level that physics teachers come in contact with the largest number of students. This has the consequence that improvements at the introductory level affect the largest number of students. Furthermore, introductory level courses consume the greatest fraction of departmental teaching resources of all courses. Consequently, any pedagogical improvement at this level increases effectiveness more substantially.

b. The introductory level is also the level farthest removed from the professional's level of expertise. Consequently, it is often difficult for the professional to recall or identify with the diversity of naive and semi-naive responses to various problems, problems which the professional, through years of training, has learned to see transparently.

This thesis is primarily concerned with difficulties in the logic of problem-solving (how students solve problems), rather than its content (what these problems are about). However, it is impossible to consider problem-solving in the abstract: the logic of problem solving necessarily interacts with the content of problem solving. Still, the difficulties with which I am concerned are somewhat more subject-independent than those implied by a naive student's preconceptions and intuitions regarding nature and the physical world.

This thesis has three separate but related components. Part I (especially Chapter 4) introduces concepts and vocabulary which make it easier to discuss the difficulties of Part II. Part II presents many examples of student difficulties, and comments on these from the perspectives outlined in Chapter 4. However, these difficulties are not logical consequences of Chapter 4, but in light of its contents, these difficulties are easier to understand. Part III makes a variety of pedagogical suggestions and raises some questions for teachers to think about, but once again, these suggestions and questions are not logical consequences of Parts I and II; rather, they are ideas which have evolved rather empirically in light of the ideas and experiences discussed in Parts I and II.

This particular study is motivated by the observation that students of introductory physics encounter similar difficulties year after year. Inexperienced teachers are often at somewhat of a disadvantage, compared to experienced ones, because they have not encountered these difficulties since their own days as beginning students. Thus, they are forced to formulate their own insights about the teaching/learning process on the fly - while actually interacting with students - usually without help from more experienced teachers.

The framework I develop should be useful to both students and teachers. Most importantly, I hope it is useful as a consciousness-raising and thought-provoking device which leads teachers to question the assumptions inherent in their pedagogical activities. A teacher might also be able to identify specific difficulties which individual students demonstrate, perhaps in their written work but more often in working problems in the teacher's presence - forewarned is forearmed. In addition, he might be able to build profiles of individual students over many problems (identifying recurrent difficulties) or of classes over one problem (identifying common difficulties). Students might benefit from an explicit awareness of some of the kinds of errors their peers make. Also, a student might be able to check his own work for some common errors.

This study is in no way exhaustive or logically rigorous. It is an exploratory study in "educational engineering" rather than an experiment in "cognitive science" - it covers a substantial but limited domain, rests fundamentally on observation rather than theoretical constructs, and probes the underlying processes only to the extent necessary to direct pedagogical practices which are aimed at enabling the student to solve specific problems. Furthermore, it focuses on a student's actual behavior under classroom circumstances, rather than what he might be capable of doing in principle.

More generally, the methodology of this study departs from conventional practice in a number of ways, and Chapter 3 discusses methodological issues in a much more detailed manner. Chapter 2, immediately following, discusses previous work related to this study.

Notes

1. Taylor (1979) has pointed out that at least 95% of the articles in the American Journal of Physics (an archival journal of physics education) fall into these categories. Only 2% might be considered "education research.".

2. The word "problem" might be used in two ways. It can mean either the text-and-picture statement which appears at the end of each chapter of most physics books, or it can mean the hardships that students encounter when they try to process these text statements. In this thesis, "problem" will be used with the first meaning, while "difficulty" will imply the second meaning.

No man is an island.

- J. Donne

Chapter 2 - Previous Work

This thesis differs in both style and content from most efforts in physics education. More specifically, most research physicists are professionally uninterested in pedagogical issues, and most philosophers, psychologists, and teachers lack sufficient familiarity with and insight into physics to generate findings of interest and utility to the teaching physicist. By bringing to bear an (I hope) adequate though non-specialized background in all these areas, I hope to shed new light on one aspect of the pedagogy of physics: the behavior of beginning students in solving physics problems.

In this specific area, not much has been done; this thesis documents an exploratory study. However, some previous work certainly bears a strong relation to the ideas contained in this thesis; this chapter discusses a substantial fraction of this work.

The most significant figure in the history of recent cognitive and developmental psychology is Piaget. Starting from a data-base of clinical observation of his own children, he has developed a theory of intellectual maturation which underlies the work of a great number of cognitive and developmental psychologists {1}.

Piaget's theory consists of a few principles which constrain intellectual development, and a description of the qualitatively different stages of thought through which children progress as they mature intellectually. He assumes that two basic functions regulate the developmental process: organization and adaptation. Organization refers to an individual's (assumed) tendency to organize mental structures or behavioral patterns (or more generally, cognitive structures) into coherent, consistent systems. One such example might be the realization that a smiling face and a frowning face may in fact be two different aspects of the <u>same</u> face.

Adaptation refers to two complementary processes: assimilation and accommodation. Assimilation is the process by which the individual understands the external environment in terms of already-existing psychological structures. Accommodation is the process by which existing psychological structures are modified in order to handle more efficiently (e.g. faster, more consistently) inputs from the external environment. Accommodation is what we normally call learning.

Piaget characterizes the increasing sophistication of cognitive structure in terms of the range of application, stability, and spatial and temporal mobility. Thus, a highly sophisticated structure would be quite broadly applicable, quite adaptable to new domains of applicability through modification (not destruction) of existing structures, and able to deal with events quite well-separated in space and time.

Intellectual development occurs when the child, at first using a variety of cognitive structures inadequate for successful adaptation, resolves the conflicts which arise from the use of different structures. The result is an integration of these structures into a more sophisticated one, which allows him to deal with a variety of new and different situations.

In addition to these developmental principles, Piaget outlines a sequence of four stages in the intellectual development of the child: sensori-motor, pre-operational, concrete-operational, and formal-operational. He emphasizes that these stages form an invariant sequence, but not that these stages emerge at any fixed age. He further emphasizes that these stages describe the maximally sophisticated behavior of which the individual is capable while in any given stage, and not his usual or "average" level of cognitive functioning.

In the sensori-motor stage, the infant lays the groundwork for all future intellectual development. More specifically, the infant develops the following notions: (a) permanent object - the child gradually learns to separate reality from immediate perception. Thus, he begins to realize that the object remains the same even though many visual changes may have taken place. (b) prediction - the child begins to deliberate and predict before performing an action, rather than finding his way by trial and error. (c) purposive behavior - the child's actions are first governed entirely by chance. He slowly learns to create and sequence new behavior with deliberate ends in mind. (d) relations - the child acquires primitive relational notions: intensity (loud vs soft noises), quantity (more vs fewer syllables in a spoken word), sequence (e.g., he must move the obstacle in front of the block before he can reach the block), and spatial relation (objects have definite locations even when invisible).

In the pre-operational stage, the child begins to use mental symbols to represent real objects and events, but is unable to perform mental operations on them. He does not ask why any particular method of solution to a problem succeeds or fails; corrective action takes the form of trial and error feedback, rather than planned, deliberate corrections. His reasoning is primitive in that he cannot separate his own desires and actions from the conclusions he draws. Alternatively, it may be based entirely on memory.

In addition to developing further the notions of the sensori-motor stage, other ideas begin to develop. These include: (a) classification - the child learns to group objects using consistent rules and defining properties for classification. He also learns to construct hierarchical orderings. (b) imagery - the child learns to handle static images, but he cannot properly represent movements or physical transformations of an object. (c) language - the child begins to use words to stand for physically absent events or things, but his use of these words does not correspond to adult usage of the same words. He will use use pronouns and demonstrative adjectives (it, he, she, this, that) without specifying their referents. He will incorrectly sequence events in relating a story, or omit causal relations.

In the concrete-operational stage, the child is for the first time able to carry out operations on symbols. He begins to focus on transformations and processes, rather than states. He is able to coordinate many aspects of a situation with each other and relate them to each other; consequently, he can identify inconsistencies. However, he is able to perform these operations only on constructs which reflect tangible objects.

In general, the concrete-operational stage is characterized by the psychological dominance of concrete objects, and increasing awareness of his own thought processes. These general tendencies result in the refinement of notions first acquired in the pre-operational stage. For example, he is quite capable of describing an event accurately, but he ascribes primary importance to the given facts, as opposed to any process that might be involved in the problem's solution. He learns to use conservation reasoning: if nothing is added or subtracted, the total amount or number is unchanged. He can now correctly represent kinematic processes. He learns to predict, but these predictions are restricted to simple extensions of familiar procedures.

In the formal-operational stage, the child (now in fact a young adult), is able to perform operations on mental constructs with no perceptual counterpart. While the concrete-operational stage involves relations between objects, formal operations include relations between relations; the individual learns to handle propositions as he does objects. He learns to coordinate operations in addition to the empirical aspects of a situation. Mental actions (logic) begin to operate on verbal statements (propositions) in the way that concrete operations operate on objects. He can create "new facts" as a result of applications of deductive logic, which serve him on equal footing as does given information. He can systematically conceive of all possibilities before attempting a solution to a problem. Consequently, he can design controlled experiments, holding all but one variable constant. He can make idealizations. He can look at a problem from many different points of view, and thereby find inconsistencies in his reasoning.

The shift away from perceptual reality gives the individual a much wider range of possibilities for actions. This leads to a propositional logic which arises from his ability to substitute propositions for concrete objects or events. Where before he made statements about individual objects, he now makes them about classes of objects. His use of propositional logic enables him to use functional relationships and construct new facts which have a truth value equal to that of the given information. As a result, he can perform thought experiments now that the deductive necessity (arising from propositional logic) is available to him.

Piaget's work is seminal in the history of cognitive and developmental psychology, but different people have different reasons for believing this. Some argue that his theory is a good representation of the intellectual development of children as they mature. Others (myself included) argue that the real significance of Piaget's work is not his particular theory, but that he is the first to introduce in a respectable manner notions of developmental sequence and qualitative differences in thought. In addition, I believe that his emphasis on action and manipulation as the precursor to operational thought is quite significant.

Much recent literature explores the application of Piaget's concrete- and formal-operational stages to the learning of physics. For example, Karplus (1977) describes an application of Piaget's theory of intellectual development to the design of an introductory college physics curriculum. In particular, Karplus cites research (cf., McKinnon and Renner (1971)) which argues that substantial numbers of college students do not function at the level of Piaget's formal operations; instead, they are concrete-operational. Karplus also points out that many introductory texts are written assuming a reader who uses formal operations. Finally, he argues that it is possible to encourage the equilibration process (thereby promoting intellectual development), by asking questions which involve the resolution of apparent paradoxes.

Karplus' work is similar to that reported in this thesis in that both attempt to deal with the general reasoning strategies necessary for learning physics. It is different in that my analysis is data-driven, and consequently is very domain-specific. By contrast, Karplus' analysis is theory-driven, and is relatively domain-independent, though the examples he offers do relate to physics.

The science education of students by teachers is comparable to the manner in which professional scientists try to understand the world; one might argue that the history of science reflects the conceptual difficulties encountered by individual students as they make the transition from beginner to expert. Kuhn (1970) focuses attention on the nature of scientific progress when he describes the radical shifts which occur in scientific progress as the new displaces the old. It is important to recognize the magnitude of these shifts because they are quite similar in magnitude to the shifts which beginning students must experience if they are to become scientific professionals.

Central to Kuhn's description of scientific progress is the notion of a paradigm which governs the scientific community's practice. A paradigm includes:

- symbolic generalizations, which the community uses without question. These generalizations can be cast in forms like (x)(y)(z)f(x,y,z), and they allow the community to employ the powerful techniques of logical and mathematical manipulation.

- beliefs in particular models, which help to determine what will be accepted as an explanation or a solution.

- values concerning prediction (e.g., accurate, quantitative) and theories (e.g., simple, self-consistent, plausible, compatible with other theories in current use).

- exemplars, which are the concrete problem solutions that students encounter from the start of their scientific education.

Radical shifts occur only when the scientific community must choose between competing paradigms. Kuhn argues that the choice between competing paradigms is a choice between incompatible modes of "scientific thinking". Consequently, the proponents of competing paradigms will often disagree about the list of problems that any candidate for paradigm must resolve, and what constitutes legitimate solution.

Kuhn further argues that the difficulty in making radical shifts is that debates over paradigm choice cannot be cast in an entirely logical form. In a logical debate, both sides start from the same premises and rules of inference. If disagreements arise, each side can retrace its steps one by one, checking each against the original premises. At the end of this process, one side must concede he has made a mistake having violated one of the premises. After this concession, he has no recourse, and his opponent's proof is compelling. However, if the two discover that they differ about the meaning or application of the premises, then there is no logical route to resolution. Reasons for theory choice (e.g., simplicity, accuracy) function as values, and these can be applied differently by people who espouse the same values.

Consequently, Kuhn argues that often scientists do not make the shift successfully. He quotes Planck's comment to the effect that "a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."

The accuracy of this view is open to debate - there are indeed historical precedents for paradigm choice on the basis of experimental data. Kuhn himself quotes two examples (the correct prediction of the phases of Venus, and the existence of Poisson's spot {2}), but nevertheless argues that such examples are neither individually nor collectively compelling.

I will not judge Kuhn's view of scientific progress. However, it appears to me that whatever its flaws or merits in a discussion concerning the history of science, his scheme does describe quite well the conceptual and epistemological shifts that beginning students must undergo in their first encounter with physics. Many of the difficulties I will document can be traced back to a sharp difference in values - an essential component of any paradigm. Indeed, I hope to demonstrate that these difficulties are not remediated by data or logic alone, just as paradigm choices cannot be made only on the basis of logic or experiment.

Kahneman (1973) reviews a variety of work in the area of attention as related to performance. He focuses on aspects of attention which are intensive (associated with mental effort) and those which are selective (associated with the allocation of mental effort). He contrasts two models of attention: a bottleneck model, in which total cognitive activity is restricted because parallel processing is assumed to be impossible (i.e., a person can focus on only one thing at a time), and a capacity model, in which limited capacity of effort can be allocated at will to various activities.

He further suggests that the intensity of effort on any given task is primarily determined by the intrinsic demands of that task. However, he points out that empirically, allocation of effort is more flexible than expected according to a bottleneck model, but more constrained than expected according to the assumption of arbitrary allocation.

Kahneman's literature review is quite useful for gaining perspective, but its specific application to the work of this thesis is limited by its focus on low-level tasks (e.g., mental arithmetic, recall of nonsense syllables, pattern recognition). Still, its focus on capacity as a limiting factor is a very important part of my ad hoc description of the architecture of the human problem solver in Chapter 4.

Perry (1970) documents a multi-year study which traces the epistemological development of students as they attend college. He finds the following general pattern of growth. The student begins with a dualistic epistemology; he divides the world into right and wrong, black and white; his knowledge derives from the word of Authority. As a result of social influences, he discovers the existence of other points of view. He thus grows from dualism into multiplicity, in which everyone has an equal right to his opinion, be it examined or unexamined, because opinion is completely arbitrary. The next stage is relativism, in which the student begins to wonder if anything is worthy of belief. The final stage is commitment to a particular point of view, with the full knowledge that equally valid alternatives exist; the student's knowledge now derives from a reasoned but personal commitment, rather than external authority. Perry's work relates to this thesis in two ways: he outlines a sound methodology for verification of a theory based inherently on "soft" data, and he provides evidence for substantial intellectual growth and maturation even after adolescence.

McDermott (1979) examines in detail conceptual difficulties that students have regarding various kinematic notions. For example, she confronts students with a demonstration in which two balls roll along aluminum tracks. One ball travels with uniform velocity from left to right along a horizontal track. A second ball starts from the left with an initial velocity greater than that of the first ball. The second ball travels up a gentle incline, slowing down and eventually coming to rest. A movie would show that the balls passed each other twice and had the same instantaneous velocity at an intermediate time. Students observed this demonstration several times, and they were asked "Do these balls ever have the same speed?" A substantial number said "Yes, when the balls pass each other." On this and other tasks, she finds confusion between velocity and acceleration, velocity and position, and acceleration as change in velocity per unit time vs change in velocity per unit distance. Further, she finds these conceptions remarkably resistant to formal instruction.

Clement (1977a) examines naive conceptions of force and its relation to motion, and documents the following phenomena:

- constant velocity implies the presence of a force
- the direction of motion is always the direction of the force
- force comes from motion
- momentum is proportional to force
- forces come only from active sources of power

McDermott (1979) and Clement (1977a) probe the student's qualitative and intuitive knowledge of the physical world. As such, they are concerned with the content of physics, rather than its problem-solving structure. By contrast, I examine my subjects as they grapple with highly quantitative problems involving many steps of analysis.

Clement (1977b) also discusses what it means to "understand physics". He argues that "understanding" requires the existence of links between four different types of knowledge: practical knowledge (observation, experience, actions), qualitative physical knowledge (models which include only causal relations and function), concrete mathematical models (quantitative representations such as counting, graphing), and symbolic manipulations (written calculations, equations). However, this analysis of understanding involves the expert's perspective, and it does not document student difficulties with these various types of knowledge.

Arons (1979) describes some of the reasoning skills expected of the college student. He includes the following: arithmetic reasoning, identification and control of variables, command of propositional logic, paraphrasing of text in one's own words, awareness of gaps in knowledge, discriminating between observation and inference, use of operational definitions, and translating between words and symbols. Arons (1976) also describes an introductory physical science course which attempts to promote these reasoning skills by giving students manipulative activities in which these skills can develop.

Arons' description of these skills seems accurate. However, they are not prescriptive; they provide labels to which we as teachers can provide meaning, but they leave much in tacit form. Still, they might have provided reasonable organizers for the difficulties described in Part II if they had been specifically oriented towards physics.

DiSessa (1979a) stresses the importance of the non-formal, non-deductive kinds of knowledge that experts employ in their professional lives. He argues that the process of science is not deductive, but rather heuristic, analogical, model-based, and open-ended. He stresses the importance of matches between curriculum content and the student's intuition, arguing that the standard deductive treatment is not simple from a student's point of view. Instead, he offers a view of education in which we teach not "science" but rather "what scientists do", emphasizing procedure more and facts less, and teaching facts with connectives to procedure; in short, we should be able to answer the student who asks "Now that I know that, what do I do next?" Finally, he argues that multiple representations of the "same" knowledge are an essential component of these connectives. DiSessa approaches the general question of pedagogy with a very different set of assumptions regarding the goals of an introductory physics course. In particular, he would not consider the solution of standard physics problems a worthwhile goal (personal communication, 1979). Thus, while I believe that he addresses some very important matters which have relevance even for problem-solving, we do not focus on the same ones.

Bloom and Broder (1950) study college students solving problems similar to those on Scholastic Aptitude and Achievement Tests, and document many of their difficulties. They generate a checklist which includes the following items (somewhat paraphrased):

- does not read directions to problem
- forgets directions
- does not translate abstract forms into more familiar terms
- does not use criteria which the solution must match
- ignores unfamiliar terms
- answers on the basis of "feelings" about the answer
- does not break a problem up into parts
- does not plan a solution
- tries to remember the answer to a similar problem
- does not follow an inferential chain to its end

To my knowledge, this is the first work on problem-solving which looks in detail at the performance of unskilled problem solvers. The checklist items are not sufficiently detailed to offer remedies for a student who exhibits such behavior; nevertheless, they do provide labels for central points of reference which can be further elaborated to provide such remedies. This thesis contains similar items, and it suffers from the same weakness. However, the main difference between these two pieces of work is that the problems with which Bloom and Broder are concerned are not primarily science-oriented. Even when they are, they are not usually problems which involve both mathematical analysis and extensive background knowledge.

Polya (1945) describes the heuristics which expert mathematicians employ when they solve problems. He groups these heuristics into four general areas:

> Understanding the Problem: What is the unknown? What is the given data? What is the condition? Is it possible to satisfy the condition? Is the condition enough to determine the unknown? Or is it redundant or contradictory? Can the condition be separated into parts?
>
> Devising a Plan: Can you reformulate the problem? Have you solved similar problems before? How are they similar? Can you introduce an auxiliary element which facilitates the solution? Did you use all the data? Can you derive something useful from the data? Have you used all essential notions in the problem?
>
> Carrying out the Plan: Doing what planning said you should do.
>
> Looking Back: Checking the result, checking each step. Can you see clearly that each step is correct? Can you prove it is correct? Can you derive the result differently? Can you see it at a glance?

This work is the work that launched the modern study of heuristics. It is carried further in Polya (1954a, 1954b). However, his work suffers from the fact that it is not prescriptive; hence, as a guide to students, it is not very useful.

Schoenfeld (1979) has attempted to carry Polya's work one step further, by constructing procedures which will allow a student to choose an appropriate heuristic. He notices that few people responsible for training students in mathematical problem-solving at the college level actually use Polya's work as a foundation for their instruction in problem-solving. People apparently enjoy reading Polya's work, but they do not solve problems more efficiently as a result, nor do they feel they have a greater number of useful problem solving techniques.

Schoenfeld observes that for the inexperienced problem-solver, generating actual methods of approaching a problem from the heuristic is not trivial. Usually, the statement of a heuristic is quite broad and contains few clues as to how one actually goes about using it. It is not, in itself, nearly precise enough to allow for unambiguous interpretation. However, for the expert, it does serve as a label attached to a closely related family of specific strategies.

Within this framework, Schoenfeld outlines three necessary conditions for the successful application of heuristics to problems:

a. a general understanding of what it means to apply a heuristic
b. sufficient grasp of the subject matter to understand the problem.
c. realization that the heuristic does in fact apply.

Schoenfeld points out that condition (c) is the most neglected aspect of teaching heuristic-based problem-solving. He therefore outlines a "managerial strategy" for the selection of heuristics which includes the following phases:

a. Analysis, which begins with reading the problem and ends with a good representation of the problem and an identification of the problem's mathematical context. Heuristics for analysis include drawing diagrams, examining special cases, and simplifying the problem.

b. Design, which keeps track of alternatives, and allocates problem-solving resources.

c. Exploration, which actually selects the heuristics to be used in the problem. This includes the consideration of essentially equivalent problems, slightly modified problems, or extensively modified problems.

d. Implementation, which actually does the calculations.

e. Verification, which checks in a variety of ways to see that the solution makes sense.

I find Schoenfeld's work quite compelling, because I believe that he does in fact touch the heart of the problem. I also believe that some of his work has definite pedagogical application in the introductory physics class. However, in general, he assumes subject material competence, and this is not something I can assume with the students with whom I am concerned. Furthermore, our knowledge bases are different.

Newell and Simon (1972) take a different approach to the study of human problem solving. They model the human problem solver as an information processor, and they identify certain invariants in this structure across all people. These invariants include the characteristics of long-term and short-term memory, the characteristics of its elementary information manipulating procedures, and the production-like and goal-seeking character of the executive program which sequences the execution of the elementary procedures {3}. In addition, they describe a computer system called GPS (General Problem Solver) which they set forth as an embodiment of their information processing model of the human problem solver.

In my view, this work is important in that it is the first to establish the utility of protocol analysis (analysis of subjects solving problems out loud) as a viable tool for investigating cognitive processes. GPS (and the theory underlying it) also provides a powerful language in which it is reasonably easy to represent certain problem solving processes. However, GPS is quite limited in the amount of domain-specific knowledge it can represent; hence it can handle only problems which do not require an extensive store of background knowledge {4}. In addition, their approach assumes that the simulation of a cognitive process (in terms of the output generated) implies an understanding of that process; this is not necessarily the case (cf., Lin (1979)).

Wickelgren (1974) reviews problem-solving heuristics derived from the fields of artificial intelligence and computer science. He discusses the following heuristics:

- drawing inferences from information (either explicit or implicit) that satisfies either or both of the following criteria: (a) successful inferences have been made in the past from the same type of information, and (b) the inferences concern the givens, the desired result, or other inferences made earlier.

- classifying transformation sequences into sets of equivalent sequences. A problem solution is a sequenced set of transformations, which act initially on the given information and result in a problem state which includes the answer. For some problems, many sets of transformations are equivalent. A determination of these equivalence classes cuts down drastically the number of transformation sequences one must examine to solve the problem.

- state evaluation and hill climbing. At any point in solving a problem, one must make decisions concerning which of the many possible "next steps" will be most fruitful. For some problems, it is possible to define a function which evaluates the fruitfulness of possible next steps; the solution then proceeds incrementally by following the path dictated by maximizing this function.

- subgoals. Many problems are too complex to be solved as they stand. Solving a simpler or less abstract or less general (but related) problem may offer insight into the appropriate method of solution. Alternatively, solving a problem in pieces may prove possible.

- contradictions. Sometimes, it is easier to disprove the contrapositive, or negate a part of the problem and show that this negation leads to a contradiction.

- working backward. Many problems ask for something specific as the answer; this may constrain the solution in such a way that it is possible to construct an inferential chain which will lead to the answer.

Wickelgren's presentation is lucid. However, it suffers from several inadequacies: the problems to which his examples apply these heuristics are, as a rule, problems which do not require much background knowledge; by contrast, the solution of physics problems requires much background knowledge. In addition, it does not address the question of how one selects the appropriate heuristic. Finally, these heuristics are themselves highly general, sometimes so general as to be useless in actual practice. Consequently, his work does not have much utility for the purposes of this thesis.

Larkin is the first to undertake detailed studies of problem solving in physics; much of this thesis is based heavily (in perspective if not in fact) on her earlier work.

For example, Larkin (1976) describes an information-processing model for expert problem-solving in physics. This model consists of a knowledge base and a strategist. The knowledge base contains procedures applicable to specific content areas; these procedures act on information given in the problem or created as a result of previous calculations, and generate new information related to the problem's solution. These procedures are organized in a net; therefore, execution of any particular procedure may lead naturally to the retrieval and execution of a number of associated procedures. The net is hierarchically organized, with a few top-level and broadly applicable "fundamental" procedures which serve as initial entry points for the strategist. Thus, the strategist need not consider the large number of individual procedures, but only the much smaller number of fundamental ones.

The strategist controls the execution of the knowledge base procedures, working at a low level of detail to select promising applications of fundamental procedures to particular aspects of the problem before becoming involved in implementing these segments.

The strategist works in three phases:

a. Analysis, during which fundamental procedures are used to redescribe the problem visually and verbally in terms of "key" quantities generally useful in the subject area. From various possibilities, the strategist selects for further consideration those fundamental procedures and aspects of the problem which seem to provide a simple and complete description of the problem.

b. Design, during which the strategist applies fundamental procedures to predict the unknown quantities which will appear during the implementation of a segment. The strategist then selects a set of segments which it predicts will produce equations which can be solved for the desired information.

c. Implementation, during which the selected segments are actually executed to produce equations, and these equations are combined to solve the problem."

In other work, this model is extended (cf., Larkin (1979a)), and some of its pedagogical implications are described (cf., Larkin (1979b)). However, as a general rule, Larkin's work concerns the analysis of expert behavior. Her analyses describe sharp contrasts to the behavior of good beginning students, who appear to work in a strictly linear fashion oriented towards direct calculation. However, the students with which this thesis is concerned cannot even be described as good beginning students.

Reif and Eylon (1979) study the relationship of the large-scale organization of knowledge to the performance of various tasks of understanding. For example, a thorough understanding of an argument would clearly involve the ability to reproduce the argument in detail; however, other abilities are encompassed as well. For example, a person should have the ability to:

- summarize the essentials of the argument
- correct mistakes in the argument

- adapt the argument to a similar but not identical problem
- state a piece of the argument given the immediately preceding piece

Reif and Eylon find that the hierarchical structuring of knowledge in net form (rather than as a simple tree or some linear structure) greatly facilitates performance on tasks which require a global or overall picture. Examples of these tasks are the first three items above.

They also find that optimal performance requires an organization of knowledge which is adapted to the specific domain from which that knowledge is drawn, and the purposes for which that knowledge is used. It turns out that top levels of the hierarchy are more easily retrieved, and therefore, it is also necessary to determine what specific knowledge belongs in which levels of the hierarchy.

It is interesting to note the direction in which a large amount of research seems to be headed {5}. The works of Polya and Wickelgren are examples of a power-based approach. Such an approach involves the application of very powerful, general procedural techniques for exploring problems too complex to be explored by trying all possible approaches. These techniques are domain-independent, and operate on many different types of large data bases. In addition, these data-bases are typically passive; information is represented in a propositional (rather than procedural) manner.

The works of Schoenfeld, Larkin, and Reif and Eylon are examples of an alternative approach. This alternative approach is knowledge-based, and it focuses on the representation of knowledge in a manner that permits and even facilitates its effective use. The problem solver must know explicitly how to use his knowledge; thus, general techniques complement domain-specific knowledge. By contrast to the power-based approach, a knowledge-based approach formulates the organizational problem as epistemological rather than as a matter of mathematical generality. It supposes, for example, that when a scientist solves a new problem, he engages a highly organized structure of especially appropriate facts, models, analogies, and planning mechanisms, as well as general problem-solving schemata.

The history of artificial intelligence illustrates a shift from the power-based approach stressing generality to the knowledge based approach stressing domain specificity. An important part of the reason for this shift has been the realization that power-based techniques are inherently slow, and knowledge-based techniques potentially much faster. Thus, since observation suggests that experts do in fact solve standard problems fairly quickly, it seems plausible that experts do in fact use knowledge based approaches. Indeed, all current work involving expert behavior of which I have knowledge does take a knowledge-based approach.

Notes

1. The discussion on Piaget which follows is taken largely from Ginsburg and Opper (1969).

2. Poisson's spot is the point of light which appears at the center of a back-lighted disk due to the constructive interference of light diffracted around its edges.

3. A production refers to a condition-action pair: when some particular condition is realized, some corresponding action is taken. Goal-seeking refers to the process of finding operators which will reduce the difference between the current problem state and the desired goal state.

4. This is not to say that no information processing model can represent domain-specific knowledge; Bhaskar and Simon (1977) describe a GPS-like system for solving problems in engineering thermodynamics.

5. The following discussion is adapted largely from Goldstein and Papert (1976).

The overall multivariate Chi-square test of the multiple regression of the 17 learning criteria on the 7 independent variables is highly significant.

- H. Walberg and A. Rothman

Chapter 3 - Research Approach

Chapter 1 (Introduction) discussed the domain which this thesis will concern: problem-solving in introductory physics courses. In addition, it gave a brief sketch of the research procedures used in this thesis. This chapter discusses these research procedures in the context of the larger picture of psychological research in general. I will discuss two approaches to psychological research (one recent, one not-so-recent), and then set forth my preferred approach against this historical perspective.

Section 3.1 - Historical Perspective

3.1.1 - A Concern with Behavior

The study of problem-solving in physics courses is fairly recent. The majority of problem-solving work in psychology over the years has concerned relatively simple problems, or problems which require creative thinking for their solution.

(By my definition, a simple problem is a puzzle-like problem which is soluble without the use of specialized factual knowledge, e.g., physics. Such problems typically specify the constraints to which the solution must conform, and provide explicitly most (if not all) of the background knowledge required for the solution. A problem which requires creative thinking is simply a problem whose solution is not straight-forward. In addition, the generation of a correct solution to such a problem requires (by definition) the use of specialized strategies not easily adaptable to other problems.)

Problem solving in general is itself a subset of cognitive psychology, which studies the entire range of human cognitive phenomena: perception, language, thinking, memory. Besides problem solving, typical tasks include paired-associate learning (memorizing unrelated pairs of words), recall of nonsense syllables, and perception of optical illusions.

A quick scan of any cognitive psychology journal will illustrate that the majority of articles concern these relatively primitive tasks {1}. Indeed, from a historical point of view, the primitive nature of these tasks lent itself to the development of an approach to psychology that mirrored the approach to the physical sciences; thus, behaviorism was born. Briefly, behaviorism asserts that only phenomena which can be directly observed have relevance to a theory of human cognition; process, as a "hidden variable", is irrelevant.

Therefore, from the behaviorist point of view, the appropriate method of psychological investigation is to search for correlations between various stimuli and responses. As such, behaviorism makes extensive use of the classical paradigm of the physical sciences: quantitative, general, abstract, detached, objective.

Behaviorism has very strong roots in the psychological community, and only within the last thirty or so years has it been seriously challenged in the United States. Its basic premises often retain strong influence, especially in fields far removed from psychology. For example, consider the following distilled, over-simplified version of what seems to be the most widely accepted theory of learning physics among physics teachers:

A student learns physics through reinforcement received when he exhibits expert behavior, e.g., has calculated the same answer to a problem that the book gives. Textbooks give answers at the back of the book, generally without mention of the problem-solving process involved. Textbook solutions of example problems are polished, with little explicit discussion of possible pitfalls, or how the student might approach a problem.

Teachers spend a large part of their time in "getting the story straight" and "showing the students the right way" to do a problem. Generally, they do not anticipate (or try to anticipate) what questions or difficulties students might have with the material. When a student asks "But how did you know to do that?", the teacher often responds that "With experience, you get to know these things." or "Practice makes perfect."

The situation is in many ways similar to the manner in which chicken farmers learn to differentiate between male and female chicks. A student chicken farmer goes to a school, where he is paired with a teacher. With the student watching, the teacher examines each chick silently, and then identifies the chick as male of female. At the end of this process, the student is able to differentiate between male and female chicks with 99% accuracy, but is completely unable to articulate the difference between male and female chicks (cf., Lunn (1948)).

Many physicists claim that it is only possible to teach the formal content of physics, and that the student must acquire the "tacit" components independently. For example:

> "An introduction to classical mechanics. Space and time: straight line kinematics; motion in a plane; forces and equilibrium; experimental basis of Newton's laws; particle dynamics; universal gravitation; collisions and conservation laws; work and potential energy; vibrational motion; conservative forces; inertial forces and non-inertial frames; central force motions; rigid bodies and rotational dynamics."

This quote from the 1978 M.I.T. catalog description of Physics I characterizes the substance of this introductory course as the formal subject matter - physics. It does NOT mention what might be called "thinking like a physicist": arguments from general principles, the reduction of a complex physical situation to a tractable mathematical form, and the formulation of reasonable questions, just to name a few items.

The approach implied by this catalog excerpt shares with behaviorism the notion that teaching is done by example - the student learns essentially by imitation. While it is difficult to argue with this notion, it begs the question from a pedagogical standpoint. What does "practice makes perfect" mean operationally? <u>How</u> does it help? An expert knows "more facts", but he can also "use them more effectively" as a result of his "better understanding". What does "more effective understanding" mean? How is this ability different from a beginner's?

3.1.2 - Process is Relevant

From time to time, an occasional physicist has suggested the importance of process as well as product (cf., Strandberg (1958)). However, it has been only in the last several years that a substantial concern with process has become discernible. In particular, a number of physicists (cf., Karplus (1977)) have recently attempted to apply Piaget's theory of intellectual development to the forms of reasoning required to understand physics in a real and meaningful way.

I take the point of view that process is relevant, and in fact is the only way to understand what a student can and cannot do. Cognitive processes can be inferred from directly observable data, just as atoms could be inferred many centuries before instruments could detect them directly.

Piaget's theory is based on his discovery that the "wrong" answers of children to certain questions exhibit remarkable regularities. From this, he infers that the manner in which children think is qualitatively different from the manner in which adults think.

Similarly, in the course of substantial involvement with students of introductory physics, I have found that they encounter the same difficulties year after year. These difficulties cluster, and they do not result from a simple deficit of equations.

This thesis is not an application of Piaget's theory to problem solving in physics. However, it is similar in that I pay close attention to process - what students are thinking as they encounter difficulties. I hope to document a claim that the manner in which beginning students think - their reasoning styles, approaches to problems, notions of what constitute explanations - differ qualitatively from the manner in which expert professionals think.

It is important to note that the difficulties I will discuss are both content-free and content-dependent. They are content-free in that these difficulties indicate failures of rather general reasoning strategies. However, they are content-dependent in that the student may well have mastered a reasoning strategy with respect to other types of knowledge, but may be unable to employ this strategy in a particular context.

The presentation contained in this thesis would be more parsimonious if these difficulties were in fact <u>either</u> content-free or content-dependent. However, I find it impossible to make such a claim. Indeed, I find that a student's proficiency in the use of a particular reasoning strategy may vary. [I will return to this point in Section 4.5 (<u>The Architecture of the Human Problem Solver</u>).]

If this statement is true, then the obvious question is "On what does the level of performance depend?" I suggest that it depends (at least to some extent) on the individual's familiarity with the subject material under consideration - somehow, the unfamiliarity with the subject material interferes with the proper functioning of the general reasoning strategy as applied to the given domain.

Here is an example.

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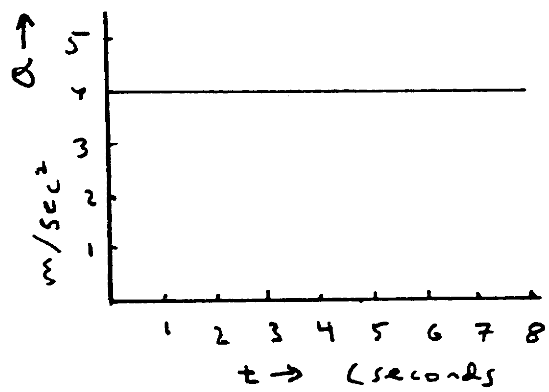
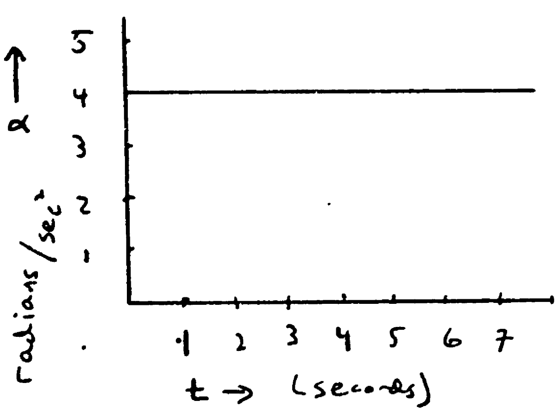
<u>Example 3.1.2.a</u>

Students were given the following problems.



The graph at the left gives the acceleration of a particle as a function of time. Write an equation describing the relationship between the position of the particle and the time t. Assume that when t = 0, x = 0 and v = 0.



The graph at the left gives the angular acceleration of a particle as a function of time. Write an equation describing the relationship between the angular position θ of the particle and the time t. Assume that when t = 0, $\theta = 0$ and $\omega = 0$.

The first problem, given quite early in the term, asks the student to write a general relationship involving the variables x, a, and t. The second problem, given several weeks later (i.e., after the weakest students dropped out, and after much more exposure to physics material), substituted the variables θ, α, and t. The average score on the second problem was half the average score on the first problem.

****

In both problems, the required reasoning strategy is the same, but the results are substantially different. The only differences between these two problems are the different symbols (Greek vs Roman letters) and the different content implied by the difference between rotational and rectilinear quantities. In either case, it is clear that something connected to the specific problem (and independent of the required reasoning strategy) affected the reasoning process students used.

3.1.3 - Two Concerns for Process

Within the concern for process, there are at least two different perspectives: I call them "deterministic" and "descriptive" {2}. Both are observationally based, but a deterministic approach requires a strongly falsifiable operating model {3} of some cognitive phenomena, whereas a descriptive approach depends on expert consensus to validate its models. In addition, deterministic models seek to explain cognitive functions; descriptive models seek to describe them.

By tradition, researchers (especially in the hard sciences) have strong preferences for deterministic models of phenomena. A deterministic model is one in which the model's behavior follows deductively with no uncertainty (or at least rigorously calculable probabilities) from the starting assumptions and initial parameter settings. As such, these models have overtones of predicting (and hence explaining) behavior.

By varying the initial parameters, one hopes that the model's behavior will change in a way that corresponds to the subject's behavior; in this way, deterministic models achieve some type of generality. They are also strongly falsifiable; they can be proved wrong or incomplete by virtue of an incorrect prediction - a mismatch between subject and model behavior.

Description offers an alternative to determinism. However, it is important to distinguish between a descriptive model (which refers to a specific set of data) and the model-building framework from which the model emerges. This framework is essentially a perspective which provides some basic vocabulary, concepts, relations, mechanisms and principles from which individual descriptive models may be constructed.

Frameworks apply to classes of problems, in the sense that a framework can be used to generate a model to deal with each class of problem, each model being structurally different. A framework makes predictions in the sense that it claims that its basic building blocks are sufficient to account for the most essential features of a subject's behavior. However, it does not make predictions concerning the specific organization of and relationships between these items. Descriptive frameworks also assist in the coherent organization of data, facilitating the discovery of principles with explanatory power.

Descriptive models apply to specific sets of data such as interviews, protocols, and exam papers, and do not necessarily make specific predictions concerning subsequent interviews, protocols, etc. From his model-building framework, an investigator selects and arranges the appropriate building blocks as he constructs a model of any interesting behavior.

On the other hand, descriptive models may be general in that they may make probabilistic predictions through the generation of profiles of one individual over a set of tasks or many individuals over a few tasks. For example, a profile may say that a student did X in 7 out of 10 protocols; hence we might guess (or "predict") that he tends to do X in most such situations.

3.1.4 - My Choice

I have chosen to take the descriptive route, for the primary reason that deterministic models cannot in general capture the influences of context, environment, and affect on the substance of learning. I am also not convinced that student difficulties are rule-governed. In short, deterministic models arbitrarily separate process from context, and I believe this separation is fundamentally incorrect.

Furthermore, the reality of psychological research is that of the existence of competing psychological paradigms. This implies that agreement on the interpretation of a given set of data is difficult if not impossible. White (1977) offers the following account:

> We are interested in what is [really] factual for children. So we subject the manifest fact to analysis to try to detect the latent fact. To picture the analytic process most clearly, imagine that you do not like.. [a particular] journal report, do not want to believe it, are committed to a very different belief... So you seek to cast doubt on the report. You might want to see if there were decent controls, if the situation was a plausible one for children, if there was anything funny about the recording of responses...
>
> Very few journal articles could survive so merciless an attack. Most psychological research is simply not that airtight.... [The research report] has told you something, but you do not know whether that something is.. about children or about the game that the experimenter played with the children or.... about the... game that investigators play [when they publish].

> [More generally,] one cannot directly observe the behaviors of children but rather must observe the properties of games reflecting a mixture of children's behavior and one's own behavior.

In short, the investigator's perspective influences significantly what he considers important. Furthermore, his own behavior is interwoven into his account of his observations. Therefore, verification becomes quite problematical. If the model makes no predictions that can be confirmed or denied, and it is likely that large groups of people may disagree substantially, then how can a given descriptive cognitive-process model be said to mirror reality in any reasonable manner?

Section 3.2 - Research Approach

I personally find this question unanswerable. Because of all the competing claims and counter-claims, it is hard to make a genuine commitment to any particular psychological paradigm. I therefore propose a different question: "What must we know about the student in order to have pedagogically useful insights into his way of thinking?"

This is an engineering approach which makes use of empirical descriptions of specific situations to guide teaching activities related to these situations. Abstractions of process can and do arise to help organize these descriptions, but these abstractions are working tools and rules of thumb and not the fundamental cornerstones of a rigorous theory.

Finally, this engineering approach (as contrasted to the approach of developing a deep theory of cognitive process) is far closer to the day-to-day concern of practicing teachers. Therefore, I have chosen to explore difficulties which actually arise in the course of real-life teaching and learning situations. As a result, this work is non-rigorous, anecdotal, and improperly controlled. However, as an exploratory work, it is quite rich in illustrating the range of difficulties that students actually encounter.

I have abandoned the classical paradigm of education research: controlled experiments, pre- and post-tests, objective, statistical analysis from a distance, and most of all, emphasis on the general and abstract. My primary reason is that emphasis on the abstract almost necessarily implies an insensitivity to local settings and individual differences; however, we as teachers are concerned precisely with our programs and specific individuals.

My approach assumes that education research should be directed primarily towards teachers who will use the findings in practice, and only secondarily to other education researchers. (After all, if education research doesn't ultimately improve education, who cares?) Any audience to which research is directed evaluates that research in the light of its own experience. I assume open-minded listeners and readers. If this assumption is warranted, then these people are concerned with the implications of my research for their own work, and will listen thoughtfully. If this assumption is not warranted, then as White (1977) points out above, very few psychological studies could survive an assault by someone determined to find fault with them; I have no hope of winning over such critics anyway.

I find that only case analyses (either of specific programs or specific individuals) result in insight, because then you learn something about those cases. Some people say that recent work in education and cognition is really getting to the fundamentals of human thought. People have been saying that since Plato, and I don't see any reason why anything is really different now: hence, my empirical engineering approach.

This thesis attempts to develop a diagnostic framework, which provides descriptors usable in the diagnosis of the problem-solving behavior of individual students. Particular collections of these descriptors, i.e., diagnostic models, provide "snapshots" of a subject's behavior, usually ignoring causal mechanisms or possible time evolution. Two examples of such frameworks are Bloom and Broder (1950) [a study of the problem-solving behavior of college students which identifies a number of their most common difficulties] and Polya (1945) [a description of many heuristics commonly employed by expert mathematicians].

This thesis is a broad-brush mapping of previously unexplored issues. As such, it covers a fairly wide range of topics, is impressionistic and anecdotal, and illuminates a few or several (but not all) interesting features of the terrain. Therefore, I have not expended much effort on questions of verification; instead, I have used the relatively cheap method of consulting with my committee and other physicists. However, I do believe that my professional training as a physicist does lead me to make observations which other physicists would also make. [See Section 10.1 for a further discussion of how this study might be made more rigorous.]

I gather data under natural conditions, i.e., in classes, lectures, tutorials, and from students and teachers of physics; therefore, this thesis is observationally based, and not experimentally based. Rather than expend effort trying to design and implement a clean, "ideal" environment for experimental data collection, I instead direct my efforts towards eliciting what data I can gather in a natural environment and attempt to generate relatively simple descriptions of behavior in this complex environment.

For example, I often maximize (hopefully judiciously) interaction between observer and subject, rather than minimize it. When I am searching for a particular response, I ask leading questions. When I want the student to elaborate without external direction, I ask open-ended questions. To minimize my students' drawing cues from my questions, I also ask them to justify even correct statements. Of course, there is the usual danger that this interaction may unduly influence the subject's thought, but I gain insights I would not have otherwise gained were I to be a passive, silent observer. In addition, I discharge my responsibilities as a teacher as I try to articulate remedial actions for the student.

I gather my data any way I can get it. When possible, I have tape-recorded discussions with students. In other instances, when I did not have access to a tape recorder, I took notes in real time, much as a reporter interviewing a person would. The examples I present in this thesis are drawn from this pool of data; the dialogue of each example is a transcript either taken from my tapes but edited to capture the essence of a discussion, or reconstructed from my notes.

The exploratory nature of this work means that I examine carefully the behavior of small numbers of subjects (tens), rather than examine superficially the behavior of many (hundreds or more). In this manner, I preserve both flexibility in my data-taking and also the essential human qualities of that data, gaining qualitative insight into processes which cannot be measured quantitatively.

The standard physics problems (e.g., one taken from an introductory physics text) which I observe students solving consist of some text and possibly a picture. They describe some physical system in a particular environment, present certain data relevant to that description, and ask the student to calculate some other piece of information related to the physical situation in question.

This calculation may be either symbolic or numerical, and it is subject to at least two constraints:

a. it must be calculated using the information contained in the problem, and knowledge about what we call physics. Therefore, guesswork and divine revelation are not legitimate approaches.

b. it must present the result(s) in terms of known information, usually what is given explicitly in the statement of the problem. However, it is usually legitimate to define additional symbols in terms of the givens, and use these symbols in the answer as presented.

These problems are also the actual problems assigned students in their introductory physics courses, typically Physics I (Newtonian mechanics) and Physics II (electricity and magnetism). These courses were taught in fairly standard ways, involving lectures and discussion sections, weekly problem sets, four exams per term, and standard textbooks such as Halliday and Resnick (1978), Sears, Zemansky, and Young (1978), and Tipler (1976). The examples I present arise from a fairly small fraction of the assigned problems, because students often have difficulty with the same problem; however, they also often have difficulties of different types. In addition, these examples are taken largely from mechanics. There is no fundamental reason for this - I have taught mechanics (Physics I) three times, and electricity and magnetism (Physics II) only once; this is approximately the ratio of examples from each domain.

Finally, the populations which I examine are largely self-selected. I draw the majority of my observations from students who seek me out because I advertised myself as a person who gives physics help. As students who freely acknowledge that they need help, these students tend to be the weaker students. This has been a deliberate choice. One good way to investigate interesting phenomena is to start with an extreme case, and work back to the middle; I have tried to do this. Furthermore, I felt that as a general rule, it is precisely these students from whom we rarely hear and about whom we know the least. These students vote with their feet, drop out of view, and often are not the central concern of teachers. It is important to know about the "losers" as well as the "winners", i.e., those who will learn physics despite anything we do to them.

On the other hand, I do not know how much weaker these students are when compared to the best, or how much better they are when compared to the worst. I am certainly dealing with a select group of students: those from the Massachusetts Institute of Technology. However, I remain uncertain about the impact of this selection; I have observed similar difficulties in students at other schools. I suspect that the difficulties I have uncovered are characteristic of the weaker students of any prestigious university; however, it might be that students with substantially weaker backgrounds would not even realize some of the pitfalls into which they might fall {4}.

Notes

1. I use the word "primitive" in two senses: "primitive" as basic or innate and hard to decompose, and primitive as simple by comparison to much more cognitively demanding tasks such as physics problems.

2. This discussion is adapted from Lin (1979).

3. Here, "falsifiable model" refers to a model which can be proven wrong in unambiguous terms.

4. An apocryphical (?) story about Wolfgang Pauli reports his comment concerning crank theories: "Crank theories aren't even wrong!".

Until it is actually demonstrated, one tends to forget the great difference between the merely competent amateur and the truly expert professional.

- Linus Van Pelt

Chapter 4 - Background

This chapter contains a large amount of the background material needed to understand the demons responsible for student difficulties. As Part II will illustrate, these demons arise from student inability to employ physicist-like reasoning approaches, and teacher inability to recognize the subtlety of many of their own assumptions. It is the function of this chapter to articulate some of the often unstated assumptions underlying the physicist's style of problem-solving analysis. More specifically, this chapter discusses the following topics:

Section 4.1 - the values of science as they relate to subject content and approach
Section 4.2 - analytic thought and the meta-principles of physics
Section 4.3 - a classification of relations useful in the solution of physics problems
Section 4.4 - on physics problems: states and time evolution
Section 4.5 - the architecture of the human being as problem solver

Sections 4.1 and 4.2 articulate some of the expert physicist's tacit knowledge, knowledge which underlies his ways of doing science, but which he has generally never been explicitly taught. Section 4.3 develops a classification scheme for the formalism of physics which the physicist uses in his professional research and which the physics teacher (often the same person!) tries to teach his students to use. Section 4.4 identifies some of the salient aspects of the problem solving domain in which students generally learn to use this formalism.

Sections 4.3 and 4.4 are similar to Sections 4.1 and 4.2 in that all involve knowledge which is generally unstated. However, Sections 4.3 and 4.4 are different in that they concern directly the tools and one domain of the discipline itself, rather than a way of looking at it. In addition, they serve as organizers for some rather specific difficulties - by contrast, Sections 4.1 and 4.2 pervade all of Part II.

Section 4.5 is completely separate from the other four - it introduces a few notions from information-processing psychology that help to interpret many of the difficulties illustrated in Part II.

Each section presents a particular perspective from which certain problem-solving difficulties can be understood. Therefore, each section should be considered a stand-alone essay. Furthermore, no section is exhaustive or totally rigorous; rather, each provides working concepts and vocabulary which will prove useful in later discussion.

Section 4.1 - The Values of Science

This thesis explores in detail some of the difficulties students of introductory physics encounter in their attempts to solve problems typically found in physics courses. I have found that many of these difficulties touch on some very deep epistemological questions, and it is the purpose of this section to discuss these questions directly {1}.

Western science places a high premium on objectivity. This necessarily implies that science must be subject to public scrutiny and criticism, for only when arguments are public can they be searched for error. This demand for objectivity imposes certain conditions on all arguments.

In particular, all arguments must follow a certain set of commonly accepted rules (e.g., accepted axioms, fundamental principles of physical law, the rules of mathematics and logic), and must involve only terms which are themselves public and well-defined. Without this kind of explicitness, arguments take on a very ephemeral quality, or as one physics professor grading an exam put it, "From what you've written, I can't tell if you're developing a physical argument, or using Divine Revelation - no credit for the latter."

The demand that knowledge should be public applies only to the verification aspect of doing science. Theory or experiment construction is usually a private affair - it is rare that one can read about the historical development of a published theory or experiment.

An example is the following: journal referees are usually critical of papers which use assumptions and procedures which are not part of the theoretical framework shared by the community of physicists. However, there is no supposition that what is contained in the papers corresponds to anything the authors actually did in first arriving at their conclusions. In other words, a paper can be (and often is) a rationalized reconstruction of what actually happened; in principle, this reconstruction enables others to convince themselves of a paper's validity. It is in this sense that science is public and objective.

This view of verification and construction has its counterpart in teaching, and teachers apply many of the same criteria for "objectivity" that they apply to professional scientific works. In particular, they insist that the final problem solutions, which are to the student as the published paper is to the professional, be public. They are far less concerned about how a student gets started and what he does in trying to achieve his final solution, and this is reflected in their relative lack of attention to teaching about the creative process. In short, problem approach is private and tacit {2}.

Insistence on public objectivity implies that physical theory must be broadly applicable, since such a theory allows assumptions and procedures to be shared to a substantial degree, thereby facilitating communication. [In many ways, a physical theory should be like a computer program which accepts different data and produces different results, but employs the same procedures for deciding which path the program should follow. This is not to say that a theory is or should be entirely algorithmic, but established physical theory should be applied in more or less the same way by different people, even in different situations.]

Correspondingly, in their teaching, physicists attempt to emphasize the broad applicability of these fundamental laws. In particular, introductory physics courses usually concern the domain of well-established theory. Labs and problem sets usually stress the application of this theory; very rarely do they test the limits of the theory.

In addition to being objective, physics should be predictive. Of course, predictions are always predictions about something, and so physics must make contact with specific situations. Indeed, experiments (which are the final test of any theory) are always experiments about certain situations. Consequently, physics teachers stress problem-solving a great deal, for it is in the problem-solving effort that they believe a student learns how the theory called physics corresponds to the real world.

However, the real world is almost infinitely complex. Physics seeks to achieve an understanding of the structure underlying that complexity, with the expectation that this structure will be appear in many different domains. Since the real world is so complex, physics must construct representations, which describe to an acceptable degree of accuracy the aspects of the real world that are of interest.

These representations may be a set of equations, a verbal description set in text, an actual scale model, a heuristic model (e.g. billiard ball model of a gas), or a set of pictures, but they all share one fundamental property: they all throw away certain aspects of the situation represent in order to achieve a simplicity not present in the original, but still strive to maintain significant resemblances to it. A particular representation of a physical situation is a statement of the key features of that situation. These features are determined by the requirements of the physical theory used to analyze the situation.

A representation has two purposes. First, by construction, it is simpler than the original. It focuses our attention on the relevant characteristics of the original, and gives us a simplified version which we can manipulate to generate results which (hopefully) correspond to other aspects of the original. In other words, to solve a problem posed by a real physical situation, we generate a representation from the real situation, solve the represented problem, and map the results thereby obtained back onto the original situation.

Second, the use of particular representations allows us to make use of general theories in the analysis of particular physical situations. A theory contains terms (for example, F, m, a, KE) which correspond to quantities (e.g., masses m_1, m_2, and m_3) specific to individual physical situations. These terms are manipulated by the theory in order to generate conclusions regarding other terms in the theory.

When faced with a specific situation, one searches for quantities of that situation which correspond to the terms of the theory applicable to that situation. The generality of the theory lies (at least partially) in the fact that this search for correspondences is essentially the same over a wide variety of situations; one follows for the most part the same set of rules.

I have emphasized the role of representation at length because representations play a very important role in the analysis of all physical problems, and yet textbooks and teachers rarely discuss its central role explicitly. Furthermore, many texts and teachers restrict their view of representation to what one might call a "mechanical" model, e.g., a billiard ball model for a gas, or a spring model for atoms in a lattice. Perhaps this is because they believe that the general notion of representation is easily accessible to most students, but as later examples shall illustrate, this is not the case.

A last aspect of physics is that it should generally be quantitative. A quantitative prediction is reasonably well-defined, its meaning precise and explicit. It allows the notion of a theory that is "more" or less" correct. In addition, one gains the ability to make fine distinctions. This emphasis on quantitative argument shows up in the problems typically assigned to beginning physics students - only rarely is a student assigned a qualitative problem.

The approach discussed above differs sharply from approaches typically employed in everyday life. Rarely in everyday life doès a person make decisions or predictions on the basis of arguments from explicit statements of general principle. This is the essence of physics, but it is a style of working to which the typical beginning student quite unaccustomed. Indeed, everyday life is quite successful with the construction of a different "mini-theory", i.e., a different set of rules, for each different life situation.

However, the success of this type of reasoning in everyday life means that there is no reason to merge a collection of disparate mini-theories under one more general theory. Consequently, contradictions can be ignored. Each seemingly contradictory situation becomes a special case with its own explanation. Contradictions and inconsistencies are meaningful only in the context of a potentially unifying theoretical structure.

Since physics stresses analysis based on the ideas discussed above (objectivity, prediction, quantification, modeling, generality), analysis that is based on descriptions which just happen to work, or knowledge grounded solely on personal experience or familiar settings is not sufficient from the physicist's point of view, though it may be quite sufficient for the lay person. The lay person may be content with an "explanation" such as "The car goes forward because you step on the gas" but the physicist wants to hear "The road pushes the car forward" {3}. Indeed, many difficulties in problem-solving among beginning students can be interpreted as differences between physicist and lay-person epistemologies.

Physics-based analysis involves a model which represents the physical situation in question. However, it is important to realize that the model that a practicing physicist creates is different from the model that a physics student creates. In the actual practice of physics, a physicist has one modeling step to perform: he must reduce a physical situation to a model, and then work with the model. By contrast, a physics student must cope with two levels of modeling. At the first level, the author of the problem takes a physical situation, models it for the student, and presents it to the student in an abstracted form (in words, pictures, and equations). At the second level, the student must pick out the key features of the physical situation from the abstracted description.

In relatively simple problems (e.g., those involving point particles) the second level is a mere formality: the abstracted form contains just those elements needed to solve the problem and the structure of the problem (or the physical situation being modeled) allows only one (or at most a few) ways of putting these elements together. However, while the abstracted form for more complex problems may still contain just the necessary elements, its structure (or corresponding situation) may admit a multitude of ways of assembling the elements.

We will see that this has concrete significance. A student may (unknowingly) focus on the physical situation which corresponds to the abstracted representation, without regard for the representation itself. In other words, he may attempt to interpret his idealized results in light of the actual physical situation without awareness of the approximations involved. If the student has not recognized conceptually the effect of modeling a physical situation, he may confuse one part with another or include parts omitted from the model.

I have discussed above some of the values of science as practiced by physicists, and emphasized the role of theory and representations. However, it is important to note that physicists share these values for good reasons, though values cannot be chosen for logical reasons alone.

In particular, physicists choose to ask certain questions. For example, they choose to ask quantitative questions about physical situations, but they realize that the answers they receive are never exact; the continuous nature of the world forces some degree of imprecision nor inaccuracy on all quantitative answers. In turn, the degree of inaccuracy or imprecision we choose to accept determines in large measure the sophistication of the representation we use.

If we are willing to accept large imprecision, we can make do with simple representations. Problems involving such representations are amenable to analyses based on general principles. These analyses are rapid and convenient, and involve low levels of cost and risk. By comparison, if we demand small imprecision, complex representations are necessary, and these require empirical as well as theoretical formulations. Empirical formulations are slow and inconvenient, and are costly as well.

For example, in a few minutes on the back of an envelope, any physicist can estimate to an order of magnitude various important parameters of a commercial airplane. The actual design of that airplane requires millions of dollars of test equipment, years of empirical test, and risk to the lives of test pilots.

This is not to say that empirical formulations do not involve general theoretical principles. Since the problem which requires empirical formulation is by assumption complex, it makes no sense to take shots in the dark, hoping to arrive by chance at an appropriate solution. Instead, the representation starts simply, and hence is tractable on a theoretical basis. As the representation evolves into a more complex entity, the initial solution provided by theory must be supplemented by empiricism. Thus, the initial theoretical solution based on general principles provides a starting point for subsequent empirical modifications.

General formulations are also a necessary component of the emphasis on public verifiability. By definition, a general formulation involves a particular theory, and correspondence rules which specify how a wide variety of physical situations should map onto that theory. By contrast, problem-dependent formulations would employ correspondence rules quite specific to particular problems. Any solution based on the latter would be suspect, for a couple of reasons:

a. it would not be publically verifiable, since the correspondence rules would be specific to the problem, and hence not part of a community-shared theory.

b. it would create a lack of confidence in the problem solver's ability to solve other, different problems, and the community could not know that the solution was not due to chance.

In short, the values that physicists share are not entirely arbitrary and independent of each other; they result from a mix of human choice and rational necessity.

Section 4.2 - Analytic Thought: On the Meta-Principles of Physics

Physics is formulated in terms of various quantitative principles such as Newton's laws and Maxwell's equations. However, these fundamental principles rest on certain meta-principles which are an intimate part of the analytic thought required to think like a physicist.

4.2.1 - Separation of System and Environment

A very useful notion in physics is the separation of the universe into a system and an environment consisting only of certain features of the system's surround. It is this separation which is at the root of many textbook descriptions of "identifying a system" as a part of doing $\Sigma F=ma$ problems: m is the system, and each F is one feature of interest in the surround of m. However, system-environment separation is apparent in all parts of classical physics, even though most textbooks stress this separation only when discussing Newton's laws. For example, it applies to all conservation law problems: to conserve X, one must isolate the system with respect to X. It also applies to classical field theory; A, a finite distance from B, sets up a field in all space. To calculate A's effects on B, we separate the universe into the system B and environment of the field around B.

4.2.2 - Cause and Effect

The separation of the universe into system and environment leads quite naturally to the idea of cause and effect. Cause A and effect B is of course an approximation to the more accurate notion of A interacting with B. Nevertheless, it is a very useful approximation in cases in which the interaction can be assumed to go in only one way: A affects B, but B does not affect A. In other words, cause-effect assumes that a decoupling of interactions is possible. This is indeed the case for the vast majority of situations posed in introductory physics. Through successive consideration of different system-environment sets, we can construct chains of cause (change in system environment) and effect (change in system behavior).

Cause and effect implies, for example, that events that happen far away from a particle do not affect its motion; only events in the immediate vicinity of the particle can do so, and the particular causes of these nearby events do not matter. Therefore, I can affect an object's motion only by pushing on it (with a contact force or a field); I can heat an object only by transferring energy across its boundaries.

It is also the inability to identify chains of cause and effect which makes thermal physics much more difficult. For example, classical thermodynamics deals with global changes in a macroscopic system, and it is exceedingly difficult to visualize how changes in one part of a thermodynamic system propagate to other parts. This difficulty is greatly alleviated by the introduction of a microscopic model which allows the identification and study of small and well defined systems.

A particularly salient example of cause and effect is the separation between dynamics and kinematics. Kinematics describes only the motion of particles; as such, it is independent of cause or environment. By contrast, dynamics concerns the reasons for that motion - thus, dynamics does concern the relation of cause (forces) to effect (kinematics).

4.2.3 - Cause and Effect vs Divine Intervention

The assumption of cause and effect leads to the point of view that effects have definite causes. A system knows that the outside world changes only through changes in its immediate environment. Conversely, the universe can affect a system only by changing the environment in the immediate vicinity of the system. Thus, in analyzing a particular situation, we must be able to choose a well-defined system on which to focus attention, throw away everything outside the system, and account for the effect of the environment by consideration of transfers of energy, momentum, mass, and charge across the system boundaries.

An alternative (and incorrect) point of view is that it is possible to know "what is really happening" from a kind of overall, privileged perspective. This point of view requires an omniscient observer, who knows everything there is to know about a given situation, and somehow transmits this knowledge to the system. In this way, changes at a distance can affect a system without immediate changes it the system's environment.

4.2.4 - Discussion and Examples

The items discussed in the last three sub-sections are elements of what I will call analytic thought. More generally, analytic thought is sequential and deductive in nature. It requires a consideration of one particular and definite aspect of a problem after another, and these aspects must be fitted together in a mutually consistent manner. [Again, this is not to say that analytic thought is necessarily the manner in which the solutions to problems are actually created; it is, however, at the very least, a requirement of the verification side of what might be quite private in origin.]

By contrast, non-analytic thought tends to be diffuse. Non-analytic thought considers a situation as an undifferentiated whole. It can result in imprecision and vagueness of expression, and vacillation (conscious or unconscious) between alternatives. A person who thinks non-analytically may never be clear or definite as to the constituents of a system he chooses. He may not construct chains of cause and effect; instead, he may blend system and environment, arguing that "everything affects everything else."

(It should be noted that the distinction between analytic and non-analytic reasoning is an idealization. In actual practice, students usually exhibit a mixture of the two styles, resulting in both the incorrect application of analytic reasoning and the application of non-analytic reasoning.)

The examples which follow illustrate various ways in which student demonstrate non-analytic thought.

****

Example 4.2.4.a

The discussion concerns a man pulling a rope tied to a wall.

H: I'm pulling on the rope. The rope isn't moving. Therefore, there must be a force on the other side.

S: When you pull, do you pull on the rope, or on the wall? When the rope is connected to the wall, are you pulling on both, or what's the reaction and what's the action.

H: Think about a chain of events. I pull on the rope and the rope isn't moving. Hence, by F = ma, another force on the other side must be pulling the other way. Since the wall pulls on the rope, by the 3rd law, the rope pulls on the wall.

S: But because the wall pulls on the rope, isn't the wall also pulling on the person?

This student's last question indicates that he does not believe in the separation of system and environment. Instead, he attempts to use the knowledge of an omniscient observer who knows that the wall is "really" pulling on the rope and therefore on the person.

****

Example 4.2.4.b

The problem under discussion required an estimation of the smallest weight a person could sense directly.

H: What's the smallest weight you can feel? A paper clip?

S: You can feel a paper clip in your hand, but you can't feel its weight.

H: How can you feel it in your hand?

S: It's pressing down.

H: Where does that weight come from?

S: Its weight??

This student draws a distinction between what makes him feel the paper clip, and its weight. He wishes to consider cause (the weight of the paper clip) and effect (what he feels in his hand) as two independent entities. While it could be argued that the student might have been thinking of mechanisms other than weight as responsible for his tactile sensation, I know from other experience with this student that he did not have this level of sophistication. In addition, he does commit himself by his statement that the paper clip presses down on his hand.

****

Example 4.2.4.c

The topic under discussion is the kinematics of circular motion.

H: An object moves with speed v in a circle of radius r. What is its acceleration?

S: v^2/r into the center.

H: OK. Now, let's say the object is a car going at speed v around a circular road banked at an angle θ at a radius r. What is its acceleration?

S: v^2/r down the slope.

H: Why not into the center?

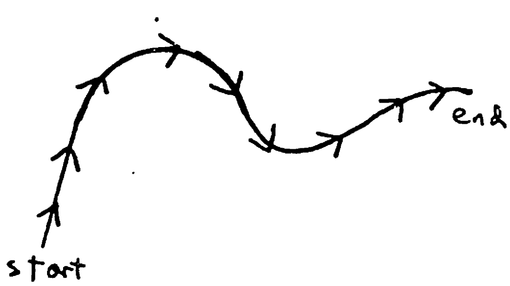
S: Because it's on a slope.

This student does not clearly separate kinematics from dynamics. He attempts to account for information obtained from the omniscient observer who knows that the car is "really" on a slope.

****

Example 4.2.4.d

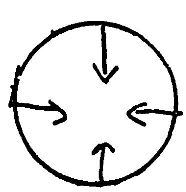
An object follows the path shown below. Draw arrows where you can say for sure forces act on the particle, where the arrows indicate the direction of the forces.



S draws the arrows as shown at the left.

S: The force here is along the path.

H: OK. What if we had a circle like this? Then what would it be?

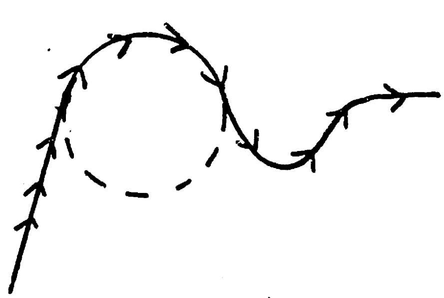


S: The force in a circle is in towards the center.

H: OK. Now, what if this curved part of the first path was like this circle here?

S: It would still be along the path.

H: Even if that part of the path is a circle?



S: Yes.

H: Why?

S: Because you know it's not really a circle.

This student also uses the omniscient observer. However, in this instance, the information provided takes a temporal form - a path includes information relating to the particle's past. She believes that the non-circular parts of the path determine what forces must act on the particle in the circular part of its path.

****

In each of the above examples, the omniscient observer is important. Each student does not clearly separate important aspects of the problem from each other, and consequently goes down dead ends.

Section 4.3 - The Relations of Physics: a Classification

The relations that can be used to solve a problem generally fall into one of four categories:

a. Fundamental relations are the relations which lie at the heart of physics. They include things like Newton's Laws, Maxwell's equations, the Schroedinger equation, the Lorentz force law, conservation of energy and momentum. These relations constitute the basic laws which the universe obeys, and have a wide range of applicability.

b. Phenomenological relations describe quantitative correlations between variables, and have relatively limited ranges of applicability. Examples include Hooke's Law, the ideal gas law, and the proportionality of friction force and the force of contact.

c. Space-time relations involve geometric and kinematic relations, and are intimately connected to the three-dimensional space in which we live. They include relations of geometric constraint, relations between lengths, angles (and their time derivatives), and time.

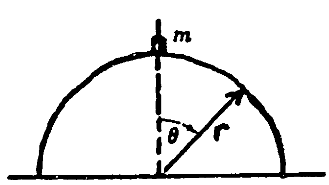
d. Definitions involve human conventions regarding the meaning of various terms used in the problem or in a theory.

A fifth class of relations - correspondence relations, which essentially specify how these general relations apply to specific problems - is documented in Section 7.3 (Local Symbol Specification).

The above listing describes the types of relations which occur in physics. However, the text statement of the problem may or may not provide these relations explicitly. In particular:

a. a problem may be a type which the student is expected to recognize as being soluble through use of a particular set of fundamental relations. Examples:

1.



A point mass m starts from rest and slides down the surface of a frictionless hemisphere of radius R. At what height does the mass leave the hemisphere?

In this problem, the student must identify conservation of energy as the principle underlying its solution.

2. An infinitely long cylinder of radius R and uniform volume charge density ρ is aligned with the z axis. Find the electric field everywhere.

In this problem, the student must identify Coulomb's law (or Gauss' law) as the principle underlying its solution.

b. a problem may contain conditions and phenomenological statements which the student is expected to translate into the appropriate mathematical form. In the first problem above, the student must translate "hemisphere" as calling for $a = v^2/r$, and "leaves the hemisphere" into $N = 0$ at the point of departure.

c. a problem may contain an explicit statement of a phenomenological relation. For example:

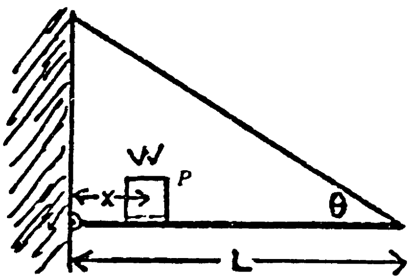
> The Yukawa potential
>
> $$U(r) = -(r_0/r)\, U_0 \exp(-r/r_0)$$
>
> gives an accurate description of the interaction between nucleons. Find the corresponding expression for the force of attraction.

I have discussed these categories of relations in order to establish a working vocabulary which I will use in later discussion.

Section 4.4 - On Physics Problems: States and Time Evolution

A problem may or may not pose a situation in which time is a relevant variable. If it does not, then the problem is static. Here are some examples:

1. Calculate the moment of inertia of a thin rod of mass M and length L about an axis perpendicular to its length.



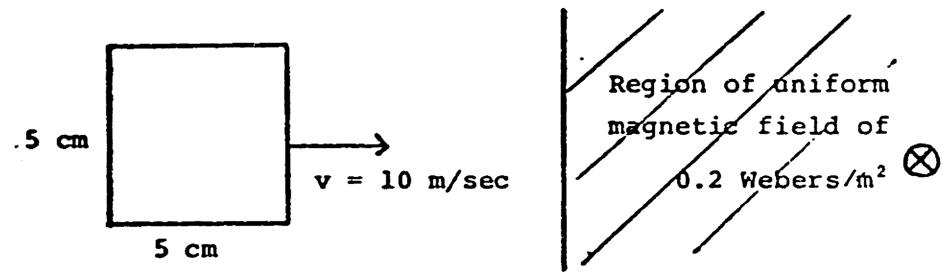
2. A thin, horizontal, massless bar of length L is pinned to a vertical wall at one end and supported at the other end by a massless wire that makes an angle θ with the horizontal. A weight W is located a distance x from the wall. Find the tension in the wire as a function of x.

3. Calculate the electric field on the axis of symmetry of a uniformly charged disk with surface charge density σ and radius a.

In each of these problems, the initial state is identical to the final state. Time is irrelevant to the problem - the system and its relation to the environment is the same every time you look at it. Consequently, only one image is required.

If time is relevant, then the system starts in some well-specified state, something happens to it, and it ends up in some other, different, final state. Here are some examples:

1. A square loop 5 cm on a side is pulled into a uniform magnetic field of 0.2 Webers/m^2 at a constant velocity of 10 m/s. The resistance of the loop is R = 10^{-4} ohms, and its self-inductance is 5 x 10^{-8} henry.



What is the variation of I with time when the loop is partly in the magnetic field? When it is totally in?

2. An 80 kg man is standing at the rear of a 400 kg iceboat that is moving at 4.0 m/s across some frictionless ice. He decides to walk to the front of the 18 m long boat and does so at a speed of 2.0 m/s relative to the boat. How far did the boat move across the ice while he was walking?

In each of these problems, the initial state is different from the final state. Consequently, a separate image is required to represent each state.

In the above discussion, the term "state" refers to the specification of the values of a set of parameters of interest - this set of parameters then defines the state. In other words, to specify a state uniquely, one must select the appropriate parameters, and also specify their values. Difficulties will occur if either of these requirements is not met {4}.

The time dependence of the problem can be specified in two ways. One way is to use time explicitly, as a state index, and ask for the state

S(t) given the initial state S(0). A second way is to use time implicitly, and ask for the state S which is characterized by particular values of the state variables, e.g., in a mechanics course, one might ask for S(x, v).

Using a movie metaphor for the time evolution of the system, the initial state is given by the first frame of the movie. The final state can be specified by giving the number (i.e., the time) of the state in which we are interested, or by specifying what the frame in which we are interested should look like {5}.

This distinction is relevant because an index-specified final state must involve relations which explicitly include time as a parameter, e.g., Newton's Laws or Maxwell's equations. By contrast, a value-specified state often involves conservation relations, which relate initial and final states in a quite direct manner. In addition, the effect of the intervening process can often be included using a sink or source term in the conservation equation.

There is a third type of problem which can fit either of the above categories, namely steady-state problems (of which constant flow problems are a subset) involving constant time derivatives. They are time-dependent in the sense that they involve something changing in time, but time independent in the sense that the rate of change is the same for all time.

Therefore, depending on the parameters which specify a state, the initial state can be very similar to the final state. For example, in a steady-state flow problem, the initial state looks exactly like the final state unless some of the flowing material is labeled artificially, and its behavior tracked as a function of time.

Section 4.5 - The Architecture of the Human Problem Solver

In this section, I will outline a few concepts from information processing and computer science which will be helpful in understanding the difficulties which students have.

My ad hoc model of the human problem solver as an information processing system has the following components:

- an executive, which supervises the operations of other components of the system. It determines the order in which various procedures are executed, manages the transfer of information between working, short-term, and long-term memories, and can itself be modified by information or procedures drawn from memory.

- working memory {6}, abbreviated WM, which contains chunks of information being manipulated by the executive at any given instant; these chunks stored in WM need not be (and often are not) related to each other.

A chunk of information can be any logical block of information (a concept, an equation, a procedure, or a pointer to more information stored in long-term memory. One difference between experts and beginners is that expert chunks contain much more information than do those of beginners (cf., Larkin (1976)). In other words, an expert tends to group information into a coherent whole (so that the resulting chunk has internal structure), while the beginner tends to work with smaller chunks, each with less internal structure.

WM is characterized by a very limited capacity of a few to several chunks; as the size and complexity of a chunk increases, the number of chunks which can fit into WM decreases from several to a few. In addition, WM has very rapid access times, on the order of a second or less (Newell and Simon

(1972)), but it decays in a matter of seconds unless transferred to short-term memory.

- short-term memory (STM), which contains information transferred from WM, and decays in minutes or hours. If this information can be related to other chunks of knowledge, it becomes a part of long-term memory, and thus becomes essentially permanent. If, instead, it remains in isolation and divorced from other knowledge, it decays. Information transfer between WM and STM is effected by rehearsal (e.g., repeating the information to yourself over and over).

- long-term memory (LTM), which stores both information and procedures for manipulating this information. LTM is highly structured, which we infer from the essentially unlimited capacity of LTM, and from the fact that most knowledge in LTM can be retrieved through a number of different routes. For example, "force" can be retrieved through "- grad U" and also through "what changes momentum."

This fact implies that LTM must have a network (rather than a tree) structure. In other words, each item of information can also have associated with it many other pointers to and from other items of related information.

Items in LTM are coded by meaning and context. In a review of memory research, Klatzsky (1975) points out that syntactic differences between sentences (e.g., "The boy hit the girl." vs "The girl was hit by the boy.") are more easily forgotten than were semantic changes (e.g., "The boy hit the girl." vs "The girl hit the boy."). She also describes a study in which points out that in subjects attempting to reproduce a story about an unfamiliar culture made systematic errors. Their errors tended to rearrange the story according to what they considered to be a normal (i.e. their own) cultural framework.

The executive does not have immediate or random access to information or procedures in LTM; rather, the executive must generate pointers to the desired information (which may itself contain pointers to other information), and retrieve it piece-by-piece into WM.

Retrieval of information is facilitated by the existence of greater numbers of pathways to that information. Thus, we would expect experts (who can retrieve relevant information much more rapidly than beginners can) to have many different ways of representing the same information, and also many different links between pieces of knowledge. The insertion of new knowledge into LTM is characterized by the construction of one or more of these anchoring links to already existing knowledge structures.

In the domain of physics, these multiple representations may include:

- verbal descriptions (in English words)
- fundamental relations which characterize the situation in question, e.g., conservation of energy or F = ma
- a perceptual image, which may or may not evolve in time
- graphs (also evolving in time)
- active, procedural knowledge; anthropomorphic intuition - feeling the pressure pushing in on me.
- kinaesthetic: I know what to do to balance the beam from my own experience as a kid on see-saws
- variants on the given situation (varying different parameters and constraints) to get a feel for the given situation as it relates to similar situations.

- input/output devices, which allow the student to communicate with the outside world. For example, a student typically reads a text description and looks at some pictures, so that one input device is visually based - a visual process transfers perceptual data (taken from the statement of the problem) into WM. This transfer process is subject to error, so that the data which actually reaches WM may not correspond to the perceptual inputs. If the student reads too quickly, he may simply overlook important key words.

An example of an output device is the student's pencil and paper. These supplement WM and STM, making permanent what he might have otherwise forgotten. The output process is also subject to error; a student might write a small m when he actually intended to write a capital M.

Since WM has a very rapid access time, pencil and paper help to expand an individual's effective WM capacity; for processing purposes, the executive can flip out of WM any item it sees on paper without fear of losing it, thereby freeing additional (and real) WM space.

In addition to the above architecture, I will make the following assumption of constraint on the executive: since WM capacity is limited (and hence costly to use), the procedures to be employed should be the least demanding (on WM) necessary to effect a solution to the problem {7}.

Another way of looking at WM capacity is to regard the fraction of WM in use at any given moment as a measure of the "mental effort" being employed. Thus, the assumption above would translate into the following: an individual will attempt to minimize his mental effort in solving a problem {8}. I will refer to this assumption later as "the minimal mental effort" assumption.

This constraint has the following consequence: a problem may require the application of several tools for its solution. A student will use (possibly without conscious choice) the simplest tools that he believes are adequate for solving it. Of course, this judgment may or may not be accurate.

If he is faced with a problem whose correct solution would not strain his cognitive resources, he can begin with his simplest tools, and increase their sophistication as appropriate. This helps to explain the fact that even experts can make mistakes on apparently simple problems which in fact contain hidden subtleties. They employ the simplest approach which they believe should work, and generate incorrect answers as the result. When these errors are pointed out to them, they often generate a more sophisticated analysis which then results in correct answers.

On the other hand, if the student is faced with a problem whose solution would strain his cognitive resources, he may trade sophistication for the resources which that sophistication would consume, leaving them free for the application of some other appropriate tool. Of course, the less sophisticated tool may be (and often is) inadequate for the task at hand.

In other words, problems require the execution of certain tasks and manipulation of certain concepts in their solution; the student must coordinate these tasks and concepts in order to effect a solution. When faced with a problem, a student may have a vague sense of these tasks and concepts, but faced with the need for coordinating them, he may well use simpler versions which individually require less mental effort. In doing so, he can retain each item in WM (and thereby coordinate them). However, the simpler version is often inappropriate, and its use can lead to error. The student will thus appear ignorant of the correct (and usually more complex) idea.

However, when he focuses all his effort on one task (e.g., because his teacher called attention to that task), that task becomes the problem. Consequently, he can bring to bear all his resources to solve the (new) problem, and is in a much better position to conclude the problem successfully.

The evidence for the process described in the last few paragraphs is familiar to many teachers. When a student asks for help in doing a problem, the teacher may identify a sub-task which the student should do next. In performing this task, the student often loses track of the reason for performing it. If asked what he is trying to find, he must refer to the original statement of the problem in order to recall his original goal.

I have never seen this happen with a student for whom sub-task performance is an easy thing; it always happens with students for whom the details are difficult and require their full attention to complete successfully.

The only conclusion I can draw is that the problem solving process is similar to learning to walk or drive. When one must pay attention to every little detail of driving, one can't do other things, e.g., talk to your companion. When driving requires little conscious attention, one can talk. Similarly, when a student must pay close attention to the details of every task, he cannot coordinate the different tasks. When he can carry out these tasks easily (i.e., when these tasks are well-internalized and they do not in themselves pose problems for the student), he can coordinate them.

The above would especially be true if the student were not sure of his judgment of the necessary sophistication. If he is sure that he needs another procedure, but feels that the procedures he has already brought to bear are possibly too sophisticated, it is reasonable to assume that he would indeed make a trade, lowering the sophistication of his procedures and thereby freeing capacity to coordinate additional procedures.

Here are a few examples of what these comments imply:

Information that can be represented perceptually will be used much more readily than information which is not so represented; hence, a picture will take precedence over a text description. Information with concrete referents will be used more readily than information without such referents (so that mass (a perceptual quantity) would be a more accessible concept than density (a derived quantity)).

A student may use inefficient methods of problem solving (even if it takes him longer) if he feels most comfortable with those methods or if they require less mental effort. For example, he may spend minutes looking for a calculator when he could do the required calculation by hand in seconds.

Knowledge learned in one context but applied to a different context may seemingly evaporate. In a new context, one must keep track of many new and different things, leaving only a limited reserve of cognitive capacity available. In other words, students may appear to master the earlier work as measured by the usual measures. However, when this earlier material appears embedded in later material, students often behave as though they had not mastered the earlier material. Nevertheless, a direct inquiry at this point often reveals that their mastery of the earlier material has not deteriorated, at least not according to the standards used at that earlier time. [Papert (personal communication, 1979) tells of an informal study in which housewives were asked to add up a list of numbers with "lb" after each number, and then to add an identical list of numbers with "kg" after each number. The results were strikingly different.]

A student may use lower levels of abstraction (e.g., he may mistake a representation of a thing for the thing itself) and less sophisticated reasoning (e.g., non-local reasoning) when a problem is complex. For example, in Example 3.1.2.a, I indicated that students more easily generated equations involving rectilinear variables than equations involving angular variables, even though the reasoning required in each case was the same. One plausible interpretation is that these students found the problem involving angular variables more complex (since they were less familiar with these variables). Consequently, they used a greater amount of cognitive resources in coordinating these notions, leaving fewer resources available for mathematical reasoning, thereby lowering performance.

A student will employ non-analytic thought in preference to analytic thought because the latter requires a great deal of differentiation between various aspects of a situation, whereas the former considers mainly undifferentiated wholes.

A final consequence of this assumption is methodologically relevant. In my investigation of problem solving difficulties, I have often been reluctant to ask questions which would focus a person's attention of one aspect or another of his difficulties. I discovered early in my investigation that doing so often eliminated the difficulty spontaneously, e.g., questions such as "Could you explain that?" or even "I couldn't hear you. Could you please repeat that?" would result in responses such as "Of course..., that's not right." and "I'm not sure about that.... I guess that's wrong... Oh, I see what to do." However, without prompting questions, those difficulties would continue to plague the student, even though he obviously had the proper knowledge stored somewhere in memory.

I conclude that my prompting questions served to focus all the student's attention on a particular aspect of the problem, i.e., a smaller, more restricted problem. In doing so, more WM was made available, and therefore the student could devote additional cognitive capabilities to the new sub-problem. The result was that he could increase the number and sophistication of tools available, and thus identify and correct his mistake.

However, often students were unable to explain why they did what they did just moments before, or they had to re-read the problem to remember their original goal. If in fact all cognitive capacity were focused on one sub-task, other parts of the problem previously held in WM would be lost, and this result is not surprising.

Notes

1. I should note that the discussion which follows below is partly a "poor man's philosophy of science" (lacking the rigor of standard philosophical treatments), and partly my account of what appear to be beliefs common among professional physicists.

2. I realize that teachers give partial credit - however, they view only the finished product; they do not care about a student's scrap paper.

3. Of course, I am speaking of a car on level ground, starting from rest.

4. It is curious to note that the term "state" does not typically appear in an introductory physics course unless the course includes thermodynamics. I first encountered "state" in quantum mechanics, in the form of a wave function which specified the "state" of a system.

5. This distinction is similar to the distinction between random access memory and content addressable memory.

6. My analysis separates working memory and short term memory in a distinction not typical of information processing theorists: most interpretations consider WM and STM as one entity.

7. This assumption is basically that of the capacity model for attention described by Kahneman (1973) in Chapter 2. However, Kahneman's assumption that the intrinsic difficulty of the task determines the level of cognitive efforts must be modified to assert that the determining factor is the individual's perception of how sophisticated a task is needed, rather than the task itself.

8. Note that this does not mean that he uses WM infrequently; it means that he tends to use smaller fractions of it.

Sooner or later, you have to stop writing all this introductory junk, and start writing something substantive. At some point, you have to do it.

- D. Roberts to H. Lin

Part II - Substance

Part II presents the fundamental substance of this thesis. It presents examples of various difficulties, and extrapolates from these examples abstracted descriptions of these difficulties. However, before turning to these difficulties, a look at the road map is in order.

Polya (1945) identifies four general categories of problem solving activity: describing the problem, planning the solution, manipulating the resulting equations, and interpreting the results of these manipulations. Polya's original discussion applies these activities in the domain of mathematics, but they appear to apply in (at least) the limited domain of standard physics problems as well.

Description refers to the process of trying to understand the problem and the situation it presents. At a superficial level, this involves extracting the given information in the problem. This problem information includes:

a. parameters - the usually known or given quantities which are assumed to remain constant throughout the duration of the problem. These are usually but not always specified explicitly, e.g., "m = mass of block = 2 kg". On occasion, they may be specified through the use of a key word, e.g., "elastic" a coefficient of restitution of 100%, and "stationary" might mean that $v_{block} = 0$ for all time.

b. spatial configuration: in other words, what the physical situation looks like - a picture. In the course of "what is happening in the problem," this picture may change (like a movie) or may not {like a still life}.

c. desired unknowns: this is what the problem asks you to calculate.

At a much deeper level, problem information can be assembled into a mental model (or image) of the situation represented in the problem. Metaphorically, one might consider this model a movie of the problem situation which runs in the problem solver's head, or perhaps some kind of anthropomorphism - an identification with some part of the objects or systems posed in the problem.

Planning refers to the process of "setting up" the problem solution. This process takes as its input the information generated in description, and itself generates a system of N equations and N unknowns, among which unknowns is the answer to be found. Planning includes the following sub-activities:

a. selecting the general relations which will be useful in solving the problem. These relations include fundamental relations, geometric and kinematic relations which relate space-time variables, phenomenological equations, and definitions.

b. formulating these relations in mathematical form. Often these relations are recalled in equation form, but a person might have a vague idea about possibly useful relations, without a sense of the corresponding mathematical statement of those relations.

c. specifying explicitly how the symbols contained in the general relations (now in mathematical form) correspond to the information given in the problem or generated as the result of intermediate calculations.

Manipulation refers to the process of solving this system of n equations and n unknowns for the answer(s) requested by the problem. Manipulation involves skills from arithmetic, algebra, trigonometry, calculus, and other appropriate mathematical techniques. In principle, manipulation can occur entirely after planning, but in practice, a person often combines manipulation with planning. In addition, manipulation involves the organizational skills necessary for keeping track of complex calculations.

Interpretation refers to the process of interpreting any intermediate or final result in terms of one's previous experience and intuition, and also in terms of the qualitative model generated in description. Interpretation answers questions like "What does this mathematical expression mean?" and "What aspect of the problem does this expression correspond to?" In other words, it interprets the formalism in terms of the physical situation posed by the problem, while planning (specifying the symbols) interprets the physical situation posed by the problem in terms of the formalism.

These activities as described above are idealizations of what a person might do when he solves a problem. Furthermore, though the sequence of description, planning, manipulation, and interpretation given above might seem to be a reasonable sequence which people would actually use, in fact students almost never do things in this order, or even all of these things.

I find it convenient to introduce a terminology for this set of activities that does not have connotations of a fixed sequence. Consequently, I group description and interpretation under the heading Understanding; I group planning and manipulation under the heading Generation.

To put it another way, understanding refers to the process of mapping writing to reality, going from the statement of the problem and all statements pertaining to its solution to the actual physical situation to which the problem and solution must refer.

Generation is the process of mapping reality to writing. From the actual physical situation, one must decide what relations to use, how these equations are formulated in symbolic form, and how particular aspects of the problem correspond to these symbols. [This is not entirely accurate - it ignores the fact that students are almost never presented with real physical problems, but rather text statements which describe problems. In this case, the mapping would be from writing (i.e., the text statement) to writing (i.e., statements which might be in the solution).]

With these definitions, I am arguing that understanding and generation pretty much exhaust the possible activities in which a problem-solver may engage, but they do not imply a fixed order of performance.

It is true that beginners often do not distinguish between these categories; indeed, they often appear to focus exclusively on generating a solution, unaware of a need to understand the problem or interpret their own work. It is even more rare that a student will distinguish between "setting up a problem" and "cranking it through". However, an expert observer can usually infer what a student is doing, and so the distinction between understanding and generation seems reasonable.

The examples of difficulties which follow are drawn from many tens of interactions with students. The same text problem will often be used to illustrate different difficulties, and I hope the reader will be struck by the tremendous variety of ways in which students understand and misunderstand the same problem.

It is important to keep in mind that my objective is to illuminate some significant difficulties which arise as students struggle with problem solving. I do so from the variety of perspectives sketched in Chapter 4, i.e., from the outside looking in. These perspectives provide organizers and vocabulary for discussing and interpreting some of these difficulties, but they do not constitute a fundamental theory of the psychology of beginners solving physics problems.

This has been a deliberate choice, and I have made this choice for the following reason: in this, my first pass through the material, I have found the matters raised by these difficulties very much intertwined - a veritable tangle which is not necessarily rule-governed. In essence, I have given up, at least for the time being, the idea of perturbing a model of competent problem solving in order to reproduce observed difficulties; I am not convinced that this is even possible. Indeed, the assumption of limited working memory capacity I made in

Section 4.5 would preclude behavior governed by simple rules. In other words, rule-governed behavior implies independence of factors. [Note that this assumption stands in contrast to much other work in this field, e.g., Brown (1978), Clement (1977a), DiSessa (1979b)].

In addition, the distinction between understanding and generation is not a sharp one - some items fall quite definitely into one category or another, but others are not so well-defined. My categories (and even individual items in each category) overlap, and I see no a priori reason to believe that any categorization could have clearly separated entries.

A second caution to keep in mind is that the physicist reader should suspend judgment in reading the excerpts which follow. In particular, he should not attempt to interpret what students say from a physicist's perspective - many things that students say might indeed make sense if uttered by a physicist; however, the context of the discussion indicates a different sort of understanding. I have tried to make clear this context in each case, but I suspect that I have not been entirely successful in this effort.

Finally, I have been convinced in the course of this study that matters of comfort, unease, unwillingness, and reluctance - all very unscientific terms - play a central role in student problem solving behavior. Often their effect is to block a student from taking further action - what he calls "getting stuck"; a student may know what he should do, but be unwilling to do it because it conflicts with his psychological preferences. In these cases, it is quite difficult to convey on paper the emotional distress which accompanies the student as he seeks out help. Furthermore, it is hard to illustrate the student who does not do a certain thing. However, I have occasionally been fortunate, and some students have explicitly discussed these feelings with me; I present these encounters when I can.

The map is not the territory!

- A. Korzybski

Chapter 5 - Understanding

In Chapter 4, I argued that physics assumes the feasibility of constructing useful representations of a physical situations and processes, and that there are a wide variety of representations. This chapter explores some of the ways in which students use and understand these representations.

The difficulties in understanding discussed in this chapter refer to two types of misunderstanding. One type can be characterized as superficial difficulties with artifactual aspects of a problem or solution (e.g., student uses inconsistent notation). A second type can be characterized as fundamental misunderstandings of a theory, either a specific theory or theories in general (e.g., student doesn't understand what a relation is).

Sections 5.1 - 5.4 will concern difficulties caused by artifactual demons. These will include:

- poor notation (Section 5.1)
  - use of the same symbol to denote similar but physically distinct quantities
  - failure to use mnemonic notation
- inadequate imagery (Section 5.2)
  - literal perceptual interpretation of schematic drawing.
  - interpretation of static image on paper as indicative of time-independent physical situation.
- incomplete extraction of information from problem statement (Section 5.3)
- poor choices of coordinate systems (Section 5.4)

Sections 5.5 - 5.9 will concern difficulties caused by demons of a more fundamental nature. These will include:

- confusion between functional and numerical equality (Section 5.5)
- poor and misleading use of calculus in physics (Section 5.6)
  - a differential Δx can indicate a change in x from some initial value to some final value, <u>or</u> a small bit of x
  - dy/dx can indicate a command to differentiate the function y(x) (e.g., $a = dv/dt$), <u>or</u> the ratio of two infinitesimally small quantities dy and dx (e.g., $i = dq/dt$)
- confusion between perceptual and representational spaces (Section 5.7)
- literal interpretation of idealized solutions (Section 5.8)
- inability to identify appropriate system states (Section 5.9)

Section 5.1 - Notation

Good notation is a working tool which provides a short-hand (preferably mnemonic) for discussing quantities and relationships of interest. Thus, like any useful language, it should be compact enough that it can be used conveniently (thereby reducing mental effort) and yet flexible enough to express a wide variety of meanings. However, once established, it should also remain invariant in meaning for the duration of a problem.

In point of fact, much notation is NOT invariant in meaning. It often happens that students are not consistent in their use and interpretation of (their own!) notation. [Perhaps more surprisingly, some of the conventionally employed notation also suffers from certain kinds of inconsistency. I will discuss this claim in Section 5.5 (Functional and Numerical Equality).]

However, these inconsistencies do not appear when a student first begins his problem solution. They emerge only as he uses this notation. Consequently, I will illustrate these inconsistencies as they appear in various stages of a problem's solution.

5.1.1 - Confusion of Personal Notation

Some students use the same symbol to denote similar but different quantities. Here are two examples:

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Example 5.1.1.a

A thin disk of dielectric material of radius R, having a total charge Q distributed uniformly over its surface, rotates with a frequency f about an axis perpendicular to the surface of the disk and passing through the center. Find the magnetic field at the center of the disk.

$$A = \pi R^2$$
$$\sigma = Q/\pi R^2$$
$$dA = 2\pi R\,dR$$
$$dQ = \frac{Q}{\pi R^2}\,2\pi R\,dR$$
$$dQ = \frac{2Q}{R}\,dR, \quad dI = f\,dQ$$
$$B = \frac{\mu_0}{2}\,I/R$$
$$\text{so}\quad B = \frac{\mu_0}{2}\int f\,\frac{2Q\,dR}{R^2}$$

S: Let's see..., the area of the disk is $A = \pi R^2$, and the total charge on the disk is Q, so $Q/(\pi R^2)$ is the charge density on the disk. To find the charge on one of these rings, we take this charge density and times it by the area of the ring, which is $dA = 2\pi R dR$, so dQ on the ring is 2Q/R dR, and so dI = f dQ, and so the total B at the center is $\mu_0/2$ times the integral dI/R ..., so I get.... wait a minute... that doesn't make sense - I can't integrate that... I'll get an undefined number.

Conceptually, this student knows very well what to do. He has made an error only because his notation does not distinguish between R (which represents a fixed parameter giving the size of the disk) and r (which represents a field coordinate indicating which ring is under consideration).

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Example 5.1.1.b

A dog sees a flowerpot sail up and then back past a window 5.0 feet high. If the total time the pot is in sight is 1.0 s, find the height above the window that the pot rises.

Student has written the equations below, and is now trying to interpret them. The transcript to this point shows that in the first equation, y refers to the distance the flowerpot travels while it is in sight through the window, i.e., the height of the window. In the second equation, y refers to the height above the bottom edge of the window to which the pot ultimately rises.

$$y = v_0 t + 1/2\,at^2$$

$$v_0 = (y - 1/2\ at^2) / t$$

$$v_y^2 = v_0^2 + 2\ a\ y$$

$$y = [\ v_y^2 - ((y - 1/2\ at^2)/t)^2\ ] / 2a$$

H: Now, you have y on both sides of your last equation. Could you explain that to me?

S: This y [on the left hand side] is probably y_0 [pointing to the top edge of the window]... nope, it's not. That y [on the right hand side] actually is y [points to equation (1)].... Why did I do that to begin with? ..... I did this [with equation (1)] to get v_0... Ah ha, this t is 0.5 seconds, and this y [on the right hand side of the last equation] is a known y, the height of the window. This other y is y at the top of the path.

This student has used the same symbol to denote two different quantities, and as a result has confused herself. Note that in order to identify the meaning of each y, she is forced to reconstruct much of the calculation. Only after some effort does she disentangle herself.

****

More generally, the failure to introduce distinctive symbolic notation can and often does cause student difficulties, since it is usually disastrous for the same symbol to have two different meanings. One offshoot of this is the fact that in the course of developing a solution, originally distinct symbols become more and more similar: M and m may look different, but M is likely to transform itself into m by the end of a long handwritten solution.

Hypothetically, one could keep track of various symbols (and just remember that this m means this mass, and this other m means the other mass); this is just what the student above attempts to do. However, in practice, it is very hard to do so, and very draining on one's cognitive resources. Indeed, the point of inventing notation in the first place is that it allows a person to focus less attention on aspects of the problem which are not fundamentally relevant.

5.1.2 - Lack of Mnemonic Notation

A second, related issue is the use of good, mnemonic notation. The proper notation can reduce mental effort substantially, and therefore increase the likelihood of correct solutions. For example, consider the following two ways of writing the relative velocities equation:

****

Example 5.1.2.a

Method 1 (from Halliday and Resnick (1977)):

$v = v' + u$

where:
- v = velocity of object with respect to S
- v' = velocity of object with respect to S'
- u = velocity of S' with respect to S

With this description, the student must remember the three different and separate definitions of v, v' and u.

Method 2 (from Sears and Zemansky (1978)):

$v_{OS} = v_{OS'} + v_{S'S}$

where:

v_{OS} = velocity of object with respect to S
$v_{OS'}$ = velocity of object with respect to S'
$v_{S'S}$ = velocity of S' with respect to S

With this method, the student has only to remember a template which says that V_{ab} = velocity of (a: first subscript) with respect to (b: second subscript). It has the further advantage that identical, adjoining subscripts "cancel", and so any number of velocities can be chained together. Finally, this type of notation is generalizable to any kinematic quantity that can be measured from different reference frames.

++

Example 5.1.2.b

A cannon is mounted inside a sealed railroad car of mass 20000 kg. It fires cannonballs of mass 20 kg with a muzzle velocity of 200 m/s. What is the velocity of the car with respect to the ground immediately after the cannon is fired?

A student said that his use of V_b stood for the muzzle velocity. Since "muzzle velocity" means "the velocity with which the cannonball leaves the gun", and the gun is attached to the car, "muzzle velocity" would seem to mean the velocity of the ball with respect to the car. However, the student then wrote:

$$m_c V_c = m_b V_b$$

H: What do those symbols mean?

S: V_c is the velocity of the car, and V_b is the velocity of the ball.

H: And so what would V_b be?

S: 200 m/s.

H: And your answer would be...

S: V_c, which would be $m_c/m_d\ V_b$

This student confused himself through use of poor notation. He apparently understands his V_c to mean the velocity of the car with respect to the ground (since that is what he gives as his answer). However, he also understands V_b to be the velocity with which the balls leaves the cannon. With the use of the second velocity notation above, he would have been forced to specify clearly the reference frames from which he measures velocities. Without it, he forgot an important part of the meaning of his symbols.

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Section 5.2 - Imagery

Section 4.1 (The Values of Science) argued that representations are central to the practice of physics. One representation used extensively in physics is diagrammatical, and a standard piece of advice given by many physics teachers is to include with every problem solution a diagram of the physical situation in question. Indeed, images are a particularly powerful form of representation, since they are transmitted directly as perceptual data, and hence require lower levels of mental effort for processing. Images are so compelling that they can easily take precedence over other forms of representation, e.g., text statements. This section will document difficulties in which visual images override text and literal interpretations of images override symbolic or abstract interpretations of images in accordance with the minimum mental effort hypothesis of Section 4.5 (The Architecture of the Human Problem Solver). Here is a preview:

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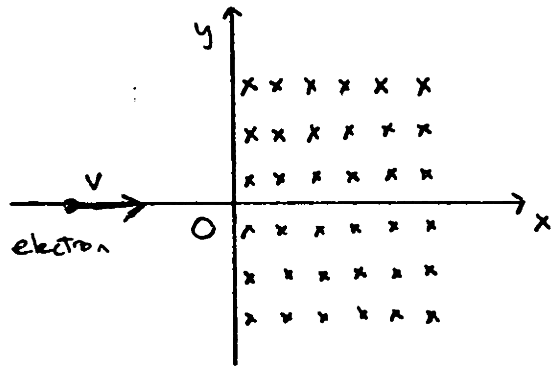
Example 5.2.a

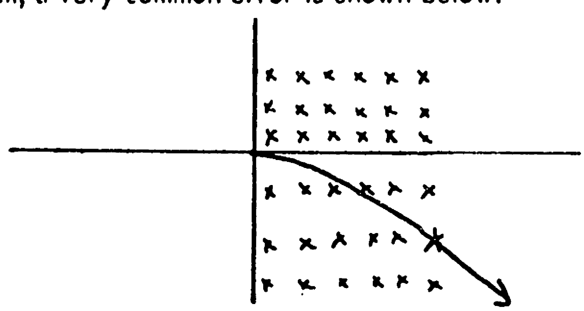


The entire x-y plane to the right of the origin O is filled with a uniform magnetic field B pointing into the page as shown. An electron of charge -e traveling with velocity v along the x - axis is injected into the magnetic field at the origin O.

(a) Draw the path of the electron on the above figure.

(b) Calculate the value of the y coordinate at which the electron leaves the region of magnetic field.

Note that the text of the problem clearly states that the magnetic field is infinite in extent to the right of the y axis. Still, a very common error is shown below:



Obviously, these students assumed that the field was limited spatially as the picture (but not the text) indicated. The statement of the problem is poor, because the diagram is misleading. Nevertheless, it does illustrate how visual images can override data from other sources.

****

In what follows below, I will discuss two aspects of this imagery, and illustrate some of the ways in which students generate and use unhelpful or incomplete images.

5.2.1 - Literal Interpretation of Schematic Drawing

A picture provides a schematic representation of a physical situation - it offers qualitative information about the spatial relationships in the problem. Therefore, quantitative irregularities in the picture itself are assumed to be insignificant in the same way that a stick figure merely represents a person, and is not meant to depict a person who really looks like the stick figure.

This type of abstraction causes difficulties for some students, who will misinterpret certain perceptual aspects of the picture (often hand drawn). To the picture's artist, these perceptual aspects are often irrelevant (e.g., he does not care about the length of the side of the cube he has drawn, or that the block is not perfectly square.). However, the perceptual image is so compelling that the student may mistakenly ascribe physical significance to these aspects.

In this case, a misinterpretation will block further action on the student's part. The physical process described by the problem will predicate a certain time evolution out of some given initial state. If the misinterpretation (usually of the initial state) occurs, the student is unable to envision the evolution of the initial state into the final state; thus, the student will be unable to begin the problem.

Here are two examples:

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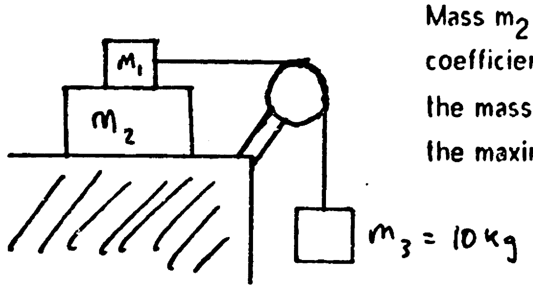
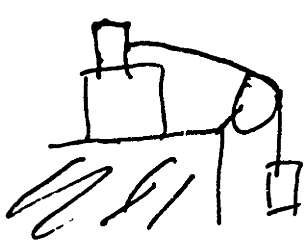
Example 5.2.1.a

A student came in for help on the following problem:



Mass m_2 = 10 kg slides on a smooth table. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. What is the maximum acceleration of m_1?

I drew by hand the picture below, and asked him to describe what would happen if he had the situation posed by the problem:



S: I'm confused... you can't do it.

H: Why not?

S: Because the blocks will fall over.

After this comment, I redrew the picture more precisely, taking care to exaggerate the factors which would give this arrangement of blocks stability, and we continued, It is interesting to note that this student was thinking like a primitive physicist in trying to interpret the diagram in terms of something related to reality.

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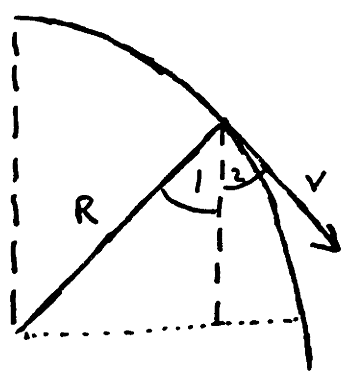
Example 5.2.1.b

The problem under discussion is a particle in uniform circular motion.

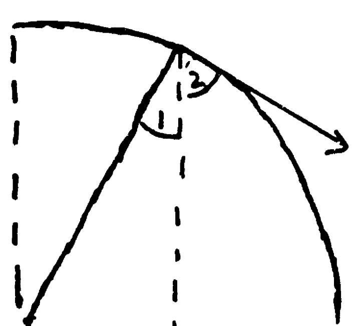


S: The vertical line bisects the perpendicular (between the radius and the velocity vector).

[In other words, angles 1 and 2 are equal.]

H: In general?

S: Yeah.



H: OK, let's try this circle. Does it in this case?

S: No, but that's a lopsided circle.

This student is unable to focus on the abstraction of a perfect circle, and is misled by inaccuracies in the drawing.

****

These examples lead me to speculate that a student who draws sloppy and/or qualitatively inaccurate pictures in doing his own work may be creating additional difficulties for himself. These examples present some fairly extreme cases of diagram misinterpretation. However, a picture far removed in appearance from the situation it represents demands additional cognitive resources to maintain the connection between picture and situation, and it is certainly possible that this would result in poorer performance.

In less extreme cases of difficulty, the student might be concerned about the picture "in the back of his mind". On problems which are non-trivial but nevertheless well within the student's capabilities, a student so concerned might not stop dead in his tracks; instead, he might continue rather ineffectively because of the additional effort required.

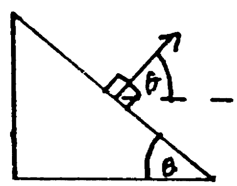
Parenthetically, it is interesting to note textbook graphics are usually good, and drawn to scale, and with reasonable perspective. Consequently, we might expect this type of diagram misinterpretation not to happen with textbook problems.

These perceptual aspects may also miscue the student into performing an action (or set of actions) inappropriate to that situation, resulting in the generation of incorrect equations.

****

Example 5.2.1.c

A student draws the following vector diagram, and attempts to resolve the vectors into components.



He concludes that the angles marked as equal are equal because they appear to be equal from his picture.

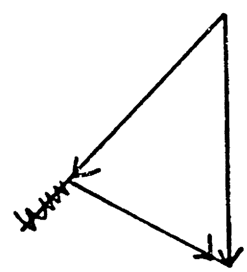
****

Example 5.2.1.d

This student is unable to resolve a vector into components when the vector is parallel to an edge of the paper. For example, given the vector to the left, the student resolves it correctly (on the right).

However, given the vector below on the left, she resolves it as shown to the right.

The vector immediately above is actually the weight vector in a problem involving a mass on an inclined plane. When faced with an abstract vector pointing vertically, she does the following:



Note that in her attempt to the right, her first line goes beyond where it should go, and the cross-out actually appeared on her paper; this occurred several times. In each case, she corrected it, but her tendency was very clearly to extend the first line to the horizontal. A look at her exam papers confirms this observation: she made several mistakes similar to the one illustrated in figure (one before last). I conclude that the new context presented by the exam increased the cognitive strain on her, thereby reducing the mental capacity available to perform the vector resolution task.

****

More generally, some conventions are much more common than others, and a student who is bound to them is unlikely to demonstrate flexibility in his approach to a problem. The above example illustrates one convention: the correspondence of x and y axes to perceptually horizontal and vertical directions.

5.2.2 - Conversion of Dynamic Problem to Static Problem

Section 4.4 (On Physics Problems: States and Time Evolution) related the notions of state and time evolution. In particular, Section 4.4 noted that often one picture (indicative of a particular state of the system) is not sufficient to characterize the problem completely. One picture characterizes a situation completely if and only if the situation is entirely time independent. By contrast, a time-dependent situation (in which an initial state evolves somehow into a final state) or even a steady-state problem (in which some time derivative is constant) requires more than one picture.

In observing student behavior, it is hard to escape the conclusion that the number of images available to the student affects substantially his understanding of what is happening in the problem. In particular, if a student has only one image available to his consciousness (thus placing less demand on available cognitive resources), he may view the problem as static, in accordance with the minimum mental effort assumption of Section 4.5.

Here are a few examples:

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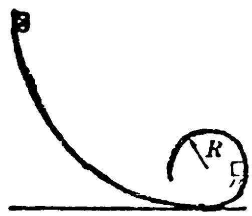
Example 5.2.2.a



A small block of mass m slides along the frictionless loop-the-loop track in the figure. At what height above the bottom of the loop should the block be released so that the force exerted on it by the track at the top of the loop is equal to its weight?

The question under discussion is why the block does not fall off the track.

S: The centripetal force keeps it on the track.

H: Which way is it pointing?

S: Outward, to keep it on. If it pushed in, the block would fall in.

[We then embark on a discussion of which way normal forces point.]

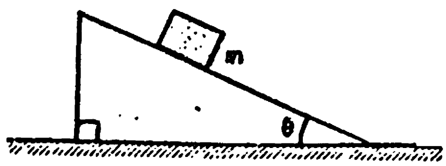
S: OK, the normal force pushes in.

H: You sure? It doesn't point out?

S: No, it goes in, because gravity would be pushing... Nope... I'm still thinking of it as standing still.

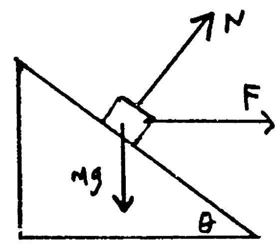
****

Example 5.2.2.b



A mass m rests on a smooth wedge which has an inclination θ as in the figure, and an acceleration a to the right such that the mass remains stationary relative to the wedge. Find a.

The student has come for help on this problem, and I have asked him to start from the beginning. He draws the diagram below, performs some vector manipulations on the vectors in his diagram



and then writes these two equations:

$$N + F \sin \theta - mg \cos \theta = 0$$

$$F \cos \theta + mg \sin \theta = 0$$

S: .... where F is the force of the wedge on the block... its velocity.

He then writes

$$F = ma$$

[Note that the F in this last equation is also the same F as in his first two equations. He then solves each equation for a, and gets two different expressions for a.]

S: I guess I made a mistake... I'm stuck.

H: Tell me what you're doing.

S: I took the forces, resolved components, summed them to zero, because the block's not moving, and I solved for a.

H: If the sum of the forces is zero, then it doesn't accelerate right?

S: The block stands still, but the wedge moves.

H: Think. What happens if the wedge moves but the block stands still?

S: Well.... Oh!! The block is stationary relative to the wedge, but the wedge accelerates.

[A second student performed exactly the same mathematical analysis. I asked him "Is the block moving?" His response: "No... I mean yes."]

Both students behave exactly as though the block were stationary for the duration of the problem. The solutions are in fact correct (except that F should point in the other direction if he really is working in an accelerated reference frame); however, it comes as a surprise to each student that he has assumed a stationary block without consciously realizing it.

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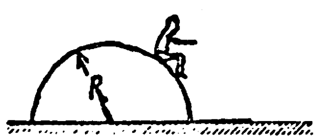
Example 5.2.2.c

A student had previously gone through the loop-the-loop problem of Example 5.2.2.a, and now tries this one:



A boy is seated on the top of a hemispherical mound of ice of radius R. He is given a very small push and starts sliding down the ice. Show that he leaves the ice at a point whose height is 2R/3.

The student has reached the point where he is attempting to understand why the boy leaves the ice.

S: The force keeping the boy on the ice is the force of the ice on the boy.

H: Why?

S: It's like before; the track kept the block going in a circle.

H: What would the boy like to do?

S: Go straight down.

H: Why can't he?

S: The ice is there.

H: What keeps the boy on the ice?

S: Gravity.

H: The normal force of the ice on the block tries to push him off.

S: So when N > gravity, he will leave the ice.

H: It seems that you want to see this thing as him standing still with one force up and another force down.

S: Yes.. it's easier that way.

This student looks at the situation statically - in terms of a balance of two forces which cancel out resulting in something standing still.

****

Example 5.2.2.d

The following problem was given on an hourly quiz.

A small child of mass M_c starts from a height H and slides down a curved slide of mass M_s onto a frictionless horizontal surface, which becomes a surface with coefficient of sliding friction μ to the right, as shown above. All other surfaces are frictionless. The slide is not attached to the horizontal surface, and is free to move horizontally as the child slides down.

(a) Find a relationship between the velocity of the child on the horizontal frictionless surface and the velocity of the slide.

The student's answer to this part was:

$$M_c V_c = M_s V_s$$

(b) Determine these velocities in terms of M_c, M_s, H and g.

The student's answer to this part was:

$$M_c gh = 1/2 M_c V_c^2$$

The following comes from a post-quiz interview.

H: How come you wrote this last equation?

S: Because energy is conserved.

H: Aren't you forgetting something?

S: ...No, I don't think so...

H: What about the slide?

S: What about it?

H: It moves, doesn't it?

S: ...Oh yeah, it does.

In answering part (a), this student has attributed a velocity to the slide. However, his response to part (b) clearly indicates that he forgot about the motion of the slide.

****

Note that in each case, the student behaves as though he views the object as standing still. In these cases, the very fact that the student has drawn an image (one image) may lock him into a static representation. Indeed, Piaget (cf., Phillips (1975)) argues that temporal awareness (which is entirely mental) occurs at a much later stage of development than does spatial awareness (which is at least partly perceptual). If we postulate that the sequence of concept development is invariant, we see a reflection of this argument in the above examples: a static picture implies a static physical situation.

The discussion on steady-state problems (Section 4.4 (On Physics Problems: States and Time Evolution)) also sheds light on this tendency. A picture that a student draws is implicitly centered on the system under consideration:

The figure on the left would represent the car before it moved.

The figure on the right would indicate the represent the car after it moved.

In addition, its position is perceptually accessible (since one sees lengths). At a higher order of abstraction, one can indicate change in position by means of an arrow. Therefore, the above diagram indicates a state implicitly characterized by x and v. However, acceleration doesn't enter this parameterization at all. Therefore, even a drawing of both initial and final states will not reflect acceleration. Furthermore, the implicit centering implies that the value of the positional parameter does not change, i.e., the system is at rest.

Section 5.3 - Extraction of Information from Problem Statement

Another area of student difficulty concerns the extraction of useful information from the problem. Most physics problems give exactly the information needed to solve the problem; nevertheless, some students are often unable to find all the relevant information in the problem. Many students make careless errors as they rush through a problem and jump to conclusions - one type of careless error is the accidental omission of some part of a problem. Indeed, these are difficulties of which students themselves are aware. Here are some self-reported examples of mistakes that students actually make:

> Sometimes I solve for the wrong things. When I'm not careful, because I've skimmed it, then I screw it up.

> The problem was deciding if they wanted a velocity. If they just said "find a velocity" I could do it, but a long word problem with all the different words thrown in would screw me up.

I'd leave something out and the whole answer would be wrong, like I'd forget a force acting on it.

They had problems with frictionless surfaces and blocks sitting on other blocks being pulled and pulleys and millions of other things that when you put them all together, I really got lost.

I forget little things along the way - I think about too much at once and start forgetting.

The first two quotes imply errors in the input process. The last three imply cognitive overloads, e.g., forgetting things, overlooking things, getting lost.

Here are a few observed difficulties which mirror some of these comments. In the example below, the student incorrectly identifies the problem (or one important aspect of it) as being similar to another problem.

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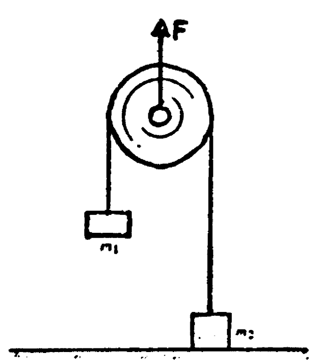
Example 5.3.a



Someone exerts a force F directly up on the axle of the pulley shown in the figure. Consider the pulley and string to be massless, and the bearing frictionless. Two bodies, m_1 of mass 1.0 kg and m_2 of mass 2 kg are attached to opposite ends of the pulley. The body m_2 is in contact with the horizontal floor. What is the largest value of F for which m_2 will remain at rest on the floor?

I have asked the student to describe what is happening in the problem. She has correctly described the motion of the pulley, and now she attempts to describe the accelerations of the blocks.

S: $a_2=-a_1$, because they're in opposite directions.

H: What are a_1 and a_2?

S: a_2 is the acceleration of m_2, and a_1 is the acceleration of m_1.

H: Is that right? What's the acceleration of m_1?

S: I'm just saying it's equal to a_1.

H: OK. What's the acceleration of m_2?

S: $-a_1$.

H: Why?

S: They're going in opposite directions.

H: Are they?

S: m_1 is going up, and m_2 is..... you don't want m_2 to move... (laughter)

H: So what's the acceleration of m_2?

S: Zero.

In this case, the student omits a part of the problem, but unknowingly replaces it with an incorrect and unconscious assumption about the motion of the masses. Consequently, her action sequence is not blocked, and she proceeds on her own, but incorrectly.

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In the next three examples, students omit part of the problem:

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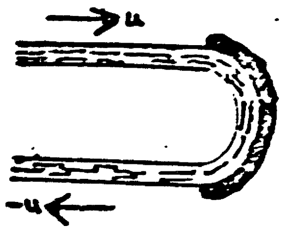
Example 5.3.b



A stream of water impinges on a stationary dished turbine blade, as shown in the figure. The speed of the water is u, both before and after it strikes the curved surface of the blade, and the mass of water striking the blade per unit time is constant at the value μ. Find the force exerted by the water on the blade.

This student came in for help, saying that she did not know how to take into account the motion of the turbine blade. I asked her to read the problem out loud, and she said:

> "A stream of water.... here's the water... impinges on a stationary dished... Oh. It's not moving, is it?"

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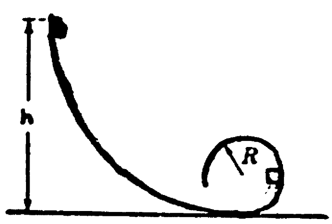
Example 5.3.c



A small mass m slides without friction along the loop-the-loop track shown in the figure. The circular loop has a radius R. The mass starts from rest at a distance h above the bottom of the loop. What is the least value of h if m is to reach the top of the loop without leaving the end of the track?

S: I'm not sure how to do this problem.... I don't have enough information..

H: Read the problem....

S: ....The circular loop has radius R... That's important, isn't it?

H: Yup.

****

Example 5.3.d

A dog sees a flowerpot sail up and then back past a window 5.0 feet high. If the total time the pot is in sight is 1.0 s, find the height above the window that the pot rises.

The student has brought me the work which she has done to this point, and she asks for help.

S: I need the velocity at the top of the window. I know the acceleration is gravity and the time is .5 sec - my problem is that I don't know the initial velocity, so I don't know v_0 or v. I know the average velocity, but I'm not sure how that's going to help.

H: What are you trying to do? You want the velocity at the top or at the bottom. Have you used all the information in the problem yet?

S: umm....

H: What have you used so far?

S: ...[long pause]...

H: You know a and you need v_0 or v. Do you have t?

S: It depends on how you look at it. If this is t=0, then this is t = .5 (top of window). But I still don't know v or v_0.......

H: Have you used all the information in the problem?

S: Not the height of the window?

H: Right. Can you use that?

****

In each of the above examples, the student is blocked from taking further action because he omits an important part of the problem.

In general, the omission of a piece of relevant information may have one of two consequences:

(a) the student may unknowingly replace the missing information with unstated assumptions (which may or may not be correct), if he can draw on previous experience with the type of problem under consideration. The use of assumptions reduces substantially the demand on cognitive resources; consequently, they are employed frequently unless there is some compelling reason why they should not be. See Section 8.3 (Intuition) for further discussion.

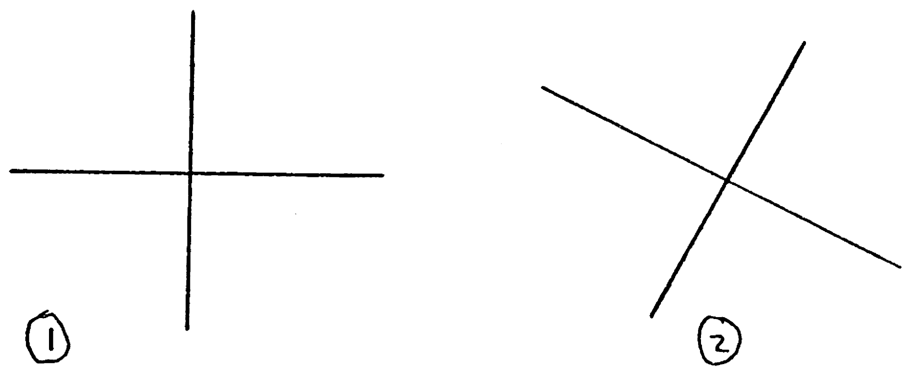
(b) If the student lacks the necessary experiential base, he does not have the information necessary to make even unstated assumptions, and therefore cannot continue.

Each of these difficulties is characterized by the fact that the student appears not to have read the problem carefully. Alternatively, the student may have forgotten an important piece of information.

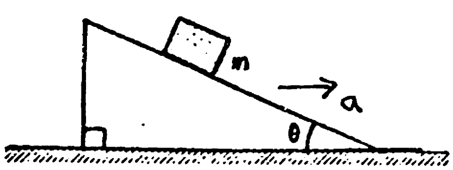
Section 5.4 - Coordinate Systems

A basic component (both explicitly and implicitly) of many physics problems is the idea of a coordinate system and points or frames of reference. Indeed, position has no meaning without the specification of an (explicit or implicit) origin; velocities have no meaning without the specification of a reference frame from which to measure velocities.

In order to be useful, a coordinate system must allow the unambiguous specification of directions and magnitudes of vector quantities. Therefore, in choosing a coordinate system, one must specify the orientation of its axes, relative to the physical situation posed in the problem. For example, the following diagram shows two possible choices of coordinate system orientation.

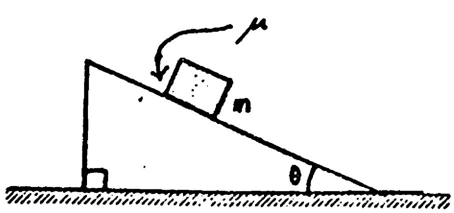


Orientation 1 is more appropriate for the following problem:



A mass m rests on a smooth wedge which has an inclination θ as in the figure, and an acceleration a to the right such that the mass remains stationary relative to the wedge. Find a.

Orientation 2 is more appropriate for the following problem:



A block of mass m rests on an immovable ramp of incline θ. The coefficient of kinetic friction between the block and the ramp is μ. The block is then released. Assuming that μ is small enough for the block to accelerate, what will be the acceleration of the block down the ramp?

Depending on other aspects of the problem, one or the other figures above might be more appropriate. In addition, the kinematic behavior of the coordinate system must be specified. For example, a particularly useful coordinate system is one which is at rest in the (inertial) lab frame. For other problems, however, it might be more appropriate to consider a coordinate system moving with uniform velocity, or one accelerating in some particular manner.

Many difficulties arise from the poor choice of a coordinate system or reference frame, or from the lack of any concrete, well-defined choice; in this way, the non-analytic thought of Section 4.2.4 manifests itself. It is curious to note that the example solutions in textbooks are quite careful to define coordinate systems and reference frames, but students have difficulty picking up these techniques.

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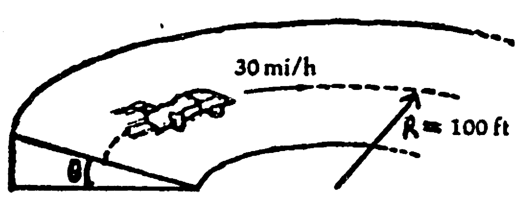
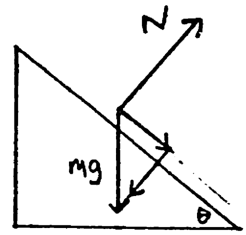
Example 5.4.a



A road is banked so that a car traveling 30 mph can round a curve of radius 100 ft even if the road is so icy that the coefficient of friction is approximately zero. Find the range of speeds at which a car can travel around this curve without skidding if the coefficient of friction between the road and the tires is

0.3.

Student has come in for help. We first start doing the problem without friction, and the student draws the following picture:



S: You have forces acting on it, mg down, and N. So I need to resolve these components into components, so gravity has a parallel and a perpendicular component. It's going in a circle, so there has to be a horizontal component, so I should have this [the parallel component] going in.

H: Wait a sec...what are you doing?

S: I want to take the horizontal component of this piece [the arrow parallel to the banked road].

H: You want to take the horizontal component of the weight?

S: Yeah.

This student vacillates between an explicit tilted frame, and an implicit untilted frame. The tilted system induces him to resolve the weight into components parallel and perpendicular to the road. However, he correctly realizes that the acceleration is horizontal, so he tries to find a horizontal force. However, he focuses on the wrong force. Had he remained in the explicit frame (taking acceleration components along the tilted x and y axes), he would not have gone astray. Better still, if he had oriented his coordinate axes so that the acceleration was along one axis, he would not have had to take acceleration components.

****

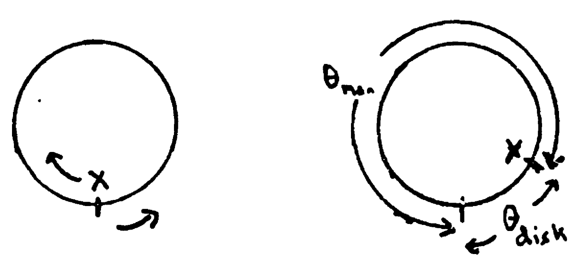
Example 5.4.b

A circular platform is mounted on a vertical frictionless axle. Its radius is $r = 5$ ft, and its moment of inertia is $I = 200$ slug-ft^2. It is initially at rest. A 150 lb man stands on the edge of the platform and begins to walk along the edge at a speed v_0 = 3 ft/sec relative to the ground. What is the angular velocity of the platform? When the man has walked once around the platform so that he is at his original position on it, what is his angular displacement relative to the ground?

> S: The man goes all the way around, so he goes 2π around the disk.
>
> H: OK, how can you formulate that?
>
> Student writes the following:
>
> $$\theta_{man} = \omega_{man} t = 2\pi$$
>
> $$I_{man}\omega_{man} = I_{disk}\omega_{disk}$$
>
> $$vt = 2\pi r$$
>
> H: Is that right?
>
> S: Well, I'm not sure, because the disk goes under as the man walks, but I'm not sure what else to do.

This student is unable to account for the movement of the disk underneath the man, even though he knows he should. His first statement involving θ_{man} is correct, for appropriate definitions of ω_{man}; if ω_{man} is ω_{man} relative to disk, then it is true. However, he conserves angular momentum in a lab frame fixed to the earth. In short, he confuses the lab frame with a frame attached to the disk.

Remaining in the lab frame would alleviate this difficulty. The following picture makes the problem transparent.



From the picture, it is clear that $\theta_{man} + \theta_{disk} = 2\pi$.

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Example 5.4.c

A dog, weighing 10.0 lb, is standing on a flatboat so that he is 20 feet from shore. He walks 8.0 feet on the boat toward shore and then halts. The boat weighs 40 lb, and there is no friction between the boat and water. How much closer is the dog to shore?

S: The position of the center of mass doesn't change, so

$$M_d X_d = M_b X_b$$

$$(10 \text{ lbs}) \ (8 \text{ feet}) = (40 \text{ lbs}) \ X_b$$

and solving for X_b, $X_b = 2$ feet, so since the dog moves 8 feet towards the shore relative to the boat, and the boat moves back 2 feet, the dog is 6 feet closer to shore.

This student correctly identifies the fact that the position of the center of mass does not change. As written, the mathematical statement corresponding to this fact implies a coordinate system fixed at the position of the center of mass. Consequently, all distances should be measured relative to this coordinate system. However, the student measures X_d relative to the boat (probably because that distance is given; see the discussion of Section 7.3.6 (Variables, Constants, Equations, and Ignorance)), and X_d relative to the center of mass. He unconsciously shifts coordinate systems.

****

In the above examples, the student vacillates in his choice of coordinate system; a definite choice must be made. This choice includes fixing the origin in a definite reference frame (most conveniently the unmoving lab frame), and orienting the coordinate axes in specific directions.

An interesting twist on this difficulty is reported by Dana Roberts (personal communication, 1978):

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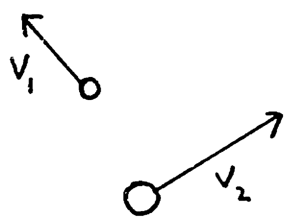
Example 5.4.d

In a lab, students were asked to find the components of v and p of two identical steel balls, before and after a collision. The balls were large enough that their radii had to be taken into account in finding the starting points of their motion. The displacements of the balls in a fixed time interval were measured, resulting in a picture like this:



The fact that the arrows did not start at the same origin confused many students when they were asked to find components. They also did not know what angles they were supposed to measure to get components, perhaps because a coordinate system was not provided. In other words, they were supposed to project the vectors along x and y directions (which, though orthogonal, are arbitrary in absolute orientation) rather than along fixed x and y axes.

****

This example raises the following issue: the difficulty was not in student inability to choose a coordinate system, or in their vacillation between different coordinate systems. Rather, it seems to lie in whether or not the student has the right to introduce a construct of his own (in this case, a coordinate system). [See also the discussion of Section 10.2 in which I suggest that there may be a correlation between dependence on external authority and poor problem solving performance.]

A third type of difficulty is the floating origin. Here are two examples:

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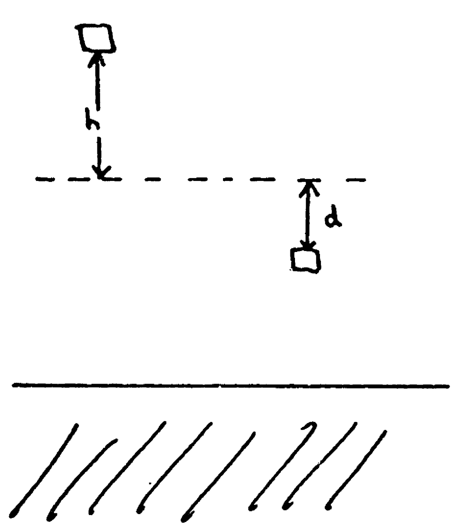
Example 5.4.e



H: What is the potential energy in the first figure?

S: mgd.

H: What is the PE in the second figure?

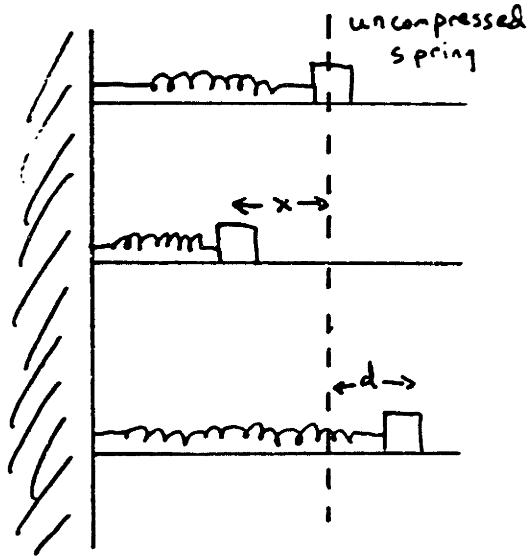
S: mg (d+h)

H: How come?

S: Because it's d+h away.

****

Example 5.4.f



A 5 kg block is held against a spring of force constant 20 N/cm, compressing it 3 cm. The block is released, and the spring expands, pushing the block along a rough horizontal surface. The coefficient of friction between the surface and the block is 0.2. If the block is attached to the spring so that the spring is extended when the block slides past the equilibrium point, by how much will the spring be extended before the block comes momentarily to rest?

H: What is the potential energy of the spring when it is compressed a distance x?

S: U(spring, before) = 1/2 kx^2

H: And how about when it is stretched a distance d?

S: U(spring, after) = 1/2 $k(x+d)^2$

In his last statement, notice that he goes to the trouble of determining the distance from the old position. He has no reason to go through this trouble, unless he really has shifted his point of reference.

****

In each case above, the student has unconsciously shifted his reference line. In the initial state, the student must measure the distance of the mass from some point - the only available reference point is the given reference line. In the final state, he must do the same thing. However, he now has two possible points from which to measure: the given reference line, or the initial position of the mass. The position of a block is less abstract than an arbitrary and abstract reference line, and so, in accordance with our minimal mental effort assumption, he chooses the former.

Section 5.5 - Functional and Numerical Equality

The formal structure of physics makes extensive use of statements of mathematical equality. However, equality can mean numerical equality (in the sense that 2+2=3+1) and it can also mean functional equality (in the sense that $y(x)=x^2$). Of these two, the latter is the more subtle, and is more demanding of mental effort. In particular, the former demands only that each side be evaluated, and one number for each side stored in memory for comparison. The latter demands recall of a relationship between the left and right hand sides which cannot be instantiated simply as one item such as a number.

Here is one student who does not understand functional equality.

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Example 5.5.a

An infinite wire (a solid conductor) of radius R_1, is supported by insulating disks on the axis of a conducting tube of inner radius R_2 and outer radius R_3. If the central conductor and the tube carry opposite currents of equal magnitude, find the magnetic field B everywhere.

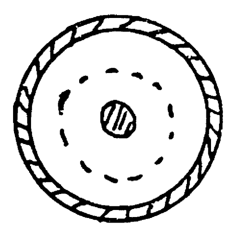
This student has come in for help, and I have asked him what the course is currently covering. She has said that the current topic is Ampere's law.

H: Do you think you can use Ampere's law for this problem?

S: $\int B\,ds = \mu_0 I$

H: how would you use it?

S: I'd draw a circle [the dotted circle inside] like this



[The shaded part represents conductor.]

and then say

$$B\,2\pi r = \mu_0 I$$

H: What is I?

S: The current.

H: Which current?

S: This one.

[With her finger, she traces a path following the dotted circle. This circle should represent an Amperian path of integration. She indicates that it represents the path of the flowing current. At this point, I realize she doesn't know what Ampere's law means.]

H: Tell me about Ampere's law. What does it mean, in words?

S: This left hand side is just the magnetic flux.

H: Tell me about this line integral. What does it mean?

S: You take a path, and you add up all the B's at every point along the path.

[She also doesn't understand what an integral is. After some discussion about the meaning of an integral, we resume.]

H: Do you see why it has to be a closed path?

S: No.

This student does not see a relationship between the right hand side and left hand side: the left hand side is a function of the path of integration, and the right hand side is also a function of the same path. Indeed, Ampere's law requires a closed path in order that the current have a well defined meaning. The current I is the charge that passes through a given surface in unit time. Without a closed path, it is impossible to define a surface, and hence impossible to have a surface through which charge must pass. This same student also did not realize the significance of the closed surface in Gauss's law - required to define $Q_{enclosed}$.

****

In short, there is a relationship between the right hand side and the left hand side which goes beyond a simple numerical one. Here is a second example:

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Example 5.5.b

The topic under discussion is Newton's Second Law.

H: What do the symbols in F = ma mean?

S: F is force, m is mass and a is acceleration.

H: Can you say more?

S: I don't understand - what more is there to say?

Contrast this to the following expert response to the same question:

Let's say we have an object, and we push on it. m is the mass of the object, F is the sum of all the external forces acting on m, and a is the acceleration of m.

Note that the expert's response explicitly ties F and a to m; the beginner's response contains no such ties. I deliberately chose not to ask about the existence of a relationship among the given symbols, because I wanted to see if the student would spontaneously mention such a relationship. If not, he would be unable to use the relationship in a meaningful way. In short, you can't use it if you don't think of it {1}.

****

More generally, the algebraic equations which physicists use in solving problems are subject to a variety of interpretations, both physical and mathematical. Here are six separate and distinct interpretations of equations used in physics problem solving {2}.

a. identity: $(x+3)^2 = x^2 + 6x + 9$. The equation holds true for all x.

b. meaning: $(x+3)^2 = x^2$. The equation implies a specific value for x; it assigns to x a specific meaning.

c. dynamic causality: F = ma. This physical interpretation is that it takes F's to cause a's.

d. dynamic constraint: $E_{init} = E_{final}$. Here, the equation constrains (but does not provide the mechanism for causing) the initial energy to be equal to the final energy. In other words, it expresses the remarkable fact that if you add up all the energies initially (according to some prescription for identifying energy) and then you add up all the energies in the final state (according to the same prescription, but independently), you get the same number.

e. global definition of symbols: $p = mv$. This equation establishes the convention that p stands for mv. For this usage, the "$\equiv$" is more appropriate, but physics books rarely use it.

f. local specification of symbols, in terms of information belonging to a particular problem or class of problems: $F = -kx$. In other words, the equation can map information specific to a particular problem or derived from a phenomenological relation onto a law of nature or a definition. In this example, it directs that $-kx$ is the characteristic of the environment relevant to a mass on a spring which enters into Newton's second law. To put it another way, the equation $F = -kx$ is a local specification (local to the particular problem under consideration) of the F which appears in $F = ma$.

Each of the above uses of equation is appropriate at some particular stage in a problem, but inappropriate (or at least unhelpful) at others. For example, causality, constraint, and mapping are useful in "setting up" the problem, i.e., in formulating (but not solving) the system of equations needed to solve the problem. Identity is used to manipulate the system of equations into a more manageable form. Meaning straddles the border between formulation and manipulation: as soon as the final equation is written (the one that closes the system of equations, thus terminating the formulation stage), all previous equations (initially written to imply causality or constraint, for example) take on the additional function of setting values.

Note well that each type of equation calls on the same symbol (=) to signal the fact that it is an equation. With the same symbol serving many different functions, mistakes in interpretation sometimes occur. For example:

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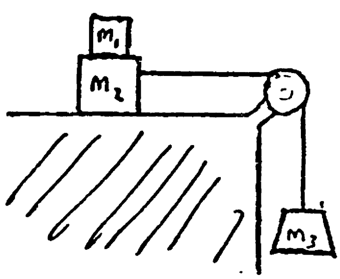
Example 5.5.c



In the figure, the mass m_2 = 10 kg slides on a smooth table. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. (a) What is the maximum acceleration of m_1? (b) What is the maximum value of the mass m_3 if m_1 moves with m_2 without slipping?

This student has come for help.

H: What laws can you use to solve this problem?

S: F = ma

H: OK, now what's F?

S: ma.

H: Not, not what is it equal to, what is it?

S: ma

H: What does F stand for?

S: The sum of the forces.

H: Good. What kinds of things go into that sum?

S: I don't know; I thought forces were ma's, since F = ma.

For this student, F is simply an abbreviation for the word "force".

****

Similarly, students consider the q in the integral form of Gauss's Law to be an abbreviation for the word "charge". These students do not bring with the symbol the rich commentary that the expert brings, e.g., F is the sum of all the external forces acting on m, or q is the charge enclosed by the surface specified by S on the other side of the equality sign. Note that the expert includes in his definition of F or q some reference to the other side of the equation, thus indicating his understanding that the equation denotes a relationship with a basis in physical reality.

Here is a second example:

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Example 5.5.d

The problem under discussion involves uniform circular motion.

H: mv^2/r is not a force.

S: Then what is it?

H: It's an ma.

S: But F = ma, so it's a force.

This student also sees F as meaning the same thing as ma.

****

In examples 5.5.c and 5.5.d, the student views the equation as meaning definition. Indeed, many physics texts "operationally define" force through the use of Newton's second law. However, the use of the term "operational definition" and the conventional (mathematical) use of "definition" are quite different. An operational definition specifies a procedure for ascertaining the meaning of a quantity, and as such, is more complex compared to the definition characteristic of standard mathematical terminology: if A is defined as B, then A can be substituted everywhere B appears.

The minimum mental effort assumption of Section 4.5 is illustrated in this belief that F is the same thing as ma. This erroneous belief also allows the student to avoid identifying aspects of the environment which should be included in F, thereby reducing the cognitive demands of the task even further. However, this belief violates the usual one of cause (F) and effect (ma).

The above two examples illustrate the use of equations as definition at too early a stage in the problem solution. Their premature use as definition or identity arises frequently: when faced with a perplexing problem, many people (beginning students among them) will attempt to carry out a small (and usually insignificant) part of the problem very well, thereby maintaining the illusion of progress. This should not be surprising in light of the fact that students are much more capable with mathematical and arithmetic manipulation (especially with calculators) than with other aspects of problem solving.

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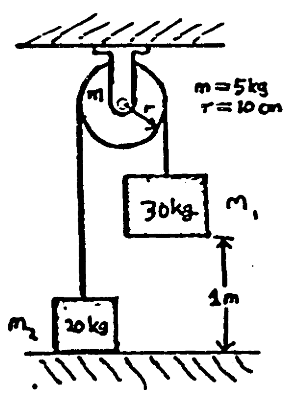
Example 5.5.e



The system in the figure is released from rest. The 30 kg mass is 1 meter above the floor. The pulley is a uniform disk of radius 10 cm and mass 5 kg. Find the velocity of the 30 kg mass just before it hits the floor. Do this problem using conservation of energy and also using F=ma with $\tau = I\alpha$.

S: I'm not sure how to use conservation of energy.

H: What does conservation of energy say?

S: That the change in KE + the change in PE add up to zero, like so:

$$\Delta PE + \Delta KE = 0$$

H: Now, how would you find ΔP?

S: If I can find ΔK, I'll know ΔP?

H: What does ΔP mean?

S: $P_f - P_i$

H: Good. What do the f and i mean?

S: After and before.

H: What's the corresponding thing for K?

S: $K_f - K_i = K_f$, since initially it's at rest.

H: OK. Now, how would you find K_f?

S: .... [pause]...

H: What is the KE of the entire system at the end?

S: You find ΔP.

H: And how will you get the velocity?

S: .... [pause]....

H: What pieces contribute to the total KE?

S: The change in PE.

H: If you have a system of particles all moving, what's the total KE of the system?

S: It's equal to the change of PE.

H: Forget PE... in terms of the individual m's and v's.

S: It's the change in PE of each one.

H: No... You add each KE up to get the KE of the whole thing.

S: OK, I get it.

From a conceptual point of view, the two sides of the equation (E_i and E_f) are separate. The student above does not keep them separate, does not attempt to decompose each side, and consequently cannot generate an expression for velocity. His actions treat the equation meaning constraint as one meaning definition or identity, and thus the student loses the physical significance of the conservation of energy.

++

In general, a dynamic equation has physical significance. In applying, foe example, the conservation of energy, one must determine the constituents of each side of the equation independently. The fact that energy is indeed conserved then allows us to equate the initial to the final energy.

By contrast, when the equation denotes definition or identity, it provides a logical but not physically significant step.

As applied to the problem above, we would define state i as in the figure above. The total energy of state i (E_i) is specified by looking at the figure to find what pieces contribute to E_i. We thus find that only the PE of m_1 contributes to E_i, so that $E_i = m_1gh$.

We define state f as the system with m_1 just before it hits the ground. We then find the energy of state f in a similar way, but independently of state i. We therefore find that

$$E_f = m_2gh + + 1/2\, I\, \omega_p^2 + 1/2\, m_1v^2 + 1/2\, m_2v^2$$

The fact that there is no energy dissipation (i.e., that energy conservation constrains the process of going from state i to state f) allows us to equate E_i to E_f.

A final comment: A few experts to whom I have spoken have noted the tendency among some students to work from left to right rather than from top to bottom. They observe that this tendency correlates with poor problem solving, and advise students to organize their written work vertically rather than horizontally. For example, the following arrangements are to be avoided:

$$\sin\theta = (1 - \cos^2\theta)^{1/2} = (1 - (b/c)^2)^{1/2}$$

$$F = ma = mv^2/r$$

However, the following arrangements are encouraged:

$\sin\theta = (1 - \cos^2\theta)^{1/2}$	$F = ma$
$\cos\theta = b/c$	
	$a = v^2/r$
$\sin\theta = (1 - (b/c)^2)^{1/2}$	$F = mv^2/r$

Note that the intermediate steps in the preferred arrangements are missing from the arrangements to be avoided. In the second example, the intermediate step is especially useful as it states explicitly that mv^2/r is an instantiation of a mass times an acceleration (ma) and not a force (F).

It is interesting to note that horizontal organization facilitates the notion of equations as implying identity or setting values. A vertical organization (in which one transforms each side of the equation separately) facilitates the notion of equations as implying causality, constraint, or mapping of problem quantities onto equation symbols.

Section 5.6 - Calculus as Used in Physics

Introductory physics courses use standard concepts of the calculus. However, the application of calculus to physics is subject to a number of subtle nuances.

5.6.1 - Derivatives as Rates of Change and Ratios of Infinitesimals

For example, the derivative of y is expressed as dy/dx. However, this can mean either that:

a. d/dx is an operator which acts on the function y(x) to result in an expression for the rate of change of y with respect to x.

b. dy/dx is the ratio of two differentially small quantities Δy and Δx in the limit as Δy and Δx approach zero (non-independently).

Interpretation (a) seems to be much easier for students to grasp. For example, I gave the following problem to students:

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Example 5.6.1.a

A particle of mass m_1 has a momentum $\underline{P}_1$ given by

$$\underline{P}_1 = (a + bt)\,\underline{i} + (ct^2)\,\underline{j}$$

A particle of mass m_2 has a momentum given by

$$\underline{P}_2 = (-a + et^2)\,\underline{i}$$

(a) What is the velocity of the center of mass of m_1 and m_2 at time t?

(b) What is the force acting on the center of mass at time t?

Though this problem is highly non-standard (a scan through the textbook in use at the time (Halliday/Resnick (1977)) reveals no problem even similar to it, and they had not been presented with a similar problem in the course), most students calculated the correct answer. Subsequent interview revealed that they simply differentiated $\underline{P}$ with respect to t to get $\underline{F}$. For these students, the procedural meaning of the derivative ($d\underline{P}/dt$ means do d/dt to $\underline{P}$) appears to be quite within their grasp.

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These same students have enormous difficulty with interpretation (b). For example:

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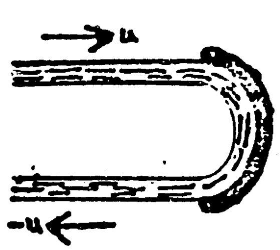
Example 5.6.1.b



A stream of water impinges on a stationary dished turbine blade, as shown in the figure. The speed of the water is u, both before and after it strikes the curved surface of the blade, and the mass of water striking the blade per unit time is constant at the value μ. Find the force exerted by the water on the blade.

Here is how an expert calculated F:

> F = dp/dt, where dp is the amount of momentum delivered to the wall in time dt. In a time dt, the stream delivers a mass dm = μ dt to the wall, which carries with it a momentum u dm. After hitting the wall, it has reversed its momentum, and so the net momentum change dp in time dt is 2 u dm = 2 μ u dt, so the dt's cancel and I have F = 2 μ u.

Notice that this approach requires a calculation of dp, which must be proportional to dt in order that F be finite.

****

This second interpretation is so hard for students to grasp that no student I observed made made a mistake in trying to use it - either a student's calculation correctly resembled the one above (very rare), or it did not resemble the above at all. Here are two examples:

****

Example 5.6.1.c

One student began this same problem by writing

$$m_1v_1 + m_2v_2 = \ldots\ldots$$

and then stopped when she was unable to complete the equation. She realized that momentum conservation was somehow related to the problem, but completely missed the force as momentum change approach.

****

Example 5.6.1.d

S: The answer is $F = \mu u$

H: How come?

S: Well, the dimensions are right.

H: Can you give me an argument for that? A proof?

S: Well..... I don't know. It looks right.

This student played around with dimensions. While this is certainly a reasonable approach in the absence of other knowledge, it is clear that he also does not have a working knowledge of force as momentum change.

****

Example 5.6.1.e

A second example of this ambiguity occurs with circuit theory. i is defined as dQ/dt. According to interpretation (a), Q is an amount of charge which should be differentiated in order to calculate the current. This is misleading because Q is everywhere the same in the steady state. However, a constant i does NOT imply a static situation; charge is flowing, but at a constant rate.

The intent of defining i as dQ/dt rests on interpretation (b) - the current is the amount of charge that flows by a given point per unit time. However, it shifts back to interpretation (a) in certain problems, e.g., when required to find the current through a capacitor:

$$Q = CV$$

Differentiating both sides with respect to time (as one does in the canonical approach) gives:

$$i = dQ/dt = C\, dV/dt.$$

As written and interpreted conventionally, V is a function to be differentiated with respect to time in order to calculate i. However, the meaning of dQ/dt has changed from "amount of charge dQ flowing in time dt" to "a function Q(t) which must be differentiated".

****

Interpretation (b) is fairly common in standard physics textbooks, and yet contains a very misleading element: it conceals an integration. For example,

$$i = dQ/dt$$

is really

$$\int i\, dt = Q$$

The interchangeable use of these two expressions actually rests on the fundamental theorem of calculus (the derivative of the integral of a function is the function), and yet this is a point which no physics textbook (to my knowledge) references.

5.6.2 - Δx as Change in x and as Infinitesimal Amount of x

A second mathematical ambiguity arises from the use of, for example, Δx. Δx can refer to either

a. the change in x from an initial value $x_{initial}$ to a final value x_{final}, or

b. the differentially small quantity Δx.

These are certainly consistent logically, but in practice, the different style in which each is used can cause confusion. Here is an example:

****

Example 5.6.2.a

The topic under discussion is the meaning of Gauss's Law.

H: Write down Gauss's Law.

S: $\int E \, dA = q$

[We begin a discussion of each symbol's meaning.]

H: What does dA mean?

S: It's the change in area.

H: What do you mean?

S: ....I guess.... It's the difference in .... I don't know.

This student interprets dA as a change rather than as an infinitesimal (as indicated by his unsuccessful search for two areas which can be subtracted to provide a difference).

****

More generally, the interpretation of Δx as a change in x requires the identification of an initial and a final value; Δx is the result of taking their difference. As a problem solving heuristic, this interpretation is most useful when a problem requires the calculation of a derivative. In the general case, the initial and final values (or corresponding spatial "boundary conditions") must be evaluated at instants of time separated only by a differentially small quantity Δt. However, in the special case that a quantity's rate of change is a constant, Δt need not be differentially small; its magnitude can be arbitrary. However, this special case so dominates all other problems which might involve non-constant rates of change that students learn to apply this heuristic improperly. Here is an example:

****

Example 5.6.2.b

A 1.5×10^6 watt locomotive accelerates a train from a speed of 10 m/s to 25 m/s in 6.0 minutes. (a) Neglecting friction, calculate the mass of the train. (b) Find the speed of the train as a function of time during this interval. (c) Find the force accelerating the train as a function of time during this interval. (d) Find the distance traveled by the train during this interval. Note: the acceleration of the train is not constant.

The student writes the following:

$$a = dv/dt = (v_f - v_i) / 6 \text{ min}$$

$$v_i = 10 \text{ m/s}, \quad v_f = 25 \text{ m/s}$$

The problem explicitly specifies that the acceleration is not constant, and yet the student pays no attention to this fact.

****

A second instance: Δx as an infinitesimal is most often required in problems involving definite integrals, and occur when one must convert a sum to an integral.

Here are two examples:

****

Example 5.6.2.c

A thin disk of dielectric material, having a total charge Q distributed uniformly over its surface, and of radius a, rotates with a frequency f about an axis perpendicular to the surface of the disk and passing through the center. Find the magnetic field at the

S: I'm not sure how to do this. I can see how there's a current, but I don't know what to do next.

H: Calculate the B due to a ring of current, and then integrate it.

S: I can do a ring... B = 2kI/r, but I don't know where to go from there.

H: Do you see how the disk is a bunch of rings?

S: Yeah...

H: Well, you can integrate that by doing $dB = 2k\ dI/r$.

S: Wait... why is that a dI?

H: dI is the current in one of those rings.

S: I mean, why shouldn't it be a dr or something.
****

Example 5.6.2.d

The topic under discussion is the notion of the center of mass.

H: For a bunch of discrete masses, we have

$$X_{cm} = (X_1 m_1 + X_2 m_2 + \ldots) / M_{total}$$

S: OK.

H: For a continuous mass distribution, we have

$$X_{cm} = \int x\ dm / M_{total}$$

S: Wait... why isn't that $\int m\ dx$?

H: Why do you want dx?

S: Because you always integrate over dx.
****

I conclude from instances such as these that students often do not see dM or dI as infinitesimal quantities with intrinsic meaning in their own right. These problems are constructed in such a way that one must eventually perform a spatial integration, but the insistence on a spatial integration at such an early stage could imply that they do not see dm or dl as being associated with (or contained in) a region of space characterized by dx or dr.

Section 5.7 - Perceptual and Representational Spaces

Section 5.2 (Imagery) discussed difficulties concerning pictures which convey a sense of the spatial relationships between objects in a given physical situation. These relationships involve what a person actually viewing the situation would see; therefore, it is reasonable to say that this diagram resides in perceptual space - for example, a Cartesian coordinate system in which diagram distances correspond to situation distances. A good example would be a blueprint of an object: one inch on the blueprint corresponds to 6 feet on the object.

This section discusses a more fundamental issue: the representation of any continuous quantity as lengths. These involve representational diagrams in which diagram lengths represent quantities other than distances: force, velocity, time.

I believe this issue is a fundamental one because it goes beyond a simple correspondence of perceptual quantities; it goes to the heart of what representation means (cf., Section 4.1 (The Values of Science) on representations). The fact that masses or forces or time can be represented as lengths is a considerable step of abstraction as compared to the representation of lengths by other lengths. [This also explains why I observed many more difficulties of the sort I document in this section, as compared to those of Section 5.2.1 (Literal Interpretations of Schematic Drawings).]

Representational diagrams come in two varieties:

a. vector representational pictures, which live in a vector space. Examples include force space (in which we draw free-body force diagrams), and momentum space (in which we draw arrows which stand for the momentum of various objects in a system).

b. mixed representational pictures, which live in a space in which each axis refers to a different quantity. The most common example is a plot representing the time evolution of anything, but other examples include the P-V diagrams of thermodynamics, and the potential well diagrams of both classical and quantum mechanics.

The most common confusion between representational and perceptual pictures is interpreting a represented quantity (e.g., a force represented as an arrow) as an actual length-like quantity.

Here is an example:

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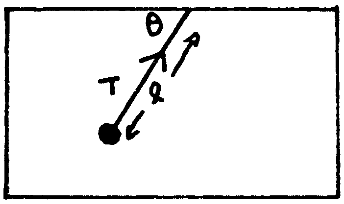
Example 5.7.a



The problem under discussion concerns a pendulum bob at the end of a string of length l, hung from the roof of an accelerated car. The string is inclined at an angle θ from the horizontal.

The student has come for help, and I begin my probe.

H: Can you express the y component of the tension in terms of T, the magnitude of the tension in the rope, and the angle.

S: It is sin θ times.... the length of the string.

H: Why? Why the length of the rope?

S: I'm not sure...

H: Do you know how to take vector components?

S: I think so.

H: OK, here is a vector A. What is its x component?

S: A cos θ.

H: Good. However, A could be any vector... F, p, V, a, etc...

S: OK.

H: Now, let's call it T. What is the y component of T?

S: T sin θ

H: What is θ?

S: I don't know... I can't draw a picture that I can see this problem in.

H: Why did you use θ?

S: Because it's in the problem.

H: What is θ in the problem?

S: θ is the angle from the horizontal.

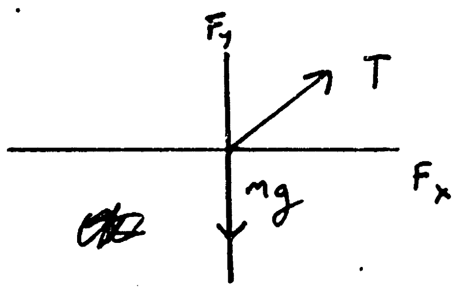
H: Point to it... Good. Where is T?

S: ....I don't know.... maybe along the string??

H: Right!

S: But how can you draw a picture and have a force on it??

Her last comment indicates an unease about the significance of the arrow. I now discuss abstract spaces with her, mentioning x-t plots, y-x plots, and v_x and v_y plots. I conclude the discussion by drawing an F_x and F_y plot, drawing what is in the figure below.



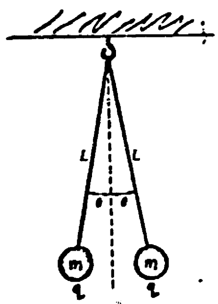
Note that the cross-out above is a mistake of mine. As soon as I cross it out, she says

"What is the object doing there?"

At this point, I immediately have to reassure her about my actions - that I was only crossing out a mistake and not putting a real object into a force-space picture.

****

Example 5.7.b



Two small spheres each of mass m = 10 gm are suspended from a common point by threads of length L = 50 cm. When each carries a charge q, they come to equilibrium, each thread making an angle θ = 10 degrees with the vertical, as shown in the figure. Find q.

S: This student draws the figure, and writes the following equations:

$$F - L \sin \theta = m\, a_x$$

$$L \cos \theta - mg = m\, a_y$$

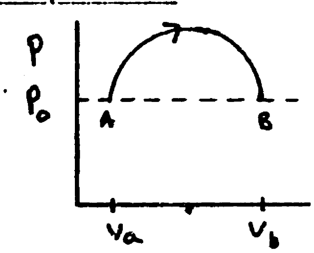
H: What is L?

S: The tension in the string... no, wait, it's the length of the string... I'm confused now.

Her first equation indicates an understanding of Newton's 2nd Law. However, she is confused by reference to her picture; she confuses the tension with the length as a result of the fact that L, a length, appears while she is thinking of something happening in force space.

****

Example 5.7.c



For the thermodynamic cycle shown at the left, what is the work done going from A to B in the direction shown? The cycle is a circle in P-V space, and you may give your answer in words or as an unevaluated integral.

The student wrote the following:

$$W = \pi r^2/2 + P_0(V_b - V_a)$$

This student is attempting to use the fact that the work done over any path in P-V space is the area under the curve. However, his use of $1/2\ \pi\ r^2$ indicates that he confuses the perceptual radius of the circle actually drawn with the other things which length represents in the problem.

****

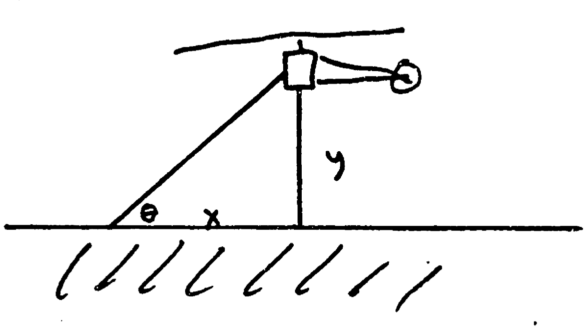
This example is a bit unfair, in that the use of graphical techniques pursued to their limit would give an appropriate answer if numbers were included. Still, the "radius" of the circle is a meaningless concept from a representational point of view, since its meaning cannot be well-defined. With a simple change of scale along one axis, the cycle would not even remain circular.

Here is another example:

****

Example 5.7.d

A helicopter is flying in a straight line over a level field at a constant speed of 4.9 m/s at a constant altitude of 4.9 meters. A package is ejected horizontally from the helicopter with an initial velocity of 12 m/s relative to the helicopter, and in a direction opposite to the helicopter's motion. What angle does the velocity vector of the package make with the ground at the instant just before impact?

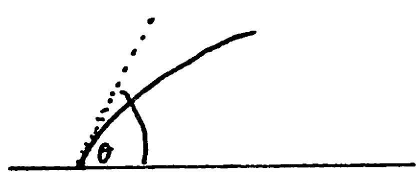


S: $\tan \theta = y/x$ [as in the figure immediately above]

H: What is θ?

S: θ is the angle with the ground that the package makes. It drops y and goes over x. It's the angle the path makes with the ground.

H: How does the particle know what path to follow?



S: It goes in a parabola like this [see figure immediately above] and the velocity is tangent to the path.

H: OK, draw in the velocity.

S: How can I draw in a velocity?

H: Draw an arrow which represents the velocity.

This student also has a slight confusion between represenlional and perceptual spaces. Her final question indicates a confusion with the idea of representing a velocity by a perceptual entity (the arrow). Her second statement indicates an understanding of the physical process involved. However, though she realizes the package travels in a parabola, she confuses the angle associated with the velocity vector at the time of impact with the angle associated with perceptual space: "dropping y and going over x". Notice that she substitutes the perceptual for the representational.

****

A final comment for this section: in addition, I have impressionistic evidence of a second type of confusion between vector representations (e.g., a plot of v_y vs v_x) and mixed representations involving time (e.g., v_x vs t). In particular, the imagery of a vector representation is static - it takes at least two "movie frames" to illustrate a change in a vector: before and after. By contrast, the imagery of a mixed representation involving time is dynamic - one need only look across the graph to determine if a quantity is changing with time.

Section 5.8 - Interpretation of Idealized Solutions

Most introductory physical problems pose situations which are idealizations of realizable physical situations. Consequently, the results which emerge are also idealizations. In Section 4.1 (The Values of Science), I pointed out that any representation is necessarily an abstraction; therefore, the mere fact of working with a representation will consume cognitive resources, which would not be consumed by more concrete manipulation. Some students have no idea of the fact that they are working with idealizations or representations. Here are two examples:

****

Example 5.8.a

An arrow while being shot from a bow was accelerated over a distance of 2.0 feet. If its speed at the moment it left the bow was 200 ft/sec what was the average acceleration imparted by the bow? Justify any assumptions you need to make.

> H: You didn't answer the question they asked. What assumptions are you making?

S: I don't understand.

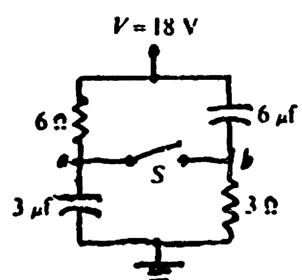
H: The problem asks for you to mention any assumptions you make in doing the problem.

S: You neglect friction?

The appropriate answer is that the arrow goes in a straight line. His last comment is a remembered statement. He has seen many problems which say "and friction can be neglected", so he says it too. However, he does not know why friction can be neglected, nor is he aware of making assumptions at all.

****

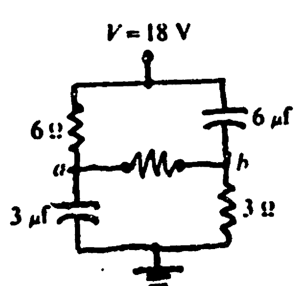
Example 5.8.b



What is the potential difference between points a and b in the figure below when switch S is open?

S: I don't understand this problem. How can you have a potential if there's an open switch there?

H: OK, let's pretend that it's a resistor, like this.



Then do you have have a potential?

S: Yes.

H: OK, now let's let R get real big. Will you still have a potential?

S: Yes.

H: Now, how about if R goes to infinity - like an open switch?

S: But if R goes to infinity, V = IR and V goes to infinity.

H: But if you have an open switch, I goes to zero.

S: But you were talking about a resistor.

From what precedes this excerpt, the student is inquiring about the meaning of a potential across an open switch, rather than about its value. (This student was also not sophisticated enough to be thinking of potential as opposed to potential difference; his use of the word "potential" was simply a short-hand for the phrase "potential difference".) This excerpt underscores his inability to model a switch as a resistor of either zero or infinite resistance.

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In other cases, students may interpret the idealized problem or the resulting answers without regard for the idealizations which lie in between them and the actual physical situation in question. Here are two examples:

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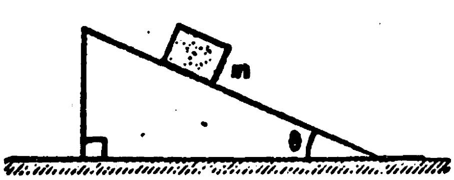
Example 5.8.c



A block of mass m rests on an immovable ramp of incline θ. The coefficient of kinetic friction between the block and the ramp is μ. The block is then released. Assuming that μ is small enough for the block to accelerate, what will be the acceleration of the block down the ramp?

Student does F = ma analysis correctly, arrives at answer:

$$a = g \sin \theta - \mu g \cos \theta$$

H: OK. Can you check it for me? For example, what happens when you check it for special cases of θ?

S: Well, when θ = 90, sin θ = 1 and cos θ = 0, so a = g.

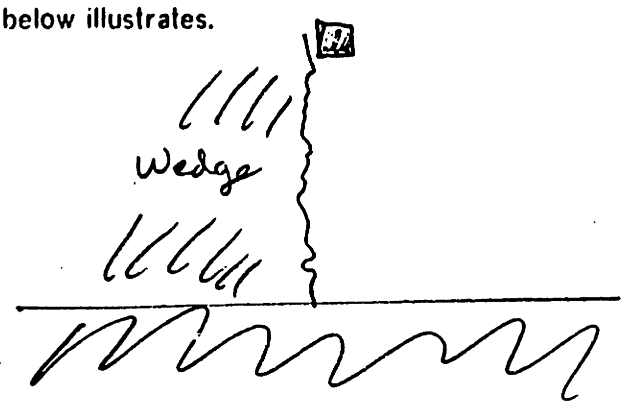
H: Does that make sense?

S: I'm not sure... shouldn't friction still affect it?

H: Why should it?

S: Because even though friction gets less, even when θ = 90, it's still touching the ramp.

This student clearly realizes that the frictional force will vanish as θ approaches 90 degrees. However, he does not realize the implications of idealizing the wedge as a perfectly straight and flat surface. With such an idealization, it is reasonable to define "touching the ramp" as the existence of a non-zero normal force between block and ramp. Without this idealization, we would have to consider the fact that a lumpy ramp might well impede the free fall of a block, as the figure below illustrates.



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Example 5.8.d

S: You know that problem with the stick that falls straight down on a frictionless surface? Well, they always say that the center falls straight down, and I never really believe it... it just seems that like that could never really happen.

H: Would you feel better if they said that it falls almost straight down, and the less friction there is the straighter it falls?

S: Yeah, but it could never really fall absolutely straight.

This student is correct in this last statement, but it indicates a substantial discomfort with even the idea of a perfectly straight fall. He is reminiscent of calculus students I have had who insist that "there is always some error left" when you go to the limit in taking derivatives.

****

More generally, the lay person's view of science (which quite a few beginning students share, at least in substantial part) is quite different from the professional's view. In particular, the lay person views science as exact, precise, and immutably logical - hence, its results are Right, if there have been no mistakes along the way.

When students first begin their study of physics, they often share such views. Indeed, Perry (1970) discusses the fact that many students entering the university have a rather dualistic view: propositions are either true or false, and it is their job as students to generate true propositions, and forego false ones. Perry further points out that students often elect to major in the physical sciences as a result of their inability to cope with the ambiguity they find in their humanities classes. For these students, this is a way of preserving their dualistic epistemology.

It is these students who are most unhappy with qualitative arguments, estimation and order of magnitude calculations, insight questions (as opposed to standard problems), and as these examples illustrate, models which only approximate the truth.

These students are the same students who believe that a numerical answer calculated on a pocket calculator to seven digit "accuracy" is more right than one estimated on the basis of $\pi = 3$ and $\pi^2 = 10$, even when the data are given to only two decimal places.

On the other hand, these students are making an (admittedly primitive) attempt to understand the problem and their own solutions in terms of the actual physical situation, and for this, they deserve credit.

Section 5.9 - Identification of System States

Section 4.4 (On Physics Problems: States and Time Evolution) related system states to time evolution; recall that time evolution is by definition an ordered succession of states infinitesmally different from those immediately preceding and following.

In the introductory physics course, the notion of state occurs (although almost never explicitly) in mechanics problems which include an inquiry about some state of the system.

Identifying a state requires freezing the situation at some instant t, and identifying the values of the variables which characterize that state. A student unaccustomed to analytic thinking (cf. the discussion of Section 4.2.4 on non-analytic thought) may be unable to pick out a definite state. Here are two examples.

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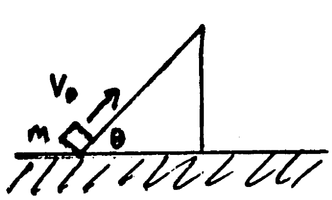
Example 5.9.a



A 2kg box is projected with initial speed of 3 m/s up a rough plane inclined at 60 degrees to the horizontal. The coefficient of friction is 0.3 How far does the box slide along the plane before it stops momentarily?

S: I can't do the problem with friction...

H: OK, can you do it without friction?

S: Sure, that's easy... all you do is set the KE equal to the potential energy, so

$$1/2\, mv_0^2 = mgh = mg\, x \sin\theta$$

[x refers to the distance along the inclined plane.]

H: OK, now with friction, what happens?

S: I'm not sure, because I'm not sure how v_0 changes.

H: Why should v_0 change?

S: Because there's friction which slows it down everywhere.

This student sees the process involved (sliding up the friction-ful hill) affecting the initial condition. He does not understand the fact that initial conditions are set arbitrarily. He does not keep process and initial conditions separate.

****

Example 5.9.b

The handle of a floor mop of mass m makes an angle θ with the vertical direction. Let μ_k be the coefficient of kinetic friction and μ_s be the coefficient of static friction between the mop and floor. Neglect the mass of the handle. Find the magnitude of the force F directed along the handle required to slide the mop with uniform velocity across the floor.

Subject writes:

$$\mu_s (F \sin\theta + mg) < F \cos\theta$$

H: Why μ_s?

S: Because you have to overcome static friction to get it moving, and as you push it harder, it'll eventually move.

Here, the student does not separate the ongoing process from the start-up process. A focus on the initial start-up from zero to F being sufficient to overcome static friction prevents him from focusing on the ongoing aspect of the problem, namely the fact that the question inquires about the system after it is already in motion.

****

Alternatively, some students identify states which are not relevant to the problem, choosing states belonging to inappropriate instants of time. These students may have trouble in keeping these states conceptually separate. Within our assumption of minimal mental effort, we can see that differentiation between and coordination among various states requires a higher level of processing than does a global consideration of everything as one.

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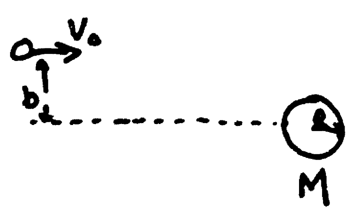
Example 5.9.c



Far away, a satellite with initial velocity v_0, mass m, and impact parameter b, travels toward a planet of mass M as in the figure to the left. What is the maximum value of b such that the satellite will still hit the planet?

H: OK, what things are conserved?

S: Well... angular momentum is, but energy isn't.

H: Why not?

S: Because when the satellite hits the planet, it stops, and that's an inelastic collision, and energy isn't conserved in inelastic collisions.

The solution of this problem requires an understanding of the term "maximum value". When b is at its maximum value, the satellite will just graze the planet; for smaller values, it will hit, and for larger values it will miss. The proper final state is the satellite just at the planet's surface.

However, the student, focusing on the collision, considers the final state to refer to the satellite after the collision, perhaps by analogy to the final state in the standard inelastic collision type of problem.

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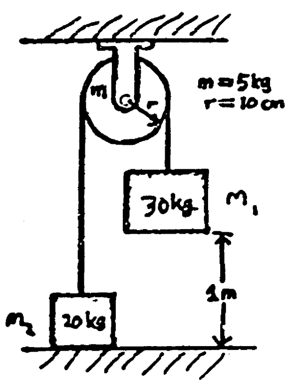
Example 5.9.d



The system in the figure is released from rest. The 30 kg mass is 1 meter above the floor. The pulley is a uniform disk of radius 10 cm and mass 5 kg. Find the velocity of the 30 kg mass just before it hits the floor. Do this problem using conservation of energy and F=ma with $\tau = I\alpha$.

H: What forces act on the system?

S: The ropes, gravity, and the ground.

H: Why the ground?

S: Because it hits the ground.

The problem sets up an initial state (with m_1 on the ground) which evolves (as m_2 falls) into a final state (with m_2 on the ground). It inquires about a process parameter: the velocity of m_2 while it is still falling. The student merges the final state with the process.

****

In Examples 5.9.c and 5.9.d, difficulties occur because each problem implies a final state of configuration in which differentially small changes in configuration would result in qualitatively different final states.

In Example 5.9.c, if b were just a bit smaller, the satellite would definitely collide with the planet, yet if b were just a bit bigger, it would definitely miss. Any satellite would have a finite radius, and it is just this finite radius that would determine if it would "really" hit or not. But the student is expected to assume a point satellite.

In Example 5.9.d, if we were to wait just an instant longer to determine the velocity of m_2, it would be easy: v_2 would be zero, since m_2 would have already collided with the ground.

Finally, here is a student who does not realize that a problem is soluble within a state variable formulation.

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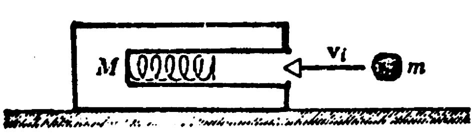
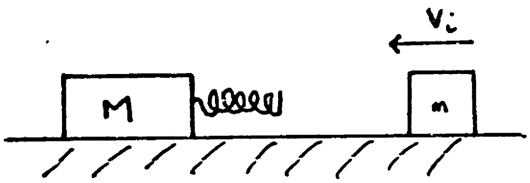
Example 5.9.e

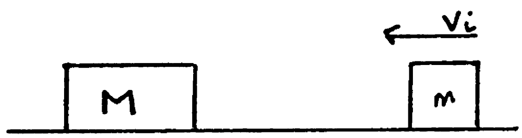


A ball of mass m is projected with a speed v_i into the barrel of a spring gun of mass M initially at rest along a frictionless surface. The mass m sticks in the gun at the point of maximum compression of the spring. No energy is lost in friction. What fraction of the initial KE of the ball is stored in the spring?

The student has come in for help on this problem, and after a while, we get onto the following problem.



Consider two blocks colliding inelastically, in one case with a spring between them that locks at the point of maximum compression, and the other case without the spring. Same initial velocities, both inelastic collisions. Find the final velocities of both blocks moving together in each case. Will they be the same in both cases, or different?



S: They're different, because it takes force to compress the spring... Let's see.. how does force relate to momentum... Well, the spring uses up energy....

H: What if the spring weren't there?

S: It would hit the block and the block would push back on the first block, and the second block would move.

H: What's different with the spring?

S: Well, the spring eats up energy.

H: And without?

S: It goes into heat.

This student focuses on the interaction between the blocks in detail, rather than on the initial and final states, even though the problem asks about the final state.

****

Notes

1. This phrase comes from Schoenfeld (1979).

2. It is possible to argue that it is the equality sign itself that carries with it different interpretations; this is the point of view taken by some researchers in mathematics education (cf., Davis (1975)). I do not disagree with this argument; however, I believe it is mostly a matter of taste whether one focuses on the equations or the equality sign.

Add little to little and there will be a big pile.

- Ovid

Chapter 6 - Generation: Problem Solutions as Arguments

Chapters 6 and 7 consider what it means to generate the solution to a problem. In actual practice, this requires both planning the solution, and manipulating the resulting equations to arrive at the desired answer(s). However, I have chosen to focus on the phase that seems to give students the most difficulty: identifying the necessary equations and formulating them in a manner useful for solving the problem. By comparison, symbolic and numerical manipulation pose fewer problems for many students.

In actual practice, generation and understanding alternate, and they cannot be rigorously separated. As a result, I might discuss some matters in Chapters 6 and 7 which other reasonable people might include in Chapter 5 (Understanding).

The quantitative form of solution which physics teachers traditionally expect implies a certain approach to solving a problem. Therefore, it is important to address two distinct questions:

a. Do students employ different approaches of understanding, and if so, what are they? The answers to these question are the focus of Chapter 6 (the current chapter).

b. Given that a student tries to work within a quantitative approach, how does he function in this attempt? Chapter 7 (Generating a Quantitative Solution) will focus on this question.

This chapter concerns the nature of problem solutions as arguments. Arguments may either qualitative or quantitative. A problem solution requires an answer and a quantitative argument for that answer (i.e., a derivation of that answer). Such a quantitative argument consists of the general principles (both physical and mathematical) relevant to the event, and the given problem information specific to the event. Once this information is mapped onto these principles, they deductively transform the given information into the desired conclusion, i.e., the answer.

A problem solution requires an understanding of the problem. The problem solver must know what is being asked, what is given. In addition, he must understand how the physical situation under consideration corresponds to the model which the problem statement represents. Finally, he must be able to identify his solution as the one desired, check its correctness, and interpret it in terms of the original physical situation; he must be able to "un-model" his solution.

Since a problem solution is essentially an argument, the remainder of this chapter discusses what argument is and is not.

Section 6.1 - Criteria for Argument

In general, a problem-solving argument must satisfy the following criteria:

a. it must employ an approach which results in a correct answer
b. it must conclude with a statement of an answer in terms of known quantities
c. it must be self-consistent
d. it must be testable
e. it must be based on general principles

Many of these criteria were first mentioned in Section 4.1 (The Values of Science), and this section will discuss these criteria and others in greater detail.

6.1.1 - Arguments Must Use An Appropriate Approach

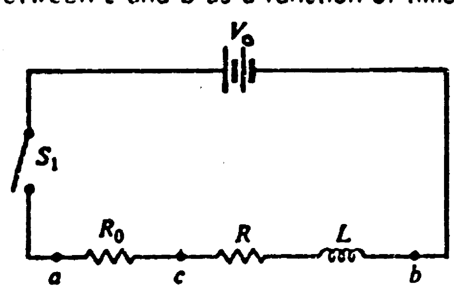
Most importantly, a good argument must employ an approach which in fact concludes with an expression for the desired unknown. Obvious this may be, but this may constitute a psychological barrier to some students, especially if a particular inappropriate approach is intuitively appealing, or if the correct approach is difficult.

Here is an example:

****

Example 6.1.1.a

An inductor of resistance R and self-inductance L is connected in series with an non-inductive resistor of resistance R_0 to a constant potential difference V_0 as in the figure below. Find the potential difference between c and b as a function of time.



S: I got the value of V at t = 0, but I don't know what to do now. How do I put in L?

H: You have to write down Kirchhoff's voltage law for the whole circuit.

S: But isn't that very long and complicated?

H: So?

S: But I don't want to do that; there should be an easier way.

In fact, KVL is not particularly complicated, but the student resists its use. [I suspect that the reason is that KVL doesn't give him the answer immediately: hence, his comment that it is complicated. See also the discussion of Section 6.1.5 (Arguments Must Be General).]

****

Both the expert and the beginner share the search for "a simpler way" to do problems. However, there are different forms of simplification. One form - the form designed by physicists - is constrained. Within this form, it is desirable to simplify by the elimination of distracting and non-essential detail from the model of the phenomena in question. However, the argument must still conform to the fundamental principles of physics, the rules of formal logic, and the values of science (e.g., science should be publicly verifiable).

By contrast, a second form - one desired by many beginning students - is not bound by such rules. These students may instead wish to use simpler reasoning strategies (e.g., non-analytic reasoning) or simpler concepts (e.g., quantities not involving ratios) with which they feel comfortable. Thus, as in the above example, students often wish to assimilate new material on the basis of old (and usually inadequate) ways of thinking. The student above apparently feels that simplicity is of paramount importance - in fact, more important than the use of an approach assured to give the correct result {1}.

6.1.2 - Arguments Must Use Only Known Information

A problem solution must conclude with a statement of the answer in terms of known information. This may be obvious, but here are two students who stumble over this point.

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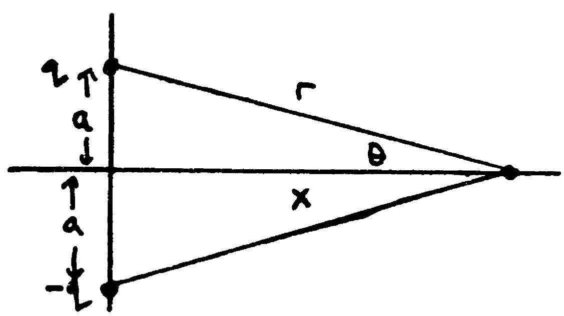
Example 6.1.2.a

Show that the field along the x-axis far from a dipole is inversely proportional to $1/x^3$.



A student has written the equation below and requests help.

$$F_1 = k\,q/r^2$$

$$x/r = \cos\theta$$

$$F = 2\,F_1 \sin\theta$$

$$F = (2\,k\,q^2 / x^2) \sin\theta \cos^2\theta$$

S: I don't know what to do now. They say $1/x^3$, and I can only get $1/x^2$. I don't know what to do now.

This student did not realize that the next step was to eliminate θ. If he had set as his goal the elimination of all but given quantities from his answer, he would have been forced to consider an algebraic expression involving x and a, at which point the problem would have been much easier.

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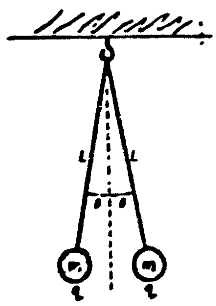
Example 6.1.2.b



Two small spheres each of mass m = 10 gm are suspended from a common point by threads of length L = 50 cm. When each carries a charge q, they come to equilibrium, each thread making an angle θ = 10 degrees with the vertical, as shown in the figure. Find q.

A student has written the following equations:

$$F = k\, q^2/r^2 = mg \tan \theta$$

$$T = mg \sin \theta$$

$$\sin \theta = r/l, \text{ so } q = r\,(rT/kl)^{1/2}$$

S: I don't know what to do next. Am I done?

Neglecting the mathematical mistakes in this solution, it is more significant that this student does not see that an answer should be given in terms of the known information.

****

6.1.3 - Arguments Must Be Self-Consistent

A theory should not predict two different outcomes for the same event. This self-consistency requirement is really a variant of the requirement that an argument use only correct statements (provided these statements themselves come from a self-consistent theory). It is different only in the sense that an argument may appear to be correct on its face, but through a different though equally correct-appearing argument, one might reach a different conclusion. This difference implies that some statement in at least one of the arguments is incorrect. However, one needs a coordinated view of both conclusions together in order to draw this inference; one cannot identify contradictions by looking only at one conclusion.

Here is one student who even identifies both conclusions, and yet does not realize the contradiction:

****

Example 6.1.3.a

The problem under discussion is a rotating space station of radius r, with a person standing inside it.

S: You have the space station moving with velocity v, and the acceleration down..

H: What's "down" mean?

S: The radial acceleration.

H: Of what?

S: ... [pause],... it's $v^2/r = a$.

H: But a is the acceleration of what?

S: Anything that might be there.

H: You mean all these things accelerate outward? Like this guy here?

S: He's not accelerating.

H: You said something's accelerating out.

S: It's like this pushing in.

H: Huh? What's "this"? What's it pushing in on?

S: In this circle, the person pushes out from a force created by mv^2/r.

H: Why?

S: Because v^2/r = acceleration outward.

H: Acceleration of what?

S: Of anything that's there.

H: So the man and the block over on the other side.. they're all accelerating out.

S: Well, they're not moving outward, so they can't accelerate out.

This student behaves as though he does not believe in classification logic: "anything" would presumably include men and dogs, and yet he argues that the latter are not accelerating.

****

I suspect that this is an example of a situation in which "physical intuition" is stronger than "faith in formal logic"; his intuitive notion of acceleration as a form of velocity overrides the formal logic of classification. Indeed, a number of studies (cf., Henle (1962)) have suggested that when people are faced with a logical argument they dislike, they tend to reject the argument rather than the conclusion.

6.1.4 - Arguments Must Be Empirically Testable

An argument must imply definite knowable consequences. One way of revealing these consequences might be observation; one question might be "How could you test that your argument is incorrect?" A second way of revealing these consequences would be the identification of a cause which leads to a given effect; one question might be "What would you be able to see if you watched the event happen?" However, Divine Revelation and intuitive feelings are not acceptable - what is "really happening" is irrelevant.

Here are two students who do not believe in observationally falsifiable arguments:

****

Example 6.1.4.a

The topic under discussion is the meaning of Newton's Third Law. I am trying to demonstrate that without an external force, the center of mass of an object cannot move if it starts from rest.

H: It is impossible to lift yourself up by your bootstraps.

S: For me, that's true.

H: But how about if you were infinitely strong? If strength weren't a factor?

S: Then sure I could.

H: How come?

S: Well, you can lift yourself up by pulling on a bar; you can do chin-ups.

H: But that's because the pipe is external to you. Your bootstraps are part of you.

S: How about pulling on a rope?

[We discuss two or three other ways; in each case, I point out that it is something external to the student's body.]

S: Well, isn't there any way of doing it?

H: No.

S: But that's not impossible - it's just that there's no real way to do it, but it's still possible.

This student draws a distinction between genuine impossibility and the individual impossibility of each and every proposed scheme; she acknowledges the latter but denies the former.

****

Example 6.1.4.b

I recall that when I first studied the Lorentz contractions in special relativity, I insisted on asking questions like "But what is its length really?" and "But which one really gets shorter?" I recall very well my discomfort with the idea of "reality" being dependent on what an observer would see. In retrospect, I was unable to accept the fact that "what was happening" depended only on what characterized the difference between two reference frames (i.e. their relative velocity) and not directly on what some "neutral observer" knew was really happening.

****

The requirement imposed by testability is quite distinct from the requirements imposed by the rules of logic. The latter need not answer to the physical world; they can stand or fall in an entirely algorithmic and deductive manner. By contrast, the former require judgment, since testability implies observation of some sort. (Modeling a physical situation falls into the same category - both must pay attention to the real world.) In turn, the necessary judgment requires belief in and acceptance of philosophical criteria which cannot be imposed by logic alone (cf., the discussion of Section 4.1 (The Values of Science) on the role of human choice in science).

To put it another way, it seems that the student above is willing to imagine a science fiction world in which a person could pull himself up by his bootstraps; however, she would repudiate any world in which 2 + 2 is not equal to 4. In short, it is the difference between science and mathematics; the latter is entirely algorithmic, while the former involves references to actual experience {2}.

6.1.5 - Arguments Must Be General

Arguments should be based on principles of broad generality., i.e., principles which are applicable to a wide variety of situations. However, students often attempt to use the relations most specifically related to the problem at hand.

For example, some students try to generate one-step solutions. I asked students to "write a few paragraphs describing your problem-solving process" and to "tell me how you plan problem solutions". One group of students said they look for "one-step" solutions by trying to find an equation that fits the problem exactly.

> I look at all the variables and try to come up with an equation which ties them all together.
>
> I try to find one equation which relates the givens and unknowns. I try to use what the book gives you and not derive everything.
>
> When I'm given several values, and I know there's a formula that will work, I look for it in the book if I don't remember it right off.
>
> Draw picture, and note givens and unknown. Try to find an equation which related the givens with the unknown.

Further discussion with students reveals that they believe finding a one-step solution will save time because they don't have to work out all kinds of "irrelevant" things. In particular, I asked the following question: "When you try to solve a problem, do you usually tend to work from general principles or from very particular cases? For example, let's say I were to give you a two-pulley problem and asked you to calculate the acceleration of some pulley. You could work it out in a number of ways. You could start with F=ma and work everything out from scratch, or you could take the one-pulley equation $a = (m_1 - m_2)/(m_1 + m_2)\ g$ and try to make that equation fit the two-pulley problem. Which would you be more inclined to do?"

More than half said they would use the one-pulley equation. Here are some typical reasons why.

> I'd think F=ma but say that there must be a specific equation so I'd try to find it.
>
> You know the specific case is directly applicable. It's more applicable to the problem and I'm less likely to forget something.
>
> It's more related to the problem. I sometimes can't see how F=ma applies, but I can see how it's a pulley.

In a quick survey of one physics recitation section, over half knew that physicists work from general principles. Nevertheless, they themselves preferred to work from special cases. Needless to say, their work suffers from this tendency. Here are a few examples.

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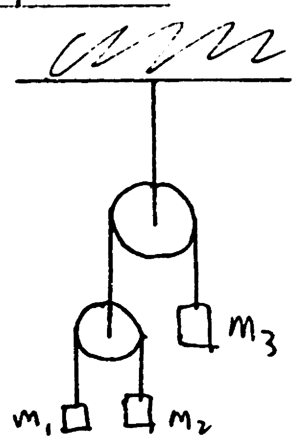
Example 6.1.5.a



Consider the pulley system shown at the left. All pulleys are massless and frictionless, and all ropes are massless and inextensible. Find the acceleration of each mass and pulley, and the tension in each rope.

One student wrote:

$$T - (m_1 + m_2)g = (m_1 + m_2)a$$

This student is pattern matching to a remembered solution to the canonical Atwood's machine problem. He tries to force fit it to the two pulley solution.

A second student wrote:

$$a = (m_1+m_2) - m_3 \; / \; (m_1 + m_2 + m_3)$$

trying to force-fit the one-pulley equation to the solution of this problem.

****

Further inquiry reveals several reasons for this behavior.

a. There are restrictions on the application of most physical principles. For example, $\Sigma F_{ext} = dP_{cm}/dt$ applies only to the center of mass of a system. In addition, though all physical laws hold true all the time, a particular law may or may not be helpful in the solution of any given problem. Finally, the information to which these principles must be applied must be precisely defined and well-formulated. One cannot just say "mass"; one must specify which mass.

As a result, there is a tension between principles which apply in general, and the conditions under which these principles apply. Both the student who wants to apply a "general principle" indiscriminately and the student who wants to apply the most specifically related equation to the problem lack the judgment necessary to balance these two issues.

In addition, one never applies a principle in exactly the same way twice. The general features of the application may be similar, but the detailed execution is never identical. Of course, this is the nature of a general principle; it is the abstraction (i.e., the process of omitting irrelevant detail) common to a large number of physical situations that gives a general physical law its broad applicability.

The result is a second conflict. On one hand, application of general principles is central to the argument of a physicist, who sees "general principles" as implying the same form of abstraction process from problem to problem. On the other hand, a student focuses on "application," which is different from situation to situation. If two problems are not exactly the same, then they are different. The student has his greatest difficulty identifying and formulating the specific information to which these general principles apply. Furthermore, teachers tend to give credit for specific application and detailed execution more easily than for identification of relevant principles.

b. Teachers often emphasize the exceptions to the rules, usually with the argument that "We want to make sure that they really understand and aren't just memorizing." While this is a useful pedagogical tool for teaching students the limits of their knowledge (and in fact is unavoidable), it leads, when carried to excess, to fragmentation of knowledge and a feeling that different rules must be applied in different situations: common sense applies in these situations, but physics applies to these other situations.

Students often say "Common sense never helps you in doing physics. If common sense is involved, it can't be right; we figure you must be trying to trick us." Further investigation reveals that some teachers often discuss paradoxical situations as examples of the theory. Indeed, there must be some situations in which "every-day" reasoning does not work, or else physics as a theory would be of no use. In order to test these students, these teachers feel they must use problems in which every-day reasoning does not work.

It is therefore no wonder that some students refuse to trust their untrained and naive intuitions. While it may be desirable from a pedagogical point of view for students to mistrust their incorrect intuition, this mistrust easily leads to a belief that all intuition is incorrect and inappropriate to use; they come to believe that intuition cannot be trained.

c. Physics is introduced as the unifier of many disparate sets of phenomena. On the other hand, as a typical course progresses, different phenomena are introduced as different physical concepts are introduced. Even within classes of phenomena, physics teachers often try to make up problems which are really the same but which appear different on the surface.

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Example 6.1.5.b

For example, the most usual textbook approach to energy is to start with work and kinetic energy. Then it continues with potential energy, and then finally to the conservation of mechanical energy. It closes by pointing out situations in which the conservation of mechanical energy does not hold. Contrast this approach to an approach which begins with conservation of energy being valid under all circumstances, except that one might not be able to write expressions for all parts of the energy. [See also the discussion of Section 9.2.9 (Identification of States).]

****

Section 6.2 - What Problem-Solving Argument Is Not

A physicist's argument refers to a set of facts and propositions, some of which are consequences of the others; this is what experts mean when they say they can derive everything from a few general principles. However, it is not necessary (or even advantageous) to employ this style of reasoning in everyday life. In fact, most people employ a non-analytic reasoning style. Instead of a chain of reasoning which says that A implies B which implies C which implies D, people might simply group A and D. This type of statement easily evolves into a statement to the effect that "A causes D" or that "D because of A". This level of understanding is sufficient for everyday life; for indeed, life is too short to require analytic thought everywhere. Nevertheless, for physics, this understanding is crucial, and students often do not understand what the physicist's argument or argument is.

6.2.1 - Correlative Argument

One common type of pseudo-argument is correlative argument, which refers to a package of facts which happen to be highly correlated in a person's experience. For example, instead of saying that contact with a hot radiator leads to heat transfer, which then leads to burns, a person may simply reason that burns and hot radiators come together. In other words, he associates those two items in a package; however, he says that "hot things cause burns" without reference to the underlying mechanism by which "hot things cause burns".

6.2.2 - Redescriptive Argument

A second form of pseudo-argument is redescriptive argument, in which a person "explains" an answer by essentially redescribing the question, or stating a definition. Here is an example:

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Example 6.2.2.a

H: Why is it harder to climb steep hills rather than shallow ones?

S: Because the angle of attack is less for shallow ones.

Indeed, this is all we need to recognize a shallow hill, and we all know that it's easier to go up a shallow hill. This student has taken a year of introductory physics at the college level, and yet he merely redescribes (or redefines) a "steep" hill. Note the absence of any mechanism or theory.

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Example 6.2.2.b

A group of students had just finished two weeks of work on geometric objects (mostly squares, circles, and triangles), measuring areas, diameters, circumferences, perimeters, and comparing them with each other. The following exchange took place in a small group discussion:

H: What is the same about all circles?

S: They're all round.

H: Is there anything else?

S: Nope...

It is important to note that these students had already encountered and even determined for themselves the fact that $C/D = \pi$.

****

Example 6.2.2.c

The situation under discussion involves the kinematics of an object moving in a circle.

H: Is there any radial velocity?

S: No.

H: Why not?

[I am looking for the explicit identification of some characteristic of a circle that would require the radial velocity to be zero.]

S: .... because it's not moving in that direction.

H: But why is the radial velocity zero?

S: Because it's moving a circle.

H: That's just saying it's a circle.

S: Let's see, it's accelerating in, because I'm pulling it in.

H: So why doesn't it come in?

S: Because it's going tangentially too.

H: So what? Why is the radial velocity zero? What is there about a circle that means it can't have a radial velocity?

S: Its radius is constant.

He redescribes the situation, saying that the object goes in a circle. I am forced to direct his attention to the mathematically significant and well-formulated properties of all circles. Only in his last statement does he discover a useful property or a circle.

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6.2.3 - Analogical Argument

A third form of pseudo-argument is analogical. In a sense, analogical argument can be characterized as reduction to the familiar. These arguments can be quite persuasive, since they make direct contact with notions with which one may feel quite at ease. Indeed, one may be quite able to transfer one's insights from one familiar domain to another more unfamiliar one. [I will return to the issue of comfort in Section 8.3 (Cognitive Risk: Fear and Courage).]

However, the cost of this comfort is often a false sense of understanding. Without a physicist's kind of (analytic) understanding, the use of an analogy means that the student has two things to explain, rather than one. Here is an example:

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Example 6.2.3.a

The topic under discussion is the fall of a man though the air. The student has already mentioned gravity.

H: What else might affect his fall?

S: Maybe something like his clothes...

H: His clothes? What will they do?

S: I was just thinking about when you're swimming and they tell you to pull up your clothes so you can float... you know... it might have something to do with the air.

H: But what does it do? I want you to discuss it. It's important to get this straight as an abstract form. You have the right idea, but say it more clearly. What does his clothing do? It's important, yes, but what does it do? Does it add something in addition to the air friction? Can his clothing hold him up by itself?

S: No.... it's going to act like a parachute.

H: But what does it do? Don't give me similes - discuss it in mechanical terms. You did it before - in order to stop his fall,

> there must be a force exerted. No other way.. no thought or anything.... only forces will affect the motion. What forces act? Gravity and what else? You said it before.

Note the analogical character of this argument. While her statements are true, they do not constitute a physical argument, but she regards them as one.

****

In order for an analogy to serve as an argument, one must first see how two situations are analogous, and also be able to explain the workings of the object of the analogy. In general, it is easier to see that an analogy exists, but much harder to tease out the abstractions common to each branch of the analogy.

6.2.4 - The Place of Pseudo-Argument

Correlation, redescription and analogy have heuristic value. The first may bring out related factors. The second can make the relevant information more clear, e.g., the fact that it is a circle, or the fact that angle is important. The third allows a transfer of intuition from one domain into another. Indeed, all of these represent some kind of non-trivial knowledge.

However, none are sufficient to replace the physicist's analytic argument (especially in a physics course!). Furthermore, their use may deceive a student into believing his task is done, or is trivial. Indeed, these pseudo-arguments often result in a certain sense of psychological comfort. Whatever the benefits of this comfort, its cost is that the student may not attempt to refine his pseudo-argument or intuition. Consequently, his level of understanding does not advance, and it is an understanding of and facility with the theoretical abstraction which is central to the physicist's sense of competence.

Notes

1. This is reminiscent of the following story taken from Weinberg (1971):

> Some people were working on a computer project. As it turned out, the method they were using was fundamentally unworkable, and they spent many hours trying to debug their system. A new person was hired to help debug the system. After a while, he saw that the old system was unworkable, and designed a new system which was workable. He presented his new system to the others, who were very skeptical. One person asked: "How long does your system take per card?" "About 20 seconds." "Well, there, you see. Ours takes only one second per card." The new person was taken aback, and responded. "But your system doesn't work. If I'm not bound by the constraint of the system working, I can design a system that takes one milli-second per card."

2. These ideas are also consistent with a common student view that the physics of physics courses is a game which has little to do with the real world.

Have a head for numbers.

- Mao Tse Tung

Chapter 7 - Generating Quantitative Solutions

Chapter 6 discussed the difficulties of students did not accept the physicist's criteria for argument. This chapter will discuss difficulties with solution generation which students encounter even when they do attempt to work within the physicist's framework.

Strictly speaking, solution generation requires an expression of the equations appropriate to the problem, and manipulation of these equations to achieve the desired result. I have chosen to concentrate on the former, as the latter is largely uncreative "busy-work". I do not deny its importance, but most students are more capable of blind manipulation than they are of setting up the problem.

Chapter 7 is broken into three sections:

Section 7.1 - selection of the appropriate relations

Section 7.2 - formulation of these relations in mathematical form

Section 7.3 - specification of the symbols in these relations in terms of quantities specific to the problem under consideration - in other words, application of equations to specific problems

Section 7.1 - Selection

Selection refers to the process of identifying general relations which are useful in the solution of a particular problem. Section 4.3 (The Relations of Physics: A Classification) discussed the various types of relation which make up the formal structure of physics. As I will illustrate below, the difficulties students have seem to depend on the type of relation which they are trying to identify. It is also interesting to note that selection seems to be the easiest of the three planning steps - however, it is by no means trivial.

This section will illustrate difficulties in the selection of fundamental relations, relations of constraint (both dynamic and geometric), definitions, and appropriate special-case relations. In addition, it will discuss a particular class of problems which causes greater than average difficulties in the selection of any type of relation.

7.1.1 - Fundamental Relations

Most students seem to be reasonably capable in identifying fundamental relations associated with given problems - it is with detailed application of these relations that they have the most difficulty. This is at the root of comments like "I understand the material but can't do the problems." and "I can follow it but I can't do it myself." I would like to argue that these comments do in fact indicate an understanding (even if vague) which is distinctly non-trivial.

Here are two examples:

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Example 7.1.1.a

I presented students with a dozen or so index cards, each with a typical mechanics problem from a typical introductory physics textbook. I also gave them the table of contents from this text; what follows below is indicative of what they received:

Particle Dynamics
- Newton's Second Law: F = ma
- frictional forces
- dynamics of uniform circular motion

Work and Energy
- work done by a constant force
- kinetic energy and work

- power

Conservation of Energy
- potential energy
- conservation of mechanical energy

Conservation of Linear Momentum
- center of mass
- linear momentum
- conservation of linear momentum

[This list also included linear and rotational kinematics, collisions, rotational dynamics, and static equilibrium.]

I then asked them to match the problem to the list: In which chapter can this problem be found?

My worst student (average course grade 15, class average 75) successfully matched more than half. Most other people were almost 100% successful. On the assumption that the identification of the appropriate chapter is an indicat[illegible]n that the student has some knowledge of the appropriate equations, I conclude that most students have some such knowledge.

****

Example 7.1.1.b

A railroad flatcar of weight W can roll without friction along a straight, horizontal track. It is initially at rest, and holds n men each of weight w. Each man jumps off the flatcar, one-by-one at a speed of v_{rel} relative to the car. What is the final velocity of the car relative to the ground?

This problem was given on a final exam; 78% correctly identified conservation of momentum as the general relation needed to solve it; in other words, 78% either mentioned "conservation of momentum" explicitly, or wrote an equation resembling $m_1 v_1 = m_2 v_2$ somewhere on the solution page. However, the average grade on that problem was 35%; note that 78% >> 35%.

****

None of this should be surprising. Students usually do enough homework problems knowing that they are assigned concurrently with a particular reading assignment. At the very least, problems are found at the end of specific chapters, or even by specific sections within that chapter. With such cues, it should not be surprising that students can recognize types of problems and the general areas of physics involved, and maybe even some vague idea of the related principles. Here are some typical student comments:

> I do problems at home, but I get screwed up on quizzes, because they give me problems that I haven't done before... it's like if I do enough pulley problems, I start to recognize a way of doing them, but if I can't recognize the problem as one I've done before, I get stuck.

> I do problems in a pattern-matching manner. I look at the problem and then look for similar examples.

On a problem set, I'd say "I have to read that section and then find three formulas and then know that this formula went with that problem.

I have tried to make the case that this knowledge is non-trivial. However, it is a superficial kind of knowledge. It is certainly useful for exam-passing purposes, but it is not a conscious or explicit type of knowledge, and hence not easily generalizable. Students "recognize" problems; they "pattern-match". They do not explicitly analyze, and they complain vocally if asked to apply their knowledge to problems whose form they have not previously encountered.

In addition, their difficulties often lie not in choosing which relations are correct (for the laws of physics are always true) but rather which relations are useful. Schoenfeld (1979) in analyzing the use of heuristics in mathematics puts it this way.

> "The fact that a student has learned to employ a series of individual heuristic does not in any way guarantee that he will solve a heterogeneous collection of problems effectively, even when he clearly demonstrates the necessary subject-matter competence discussed in (B). The student needs an efficient means of sorting through the heuristics at his disposal and determining within a reasonable length of time which heuristic is appropriate for approaching the problem -- a means of assessing and allocating his resources which we will call a managerial strategy. Lacking a competent managerial strategy, the student may squander his heuristic resources so badly that he loses the benefits he might obtain from them....

"[The problem]... is not that they have difficulty learning to apply each of the particular techniques. Most students can learn to apply each of the standard techniques -- when they know it is the techniques they are supposed to be using.... Rather, the obstacle is that, when faced with a problem out of context {say on a test}, students often find it quite difficult to SELECT the technique appropriate for employing on the problem."

At times, students will attempt to use valid but inappropriate relations. For example, conservation of energy is always valid. However, it may not always be useful to apply this principle.

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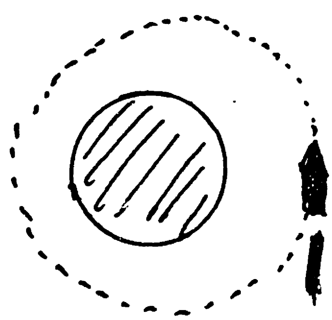
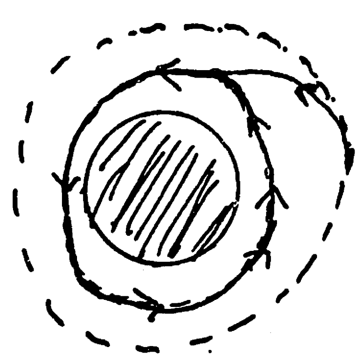
Example 7.1.1.c (This example is inspired by DiSessa (personal communication, 1979))



A rocket is in orbit around a planet. It fires its rocket in a direction opposite to its direction of motion. What does the resulting orbit look like?

S: Energy is conserved, so if its goes faster, its kinetic energy goes up, and so it must lose some PE and the only way to do that is to get closer to the planet, so it must do this.



[Though the student uses the word "energy", his statement above implies that he means "mechanical energy". My statement below uses "energy" the way he used it.]

H: But energy isn't conserved in this problem.

S: What do you mean? I thought energy was always conserved.

In his last statement, the student has reinterpreted "energy" as "total energy", and so the student and I are talking without a shared meaning for "energy". What I am trying to say is that mechanical energy is not conserved, and that the student has applied the conservation of total energy incorrectly; conservation of energy does in fact apply to this problem, but is quite difficult to use. At the very least, it requires the inclusion of energy supplied by the rocket, which the student omits.

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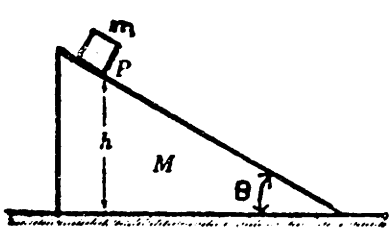
Example 7.1.1.d



A very small block of mass m rests on a wedge of mass M, which, in turn rests on a horizontal table. All surfaces are frictionless. If the system starts from rest with the block a distance h above the table, what is the velocity of the wedge when the block hits the table?

Some students attempt an F=ma analysis of this problem, especially when deprived of external cues such as knowledge of the chapter from which it was taken. F = ma certainly applies to the problem, but a solution formulated on this basis would be quite intricate, and the problem is most easily solved through use of the conservation of momentum.

****

In each of these problems, the student selects a principle which certainly applies to the problems in question; however, these principles are not helpful in their solution. To generalize, the above examples point to the need for a managerial or executive strategy which will help the student to identify useful equations. I will discuss aspects of a possible managerial strategy in Section 9.2.12.

7.1.2 - Relations of Geometric and Dynamic Constraint

Relations of geometric or dynamic constraint are also difficult for students to identify. These relations are always mentioned explicitly in the statement of the problem, but they are usually given in such a form that their significance is hidden.

Here are some examples:

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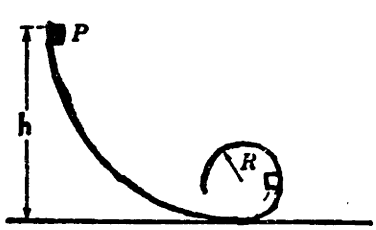
Example 7.1.2.a



A small mass m slides without friction along the loop-the-loop track in the figure. The circular loop has a radius R. The mass starts from rest at point P a distance h above the bottom of the loop. What is the least value of h if m is to reach the top of the loop without leaving the track?

S: I know I have to use conservation of energy, but I'm not sure what else to do.

$$mgh = mg\ 2\ R + 1/2\ mv^2$$

H: Where should you go from here? Can you use anything else that the problem gives you?

S: Well, they say it goes in a circle, but I'm not sure how to use that.

H: Do you know anything that involves circles and what you've already written?

S: Oh... maybe I can use acceleration.

This problem involves two constraints: The geometric constraint that the object must travel in a circular path, and the dynamical constraint that the normal force between the object and the track at the top of the loop is zero. However, the first constraint results not in a statement about path, but in one about the acceleration of a particle moving in a circular path. The first student (below) has a hard time relating the geometric constraint to the kinematic constraint that $a = v^2/r$.

****

Example 7.1.2.b

Here is a second student working on the same problem, who has written the following equations:

$$mgh = mg\ 2\ R + 1/2\ mv^2$$

$$a = v^2/r$$

S: But I'm not sure what to do now.

H: Where do you want the acceleration?

S: At the top.

H: Good. What do you know that refers to acceleration?

S: Forces?

H: Good. What forces act on it?

S: Gravity pulls it down at the top, and the track pulls it up.

H: Really? How come?

S: Because if it didn't, it would fall.

H: The problem says that you release it from the minimum height necessary to keep it going in a circle. What does that mean?

S: If you release it any lower, it will fall off.

H: Good. What does that mean?

S: It won't go in a circle anymore.

H: I mean in terms of forces.

S: What do forces have to do with it? I already told you about the forces.

The second constraint must be inferred from the fact that release from the minimum height means that the object will just barely graze the top of the loop, and hence the contact force will be zero. The second student completely misses this connection between forces and the release height.

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7.1.3 - Definitions

I have discussed above the identification of fundamental principles and equations of constraint. These relations are retrieved as the result of a problem scan that associates these relations with pieces (or the whole) of the problem.

A third set of relations - definitions - are retrieved through a quite different process. Definitions provide formal mathematical statements which substitute for (rather than associate with) various terms contained in the problem statement.

As such, definitions provide meaning for the terms in the problem. If a student believes he understands what any particular term means, he will often not stop to give it a precise meaning, and consequently will not invoke its formal definition. This often leads to error. Here is one example:

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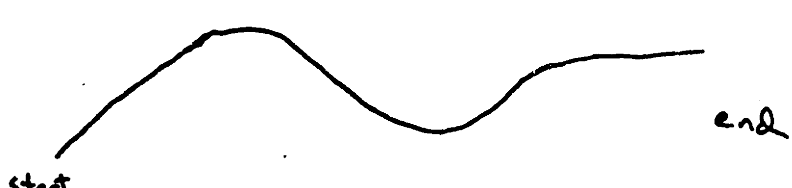
Example 7.1.3.a

An object follows the path shown below. Draw arrows where you can say for sure forces act on the particle, where the arrows indicate the direction of the forces.



S: The forces point along the direction of motion.

H: What do you mean by the direction of motion?

S: What do you mean, what do I mean. It's going that way. The direction of motion is ... which way it's going.

H: I mean in terms of its position vector, or its velocity vector, or its acceleration vector.

This student is quite inarticulate about what he understands direction of motion to be. He apparently considers direction of motion to be a primitive, a basic unit not worthy of further analysis. As a result, he cannot proceed with the task at hand. His superficial familiarity with the term "direction of motion" conceals his inability to define it in terms useful for doing physics problems.

****

Example 7.1.3.b

A thin disk of dielectric material, having a total charge Q distributed uniformly over its surface, and of radius a, rotates with a frequency f about an axis perpendicular to the surface of the disk and passing through the center. Find the magnetic field at the center of the disk.

The excerpt below begins after we have established the fact that a disk is a bunch of concentric rings. In addition, the student has already calculated the magnetic field at the center of a ring of current.

S: I'm not sure what to do next. I can see how the charge going around is like a current and that creates a B field, but I don't know how to calculate it.

H: Well, what is current?

S: It's the flow of charge.

H: What does that mean?

S: I don't understand? What do you mean?

H: Can you tell me the definition of current?

S: I just did.

H: No, I mean the mathematical definition of it.

S: I guess not.

This student has an intuitive notion of current; indeed, it is the "flow of charge". However, he is unable to formalize it into "the charge flowing past a point per unit time". Consequently, he is unable to make progress.

****

Example 7.1.3.c

The following excerpt was taken from a final examination.

(a) Define average speed.

S: Average speed is total distance traveled divided by the total time of travel.

(b) I drive from here to Lexington at 40 mph, and return along the same route at a speed of 20 mph. What is my average speed for the total round trip?

S: The average speed is (40 + 20)/2 = 30 mph.

I find this example quite startling. This student first defines average speed correctly, and then fails to use this definition when calculating the average speed for a particular case. From this, it appears that his intuitive notion of average speed takes precedence over the formal definition. Since he thought he knew what it meant, he used his intuitive notion without retrieving the formal definition, even though he had written it just minutes before.

****

7.1.4 - Special Case Equations

A third class of difficulties stems from the fact that the same special case equations (e.g., $U = 1/2\ kx^2$, $a = -v^2/r$, $x = 1/2\ at^2$) occur in many problem solutions. Consequently, it makes good sense to commit those equations to memory, thereby avoiding the need for deriving them repeatedly. However, the cost of these short-cuts is that one must also remember the conditions under which these equations apply.

Here is one example:

****

Example 7.1.4.a

A 1.5×10^6 watt locomotive accelerates a train from a speed of 10 m/s to 25 m/s in 6.0 minutes. (a) Neglecting friction, calculate the mass of the train. (b) Find the speed of the train as a function of time during this interval. (c) Find the force accelerating the train as a function of time during this interval. (d) Find the distance traveled by the train during this interval. Note: the acceleration of the train is not constant.

One typical form of student solution involved these equations:

$$a = (v_f - v_i)/t$$

$$a = (25 - 10)/360$$

$$x = 1/2\ at^2$$

Notice that even though the statement of the problem includes a warning that the acceleration is not constant, the student still uses the constant acceleration equations. [When questioned, one student said "I know it wasn't constant acceleration, but I didn't know what else to do. Others said they simply forgot.]

****

The characteristic feature of difficulties in this class is that the equations used do bear some relation to the problem in that they are derived from the same definitions and physical principles which govern the fundamental physics of the problem. However, the conditions under which these equations are derived are not present in the problem at hand.

7.1.5 - Arbitrary Parameter Problems

Students seem to have tremendous difficulty with problems which require (or are greatly facilitated by) the introduction of an arbitrary parameter which ultimately drops out of the solution, i.e., cancels from both sides of some equation which involves this parameter.

Note that this is NOT the same as introducing an intermediate unknown which would have physical significance (thus, a definite value) if one simply chose to calculate it. This arbitrary parameter is introduced solely to facilitate calculation, and is itself fundamentally uncalculable. As a rule of thumb, these arbitrary parameters always drop out of an answer by dividing through; they are never additive.

Here is an example:

****

Example 7.1.5.a

Consider a conducting sphere of radius R, mounted on a long, insulated pole, far from all other conductors. What is the capacitance of the sphere?

S: I don't know how to calculate C.

H: Do you know its definition?

S: Yeah. It's Q/V = C.

H: So you can do it.

S: But I don't have a Q, and don't I need a Q to have a C?

H: If you discharge a capacitor, is it still a capacitor?

S: Yeah.

H: So why do you need Q?

S: Oh! You mean it will cancel out?

H: Yeah.

This student is unable to identify Q/V = C as a relation useful for the calculation of C. He does not see that Q is irrelevant to the value for C, that Q is proportional to V and thus will eventually drop out. Indeed, the notion that Q will drop out seems to be unfamiliar to him. Of course, an argument can be made that Q must drop out, since C can depend only on the material and the shape and size of the capacitor. However, this is a very sophisticated argument, and though the student realizes the point of it, he does not come to this realization spontaneously.

****

Example 7.1.5.b

A mass is between two springs attached to walls. With the left spring alone, the system oscillates at ω_1. With the right spring alone, the system oscillates at ω_2. What is the frequency with both springs?

In trying to solve this problem, the student wrote equations involving x, v, a, and k, but not m. [She also did not write F = ma.] In addition, she writes $\omega_1 + \omega_2 = \omega$, but is unable to justify it. When I ask her to describe her difficulty, she says

> S: If I could introduce m as a number (i.e., as a given), it would be trivial, but since m is not given, I have to do it without the mass.
>
> H: Why can't you just use m for the mass?
>
> S: Because I would wind up going in circles needing one variable or another, but not having either.

This student also illustrates a reluctance to introduce an auxiliary variable, in this case m. This problem is slightly different from the first example, because in fact ω_1 and ω_2 do depend implicitly on m, whereas the first problem was independent of Q. However, the introduction of m might ultimately have forced her attention to F = ma, and a consideration of the forces on the mass.

****

More generally, the introduction of an arbitrary parameter which drops out of the solution is common to many problems. It aids the solution by allowing one to consider quantities which have more concrete referents, thereby reducing the required mental effort.

For example, problems involving rates of flow often involve a definition such as F = dp/dt, and might ask something about F. To solve this, one would calculate dp as (stuff) times dt, so that dt would cancel. [For further discussion, see Section 5.6.1 (Derivatives as Rates of Change and Ratios of Infinitesimals).]

The introduction of an arbitrary length, area, or volume parameter can also convert an intensive quantity into an extensive one, which is often easier to visualize. For example, instead of calculating pressure, it is often easier to consider the force acting on a small area, and as the last step, divide through by that area.

[It is often easier to work with extensive quantities because extensive quantities are closely related to amount, a notion with perceptual roots. By contrast, intensive quantities involve ratios, and ratios are not perceptually accessible.]

In many cases, this reluctance to introduce an arbitrary parameter results in an inability to identify as useful relations involving this parameter, which is the reason I have included it under Selection. However, it is also related closely to Section 7.3.6 (Variables, Constants, Equations, and Ignorance), in which I discuss at much greater length the common feeling that any symbol or variable introduced must have a definite and preferably numerical value for it to be useful in making progress towards the solution.

Section 7.2 - Formulation

Formulation refers to the generation of the mathematical equations from the relations as being relevant in the selection process. This is an important intermediate step because the mere identification of a relation as being important does not guarantee its correct formulation in mathematical terms. The difficulties in formulation can be either:

- translation from English to equations (Section7.2.1)
- imprecise vocabulary (Section 7.2.2)

Here is a preview.

****

Example 7.2.a

A dog, weighing 10.0 lb, is standing on a flatboat so that he is 20 feet from shore. He walks 8.0 feet on the boat toward shore and then halts. The boat weighs 40 lb, and there is no friction between the boat and water. How much closer is the dog to shore?

Many students identify the governing principle, namely the fact that the center of mass of the dog+boat system does not move. However, they are unable to formulate this statement correctly in mathematical terms. Here are two sample responses:

#1

$$m_d x_d + m_b x_b = \ldots\ldots$$

Here, the student attempts to write an expression involving the center of mass, and then stops. He is unable to identify initial and final states, and also unable to specify meanings for x_d and x_b.

*2

$$\Delta x = 0...$$

This student tries to express the fact that the change in the position of the center of mass is zero. However, he is also unable to identify initial and final states, and is also blocked from further progress.

****

7.2.1 - Translation From English to Equations

English is a language - a device used to communicate ideas. Mathematics is also a language (even if it does include rules of logic not embedded in any other language). The domains of discourse in which each language is applicable are not identical, but they do overlap. In this overlap region, it should be possible to translate from one language to the other. However, this is often not the case, even when a student is conversant in each language individually.

Here is an example:

****

Example 7.2.1.a

Write an equation using the variables S and P to represent the following statement: There are six times as many students as professors at this university. Use S for the number of students, and P for the number of professors.

Clement (1979) reports that only 63% of a 1500 person freshman engineering class got this problem right, whereas more than 90% were able to solve simple algebraic problems involving proportions. The following are his words.

> A standard error was 6S = P. They document two patterns of reasoning. In the first, a syntactic form, the student assumes that the key words in the problem will map directly onto the symbols, in the same order, e.g.:
>
> Student: The problem tells you. It says "6 times as many students" so it should be 6 S equals number of professors.
>
> A second reasoning pattern, a semantic form, indicates that the equation is used as a description of relative size, rather than as an expression of equivalence. Consequently, 6S means that you have 6 students, so 6S = P means you have six students for every professor. In other words, P and S stand for one "professor" and "student" respectively, and the equality sign indicates "association with" rather than "equivalence to". One student explains this his incorrect equation this way:
>
> Student: There's six times as many students, which means that 6 students [points to 6S] equal one professor [points to P].
>
> These students have an accurate understanding of the problem as posed, but are unable to translate their understanding into mathematical form. With results such as these, it should not be surprising that students may be unable to make more subtle translations (involving meaning) which involve sentences which carry the sense of an idea but not a literal statement of it."

****

Example 7.2.1.b

A small child of mass M_c starts from a height H and slides down a curved slide of mass M_s onto a frictionless horizontal surface, which becomes a surface with coefficient of sliding friction μ to the right, as shown above. All other surfaces are frictionless. The slide is not attached to the horizontal surface, and is free to move horizontally as the child slides down.

(a) Find a relationship between the velocity of the child on the horizontal frictionless surface and the velocity of the slide.

(b) Determine these velocities in terms of M_c, M_s, H and g.

A student correcting his work in a post-quiz interview says the following:

S: Let's see... On my quiz, I had $m_c gh = 1/2\ m_c\ v^2$, but I know that's wrong, because the KE_c is really less, so I have to subtract KE_s from KE_c, so that $m_c gh = 1/2\ m_c\ v_c^2 - 1/2\ m_s\ v_s^2$.

H: OK, why is that a minus? Why can't it be a plus?

S: Because you have to subtract from the KE_c, since you want it to be less.

H: Let's try numbers. You have 10 dollars, and you go to the bank. How much do you have at the bank?

S: 10

H: What would happen if you lost 3 dollars on the way?

S: I'd have 7.

H: So you start with 10, you end up with 7, and you lose 3. Are you saying that 7-3 = 10?

S: No.

H: OK, then what corresponds to the 10, the 7 and the 3?

S: mgh = 10, KE_C = 7, and KE_S = 3.
****

Example 7.2.1.c

The question under discussion is the derivation of the equation $v = v_0 + at$.

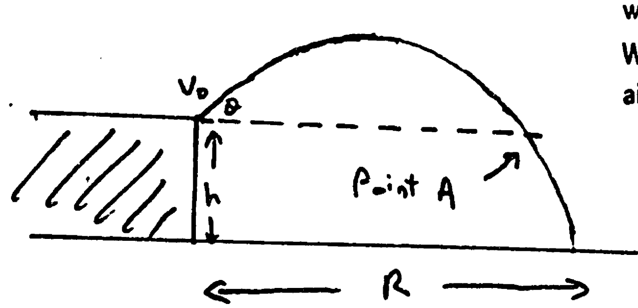
H: how do you find the new velocity?

S: Well, you start out at some velocity, and you accelerate for some time, so you change your velocity, and your final velocity is your initial velocity plus the change in velocity.

H: Good. Now, can you write that in equation form?

S: ....[long pause]....... No.
****

Example 7.2.1.d



A projectile is fired from the edge of a cliff of height h at an angle θ from the horizontal with a velocity v_0 as in the figure shown. What is the range of the projectile? Neglect air resistance.

The student has decided he needs the time it takes to go from point A on the figure above to the ground, and has written the following:

$$h = V_0 \sin\theta \, t + \frac{1}{2} g t^2$$

$$t = \frac{-V_0 \sin\theta \pm \sqrt{V_0^2 \sin^2\theta + 2gh}}{g}$$

H: How do you choose the sign?

S: I don't know.

H: Well, what happens if h goes to zero?

S: So?

H: What do you get?

S: I don't know.

H: Look. If h, the height of this cliff, goes to zero, how long should it take to go from here [point A] to the ground?

S: No time.

H: Right. Does that help you choose the sign?

S: No.

H: Well, let's say it this way. If this distance is zero [A to ground], how long will it take to go from here to here [A to ground]?

S: No time.

H: Good. What does that tell you about this expression for t here?

S: I don't know.

H: Remember, you're trying to decide what sign to use.

S: I know, but I still don't know what to do.

H: What value should t have if this height were zero?

S: I don't know.

This student has a very difficult time translating the statement of "no time" into "time should be zero". His verbal statements are correct, but he is unable to convert them into a usable mathematical form.

++

In each of these cases, the student has correctly expressed the appropriate idea in words, but is unable to express the same idea in mathematical terms. It is interesting to note that in many cases the use of algebraic symbols increases the frequency of these difficulties. Conversely, the use of simple numbers often provides a quick and dirty check on the correctness of a mathematical statement.

7.2.2 - Imprecision in Vocabulary

A second difficulty form of translation difficulty is that certain words in physics are also common in everyday speech. Thus, the student develops a mistaken sense of familiarity with these terms. The result is that the student often does not feel a need to specify his terms more precisely. Since words in physics have very precise meanings compared to words in everyday language, this can lead to difficulties when he must use terms with definite and unambiguous meanings. Consequently, he cannot formulate a problem in precise mathematical terms.

Here is an example:

****

Example 7.2.2.a

H: I throw a ball up and it comes down. What is its acceleration at the top?

S: Zero.

H: How come?

S: It's decelerating going up and accelerating coming down, so it's zero at the top.

H: What does decelerating mean?

S: Going slower and slower.

H: What is acceleration?

S: Change of velocity.

I can construct at least two explanations for this excerpt. The first is that he confuses velocity and acceleration. The second is that he considers acceleration and deceleration to be different things, not the same thing with different signs. In either case, it is clear that he does not use a precise notion of acceleration. [McDermott et al (1979) discuss confusions between acceleration and velocity in much greater detail.]

****

Example 7.2.2.b

S: Friction acts to oppose the motion.

H: So if I start my car and drive off to the right, which way does the friction between the road and the tires point?

S: Left.

H: So how does the car go right?

S: But friction opposes the motion.

The correct statement is that friction acts to oppose the relative motion. Students tend to ignore the adjective "relative" and focus on the word "motion". While "motion" is certainly important, "relative" contains much information which is lost when ignored. In particular, it contains the fact that "relative" implies the relation of one thing to another. Unpacked further, it implies two entities which are next to each other.

****

Example 7.2.2.c

The topic under discussion is the transformation of energy from one form to another. We are discussing the rise and fall of a ball in energy terms.

S: Potential energy is the opposite of kinetic energy.

H: But what does that mean? The opposite?

S: I'm not sure, I guess it means they're the the same, only opposites, like + and -. It's like these problems where the PE turns into the KE.

Indeed, many problems involve total conversion of PE into KE (and vice-versa). The notion of these as "opposites" is sufficient to deal with many of these problems (you lose PE, you gain KE), but is inadequate to deal with problems which are somewhat more complicated, e.g., those involving work, PE, and KE, or partial rather than total conversions between PE and KE.

****

Example 7.2.2.d

A 2kg box is projected with initial speed of 3 m/s up a rough plane inclined at 60 degrees to the horizontal. The coefficient of friction is 0.3 How far does the box slide along the plane before it stops momentarily?

S: Friction on the incline speeds up the conversion of KE to PE.

Further discussion reveals that he really understood the role of friction as an energy dissipator: friction takes energy out of the system, so it goes up less, so it gets there sooner. Nevertheless, how his inability to specify what he means by "speed up the conversion of KE to PE" cripples his solution of the problem, since it leaves him unable to write an equation involving all energies in the problem.

****

Example 7.2.2.e

The topic under discussion concerns the kinematics of circular motion.

H: What is "period"? What does it mean?

S: It's the distance that it takes for something to go once around a circle.

On the surface, this is obviously wrong. However, as he says it, he draws a circle in the air with his finger. This student clearly understands the fundamental notion of period, but is unable to formulate this notion clearly.

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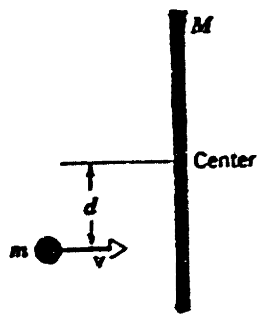
Example 7.2.2.f



A hockey puck of mass m slides on frictionless ice with velocity v. It collides elastically with a stick of mass M and length L a distance x from its midpoint. The stick is initially at rest, and the puck is observed to be at rest after the collision.

S: I'm not sure how to do this problem.

H: OK, what have you done?

S: Well, I have here

$$P_i = m_1 v = m_2 V + I\omega$$

H: How come?

S: Because the initial momentum of the puck goes into the momentum of the stick plus its angular momentum.

H: What do you mean?

S: Well, the stick is initially moving, and then it stops, and then the stick goes off.

[The last equation is correct, if "energy" or "motion" are substituted for "momentum". The last statement indicates that he is thinking of the problem in "motion" terms; he then chooses "momentum" to indicate motion.]

H: But how can you add linear and angular momentum?

S: They're both momentum, aren't they?

The student has now been thrown off the track by use of the word "momentum" in each quantity. His guess is not a bad one - after all it is reasonable to add translational and rotational kinetic energy, so why not momentum? A careful reflection about the units of each "momentum" would of course reveal his error. However, since he believes that "linear" and

"angular" momentum are two different kinds of momentum, why should he know to check units?
****

In general, difficulties in translation between English and equations or other formally defined terms are generally symptoms of other difficulties. Thus, they should serve to warn the student that his understanding is inadequate. Since students are often unaware of these difficulties, they must be made to confront these difficulties explicitly. [See Section 9.2.12 for further discussion concerning the pedagogical implications of this statement.]

Section 7.3 - Local Symbol Specification

Local symbol specification {I} is the process by which specific problem information must be mapped onto the general equations which result from the selection of useful relations and the formulation of these relations in mathematical form. It is this step - local symbol specification - with which students have the most difficulty; many students are able to identify the appropriate relations, and even articulate a mathematical form, but they fail when they must apply these equations to specific problems. I asked some students to relate their experience in a freshman physics course. Here are some responses:

I concentrate on the example problems. If I have trouble on the problem set, I look at the examples again. If I can't find it there, I look at the section that the problem came from. The examples are the most important because it is pure application.

I didn't do enough problems and so couldn't perform on tests.

Panicked on exams.

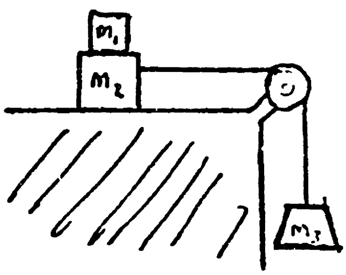
Trouble in applying formulas in the correct manner on exams.

I thought I understood it but had difficulty applying my knowledge of 8.01 to quiz problems.

Note that the emphasis of these comments revolves around the solution of specific problems. A look at the numbers involved gives some insight into this fact. There are some tens of general or widely used relations in, for example, Newtonian mechanics. The typical student faces some some hundreds of different textbook problems and perhaps thousands of different problem-specific variables to which these relations apply. Thus, it is not hard to see that without well-formulated and effective procedures for symbol specification, the probability for a successful mapping from relation to problem is quite low.

An equation such as $F = ma$ is a short-hand expression which summarizes the dynamics of a large variety of situations. In order to use this expression properly, one must be able to perform a correct mapping of quantities specific to the particular problem onto each symbol in this general equation. An equally correct view is that one must break down each symbol or quantity in the equation into its constituent parts.

For example, consider the following problem:



The mass m_2 = 10 kg slides on a smooth table as in the figure to the left. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. (a) What is the maximum acceleration of m_1? (b) What is the maximum value of the mass m_3 if m_1 moves with m_2 without slipping?

Here is how a student approached the problem:

****

Example 7.3.a

S: I'm stuck.

H: OK, do you know a law that relates forces, masses and accelerations?

S: Yes. F = ma.

H: Write it down.

S: OK.... (student writes down F = ma)

[silence...]

H: Does that help?

S: No.

H: OK, can you apply F = ma to any particular object in the problem?

S: Nope, I can't see it.

H: OK, how about m_1?

Note that "F = ma" is a general equation. It is of no use until the forces, masses, and accelerations of a particular problem are identified with F, m, and a. This student never identifies m_1 as the appropriate value to put into F = ma, and hence cannot continue.

****

By contrast, an expert would apply F = ma to a system consisting of m_1. He would decompose the symbol F into the tension of the rope and the friction of the bottom block, each of which acts on m_1. He would specify that the m in F=ma should be m_1.

Even a computer cannot apply an equation without specifying the meaning or values of the symbols it manipulates. For example, a computer language may allow the use of user-defined subroutines. When writing this subroutine, the programmer will use dummy variables; these variables are input arguments to the subroutine.

For example, a user-written subroutine might look like this:

```
SUBROUTINE X {x_0, v_0, A, T)
X = x_0 + v_0T + AT^2/2
RETURN
END
```

However, standing alone, the subroutine X is completely impotent, as illustrated by the following subroutine call:

```
BEGIN:
    CALL SUBROUTINE X{x_0, v_0, A, T}
    PRINT X
END
```

In response to this subroutine call, the computer would say "EXECUTION-TIME ERROR, CAN'T DO IT: x_0, v_0, a,t ARE UNDEFINED." In other words, the subroutine call must be embedded in a higher-level program which supplies a meaning for the previously undefined variables. Similarly, the student illustrated above does not supply values for the parameters of his F=ma subroutine.

By contrast, consider the following program:

```
BEGIN:
    x_0 = INITIAL POSITION OF CAR
    v_0 = INITIAL VELOCITY OF CAR
    A = ACCELERATION OF CAR
    T = TIME OVER WHICH CAR ACCELERATES

    CALL SUBROUTINE X {x_0, v_0, A, T
    PRINT X
END
```

This time, the computer would print the final position of the car, rather than giving an error message. A second program, specifying different meanings for input arguments might be the following:

```
BEGIN:
    x0 = INITIAL POSITION OF BIKE
    v0 = INITIAL VELOCITY OF BIKE
    A = ACCELERATION OF BIKE
    T = TIME OVER WHICH BIKE ACCELERATES

    CALL SUBROUTINE X {x0, v0, A, T}
    PRINT X
END
```

In each subroutine call, specific values are passed to the subroutine. Consequently, no error message will result. In addition, since the referents of x_0, v_0, a, and t are different in the second program than in the first, the computer will print a different X the second time than it does the first.

The latter two programs resemble the expert's approach given above, in that both do specify values for previously undefined variables. By itself, the subroutine represents the general equation. The two programs above represent that general equation as applied to specific cases.

Conceptually, symbol specification may be a straightforward process. However, it requires the identification of a symbol's constituents, and the correct inclusion (usually additive) of these constituents into the symbol. This section offers examples of difficulties with symbol specification which can and do arise {2}.

In what follows below, I will discuss other aspects of symbol specification. I have devoted such a large amount of space to a general discussion of this topic, because in the view of most teachers, it is symbol specification - the detailed application of equations to specific problems - which is the crucial test of understanding. More specifically, I will address the following items:

- types of physical quantity (additive, non-additive) (Section 7.3.1)
- choice of a system, choice of directions (to generate new equations) (Section 7.3.2)
- special cases vs general relations (Section 7.3.3)
- cause-effect chains (Section 7.3.4)
- magnitude vs quantity (Section 7.3.5)
- variables, coupled equations, field relations (Section 7.3.6)

7.3.1 - Physical Quantities

One familiar distinction between various physical quantities is that between intensive and extensive. Intensive quantities are attributes of a system which have well-defined values at a point: examples include pressure, density, electric field, and temperature. Extensive quantities are attributes of a system which are proportional to the size of the system: examples include energy, mass, momentum, volume, and charge.

One consequence of this definition is that intensive quantities must in general be specified by mathematical functions of space, e.g., f(r), where r specifies the spatial point at which the property has the value f(r). Another way of saying it is that intensive quantities are vector or scalar field properties. In addition, understanding intensive quantities requires the use of proportional reasoning. Note that a large majority of problems in elementary Newtonian mechanics do NOT depend on field notions. However, classical electricity and magnetism absolutely demand such notions. I suspect that this difference alone accounts for much of the difficulty students have with electricity and magnetism even after doing well with Newtonian mechanics.

I have found it useful to make a second distinction between quantity types: additive vs non-additive. An additive quantity is one which may consist of independent, physically distinguishable pieces. In more common language, a quantity X is additive if it is reasonable to use the phrase "the total X". Examples include:

a. mass, energy, and momentum. All extensive quantities are additive, since doubling the size of the system would double the amount of the extensive quantity, with each part (or extensive attribute) of the newly increased system physically distinguishable from other parts. In other words, one can associate the newly created mass, energy, etc, with a well-defined piece of the system.

b. electric field, potential, force and other quantities which originate from physically distinguishable sources. In general, any quantities which superpose are additive. Note well that such quantities may be intensive or extensive.

A non-additive quantity is one which does not have constituents which are physically distinguishable. In more common language, a quantity X is non-additive if it is unreasonable to use the phrase "the total X." Examples include:

a. kinematic quantities such as velocity and acceleration which describe the motion of a single point. One can certainly discuss the total velocity as being the sum of two velocity components in different directions, but vector resolution into components is arbitrary, and one velocity component cannot be distinguished physically from another. The same argument can be made concerning velocity transformations from one reference frame to another.

b. quantities which name single entities: normal force, tension.

c. intensive quantities which are simply not specified as having constituent parts. Under certain circumstances, it might be reasonable to treat pressure (for example) as an additive (though intensive) quantity. In fact, this does happen in advanced kinetic theory courses which make reference to partial pressures. However, note that each partial pressure is due to the presence of a separate kind of gas, i.e., they are physically distinguishable from one another.

Confusion between additive and non-additive quantities can lead to difficulties. For example, a student who treats non-additive quantities as though they are additive tries to find the physically distinguishable constituents of these quantities.

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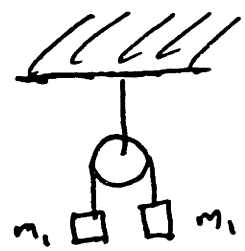
Example 7.3.1.a



Consider the balanced Atwood's machine at the left. What is the tension in the rope at the top of the pulley?

S: Zero, because when the masses are equal, the total tension would be zero if the same force acts on both sides.

This student does not distinguish between tension (the name of a force which acts on one side of the rope due to the other side) and total force (a legitimate concept). Notice that tension is not an additive quantity: one cannot identify one part of the tension as coming from one side of the rope. A question which inquired as to the possible meaning of the phrase "total tension" might have raised a danger signal, and forced the student to confront the difference between tension (the name of a single force) and (net or total) force.

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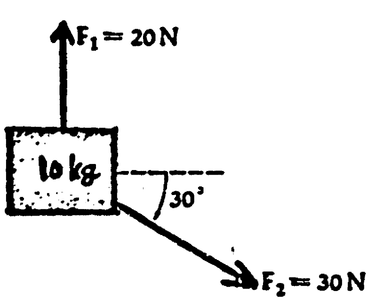
Example 7.3.1.b



A 10 kg object is subjected to the two forces F_1=20 Nt and F_2=30 Nt, as shown in in the figure. Find the acceleration of the object.

S: F_1 creates an acceleration $a_1 = 2\ m/s^2$, and F_2 creates an acceleration $a_2 = 3\ m/s^2$. Hence the total acceleration is the sum of these two.

This answer is obviously incorrect, since the appropriate procedure would be to add these

vectorially. However, I would argue that even if the student added these vectorially, he would be incorrect conceptually, even though he would have arrived at the correct answer.
****

More generally, the addition of acceleration vectors as illustrated above would violate the usual notions of cause (sum of the forces) and effect (acceleration). Normally, Newton argues that it is only the sum of the forces that affects a body's motion, and not each of the forces separately. Viewing it in the manner described by this student leads too easily to the view that since F = ma, ma must be the same thing as force; it is not, as I will discuss in the next section.

The second reason is that the resolution of this vector into components is arbitrary, depending on an artificial choice of coordinate system orientation. Physicists would generally prefer techniques which stress physical properties (such as magnitude) over mathematical constructs (such as the specific values of each component). Premature focus on vector components would emphasize arbitrary mathematical constructs.

Students also have difficulty in identifying a quantity as additive, identifying its constituent parts, and isolating each part from irrelevant aspects of the situation; this is closely related to the separation of system and environment discussed in Section 4.2.1. Here are some examples:

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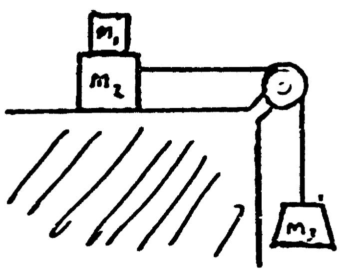
Example 7.3.1.c



In the figure, the mass m_2 = 10 kg slides on a smooth table. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. (a) What is the maximum acceleration of m_1? (b) What is the maximum value of the mass m_3 if m_1 moves with m_2 without slipping?

Student writes the following:

$$F = ma$$

$$F = (15 \text{ kg})\ g$$

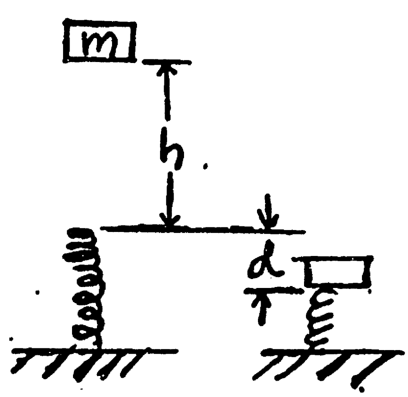
H: Why 15?

S: Well, I'm not sure if it should be 5 or 15... I'm looking at the top block, but they move together as though they were one, so it could be 5 + 10 = 15.

This difficulty arises from a lack of clarity about the components of his system.

****

Example 7.3.1.d



H: What is the total PE of the figure at the left?

S: mgh

H: And of the one on the right?

S: $1/2\ kd^2$

This student simply forgets that gravity is still acting on the block even when the spring is compressed; the problem focuses his attention on the spring, and he loses gravity.

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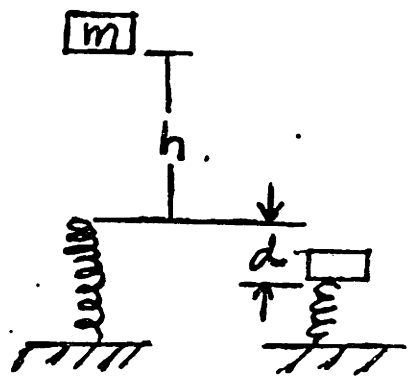
Example 7.3.1.e



H: What is the total PE of the figure at the left?

S: mgh

H: And of the one on the right?

S: $-1/2\ kd^2$

H: How come?

S: Because the spring is compressed d, so the PE is $1/2\ kd^2$.

H: And why minus?

S: Because it's below the line.

Here is a failure to consider the spring analytically, i.e., by itself, apart from its environment. The student knows that the PE of a spring compressed a distance d is $1/2\ kd^2$, but then focuses on another aspect of the entire system: the orientation of the spring in a gravitational field. However, the PE stored in an ideal spring is $1/2\ kx^2$ where x is the displacement from equilibrium, independently of any other consideration.

****

[As a parenthetical comment, it is curious that in problems involving two different conservative forces (and hence two different potential energies), students will often forget one or the other potential energy, even though they may well have demonstrated competence at working problems involving one of these forces, and even though they profess belief in the superposition principle which requires a simple addition of potential energies.]

****

Example 7.3.1.f

A circular platform is mounted on a vertical frictionless axle. Its radius is $r = 5$ ft, and its moment of inertia is $I = 200$ slug-ft^2. It is initially at rest. A 150 lb man stands on the edge of the platform and begins to walk along the edge at a speed $v_0 = 3$ ft/sec relative to the ground. What is the angular velocity of the platform? When the man has walked once around the platform so that he is at his original position on it, what is his angular displacement relative to the ground?

$$I_1 \omega_1 = I_2 \omega_2$$

$$I_1 = m_1 R^2$$

$$I_2 = I_{disk} + I_1$$

H: Why do you include I_1?

S: Because the man is still standing on the disk.

This example is similar to the preceding example. The student properly attempts to use a conservation law. However, he does not correctly isolate pieces of the system, and includes the man twice.

****

Example 7.3.1.g

A cannon is mounted inside a sealed railroad car of mass 20000 kg. If fires cannonballs of mass 20 kg with a muzzle velocity of 200 m/s. What is the velocity of the car with respect to the ground immediately after the cannon is fired?

The student writes the following equation:

$$m_b v_b = (m_c + m_b) v_c$$

H: Why did you write $m_c + m_b$?

S: Because the ball's inside the car.

Again, here is a student who does not isolate systems consistently.

****

In each instance above, the student takes a privileged view of the physical situation: "I look at one part of the system, but since I know what components really make up the system, I can use that knowledge too." [See the discussion of Section 4.2.3 (Divine Intervention).] Consequently, the student includes too many or too few items in his system.

7.3.2 - Application of Relations to Different Aspects of Problem

Students are often unable to generate all the equations they need in the solution of a problem, even if they have correctly identified the necessary physical concepts. In other words, they are unable to apply a useful physical relation to different aspects of the problem, e.g., a second system, or a different direction; this results in a deficit of equations. Indeed, breaking a problem into smaller parts and considering each part in turn is a very important aspect of analytic thought (cf., Section 4.2).

Here are examples:

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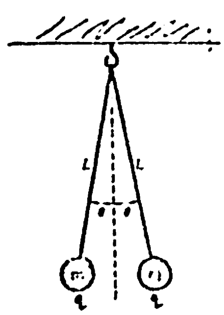
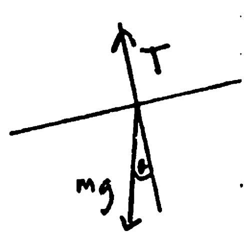
Example 7.3.2.a



Two small spheres each of mass m = 10 gm are suspended from a common point by threads of length L = 50 cm. When each carries a charge q, they come to equilibrium, each thread making an angle θ = 10 degrees with the vertical, as shown in the figure. Find q.

The student appears with the picture below on his paper

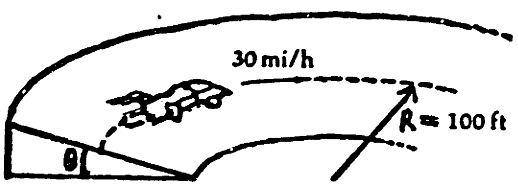


$T = mg \cos \theta$

and asks for help. He has applied F = ma correctly, but only along the rope. He has not generated another equation by applying F = ma perpendicular to the rope.

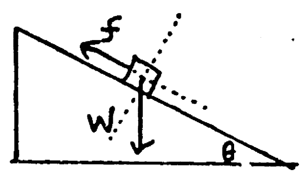
****

Example 7.3.2.b



A road is banked so that a car traveling 30 mph can round a curve of radius 100 ft even if the read is so icy that the coefficient is approximately zero. Find the range of speeds at which a car can travel around this curve without skidding if the coefficient of friction between the road and the tire is 0.3.

The student asks for help with the following on his paper:



$$f - W \sin \theta = mv^2/r \cos \theta$$

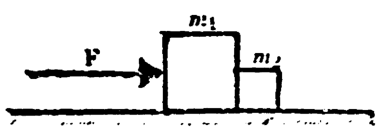
This student resolved vector components parallel to the road, but also forgot to apply F = ma in the perpendicular direction.

****

In each case, the student has correctly identified the fact that the external forces must sum to ma; however, he has performed this operation only for one direction.

****

Example 7.3.2.c



Two blocks are in contact on a frictionless table. A horizontal force is applied to one, as shown in the figure. If the applied force is 3 Nt, and m_1 = 1 kg and m_2 = 2 kg, what is the force of contact between the blocks?

A student appears for help with the following on his paper:

$$F=ma$$

$$F = (m_1 + m_2)\, a$$

Here the student has correctly identified F=ma as the appropriate dynamical law to use in the problem's solution. However, he has only applied it to the system consisting of both blocks. The problem requires an additional application of F = ma to one or the other blocks individually.

****

7.3.3 - Special Cases vs General Relations

A general relation is always true. An instantiation of that general relation is only true for the particular application of the relation to a specific situation. Students often mistake a frequently occurring instantiation for the general relation itself.

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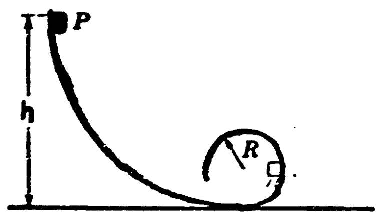
Example 7.3.3.a



A small mass m slides without friction along the loop-the-loop track in the figure. The circular loop has a radius R. The mass starts from rest at point P a distance h above the bottom of the loop. What is the least value of h if m is to reach the top of the loop without leaving the track?

H: Write the conservation of energy between these two points A and B.

S: $mgh = 1/2\ mv^2$

H: What does that v stand for?

S: The velocity.

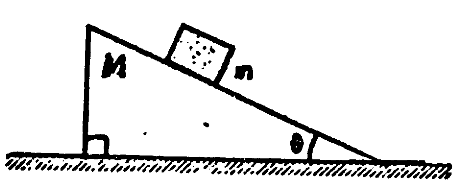
H: Where?

[S points to B]

Note how this student confuses the frequent special case where PE (at the top) = KE (at the bottom) with the more general relation PE (A) + KE (A) = PE (B) + KE (B).

****

Example 7.3.3.b



A right triangular wedge of mass M and angle θ, supporting a cubical block of mass m, rests on a horizontal table, as in the figure. What horizontal acceleration must M have relative to the table to keep m stationary relative to the wedge? The coefficient of friction between the block and the wedge is μ.

H: What is the frictional force?

S: μ times the normal force.

H: And what is the normal force?

S: It's equal in magnitude and opposite in direction to mass times gravity.

H: Always? Tell me what you mean by normal force. Is it always equal to mg? What does it mean, not what is it equal to. I want a definition.

S: I can't define it. For example, if it were here, it would be the force of the table on the slab.

H: That's a special case. I want a definition. What does N mean?

S: I don't know.

H: OK. N is the force between two surfaces one on another acting perpendicular to the surface, in general.

S: That's essentially what I said, but I didn't know how to say it.

H: No. That's not what you said.

This student is unable to give a universally applicable definition of normal force. However, she does recall one frequently occurring example of normal force. In addition, note how she confuses her instantiation with the definition I offer.

****

Even textbooks violate this distinction between instantiation and the general relation. $g = 9.8\ m/s^2$ is universally known as the "acceleration due to gravity". While textbooks usually qualify this phrase in their initial discussion of the term (mentioning free fall), "acceleration due to gravity" is used continually throughout the remainder of the text. As a result, one has examples like the following:

****

Example 7.3.3.c

The situation under discussion is the meaning of the phrase "acceleration due to gravity".

S: mg is the force due to gravity.

H: What's m?

S: The mass of the object.

H: What's g?

S: The acceleration due to gravity.

H: Does that mean that any object with a force mg on it has an acceleration g?

S: Yes.

H: Well, you're sitting in that chair now. Is gravity acting on you?

S: Yes.

H: Are you accelerating?

S: Sure.

H: How come?

S: Because gravity is acting on me, and F = ma, and F is the gravity force mg, and it's equal to mass times acceleration, and m is the mass, so g is the acceleration due to gravity.

While it may be tempting to consider g as an acceleration from a general relativistic point of view, and argue that the student is grappling with the unexplained mystery of the strong equivalence principle, students are generally not this sophisticated. In the most cases, it seems to be far more productive to treat g as an approximate conversion factor between weight and mass. In other words, g should be treated as a force per unit mass and not as an acceleration {3}.

****

To carry this discussion further, the use of a general relationship is essentially the use of a template laid out by the general relation. It consists of a number of conditions or characteristic features associated with the general relation, each of which must apply or be present if the relation is to be instantiated meaningfully. The same might be said about concepts as well.

7.3.4 - Inability to Isolate Cause-effect Chain

It is necessary to identify cause-effect chains only with time-evolution problems in which process is important. These chains arise mostly in the context of doing problems which require application of $F = ma$ or $\tau = I\alpha$. Such problems require the (often repeated) isolation of a system from its environment and the application of $F = ma$ or $\tau = I\alpha$ to that system.

Without identifying a cause-effect chain, one incorrectly takes the privileged perspective of knowing that what is "really" causing an object to move is somehow different than what it "appears" to be. The weight pulls on the rope, the rope pulls on the block, so the weight pulls on the block. A local view would argue that the block knows only what the rope tells it, and locally it does not matter what makes the rope influence the block.

Students often have difficulty in isolating the system and matching the separate components of the system and environment with their appropriate counterparts in the fundamental equations.

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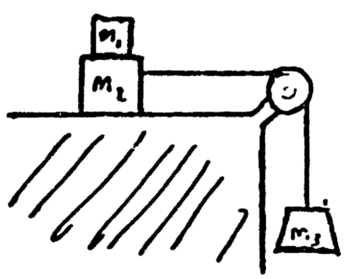
Example 7.3.4.a



In the figure, the mass m_2 = 10 kg slides on a smooth table. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. (a) What is the maximum acceleration of m_1? (b) What is the maximum value of the mass m_3 if m_1 moves with m_2 without slipping?

S: I want to apply F = ma to m_1.

H: OK, what is F?

S: The sum of all the forces acting on m_1.

H: Good. What forces act on m_1?

S: Gravity, the normal force of m_2 on m_1, and friction to the left.

H: Why left?

S: Because it accelerates right, and friction opposes the motion.

H: Does anything else acts right?

S: The force that pulls it, namely m_3g.

H: That force acts on m_1?

S: Well, not directly, but m_1 is connected to m_2 by friction and it acts on m_2.

[Note his failure to isolate a system from its environment. This leads to his consideration of direct and indirect causes in an identical manner.]

H: How do you know what acts on an object? What touches it, plus gravity. Reasonable? I can't act on you unless I touch you.

S: Sure.

H: What touches m_1?

S: Friction, gravity, the normal, and the force pulling it to the right.

H: Really?

S: Sure.

H: What touches it?

S: All except the last.

The student apparently accepts the rule of "what touches the system." However, it is hard for him to accept this rule as a rule not to be violated, rather than as a rule of thumb. [Section 10.3 poses as a topic for future work the belief of non-scientists in rules of thumb vs rules which are inviolate.]

****

Example 7.3.4.b

The following person is doing the problem of the previous example, and has already heard the "what touches it" prescription.

S: I want to use $F = m_3 a$

H: What acts on m_3?

S: Gravity and the weight of m_1 and m_2.

[Once again, we see an unwillingness to isolate the body and ask what forces act on it directly.]

H: What touches it?

S: T and gravity, so $-T + m_3 g = m_3 a$

H: Good. To review, how does m_3 know m_1 and m_2 are there?

S: They're connected by the string.

H: So how do m_1 and m_2 affect m_3?

S: Through the rope.

H: Good. The rope is next to m_3. m_1 and m_2 affect the rope, but only the rope affects m_3.

S: So I have to do like a chain all the way around? I have to follow it around? I can't just assume the rope doesn't exist? No one ever told me that before.

Note his surprise at the need for the construction for a cause-effect chain.

****

7.3.5 - Magnitude of Quantity vs Quantity

The quantities of elementary physics are scalars or vectors. However, scalars may be vector-like in that they may have a plus or minus sign attached to them. Vectors, of course, have magnitude and direction. For many students, a quantity is most easily (and incorrectly) represented as a magnitude without consideration of algebraic sign or spatial direction {4}.

This difficulty can only be inferred, since the omission of a sign or direction might be attributed to simple carelessness. When a student's attention is called to sign or direction (e.g., through teacher intervention), most students will determine sign or direction correctly. An unassisted student may never realize that sign or direction are important, and thus his work will contain only magnitudes.

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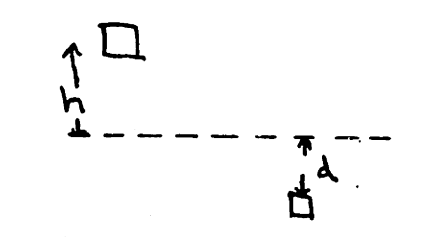
Example 7.3.5.a



H: What is the gravitational potential energy of the mass in the figure on the left? The zero of PE is the dotted line.

S: mgh

H: Now, what is the gravitational potential energy in the figure on the right?

S: mgd

H: How come?

S: Because this distance [between the reference line and the block] is d.

Note the omission of the appropriate algebraic sign. The student's concept of gravitational PE appears to be simply U_{grav} = mg (distance).

****

Example 7.3.5.b

People often have difficulty in determining the correct sign of potential energy. For example, consider the following problem:

A charge of -e is fixed at the origin. A second charge of -e is moved from x = -a to x = -b, where a > b. What is the change in the potential energy of the system?

A person who chose to solve this problem by evaluating the integral which defines potential energy would have to make an even number of mistakes in order to get the correct sign. However, the most common non-zero number of mistakes is one, and one is odd. I have evaluated this integral improperly many times, and many quite competent physicists have also been confused by the multitude of minus signs.

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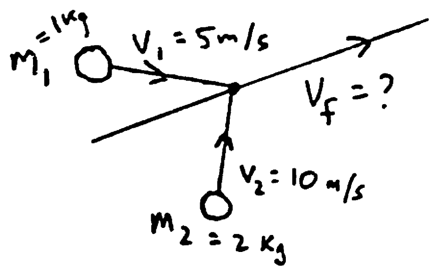
Example 7.3.5.c



The figure at the left represents a totally inelastic collision. Find the error in the following statement of the conservation of momentum:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_{final}$$

$$(1)(5) + (2)(10) = (3) v_{final}$$

About half of one recitation were unable to find the error in this statement, even when told that an error existed. Once again, for these students, the magnitude (but not the vector nature) of the momentum is significant.

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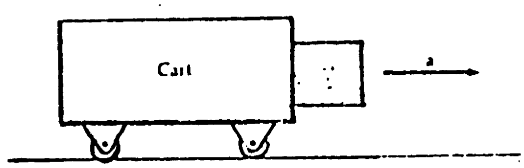
Example 7.3.5.d



The coefficient of friction between the block and the cart is 0.6. The block has a mass of 2.0 kg. Find the minimum acceleration of the cart such that the block will not fall.

H: Let's call the force of gravity on the block W, the normal force of the first block on the second block N, and the force of friction on it f. Write me ΣF = ma for the second block.

S: W + N + f = ma.

H: Now, continue.

S: W + N + f = ma

H: How come?

S: Because that's what I have here.

Note that this student also ignores the vector nature of the forces.

****

In general, the symbol to which units are attached is considered "the real thing" - signs and directions are considered secondary. This attitude is characterized by the student who argues "But all I got wrong was the sign." and teachers who take off fewer points for an incorrect sign than for most other mistakes.

The problem I describe here is made worse by the fact that teachers themselves are often inconsistent in their illustrations of problem solving. Here is an example of a teacher presentation:

****

Example 7.3.5.e

A weight W hangs vertically from a weighted rope of length L and mass per unit length μ. Find the tension T in the rope as a function of x, where x is the distance from the weight.

> Let's consider a simpler problem first. What if the rope is weightless? Then we have a very simple situation. As long as the rope doesn't break, the weight doesn't move, and the forces balance: the force up equals the weight down, so T = W.

****

On the face of it, this presentation is a reasonable one, and would be accepted as correct by most physicists. However, note that algebraic signs do not enter into this presentation.; the use of "up" and "down" allow the use of magnitudes alone. The use of F = ma (i.e., T - W = 0, which is logically prior to T = W) is hidden in the statement that the weight doesn't move.

In other words, saying that "Force up equals force down in equilibrium" or "Forces balance" may be an intuitively appealing statement, but is also a complex statement, because it implies a capability for switching easily between vector and scalar modes.

7.3.6 - Variables, Constants, Equations, and Ignorance

The use of an algebraic symbol to stand for a specific value is assumed in all of physics. In fact, this is a step of great abstraction, because it allows at least two additional possibilities:

a. deferring the calculation of a specific value to some subsequent step, while using that symbol as if its value were known.

b. using the symbol to stand for some arbitrary value which may be of no physical significance as far as the desired result is concerned.

The use of algebraic symbols can leave students quite uneasy, and they often avoid these uses even when it is inappropriate to do so. For example, I asked some students if they generally wrote symbolic or numerical solutions. Here are some reasons for preferring numerical ones.

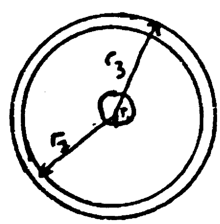
> It makes me feel I'm making headway to the answer, since the problem wanted a number I said I should do it now.
>
> I feel I'm making progress with numbers. Given a long formula, I'd rather put in the numbers rather than simplify it algebraically. A long formula becomes shorter.
>
> Problem is hard when the information you plug into your equations has to be derived.
>
> I had a hard time understanding that you can form two simultaneous equations with two variables. Before I saw that I wasn't sure what to do.
>
> Numbers are more concrete.

These students try to work "sequentially", calculating a number from each equation which they can then insert into another equation. Leaving an equation in symbolic form (as one of a pair of coupled equations or even as a collection of known but unevaluated quantities) indicates a lack of well-defined knowledge (i.e., a lack of definite values), and their work often shows that they are quite uneasy with such situations.

Student mistakes will often involve the substitution of definite and concrete values for symbols not assigned definite meanings - it often appears that the student is in a hurry to substitute a value for an unknown, and is distressed by leaving it as an unknown.

****

Example 7.3.6.a



A long coaxial cable consists of an inner cylindrical conductor of radius r_1 and an outer coaxial cylinder of inner radius r_2 and outer radius r_3. The outer cylinder is mounted on insulating supports and has no net charge. The inner cylinder has a uniform line charge density λ. Calculate the electric field at any point in between the cylinders.

The student has asked for help, and I have asked him to start over. He begins by writing Gauss' law:

$$\int E\, dA = q_{enclosed}$$

and then he writes:

$$E\, 2\pi r l = q$$

S: Now, let's see, is r going to be r_1 or r_2?

H: r is a variable.

S: I don't understand.

H: What is E? What do you want to find?

S: E in between here.

[Student points to space between the coaxial cylinders.]

H: So r is a variable that ranges from r_1 to r_2.

S: You mean E 2 π r_2 1?

H: No. r tells you where you're finding E.

This student insists on a known value for r (namely r_2). He does not see r as a field coordinate which he must specify in order to locate the point at which he must find the field.

****

More generally, the idea of a field seems inaccessible to many students. It is closely related to the idea of an intensive quantity, since a field specifies point properties, and to the mathematical idea of function; I give the function an x and it tells me the value y associated with that x, i.e., x labels the point in which I might be interested. [See also the discussion of Section 5.5 (Functional and Numerical Equality) and Section 10.10 on a possible exploration of functional relationships.]

[It is curious to note that field concepts generally do not enter introductory physics until the study of electricity and magnetism - usually second term. Field concepts appear only rarely in introductory mechanics.]

A second example of discomfort with an unknown left as an unknown:

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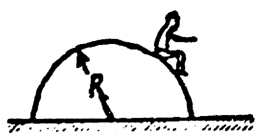
Example 7.3.6.b



A boy is seated on a hemispherical mound of ice of radius R. He is given a very small push and starts sliding down the ice. Show that he leaves the ice at a point whose height is 2R/3 if the ice is frictionless.

H: Write F = ma for the radial direction.

S writes:

$$mv^2/r = mg\cos\theta - N$$

S: Is N equal to mg sin θ or cos θ?

H: Just leave it as N?

S: But I have mv^2/r, and I don't have a velocity. How about if I use $v^2 = 2ah$?

Note how this student wants to put in a value for N immediately (by calling on an inappropriately remembered expression for N) rather than by searching the rest of the problem and/or physical situation for a second equation involving N.

****

Example 7.3.6.c

A 1.5×10^6 watt locomotive accelerates a train from a speed of 10 m/s to 25 m/s in 6.0 minutes. (a) Neglecting friction, calculate the mass of the train. (b) Find the speed of the train as a function of time during this interval. (c) Find the force accelerating the train as a function of time during this interval. (d) Find the distance traveled by the train during this interval.

$$P = F v$$

$$v = 25 \text{ m/s}$$

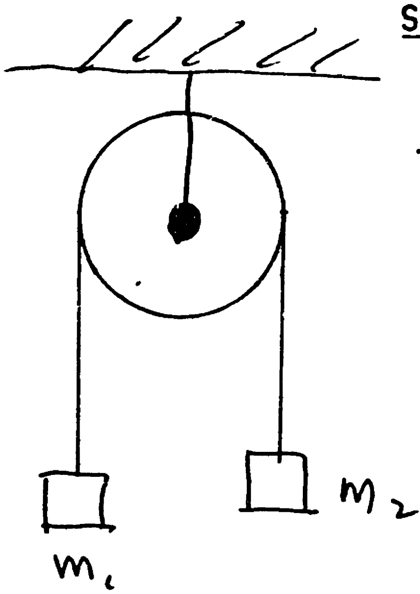
This student has not grasped the fact that P = F v means P(t) = F(t) v(t), i.e., that v changes

with time. Instead, he substitutes a definite (and incorrect) value for v.
****

More generally, variables (e.g., x) which change with time cannot be identified as "the x", a phrase with definite connotations of constancy. A variable which changes with time has no one value unless a particular time is specified as well. However, students will often behave as though it does.

It is also important to realize that the solution to most physics problems requires an inferential chain of more than one step - it is a rare problem that can be solved by writing an equation which directly relates the givens to the unknown. In mathematical terms, this often means a set of coupled equations.

Given the analytic approach of the physicist discussed in Section 4.2, coupled equations should not be surprising. In the solution of a problem, we often decompose a given physical situation into a number of different sets of system and environment (cf. Section 4.2.1). When we write down an equation for one such set, we may be forced to use quantities which also appear in a second system-environment set.



For example, consider the Atwood's machine shown at the left. The pulley is massless, and the bearing frictionless, and the rope is inextensible. What is the acceleration of mass m_1?

Applying F = ma to m_1, we see that the environment of m_1 includes the tension in the rope T and the force of gravity $m_1 g$. Thus, we may write

$$m_1 g - T = m_1 a_1$$

However, notice that T is also part of m_2's environment. Applying F = ma to m_2, we get

$$m_2 g - T = m_2 a_2$$

which involves T again.

Obvious though this may be to experts, students often do not understand how repeated consideration of system-environment sets generates equations.

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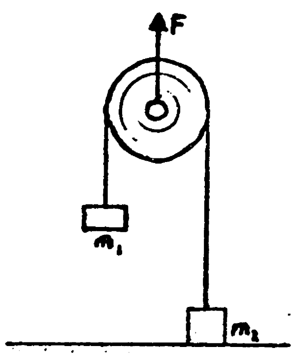
Example 7.3.6.d



Someone exerts a force F directly up on the axle of the pulley shown in the figure. Consider the pulley and string to be massless, and the bearing frictionless. Two bodies, m_1 of mass 1.0 kg and m_2 of mass 2 kg are attached to opposite ends of the pulley. The body m_2 is in contact with the horizontal floor. What is the largest value of F for which m_2 will remain at rest on the floor?

Student has written

$$(m_1 + m_2 + m_p)\, a = F - (m_1+m_2)g$$

H: You looked at m_1 and m_2 and the pulley at the same time.

S: It makes sense to look at the pulley because that's what F is acting on, but since T_1 and T_2 depend on the masses, you have to look at them too.

H: Look at the pulley. What touches it? T_1 and T_2. Now, can you find <u>other</u> expressions involving T_1 and T_2? sure, because the other end of the rope goes to m_1 and m_2. The trouble is you're not breaking it up into pieces.

S: Yes, I'm trying to do it as a whole.

This student wants to consider the entire assembly as a system. While this may be reasonable under some circumstances, it helps very little in learning about the motion of some piece of the assembly. Notice how she considers the situation globally ("as a whole") and is unable to generate equations appropriate to the problem's solution.

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The previous excerpts are indicative. However, the following is the clearest indication I have found of the reluctance to use coupled equations.

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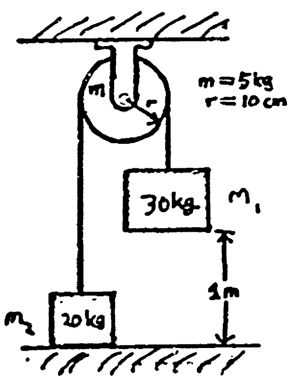
Example 7.3.6.e



The system in the figure is released from rest. The 30 kg mass is 1 meter above the floor. The pulley is a uniform disk of radius 10 cm and mass 5 kg. Find the velocity of the 30 kg mass just before it hits the floor. Do this problem using conservation of energy and F=ma with $\tau = I\alpha$.

S: $T_1R - T_2R = I\alpha$. OK, so now I need T_1, and so $T_1 = m_1g$.

H: Nope. What does the sum of the forces add up to?

S: It seems you should put in something concrete for T.

H: The sum of the forces add up to ma, so you have to write $T_1 - m_1g = m_1a$.

S: I don't see where this is going. Now I don't know a, and I keep on using more variables, and I don't know where it's going to stop.

H: Really?

At this point, a second student who has been listening says:

S: Yeah, that's really true. I'm always afraid I won't be able to get rid of anything new.

****

Indeed, I remember from my own undergraduate days feeling that I might always have n equations and n+1 unknowns. Introducing additional equations is in fact contrary to the (admittedly simplistic) view of "reducing the number of unknowns or the state of ignorance" or "getting to the answer with a minimum of wasted effort"; when I write down

$$T - m_1 g = m_1 a$$

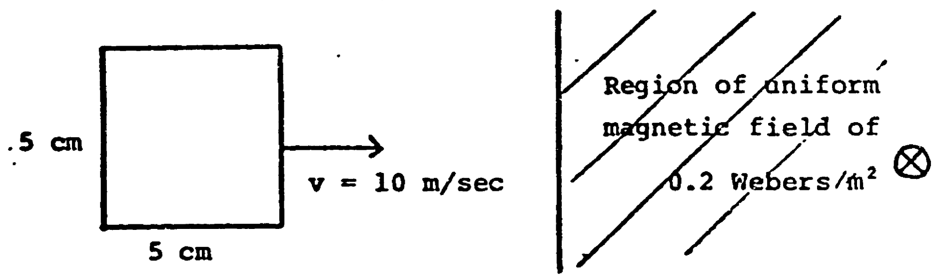
I am introducing additional ignorance (in the form of T) into the problem. Furthermore, a value for T is not directly obtainable from the given information in the problem in any way. However, I can use the symbol T to stand for the tension in the rope as if I know its value.

Here is an example of a student who finds this hard to appreciate:

****

Example 7.3.6.f

A square loop 5 cm on a side is pulled into a uniform magnetic field of 0.2 Webers/m^2 at a constant velocity of 10 m/s. The resistance of the loop is $R = 10^{-4}$ ohms, and its self-inductance is 5×10^{-8} henry.



What is the variation of I with time when the loop is partly in the magnetic field? When it is totally in?

S: How do I calculate the flux through this loop?

H: You have to account for the external flux, and then the induced flux.

S: But aren't they functions of each other?

H: You subtract one from the other.

S: But one depends on the other.

H: Are you worrying about a mirror. in a mirror in a mirror?

S: Yeah..., sort of..

This student is thinking that the external B field creates a current, which creates a back EMF, which means an opposing flux, which means another change in current, which means another back EMF, ... and so on. He does not know how to handle this iterative process. Indeed, the specification of L means this series need not be summed explicitly - it allows the use of $LI = \Phi_{induced}$, where $\Phi_{induced}$ is the actual flux induced through the loop. It is interesting to note that this question is rarely (if ever) addressed in texts.

****

This iterative approach also occurs in some treatments of electro-magnetic waves: a changing B causes a changing E which causes a changing B, ... and so on. While intuitively appealing, it is hard to make theoretical progress with such an approach. It is better to write down Maxwell's equations with the appropriate boundary conditions, and argue that there is a definite relationship between E and B, reflected in the coupling of the equations involving curls.

Larkin (1976) finds that experts routinely block out solutions to problems before attempting calculations of any sort. They engage in a qualitative analysis which guides their detailed calculations from beginning to end.

This implies that experts can tell before engaging in detailed calculation if a problem is well-posed; i.e., if the statement of the problem gives information sufficient to constrain all relevant variables to have well-defined values.

By contrast, she find that novices do not engage in qualitative analysis, but rather proceed to calculate immediately. The implication is that novices must test their approaches to the problem by direct calculation, and it can only be after calculation that they know if they need more information. If in fact this is how novices work, it is not surprising that they worry about being able to close the loop.

In other words, if the problem solver realizes that the problem statement describes a real physical situation, then all physical parameters of that situation have, by definition, unique values - the physical universe does not permit otherwise {5}. However, if one does not see the real physical counterpart of a model, then there is no reason to suppose that the system under consideration is well-defined.

Checking with a few colleagues supports this view: experts attempt to reconstruct the physical situation from information in the problem statement, and ask themselves questions like "Do I have enough information?" and "Should what it says is happening actually happen?"

A second piece of evidence is the fact that beginners often fail to understand that once certain physical parameters are set, other measurable quantities cannot be independently varied. For example, consider a mass m oscillating on a spring of spring constant k. Once k and m are set, ω is fixed, and yet beginners often try to say that ω can be independently varied too.

Notes

1. The word "local" implies a specification of symbols as they relate to a specific problem; hence, the word "local".

2. While the label "symbol specification" could reasonably be applied to the majority of student difficulties (since in some sense symbol specification is "all" that is required in the application of equations), I will use it to refer to situations in which it is clear from context that a student has at least a vague sense of the physical parameters that he must insert into the mathematical statement.

3. Treating g as an acceleration also confuses the distinction between dynamics (force per unit mass) and kinematics (acceleration).

4. This tendency is not surprising in light of the difficulty grade school children have with negative numbers. We see here a situation in which difficulties at a low level parallel ones at a much higher level.

5. Purists can talk about distributions, but that point is irrelevant here - one can define distributions as functions of space and time, but these usually don't arise in introductory physics anyway.

I have learned a great deal by stumbling upon it while I was looking for something else.

- Mark Twain

Chapter 8 - Other Ideas

This chapter is somewhat more speculative than those preceding, in that I draw conclusions which do not emerge directly from the data - they must be inferred. It is different in another sense as well: the perspectives laid out in Chapter 4 (Background) emerged from an analysis of the difficulties contained in Chapters 5 - 7; however, the perspectives of Chapter 4 are independent of (and not inherently derived from) the examples of Chapters 5 - 7. To put it another way, an introspective and reflective physicist could have made explicit his tacit knowledge involving the themes of Chapter 4 without any reference at all to student behavior.

By contrast, the ideas raised in this chapter do not, for the most part, involve the tacit knowledge of the physicist. In other words, the physicist's perspective does not provide a good background against which these ideas can be analyzed - they must be analyzed in their own right, and it is to this analysis to which I now turn.

Section 8.1 - Cognitive Risk: Fear and Courage

The words which title this section are words of affect, and consequently, many may feel that they are inappropriate in a study which concerns the cognitive. However, I believe that the distinction between cognitive and affective is not always a useful one; affective issues do significantly influence the performance of cognitive tasks. In the section which follows, I will discuss some of the difficulties illustrated in Chapters 5 - 7 from an "affective" point of view.

A student who attempts to solve a non-trivial problem faces a situation involving some risk. Literally speaking, risk is the chance that something undesirable will happen, and this term is indeed appropriate to the situation. Possible undesirable outcomes include making incorrect statements, getting stuck, wasting time, and going down dead ends. [Note well that this list has not included external evaluations, though it could reasonably have done so.]

If we assume that the problem solving student does in fact face some risk, then we must consider how people in general deal with risky situations.

Some people face these situations with courage and conviction, and try to take charge of the situation in a way that minimizes the risk while allowing its resolution. Other people react to risky situations by running away or refusing to acknowledge the risk. Still others are paralyzed into inaction or vacillate between alternatives by the uncertainty present by definition in risky situations. All of these have their counterpart in the problem solver.

The problem solver with courage and conviction will take charge of the problem, changing, transforming, manipulating, simplifying the problem into a form more easily amenable to solution; in doing so, he is willing to take some risk and deal with the problem on his own terms. He will explore possible alternatives with the realization that some may not work out successfully. He will approach a problem with flexibility but not vacillation. While exploring a particular approach, he will demonstrate commitment to that route and ignore other possibilities until or unless he decides that further pursuit will be unfruitful. In addition, he will make definite statements within the framework of the approach under exploration, even though these statements might turn out to be in error.

Needless to say, the problem solver described above does not resemble the typical poor problem solver. Indeed, Lochhead and Whimbey (1979) have noted that one salient characteristic of poor problem solvers is their extreme passivity in their problem solving approaches - they minimize cognitive risk by altering their approach to the problem; consequently they deal with the problem "on the problem's turf", rather than their own. In parts of Chapters 5 - 7, I discussed the following types of behavior:

- vacillation in choices of or inability to choose coordinate system orientation, origin, and reference frame (Section 5.7)
- search for equation very specifically related to problem (Section 6.1.5)
- reluctance to introduce arbitrary parameters which would facilitate a problem's solution (Section 7.1.5)
- inability to specify the components of a system precisely (Section 7.3.1)
- unwillingness to defer the resolution of ignorance (Section 7.3.6)

Poor problem solvers often exhibit these types of behavior, and their difficulties can be understood if one understands the fear of the problem which often dominates them. In behaving as described above, the student tries to minimize the uncertainty in the problem by taking a very conservative approach to the problem which he feels maximizes his state of knowledge at all times - he attempts to maintain maximal proximity to the problem as stated. However, in doing so, he allows the form of the problem (rather than its substance) to control his actions. At times, this may be a prudent approach, but it is certainly unhealthy to follow this strategy under all circumstances.

Other difficulties also appear in this type of student. For example, after much work, students often have all the elements necessary for a problem's solution, but are unable to put them all together. They do not stop to summarize or consolidate what they know; their work is scattered in many different places. As their work spreads out over a larger area, they must devote more mental effort to the task of remembering where things are, and as a result, get lost in the detail of their solution.

Other students will not algebraically simplify their equations without external directions to do so. They do not factor expressions, group similar terms, or introduce simplifying notation (e.g., if the combination v^2/g appears repeatedly, it makes sense to call this combination l_0 and use l_0 when v^2/g appears.). These same students will substitute numerical values and do any arithmetic which can be done - saying that "it's simpler that way" - but this is not simplification in the same sense. Numbers separated by arithmetic operators cry out for evaluation, and have an immediacy that algebraic manipulations do not have.

A student who will not simplify is taking a passive attitude, allowing the form and not the content of his solution to control his actions. He may not believe that there is utility in transforming his solution into an equivalent but perhaps more transparent form. Alternatively, he may not believe that this attempt would be successful. By contrast, the expert will attempt to simplify, with the full knowledge that he may not be able to do so - he tries, and if he fails, that is OK; on the basis of the information available at the beginning, his time was well-spent.

A third example is the student who is reluctant to use external aids to memory such as pencil and paper. At first glance, this might seem strange - most people realize that they cannot solve all problems in their heads. However, writing things down is a well-known technique in psychotherapy for establishing a commitment to a definite point of view. The cost of such definitiveness is that the point of view is then open to criticism and scrutiny. A student lacking in confidence may well prefer the security of ambiguity (which cannot be evaluated) to the threat of definitiveness (which can be evaluated).

Here is one example. One student who came to me for help regularly appeared quite unsure of herself. Discussion revealed that she often knew much more than was indicated by the contents of her papers and problem solutions. In talking her through a problem, I would try to ask leading questions. Often she would know the answer, but would not write anything until I told her that it was all right to do so. I inquired about her reluctance to write things down, and she said "Because I'm scared to make mistakes." {1}

This reluctance to make a commitment also helps to explain the student's lack of coherence when asked to discuss some particular phenomena. His use of vague words and ambiguously defined concepts may be his attempts to cover all bases so that the possibility for error is minimized.

Here is an example:

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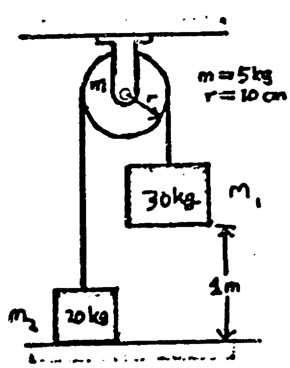
Example 8.1.a (previously presented as Example 5.8.d)



The system in the figure is released from rest. The 30 kg mass is 1 meter above the floor. The pulley is a uniform disk of radius 10 cm and mass 5 kg. Find the velocity of the 30 kg mass just before it hits the floor. Do this problem using conservation of energy and F=ma with $\tau = I\alpha$.

H: What forces act on the system?

S: The ropes, gravity, and the ground.

H: Why the ground?

S: Because it hits the ground.

I previously analyzed this excerpt from the standpoint of an inability to distinguish between final state and intermediate process. One could also argue that the student tries to cover all possibilities. I do not know which of these interpretations is "really responsible" for his behavior - I am not even sure that it is reasonable to make this distinction. It is certainly plausible that he in fact had the ability to identify the forces on the block and should have used this ability, but for some reason he chose not (or was afraid) to do so.

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Section 8.2 - Binary Logic

In Section 4.1 (The Values of Science), I pointed out that quantitative analysis was an important part of science. Its importance stems from the fact that quantification allows a much greater degree of precision in making predictions - with the notion of quantity, it is much easier to make fine distinctions. For example, the most primitive form of quantification is binary: this allows the classification of objects with respect to a metric corresponding to the presence or absence of some attribute. A far more sophisticated form of quantification is the real number continuum, which allows an infinite number of possibilities.

A continuous quantification is more sophisticated than a binary quantification, and consequently demands greater mental effort. In particular, it requires that one keep track of an infinity of possibilities, rather than just two. Hence, on the basis of this increased cognitive demand, we would expect that students might improperly resort to binary quantification. These students would be unable to translate between the two metrics, or else behave as though they had no sense of "less" or "more". One very special case of binary logic is the distinction between quantities which are zero and non-zero.

Zero is a very special number, and students often make sharp distinctions between a quantity whose value is zero and one whose value is not zero. Here is one example:

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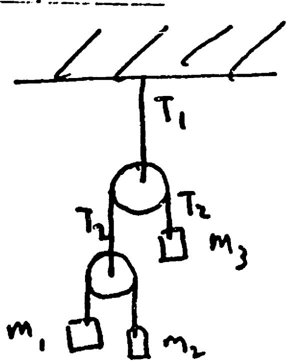
Example 8.2.a



Consider the pulley system shown to the left. All pulleys are massless and frictionless, and all ropes are massless and inextensible. Find the acceleration of each mass and pulley, and the tension in each rope.

A student writes

$$T_1 - m_1 g = m_1 a_1$$

$$T_2 - m_2 g = m_2 a_2$$

$$T_3 - m_3 g = m_3 a_3$$

S: I need a relation between T_2 and T_1, but T_1 has something to do with m_1 and m_2, and T_2 has something to do with the whole thing, since it's connected to everything...

H: Can you write me a relation involving T_1 and T_2?

S: No...

H: What do they touch?

S: The pulley, but the pulley is massless, so I can't do it.

H: Write it anyway.

S: $T_1 - 2\,T_2$ = I don't know what.

H: Write ma.

S: But m = 0

H: Do it!

S: $T_1 - 2\,T_2 = 0$

Notice the extreme reluctance with which this student acknowledges the fact that $m = 0$ implies that $\Sigma F = 0$. If instead the problem had specified values of $m_1 = 10$ kg, $m_2 = 20$ kg, $m_3 = 40$ kg, and $m_{pulley} = 1$ kg, I am quite sure she would not have exhibited this reluctance (but see the discussion at the end of this section).

****

Example 8.2.b

The problem under discussion is the meaning of Ampere's Law in integral form:

$$\int B\, ds = \mu_0\, I_{enclosed}$$

The student has already identified $I_{enclosed}$ as the net current which passed through the surface bounded by the curve C. We then turn to the following problem:

An infinite wire (a solid conductor of radius R_1, is supported by insulating disks on the axis of a conducting tube of inner radius R_2 and outer radius R_3. If the central conductor and the tube carry opposite currents of equal magnitude, find the magnetic field B everywhere.

H: So let's go back to the problem we were doing before. What is the left hand side of Ampere's law?

S: B 2 π r

H: And the right hand side?

S: The net I enclosed by the path.

H: And in this case, that is....

S: The I in the outside conductor.

H: No. what did you say I was before?

S: The I enclosed by the path.... but that's zero, so that can't be right.

H: Why not?

S: But.... oh, I see. It's what is actually enclosed by the path.

In this problem, I am confident that if the problem had involved a current of four amps in the inner conductor, and one of three amps in the opposite direction in the outer conductor, he would have subtracted these currents. However, the net current of zero appears to deflect his reasoning, as indicated by his use of the word "actually" in his last statement.

****

More generally, mathematical manipulation does not usually distinguish between zero and non-zero quantities; for the most part, all real numbers are equivalent for algebraic purposes {2}, and the physicist usually treats zero like any other number {3}.

However, some students do notice the distinction between zero and non-zero quantities, with the result that they employ two qualitatively different approaches. The first of these approaches works correctly with both zero and non-zero quantities, but these students apply it only to non-zero quantities. The second of these approaches is invoked only when a zero quantity is involved, and it does not work properly.

For instance, some students seem to react to a massless object as though it were not present at all. Consider that most objects have both functions and properties. Function refers to what the object does, while property refers to what the object is. Property comes in two flavors: perceptual properties refer to an attribute of an object that is accessible to the human senses: mass can be seen or felt, length can be seen, force can be felt. Non-perceptual properties are derived attributes, e.g., algebraic combinations of other properties: density is a ratio between perceptual properties of mass and volume, momentum is the product of two perceptual quantities: mass and velocity, electric field is a ratio between a perceptual force and a non-perceptual charge.

If we argue that less abstract data will take precedence over more abstract data, we would expect that the non-existence of a perceptual property (e.g., mass) would override other information about function and non-zero perceptual properties. In other words, a non-existent object (conceptualized in this manner by the student due to the vanishing of some perceptual property) would have no function as well.

Here are two examples:

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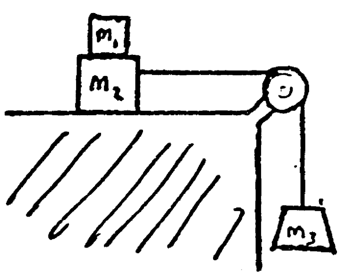
Example 8.2.c (previously presented as Example 7.3.4.b)



In the figure, the mass m_2 = 10 kg slides on a smooth table. The coefficients of static and kinetic friction between m_2 and the mass m_1 = 5 kg are μ_s = 0.6 and μ_k = 0.4. (a) What is the maximum acceleration of m_1? (b) What is the maximum value of the mass m_3 if m_1 moves with m_2 without slipping?

S: I want to use $F = m_3 a$

H: What acts on m_3?

S: Gravity and the weight of m_1 and m_2.

H: What touches it?

S: T and gravity, so $-T + m_3 g = m_3 a$

H: Good. To review, how does m_3 know m_1 and m_2 are there?

S: They're connected by the string.

H: So how do m_1 and m_2 affect m_3?

S: Through the rope.

H: Good. The rope is next to m_3. m_1 and m_2 affect the rope, but only the rope affects m_3.

S: So I have to do like a chain all the way around? I have to follow it around? I can't just assume the rope doesn't exist? No one ever told me that before.

My previous analysis took the standpoint of isolating causal chains. Indeed, this student already knows about the the "what touches it" prescription for identifying forces on an object: look at what touches the object, and then include any fields. From the standpoint of the function/property distinction, his last statement is crucial. He ignored the massless rope entirely. I believe that if the problem had specified a mass for the rope, he would not have ignored it. It seems reasonable to conclude that his inability to attribute function to the massless rope contributed to his inability to identify the appropriate causal chain.

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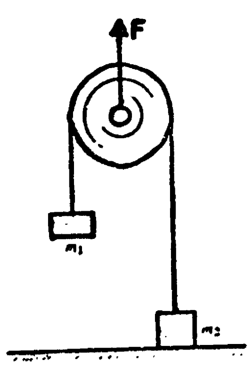
Example 8.2.d



Someone exerts a force F directly up on the axle of the pulley shown in the figure. Consider the pulley and string to be massless, and the bearing frictionless. Two bodies, m_1 of mass 1.0 kg and m_2 of mass 2 kg are attached to opposite ends of the pulley. The body m_2 is in contact with the horizontal floor. What is the largest value of F for which m_2 will remain at rest on the floor?

S: I have to look at each object.

H: Right.

S: [points to his equations] This is the sum of the forces on m_1 and this on m_2. [These equations were correct.] Now, I look at the pulley. How does F on the pulley affect the masses?

H: Is the pulley massless?

S: Yes, so it affects the tension in the string.

H: Let's say you have forces acting on a massless object. What do the masses add up to?

S: Zero... I know you're giving me a hint, but I can't see it.

H: What are you looking for?

S: The maximum force on the pulley such that $T_2 < m_2g$.

H: Point to the symbol on the paper which indicates what you want - the thing you're looking for.

S: F. [He points to the diagram.]

H: Can you write a relation that involves F?

S: $F = (m_1+m_2)a$

H: No. What does F touch?

S: Well, it touches the pulley, but the pulley's massless.

H: Good. What else touches the pulley?

S: The string, but that's also massless.

H: Right, but I'm asking you what else acts on the pulley.

S: The string.

H: What do the forces add up to?

S: Are you saying that $F - T_1 - T_2 = 0$?

H: Yes.

To summarize, the force F acts on the pulley, but the pulley is massless, so the appropriate mass to insert in F = ma is the mass of the blocks. This student is apparently making the argument that the effect of F is transmitted to the blocks as if there were nothing in between. Again, I believe that if the numbers involved had been m_1=10 kg, m_2=20 kg, and m_{pulley}= 10 grams, she would not have been confused.

****

I should point out that the difficulty does not lie in the approximations required to assume a massless pulley, at least not in the conventional sense of "approximation". In this sense, the massless approximation would mean that the mass of the pulley (10 grams) is small compared all other masses of interest in the problem. I believe that some of these students would be quite willing to say "Well, m_1 and m_2 are so much larger than the mass of the rope (which is only 10 grams) that we can consider the pulley massless." However, I also believe that having made the approximation of a massless pulley, they would still make the same mistake of attributing no function to the pulley.

To carry this point further, the conventional sense of "approximation" implies (but does not explicitly state) an independence of attributes, e.g., the ability of a rope to transmit a force is not dependent on the mass of the rope. In everyday life, this independence (of separation of function and property) is not the rule - a 10 lb test fishline is heavier than a 6 lb test fishline.

In short, a student might have a conventional sense of approximation, but lack the ability to keep function and property separate; these are skills are not necessarily one and the same.

The confusion between zero and non-zero quantities may also represent a linguistic confusion: does the statement "Consider the current I." imply that the current I exists and hence cannot be zero? Does the student's usage of the phrase "the current I" necessarily attribute a non-zero value to I?

Finally, the following example illustrates a student who does not understand limiting cases as differing only qualitatively from other cases.

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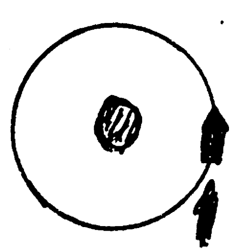
Example 8.2.e (This example is stolen from DiSessa (personal communication, 1979.))

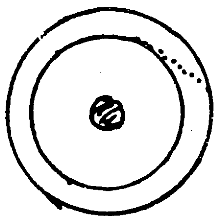
The problem under discussion refers to a rocket in orbit around the earth.



H: You fire the rocket in the direction shown. What happens to the orbit?

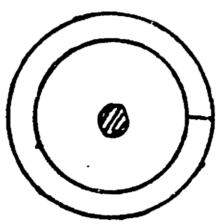
S: Well, energy is conserved, so the KE

goes up and the PE goes down, and the only way that can happen is if the rocket gets closer to the earth, so it goes like this.



[Student draws a path from the outer circle to the inner circle, along the dotted line]

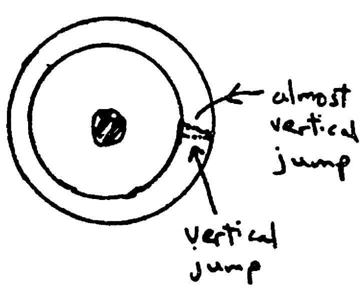
H: You say that energy is conserved, and if the KE goes up, the PE goes down, so it will have to look like this.



[Student points to discontinuous jump.]

S: Hmmmm.... well, there's no such thing as instantaneous increases, and it will take a while, so it should go off like the way I drew it.

H: But I can say that I fire the rocket for a time short compared to the orbital period, so then instead of jumping, it will do this, according to your argument.



S: But that's like what I drew.

H: How?

S: That one is a sudden jump, and that one isn't.

H: You don't call that one a sudden jump?

S: No.

H: How come?

S: It's gradual.

This student uses an incorrect version of conservation of energy. In an attempt to force the student to confront his mistake, the instructor presents him with the logical implication of his argument, with the expectation that the student will see the inconsistency. However, the student patches the argument (in a fairly sophisticated way!), and the instructor's attempt to pursue the implications of the patched argument runs into binary quantification. This student considers "jumps" and "other trajectories", and has little, if any, sense of all paths illustrated above being only quantitatively different.

****

To generalize, there are many instances in which the trained eye of the physicist sees instances of more and less, where the untrained eye of the novice sees qualitative differences. For example, bouncing balls either deform or don't deform, weighted beams either bend or don't bend, objects are either in or out of the gravitational field of the earth, and similar instances of "If it's small, then it must be zero."

I also have memories of (but no notes or transcripts for) students confusing effects differing only in amount with effects differing in quality. The example I remember concerned falling objects including air resistance. Air resistance and gravity have the same dynamic effect on a falling body - these are forces which alter velocity of the falling body. However, the amounts (and sign) of these effects differ. The student I recall was unable to see these two effects as similar in any way; she was unable to abstract their common function without much prompting.

Finally, it often seems as though students draw qualitative distinctions between the limiting case and other cases. If an expression for a distance x involves a parameter h, some students will not realize that as you take the limit of x as h approaches zero, the value of x for h = 0 is only slightly different from the value of x for h just a little bit different from zero. Here is an example:

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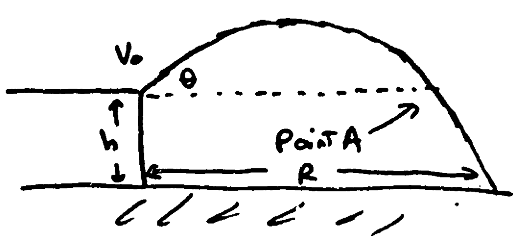
Example 8.2.f (previously presented as Example 7.2.1.d)



A projectile is fired from the edge of a cliff of height h at an angle θ from the horizontal with a velocity v_0 as in the figure shown. What is the range of the projectile? Neglect air resistance.

The student has decided he needs the time it takes to go from point A on the figure above to the ground, and has written the following:

$$h = V_0 \sin\theta + \frac{1}{2} g t^2$$

$$t = \frac{-V_0 \sin\theta \pm \sqrt{V_0^2 \sin^2\theta + 2gh}}{g}$$

H: How do you choose the sign?

S: I don't know.

H: Well, what happens if h goes to zero?

S: So?

H: What do you get?

S: I don't know.

H: Look. If h, the height of this cliff, goes to zero, how long should it take to go from here [point A] to the ground?

S: No time.

H: Right. Does that help you choose the sign?

S: No.

H: Well, let's say it this way. If this distance is zero [A to ground], how long will it take to go from here to here [A to ground]?

S: No time.

H: Good. What does that tell you about this expression for t here?

S: I don't know.

H: Remember, you're trying to decide what sign to use.

S: I know, but I still don't know what to do.

H: What value should t have if this height were zero?

S: I don't know.

This student has a very difficult time translating the statement of "no time" into "time should be zero". I conclude that he is working within a "no time" vs "some time" framework; the statement that "time should be zero" is one which implies a continuum of values for t, a necessary concept if one takes limits.

****

Section 8.3 - Intuition

Operationally, intuition refers to a prediction about the outcome of a problem without detailed analysis. However, the views of beginners and experts on the nature of intuition tend to diverge at this point.

Introductory physics students usually feel that an intuitive answer is simply their first instinctive reaction to a problem, and feel that intuition is something one either has or doesn't have, and furthermore, they mistrust it. Here are two representative comments:

> Intuition is your first guess.
>
> Intuition can't be trained... you either have it or you don't.

Other students believe that intuition and physics are incompatible. For example:

> I'm never sure about what's right because my instructor always says "This answer isn't really correct because we ignore this and that."
>
> If my intuition says one thing, I'll always say the opposite,

because it's always wrong.

Intuition never helps because you know they're always trying to trick you.

These comments indicate an attitude quite different from that of the expert, who believes that good intuition is a "practiced way of seeing" {4}; it can be trained. Students who believe that intuition is nothing more than their first reaction will not stop to analyze their reactions, and thus are led to mistrust their intuition. This section comments on intuition in light of some of the matters raised in preceding chapters.

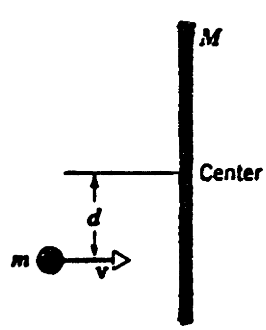
One necessary component of sound, trustworthy intuition is a model that faithfully reflects the key features of the problem. This model can then be manipulated and transformed easily using statements of fundamental principle. By contrast, unsuccessful intuitive leaps often fail because the user has jumped to conclusions or has left out essential features.

The rapid identification of the key features of a physical situation requires a facility with the modeling process. Without this facility, one at best stumbles upon these features only after much calculation. For example, consider the following problem:

An 8.0 lb block and a 16 lb block connected by a massless, inextensible rod slide together down an inclined plane angled at 30 degrees. The coefficient of kinetic friction between the 8.0 lb block and the plane is 0.1, and between the 16 lb block and the plane is 0.2. What is the acceleration of the 16 lb block?

Most experts will notice the fact that the blocks move together, and identify the total external forces on the system as the relevant quantity. Poor students with bad intuition will analyze each block separately, using equations involving the tension in the rod, a quantity irrelevant to the problem as stated.

A second example:



A hockey puck of mass m slides on frictionless ice with velocity v. It collides elastically with a stick of mass M and length L a distance x from its midpoint. The stick is initially at rest, and the puck is observed to be at rest after the collision. What will happen?

Experts will identify the x momentum as a key feature of the situation - they can then apply conservation of momentum to see that the center of mass of the stick will move in the x direction alone. They will also identify the rotational and linear motions as important but separate features of the problem.

Students with bad intuition will do neither. Their intuition will not separate the linear and rotational motions; consequently, their intuition will mislead them into believing that the stick goes off up and to the right.

A second component of intuition is an experiential background on which the individual can draw. The physicist solves thousands and perhaps tens of thousands of different problems over the course of his training. Many of these problems occur in a variety of contexts; the harmonic oscillator is a most ubiquitous example: it appears in classical mechanics, quantum mechanics, electromagnetism, circuit theory, and a variety of other domains.

Since many of these problems appear frequently, it would seem reasonable for the individual to catalog information by problem type; Minsky (1974) would call each catalog entry a frame: a data structure containing top-level descriptors about the key features of a problem, and low-level descriptors which must be filled by specific instances of data. These low-level descriptors come with default assignments, which are changed only through explicit reference to the specific problem in question.

A person would retrieve a frame if he was able to match features of the situation in question to the top-level descriptors of some frame in memory.

In this way, he could generate a prediction without engaging in detailed analysis by retrieving the required information from the default assignments of the low-level descriptors.

However, *if* the default assumptions contradicted the reality of the situation, an incorrect prediction would result. When a student "jumps to conclusions", he has not checked his model against other possible outcomes. In the language of frames, he has not explicitly addressed the low-level descriptors in terms of the specific problem at hand.

Here are two examples:

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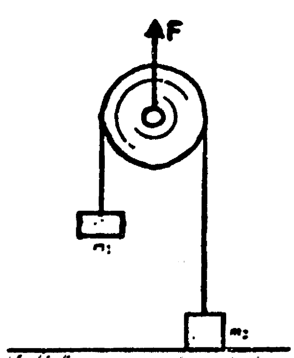
Example 8.3.a (previously presented as Example 5.4.a)



Someone exerts a force F directly up on the axle of the pulley shown in the figure. Consider the pulley and string to be massless, and the bearing frictionless. Two bodies, m_1 of mass 1.0 kg and m_2 of mass 2 kg are attached to opposite ends of the pulley. The body m_2 is in contact with the horizontal floor. What is the largest value of F for which m_2 will remain at rest on the floor?

S: $a_2=-a_1$, because they're in opposite directions.

H: What are a_1 and a_2?

S: a_2 is the acceleration of m_2, and a_1 is the acceleration of m_1.

H: Is that right? What's the acceleration of m_1?

S: I'm just saying it's equal to a_1.

H: OK. What's the acceleration of m_2?

S: $-a_1$.

H: Why?

S: They're going in opposite directions.

From the point of view currently under discussion, this student has matched the above problem to a canonical "pulley problem" frame. From the excerpt, her top-level descriptors include only a pulley with a mass on either side. The relationship between the accelerations involves low-level descriptors; she does not explicitly address this descriptor, and consequently retrieves a default value which is mistaken.

****

Example 8.3.b (previously presented as Example 7.3.1.c)

The following excerpt was taken from a final examination.

(a) Define average speed.

S: Average speed is total distance traveled divided by the total time of travel.

(b) I drive from here to Lexington at 40 mph, and return along the same route at a speed of 20 mph. What is my average speed for the total round trip?

S: The average speed is (40 + 20)/2 = 30 mph.

In this example, the student draws on a canonical "average" frame; in the absence of other information, the student's default definition of average is $(x_1 + \ldots + x_n)/n$. Since he does not

apply the formal definition of average to the problem, an incorrect answer results.
****

The above discussion has side-stepped a crucial question: how is a canonical problem frame constructed? This question is fundamental to pedagogy, and I will return to this question in the next chapter. However, from the standpoint of the frame's existence, it is irrelevant: the required data could be derived from actual personal experience with the system in question, or from mathematical analysis, or any other process.

Notes

1. If I had responsibilities for grading her, this might be understandable, but I had no such responsibilities.

2. There are exceptions (e.g., the square root of a number slightly greater than zero is real, while the square root of a number slightly less than zero is imaginary), but these are few.

3. A striking exception is the physicist's emphasis on the zero of div B = 0. In this case, they make a big deal of the zero representing the complete non-existence of magnetic monopoles.

4. This wonderful phrase comes from Taylor and Wheeler (1972).

I understand what's making me feel bad, but I don't feel any better for knowing it - I want to know what I can do about it!

- client to therapist

Part III - Conclusions

Part I (Preliminaries) provided a variety of perspectives from which the examples of Part II could be understood. Part III discusses some of the possible implications of Part II for teaching physics, and sketches out a number of areas in need of more exploration.

More specifically, Chapter 9 (Pedagogy) discusses a wide range of alternatives to various aspects of standard practice, alternatives which explicitly take into account the difficulties and demons I have uncovered in this study. However, they are not guaranteed to work, and they are not rigorous consequences of Parts I and II. [Indeed, their efficacy remains to be tested in future work.] Instead, they are suggested in the light of the preceding chapters.

All explorations raise more questions than they answer, and this thesis is no exception. Chapter 10 (Work to Come) discusses several areas I have only touched in this study and which warrant further investigation.

If a student needs a good teacher, he shouldn't be at M.I.T. in the first place.

- M.I.T. professor

Chapter 9 - Pedagogy

The pedagogical implications of this thesis are both general and specific. In this chapter, I will discuss first the global implications by focusing on some of the broad-ranging matters raised by the difficulties discussed in Part II. I will then turn to some of the specific implications; for the most part, each of these will be local in scope (rather than global) in that each will address a specific item of difficulty.

However, I must remind the reader that I am primarily concerned with physics students in difficulty. Many of these suggestions would not necessarily benefit the best students, but then again, the best students don't need teachers either. I am interested in raising the bottom levels of performance, rather than pushing up the top levels.

This chapter is divided into six sections. Section 9.1 offers a few global suggestions for altering the introductory physics course. Sections 9.2 - 9.4 make some specific suggestions which attempt to address specific difficulties outlined in Part II. Section 9.5 is a collection of miscellaneous bits of advice which don't fit elsewhere. Section 9.6 offers some general discussion concerning the suggestions of Sections 9.1 - 9.5.

It is very important to emphasize that the pedagogical suggestions of Chapter 9 are coupled only weakly to Parts I and II. I believe these suggestions are motivated and made plausible by the contents of Parts I and II, but these suggestions are not intended to be prescriptive. Rather, they serve as focal points to which each individual teacher can direct his own consideration. In other words, these suggestions are just that - suggestions - which I find have actually helped, or which I believe would have been helpful had I thought of them at the time. They are not guaranteed to work, but they do take into account the difficulties some students seem to have. In any case, they do represent departures from present practice, for whose failure (at least among these students) the difficulties of Part II are prima facie evidence.

Section 9.1 - Global Suggestions

This section discusses a number of global suggestions for changing certain emphases of and approaches to the introductory course. However, putting these suggestions into practice, it must be kept in mind that by definition, a global suggestion is not limited in scope to a particular domain. Therefore, it makes little sense to implement these suggestions as a stand-alone package at the beginning of a course. They can and should be parts of the first few weeks, but they must also be integrated throughout the rest of the course.

It is true that this effort will often require extra time, and hence leave less time available for the coverage of other material. My only response is that sometimes less can be more {1}.

I would like to be able to claim that with this approach, the standard topics could be covered in less time, but this claim needs to be substantiated. [In fact, I suspect that it is not true.] On the other hand, I am primarily interested in increasing student performance, and only secondarily interested in increasing efficiency; running through a museum allows a student to say he has "seen" a great deal of art, but the value of this enterprise is another matter.

9.1.1 - Socializing and Teaching Physics Students to be Community Members

I believe that the most important pedagogical function of this thesis lies in what I hope is the consciousness it raises. I have illustrated a wide variety of difficulties; if the reader's reaction is one of "That's ridiculous! No one could have looked at it that way!", then to a large extent, I will have achieved one of my major goals.

Indeed, I found many of the difficulties I observed quite surprising. Underlying these difficulties have been assumptions I as a person well-socialized into the community of physicists had long taken for granted. The foremost of these assumptions has been the proposition that our first-year introductory physics student is essentially a junior physicist, who thinks like a physicist but lacks the factual knowledge of the more experienced physicist. Few physics teachers phrase this assumption as bluntly as this, but stripped to essentials, this is the essence of a quite major assumption.

As I hope my examples illustrate, this proposition is blatantly false. A beginning student is much more like a lay person than a professional scientist in the way he approaches and thinks about problems. Furthermore, many difficulties of the beginning student appear to arise from a divergence between his problem solving approaches and philosophies and those of the expert professional. [I believe this claim had added validity in view of the fact that I performed this study at the Massachusetts Institute of Technology, an institution which attracts students of presumably high scientific ability.]

Therefore, one quite general pedagogical implication is that teachers must pay attention to and concern themselves with the assumptions inherent in their views of doing and teaching science and physics. Such a task is quite difficult, because it requires the explicit articulation of ways of knowing and thinking that are second nature to most practicing physicists; in certain ways, it requires the physicist to turn philosopher. I offer the material of Chapter 4 as a start, which provides general themes to be elaborated in greater detail in future work.

It is particularly important that this philosophy of science be articulated by scientists. Even though it is philosophy, it is applied philosophy, and it will be useful only to the extent that it reflects the actual practice (even if unstated) of the scientific community. Without this connection to real science and scientists, it will be too vague to be useful.

Some students do not fully accept the physicist's definition of understanding a problem or a solution. In particular, an acceptable problem solution consists of a set of general relations which deductively transform the givens of the problem into the desired results. These results must be given in terms of known information. In addition, arguments must be consistent, testable, and based on relations of broad applicability. Argument is not correlative, redescriptive, or analogical. Students often do not work within these constraints.

From a pedagogical standpoint, this is of course frustrating. However, I believe we could take a significant step forward if we shifted the emphasis of the introductory course away from "learning physics" and toward "being socialized into the community of physicists". [Recall that this thesis addresses primarily the introductory physics course for those who will need physics as a working tool.]

The primary advantage of such a change is that an emphasis on socialization forces the student to confront explicitly the fact that he has something new to learn. All students come into an introductory physics course with preconceptions about nature and about methods of analysis. The former are often quite strong (cf., Clement (1977a)), and in fact do often have an observational base in reality. It is not hard to imagine a student who says "Yeah, I know that's the way the problem works out, but that's not like what I saw when I actually saw it happen." Indeed, the idealized physics problem lacks an exact counterpart in the "real world", and Aristotle was not stupid when he maintained that a force was necessary to maintain motion.

The sentiment which underlies the "learning physics" approach is the "student as ignorant physicist" view: if students look at the data, then they must invariably come to see the world as physicists do. What this attitude fails to take into account is that this conclusion is inevitable only if you choose to and are capable of looking at the world through a particular pair of glasses.

Teaching students how to look through this pair of glasses is the task of the socialization process. This socialization process can and often does involve arbitrary things. Nature does not impose these rules and ways of knowing; people do. Similarly, nature may judge the work of the scientist, but teachers judge the work of the student. Furthermore, students lack observational backing for their preconceptions about ways of knowing, backing which they do have for their preconceptions regarding nature.

With an emphasis on socialization, it is reasonable to impose on students notions which might appear to them counter-intuitive or arbitrary. As teachers, we may know these ideas have great power, and that some of these ideas may even follow quite cogently from the questions we ask, but students do not share this knowledge.

Put another way, an emphasis on socialization means that teacher can say to students "I know you don't do it using the method of Y, but physicists do, and so that is what we want you to learn." An emphasis on "learning physics" means that teachers say to students "I know you don't think Y is true, but it is, so you have to believe it." In short, I believe it is easier and more reasonable to impose ways of knowing rather than specific ideas about the physical world, because the latter involves objective data, while the former does not.

It might be argued that an emphasis on socialization makes the introductory physics course a game in which the student must follow completely arbitrary rules, and does not learn about the spirit in which physicists do physics -

the spirit in which nature is the final arbiter.

I take this concern seriously, but I believe it is unwarranted if handled judiciously. In the first place, a typical course sets up the teacher as final arbiter - not surprisingly, since the purpose of a course is to pass on to the student known material in a useful manner. A student in a course does not confront nature; he confronts his teacher's view of nature, and if the student's view is different from his teacher's, it is the student who must defend and justify his views to the teacher.

Secondly, the ways of knowing implied by the physicist's philosophy of science are not without reason, though they cannot be made logically compelling. There are other ways of knowing, but mankind has found a particular set of rules useful for answering certain questions (cf., the discussion of Section 4.1 (The Values of Science)).

Socialization enters into the choice of questions that physicists ask and in the issues they consider important. They insist on public verifiability; they ask quantitative questions; they acknowledge the approximate nature of all representations, and confront the question "How good must a representation be?"

The answer is "It depends on what you want to use it for." Note that this answer depends on human choice, rather than logic or nature.

Rational necessity enters in the physicist's response to these questions and issues. Public verifiability requires an acceptance of general whose correspondence rules are not problem-specific. The complexity of the problem determines the need for empirical input.

In the introductory course, socialization dictates that fairly simple problems are appropriate. Through long experience, teachers know that general formulations provide the most reliable approach to the solutions of these problems; indeed, it is hard to imagine any other approach with a reasonable hope for success.

Socialization also determines the range of the "general relations" which are lie at the heart of general formulations. $\Sigma F = ma$ is certainly a general relation. Probably $U_{spring} = 1/2\ kx^2$ and $x = 1/2\ at^2$ also fall into the class of general relations. Probably $a = (m_1 - m_2)/(m_1 + m_2)\ g$ does not. The generality of relations are determined by human judgments, and not by nature. However, nature does dictate that these relations are what they are.

My work illustrate that students are often unwilling to use general formulations. Though teachers may have learned through experience the power and utility of these formulations, the pedagogical enterprise is predicated on the assumption that students need not reconstruct everything for themselves. I believe that an emphasis on a socialization process which requires, for example, general formulations, could overcome student reluctance to use these formulations.

To summarize, an emphasis on socialization provides a way of bypassing student hesitation to employ physicist ways of knowing by explicitly introducing new rules and new conventions. The reasons underlying the choice of these particular rules and conventions should be made clear, but the student should realize that a combination of human choice, rational necessity, and physical reality is involved.

It is also interesting to note that the socialization process I am advocating mirrors in certain ways the practices of the community of professional physicists. In particular, the problems teachers present students in the introductory course and the form of solutions they demand do not represent physics as it is "really" done, but rather a distorted picture of it. Students often do not see the connection between the toy problems they are asked to solve and the real world about which physics is supposedly concerned. Furthermore, teachers draw a quite artificial line between the process of solution generation and the formal language in which these solutions must be expressed; they care mostly about the latter and only rarely about the former.

Similarly, many elements of writing papers for publication are also rooted in convention. This does not mean that an author lies or plagiarizes (this word really is spelled this way!), but it does mean that the papers which appear in the professional journals also offer a distorted picture of what physics "really" is. A paper (especially one concerning an experimental project) may give the reader some idea of what its author is trying to accomplish, but it often leaves the reader without a clear idea of what he (the reader) must do in order to reproduce the results documented in the paper. In addition, papers almost never contain an account of the process by which their authors decided to take the course they did.

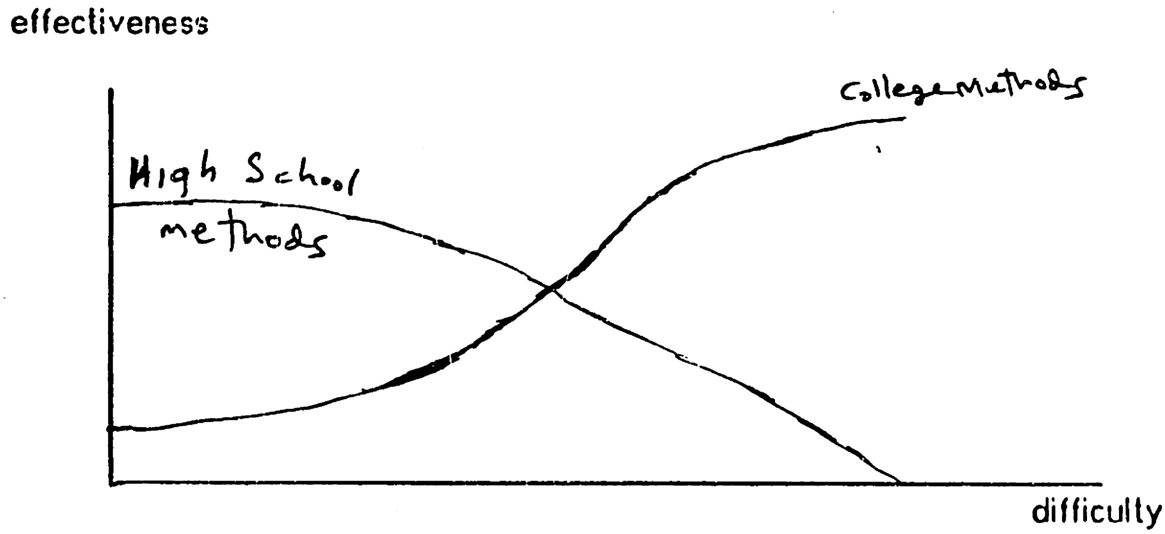
This similarity goes only so far - beginning students in introductory physics are not beginning assistant professors of physics. Still, I believe the analogy captures certain basic parallels, and provides a starting point for reassessing the basic introductory physics course.

9.1.2 - Teaching Methods of Problem-Solving Analysis Explicitly

A second major conclusion of this thesis is that many students do not learn new techniques of analysis or approaches to problem-solving from their introductory course. I draw this conclusion from the difficulties I have uncovered; they illustrate that certain inappropriate approaches to problem solving are very deeply ingrained in many students. Nevertheless, these difficulties emerge mostly through interactive discussion between teacher and student; only a few would have emerged through standard assessment techniques (e.g., written problem-sets, final exams). In other words, students often survive an introductory physics course without adopting new ways of doing and knowing things. Some students are even quite articulate about this fact:

> I look at all the variables and try to come up with an equation which ties them all together.
>
> I try to find one equation which relates the givens and unknowns. I try to use what the book gives you and not derive everything.
>
> When I'm given several values, and I know there's a formula that will work, I look for it in the book if I don't remember it right off.
>
> Draw picture, and note givens and unknown. Try to find an equation which relates the givens with the unknown.

Note the emphasis on a "one-step" solution, and how these answers contrast with the methods of analysis teachers believe they are teaching. Here is a sketch of one possible theory for this phenomenon.



These are curves of effectiveness vs difficulty for two types of analysis. Methods that are typical of high school physics work well with relatively simple problems, but lose for harder problems. More advanced methods work well on harder problems, but they are overkill for simple problems.

When M.I.T. students come into physics, they are reasonably proficient with high school methods; if they are not, they don't take physics or don't go to M.I.T. In addition, these students often perceive their high school experience with physics as mere "formula-plugging". Here are a few indicative comments:

Special cases are what we did in high school.

In high school, you're given this number and you find the answer - that's the way it's taught - you're not expected to derive general formulas but just get answers.

I grew up with Regents physics which was given the numbers, don't derive the formula, use what you know and the solution pops out, neat and simple.

The problems of high school physics typically come from the easier end - a perfectly reasonable approach, since one's first contact with physics should not be overly difficult.

On the other hand, college physics stresses more advanced analytical techniques which require more mental effort and greater fractions of working memory. Therefore, teachers assign problems that are hard enough to challenge student abilities, and encourage them to adopt these more advanced methods. However, these problems must remain tractable to beginning students, and so graduate level problems are inappropriate in an introductory course. Consequently, teachers assign problems in the middle domain.

Note well that this domain includes a cross-over point; these problems are also at the upper limits of what can be handled with high school techniques. Consequently, students who feel comfortable with their old techniques (which require less mental effort) feel no need for a qualitative change of approach, and continue to use techniques which worked quite well before. They attribute any difficulties with college physics to the fact that college physics is simply harder than high school physics, and resolve to push their techniques still further.

What can be done about the disparity between what teachers try to teach and what students actually use? I have already mentioned one necessary component: the explicit articulation of the techniques teachers employ when they solve problems.

However, this is not enough. It is important to understand the distinction between the learning of a skill and the actual use of a skill. It does no good to teach a student a new method or approach if he does not use it, and the good teacher will address both issues.

It is in dealing with the latter issue - actual use - that teachers must confront the notion of psychological comfort. In Section 8.1 (Cognitive Risk:

Fear and Courage), I discussed the notion of commitment. In general, commitment arises only to things which provide a sense of comfort. Student resistance to the adoption of new ways of knowing arises because students are comfortable with (and hence committed to) the techniques and ways of knowing which have proven successful prior to their encounter with our physics courses. Thus, the idea that the student is not an empty mold into which physics knowledge can be poured comes again to the forefront.

I argued in Chapter 2 that the separation between expert teacher and beginning student is similar in many ways to the difference between competing paradigms. Kuhn (1970) argues that translation between paradigms is necessary for a paradigm shift, and this provides us with one way of looking at the pedagogical process. In particular, translation is the process by which each party attempts to understand the other's point of view. Such an understanding is possible when (as is usually the case) the bulk of their language is shared. If they can sufficiently refrain from explaining anomalous behavior as the consequence of mere error or madness, they may in time become very good predictors of each other's behavior {2}.

However, in the pedagogical domain, the particular choice of paradigm is not open to question - the paradigm that the teacher sets forth is the preferred paradigm; thus the analogy holds in only one direction. On the other hand, teachers cannot expect students to share their professional language - if they could, the relationship would not be that of teacher to student. Therefore, if the teacher is to convert the student to the appropriate paradigm, he must use the language of the student to translate his own (the teacher's) world view to the student, in order to bring him into the professional fold.

In short, teachers face the task of persuading the student to adopt desirable techniques and to cast out his old and inadequate approaches. This usually requires that the teacher find a way of easing the transition from his old state of knowledge to his new state; this in turn requires that a teacher take seriously a student's old state.

However, as teachers, we have a second tool at our disposal. The relationship between teacher and student is not that of equals, even excepting the difference in knowledge. The teacher does hold a club over the student's head - the teacher has the responsibility of evaluation as well, and the second way of increasing the usage of appropriate ways of knowing is to hold the student explicitly accountable for the use of procedures already determined to be generally useful.

I am aware that there exists considerable skepticism concerning the utility of teaching the procedural knowledge of physics explicitly. This is similar to the doubt about the value of explicit instruction concerning physical skills, and so let me describe this argument.

One traditional view of learning to ride a bike is that explicit instruction in this process is useless; the only way to learn to ride a bike is to try it a few times, falling in the process. This claim can only arise out of many unsuccessful attempts to teach beginners to ride a bike. However, these attempts may have failed for one of two reasons: all explicit instruction is useless, or the particular instructions employed were useless.

However, it is an empirical fact that a bike in motion is inherently stable. A bike with no rider pushed down a hill will stay upright for a reasonably long time: bike plus novice rider is less stable than bike alone {3}. This fact is quite significant. Our outlook must shift from the student learning to balance the bike to the student learning to not unbalance the bike; to first order, the student rider must learn to do nothing!

Notice that if this fact were unavailable, a teacher might tell the student to do X, where in fact, he should have told him to not do Y. Quite possibly, this might have done more harm than good, and thus created support for the idea that explicit instruction is a bad thing. In addition, it is also possible to see how "Don't say anything" might in fact be a useful instructional strategy: the teacher's theory of bike riding might be wrong (and harmful as well). Also, if the student hears nothing, he might not do anything, thereby not unbalancing the bike.

The analogy goes only so far. However, I am not yet ready to conclude that all explicit instruction is useless. I am sure that some instruction is less than optimally helpful, and I can only hope that future work will reveal more productive methods of teaching students what it means to think as physicists do.

9.1.3 - Teaching Micro-Skills

Problem-solving often requires the performance of "micro-skills" which are quite specialized but which recur frequently. Examples might include the resolution of vectors into components, the identification of forces acting on a mass, and the separation of total system energy into component parts.

These micro-skills are of most value when the student can perform them automatically, without conscious thought or effort. [The reason is that when they are automatic, they demand less mental effort, leaving more resources available to do more interesting things.] A student who takes several minutes to identify the forces acting on an object almost assuredly will have difficulty solving problems involving F=ma.

In addition, the mastery of a micro-skill is not binary - it is not true that one either knows or doesn't know how to do something; intermediate stages exist as well. We know that for a person with a high degree of proficiency in micro-skill X, differences in problems requiring X make little difference in his performance in using X. Hence, it is not unreasonable to suppose that for a person with an intermediate degree of mastery, small differences in problems requiring X might make large differences in his performance in using X.

This leads to the suggestion that the student should exercise these micro-skills in the various contexts in which they will actually be required, until he develops a high degree of proficiency in their use. In short, drill has its place.

For example, it would be preferable to drill students in the resolution of vectors oriented in different directions and in situations with non-essential detail so that they could develop this micro-skill. It makes less sense to assign exercises involving vector manipulations such as cross products and the addition of many vectors when these skills are not often used in the standard introductory course.

9.1.4 - Tutorials as a Complement to Lectures and Recitations

For many years, physics courses have been taught in the lecture/recitation format and opportunities for one-to-one instruction have been limited. However, the difficulties I have documented in Part II (some of which are quite fundamental and also not at all isolated) emerged only after long, extensive, and personal discussions with students. I believe that if teachers are to respond to these difficulties in a meaningful way, personal interaction is absolutely essential; only through personal interaction will many of these difficulties become apparent.

Consequently, I suggest that one-to-one tutorials be made a part (perhaps even a required part) of the introductory physics course.

However, I also realize that personal interaction is a very labor-intensive process. Nevertheless, it is not necessary for a tutor to have a Ph.D in physics or even an S.B. While the tutor should be competent in physics, what is most necessary it that he be willing to listen, and not be so quick to "give the student a hint". Indeed, if I had to point out the most significant failing of this study, I would say that I talked too much and didn't listen enough.

Section 9.2 - Specific Suggestions: Understanding (Chapter 5)

Sections 9.2.1 - 9.2.9 follow the sequence of Chapter 5 (Understanding). Each of the nine sections below gives some suggestion(s) which I believe should help at least some students in their efforts to describe the problem and interpret their solutions and the theories they use.

9.2.1 - Elements of Good Notation (Section 5.1)

To prevent the use of the same symbol to denote similar but physically distinct quantities, and to maximize the effectiveness of notation, here are some helpful guidelines:

a. use subscripts to distinguish between similar quantities (e.g., W_1 and W_2).

b. refrain from using upper and lower case versions of a letter together in a problem if they differ only in size. In other words, C-c, F-f, I-i, J-j, K-k, M-m, O-o, P-p, S-s, U-u, V-v, W-w, X-x, Y-y, and Z-z are inappropriate, because the lower-case version invariably turns into the upper-case version by the end of a long solution.

c. use mnemonic notation. In general, this includes adherence to standard conventions for the meaning for various symbols, e.g., p usually stands for momentum, W for weight, t for time, m for mass, E for energy. In addition, a symbol should cue one's memory regarding its meaning; hence, x should be used to denote lengths (known or unknown), rather than everything that is unknown. Greek letters are generally used to represent dimensionless quantities (quantities without units attached), at least in introductory courses, so we get Δx for "change in x," θ and α for angles, μ for coefficients of friction, π for the ratio of a circle's circumference to diameter.

Finally, the judicious use of subscripts can be very helpful. Appropriate placement of subscripts can provide a reminder of other important information associated with the primary symbol. For example, the subscript can stand for a reference frame (e.g., $V_{ab} \equiv$ velocity of a with respect to b), or objects (e.g., $F_{ab} \equiv$ force of a on b).

9.2.2 - Imagery: Good Diagrams and Other Tips (Section 5.2)

To prevent literal interpretations of hand-drawn pictures, these pictures should be drawn with some care until the student is capable of ignoring the unessential features of the picture. In general, the following guidelines are useful:

a. angles are usually more important than lengths, so pay more attention to orientation.

b. don't use unwarranted special cases.

c. do use pictures that are mnemonic: if a situation requires that two angles be equal, draw them that way. If an object is far away, draw it at least two times as far away as the largest object size given in the problem.

d. make your pictures BIG.

A second suggestion is that some students might benefit from problems in the form of the actual physical situation, rather than a text statement of it {4}. In this way, confusion between the actual drawing and the physical reality it represents might be reduced.

Finally, to emphasize the non-static aspects of a problem, the following tips seem to be helpful:

a. draw two or more pictures, each with a picture frame around it (cf., Section 5.3.2 (Conversion of Dynamic Problem into Static Problem)). In this way, the movement of the system can be visualized.

b. visualize the physical situation evolving in time by playing an imaginary movie in your head.

9.2.3 - Extraction of All Relevant Information from Problem (Section 5.3)

When a student is "stuck", sometimes the reason is that he has not extracted all relevant information from the problem statement. Consequently, he may be blocked from taking further action. One thing he can do is compare his own summary of the problem to the text statement of the problem. If he has no summary, he should create one. In general, one useful trick for identifying useful information is to read the problem slowly, supplying a meaning for every phrase. Then he should paraphrase the entire problem.

Alternatively, students may supply their own (and incorrect) default information in place of the missing information. Such errors are hard to realize at the time of the error, so checking is necessary; this check should be performed at many points in the course of a solution, and it should always include all work done up to that point.

I have often found it useful to give students the following two exercises:

a. Here is a text statement of a problem. Take any notes you need to solve the problem without this text, but do not copy the problem over.

b. Here is a problem. Write 10 (or other appropriate number) true things about the problem, which involve essentials and not irrelevant detail.

Finally, it is often useful to read the problem out loud. This is even more effective at uncovering overlooked information if a second party reads the problem to the problem solver.

9.2.4 - Choosing Good Coordinate Systems (Section 5.4)

To assist the selection of a definite coordinate system orientation, reference frame, or origin, the following rules of thumb are helpful.

a. Unless otherwise directed, always select an inertial reference frame at rest with respect to the lab. If the reference frame represents you, an observer of what is happening, you should not be moving.

b. If possible, orient the coordinate system so that one axis is aligned with the acceleration of the system under consideration. In this way, only forces and velocities must be resolved into components. The acceleration vector will be zero in all components but one, and life is made much easier.

c. Label the origin or reference line from which distances will be measured. For each axis, include a direction which represents positive, and maintain that convention for all vector components.

9.2.5 - Clarifying Functional and Numerical Equality (Section 5.5)

To emphasize functional equality, the notation used to write equations should reflect explicitly the connection between the right and left hand sides of the equation. Thus, we would write:

$$y(x) = e^{-x} \quad \text{and} \quad \Sigma F_{\text{on } m} = m\, a_m$$

instead of

$$y = e^{-x} \quad \text{and} \quad F = ma$$

In this way, the student's attention can be called to the existence of an explicit connection between the left and right sides of the relation. [This is not to say that nothing more should be done; I am simply arguing for a visual reminder for the meaning of relational statements; certainly we should do anything we can to make their meaning clearer.]

To distinguish between the various types of equation, it is reasonable to require one line of justification for each and every equation which would specify its type: meaning, identity, causality, etc.

9.2.6 - Calculus as Used in Physics

In Section 5.6, I discussed difficulties which stemmed from the use of derivative as rate of change and ratio of infinitesimals; I pointed out that the latter is a concealed integration. I suggest that there is a significant difference between $F = dP/dt$ and $F\,dt = dP$, between $i = dQ/dt$ and $i\,dt = dq$; both forms should be presented and discussed; teachers should not assume that they are equivalent.

9.2.7 - Perceptual and Representational Spaces (Section 5.7)

To deal with confusions between perceptual and vector representational spaces (i.e., the actual drawing and a higher-order abstraction), it is necessary to emphasize the difference between the two manifolds. This can be done by drawing two separate pictures (one in perceptual space and one in a vector representational space). The connection between these pictures lies in the isomorphism of each space's directions, possible only because directions are specified by angles, which themselves result from dimensionless ratios - the only means by which quantities from different spaces can be compared.

9.2.8 - Idealizations (Section 5.8)

Students may interpret idealized answers without considering the approximations involved in the construction of the model which led to those answers. Again, I suggest that demonstration-problems might help.

9.2.9 - Identification of States (Section 5.9)

Most introductory physics textbooks defer the notion of state until thermodynamics (typically after 15 or so chapters). However, this notion is useful much earlier, and I believe it should be introduced much earlier.

For example, the use of conservation laws requires the notion of state. Even Newton's laws concern state: the state of a particle is characterized by its position and velocity. Force can be conceptualized as an operator which changes the value of one state variable (velocity) but not the other (position).

[Indeed, this characterization of a particle's state underlies the Lagrangian formulation of Newtonian dynamics - specifying x and v (the only variables which enter the Lagrangian) is much easier than specifying the acceleration a.]

An earlier discussion of state would imply a change in the way much of mechanics is presented. For example, discussions about kinetic energy, potential energy, work, and heat would no longer be separate. Instead, kinetic and potential energy would be introduced as state variables. Work and heat would be introduced as source and sink terms. As new forms of energy would be introduced, so would new state variables.

One caveat: students often do have an intuitive notion of state as "what the system is", and this notion can serve as a base on which to build explicit considerations of state as understood by physicists. However, by itself, it is inadequate, as the examples of Section 5.9 illustrate.

Section 9.3 - Specific Suggestions: Problem Solutions as Arguments (Chapter 6)

Chapter 6 discussed some of the ways in which students do not accept the criteria for scientific argument. This difficulty is a very thorny one, and I see no easy way of addressing it.

However, I am sure I know one way NOT to address it. Teachers should NOT present their students with an introductory unit on "the nature of scientific argument" and then proceed as before. Doing so would ensure that the introductory unit would be forgotten or ignored. Students must explicitly see it as something which pervades the entire course. In other words, it must be raised explicit time and time again, in a variety of contexts. If it is not, students will revert to their old styles of argument.

I also believe the following exercises would be helpful.

1. Assignment (a): convince me of something - anything - but make your argument as air tight as possible. Xerox these assignments, distribute them to students. Then give assignment (b): read your classmates' papers. Sort them into three piles: I am definitely convinced, I am definitely not convinced, I'm not sure. Then answer the following question: what is common among the papers in each of these piles?

The purpose of this exercise is to develop facility with a generalized version of the physicist's analytic thought, and a critical sense which will enable students to pinpoint reasoning errors.

2. Return to students their homework and problems sets ungraded. Then ask them to grade it themselves, and grade them on their own grading of their work. In this way, they learn that it is important to be able to check their work.

3. Give students erroneous problem solutions, or arguments which are correlative, analogical, or redescriptive, and ask them to criticize these papers from the perspective of a physicist.

Section 9.4 - Specific Suggestions: Generating a Quantitative Solution (Chapter 7)

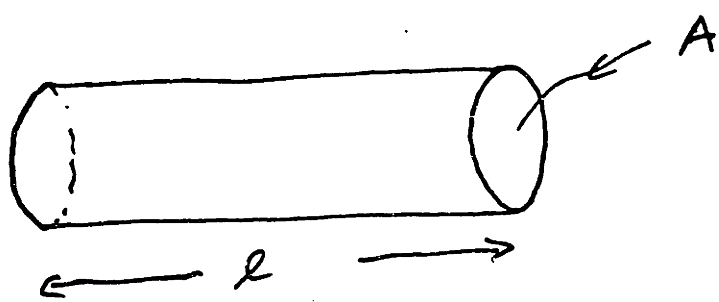
Section 9.4.1 makes a few suggestions regarding selection, but there is currently much work being devoted to this area (e.g., Reif and Eylon

(1979), Larkin (1979b)), and so I will not discuss this area extensively. Section 9.4.2 offers one suggestion for assisting the proper formulation of equations from a given relation or idea. Sections 9.4.3 discusses some suggestions applicable to all of Section 7.3 (Local Symbol Specification). Sections 9.4.4 - 9.4.8 discuss a few of the specifics in Section 7.3 and Chapter 8.

9.4.1 - Identification and Selection of Useful Relations (Section 7.1)

There are different difficulties associated with the selection of the different types of relation discussed in Section 4.3. A generally useful strategy in the selection of relations is the construction of a qualitative model from the statement of the problem. Larkin (1976) finds experts solving problems routinely generate a qualitative model before engaging in detailed calculations - an intermediate representation of the problem which can be easily checked against the original statement of the problem and also against any equations which they subsequently use. This model often helps to guide the selection of relations useful in solving the problem.

Whatever the form of the model, it is essentially qualitative, and functions at a low level of detail. Reif (personal communication, 1976) uses the example that the electrical resistance of an object is a function of the particular material and the geometry of the object. This highly abstract and qualitative description contrasts with the quite specific and detailed description which uses the relation $R = \rho l/A$. This relation very easily calls to mind the following object:



A managerial strategy which helps to choose useful relations is also necessary. An example might be the following: if the problem asks about initial or final state (specified by position or velocity), try to apply a conservation law. If it asks about time explicitly, use a dynamical law. This requires knowledge about (a) specifying states, and (b) use of conservation laws which include sink or source terms.

A second strategy might be for the student to search for relations of each type of the Section 4.3 taxonomy of relations: fundamental, phenomenological, constraining, and definitional.

Finally, it is often possible to generate additional equations by applying the same general relation to different aspects of a problem, e.g., to another system, or in a different direction.

9.4.2 - Formulation (Section 7.2)

Once a relation is selected, it must be translated into mathematical form. If the initial statement is already in mathematical form, then all is well, but there are many cases in which it is not. In order to assist in the formulation of these mathematical statements, it is useful to use numerical tests which are cheap to use and easy to check. For example, in many cases, one may know that a quantity X is related to two others Y and Z as a ratio, but be unsure if the appropriate ratio is Y/Z or Z/Y. In these cases, it is useful to insert simple numbers for Y and Z checking to see what result is obtained.

Here is an example:

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Example 9.4.2.a

The problem is the students and professors problem from Example 7.2.1.a. What follows below is an advanced graduate student solving the problem.

Let's see. The answer is either 6S = P or 6P = S. Let's try it... if P = 1, then S should be 6. Which one works?

6 x (S=6) = (P=1) ..... nope.., 36 is not 1

6 x (P=1) = (S=6) ..... OK, that works, 6 = 6, so it must be 6P=S.

Notice the way in which he realizes that a ratio is involved: the only thing about which he is uncertain is which way it goes. So, he generates a quick and dirty check on his first proposed answer.

****

I also pointed out that an imprecise vocabulary creates difficulties in formulating the appropriate relations. However, the student is most often unaware of (or unwilling to confront) his vagueness. Thus, it is incumbent upon the teacher to raise the issue of precise articulation explicitly. Raising this issue may not necessarily help everyone (and certainly other things are necessary too) but NOT raising it guarantees that it will remain hidden from the view of most students. In practical terms, this means mentioning explicitly the possible confusions that might arise between terms, and placing students in situations in which the lack of differentiation between concepts is likely to lead to error. This also means requiring the student to articulate correctly in his own words the meanings of various concepts.

9.4.3 - Local Symbol Definition (Section 7.3)

Local symbol definition is the process by which the information given in the problem is mapped onto the general relations previously identified and formulated. This step is the most difficult for students, both by their own admission, and by my observation. A general piece of pedagogical advice is the is the following:

Teachers can require that problem solutions should begin with general relations (or at least very widely applicable special cases). For example, the following list of basic relations on the following page should be sufficient to begin 95% of all typical introductory mechanics problems:

Kinematics

$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2 \qquad v_f^2 = v_0^2 + 2ax \qquad \Big\}\ a, \alpha \text{ constant} \qquad \Big\{\ \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad \omega_f^2 = \omega_0^2 + 2\alpha\theta$$

$$v \equiv \frac{dx}{dt}, \quad a \equiv \frac{dv}{dt} = \frac{d^2x}{dt^2} \qquad s = r\theta \quad v = r\omega \quad a = r\alpha \qquad \omega \equiv \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}$$

circular motion: $a_{radial} = \frac{-v^2}{r} = -\omega^2 r$

Dynamics: causal

$$\sum \vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \qquad \vec{p} \equiv m\vec{v}$$

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} = I\alpha \text{ (special case)} \qquad \vec{L} \equiv \vec{r} \times \vec{p} = I\vec{\omega} \text{ (special case)}$$

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

$$\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{M} = \frac{\sum_i m_i \vec{r}_i}{M} \qquad I \equiv \int r^2 dm \qquad I = I_{cm} + M_{tot} R_{cm}^2$$

$$\vec{F}_{earth} = \frac{G M_e m}{r^2}\hat{r} \qquad F_{wt} = mg \qquad F_{fric} = \mu_{k,s} N \qquad F_{spring} = -kx$$

Dynamics: Conservation

$$E_{init}(K+U) = E_{final}(K+U) - \text{energy in} + \text{energy lost}$$

$$K \equiv \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \qquad \text{Work} = \int \vec{F}\cdot d\vec{r}$$

$$U_{ab} \equiv -\int_a^b \vec{F}\cdot d\vec{r} \qquad \text{Power} = \frac{dE}{dt} = \vec{F}\cdot\vec{v} \text{ (special case)}$$

$$\vec{P}_{init} = \vec{P}_{final} - \text{momentum in} + \text{momentum out}$$

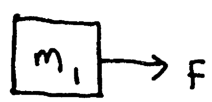
$$\vec{L}_{init} = \vec{L}_{final} + \vec{L}_{in\ or\ out} \text{ (if external torques present)}$$

$$U_{sp} = \tfrac{1}{2}kx^2 \qquad U_{wt} = mgh \qquad U_{earth} = \frac{-G m_1 m_2}{r^2}$$

The advantage of this requirement is that students then acquire familiarity with the idea that basic relations underlie the solution of problems. In addition, teachers should also discuss explicitly the correspondence rules {6} which describe how to map the elements of the problem onto the applicable general relations. Students should be drilled and tested explicitly on the use of these rules, in a variety of circumstances with varying amounts of irrelevant detail. In this way, the student can learn to ignore irrelevant detail in the application of these rules.

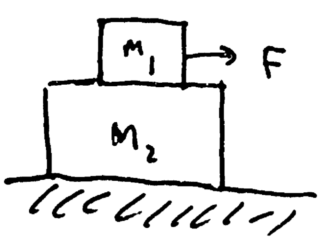
Here is one example:

#1.



The problem is to find the acceleration of m_1? If we apply F = ma to block m_1, what is the appropriate value for m in F = ma?

#2



Now, replace the figure in #1 with the figure at the left, and pose the same problem. Applying F = ma to block m_1, what is the appropriate value for m in F = ma?

The second situation is essentially the same as the first; however, it has added some additional and irrelevant detail.

In formulating these correspondence rules, it is helpful to draw on the requirements of the physicist's form of analytic thought. For example, the separation of system and environment dictates that the application of an equation to a system requires that attention be focused on that system, and that the world outside that system is replaced by transfers of stuff across the system boundary. A useful metaphor: look at the system through a tube. While this advice has been offered before in other forms (e.g., draw a box around the system), it has the advantage that it is particularly graphic; I have found this to be significant in a number of cases.

Cause and effect (Section 4.2.2) leads to a second rule - to determine what forces act on a system, as the following question: what touches the system? If fields are allowed to "touch" the system (e.g., by bathing it in the field), this rule always leads to a complete and accurate identification of the forces which act on the system.

More generally, teachers must attempt to formulate correspondence rules explicitly and in a way that is useful to the student trying to apply general relations to specific problems. Furthermore, teachers must explicitly apply these rules whenever they might apply, and not take "obvious" short-cuts.

I am aware of three objections concerning the use of these rules. The first is that these rules are intrinsically inarticulable. The second is that students who are "rule-bound" use the rules mechanically, without "real understanding" and in cases in which they are inappropriate. The third is that they suppress the creativity of the good student, by demanding that he conform to an approach which is not his own.

My response is the following. The first is an evasion of responsibility; perhaps no one has been able to articulate appropriate rules in a coherent manner, but that is quite different from saying that it is impossible to do so. Indeed, I believe that it _is_ possible to a much greater extent than most teachers would believe. The second merely states that the rules have been formulated poorly; a proper formulation includes caveats, warnings, and heuristics for determining if any given rule is appropriate. The third is that the good student will learn despite anything we do to him. Explicit statements of these rules are necessary for the weaker students, who lack even one way of approaching a problem. Furthermore, teachers can give extra credit for solutions worked out in original ways, as long as the student demonstrates competence in our way as well.

9.4.4 - Determination of Sign (Section 7.3.5)

Section 7.3.5 discussed the omission of signs and directions even when these are important aspects of a problem. This difficulty can be interpreted as an incorrectly invoked default assumption that only magnitudes are important or that problems are one dimensional. Thus, it is necessary to call a student's attention explicitly to these items.

One way to do it involves a bureaucratic directive that any problem which incorrectly omits sign or direction will be heavily penalized. The result will be many student complaints, along with a reduction in the frequency of this type of mistake.

A second way is to realize that a student often does not recognize when sign or direction is important. An appropriate set of exercises might offer the student many problems to do in a short period of time, drilling him in the identification of problems as uni- or multi-dimensional, or as problems in which sign was important.

There are also several methods for reducing confusion about signs. Each of these has its place, but confusion between them is often disastrous. In particular, when a problem involves the existence of an additive term, there are usually are two ways to assign to it the appropriate sign: through strict use of the formalism, or through a consideration of physical grounds. The former is far more prone to error, because it involves a multi-step process.

For example, the determination of the sign of electrostatic potential energy on purely formal grounds requires knowledge of (a) the limits of integration (is it a to b or b to a?), (b) the minus sign in front of the integral, (c) the signs of the charge to be moved and the field source, and (d) if the source is a point source (for which $E \sim 1/r^2$), the appearence of a minus sign upon integration of the electric field.

By contrast, with the latter method, these four items collapse into the answer to one question: Do I have to expend energy to move my charge from a to b? If yes, the potential energy is positive. If no, then it is negative.

9.4.5 - Use of Coupled Equations (Section 7.3.6)

Section 7.3.6 discussed the fear and hesitation to use coupled equations or symbolic variables. This issue must be raised explicitly, because an explicit acknowledgment of this kind of fear would help reassure them. Indeed, as I mentioned in Chapter 6, this fear is not unreasonable if you believe that the problem is just a mathematical problem.

In addition, the root of the physicist's faith in the eventual closure of the system of equations should be discussed. Thus, the idea of model must be addressed; perhaps this is another area in which demonstration-problems would be helpful. This discussion should also include the concepts of a well-posed problem, and constrained solutions.

9.4.6 - Reducing Cognitive Fear and Risk (Section 8.1)

The fear of being wrong can be a paralyzing one; I raised this issue in Section 8.1. When this fear is an issue, brainstorming techniques can be useful (e.g., Woods (1975)). The essence of this approach is to create an environment in which all ideas (whether ultimately useful or not) can be expressed freely without fear, since most ideas will be thrown out anyway. [This approach is similar to the advice given by many writing teachers: throw away the first draft of anything you write. In this way, you don't worry about whether or not the first draft sounds right.]

9.4.7 - Zero Property and Non-Existence (Section 7.2)

Section 7.2 discussed the distinction between quantities which are zero and quantities which are not. Students often behave very differently when confronted with a zero quantity - they abandon procedures which they have used effectively in other situations involving non-zero quantities. In addition, they are unable to separate an object's function from its property.

I have found it necessary to raise explicitly the fact that ma can be zero if a is zero <u>or</u> if m is zero.

This distinction does not depend on notions of quantitative approximation; many students are aware of the fact that "massless" means "mass is small compared to other masses in the system". Rather, it is a distinction based on structural characteristics of the model: attributes are separable, and can be considered independently. Hence, it is important to discuss the qualitative notions of modeling as well.

9.4.8 - Intuition

The development of reliable physical intuition is a worthy enterprise, even if the primary goal of a physics course is to develop problem solving ability. Indeed, I have found (not surprisingly) that a reliable intuition facilitates correct problem solutions.

My primary suggestion regarding the development of intuition is the following: encourage students to use less abstract arguments when possible as a heuristic for understanding the situation. If the assumption of limited capacity in the human problem solver is valid, then the use of less abstract arguments will increase the effort available for coordinating various pieces of information.

One particularly powerful form of less abstract argument is anthropomorphic. For example:

Proposition: the electric field due to a uniform sheet of charge is independent of the distance from the sheet.

Anthropomorphic argument: Teacher directs student to look at a uniformly lit blackboard through a tube. "Pretend you are a charge. Go closer to the board; can you tell if you're any closer? Does it look any different? Now move back. Does what you see change?

Of course, this "argument" does not explain anything, and its plausibility rests on the fact that the intensity of a point source of light drops off as $1/r^2$, just like the Coulomb field. However, when such analogies are judiciously constructed, they give the student the opportunity to use concrete and personally oriented processes. I have found anthropomorphic arguments particularly useful in two domains:

a. in illustrating the local nature of classical physics, because all people realize that their communication with the outside world stops at their extremities; if you close your eyes and ears (thereby mimicking bricks and wedges which also do not possess sight or hearing), you can't know what is "really" going on around you.

b. in clarifying topics dealing with relative motion: what do you see? Halfman (1968) employs observers fixed in various reference frames; he comments (personal communication) that it was only through using these observers that he himself was able to understand relative motion.

It may also be useful to construct teleological arguments, though I have never done so. A teleological argument would take advantage of the fact that notions of function are more easily accessible than notions of property. For example, Clement (1977b) reports that students are often quite capable of making qualitative and causal arguments before they can work out a problem's solution in quantitative detail.

Of course, it is important to stress the limits of such arguments; alone, they are not sufficient, and they must be made more precise and eventually turned into first principle arguments. Nevertheless, they do provide a low-barrier entry point into analyzing a problem, and allow the student to make progress he might not otherwise make; teachers would do well to recognize that these arguments to have value, and grade accordingly.

A second general suggestion for developing intuition is to teach students to use top-down modeling. This suggestion is related to my first suggestion, in that a good hierarchical structuring would almost certainly include qualitative and less abstract information in its higher levels, and more quantitative and detailed information in its lower levels.

Section 9.5 - Miscellaneous Suggestions

The suggestions which follow seem to be generally useful; however, they are related only marginally to the difficulties documented in Part II.

9.5.1 - Periodic Summary and Simplification

When a student gets stuck, he should summarize everything he knows about the problem. In fact, he should do so periodically in order to keep his solution in some coherent form. In observing students, I have been struck often by the lack of organization and structure in their problem-solving approaches. For example, I have seen student scrap paper which contained all the necessary information, but scattered about in a way that the student was unable to put it all together into a coherent answer.

In addition, he should simplify all algebraic expressions; doing so will often result in a new way of looking at the problem, or bring out what is really known more clearly.

9.5.2 - Frequent Checks

Many student mistakes result from sloppy and careless thinking (e.g., Section 5.4 - Extraction of Information from Problem Statement). Consequently, students should learn to check their work frequently. Indeed, Lochhead and Whimbey (1979) point out that many weak students believe good problem solvers rarely make mistakes. On the contrary, good problem solvers make mistakes frequently. However, their problem solving techniques are such that they can catch their mistakes with a high probability of success, i.e., they check their work at many intermediate stages of solution where it is not costly to back up and try another approach. [If one goes too long without checking, one may become psychologically locked into a particular solution in which one has invested much effort - thus making it harder to discard such a solution.]

9.5.3 - Alternative Checks

Many students believe that the only way to check an answer is to go over the solution line by line. While this is certainly one aspect of a complete check, it is time consuming (and hence will not be employed frequently), and is likely to lead to a repetition of any mistake already present in the solution.

By contrast, the checking techniques which follow are fairly cheap, and take little effort compare to a line-by-line check.

a. Checking algebraic expressions by taking limiting or special values for the parameters in the problem. Note that this method of checking depends on an algebraic expression for the desired quantity, in terms of symbols for known quantities before putting in numerical values.

b. Checking units of expressions.

c. Using qualitative graphical techniques and continuity arguments to assure at least approximately correct functional dependence of the answer on the given parameters.

d. Checking the direction or sign of the answer.

e. Checking the magnitude of the answer. This requires knowledge of benchmark values, and a sense of the numbers involved in a problem. For example, students should know that paper clips weigh about a gram, and that a penny doesn't weigh an ounce. This in turn requires that the problems we give the student bear some connection to his experience.

9.5.4 - Grading

Teachers and students often do not share basic convictions concerning what parts of a solutions are important and what parts are not. In particular, teachers don't usually give credit for the "trivial things". For example, teachers may deduct a few points out of many on a solution which includes an incorrect sign, because they have decided that a correct sign is worth only a few points. Indeed, the sign of the answer may be the easiest part of the problem; however, in this case, an incorrect sign should be penalized heavily. Otherwise, the student learns from the teacher's response that signs are unimportant, and begins to concentrate on the "real stuff".

I believe the consequence of this grading policy would be to reduce sharply the number of careless mistakes. However, this approach implies a genuine distinction between performance and competence, and in my experience, this is not one which teachers usually make consistently or even consciously.

Section 9.6 - Pedagogical Suggestions: Caveats and Comments

Sections 9.2 - 9.5 make specific suggestions which attempt to deal explicitly with the difficulties outlined in Part II. However, I believe these suggestions are no more than a set of tactical hints and ideas which many students would find useful to keep in mind.

Indeed, the difficulties of Part II are not independent of each other, and a simple attempt to patch each difficulty piece by piece is not likely to be successful in and of itself. In other words, there may be deep-seated (and inadequate) epistemologies and conceptual strategies (which may or may not be rule-governed) underlying these "surface-structure" difficulties.

A naive pedagogical approach would be to proceed on the assumption that the underlying structure must be modified, and to modify it directly. For example, a teacher might discuss appropriate ways of knowing and the values of a physicist at the beginning of a course, and then continue as before with the standard presentations.

My experience leads me to believe that this approach would be completely inadequate. The deep-seated nature of these conceptual structures implies that they will be hard to midify, and consequently, the mere suppression of certain difficulties may well cause difficulties to appear elsewhere.

This is not to say that teachers should not discuss possible difficulties and confusions explicitly; I believe they should, because the student's difficulty is often not that he cannot find X, but that he has not thought of looking for X; students often do not know what they need. Explicit identification of possible pitfalls can help give shape to a student's "being stuck", and may give the student enough knowledge to ask a well-formed question. For example, the categorization of relations would enable a student to search for various types of relation systematically: Do I need to apply a definition? Are there relations of constraint? Would it help to introduce an arbitrary parameter?

In addition, I also that many of these hints would not have much effect if offered before the student confronts the difficulty himself. The reason is that many of the ideas which underlie his difficulties are familiar to the student, and he believes he understands them; in fact, he does not, but the student who feels no reason to pay closer attention to these issues will not do so. Therefore, from a pedagogical standpoint, it is desirable to design problems in which the student is likely to encounter these difficulties. In this way, he has some incentive to change, and is therefore more likely to accept his teacher's advice about things to which he should pay attention. Here is some evidence for this point of view. The following quotes are student responses to an inquiry concerning reasons for past failure:

> I used to attack each problem in a separate manner. Only near the end of the year did I realize that particular problems were solved in a general fashion, by fitting them into a more general situation. [Me: Why did you change?] I was failing.

> My approach on the first two exams was to try to remember a single equation and plug in things. It didn't work, so I had to change. Things got better, but not enough to pass.

> I used to try to force what I knew into the problem, rather than use the problem to determine what I needed to know. I changed when it didn't work.

> I used to read a chapter just once to get the equations I needed for the problems. [Since I flunked,] I'm going to do the reading much more carefully.

Many of the suggestions I have made take the form of a rule, a template, or a verbal overlay. While it can be argued that verbal templates do not address the question of "real understanding", I believe they should be treated as mnemonic, rather than as substantive in themselves. Of course, the substance of what falls underneath these mnemonic umbrellas is a central issue, but I suggest that it is this question whose answer is commonly the focus of the standard introductory course.

Finally, I have made the suggestions of Sections 9.1 - 9.5 are made from the following perspective: I assume that it is appropriate for a teacher whose goal is to teach problem solving of the type characteristic of the standard introductory physics course to advise his students explicitly about what to do and look for when they solve problems.

Note well what am not saying. I am not passing judgment on the merits or pitfalls of "discovery learning", or project-based courses, or the use of open-ended problems. I am saying that if the goal is to teach students to solve the problems in the typical textbook, then I assume teachers should be prescriptive; saying "You get better with experience" is no help to the student.

Of course, this requires that teachers admit that teaching problem solving is a primary goal of an introductory physics course, and many teachers are quite reluctant to do so. Indeed, many teachers argue that problem solving is only a means to the end of teaching "real understanding". I believe that this is an abdication of responsibility if they do not define "real understanding", especially since many of them in practice do define understanding as good problem-solving ability.

Problem-solving does have its advantages as a major focus of the typical course. In particular, a student is responsible for producing an answer, and it is readily apparent if he cannot do so. By contrast, students are often unable to criticize their own verbal or qualitative explanations. A student solving a problem confronts his ignorance when he cannot do anything else. A student asked to write a qualitative explanation can almost always write down something which allows at least the illusion of understanding. The beginning student can take refuge in vague and ambiguous statements; the ability to criticize these statements comes much later {6}.

On the other hand, it is far from clear that standard instruction and problem solving is actually relevant to real understanding - work by DiSessa

(1979b), McDermott (1979), and Clement (1977a, 1977b) would suggest that it is not. Hence, this matter is also one to be addressed in future work. However, I have no idea how to address it, so I make no comment on it in Chapter 10 (Work to Come).

Notes

1. In a wonderful article entitled "Less May Be More", Morrison (1964) argues that a course which covers less material has a better chance of giving students a meaningful learning experience. While his article is directed essentially towards teachers of science for non-scientists, I believe its lessons applicable to the students with which this thesis is concerned as well.

2. This last sentence is a direct quote from Kuhn (1970).

3. I can personally testify to this fact. I verified it experimentally when my bike went down a hill without me - it fell over when it hit a car.

4. Prigo (1977) calls these demonstration-problems.

5. Correspondence rules should be made as algorithmic as possible. They should, to the extent possible, specify a completely deterministic procedure which will result in appropriate symbol specifications. However, there will invariably occur instances (maybe in perhaps the majority of cases) in which these procedures can be specified only incompletely. Such rules will then be "only" heuristic, in that they may provide "only" rules-of-thumb and starting points.

6. I am not hiding behind the claim that "experience helps"; I merely state this as a matter of empirical fact. Unpacking this phenomenon (that only later can one criticize qualitative explanations) is a subject for another thesis.

The longest journey starts with but a single step.

- Lao-Tze

Chapter 10 - Work to Come

In this thesis, I have illuminated some of the difficulties students have in solving problems. However, I suspect I have raised more questions than I have answered. The purpose of this chapter is to set down some of these questions explicitly.

The areas of future work fall into two general categories: areas concerned with an improved understanding of the learning and problem-solving process, and areas concerned with an improved pedagogy. I lean towards the belief that without an understanding of the learning process, pedagogical structures are built on shifting foundations, and though it may be possible to design a structure which is stable even when resting on shifting foundations, it is harder to do so.

Of course, it might well be the case that instruction which is sufficiently prescriptive would be quite insensitive to the structure of learning processes. However, since this seems to be the point of view taken by most teachers, it is an approach I need not address; others will address it with or without me. Hence, the discussion which follows will focus primarily on understanding learning process.

In the sections which follow, I will sketch several areas which I believe warrant further investigation; most follow the broad-brush exploration technique which has guided this work. I should add that I have a definite preference for broad-brush explorations. The alternative, a very sharp and hard pencil, results in studies which tend to cover a very narrow range of issues, are precise and detailed, and take a very close look at quite definite features of the terrain. While I am willing to believe that these studies can be useful, I have a gut feeling that knowledge of detail can hinder as much as help the pedagogical process.

Section 10.1 - This Thesis Made More Rigorous

One obvious piece of future work is the repetition of the exploration documented by this thesis from a more rigorous point of view. In a more rigorous study, the descriptive framework would be verified through the use of independent judges trained in the use of this framework. These judges would observe the behavior of a subject from the perspective of this framework, and then construct a description of the student's actions from the basic building blocks provided by the framework. To the extent that the judges construct similar descriptions among themselves in a variety of cases, it would be reasonable to assume that the framework was a reasonable way of describing student behavior.

A good example of this verification process is offered by Perry (1970):

> Any ... inferential construct, drawn from ... varied data, faces the question of being no more than the observer's way of imposing an order where it does not exist. We therefore endeavored to reduce our [original] scheme from its broadly discursive expression into a representation sufficiently condensed, rigorous and denotative to be susceptible to a test of reliable use by independent observers.... The extent to which the several observers' placements or "ratings" might agree with one another, beyond the level of chance, would then be the measure of the validity or "existence" of our scheme.

Judges would be chosen on the basis of the assumption that expertise in any scientific field is defined by the consensus of the community of expert professionals in that field. Since these community members usually have a dual role as researcher and teacher (at least at the college level), their professional judgment necessarily includes a pedagogical as well as a research component; they are therefore the judges of the state of a student's knowledge and abilities. Therefore, judges should be members of this community {1}.

Such a procedure is appropriate since these experts are the ones who answer questions such as: "What is the student doing incorrectly?"; "What does the student know?"; "What should the student be doing?" Indeed, it is only by comparison to their own shared background as recognized experts that they are able to answer such questions at all; if the student does something that experts would not do, then the student is "in error."

Thus, in general, judges would require membership in a community which almost unanimously shares a subject discipline paradigm, and instruction in the use of my diagnostic framework. The time scales relevant to these two parts differ substantially. An individual adopts and internalizes the subject-discipline paradigm over many years. The plausible articulation is another matter - if after substantial discussion (empirically speaking, days or weeks, maybe months, certainly not years), the articulation seems implausible or unreasonable to a large number of other experts, it must be discarded or substantially reworked.

These judges are only one part of a cleaner research approach. A clean approach would more generally involve the following:

1. Observe and tape-record student's performing some tasks related to an introductory physics course (problem-solving, explaining a problem or a text section in their own words, participating in interviews concerning their approaches to and difficulties with physics).

2. Abstract and idealize that which doesn't correspond to the perspectives of experts, thus generating the diagnostic framework.

3. Present abstractions to experts, and ask their opinions; this provides a low-cost verification of the framework.

4. Present expert judges with the raw data collected in item (1), and ask them to analyze the data within the framework.

5a. If judges cannot identify the cases from which the framework was abstracted, then rework the list.

5b. If experts correctly identify these cases, then conclude that the framework is a reasonable idealization of the difficulties listed.

The primary difference between step (3) and steps (4-5) is one of rigor and formality; steps (4-5) would not involve directly the author(s) of the lists (i.e., me), they would involve blind experiments, and so on. In this way, the map I have made can be made more reliable and trustworthy.

A second aspect of this more rigorous study would be the development of questions or problems in which the difficulties of Part II appear frequently. With this set of problems, an investigator probing these difficulties would not need to rely on chance in order to observe them.

A third aspect of a more complete study would investigate problem-solving difficulties in other subjects: electricity and magnetism, quantum mechanics, statistical mechanics, thermodynamics. [Separate investigations are warranted because of my assumption that even difficulties in the logic of problem solving will manifest themselves in different ways in different fields. In addition, examples in other areas of physics would be helpful to those teaching those areas.]

Finally, it remains to be seen if any of the pedagogical suggestions made in Chapter 9 are indeed helpful to students actually engaged in problem solving.

Section 10.2 - Revolutionary Changes in Thought

As an individual makes the transition from beginning student to expert professional, his ways of thinking (and not just his factual knowledge) go through radical changes - this is the primary lesson of this thesis. Piaget, Kuhn,

mirrored in these four stages.

While much work has focused on a literal application of Piaget's theory (especially the last two stages) to the assimilation of physics content, it is plausible that the sensori-motor and pre-operational stages also describe certain aspects of the physics learning process as well. For example, the following things happen in the sensori-motor stage:

- acquisition of specific behavioral acts which are the basic building blocks of complex solutions.

- purposive, deliberate, goal-seeking behavior (as opposed to repeated behavior first instantiated by chance).

- the necessity of sequencing actions to achieve a desired result.

- the beginning of representational thought.

- the concept of permanent object.

The parallels in the introductory physics student could be:

- acquisition of specific microskills (e.g. resolution of vectors into component forms, identification of all forces acting on a body) which are the basic building blocks of complex solutions to typical course problems.

- self-initiated application of these microskills (as opposed to following a textbook or instructor solution).

- the realization of a need (as opposed to the ability) to plan a coherent solution.

- the beginning of representational thought (e.g. allowing a static picture to represent a moving object).

- the concept of permanent process: The notion that the same physical process underlies very different physical situations.

The pre-operational stage also reflects much of the behavior demonstrated by novice problem solvers. The pre-operational stage marks the beginning of symbolic thought, but that this thought usually appears to the outsider to be random and unorganized. For example the child often uses words in a way that does not correspond to adult usage. Similarly, the beginner's usage of physics terminology does not correspond to the physicist's use. For example, a novice will often use "velocity" and "acceleration" and "force" and "power" and "momentum" interchangeably when describing motion. He may always use the phrase "centrifugal force" in discussing anything about motion in a circle: "centrifugal force" is a distinct and real force, which pulls the ball out, just as the tension in the rope is a distinct and real force that pulls the ball in.

By contrast, the expert sees clear distinctions between force and power and momentum, i.e., he groups them differently than does the novice. He sees "centrifugal force" as just another <u>name</u> for the rope's force. He uses the same words, but for him, they are associated with different things.

A second example: when a beginning student solves a problem orally, he is often very vague. As the pre-operational child, the beginning student also uses "it" and "this" without specifying their referents, and he also omits causal relations. By contrast, the expert takes care to be precise.

10.2.2 - Kuhn and Physics

I have already argued that Kuhn's 1970 description of revolutionary scientific progress might be better suited to a description of the beginner casting out his inadequate intuitions and problem-solving approaches in favor of the more sophisticated ones of the expert. However, my argument was not much more than a noticing of certain parallels, and much territory remains for exploration.

As one example, Kuhn (1970) notes that

> Two men whose discourse had previously proceeded with apparently full understanding may suddenly find themselves with incompatible descriptions and generalizations. Those difficulties will not be felt in all areas..., but will arise and then cluster about the phenomena upon which the choice of theory most centrally depends.... Such problems, though they first become evident in communication, are not merely linguistic, and they cannot be resolved simply by stipulating the definitions of the troublesome words. Because the words about which difficulties cluster have been learned in part from

> direct application to exemplars, the participants cannot say "I use the word element in ways determined by the following criteria." They cannot...resort to a neutral language which both use in the same way and which is adequate to the statement of both their theories.

A clearer description of teachers talking to uncomprehending students is hard to imagine. A physicist might be able to understand what a student means when the student discusses the dynamics of motion using force and momentum interchangeably. However, the physicist uses force and momentum in the Newtonian dynamics sense, and thus conflict seems inevitable.

10.2.3 - Perry and Physics

Finally, Perry (1970) offers his scheme of ethical and epistemological development. At first glance, such a scheme seems far removed from the development of problem solving ability. However, his scheme describes a progression from a rigid dualism to a flexible relativism followed by reasoned commitment. Similarly, a student who cannot approach a problem flexibly, who is dependent of external guidance, and who refuses to take an assertive attitude towards a problem is unlikely to solve anything but the most routine problem. Many students are disturbed by instructors who qualify their statements, restricting their statements' range of validity. They insist that statements are right without qualification, or are totally wrong.

Perry notes that dualistic students focus on the rules of the game which external authority imposes. In the physics classroom, students may also assume that these rules govern what they may or may not do in the course of problem solving. Teachers will often claim they value creativity, self-reliance, independence of thought, and initiative, while in fact they really demand conformity and obedience. What does the student have the right to do? Is he allowed to introduce his own notation? Can he make simplifying assumptions? In high school, the answers to these and similar questions is often no, and it may not be surprising when, for example, a student cannot decide on a particular orientation of coordinate system when the problem does not offer explicit guidance in this choice (cf., Example 5.4.d).

Section 10.3 - Attitudes of the Lay Person

I have tried to argue that many student difficulties can be traced to value differences between experts and beginners. On the assumption that beginning students are likely to be more similar to lay people than to experts along these lines, it would be interesting to conduct a broad-brush investigation would probe the views and attitudes of lay people concerning scientific ways of knowing. This investigation might include the following points:

I have heard of intelligent lay people who will argue that "what you say about exponential growth [for example] may be true, but that can't happen in this community." The physicist will argue that if a rule holds in general, then it must hold for every case. By contrast, lay people argue that though a rule may hold in general, there can be exceptions. ("This is the exception that proves the rule...") An additional question: to what extent does the lay person consider laws of nature in the same spirit as laws of Congress? Are they arbitrary? Can they be repealed? Can they be violated?

David Hawkins (personal communication, University of Colorado at Boulder, 1979) tells of an elementary school teacher who finished a several-week workshop on geometry. Her end-of-workshop reaction was "I never realized that there were things in life that you could be SO sure of". Other people speak of "not believing in logic", that it is inhuman, that analysis destroys the essential reality of actual situations. Similarly, physics students often see two worlds: the classroom world in which physics is valid, and the real world in which their intuition is valid (and never the twain shall meet!) {2}.

I have a friend, a drama teacher, who said to me that he had no confidence in the ability of science to predict anything: if the specific event had not yet happened or yet been tested, you couldn't know anything about it. Further discussion revealed that he had no sense of magnitude of effect; to him, small, detailed effects had as much significance as large, aggregate effects.

Many lay people have a very strong sense that science is exact and precise. To what extent does this view affect their capacity to accept and/or use approximate representations and models?

Section 10.4 - Computers and Problem Solving

Example 7.3.a illustrated a student who attempted to use a general equation without specifying its arguments. If this student had attempted to implement his solution as a computer program, he would have been forced to search for the cause of the error that the computer would have indicated. Two types of computer-oriented study would be interesting. One would examine differences with respect to local symbol specification (Section 7.3) among students with and without computer programming experience. A second study might examine the efficacy of teaching programming to introductory physics students and demanding that problem solutions be formulated as computer programs.

A third somewhat related study would explore the effects of the hand calculator on problem solving. I suspect that it has had at least some derogatory effects on problem solving performance. In particular, examination of homework solutions in which symbolic as well as numerical solutions are required reveals that students with calculators have no incentive to simplify algebraic expressions, since calculation becomes trivial. Nevertheless, it is in the simplification of algebraic expressions that we often obtain insights into the physics underlying the problem in question.

Section 10.5 - Students and Real Physical Situations

I have found that some students are unable to cope with the modeling assumptions inherent in a pencil and paper text problem. I have also suggested that these students might demonstrate better performance if they were confronted by the real physical situation to which that text problem corresponds; however, I do not know this to be true. It would be very interesting to take protocols of and record interviews with students trying to solve these "real-life" problems.

Section 10.6 - Representation of Motion

Section 5.2.2 discussed the static interpretation of pictures intended to represent time-dependent situations. The increasing availability of good computer graphics makes it practical to investigate the implications of presenting students with text problems and time-evolving pictures. I suspect that the availability of pictures which actually change would reduce sharply the tendency to freeze a dynamic situation.

I also believe it would be useful to investigate student's intuitive representations of motion. For example, teachers conventionally represent motion (velocity and acceleration) by means of an arrow on a picture {3}. If this representation does not match a student's intuitive representation, the student may be unable to grasp the full implications of the arrow.

Section 10.7 - Confidence and Problem Solving

Section 8.1 raised the issue of cognitive risk. Closely related is the issue of confidence. In solving a problem, a person may start with the givens. However, as the beginning student gets further and further from the givens (i.e., as his solution develops), his confidence in each step often diminishes, and he is less and less sure of what information is and is not reliable. By contrast, an expert can, through the use of reliable physical and mathematical theories, create new information in which he has as much confidence as he has in the given information. It would be interesting to explore both domains.

Section 10.8 - Hierarchical and Linear Organization and Approaches

Section 4.5 argued that the limitations on working memory and the total mental effort possible at any given instant were significant factors in problem-solving performance. Section 9.1.3 concluded that lack of facility with basic micro-skills led to poor problem solving performance. So, if the student were freed of the need to perform these micro-skills themselves (non-trivial for some students), to what extent would the additional mental capacity thereby released enable the student to complete his solution successfully?

Here is one possible experiment: put on index cards descriptions of the various subtasks which enter the solution of typical problems in a given domain, and ask the student to choose from these cards what he should do next. When he chooses a card, an expert would then execute the procedure described on that card. Thus, the student would block out a solution, but not be bothered by its details.

Another experiment might instruct the student to solve a problem, allowing him at any time to ask an expert any question to which a precisely formulated answer could be given. (Of course, the amount of the information which might be passed on depends critically on the expert's judgment.) In this way, the student's questions would provide a way of seeing what knowledge he thought was important.

These experiments would shed light on the following question: Reif and Eylon (1979), Larkin (1976), and others have suggested that hierarchical approaches are a very important attribute of expert problem solving behavior. To what extent does this approach presuppose a basic competence with the building blocks of the hierarchy? My own teaching experience indicates that my weakest students are unable to perform any sort of hierarchical organization, but I do not know if this is the effect of their inability to do the small things, or if it is simply correlated with this inability. On the other hand, the top levels of hierarchical organizations are usually vague, and detail becomes important only at lower levels. Many student approaches to problem solving also begin in a vague manner; does this provide a good starting point to teach students to use hierarchical tasks?

Section 10.9 - One-Idea vs Multi-Idea Problems

Chapter 7 discussed the generation of quantitative solutions to physics problems. In Section 6.1.5 (Arguments Must Be General), I drew a distinction between "one-step" solutions and "multi-step" solutions; I pointed out that multi-step solutions required more than one equation. However, I did not distinguish between one-idea problems and multi-idea problems. Some problems require only (for example) kinematic ideas (cf., Example 5.1.1.b: A dog sees a flowerpot sail up and then back past a window 5.0 feet high. If the total time the pot is in sight is 1.0 s, find the height above the window that the pot rises.). Others require (for example) kinematic ideas and also dynamic ideas such as conservation of energy (cf., Example 5.2.2.a: A small block of mass m slides along the frictionless loop-the-loop track. At what height above the bottom of the loop should the block be released so that the force exerted on it by the track at the top of the loop is equal to its weight?)

To what extent, if any, are one-idea problems easier than multi-idea problems? Do students distinguish between independent equations from one domain and independent equations from several domains? My intuitive response if yes to the first question and no to the second, and yet one seems to be merely a rephrasing of the other. This is a matter to be explored in further research.

Section 10.10 - Functional Relationships

Section 5.5 (Functional and Numerical Equality) discussed student difficulties with the notion of functional equality. Example 7.3.6.b discussed field quantities. A matter very closely related to both sections is the understanding of a function such as y(x). A function y(x) takes as its input some value of x, and for every value of x, gives back some definite value for y(x). Field quantities (and more generally, intensive quantities) are similarly specified - at every point in space X, they have a definite value; they specify point properties. However, various studies (e.g., McKinnon and Renner (1971)) have documented the inability of many students to understand intensive variables (because they require proportional reasoning). Thus, it would be useful to probe student understanding of functions in particular and relationships in general.

Notes

1. Membership in this community usually means that they share largely similar educations, share many of the same values, use largely similar techniques, and make largely similar though independent professional judgments (cf., Kuhn (1970)).

2. Walker (1977) suggests that Ph.D. candidates in physics should not take comprehensive or qualifying exams. He proposes instead a requirement that graduate students walk barefoot over a bed of hot coals, relying on the Leidenfrost effect to avoid injury, to receive their degree from the chairman of the department. In this way, their faith in the applicability of physics to the real world could be tested. [The Leidenfrost effect is the vaporization of surface perspiration which forms an insulating barrier between the hot coals and your feet.]

3. They also represent forces, momentum, and all other vector quantities with arrows on a picture. The potential for confusion is enormous.

When you steal material from one author, it's plagiarism. When you steal from many, then it's research.

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Bibliography and References

The references in this chapter are divided into two sections. The first section contains the references to citations made by the thesis itself. The second section contains other references which may be useful in acquiring an overview of certain areas, or developing an understanding in greater depth; these references are not explicitly cited in the thesis.

Thesis References

A. Arons, "Cultivating the Capacity for Formal Reasoning: Objectives and Procedures in and Introductory Physical Science Course", American Journal of Physics, Volume 44, page 834, 1976

A. Arons, "Some Thoughts on Reasoning Capacities Implicitly Expected of College Students", in Cognitive Process Instruction: Research on Teaching Thinking Skills, eds., J. Lochhead and J. Clement, Franklin Institute Press, Philadelphia, 1979

R. Bhaskar and H. Simon, "Problem Solving in Semantically Rich Domains: An Example From Engineering Thermodynamics", Cognitive Science, 1977, Volume 1, page 193

RB and HS describe a study in which a GPS-like system makes "extensive use of means-ends analysis, similar to that observed in semantically less rich domains [e.g., cryptarithmetic], supplemented by recognition mechanisms for accessing information in semantic memory. An interesting paper, but not particularly relevant to understanding how beginners think.

B. Bloom and L. Broder, Problem Solving Processes of College Students, University of Chicago Press, 1950

J. Brown, "Diagnostic Models for Procedural Bugs in Basic Math Skills", Cognitive Science, Volume 2, page 155, 1978

JB describes a computer model for various systematic errors which grade school children make in doing arithmetic. The model is abstracted from extensive observation of grade school children.

J. Clement, "Catalogue of Students' Conceptual Models in Physics, Section 1, Movement and Forces", working paper, 1977a, available from the author at Physics Department, University of Massachusetts, Amherst, Mass., 01002

J. Clement, "Some Types of Knowledge Used in Understanding Physics", working paper, 1977b, available from the author at Physics Department, University of Massachusetts, Amherst, Mass., 01002

J. Clement et al, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Word Problems", unpublished working paper, Cognitive Development Project, Department of Physics, University of Massachusetts, Amherst, Mass., 01002, 1979

R. Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations", Journal of Children's Mathematical Behavior, Volume 1(3), 1975

RD documents some of the cognitive processes necessary to perform algebra successfully. For example, he illustrates syntactic and semantic meanings for the equality sign.

A. DiSessa, "On Learnable Representations of Knowledge: A Meaning for the Computational Metaphor", in Cognitive Process Instruction: Research on Teaching Thinking Skills, eds., J. Lochhead and J. Clement, Franklin Institute Press, Philadelphia, 1979a

A. DiSessa, "When Almost Everybody Does Essentially the Same Thing", unpublished working paper, available from the author at the Division for Study and Research in Education, M.I.T., Cambridge, Massachusetts, 02139, 1979b

AD describes reactions to a computer simulation of a world governed by Newton's First and Second Laws. Subjects were asked to hit a target firing from a moving platform; the simulation provided the capability for real-time guidance (by altering the projectile velocity). Subjects uniformly reacted as though the world represented were Aristotelean, independently of formal physics background; they react by saying "That's absurd! The machine must be broken."

H. Ginsburg and S. Opper, Piaget's Theory of Intellectual Development, Prentice-Hall, 1969

This book is an introductory exposition of Piaget's work with minimal evaluation and commentary.

I. Goldstein and S. Papert, Artificial Intelligence, Language, and the Study of Knowledge, A.I. Memo 337, Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., 02139

H. Gruber, Darwin on Man: A Psychological Study of Scientific Creativity, E. P. Dutton, New York, 1974

HG traces the development of Darwin's ideas from a thorough analysis of his notebooks.

H. Gruber, The Essential Piaget, Basic Books, 1977

This is a huge book, containing an excellent and extensively annotated collection of lengthy excerpts from Piaget's writings Rather sophisticated.

R. Halfman, Dynamics, Addison-Wesley, 1968

A rather light-hearted and very readable (but also very professional) book on dynamics for students with some experience with introductory physics.

D. Halliday and R. Resnick, Physics, John Wiley and Sons, 1977

M. Henle, "On the Relation Between Logic and Thinking", Psychological Review, Volume 69, page 366, 1962

G. Holton, Thematic Origins of Scientific Thought, Harvard University Press, 1973

GH describes the manner in which "themata" relate to scientific progress. Themata are world-views and meta-principles which give qualitative structure to a wide set of ideas.

B. Inhelder and J. Piaget, Growth of Logical Thinking from Childhood to Adolescence, Basic Books, 1958

This books is a detailed epistemology of logical structures of thought present in children of various ages. It makes little mention transition mechanisms from one form of thought to another. Of particular interest to college instructors are his discussions of concrete and formal thought; it will offer perspective on the various "Piagetian-based" programs beginning to appear on the scene.

D. Kahneman, Memory and Attention, Prentice Hall, 1973

R. Karplus, et al, Science Teaching and the Development of Reasoning: Physics, 1977, available from the Lawrence Hall of Science, University of California, Berkeley, California, 94720

R. Klatzky, Human Memory: Structures and Processes, W. H. Freeman, 1975

T. Kuhn, The Structure of Scientific Revolutions, 2nd edition, University of Chicago Press, 1970

J. Larkin, "Human Problem Solving in Physics: Global Features of an Information Processing Model", working paper, 1976, Group in Science and Mathematics Education, University of California, Berkeley, California, 94720

J. Larkin and F. Reif, "Understanding and Teaching Problem Solving in Physics", European Journal of Science Education, 1979a

J. Larkin, "Information Processing Models and Science Instruction", in Cognitive

Process Instruction: Research on Teaching Thinking Skills, eds., J. Lochhead and J. Clement, Franklin Institute Press, Philadelphia, 1979b

On the basis of recent work in artificial intelligence research, JL suggests three things which science education should emphasize more: the large-scale organization of knowledge, when to perform certain actions as well as how to perform them, and the importance of low-level, qualitative thought to plan solutions.

H. Lin, "Approaches to Clinical Research in Cognitive Process Instruction", in Cognitive Process Instruction: Research on Teaching Thinking Skills, eds., J. Lochhead and J. Clement, Franklin Institute Press, Philadelphia, 1979

HL describes deterministic and descriptive approaches to investigating cognitive processes, and reasons for preferring the latter.

J. Lochhead and A. Whimbey, Problem Solving and Comprehension: A Short Course in Analytic Reasoning, Franklin Institute Press, 1979

Through working problems out loud, this book presents a course intended to develop skills useful for solving sequential reasoning problems.

J. Lunn, Chick Sexing, American Scientist, 1948, Volume 36, page 280

L. McDermott and D. Trowbridge, "Difficulties with Kinematical Concepts among Introductory Calculus Physics Students", paper presented at American Association of Physics Teachers meeting in New York, February 1, 1979, available from the authors at Physics Department, University of Washington, Seattle, Washington, 98195

J. McKinnon and J. Renner, "Are Colleges Concerned With Intellectual Development?", American Journal of Physics, Volume 39, page 1047, 1971

M. Minsky, "A Framework for Representing Knowledge", in The Psychology of Computer Vision, ed., P. Winston, McGraw Hill, 1975

MM reformulates old ideas about memory organized around stereotypes in a manner useful for computer implementation. The computer part of this paper is not particularly useful, but it does present a good overview of a particular theory of memory.

P. Morrison, "Less May Be More", American Journal of Physics, Volume 32(6), 1964, page 441

A. Newell and H. Simon, Human Problem Solving, Prentice-Hall, 1972

W. Perry, Forms of Intellectual and Ethical Development in the College Years: A Scheme, Holt, Reinhart and Winston, 1970

J. Phillips, The Origins of Intellect: Piaget's Theory, W. H. Freeman, 1975

This is a somewhat less sophisticated treatment of Piaget than that contained in Ginsburg and Opper (1969). It does, however, provide an overview correct to zeroth-order by a person without extensive first-hand experience working with Piaget. This means that JP lacks the robust intuition for what Piaget actually preaches. However, JP does represent what an insightful outsider might determine from reading Piaget's work.

R. Prigo, "A New Addition to Homework Assignments: Demonstration-problems", American Journal of Physics, Volume 45(5), 1977, page 433

RP discusses advantages to demonstration-problems (as compared to text problems), and provides examples of several.

G. Polya, How To Solve It, Doubleday Books, 1945

G. Polya, Induction and Analogy in Mathematics, Princeton University Press, 1954a

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F. Reif and B. Eylon, "Effects of Internal Knowledge Organization on Task Performance", expanded version of paper presented at 1979 meeting of the American Educational Research Association, available from the authors at the Group in Science and Mathematics Education, University of California, Berkeley, California, 94720

A. Schoenfeld, "Can Heuristics Be Taught?", in Cognitive Process Instruction: Research on Teaching Thinking Skills, eds., J. Lochhead and J. Clement, Franklin

Institute Press, Philadelphia, 1979

F. Sears, M. Zemansky, H. Young, University Physics, 5th edition, Addison-Wesley, 1978

M. Strandberg, "Design of Examinations and Interpretations of Grades", American Journal of Physics, Volume 26(8), 1958, page 555

MS describes certain aspects of what makes a problem hard, and frames them in terms of what the student must do to solve it.

E. Taylor, "What is the American Journal of Physics", American Journal of Physics, Volume 47(4), 1979, page 300

E. Taylor and J. Wheeler, Spacetime Physics, W.H. Freeman, 1970

The best introductory book on relativity!!

P. Tipler, Physics, Worth Publishers, Inc., 1976

G. Weinberg, The Psychology of Computer Programming, Van Nostrand/Reinhold, 1971

S. White, "Social Proof Structure: the Dialectic of Method and Theory in the Work of Psychology", in Life Span Developmental Psychology, Academic Press, 1977

W. Wickelgren, How To Solve Problems, W. H. Freeman, 1974

D. Woods, et al, "Teaching Problem Solving Skills", Engineering Education, December 1975, page 238

DW describes a four year longitudinal study of freshman engineering students designed to shed light on their problem solving techniques The primary investigative technique is to observe students during a non-credit seminar in which students solve problems with heuristic hints from a professor.

Background References

These references are very incomplete, as they have been chosen on the basis of what I can find in my office in the last few days of writing this thesis. On the other hand, these references aren't bad, and will provide some a reasonable degree of perspective.

J. Larkin, "Cognitive Structures and Problem Solving Ability", unpublished working paper, available from SESAME, c/o Physics Department, University of California, Berkeley, California, 94720, 1975

> JL discusses the cognitive structures used by expert physics problem solvers, noting that (a) their structures are hierarchical, with general principles forming basic building blocks of a problem's solution, (b) these principles are often encoded pictorially, and (c) very specific relations are usually reconstructed from general principles. She also describes several techniques for investigating cognitive structures for physics knowledge: structured protocols, free protocols, measurement of observables, retrospection. A fine introduction to this area of research.

H. Lin, "The Hidden Curriculum of the Introductory Physics Classroom", unpublished (sigh!) working paper, 1979

> This paper documents some effects on student learning induced by the mismatch of teacher and student views of physics courses. Many of the quotes in this thesis were taken from the study documented in this paper.

J. Ogborn, et al, Higher Education Learning Project: Physics, Heinemann Educational Books, London, 1977

> This is a set of books covering small group teaching in science, lab work, student reactions to university science, and alternatives to the lecture format. These books offer a great deal of sound insight and advice regarding the processes

involved in teaching and learning physics. Their methodology is also to study what students actually say and do in realistic settings.

M. Parlett, An Introduction to Illuminative Evaluation, Pacific Soundings Press, Berkeley, 1977

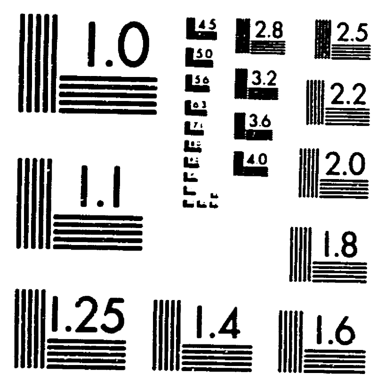
MP describes an alternative to the statistically oriented paradigm of program evaluation. In particular, he offers an approach rooted in the techniques of the anthropologist which studies individual programs in their natural settings. Though this book is slanted towards the program evaluator, much of its methodology is directly applicable to people investigating cognitive processes and learning. This book's publisher is now defunct; the book can be obtained through Higher Education Study Group, Education Development Center, Newton, Massachusetts.

M. St. John, "Thinking Like a Physicist: New Goals and Methods for the Introductory Lab", Ph.D thesis, Group in Science and Mathematics Education, University of California, Berkeley, California, 94720

MSJ presents new lab materials which explicitly teach lab skills such as making order of magnitude calculations, developing intuitive notions of statistics, and making reliable estimates of of physical quantities. In addition, they teach students to give clear and coherent descriptions of their experiments.

F. Reif, et al, "Teaching General Learning and Problem Solving Skills", American Journal of Physics, March 1976 Volume 44(3), page 212

To my knowledge, this is the first (and I think only) article in the physics literature concerning the teaching of specific cognitive skills. FR describes skills necessary for understanding equations and for effective problem-solving. Emphasis is on organization of thoughts of students. Good paper, although their unpublished working papers on the same subject are much more useful.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