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(1971)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
at the
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February 1986

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Submitted to the Department of Earth and Planetary Sciences on January 1l, 1980 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Earth and Planetary Sciences

## ABSTRACT

The differential equations of motion for the Moon's rotation are derived in terms of Euler angles referenced to an inertial coordinate system. Modifications are made to the rigid-body equations of motion to allow for lunar elasticity, as well as for two possible forms of lunar anelasticity. The two models of anelasticity correspond to different assumptions regarding the frequency dependence of the underlying dissipative mechanisms. The possibilities considered are a constant $Q$, and $a \operatorname{Q}$ which is inversely proportional to the strain oscillation frequency. Also developed are variational equations of the rotational motion with respect to six Euler angle initial conditions, the lunar moment of inertia ratios $\beta$ and $\gamma$, the coefficients of the third and higher-degree gravity harmonics, the potential Love number $k$, and two different parameters describing dissipation. The equations of motion and the variational equations were integrated numerically within the framework of the M.I.T. Planetary Ephemeris Program.

The numerical rotation model has been fit by weighted-least-squares to $\overline{7} 1 / 2$ years of lunar laser range data. The rms of the postfit range residuals was 27 cm . Some of the parameter estimates obtained are presented and compared to other determinations that have been published. Especially interesting are the results for the lunar dissipation parameters. Both models of anelasticity yielded similar estimates of $Q$ : $27 \pm 4$ for the constant $Q$ model, and $22 \pm 4$ for the $Q$ at a frequency of one cycle per month, with the model in which $Q$ was inversely proportional to frequency. The apparent discrepancy between these estimates of $Q$ and the much higher estimates obtained by lunar seismologists may be explained by the six-order-of-magnitude difference between the relevant strain frequencies. However, several possible deficiencies in the present model of the Moon's rotation are noted; these deficiencies may have significantly biased the estimates of $Q$, and warrant further study.

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Several people have played integral roles in both my development as a scientist and in the research presented here.

My thesis adviser, Chuck Counselman, has contributed greatly to my technical development. Numerous discussions with him in the course of this research have given me new insights into the problems with which I grappled. His close supervision of my writing has been most helpful in learning how to express myself clearly.

Of the many teachers $I$ have had in 22 years of schooling, Irwin Shapiro has taught me most. He inas done so not only by his singular ability to reduce a problem to its fundamentals, but even more importantly, by being a model of scientific integrity and precision. I can only hope that my years of association with him will result in my acquiring some of his traits $I$ so admire.

Bob King laid most of the groundwork for the lunar rotation model herein described. He is also responsible for the flexibility and sophistication of much of the software we use to process lunar range data. His previous analyses of the data facilitated my application of the new rotation models to observations. We have shared innumerable discussions concerning the modelling and interpretation of the data, out of which we have both gained understanding.

I thank Don Eckhardt for helpful conversations about analytic lunar rotation theories, and also for generating some specific rotation models that $I$ have employed in my research.

Finally, $I$ would like to give to my wife, Nancy, the public tribute she aeserves. Not only has she been an understanding companion while this research has pre-empted my time and energy, but she has typed this thesis expertly, and taken an active part in its production.

This research was supported in part through contract F19628-79-C-0064 with the Air Force Geophysics Laboratory, and in part by grant NSG 7010 from the Lunar Programs Office of the National Aeronautics and Space Administration.

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I. Introduction

As our nearest celestial neighbor and the second-most prominent feature of the heavens, the Moon has been scrutinized by Man since prehistoric times. However, precise observations of the Moon's rotation have been possible only since the advent of the telescope in the 17 th Century. Based on such telescopic observations, in 1693 the French astronomer G. D. Cassini proposed two simple laws describing the rotation of the Moon:

1) The Moon rotates eastward about its polar axis with constant angular velocity in a period of rotation equal to the time of the revolution about the Earth, i.e., one sidereal month.
2) The inclination of the Moon's equator to the plane of the ecliptic is constant and equal to $1.5^{\circ}$. The poles of the Moon's axis of rotation, of the ecliptic, and of the lunar orbit, lie in one great circle in the order given; i.e., the planes of the lunar equator, lunar orbit, and ecliptic meet in one line, the so-called line of nodes, with the descending node of the equator coinciding with the ascending node of the orbit.

An interesting dynamical implication of Cassini's second law is that as the node of the lunar orbit on the ecliptic
undergoes its 18.6 year regression cycle, the lunar pole regresses about the ecliptic pole, keeping the geometry constant. It is well known (see, for example, Peale, 1969) that the Cassini state corresponds to the dynamical solution that has minimum dissipation of energy by internal friction.

Many great mathematicians have studied the rotation of the Moon: Newton, D'Alembert, Euler, and particularly Lagrange made successive contributions to the theory. But it was Newton (1686) who first realized that there might exist small departures from uniformity in the lunar rotation, which have since become known as "physical librations". These physical librations, whose amplitudes are at most a few hundred seconds of selenocentric arc, would be even smaller were it not for a much larger, apparent nonuniformity of the Moon's rotation called "optical librations." The optical librations are due to observing geometry: Since the lunar orbital angular velocity is non-uniform (a result of the $5 \%$ orbital eccentricity), the Moon appears to librate in longitude by about $\pm 6^{\circ}$, as seen from the Earth. Similarly, the $6.5^{\circ}$ inclination of the lunar orbit to the lunar equator causes a monthly variation of similar size in the apparent orientation of the Moon, but in a (selenocentric) latitude sense. When viewed from a frame fixed in the lunar body, the optical librations cause the Earth to oscillate about some mean position, with
amplitudes of about $6^{\circ}$ in both longitucie and latituae, and predominantly monthly periods. Since the lunar gravity field is not spherically symmetric, this oscillation causes time-varying torques on the Moon. The resulting forced motions of the lunar globe are the major constituents of the physical librations. There are also physical librations due to the torque exerted by the Sun, but these are smaller by more than two orders of magnitude.

The differential equations of motion of the Moon (treated as a rigid body) about its center of mass were first written by Euler, but due to the complexity of the forcing terms, they, have been solved only approximately. Formulations of these equations in which the orbital motions of the Earth and Moon are approximated by functions composed of terms secular and periodic in time (such as the Brown lunar theory) are amenable to algebraic solution; the theories of motion so derived are called analytic theories. Several twentieth-century investigators have been involved in the development of analytic theories, culminating in the computer-assisted developments of Eckhardt (1970) and Migus (1976). Such modern analytic theories are invaluable for their concise description of different modes of physical libration, but have the dual disadvantages of reliance on relatively inaccurate analytic orbit theories, and an astounding complexity of algebraic manipulation that involves approximation at many steps.

Approximate solution of the equations of motion by numerical integration has become a feasible alternative to analytic theories only since the advent of digital computers. The solution by numerical integration offers many advantages, perhaps the greatest of which is simplicity. Numerous computer programs that integrate differential equations have already been written; the choice of which to use is mostly a matter of efficiency and personal taste. Since the theorist need only write the equations of motion and present them (along with ancillary control information) to the integration routine, more effort can be spent modelling small effects of interest. Also, for many of the same reasons given above, modern high-accuracy lunar orbit theories are also numerically integrated, so that the numericallyintegrated rotation theory can be set in a consistent framework with the orbit to which it is intimately tied.

This dissertation describes a model of the lunar rotation based upon the numerical integration of Euler's equations, written in terms of Euler angles referenced to an inertial coordinate system. The model is implemented as part of the M.I.T. Planetary Ephemeris Program (PEP), and is used in the reduction of lunar laser range data. Unknown, or poorly determined, physical constants in our model are estimated by weighteá-least-squares fitting to the range data.

The high precision of the current laser data, with typical standard erors of 10 cm in range to the Moon, imposes exacting demands upon the theoretician. It has become necessary to include many small, previously ignorable effects in models of the lunar rotation and orbit. The gravity fields of Earth and Moon must be described to a higher order, and the interaction of their figures considered. Even when these refinements are incorporated into the lunar orbital and rotational motions, the discrepancies between the theoretically calculated ranges and the observed ranges are apt to be several times the uncertainties quoted for the latter.

Yoder and Williams (Yoder, 1979; Ferrari et al., 1980), seeking to lessen the discrepancy, included the effects of lunar solid-body elasticity and dissipation in their model of the Moon's rotation. The reasoning is simple: since the Moon has only finite rigidity, it deforms in response to the tidal forces exerted by the Earth, and to the centripetal forces, which vary as the moon undergoes physical librations. The response is not perfectly elastic, so there is a loss of elastic energy into heat due to internal friction. Thus, the response is retarded relative to that which a perfectly elastic "Moon" would have. The exact manner in which this delay is evidenced is dependent upon the dissipative mechanisms involved. These time-varying perturbations to the
lunar inertia tensor affect the rotation, and thus, the range to locations fixed on the lunar surface. Yoder obtained the rather startiing (at least for lunar seismologists) result that, for the lunar interior, a dissipative quality factor of $Q=14 \pm 10$ allows a significant improvement in the fit to data. Studies of seismic wave propagation within the Moon indicate $Q$ to be much greater, in the range of $200-10,000$, but it is argued in Chapter $V$ that a direct comparison of seismic and rotationally-inferred results may not be meaningful.

One of the goals of the research here reported is to ascertain the validity of such a small value for the lunar quality factor. We do so by attempting to verify Yoder's results using two different global models of the dissipative mechanism. The two models, one of which Yoder adopted, are used in fits to lunar laser range data. Our results and their dependence upon the choice of dissipation model are presented and discussed.
II. Model of the Moon's Rotation
A. Development of the Equations of Motion Treating the Moon as a Rigid Body

As a first approximation to the equations governing the real Moon's rotation, we make the simplifying (and very accurate) assumption that the Moon is a perfectly rigid body. We will find it convenient to adopt a right-handed body-fixed coordinate system with its origin at the lunar center of mass and its axes aligned along the Moon's principal axes of inertia, $x_{i}(i=1,2,3)$, with $x_{1}$ being the axis of least moment and $x_{3}$ the axis of greatest moment. The positive $x_{1}$ axis points toward the mean-Earth direction; $x_{3}$ is positive toward the lunar North pole. In this selenocentric principal axis system the rigid Moon's inertia tensor, $\overline{\bar{I}}_{o}$, is diagonal and constant; we will follow convention and refer to the diagonal elements of $\overline{\bar{I}}_{0}$ as $A, B$, and $C(A<B<C)$. As our fundamental reference frame we will use an inertial coordinate system, $\xi_{i}$, referred to the Earth's mean equinox and equator of 1950.0. The $\xi_{3}$ axis is perpendicular to the mean equator of 1950.0 and is positive northward, the $\xi_{1}$ axis lies along the mean equinox and is positive in the direction of the constellation Aries, and the $\xi_{2}$ axis completes the right-handed system. The origin of the inertial frame is the solar
system barycenter. The orientation of the seienocentric principal axis system relative to the 1950.0 inertial system is defined by the Euler angles $\psi, \theta$, and $\phi$ as shown in Figure 1.

The instantaneous state of rotation of a rigid body may be defined completely by six quantities; for this purpose we choose the above-defined Euler angles and their rates of change. Since the dynamical equations are written more easily in the body-fixed system, we will need the kinematic relations between motions in the inertial and body-fixed systems. The components of the angular velocity vector in the body-fixed system are easily expressed in terms of the reference Euler angles (see Goldstein, 1950 for example):

$$
\begin{align*}
& \omega_{1}=\dot{\theta} \cos \phi+\dot{\psi} \sin \phi \sin \theta \\
& \omega_{2}=-\dot{\theta} \sin \phi+\dot{\psi} \cos \phi \sin \theta \\
& \omega_{3}=\dot{\psi} \cos \theta+\dot{\phi} \tag{II-1}
\end{align*}
$$

If we differentiate equations (II-1) with respect to time we obtain a system of equations linear in $\ddot{\psi}, \ddot{\theta}$, and $\ddot{\phi}$, for which we (algebraically) solve, obtaining:

$$
\begin{align*}
& \ddot{\psi}=\csc \theta\left(\dot{\omega}_{1} \sin \phi+\dot{\omega}_{2} \cos \phi+\dot{\theta} \dot{\phi}\right)-\dot{\psi} \dot{\theta} \cot \theta \equiv F_{1} \\
& \ddot{\theta}=\dot{\omega}_{1} \cos \phi-\dot{\omega}_{2} \sin \phi-\dot{\psi} \dot{\phi} \sin \theta \equiv F_{2} \\
& \ddot{\phi}=\dot{\omega}_{3}-F_{1} \cos \theta+\dot{\psi} \dot{\theta} \sin \theta \equiv F_{3} \tag{II-2}
\end{align*}
$$

The above equations constitute our Euler angle equations of motion, with the dynamics of the problem entering through the body-fixed angular acceleration components, $\dot{\omega}_{i}$, whicn we will now derive.

In an inertial coordinate system the relation governing the rate of change of angular momentum of a body is simply

$$
\begin{equation*}
\frac{d}{d t}(\overline{\bar{I}} \vec{\omega})=\vec{N}, \tag{II-3}
\end{equation*}
$$

where the time-derivative is taken in the inertial reference frame, and $\vec{N}$ represents the total of all externally applied torques. The operator $d / d t$ in an inertial frame is equivalent to $(d / d t+\vec{\omega} \times)$ in a frame rotating at angular velocity $\vec{\omega}$ relative to an inertial frame, so we have

$$
\begin{equation*}
\frac{d}{d t}(\overline{\bar{I}} \vec{\omega})+\vec{\omega} \times \overline{\bar{I}} \vec{\omega}=\vec{N} \tag{II-4}
\end{equation*}
$$

when the time-derivative is evaluated in the selenocentric principal axis system. Solving equation (II-4) for $\dot{\vec{\omega}}$ we find that the resultant angular acceleration takes a simple form due to the rigid-body assumption:

$$
\begin{equation*}
\stackrel{\vec{\omega}}{ }=\bar{I}_{0}^{-1}\left(\vec{N}-\vec{\omega} \times \overline{\bar{I}}_{0} \vec{\omega}\right) \tag{II-5}
\end{equation*}
$$

By utilizing the fact that

$$
\overline{\bar{I}}_{0}^{-1}=\left[\begin{array}{ccc}
A^{-1} & 0 & 0  \tag{II-6}\\
0 & B^{-1} & 0 \\
0 & 0 & C^{-1}
\end{array}\right]
$$

and by introducing the lunar moment of inertia ratios $\alpha \equiv \frac{C-B}{A}$, $\beta \equiv \frac{C-A}{B}$, and $\gamma \equiv \frac{B-A}{C}$, we can write equation (II-5) in component form as the familiar set of Euler's equations:

$$
\begin{align*}
& \dot{\omega}_{1}=-\alpha \omega_{2} \omega_{3}+N_{1} / A \\
& \dot{\omega}_{2}=\beta \omega_{1} \omega_{3}+N_{2} / B \\
& \dot{\omega}_{3}=-\gamma \omega_{1} \omega_{2}+N_{3} / C \tag{II-7}
\end{align*}
$$

The total torque, whose components in the selenocentric principal axis system are represented in equation (II-7) by $N_{i}$, can be approximated to the desired level of accuracy by considering only the gravitational effects of an oblate Earth and a point-mass Sun (see Appendix Efl). Specifically, we will represent the total torque as the sum of the torques caused by the Earth $\left(\vec{N}_{\oplus}\right)$ and the $\operatorname{sun}\left(\mathbb{N}_{\odot}\right)$ treated as point masses, and the Earth-figure torque $\left(\vec{N}_{\oplus \mathrm{F}}\right)$ :

$$
\begin{equation*}
\vec{N}=\vec{N}_{\oplus}+\vec{N}_{0}+\vec{N}_{\oplus \mathrm{F}} \tag{II-8}
\end{equation*}
$$

The Earth-figure torque, originally referred to by Breedlove (1977) as the "figure-figure interaction", is derived in Appendix B. This torque results from the interaction of the $2^{\text {nd }}$ zonai harmonic of the gravity field of the Earth (i.e. the $J_{2}$ term in the potential) with the complete $2^{\text {nd }}$ degree gravity field of the Moon. Since the amplitude of the lunar librations induced by the Earth-figure interaction is less than 0.1" (Yoder, 1979), the longitudinal inhomogeneities of the Earth's gravity field, which are more than two orders of magnitude smaller than the oblateness, are negligible. For the purpose of calculating $\vec{N}_{\oplus F}$ the orientation of the Earth is adequately represented by the customary expressions for precession and nutation, with polar motion neglected. We now outline the derivation of the expressions for the point-mass torques; the detailed equations omitted here are in Appendix A. The lunar gravitational potential is expressed as an expansion in spherical harmonics. The force exerted by the Moon on body $b(b=\oplus$ or 0$)$ is then given by

$$
\begin{equation*}
\vec{F}_{b}=M_{b}\left(-\nabla U_{b}\right), \tag{II-9}
\end{equation*}
$$

where $\nabla U_{b}$ denotes the gradient of the lunar potential evaluated at the position of body b. By Newton's Third Law of Motion, there is a reaction force of $-\vec{F}_{b}$, which gives rise to a torque on the Moon:

$$
\begin{align*}
\vec{N}_{b} & =\vec{r}_{b} \times\left(-\vec{F}_{b}\right) \\
& =M_{b}\left(\vec{r}_{b} \times \nabla U_{b}\right) \tag{II-10}
\end{align*}
$$

where $\vec{r}_{b}$ is the vector from the center of mass of the Moon to that of body $b$. The potential gradient, $\nabla U_{b}$, is evaluated as shown in equation ( $A-3$ ), except for the infinite summations. PEP is programned to handle a maximum degree of $n=20$ in the case of zonal harmonics ( $J_{n}$ terms), and $n=10$ for the tesseral harmonics $\left(C_{n m}\right.$ and $s_{n m}$ terms). Due to limited data sensitivity, we used a maximum of $n=3$ for both zonal and tesseral terms in the research here reported. However, it is possible (see Appendix E.2) that some fourth degree harmonic terms should be included in the potential.
B. Alterations to the Equations of Motion to Incorporate Lunar Solid-Body Elasticity and Dissipation

In section II-A we made the simplifying approximation that the Moon is a perfectly rigid body. Of course this is not true -- the Moon is made of material with finite strength and thus yields to the stresses of tidal forces and rotational (centripetal acceleration) forces. Fortunately for our analysis, the deformation so induced is small compared to the permanent asymmetry of the moon's figure (about two orders of magnitude smaller even for the limiting case of a
fluid Moon), so the formulation can be somewhat simplified by retaining only leading-order effects. In particular we will assume that the effects of elastic deformation can be modelled by adding a small perturbation to the rigid Moon's inertia tensor

$$
\begin{equation*}
\overline{\bar{I}}(t)=\overline{\bar{I}}_{0}+\overline{\overline{\delta I}}(t), \tag{II-11}
\end{equation*}
$$

where we have explicitly shown that $\overline{\bar{I}}$ is no longer constant, but rather a function of time. We also note that in general $\overline{\bar{I}}$ is no longer diagonal in our (unperturbed) "principal axis" system, though the products of inertia should have a nearly zero time average.

We again use equations (II-3) and (II-4) to find the angular acceleration, but now $\overline{\bar{I}}$ is the inertia tensor of an elastic Moon. Since $\vec{\omega}$ can vary with position in an elastic Moon, we will define $\vec{\omega}$ to be the angular velocity of our "body-fixed" reference frame, which coincidus with the unperturbed principal axis system. In Appendix E. 3 we show that the angular momentum of a homogeneous elastic (or slightly dissipative) sphere is still $\overline{\bar{I}} \vec{\omega}$. The extent to which the Moon deviates from these ideal conditions and the errors introduced by this assumption are beyond the scope of this paper.

When we now solve equation (II-4) for $\dot{\vec{\omega}}$ we get a somewhat more complicated expression:

$$
\begin{equation*}
\dot{\vec{\omega}}=\overline{\bar{I}}^{-1}(\overrightarrow{\mathbf{N}}-\vec{\omega} \times \overline{\bar{I}} \vec{\omega}-\stackrel{\dot{\bar{I}} \vec{\omega}}{\vec{\omega}}) \tag{II-12}
\end{equation*}
$$

For reasonable values of the lunar rigidity the elements of $\overline{\overline{\delta I}}$ are no larger than $10^{-6}$ of the diagonal elements in $\overline{\bar{I}}_{0}$, so it is reasonable to express $\overline{\bar{I}}^{-1}$ in terms of $\overline{\bar{I}}_{0}^{-1}$ plus a small perturbation:

$$
\begin{equation*}
\overline{\bar{I}}^{-1}=\overline{\bar{I}}_{0}^{-1}+\overline{\bar{\Gamma}} \tag{II-13}
\end{equation*}
$$

To first order in the ratios $\delta I_{i j} / C$ we have

$$
\dot{\Gamma}_{i j}=-\left[\begin{array}{lll}
\delta I_{11} / A^{2} & \delta I_{12} / \mathrm{AB} & \delta I_{13} / \mathrm{AC}  \tag{II-14}\\
\delta I_{12} / \mathrm{AB} & \delta I_{22} / \mathrm{B}^{2} & \delta I_{23} / \mathrm{BC} \\
\delta I_{13} / \mathrm{AC} & \delta I_{23} / \mathrm{BC} & \delta I_{33} / C^{2}
\end{array}\right]
$$

or simply

$$
\begin{equation*}
\Gamma_{i j}=\frac{-\delta I_{i j}}{\left(I_{o}\right)_{i i}{ }^{\left(I_{0}\right)_{j j}}} \tag{II-15}
\end{equation*}
$$

By dropping terms that are second order in the elastic perturbation we can rewrite (II-12) such that $\dot{\vec{\omega}}$ is linear in the perturbation $\overline{\delta \mathrm{II}}$ :

$$
\begin{align*}
\stackrel{\rightharpoonup}{\omega}= & \overline{\bar{I}}_{0}^{-1}\left(\vec{N}-\vec{\omega} \times \overline{\bar{I}}_{0} \vec{\omega}\right)+\overline{\bar{\Gamma}}\left(\vec{N}-\vec{\omega} \times \overline{\bar{I}}_{0} \vec{\omega}\right) \\
& +\overline{\bar{I}}_{0}^{-1}(\delta \vec{N}-\vec{\omega} \times \overline{\overline{\delta I}} \vec{\omega}-\overline{\overline{\delta I}} \vec{\omega}) \tag{II-16}
\end{align*}
$$

Note that the first term on the right-hand side is the righthand side of the rigid body formulation, given in equation (II-5). We have also introduced $\overrightarrow{\delta N}$, a perturbation torque which arises from the elastic changes in the Moon's figure. Again to first order in the elastic perturbation we can write $\overrightarrow{\delta N}$ as the interaction of the change in the lunar inertia tensor with the Earth, treated as a mass point; so by Eckhardt (1967) we have:

$$
\begin{equation*}
\overrightarrow{\delta N}=\frac{\mathbf{3 G M}_{\oplus}}{\mathbf{r}_{\oplus}^{5}} \overrightarrow{\mathbf{r}}_{\oplus} \times \overline{\overline{\delta I}} \overrightarrow{\mathbf{r}}_{\oplus} \tag{II-17}
\end{equation*}
$$

The rationale for ignoring the torque caused by the Sun acting on the elastic perturbation to the lunar figure is given in Appendix E. 4.

If the Moon is perfectly elastic then $\overline{\delta \bar{I}}$ takes the following form (Peale, 1973):

$$
\begin{equation*}
\delta I_{i j}=k R_{i}^{5}\left[\frac{\left(\omega_{i} \omega_{j}-\frac{\omega^{2}}{3} \delta_{i j}\right)}{3 G}-\frac{M_{\oplus}\left(u_{i} u_{j}-\frac{1}{3} \delta_{i j}\right.}{r^{3}}\right] \tag{II-18}
\end{equation*}
$$

with $G$ the Newtonian gravitational constant, $u_{i}$ the direction cosines of the Earth, and $k$ the (dimensionless) second order lunar potential Love number. The Kronecker delta is defined by $\delta_{i j}=1$ for $i=j$ and 0 otherwise. The elastic distortion of the Moon's figure represented by equation (II-18) has been derived by considering only the equilibrium distortion of a homogeneous elastic sphere. Terms of higher order than the second in $\left(R_{\mathbb{G}} / r_{\theta}\right)$ in the Earth's tide-raising potential have been neglected. We have also left out of equation (II-18) the tidal bulges raised by other bodies. The largest tidal perturbation omitted is due to the Sun, which is smaller than the tide raised by the Earth by a factor of

$$
\frac{M_{\odot}}{M_{\oplus}}\left(\frac{r_{\oplus}}{r_{\odot}}\right)^{3} \simeq 1 / 180
$$

It should be noted that the time-varying part of the rotational deformation is about three orders of magnitude smaller than the time-varying part of the tidal bulge raised by the Earth; in retrospect, it was perhaps inconsistent to have included the rotational effects, in view of the other approximations inherent in equation (II-18).

As a matter of convenience we have adopted a slightly altered version of equation (II-18) in our elastic model, wherein we replace $\left(\omega^{2} / 3\right) \delta_{i j}$ by $\left(\omega^{2} / 3-a_{i}\right) \delta_{i j}$ in the rotational part, with

$$
a_{i}=\left\{\begin{array}{lr}
+(1 / 3) n^{2} & \text { for } i=1,2  \tag{II-19}\\
-(2 / 3) n^{2} & i=3
\end{array}\right.
$$

where $n$ is the mean motion of the Moon. The effect of this change is to make the diagonal elements of the rotationallyinduced elastic perturbation nearly time-average to zero. Otherwise our estimates of the moment of inertia ratios, especially $\beta$ (since the main effect of rotation is an equatorial bulge), would change from those obtained with the rigid-Moon model. We have not altered the tidal part of (II-18), since to do so would induce ron-zero perturbation torques ( $\overline{\delta N}$ ), even in the case of perfect elasticity. Such torques would exist because the tidal bulge would no longer "follow" the Earth exactly. Of course, the different value of $\gamma$ that one would find and use would result in rigid body torques offsetting the $\overrightarrow{\delta N}$, but for aesthetic reasons we left in the time-average tidal deformation.

$$
\begin{aligned}
& \text { We obtain } \dot{\overline{\delta I}} \text { by differentiating (II-18): } \\
& \begin{aligned}
\frac{d}{d t}\left(\delta I_{i j}\right)= & k R_{\mathbb{Q}}^{5}\left(\frac{\dot{\omega}_{i} \omega_{j}+\omega_{i} \dot{\omega}_{j}-\frac{2}{3} \delta_{i j} \vec{\omega} \cdot \vec{\omega}}{3 G}\right. \\
& \left.-M_{\oplus}\left[\frac{\dot{r}_{i} r_{j}+r_{i} \dot{r}_{j}+\delta_{i j} \vec{r} \cdot \overrightarrow{\vec{r}}}{r_{\oplus}^{5}}-\frac{5 r_{i} r_{j} \vec{r} \cdot \dot{\vec{r}}}{7}\right]\right)(I I-20)
\end{aligned}
\end{aligned}
$$

Up to now we have considered the ideal case of a purely elastic Moon; now we consider the effect of a slight anelasticity. There must be some energy dissipated as the lunar material undergoes its (predominantly monthly) cycle of strain. The physical mechanisms underlying this anelasticity of the Moon are unknown: Indeed, one result of the research here reported may be to shed some light on the nature of the lunar interior. With this possibility in mind, we have incorporated in our lunar rotation model two different models of lunar anelasticity: a constant-time-lag model and a constant-Q (or constant-phase-lag) model.

The constant-time-lag model is based upon the assumption that the mechanism of dissipation is "viscous". As a result of slight viscous dissipation, the normal elastic response of a body to tidal and rotational stresses is simply delayed in time (Munk \& MacDonald, 1960). This leads to a simple computational form: we merely replace all occurrences of $\overline{\delta I}(t)$ in the equations of motion (II-16) by $\overline{\overline{\delta I}}(t-T)$, with

$$
\begin{align*}
\overline{\overline{\delta I}}(t-T) & \simeq \overline{\overline{\delta I}}(t)-T \overline{\overline{\delta I}}(t) \\
& =\overline{\overline{\delta I}}(t)-D(t) / k)  \tag{II-2l}\\
\text { and } \dot{\overline{\delta \delta I}}(t-T) & \simeq \dot{\overline{\delta I}}(t)-D(t) / \overline{\overline{\delta I}}(t) / k) .
\end{align*}
$$

We have parameterized our dissipative model by $D \equiv k T$, since this allows easier separation of the effects of elasticity and dissipation. We retard $\bar{\Gamma}$ and $\overrightarrow{\delta N}$ by using $\overline{\overline{\delta I}(t-T) ~ f o r ~}$ their computation. We find $\overline{\overline{\delta I}}$ by differentiating (II-20).

$$
\begin{aligned}
\frac{d^{2}\left(\delta I_{i j}\right)}{d t^{2}}= & k R_{0}^{5}\left\{\frac{\ddot{\omega}_{i} \omega_{j}+2 \dot{\omega}_{i} \dot{\omega}_{j}+\omega_{i} \ddot{\omega}_{j}-\frac{2}{3} \delta_{i j}(\vec{\omega} \cdot \ddot{\vec{\omega}}+\dot{\vec{\omega}} \cdot \dot{\vec{\omega}})}{3 G}\right. \\
& -M_{\oplus}[
\end{aligned}
$$

See Appendix E. 5 for a discussion of the approximations used in the evaluation of $\ddot{\vec{\omega}}$ and $\ddot{\vec{r}}$.

A useful parameter in the description of slightly dissipative mechanical systems is $Q$, a dimensionless quantity defined as the ratio of the peak stored elastic energy to the energy dissipated per radian of the strain oscillation ( $2 \pi$ radians $=1$ cycle). The reciprocal of $Q$ is sometimes referred to as the specific dissipation. The assumption of linear viscosity as the dissipative mechanism corresponds to a specific dissipation, $1 / Q$, that is proportional to frequency.

There is no evidence to suggest that solid-body dissipation in the Moon actually occurs through a viscous mechanism. If anything, the combination of seismic and rotational evidence suggests that $Q$ increases with frequency, rather than decreases as the "viscous", or constant-time-lag model predicts. However, the range of frequencies of the principal strain components is so narrow that this simple model still might be expected to produce a good representation of the actual effect of dissipation on the rotation. On the
other hand, it is of interest to determine, if possible, the form of the frequency-dependence of the dissipation in the Moon. Thus, we have also implemented a constant-Q model, which represents a mechanism whose specific dissipation is independent of frequency. The motivation for such an assumption is given by the large body of laboratory and seismic studies which suggest that $Q$, at least for rocks of the Earth's mantle and crust, remains relatively constant for oscillation periods from microseconds to minutes. It is difficult to justify the extrapolation of such terrestrial experience to the Moon, where the strain cycle has appreciable components with periods from 9 days to 6 years; one can only treat the constant-Q model of lunar dissipation as a hypothesis, whose predictions are to be compared (along with those of other models) with the observations.

A dissipation mechanism characterized by a constant $Q$ is dispersive. If the elastic inertia tensor is harmonically decomposed, i.e. expressed in the frequency domain, then the phase of each spectral component lags by a constant amount. (In contrast, the constant-time-lag model yields phase lags which are proportional to the frequency. For an angular frequency, $w$, the phase lag corresponding to a time lag, $T$, is just $\omega T$.) In the constant-Q model the phase lag, $\varepsilon$, is related to $Q$ by (Melchior, 1978):

$$
\begin{equation*}
\sin E=1 / Q \tag{II-22}
\end{equation*}
$$

Our numerical model of the lunar rotation does not lend itself easily to a harmonic decomposition of $\overline{\overline{\delta I}}$ as given in (II-18). Therefore we obtained from D. H. Eckhardt (private communication, 1979) -uch a decomposition based upon an analytic rotation theory. It is tabulated in terms of sines and cosines of linear combinations of the classical Delaunay arguments, $\ell, \ell^{\prime}, F$, and $D$. In particular if we define

$$
\begin{equation*}
\mu(t) \equiv p \ell+q l^{\prime}+r F+s D \tag{II-23}
\end{equation*}
$$

where $p, q, r$, and $s$ are integers, then $\overline{\bar{\delta} \bar{I}}$ can be written as

$$
\begin{equation*}
\overline{\overline{\delta I}}=k \sum_{p q r s}\left\{\overline{\bar{s}}_{p q r s} \sin \mu+\overline{\bar{C}}_{\text {pqrs }} \cos \mu\right\} \tag{II-24}
\end{equation*}
$$

In the above expression, $p, q, r$, and $s$ take on all values for which a significant $\overline{\bar{C}}_{\text {pars }}$ or $\overline{\bar{S}}_{\text {pg rs }}$ exists; in our analysis we included all terms greater than $1 \%$ of the largest time-varying term. The various $\overline{\bar{C}}$ and $\overline{\bar{S}}$ matrices can be found in Table 1. (We shall henceforth dispense with the pars subscript notation on $\overline{\bar{C}}$ and $\overline{\bar{S}}$.) If we retard the response by a phase $\varepsilon \ll 1$, then

$$
\begin{aligned}
\overline{\delta \bar{I}}(\varepsilon) & =k \sum_{\mathrm{pqrs}}\{\overline{\bar{S}} \sin (\mu-\varepsilon)+\overline{\overline{\mathrm{C}}} \cos (\mu-\varepsilon)\} \\
& \simeq k \sum_{\text {pqrs }}\{\overline{\bar{S}}(\sin \mu-\sin \varepsilon \cos \mu)+\overline{\bar{C}}(\cos \mu+\sin \varepsilon \sin \mu)\}
\end{aligned}
$$

$$
\begin{equation*}
=\overline{\overline{\delta I}}-(k / Q) \sum_{\text {pqrs }}\{\overline{\bar{S}} \cos \mu-\overline{\bar{C}} \sin \mu\} \tag{II-25}
\end{equation*}
$$

This expression has been implemented in our equations of motion, with $k / Q$ now becoming the natural parameter with which to model dissipation in the case of constant $Q$. Once
 differentiating (II-25) and (II-24):
$\dot{\overline{\delta I}}=k \sum_{\text {pqrs }}(\overline{\bar{S}} \cos \mu-\overline{\bar{C}} \sin \mu) \dot{\mu}+(k / Q) \sum_{\text {pqrs }}(\overline{\bar{S}} \sin \mu+C \cos \mu) \dot{\mu}$
C. Solution of the Equations by Numerical Integration

The rotational equations of motion developed in the previous two sections are non-linear and cross-coupled, and are not easily solved to determine the Euler angles as functions of time. In the implementation of our rotation model in PEP, we solve them numerically via an Adams-Moulton predictorcorrector numerical integration (Smith, 1968). We have found that adequate accuracy is maintained by taking eight steps/day using eleventh differences in the integrating polynomial. Since the lunar orbital and rotatio:al equations are strongly coupled, we integrate them simultaneously (the lunar orbital equations will be described in Chapter
III). The position of the Sun is found by interpolation of ephemerides of the orbital motion of the Earth-Moon barycenter, also the result of numerical integration within PEP. Since the orbit of the Earth-Moon barycenter is only weakly affected by the lunar orbit, we are able to simplify matters somewhat by integrating the motion of the Earth-Moon barycenter about the Sun separately, prior to the Iunar orbit and rotation integration (see Ash, 1965a, for a derivation of the equations of motion for the Earth-Moon barycenter).

The Adams-Moulton integrator is configured for a set of simultaneous first-order differential equations, so we rewrite the three second-order rotational equations of motion as a set of six first-order equations. Defining

$$
\begin{array}{lll}
\mathbf{Y}_{1} & \equiv \psi & Y_{4} \equiv \dot{\psi} \\
\mathbf{Y}_{2} \equiv \theta & Y_{5} \equiv \dot{\theta} \\
y_{3} \equiv \phi & y_{6} \equiv \dot{\phi}
\end{array}
$$

we obtain:

$$
\begin{array}{ll}
\dot{Y}_{1}=Y_{4} & \dot{\mathbf{Y}}_{4}=\mathrm{F}_{1} \\
\dot{\mathbf{Y}}_{2}=\mathbf{Y}_{5} & \dot{\mathbf{Y}}_{5}=\mathrm{F}_{2} \\
\dot{\mathbf{Y}}_{3}=\mathbf{Y}_{6} & \dot{\mathbf{Y}}_{6}=\mathrm{F}_{3} \tag{II-27}
\end{array}
$$

where the $F_{i}$ are given by equations (II-2). The six initial conditions of the state vector $\bar{y}$ are the three Euler angles and their rates at some initial epoch. The six rotational equations of motion are integrated in parallel with six orbital equations of motion and six variational equations for each adjustable parameter that affects the orbit or rotation.
D. Development of the Variational Equations

In order to fit our rotation model to data we rely upon the iterative use of a linear least-squares estimator, as elaborated in Chapter IV. Thus, in addition to the model for motion, it is necessary to generate partial derivatives of the state vector with respect to the libration parameters at all times. In order to derive the partial derivatives we must first determine which members to include in our set of adjustable rotation parameters. In principle any unknown parameter affecting the rotation should be modelled as such and have derivatives generated for it; practically, we need only model the small set of parameters whose uncertainties influence the rotation to a measurable extent. We are left with the following (significant) parameters, all of which refer to the Moon:

1) six initial conditions of the rotation state vector;
2) $J_{2}$;
3) $B$ and $\gamma_{8}$
4) third and higher order harmonic coefficients; and
5) the elastic parameters $k$ and either $D$ (constant-time-lag), or $\mathrm{k} / \mathrm{Q}$ (constant-Q).

Note that we have arbitrarily chosen $J_{2}, B$, and $\gamma$ as our independent second-degree coefficients. $C_{22}$ is a combination of all three, whereas $\alpha$ depends only on $\beta$ and $\gamma$. Since we choose the lunar principal axes of inertia to define our selenocentric coordinate system, $C_{21}, S_{21}$, and $S_{22}$ are identically zero. (This relation is not strictly true in the presence of elasticity, but in that case we model the second degree gravity field through the inertia tensor.)

Referring back to the defining equations (II-2) for the driving terms, we note that the $F_{k}$ can be written for convenience as

$$
F_{k} \equiv F_{k}(t, \bar{p}, \bar{y})
$$

since the $F_{k}$ are explicit functions of time, the set of adjustable parameters $\overline{\mathrm{p}}$, and the Eulerian state vector $\overline{\mathrm{Y}}$. We differentiate the equations of motion (II-27) with respect to any specific time-independent parameter $p_{i}$ and interchange the order of differentiation to obtain the variational equations:

$$
\begin{aligned}
& \frac{d}{d t}\left(\frac{\partial y_{k}}{\partial p_{i}}\right)=\frac{\partial y_{k+3}}{\partial p_{i}} \\
& \frac{d}{d t}\left(\frac{\partial y_{k+3}}{\partial p_{i}}\right)=\left.\frac{\partial F_{k}}{\partial p_{i}}\right|_{t, \bar{p} \neq i}
\end{aligned}
$$

with $\bar{p} \neq i$ denoting the parameter set $\bar{p}$ exclusive of $p_{i}$. The initial conditions for these equations are all zero except when $p_{i}$ is one of the stace vector initial conditions, in which case we have

$$
\left.\frac{\partial y_{k}}{\partial p_{i}}\right|_{t=t_{0}}=\delta_{k \ell} \quad \text { for }\left.p_{i} \equiv y_{\ell}\right|_{t=t_{0}}
$$

In general a change in $p_{i}$ will affect the $F_{k}$ both through an explicit dependence of $F_{k}$ upon $p_{i}$, and also implicitly through a change evoked in the state vector $\bar{Y}$ by the integrated effect of the $p_{i}$ perturbation. Namely,

$$
\begin{equation*}
\left.\frac{\partial \mathrm{F}_{k}}{\partial p_{i}}\right|_{t, \bar{p} \neq i}=\left.\frac{\partial F_{k}}{\partial p_{i}}\right|_{t, \bar{p} \neq i, \bar{y}}+\left.\sum_{\ell=1}^{6} \frac{\partial F_{k}}{\partial y_{\ell}}\right|_{t, \bar{p}, \bar{y} \neq \ell} \cdot \frac{\partial y_{\ell}}{\partial p_{i}} \text { for } k=1,2,3 . \tag{II-29}
\end{equation*}
$$

The vector $\partial F_{k} /\left.\partial p_{i}\right|_{t, \bar{Y}, \bar{p} \neq i}$ is found through differentiation of the explicit dependence of the $F_{k}$ upon $p_{i}$; therefore when $p_{i}$ is one of the rotation initial conditions this term vanishes. For the rest of the parameters, the explicit dependence is found from equations (II-2):

$$
\begin{aligned}
& \frac{\partial \mathrm{F}_{1}}{\partial \mathrm{p}_{i}}=\csc \theta\left(\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \mathrm{p}_{i}}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \mathrm{p}_{i}}\right) \\
& \frac{\partial \mathrm{F}_{2}}{\partial \mathrm{p}_{i}}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \mathrm{p}_{i}}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \mathrm{p}_{i}} \\
& \frac{\partial \mathrm{~F}_{3}}{\partial \mathrm{p}_{i}}=\frac{\partial \dot{\omega_{3}}}{\partial \mathrm{p}_{i}}-\cos \theta \frac{\partial \mathrm{F}_{1}}{\partial \mathrm{p}_{i}}
\end{aligned}
$$

To evaluate $\partial \dot{w}_{k} / \partial p_{i}$ we differentiate (II-16) with respect to $p_{i}$ :

$$
\begin{aligned}
& \frac{\partial \stackrel{\rightharpoonup}{\omega}}{\partial p_{i}}=\frac{\partial \overline{\bar{I}}_{0}^{-1}}{\partial p_{i}}\left(\vec{N}-\vec{\omega} \times \overline{\bar{I}}_{0} \vec{\omega}\right)+\overline{\bar{I}}_{0}^{-1}\left(\frac{\partial \vec{N}}{\partial p_{i}}-\vec{\omega} \times \frac{\partial \overline{\bar{I}}_{0}}{\partial p_{i}} \vec{\omega}\right) \\
& +\frac{\partial \overline{\bar{\Gamma}}}{\partial p_{i}}\left(\vec{N}-\vec{\omega} \times \overline{\bar{I}}_{o} \vec{\omega}\right)+\overline{\bar{\Gamma}}\left(\frac{\partial \vec{N}}{\partial p_{i}}-\vec{\omega} \times \frac{\partial \overline{\bar{I}}}{\partial p_{o}} \vec{\omega}\right) \\
& +\frac{\partial \overline{\bar{I}}_{0}^{-1}}{\partial \mathrm{P}_{i}}(\overrightarrow{\delta N}-\vec{\omega} \times \overline{\overline{\delta I}} \vec{\omega}-\dot{\overline{\delta I}} \vec{\omega}) \\
& +\overline{\bar{I}}_{0}^{-1}\left(\frac{\partial \overrightarrow{\delta N}}{\partial p_{i}}-\vec{\omega} \times \frac{\partial \overline{\delta \bar{I}}}{\partial p_{i}} \vec{\omega}-\frac{\partial \overline{\overline{\delta I}}}{\partial p_{i}} \vec{\omega}\right)
\end{aligned}
$$

We have

$$
\frac{\partial \overline{\bar{I}}_{0}^{-1}}{\partial \mathrm{p}_{i}}=\left[\begin{array}{ccc}
\frac{\partial A^{-1}}{\partial \mathrm{p}_{i}} & 0 & 0 \\
0 & \frac{\partial \mathrm{~B}^{-1}}{\partial \mathrm{p}_{i}} & 0 \\
0 & 0 & \frac{\partial \mathrm{C}^{-1}}{\partial \mathrm{p}_{i}}
\end{array}\right]
$$

$$
(I I-32)
$$

with the diagonal elements presented in Appendix C. Similarly, since $A=\left(A^{-1}\right)^{-1}$, etc., we get

$$
\frac{\partial \overline{\bar{I}}_{o}}{\partial p_{i}}=\left[\begin{array}{ccc}
-A^{2} \frac{\partial A^{-1}}{\partial p_{i}} & 0 & 0 \\
0 & -B^{2} \frac{\partial B^{-1}}{\partial p_{i}} & 0 \\
0 & 0 & -C^{2} \frac{\partial C^{-1}}{\partial p_{i}}
\end{array}\right]
$$

From equation (II-17) it follows that

$$
\begin{equation*}
\frac{\partial \overrightarrow{\delta N}}{\partial \mathbf{p}_{i}}=\frac{3 G M_{\oplus}}{\mathbf{r}_{\oplus}^{5}} \overrightarrow{\mathbf{r}}_{f} \times \frac{\partial \overline{\delta I}}{\partial \mathbf{p}_{i}} \overrightarrow{\mathbf{r}}_{\oplus} \tag{II-34}
\end{equation*}
$$

Upon differentiating equation (II-8) we find

$$
\begin{equation*}
\frac{\partial \vec{N}}{\partial p_{i}}=\frac{\partial \vec{N}_{\oplus}}{\partial p_{i}}+\frac{\partial \vec{N}_{o}}{\partial p_{i}}+\frac{\partial \vec{N}_{\oplus F}}{\partial p_{i}} \tag{II-35}
\end{equation*}
$$

and by $(I I-10)$ we get

$$
\begin{equation*}
\frac{\partial \vec{N}_{b}}{\partial \mathrm{p}_{i}}=M_{b}\left[\vec{r}_{b} \times \frac{\partial}{\partial \mathrm{p}_{i}}\left(\nabla \mathrm{U}_{\mathrm{b}}\right)\right] \tag{II-36}
\end{equation*}
$$

The formulae for $\partial / \partial p_{i}\left(\nabla U_{b}\right)$ as well as the elastic perturbation partials, $\partial \overline{\overline{\delta I}} / \partial p_{i}, \partial \overline{\overline{\delta I}} / \partial p_{i}$, and $\partial \overline{\bar{\Gamma}} / \partial p_{i}$, can ail be found in Appendix C. Finally, from equation (B-19) for $N_{\oplus F}$ we derive:

$$
\begin{align*}
\frac{\partial \vec{N}_{\oplus F}}{\partial p_{i}}= & \hat{x}_{1} g_{23}\left[\frac{\partial C}{\partial p_{i}}-\frac{\partial B}{\partial p_{i}}\right]+\hat{x}_{2} g_{13}\left[\frac{\partial A}{\partial p_{i}}-\frac{\partial C}{\partial p_{i}}\right] \\
& +\hat{x}_{3} g_{12}\left[\frac{\partial B}{\partial p_{i}}-\frac{\partial A}{\partial p_{i}}\right] \tag{II-37}
\end{align*}
$$

The only quantity in (II-29) that remains to be formulated is the $3 \times 6$ matrix $\partial F_{k} / \partial y_{\ell}$. First we differentiate the defining equations (II-2) for the $F_{k}$ with respect to each component of the state vector:

$$
\begin{aligned}
& \frac{\partial F_{1}}{\partial \psi}=\csc \theta\left[\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \psi}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \psi}\right] \\
& \frac{\partial F_{2}}{\partial \psi}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\psi}}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \psi} . \\
& \frac{\partial F_{3}}{\partial \psi}=\frac{\partial \dot{\omega}_{3}}{\partial \psi}-\cos \theta \frac{\partial F_{1}}{\partial \psi} \\
& \frac{\partial F_{1}}{\partial \theta}=\csc \theta\left[\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \theta}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \theta}\right]-\cot \theta \csc \theta\left[\dot{\omega}_{1} \sin \phi+\dot{\omega}_{2} \cos \phi\right] \\
& -\dot{\theta} \dot{\phi} \cot \theta \csc \theta+\dot{\psi} \dot{\theta} \csc ^{2} \theta \\
& \frac{\partial F_{2}}{\partial \theta}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \theta}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \theta}-\dot{\psi_{\phi}} \cos \theta \\
& \frac{\partial F_{3}}{\partial \theta}=\frac{\partial \dot{\omega}_{3}}{\partial \theta}-\cos \theta \frac{\partial F_{1}}{\partial \theta}+\sin \theta F_{1}+\dot{\psi} \dot{\theta} \cos \theta \\
& \frac{\partial F_{1}}{\partial \phi}=\csc \theta\left[\dot{\omega}_{1} \cos \phi+\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \phi}-\dot{\omega}_{2} \sin \phi+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \phi}\right] \\
& \frac{\partial F_{2}}{\partial \phi}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \phi}-\sin \phi \dot{\omega}_{1}-\cos \phi \dot{\omega}_{2}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \phi} \\
& \frac{\partial F_{3}}{\partial \phi}=\frac{\partial \dot{\omega}_{3}}{\partial \phi}-\cos \theta \frac{\partial F_{1}}{\partial \phi} \\
& \frac{\partial F_{1}}{\partial \dot{\psi}}=\csc \theta\left[\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\psi}}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\psi}}\right]-\dot{\theta} \cot \theta \\
& \frac{\partial F_{2}}{\partial \dot{\psi}}=\cos \phi \frac{\partial \dot{w}_{1}}{\partial \dot{\psi}}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\psi}}-\dot{\phi} \sin \theta \\
& \frac{\partial F_{3}}{\partial \dot{\psi}}=\frac{\partial \dot{\omega}_{3}}{\partial \dot{\psi}}-\cos \theta \frac{\partial F_{1}}{\partial \dot{\psi}}+\dot{\theta} \sin \theta
\end{aligned}
$$

$$
\begin{align*}
& \frac{\partial F_{1}}{\partial \dot{\theta}}=\csc \theta\left[\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\theta}}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\theta}}\right\}+\dot{\phi} \csc \theta-\dot{\psi} \cot \theta \\
& \frac{\partial F_{2}}{\partial \dot{\theta}}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\theta}}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\theta}} \\
& \frac{\partial F_{3}}{\partial \dot{\theta}}=\frac{\partial \dot{\omega}_{3}}{\partial \dot{\theta}}-\cos \theta \frac{\partial F_{1}}{\partial \dot{\theta}}+\dot{\psi} \sin \theta \\
& \frac{\partial F_{1}}{\partial \dot{\phi}}=\csc \theta\left[\sin \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\phi}}+\cos \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\phi}}\right]+\dot{\theta} \csc \theta \\
& \frac{\partial F_{2}}{\partial \dot{\phi}}=\cos \phi \frac{\partial \dot{\omega}_{1}}{\partial \dot{\phi}}-\sin \phi \frac{\partial \dot{\omega}_{2}}{\partial \dot{\phi}}-\dot{\psi} \dot{\sin } \theta \\
& \frac{\partial F_{3}}{\partial \dot{\phi}}=\frac{\partial \dot{\omega}_{3}}{\partial \dot{\phi}}-\cos \theta \frac{\partial F_{1}}{\partial \dot{\phi}} \tag{II-38}
\end{align*}
$$

To obtain the quantities $\partial \stackrel{\rightharpoonup}{\omega} / \partial y_{\ell}$ we should again aifferentiate equation (II-16), this time with respect to the Euler angles. Instead, we make the simplifying approximation (elaborated in Appendix E.6) that the rigid Moon formulation will here suffice. So we differentiate ti:e rigid-body dynamic equation (II-5) to get:

$$
\begin{equation*}
\frac{\partial \stackrel{\rightharpoonup}{\omega}}{\partial Y_{\ell}}=\overline{\bar{I}}_{0}^{-1}\left[\frac{\partial \vec{N}}{\partial Y_{\ell}}-\frac{\partial \vec{\omega}}{\partial Y_{\ell}} \times \overline{\bar{I}}_{0} \vec{\omega}-\vec{\omega} \times \overline{\bar{I}}_{0} \frac{\partial \vec{\omega}}{\partial Y_{\ell}}\right] \tag{II-39}
\end{equation*}
$$

A few words clarifying the notation might be helpful here. For consistency within equation (II-39) we must evaluate the vector components in the selenocentric principal axis
system. Thus, for example, $\partial \vec{N} / \partial y_{2}$ means the column matrix $\left(\partial N_{1} / \partial \theta, \partial N_{2} / \partial \theta, \partial N_{3} / \partial \theta\right)^{T}$. Now, $\partial \vec{\omega} / \partial Y_{\ell}$ can be derived simply frca the kinematic equation (II-1), and is presented below.

$$
\begin{array}{lll}
\frac{\partial \omega_{1}}{\partial \psi}=0 & \frac{\partial \omega_{2}}{\partial \psi}=0 & \frac{\partial \omega_{3}}{\partial \psi}=0 \\
\frac{\partial \omega_{1}}{\partial \theta}=\dot{\psi} \sin \phi \cos \theta & \frac{\partial \omega_{2}}{\partial \theta}=\dot{\psi} \cos \phi \cos \theta & \frac{\partial \omega_{3}}{\partial \theta}=-\dot{\psi} \sin \theta \\
\frac{\partial \omega_{1}}{\partial \phi}=-\dot{\theta} \sin \phi+\dot{\psi} \sin \theta \cos \phi & \frac{\partial \omega_{2}}{\partial \phi}=-\dot{\theta} \cos \phi-\dot{\psi} \sin \phi \sin \theta & \frac{\partial \omega_{3}}{\partial \phi}=0 \\
\frac{\partial \omega_{1}}{\partial \dot{\psi}}=\sin \phi \sin \hat{\theta} & \frac{\partial \omega_{2}}{\partial \dot{\psi}}=\alpha \sin \phi \sin \theta & \frac{\partial \omega_{3}}{\partial \dot{\psi}}=\cos \theta \\
\frac{\partial \omega_{1}}{\partial \dot{\theta}}=\cos \phi & \frac{\partial \omega_{2}}{\partial \dot{\theta}}=-\sin \phi & \frac{\partial \omega_{3}}{\partial \dot{\theta}}=0 \\
\frac{\partial \omega_{1}}{\partial \dot{\phi}}=0 & \frac{\partial \omega_{2}}{\partial \dot{\phi}}=0 & \frac{\partial \omega_{3}}{\partial \dot{\phi}}=1
\end{array}
$$

Once again we simplify (see Appendix E.7) by assuming $\partial \overrightarrow{\mathbb{N}}_{\oplus \mathrm{F}} / \partial \mathrm{y}_{\ell}$ is small enough to be ignored, and irom equations (II-8) and (II-1ㄱ) we derive

$$
\begin{align*}
\frac{\partial \vec{N}}{\partial y_{\ell}} & =\sum_{b=\oplus, \odot} M_{b}\left[\frac{\partial \vec{r}_{b}}{\partial Y_{\ell}} \times \nabla U_{b}+\vec{r}_{b} \times \frac{\partial}{\partial Y_{\ell}} \nabla U_{b}\right] \\
& =\sum_{b=\oplus, \odot} M_{b}\left[\frac{\partial \vec{r}_{b}}{\partial Y_{\ell}} \times \nabla U_{b}+\vec{r}_{b} \times\left\{\sum_{i=1}^{3} \frac{\partial x_{i}}{\partial Y_{\ell}} \frac{\partial}{\partial x_{i}} \nabla U_{b}\right\}\right] \tag{II-41}
\end{align*}
$$

by invoking the chain rule. The $k^{\text {th }}$ component of $\partial / \partial x_{i}\left(\nabla U_{b}\right)$ can be found in Appendix $A$ (equation $\left.A-7\right)$. We should make explicit the meaning of $\partial \vec{r}_{b} / \partial y_{\ell}$ : If the selenocentric components of $\vec{r}_{b}$ are ( $x_{1}, x_{2}, x_{3}$ ), then $\partial \vec{r}_{b} / \partial y_{\ell} \equiv\left(\partial x_{1} / \partial y_{\ell}, \partial x_{2} / \partial y_{\ell}, \partial x_{3} / \partial y_{\ell}\right)^{T}$.

The terms $\partial x_{i} / \partial y_{\ell}$ represent the change in the selenocentric coordinates of the perturbing body with respect to changes in the Euler angles. In PEP, the positions of both Earth and Sun are integrated and tabulated in the 1950.0 inertial frame. Let $\bar{\xi}$ be the coordinates of body $b$ in this frame, relative to the lunar center of mass. Then the selenocentric coordinates are obtained from the inertial coordinates by a rotation matrix $\overline{\bar{R}}$ defined by

$$
\overline{\mathbf{x}}=\overline{\bar{R}}(\psi, \theta, \phi) \bar{\xi} .
$$

Goldstein (1950) gives the elements of $\overline{\bar{R}}$ as:
$\left[\begin{array}{lll}\cos \phi \cos \psi-\sin \phi \cos \theta \sin \psi & \cos \phi \sin \psi+\sin \phi \cos \theta \cos \psi & \sin \phi \sin \theta \\ -\sin \phi \cos \psi-\cos \phi \cos \theta \sin \psi & -\sin \phi \sin \psi+\cos \phi \cos \theta \cos \psi & \cos \phi \sin \theta \\ \sin \theta \sin \psi & -\sin \theta \cos \psi & \cos \theta\end{array}\right]$

Now $\bar{\xi}$ does not depend on the orientation of the selenocentric coordinate system, so:

$$
\begin{gathered}
\frac{\partial x_{1}}{\partial \psi}=(-\cos \phi \sin \psi-\sin \phi \cos \theta \cos \psi) \xi_{1}+(\cos \phi \cos \psi-\sin \phi \cos \theta \sin \psi) \xi_{2} \\
\frac{\partial x_{1}}{\partial \theta}=\sin \phi \sin \theta \sin \psi \xi_{1}-\sin \phi \sin \theta \cos \psi \xi_{2}+\sin \phi \cos \theta \xi_{3} \\
\frac{\partial x_{1}}{\partial \phi}=(-\sin \phi \cos \psi-\cos \phi \cos \theta \sin \phi) \xi_{1}+(-\sin \phi \sin \psi+\cos \phi \cos \theta \cos \psi) \xi_{2} \\
\\
+\cos \phi \sin \theta \xi_{3}
\end{gathered}
$$

$$
\begin{align*}
& \frac{\partial x_{2}}{\partial \psi}=(\sin \phi \sin \psi-\cos \phi \cos \theta \cos \psi) \xi_{1}+(-\sin \phi \cos \psi-\cos \phi \cos \theta \sin \psi) \xi_{3} \\
& \frac{\partial x_{2}}{\partial \theta}=\cos \phi \sin \theta \sin \psi \xi_{1}-\cos \phi \sin \theta \cos \psi \xi_{2}+\cos \phi \cos \theta \xi_{3} \\
& \frac{\partial x_{2}}{\partial \phi}=(-\cos \phi \cos \psi+\sin \phi \cos \theta \sin \psi) \xi_{1}+(-\cos \phi \sin \psi-\sin \phi \cos \theta \cos \psi) \xi_{2} \\
& \frac{\partial x_{3}}{\partial \psi}=\sin \phi \cos \psi \xi_{1}+\sin \theta \sin \psi \xi_{2} \\
& \frac{\partial x_{3}}{\partial \theta}=\cos \theta \sin \psi \xi_{1}-\cos \theta \cos \psi \xi_{2}-\sin \theta \xi_{3} \\
& \frac{\partial x_{3}}{\partial \phi}=0
\end{align*}
$$

This completes the derivation of the variational equations for our lunar rotation model. For the sake of completeness, we have presented most of the important terms in the variational equations. However, some of the preceding formulae have not yet been implemented in our lunar rotation model. In equation (II-31) the elastic perturbation terms (all terms except the first two on the right-hand side) have been included only for the elastic ( $k$ ) and the dissipative ( $D$ or $k / Q$ ) parameters. Therefore all the other parameters
will have errors of order $\delta I /(C-A) \simeq 10^{-3}$ in their variational. equations. The derivative of the Earth-figure torque, given in equation (II-37), is ignored for all parameters, thus committing an error of about 1 part in $10^{6}$ (see Appendix E.7). Finite differencing of the equations of motion over the relatively short (for economic reasons) time-span of 40 days has proven the variational equations for each adjustable rotation parameter to be consistent with the equations of motion to the level expected (0.1\%).
III. Model of the Moon's Orbit

The rotation of the Moon is strongly coupled to the orbital motion of the Moon, as a result of the spin-orbit synchronism. Any deviation from uniform orbital motion that is not accompanied by a similar excursion from uniform rotational motion will result in a torque which forces physical librations of the Moon. For this reason, the orbital model used in the generation of a description of the lunar rotation is of central importance. In this chapter we give an overview of the treatment of the lunar orbit in PEP; more complete detail can be found in Slade (1971), and Ash (1965b). The differential equations governing the motion of the Moon's center of mass about the Earth are written and integrated in the inertial 1950.0 Cartesian coordinate system, defined in Chapter II. The principal force terms are the Newtonian $1 / r^{2}$ attractions of the Earth and Sun, treated as point masses. Smaller perturbing furces due to the other eight planets are included, with their positions interpolated from a numerical ephemeris. The figures of the Earth and Moon measurably affect the lunar orbit; Slade's treatment of this effect has been modified, as described later in this chapter. General relativistic correction terms to the equations of motion, based on the post-Newtonian approximation, are also applied. Tidal friction in the Earth is
modelled by retarding the elastic response of the Earth to the tides raised by the Moon by a fixed lag angle, $\delta$; the retarded tidal bulge is then used to calculate the tidal acceleration of the Moon. This parameterization is designed not so much to study dissipation within the Earth as it is to account semi-empirically for an observed secular acceleration in the Moon's longitude. (The Earth's interior may be better studied by seismology and in situ tidal measurements, than by indirect astronomical inference.) The tidal bulge due to the Sun, though nearly half the size of the lunar bulge, is ignored since it causes a nearly sinusoidal perturbing force on the Moon, and tends to average to zero over the synodic month. However, this assumption should be subjected to more rigorous analysis, due to the possibility that the non-zero eccentricity and inclination of the lunar orbit could significantly affect the longitudinal symmetry. Also, the solar perturbations of the lunar orbit may interact importantly with the solar tidal bulge.

Because of the existence of high precision laser range data to which we can compare our models, the original slade formulation has been modified slightly. Slade included the effects of the second and third-degree zonal terms of the Earth's gravity field and the complete second-degree gravity field of the Moon. To this we have added the effect of the Earth's fourth-degree zonal field and the complete third-
degree gravity field of the Moon. According to estimates presented by Slade (1971), the Earth's fourth-degree zonal field can cause a secular change in some of the lunar orbital elements amounting to about $1 \mathrm{~cm} / \mathrm{yr}$ change in lunar position. Since nearly ten years of laser range data are now available for analysis, this term should be added.

The necessity of including the third-degree gravity field of the Moon is somewhat more subtle. primarily it is done to ensure consistency between the orbital and rotational equations of motion. For example, when one includes the $S_{31}$ third-degree harmonic term in a rotation model, a longitude offset of approximately $110^{\prime \prime}$ arises between the mean-Earth direction and the Moon's principal axis of least moment. With this orientation offset a balance is struck between the torques caused by the lunar $C_{22}$ and $S_{31}$ fields interacting with the Earth. Hence the transverse forces applied to the Earth are also balanced when averaged over time. If one includes only second-degree lunar gravity harmonics in the orbital equations, while using a thirddegree model to find the (rotational) orientation of the Moon, the offset of the $C_{22}$ field will cause a spurious secular acceleration of the Moon along its orbit. The formulae for the higher order gravitational harmonic effects have been adopted directly from the potential gradient formulation of Appendix A.

A major alteration to Slade's model was the parallel integration of orbital and rotational equations of motion. Since the equations are coupled, one must integrate the equations together to be rigorously correct. Owing to the much stronger dependence of the rotation on the orbit than vice versa, one can approximate simultaneity by first integrating the lunar orbit (using a previously integrated rotation model), then integrating the rotation using the orbit just integrated, and if necessary, iterating the procedure. We feel that the procedure of simultaneous integration is safer and more efficient, as well as being simpler conceptually and easier to use.

The direct orbital effects of the Moon's figure interacting with the Earth's figure, and those of lunar dissipation are very small, and thus are neglected (see Appendices E. 8 and E.9).
IV. Fitting the Models to Laser Range Observations of the Moon

The impetus for the refinements to the models of lunar rotational and orbital motion described in previous chapters was the availability of high precision range data, taken between an Earth-based observatory and the retroreflectors on the Moon. The retroreflectors are arrays of quartz cornercubes, designed so that incident light is reflected back toward the source. The Apollo astronauts on Missions 11, 14, and 15 left a retroreflector at each landing site. Also, the Soviet Union has equipped with retroreflectors two unmanned vehicles on the lunar surface, Lunakhods 1 and 2. All of these reflectors, except Lunakhod 1 , have been ranged regularly by pulsed laser systems on the Earth, using telescope optics for transmission and reception. Although the first detection of a reflected pulse occurred within a month of the Apollo 11 landing (July 1969), range measurements sufficiently accurate for geodetic and selenodetic studies were not obtained until October 1970.

The great majority of scientifically useful ranges (and a].1 the data used in the research here reported) have been taken at the McDonald Observatory on the 2.7 m instrument, owned and operated by the Unversity of Texas. In spite of the vast number of photons emitted by the laser in each
pulse, only $10-20 \%$ of such firings result in even one returned photon being detected. The McDonald team generally creates "normal points" consisting of perhaps 4-10 photon detections each. The statistical errors assigned to the normal points are based on scatter of the individual returns, and also reflect system calibration uncertainties. Due to continuing system improvements during the nine years of regular ranging since the initial shakedown period, the typical estimated uncertainties of the normal points have decreased from around 15 cm to 10 cm . The distribution of the data is non-uniform, due not only to the vagaries of the weather and telescope availability, but also, in a more worrisome systematic manner, as a result of an inability to range close to new Moon (a limitation which is due primarily to the difficulty of pointing the telescope when there is little contrast on the visible lunar disk).

In order to analyze the laser range data to improve determinations of physical constants of the Earth-Moon system, we construct a theoretical model of the observable, from which the value of the range can be calculated, given the time of transmission of the pulse. We calculate the orientation of the Moon and its position relative to the Earth using the previously described lunar rotational and orbital models. The Earth's orientation is found by applying the conventional expressions for the Earth's precession and nutation (Woolard, 1953). The "old" IAU
value of the precession constant (5026.75"/cy) was used; our estimate of $D$ may be affected by this choice (see Chapter
 the effects of the Earth's elasticity on nutation (Melchior, 1971) and for the nutation-induced diurnal polar motion (Woolard, 1953; McClure, 1973). Polar motion values were interpolated from tables circulated by the Bureau International de l'Heure (BIH), and given in their "1968 system." Universal Time (UTI) also was interpolated from the BIH circulars. However, to the interpolated BIH values of UTl we added fortnightly and monthly periodic corrections, each of about 0.7 ms amplitude (Woolard, 1959; Guinot, 1970, 1974), to account for tidal effects which had been mostly removed by the BIH's smoothing procedure. After applying these corrections, we augmented the BIH values for UTl with a parameterized piece-wise linear continuous function of atomic time (King et al., 1978). The displacement of the McDonald Observatory due to solid-Earth tides was not modelled; other analyses we have performed indicate this neglect to have no significant effect on our results. Specifically, when the effects of Earth tides were included in fits to laser range data, the parameter estimates presented here changed by only a small fraction of their formal errors.

There are a number of parameters in our theoretical model of the round-trip light travel time whose values have
not been determined as well from other measurements as they can be determined from lunar laser ranging. These parameters have been incorporated as "free" parameters in the model, so that their values may be adjusted to create a "best-fit" of the model to the data. The number of free parameters is not constant. We have varied the set of estimated parameters to perform sensitivity studies, and at times have estimated values for parameters to which the data are only weakly sensitive. However, the following list can be considered representative of the set of free parameters usually estimated in our analyses:

1) eight orbit parameters: the mass of the Earth-Moon system, the Earth's tidal friction parameter, and six initial conditions
2) 17 rotational parameters: six initial conditions, $\beta, \gamma$, all seven third-degree harmonic coefficients, the lunar Love number $k$, and a dissipation parameter ( $D$ or $k / Q$ )
3) 12 lunar coordinates: three coordinates for each of four reflectors
4) three McDonald coordinates
5) five orbital elements of the Earth-Moon barycenter (the longitude of the ascending node is kept fixed to define an origin of right ascension)
6) 186 parameters of the piecewise-linear function describing the excursion of UTl from the BIH values (the number varies with the length of the data span and the distribution of the data)
7) one parameter to model an instrumental bias at McDonald for 1972

The parameters listed above were fitted to the data by iterative use of a linearized weighted least-squares estimator. The weight was proportional to the inverse square of each observition error, so the estimator minimizes $\sum_{i}\left(\left(O_{i}-C_{i}\right) / \sigma_{i}\right)^{2}$, where $O_{i}, C_{i}$, and $\sigma_{i}$ are respectively the observed value, the computed value, and the standard error of the $i^{\text {th }}$ observation. A more detailed account of the method of least squares as here applied can be found in Ash (1972). Under the assumption that the observation errors are zero mean, inclependent random variables obeying Gaussian statistics, the weighted least-squares estimator is also the maximum-likelihood estimator. The formal standard errors of the parameter estimates are based upon the truth of the above assumption. Of course, the premise is not strictly valid; many error sources are correlated, e.g. errors in electronics calibration. In view of this fact, we often quote errors several times larger than the formal errors, . especially when sensitivity studies or other information indicates the presence of systematic errors.

Due to the non-linear dependence of our theoretical model on most of the estimated parameters we iterated our solutions to achieve convergence. The process by which we accomplished this goal was as follows:
l) We began with a priori values taken from the literature for all free parameters, except the UTl corrections and the McDonald bias, which we assumed to be zero.
2) We integrated the equations of motion of the farth-Moon barycenter orbit about the Sun, and the variational equations for the corresponding initial conditions.
3) Using the ephemeris created in step 2, we integrated the lunar orbital and rotational equations of motion, and all relevant lunar variational equations.
4) We computed ( $0-C$ ) for each observation and partial derivatives of $C$ with respect to all relevant parameters, formed and solved the least-squares normal equations, and obtained new estimates for the adjustable parameters.
5) We repeated steps 2 through 4 until the model had converged.

Due to the high cost of the integrations in steps $2 \& 3$ we stopped iterating when it appeared that the next round of adjustments would change the parameter estimates by only small fractions of the formal parameter errors. This approximate convergence occurred after only a few iterations since our a priori parameter values placed us within a domain in parameter-space where the theoretical value of the observa-

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V. Discussion of Results

We have analyzed 2317 normal points as described in Chapter IV; these data spanned 7 1/2 years, from October 1970 to May 1978. Ranges between McDonald Observatory and all four "functioning" reflectors were included. The errors assigned to the majority of the observations were those assigned by the University of Texas observing team (Abbot et al., 1973; Shelus et al., 1975; Muiholland et al., 1975; P. J. Shelus, private communication, 1976). However, for about one quarter of the normal points we increased the standard errors derived by the University of Texas by factors of \(1.5-3.0\) to reflect unexpectedly large scatter in the corresponding residuals. Such scatter occasionally appeared in the residuals for isolated days but usually persisted for a week or more.

We performed a number of fits to the data, varying both the physical models employed and the membership of the parameter set estimated. Of these fits, two are of particular interest and will be discussed here. The only difference between these two fits was in the model employed for dissipation: in one fit we used the constant-time-lag model and estimated \(D\) (三kT); in the other fit we used the constant-Q model and estimated \(k / Q\). Along with each dissipation parameter, we simultaneously estimated the 231 other parameters listed in Chapter IV. Both fits yielded a postfit rms range residual of 27 cm .

The estimated values for parameters of geophysical interest are presented for both fits in Table 2. The quoted errors are the formal standard deviations of the parameter estimates obtained by uniformly scaling (by a multiplicative factor of 1.8 ) all the range measurement errors so that the weighted mean-square postfit range residual was unity. As mentioned in Chapter IV, however, the true parameter uncertainties are probably greater than the formal errors, due to systematic errors in the observations and the theoretical model of the observations. Thus the formal errors given in Table 2 should be treated as lower bounds on the true uncertainties. Previous, sensitivity studies of lunar laser range data analyses, in which both data and parameter sets were varied, have led to estimates of uncertainties in parameter estimates of the order of three or four times the formal errors (see, for example, Shapiro et al., 1976 or Ferrari et al., 1980).

In order to allow comparison with an independent determination, we have also included in Table 2 results recently obtained by Ferrari et al. (1980). In addition to seven years (through August 1977) of lunar range data, their analysis included Doppler tracking data from Lunar Orbiter IV. The main contribution of the Doppler tracking data to their analysis was an enhanced sensitivity to the lunar gravity field. They down-weighted the laser range data by
scaling the observation errors by about a factor of five relative to the errors we assigned, resulting in formal errors larger than ours for some of the parameter estimates. In their analysis they used a constant-time-lag model for the lunar dissipation. Two important differences between their analysis and ours should be emphasized. First, their model of the Earth's orbit about the sun was based on previous analyses of interplanetary radar and spacecraft tracking data, and was not adjusted in the fit to laser range data. Second, they used a different model for corrections to the BIH UTl data. Their UTl model had only four degrees of freedom: the amplitude and phase of an annual term, the coefficient of a term linear in time, and a coefficient for an ad hoc sinusoidal term having the lunar nodal period ( 1.8 .6 yr ). The simple model of UTl corrections they employed probably contributed greatly to their postfit rms range residual of 38 cm ; we have gotten similar results when using a simpler UTl model having only annual and semiannual correction terms.

Comparing our results with theirs, we see that the estimates of the mass of the Earth-Moon system and the Earth's tidal dissipation parameter agree very well. This is reassuring, since the satellite tracking data, which were obtained over a time span of less than one month in 1967, are quite insensitive to both of these parameters. The
parameters describing the lunar gravity field are somewhat discrepant. This is probably attributable to their addition of the Doppler data; for instance, when Ferrari et al used laser range data alone, they obtained for \(J_{3}\) an estimate of \((8.5 \pm 2.3) \times 10^{-6}\), in excellent agreement with our result. The source of disagreement between estimates of gravity field parameters based on the two different data types is unknown.

Of particular interest are the dissipation parameters, D and \(k / Q\), for which our estimated values are about 25 times the formal errors. Even taking into account the abovediscussed possibility of systematic errors, this still seems a significant result. It is interesting that the dissipation parameters seem better determined than the elastic Love number \(k\). The magnitudes of the correlation coefficients between \(k\) and \(D\) (or \(k / Q\) ) and the rest of the parameters (as well as each other) are all less than 0.5 , indicating that no single parameter mimics the signatures of \(k\) and \(D\). \(A\) gauge of the extent to which the signature of a given parameter can be separated from the signatures of the ensemble of other estimated parameters is the so-called "masking factor"; we define it here as the ratio of the given parameter's formal error when estimated simultaneously with the complete set of parameters, to the given parameter's formal error when estimated alone. For both dissi-
pation models the masking factor for the dissipation parameter is about seven, while the masking factor for \(k\) is about 30 -- which shows the greater separability of the effects of D and \(k / Q\), as compared to \(k\). According to Yoder (1979), the unique observable signature of internal dissipation is a constant offset of the direction of the lunar pole from that predicted by the Cassini state. The sense of the offset is such that it represents a lag in the precession of the lunar pole about the pole of the ecliptic; its magnitude is 0.16", given our estimate of \(D\) (D. H. Eckhardt, private communication).

It can be seen from Table 2 that our estimate of \(D\) is disparate with the estimate of Ferrari et al., theirs being about \(80 \%\) greater than ours. We know of no satisfactory explanation for this result. For instance, when we fit to the range data using the "new" IAU value for the precession constant, the estimate of \(D\) was smaller by about \(15 \%\). The effect of the difference in models for the Earth's rotation has also been investigated: When we estimated \(D\) using an Earth rotation model similar to theirs, the estimate of \(D\) increased, but only by \(10 \%\).

As discussed in Chapter II-B, a useful quantity for describing slightly dissipative phenomena is the quality factor, Q. Since the constant-time-lag model corresponds
to a frequency-dependent \(Q\), one must make assumptions concerning the frequency at which most of the energy is being dissipated in order to speak of an "equivalent Q" for this model. An examination of Table 1 shows that the greatest elastic perturbations occur at near-monthly periods, so it is not unreasonable to assume this to be the dominant time scale for dissipation as well. The relation between \(D\) and \(Q\) is then simply \(Q \simeq k /(n D)\), where \(n\) is the lunar mean motion. The consistency between our estimate of \(Q\) derived in this manner \((22 \pm 4)\) and that derived more directly via the constant-Q model \((27 \pm 4)\), tends to validate the above assumptions a posteriori.

Our estimate of about 25 for the lunar \(Q\) disagrees sharply with the values inferred by lunar seismologists. The seismic \(Q\) values at depths of less than 500 km must be on the order of 10,000 to account for the characteristics of high-frequency teleseismic events (Nakamura, 1974). It is likely, though, that most of the dissipation of tidal energy takes place deeper in the lunar interior. From a numerical model of the lunar interior based on seismic data, Cheng and Toksöz (1978) find that a broad maximum of (tidal) shear stress in the Moon occurs at depths of \(600-1200 \mathrm{~km}\). Given the hypothesis that moonquakes are triggered by tidal shear stress, as suggested by the coincident timing of moonquakes and periods of maximum shear stress, the validity of their
numerical model is supported by the observation that the given depth range corresponds to the region of the lunar interior from which most moonquakes originate. In an earlier paper Toksöz et al. (1974) inferred a seismic \(Q\) in the range 200-700 for regions 800-1100 km deep.

This determination of \(Q\) still fails to agree with our tidal \(Q\) by at least a factor of eight, but there is really nc good reason to expect agreement. The lowest frequency at which the seismic \(Q\) is determined is about one Hertz. The shortest period represented in our elastic perturbation to the inertia tensor is a third of a month, or about nine days. Thus the frequencies at which \(Q\) is determined are six orders of magnitude apart, and extrapolation of the seismic results to monthly periods cannot be relied upon. The dissipative mechanisms involved could be totally different.

Perhaps the closest analog for the lunar tidal \(Q\) is the Q inferred for the Earth at the Chandler wobble frequency. A value for the Earth's \(Q\) of \(\approx 30-60\) has been deduced from the apparent width of the spectral peak at the Chandler frequency in the Earth's polar motion data. Markowitz (1980), among others, has argued that this range of \(Q\) values can only be interpreted as a lower bound on the Earth's \(Q\), since the spectrum of the excitation inechanisms remains unknown and it is not clear that any damping of the Chandler wobble has been observed. The role of the oceans
in the dissipation of Chandler wobbie energy is also unknown. We can only conclude that our estimates of the lunar \(Q\) are consistent with the lower bounds placed upon the \(Q\) of the Earth's mantle.

The value we estimate for the global lunar Love number, \(k\), varies somewhat, depending on which dissipation model is used; we are still unsure of the reason why this is so. There is no convincing evidence for favoring either model of dissipation: both fit the data equally well, and the actual lunar dissipative mechanisms involved are unknown. Cheng and Toksöz (1978), using their numerical model of the lunar interior which was constrained to agree with the seismic velocity profiles, arrived at a surface potential Love number value of \(k=0.029\). This agrees better with the constant -0 model's estimated value of \(k=0.026 \pm 0.003\) than with the constant-time-lag result of \(k=0.020 \pm 0.003\). In the absence of any other indications that the constant \(-Q\) model better describes the anel.asticity of the Moon, we feel there is no basis for favoring either model.
VI. Sugggestions for Further Research

The apparent detection of a significant amount of dissipation in the Moon's interior is extremely interesting and warrants continued study. Of particular concern as a possible source of error is some unmodelled effect which is mimicking the "signature" of the solid-body dissipation parameter. For example, Yoder (Ferrari et al. 1980) has investigated the possibility that a viscous core-mantle coupling in the Moon is providing dissipation by damping the non-uniformity of the mantle rotation. He found that the signatures are similar at about the \(10 \%\) level, but to explain the estimated value of \(k T\) in this way would seem to require an unreasonably large value of \(r\) ther the radius of the core or the kinematic viscosity at the core-mantle interface. The question of whether significant unmodelled effects exist that mimic solid-body dissipation, will possibly be resolved by the addition of new data to the analysis, as they become available. In general, as the span of time covered by lunar ranging data lengthens, the similarity between the signature of dissipation and the signatures of possible unmodelled effects is lessened. Thus, if the effects of some unmodelled parameter are being compensated for by the dissipation model, the estimate of the dissipation parameter might vary as more data are added, and the "mapping function" changes.

Sizable effects have probably been omitted from our model of the lunar range, since the rms of the postfit residuals is 27 cm , much larger than the \(10-15 \mathrm{~cm}\) errors typically assigned by the observers. One possible source of error is in the empirical model we use for the motion of the Earth's pole of figure about the spin-axis. As explained in Chapter IV, we have been using the polar motion data circulated by the BIH; these are smoothed tabular points based upon optical observations of stars, and (since 1972) Doppler observations of Earth satellites. Lunar range data may determine the Earth's polar motion more accurately, although only one component of the pole position can be estimated from observations by a single ranging station. (See, for example, Harris and Williams (1977).) We are in the process of implementing in PEP a piece-wise linear correction function for the BIH polar motion values, in a manner very similar to our treatment of the BIH UTl data. Preliminary results indicate that a substantial improvement in the fit to data may be achieved.

As more range data become available it may be possible to determine the frequency dependence of the lunar \(Q\). The two dissipation models we have implemenred correspond to \(Q \propto \mathrm{E}^{0}\) (constant-Q) and \(Q \propto \mathrm{E}^{-1}\) (constant-time-lag). One might consider a model in which the lunar \(Q\) is proportional to an arbitrary power of frequency, say \(Q \alpha f^{n}\); the parameter \(n\) might then be estimated from the data. This
could be done using the technique of harmonic decomposition of the elastic perturbation to the inertia tensor, in a manner similar to our constant-Q formulation. Instead of a constant phase shift, though, one would use a phase shift proportional to frequency to the \(-n\) power.

Finally, it would be interesting to determine whether or not free librations of the Moon have really been detected, as has been asserted by Calame (1976). The Moon's free librations represent sinusoidal motions about the Cassini state. There are three modes of free libration, since there are three rotational degrees of freedom. Peale (1973) describes them as the free libration in longitude, the free wobble (of the polar principal axis about the spin-axis), and the free precession (of the spin-axis about the Cassini direction); the periods of oscillation are, respectively, \(2.9 y, 75 y\), and 24y. The free librations are called "free" because the amplitudes and phases of the three modes are not fixed, in contradistinction to the "forced" librations of the Moon, where the response is totally determined, given the driving torques and the lunar inertia ellipsoid. The amplitudes and phases of the free librations are dependent on the practically unknown past history ( \(\sim 10^{6}\) years) of the Moon, since they can be stimulated by such events as meteorite impacts. The rate of decay of the free librations is determined by the dissipative properties of the Moon.

Using our numerical rotation model, we have taken free librations into account implicitly by estimating six initial conditions of rotation at an arbitrary epoch. In a sense, the free librations are "buried" within the forced rotational motion, since we have made no attempt to distinguish them within our numerical model (in fact, we know of no practical way to do sol. In an analytic theory, however, the distinction between free and forced librations is explicit.

Calame modelled free librations by adding the appropriate sinusoidal terms to an Eckhardt analytic model that represented only the forced librations. She then estimated the amplitudes and phases of the three modes of free libration, along with other parameters, by least-squares fit to laser range data; she obtained an amplitude of 1.7" for the free libration in longitude. Perhaps without justification, some have voiced skepticism concerning these results; to date, no one has published an opposing viewpoint.

We have compared our best-fit constant-time-lag rotation model with a more recent analytic model (D. H. Eckhardt, private communication) which was generated using the parameter values from our fit. The results of this preliminary comparison seem in good agreement with those of calame, especially for the free libration in longitude. Since our rotational model gives rms residuals of under 30 centi-
meters, it is hard to believe that it is in error by \(1.7^{\prime \prime}\), a lunar surface displacement of over 12 meters. Rather than a detection of free librations of the Moon, however, this result might be merely an indication of a large error in the analytic theory in a nearly resonant term. Very small errors (probably of omission) in the analytic lunar orbit theory used to generate the rotation theory, can be magnified enormously in the rotational response due to near-resonance. Given a value for the lunar dissipation, though, one can place an upper bound on this magnification factor. In subsequent work we hope to continue our comparisons with an analytic theory, and further explore the extent to which analytic theories can be relied upon at frequencies near resonance.

In an attempt to identify possible sources of error in our model of the lunar range, we have compiled a list of neglected or erroneously implemented effects. They are given below in approximate order of importance, along with page numbers of relevant discussions in this thesis, where applicable.
1) BIH polar-motion data for the Earth inadequate (61)
2) neglected effect of the Earth tide raised by the Sun acting on the lunar orbit (43)
3) elastic perturbations not incorporated in all of the variational equations for rotation \((40,96)\)
4) precession constant error (56); elastic nutation corrections possibly in error
5) neglected effect of the fourth-degree lunar harmonics on the lunar rotation (90)
6) neglected effect of lunar non-homogeneity on the elastic and dissipative rotational model (91)
7) effect of an error in the lunar \(J_{2}\) coefficient on the lunar orbit
8) effect of errors in the planetary masses on the lunar orbit
9) neglected effect of asteroids on the lunar orbit
10) questionable validity of diurnal polar motion model
11) neglected effect of solar torques on the retarded lunar tidal bulge (94)
12) neglected effect of figure-figure forces on the orbit (97)
13) neglected effect of lunar dissipation on the lunar orbit (99)
14) neglected effect of torques exerted by other planets on the lunar rotation
15) neglected effect of Earth-figure torques in the variational equations for rotation (96)

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\section*{APPENDIX A}

The Lunar Potential

In the derivation of the lunar rotational and orbital equations of motion and the variational equations, it is useful to write equations for the lunar gravitational potential gradient, as well as for the partial derivatives of the potential gradient with respect to position along Cartesian axes. We will now derive the necessary formulae for the lunar potential, though the results are equally applicable to other bodies. The lunar potential can be expanded in spherical harmonics as follows (Kaula, 1966):
\[
\begin{align*}
U(r, L, \theta)= & -G M\left\{\frac{1}{r}-\sum_{n=2}^{\infty} J_{n} a^{n} r^{-(n+1)} P_{n}(\sin L)\right. \\
& \left.+\sum_{n=2}^{\infty} \sum_{m=1}^{n} a^{n} r^{-(n+1)} \theta_{n m}(\theta) P_{n m}(\sin L)\right\} \tag{A-1}
\end{align*}
\]
with
\[
\begin{equation*}
\theta_{n m}(\theta) \equiv C_{n m} \cos m \theta+s_{n m} \sin m \theta \tag{A-2}
\end{equation*}
\]
and \(\mathbf{G}=\) Newtonian gravitational constant
\(M=\) mass of the Moon
\(a=\) lunar radius
\(r=\) distance from the lunar center of mass
\(L=\) selenocentric latitude
\(\theta=\) selenocentric East longitude (for this appendix only)
\(P_{n}\) are the Legendre polynomials.
\(P_{n m}\) are the associated Legendre functions.
\(J_{n}, C_{n m}\), and \(S_{n m}\) are dimensionless coefficients.
The lunar radius, a, serves here only to scale the harmonic coefficients of different degrees. The component of the potential gradient along the \(k^{\text {th }}\) Cartesian axis is found simply by differentiating equation ( \(A-1\) ) with respect to \(x_{k}\) :
\(\frac{\partial U}{\partial x_{k}}=-G M\left\{\frac{-x_{k}}{r^{3}}+\sum_{n=2}^{\infty} J_{n}\left(\frac{a}{r}\right)^{n}\left[\frac{(n+1) x_{k}}{r^{3}} P_{n}-P_{n}^{\prime} \frac{1}{r} \frac{\partial \sin L}{\partial x_{k}}\right]\right.\)
\[
\begin{align*}
+\sum_{n=2}^{\infty} \sum_{m=1}^{n}\left(\frac{a}{r}\right)^{n} & {\left[\frac{1}{r} \theta_{n m}^{\prime} \frac{\partial \theta}{\partial x_{k}} P_{n m}+\frac{1}{r} \theta_{n m} P_{n m}^{\prime} \frac{\partial \sin L}{\partial x_{k}}\right.} \\
& \left.\left.-\frac{(n+1) x_{k}}{r^{3}} \theta_{n m} P_{n m}\right]\right\} \tag{A-3}
\end{align*}
\]

We have suppressed the arguments of \(P_{n}, P_{n m}\), and \(\theta_{n m}\) in (A-3) for notational simplicity. Differentiation with respect to. a function's argument is represented by primes, as in \(P_{n m}^{\prime} \equiv d P_{n m} / d(\sin L)\), and \(\theta_{n m}^{\prime} \equiv d \theta_{n m} / d \theta\); hence
\[
\begin{equation*}
\theta_{\mathrm{nm}}^{\prime}=m\left[S_{n m} \cos m \theta-C_{n m} \sin m \theta\right] \tag{A-4}
\end{equation*}
\]

If we choose the \(x_{k}\) axis to lie along the corresponding selenocentric principal axis, then it can be easily demonstrated that
\[
\begin{equation*}
\frac{\partial \sin L}{\partial x_{k}}=\frac{\delta_{3 k}}{r}-\frac{x_{3} x_{k}}{r^{3}} \tag{A-5}
\end{equation*}
\]
\[
\begin{equation*}
\frac{\partial \theta}{\partial x_{k}}=\frac{x_{1} \delta_{2 k}-x_{2} \delta_{1 k}}{x_{1}^{2}+x_{2}^{2}} \tag{A-6}
\end{equation*}
\]

For the development of the variational equations we need to know how the potential gradient varies with position, as embodied in the expression for \(\partial^{2} U / \partial x_{i} \partial x_{k}\); this can be found by differentiating equation (A-3) with respect to \(x_{i}\) :
\[
\begin{aligned}
\frac{\partial^{2} U}{\partial x_{i} x_{k}}= & -G M\left\{\frac{3 x_{i} x_{k}}{r^{5}}-\frac{\delta_{i k}}{r^{3}}\right. \\
& +\sum_{n=2}^{\infty} J_{n}\left(\frac{a}{r}\right)^{n}\left(( n + 1 ) \left[\frac{\delta_{i k}}{r^{3} P_{n}}+\frac{1}{r^{3}} P_{n}^{\prime}\left(x_{k} \frac{\partial \sin L}{\partial x_{i}}+x_{i} \frac{\partial \sin L}{\partial x_{k}}\right)\right.\right. \\
& -(n+3) \frac{x_{i} x_{k}}{\left.\left.r^{5} P_{n}\right]-\frac{1}{r}\left[P_{n}^{\prime} \frac{\partial^{2} \sin L}{\partial x_{i} x_{k}}+P_{n}^{\prime \prime} \frac{\partial \sin L}{\partial x_{i}} \frac{\partial s i n L}{\partial x_{k}}\right]\right)} \\
& +\sum_{n=2}^{\infty} \sum_{\sum_{n=1}^{n}\left(\frac{a}{r}\right)^{n}\left(-(n+1)\left[\frac{1}{r^{3}} \theta_{n m}^{\prime} P_{n m}\left(x_{i} \frac{\partial \theta}{\partial x_{k}}+x_{k} \frac{\partial \theta}{\partial x_{i}}\right)\right.\right.} \\
& +\frac{1}{r^{3}} \theta_{n m} P_{n m}^{\prime}\left(x_{i} \frac{\partial s i n L}{\partial x_{k}}+x_{k} \frac{\partial s i n L}{\partial x_{i}}\right)+\frac{\delta_{i k}}{r^{3}} \theta_{n m} P_{n m} \\
& \left.-(n+3) \frac{x_{i} x_{k}}{r^{5}} \theta_{n m} P_{n m}\right]+\frac{1}{r}\left[\theta_{n m}^{\prime \prime} \frac{\partial \theta}{\partial x_{i}} \frac{\partial \theta}{\partial x_{k}} P_{n m}+\theta_{n m}^{\prime} \frac{\partial^{2} x_{i}}{\partial x_{i} x_{k}} P_{n m}\right.
\end{aligned}
\]
\[
+\theta_{n m}^{\prime} P_{n m}^{\prime}\left(\frac{\partial \theta}{\partial x_{k}} \frac{\partial \sin L}{\partial x_{i}}+\frac{\partial \theta}{\partial x_{i}} \frac{\partial \sin L}{\partial x_{k}}\right)
\]
\[
\left.\left.\left.+\theta_{n m} P_{n m}^{\prime \prime} \frac{\partial \sin L}{\partial x_{i}} \frac{\partial \sin L}{\partial x_{k}}+\theta_{n m} P_{n m}^{\prime} \frac{\partial^{2} \sin L}{\partial x_{i} \partial x_{k}}\right]\right)\right\} \quad(A-7)
\]

By differentiating equations \((A-4),(A-5)\), and \((A-6)\) we obtain:
\[
\begin{align*}
\theta_{\mathrm{nm}}^{\prime \prime} & =-m^{2}\left[S_{\mathrm{nm}} \sin m \theta+c_{n m} \cos m \theta\right] \\
& =-m^{2} \theta_{\mathrm{nm}} \tag{A-8}
\end{align*}
\]
\(\frac{\partial^{2} \sin L}{\partial x_{i} x_{k}}=\frac{3 x_{3} x_{k} x_{i}}{r^{5}}-\frac{\delta_{3 k} x_{i}+\delta_{3 i} x_{k}+\delta_{i k} x_{3}}{r^{3}}\)
\(\frac{\partial^{2} \theta}{\partial x_{i} \partial x_{k}}=\frac{\left(x_{1} \delta_{1 k}+x_{2} \delta_{2 k}\right)\left(x_{2} \delta_{1 i}-x_{1} \delta_{2 i}\right)+\left(x_{1} \delta_{1 i}+x_{2} \delta_{2 i}\right)\left(x_{2} \delta_{1 k}-x_{1} \delta_{2 k}\right)}{\left(x_{1}^{2}+x_{2}^{2}\right)^{2}}\)

\section*{APPENDIX B}

Derivation of the Earth-figure Torque

The distribution of mass within the Earth deviates from spherical symmetry enough so that the lowest-order effects of the Earth's figure must be accounted for in the calculation of the torque acting on the Moon. As discussed in section II-A, the largest torque due to the Earth's nonsphericity arises from the interaction of the Earth's oblateness (corresponding to the \(J_{2}\) term in the potential) with the lunar second-degree gravity field. The resultant torque, which we call the Earth-figure torque, we will now derive.

The second-degree term in the gravitational potential of the Earth, when the Earth is approximated as an oblate spheroid, is
\[
\begin{equation*}
\mathrm{U}_{\oplus}=\frac{\mathrm{GM}_{\oplus} \mathrm{R}_{\oplus}^{2}}{\mathrm{r}^{3}} \mathrm{~J}_{2 \oplus}\left(\frac{3}{2} \sin ^{2} \delta-\frac{1}{2}\right) \tag{B-1}
\end{equation*}
\]
where \(r\) is the distance from the Earth's center of mass, and \(\delta\) is the declination with respect to the equator of date. The Force per unit mass due to this field is then
\[
\begin{align*}
\vec{f}(r, \delta) & =-\nabla U_{\oplus} \\
& =-K r^{-4}\left[\left(-3 \sin ^{2} \delta+1\right) \hat{r}+\sin 2 \delta \hat{\delta}\right] \tag{B-2}
\end{align*}
\]
where \(K \equiv 3 / 2 G M_{\oplus} J_{2 \oplus} R_{\oplus}^{2}\). Since the lunar mass distribution is expressed relative to the selenocentric principal axis system, we now express \(\overrightarrow{\mathbf{E}}\) in this system and linearize about the lunar
center of mass via the Taylor expansion:
\[
\begin{equation*}
\left.\vec{f}\left(x_{1}, x_{2}, x_{3}\right) \simeq \vec{f}\right|_{0}+\left.\sum_{i=1}^{3} x_{i} \frac{\partial \vec{f}}{\partial x_{i}}\right|_{0} \tag{B-3}
\end{equation*}
\]
lo denotes evaluation at the lunar center of mass. The quantivies \(\partial \vec{f} / \partial x_{i}\) can be found via the chain rule:
\[
\begin{equation*}
\frac{\partial \vec{f}}{\partial x_{i}}=\frac{\partial \vec{f}}{\partial r} \frac{\partial r}{\partial x_{i}}+\frac{\partial \vec{f}}{\partial \delta} \frac{\partial \delta}{\partial x_{i}}+\frac{\partial \vec{f}}{\partial \phi} \frac{\partial \phi}{\partial x_{i}} \tag{B-4}
\end{equation*}
\]
where \(\phi\) is the right ascension of the lunar mass element. We find
\[
\begin{align*}
& \frac{\partial \vec{f}}{\partial r}=4 \mathrm{Kr}^{-5}\left[\left(-3 \sin ^{2} \delta+1\right) \hat{r}+\sin 2 \delta \hat{\delta}\right]  \tag{B-5}\\
& \frac{\partial \vec{f}}{\partial \delta}=-\mathrm{Kr}^{-4}\left[-4 \sin 2 \delta \hat{r}+\left(3-7 \sin ^{2} \delta\right) \hat{\delta}\right]  \tag{B-6}\\
& \frac{\partial \vec{f}}{\partial \phi}=-K r^{-4}\left[\cos \delta\left(1-5 \sin ^{2} \delta\right) \hat{\phi}\right] \tag{B-7}
\end{align*}
\]

At the lunar center of mass, we have
\[
\begin{align*}
& \frac{\partial r}{\partial x_{i}}=\hat{r} \cdot \hat{x}_{i}  \tag{B-8}\\
& \frac{\partial \delta}{\partial x_{i}}=\frac{1}{r} \hat{\delta} \cdot \hat{x}_{i}  \tag{B-9}\\
& \frac{\partial \phi}{\partial x_{i}}=\frac{1}{r \cos \delta} \hat{\phi} \cdot \hat{x}_{i} \tag{B-10}
\end{align*}
\]

After substitution of equations ( \(B-5\) ) through ( \(B-10\) ) equation (B-4) becomes
\[
\begin{align*}
& \frac{\partial \vec{f}}{\partial \mathrm{x}_{\mathrm{i}}}=\mathrm{Kr}^{-5}\left\{\left[4\left(-3 \sin ^{2} \delta+1\right) \hat{r} \cdot \hat{\mathrm{x}}_{i}+4 \sin 2 \delta \hat{\delta} \cdot \hat{\mathrm{x}}_{\mathrm{i}}\right] \hat{r}\right. \\
&+\left[4 \sin 2 \delta \hat{r} \cdot \hat{\mathrm{x}}_{i}+\left(-3+7 \sin ^{2} \delta\right) \hat{\delta} \cdot \hat{x}_{i}\right] \hat{\delta} \\
&\left.+\left[\left(-1+5 \sin ^{2} \delta\right) \hat{\phi} \cdot \hat{x}_{i}\right] \hat{\phi}\right\} \tag{B-11}
\end{align*}
\]
where it must be kept in mind that \(r, \delta, \hat{r}, \hat{\delta}\), and \(\hat{\phi}\) are all to be evaluated at the lunar center of mass. Given \(\hat{n}\), the unit vector along the Earth's North pole of date, we can find \(\hat{\delta}\) and \(\hat{\phi}\) by the relations
\[
\begin{equation*}
\hat{\delta}=\frac{\hat{r} \times(\hat{n} \times \hat{r})}{|\hat{r} \times(\hat{n} \times \hat{r})|} \tag{B-12}
\end{equation*}
\]
and
\[
\begin{equation*}
\hat{\phi}=\hat{\delta} \times \hat{r} . \tag{B-13}
\end{equation*}
\]

The force acting on a lunar mass element is \(\vec{f} d M\), and the contribution of this element to the torque about the lunar center of mass is \(\overrightarrow{d N}=\vec{\rho} \times \vec{f} d M\), where \(\vec{\rho} \equiv\left(x_{1}, x_{2}, x_{3}\right)\) is the vector to the element from the lunar center of mass. Thus the total Earth-figure torque is found by integrating \(\overrightarrow{\mathrm{dN}}\) over the entire mass distribution of the moon. So
\[
\begin{align*}
\overrightarrow{\mathrm{N}} & =\int \mathrm{dM} a \vec{\rho} \times \overrightarrow{\mathrm{f}} \\
& \simeq \int \mathrm{dM}_{a} \vec{\rho} \times\left.\overrightarrow{\mathrm{f}}\right|_{0}+\int \mathrm{dM}_{\sigma} \vec{\rho} \times\left.{\underset{i=1}{3}}^{3} x_{i} \frac{\partial \vec{f}}{\partial \mathrm{x}_{\mathrm{i}}}\right|_{0} \tag{B-14}
\end{align*}
\]
by equation ( \(B-3\) ). The first integral vanishes since our origin is at the lunar center of mass and all integrals of the form \(\int d M c x_{i}\), where \(c\) is a constant, vanish by definition of the center of mass. Thus \((B-14)\) becomes
\[
\begin{equation*}
\overrightarrow{\mathrm{N}}=\int \mathrm{dM} \sum_{i=1}^{3}\left(x_{i} \vec{\rho} \times\left.\frac{\partial \vec{f}}{\partial \mathrm{x}_{i}}\right|_{0}\right) \tag{B-15}
\end{equation*}
\]

Define the \(j^{\text {th }}\) component (in the selenocentric principal axis system) of \(\left.\frac{\partial \vec{f}}{\partial x_{i}}\right|_{0}\) to be \(g_{i j}\) so that
\[
\begin{equation*}
g_{i j}=\left.\frac{\partial \vec{f}}{\partial \mathbf{x}_{i}}\right|_{0} \cdot \hat{x}_{j} \tag{B-16}
\end{equation*}
\]

Note that when we perform the indicated dot product on equation ( \(B-11\) ) we see that \(g_{i j}\) is symmetric, that is \(g_{i j}=g_{j i}\). Working in the selenocentric principal axis system we evaluate the cross product of equation ( \(\mathrm{B}-15\) ):
\(\vec{\rho} \times\left.\frac{\partial \vec{f}}{\partial x_{i}}\right|_{0}=\hat{x}_{1}\left(x_{2} g_{i 3}-x_{3} g_{i 2}\right)+\hat{x}_{2}\left(x_{3} g_{i 1}-x_{1} g_{i 3}\right)+\hat{x}_{3}\left(x_{1} g_{i 2}-x_{2} g_{i 1}\right)\)

Substituting ( \(B-17\) ) into \((B-15)\) and rearranging, we have
\[
\begin{align*}
\stackrel{\rightharpoonup}{N}= & \hat{x}_{1} \int \mathrm{dm}_{a_{i=1}} \sum_{i}^{3}\left(x_{2} g_{i 3}-x_{3} g_{i 2}\right) \\
& +\hat{x}_{2} \int \mathrm{dm}_{Q_{i=1}} \sum_{i}^{3} x_{i}\left(x_{3} g_{i 1}-x_{1} g_{i 3}\right) \\
& +\hat{x}_{3} \int \mathrm{dm}_{\mathbb{m}_{i=1}}^{3} x_{i}\left(x_{1} g_{i 2}-x_{2} g_{i 1}\right) \tag{B-18}
\end{align*}
\]

But in a principal axis system the products of inertia are (by definition) zero, so \(\int \mathrm{SM}_{\mathbb{Q}} \mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}}=0\) for \(\mathrm{i} \neq \mathrm{j}\). Using this proparty and the symmetry of \(g_{i j}\), it follows that
\[
\begin{aligned}
& \overrightarrow{\mathrm{N}}=\hat{\mathrm{x}}_{1} \mathrm{~g}_{23} \int \mathrm{dM}_{\mathrm{Q}}\left(\mathrm{x}_{2}{ }^{2}-\mathrm{x}_{3}^{2}\right)+\hat{\mathrm{x}}_{2} \mathrm{~g}_{13} \int \mathrm{dM}\left(\mathrm{x}_{3}{ }^{2}-\mathrm{x}_{1}{ }^{2}\right) \\
& +\hat{x}_{3} g_{12} \int \mathrm{dM}_{\mathrm{d}}\left(\mathrm{x}_{1}^{2}-\mathrm{x}_{2}^{2}\right) \\
& =\hat{x}_{1} g_{23}(C-B)+\hat{x}_{2} g_{13}(A-C)+\hat{x}_{3} g_{12}(B-A)(B-19)
\end{aligned}
\]
by the definitions of the moments of inertia \(A, B\), and \(C\).

\section*{APPENDIX C}

\section*{Partial Derivatives with Respect to}

\section*{Specific Rotation Parameters}

In this appendix we present the somewhat tedious and often simple derivations of partial derivatives with respect to the specific rotation parameters appearing in our model. These derivative sub-expressions originate in Chapter II-D, equations (II-31), (II-32), and (II-36).
\(\partial A^{-1} / \partial p_{i}, \partial B^{-1} / \partial p_{i}\), and \(\partial C^{-1} / \partial p_{i}\) are all zero for \(p_{i}\) other than the second degree parameters \(\beta, \gamma\), and \(J_{2}\), for which the following obtain:
\[
c^{-1}=\frac{1}{M_{q} a^{2}} \frac{2 \beta-\gamma+\beta \gamma}{2 J_{2}(1+\beta)}
\]
\[
\frac{\partial C^{-1}}{\partial J_{2}}=\frac{-1}{J_{2}} C^{-1}
\]
\[
\frac{\partial C^{-1}}{\partial \beta}=\frac{1+\gamma}{M_{Q} a^{2} J_{2}(1+\beta)^{2}}
\]
\[
\frac{\partial C^{-1}}{\partial \gamma}=\frac{\beta-1}{2 M_{Q} a^{2} J_{2}(1+\beta)}
\]
\[
A^{-1}=c^{-1} \frac{1+\beta}{1-\beta \gamma}
\]
\[
\begin{align*}
& \frac{\partial A^{-1}}{\partial J_{2}}=\frac{1+\beta}{1-\beta \gamma} \frac{\partial C^{-1}}{\partial J_{2}} \\
& \frac{\partial A^{-1}}{\partial \beta}= \frac{1+\beta}{1-\beta \gamma} \frac{\partial C^{-1}}{\partial \beta}+\frac{1+\gamma}{(1-\beta \gamma)^{2}} C^{-1} \\
& \frac{\partial A^{-1}}{\partial \gamma}=\frac{1+\beta}{1-\beta \gamma} \frac{\partial C^{-1}}{\partial \gamma}+\frac{\beta(1+\beta)}{(1-\beta \gamma)^{2}} C^{-1} \\
& B^{-1}=\frac{1+\beta}{1+\gamma} C^{-1} \\
& \frac{\partial B^{-1}}{\partial J_{2}}=\frac{1+\beta}{1+\gamma} \frac{\partial C^{-1}}{\partial J_{2}} \\
& \frac{\partial B^{-1}}{\partial \beta}=\frac{1+\beta}{1+\gamma} \frac{\partial C^{-1}}{\partial \beta}+\frac{1}{1+\gamma} C^{-1} \\
& \frac{\partial B^{-1}}{\partial \gamma}+\frac{1+\beta}{1+\gamma} \frac{\partial C^{-1}}{\partial \gamma}-\frac{1+\beta}{(1+\gamma)^{2}} C^{-1} \tag{c-1}
\end{align*}
\]

The \(k^{\text {th }}\) component of \(\partial / \partial p_{i}\left(\nabla U_{b}\right)\) when \(p_{i}\) is one of the harmonic coefficients is simply the term by which the coefficient is multiplied in \(\nabla \mathrm{U}_{\mathrm{b}}\) (equation (A-3)):
\(\frac{\partial}{\partial J_{n}}\left(\frac{\partial U_{b}}{\partial x_{k}}\right)=-G M_{b}\left(\frac{a}{r}\right)^{n}\left[\frac{(n+1) x_{k}}{r^{3}} P_{n}-P_{n}^{\prime} \frac{l}{r} \frac{\partial \sin L}{\partial x_{k}}\right]\)
\[
\begin{align*}
\frac{\partial}{\partial C_{n m}}\left(\frac{\partial U_{b}}{x_{k}}\right)= & -G M_{b}\left(\frac{a}{r}\right)^{n}\left[-m \sin m \theta \frac{1}{r} \frac{\partial \theta}{\partial x_{k}} P_{n m}\right. \\
& \left.+\cos m \theta \frac{1}{r} P_{n m}^{\prime} \frac{\partial \sin L}{\partial x_{k}}-\cos m \theta \frac{(n+1) x_{k}}{r^{3}} P_{n m}\right] \\
\frac{\partial}{\partial S_{n m}}\left(\frac{\partial U_{b}}{\partial x_{k}}\right)= & -G M_{b}\left(\frac{a}{r}\right)^{n}\left[m \cos m \theta \frac{1}{r} \frac{\partial \theta}{\partial x_{k}} P_{n m}\right. \\
& \left.+\sin m \theta \frac{1}{r} P_{n m}^{\prime} \frac{\partial \sin L}{\partial x_{k}}-\sin m \theta \frac{(n+1) x_{k}}{r^{3}} P_{n m}\right] \tag{C-2}
\end{align*}
\]

For \(p_{i}\) one of \(\beta, \gamma\), or \(J_{2}\) we apply the chain rule to find the (additional, in the case of \(J_{2}\) ) change in the potential gradient due to the change induced in \(\mathrm{C}_{22}\) :
\[
\begin{equation*}
\frac{\partial}{\partial P_{i}}\left(\frac{\partial U_{b}}{\partial x_{k}}\right)=\frac{\partial C_{22}}{\partial p_{i}} \frac{\partial}{\partial C_{22}}\left(\frac{\partial U_{b}}{\partial x_{k}}\right) \tag{C-3}
\end{equation*}
\]

Since
\[
c_{22}=J_{2} \frac{\gamma}{2} \frac{1+\beta}{2 \beta-\gamma+\beta \gamma}
\]
we have
\[
\frac{\partial C_{22}}{\partial J_{2}}=\frac{C_{22}}{J_{2}}
\]
\[
\begin{align*}
& \frac{\partial C_{22}}{\partial \beta}=-J_{2} \frac{\gamma(1+\beta)}{(2 \beta-\gamma+\beta \gamma)^{2}} \\
& \frac{\partial C_{22}}{\partial \gamma}=J_{2} \frac{\beta(1+\beta)}{(2 \beta-\gamma+\beta \gamma)^{2}} \tag{C-4}
\end{align*}
\]

The second factor on the right hand side of equation (C-3) is given by the second equation of \((C-2)\) with \(n=m=2\). The partial derivatives of \(\nabla \mathrm{U}_{\mathrm{b}}\) with respect to the elastic parameters are all zero.

From equation (II-15) we obtain
\[
\begin{aligned}
\frac{\partial \Gamma_{j k}}{\partial p_{i}}= & \frac{-1}{\left(I_{0}\right)_{j j}\left(I_{0}\right)_{k k}} \cdot \frac{\partial \delta I_{j k}}{\partial p_{i}} . \\
& +\frac{\delta I_{j k}}{\left[\left(I_{0}\right)_{j j}\left(I_{0}\right)_{k k}\right]^{2}}\left[\left(I_{0}\right)_{j j} \frac{\partial\left(I_{0}\right)_{k k}}{\partial p_{i}}+\frac{\partial\left(I_{0}\right)_{j j}}{\partial p_{i}}\left(I_{o}\right)_{k k}\right]
\end{aligned}
\]

The quantities \(\partial\left(I_{0}\right)_{j j} / \partial p_{i}\) are simply the appropriate diagonal elements of \(\partial \overline{\bar{I}}_{o} / \partial p_{i}\), as given in equation (II-33).

The two quantities from equation (II-31) which remain unformulated, \(\partial \overline{\overline{\delta I}} / \partial p_{i}\) and \(\partial \overline{\overline{\delta I}} / \partial p_{i}\), are dependent upon which dissipation model is invoked. With the adoption of the con-stant-time-lag model, the only non-zero partials are with respect to \(k\) and \(D\). We then have (from equations (II-18)
through (II-2l)):
\[
\begin{align*}
& \frac{\partial \overline{\overline{\delta I}}}{\partial \mathrm{k}}=\frac{1}{\mathrm{k}}=\overline{\overline{\delta I}} \\
& \frac{\partial \overline{\overline{\delta I}}}{\partial \mathrm{D}}=\frac{-1}{\mathrm{k}} \dot{\overline{\delta I}} \\
& \frac{\partial \dot{\overline{\delta I}}}{\partial \mathrm{k}}=\frac{1}{\mathrm{k}} \dot{\overline{\delta I}} \\
& \frac{\partial \overline{\delta I}}{\partial \overline{\mathrm{D}}}=-\frac{1}{\mathrm{k}} \overline{\overline{\delta I}} \tag{c-6}
\end{align*}
\]

For computational reliability in the event of small (or zero) \(k\), we form \(\partial \overline{\overline{\delta I}} / \partial k\), etc., first, and then multiply by \(k\) to find \(\overline{\overline{\delta I}}\).

The constant-Q model, due to its linear formulation, leads also to simple derivatives. All partials are zero, except those with respect to \(k\) and \(k / Q\), when by equations (II-24) through (II-26) we get:
\[
\begin{aligned}
& \frac{\partial \overline{\overline{\delta I}}}{\partial \mathrm{k}}=\sum_{\mathrm{pqrs}}(\ddot{\bar{S}} \sin \mu+\overline{\overline{\mathrm{C}}} \cos \mu) \\
& \frac{\partial \overline{\overline{\delta I}}}{\partial \overline{(k / Q)}}=-\sum_{\mathrm{pqrs}}(\overline{\bar{S}} \cos \mu-\overline{\overline{\mathrm{C}}} \sin \mu) \\
& \frac{\partial \overline{\overline{\delta I}}}{\partial \bar{k}}=\sum_{\operatorname{pyrs}}(\overline{\bar{S}} \cos \mu-\overline{\overline{\mathrm{C}}} \sin \mu) \dot{\mu}
\end{aligned}
\]
\[
\begin{equation*}
\frac{\partial \overline{\overline{\delta I}}}{\partial(k / Q)}=\sum_{p q r s}(\overline{\bar{S}} \sin \mu+\overline{\bar{C}} \cos \mu) \dot{\mu} \tag{c-7}
\end{equation*}
\]

\section*{APPENDIX D}

Tests Performed on the Rotation Model to Verify Correctness

Several tests have been performed on the programmed version of the lunar rotation model (l) to verify that the equations were properly programmed, and (2) to demonstrate that the equations yield the correct solution for problems to which the solution is already known. Falling into the first category are tests such as printing out selected variables and checking derivative expressions by finite differencing of the parent expressions. Into the second category fall two tests, the details of which we shall further describe. They are (1) the comparison of our rigidbody model with a similar model derived elsewhere, and (2) a test of the elastic and dissipative models, for a geometrically simple case in which the solution was easily derived analytically.

We compared our rigid-body lunar rotation model to the widely-available "LLB-5" rotation model developed by J. G. Williams of the Jet Propulsion Laboratory (private communication): a detailed description of the procedure used and the results obtained can be found in Cappallo et al. (1980). The LLB-5 model represents a numerical solution of rigidbody equations of rotational motion for the Moon, but the equations were integrated in a non-inertial frame and the
dependent variables were small Encke-like variables that described the departure of the Moon's rotation from a uniform Cassini motion. We adjusted our rotation initial conditions to obtain a best fit, over a six-year integration span, of the lunar orientation given by the two models. After removing three small bias angles representing coordinate system offsets, the postfit rms difference between the two models was 24 cm , when expressed as a lunar surface displacement. The size of this difference is consistent with the hypothesis that the discrepancy is caused by the known differences in lunar orbital. ephemerides used to create the models. .

The elastic and constant-time-lag dissipative code was alsc checked, albeit in a non-comprehensive manner. For this purpose we created a simplified and isolated twodimensional Earth-Moon system by appropriately choosing the orbital and rotational initial conditions, and by setting the masses of all the other bodies in the solar system to zero. The lunar gravity field was that of a tri-axial ellipsoid, and all orbital forces other than the Newtonian \(1 / r^{2}\) attraction of the centers of mass were suppressed. The Moon was put in a circular orbit about the Earth, with the lunar equatorial plane coincident with the orbital plane. At the initial epoch of integration the \(x_{1}\) principal axis was purposefully offset by about two arc-seconds
in longitude from the Earthward direction, so that free librations would ensue. We then introduced a known amount of damping and observed the decaying sinusoidal oscillations. The rate of decay was predicted using simple formulae not based upon the equations in our model, and was found to agree with the observed decay rate to within the accuracy expected of the simple formulae (a few percent).

APPENDIX E
Approximations

In the course of the derivation of the lunar rotation model and of the alterations to the lunar orbital model we have made several approximations, as noted in the text. Generally, the motivation for these approximations was to simplify the derivation and computer coding, to save our time and the computer's. The lone exception is the approximation explained in E.3, where we don't know enough about the composition of the lunar interior to warrant a more rigorous treatment. In many cases the estimates of orbital and rotational errors are based upon simplifying assumptions such as two-dimensional geometry. It should be kept in mind that these are not rigorous error analyses; instead, they represent the current state of our investigation into possible sources of error.

\section*{E.1: Neglect of torques exerted by other planets (see \\ page 15)}

The far-field torque exerted on the Moon by an external body is directly proportional to the body's mass, and inversely proportional to the cube of its distance. A calculation of \(\mathrm{M} / \mathrm{r}_{\text {minimum }}^{3}\) for the principal bodies of the solar system demonstrates that the torque on the Moon
may be approximated as the sum of the torques induced by Earth and Sun to an accuracy of 1 part in \(10^{6}\), with the largest neglected torque being due to Venus. An examination of a table of planetary synodic periods shows no periods that are near resonance with the lunar libration periods, so in order to establish an upper bound on the effects of neglected planets, we will only consider here the neglected librations due to Venus.

Assume for the moment that the orbit of venus, the orbit of the Moon, and the lunar equator all lie in the same plane. Then the longitudinal angular acceleration of the Moon due to the torque exerted by Venus is
\[
\begin{equation*}
\frac{3 \gamma G M_{f}}{r_{q \text { min }}^{3}} \sin 2 \lambda \simeq 1.9 \times 10^{-6} \sin 2 \lambda \mathrm{arc}-\mathrm{sec} / \mathrm{d}^{2} \tag{E.1-1}
\end{equation*}
\]
where \(\lambda(\approx n t)\) is the angle between the the \(x_{1}\) principal axis and the direction of Venus. Since \(\sin 2 \lambda\) varies with a period of about 14 days, during which the Moon-Venus relative orbital geometry stays roughly constant, the time-average of displacement tends to average to zero. of course, there are fluctuations of the Moon's rotation with this 14-day period. If we integrate (E.1-1) twice with respect to time, we find fortnightly variations in longitude with amplitude \(9 \times 10^{-6}\) arc-sec, far below our observational limit.
we now estimate the size of the latitudinal librations caused by Venus. Since the lunar equator is inclined by only \(1.5^{\circ}\) to the ecliptic, the size of the maximum excursion of Venus from the lunar equatorial plane is set approximately by the orbital inclination of Venus relative to the ecliptic. When the passage of Venus through the point farthest from the ecliptic ( \(90^{\circ}\) from the line of nodes) coincides with opposition of the Earth, Venus is about \(10^{\circ}\) from the ecliptic, as viewed from the Moon. The magnitude of the torque exerted on the Moon due to the attraction of the lunar equatorial bulge by Venus can be found easily from equations \((A-3)\) and (II-10):
\[
\begin{equation*}
|\vec{N}|=\frac{3 G M_{Q} M_{q} J_{2 q} a^{2}}{2 r_{q}^{3}}\left|\dot{\sin } 2 L_{q}\right| \tag{E.1-2}
\end{equation*}
\]

Adopting the worst-case values of \(L_{p} \simeq 10^{\circ}\) and \(r_{p} \simeq 0.26 \mathrm{AU}\), we \(f\) ind \(|\vec{N}| \simeq 1.3 \times 10^{21}\) dyne-cm. The precession of the lunar spin-axis due to this torque can be found from equation (II-3); after a short time, \(\delta t\), the change in the direction of the spin-axis with respect to an inertial frame is simply \(|\vec{N}| \delta t C^{-1} \omega^{-1}\). The time interval over which the Earth-Venus orbital configuration roughly maintains the worst-case geometry is about l00d. During this interval the lunar spin-axis thus suffers an "impulsive" orientation change of \(1.0 \times 10^{-3}\) arc-sec, the equivalent of a lunar-
surface displacement of about 8 mm . Such spin-axis displacements are cumulative over successive conjunctions of Venus. However, the lunar spin-axis, having been displaced, will precess about the Cassini position with a period of 24 years, so that the "impulsive" displacements do not accumulate indefinitely. Furthermore, since we estimate initial conditions for the lunar spin-axis, we are sensitive only to the change in the direction of the spin-axis over the limited time spanned by observations. Over the decade for which lunar range data are available, at most only a few occurrences of this worst-case geometry are possible, and this effect is negligible. Over long time scales, however, there could be a significant departure of the lunar spin-axis from the Cassini state; perhaps analytic theories should account for this effect.

\section*{E.2: Neglect of torques due to fourth-degree harmonics} (see page 17)
Eckhardt (1973) has used his semi-analytic theory to find the rotational pertubations due to the fourth-degree harmonics of the lunar gravity field. Except for some unobservable (by Earth-based ranging) constant offsets, he finds only one libration term with an amplitude over 0.1". The period of this longitude term is six years, while its amplitude is 0.11", based upon the fourth-degree harmonic
coefficients of Ferrari et al. (1980). Williams (private communication) believes that this term, and others that result in libration amplitudes <0.l", can be masked by small changes in the estimated values of third-degree coefficients. Using laser range data, we have estimated various subsets of the fourth-degree harmonic coefficients, and find their values poorly determined. When the coefficients ( \(S_{4 .}\) and \(S_{43}\) ) to which we appear to have marginal sensitivity are estimated, the values of the third-degree coefficients change by as much as \(40 \%\), but \(k\) and \(D\) are only slightly affected.
E.3: Contribution of deformation velocity to angular momentum (see page 18)

For an elastic Moon the angular momentum is, in general, no longer given by \(\vec{L}=\overrightarrow{\bar{I}} \vec{\omega}\), since \(\vec{\omega}\) will be non-uniform as the Moon deforms. This relation will hold, however, under certain restricted conditions: if we let \(\vec{\omega}\) refer to the rigid-body principal axis system (since this is the frame in which \(\overline{\bar{I}}\) is evaluated), and if we consider only the equilibrium (quasi-static) distortion of a homogeneous elastic sphere. The adoption of an equilibrium distortion is reasonable since the tidal and rotational stresses occur mostly at monthly and fortnightly periods, which are much longer than the free oscillation period of about an hour for lunar-sized bodies. Hence the strain field will be able to follow the stress, without sizable non-equilibrium elastic
waves being stimulated. Of course the Moon does not conform to these idealized conditions; a treatment of a more realistic model of the lunar interior is beyond the scope of this paper. In this appendix we attempt only to demonstrate the consistency of our representation of the effects of lunar solid-body elasticity and dissipation.

Given these idealized properties of the Moon, it is easily seen that the distortion caused by centrifugil force doesn't alter the uniformity of \(\vec{\omega}\), since the displacement field has cylindrical symmetry about the instantaneous rotation axis. In general a tidal bulge will cause longitudinal (used here to denote rotation about the spin-axis) displacements, but the integrated longitudinal displacement will vanish for a homogeneous Moon. This follows from the reflection symmetry of the bulge about the plane containing the Earth and the spin-axis of the Moon. In general, symmetry of displacement about a plane containing the spin-axis is a sufficient condition for the longitudinal displacement to vanish when integrated throughout the spherical Moon. Such symmetry in the time-derivative of displacement ensures that the excursions of angular velocity from the spin-axis value also average to zero. As the position of the Earth changes in the principal axis frame, so does the position of the tidal bulge. That this motion
doesn't break the symmetry inherent in the time-derivative of the bulge can be seen by considering the Earth's motion stroboscopically as a succession of appearances of a massive body, each of which raises a tidal bulge symmetric about the plane containing the body and the lunar spin-axis. As the interval between appearances approaches zero the planes of symmetry approach one another, and so the time-derivative of displacement is symmetric. This result has been verified somewhat more tediously by the direct calculation of the changes in the longitudinal tidal displacement field, and by then demonstrating that the planar symmetry does indeed obtain.
"Constant-time-lag" dissipation will not change the above conclusions concerning tides since there will still be a plane of symmetry defined by the spin-axis and the axis of symmetry of the bulge, even though it no longer contains the center of mass of the Earth. The offset of the symmetryaxis of the rotational bulge from the spin-axis due to slight dissipation also displays symmetry about the (tangent) plane passing through the spin-axis and the symmetry axis of the bulge ("old" spin-axis). The rate of change of the rotational bulge is also symmetric about this plane.

\section*{E.4: Neglect of solar torque on elastic perturbation (see} page 20)

The torque due to the Sun's attraction of the elastic perturbation to the lunar figure is neglected. As staited in Chapter \(I I-B\), the time-varying part of the rotational deformation is so small that it could safely have been ignored; we will do so here, and just consider the tidal bulge.

Using equation (II-17) and the tidal part of equation (II-18) we find the magnitude of the solar torque on the tidal bulge to be
\[
\begin{equation*}
|\vec{N}|=\frac{3}{2} k \frac{\mathrm{GM}_{\odot}}{r_{\odot}^{3}} \frac{{ }^{\mathrm{M}_{\oplus}}}{\mathrm{r}_{\oplus}^{3}} \mathrm{R}_{\mathrm{Q}}^{5} \sin 2 \zeta \tag{E.4-1}
\end{equation*}
\]
where \(\zeta\) is the angle subtended by the Earth and the Sun, as viewed from the Moon. The direction of the torque is normal to the plane defined by the centers of mass of the Earth, Moion, and Sun. A second neglected torque is that due to the Earth attracting the lunar tidal bulge raised by the Sun; this torque was ignored implicitly by omitting the solar tides in equation (II-18). As can be seen by the (EarthSun) symmetry of equation (E.4-1), the magnitude of this second torque will match that of the first. Furthermore, the direction of the second torque is opposite that of the first, so that the net torque is zero for an elastic Moon.

Under the presence of lunar dissipation, however, the tidal bulges no longer follow the Earth and Sun exactly, so that the symmetry is broken, and there may be a net torque. Although the ensuing librations are not easily derived, we note that the nearly-cancelling torques being discussed are quite small; using a \(k=0.03\) in equation (E.4-l), we find the maximum resultant rate of change of the lunar angular velocity to be \(5 \times 10^{-5}\) arc-sec \(/ d^{2}\). The only appreciable offset of the tidal bulges due to dissipation will be in a longitude sense (due to the size of \(\omega_{3}\) ), and the longitudinal geometry of the Earth-Moon-sun system is roughly periodic, with a period of 29.5 days. Thus even this small net torque due to the asymmetry of the dissipative bulges will tend to average to zero.
E.5: Effect of approximations in \(\ddot{\overline{\delta I}}\) (see page 24)

Since \(\ddot{\vec{\omega}}\) is used in our rotation model only for the calculation of the rotational part of \(\overline{\overline{\delta I}}\), a sufficiently accurate expression is obtained by differentiating the rigid body angular acceleration in equation (II-5). The timederivative of the torque is evaluated by differentiating the torque exerted on a tri-axial Moon. The acceleration of the

Earth relative to the Moon, \(\ddot{\vec{r}}\), is approximated in \(\ddot{\overrightarrow{\delta I}}\) by considering only the Newtonian \(1 / r^{2}\) mutual attraction of Earth and Moon, and the disturbing acceleration of the Sun.

\section*{E.6: Neglect of elastic effects in implicit terms of variational equations (see page 36)}

The ratio of the magnitudes of the neglected terms in equation (II-16) to the included rigid-body terms is roughly the same as the ratio of the elastic perturbation to the permanent asymuetry, or about \(S I /(C-A) \approx 10^{-3}\). Errors of this size in the variational equations are tolerable since our solutions are iterated until convergence is achieved, although the formal errors for highly-correlated parameters could be affected.

\section*{E.7: Neglect of Earth-figure torques in variational equations (see pages 37 and 41)}

The Earth-figure torque is smaller than the torque exerted by the Earth's center of mass by a factor of about \(J_{2 \oplus}\left(R_{\Theta} / r_{\Theta}\right)^{2} \approx 10^{-6}\). For the reasons given in Appendix E. 6 a relative error of this size is tolerable in the variational equations.
E.8: Neglect of figure-figure forces on orbit (see page 45)

The extra force term due to the interaction of the
Earth's figure with the lunar figure has been left out of the orbital equations of motion due to its small size. We now derive an estimate of this force.

Let \(\vec{f}\) be the force field due to the Earth's oblateness, so that \(\vec{f}=-\nabla\left(J_{2 \theta}\right.\) potential term). We expand \(\vec{f}\) about the lunar center of mass, using the selenocentric principal axis system as a local orthogonal coordinate system:
\[
\begin{align*}
&\left.\vec{f}\left(x_{1}, x_{2}, x_{3}\right) \stackrel{\rightharpoonup}{f}\right|_{0}+\left.\sum_{i=3}^{3} x_{i} \frac{\partial \vec{f}}{\partial x_{i}}\right|_{0}+\left.\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^{2} \vec{f}}{\partial x_{i} \partial x_{j}}\right|_{0} x_{i} x_{j} \\
& \cdots \\
&==\left.\vec{f}\right|_{0}+\sum_{i=1}^{3} \vec{a}_{i} x_{i}+\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \vec{b}_{i j} x_{i} x_{j}
\end{align*}
\]
if we define
\[
\begin{equation*}
\left.\vec{a}_{i} \equiv \frac{\partial \vec{k}}{\partial x_{i}}\right|_{0} \quad \text { and }\left.\quad \vec{b}_{i j} \equiv \frac{\partial^{2 \overrightarrow{1}}}{\partial x_{i} \partial x_{j}}\right|_{0} \tag{E.8-2}
\end{equation*}
\]

The total force acting on the Moon is found by integrating声dM over the lunar mass distribution.
\[
\begin{equation*}
\vec{F}_{e}=\left.M_{\mathbb{C}} \vec{f}\right|_{0}+\sum_{i=1}^{3}\left[\vec{a}_{i} \int d M_{d} x_{i}\right]+\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3}\left[\vec{b}_{i j} \int d M_{d} x_{i} x_{j}\right] \tag{E.8-3}
\end{equation*}
\]

The first term of ( \(\mathbf{E} .8-3\) ) is the interaction between the lunar center of mass and the Earth's oblateness, and was modelled by Slade. The second term vanishes, by definition of the center of mass. By definition of the moments and products of inertia we have
\[
\begin{align*}
\int \mathrm{dM}_{\mathrm{d}} \mathrm{x}_{\mathrm{i}} \mathrm{x}_{j} & =-I_{i j}+\delta_{i j} \frac{1}{2}[A+B+C] \\
& =\delta_{i j}\left[\frac{1}{2}(A+B+C)-I_{i i}\right] \tag{E.8-4}
\end{align*}
\]
since the products of inertia are zero in the principal axis system. It can be shown that \(\vec{b}_{11}+\vec{b}_{22}+\vec{b}_{33}=0\) by expressing the \(\vec{b}_{i i}\) in terms of the potential and invoking Laplace's equation. Using this relation and equation (E.8-4) we can simplify the second term of (E.8-3):
\[
\begin{align*}
\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3}\left[\vec{b}_{i j} \int d M_{a} x_{i} x_{j}\right] & =\frac{1}{2} \sum_{i=1}^{3} \vec{b}_{i \dot{1}}\left[\frac{1}{2}(A+B+C)-I_{i i}\right] \\
& =-\frac{1}{2}\left[A \vec{b}_{11}+B \vec{b}_{22}+C \vec{b}_{33}\right] \\
& =\frac{1}{2}\left[(C-A) \vec{b}_{11}+(C-B) \vec{b}_{23}\right] \tag{E.8-5}
\end{align*}
\]

It can be shown that \(\vec{b}_{11}\) and \(\vec{b}_{22}\) are expressible as linear combinations of the various second partial derivatives \(\partial^{\mathbf{2}} \mathrm{f} / \partial \mathrm{r}^{\mathbf{2}}, \partial^{\mathbf{2}} \mathrm{f} / \partial \delta \partial \phi\), etc. and furthermore, that
the magnitude of the coefficients are no greater than unity if we scale the partial derivatives so that they represent derivatives with respect to linear distance in the indicated directions. A derivation of the nine partial derivatives shows \(\partial^{2} f / \hat{S} r^{2}\) to be potentially the largest, with a maximum magnitude bounded by \(30 \sqrt{5} \mathrm{GM}_{\oplus} J_{2 \oplus} \mathrm{R}_{\oplus}{ }^{2} / \mathrm{r}_{\oplus}^{6}\). Using (E.8-5) this corresponds to a figure-figure acceleration of \(a_{f f}<12 \sqrt{5} \beta G M_{\oplus} J_{2 \oplus} R_{\oplus}{ }^{2} R_{\mathbb{Q}}{ }^{2} / r_{\oplus}^{6} \simeq 2 \times 10^{-4} \mathrm{~cm} / \mathrm{d}^{2}\), smaller than some other neglected effects on the lunar orbit.
\[
\begin{array}{ll}
\text { E.9: } \quad \text { Direct orbital effects of lunar elasticity and } \\
& \text { dissipation (see page 45) }
\end{array}
\]

The external potential of the lunar tidal bulge raised by the Earth when offset by an angle \(\phi\) (due to dissipation) and evaluated at the center of mass of the Earth is:
\[
\begin{equation*}
\mathrm{V}=-\mathrm{k}\left(\frac{\mathrm{R}_{a}}{\mathrm{r}_{\oplus}}\right)^{5} \frac{\mathrm{GM}_{\oplus}}{\mathrm{r}_{\oplus}}\left(\frac{3}{2} \cos ^{2} \phi-\frac{1}{2}\right) \tag{E.9-1}
\end{equation*}
\]

The force exerted on the Earth (treated as a point mass) is:
\[
\begin{align*}
\vec{F} & =-M_{\oplus} \nabla V \\
& =-\frac{3}{2} k\left(\frac{R_{q}}{r_{\oplus}}\right)^{5} \frac{\mathrm{GM}_{\oplus}^{2}}{r_{\oplus}^{2}}\left[\left(3 \cos ^{2} \phi-1\right) \hat{r}+\sin 2+\hat{\phi}\right] \tag{E.9-2}
\end{align*}
\]

There is a reaction force of \(\overrightarrow{-}\) acting on the Moon, so the acceleration of the Moon relative to the Earth is:
\(\vec{a}_{\oplus}=\frac{3}{2} k\left(\frac{R_{\Theta}}{r_{\oplus}}\right)^{5} \frac{G\left(M_{\oplus}+M_{Q}\right)}{r_{\oplus}^{2}}\left(\frac{M_{\oplus}}{M_{G}}\right)\left[\left(3 \cos ^{2} \phi-1\right) \hat{r}+\sin 2 \phi \hat{\phi}\right]\)
(E.9-3)

Since \(\phi\) is quite smail the coefficient of \(\hat{r}\) is about two, and using a value of \(k=0.03\) we find the maximum radial acceleration to be \(\sim 4 \times 10^{-2} \mathrm{~cm} / \mathrm{d}^{2}\). Due to the inverse seventh-power dependence on distance, the minimum radial acceleration is about half this value. The effect of these variations in the radial acceleration is to cause an advance of the argument of perigee at a rate of about \(6 \times 10^{-4}\) arc-sec/year. This slight advance of the line of apsides is not currently detectable with the lunar range data.

The effect of the lunar elasticity is included in our orbital model, but there is no allowance made for dissipation. The effect of dissipation is much smaller than the elastic effect discussed above. The amplitude of the optical librations is about 0.1 radian, so the maximum \(\phi\) that results is given by \(\phi_{\max } \sim \frac{1}{10 Q}\) radians. Thus a lunar \(Q \approx 20\) would cause a maximum transverse acceleration \(\sim 1 / 200\) the size of the radial acceleration. The only way in which an acceleration of this small magnitude could become detectable would be through a secular acceleration in longitude, in which case it would be hopelessly submerged within the effect of the Earth's tidal bulge.

TABLE 1
Harmonic decomposition of the elastic inertia tensor
\begin{tabular}{llllllllll}
p & q & r & S & \(\mathrm{C}_{11}\) & \(\mathrm{~S}_{12}\) & \(\mathrm{~S}_{13}\) & \(\mathrm{C}_{22}\) & \(\mathrm{C}_{23}\) & \(\mathrm{C}_{33}\) \\
0 & 0 & 0 & 0 & -3.025 & 0.0 & 0.0 & 0.852 & 0.0 & 2.174 \\
0 & 0 & 0 & 2 & -.079 & -.058 & 0.0 & 0.043 & 0.0 & 0.035 \\
0 & 0 & 2 & 0 & -.026 & 0.014 & 0.0 & 0.0 & 0.0 & 0.026 \\
1 & 0 & 0 & -2 & -.076 & 0.085 & 0.0 & 0.036 & 0.0 & 0.040 \\
1 & 0 & 0 & 0 & -.422 & -.426 & 0.0 & 0.211 & 0.0 & 0.211 \\
2 & 0 & 0 & 0 & -.058 & -.049 & 0.0 & 0.041 & 0.0 & 0.017 \\
0 & 0 & 1 & 0 & 0.0 & 0.0 & 0.455 & 0.0 & 0.0 & 0.0 \\
1 & 0 & 1 & 0 & 0.0 & 0.0 & 0.062 & 0.0 & -.025 & 0.0 \\
1 & 0 & -1 & 0 & 0.0 & 0.0 & -.009 & 0.0 & 0.025 & 0.0 \\
0 & 1 & 0 & -2 & -.005 & 0.004 & 0.0 & 0.0 & 0.0 & 0.0 \\
1 & -1 & 0 & 0 & -.004 & -.004 & 0.0 & 0.0 & 0.0 & 0.0 \\
1 & 0 & 2 & 0 & -.004 & 0.0 & 0.0 & 0.0 & 0.0 & 0.005 \\
1 & 1 & 0 & -2 & -.003 & 0.004 & 0.0 & 0.0 & 0.0 & 0.0 \\
1 & 1 & 0 & 0 & 0.004 & 0.003 & 0.0 & 0.0 & 0.0 & 0.0 \\
2 & 0 & 0 & -2 & 0.006 & 0.005 & 0.0 & -.008 & 0.0 & 0.0 \\
3 & 0 & 0 & 0 & -.006 & -.006 & 0.0 & 0.005 & 0.0 & 0.0 \\
0 & 1 & 0 & 0 & 0.0 & 0.014 & 0.0 & 0.0 & 0.0 & 0.0 \\
1 & 0 & 0 & 2 & 0.0 & -.014 & 0.0 & 0.012 & 0.0 & 0.006 \\
0 & 0 & 1 & -2 & 0.0 & 0.0 & -.007 & 0.0 & 0.0 & 0.0 \\
0 & 0 & 1 & 2 & 0.0 & 0.0 & .010 & 0.0 & -.004 & 0.0 \\
1 & 0 & -1 & -2 & 0.0 & 0.0 & -.012 & 0.0 & -.005 & 0.0 \\
2 & 0 & 1 & 0 & 0.0 & 0.0 & 0.007 & 0.0 & -.004 & 0.0
\end{tabular}

\section*{Notes}

Used in equation (II-24). Since the inertia tensor is symmetric, only the upper diagonal half is given. The tensor element coefficients not given \(\mathrm{S}_{11}, \mathrm{C}_{12}, \mathrm{C}_{13}, \mathrm{~S}_{22}\), \(S_{23}\), and \(S_{33}\) ) are zero for all arguments. The constant parts of the tidal and rotational deformations have not been removed. The units are \(4.848 \times 10^{-6} C\), where \(C\) is the Moon's polar moment of inertia.

TABLE 2

\section*{Comparisons of parameter estimates}

Constant-time-lag Constant-Q. Ferrari et al. dissipation model dissipation model
\begin{tabular}{|c|c|c|c|}
\hline \(M_{\odot} /\left(M_{\oplus}+M_{\mathbb{C}}\right) 3\) & \(900.529 \pm .002^{(3)}\) & 9900.529 \(\pm .002328\) & \(00.540 \pm .021\) \\
\hline \(\sin 2 \delta(1)\) & \(0.081 \pm .002\) & \(0.081 \pm .002\) & \(0.082 \pm .014\) \\
\hline \(B\left(\times 10^{6}\right)\) & \(631.3 \pm .4\) & \(631.0 \pm .4\) & \(631.59 \pm .13\) \\
\hline \(\gamma \quad 0 \quad(2)\) & \(227.92 \pm .06\) & \(227.56 \pm .07\) & \(228.02 \pm .10\) \\
\hline \(J_{3}\left(\times 10^{6}\right)\) & \(8.7 \pm .4\) & \(8.6 \pm .4\) & \(12.1 \pm 1.8\) \\
\hline \(\mathrm{C}_{31}{ }^{\prime \prime}\) & \(25.4 \pm 6.0\) & \(23.4 \pm 6.0\) & \(30.7 \pm 1.9\) \\
\hline \(\mathrm{C}_{32}{ }^{\prime \prime}\) & \(4.867 \pm .010\) & \(4.878 \pm .010\) & \(4.888 \pm .05\) \\
\hline \(\mathrm{C}_{33}{ }^{\prime \prime}\) & \(1.90 \pm .14\) & \(2.59 \pm .14\) & \(1.44 \pm .17\) \\
\hline \(\mathrm{S}_{31}{ }^{\prime \prime}\) & \(14.6 \pm 1.6\) & \(17.8 \pm 1.6\) & \(5.6 \pm 2.5\) \\
\hline \(\mathrm{S}_{32}{ }^{\prime \prime}\) & \(1.612 \pm .007\) & \(1.605 \pm .007\) & \(1.69 \pm .04\) \\
\hline \(\mathrm{S}_{33}\) & \(-0.90 \pm .07\) & \(0.79 \pm .07\) & \(-0.33 \pm .17\) \\
\hline \(k\) & \(0.020 \pm .003\) & \(0.026 \pm .003\) & \(0.022 \pm .013\) \\
\hline \(D \equiv k T\) & \(0.00401 \pm .00016\) & --- & \(0.0072 \pm .0010\) \\
\hline k/Q & --- & \(0.000953 \pm .000036\) & --- \\
\hline Q (inferred) & \(22 \pm 4\) & \(27 \pm 4\) & \(14+\frac{11}{9}\) \\
\hline
\end{tabular}

\section*{Notes}
(1) Ferrari et al. didn't estimate sin \(2 \delta\) directly, but rather the product \(k_{2} \delta\). The value listed in the table is computed using \(k_{2}=0.3\) for the Earth, which is the value used in our tidal friction model.
(2) Our estimate of \(\gamma\) has been modified to incorporate the constant part of the tidal bulge raised by the Earth.
(3) The errors given with our parameter estimates are the formal errors, while Ferrari et al. have scaled their errors to reflect possible systematic errors (see text).
(4) This was the postfit rms range residual when laser data alone were fit; with the addition of tracking data, their postfit rms range residual was 39 cm .


FIGURE 1. The Euler Angles define the orientation of the selenocentric principal axis system ( \(x_{i}\) ) relative to the inertial 1950.0 Earth equator-equinox system ( \(\xi_{i}\) ).```


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