CAUSAL STRUCTURES IN LIE GROUPS AND APPLICATIONS TO STABILITY OF DIFFERENTIAL EQUATIONS

by

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ABSTRACT

According to results of Kostant, a real simple Lie algebra \( \mathfrak{g} \) has an invariant convex cone if and only if \( \mathfrak{g} \) is Hermitian symmetric. We classify and explicitly describe such cones in the classical algebras, specifically, all such cones in \( \text{sp}(n,\mathbb{R}) \) and all open or closed invariant convex cones in \( \text{su}(p,q) \), \( \text{so}^*(2n) \), and \( \text{so}(2,n) \).

The universal covering group \( \tilde{G} \) receives a left- and right-invariant causal structure from any such cone. In the tube-type cases \( \tilde{G} \) is shown to be globally causal under all such structures, and likewise for certain of the structures in the other cases.

The open cones are found to consist of elliptic elements, each within a unique maximal compact subalgebra. This mapping of positive elliptics to maximal compacts (Kählerization) is shown to be real-analytic. The same results hold for infinite-dimensional analogues of \( \text{sp}(n,\mathbb{R}) \) and \( \text{so}^*(2n) \). In these two cases the Cayley transform is a causal mapping of an open subset of \( \mathfrak{g} \) into the group. Some of the mapping's other properties are collected.

Let \( \mathcal{S} = \{ g \in \text{Sp}(n,\mathbb{R}) : g \text{ within a unique maximal compact subgroup} \} \), an open set having \( 2^n \) connected components \( \mathcal{S}_j \). The inverse images of the \( \mathcal{S}_j \) in the universal covering \( \tilde{\text{Sp}}(n,\mathbb{R}) \) correspond naturally to the well-known regions of strong stability of linear periodic canonical differential equations. Two stability criteria are found for certain such equations of positive type, applicable in finite or infinite dimensions. One involves the operator bounds of the coefficients, the other an invariant Hilbert-Schmidt bound.

Thesis Supervisor: Irving E. Segal
Title: Professor of Mathematics
To the memory of

my grandfather,

Ernest W. Engelkemier

(1895 - 1978)

Let us then provisionally adopt the unispace cosmos, and seek to analyze free propagation over very long times, and its effect on the measurement of frequency of light.

I. E. Segal

Let us go then, you and I,
When the evening is spread out against the sky...

T. S. Eliot
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The MIT math dept. has been a friendly and intriguing place to work and converse, and it will not be easy to leave. And I have appreciated the Pleasant Folks of Cambridge.

Do I dare
Disturb the universe?...
I have measured out my life
with coffee spoons.

-Prufrock again
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INTRODUCTORY REMARKS

The main ideas of chapters one to five on the classification of invariant convex cones in the classical Hermitian symmetric Lie algebras, seem to be the following.

1) We use the orbit classification of Burgoyne and Cushman [2] from time to time, particularly to deal with the boundaries of the cones and so(2,n). These are the only really technical results we quote, and the above authors claim they "only need quote one result, Sylvester's theorem on the signature."! Also, the form of the orbits in the open maximal cones on su(p,q) could have been determined by the idea of Theorem 14.2, just as was done for the corresponding open maximal cones in so*(2n) (Theorem 17.3), independently of classification.

2) The quadratic mapping

\[ \mathbf{v} \in V \rightarrow \mathcal{A}(\mathbf{v},\mathbf{v}) \in \mathfrak{g}^* \]

where \( V \) is a fundamental representation space and \( \mathcal{A} \) an invariant symplectic form (and a rather different expression, \( \tau(\cdot,e,f) \), for so(2,n); see section 21) is shown to single out orbits with particular positivity properties (Theorems 6.1, 10.1, and 15.1), connecting the geometries of the Lie algebra and the space \( V \), and is essential to establishing the maximality of certain
explicitly described cones (Theorems 8.2, 12.3, 18.2).

3) The argument of sections 7, 11, and 16 seems well-known but appears in other context, such as the Klein-Gordon equation in Minkowski space

$$\Box \varphi + m^2 \varphi = 0 , \ m > 0 .$$

In section 7 the form $\mathcal{E}(\cdot, \cdot)$ is the energy norm

$$\langle \langle \varphi \rangle, (\varphi) \rangle_{E} = \int_{\mathbb{R}^3} \left\{ |\hat{\varphi}|^2 + m^2 |\varphi|^2 + |\nabla \varphi|^2 \right\} d\mathbf{x}$$

and $\mathcal{L}(\cdot, \cdot)$ is the Lorentz-invariant norm

$$\langle \langle \varphi \rangle, (\varphi) \rangle_{L} = \int_{\mathbb{R}^3} \left\{ |\varphi|^2 + |\nabla^{-1} \varphi|^2 \right\} d\mathbf{x}$$

where $C = (m^2 - \Delta)^{1/4}$ and say $\varphi, \hat{\varphi} \in C_0^\infty(\mathbb{R}^3)$. 

4) The "noncompact convexity" theorems 14.2, 17.3, 21.3, and possibly 9.2 for $\text{sp}(n, \mathbb{R})$ (a "discrete" version), are in the direction of generalizing to noncompact groups Horn's theorem for Hermitian matrices (see section 14) and Kostant's generalizations to maximal compact subgroups of semisimple Lie groups, within [16]. The classification of the open and closed invariant convex cones and all invariant convex cones for $\text{sp}(n, \mathbb{R})$ (Theorems 8.1, 14.4, 19.3, and 22.2) is a fairly immediate consequence, just as Horn's theorem easily classifies the invariant convex cones in $\text{u}(n)$.

Chapter six is a fairly direct application of the preceding chapters. The uniqueness and analyticity of
Kählerization (for \( \text{sp}(\mathcal{N}) \), at least), shown in chapter seven, 25.2, 25.4, and 27.2, are valid in infinite dimensions and, like chapter nine, suggested by problems in quantum field theory. Chapter eight is a rather non-systematic collection of properties of the mapping

\[
C: X \to \frac{I + X}{I - X}
\]

of the Lie algebra to the Lie group. One remark that should have been made there, is that all group elements in the range of the Cayley transform restricted to the cones \( \pm \mathbb{C}_1 \), described in Proposition 30.3, commute with a unique complex structure, by Theorem 25.2. The proof of the latter applies equally to \( \text{SO}^*(\mathcal{N}) \), defined in section 16.

One comment on notation: \( A \subseteq B \) means \( A \) is contained in \( B \) and \( A \neq B \).
1. Preliminaries on Convex Cones.

We need some standard facts about convex cones as, for example, in [6], to which we refer for proofs. In this section all vector spaces are finite-dimensional.

Definition. Let $\mathbb{E}$ be a real vector space. $C \subseteq \mathbb{E}$ is convex if $\lambda x + (1-\lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in [0, 1]$. $C \subseteq \mathbb{E}$ is a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. A non-trivial cone is a nonempty cone $C$ satisfying $\{0\} \neq C \neq \mathbb{E}$.

Elementary properties of convex cones are, for example: the closure or interior of a convex cone is also a convex cone. Less intuitive is the following

Lemma 1.1. A convex cone which is dense in $\mathbb{E}$ is equal to $\mathbb{E}$.

Let $\hat{\mathbb{E}}$ be the vector space dual to $\mathbb{E}$. There is a "separating hyperplane theorem" for convex cones.

Lemma 1.2. If $C$ is a closed convex cone in $\mathbb{E}$ and $x \notin C$, then there exists $f \in \hat{\mathbb{E}}$ such that $f(x) > 0$ and $f(y) \leq 0$ for all $y \in C$.

Corollary 1.3. A convex cone, not all of $\mathbb{E}$, is in some closed half-space.
It is convenient to introduce a real positive-definite scalar product $\langle \cdot, \cdot \rangle$ in $E$ to state the duality theorem for cones, but this is not strictly necessary as $\mathbb{R}^E \cong E$ canonically.

**Definition.** Given any cone $C$ in $E$, let $C^* = \{ y : \langle y, x \rangle \leq 0 \text{ for all } x \in C \}$. $C^*$ is a closed convex cone, called the dual cone of $C$.

**Theorem 1.4.** For any cone $C$ in $E$, $(C^*)^*$ is the closed convex hull of $C$. In particular, if $C$ is closed and convex, $(C^*)^* = C$.

**Proof.** It follows from the definitions and Lemma 1.2.

2. **Existence of the Minimal Cone.**

In this section we recall the results of Kostant on the existence of invariant convex cones in semisimple Lie algebras.

**Definition.** A causal cone in a real Lie algebra $\mathfrak{g}$ is a non-zero closed convex cone $C$ in $\mathfrak{g}$ invariant under the adjoint group of $\mathfrak{g}$ and satisfying $C \cap -C = \{0\}$.

**Proposition 2.1.** Any non-trivial invariant convex cone $C$ in a simple real Lie algebra satisfies $C \cap -C \neq \{0\}$.

**Proof.** $C \cap -C$ is an ideal in $\mathfrak{g}$ by the invariance.
Theorem 2.2 (Kostant). Let $G$ be a semisimple Lie group acting on a real finite-dimensional space $V$. Assume that $K$ is a maximal compact subgroup of $G$. Then there exists a non-trivial $G$-invariant convex cone $C$ in $V$ satisfying $C \cap -C \subseteq \{0\}$ iff $V$ has a non-zero $K$-invariant vector.

Proof. The proof is given in [19], but the ideas are useful later so we repeat them here. Let $C$ be a cone with the given properties. By Lemma 1.2, there exists $f \in \hat{V}$ such that $f(x) > 0$ for some $x \in C$ and $f(z) > 0$ for some $z \in C$. Then

$$w = \int_K k(z)dk$$

is $K$-invariant and $f(w) > 0$ so $w \neq 0$.

Conversely, let $w \neq 0$ be $K$-invariant. Let $k$ be the Lie algebra of $K$ and $\mathfrak{k} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. As $G$ is a matrix group, its complexification $G_c$ acts on $V_c = V + iV$, and any subgroup $G_u$ corresponding to $\mathfrak{k} + i\mathfrak{p}$ is compact. We can assume $K \subseteq G_u$, and $G_u$ leaves invariant some complex Hilbert structure $<\cdot,\cdot>$. All $X \in \mathfrak{k} + i\mathfrak{p}$ are skew-Hermitian on $V_c$, so all $X \in \mathfrak{p}$ are Hermitian and $\exp \mathfrak{p}$ is positive-definite Hermitian.

Any $g \in G$ can be written uniquely as $(\exp X)k$ for $X \in \mathfrak{p}$, $k \in K$. Thus $<gw, w> = <(\exp X)w, w> > 0$. Now $<gu, v> = <u, \Theta(g^{-1})v>$ $\forall g \in G$, $u, v \in V$, where $\Theta: G \to \Theta: (\exp X)k \to (\exp -X)k$ is the Cartan involution corresponding to $\Theta: X + Y \to X - Y$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$. Letting $C_0$ denote the convex cone generated by the $gw$ for $g \in G$, it is clear
that $C_o$ is a non-trivial $G$-invariant convex cone in $V$, and $\langle u,v \rangle > 0 \ \forall u,v \in C_o$.

**Corollary 2.3.** Let $\mathfrak{g}$ be a semisimple Lie algebra and $G$ the adjoint group of $\mathfrak{g}$. Let $K$ be any maximal compact subgroup of $G$, with Lie algebra $\mathfrak{k}$. Then there is a non-trivial $G$-invariant convex cone $C$ in $\mathfrak{g}$ satisfying $C \cap -C = \{0\}$ if $\mathfrak{k}$ has a non-trivial center.

**Proof.** Take $V = \mathfrak{g}$ in Theorem 2.2. As $K$ acts irreducibly on the $\mathfrak{p}$'s of the simple components of $\mathfrak{g}$, any non-zero $K$-fixed vector must be in $\mathfrak{k}$.

It is clear that any invariant convex cone in a semisimple $\mathfrak{g}$ is contained in the direct sum of invariant cones in the simple factors, and we restrict to the simple case from now on. It is well known that if $\mathfrak{g}$ is a real simple Lie algebra the dimension of the center $Z(\mathfrak{k})$ is either 0 or 1. Thus by Proposition 2.1, a simple $\mathfrak{g}$ admits a non-trivial invariant convex cone iff $\dim Z(\mathfrak{k}) = 1$. The integration argument above shows that in this case there are unique minimal causal cones $\pm C_o$. By the simplicity, the positive-definite $K$-invariant form used in the proof of Theorem 2.2 was on $\mathfrak{g}B_\theta(\cdot,\cdot) = -B(\cdot,\theta \cdot)$ up to a scalar, where $B$ is the killing form on $\mathfrak{g}$. We use later the fact that $B_\theta(X,Y) > 0 \ \forall X,Y \in \overline{C_o}$.

We see that no compact or complex simple Lie algebra admits a causal cone. By the classification the only
algebras that do are \( \mathfrak{sp}(n, \mathbb{R}) \) \((n \geq 1)\), \( \mathfrak{su}(p,q) \) \((p \geq q \geq 1)\), \( \mathfrak{so}^*(2n) \) \((n \geq 3)\), \( \mathfrak{so}(2,n) \) \((n \geq 3)\), and two exceptional algebras. We list information about the dimensions of these algebras below, using the notation of \([9]\). Recall that \( \text{char} \, \mathfrak{g} = \text{dim} \, \mathfrak{p} - \text{dim} \, \mathfrak{k} \) where \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) is any Cartan decomposition, and \( \text{rank} \, \mathfrak{g} \) is the dimension of any maximal abelian subalgebra of \( \mathfrak{p} \).

\[
\begin{align*}
\mathfrak{sp}(n, \mathbb{R}) & \quad \text{dim} \, \mathfrak{k} = n^2, \text{dim} \, \mathfrak{p} = n^2 + n; \text{rank} \, \mathfrak{g} = n; \text{char} \, \mathfrak{g} > 0 \forall n. \\
\mathfrak{su}(p,q) & \quad \text{dim} \, \mathfrak{k} = p^2 + q^2 - 1, \text{dim} \, \mathfrak{p} = 2pq; \text{rank} \, \mathfrak{g} = q; \text{char} \, \mathfrak{g} > 0 \text{ when } p = q, \text{ char} \, \mathfrak{g} = 0 \text{ when } p = q + 1, \text{ and char} \, \mathfrak{g} < 0 \text{ otherwise.} \\
\mathfrak{so}^*(2n) & \quad \text{dim} \, \mathfrak{k} = n^2, \text{dim} \, \mathfrak{p} = n^2 - n; \text{rank} \, \mathfrak{g} = \lfloor n/2 \rfloor; \text{char} \, \mathfrak{g} < 0 \forall n. \\
\mathfrak{so}(2,n) & \quad \text{dim} \, \mathfrak{k} = \frac{1}{2}n(n-1) + 1, \text{dim} \, \mathfrak{p} = 2n; \text{rank} \, \mathfrak{g} = 2; \text{char} \, \mathfrak{g} > 0 \text{ for } n = 3, 4 \text{ and char} \, \mathfrak{g} < 0 \text{ otherwise.} \\
\mathfrak{e}_{	ext{III}} & \quad \text{(a real form of } \mathfrak{e}_6 \text{)} \quad \text{dim} \, \mathfrak{k} = 46, \text{dim} \, \mathfrak{p} = 32; \text{rank} \, \mathfrak{g} = 2. \\
\mathfrak{e}_{	ext{VII}} & \quad \text{(a real form of } \mathfrak{e}_7 \text{)} \quad \text{dim} \, \mathfrak{k} = 79, \text{dim} \, \mathfrak{p} = 54; \text{rank} \, \mathfrak{g} = 3.
\end{align*}
\]

A glance at the list on p. 346 of \([9]\) proves the following Corollary. A complex simple Lie algebra \( \mathfrak{g} \) admits a non-compact real form having a causal cone iff the adjoint group corresponding to a compact real form of \( \mathfrak{g} \) is not simply connected.

Let $\mathfrak{g}$ be one of the Lie algebras in the list in Section 2, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a fixed Cartan decomposition. Let $\mathfrak{g}^\mathbb{C}$ be its complexification, and set $\mathfrak{k}^\mathbb{C} = \mathfrak{k} + i\mathfrak{k}$, $\mathfrak{p}^\mathbb{C} = \mathfrak{p} + i\mathfrak{p}$. Let $c$ be the one-dimensional center of $\mathfrak{k}$, $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{k}$, and set $\mathfrak{h}^\mathbb{C} = \mathfrak{h} + i\mathfrak{h}$. Then $c \subseteq \mathfrak{h} \subseteq \mathfrak{h}^\mathbb{C}$. We have $\mathfrak{k} = \mathfrak{k}_s \oplus c$, where $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ is a compact semisimple Lie algebra.

We recount the notation and analysis in [9], p. 312-16. It is shown there that $\mathfrak{h}^\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Let $\Delta$ denote the set of roots of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}^\mathbb{C}$. (A root is always nonzero.) If

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

is the root space decomposition, then each $\mathfrak{g}_\alpha$ satisfies either $\mathfrak{g}_\alpha \subseteq \mathfrak{k}^\mathbb{C}$, in which case the root $\alpha$ is called compact, or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}^\mathbb{C}$, in which case $\alpha$ is called noncompact. We have the decompositions

$$\mathfrak{k}^\mathbb{C} = \mathfrak{h}^\mathbb{C} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$
$$\mathfrak{p}^\mathbb{C} = \sum_{\beta \in \Xi} \mathfrak{g}_\beta$$

$\alpha$ running over the compact roots and $\beta$ over the noncompact roots.
All roots are real-valued on $\mathfrak{h}$, and a root is compact iff it vanishes on $\mathfrak{c}$. Let $B$ be the restriction of the killing form to $\mathfrak{h}$, a real positive-definite inner product. (For our purposes, any positive multiple of $B$ would serve equally as well.)

Choose compatible orderings in the real duals of $\mathfrak{i}c$ and $\mathfrak{i}h$. $\forall \alpha \in \Delta$ take $H_\alpha \in \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$ and $B(H_\alpha, \cdot)$ proportional to $\alpha$. Also take $E_\alpha \in \mathfrak{c}$ such that $i(E_\alpha + E_\alpha^{-})$, $E_\alpha - E_\alpha^{-} \in \mathfrak{k} + i\mathfrak{p}$, and $[E_\alpha, E_\alpha^{-}] = H_\alpha \forall \alpha \in \Delta$. Let $Q_+$ be the set of positive and noncompact roots. As in [15] $\forall \alpha \in Q_+$ set

$$X_\alpha = E_\alpha + E_\alpha^{-}$$
$$Y_\alpha = -i(E_\alpha - E_\alpha^{-})$$

These vectors are a basis for $\mathfrak{p}$. We have

$$[H_\alpha, X_\alpha] = 2iY_\alpha$$
$$[H_\alpha, Y_\alpha] = -2iX_\alpha$$
$$[X_\alpha, Y_\alpha] = 2iH_\alpha$$

for all $\alpha \in Q_+$.

Let $\Delta^+ \subset \Delta$ be the positive roots with respect to the ordering, and let $\Delta^+_0 = \{\alpha_1, \ldots, \alpha_p\}$ be the simple positive roots. The $\alpha_i$ are a basis for the dual of $\mathfrak{h}$, and any $\alpha \in \Delta^+$ is a nonnegative integral linear
combination of the \( \alpha_1 \). By the compatibility of the ordering, \( \exists Z \in c \) such that \( \alpha(iZ) > 0 \) iff \( \alpha \) is noncompact and positive.

The following is well known.

**Proposition 3.1.** \( \alpha(iZ) = \beta(iZ) \forall \alpha, \beta \in Q_+ \), and all but exactly one of the \( \alpha_i \in \Delta_0^+ \) are compact.

**Proof.** As \( [p, p] \leq k \), if \( \alpha, \beta \) are noncompact roots,

\[
0 = [Z, [E_\alpha, E_\beta]] = (\alpha + \beta)(Z)[E_\alpha, E_\beta]. \]

Thus \( \alpha + \beta \in \Delta \) implies \( (\alpha + \beta)(Z) = 0 \). We recall the lemma on root systems ([13, p. 45]): \( B(\alpha, \beta) < 0 \) implies \( \alpha + \beta \) is a root (so \( B(\alpha, \beta) > 0 \) implies \( \alpha - \beta \) is a root). Therefore \( |\alpha(Z)| \neq |\beta(Z)| \) implies \( \alpha \perp \beta \).

Thus one can put the noncompact roots into mutually perpendicular subspaces, each singled out by a particular value of \( |\alpha(Z)| \). As shown above, root spaces corresponding to different values must commute (and recall \( [E_\alpha, E_\beta] = 0 \) for \( \alpha, \beta \in Q_+ \)).

We wish to show that each compact root \( \alpha \) is in one of these subspaces. Now there must be some noncompact \( \beta_1 \) such that \( \alpha + \beta_1 \) is a root; otherwise \( [E_\alpha, p] = 0 \) and \( \{X \in k: [X, p] = 0\} \) would be a non-trivial ideal in \( \mathfrak{g} \), contradicting simplicity. Thus \( \alpha + \beta_1 = \beta_2 \) for
noncompact $\beta_1, \beta_2$, $\alpha = \beta_2 - \beta_1$ is a root, and $\beta_1$ and $\beta_2$ must be in the same subspace.

The root system is then decomposed into non-trivial mutually orthogonal systems, contradicting simplicity, unless all $|\alpha(Z)| = |\beta(Z)|$ for $\alpha, \beta$ noncompact. Assume then that $\alpha(iZ) = 1 \forall \alpha \in Q_+$. Let $\beta_1, \ldots, \beta_n$ be the noncompact roots in $\Delta_o^+$. Any noncompact root involves one $\beta_i$ but no others, as $(\beta_i + \beta_j)(iZ) = 2$. If $\gamma, \delta$ are noncompact and involve different $\beta_i$'s, then $[E_\gamma, E_\delta] = 0$, and the noncompact roots are again partitioned non-trivially unless $n = 1$.

If $\alpha(iZ) = 1 \forall \alpha \in Q_+$, we have $[Z, E_\alpha] = -iE_\alpha$ and $[Z, E_{-\alpha}] = iE_{-\alpha}$, so $[Z, X_\alpha] = Y_\alpha$ and $[Z, Y_\alpha] = -X_\alpha$ $\forall \alpha \in Q_+$. Thus $\text{ad } Z : \mathfrak{p} \rightarrow \mathfrak{p}$ is a complex structure on $\mathfrak{p}$.

In [9] it is shown that there exists a set $\Sigma_0 \subseteq Q_+$ such that

$$\alpha = \sum_{\delta \in \Sigma_0} RX_\delta$$

is maximal abelian in $\mathfrak{p}$, and $\gamma \pm \delta$ are not roots $\forall \gamma, \delta \in \Sigma_0$. Let $\mathfrak{h}^-$ be the real span of $iH_\alpha$, $\alpha \in \Sigma_0$. Then $\mathfrak{h}^- \subseteq \mathfrak{h}$ and these $iH_\alpha$ are orthogonal, so $\dim \mathfrak{h}^- = \dim \alpha$. Let $\mathfrak{h}^+$ be the orthocomplement of $\mathfrak{h}^-$ in $\mathfrak{h}$ with respect to the killing form. In [15] it is
shown that $h^+$ is the centralizer of $\alpha$ in $\mathfrak{g}$, and that the complexification of $h^+ \oplus \alpha$ is also a Cartan subalgebra of $\mathfrak{g}^c$. It may be shown that the centralizer of $\alpha$ in $\mathfrak{k}$, usually denoted $\mathfrak{m}$, is spanned by $h^+$ and all $i(E_\alpha + E_{-\alpha})$, $E_\alpha - E_{-\alpha}$ as above, where $\alpha$ is compact and $iH_\alpha \in h^+$.

We introduce one more bit of notation; as is standard, let $Z_0 = -(1/2) \sum_{\alpha \in \Sigma_0} H_\alpha \in h^-$. Then

$$[Z - Z_0, X_\alpha] = Y_\alpha - [Z_0, X_\alpha] = Y_\alpha - Y_\alpha = 0,$$

so $Z - Z_0 \in h^+$, and we have decomposed

$$Z = (Z - Z_0) + Z_0 \in h^+ \oplus h^-.$$

Only in the symplectic case is $h^+ = 0$, but often $Z = Z_0$ even if $h^+ \neq 0$. Whether $Z = Z_0$ or not depends only on $\mathfrak{g}$ and not on the choices we have made, as is proven in [15]:

**Proposition 3.8.** $Z = Z_0$ iff $Z$ is contained in a three-dimensional simple subalgebra of $\mathfrak{g}$.

Finally, we compute the action of $A = \exp \alpha$ on $Z$. (The action of $A$ on $h$ is used a great deal in
the next several sections.) Let $X = \sum_{\alpha \in \Sigma} t^a x_\alpha \in \mathfrak{a}$. The relations (1) imply

$$\text{Ad}(\exp X)(Z) = Z - Z_0 + \text{Ad}(\exp X)(-(i/2) \sum_{\alpha \in \Sigma} H_\alpha)$$

$$= Z - Z_0 - (i/2) \sum_{\alpha \in \Sigma_0} (\cosh 2t^a) H_\alpha$$

$$- (1/2) \sum_{\alpha \in \Sigma_1} (\sinh 2t^a) Y_\alpha.$$

**Corollary 3.2.** Any invariant convex cone in $\mathfrak{a}$ containing $Z$ contains all

$$c_0(Z - Z_0) + \sum_{\alpha \in \Sigma_0} c_\alpha (-iH_\alpha) \in \mathfrak{h}$$

for $c_0, c_\alpha$ satisfying $0 < c_0 < 2 \min_{\alpha \in \Sigma_0} c_\alpha$.

**Proof.** Average over positive and negative $t^a$ in the above.


To classify some of the invariant convex cones in the classical Hermitian Lie algebras we need information about their orbits, and the work [2] of Burgoyne and Cushman, from the point of view of the lowest-dimensional representation, is ideal for our purposes. The exceptional algebras will probably require a more abstract approach,
which is currently being suggested by the similarities and analogies seen in the following treatments of the four classical series.

In this section we review their description of the classical linear groups, and in the next describe their classification of the orbits. It would not simplify matters at this stage to restrict to the Hermitian case, and we refrain from doing so until the end of section five.

Let \( V \) be a finite-dimensional complex vector space, always non-zero. If \( A: V \to V \) is linear, \( \det A \) denotes the usual complex determinant. Let \( G = \text{GL}(V) \). Let \( \sigma \) be any conjugate-linear operator \( \sigma: V \to V \) such that \( \sigma^2 = \pm I \). \( \sigma \) induces an automorphism of \( G \) of order two.

We consider the real groups \( H_\sigma = \{ g \in H : g\sigma = \sigma g \} \) for \( H = G \) and certain special proper subgroups \( H \) of \( G \). Such \( \sigma \) and \( \sigma' \) are called equivalent with respect to \( H \) if \( \sigma = g\sigma'g^{-1} \) for some \( g \in H \). If this happens \( H_\sigma \) and \( H_{\sigma'} \) are isomorphic.

The two cases \( \sigma^2 = \pm I \) induces quite different structures on \( V \).

(a) \( \sigma^2 = I \). \( V_\sigma^+ = \{ v \in V : \sigma v = v \} \) is a real form of \( V \), and any \( g \in H_\sigma \) leaves \( V_\sigma^+ \) invariant.

(b) \( \sigma^2 = -I \). In this case \( V \) must have even dimension, and \( V \) receives the structure of a quaternionic vector space as follows. Let
\[ \mathbb{Q} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\} \]
denotes the quaternions as usual. If \( a + \beta j \in \mathbb{Q} \) for \( a, \beta \in \mathbb{C} \) and \( v \in V \), define \((a + \beta j)v = av + \beta \sigma(v)\). (Note \( aj = j\bar{a} \) for \( a \in \mathbb{C} \).) We use the anti-automorphism
\[ a + bi + cj + dk \in \mathbb{Q} \rightarrow a + bi - cj + dk \in \mathbb{Q}, \]
denoted \( \lambda \rightarrow \lambda^q \); it satisfies \((\lambda \mu)^q = \mu^q \lambda^q \) \( \forall \lambda, \mu \in \mathbb{Q} \). Note \( \lambda \lambda^q \) is not necessarily real, and usually \( \lambda \lambda^q \neq \lambda^q \lambda \).
Again, \( \det g \in \mathbb{R} \) for any \( g \in H_\sigma \).

**Real forms of GL(V).**

The first case involves no \( \sigma \); we assume only a nondegenerate Hermitian form \( \tau_*(\cdot, \cdot) \) on \( V \), and define
\[ G_{\tau_*} = \{g \in G: \tau_*(g\cdot, g\cdot) = \tau_*(\cdot, \cdot)\} \] \( |\det g| = 1 \ \forall g \in G_{\tau_*} \), and \( G_{\tau_*} \) is connected. The subgroup of \( \det g = 1 \) is connected and isomorphic to an \( SU(p, q) \).

(a) \( \sigma^2 = I \). All \( \sigma \) are equivalent with respect to \( G \).
\[ G_{\sigma} \cong GL(n, \mathbb{R}) \] where \( n = \dim V \). \( G_{\sigma} \) has two components, and \( SL(n, \mathbb{R}) \) is connected.

(b) \( \sigma^2 = -I \). All \( \sigma \) are equivalent with respect to \( G \).
\[ G_{\sigma} \cong U^*(2n) \], where \( 2n = \dim V \). \( G_{\sigma} \) has two components, and is isomorphic to the general linear group on a quaternionic vector space. \( SU^*(2n) \) is connected.
Real Forms of $O(V, \tau)$.

Let $\tau$ be a nondegenerate symmetric complex-bilinear form on $V$. We consider only those $\sigma$ as above such that $\tau(\sigma \cdot, \sigma \cdot) = \overline{\tau(\cdot, \cdot)}$. Let $H = \{ g \in G : \tau(g \cdot, g \cdot) = \tau(\cdot, \cdot) \}$. 

(a) $\sigma^2 = I$. There are several equivalence classes of the $\sigma$'s with respect to $H$, depending on the (real) signature of $\tau$ restricted to $V^+$, denoted $\tau_+$. $H_\sigma$ has either two or four components, and $\det g = \pm 1$ for each $g \in H_\sigma$. Each $H_\sigma$ is isomorphic to some $O(p, q)$.

(b) $\sigma^2 = -I$. Let $\dim V = 2n$. $V$ becomes a quaternionic vector space as above, and defining $\tau_-(u, v) = \tau(u, v) + \tau(u, \sigma v)j$ for $u, v \in V$, $\tau_-$ is a nondegenerate $\mathbb{Q}$-valued form on $V$. It satisfies

$$(1) \quad \tau_-(\lambda u, \mu v) = \lambda \tau_-(u, v)\mu^q$$

and $\tau_-(u, v) = \tau_-(v, u)^q$ for all $u, v \in V$, $\lambda, \mu \in \mathbb{Q}$. There is one equivalence class for $\sigma$, and $\mathbb{Q}$-bases $\{e_i\}$ exist for $V$ such that $\tau_-(e_i, e_j) = \delta_{ij}$ [5]. Each $g \in H_\sigma$ automatically has determinant 1, just as for the symplectic case, and $H_\sigma \simeq SO^*(2n)$ is connected.

Real Forms of $Sp(V, \tau)$.

Let $\tau$ be a nondegenerate anti-symmetric complex
bilinear form on $V$. $\dim V$ is necessarily even. We consider only those $\sigma$ such that $\tau(\sigma, \sigma) = \overline{\tau(\sigma, \sigma)}$.

Let $H = \{ g \in G : \tau(g \cdot, g \cdot) = \tau(\cdot, \cdot) \}$.

(a) $\sigma^2 = I$. There is a unique equivalence class for $\sigma$ with respect to $H$, and $H_\sigma \cong \text{Sp}(n, \mathbb{R})$ is connected. Each $g \in H_\sigma$ has determinant 1.

(b) $\sigma^2 = -I$. Again $V$ becomes a quaternionic vector space, and $\tau_-$ is defined on $V$ by the same expression as above. (2) still holds, but now $\tau_-(u, v) = -\tau(v, u)$, and there are various equivalence classes for $\sigma$ with respect to $H$, classified by the signature of $\tau_-$. That is, there are $\mathbb{Q}$-bases $\{e_r\}$ such that $\tau_-(e_r, e_s) = \pm \delta_{rs} j$. Each $g \in H_\sigma$ has determinant 1, and $H_\sigma \cong \text{Sp}(p, q)$ is connected.

5. **Classification of the Orbits.** (Burgoyne and Cushman)

One may designate any one of the groups (real or complex) of the previous section by $G(V, \sigma, \tau)$, using the notation there. The Lie algebra of $G(V, \sigma, \tau)$, written $L(V, \sigma, \tau)$, is just those linear endomorphisms of $V$ commuting with $\sigma$ and skew with respect to $\tau$. However, as in [2], either $\tau$ or $\sigma$ may not actually occur in the definition of the group or algebra. Moreover, $\tau$ may denote $\tau_*$, in which case $\sigma$ is absent.
There is the notion of a type $\Delta$, an equivalence class of pairs $(A,V)$ where $A \in L(V,\sigma,\tau)$, under the obvious notation of equivalence. $\dim \Delta (= \dim V$ for all $(A,V) \in \Delta$) is well defined. The connection between types and orbits is the following

**Proposition 5.1.** Let $A,B \in L(V,\sigma,\tau)$. Then there exists $g \in G(V,\sigma,\tau)$ such that $g^{-1}Ag = B$ iff $(A,V)$ and $(B,V)$ belong to the same type.

The sum of two types is defined to be just the obvious $\tau$-orthogonal, $\sigma$-invariant direct sum. A type $\Delta$ is indecomposable if it cannot be written as the sum of two other types.

Given any $A \in L(V,\sigma,\tau)$, one can write it uniquely as $A = S + N$, where $S,N \in L(V,\sigma,\tau)$, $S$ is semisimple, $N$ is nilpotent, and $SN = NS$.

**Definitions.** 1) Let $m \geq 0$ be the unique integer such that $N^m \neq 0$ and $N^{m+1} = 0$ in the above. $m$ is called the height of $(A,V)$, and the notation $ht \Delta$ for any type $\Delta$ is well defined.

2) Let $K = \ker N^m$; then $K \supseteq NV$. If $K = NV$, we say $(A,V)$ is uniform, and clearly one can speak of uniform types.

3) If $ht \Delta = 0$ call $\Delta$ a semisimple type. A semisimple type is uniform.
There is a natural mapping of uniform types to semisimple types, defined as follows. Let $\Delta$ be uniform and $m = \text{ht} \Delta$. If $(A, V) \in \Delta$ put $\overline{V} = V/NV$, and for $v \in V$ put $\overline{v} = v + NV$. Define $\overline{A}$, $\overline{\sigma}$, and $\overline{\tau}$ on $\overline{V}$ by $\overline{A} \overline{v} = A \overline{v}$, $\overline{\sigma} \overline{v} = \overline{\sigma v}$, and $\overline{\tau}(\overline{u}, \overline{v}) = \tau(u, N^m v)$. Since $\tau$ is nondegenerate on $V$ and $(A, V)$ is uniform, $\overline{\tau}$ is nondegenerate on $\overline{V}$. $G(\overline{V}, \overline{\sigma}, \overline{\tau})$ and $L(\overline{V}, \overline{\sigma}, \overline{\tau})$ are well defined, and $A \in L(\overline{V}, \overline{\sigma}, \overline{\tau})$. Let $\overline{\Delta}$ denote the type containing $(\overline{A}, \overline{V})$; $\overline{\Delta}$ is semisimple. Note $\overline{\Delta}$ has nothing to do with complex conjugation in $G$.

Remark. $G(\overline{V}, \overline{\sigma}, \overline{\tau})$ may be a group in a different class than $G(V, \sigma, \tau)$: if $\tau$ is complex bilinear and $\tau(u, v) = \lambda \tau(v, u)$, $\lambda = \pm 1$, then $\overline{\tau}$ satisfies $\overline{\tau}(\overline{u}, \overline{v}) = \lambda(-1)^m \overline{\tau}(\overline{v}, \overline{u})$. If $\tau$ denotes $\tau_*$ one can assume that $\overline{\tau}$ is again an Hermitian form by replacing $\overline{\tau}$ by $i \overline{\tau}$ if necessary.

They prove the following results.

**Theorem 5.2.** The decomposition of a type $\Delta = \Delta_1 + \ldots + \Delta_s$ into indecomposable types (which clearly exists) is unique.

**Proposition 5.3.** If $\Delta$ is uniform it is uniquely determined by $\text{ht} \Delta$ and $\overline{\Delta}$.

**Proposition 5.4.** If $\Delta$ is indecomposable then $\Delta$ is uniform and $\overline{\Delta}$ is indecomposable.

We next give the simple construction recovering a
uniform type $\Delta$ from $\text{ht } \Delta$ and the semisimple type $\bar{\Delta}$, as in Proposition 5.3. Let $(S,E) \in \bar{\Delta}$ and $S \in \text{L}(E,\sigma,\bar{\tau})$. Let $V$ be the direct sum of $m+1$ copies of $E$, written conveniently as $E + NE + \ldots + N^m E$. Extend $\sigma$ and $S$ to $V$, acting componentwise on this decomposition, and let $N$ act on $V$ in the obvious way. Then $N^m \neq 0$, $N^{m+1} = 0$, and $\ker N^m = NV$. To define $\tau$ on $V$, let $\tau(N^r E, N^s E) = 0$ unless $r+s = m$, and in that case let $\tau(N^r e_1, N^s e_2) = (-1)^{r+s}(e_1, e_2) \forall e_1, e_2 \in E$. Then clearly $G(V, \sigma, \tau)$ is one of the above groups, $A = S+N \in \text{L}(V, \sigma, \tau)$, and $(A, V) \in \Delta$.

By the above, the problem of the classification of the orbits under the adjoint group of any of these algebras reduces to the determination of the semisimple indecomposable types, at least in the case where $G$ is connected. This description is straightforward for the complex groups and the $\text{U}(p,q)$. The results are as follows. Let $\Delta$ be a semisimple indecomposable type for either a complex $G$ or $\text{U}(p,q)$ and $(S,W) \in \Delta$.

$G = \text{GL}(W)$. $W$ is one-dimensional and $\Delta$ is determined by a single eigenvalue $\zeta \in \mathbb{C}$. Denote this type by $\Delta(\zeta)$.

$G = \text{O}(W, \tau)$. If $S = 0$ $W$ is one-dimensional. There is a basis element $e$ such that $\tau(e, e) = 1$. Denote this type by $\Delta(0)$. 
If \( S \neq 0 \) the set of eigenvalues for \( S \) is
\[ \{ \zeta, -\zeta \} \]
for some \( 0 \neq \zeta \in \mathbb{C} \). \( W \) is two-
dimensional and there is a basis \( \{ e, f \} \) such
that \( Se = \zeta e, Sf = -\zeta f \), \( \tau(e,e) = \tau(f,f) = 0 \)
and \( \tau(e,f) = 1 \). Denote this type by \( \Delta(\zeta, -\zeta) \).

**\( G = \text{Sp}(W, \tau) \).** Whether \( S \neq 0 \) or not, \( \dim W = 2 \), and there
is a basis \( \{ e, f \} \) and \( \zeta \in \mathbb{C} \) such that
\( Se = \zeta e, Sf = -\zeta f \), and \( \tau(e,f) = 1 \). Denote
this type also by \( \Delta(\zeta, -\zeta) \).

**\( G = \text{U}(p, q) \).** \( \dim W \) is either 2 or 1. When \( \dim W = 2 \)
there is a basis \( \{ e, f \} \) and \( \zeta \in \mathbb{C} \) such that
\( \zeta \neq -\zeta \) where \( \tau_*(e,e) = \tau_*(f,f) = 0 \),
\( \tau_*(e,f) = 1 \), and \( Se = \zeta e, Sf = -\zeta f \). Denote
this type by \( \Delta(\zeta, -\zeta) \).

When \( \dim W = 1 \) there is \( \zeta \in \mathbb{C} \) such that
\( \zeta = -\zeta \) and a basis element \( e \) such that
\( Se = \zeta e \) and \( \tau_*(e,e) = \pm 1 \). The two signs
give different types. Denote these by \( \Delta^\pm(\zeta) \).

For the real groups \( G(V, \sigma, \tau) \) one proceeds as
follows. An indecomposable semisimple type \( \Delta \) for
\( G(V, \sigma, \tau) \) clearly gives rise to a type \( \Delta^c \) for the corre-
sponding complex group \( G(V, \tau) \); just omit \( \sigma \). Let
\( (S,V) \in \Delta \). \( \Delta^c \) is a sum of semisimple indecomposable types
for \( G(V, \tau) \), so let \( \Delta^c_1 \) be an indecomposable component
of \( \Delta^c \). Let \( (S,W) \in \Delta^c_1 \) where \( W \subseteq V \). Clearly \( (S, \sigma W) \)
is also a type for $G(V, \tau)$, and we have either $\sigma W = W$ or $\sigma W \cap W = \{0\}$. Since $\sigma^2 = \pm I$ and $\Delta$ is indecomposable we have three possible decompositions for $\Delta^c$:

(a) $\Delta^c = \Delta^c_1 + \sigma \Delta^c_1$ and $\Delta^c_1 \neq \sigma \Delta^c_1$;
(b) $\Delta^c = \Delta^c_1 + \sigma \Delta^c_1$ and $\Delta^c_1 = \sigma \Delta^c_1$;
(c) $\Delta^c = \Delta^c_1$ and $\Delta^c_1 = \sigma \Delta^c_1$.

Let $\text{eig} \Delta^c_1$ denote the set of eigenvalues of $S$ on $W$. They prove the following

**Lemma A.1.** (a) occurs iff $\text{eig} \Delta^c_1 \neq \text{eig} \Delta^c_1$.

**Lemma A.2.** Suppose $S \neq 0$. Then (b) occurs iff $\sigma^2 = -I$ and all elements of $\text{eig} \Delta^c_1$ are real.

We now consider each of the six classes of fixed-point groups and indicate the classification of and notation for the semisimple indecomposable types, based mostly on Lemmas A.1 and A.2.

$G = GL(V)$

$\sigma^2 = I$. Let $\Delta^c = \Delta(\zeta)$.

1) $\zeta \neq \zeta$; (a) by A.1; type $\Delta(\zeta, \zeta)$.
2) $\zeta = \zeta \neq 0$; (c) by A.1 and A.2; type $\Delta(\zeta)$.
3) $\zeta = 0$; (c) because if (b) and $W = \mathcal{G}e$, $\mathcal{G}(e+\sigma e)$ is $\sigma$-invariant, contrary to indecomposability; type $\Delta(0)$.

$\sigma^2 = -I$. Let $\Delta^c = \Delta(\zeta)$.

1) $\zeta \neq \zeta$; (a) by A.1; type $\Delta(\zeta, \zeta)$. 

2) \( \zeta = \xi \neq 0 \); (b) by A.2; type \( \Delta(\xi,\xi) \).

3) \( \zeta = 0 \); (b) as \( \dim V \) must be even;
type \( \Delta(0,0) \).

\[ G = \text{O}(V, \tau) \]

\( \sigma^2 = I \). Let \( \Delta_1^c = \Delta(0) \).

1) Clearly (c); \( \text{sig } \tau_+ = (l,0) \) or \( (0,1) \);
two types \( \Delta^\pm(0) \).

Let \( \Delta_1^c = \Delta(\xi,-\xi) \), \( \xi \neq 0 \).

2) \( \zeta \neq \pm \xi \); (a) by A.1; \( \text{sig } \tau_+ = (2,2) \); type
\( \Delta(\xi,-\xi,\zeta,-\zeta) \).

3) \( \zeta = \xi \); (c) by A.1, A.2; \( \text{sig } \tau_+ = (1,1) \);
type \( \Delta(\xi,-\xi) \).

4) \( \zeta = -\xi \); again (c); \( \text{sig } \tau_+ = (2,0) \) or \( (0,2) \);
two types \( \Delta^\pm(\xi,-\xi) \).

\[ \sigma^2 = -I \]. Let \( \Delta_1^c = \Delta(0) \).

1) (b) as \( V \) must be even-dimensional; type
\( \Delta(0,0) \).

Let \( \Delta_1^c = \Delta(\xi,-\xi) \), \( \xi \neq 0 \).

2) \( \zeta \neq \pm \xi \); (a) by A.1; type \( \Delta(\xi,-\xi,\zeta,-\zeta) \).

3) \( \zeta = \xi \); (b) by A.2; type \( \Delta(\xi,-\xi,\zeta,-\zeta) \).

4) \( \zeta = -\xi \); (c) by A.1, A.2; two types \( \Delta^\pm(\xi,-\xi) \)
depending on the signature of \( \tau_-(\xi,-\xi) \)
(\( V \) is one-dimensional over \( \mathbb{Q} \).

\[ G = \text{Sp}(V, \tau) \]

\( \sigma^2 = I \). Let \( \Delta_1^c = \Delta(\xi,-\xi) \).
1) $\zeta \neq \pm \zeta$; (a) by A.1; type $\Delta(\zeta, -\zeta, \zeta, -\zeta)$.

2) $\zeta = \zeta \neq 0$; (c) by A.1, A.2; type $\Delta(\zeta, -\zeta)$.

3) $\zeta = -\zeta \neq 0$; (c) by A.1, A.2; $\tau(S_{-}, \cdot)$ either positive or negative definite on $V_{\sigma}^+$; two types $\Delta^\pm(\zeta, -\zeta)$.

4) $\zeta = 0$; clearly (c); type $\Delta(0,0)$.

Let $\sigma^2 = -I$. Let $\Delta^c = \Delta(\zeta, -\zeta)$.

1) $\zeta \neq \pm \zeta$; (a) by A.1; type $\Delta(\zeta, -\zeta, \zeta, -\zeta)$.

2) $\zeta = \zeta \neq 0$; (b) by A.2; type $\Delta(\zeta, -\zeta, \zeta, -\zeta)$.

3) $\zeta = -\zeta \neq 0$; (c) by A.1, A.2; two types $\Delta^\pm(\zeta, -\zeta)$ depending on $\operatorname{sig} \tau(-, \cdot)$.

4) $\zeta = 0$; clearly (c); again two types $\Delta^\pm(0,0)$.

This information is summarized in the following table taken from [2].
The Semisimple Indecomposable Types

<table>
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<th>$r$</th>
<th>Type</th>
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<td></td>
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<td>$\Delta(\xi)$</td>
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</tr>
<tr>
<td></td>
<td>alt</td>
<td>$\Delta(0, 0)$</td>
<td></td>
</tr>
<tr>
<td>+I</td>
<td>alt</td>
<td>$\Delta(\xi, -\xi, \xi, -\xi)$</td>
<td>$\xi \neq \pm \xi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta(\xi, -\xi)$</td>
<td>$\xi = \xi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta(\xi, -\xi)$</td>
<td>$\xi = -\xi \neq 0$</td>
</tr>
<tr>
<td>-I</td>
<td>alt</td>
<td>$\Delta(\xi, -\xi, \xi, -\xi)$</td>
<td>$\xi \neq -\xi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta(\xi, -\xi)$</td>
<td>$\xi = -\xi$</td>
</tr>
</tbody>
</table>

Using the construction outlined before, it is now a simple matter to list the indecomposable types. We do so
for the Hermitian symmetric Lie algebras, and in fact for all the $o(p,q)$. Write $\sigma = \sigma_\pm$ according to whether $\sigma^2 = \pm I$, and set $\delta = (-1)^{m/2} \epsilon$ for $m$ even where $\epsilon = \pm 1$.

### Indecomposable Types for Hermitian Lie Algebras

<table>
<thead>
<tr>
<th>Group</th>
<th>Type</th>
<th>Conditions</th>
<th>Signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL(V,\tau_*)$</td>
<td>$\Delta_m(\zeta,-\zeta)$</td>
<td>$\zeta \neq -\zeta$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td>$[\geq SU(p,q)]$</td>
<td>$\Delta_m^e(\zeta)$</td>
<td>$\zeta = -\zeta$</td>
<td>$\begin{cases} m \text{ even} &amp; (\frac{1}{2}(m+1+\delta),\frac{1}{2}(m+1-\delta)) \ m \text{ odd} &amp; (\frac{m+1}{2},\frac{m+1}{2}) \end{cases}$</td>
</tr>
<tr>
<td>$O(V,\sigma_+,\tau)$</td>
<td>$\Delta_m(\zeta,-\zeta,\zeta,-\zeta)$</td>
<td>$\zeta \neq \pm \zeta$</td>
<td>$(2m+1,2m+1)$</td>
</tr>
<tr>
<td>$[= o(p,q)]$</td>
<td>$\Delta_m(\zeta,-\zeta)$</td>
<td>$\zeta = \zeta \neq 0$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_m^e(\zeta,-\zeta)$</td>
<td>$\zeta = -\zeta \neq 0$</td>
<td>$\begin{cases} m \text{ even} &amp; (m+1, m+\delta) \ m \text{ odd} &amp; (m+1, m+1) \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_m(0,0)$</td>
<td>m even</td>
<td>$(\frac{1}{2}(m+1+\delta),\frac{1}{2}(m+1-\delta))$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_m(0,0)$</td>
<td>m odd</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td>$O(V,\sigma_-,\tau)$</td>
<td>$\Delta_m(\zeta,-\zeta,\zeta,-\zeta)$</td>
<td>$\zeta \neq -\zeta$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td>$[= SO^*(2n)]$</td>
<td>$\Delta_m^e(\zeta,-\zeta)$</td>
<td>$\zeta = -\zeta \neq 0$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_m(0,0)$</td>
<td>m even</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Delta_m^e(0,0)$</td>
<td>m odd</td>
<td></td>
</tr>
<tr>
<td>$Sp(V,\sigma_+,\tau)$</td>
<td>$\Delta_m(\zeta,-\zeta,\zeta,-\zeta)$</td>
<td>$\zeta \neq \pm \zeta$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td>$[=Sp(n,\mathbb{R})]$</td>
<td>$\Delta_m^e(\zeta,-\zeta)$</td>
<td>$\zeta = \zeta \neq 0$</td>
<td>$(m+1,m+1)$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_m(0,0)$</td>
<td>m even</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Delta_m(0,0)$</td>
<td>m odd</td>
<td></td>
</tr>
</tbody>
</table>
The indecomposable types which can contribute to a type for $o(2,n)$ are quite limited, and listed below.

\[ \Delta_m(\zeta, -\zeta, \xi, -\xi) \quad m = 0 \]

\[ \Delta_m(\zeta, -\zeta) \quad m = 0, 1 \]

\[ \Delta_m^e(\zeta, -\zeta) \quad m = 0, \epsilon = \pm 1; \quad m = 2 \text{ with } \epsilon = 1 \]
\[ \quad m = 1, \epsilon = \pm 1 \]

\[ \Delta_m^e(0) \quad m = 0, \epsilon = \pm 1; \quad m = 2, \epsilon = \pm 1; \quad m = 4 \text{ with } \]
\[ \quad \epsilon = -1 \]

\[ \Delta_m(0, 0) \quad m = 1 \]
CHAPTER II. Invariant Convex Cones in $\mathfrak{sp}(n,\mathbb{R})$.

6. Conventions for $\mathfrak{sp}(n,\mathbb{R})$.

It will be convenient to work with fixed models for the classical Hermitian Lie algebras, fixed Cartan decompositions, etc. The notation is as in section 3.

We take

\[ \mathfrak{g} = \begin{cases} \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : & A,B,C \text{ real nxn matrices,} \\ & B,C \text{ symmetric, } A \text{ arbitrary} \end{cases} \]

\[ \mathfrak{k} = \begin{cases} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : & A \text{ skew, } B \text{ symmetric} \end{cases} \]

\[ \mathfrak{p} = \begin{cases} \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : & A,B \text{ symmetric} \end{cases} \]

\[ \mathfrak{h} = \begin{cases} \begin{pmatrix} O & D \\ -D & O \end{pmatrix} : & D \text{ diagonal} \end{cases} \]

\[ c = \mathbb{R} \begin{pmatrix} O & -I \\ I & O \end{pmatrix}. \]

Let $d_j$ ($j = 1, \ldots, n$) be linear functionals spanning the dual of $\mathfrak{h}$:

\[ d_j : \begin{pmatrix} O & D \\ -D & O \end{pmatrix} \mapsto D_{jj}. \]

Then

\[ \{\text{noncompact roots}\} = \{ \pm i(d_j + d_k) : 1 \leq j < k \leq n; \pm i2d_j : 1 \leq j \leq n \} \]

\[ \{\text{compact roots}\} = \{ \pm i(d_j - d_k) : 1 \leq j < k \leq n \} \]
and the positive roots we take to be the above with "+" only. Also,

\[ \{ \text{simple positive roots} \} = \Delta_o^+ = \{ i(d_1-d_2), \ldots, i(d_{n-1}-d_n), i2d_n \} . \]

We give the \( H_\alpha, X_\alpha, Y_\alpha \) for \( \alpha \in Q_+ \). For \( \alpha = i(d_j+d_k) \) (1 \( \leq j < k \leq n \))

\[
H_\alpha = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},
\]

where \( D = -iE_{jj} - iE_{kk}, A = B = E_{jk} + E_{kj} \).

For \( \alpha = i2d_j \) (1 \( \leq j \leq n \))

\[
H_\alpha = \begin{pmatrix} 0 & D_j \\ -D_j & 0 \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},
\]

where \( H_\alpha = -iE_{jj}, A = B = E_{jj} \).

We have then \( Z = \begin{pmatrix} 0 & -\frac{1}{2}I \\ \frac{1}{2}I & 0 \end{pmatrix} = Z_0, h = h^-, \) and take

\[ \Sigma_o = \{ i2d_1, \ldots, i2d_n \}, \quad \sigma = \left\{ \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \right\} \in \sigma_j: \Lambda \text{ diagonal} \} . \]

The cone in \( h \) obtained from Corollary 3.2 is

\[ \left\{ \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \in h: D \text{ has positive diagonal entries} \} . \]

Define \( B(X,Y) = \text{Tr}(XY) \) for \( X,Y \in \sigma_j \). If

\[
X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad Y = \begin{pmatrix} D & G \\ H & -D^t \end{pmatrix} \in \sigma_j,
\]

we have

\[
B(X,Y) = 2 \sum_{i,j} A_{ij}D_{ji} + \sum_{i,j} B_{ij}H_{ji} + \sum_{i,j} C_{ij}G_{ji} .
\]

The Cartan involution is \( \theta: X \rightarrow -X^t \) and \( B_\theta(X,Y) = -B(X,\theta Y) \),
Define the symplectic form \( \sigma(\cdot, \cdot) \) on \( \mathbb{R}^{2n} \) by

\[
\sigma\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = y \cdot u - x \cdot v
\]

for \( x,y,u,v \in \mathbb{R}^n \).

**Theorem 6.1.** Let \( u \in \mathbb{R}^{2n} \) and \( X \in \mathfrak{g}_y \). Then \( \sigma(Xu, u) = B_\theta(X,Y) \) where \( \sigma(Yv, v) > 0 \) for all \( v \in \mathbb{R}^{2n} \).

**Proof.** Let \( u = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \) and \( X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \). Then

\[
\sigma(Xu, u) = (Cv) \cdot v - 2(Av) \cdot w - (Bw) \cdot w = \sum_{i,j} C_{ij} v_i v_j - \sum_{i,j} B_{ij} w_i w_j
\]

\[
- 2 \sum_{i,j} A_{ij} w_i v_j = B_\theta\left(\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \begin{pmatrix} D & G \\ H & -D^t \end{pmatrix}\right) = B_\theta(X,Y),
\]

where

\[
D_{ij} = -w_i v_j, \quad G_{ij} = -w_i w_j, \quad \text{and} \quad H_{ij} = v_i v_j.
\]

Clearly \( Y \in \mathfrak{g}_y \).

Secondly, let \( \begin{pmatrix} x \\ y \end{pmatrix} = v \in \mathbb{R}^{2n} \). Then

\[
\sigma(Yv, v) = (Hx) \cdot x - 2(Dx) \cdot y - (Gy) \cdot y
\]

\[
= \sum_{i,j} x_i H_{ij} x_j - 2 \sum_{i,j} x_j D_{ij} y_i - \sum_{i,j} y_i G_{ij} y_j
\]

\[
= (x \cdot v)(x \cdot v) + 2(x \cdot v)(w \cdot y) + (y \cdot w)(y \cdot w)
\]

\[
= (x \cdot v + w \cdot y)^2 \geq 0.
\]
7. Conjugacy of Positive Elliptics in $\text{sp}(\mathcal{H})$.

Let $\mathcal{H}$ be a real Hilbert space, possibly infinite dimensional, with inner product $\langle \cdot, \cdot \rangle$. We assume that $\mathcal{H}$ has a complex structure $J$, i.e., an orthogonal with square $-I$. Define the symplectic form $\sigma(x,y) = \mathcal{J}_0(J^{-1}x,y)$ for $x,y \in \mathcal{H}$. $\sigma$ is skew and nondegenerate. $(\mathcal{H},\langle \cdot,\cdot \rangle)$ is then a complex Hilbert space, where $\langle \cdot,\cdot \rangle = \mathcal{J}_0(\cdot,\cdot) + i\sigma(\cdot,\cdot)$; note $\mathcal{J}_0(\cdot,\cdot) = \sigma(J\cdot,\cdot)$. $\sigma$ is really the primary bilinear form here, but we need some $\mathcal{J}_0$ to define the topology.

Write $\| \cdot \|^2 = \langle \cdot,\cdot \rangle$ or sometimes $\| \cdot \|^2_{\mathcal{J}_0}$ for clarity. In this section all operators are bounded. Given a (real-linear) operator $B: \mathcal{H} \to \mathcal{H}$, $\|B\|$ denotes the norm of $B$. Let $\text{Sp}(\mathcal{H})$ be the group of invertible linear operators $\mathcal{H} \to \mathcal{H}$ preserving $\sigma$. Call $B: \mathcal{H} \to \mathcal{H}$ infinitesimally symplectic if $B$ is skew with respect to $\sigma$. The set of infinitesimally symplectic operators we denote $\mathcal{J}_\sigma$, or $\text{sp}(\mathcal{H})$.

**Theorem 7.1.** Let $B \in \mathcal{J}_\sigma$ satisfy $\sigma(Bv,v) \geq k\langle v,v \rangle \forall v \in \mathcal{H}$, for some $k > 0$. Then there exists $S \in \text{Sp}(\mathcal{H})$ such that $SB^{-1} = JH$ where $H$ is a positive self-adjoint operator on $(\mathcal{H},\langle \cdot,\cdot \rangle)$ (i.e., $H$ commutes with $J$). Furthermore, the spectrum of $H$ is bounded as follows:

$$(\|B\|^{3/2}/k^{1/2})\|v\|^2 \geq \langle Hv,v \rangle \geq (k^{3/2}/\|B\|^{1/2})\|v\|^2$$
for all \( v \in \mathcal{V} \).

**Proof.** Clearly \( \|B\| \geq \sup \|Bv,v\| \), and in fact by polarization one can prove equality, but we do not need this. It follows from the identity \( \mathcal{S}_0(Bv,w) = \mathcal{A}(Bu_-,u_-) - \mathcal{A}(Bu_+,u_+) \) where \( u_\pm = (v \pm Jw)/2 \) and the positivity of \( \mathcal{A}(B^\ast,\cdot) \).

Define \( \mathcal{S}(v,w) = \mathcal{A}(Bv,w) \) for all \( v,w \in \mathcal{V} \). \( \mathcal{S} \) is symmetric as \( B \in \mathcal{G} \) and \( \mathcal{A} \) is skew. We have

\[
\|B\| \|v\|^2 \geq \mathcal{S}(v,v) \geq k \|v\|^2 \ \forall \ v \in \mathcal{V}
\]

so \( \mathcal{S}_0 \) and \( \mathcal{S} \) are real Hilbert structures defining equivalent norms on \( \mathcal{V} \). We have \( \mathcal{S}(\cdot,\cdot) = \mathcal{S}_0(J^{-1}B^\ast,\cdot) \), and \( J \) is orthogonal for \( \mathcal{S} \) iff \( B \) commutes with \( J \).

\( J^{-1}B \) is a symmetric operator with respect to \( \mathcal{S}_0 \), bounded below by \( k \) and above by \( \|B\| \), so \( B \) is bijective. Hence it has a bounded inverse, and \( \|B^{-1}\| \leq 1/k \).

Now \( B \) is skew with respect to \( \mathcal{S} \), so if \( B = K \sqrt{-B^2} = K \sqrt{-B^2} \) is the polar decomposition in \( (\mathcal{V},\mathcal{S}) \), all three operators \( B,K, \) and \( \sqrt{-B^2} \) commute. We have \( K^2 = -I \), and \( \sqrt{-B^2} \) is invertible and positive-definite symmetric in \( (H,\mathcal{S}) \).

Furthermore, \( K \in \text{Sp}(\mathcal{V}) \) as

\[
\mathcal{A}(Kv,Kw) = \mathcal{A}((\sqrt{-B^2})v, (\sqrt{-B^2})w) = \mathcal{S}(1,\sqrt{-B^2})v, (\sqrt{-B^2})w) = \mathcal{S}(v,-B^{-1}w) = \mathcal{A}(v,w).
\]
Setting $g(x, y) = \mathcal{A}(Kx, y) \forall x, y \in \mathcal{H}$, we have

\[(4) \quad g_1(\cdot, \cdot) = g((-B^2)^{-\frac{1}{4}} \cdot, \cdot) \],

and $g_1$ defines yet another equivalent norm on $\mathcal{H}$. Its advantage over $g$ is that

\[\langle \cdot, \cdot \rangle_1 = g_1(\cdot, \cdot) + i\mathcal{A}(\cdot, \cdot)\]

defines a complex Hilbert structure on $\mathcal{H}$ (with complex structure $K$) with the same imaginary part as that of $\langle \cdot, \cdot \rangle$.

It is not difficult to see that the topological equivalence of $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $(\mathcal{H}, \langle \cdot, \cdot \rangle_1)$ implies there exists a unitary equivalence

\[S: (\mathcal{H}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathcal{H}, \langle \cdot, \cdot \rangle) .\]

We have $JS = SK$, $S \in \text{Sp}(\mathcal{H})$, and $SBS^{-1} = J(S\sqrt[-4]{-B^2} S^{-1})$. So, we set $H = S\sqrt[-4]{-B^2} S^{-1}$, and $HJ = JH$ because $K$ commutes with $\sqrt[-4]{-B^2}$.

By $(4)$ we have orthogonal equivalences

\[ (\mathcal{H}, g) \xrightarrow{(-B^2)^{1/4}} (\mathcal{H}, g_1) \xrightarrow{S} (\mathcal{H}, g_0) \]

of real Hilbert spaces. As $(-B^2)^{1/4}$ commutes with $\sqrt[-4]{-B^2}$, $\sqrt[-4]{-B^2}$ is positive-definite symmetric in $(\mathcal{H}, g_1)$ also. By the equivalence, $H$ has the same property in $(\mathcal{H}, g_0)$, and is self-adjoint in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ as $HJ = JH$.

It remains to prove the bounds on $H$. They are equivalent to
\[(\|B\|^{3/2}/k^{1/2}) \cdot \|w\|_{\mathfrak{g}}^2 \geq s(B^2, \cdot) \geq (k^{3/2}/\|B\|^{1/2}) \cdot \|w\|_{\mathfrak{g}}^2\]
or
\[(\|B\|^{3/2}/k) \cdot \|w\|_{\mathfrak{g}}^2 \geq s(-B^2, \cdot) \geq (k^{3/2}/\|B\|^{1/2}) \cdot \|w\|_{\mathfrak{g}}^2\]

For the upper bound of (5),

\[s(-B^2w, w) = s(Bw, Bw)\]
\[\leq \|B\|_{\mathfrak{g}_o} s_o(Bw, Bw)\]
\[\leq \|B\|_{\mathfrak{g}_o}^{3/2} \|w\|_{\mathfrak{g}_o}^2 \leq (\|B\|_{\mathfrak{g}_o}^{3/2}/k) \|w\|_{\mathfrak{g}_o}^2,\]

the first and third inequalities following from (3). For the lower bound,

\[s(Bw, Bw) \geq k s_o(Bw, Bw)\]
\[\geq k \|B^{-1}\|_{\mathfrak{g}_o}^{-2} \|w\|_{\mathfrak{g}_o}^2\]
\[\geq k^3 (1/\|B\|_{\mathfrak{g}_o}) \|w\|_{\mathfrak{g}_o}^2\]

by (3) and \(\|B^{-1}\|_{\mathfrak{g}_o} \leq 1/k\) shown earlier.

8. **Uniqueness of Causal Cones in \(sp(n, \mathbb{R})\).**

The goal of this section is to prove Theorem 8.1. We use the notation of section 6, and in order to apply Theorem 7.1 we indicate the connection between the notations of sections 6 and 7. Our real Hilbert space \(\mathfrak{g}\) is \(\mathbb{R}^{2n}\) with the standard inner product. The complex
structure \( J \) is \(
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}
\); note that \(
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}
= \begin{pmatrix}
-w \\
v
\end{pmatrix}
\) corresponds to \( i(v+iw) = -w + iv \) for \( v, w \in \mathbb{R}^n \). Also \( \langle v+iw, x+iy \rangle = v \cdot w + w \cdot y + i(w \cdot x - v \cdot y) \), so the two definitions of the symplectic form \( \mathcal{J} \) agree.

A short computation shows that \( g = \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \text{Sp}(n, \mathbb{R}) \) iff \( P^t M = M^t P \), \( N^t Q = Q^t N \), and \( Q^t M - N^t P = I \), or equivalently \( MN^t = NM^t \), \( QP^t = PQ^t \), and \( MQ^t - NP^t = I \).

Thus \( g^{-1} = \begin{pmatrix} Q^t & -N^t \\ -P^t & M^t \end{pmatrix} \). Also \( X \in \text{gl}(2n, \mathbb{R}) \) commutes with \( J \) iff \( X \) has the form \( \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \) for \( A, B \in \text{gl}(n, \mathbb{R}) \). We find \( g \in \text{Sp}(n, \mathbb{R}) \) in the subgroup \( K \) corresponding to \( K \) iff \( g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \) where \( AA^t + BB^t = I \) and \( AB^t = BA^t \), or \( A^t A + B^t B = I \) and \( A^t B = B^t A \). As shown on p. 343 of [9], \( K \) is algebraically isomorphic to \( U(n) \) acting on \( \mathbb{C}^n \), i.e. \( I = (A+iB)(A+iB)^* \) gives the same equations.

Let \( C \) be the set of all non-trivial invariant convex cones in \( \mathbb{R}^n \), and let \( C_0 = \{ X \in \mathcal{J} : \mathcal{J}(Xv, v) > 0 \ \forall \ v \neq 0 \} \) in \( \mathbb{R}^{2n} \).

**Theorem 8.1.** \( C_0 \) is nonempty and in \( C \). Any \( C \in C \) contains either \( C_0 \) or \(-C_0 \) but not both. \( \pm C_0 \) are the unique open cones in \( C \). \( \overline{C}_0 = \{ X \in \mathcal{J} : \mathcal{J}(Xv, v) \geq 0 \ \forall \ v \in \mathbb{R}^n \} \), and \( \pm \overline{C}_0 \) are the unique closed cones in \( C \).

**Proof.** Clearly \( J \in C_0 \) and \( C_0 \in C \). Let \( C \in C \). By Lemma 1.1 \( \overline{C} \in C \) also, and by the proof of Theorem 2.2 \( \overline{C} \) contains either \( J \) or \(-J \). Assume \( J \in \overline{C} \). Now
\(\sigma(X,\cdot)\) is a symmetric form on \(\mathbb{R}^{2n}\) for all \(X \in \mathcal{Q}\). Since all norms are equivalent in a finite-dimensional space \(\mathcal{C}\) is open. Thus \(J \in \mathcal{C} \cap \mathcal{C}_0\) implies \(\exists W \in \mathcal{C} \cap \mathcal{C}_0\).

Let \(W\) be the \(B\) in Theorem 7.1. The invariance of \(C\) implies \(JH \in C\) as in the theorem, and the diagonalization of \(JH\) by a unitary implies that \(C\) also contains

\[
J\tilde{D} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \tilde{D} & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}
\]

for some diagonal matrix \(D\) with positive entries.

Now conjugation by \(a, a^{-1} \in A\) and averaging (\(C\) is convex) shows that \(C\) also contains any \(\begin{pmatrix} 0 & -D' \\ D' & 0 \end{pmatrix}\) where \(D'_{ii} \geq D_{ii}\) for \(i = 1, \ldots, n\). As \(C\) is a cone it must contain all \(\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}\), \(D\) positive diagonal. The argument can now be repeated to show \(\mathcal{C}_0 \subset C\). But \(C\) cannot contain \(\mathcal{C}_0\) and \(-\mathcal{C}_0\) by Proposition 2.1.

Of the remaining three statements we show how the first follows from the last two. Let \(C\) be an open cone in \(\mathcal{C}\), and assume \(\overline{C} = \mathcal{C}_0\). As \(\overline{C}_0 \cap -\overline{C}_0 = \{0\}\) we must have \(\mathcal{C}_0 \subset C\). If \(X \in C - \mathcal{C}_0 \subset \overline{C}_0 - \mathcal{C}_0\) \(\exists v \in \mathbb{R}^{2n}\) such that \(\sigma(Xv,v) = 0\) and \(v \neq 0\). \(C\) being open implies \(\exists \varepsilon > 0\) such that \(Y = X - \varepsilon J \in \mathcal{C} \subset \overline{C}_0\) but \(\sigma(Yv,v) < 0\), a contradiction.

To prove \(\overline{\mathcal{C}_0} = \{X \in \mathcal{Q}: \sigma(Xv,v) \geq 0 \ \forall \ v \in \mathbb{R}^{2n}\}\) \(\equiv S\), note first that clearly \(\overline{\mathcal{C}_0} \subset S\). Conversely, if \(X \in S\) then \(X + \varepsilon J \in \mathcal{C}_0 \ \forall \ \varepsilon > 0\), so \(X \in \overline{\mathcal{C}_0}\).

Finally, let \(C_1\) be closed and in \(\mathcal{C}\). We may assume
$C_0 \subset C_1$ so that $\overline{C}_0 \subset \overline{C}_1 = C_1$. If $\overline{C}_0 \subset C_1$ then $C_1^* \subset \overline{C}_0^* = -\overline{C}_0$ by Theorem 8.2 below. But we have proven that then $-C_0 \subset C_1^*$ must occur, hence $-\overline{C}_0 \subset C_1^*$, a contradiction.

Theorem 8.2. $\overline{C}_0^* = -\overline{C}_0$, i.e., the closed cone $\overline{C}_0$ is self-dual with respect to any $B_0$ form.

Proof. We saw in section 2 that $B_0(X,Y) \geq 0 \forall X,Y \in \overline{C}_0$, which is just $-\overline{C}_0 \subset \overline{C}_0^*$. If now $X \not\in \overline{C}_0 \exists \nu \in \mathbb{R}^{2n}$ such that $\alpha(X\nu,\nu) > 0$. By Theorem 6.1 $B_0(X,Y) > 0$ for some $Y \in \overline{C}_0$, so $X \not\in \overline{C}_0^*$.

Corollary 8.3. $C_0$ is the topological interior of $\overline{C}_0$.

Proof. The interior of $\overline{C}_0$ contains $C_0$ and is an invariant convex cone, and so equals $C_0$ by the uniqueness.

9. Classification of Cones in $\text{sp}(n,\mathbb{R})$.

We have seen that any invariant convex cone $C$ (or its negative) in $\text{sp}(n,\mathbb{R})$ must sit between $C_0$ and $\overline{C}_0$. Let $C_+ = \{C \in C : C_0 \subset C \subset \overline{C}_0\}$. It turns out that $C_+$ is a finite collection, and we will be able to classify its elements. For example, in $\text{sp}(2,\mathbb{R})$ there are 12 such cones in $C_+$.

Any $C \in C_+$ can be considered a collection of types $\Delta$, each of which is a sum of indecomposable types $\Delta = \Delta_1 + \ldots + \Delta_m$. By Theorem 8.1 it is clear that $\Delta$
is in \( \overline{C}_0 \) iff each \( \Delta_j \) is in the \( \overline{C}_0 \)'s for the (possibly) lower-dimensional \( \sigma \)-orthogonal subspaces. Thus to determine the types on the boundary of \( C_0 \) it suffices to consider the indecomposable types.

Using the notation and analysis in Section 5, it is clear that \( X \in C_0 \) determines a type whose indecomposable components are all of the form \( \Delta_0^+(\zeta,-\zeta) \) for \( 0 \neq \zeta \in \mathbb{IR} \). \( \pm \zeta \) are the eigenvalues of this type, so the eigenvalues of any \( X \in \overline{C}_0 \) must also lie on the imaginary axis. Thus it suffices to consider only \( \Delta_m^+(0) \) (\( m \) odd), \( \Delta_m(0,0) \) (\( m \) even), and \( \Delta_m^+(\zeta,-\zeta) \) (\( \zeta = -\xi \neq 0 \)).

**Theorem 9.1.** Only the types \( \Delta_1^+(0) \), \( \Delta_0(0,0) \), and \( \Delta_0^+(\xi,-\xi) \) are in \( \overline{C}_0 \).

**Proof.** Let \( A = S + N \) be a representative of one of the above types of height \( m \). The representation space is a direct sum \( E + NE + \ldots + N^m E \), where \( \sigma(\cdot,N^m \cdot) \) is non-degenerate on \( E \).

Assume first \( m \geq 1 \), take \( v,w \in E \), and consider

\[
\sigma(A(N^{m-1}v+w),N^{m-1}v+w)
\]

\[
= \sigma(Aw,w) + (-1)^{m-1} \sigma(A(N^{2m-2}v,v) + 2 \sigma(N^mv,w)
\]

2m-2 \( \geq m+1 \) iff \( m \geq 3 \), in which case the first and second terms vanish, and the expression is negative for the proper choice of \( v \) and \( w \).

If \( m = 2 \) the expression is \( -\sigma(SN^2v,v) + 2 \sigma(N^2v,w) \),
and this is negative if \( w \) is sufficiently large and of the proper sign.

Finally, if \( m = 1 \) take \( v, w \in E \) and consider

\[
\mathcal{A}(A(Nv+w),Nv+w) = -2\mathcal{A}(NSw,v) + \mathcal{A}(Nw,w).
\]

If \( S \neq 0 \) it is injective, and again this is not positive-definite.

The remaining cases are just those stated in the theorem. All are orbits in \( \text{sp}(1,\mathbb{R}) \sim \mathfrak{sl}(2,\mathbb{R}) \). \( \Delta_0(0,0) \) is just \((0 0',0 0')\), \((a b, c -a)\) is in the orbit of \( \Delta_0(\xi, -\xi) \) iff

\[
-a^2 - bc > 0, \quad b < 0, \quad \text{and} \quad c > 0.
\]

\((a b, c -a) \neq 0\) is in \( \Delta_0^+(0) \) iff \( -a^2 - bc = 0, \quad b < 0, \quad \text{and} \quad c > 0 \). This completes the proof.

To classify the cones we need some further machinery and a convexity lemma. Let \( E_n = \{(x_1, y_1), \ldots, (x_n, y_n)\} : \) all \( x_i, y_i \geq 0 \). We consider it a cone in \( \mathbb{R}^{2n} \). Define a semigroup \( \mathcal{G} \) acting on \( E_n \) generated by

1) \([(x_1, y_1), \ldots] \rightarrow [(y_1, x_1), \ldots] \),

2) \([(x_1, y_1), \ldots, (x_i, y_i), \ldots, (x_n, y_n)] \rightarrow
   \[(x_1, y_1), \ldots, (x_i, y_i), \ldots, (x_n, y_n)]\) for \( i=2, \ldots, n \),

3) \([(x_1, y_1), \ldots] \rightarrow [(\lambda x_1, (1/\lambda)y_1), \ldots]\) for \( \lambda > 0 \),

and

4) \([(x_1, y_1), (x_2, y_2), \ldots] \rightarrow [(x_1 + y_2, (x_1 + x_2, y_2), \ldots] \).

Let \( \mathcal{C} \) be the collection of all Schrödinger bases in \( \mathbb{R}^{2n} \) on which \( G \) acts, i.e.
\[ \xi = \{(a_i,b_i)_{i=1}^n: \sigma(a_i,a_j) = \sigma(b_i,b_j) = 0, \sigma(a_i,b_j) = \delta_{ij}\} , \]

and let \( \tilde{\mathcal{C}}_+ \) be the collection of all convex cones in \( E_n \) which are invariant under \( \varphi \) and contain \( \mathcal{C}_n = \{(x_1,y_1),\ldots,(x_n,y_n)\}: \text{all } x_i,y_i > 0\) . We will put \( \mathcal{C}_+ \) in 1-1 correspondence with \( \tilde{\mathcal{C}}_+ \) by means of the following maps. Given \( X \in \mathcal{C}_0 \) and \( \varphi = (a_i,b_i)_{i=1}^n \in \xi \), define

\[ M_\varphi(X) = [(\sigma(Xa_1,a_1),\sigma(Xb_1,b_1)),\ldots,(\sigma(Xa_n,a_n),\sigma(Xb_n,b_n))] \in E_n . \]

For any \( C \in \mathcal{C}_+ \) let \( E_C \) be the convex cone generated by all \( M_\varphi(X) \) for \( X \in C \), \( \varphi \in \xi \).

**Lemma 9.2.** Let \( X \in \mathcal{C}_0 \) and let the decomposition of the type of \( X \) be \( P \) summands of the form \( \Delta^+(\zeta,-\zeta) \) \((\zeta = -\zeta \neq 0)\), \( N \) summands of the form \( \Delta_1^+(0) \), and \( Z \) of the form \( \Delta(0,0) \). Then \( n = P + N + Z \). Let \( \varphi = (a_i,b_i) \in \xi \) exhibit this decomposition, so that \( M_\varphi(X) \) has \( P \) pairs of the form \((e,f)\), \( e,f > 0 \), \( N \) of the form \((\lambda,0)\) or \((0,\lambda)\), \( \lambda > 0 \), and \( Z \) of the form \((0,0)\).

Let \( \psi = (c_i,d_i) \in \xi \) be arbitrary. We can assume \( \psi \) is ordered so that \( \sigma(Xc_i,c_i),\sigma(Xd_i,d_i) > 0 \) for \( i = 1,\ldots,P' \), \( \sigma(Xc_i,c_i) > 0 \), \( \sigma(Xd_i,d_i) = 0 \) for \( i = P'+1,\ldots,P'+N' \), and \( \sigma(Xc_i,c_i) = \sigma(Xd_i,d_i) = 0 \) for \( i = P'+N'+1,\ldots,n \).

Then \( Z' \leq Z \) and \( N'+Z' \leq N+Z \).

**Proof.** Assume that the diagonalizing basis \( \varphi \) has been ordered so that the sentence above is true with \( (a_i,b_i) \) replacing \( (c_i,d_i) \), and \( P,N,Z \) replacing \( P',N',Z' \).
respectively. Now each \( c_j \) and \( d_j \) is a linear combination of the \( a_i \) and \( b_i \). If \( j = P'+N'+1, \ldots, n \), \( c_j \) and \( d_j \) can involve no \( a_i \) or \( b_i \) with \( i = 1, \ldots, P \) or \( a_i \) with \( i = p+1, \ldots, P+N \). The \( 2Z' \times 2Z' \) matrix

\[
\begin{pmatrix}
\sigma(c_k, c_{k'}) & \sigma(c_k, d_{k'}) \\
\sigma(d_k, c_{k'}) & \sigma(d_k, d_{k'})
\end{pmatrix}_{k, k'=P'+N'+1, \ldots, n}
\]

(which depends only on the combinations of the \( a_i, b_i \) for \( i = P+N+1, \ldots, n \)) is of rank \( 2Z' \) as \( \forall \in \xi \), but is equal to \( \text{At} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A \) where \( A \) is of order \( 2Z \times 2Z' \), hence \( Z' \leq Z \).

Express likewise \( d_j \) for \( j = P'+1, \ldots, n \). These \( d_j \) can involve no \( a_i, b_i \) for \( i = 1, \ldots, P \). Now the maximal dimension of an isotropic subspace in a \( 2(N+Z) \)-dimensional symplectic space is \( N+Z \), and \( N'+Z' \leq N+Z \) follows.

**Definition.** If \( e \in E_n \) has exactly \( N \) pairs of the form \((0, \lambda)\) or \((\lambda, 0)\), \( \lambda > 0 \), and \( Z \) pairs of the form \((0, 0)\), say \( e \) gives \((N, Z)\).

**Theorem 9.3.** \( E_C \in \mathbb{C}_+ \) for each \( C \in \mathbb{C}_+ \), and the mapping

\[
E: \mathbb{C}_+ \to \mathbb{C}_+: C \to E_C
\]

is a 1-1 correspondence.

**Proof.** \( E_C \) clearly contains \( C_n \) since each \( C \) contains \( C_0 \). Invariance under the generators 1) - 3) follows from
the trivial basis changes \( a_1 \to b_1 \), \( b_1 \to -a_1 \) and \( a_1 \to \lambda a_1 \), \( b_1 \to (1/\lambda) b_1 \).

Invariance under 4) follows from an \( \text{Sp}(2,\mathbb{R}) \) symmetry: given \( (a_1, b_1, a_2, b_2) \), make the change of basis

\[
\begin{align*}
    a_1 &\to c_1 = a_1 & & a_2 \to c_2 = a_2 + ka_1 \\
    b_1 &\to d_1 = b_1 - kb_2 & & b_2 \to d_2 = b_2
\end{align*}
\]

for \( k \in \mathbb{R} \), which is symplectic. If \( (a_1, b_1, a_2, b_2) \) is part of a \( \varphi \in \mathfrak{q} \) and \( X \in \mathfrak{c} \) so \( M_\varphi(X) \in E_\mathfrak{c} \), do (6) for \( k = \pm 1 \) to obtain \( \varphi_+, \varphi_- \), and \( \frac{1}{2}(M_{\varphi_+}(X) + M_{\varphi_-}(X)) \in E_\mathfrak{c} \) exhibits the invariance under 4). Thus \( E_\mathfrak{c} \in \widehat{\mathfrak{c}}_+ \).

Before proving the 1-1 correspondence, we first make a simplifying remark about the \( D \in \widehat{\mathfrak{c}}_+ \). If \( e \in D \), then \( D \) also contains any other \( e' \) obtained from \( e \) by changing any positive \( x_1 \) or \( y_1 \) in \( e \) to any other positive \( x'_1 \) or \( y'_1 \). This is trivial for the \( (\lambda,0) \) and \( (0,\lambda) \) \( (\lambda \neq 0) \) pairs by (3), and accomplished for the \( (x,y) \) \( (x,y > 0) \) pairs by initially multiplying by a small constant, and then applying (3) twice so

\[
(x,y) \to (\lambda x, (1/\lambda)y) \quad \text{and} \quad (x,y) \to ((1/\lambda)x, \lambda y)
\]

and averaging, to "raise" \( (x,y) \) by any desired amount.

Now let \( D \in \widehat{\mathfrak{c}}_+ \). By the above \( D \) is characterized by the set of pairs \( (N,Z) \) which the \( e \in D \) give. (Of course, the \( (N,Z) \) cannot be specified arbitrarily.)
Each \((N, Z)\) determines many types \(\Delta\), as in the statement of Lemma 9.2; let \(C\) be the convex cone in \(\mathcal{C}_0\) generated by the representatives of these types determined by \(D\).

That \(E_C = D\) follows from the fact that \(D\) is convex, Lemma 9.2, and the observation that if \(D\) contains \(e\) giving \((N, Z)\) where \(Z < n \ (e \neq 0)\), then it also contains all points giving \((N', Z')\) where \(Z' \leq Z\) and \(N' + Z' \leq N + Z\), by invariance under \(1\) to \(4\). Thus \(C \rightarrow E_C\) is surjective.

Finally, suppose \(E_{C_1} = E_{C_2}\) for \(C_1, C_2 \in C_+\). We must show that each type in \(C_1\) is in \(C_2\) also. Let \(\Delta\) be a type in \(C_1\) and \(X \in \mathcal{C}_0\) representing \(\Delta\). If \(\varphi_1 \in \mathfrak{e}\) is a diagonalizing basis as in Lemma 9.2, \(e = M_{\varphi_1}(X) \in E_{C_1}\) and gives a pair \((N, Z)\). By hypothesis \(e \in E_{C_2}\), so \(e\) is a convex combination of \(M_{\varphi_1}(Y_i)\) for \(Y_i \in C_2\), \(\varphi_1 \in \mathfrak{e}\).

Observe that if \(x, y \in E_n\), and \(x, y\), and \(x+y\) give \((N_1, Z_1), (N_2, Z_2)\), and \((N_3, Z_3)\) respectively, then \(Z_3 \leq \min(Z_1, Z_2)\), and \(N_3 + Z_3 \leq \min(N_1 + Z_1, N_2 + Z_2)\). Thus for some \(Y \in C_2\) and \(\varphi \in \mathfrak{e}\), \(M_{\varphi}(Y)\) gives \((N_1, Z_1)\) where \(Z \leq Z_1\) and \(N + Z \leq N_1 + Z_1\). Now \(Y\) may be diagonalized by some \(\varphi \in \mathfrak{e}\), and \(M_{\varphi}(Y)\) gives \((N_2, Z_2)\). By the lemma, \(N_2 + Z_2 \geq N_1 + Z_1\) and \(Z_2 \geq Z_1\), so \(N_2 + Z_2 \geq N + Z\) and \(Z_2 \geq Z\).

It remains to see that these inequalities and \(Y \in C_2\) imply \(X \in C_2\).

Via the basis \(\mathfrak{e}\), \(Y\) acts on \(n\) copies of \(\mathbb{R}^2\); on \(Z_2\) of them \(Y\) acts as \((0, 0)\), on \(N_2\) like \((0, 0)\),
and on $n - Z_2 - N_2$ copies like a representative of a
$\Delta^+(\zeta, -\zeta)$, $\zeta = -\zeta \neq 0$. $\text{Sp}(n, \mathbb{R})$ includes $\text{SL}(2, \mathbb{R})$ on
each $\mathbb{R}^2$ and all permutations of the $\mathbb{R}^2$'s. The average
of, say $(0 0)$ with its transform $(0 -2)$ is $(0 -1)$,
so a $\Delta^+_1(0)$ can be changed to a $\Delta^+(\zeta, -\zeta)$ easily. Also,
the average of $(0 0) \oplus (0 0)$ with its transform
$(0 0) \oplus (0 0)$ gives $(0 0) \oplus (0 0)$, etc. In this way
one can see that $X$ is in the invariant convex cone
generated by $Y$, provided $N_2 > 0$.

If $N_2 = 0$, $Z_2 < n$ (otherwise $Y = 0$), and we
must show how $(0 1) \oplus (0 0)$ can give rise to an
$(0 1) \oplus (0 0)$. This is done by the $\text{Sp}(2, \mathbb{R})$ symmetry
(6). If $Y a_1 = -b_1$, $Y b_1 = a_1$, $Y a_2 = Y b_2 = 0$, we consider
the action on the basis $(c_1, d_1, c_2, d_2)$ from (6):

$$Y c_1 = -d_1 - k d_2 \quad Y c_2 = -k d_1 - k^2 d_2$$
$$Y d_1 = c_1 \quad Y d_2 = 0$$

The average for $k = \pm 1$ is

$$Y c_1 = -d_1 \quad Y c_2 = -d_2$$
$$Y d_1 = c_1 \quad Y d_2 = 0$$

as desired.

Thus $X \in C_2$ also, so $C_1 = C_2$ and $E \to E_0$ is
injective. This completes the proof.

We see from the proof that any cone in $C_+$ may be
represented graphically by a set of lattice points \((N,Z)\) in \([x,y] \in \mathbb{Z} \oplus \mathbb{Z} : x, y \geq 0, x+y \leq n\), having the general form

\[
\begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]

or

\[
\begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]

(in \(\text{sp}(5,\mathbb{R})\)), for example. The segments bounding the region containing the \((N,Z)\) are either horizontal or have slope \(-1\). \((0,0)\) is always included \((C_0 \subseteq C\) always), and \((0,n)\) is included optionally according to whether \(0 \in C\) or not.

It is clear anyway that \(\text{sl}(2,\mathbb{R})\) has 4 cones in \(C_+\):

\[
\begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]

(closed) \hspace{1cm} \begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]

(open),

and there are 6 cones in \(\text{sp}(2,\mathbb{R})\) which do not contain \(0\):

\[
\begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]

\[
\begin{array}{c}
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\ \cdots \\
\end{array}
\]
They are listed this way to exhibit a duality. Letting $C_+^0$ denote the cones in $C_+$ not containing 0, we have the mapping

$$C \in C_+^0 \rightarrow \hat{C} = \{X \in \mathfrak{g}: B_0(X,Y) > 0 \ \forall Y \in C\} \in C_+^0,$$

and perhaps there is a corresponding explicit duality in these graphs. Are there any "self-dual" cones?

Since the previous was written, Bob Proctor pointed out that the enumeration of the above graphs involves no complicated combinatorics. The positive cones not containing 0, in 1-1 correspondence with the above graphs not containing $(0,n)$, are also in 1-1 correspondence with the set of nonempty and proper subsets of $S_n = \{1,2,...,n+1\}$, as follows. Notice that each row in a graph contains strictly fewer points than any rows below it, and the graph is faithfully represented by the collection of positive integers each the number of points in some row of the graph. For example, in the $sp(5,\mathbb{R})$ examples above, the subsets are $\{5,4,3,1\}$ and $\{6,4,2\}$, respectively.

Thus there are $2(2^n-1)$ positive invariant convex cones not containing 0 in $sp(n,\mathbb{R})$. And it appears that the geometrical duality above corresponds simply to complementation in $S_n$ of these subsets, in which case there are evidently no cones $\hat{C}$ satisfying $\hat{\hat{C}} = C$. 
CHAPTER III. Invariant Convex Cones in $\text{su}(p,q)$.

10. Conventions for $\text{su}(p,q)$.

We take $p \geq q \geq 1$ and $n = p+q$. As in section 3,

$$\mathfrak{g}_j = \begin{cases} 
(B & A) : A, B, C \text{ complex matrices; } B \text{ and } C \text{ skew-Hermitian of order } p \text{ and } q, \text{ resp.}; \\
(A^* & C) : A \text{ arbitrary; } \text{Tr} B + \text{Tr} C = 0
\end{cases}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{g}_j \right\}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \in \mathfrak{g}_j \right\}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{k} : B \text{ and } C \text{ diagonal} \right\}$$

and if $Z = \begin{pmatrix} -(q/(p+q))ii & 0 \\ 0 & (p/(p+q))ii \end{pmatrix}$, $c = \mathbb{R}Z$.

Let $d_j$ ($j = 1, \ldots, n$) be linear functionals spanning the dual of $\mathfrak{h}$:

$$d_j : \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \to \begin{pmatrix} B_{jj} \text{ for } j \leq p \\ C_{j-p,j-p} \text{ for } j \geq p \end{pmatrix}$$

Then \{noncompact roots\} = \{\pm(d_j - d_k) : 1 \leq j \leq p, p+1 \leq k \leq n\}
{compact roots} = {±(d_j-d_k): 1 \leq j < k \leq p \text{ and } p+1 \leq j < k \leq n}

and the positive roots we take to be the above with "+" only.

\[ \Delta^+_0 = \{d_1-d_2, \ldots, d_p-d_p, d_p-d_p+1, d_p+1-d_p+2, \ldots, d_n-1-d_n\} \]

and \( d_p-d_p+1 \) is the only noncompact simple positive root.

For \( 1 \leq j \leq p \) and \( p+1 \leq k \leq n \), \( \alpha = d_j-d_k \in \mathbb{Q}_+ \), and

\[ H_\alpha = E_{jj} - E_{kk}, \quad X_\alpha = E_{jk} + E_{kj}, \text{ and } Y_\alpha = -iE_{jk} + iE_{kj}. \]

We take \( \Sigma_0 = \{d_i-d_{p+i}: 1 \leq i \leq q\} \) so

\[ \alpha = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ D & 0 & 0 \end{pmatrix} : D \text{ real and diagonal} \]

\[ Z_0 = -\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

\[ h^- = \begin{pmatrix} D_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_i \end{pmatrix} \in h \]

and

\[ h^+ = \begin{pmatrix} D_i & 0 & 0 \\ 0 & F_i & 0 \\ 0 & 0 & D_i \end{pmatrix} \in h. \]

Thus \( Z = Z_0 \) iff \( p = q \), and \( h^+ = 0 \) iff \( p = q = 1 \).

\[ Z - Z_0 = \begin{pmatrix} (-q/(p+q)+\frac{1}{2})iI & 0 & 0 \\ 0 & -(q/(p+q))iI & 0 \\ 0 & 0 & (p/(p+q)-\frac{1}{2})iI \end{pmatrix} \]

and the cone from Corollary 3.2 is the set of rays containing
Define $B(X,Y) = \text{Tr}(XY)$ for $X,Y \in \mathfrak{g}$. If $X = (B^\ast A)$, $Y = (G \ D^\ast H) \in \mathfrak{g}^\ast$, we have

$$B(X,Y) = -\sum_{i,j} B_{ij}G_{ij} + 2\text{Re} \sum_{i,j} A_{ij}D_{ij} - \sum_{i,j} C_{ij}H_{ij}.$$ 

The Cartan involution is $\theta: X \rightarrow (I \ 0)X(I \ 0)^{-1}$, and

$$B_\theta \left( \begin{pmatrix} B \\ A^\ast \\ C \end{pmatrix}, \begin{pmatrix} G \\ D^\ast \\ H \end{pmatrix} \right) = \sum_{i,j} B_{ij}G_{ij} + 2\text{Re} \sum_{i,j} A_{ij}D_{ij} + \sum_{i,j} C_{ij}H_{ij}.$$ 

Define the Hermitian form $H(\cdot, \cdot)$ on $\mathfrak{g}^n$ by

$$H((x^i, y^j)) = \sum_{i=1}^p x^i u_i - \sum_{j=1}^q y^j v_j,$$ 

for $x,y \in \mathfrak{g}^p$, $y,v \in \mathfrak{g}^q$. $H(x^i, v^j)$ is purely imaginary $\forall x \in \mathfrak{g}^p$, $v \in \mathfrak{g}^q$.

**Theorem 10.1.** Let $u \in \mathfrak{g}^n$ and $X \in \mathfrak{g}^p$. Then $iH(Xu, u) = B_\theta(X,Y)$ where $Y \in u(p,q)$ (using the same formula for $B_\theta$).

$\text{Tr} \ Y = -iH(u, u)$, and $iH(Yv, v) \geq 0 \ \forall v \in \mathfrak{g}^n$.

**Proof.** Let $u = (v^i)$ and $X = (B^\ast A)$.

$$iH(Xu, u) = i \left\{ \sum_{i,j} v_i B_{ij} v_j^i + \sum_{i,j} w_i A_{ij} v_j^i - \sum_{i,j} w_i A_{ij} v_i^j - \sum_{i,j} w_i C_{ij} v_i^j \right\}$$

$$= B_\theta \left( \begin{pmatrix} B \\ A^\ast \\ C \end{pmatrix}, \begin{pmatrix} G \\ D^\ast \\ H \end{pmatrix} \right) = B_\theta(X,Y).$$
where \( G_{ij} = -iv_i^j \), \( H_{ij} = i w_i^j \), and \( D_{ij} = -iv_i^j \).

Clearly \( Y \in u(p,q) \) and \( \text{Tr} Y = -iH(u,u) \).

If \( v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^{p+q} \),

\[
iH(Yv,v) = |<x,v>|^2 + 2 \text{Re} (\langle y,w\rangle\langle v,x\rangle) + |\langle y,w\rangle|^2
\]

\[
= |<x,v>|^2 + |\langle y,w\rangle|^2 \geq 0 .
\]

11. **Conjugacy of Positive Elliptics in \( u(p,q) \)**

Let \( \mathcal{H} \) be a complex Hilbert space with complex structure \( i \) as usual and complex inner product \( < , >_0 \).

"Complex-linear" (or \( i \)-linear) in this section means commuting with \( i \). Let \( \mathcal{J} \) be an Hermitian operator on \( \mathcal{H} \) with spectrum \( \{+1,-1\} \) and define \( H(\cdot,\cdot) = \langle \mathcal{J}\cdot,\cdot \rangle_0 \). Let \( \mathcal{A}(\cdot,\cdot) = -\text{Im} H(\cdot,\cdot) \), a (real) symplectic form on \( \mathcal{H} \). If we set

\[
<\cdot,\cdot> = \text{Re}<\cdot,\cdot>_0 + i \mathcal{A}(\cdot,\cdot)
\]

and \( J = -\mathcal{J} = -i\mathcal{J} \), then \( (\mathcal{H},<,>) \) is a complex Hilbert space with complex structure \( J \).

Our notation here is chosen carefully to correspond with section 7; in particular, the \( J \) and \( <,> \) correspond, and the earlier remarks on \( \| \cdot \| \) apply. All operators are bounded.

Let \( U(\mathcal{H},\mathcal{J}) = \{ S : \mathcal{H} \rightarrow \mathcal{H} | \text{S an } i\text{-linear invertible operator preserving } H \text{ (or } \mathcal{A}) \} \), and \( \mathcal{B} = \{ B : \mathcal{H} \rightarrow \mathcal{H} | \text{B i-linear and skew for } H \} \). The following lemma is clear.
Lemma 11.1. For all $A \in \mathcal{H}$, $\sigma(A \cdot , \cdot ) = \text{Re}(iH(A \cdot , \cdot ))$ is a symmetric real bilinear form on $\mathcal{H}$ for which $i$ is orthogonal. $iH(Ax,x)$ is real for all $A \in \mathcal{H}$.

Theorem 11.2. Let $B \in \mathcal{H}$ satisfy $\sigma(Bv,v) > k <v,v>$ for some $k > 0$. Then there exists $S \in U(\mathcal{H}, \mathcal{G})$ such that $SBS^{-1} = JH_1$ where $H_1$ satisfies $H_1 \mathcal{G} = \mathcal{G}H_1$ and is a positive self-adjoint operator on $(\mathcal{H}, < , >)$ (and $(\mathcal{H}, < , >_0)$). Furthermore, $H_1$ is bounded as follows:

$$(||B||^{3/2}/k^{1/2})||v||^2 > <H_1v,v> > (k^{3/2}/||B||^{1/2})||v||^2$$

for all $v \in \mathcal{H}$.

Proof. The situation is the same as in section 7 except that we have included an $i$ which is intended to commute with all operators we consider and be orthogonal for every bilinear form. The proof is virtually the same.

We define $\mathcal{G}(\cdot , \cdot ) = \sigma(B \cdot , \cdot )$ and (3) still holds. We decompose $B = K\sqrt{-B^2}$ in the real space $(\mathcal{H}, \mathcal{G})$ in which $i$ is orthogonal. $Bi = iB$ so $K$ and $\sqrt{-B^2}$ also commute with $i$, and $K \in \mathcal{H} \cap U(\mathcal{H}, \mathcal{G})$. Letting $\mathcal{G}_1(\cdot , \cdot ) = \sigma(K \cdot , \cdot ) = \mathcal{G}(\sqrt{-B^2}, \cdot )$, $<\cdot , \cdot >_1 = \mathcal{G}_1(\cdot , \cdot ) + i\sigma(\cdot , \cdot )$ is a complex Hilbert structure on $\mathcal{H}$ with complex structure $K$.

Any unitary equivalence

$S : (\mathcal{H}, < , >_1) \rightarrow (\mathcal{H}, < , >)$
satisfies $SKS^{-1} = J = -i \mathcal{G}$ and preserves $\text{Im } H$, but we want $S$ to be also $i$-linear. This is clearly equivalent to $S^{-1} \mathcal{G} S = iK$ if $S$ is a unitary equivalence, which we now assume. Now $\mathcal{G}$ and $iK$ are Hermitian with spectra $\{+1,-1\}$ in the complex spaces $(\mathcal{U}, \langle , \rangle)$ and $(\mathcal{U}, \langle , \rangle_1)$ respectively. Thus we need an $S$ taking the $\pm 1$ eigenspaces of $iK$ to the $\pm 1$ eigenspaces of $\mathcal{G}$. Clearly such an $S$ exists iff the $L_2$-dimensions of these subspaces match.

Observe

$$\langle \cdot , \cdot \rangle = \text{Re } H(\mathcal{G}, \cdot ) - i \text{Im } H(\cdot , \cdot )$$

and

$$\langle \cdot , \cdot \rangle_1 = \text{Re } H(iK, \cdot ) - i \text{Im } H(\cdot , \cdot ).$$

Thus if $L \subseteq \mathcal{U}$ is in the positive eigenspace of $iK$, $H(\cdot , \cdot )$ is strictly positive-definite on $L$. We need an infinite-dimensional "Sylvester's law of inertia," and the lemma below is sufficient for this.

The rest of the theorem is proven as in section 7.

**Lemma 11.3.** Let $(\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2, \langle , \rangle)$ be a nontrivial direct sum of complex Hilbert spaces. Let $\mathcal{G}$ be the Hermitian operator on $\mathcal{U}$ equal to $+I$ on $\mathcal{U}_1$ and $-I$ on $\mathcal{U}_2$, and set $H(\cdot , \cdot ) = \langle \mathcal{G}, \cdot \rangle$. Let $\alpha$ be the cardinality of the dimension of $\mathcal{U}_1$. If $S \subseteq \mathcal{U}$ is a closed complex-linear subspace such that $H|_S$ is strictly positive-definite on $S$, then $\text{card}(\text{dim } S) \leq \alpha$. 
Proof. Let \{e_\lambda\}_{\lambda \in \Gamma} be an $H$-basis of $S$, i.e.
$H(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. By the open mapping theorem

$M: \ell_2(\Gamma) \to S: \eta_\lambda \mapsto e_\lambda$

is an embedding. Clearly $\eta_\lambda = H(M(\eta), e_\lambda) \forall \eta \in \ell_2(\Gamma)$,
$\lambda \in \Gamma$.

If $\text{Card}(\dim S) > \infty$, then $\exists \eta \in \ell_2(\Gamma)$ where $\eta \neq 0$ and
$(\text{Pr}_{\nu_1} \circ M)(\eta) = 0$. (Otherwise the adjoint of \text{Pr}_{\nu_1} \circ M has
dense range from $\nu_1$ to $\ell_2(\Gamma)$, and the Gram-Schmidt
process gives a contradiction.) Then

$0 < |\eta|^2 = \sum_{\mu \in \Gamma} H(\text{Pr}_{\nu_1} \circ M(\eta), e_\mu) \overline{\eta_\mu}$

$= \sum_{\mu \in \Gamma} H(\text{Pr}_{\nu_2} \circ M(\eta), \text{Pr}_{\nu_1} \circ M(\eta)) \leq 0$

as $H \leq 0$ on $\nu$.

12. Minimal and Maximal Causal Cones in $\text{su}(p,q)$

Let $\mathcal{C}$ be the set of all nontrivial invariant convex
cones in $\text{su}(p,q)$. For convenience we now exclude the
\text{case } p = q = 1. We use the notation of section 10, and
now indicate the connection with the notation of section 11.
The complex Hilbert space $(\nu, \langle \cdot, \cdot \rangle_0)$ is $\mathbb{C}^n$ ($n = p+q$)
with the usual Hilbert structure. Writing $x \in \mathbb{C}^n$ as
$(y \ z)^T$ where $y \in \mathbb{C}^p$, $z \in \mathbb{C}^q$, $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $H(\cdot, \cdot)$ is an
Hermitian form on $\nu$ with signature $(p,q)$. $J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$
and $J \in \mathbb{C}^\nu$ iff $p = q$, but this is not a problem; recall
the closely related $Z \in \mathcal{O}_\gamma$. Let $G = SU(p,q)$.

One computes that $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,q)$ iff $AA^* - BB^* = I$, $DD^* - CC^* = I$, and $AC^* = BD^*$, or equivalently $A^*A - C^*C = I$, $D^*D - B^*B = I$, and $A^*B = C^*D$. It follows that $g^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}$.

Now $O \in \text{gl}(n,\mathbb{C})$ commutes with $J$ iff $OZ = ZO$ iff $O$ has the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Our maximal compact $K$ is all $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$, $U_1, U_2$ unitary with $(\det U_1)(\det U_2) = 1$.

**Definition.** Let

$$C_0 = \{ X \in \mathcal{O}_\gamma: iH(Xv,v) > 0 \forall \text{ nonzero } v \in \mathbb{C}^n \}$$

(Note Lemma 11.1) and

$$C_1 = \{ X \in \mathcal{O}_\gamma: iH(Xv,v) > 0 \forall \text{ nonzero } v \in \mathbb{C}^n \text{ such that } \nu(v,v) = 0 \} .$$

**Theorem 12.1.** $C_0$ and $C_1$ are nonempty, open, and in $\mathcal{C}$. Any $C \in \mathcal{C}$ contains either $C_0$ or $-C_0$ but not both. $\overline{C_0}$ and $\overline{C_1}$ are obtained by changing $>$ to $\geq$ in the definitions of $C_0$ and $C_1$, and $C_0 \subset C_1$, $\overline{C_0} \subset \overline{C_1}$.

$\overline{C_0}, \overline{C_1} \in \mathcal{C}$, and any closed (open) cone in $\mathcal{C}$ which contains $C_0$ is contained in $\overline{C_1}$ (resp. $C_1$).

**Proof.** It is clear that $Z \in C_0 \subset C_1$ and $C_0, C_1 \in \mathcal{C}$. $C_0$ is open by the same argument as in Theorem 8.1; we will return to $C_1$ later.

Let $C \in \mathcal{C}$. As before, we may assume $Z \in \overline{C}$, so
again $A B \in C \cap C_0$. Apply Theorem 11.2 and note that a conjugation by $S \in U(p,q)$ can be achieved by an element of $G$. Diagonalizing further by means of unitaries, it follows that $C$ contains $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ where $\lambda$ and $\sigma$ are positive diagonal matrices. Averaging over permutations of the $\lambda_i$ and $\sigma_j$ (obtained from conjugations from $K$), we see $Z \in C$.

Now reverse the argument to show $C_0 \subseteq C$. The unclear step is that any $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ as above can be obtained from $Z$ by conjugations, convex combinations, and scalar multiplication. Now the effect of conjugating $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ as above, where all $\lambda_i + \sigma_j > 0$, by $a, a^{-1} \in A$ and averaging, is as follows. Any $\begin{pmatrix} -i\tilde{\lambda} & 0 \\ 0 & i\tilde{\sigma} \end{pmatrix}$ can be so obtained from $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$, where $\tilde{\lambda}$ and $\tilde{\sigma}$ are related to $\lambda$ and $\sigma$ as follows. Take $i, j$ such that $1 \leq i \leq p$, $1 \leq j \leq q$, and $c > 0$. Then $\tilde{\lambda}_l = \lambda_l$, $\tilde{\sigma}_k = \sigma_k$ for $l \neq i$ and $k \neq j$, and $\tilde{\lambda}_i = \lambda_i + c$, $\tilde{\sigma}_j = \sigma_j + c$. (We emphasize this only depends on $\lambda_i + \sigma_j > 0$.) It is now clear how any $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$, $\lambda_i, \sigma_j > 0$, can be obtained from $Z$, so $C_0 \subseteq C$.

That $\overline{C_0}$ and $\overline{C_1}$ have the asserted forms follows exactly in the symplectic case.

$C_0 \neq C_1$ follows from the exclusion of the case $p = q = 1$ earlier. The situation is clarified by the following

**Lemma 12.2.** If $X = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix} \in h$, $X \in C_1$ iff $\forall i, j$, $\lambda_i + \sigma_j > 0$. This is clearly equivalent to: $\exists c \in \mathbb{R}$
such that $\lambda_i > c > -\sigma_j \forall i, j$. Thus either all $\lambda_i > 0$
or all $\sigma_j > 0$, but not necessarily both. $X \in C_0$ if$c = 0$ may be taken.

Proof. Let $y = (u \atop v) \in \mathbb{C}^n$ satisfy $y \neq 0$ and $\Psi(y, y) = 0$.
Then

$$iH(Xy, y) = \sum_{i=1}^{p} \lambda_i |u_i|^2 + \sum_{j=1}^{q} \sigma_j |v_j|^2$$

$$> c(\sum_{i=1}^{p} |u_i|^2 - \sum_{j=1}^{q} |v_j|^2) = 0,$$
the inequality being strict as $y \neq 0$.

Thus $C_0 \neq C_1$ as, for example, in $\text{su}(2,1)$

$$\begin{pmatrix}
-3i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2i
\end{pmatrix} \in C_1 - C_0,$$
and other examples show $\overline{C}_0 \neq \overline{C}_1$.

$\overline{C}_0, \overline{C}_1 \in \mathbb{C}$ is clear from Lemma 1.1.

To show $C_1$ is open we need the fact, proven by
classification of the orbits in the next section, that each
$X \in C_1$ is conjugate to a $Y = \begin{pmatrix}
-i\lambda & 0 \\
0 & i\sigma
\end{pmatrix} \in \mathfrak{h}$ where all
$\lambda_i + \sigma_j > 0$. Then $\exists \varepsilon > 0$ such that $Y - \varepsilon Z \in C_1$ also,
so $Y \in (Y - \varepsilon Z) + C_0 \subset C_1$, so $Y$, hence $X$, is in the
interior of $C_1$.

The last assertion about open cones follows just as
in the symplectic case from the maximality of $\overline{C}_1$, which we
now prove. Let $C \in \mathbb{C}$ be closed and contain $C_0$. Clearly then
$-C_0 \subset C^*$ so $-\overline{C}_0 \subset C^*$, hence $C \subset -\overline{C}_0 = C_1$ by Theorem 12.3
below and Theorem 1.4.
Theorem 12.3. \( \overline{C}_0^* = -\overline{C}_1 \).

Proof. \( \overline{C}_0^* = -\overline{C}_1 \) follows from the minimality of \( C_0 \):

\( \overline{C}_1 \in C \) and \( -C_0 \subset \overline{C}_1^* \), so \( -\overline{C}_0 \subset \overline{C}_1^* \), hence \( -\overline{C}_1 \subset \overline{C}_0^* \) by duality.

If now \( X \not\in -\overline{C}_1 \) \( \forall v \neq 0 \) such that \( \kappa(v,v) = 0 \) and \( iH(Xv,v) > 0 \). By Theorem 10.1 \( B_\theta(X,Y) > 0 \) for some \( Y \in \overline{C}_0 \), so \( X \not\in \overline{C}_0^* \) also.

Corollary 12.4. \( C_1 \) (resp. \( C_0 \)) is the topological interior of \( \overline{C}_1 \) (resp. \( \overline{C}_0 \)).

Proof. The assertion for \( C_1 \) follows as before (Corollary 8.3) from the maximality of \( C_1 \). We cannot use this argument for \( (\overline{C}_0)^0 \), however. But \( (\overline{C}_0)^0 \subset C_1 \), and if \( X \in (\overline{C}_0)^0 \subset \overline{C}_0 \), \( X \) is conjugate to a \( \begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix} \) such that all \( \lambda, \sigma \geq 0 \), again by the fact we prove in the next section. But in fact all \( \lambda, \sigma \) must be positive as \( (\overline{C}_0)^0 \) is open and in \( \overline{C}_0 \).

Remark. One can see already that \( \overline{C}_0 \) and \( C_1 \) intersect nontrivially by examining particular diagonal matrices.

13. Orbits in \( \overline{C}_1 \) of \( su(p,q) \).

It is clear from our description of \( \overline{C}_1 \) in the last section that, in examining which types \( \Delta \) can lie in \( \overline{C}_1 \), we obtain only necessary conditions on \( \Delta \) by considering the possible indecomposable types in \( \overline{C}_1 \). Also, we must
examine all the orbits as it is not yet clear that all elements of \( C_1 \) are elliptic. Still, the necessary conditions are quite restrictive on the possible indecomposable types.

**Theorem 13.1.** Among the indecomposable types for \( u(p,q) \), only \( A^\pm(\zeta) \) and \( A^\pm(\zeta) \) \((\zeta = -\zeta)\) are in \( \pm C_1 \). (We define \( C_1, C_0 \), etc. for \( u(p,q) \) the same as for \( su(p,q) \).)

**Proof.** Consider first a type \( \Delta_m(\zeta,-\zeta) \) for \( \zeta \neq -\zeta \), containing \((A = S+N,V)\) where \( V = E + N\varepsilon + \ldots + N^m \varepsilon \).

\( E \) is two-dimensional, and there is a basis \( \{e,f\} \) for \( E \) such that \( H(f,N^m f) = H(e,N^m e) = 0 \) and \( H(e,N^m f) = 1 \).

Let \( a, b \in \mathbb{C} \). If \( m > 1 \), then only the cross-terms in the following expression contribute:

\[
H(A(aN^{m-1}f + be),aN^{m-1}f + be) = -21 \text{Im} \, ba
\]

and this is of indeterminant sign.

If \( m = 0 \), let \( A e = \zeta e, A f = -\zeta f \), \( H(e,e) = H(f,f) = 0 \), and \( H(e,f) = 1 \). Then \( H(ae+bf,ae+bf) = 2 \text{Re} \, ab \), so let \( a\overline{b} = ci, c \in \mathbb{R} \). Then \( H(A(ae+bf),ae+bf) = 2ci \text{Re} \, \zeta, \text{Re} \, \zeta \neq 0 \), and \( c > 0 \) or \( c < 0 \) are possible.

Finally, consider \( \Delta_m^e(\zeta) \) for \( \zeta = -\zeta \), \( V \) as above. \( E \) is one-dimensional, \( A = S+N \), and \( S \) has eigenvalue \( \zeta \). The case \( m \geq 3 \) is quickly ruled out just as in the symplectic case. Next let \( m = 2 \). If \( u,v,w \in E \) set \( y = N^2 v + Nw + u \), and note \( H(y,y) = 2 \text{Re} \, H(u,N^2 v) - H(w,N^2 w) \)
and \( H(Ay,y) = \zeta(2 \Re H(u,N^2v) - H(w,N^2w)) + 2\Im H(N^2w,u) \).

Take \( u,v,w \) all nonzero such that \( H(y,y) = 0 \). As the phase of \( w \) is rotated this is maintained but \( iH(Ay,y) \) takes both positive and negative values.

The remaining cases are indeed in \( \pm \mathfrak{C} \), and lie in \( u(l,1) \) or \( u(1) \).

To aid in understanding these types \( \Delta^{\pm}_{0,1}(\zeta) \), we record a convenient isomorphism between \( \mathfrak{sp}(1,\mathbb{R}) \) and \( \mathfrak{su}(1,1) \). It is given by conjugating by a particular 2x2 unitary, coming from a "Cayley transform". In fact, we lose nothing in letting our 2x2 matrices have square matrix entries.

Let \( A, B, \) and \( C \) be real nxn matrices, \( B \) and \( C \) symmetric, and let \( D = A + \lambda iI, \lambda \in \mathbb{R} \). Then
\[
\begin{pmatrix}
D & B \\
C & -D^*
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & -A^t
\end{pmatrix}
+ \lambda i \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
= \mathfrak{C}.
\]

Let \( U = (1/\sqrt{2}) \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \), so \( U^{-1} = U^* = (1/\sqrt{2}) \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \).

Then
\[
\mathfrak{sp}(n,\mathbb{R}) \text{ in } \mathfrak{su}(n,n).
\]

If \( n = 1 \) then \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) + \mathbb{R}iI \), corresponds, under a Lie algebra isomorphism, to
\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix} + \frac{1}{2}\begin{pmatrix}
1(b-c) & -2a+1(b+c) \\
-2a-1(b+c) & 1(b-c)
\end{pmatrix} = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix} + \begin{pmatrix}
di & \beta \\
\beta & di
\end{pmatrix}.
\]

Recall \( \begin{pmatrix} a \\ b \\ c \\ -a \end{pmatrix} \) represents a positive elliptic type iff \( b < 0, c > 0 \), and \( -a^2 - bc > 0 \), and a positive nilpotent type iff \( b > 0, c > 0, c-b > 0 \), and \( -a^2 - bc = 0 \). These conditions correspond for

\[
\begin{pmatrix}
-di \\ \beta \\
\beta & di
\end{pmatrix} \in \text{su}(1,1) \] to \( d > 0, d > |\beta|, \) and \( d > 0, d = |\beta| \). Note \( \begin{pmatrix} -di \\ \beta \\ \beta & di \end{pmatrix} \) is conjugate under \( \text{SU}(1,1) \) to \( \begin{pmatrix} -d' \\ 0 \\ 0 & d' \end{pmatrix} \) iff \( |d| > |\beta| \).

We relate the types \( \Delta_{0,1}(i\lambda) (\lambda \in \mathbb{R}) \) to particular matrix representatives. Recall the notation \( \Delta_{0,1}(\zeta) \) means \( E \) is spanned by \( e \) with \( H(e,e) = \pm 1 \).

**In** \( \mathbb{C}_0 \). \( \Delta^+(i\lambda) \) for \( \lambda < 0 \)

and \( \Delta^-(i\lambda) \) for \( \lambda > 0 \).

Both correspond to the 1x1 \( (i\lambda) \).

**In** \( \mathbb{C}_0 - \mathbb{C}_0 \) and \( \mathbb{C}_1 \) (vacuously). \( \Delta(0) \); (0).

**In** \( \mathbb{C}_0 - \mathbb{C}_1 \). \( \Delta^+_0(0) \); corresponds to, say \( \begin{pmatrix} -i \\ i \\ -i \\ i \end{pmatrix} \).

**In** \( \mathbb{C}_1 - (\mathbb{C}_1 \cup \mathbb{C}_0) \). \( \Delta^+_1(i\lambda) \) for \( \lambda \neq 0 \) corresponds to

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix} + \begin{pmatrix}
-i \\ i \\
-i \\ i
\end{pmatrix}.
\]

The above facts follows from the lemma below, proved by computation.
Lemma 13.2. Let $X = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} -d & \beta \\ \beta & d \end{pmatrix} \in \mathfrak{u}(1,1)$. Then $X \in \mathbb{C}_0$ iff $d > 0$ and $d^2 > \lambda^2 + |\beta|^2$ (a consequence is $d-\lambda, d+\lambda > 0$), $X \in \overline{\mathbb{C}}_0$ iff $d \geq 0$ and $d^2 \geq \lambda^2 + |\beta|^2$, $X \in \mathbb{C}_1$ iff $d > 0$ and $d > |\beta|$, and $X \in \overline{\mathbb{C}}_1$ iff $d \geq 0$ and $d \geq |\beta|$.

Finally, we list the types in $\overline{\mathbb{C}}_1$ for $\mathfrak{su}(p,q)$.

Proposition 13.3. The types $\Delta$ in $\mathbb{C}_1$ are of the form

$$(7) \quad \Delta = \Delta^+(-i\lambda_1) + \ldots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_1) + \ldots + \Delta^-(i\sigma_q)$$

where all $\lambda_i + \sigma_j > 0$ and $\Sigma_1 \lambda_1 = \Sigma_j \sigma_j$. $\Delta$ is furthermore in $\mathbb{C}_0$ iff all $\lambda_i, \sigma_j > 0$.

The types $\Delta$ in $\overline{\mathbb{C}}_0$ are of either of two forms:

1) (7) with all $\lambda_i, \sigma_j > 0$ (and such a $\Delta$ may also be in $\mathbb{C}_1$), or

2)

$$(8) \quad \Delta = \Delta^+(-i\lambda_1) + \ldots + \Delta^+(-i\lambda_q) + \Delta^+_1(-i\lambda) + \ldots + \Delta^+_1(-i\lambda)$$

$$+ \Delta^-(i\sigma_1) + \ldots + \Delta^-(i\sigma_q - \lambda)$$

for $\lambda = 0$, some $\lambda$ with $1 \leq \lambda \leq q$, all $\lambda_i, \sigma_j > 0$ and $\Sigma_1 \lambda_1 = \Sigma_j \sigma_j$. Such a type is on the boundaries of $\overline{\mathbb{C}}_0$ and $\overline{\mathbb{C}}_1$.

The remaining types in $\overline{\mathbb{C}}_1$ are those either of the form (7) with all $\lambda_i + \sigma_j > 0$, $\Sigma_1 \lambda_1 = \Sigma_j \sigma_j$, and some $\lambda_i = -\sigma_j \neq 0$, or (8) for $\lambda \neq 0$, some $\lambda$ with $q \geq \lambda \geq 1$. 
\( \lambda_i \) and \( \sigma_j \) satisfying \( \lambda_i \geq \lambda \geq -\sigma_j \) \( \forall i,j \), and
\[
2\lambda + \sum_{i=1}^{p-l} \lambda_i = \sum_{j=1}^{q-l} \sigma_j .
\]

Proof. It is clear from the classification of indecomposable types in \( \mathfrak{C}_1 \) and the computation in the proof of Lemma 12.2.

14. Classification of Open and Closed Cones in \( \mathfrak{su}(p,q) \).

To classify some of the invariant convex cones in \( \mathfrak{a}_j \) we need a convexity result (Theorem 14.2), the proof of which depends on the

Lemma 14.1. All orbits in \( \mathfrak{C}_1 \) are closed.

Proof. Let \( Y \in \mathfrak{C}_1 \), and suppose
\[
X = \begin{pmatrix} -i\lambda & \beta \\ \beta^* & i\sigma \end{pmatrix}
\]
is in the \( K \)-orbit of \( Y \), for Hermitian \( \lambda, \sigma \). We need some measure of the finite distance of the \( K \)-orbit of \( Y \) from the boundary of \( \mathfrak{C}_1 \) : clearly \( \exists k > 0 \) such that
\[
iH(Yv,v) \geq k\|v\|^2 \quad \forall v \in \mathfrak{g}^N \text{ with } H(v,v) = 0 , \quad \| \cdot \| \text{ the usual (} K \text{-invariant) norm on } \mathfrak{g}^N . \text{ The same holds for } X , \text{ and a simple computation shows that a consequence is }
\[
(\lambda_{ii} + \sigma_{jj})/2 - |\beta_{ij}| \geq k
\]
for \( i,j \) satisfying \( 1 \leq i \leq p , \ 1 \leq j \leq q . \)

Now let \( \text{Ad}(g_m)Y \to \mathfrak{g} \in \mathfrak{a}_j \) as \( m \to \infty \). We can write
according to $G = K\mathbb{A}^+K$. Let

$$a = \left( \begin{array}{c} \cosh \theta_j \\ \sinh \theta_j \end{array} \right) \in \mathbb{A}^+ \ , \ \theta_j \geq 0 \ .$$

Letting $\text{Ad}(a)X = \left( \begin{array}{c} L \\ M \end{array} \right)$, then for $1 \leq j \leq q$,

$$L_{jj} = -i\lambda_{jj} \cosh^2 \theta_j - i\sigma_{jj} \sinh^2 \theta_j - 2i \sinh \theta_j \cosh \theta_j \text{Im} \theta_{jj}$$

and

$$M_{jj} = i\sigma_{jj} \cosh^2 \theta_j + i\lambda_{jj} \sinh^2 \theta_j + 2i \sinh \theta_j \cosh \theta_j \text{Im} \theta_{jj} .$$

Also,

$$iL_{jj} - iM_{jj} \geq \cosh 2\theta_j (\lambda_{jj} + \sigma_{jj}) - 2|\sigma_{jj}| \sinh 2\theta_j$$

$$\geq (\sinh 2\theta_j)2k \ ,$$

and note $|x|^2 + |y|^2 \geq \frac{1}{2}|x \pm y|^2 \forall \ x, y \in \mathbb{C}$.

Now return to the $Y_m = \text{Ad}(g_m)Y$. The $\text{B}_\theta$-norms of the $Y_m$ remain bounded by some fixed number as $m \to \infty$, and $\text{B}_\theta$ is $K$-invariant. The above estimate shows that the allowable $\mathbb{A}^+$ components in $g_m$ must lie in some compact set. As $K$ is compact the $g_m$ have some convergent subsequence, proving the lemma.

We recall a result of Horn [11]: given an Hermitian matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, the set of diagonal entries of all conjugates $UAU^{-1}$, $U$ unitary, is precisely the convex hull of the points $(\lambda_p(1), \ldots, \lambda_p(n))$, $p$ a permutation of $(1, \ldots, n)$. We will only need the (presumably) easier half of this theorem, where "precisely"
is replaced by "contained in".

**Definitions.**

1) Let \( V, W = (\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q) \in \mathbb{R}^{p+q} \). say \( V \) is in the **noncompact convex hull of** \( W \) if \( V \) is equal to a vector in the convex hull of the set

\[
\{(\lambda_r(1), \ldots, \lambda_r(p), \sigma_s(1), \ldots, \sigma_s(q)) : \text{r,s permutations of } (1, \ldots, p), (1, \ldots, q) \text{, respectively}\}
\]

plus a vector in the closed positive orthant of \( \mathbb{R}^{p+q} \).

Clearly this relation is transitive, and the noncompact convex hull of a point is closed.

2) If \( X = (\mathbf{a}, \mathbf{b}) \in \mathcal{F} \), call \( R = \sqrt{\sum_i \sum_j |a_{ij}|^2} \) the \( p \)-norm of \( X \) (although it is not a norm on \( \mathcal{F} \)).

**Theorem 14.2.** Let \( X \in C_1 \), and let \( X \) represent the type

\[
\Delta = \Delta^+(-i\lambda_1) + \ldots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_1) + \ldots + \Delta^-(i\sigma_q)
\]

as in Proposition 13.3. Let the diagonal elements of \( X \) be \((-i\lambda_1, \ldots, -i\lambda_p, i\sigma_1, \ldots, i\sigma_q)\). Then \((a_1, \ldots, a_p, b_1, \ldots, b_q)\) is in the noncompact convex hull of \((\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q)\).

The proof is dependent on the following

---

*I thank Reneé Chipman for an accidental wish for a "good flow" which helped to get this theorem and its lemma unstuck.*
Lemma 14.3. Let $X \in C_1 - k$. Then there exists $Y$ in the orbit of $X$ with strictly smaller $p$-norm such that the diagonal of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}X$ is in the non-compact convex hull of the diagonal of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Y$.

Proof. By Horn's theorem we may assume

$$X = \begin{pmatrix} -i\lambda & \beta \\ \gamma^* & i\sigma \end{pmatrix}$$

where $\lambda$ and $\sigma$ are diagonal. (The $p$-norm is $K$-invariant.) As in the proof of Lemma 14.1, $\lambda_i + \sigma_j > 2|\beta_{ij}| \ \forall i,j$. Set $\gamma_{ij} = i\delta_{ij}/(\lambda_i + \sigma_j)$, and consider

$$\text{Ad}(\exp \varepsilon P)X = Y(\varepsilon) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & \gamma \\ \gamma^* & 0 \end{pmatrix} \in \mathcal{S}$$

We have

$$A = -i\lambda + \varepsilon(\gamma\beta^* - \beta\gamma^*) + O(\varepsilon^2),$$

$$B = \varepsilon(\gamma^*\sigma - \sigma^*\gamma) + O(\varepsilon^2),$$

and

$$C = i\sigma + \varepsilon(\gamma^*\beta - \beta^*\gamma) + O(\varepsilon^2).$$

The diagonal entries of $A$ and $C$ are

$$-i\lambda_i + 2i\varepsilon \sum_{j=1}^{q} (|\delta_{ij}|^2/(\lambda_i + \sigma_j)) + O(\varepsilon^2),$$

and

$$i\sigma_j - 2i\varepsilon \sum_{i=1}^{p} (|\delta_{ij}|^2/(\lambda_i + \sigma_j)) + O(\varepsilon^2).$$
and we have $B_{1j} = (1-\varepsilon)\beta_{1j} + O(\varepsilon^2)$.

Note that if $\sum_{j=1}^{q} |\beta_{1j}|^2 = 0$ for some $1$, then the $i$th diagonal element of $A$ is equal to $-i\lambda_i$ for all $\varepsilon$, and likewise for the $j$th diagonal element of $C$ if $\sum_{i=1}^{p} |\beta_{ij}|^2 = 0$. Thus if $\varepsilon > 0$ is small enough $Y(\varepsilon)$ satisfies the requirements of the lemma.

Proof of Theorem 14.2. It is not clear that the lemma above can be strengthened to produce a sequence converging to an element of $\mathfrak{k}$, but Zorn's lemma can be used.

First define some notation. If $Y, Z \in \mathfrak{g}$ write $Y \geq Z$ if the $p$-norm of $Y$ is less than the $p$-norm of $Z$, and the diagonal of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Z$ is in the noncompact convex hull of the diagonal of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Y$. Let $\Omega = \{Y \in \text{Ad}(G)X: Y \geq X\}$. We apply Zorn's lemma to $(\Omega, \geq)$. By Lemma 14.3 it is clear that a $\mathfrak{z}$-maximal element $Y$ of $\Omega$ must be in $\mathfrak{k}$, and if such an element exists the $K$-diagonalization of $Y$ exhibits the type of $X$, proving the theorem.

To prove the hypothesis of Zorn's lemma, first note that $\Omega$ is a bounded subset of $\mathfrak{g}$, as the Killing form is constant on the orbit of $X$. Let $\hat{\xi} \in \Gamma$ be a simply ordered subset of $\Omega$, considered as a net indexed by itself. The closure of $\hat{\xi}$ is compact, so $\hat{\xi}$ has a subnet $\hat{\xi}$ converging to $Y$. $Y$ is in the orbit of $X$ by
Lemma 14.1, and clearly the $p$-norm of $Y$ is \leq the $p$-norm of any $Z \in \tilde{\mathcal{T}}$, hence any $Z \in \tilde{\mathcal{T}}$ by the cofinality of $\tilde{\mathcal{T}}$.

Finally, if $Z \in \tilde{\mathcal{T}}$ express the diagonals of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Z$ as a vector in the noncompact convex hull of the diagonals of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}W$, where $W$ is any element of $\tilde{\mathcal{T}}$ where $W \geq Z$. The set of possible nonnegative vectors in these expressions is bounded as $\Omega$ is bounded, so the expressions must have a convergent subnet. The limit expresses the diagonals of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Z$ as a vector in the noncompact convex hull of the diagonals of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Y$. Thus $Y \in \Omega$ and $Y \geq Z \forall Z \in \tilde{\mathcal{T}}$, proving the hypothesis of Zorn's lemma.

It is now not difficult to classify the open and closed invariant convex cones in $\mathcal{F}$. We need some notation.

Let

$$E_1 = \{ (\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q) \in \mathbb{R}^{p+q} : \sum_i \lambda_i = \sum_j \sigma_j \text{ and } \lambda_i + \sigma_j \geq 0 \forall i, j \}$$

and $E_0 = \{ e \in E_1 : \text{all components of } e \geq 0 \}$. Let $\varphi$ be the group of permutations of the $\lambda$'s and permutations of the $\sigma$'s acting on $E_1$. Let $C_+ = \{ C \in \mathcal{C} : C_0 \subset C \subset \overline{C}_1 \}$, $C^0_+ = \{ C \in \mathcal{C}_+ : C \text{ is open} \}$, $C^c_+ = \{ C \in \mathcal{C}_+ : C \text{ is closed} \}$, $\delta^0 = \{ \text{all } \varphi\text{-invariant open convex cones in } E_1 \text{ containing } (E_0)^0 \}$, and let $\delta^c = \{ \text{all } \varphi\text{-invariant closed convex cones in } E_1 \text{ containing } E_0 \}$.
Let $\mathcal{I}$ be the collection of all bases $\{(e_i, f_j): i = 1, \ldots, p, j = 1, \ldots, q\}$ of $\mathbb{T}^n$ such that $H(e_i, e_j) = \delta_{ij}$, $H(f_j, f_k) = -\delta_{jk}$ and $H(e_i, f_j) = 0 \forall i, j$. Given $\varphi = (e_i, f_j) \in \mathcal{I}$ define

$$M_\varphi: \mathbb{R}^p \to \mathbb{R}^{p+q}:$$

$$X \to (iH(Xe_1, e_1), \ldots, iH(Xe_p, e_p), iH(Xf_1, f_1), \ldots, iH(Xf_q, f_q)).$$

As noted previously, $M_\varphi(X) \in E_1$ if $X \in C_1$, and $M_\varphi(X) \in (E_1)^\circ$ if $X \in C_1$.

**Theorem 14.4.** Let $C_3 \in C^O_+$ and $C_4 \in C^C_+$. Let $E_{C_i}$ be the convex cone generated by $\{M_\varphi(X): X \in C_1, \varphi \in \mathcal{I}\}$ for $i = 3, 4$. Then $E_{C_3} \in \mathfrak{S}^O$, $E_{C_4} \in \mathfrak{S}^C$, and these mappings

$$E: C_+^O \to \mathfrak{S}^O$$

$$E: C_+^C \to \mathfrak{S}^C$$

are bijections.

**Proof.** We deal with the open cones first. To see that $E_{C_3}$ is open we show that $E_{C_3}$ contains a neighborhood of each $M_\varphi(X), X \in C_3, \varphi \in \mathcal{I}$. By Theorem 14.2, $M_\varphi(X)$ is a nonnegative vector plus a convex combination of other vectors $S = \{\lambda_r(1), \ldots, \lambda_r(p), \sigma_s(1), \ldots, \sigma_s(q)\}$ in $E_{C_3}$ coming from the type of $X$. $E_{C_3}$ must contain a neighborhood of each element of $S$ as the intersection of $C_3$ with $h$ must be open in $h$. $E_{C_3}$ also contains
all positive vectors as $C_0 \subseteq C_3$. A simple picture now shows $M_\phi(X)$ to be in the interior of $E_{C_3}$. Thus $E_{C_3}$ is open and clearly then $E_{C_3} \in \mathcal{S}^O$.

To show surjectivity, let $D \in \mathcal{S}^O$, and let $C_3 = \{X \in C_1: M_\phi(X) \in D \ \forall \ \phi \in \Phi\}$. Clearly $C_3$ is invariant, convex, and contains $C_0$. To show $C_3$ is open, we may take $X \in \mathcal{h} \cap C_3$. If $\varepsilon > 0$ such that the diagonal of $(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})(X - \varepsilon Z)$ is in $D$. By Theorem 14.2 $M_\phi(X - \varepsilon Z) \in D \ \forall \ \phi \in \Phi$, so $X - \varepsilon Z \in C_3$, and $X \in (X - \varepsilon Z) + C_0 \subseteq C_3$ proves $X \in (C_3)^O$. Thus $C_3 \in C^O$. That $E_{C_3} = D$ follows from Theorem 14.2.

If now $C_2 \in C^O$ and $E_{C_2} = D$ then clearly $C_2 \subseteq C_3$. By Theorem 14.3 $E_C$ is simply the set of diagonals of $(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})(C \cap \mathcal{h})$ for any open cone $C$, so $C_2$ and $C_3$ clearly contain the same elliptic orbits and are equal. Thus $E$ is a bijection for the open cones.

Now let $C_4 \in C^O_+$. The set $S$ of diagonals of $(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})(C_4 \cap \mathcal{h})$ is closed and in $E_{C_4}$; we will see that it is all of $E_{C_4}$. If $M_\phi(X) \notin S$ for some $X \in C_4$, $\phi \in \Phi$, the same is true for $X + \varepsilon Z \in C_4 \cap \mathcal{C}_1$ for some $\varepsilon > 0$, and Theorem 14.2 may be applied to reach a contradiction. Thus $E_{C_4} \in \mathcal{S}^C$.

To show surjectivity, take $D \in \mathcal{S}^C$ and let $C_4 = \{X \in \mathcal{C}_1: M_\phi(X) \in D \ \forall \ \phi \in \Phi\}$. Then $C_4 \in C^C_+$ is automatic, and clearly $E_{C_4} \subseteq D$. Further, equality holds again by an application of 14.2 to $X + \varepsilon Z$. 
Finally, suppose $E_{c_2} = D$ for $c_2 \in c_2^c$. $c_2 \subset c_4$ again, and $E_{c_2} = E_{c_4}$ implies $c_2 \cap h = c_4 \cap h$, so $c_2$ and $c_4$ have the same elliptic orbits. Suppose then

$$\Delta = \Delta^+(-i\lambda) + \ldots + \Delta^+(-i\lambda) + \Delta_e$$

where

$$\Delta_e = \Delta^+(-i\lambda_{l+1}) + \ldots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_{l+1}) + \ldots + \Delta^-(i\sigma_q)$$

as in Proposition 13.3, is a nonelliptic type in $c_4$. Then

$$\tilde{\Delta} = \Delta^+(-i\lambda) + \ldots + \Delta^+(-i\lambda) + \Delta_e$$

is an elliptic type in $c_4$, as $c_4$ is closed, so $\tilde{\Delta}$ is in $c_2$, too. But then $\Delta$ is obtained back from $\tilde{\Delta}$ by adding on $\Delta^+_l(0) + \Delta^+_l(0) + \ldots + \Delta^-(0)$ (which is in $c_0 \subset c_2$) the appropriate number of times. Thus $c_2 = c_4$, and the proof is complete.

The classification of all invariant convex cones in $\mathcal{C}$ does not seem hopeless, and probably involves ideas similar to those successful for $sp(n,R)$. Just as for those groups, the set of elliptic orbits in the closed maximal cone do not form a convex cone: we saw that a convex combination of $sp(2,R)$ transforms of

$$(0 -1) \oplus (0 0)$$

is

$$(0 -1) \oplus (0 0).$$

There is a similar symmetry here, existing in any $SU(2,2)$ subgroup of $SU(p,q)$. If $(e_1, e_2, f_1, f_2) \in \mathcal{F}$, then one may check that
\[ \begin{align*}
E_1 &= e_1 - k(e_2 - f_2) & \quad & E_2 = e_2 + k(e_1 + f_1) \\
F_1 &= f_1 + k(e_2 - f_2) & \quad & F_2 = f_2 + k(e_1 + f_1)
\end{align*} \]

for \( k \) real, is also orthonormal in \((\mathbb{C}_4, \mathbb{H})\). It comes from the nilpotent

\[
\begin{pmatrix}
0 & k & 0 & k \\
-k & 0 & k & 0 \\
0 & k & 0 & k \\
k & 0 & -k & 0
\end{pmatrix}
\]

in \( \text{su}(2,2) \), which has square zero.

Furthermore, one may calculate that averaging the transforms of the elliptic type \( \Delta^+(-i\lambda) + \Delta^+(-i\lambda) + \Delta^-(i\lambda) + \Delta^-(i\lambda) \) for \( k \) positive and negative gives \( \Delta^+(-i\lambda) + \Delta^+(-i\lambda) + \Delta^-(i\lambda) \). There may be other \( \text{SU}(p,q) \) symmetries involved in classifying the invariant cones in \( \text{su}(p,q) \), but based on the analogy with \( \text{sp}(n,\mathbb{R}) \) I suspect not.

Finally, there is a simple geometric picture that one can use to see how the various closed and open invariant cones in \( \text{su}(2,2) \) overlap. We consider the slice \( \lambda_1 + \lambda_2 = \sigma_1 + \sigma_2 = 1 \) in \( E_1 = \{ (\lambda_1, \lambda_2, \sigma_1, \sigma_2) : \lambda_1 + \sigma_j \geq 0, \lambda_1 + \lambda_2 = \sigma_1 + \sigma_2 \} \), perpendicular to \((1,1,1,1)\), regarded as a \( z \)-axis.

Define \( x = \lambda_1 - \lambda_2 \) and \( y = \sigma_1 - \sigma_2 \). As closed convex cones, \( E_{O_1}^{-\sigma_1} \) is generated by \((1,0,1,0), (1,0,0,1), (0,1,1,0), \) and \((0,1,0,1)\), and \( E_{C_1}^{-\sigma_1} \) is generated by
\( \frac{1}{3}(3,-1,1,1) \), \( \frac{1}{3}(-1,3,1,1) \), \( \frac{1}{3}(1,1,3,-1) \), and \( \frac{1}{3}(1,1,-1,3) \).

(In general, \( C_0 \) has \( pq \) generators and \( C_1 \) has \( p+q \) generators.) They correspond in the \((x,y)\) plane to \( \{(x,y): x = \pm 1, y = \pm 1\} \) and \( \{(0,\pm 2), (\pm 2,0)\} \), respectively.

We have

\[
\begin{array}{c}
\text{\( S_1 \)} \\
\text{\( S_2 \)}
\end{array}
\]

The minimal cone corresponds to the inner square \( S_1 \), the maximal to the outer \( S_2 \). Closed invariant convex cones correspond to closed convex sets containing \( S_1 \) and contained in \( S_2 \) which are invariant under reflections about the \( x \) and \( y \) axes.

The outer automorphisms of \( su(2,2) \) are generated by the inner as well as "time reversal" \( X \to \bar{X} \) and "space reversal" \( X \to (0 \ I) \bar{X} (0 \ I) \). The former takes \( C_0 \) to \( -C_0 \), and the latter has the effect in the diagram of a reflection about \( x=y \), or, up to inner automorphisms, a 90° rotation. Thus there are clearly lots of invariant convex cones also invariant under that second outer automorphism.
CHAPTER IV. Invariant Convex Cones in $\mathfrak{so}^*(2n)$.

15. Conventions for $\mathfrak{so}^*(2n)$.

We take $n \geq 3$, and let $\lambda = \lfloor n/2 \rfloor$, the greatest integer $\leq n/2$. As in section 3,

$$\sigma = \left\{ \begin{array}{c} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : \text{A skew, B Hermitian} \\ \end{array} \right\}$$

$$\mathfrak{k} = \left\{ X \in \sigma : \text{X real} \right\} = \left\{ \begin{array}{c} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : \text{A skew, B symmetric} \\ \end{array} \right\}$$

$$\mathfrak{p} = \left\{ X \in \sigma : \text{iX real} \right\} = \left\{ \begin{array}{c} \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \end{array} \right\}$$

$$\mathfrak{h} = \left\{ \begin{array}{c} \begin{pmatrix} O & D \\ -D & O \end{pmatrix} \end{array} : \text{D diagonal} \right\}$$

$$\mathfrak{c} = \mathbb{R} \mathbb{Z} \text{ where } Z = \begin{pmatrix} 0 & -\frac{1}{2} \mathbb{I} \\ \frac{1}{2} \mathbb{I} & 0 \end{pmatrix}.$$  

Let $d_j$ $(j = 1, \ldots, n)$ be linear functionals spanning the dual of $\mathfrak{h}$:

$$d_j : \begin{pmatrix} O & D \\ -D & O \end{pmatrix} \rightarrow D_{jj}.$$  

Then $\{\text{noncompact roots}\} = \{ \pm i(d_j + d_k) : 1 \leq j < k \leq n \}$

$\{\text{compact roots}\} = \{ \pm i(d_j - d_k) : 1 \leq j < k \leq n \}$

and the positive roots we take to be the above with $"+"$ only.

$$\Delta^+_0 = \{ i(d_1 + d_2), i(d_1 - d_2), \ldots, i(d_{n-1} - d_n) \}.$$
Let \( 1 \leq j < k \leq n \). For \( \alpha = i(d_j + d_k) \),
\[
H_\alpha = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad \text{and} \quad Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},
\]
where \( D = -iE_{jj} - iE_{kk}, A = -iE_{jk} + iE_{kj} \), and
\( B = -iE_{jk} + iE_{kj} \).

We take \( \Sigma_0 = \{i(d_1 + d_2), i(d_3 + d_4), \ldots, i(d_{2\ell-1} + d_{2\ell})\} \),
so
\[
h^- = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in h : \begin{array}{l}
D_{11} = D_{22}, \ldots, D_{2\ell-1,2\ell-1} = d_{2\ell-1,2\ell} \\
D_{2\ell+1} = 0 \ (\text{if it exists})
\end{array} \right\}
\]
\[
h^+ = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in h : \begin{array}{l}
D_{11} = -D_{22}, \ldots, D_{2\ell-1,2\ell-1} = -D_{2\ell-1,2\ell} \\
D_{2\ell+1} = 0 \ (\text{if it exists})
\end{array} \right\}
\]
\[
Z_0 = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in h \right\}
\]
where all \( D_{jj} = -\frac{1}{2} \) if \( n \) even;
\( D_{jj} = -\frac{3}{2} \) for \( j = 2\ell \), \( D_{nn} = 0 \),
if \( n \) odd,
so
\[
\alpha = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \in \mathfrak{h} : \begin{array}{l}
A = \begin{pmatrix} 0 & -c_i \\ c_i & 0 \end{pmatrix} \\
\ldots \\
\begin{pmatrix} 0 & -c_i \\ c_i & 0 \end{pmatrix}, c_j \in \mathbb{R}
\end{array} \right\}.
\]

Thus \( Z = Z_0 \) iff \( n \) even, and \( h^+ \neq 0 \) always.

The cone in \( h \) obtained from Corollary 3.2 is all
\[
\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in h\)
where \( \text{diag} \ D = (-d_1, -d_1, \ldots, -d_{2\ell}, -d_{2\ell}) \) for
\( d_j > 0 \) if \( n \) is even, and \( \text{diag} \ D = (-d_1, -d_1, \ldots, -d_{2\ell}, -d_{2\ell}, -e) \)
where \( 0 < e \leq d_j \) if \( n \) is odd.
Define \( B(X,Y) = \frac{1}{2} \text{Tr}(XY) \) for \( X, Y \in \mathfrak{g} \). If
\[
X = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \quad Y = \begin{pmatrix} C & D \\ -\overline{D} & \overline{C} \end{pmatrix} \in \mathfrak{g},
\]
\[
B(X,Y) = -\text{Re} \sum_{i,j} A_{ij} \overline{C}_{ij} - \text{Re} \sum_{i,j} B_{ij} \overline{D}_{ij}.
\]
The Cartan involution is \( X \rightarrow \overline{X} \), and
\[
B_\epsilon \left( \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \begin{pmatrix} C & D \\ -\overline{D} & \overline{C} \end{pmatrix} \right) = \text{Re} \left( \sum_{i,j} A_{ij} \overline{C}_{ij} \right) + \text{Re} \left( \sum_{i,j} B_{ij} \overline{D}_{ij} \right).
\]
Define the complex symmetric form \( \tau(\cdot,\cdot) \) on \( \mathbb{C}^{2n} \) by
\[
\tau(x,y) = \sum_{j=1}^{2n} x_j y_j
\]
and the symplectic form \( \sigma(\cdot,\cdot) \) on \( \mathbb{C}^{2n} \) by
\[
\sigma\left( (x,y), (u,v) \right) = \text{Re} \left( \sum_{j=1}^{n} (y_j \overline{u}_j - x_j \overline{v}_j) \right)
\]
for \( x,y,u,v \in \mathbb{C}^n \). In fact \( \mathfrak{g} \) is precisely the complex-linear endomorphisms of \( \mathbb{C}^{2n} \) skew with respect to \( \tau \) and \( \sigma \). Also, \( \sigma(\cdot,\cdot) = \text{Re} \tau(\cdot,\cdot,\sigma) \), where \( \sigma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot \mathcal{C} \) where \( \mathcal{C} \) is complex conjugation on \( \mathbb{C}^{2n} \). All elements of \( \mathfrak{g} \) commute with the anti-linear \( \sigma \).

Theorem 15.1. Let \( w \in \mathbb{C}^{2n} \) and \( X \in \mathfrak{g} \). Then \( \sigma(Xw,w) = B_\epsilon(X,Y) \) where \( Y \in \mathfrak{g} \), and \( \sigma(Yv,v) = 0 \) for all \( v \in \mathbb{C}^{2n} \).

Proof. Note \( \tau(Xu,\sigma u) \) is automatically real for \( X \in \mathfrak{g}, \)
\( u \in \mathbb{C}^{2n} \). Let \( w = \begin{pmatrix} u \\ v \end{pmatrix} \) and \( X = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \). Then
\[ a(xw, w) = \sum_1 \left\{ -(Bu)_1 u_1 + (Av)_1 u_1 - (Au)_1 v_1 - (Bv)_1 v_1 \right\} \]

\[ = \sum_{i,j} B_{ij} (-u_i u_j - v_i v_j) + Re \left\{ \sum_{i,j} A_{ij} (u_i v_j - u_j v_i) \right\} \]

\[ = B_\theta \left( \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \right) = B_\theta (X, Y) \]

where

\[ C_{ij} = u_i v_j - u_j v_i, \quad D_{ij} = -v_i v_j - u_i u_j. \]

Clearly \( Y \in q. \)

If \( v = (x, y) \in \mathbb{R}^{2n}, \)

\[ a(yv, v) = -\sum_{i,j} x_i D_{ij} y_j - \sum_{i,j} y_i C_{ij} x_j + \sum_{i,j} x_i C_{ij} y_j - y_i D_{ij} y_j \]

\[ = |<x, \tilde{v}>|^2 + |<x, u>|^2 + |<v, y>|^2 + |<y, \bar{u}>|^2 \]

\[ + 2 Re (\langle y, v \rangle <u, x>) - 2 Re (\langle x, \bar{v} \rangle <\bar{u}, y>) \]

\[ = |<x, \tilde{v}> - <u, \bar{y}>|^2 + |<y, v> + <x, u>|^2 \geq 0, \]

\( \langle \cdot, \cdot \rangle \) being the usual complex Hilbert structure on \( \mathbb{C}^n. \)

16. **Conjugacy of Positive Elliptics in so*(\mathbb{R}).**

Let \((H, \langle , \rangle_0)\) be a real Hilbert space and \(J: H \to H\) an orthogonal with square \(-I.\) Let \(\mathcal{H} = H \oplus iH\), and extend \(J\) to \(\mathcal{H}\) the obvious way so that \(iJ = Ji.\)

Extend \(\langle , \rangle_0\) to \(\mathcal{H}\) so that \((\mathcal{H}, \langle , \rangle_0)\) is a complex Hilbert space with complex structure \(i.\) We write

\[ ||x||^2 = \langle x, x \rangle_0 \] for \(x \in \mathcal{H},\) and \(||B||\) for the usual norm with respect to \(\langle , \rangle_0\) for any linear operator \(B.\) In
this section all operators are bounded and commute with \( i \).

Finally, extend \(< \cdot, \cdot >_o\) on \( H \) to \( \tau(\cdot, \cdot) \) on \( \mathcal{H} \) by complex bilinearity so that \( \tau \) is a complex symmetric form on \( \mathcal{H} \). If \( \mathcal{C} \) is the anti-linear conjugation of \( \mathcal{H} \) with respect to the decomposition \( H \oplus iH \) (\( \mathcal{C} = +I \) on \( H \), \(-I \) on \( iH \)), then \( \tau(\cdot, \cdot) = <\cdot, \mathcal{C}\cdot >_o \).

Letting \( \mathcal{J} = iJ \), \( \mathcal{J} \) is an Hermitian operator on \((\mathcal{H}, < \cdot, \cdot >_o)\) with spectrum \([+1, -1]\). In fact, the \(+1\) eigenspace of \( \mathcal{J} \) is \( \mathcal{H}_+ = \{ x + iJx : x \in H \} \) and the \(-1\) eigenspace is \( \mathcal{H}_- = \{ x - iJx : x \in H \} \). One can check that \( \tau \) vanishes when restricted to either \( \mathcal{H}_+ \) or \( \mathcal{H}_- \).

Our notation is chosen to correspond with sections 7 and 11, and this section is a further restriction on the situation in section 11. In particular, the \( J \), \( \mathcal{J} \), and \( < \cdot, \cdot >_o \) correspond.

Let \( \mathcal{C} = J\mathcal{C} = CJ \), an anti-linear operator with square \(-I\). We have \( \tau(\mathcal{C}\cdot, \mathcal{C}\cdot) = \overline{\tau(\cdot, \cdot)} \), and note that \( \mathcal{C} \) switches \( \mathcal{H}_+ \) and \( \mathcal{H}_- \). As before, we define

\[ a(\cdot, \cdot) = -\text{Im} <\cdot, \cdot >_o \]

a real symplectic form on \( \mathcal{H} \). Furthermore,

\[(9) \quad a(\cdot, \cdot) = \text{Re} \, \tau(\cdot, \mathcal{C}\cdot) \]

Defining \( <\cdot, \cdot>_o = \text{Re}<\cdot, \cdot>_o + ia(\cdot, \cdot) \), \((\mathcal{H}, < \cdot, \cdot>_o)\) is a complex Hilbert space with complex structure \( J \).

Define
\[ \text{SO}^*(\mathcal{V}) = \{ g : \mathcal{V} \to \mathcal{V} \mid g \text{ invertible, } g \text{ commutes with } i \text{ and } \sigma, \text{ and } \tau(g \cdot, g \cdot) = \tau(\cdot, \cdot) \} \]

and

\[ \sigma' = \{ B : \mathcal{V} \to \mathcal{V} \mid B \text{ commutes with } i \text{ and } \sigma \text{ and is skew for } \tau = \text{so}^*(\mathcal{V}) \} . \]

Note \( \tau \) could be replaced by the symplectic form \( \sigma \) in the above two definitions by (9). Also note \( J \in \sigma' \cap \text{so}^*(\mathcal{V}) \).

The condition defining the "maximal compact subgroup" \( K \) of \( \text{so}^*(\mathcal{V}) \) is the same as in the previous sections, i.e. \( g \in K \iff gJ = Jg \), which here is equivalent to \( g \) leaving the space \( H \) invariant. (Not the "K" in 16.2.)

**Lemma 16.1.** For all \( X \in \sigma' \), \( \sigma(A \cdot, \cdot) = \text{Re} \tau(A \cdot, \sigma \cdot) \) is a real symmetric form on \( \mathcal{V} \) for which \( i \) and \( \sigma \) are orthogonal. \( \tau(Ax, \sigma x) \in \mathbb{R} \forall x \in \mathcal{V} \).

**Theorem 16.2.** Let \( B \in \sigma' \) satisfy \( \sigma(Bx, x) \geq k <x, x> \forall x \in \mathcal{V} \), for some \( k > 0 \). Then there exists \( S \in \text{so}^*(\mathcal{V}) \) such that \( SBS^{-1} = JH_1 = H_1J \), where \( H_1 \) is a positive self-adjoint operator in \((\mathcal{V}, <, >)\) (and \((\mathcal{V}, <, >_\circ)\)). Also,

\[ (||B||^{3/2}/k^{1/2})||x||^2 \geq <H_1 x, x> \geq (k^{3/2}/||B||^{1/2})||x||^2 \forall x \in \mathcal{V} . \]

**Proof.** Again we follow the proof in section 7. Defining \( \sigma(\cdot, \cdot) = \sigma(B \cdot, \cdot) \), we polar-decompose \( B = K, B^2 \) in \((\mathcal{V}, \sigma)\). As \( i \) and \( \sigma \) are orthogonal for \( \sigma \) and commute with \( B \),
$K$ and $\sqrt{-1}$ also commute with $i$ and $\sigma$, and again $K \in \mathfrak{so}(\mathfrak{h}) \cap \mathfrak{so}^*(\mathfrak{h})$.

Defining $g_1(\cdot,\cdot) = \mathcal{A}(K \cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_1 = g_1(\cdot,\cdot) + i\mathcal{A}(\cdot,\cdot)$, $(\mathcal{H}, \langle \cdot, \cdot \rangle_1)$ is a complex Hilbert space with complex structure $K$. It suffices to find a unitary equivalence

$$S: (\mathcal{H}, \langle \cdot, \cdot \rangle_1) \to (\mathcal{H}, \langle \cdot, \cdot \rangle)$$

which commutes with both $i$ and $\sigma$. As before, $i$-linearity follows if $S$ takes the $+1$ eigenspace $\mathcal{H}_+$ (resp. $-1$ eigenspace $\mathcal{H}_-$) of $iK$ to $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Note $\sigma$ switches $\mathcal{H}_+$ and $\mathcal{H}_-$ also as $\sigma$ is anti-linear and commutes with $K$. Thus clearly all of $\mathcal{H}_\pm$ and $\mathcal{H}_\pm$ have the same dimension, and the desired $S$ is obtained by taking any unitary equivalence $(\mathcal{H}_+, \langle \cdot, \cdot \rangle_1) \to (\mathcal{H}_+, \langle \cdot, \cdot \rangle)$ and extending it to $\mathcal{H}$ by requiring it to commute with $\sigma$.

The rest of the theorem follows as before.

17. **Noncompact Convexity in $\mathfrak{so}^*(2n)$**.

Just as the orbits of interest in $\mathfrak{su}(p,q)$ were associated with the non-simple $\mathfrak{u}(1,1)$ and $\mathfrak{u}(1)$, we will have occasion to examine the orbits of $\mathfrak{so}^*(4) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}^*(2) \cong \mathfrak{so}(2)$ in classifying invariant convex cones in $\mathfrak{so}^*(2n)$. However, $n \geq 3$ is assumed in what follows unless otherwise stated.
To relate the notations of sections 15 and 16, we let $H = \mathbb{R}^{2n}$ with the usual inner product, $\mathcal{V} = \mathbb{C}^{2n}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ (regarding $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$), $C$ complex conjugation in $\mathbb{C}^{2n}$, and $\sigma = JC$. There are basically three forms on $\mathbb{C}^{2n}$ invariant under $SO^*(2n)$, all closely related:

1) the complex symmetric form $\tau$,  
2) the symplectic form $\mathcal{A}(:,\cdot) = \text{Re} \, \tau(:,\sigma \cdot)$ (both defined in section 15), and  
3) the Hermitian form $H_0(:,\cdot) = -i\tau(:,\sigma \cdot)$ with signature $(n,n)$ alluded to in section 16.

As with the earlier groups, we define a class of bases that are transformed by $G = SO^*(2n)$. Call

$$\mathcal{G} = \{(e_j, f_j)_{j=1}^n : \tau(e_k, e_j) = \tau(f_k, f_j) = 0 \forall k, l, \tau(e_j, f_k) = \delta_{jk}, \text{ and } \sigma e_j = if_j$$ 
$$\sigma f_j = -ie_j \}$$

the set of $G$-bases. Our standard basis is $e_j = (\tilde{e}_j + ie_{n+j})/\sqrt{2}$, $f_j = (\tilde{e}_j - ie_{n+j})/\sqrt{2}$ where $\tilde{e}_1, \ldots, \tilde{e}_{2n}$ are the standard unit vectors in $\mathbb{C}^{2n}$. For $\varphi = (e_j, f_j) \in \mathcal{G}$, note $\varphi$ is a basis for $H_0$ in the sense of section 14 for $su(p,q)$, and our notation is consistent.

We will need several observations about "matrix elements" $\mathcal{A}(\cdot, \cdot)$. Note that $\mathcal{A}(\sigma \cdot, \sigma \cdot) = \mathcal{A}(\cdot, \cdot)$, so $\forall X \in \mathcal{G}$ and $\forall (e_j, f_j) \in \mathcal{G}$ we have $\mathcal{A}(xe_i, e_i) = \mathcal{A}(xf_i, f_i)$.  

Also, if \( a_i = (e_i + y f_i)/\sqrt{2} \) for \(|y| = 1\), then
\[
\sigma(Xa_i, a_i) = \sigma(Xe_i, e_i).
\]
(Such \( \{a_i\} \) are quaternionic bases in the sense of section 4.)

In particular, if \( X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \subseteq \mathcal{G} \) and \( a_i \) is the unit vector \( \tilde{e}_i \), then \( \sigma(Xa_i, a_i) = -B_{ii} \). (Also, \( \sigma((-D)B, v, w) = \sum_j \text{D}_j (|v_j|^2 + |w_j|^2) \) where \( D \) is diagonal.) If \( e = \tilde{e}_1 \pm i\tilde{e}_2 \), then \( \tau(e, e) = \tau(e, se) = 0 \), and \( \sigma(Xe, e) = -B_{11} - B_{22} + 2 \text{Im} B_{12} \).

Any \( g \in SO^*(2n) \) has the form \( g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \) where \( A^t B = B^t A \) and \( A^t A + B^t B = I \), or equivalently \( AA^t + BB^t = I \) and \( AB^t = BA^t \). We have \( g^{-1} = \begin{pmatrix} A^t & -B^t \\ B^t & A^t \end{pmatrix} \), and \( g \) is in the maximal compact \( K \approx U(n) \) iff \( g \) is a real matrix.

**Definitions.**
1) Let
\[
C_2 = \{ X \in \mathcal{G} : \sigma(Xe, e) > 0 \forall e \neq 0 \text{ such that } \tau(e, e) = \tau(e, se) = 0 \}.
\]

2) If \( X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \subseteq \mathcal{G} \) call \( (-B_{11}, \ldots, -B_{nn}) \in \mathbb{R}^n \) the \underline{G-diagonal of} \( X \). (It is essentially the projection of \( X \) on the maximal torus \( h \).

3) Let
\[
E_0 = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : 0 \leq \lambda_i \leq \sum_{j=1}^{n} \lambda_j, i = 1, \ldots, n \}
\]
and \( E_2 = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i + \lambda_j \geq 0 \forall i \neq j \} \).
4) If $V, W = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, say $V$ is in the noncompact convex hull of $W$ if $V$ is equal to an element of $E_0$ plus a vector in the convex hull of $\{(\lambda_{p(1)}, \ldots, \lambda_{p(n)}) : p \text{ permutation of } (1, \ldots, n)\}$.

5) If $X = Y + Z$, $Y \in \mathbb{R}^n$, $Z \in \mathbb{P}$, call $\sqrt{B(Z, Z)}$ the $p$-norm of $X$.

Lemma 17.1. $X = (0 - D) \in C_2$ if $D$ is diagonal and $D_i + D_j > 0 \ \forall i \neq j$.

Proof. Let $e = (c + id, f + ig) \neq 0$ satisfy $\tau(e, e) = \tau(e, se) = 0$, where $c, d, f, g \in \mathbb{R}^n$. The conditions are equivalent to $c \cdot g = f \cdot d$, $c \cdot c + f \cdot f = d \cdot d + g \cdot g$, and $c \cdot d + f \cdot g = 0$.

In terms of vectors in $\mathbb{C}^n$, they become $||c + if||^2 = ||d + ig||^2$ and $\langle c + if, d + ig \rangle = 0$ ($\langle , \rangle$ the sesquilinear form on $\mathbb{C}^n$).

Recalling the isomorphism $\mathbb{R}^{2n} \approx \mathbb{F}^n$ and $K \approx U(n)$, there clearly exists $g \in K$ and $s > 0$ such that

$$g(s(\begin{bmatrix} c \\ f \end{bmatrix}) + is(\begin{bmatrix} d \\ g \end{bmatrix})) = \tilde{e}_1 + i\tilde{e}_2 = f.$$ 

Then $\sigma(Xe, e) = s^{-2} \sigma((gXg^{-1})f, f)$. $gXg^{-1}$ is a real matrix, so $\sigma(Xe, e) > 0$ follows from the remark above (on the $\sigma(Xe, e)$) and Horn's theorem.

Lemma 17.2. The $G$-orbit of any $Y \in C_2$ is closed.

Proof. $\exists k > 0$ such that $\sigma(Ye, e) \geq k||e||^2 \ \forall e$ such that $\tau(e, e) = \tau(e, se) = 0$. (Note $-i\tau(e, se) = H_0(e, e)$, and the proof is the same as for $su(p, q)$.) The inequality
is also true for any \( X \) in the \( K \)-orbit of \( Y \). We compute \( \text{Ad}(a)X \) for \( a \in \mathbb{A}^+ \). As
\[
a = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix},
\]
where \( S = \begin{pmatrix} \cosh \theta_1 & -i \sinh \theta_1 \\ i \sinh \theta_1 & \cosh \theta_2 \end{pmatrix} \)...
\[
\begin{pmatrix} \cosh \theta_2 & -i \sinh \theta_2 \\ i \sinh \theta_2 & \cosh \theta_3 \end{pmatrix}
\]
\( \theta_j \geq 0 \), it suffices to multiply
\[
\begin{pmatrix} \cosh \theta & -i \sinh \theta \\ i \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} -c & e^{i r} \\ e^{-i r} & d \end{pmatrix} \begin{pmatrix} \cosh \theta & -i \sinh \theta \\ i \sinh \theta & \cosh \theta \end{pmatrix}
\]
where \( c,d,e,r \) are real. We have \( c+d-2|r| \geq 2k \) by our choice of \( k \) and a remark earlier. The diagonal elements of this matrix are \( -c \cosh^2 \theta - d \sinh^2 \theta - r \sinh 2\theta \) and \( -d \cosh^2 \theta - c \sinh^2 \theta - r \sinh 2\theta \), and their sum is
\[
-(c+d) \cosh 2\theta - 2r \sinh 2\theta \leq (-(c+d)+2|r|) \sinh 2\theta
\leq -2k \sinh 2\theta .
\]

Now argue as in the proof of Lemma 14.1, using the estimate above.

**Theorem 17.3.** Let \( Y \in C_2 \). Then \( Y \) is conjugate to an \( X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \) as in the statement of 17.1, where the \( G \)-diagonal of \( Y \) is in the noncompact convex hull of the \( G \)-diagonal of \( X \).
The proof runs exactly as the proof of Theorem 14.2, and is dependent only on Lemma 17.2 and the following

**Lemma 17.4.** Let \( X \in C_2 - k \). Then there exists \( Y \) in the orbit of \( X \) with strictly smaller \( p \)-norm such that the \( G \)-diagonal of \( X \) is in the noncompact convex hull of the \( G \)-diagonal of \( Y \).

**Proof.** By Horn's theorem we may assume \( X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} + i \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \) where \( A \) and \( B \) are real and skew, and \( D \) is diagonal and satisfies the hypothesis of Lemma 17.1.

If \( G \) and \( H \) are \( n \times n \) real and skew,

\[
\text{Ad}(\exp(\epsilon \begin{pmatrix} iG & iH \\ iH & -iG \end{pmatrix}))X = \begin{pmatrix} iA + \epsilon(S+iM) & iB - D + \epsilon(T+iL) \\ iB + D + \epsilon(-T+iL) & iA + \epsilon(S-iM) \end{pmatrix} + O(\epsilon^2)
\]

where \( S, T, L, M \) are real and \( n \times n \):

\[
S = AG - GA + BH - HB \\
T = HA + AH - BG - GB \\
L = -GD - DG \\
M = HD + DH
\]

Therefore we set

\[
G_{ij} = \left( \frac{B_{ij}}{(D_i + D_j)} \right) \quad \text{and} \quad H_{ij} = \left( -\frac{A_{ij}}{(D_i + D_j)} \right)
\]

so that

\[
(B + \epsilon L)_{ij} = (1 - \epsilon)B_{ij}
\]
As before, if $T_{ii} = 0$ the corresponding $O(\varepsilon^2)$ terms are zero $\forall \varepsilon$. It is clear that $\{T_{ii}\}_{i=1}^n \in E_0$ as $A$ and $B$ are skew. Thus the desired $Y$ is obtained if $\varepsilon$ is positive and sufficiently small.

18. Minimal and Maximal Causal Cones in $\text{so}^*(2n)$.

In addition to the definition of $C_2$ in the last section, let

$$C_1 = \{X \in : \sigma(Xv,v) > 0 \forall v \neq 0 \text{ in } \mathbb{C}^{2n}\},$$

$$C_0 = \{X \in C_1: \sum_{j=2}^n \sigma(Xe_j,e_j) > \sigma(Xe_1,e_1) \forall (e_j,f_j) \in \mathfrak{g}\},$$

and now let $\mathcal{C}$ be the collection of all nontrivial invariant convex cones in $\mathfrak{g}^\perp$.

**Theorem 18.1.** $C_i$ $(i = 0,1,2)$ are nonempty, open, and in $\mathcal{C}$. Any $C \in \mathcal{C}$ contains either $C_0$ or $-C_0$ but not both. $C_0$, $C_1$, and $C_2$ are obtained by changing "$>$" to "$\geq$" in the definitions. Any closed (open) cone in $\mathcal{C}$ containing $C_0$ is contained in $C_2$ (resp. $C_2$).

**Proof.** Clearly $C_0 \subseteq C_1 \subseteq C_2$ and that the $C_i$ are in $\mathcal{C}$ provided they are nonempty. In fact the inclusions are proper: if $X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, $D$ diagonal, and diag $D$ is
(n,1,...,1) then \( X \in C_1 - C_0 \), and if it is \((-1,2,...,2)\) then \( X \in C_2 - C_1 \) by Lemma 17.1.

We show \( J \in C_0 \) to prove \( C_0 \neq \emptyset \). Using \( G = K\mathbb{A}^+K \), it suffices to see that

\[
\sum_{j=2}^{n} \sigma(Ja_{j},a_{j}) > \sigma(Ja_{1},a_{1})
\]

where \( a \in \mathbb{A}^+ \) and \( a_{j} = \left( \frac{c_{j}}{\alpha_j} \right) \in \mathbb{R}^2n \) \((1,\alpha = 1,...,n)\), the \( a_{j} \) corresponding under \( \mathbb{R}^2n \approx \mathbb{C}^n \) to an orthonormal basis of \( \mathbb{C}^n \). Setting \( N_{i}^{\alpha} = |c_{i}|^2 + |d_{i}|^2 \), we have

\[
\sum_{i=1}^{n} N_{i}^{\alpha} = 1 \quad \text{and} \quad \sum_{i=1}^{n} N_{i}^{\alpha} = 1 \quad \forall \, i,\alpha.
\]

Taking an \( a \) as in the proof of 17.2, we have

\[
\sigma(Ja_{1},a_{1}) = \sum_{\alpha=1}^{n} (\cosh 2\theta_{[\alpha+1/2]})N_{i}^{\alpha}
\]

(where \( \theta_{[\alpha+1/2]} = 0 \)), and, dropping the factor of 2 from \( \theta_{1},...\), \( \theta_{2} \), we must show

\[
\sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})N_{i}^{\alpha} < \sum_{i=2}^{n} \sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})N_{i}^{\alpha} = \sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})(1-N_{i}^{\alpha})
\]

or

\[
2 \cosh \theta_{1} + ... + 2 \cosh \theta_{n} + 1 > 2(N_{1}^{1} + N_{1}^{2}) \cosh \theta_{1} + ... + 2(N_{1}^{2n-1} + N_{1}^{2n}) \cosh \theta_{n} + 2N_{1}^{2n} + 1
\]

if \( n \) is odd. (If \( n \) is even the same with the \( 1 \) and \( 2N_{1}^{2n} + 1 \) removed must be shown, but this is now clear from
\[ \sum_{c=1}^{n} N_1^c = 1 \text{ and } n \geq 3. \) In that case it suffices to show
\[ 2 \cosh \theta + 1 > 2(N_1 + N_2) \cosh \theta + 2N_3 \text{ where } 0 \leq N_1, N_2, N_3 \]
and \( N_1 + N_2 + N_3 \leq 1 \). But the r.h.s. is
\[ \leq 2(1 - N_3) \cosh \theta + 2N_3 \text{ and } 1 > 2(1 - \cosh \theta)N_3 \text{ is clear.} \]
Thus \( C_0 \) is nonempty. \( C_1 \) is open by the same proof as that for \( C_0 \) in \( \text{sp}(n, \mathbb{R}) \). By Theorem 16.2, any \( X \in C_1 \) is conjugate to a \( \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \), D diagonal, where all \( D_{jj} > 0 \). D is clearly unique up to a permutation as D essentially gives the eigenvalues of \( X \) acting on \( C^{2n} \). By 17.3 \( X \in C_0 \) iff it is conjugate to a \( Y = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \), D diagonal, where \( \text{diag} D \) is in \( (E_0^0) \). \( C_0 \) is then open by the continuity of eigenvalues. We have already seen that any \( Y \in C_2 \) is conjugate to an \( X \) as in 17.1, so \( C_2 \) is open by the same argument as for \( C_1 \) in \( \text{su}(p, q) \).

Now let \( C \in C \); we may assume \( J \in \overline{C} \). As before \( C_0 \) being open implies \( \forall W \in C \cap C_0 \), so \( J \in C \) by the above conjugacy in \( C_0 \). (Average over permutations of the G-diagonal of \( \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \).) Now \( \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \) (where \( \text{diag} D = (1, 1, \varepsilon, \ldots, \varepsilon) \) for \( 1 > \varepsilon > 0 \) can be obtained from averaging the transforms of \( J \) by \( a, a^{-1} \in A \), as in Corollary 3.2. It may be seen without much difficulty that the convex cone generated by permutations of \( (1, 1, 0, \ldots, 0) \) is precisely \( E_0 \), and likewise the convex cone generated by permutations of the above \( (1, 1, \varepsilon, \ldots, \varepsilon) \) is just \( (E_0^0) \). Thus \( C_0 \) is the invariant convex cone generated by \( J \).
and \( C_0 \subset C \).

That \( \tilde{C}_0 \), \( \tilde{C}_1 \), and \( \tilde{C}_2 \) have the asserted forms follows exactly as in the symplectic case.

Finally, the last two statements follow, as in the proof for \( \text{su}(p,q) \), from

**Theorem 18.2.** \( \tilde{C}_1^* = -\tilde{C}_1 \) and \( \tilde{C}_0^* = -\tilde{C}_2 \).

**Proof.** \( B_\delta(X,Y) > 0 \ \forall \ X,Y \in C_1 \) follows from Theorem 16.2 and the definitions of \( C_1 \) and \( B_\delta \), so \( -\tilde{C}_1 \subset \tilde{C}_1^* \). The converse proceeds in the by now familiar way from Theorem 15.1.

\( -\tilde{C}_2 \subset C_0^* \) follows from the minimality of \( \pm C_0 \).

Conversely, if \( X \not\in \tilde{C}_2 \) then, as in the proof of 17.1, there is some \( K \)-conjugate of \( X \) of the form \( Y = \begin{pmatrix} * & B \\ -\lambda & \alpha \end{pmatrix} \) where \( B \) has \( \begin{pmatrix} -\lambda_1 & \alpha \\ \alpha & -\lambda_2 \end{pmatrix} \) in the upper left corner, and \( \lambda_1 + \lambda_2 - 2 \text{Im} \alpha < 0 \). But now \( W = \begin{pmatrix} 0 & L \\ -1 & 0 \end{pmatrix} \) where \( L \) is all zeros except for \( \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \) in the upper left corner, is in \( \tilde{C}_0 \), and \( B_\delta(Y,W) = \lambda_1 + \lambda_2 - 2 \text{Im} \alpha < 0 \). Thus \( Y \), and hence \( X \), is not in \( -\tilde{C}_0^* \).

**Corollary 18.3.** \( C_0 \), \( C_1 \), and \( C_2 \) are the interiors of their closures.

**Proof.** As we have shown that all \( X \in C_2 \) are conjugate to a \( Y \in \mathfrak{h} \), the proof follows that of Corollary 12.4.
19. **Classification of Orbits and Cones in $\mathfrak{so}^*(2n)$**.

We shall see that if $\Delta$ is a type in some $\mathfrak{g}_2$ and $\Delta = \Delta_1 + \ldots + \Delta_m$ its decomposition into indecomposable types, then each $\Delta_j$ lies in an $\mathfrak{so}^*(4)$ or $\mathfrak{so}^*(2)$.

It will be useful to examine the orbits in $\mathfrak{so}^*(4)$, as the same definitions of $\mathfrak{C}_i$ ($i = 0,1,2$) make sense there (and only $\mathfrak{C}_1$ for $\mathfrak{so}^*(2)$), and the $\mathfrak{C}_i$ can be explicitly described.

The $\mathfrak{sl}(2,\mathbb{R})$ component of $\mathfrak{so}^*(4)$ is spanned by

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}
\]

and the $\mathfrak{su}(2)$ component is spanned by

\[
\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}
\]

We let $B_1$ and $B_2$ be $-\text{Tr}(\cdots)$ restricted to the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ components, respectively.

By the uniqueness of the decomposition it is clear that if $Y \in \mathfrak{sl}(2,\mathbb{R})$, $Z \in \mathfrak{su}(2)$, and $(e_1, e_2, f_1, f_2) \in \xi$, $\mathcal{A}(Ye_1, e_1) = \mathcal{A}(Ye_2, e_2)$ and $\mathcal{A}(Ze_1, e_1) = -\mathcal{A}(Ze_2, e_2)$.

Lemma 17.1 shows that $\forall e \neq 0$ such that $0 = \tau(e, e) = \tau(e, se)$, $\mathcal{A}(Ze, e) = 0$ and $\mathcal{A}(Ze, e) > 0 \forall Y$ in the positive cone of $\mathfrak{sl}(2,\mathbb{R})$. Note also
the $4 \times 4$ matrix being in the positive nilpotent orbit.

We see that $C_0$ is empty and "$\tilde{C}_0$", so to speak, is the closed positive cone in $\mathfrak{sl}(2, \mathbb{R})$. $C_2$ is the direct sum of the open positive cone in $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$.

The cone $C_1$ with self-dual closure (and 18.2 is valid here, too) is

$$C_1 = \{(Y, Z) : Z \in \text{open positive cone in } \mathfrak{sl}(2, \mathbb{R}), \quad B_1(Y, Y) > B_2(Z, Z)\}.$$ 

That $C_1$ is convex follows from an elementary estimate.

**Theorem 19.1.** Only the indecomposables $\Delta_0(0, 0)$, $\Delta_1^+(0, 0)$, $\Delta_0^+(\zeta, -\zeta)$, and $\Delta_1^+(\zeta, -\zeta)$ where $\zeta = -\zeta \neq 0$, lie in $\pm \tilde{C}_2$.

**Proof.** We consider first any of the types $\Delta_m^+(\cdots)$ or $\Delta_m(\cdots)$ with $m \geq 3$. As in the earlier proofs we considered $\mathcal{A}(Aa, a)$ where $a \in N^{m-1}E + E$, and found it to be of indeterminant sign. $\tau$ vanishes on the $\sigma$-invariant $N^{m-1}E + E$ so clearly no such type can be in $\pm \tilde{C}_2$.

For the other orbits we take $A = S + N$ as usual.

To rule out the remaining types excluded by the statement of the theorem, it is sufficient to let $S = 0$, as that orbit is in any closed invariant cone containing the original type with $S \neq 0$. 
For $\Delta_2(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$ ($\zeta \neq -\bar{\zeta}$), $\mathcal{O}(N(Ne+w),Ne+w) = -2\mathcal{O}(w,N^2e)$. NE has dimension 4, so $e \neq 0$ can be found with $\tau(Ne,Ne) = \tau(Ne,\sigma Ne) = 0$. $w + Ne$ has the same properties, and $\mathcal{O}(w,N^2e)$ can take any value.

For $\Delta_2^+(\zeta,-\zeta)$ ($\zeta = -\bar{\zeta} \neq 0$) or $\Delta_2(0,0)$ again let $S \to 0$. $V = N^2E + NE + E$ so let $T = +\text{Id.}$ on $N^2E + E$ and $T = -\text{Id.}$ on $NE$. Then $T \in \text{SO}^*(6)$ and $TNT = -N$, showing that $N \notin \mathcal{O}_2$ as $\mathcal{O}_2 \cap -\mathcal{O}_2 = \{0\}$.

For $\Delta_1(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$ ($\zeta \neq -\bar{\zeta}$) again let $S \to 0$. $E$ has a basis $e,f,se,\sigma f$ where $\tau(e,Nf) = 1$, $\tau(e,Ne) = \tau(f,Nf) = 0$, and $Se + \sigma f$ is $\tau$-orthogonal to $N[Se + \sigma f]$. Let $Te = f$, $Tf = e$, $T(Ne) = -Nf$, $T(Nf) = -Ne$, and extend it to $V$ by requiring it to commute with $\sigma$. Then $T \in \text{SO}^*(8)$ and $TNT = -N$, excluding this type.

The remaining types are in $\text{so}^*(2)$ or $\text{so}^*(4)$. $\Delta_0(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$ has a hyperbolic $\text{sl}(2,\mathbb{R})$ component so is never in a $\mathcal{O}_2$. $\Delta_1^+(\zeta,-\zeta)$ ($\Delta_1^+(0,0)$) correspond to a nilpotent $\text{sl}(2,\mathbb{R})$ component and a nonzero (resp. zero) $\text{su}(2)$ component, so are always on the boundary of $\mathcal{O}_2$.

From now on reserve the notation $\Delta_1^+(\zeta,\bar{\zeta})$ for the case $\zeta = -\bar{\zeta} \neq 0$ only; this is the only ambiguity in the notation for the indecomposables left. We note that $\Delta_1^+(0,0)$ is on the boundary of $\mathcal{O}_0$ and $\mathcal{O}_2$, but $\Delta_1^+(\zeta,-\zeta)$ is on the boundary of $\mathcal{O}_2$ but not that of $\mathcal{O}_1$ or $\mathcal{O}_0$. 
This is quite analogous to the types $\Delta_1^+(0)$ and $\Delta_1^+(i\lambda)$ ($\lambda \neq 0$) for $\text{su}(p,q)$. Finally, just as for $\text{su}(p,q)$, representatives of $\Delta_1^+(\zeta,-\zeta)$ are a sum within $\text{so}^+(4)$ of a representative of $\Delta_1^+(0,0)$ and a representative of the elliptic type $\Delta^+(\zeta,-\zeta) + \Delta^-(\zeta,-\zeta)$, which is on the boundary of $\mathfrak{g}_2$ but not that of $\mathfrak{g}_1$ or $\mathfrak{g}_0$.

Finally, as a consequence of the results of the last three sections, we list the types in $\mathfrak{g}_2$ for $\text{so}^*(2n)$, $n \geq 3$.

**Proposition 19.2.** Types in $\mathfrak{g}_1$ are those of the form

(10) $\Delta = \Delta^+(\zeta_1,-\zeta_1) + \cdots + \Delta^+(\zeta_n,-\zeta_n)$

$\Delta$ is furthermore in $\mathfrak{g}_0$ if $(|\zeta_1|, \ldots, |\zeta_n|) \in (E_0)^O$. Other types in $\mathfrak{g}_2$ have either the form

(11) $\Delta = \Delta(0,0) + \Delta^+(\zeta_2,-\zeta_2) + \cdots + \Delta^+(\zeta_n,-\zeta_n)$

or

(12) $\Delta = \Delta^-(\zeta_1,-\zeta_1) + \Delta^+(\zeta_2,-\zeta_2) + \cdots + \Delta^+(\zeta_n,-\zeta_n)$

for $(-|\zeta_1|, |\zeta_2|, \ldots, |\zeta_n|) \in (E_2)^O$.

Types in $\mathfrak{g}_0$ have either the form (10) with $(|\zeta_1|, \ldots, |\zeta_n|) \in E_0$ or

(13) $\Delta = \underbrace{\Delta_1^+(0,0) + \cdots + \Delta_1^+(0,0)}_{\ell \text{ times}} + \underbrace{\Delta(0,0) + \cdots + \Delta(0,0)}_{m \text{ times}}$

$+ \Delta^+(\zeta_2\ell+m+1,-\zeta_2\ell+m+1) + \cdots + \Delta^+(\zeta_n,-\zeta_n)$
with $(0,\ldots,0,|\xi_{2l+m+1}|,\ldots,|\xi_n|) \in E_0$ and $2l+m \geq 1$.

Types in $\partial C_1$ have the form (13) with $2l+m \geq 1$.

Finally, types in $\partial C_2$ have either the form (12) with $(-|\xi_1|,|\xi_2|,\ldots,|\xi_n|) \in \partial E_2$, (13) with $l \geq 1$ or $m \geq 2$, or

$$\Delta = \Delta_1^+(\xi_1,-\xi_1) + \Delta_3^+(\xi_3,-\xi_3) + \ldots + \Delta_1^+(\xi_n,-\xi_n)$$

where $(-|\xi_1|,|\xi_2|,\ldots,|\xi_n|) \in \partial E_2$.

It is now easy to classify the open and closed invariant convex cones in $\mathcal{Q}$. The proof involves no new ideas over those for $su(p,q)$, but one must use the remarks about the nilpotents above for the closed cones. The proof uses the maps

$$M_\varphi: \mathcal{Q} \to \mathbb{R}^n: \mathcal{X} \to (\mathcal{A}(Xe_1,e_1),\ldots,\mathcal{A}(Xe_n,e_n))$$

for $\varphi = (e_j, f_j) \in \xi$. We have seen that $M_\varphi(X) \in E_2$ if $X \in \partial C_2$, $M_\varphi(X) \in (E_2)^o$ if $X \in C_2$, etc.

Theorem 19.3. The open (closed) non-trivial invariant convex cones in $so^*(2n)$ ($n \geq 3$) containing $C_0$ are in 1-1 correspondence with the open (resp., closed) cones in $(E_2)^o$ (resp., $E_2$) containing $(E_0)^o$ (resp., $E_0$) and invariant under the finite group of permutations of the Euclidean coordinates of $\mathbb{R}^n$. 
CHAPTER V. Invariant Convex Cones in $\mathfrak{so}(2,n)$.

20. Conventions and Preliminaries for $\mathfrak{so}(2,n)$.

$\mathfrak{so}(2,n)$ is simple if $n \geq 3$. Let $t = \lfloor n/2 \rfloor$.

We take
\[
\mathcal{Q} = \left\{ \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} : A,B,C \text{ real; } B \ 2 \times n; \ A,C \text{ skew} \right\}
\]
\[
\mathcal{K} = \left\{ X \in \mathcal{Q} : X \text{ skew} \right\} , \quad \mathcal{P} = \left\{ X \in \mathcal{Q} : X \text{ symmetric} \right\}
\]
\[
\mathcal{H} = \left\{ \begin{pmatrix} 0 & d_0 & 0 & \cdots & 0 \\ -d_0 & 0 & d_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -d_{t-1} & 0 & d_t \\ & & & -d_t & 0 \end{pmatrix} \in \mathcal{Q} \right\}
\]
\[
c = \mathbb{R}Z \quad \text{where } \quad Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Let $d_j$ ($j = 0, \ldots, t$) be the linear functionals spanning the dual of $\mathcal{H}$, whose definition is clear from the above notation for $\mathcal{H}$. Then

\[
\{\text{noncompact roots}\} = \{ \pm i(d_0 + d_j), \pm i(d_0 - d_j) : j = 1, \ldots, t \}
\]

if $n$ is even; if $n$ is odd one has the additional noncompact roots $\pm id_0$. 

{compact roots} = \{\pm i(d_j+d_k), \pm i(d_j-d_k) : 1 \leq j < k \leq \ell \}

if \( n \) is even; if \( n \) is odd one has the additional compact roots \( \pm id_1, \ldots, \pm id_\ell \). The positive roots we take to be the above with "+" only, and

\[ \Delta_o^+ = \{i(d_0-d_1), i(d_1-d_2), \ldots, i(d_{\ell-1}-d_\ell), i(d_{\ell-1}+d_\ell)\} \]

if \( n \) is even, and

\[ \Delta_o^+ = \{i(d_0-d_1), i(d_1-d_2), \ldots, i(d_{\ell-1}-d_\ell), id_\ell\} \]

if \( n \) is odd. We forego listing the \( H_\alpha, X_\alpha, Y_\alpha \) for \( \alpha \in Q_+ \) as we will not really need them.

We let \( \Sigma_o = \{i(d_0+d_1), i(d_0-d_1)\} \), so \( Z = Z_0 \) always, and

\[ \sigma_\alpha = \begin{pmatrix}
0 & 0 & a & 0 & 0 & \cdots \\
0 & 0 & 0 & b & 0 & \cdots \\
0 & a & 0 & 0 & 0 & \cdots \\
0 & 0 & b & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} : a, b \in \mathbb{R} \}

Then \( h^- = \{X \in h : d_j(X) = 0 \text{ for } 2 \leq j \leq \ell \} \) and

\( h^+ = \{X \in h : d_0(X) = d_1(X) = 0\} \), so \( h^+ = 0 \) iff \( n = 3 \).

The cone in \( h^- \) given by Corollary 3.2 is (in \( \mathfrak{o}(2,2) \)) generated by

\[-\frac{1}{2} \sum_{\alpha \in \Sigma_o} d_\alpha h_\alpha = \begin{pmatrix}
0 & \frac{1}{2}(d_1+d_2) & 0 & 0 \\
\frac{1}{2}(d_1+d_2) & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}(d_1-d_2) \\
0 & 0 & \frac{1}{2}(d_1-d_2) & 0 \\
\end{pmatrix} \]
for $d_e \geq 1$, or all
\[
\begin{pmatrix}
0 & -d & 0 \\
d & 0 & 0 \\
0 & 0 & -g \\
g & 0 & 0
\end{pmatrix}
\]
for $|g| \leq d-1$, $d \geq 1$. The cone generated by this is all of the form (15) where $d > 0$, $|g| < d$.

We take $B(X,Y) = \text{Tr}(XY)$ for $X,Y \in q_f$, so if $X = \begin{pmatrix} A \\ B \end{pmatrix}$, $Y = \begin{pmatrix} P \\ Q \end{pmatrix} \in q_f$
\[
B(X,Y) = -\sum_{i,j} A_{ij}P_{ij} - \sum_{i,j} C_{ij}R_{ij} + 2\sum_{i,j} B_{ij}Q_{ij}
\]
and $B_e(X,Y)$ is the same with "+" replacing "-" in the above. ($\theta(X) = -X^t$).

Given $V = (d_0, d_1, \ldots, d_e) \in \mathbb{R}^{e+1}$, for concreteness we associate
\[
(16) \quad X = \begin{pmatrix}
0 & -d_0 \\
d_0 & 0 \\
0 & -d_1 \\
d_1 & 0 \\
\vdots & \ddots & \ddots & \vdots & \ddots
\end{pmatrix} \in h
\]
In what follows we replace $Z$ by the more comfortable $J$, so $J \in h$ corresponds to $(1,0,\ldots,0)$ as in (15).

Note $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(2,n)$ iff $A^tA - C^tC = I$, $D^tD - B^tB = I$, and $A^tB = C^tD$, or equivalently $AA^t - BB^t = I$, $DD^t - CC^t = I$, and $AC^t = BD^t$, so
\[ g^{-1} = \begin{pmatrix} A^t - C^t \\ -B^t \\ D^t \end{pmatrix} \cdot \text{g is in the maximal compact K (of SO}_0(2,n)) \text{ iff } B = 0, C = 0, \det A = \det D = 1. \text{ O(2,n) preserves the symmetric form } \tau \text{ on } \mathbb{R}^{2+n} \]

\[ \tau((x, y), (z, w)) = x_1z_1 + x_2z_2 - \sum_{j=1}^n y_jw_j, \]

where \( x, z \in \mathbb{R}^2, y, w \in \mathbb{R}^n. \)

We characterize the identity component \( G = \text{SO}_0(2,n) \) as follows. Let \( e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^{2+n}, \) and let \( p: \mathbb{R}^{2+n} \rightarrow \mathbb{R}^2 \) be the projection. If \( g \in \text{O}(2,n), \) then \( g \in G \text{ iff } \det g = 1 \text{ and the linearly independent vectors } p(g(e_1)) \text{ and } p(g(e_2)) \) have the same orientation as \( e_1 \) and \( e_2. \)

Recall that the reflections of \( h \) coming from conjugations by \( K \) (or \( G \)) are, for \( n \) odd, all permutations and sign changes of the \( d_1, \ldots, d_n, \) and all permutations and even numbers of sign changes of the \( d_1, \ldots, d_n \) for \( n \) even. This finite group we call \( W. \)

There is an analogue of Horn's theorem (see section 14) for any semisimple Lie group, one side of which we state here for \( K \) acting on its Lie algebra \( \mathfrak{k} \) via \( \text{Ad}. \) Let \( \Gamma: \mathfrak{k} \rightarrow \mathfrak{h} \) be the orthogonal projection.

**Proposition 20.1.** Let \( x \in \mathfrak{h}. \) Then \( \Gamma(\text{Ad}(K)X) \) is contained in the convex hull of the points \( \sigma(X) \) for \( \sigma \in W. \)

**Proof.** The proof is essentially word for word that of
Kostant's proof on p. 452 in [16], and is the standard maximization of a continuous function over the compact argument. In fact, equality holds in the above, but we do not need this.

We record how \( A \) acts on \( h \) by conjugation. If

\[
\begin{bmatrix}
\theta_1 & 0 & 0 \\
0 & \theta_1 & 0 \\
0 & 0 & \theta_2
\end{bmatrix}
\]

then

\[
\text{Ad}(e^a) = \begin{bmatrix}
0 & -d_0 & 0 \\
d_0 & 0 & 0 \\
0 & 0 & -d_1
\end{bmatrix}
\]

where

\[
f_0 = d \cosh \theta_1 \cosh \theta_2 - d_1 \sinh \theta_1 \sinh \theta_2,
\]

\[
f_1 = d_1 \cosh \theta_1 \cosh \theta_2 - d_0 \sinh \theta_1 \sinh \theta_2.
\]

Note the * changes sign if \( -a \) is used instead.

Define \( \mathcal{B} \) to be the collection of all ordered bases of \( \mathbb{R}^{2+n} \) which are transforms by a \( g \in G \) of the standard basis \( e_1, \ldots, e_{2+n} \). Let \( \Omega \) be the set of all pairs \( (e,f) \) of linearly independent vectors in \( \mathbb{R}^{2+n} \) satisfying \( \tau(e,e) = \tau(f,f) = \tau(e,f) = 0 \), and such that the \( \mathbb{R}^2 \) components of \( e \) and \( f \) are oriented the same as \( e_1 \) and \( e_2 \). It is clear that \( G \) acts on \( \Omega \), and given any \( (e,f) \in \Omega \), there exists \( g \in K \) such that \( g(re) = (1,0,1,0,\ldots,0) \) and \( g(f) = (a,b,a,b,0,\ldots,0) \) for \( a,b \in \mathbb{R} \) with \( b > 0 \).

Finally, we define two closed cones in \( \mathbb{R}^{1+2} \):
The representation of $\mathcal{G}$ on $\mathbb{R}^{2+n}$ which we use here is quite different from the defining representations of the three previous series of simple causal groups. (Their analogue for $\text{so}(2,n)$ is the spin representation.) Here we do not have available Hilbert space arguments to prove ellipticity of the orbits in the interior of the minimal cone, so we rely more heavily on the classification of the orbits. In this section and the next $n \geq 4$ for convenience.

Let $\mathcal{C}$ be the collection of all nontrivial invariant convex cones in $\mathcal{G}$. We define

$$\mathcal{C}_1 = \{ X \in \mathcal{G} : \tau(Xe,f) > 0 \ \forall \ (e,f) \in \Omega \}$$

and

$$\mathcal{C}_0 = \{ X \in \mathcal{C}_1 : \tau(Xv_1,v_2) > \sum_{j=1}^l \mid \tau(Xv_{2j+1},v_{2j+2}) \mid \ \forall \ (v_1,\ldots,v_{2+n}) \in \mathcal{G} \}$$

We will see in the next section that $\mathcal{C}_0$ and $\mathcal{C}_1$ are, respectively minimal and maximal open invariant convex cones in $\mathcal{G}$. 
**Proposition 21.1.** Let \((d_0, \ldots, d_L) \in \mathbb{R}^{l+L}\) such that 
\[ d_0 > |d_j| \quad \text{for} \quad j = 1, \ldots, L. \] 
Then \(X\) as in (16) is in \(C_1\). If also \(d_0 > \sum_{j=1}^{L} |d_j|\), then \(X\) is in \(C_0\).

**Proof.** By 20.1 and the remark following the definition of \(\Omega\), it suffices to consider a \(K\)-conjugate \(Y\) of \(X\) whose \(\sigma(2,2)\) part is
\[
\begin{bmatrix}
0 & -d_0 \\
d_0 & 0
\end{bmatrix}
\] where \(|d| > \max_{j > 0} |d_j|\),
\[
\tilde{e} = (1, 0, 1, 0, \ldots, 0), \quad \text{and} \quad \tilde{p} = (a, b, a, b, 0, \ldots, 0) \quad \text{where} \quad b > 0.
\]
One computes \(\tau(Y\tilde{e}, \tilde{p}) = b(d_0 - d) > 0\), so \(X \in C_1\).

To prove the second statement, we first note that \(J \in C_0\). This is clear from \(\text{Ad}(G)J = \text{Ad}(K)\text{Ad}(A)J\), the computation of the \(A\)-action earlier, and Proposition 20.1. But the \(X\) above is clearly in the invariant convex cone generated by \(G\)-transforms of \(J\), as \(\exists \epsilon > 0\) such that 
\[
d_0 = \sum_{j=1}^{L} (|d_j| + \epsilon), \quad \text{and} \quad (d_0, d_1, \ldots, d_L) = (|d_1| + \epsilon, d_1, 0, \ldots, 0) + \ldots + (|d_L| + \epsilon, 0, \ldots, 0, d_L).
\]
Thus \(X \in C_0\) as \(C_0\) is \(G\)-invariant.

**Lemma 21.2.** The \(G\)-orbit of an \(X\) as in (16) where \(d_0 > |d_j| \quad (j = 1, \ldots, L)\) is closed.

**Proof.** Again we follow the proofs in Lemmas 14.1 and 17.2, and give the necessary estimate. \(\exists k > 0\) such that \(d_0 - |d_j| \geq k \forall j > 0\), and this inequality also holds for the \((d_0, h_1, \ldots, h_L)\) corresponding to any \(Y\) in the
K-orbit of $X$. We may assume $\theta_1, \theta_2 \geq 0$, and using the earlier computation and the notation there,

$$f_0 - f_1 = d_0 \cosh (\theta_1 + \theta_2) - h_1 \cosh (\theta_1 + \theta_2) = k \cosh (\theta_1 + \theta_2).$$

Thus $\theta_1 + \theta_2$ cannot grow too large, and likewise $\theta_1$ and $\theta_2$.

**Remark.** As $n \geq 4$, $(v_1, \ldots, v_{2+n}) \in \mathfrak{t}$ implies $(v_1 \pm v_3, v_2 \pm v_4) \in \Omega$, and so if $X \in C_1$ we have $\tau(Xv_1, v_2) > |\tau(Xv_3, v_4)|$. It will turn out by classification that $C_1$ consists of conjugates of the $X$ considered in the previous lemma.

**Definitions.** 1) Given $X \in \mathfrak{g}$, let $X = Y + Z$, $Y \in \mathfrak{k}$, $Z \in \mathfrak{p}$. Call $\sqrt{B(Z, Z)}$ the $p$-norm of $X$. Call the vector $(d_0, \ldots, d_\ell)$ obtained from $\Gamma(Y) \in \mathfrak{h}$ as in (16) the G-diagonal of $X$.

2) If $V, U \in \mathbb{R}^{1+\ell}$, say $V$ is in the noncompact convex hull of $U$ if $V$ is equal to a vector in $E_0$ (see section 20) plus a vector in the convex hull of $\{\sigma(U): \sigma \in \mathfrak{w}\}$.

**Theorem 21.3.** Let $Y \in C_1$ be in the orbit of an $X$ as in the statement of Lemma 21.2. Then the G-diagonal of $Y$ is in the noncompact convex hull of the $(d_0, \ldots, d_\ell)$ there.

**Proof.** It is exactly the same argument as for $\text{su}(p,q)$...
and so \( * (2n) \), using the following.

**Lemma 21.4.** Let \( X, Y \) be as in the statement of 21.3, and assume \( Y \notin k \). Then there exists \( Z \) in the orbit of \( Y \) such that \( p \)-norm of \( Z \) is strictly less than the \( p \)-norm of \( Y \), and the \( G \)-diagonal of \( Y \) is in the noncompact convex hull of the \( G \)-diagonal of \( Z \).

**Proof.** The proof is computationally more difficult because the noncompact root spaces are not as easily expressed in terms of our representation. By Proposition 20.1 we may assume the \( k \)-component of \( Y \) is in \( h \). Let \( Y = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \), \( B \neq 0 \). For \( D \) \( 2 \times n \) we compute

\[
(I + \varepsilon D)(A \ B)(I - \varepsilon D) =
(I + \varepsilon D_t)(A \ B)(I - \varepsilon D_t) =

\begin{pmatrix}
A + \varepsilon(DB^t - BD^t) & B + \varepsilon(DC - AD) \\
B^t + \varepsilon(D^tA - CD^t) & C + \varepsilon(D^tB - BD^t)
\end{pmatrix} + O(\varepsilon^2).
\]

Assume \( \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \) gives rise to \( (c_0, c_1, \ldots, c_l) \) as in (16). Break \( B \) and \( D \) up into \( 2 \times 2 \) blocks plus possibly \((n \text{ odd})\) a \( 2 \times 1 \) vector:

\[
B = \begin{pmatrix} (a \ b) \\ (c \ d) \end{pmatrix}, \quad D = \begin{pmatrix} (x \ y) \\ (z \ w) \end{pmatrix}.
\]

Then

\[
DC - AD = \begin{pmatrix} yc_1 + zc_0 & -xc_1 + wc_0 \\ -c_0x + wc_1 & -c_0y - zc_1 \end{pmatrix} \begin{pmatrix} c_0v_1 \\ c_0u \end{pmatrix}.
\]

Now \( c_0 > |c_j| \) \((j = 1, \ldots, l)\) by the Remark in this section, so \( x, y, z, w \), etc. and \( u, v \) are determined
uniquely by

\[
\frac{(c_0+c_1)}{2}(z+y) + \frac{(c_0-c_1)}{2}(z-y) = -a
\]

\[
\frac{(c_0+c_1)}{2}(w-x) + \frac{(c_0-c_1)}{2}(w+x) = -b
\]

\[-\frac{(c_0+c_1)}{2}(w-x) + \frac{(c_0-c_1)}{2}(w+x) = c
\]

\[
\frac{(c_0+c_1)}{2}(z+y) - \frac{(c_0-c_1)}{2}(z-y) = d, \text{ etc.,}
\]

and \( \alpha = -c_0v, \beta = c_0u \). Then

\[
DC - AD = \begin{bmatrix}
-a & -b \\
-c & -d
\end{bmatrix}
\]

We will take \( \varepsilon \) small and positive.

Then

\[
DB^t - BD^t = \begin{bmatrix}
0 & P \\
-P & 0
\end{bmatrix}
\]

where \( P = -2a - wb + cx + dy + \ldots -\alpha v + \beta u = \)

\[
\frac{(c_0+c_1)}{2}((z+y)^2 + (x-w)^2) + \frac{(c_0-c_1)}{2}((y-z)^2 + (x+w)^2)
\]

\[+ \ldots + c_0(u^2 + v^2) > 0, \text{ and}
\]

\[
D^tB - B^tD = \begin{bmatrix}
0 & -Q & \ldots \\
Q & 0 & \ldots \\
\vdots & \ddots & \ddots
\end{bmatrix}
\]

where \( Q = ya + wc - bx - dz = -\left(\frac{(c_0+c_1)}{2}\right)((z+y)^2 + (w-x)^2) \]

\[+ \left(\frac{(c_0-c_1)}{2}\right)((z-y)^2 + (w+x)^2) \], which is of indeterminant sign, but it is clear that the G-diagonal of the \( \varepsilon \) terms
of (17) are in \(-E_0\).

22. **Classification of Orbits and Cones in \(\text{so}(2,n)\).**

Just as the types in the maximal cone of \(\text{so}^*(2n)\) \((n \leq 3)\) were composed of indecomposable types coming from \(\text{so}^*(4)\), we will see that the corresponding orbits for \(\text{so}(2,n)\) come from the orbits of \(\text{o}(2,2) \cong \text{sl}(2,\mathbb{R}) \oplus \text{sl}(2,\mathbb{R})\).

In fact, all but two of the indecomposable types for the series \(\text{o}(2,n)\), \(n \geq 1\) (for the entire algebra), are in \(\text{o}(2,2)\) or a subalgebra. (Of these two, one is in \(\text{o}(2,3)\) and one in \(\text{o}(2,4)\).)

Temporarily let \(C_{-1}\) be the minimal closed convex invariant cone containing \(J\), and let \(C_2\) be the negative of the cone dual to \(C_{-1}\) under any \(E_0(\cdot,\cdot)\). (By the \(G\)-invariance \(C_2\) is clearly independent of the particular \(\theta\).) We have already seen that \(C_2\) contains \(C_1\), which in turn contains all \(X\) as in (16) with \(d_0 \geq |d_j|\) for \(j > 0\). Also, we know already that \(C_{-1} \subset C_0\), and that \(C_{-1}\) contains all \(X\) as in (16) with \(d_0 \geq \sum_{j=1}^l |d_j|\). As \(-(E_0)^* = E_1\) (Euclidean metric), a necessary condition for \(X \in C_2\) is that all \(G\)-diagonals of all conjugates of \(X\) lie in \(E_1\).

It is appropriate to first consider \(\text{o}(2,2)\). One \(\text{sl}(2,\mathbb{R})\) component is spanned by
$J_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and all $\begin{pmatrix} 0 & x & y \\ y & -x & 0 \end{pmatrix}$, 
the other is spanned by 
$J_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and all $\begin{pmatrix} 0 & x & y \\ -y & x & 0 \end{pmatrix}$.

Of the outer automorphisms from $O(2,2)$, conjugation by
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ exchanges the two subalgebras (and exchanges $J_1$
and $J_2$), whereas conjugation by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ "inverts"
each $sl(2,\mathbb{R})$ separately, exchanging the positive and
negative cones. In $Ad(G)$ the latter does not exist, but
the former does in the sense that $o(2,2) \subset o(2,n)$ and
$n \geq 4$. Note $J = \frac{1}{2}(J_1 + J_2)$ with this inclusion.

Recall each indecomposable type for $so(2,n)$ carries
a certain signature $(p,q)$, and we find $(p+q) \times (p+q)$
matrices representing such types.

$\Delta_m^\epsilon(0)$, $m$ even.

A) $m = 0$
1) $\epsilon = +1$; $\text{sig}(1,0); (0)$.
2) $\epsilon = -1$; $\text{sig}(0,1); (0)$.

B) $m = 2$
1) $\epsilon = +1$; $\text{sig}(1,2)$; $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$.
In $o(2,2)$ it is the sum of two nonzero nilpotents in the $sl(2,\mathbb{R})$'s, one with positive $J_1$ component, the other with negative such. It cannot contribute to a type in $C_2$ for $so(2,n)$.

2) $\epsilon = -1$; $\text{sig}(2,1)$; \[ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]; the other sum of two nilpotents; it can contribute to $C_2$.

C) $m = 4$

Only $\epsilon = -1$ gives a type for $so(2,n)$; $\text{sig}(2,3)$;

\[
\begin{pmatrix}
0 & -1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & \sqrt{2} \\
0 & 1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & \sqrt{2} \\
0 & \sqrt{2} & 0 & \sqrt{2} & 0
\end{pmatrix}
\]

It is conjugate under $K$ to an element with $o(3)$ component $\begin{pmatrix} 0 & \sqrt{3} & 0 \\
\sqrt{3} & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}$, so cannot contribute.

$\Delta_m(0,0), \ m \text{ odd}$

Only $m = 1$ is relevant; $\text{sig}(2,2)$;

\[
\begin{pmatrix}
0 & -1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0
\end{pmatrix}
\]

It is a nonzero nilpotent in one $sl(2,\mathbb{R})$ and $0$ in the other; it can contribute.

$\Delta_m^\epsilon(\zeta, -\bar{\zeta}) \ (\zeta = -\bar{\zeta} \neq 0)$

A) $m = 0$

1) $\epsilon = 1$; $\text{sig}(2,0)$; \[ \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \] .

2) $\epsilon = -1$; $\text{sig}(0,2)$; the same matrix.
B) \( m = 2 \)

Only \( \epsilon = 1 \) is relevant; \( \text{sig}(2, 4); (\zeta = 1 \lambda) \)

\[
\begin{pmatrix}
0 & -\lambda & 0 & 0 & 1/2 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & -\lambda & -1/2 & 0 \\
0 & 0 & \lambda & 0 & 0 & -1/2 \\
1/2 & 0 & 1/2 & 0 & 0 & -\lambda \\
0 & 1/2 & 0 & 1/2 & \lambda & 0 \\
\end{pmatrix}
\]

It is conjugate under \( S_0(2,4) \) to a matrix with unchanged \( \lambda \)-entries and non-\( \lambda \) entries multiplied by an arbitrarily large factor, so clearly cannot contribute to \( C_2 \).

C) \( m = 1 \)

\( \text{sig}(2,2) \). The \( \epsilon = \pm 1 \) cases are the two possibilities for an elliptic element in one \( \text{sl}(2,\mathbb{R}) \) and a nonzero nilpotent in the other. One can contribute to \( \pm C_2 \), the other cannot. Let us say \( \Delta_1^1(\zeta,-\zeta) \) can contribute.

A typical representative is

\[
X = \begin{pmatrix}
0 & -\lambda & 0 \\
\lambda & 0 & 0 \\
0 & 0 & -\lambda \\
\end{pmatrix} + \begin{pmatrix}
0 & -1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0 \\
\end{pmatrix}
\]

The remaining cases \( \Delta_0,1(\zeta,-\zeta) \) (0 \( \neq \zeta \) real) and \( \Delta_0(\zeta,-\zeta,2,-\zeta) \) \( (\zeta \neq \pm \xi) \) are in \( \text{o}(2,2) \), and correspond to hyp. \( \Theta \) (0 or hyp.), hyp. \( \Theta \) nilp., and hyp. \( \Theta \) elliptic.
None can contribute to $C_2$.

Now a type involving $\Delta_1^(-,\xi)$ above cannot contribute to $C_1$, as $(e,f) = ((1,0,-1,0),(0,1,0,-1)) \in \Omega$, and $Xe = (0,\lambda,0,-\lambda)$, which is $\tau$-orthogonal to $f$. It follows that $\Delta_2^-(0)$ (nilp. $\Theta$ nilp.) and $\Delta_1(0,0)$ (nilp. $\Theta$ 0) cannot contribute to $C_1$, as $\Delta_1^+(\xi,-\xi)$ is in the invariant convex cone generated by $\Delta_2^-(0)$, and $\Delta_1(0,0)$ is just $\Delta_1^-(\xi,-\xi)$ for $\xi = 0$, so to speak; the same $(e,f)$ rules out $\Delta_1(0,0)$.

By this analysis and Proposition 21.1, we see that $C_1$ consists of conjugates of $X$ as in (16) with $(d_0,\ldots,d_k) \in (E_1)^0$, and $C_2 = \mathcal{C}_1$. Also, we see that $C_0$ contains nothing else besides what we already knew it contained, namely, conjugates of $X$ as in (16) with $(d_0,\ldots,d_k) \in (E_0)^0$. Thus $C_{-1} = \mathcal{C}_0$.

**Theorem 22.1.** $C_0$ and $C_1$ are open cones in $C$, and any $C \in C$ contains either $C_0$ or $-C_0$ but not both. $\mathcal{C}_0$ and $\mathcal{C}_1$ are obtained by changing "\" to "\" in the definitions. Any closed (open) cone in $C$ containing $C_0$ is contained in $\mathcal{C}_1$ (resp. $C_1$).

**Proof.** From the form of $C_1$ and $C_0$ exhibited in the paragraph above, to prove $C_0$ and $C_1$ open it suffices to show there is a neighborhood of $J$ in $C_0$. By the continuity of the eigenvalues there is a neighborhood $N$
about \( J \) such that \( \forall X \in N \) \( X \) has one eigenvalue near \( i \), one near \(-i\), and the rest in a small disc about \( 0 \).
By the classification of types in \( \mathcal{S} \) the ones near \( i \) must be on the imaginary axis and be negatives of each other. If then \( X(v_1 + iv_2) = \lambda i(v_1 + iv_2) \), \( \lambda > 0 \), then
\[
Xv_1 = -\lambda v_2, \quad Xv_2 = \lambda v_1, \quad \tau(v_1, v_2) = 0, \quad \text{and}
\]
\[
\tau(v_1, v_1), \tau(v_2, v_2) > 0 \quad \text{are evident if} \quad N \quad \text{is small enough.}
\]
\( X \) leaves the \( \tau \)-complement of \( \mathbb{R}v_1 + \mathbb{R}v_2 \) invariant, and by the classification of types in \( o(n) \) and the continuity of eigenvalues it is clear that \( X \in C_0 \).
Thus \( C_0 \) is open, and any \( C \in \bar{C} \) with \( J \in \bar{C} \) must contain \( U \in C_0 \), so \( J \in C \). (The sum of all \( W \)-conjugates of \( X \in h \) is a multiple of \( J \).) Thus \( C_0 \subseteq C \) as before.
The statements about the form of \( \bar{C}_0 \) and \( \bar{C}_1 \) follow as before. Finally, we have already noted \( \bar{C}_0^* = -\bar{C}_1 \), so the remaining statements follow as before.

It is clear from our classification and Theorem 21.3 that types in \( \bar{C}_1 \) are composed of \( \Delta_0^+(0) \) (a 1x1 zero), \( \Delta_2(0) \) (nilp. \( \theta \) nilp.), \( \Delta_1(0, 0) \) (nilp. \( \theta \) zero), \( \Delta_1^+(0, -\xi) \) (nilp. \( \theta \) elliptic), and \( \Delta_0^-(0, -\xi) \) (the \( \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \)'s). Furthermore, representatives of \( \Delta_2(0) \) and \( \Delta_1(0, 0) \) are conjugate to arbitrarily small multiples of themselves, so they can appear in a type in \( \bar{C}_1 \) only if they are followed by all \( \Delta_0^-(0) \)'s. These types are in the minimal closed cone \( \bar{C}_0 \). Also, representatives of
\( \Delta_1^\pm(\xi, -\xi) \) are a sum of representatives of the elliptic types \( \Delta_0^\pm(\xi, -\xi) \) and those of \( \Delta_1(0, 0) \) (or \( \Delta_2^-(0) \)).

Using the maps

\[
X \in \mathfrak{g} \rightarrow (\tau(Xv_1, v_2), \tau(Xv_3, v_4), \ldots, \tau(Xv_{2n+1}, v_{2n+2}))
\]

for \((v_1, \ldots, v_{2n+1}) \in \mathfrak{g}\), the proof of the following parallels the earlier proofs.

**Theorem 22.2.** The closed (open) non-trivial invariant convex cones in \( \mathfrak{so}(2, n) \) (\( n \geq 4 \)) containing \( C_0 \) are in 1-1 correspondence with the closed (resp. open) convex cones in \( E_1 \) (resp. \( (E_1)^0 \)) containing \( E_0 \) (resp. \( (E_0)^0 \)) and invariant under the finite permutation and sign-changing group \( W \) acting on \( E_1 \).
CHAPTER VI. Global Causality of the Covering Groups.

23. Definitions and Isomorphisms.

If a Lie group $G$ acts transitively on a manifold $M$, then it is well known that the universal covering group $\tilde{G}$ acts on the universal cover $\tilde{M}$. In fact, if $M = G/H$ for some subgroup $H$, and if $0 \to D \to \tilde{G} \xrightarrow{\pi} G \to 0$ is the exact sequence of groups, then there is the sequence of covering maps

$$\tilde{G}/\tilde{H}' \to \tilde{G}/\tilde{H} \xrightarrow{\pi} G/H_0 \to G/H$$

where $H_0$ is the identity component of $H$, $\tilde{H} = \pi^{-1}(H_0)$, and $\tilde{H}'$ is the identity component of $\tilde{H}$. $\tilde{G}/\tilde{H}'$ is simply connected, so $\tilde{M} \cong \tilde{G}/\tilde{H}'$. (See for example [19, p. 33].) Furthermore, the composition $\tilde{H}' \to \pi^{-1}(H_0) \xrightarrow{\pi} H_0$ is also a covering map.

Recall that $G$ is said to act effectively on $M$ if the identity element $e \in G$ is the only element of $G$ acting trivially on $M$. It is easily seen that if $M = G/H$, then the subgroup of $G$ acting trivially on $M$ is precisely the largest normal subgroup of $G$ contained in $H$.

The proof of the following is not difficult, and we will not use it in this chapter.
Proposition 23.1. With the above notation, if \( G \) acts effectively on \( M \), then \( \widetilde{G} \) acts effectively on \( \tilde{M} \) if and only if \( \pi: \widetilde{H'} \to H_0 \) is an isomorphism. In any case, the subgroup of \( \widetilde{G} \) acting trivially on \( \tilde{M} \) is \( D \cap \tilde{H}' \).

We recall now some of the notions and definitions in Chapter 2 of [19]. A causal structure on a manifold \( M \) is a smooth assignment to each \( p \in M \) of a non-trivial closed convex cone in the tangent space \( T_p(M) \), defined in a neighborhood of each point by a finite number of functions on points and the components of tangent vectors. One say that a Lie group acts causally on \( M \) if the group transformations preserve this structure. A causal structure on a Lie group is said to be invariant if it is invariant under both left and right translations. Clearly such a structure is completely determined by a causal cone in the Lie algebra.

In [19] causal structures on groups acting causally on causally oriented manifolds were defined implicitly (Scholium 2.4). As we now have more detailed information on the possibilities for causal structures on simple Lie groups, it seems worthwhile to make the following

Definition 23.2. Given a Lie group \( G \) with a causal cone \( C \) in its Lie algebra, and a manifold \( M \) with causal structure \( \{C_p\}_{p \in M} \) on which \( G \) acts, say the group action is compatible with the causal orientations, or briefly,
G acts temporally, if G acts causally on M and if

\[(d/dt) \exp tX \cdot y\] \[t=0 \in C_y\]

for all \(X \in C, y \in M\). If \(\theta: G \times M \to M\) is the group action, these conditions are equivalent to requiring that the differential \(d\theta(g, p)\) map the direct sum of cones \(C_g \times C_p\) into \(C_{g(p)}\) for all \(g \in G, p \in M\). (Of course \(C_g = dL_g C = dR_g C\).)

**Remark.** In [19] the cones \(C_g\) were essentially defined by projecting the inverse images \((d\theta(g, p))^{-1}(C_{g(p)})\) onto \(T_g(G)\).

We now specialize to the cases of the simple groups considered thus far. We have sometimes regarded \(su(p, q)\) as a subalgebra of \(sp(p+q, \mathbb{R})\) (section 11) and \(so^*(2n)\) as a further restriction of \(su(n,n)\). Here we keep the second embedding \(so^*(2n) \hookrightarrow su(n,n)\) (detailed below) and consider it in parallel with another map \(sp(n, \mathbb{R}) \hookrightarrow su(n,n)\), which was in fact already defined in section 13. This last was accomplished by a linear operator that we now call \(C: \mathfrak{g}^{2n} \to \mathfrak{g}^{2n}\), originally denoted \(U\). Regarding \(sp(n, \mathbb{R})\) acting as complex-linear operators on \(\mathfrak{g}^{2n}\), we have

\[C \cdot sp(n, \mathbb{R}) \cdot C^{-1} = sp(n, \mathbb{R}) \subseteq su(n,n)\]

with equality only if \(n = 1\).
We extend the original symplectic form $\sigma(\cdot,\cdot)$ on $\mathbb{R}^{2n}$ to a real symplectic form $\sigma(\cdot,\cdot)$ on $\mathfrak{c}^{2n}$ by $\sigma(ix,y) = 0$, $\sigma(ix,iy) = \sigma(x,y)$ $\forall x,y \in \mathbb{R}^{2n}$. This form is to be regarded as the $\sigma$ appearing in the treatment of $\mathfrak{so}^*(2n)$ (section 17): $\sigma(x,y) = \tau(x,\sigma y) = \text{Re}<x,Jy>_0$, $<\cdot,\cdot>_0$ the standard complex Hilbert structure on $\mathbb{C}^{2n}$. One checks that $\sigma(v,w) = \text{Re} \{iH(Cv,Cw)\}$ for $v,w \in \mathbb{C}^{2n}$, where $H(\cdot,\cdot)$ is the Hermitian form on $\mathbb{C}^{2n}$ defined in section 12, and in fact

$$C \mathfrak{so}^*(2n) C^{-1} = \mathfrak{so}^*(2n) \subset \mathfrak{su}(n,n)$$

for all $n \geq 1$. Thus we can compare directly the matrix elements $\sigma(Xv,v)$ for $X$ in $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{so}^*(2n)$ with the corresponding $\text{Re} \{iH(Yu,u)\}$ for $Y$ in $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{so}^*(2n)$.

It is useful to have the explicit forms of these new algebras and their corresponding analytic groups in $\text{SU}(n,n)$. We have

$$\mathfrak{Sp}(n,\mathbb{R})_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^t & -\alpha^t \end{pmatrix} : \alpha,\beta \text{ n\times n complex matrices,} \right\}$$

where $\alpha = \frac{1}{2}(A-A^t+i(B-C))$ and $\beta = \frac{1}{2}(-A-A^t+i(B-C))$.

Also, $\mathfrak{Sp}(n,\mathbb{R})_1 = \left\{ \begin{pmatrix} F \\ G \\ \bar{F} \\ \bar{G} \end{pmatrix} : G^*F \text{ symmetric, } F^*F-G^t\bar{G} = I \right\}$,
the two conditions being equivalent to $FG^t$ symmetric and $FF^* - GG^* = I$.

Analogously,

$$so^*(2n)_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \begin{matrix} \alpha, \beta \text{ nxn complex matrices,} \\ \alpha \text{ skew Hermitian, } \beta \text{ skew} \end{matrix} \right\}$$

and

$$c\left(\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}\right) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

where $\alpha = \frac{1}{2}(A + \overline{A} + i(B + \overline{B}))$ and $\beta = \frac{1}{2}(-A + \overline{A} + i(B - \overline{B}))$.

Finally,

$$SO^*(2n)_1 = \left\{ \begin{pmatrix} F & G \\ -\overline{G} & \overline{F} \end{pmatrix} : \begin{matrix} G^*F \text{ skew,} \\ F^*F - G^*G = I \end{matrix} \right\} ,$$

the two conditions equivalent to $FG^t$ skew and $FF^* - GG^* = I$.

**Remark.** By examining the maximal tori $h_1$ in $sp(n, \mathbb{R})_1$ and $so^*(2n)_1$ corresponding to the $h$ for $sp(n, \mathbb{R})$ and $so^*(2n)$, and comparing them with the $h$ for $su(n,n)$, we have the following: the invariant positive cone in $sp(n, \mathbb{R})_1$ and the "middle" (self-dual) cone $C_1$ in $so^*(2n)_1$ are contained in the minimal cone $C_0$ for $su(n,n)$. However, the maximal cone $C_2$ of $so^*(2n)_1$ extends outside the maximal cone $C_1$ of $su(n,n)$.

24. **Causal Actions on Shilov Boundaries.**

We recall the standard action of $SU(n,n)$ on the group of unitaries $U(n)$ by fractional linear trans-
formations [19, p. 35]. For \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,n) \) and \( U \in U(n) \), define

\[
(\rho(g))U = (AU + B)(CU + D)^{-1} \in U(n)
\]

\( \rho \) defines a (left) group action, and the action is causal. \( (U(n) \) has the standard invariant causal structure determined by positive-definite Hermitians.)

Let

\[
U_s(n) = \{ U \in U(n) : U \text{ symmetric} \}
\]

and (only for \( n \) even!)

\[
U_k(n) = \{ U \in U(n) : U \text{ skew} \}
\]

One checks that \( Sp(n, \mathbb{R})_1 \) and \( SO^*(2n)_1 \) transform \( U_s(n) \) \( U_k(n) \), respectively. Restricting the causal structure of \( U(n) \) to these closed submanifolds, one finds that the resulting cone fields on \( U_s(n) \) and \( U_k(2n) \) are non-trivial for all \( n \geq 1 \). Thus \( Sp(n, \mathbb{R})_1 \) and \( SO^*(4n)_1 \) act causally on \( U_s(n) \) and \( U_k(2n) \), respectively. (These two actions and the action of \( SU(n,n) \) on \( U(n) \) are individually isomorphic to the actions of each group \( G \) on the Shilov boundary of its associated Hermitian symmetric domain \( G/K \).)

**Theorem 24.1.** \( SU(n,n) \) acts temporally on \( U(n) \), with the causal structure on \( SU(n,n) \) obtained from the maximal cone \( \mathcal{C}_1 \) in \( su(n,n) \) described earlier.
Proof. As $SU(n,n)$ acts causally on $U(n)$ and its causal structure is invariant, it suffices to check the second condition in Definition 23.2 for $g = e$; we compute the differential of $p$ at this point. For $X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ in the Lie algebra, \((U + eAU + eB)(eB^*U + I + eC)^{-1} = U(I + eL) + o(e^2)\), where $L = U^{-1}AU - C - B^*U + U^{-1}B$. We check that $iL$ is positive-definite Hermitian if $X \in \mathcal{O}_1$.

Given $y \in \mathbb{C}^n$ set $x = Uy$, and note

\[
<iL, y> = i<A, x> - i<C, y> \\
- i<B^*, y> + i<B, x>
\]

In fact, this expression is just $iH(Xw, w)$ for $w = \begin{pmatrix} x \\ y \end{pmatrix}$ and $H(w, w) = 0$, so $<iL, y> \geq 0 \forall y \in \mathbb{C}^n$ by the definition of $\mathcal{O}_1$.

In particular, $SU(n,n)$ acts temporally with the causal structure obtained from the minimal cone $\mathcal{O}_0$, and by the last remark in Section 23 we have the

Corollary 24.2. For all $n \geq 1$, $Sp(n, \mathbb{R})_1$ acts temporally on $U_s(n)$ (with the essentially unique causal structure on $Sp(n, \mathbb{R})_1$). For the causal structures on $SO^*(2n)_1$ obtained from the self-dual cones $\mathcal{C}_1$, $SO^*(2n)_1$ acts temporally on $U_k(n)$ (for $n \geq 4$ even) and on $U(n)$ (for all $n \geq 3$).

The case above of $SO^*(2n)$, $n$ even, can be strengthened.
Theorem 24.3. For $n \geq 2$, $SO^*(4n)$ acts temporally on $U_k(2n)$, with the causal structure on $SO^*(4n)$ obtained from the maximal cone $\mathcal{C}_2$.

Proof. As in the previous proof, we must show that $iL = i(U^{-1}aU - \bar{a} + \bar{\xi}U + U^{-1}b)$ is positive-definite Hermitian if $U$ is unitary and skew and $C^{-1}(\begin{pmatrix} a & \beta \\ -\bar{\beta} & \bar{a} \end{pmatrix}) e \in \mathcal{C}_2$ of $so^*(2n)$.

To simplify the argument we use the fact that any $U \in U_k(2n)$ may be written as $U_1 D U_1^t$ where

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(as in [12, p. 6]) and $U_1$ is unitary. (This can be proven by the usual diagonalization arguments, but abstractly one knows that $K$ acts transitively on the Shilov boundary [15, p. 270].) Then

$$iL = i(U^{-1}aU - \bar{a} + \bar{\xi}U + U^{-1}b)$$

where $w = U_1^t v$, $a_1 = U_1^{-1}aU_1$, $\beta_1 = U_1^{-1}\beta U_1^{-1}$. Then

$$C^{-1}(\begin{pmatrix} a_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{a}_1 \end{pmatrix}) e \in \mathcal{C}_2$$

also, and we drop the subscript.

Given any $w \in \mathfrak{a}^n$ let $e = \begin{pmatrix} w \\ -iDw \end{pmatrix}$, and check that $\tau(e,e) = \tau(e,\beta e) = 0$ (as in Section 17), so $\mathcal{A}(Xe,e) \geq 0$ for all $X \in \mathcal{C}_2$. By the identity of the last section

$$iH(\begin{pmatrix} a & \beta \\ -\bar{\beta} & \bar{a} \end{pmatrix}) e, e) \geq 0.$$
Now \( e = \left( \begin{array}{c} Dv \\ v \end{array} \right) \) for some \( v \), and in fact (19) equals (18) with \( \alpha, \beta, \gamma \) replacing \( \alpha_1, \beta_1, \gamma \) respectively, which completes the proof.

Before stating the consequences of the above for global causality we consider \( SO_o(2, n) \) and its action on the projective quadric

\[
\mathbb{Q} = \{ [x \in \mathbb{R}^{2+n}: \tau(x, x) = 0] \}
\]

\([x]\) meaning the line determined by \( x \). (See [19], Section II.4.) The tangent space to \( [x] \in \mathbb{Q} \) can be identified with the projective space of vectors \( y \in \mathbb{R}^{2+n} \) such that \( \tau(y, x) = 0 \), modulo \( \mathbb{R}x \). \( \tau \) factors to a linear conformal structure \( \tau_1 \) on \( \mathbb{Q} \) with signature \((1, n)\). As \( S^1 \times S^n \) is a double cover of \( \mathbb{Q} \) a system of forward cones can easily be chosen.

**Theorem 24.4.** \( SO_o(2, n) \) acts temporally on \( \mathbb{Q} \), with the causal structure on the group coming from a closed maximal invariant convex cone in the Lie algebra.

**Proof.** Let \( \mathcal{C}_1 \) be the cone defined earlier. (See Sections 20 and 21 for notation.) Then \( X \in \mathcal{C}_1 \) iff \( \tau(Xe, f) \geq 0 \) \( \forall (e, f) \in \mathcal{Q} \).

This condition has a pleasant geometric interpretation. Given any \( [e] \in \mathbb{Q} \) and \( X, f \in \mathcal{Q} \), \( Xe \in T_e(\mathbb{R}^{2+n}) \) as usual, and with care we can write \( [Xe] \in T_\mathcal{Q}(\mathcal{Q}) \), as \( \tau(Xe, e) = 0 \) always. Now the projective space of \( f \in \mathbb{R}^{2+n} \) such that...

\[ \tau(f,f) = \tau(f,e) = 0 \text{ and } (e,f) \in \Omega, \text{ modulo } \Re, \text{ gives one of the cones } \overline{C} \text{ (not convex) in } T_{[e]}(\mathcal{Q}) \text{ of } \tau_1\text{-isotropic vectors. Then } \tau(Xe,f) \geq 0 \forall (e,f) \in \Omega \text{ implies } \tau_1([Xe],w) \geq 0 \forall w \in \overline{C}, \text{ which implies (as } \tau_1 \text{ has signature } (1,n)) \text{ that } [Xe] \text{ is a tangent vector in the convex cone generated by } \overline{C}. \text{ Thus } [Xe] \text{ is a forward-pointing tangent vector at } [e] \text{ for all } [e] \in \mathcal{Q}. \]

**Definition.** A manifold \( M \) with causal structure is called **globally causal** if there exists no closed non-trivial piece-wise \( C^1 \) curve in \( M \) the differential of which lies in the causal cone at each point.

**Theorem 24.5.** For all \( n \geq 1 \), \( \widehat{Sp(n,\mathbb{R})} \), \( \widehat{SU(n,n)} \), \( \widehat{SO^*(4n)} \), and \( \widehat{SO_0(2,n+2)} \) are globally causal with respect to their causal structures coming from a maximal causal cone in their Lie algebras. \( \widehat{SU(p,q)} \) \( (p > q \geq 1) \) is globally causal with respect to the minimal cone \( \overline{C}_0 \) causal structure, and \( \widehat{SO^*(4n+2)} \) \( (n \geq 1) \) is globally causal with respect to the causal structure coming from the self-dual cone \( \overline{C}_1 \).

**Proof.** The first four series of groups act temporally on \( \widehat{U_s(n)} \), \( \widehat{U(n)} \), \( \widehat{U_k(2n)} \), and \( \mathbb{R} \times S^n \), respectively, by the initial remarks in section 23 and Theorems 24.1-4. \( \widehat{U(n)} \) and \( \mathbb{R} \times S^n \) are globally causal by Corollary 2.3.1 and Scholium 2.11 in [19].

To identify \( \widehat{U_s(n)} \) and \( \widehat{U_k(2n)} \), we set
\[ M_s(n) = \{ U \in U_s(n) : U \in SU(n) \} = \{ UU^\dagger : U \in SU(n) \} \]

and

\[ M_k(2n) = \{ UDU^\dagger : U \in SU(2n) \} , \]

D as in 24.3. They are coset spaces, isomorphic to
SU(n)/SO(n) and SU(2n)/Sp(n), respectively, and are
simply connected because SU(n) is simply connected and
the factor spaces are connected.* Thus

\[ \widetilde{U_s(n)} \equiv M_s(n) \times \mathbb{R} \quad \text{and} \quad \widetilde{U_k(2n)} \equiv M_k(2n) \times \mathbb{R} , \]

and the spaces on the r.h.s. are contained component-wise
in SU(n) \times \mathbb{R} \cong \widetilde{U(n)} . Clearly then \( \widetilde{U_s(n)} \) and \( \widetilde{U_k(2n)} \)
are also globally causal.

We can assume that any closed time-like curve \( g(t) \)
\((0 \leq t \leq 1)\) in any of the covering groups \( \tilde{G} \) starts and
ends at \( e \in \tilde{G} \). As the action of \( \tilde{G} \) on the relevant
manifold \( \tilde{S} \) is temporal, \( g(t)p \ (\forall p \in \tilde{S}) \) is a closed
time-like curve in \( \tilde{S} \), contradicting the global causality
unless \( g(t)p = p \forall t \), so \( g(t) = e \forall t \).

The last statement follows in the same way by
Corollary 24.2, and the next-to-last by the embedding
su(p,q) \( \hookrightarrow \) su(p,p), under which the minimal cone of
su(p,q) goes into the minimal cone of su(p,p).

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*I thank Peter Greenberg for these observations.
Corollary 24.6. If \( \tilde{G} \) is any of the above \((24.5)\) groups with the causal cone \( C \) indicated, then there exists a semigroup \( S \) in \( \tilde{G} \) such that \( S \cap S^{-1} = \{e\} \) and \( gSg^{-1} = S \) for all \( g \in \tilde{G} \), generated by \( \{ \exp X : X \in C \} \).

Proof. If \( \exp X \cdots \exp X_n = e, X_j \in C \), the expression gives a piecewise \( C^1 \) time-like curve in \( \tilde{G} \), so all \( X_j \) must be 0. This suffices to get \( S \cap S^{-1} = \{e\} \), and \( gSg^{-1} = S \) is clear.

Remarks. In the theory of Hermitian symmetric spaces the first four series of groups are referred to as of "tube type", the latter two of "non-tube type". (The \( e_7 \) domain is also of tube type, the \( e_6 \) domain not.) Only in the case of tube type domains does the Šilov boundary unravel and have a globally causal universal covering. (For example, the Šilov boundary for \( SU(2,1) \) is isomorphic to \( S^3 \).) Thus proving the global causality with the maximal causal structure may require some new ideas in the non-tube type case.

In the former cases one can ask whether the semigroup of Corollary 24.6 is closed and in fact the entire set of forward displacements of the universal covering of the Šilov boundary coming from the group action. I believe that it is, but this question may require a consideration of the classification of the conjugacy classes in the groups. [2] also gives a treatment of
this, a simple extension of the Lie algebra classification. It is interesting that here the Cayley transform (see section 29) appears naturally.

Based on the analogy with $\text{SL}(2,\mathbb{R})$ and some other cases, I would conjecture that any forward displacement is a product of two exponentials as in 24.6, one of which is some $\exp tZ$, $Z \in Z(k)$, $t \geq 0$, the other in the range of the Cayley transform restricted to the forward maximal cone $C$. (Any such transform is also an exponential of an $X \in C$.)
CHAPTER VII. Kählerization in Causal Lie Algebras.

25. Uniqueness of the Complex Structure.

We have seen that each $X$ in the interior of a maximal open invariant convex cone in the Lie algebras thus far considered, is conjugate to an element in an open cone in the maximal torus $\mathfrak{h}$ of $\mathfrak{k}$. In particular, such $X$ are elliptic, i.e., in some maximal compact subalgebra. In fact, this maximal compact is uniquely determined by $X$. Such uniqueness does not hold in general for non-positive elliptics, but indications are that it holds for at least a "generic" set (Proposition 25.3). We call this mapping of elliptics to maximal compacts Kählerization*.

Uniqueness in the symplectic case, from the standpoint of linear quantum fields, was first shown by Weinless in his thesis [20]. His proof involved holomorphic functions in the upper half-plane and particularly some results of Goodman [8] on such functions. Here we give two fairly straightforward proofs of the above claim. The first is an argument in terms of real Hilbert spaces, finite- or infinite-dimensional, applicable to the open invariant cone in $\text{sp}(n,\mathbb{R})$, the minimal open cone in $\text{su}(p,q)$, and the

*Although, this is a term perhaps best reserved for only the symplectic groups.
"middle" cone $C_1$ in $\text{so}^*(2n)$. The second proof is more non-linear, involving the action of $G$ on the symmetric space $G/K$, but covers all the maximal open cones.

Consider first $\text{sp}(\mathcal{V})$, where $(\mathcal{V}, \mathfrak{g}_0)$ is a real Hilbert space possessing an orthogonal $J$ with square $-I$. (We use the notation in section 7, and again consider only bounded operators.) The extension appropriate for our purposes of the notion of "maximal compact subalgebra" of $\text{sp}(\mathcal{V})$ to infinite dimensions* is as follows. Such a subalgebra is all those elements skew with respect to a complex Hilbert structure on $\mathcal{V}$, topologically equivalent to and having the same imaginary part $\sigma(\cdot, \cdot)$ as the original one. A complex structure in $\text{sp}(\mathcal{V})$ is a $J_1 \in \text{sp}(\mathcal{V})$ such that $J_1^2 = -I$ and $\sigma(J_1 \cdot, \cdot)$ is a positive-definite symmetric form on $\mathcal{V}$ defining an equivalent norm. Denote the set of complex structures by $\Gamma$. One checks that $\Gamma \subseteq \text{Sp}(\mathcal{V})$ also, and that $\Gamma$ is in 1-1 correspondence with the afore-mentioned set of complex Hilbert structures. Furthermore, the correspondence extends to the set of maximal compacts: given $J_1 \in \Gamma$, the corresponding maximal compact $\mathfrak{k}$ is all $X \in \text{sp}(\mathcal{V})$ commuting with $J_1$. That $\mathfrak{k}$ in fact determines $J_1$ is

*"Quasi-compact" in [18], but we shall refrain from introducing this term as the compactness of the corresponding group in the finite-dimensional case is not used here.
clear from the observation that no two distinct complex structures commute: if say $J_1$ commuted with the original $J$, then $J^{-1}J_1 = D$ is positive-definite symmetric in $(\mathcal{X}, g_0)$ and has square $I$, hence equals $I$ and $J_1 = J$. (This argument is a special case of 25.2.)

By the unitary equivalence of Hilbert spaces with the same dimension, all maximal compacts in $\text{sp}(\mathcal{X})$ are conjugate under $\text{Sp}(\mathcal{X})$, and $\Gamma$ is a single orbit in the positive cone in $\text{sp}(\mathcal{X})$.

Now $U(\mathcal{X}, \mathcal{J})$ and $\text{SO}^*(\mathcal{X})$ (see sections 11 and 16) are further restrictions of this situation. To pass to $u(\mathcal{X}, \mathcal{J})$ from $\text{sp}(\mathcal{X})$, one takes an $i \in \text{sp}(\mathcal{X})$ such that $i^2 = -I$, $ij = ji = \mathcal{J}$, and $\mathcal{A}(\cdot, \cdot, \cdot)$ not positive- or negative-definite (in finite dimensions the (real) signature being $(2p, 2q)$, $p, q \geq 1$), and defines $u(\mathcal{X}, \mathcal{J}) = \{ X \in \text{sp}(\mathcal{X}) : iX = iX \}$. (All $i \in \text{sp}(n, \mathbb{R})$ such that $i^2 = -I$ and $\mathcal{A}(\cdot, \cdot, \cdot)$ has a particular signature, also form a single $\text{sp}(n, \mathbb{R})$-orbit.) This embeds $u(p, q) \subset \text{sp}(p+q, \mathbb{R})$, and one sees that maximal compact subalgebras of $\text{su}(p, q)$ are in 1-1 correspondence with $\Gamma \cap u(p, q)$, a single $u(p, q)$-orbit, which lies in a minimal open invariant cone in $\text{su}(p, q)$ iff $p = q$. If $p \neq q$, $\Gamma \cap u(p, q)$ projects injectively onto an $\text{su}(p, q)$-orbit, each element of which spans the center of some maximal compact.

As shown in section 16, $\text{so}^*(\mathcal{X})$ is a further restriction on $u(\mathcal{X}, \mathcal{J})$ in the case when the dimensions of
the eigenspaces of $\mathfrak{g}$ are equal. In finite dimensions one has $\mathfrak{so}^*(2n) \subset \mathfrak{su}(n,n) \subset \mathfrak{sp}(2n,\mathbb{R})$, as follows. $\mathfrak{so}^*(2n)$ is defined by taking $i, \sigma \in \mathfrak{sp}(2n,\mathbb{R})$ which arise as in section 16, and letting

$$\mathfrak{so}^*(2n) = \{ X \in \mathfrak{sp}(2n,\mathbb{R}) : X_i = iX, X\sigma = \sigma X \} .$$

The intersection of the open positive invariant cone in $\mathfrak{sp}(2n,\mathbb{R})$ with $\mathfrak{so}^*(2n)$ is the cone $C_1$ above, and furthermore maximal compacts of $\mathfrak{so}^*(2n)$ are in 1-1 correspondence with $\Gamma \cap \mathfrak{so}^*(2n)$, a single $SO^*(2n)$-orbit.

The purpose of the preceding two paragraphs is to make the uniqueness argument of 25.2 applicable to the indicated cones in $\mathfrak{su}(p,q)$ and $\mathfrak{so}^*(2n)$. We see that maximal compacts in these two algebras are contained in unique maximal compacts in a symplectic algebra, so uniqueness for $\mathfrak{sp}(n,\mathbb{R})$'s open cone $C_0$ implies uniqueness for the cones obtained by intersecting $C_0$ with $\mathfrak{su}(p,q)$ and $\mathfrak{so}^*(2n)$.

There is a simple parametrization of the set of positive complex structures $\Gamma$ for $\mathfrak{sp}(\mathbb{H})$. Let $\mathcal{L}(\mathbb{H})$ denote the algebra of bounded operators on $\mathbb{H}$, and $A^t$ the transpose, with respect to $S_0(\cdot,\cdot) = \langle J\cdot,\cdot \rangle$, of $A \in \mathcal{L}(\mathbb{H})$. We have

$$\mathfrak{Sp}(\mathbb{H}) = \{ g \in \mathcal{L}(\mathbb{H}) : g^tJg = J \} \quad \text{and} \quad \mathfrak{sp}(\mathbb{H}) = \{ X \in \mathcal{L}(\mathbb{H}) : X^tJ + JX = 0 \} = \{ JH : H \in \mathcal{L}(\mathbb{H}), H^t = H \} .$$
If \( K \in \Gamma \) then \( K = JD \) where \( D^t = D \) is positive-definite symmetric (with respect to \( g_0 \)). \( K^2 = -I \) implies \( J = DJD \), so \( D \in \text{Sp}(\mathcal{K}) \).

In the finite-dimensional case we have in fact \( D \in \exp \mathcal{P} \). \( D \) is the exponential of a unique symmetric \( X \), and \( D^{-1} = JDJ^{-1} \) is equivalent to \(-X = J^{-1}XJ\); together with \( X^t = X \) this implies \( X \in \mathcal{P} \). Reversing the steps, we have

\[
\Gamma = \{ J \exp X : X \in \mathcal{P} \}.
\]

Returning to the general case, note that (positive-definite) symmetric forms \( g(\cdot,\cdot) \) on \( \mathcal{K} \) are in linear 1-1 correspondence with (positive-definite, resp.) symmetric operators \( \tilde{g} \) on \( (\mathcal{K},g_0) \), via \( g(\cdot,\cdot) = g_0(\tilde{g}(\cdot,\cdot)) \).

**Proposition 25.1.** A positive-definite symmetric form \( g(\cdot,\cdot) \) can be the real part of a topologically equivalent complex Hilbert structure on \( \mathcal{K} \) with imaginary part \( \sigma(\cdot,\cdot) \), if and only if \( \tilde{g} \in \text{Sp}(\mathcal{K}) \).

**Proof.** If \( R \in \Gamma \) and \( g(\cdot,\cdot) = g(R\cdot,\cdot) \), then \( \sigma(R\cdot,\cdot) = g(\tilde{g}\cdot,\cdot) = g(J\tilde{g}\cdot,\cdot) \), so \( \tilde{g} = J^{-1}R \in \text{Sp}(\mathcal{K}) \). Conversely, \( \tilde{g} \in \text{Sp}(\mathcal{K}) \) implies \( R = J\tilde{g} \in \text{Sp}(\mathcal{K}) \) and \( R^2 = J(\tilde{g}J\tilde{g}) = J^2 = -I \).

**Theorem 25.2.** Suppose \( X \in \text{sp}(\mathcal{K}) \) and \( \sigma(X\nu,\nu) \geq k g_0(\nu,\nu) \) \( \forall \nu \in \mathcal{K} \), for some \( k > 0 \). Then \( X \) commutes with a unique complex structure.
Proof. The existence of one such complex structure is clear from Theorem 7.1, and so it suffices to assume $JX = XJ$.

If now $XK = KX$ for some $K \in \Gamma$, write $K = JD$ and $X = JH$ as above. The hypothesis on $X$ implies that $H$ is positive-definite symmetric in $(X, S_0)$. Further, we have $(JH)D = D(JH)$ as $JH = HJ$, so $H^2$ commutes with $D$. Thus $H$ commutes with $D$ by the functional calculus, and $D(JH) = (JH)D = J(HD) = J(DH)$. As $H$ is surjective, $JD = DJ$ also. Now $J = DJD$ and $D$ is positive-definite, so $D^2 = I$, hence $D = I$ and $K = J$.

Mostly for the sake of convenience, let us restrict to the finite-dimensional case for the rest of this section. It is not difficult to see that $X \in \text{sp}(n, \mathbb{R})$ is elliptic iff $X$ is diagonalizable over $\mathbb{C}$ with purely imaginary eigenvalues. The condition that $X$ is in a unique maximal compact subalgebra is fairly simple.

Proposition 25.3. Let $X = JH = HJ \in \text{sp}(n, \mathbb{R})$. Then $H$ is in a unique maximal compact subalgebra iff $H$ and $-H$ have no eigenvalues in common.

Proof. Let $\mathbb{R}^{2n} = V_1 \oplus \ldots \oplus V_m$ be the eigenspace decomposition of $H$, where $H = \alpha_j$ on $V_j$, $\alpha_j \in \mathbb{R}$. The $V_j$ are $S_0$- and $\alpha$-orthogonal and $J$-invariant, so all dim $V_j$ are even. Let $JD$, $D$ positive-definite symmetric and symplectic, be a candidate for an additional complex structure commuting with $JH$. As above, $D$
commutes with $H^2$, hence with $|H|$, so the sufficiency is clear as in the proof of 25.2. Conversely, if say $\alpha_1 = 0$, then any $D \in \exp \mathfrak{p}$ on $V_1$ is possible, and there is no uniqueness. Secondly, if say $\alpha_1 = -\alpha_2 \neq 0$, it suffices to see that there are lots of complex structures in $\text{sp}(2,\mathbb{R})$ commuting with the elliptic

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

These can be determined, and are a two-parameter family: $L$ commutes with all

$$JD = \begin{pmatrix} 0 & a & x & b \\ a & 0 & b & x \\ -x & b & 0 & -a \\ b & -x & -a & 0 \end{pmatrix}$$

where $x > 0$ and $1 + a^2 + b^2 = x^2$. (The eigenvalues of $D$ are $x \pm \sqrt{x^2 - 1}$, each with multiplicity two.)

We next give the second proof of uniqueness mentioned at the start. The idea is to note that each maximal compact subgroup leaves fixed a unique point in $G/K$, and also that one-parameter groups generated by an $X$ in an open maximal cone also have this property. The first statement is a general fact about symmetric spaces, and is sharpened in the Hermitian symmetric case as follows (Theorem 3.2, Chapter IX [9]). Let $G$ be a simple Lie group and $K$ a maximal compact subgroup, such that $G/K$ is Hermitian symmetric, and $\gamma^\infty_k$ as usual. Let $x_0$ be
the identity coset in \( G/K \) and \( J \in Z(\mathfrak{g}) \) such that
\( \text{ad}(J) : \mathfrak{p} \to \mathfrak{p} \) is a complex structure (see section 3).
Then \( \text{Ad}(\exp \pi J) \) is the Cartan involution for \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \),
and the geodesic symmetry (reflection) about \( x_0 \) is the
action of \( \exp(\pi J) \) on \( G/K \). \( x_0 \) is the unique fixed
point of the one-parameter group generated by \( J \).

Now consider the standard actions of \( SU(p,q) \),
\( \text{Sp}(n,\mathbb{R}) \), and \( \text{SO}^*(2n) \) (see section 23) on the appro-
priate bounded domain in some \( \mathbb{C}^m \) by fractional linear
transformations [14, p. 215]. To prove the second
assertion above, it suffices to consider \( X \) in the
maximal torus \( \mathfrak{h} \).

\[ \text{SU}(p,q) \] Take \( (\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q) \in \mathbb{R}^{p+q} \) such that
\( \lambda_i + \sigma_j > 0 \) \( i, j \). If \( Z \) is any \( pxq \) complex matrix
then clearly
\[
\begin{pmatrix}
-e^{-t\lambda_1} & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -e^{-t\lambda_p}
\end{pmatrix}
Z
\begin{pmatrix}
e^{-t\sigma_1} & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -e^{-t\sigma_q}
\end{pmatrix}
= Z \quad \forall \ t \in \mathbb{R}
\]
iff \( Z = 0 \).

\[ \text{Sp}(n,\mathbb{R}) \] Let \( (d_1, \ldots, d_n) \in \mathbb{R}^n \) such that all \( d_j > 0 \).
If \( Z \) is any \( nxn \) symmetric complex matrix then
\[
(20) \quad 
\begin{pmatrix}
e^{-tid_1} & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -e^{-tid_n}
\end{pmatrix}
Z
\begin{pmatrix}
e^{-tid_1} & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -e^{-tid_n}
\end{pmatrix}
= Z \quad \forall \ t \in \mathbb{R}
\]
iff \( Z = 0 \).
Let \( (d_1, \ldots, d_n) \in \mathbb{R}^n \) such that \( d_j + d_k > 0 \) \( \forall j \neq k \). If \( Z \) is any \( n \times n \) skew-symmetric complex matrix, then clearly (20) holds \( \forall t \in \mathbb{R} \) iff \( Z = 0 \).

Finally, consider \( \text{so}(2,n) \). The action of \( \text{SO}_0(2,n) \) on its bounded domain is a bit of a mess [21, p. 168], but the \( K = \text{SO}(2) \times \text{SO}(n) \)-action is simple in a particular representation. In fact it is equivariant with the linear adjoint action of \( K \) on \( \mathfrak{p} \) [15]. So, let \( d_0 > |d_j| \) \( (j = 1, \ldots, [n/2]) \) give an element in a maximal open cone as in (16), set \( R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \), and let \( B \) be \( 2 \times n \). Then
\[
\begin{pmatrix} R_{d_0} & 0 \\ 0 & R_{d_j} \end{pmatrix} \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \begin{pmatrix} R_{-d_0} & 0 \\ 0 & R_{-d_j} \end{pmatrix} = \begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}
\]
where \( C = R_{d_0} B R_{-d_j} \). One checks easily that
\[
\frac{d}{dt} (BR_{d_0} B)_{t=0} = \frac{d}{dt} (R_{d_0} B)_{t=0}
\]
iff \( B = 0 \).

As a result, we have proven

**Theorem 25.4.** Let \( \mathfrak{l}_j \) be one of the simple causal Lie algebras \( \text{sp}(n, \mathbb{R}) \), \( \text{su}(p,q) \), \( \text{so}^*(2n) \), \( \text{so}(2,n) \). Then each \( X \in \mathfrak{l}_j \) in an open maximal invariant convex cone is contained in a unique maximal compact subalgebra.

In this section we extend the discussion in section 7 to some unbounded infinitesimally symplectic operators. Using the notation there, we consider real-linear, possibly unbounded $S: \mathcal{D}_S \rightarrow \mathcal{V}$ in $(\mathcal{V}, \langle \cdot , \cdot \rangle)$ having dense domain $\mathcal{D}_S \subseteq \mathcal{V}$ and satisfying $c(Sx,y) + c(x,Sy) = 0 \ \forall \ x,y \in \mathcal{D}_S$.

We raise the question: when can we regard $S$ as a skew-adjoint operator on some complex Hilbert space $(\widetilde{\mathcal{V}}, \langle \cdot , \cdot \rangle_\perp)$ having a "dense intersection" with $\mathcal{V}$ and such that $\text{Im}\langle \cdot , \cdot \rangle = \text{Im}\langle \cdot , \cdot \rangle_\perp$ on that intersection? On this question mention should be made of [3] and [18].

When $S$ satisfies a positivity condition on its domain, we find a canonical $\widetilde{\mathcal{V}}$, and $\mathcal{V} \cap \mathcal{V}$ contains a certain "finite-energy" subspace, in complete analogy with the positive-mass Klein-Gordan wave equation in Minkowski space. However, the formulation of a uniqueness statement like Theorem 25.2 is not yet clear.

We first recall the Friedrichs extension. Let $(\mathcal{V}, \langle \cdot , \cdot \rangle)$ be a real or complex Hilbert space and $H$ an Hermitian operator on $\mathcal{V}$ with dense domain $\mathcal{D}_H$, bounded from below by $k > 0$:

$$\langle Hv,v \rangle \geq k\langle v,v \rangle \ \forall \ v \in \mathcal{D}_H.$$

(If $\mathcal{V}$ is complex we assume $H$ is complex-linear.) $H$ extends to a surjective self-adjoint $A: \mathcal{D}_A \rightarrow \mathcal{V}$, also bounded below by $k$, as follows. Let $[\mathcal{D}_H]$ be the
completion of $\mathcal{B}_H$ in $<\cdot,\cdot>_{\varepsilon} = <H\cdot,\cdot>$, and extend $<\cdot,\cdot>_{\varepsilon}$ to a real Hilbert structure on $[\mathcal{B}_H]$. $([\mathcal{B}_H],<\cdot,\cdot>_{\varepsilon})$ is the so-called "finite-energy" subspace. Let $\mathcal{B}_A = \mathcal{B}_H^* \cap [\mathcal{B}_H]$ and $A = H^*|_{\mathcal{B}_A^*}$. A consequence of the proof is $<x,y>_{\varepsilon} = <Ax,y> \forall x \in \mathcal{B}_A, y \in [\mathcal{B}_H]$, and it is not hard to see that $[\mathcal{B}_H] = [\mathcal{B}_A]$. Typically, the inclusion $\mathcal{B}_A \subseteq [\mathcal{B}_A]$ is proper.

**Lemma 26.1.** $([\mathcal{B}_A],<\cdot,\cdot>_{\varepsilon})$ is unitarily equivalent to $(\mathcal{N},<\cdot,\cdot>)$ via the densely defined unitary map $U = (A^{-1})^{\frac{1}{2}} \cdot A : \mathcal{B}_A \to \mathcal{N} \to (A^{-1})^{\frac{1}{2}}$ where $\mathcal{N} = (A^{-1})^{\frac{1}{2}} \supseteq \mathcal{B}_A$ is the range of $(A^{-1})^{\frac{1}{2}}$. Thus $\mathcal{B}_A = [\mathcal{B}_A]$ iff $A$ is bounded.

**Proof.** By a remark above, if $x,y \in \mathcal{B}_A$

$$<x,y>_{\varepsilon} = <Ax,y> = <Ax,A^{-1}Ay> = <Ux,Uy> \ .$$

Let $\mathcal{B}_{A_1} = A^{-1}(\mathcal{B}_A)$ and $A_1 = A|_{\mathcal{B}_{A_1}}$. Then $A_1 : \mathcal{B}_{A_1} \to [\mathcal{B}_A]$ is surjective.

**Lemma 26.2.** In $([\mathcal{B}_A],<\cdot,\cdot>_{\varepsilon})$, $\mathcal{B}_{A_1}$ is dense (-also in $(\mathcal{N},<\cdot,\cdot>)$) and $A_1$ is self-adjoint and bounded from below also by $k$.

**Proof.** That $\mathcal{B}_{A_1}$ is dense and $A_1$ Hermitian is easily shown. To see that $A_1$ is self-adjoint, let $y \in \mathcal{B}_{(A_1)^*} \subseteq [\mathcal{B}_A]$, i.e. $\exists y^* \in [\mathcal{B}_A]$ such that $<x,y^*>_{\varepsilon} = <A_1x,y>_{\varepsilon}$.
∀ x ∈ 𝐹_{A_1}. Let A_1 x = x* ∈ [𝐹_A], and obtain <x*, y*> = <x*, y>e for all x* ∈ [𝐹_A]. If we assume only x* ∈ 𝐹_A, then <x*, y*> = <Ax*, y>, so y ∈ 𝐹_A* = 𝐹_A, and Ay = y* ∈ [𝐹_A], implying y ∈ 𝐹_{A_1} as desired.

Finally, <A_1 x, x>e ≥ k<x, x>e ∀ x ∈ 𝐹_A iff <x*, x*> ≥ k<x*, A_1⁻¹ x*> ∀ x* ∈ [𝐹_A], which is clear as ∥A_1⁻¹∥ ≤ 1/k.

Now take S: 𝐹_S → 𝗦 as in the first paragraph of this section, where (𝗦, <•, •> = 𝑆_0(•, •) + iΩ(•, •)) is a complex Hilbert space with complex structure J. Assume that Ω(Sv, v) ≥ k<v, v> ∀ v ∈ 𝐹_S, for some k > 0. We define H = J⁻¹S so that 𝐹_H = 𝐹_S and note that H is symmetric in (𝗦, 𝑆_0) and bounded from below by k.

Now extend H to a surjective (real) self-adjoint operator A: 𝐹_A → 𝗦 à la Friedrichs as above, and S likewise, so that JA = S: 𝐹_A → 𝗦. As previously, define the surjective restriction A_1: 𝐹_{A_1} → [𝐹_A], and likewise let 𝐹_{S_1} = S⁻¹([𝐹_A]), S_1 = S|_{𝐹_{S_1}} : 𝐹_{S_1} → [𝐹_A].

Then Lemma 26.2 applies, and the following is the analogue for S_1, proven the same way.

**Lemma 26.3.** In ([𝐹_A], <•, •>e), 𝐹_{S_1} is dense (also in (𝗦, <•, •>)), and S_1 is skew-adjoint.

Note that we have 𝐹_{A_1}, 𝐹_{S_1} ⊆ 𝐹_A ⊆ [𝐹_A] ⊆ 𝗦; in general 𝐹_{A_1} and 𝐹_{S_1} overlap non-trivially, and equality holds in any of these inclusions iff A is bounded.

We apply the polar decomposition in ([𝐹_A], <•, •>e) for
closed operators in a real space, writing \( S_1 = RT \), where
\( R \) is orthogonal, \( T \) is non-negative self-adjoint, and \( R \) and \( T \) commute with all orthogonals commuting with \( S_1 \).
\((R \) is orthogonal as \( \ker S_1 = \{0\} \) and \( \text{range } S_1 = [\mathcal{A}] \).

We have \( S_1 = -S_1^* = -TR^{-1} = (-R^{-1})(RTR^{-1}) \) by 26.3, so by
uniqueness \( R^2 = -I \), \( R(S_1) = S_1 = S_1 = S_T \), and \( RT = TR \).

\( T \) is also surjective, so \( T_1: [\mathcal{A}] \to S_1 \) is
bounded and positive-semidefinite. \( \mathcal{A}(Rx, Ry) = \mathcal{A}(x, y) \)
\( \forall x, y \in [\mathcal{A}] \) is shown using the equation \( JAx = RTx \)
\( \forall x \in S_1 \).

Now define
\[
S_1(x, y) = \mathcal{A}(Rx, y) \quad \forall x, y \in [\mathcal{A}].
\]

We have
\[
S_1(x, y) = S_0(J^{-1}Rx, y)
\]
(21)
\[
\quad = S_0(AT^{-1}x, y)
\]
\[
\quad = \langle T^{-1}x, y \rangle \quad \forall x, y \in [\mathcal{A}],
\]
so \( S_1 \) is a positive-definite form on \( [\mathcal{A}] \). Let \( \widetilde{\mathcal{H}} \)
be the completion of \( [\mathcal{A}] \) with respect to \( S_1 \). \( R, S_1 \),
and \( \mathcal{A} \) extend continuously to \( \widetilde{\mathcal{H}} \) so that
\( (\widetilde{\mathcal{H}}, S_1(\cdot, \cdot) + i\mathcal{A}(\cdot, \cdot) = \langle \cdot, \cdot \rangle) \) is a complex Hilbert space
with complex structure \( R \).

**Proposition 26.4.** 1) \( \|x\|^2_1 \leq \|T^{-1}\| \epsilon \|x\|^2_\epsilon \) \( \forall x \in [\mathcal{A}] \).
2) In \((\widetilde{\mathcal{H}}, \langle \cdot, \cdot \rangle)\), \( S_1 \) is dense and \( T \) is Hermitian
(complex-linear) and bounded from below by \( 1/\|T^{-1}\| _\epsilon \).
3) Making the Friedrichs extension (in the complex case) as earlier, $T$ and $S_1$ extend to surjective self-adjoint and skew-adjoint operators $\tilde{T}, \tilde{S}$ respectively, so that $R\tilde{T} = \tilde{T}R = \tilde{S}$.

**Proof.** 1) follows from (21) and 3) follows from 2). The density of $S_1$ is not difficult, and the bound on $T$ follows so: $\forall x \in S_1$, 

$S_1(Tx, x) = \langle x, x \rangle \geq (1/\|T^{-1}\|_\epsilon)\langle T^{-1}x, x \rangle_\epsilon$

$= (1/\|T^{-1}\|_\epsilon)S_1(x, x)$.

27. **Analyticity of Kählerization in** $\text{sp}(\mathcal{V})$.

As in section 7 let $(\mathcal{V}, S_0(\cdot, \cdot) + \text{i}G(\cdot, \cdot) = \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with complex structure $J$. We return to bounded operators only, and recall the definitions of $\Gamma$ and $\mathcal{L}(\mathcal{V})$ from section 25, using only the $S_0$-norm topology on $\mathcal{V}$.

Let $C_0 = \{X \in \text{sp}(\mathcal{V}): G(Xv, v) = k\langle v, v \rangle \forall v \in \mathcal{V} \text{ for some } k > 0\}$, an open convex cone in $\text{sp}(\mathcal{V})$, and $\Gamma \subseteq C_0$. To define an analytic structure in $\Gamma$ we use the following.

**Lemma 27.1.** $\Gamma$ is closed in $\mathcal{L}(\mathcal{V})$.

**Proof.** We saw in section 25 that
\[ \Gamma = \{ D \in \mathcal{L}(\mathcal{Y}) : D^t = D, DJD = J, \text{ and } D \text{ positive-definite} \}. \]

The first two conditions are obviously preserved on passing to limits, and the third follows from \( D^{-1} = JDJ^{-1} \).

If \( X \in C_0 \) determines a complex structure \( J_1 \) as in 25.2, we have \( X = J_1 H = HJ_1 \), where \( H \) is symmetric and positive-definite with respect to \( g_X(\cdot,\cdot) = \sigma(X,\cdot,\cdot) \).

Then clearly

\[
(22) \quad J_1 = X(-X^2)^{-1/2} = \varsigma(X),
\]

the square root being well-defined in \( (\mathcal{Y}, g_X) \).

**Theorem 27.2.** The mapping \( \varsigma : C_0 \to \Gamma \) defined by (22) is real-analytic.

**Proof.** This is a local property, and as \( \varsigma \) is constant on rays it suffices to check analyticity at \( X \in C_0 \) such that \( 0 < I + X^2 < I \) with respect to \( g_X \). \( (\|Y\|_{g_X}, Y \in \mathcal{L}(\mathcal{Y}), t > 0) \) is independent of \( t \). Now if \( Y \) is in a small neighborhood \( N \) of \( 0 \) in \( \text{sp}(\mathcal{Y}) \), \( I + (X + Y)^2 = M \) is longer symmetric in \( (\mathcal{Y}, g_X) \), but \( \|M\|_{g_X} \leq 1 \) may be assumed, and the power series

\[
\sqrt{(X+Y)^2} = \sqrt{I-M} = I - (1/2)M - (1/2)(3/2)M^2 - \ldots
\]

is valid for \( \|M\| \leq 1 \). \( M \) is also analytic in \( Y \), and thus \( \varsigma(X+Y) \) is a power series in \( Y \) for \( Y \in N \).
The theorem is no doubt true for the maximal open cones in the other algebras, too. It would follow if one could show that the unique fixed point in \( G/K \) of \( \{ e^{tX} \} \) (see section 25, second proof) depended analytically on \( X \).

28. An Example and Analytic Continuation: \( SU(1,1) \).

It is not only customary but also usually quite worthwhile to illustrate general theory in Lie groups by considering in detail the case of \( SL(2,\mathbb{R}) \) [10]. For this group we can analytically continue the map \( \chi \) in the last section to the whole Lie algebra minus zero (the range space no longer being merely complex structures), give an interpretation in the symmetric space, and complex-analytically extend \( \chi \) to \( SL(2,\mathbb{C}) - \{ \pm I \} \). Some indications for \( SU(2,1) \) will also be given, but an overall picture of \( \text{Kählerization} \) there has not yet emerged.

It will be useful to give \( SL(2,\mathbb{R}) \) some global coordinates in order to visualize the various regions of ellipticity, etc. (Compare with the picture in [7].) This we do by the Cartan decomposition \( G = K \exp p \), where

\[
K = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\}, \quad P = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right\}.
\]

Conjugation by \( K \) is merely orthogonal rotations in the \( A,B \)-planes, with respect to the coordinates \( \theta,A,B \).

Setting \( R = \sqrt{A^2 + B^2} \), the coordinates for the subgroup \( \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\} \) satisfy \( \cosh R \cos \theta = 1 \), and a two-dimensional
slice of the group is as follows.

The four conjugacy classes containing \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \), and \( \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \) are each homeomorphic to \( \mathbb{R}^2 - \{0\} \), and form boundaries between the simply connected elliptic regions \( e_1, e_2 \) and the two hyperbolic regions \( h_1, h_2 \) homeomorphic to \( S^1 \times \mathbb{R}^2 \). The orbit of complex structures \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \exp \theta \) is clearly just the plane \( \theta = \pi/2 \).

Turning to the Lie algebra, in Theorem 9.1 we saw that \( X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) is elliptic iff \( \det X = -a^2 - bc > 0 \), in which case \( b \) and \( c \) must be non-zero and of opposite sign.
The positive cone \( C_0 \) is where \(-b, c > 0\), and with our coordinates \( C_0 \) exponentiates into \( e_1 \).

One can see that \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) (\( ad - bc = 1 \)) is in \( e_1 \cup e_2 \) iff \( |\text{Tr} \, g| = |a+d| < 2 \), and we have

\[
e_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : |a+d| < 2, -b, c > 0 \right\},
\]

\[
e_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : |a+d| < 2, -b, c < 0 \right\}.
\]

These restrictions are closely related to formulas for \( \kappa \), which can be computed. \( X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in C_0 \) commutes with the unique positive complex structure

\[
\kappa(X) = \left( \frac{1}{\sqrt{-a^2 - bc}} \right) a \begin{pmatrix} a & b \\ c & -a \end{pmatrix},
\]

and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in e_1 \) is in the unique maximal compact determined by

\[
\left( 1 / \sqrt{4 - (a+d)^2} \right) \begin{pmatrix} a-d & 2b \\ 2c & -a+d \end{pmatrix}.
\]

Now \( \kappa(X) \) \( (X \in C_0) \) is just the unique complex structure in the one-parameter group generated by \( X \), so clearly \( \kappa \) could be analytically extended to \( \text{sl}(2, \mathbb{R}) - 0 \) with range \( S^2 \), by mapping each \( X \) to the ray in \( \text{sl}(2, \mathbb{R}) \) containing \( X \). However, this prescription does not give much insight into all the other cases, where \( \dim K > 1 \).

We pass next to the bounded domain, where \( G = \text{SU}(1,1) \) acts on \( D^1 = \{ |z| < 1 \} \) and nonsingularly on the entire Šilov boundary \( \mathcal{S} = S^1 = \{ |z| = 1 \} \). The complex structures go over to
\[ \Gamma_1 = \left\{ \begin{pmatrix} ic & D \\ D & -ic \end{pmatrix} : c < 0, D \in \mathbb{C}, c^2 = |D|^2 + 1 \right\} \]

which act as reflections about points in \( D' \). Let 
\( g = \text{su}(1,1) \), recall

\[ G_0 = \left\{ \begin{pmatrix} ia & B \\ B & -ia \end{pmatrix} \in g : a^2 - |B|^2 > 0, a < 0 \right\} , \]

and note that the corresponding elliptic regions are

\[ E_1 = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G : |\text{Re} A| < 1, \text{Im} B > \text{Im} A \right\} \]

and

\[ E_2 = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G : |\text{Re} A| < 1, \text{Im} B < \text{Im} A \right\} , \]

the two latter conditions being equivalent to \( \text{Im} A + \text{Im} B < 0 \), \( \text{Im} A + \text{Im} B > 0 \) respectively.

In what follows we define Kählerization \( \kappa_1 \) on the group; this is a bit more general as the corresponding map on the Lie algebra can be recovered by evaluating \( \kappa_1 \) on exponentials.

One computes that \( T = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in E_1 \) determines

\[ J_1 = \begin{pmatrix} ic & D \\ D & -ic \end{pmatrix} \in \Gamma_1 \]

where \( c = (\text{Im} A)/L, D = B/L, L = \sqrt{1-(\text{Re} A)^2} \), and the unique fixed points of \( J_1 \) are

\[ z_T = i \frac{c+1}{D} \hspace{1em} \text{(interior)} \]

and

\[ w_T = i \frac{c-1}{D} \hspace{1em} \text{(exterior)} \].
But $z_T, w_T$ are not analytic functions of $T$ near a nilpotent boundary. To remedy this, we assign to $z_T$ two points $y_T, m_T$ on $\mathcal{S}$ which will be analytic functions of $T$ as $T$ crosses a boundary, and seem to have a chance to generalize to higher dimensions. We will see that these points are closely related to the ordinary polar coordinates on $S^2$. At least in the tube-type cases $\dim \mathcal{S} = \frac{1}{2} \dim \mathcal{P}$, so two points on $\mathcal{S}$ would seem appropriate to determine a point in the bounded domain.

Of course this assignment is not made completely invariantly, but relative to prior choices of a maximal compact $K$, a maximal abelian $\alpha \subset \mathcal{P}$, and a Weyl chamber $\alpha_+ \subset \alpha$. We take

$$\alpha_+ = \left\{ \begin{pmatrix} 0 & -s \\ -s & 0 \end{pmatrix} : s \leq 0 \right\}$$

and then $A^+ = \exp \alpha_+$ moves the real line in $D'$ to the left with the unique fixed points $\pm 1$.

Given now $T \in E_1$, $z_T = S(0)$ for a unique transvection $S$ from $0$. $S$ is conjugate under $K$ to a unique $a \in A^+$, so let $m_T = a^2(i)$. As $T$ moves to a boundary $m_T$ heads toward $-1$; only with the square power does $m_T$ extend analytically! If $T \in K$, $z_T = 0$ and $\lambda_T$ is undefined; if $T \not\in K$, let $y_T$ be the unique point on $S^1$ on the same ray from $0$ as $z_T$. ("Ray from $0$" is a $K$-invariant notion.)

With $T = \begin{pmatrix} A & B \\ D & A \end{pmatrix} \in E_1$,
\[ \lambda_T = -i(B/|B|) \quad (\text{if } T \not\in K) \]  

\[ m_T = \frac{2|B| \text{Im} A + i((\text{Im} A)^2 - |B|^2)}{(|B|^2 + |\text{Im} A|^2) \sqrt{2}}. \]

The \( a \in \mathbb{A}^+ \) above is \((r \ s \ r)\) where \( r = |D|/M, s = (c+1)/M, M = \sqrt{-2-2c}, c, D \) as in (23).

\( m_T \) now extends to all of \( G - (\pm I) \), and \( \lambda_T \) to \( G - K \). We interpret \( \lambda_T, m_T \) for \( T \) ± hyperbolic, ± unipotent below, but first indicate the connection with polar coordinates \((\varphi, \theta)\) on \( S^2 \). Set \( \lambda_T = e^{i\varphi} \) and

\[
\cos \theta = (-\text{Im} A)/\left(\sqrt{(\text{Im} A)^2 + |B|^2}\right), \quad \sin \theta = |B|/\left(\sqrt{(\text{Im} A)^2 + |B|^2}\right)
\]

for \( 0 \leq \theta \leq \pi \). Then \( m_T = ie^{2i\theta} \). Clearly the mapping \( \chi_1: G - (\pm I) \to S^2 \) factors to \( \chi_1: (G/\pm I) - \{\pm I\} \to \mathbb{R}P^2 \) in the obvious way. \( \chi_1(E_1) \) is the set of all points with latitude \( \theta > +45^\circ \).

If \( T \in G \) is on the boundary of \( E_1 \), \( \lambda_T \) is the unique fixed point of \( T \); if on the boundary of \( E_2 \), the negative of that point. \( m_T = -1, +1 \) respectively in the two cases.

Any ± hyperbolic \( T \in G \) has two unique fixed points on \( S \), and is a sort of finite "flow" from one to the other. Then \( \lambda_T \) is one of the two points midway between these two fixed points: the midpoint on the "left" side of the flow if \( T \) is in the hyperbolic region with \( -I \) in its closure (example below), and the opposite midpoint if \( T \) is in the other region. \( m_T \) measures the
relative distance of the two fixed points from each other, but we will not make this precise.

With our polar coordinates, it is clear that
\[ \chi_1 \left( \begin{array}{cc} x+iy & z+iw \\ z-iw & x-iy \end{array} \right) = \frac{(y,z,w)}{\sqrt{y^2+z^2+w^2}} \in S^2 \]
for \( x,y,z,w \in \mathbb{R} \) satisfying
\[ (25) \quad x^2 + y^2 - z^2 - w^2 = 1, \quad |y|^2 + |z|^2 + |w|^2 \neq 0, \]
and it is clear why \( \pm I \in G \) is excluded. The mapping extends complex-analytically to
\[ \chi^C_1 : SL(2,\mathbb{C}) - (\pm I) \to \mathbb{CP}^2 : \left( \begin{array}{cc} x+iy & z+iw \\ z-iw & x-iy \end{array} \right) \to [y,z,w] \]
for \( x,y,z,w \in \mathbb{C} \) satisfying (25). At least for \( SU(1,1) - (\pm I) \), \( \chi_1 \) is clearly equivariant with respect to conjugation in \( G \) and some representation of \( G \) by analytic transformations of \( S^2 \) leaving the "polar regions" \( \theta > 45^\circ, \theta < -45^\circ \) invariant (hence did not really depend on the original choice of \( K \) and \( \alpha_+ \)), but this has not been worked out.

We conclude this section with a few remarks for Kählerization in \( su(2,1) \), where most of the real work related to this chapter has gone! From [2] one sees that
all orbits in $\text{su}(2,1)$ come from elements of $u(1) \oplus u(1,1)$, except for the six-dimensional "principal nilpotent" $N$ orbit, where $N^2 \neq 0, N^3 = 0$ as in [2]. It seems that $\chi$ should extend to all of $\text{su}(2,1)$, minus the union of all the four-dimensional elliptic orbits of real multiples of $X_e = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which we recognize as lying on the boundary of a closed maximal invariant cone. If this is the case, $\chi$ would be singular on a closed five-dimensional submanifold of $\text{SU}(2,1)$. Note that orbits of elements like $X_e + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & -i \\ 0 & 1 & i \end{pmatrix}$ are six-dimensional, and their multiples make up a seven-dimensional union of orbits composing the rest of the boundary of the maximal cones.
CHAPTER VIII. The Cayley Transform.

29. General Properties.

If \( H \) is a self-adjoint operator on some complex Hilbert space, the Cayley transform \( C(iH) = (I+iH)(I-iH)^{-1} \) is a unitary operator, and \( H \) can be recovered by the equation

\[
H = -i \frac{U-I}{U+I}.
\]

We have seen that, among the algebras enumerated in section 2, only \( \text{sp}(n,\mathbb{R}) \), \( \text{so}^*(2n) \), and \( \text{su}(n,1) \) have maximal compact subalgebras isomorphic to all skew-Hermitian operators on some complex Hilbert space. However, \( \text{sp}(n,\mathbb{R}) \) and \( \text{so}^*(2n) \) are distinguished as there this complex space can be naturally taken to be a complexification of the representation space of the algebra's fundamental (defining) representation. We will see that in the former two cases (but not in the case of \( \text{su}(n,1) \)) \( C \) extends to an open subset of the noncompact algebra, mapping into the corresponding group, and has several interesting algebraic and "causal" properties. We discuss features specific to \( \text{sp}(n) \) and \( \text{so}^*(n) \) in the next section, but in this section consider the mapping

\[
(26) \quad C(X) = \frac{I+X}{I-X}.
\]
in the general context of a Lie group acting on a finite-
dimensional vector space.

So, let the Lie group $G$ and its Lie algebra $\mathfrak{g}$ act
on $V$.

**Proposition 29.1.** $C$ maps a neighborhood of $0$ in $\mathfrak{g}$ into
$G$ iff $\tanh$ (given by the power series) maps a neighborhood
of $0$ in $\mathfrak{g}$ into $\mathfrak{g}$.

**Proof.** As $C$ is nonsingular near $0$, the condition is
equivalent to $C^{-1}(e^Y) \in \mathfrak{g}$ for all $Y$ in a neighborhood
of $0$, and the equivalence follows from the identity

$$\frac{e^Y - I}{e^Y + I} = \frac{e^{Y/2} - e^{-Y/2}}{e^{Y/2} + e^{-Y/2}} = \tanh \frac{Y}{2}.$$  

As the series for $\tanh$ has only odd powers, it is
clear that $C$ locally maps $\mathfrak{g}$ into $G$ if $G$ is a
subgroup of $\text{GL}(V)$ defined by conditions of commuting with
a family of operators and/or preserving some bilinear forms.

Let us now consider only those $G$ satisfying 29.1,
a fairly wide class. Nevertheless, $\{C(tX): t \in \mathbb{R}\}$ is
rarely a one-parameter subgroup of $G$ (even with a
different parameter than $t$).

**Proposition 29.2.** The mapping $t \to C(tX)$ is (locally) a
a homomorphism iff $X^3 = 0$. If $X \neq 0$, $\{C(tX)\}$ is a
(local) group iff $X^3$ is a multiple of $X$.

**Proof.** If $s,t,u$ are small, one checks that
\[ c(tX)c(sX) = c(uX) \iff (t+s-u)X = stux^3. \] If \( x^3 = 0X \), \( u \) is uniquely determined by
\[ u = \frac{t+s}{1+cts}. \]

\( c \) is defined on an open subset of \( \sigma_f \), and the following is easily shown.

**Proposition 29.3.** Let \((dc)_X\) be the differential of \( c \) at \( X \in \sigma_f \). If \( c \) is defined at \( X \),
\[ [dL_{C(X)}^{-1} \circ (dc)_X](y) = 2(I+X)^{-1}y(I-X)^{-1} \]
for all \( Y \in \sigma_f \), where \( L_g (g \in G) \) is left translation by \( g \).

**Remark.** Let \( R_X = \frac{1}{2}dL_{C(X)}^{-1} \circ (dc)_X : \sigma_f \to \sigma_f \) be the linear map above. Then
\[ \text{Ad}(C(X)) = (R_X)^{-1} R_{-X}, \]
so \( R(\cdot) \) is something of a "square root" of the adjoint action.

We record some easy identities, in the context of \( c \) as a partially defined map from \( gl(V) \) to \( gl(V) \):

1) \( c(-X) = c(X)^{-1} \)
2) \( c(X^{-1}) = -c(X) \)
3) \( c^2 = -I \)
4) \( c(\tanh(X+Y)) = c(\tanh X)c(\tanh Y) \)

hold whenever all \( c \)-transforms make sense, and in 4),
when $X$ and $Y$ commute.

30. **Cayley Transforms for Causal Groups.**

We consider simultaneously the algebras of bounded operators $\text{sp}(\mathcal{W})$ and $\text{so}^*(\mathcal{W})$ and their corresponding groups $\text{Sp}(\mathcal{W})$ and $\text{SO}^*(\mathcal{W})$ as defined in sections 7 and 16. In each case the group acts as continuous automorphisms of the symplectic space $(\mathcal{W}, \mathcal{A}(\cdot, \cdot))$, and additionally, elements of $\text{SO}^*(\mathcal{W})$ commute with the $i$ and $\sigma$ which anti-commute, have squares $-I$, and preserve $\mathcal{A}$. The propositions in this section are valid for both classes of groups; we let $G$ denote the group, $\mathcal{J}$ the algebra, and $\tau$ the topology on $\mathcal{W}$.

**Definitions.** Let

$$C_1 = \{ X \in \mathcal{J} : \mathcal{A}(X\cdot, \cdot) \text{ is a positive-definite form defining the topology } \tau \} ,$$

and $\Gamma = \{ X \in C_1 : X^2 = -I \}$.

**Proposition 30.1.** $C(X) \in G$ for $X \in \mathcal{J}$ whenever it is defined, and $C$ is causal with respect to the cone $C_1$.

**Proof.** Let $v_i \in \mathcal{W} (i = 1, 2)$ and $X \in \mathcal{J}$ such that $(I-X)^{-1}$ exists. Thus there exists unique $w_i$ such that $w_i - Xw_i = V_i$. Then
\[ \sigma(C(X)v_1, C(X)v_2) = \sigma(w_1 + Xw_2, w_2 + Xw_2) \\
= \sigma(w_1 - Xw_1, w_2 - Xw_2) \\
= \sigma(v_1, v_2) \]

as \( X \) is skew with respect to \( \sigma \), so the first statement is clear. The second follows from the computation of the differential of \( C \) in the last section and the identity

\[ \sigma((I+X)^{-1}Y(I-X)^{-1}v, v) = \sigma(Y(I-X)^{-1}v, (I-X)^{-1}v), \]

\( X, Y \in \mathfrak{g} \), \( v \in \mathfrak{u} \).

**Remark.** (Refer to the remark in the last section.) The proof of 30.1 shows that the \( R_X \), although not automorphisms of \( \mathfrak{g} \), nevertheless preserve the causal cone. In the case of \( \text{sl}(2, \mathbb{R}) \), the \( R_X \) turn out to be "conformal automorphisms", that is, positive multiples of automorphisms; this is a low-dimensional "accident". In fact, if \( X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) and \( \langle \cdot, \cdot \rangle \) is Killing form, a lengthy computation shows that

\[ B(R_X(Y), R_X(Z)) = (1/(1-a^2-bc)^2)B(Y,Z) \]

for all \( Y, Z \in \text{sl}(2, \mathbb{R}) \); of course \( a^2+bc \) is a multiple of \( B(X,X) \).

It is clear that \( C(J) = J \) and \( \exp(\pi/2)J = J \) for all \( J \in \Gamma \). A generalization of this relationship between \( C \) and \( \exp \) holds for all of \( C_1 \). (By Theorems 7.1 and 16.2 \( C \) is defined on all of \( C_1 \).)
Proposition 30.2. Let \( X \in C_1 \) so that \( X = JH = HJ \) for a unique \( J \in \Gamma \) and \( H \) positive-definite with respect to \( J_J(\cdot, \cdot) = J(J\cdot, \cdot) \). Then

\[
C(JH) = \exp(JT)
\]

where \( JT = TJ \), \( H = \tan \frac{1}{2}T \), and \( \pi > T > 0 \) in \((\mathcal{N}, S_J)\).

Proof. \( I \pm X \) are invertible, and

\[
C(X) = \frac{(I+X)}{(I-X)} = \frac{(I+X^2)}{(I-X^2)} + 2X/(I-X^2)
\]

\[
= \frac{(I-H^2)}{(I+H^2)} + J(2H/(I+H^2))
\]

Take \( T \) positive and symmetric in \((\mathcal{N}, S_J)\) such that \( \pi > T > 0 \) and \( (I-H^2)/(I+H^2) = \cos T \). Then \( \sin T = 2H/(I+H^2) \) and \( C(X) = \exp JT \). Also,

\[
\tan^2 \frac{1}{2}T = \frac{I-(I-2 \sin^2 \frac{1}{2}T)}{(I+2 \cos^2 \frac{1}{2}T-I)}
\]

\[
= \frac{I-\cos T}{I+\cos T} = H^2
\]

so that \( H = \tan \frac{1}{2}T \).

Proposition 30.3. Let \( J \in \Gamma \) be arbitrary. The image of \( C_1 \) under \( C \) is

\[
\{ g \in G: \varphi(gv, v) \geq k\varphi(Jv, v) \forall v \in \mathcal{N}, \text{ for some } k > 0 \}
\]

Proof. Use the identity

\[
\varphi((g-I)(g+I)^{-1}v, v) = 2\varphi(gv, w)
\]

where \( w = (g+I)^{-1}v \). The inequality above for a \( g \in G \)
implies first that \( g+I \) has dense range, and that \((g+I)^{-1}\) exists quickly follows.

Naturally many mathematical questions arise concerning the applicability and causality of \( C \) for the other causal groups and structures. For the \( su(p,q) \) series \( C \) just does not map into \( SU(p,q) \), only \( U(p,q) \); \( \text{tr}(\cdot) = 0 \) implies \( \det C(\cdot) = 1 \) holds in general only for \( 2 \times 2 \) matrices. \( C \) does map \( so(2,n) \) into \( S_{0,0}(2,n) \) but is not causal; it suffices to examine \( (dC)_{x}(Y) \) for \( x, Y \) in the two-dimensional maximal torus \( h \) of \( o(2,2) \). Finally, an example in \( so^*(4) \) shows that \( C \) is not causal with respect to the minimal invariant cone, and causality seems doubtful for the maximal cone, too. However, it does not yet seem unlikely that causal mappings from algebra to group exist. (The exponential mapping is not causal, but note [19, p. 31].) Hopefully the isomorphism \( sp(2,\mathbb{R}) \approx so(2,3) \) can suggest a definition for the \( so(2,n) \), and the failure for \( su(2,1) \) should be examined more closely.

Finally, our \( C \) seems to have but loose ties with the Cayley transform for Hermitian symmetric spaces [15]. However, the conformal mapping \( z \rightarrow \frac{1+z}{1-z} \) in \( \mathbb{C} \), which takes the unit disc to the right half-plane and fixes \( \pm i \), is a Cayley transform for \( Sl(2,\mathbb{R})/SO(2) \).
CHAPTER IX. Applications to Differential Equations.

31. Preliminaries.

Let \((H_0, \langle \cdot, \cdot \rangle)\) be a real Hilbert space possessing an orthogonal \(J\) with square \(-I\). Define the symplectic form \(\mathcal{A}(\cdot, \cdot) = \langle J^{-1} \cdot, \cdot \rangle\) as usual. The norm \(\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}\) gives the only topology on \(H_0\) that we use. In this final chapter we consider linear differential equations of the form

\[
\frac{d}{dt} X(t) = A(t)X(t)
\]

with periodic coefficients, i.e. for some \(T > 0\)

\[
A(t+T) = A(t), \quad t \in [0, \infty)
\]

where \(X(t) \in H_0\), and \(A(t)\) is a strongly continuous curve in \(\text{sp}(H_0)\). Recall that the latter means that for all \(v \in H_0\), \(t \to A(t)v\) is a continuous curve in \(H_0\).

The usual existence and uniqueness statements hold for such equations. However in the infinite-dimensional case a solution to (27) is not necessarily (Fréchet) differentiable; we define the solutions of (27) to be those \(X(t)\) satisfying the integrated form of the equation

\[
X(t) = X(0) + \int_0^t A(s)X(s)ds
\]

Recall (section 25) that \(u(H_0, \mathcal{A})\) and \(\text{sc}^*(H_0)\) can be regarded as subalgebras of \(\text{sp}(H_0)\).
(27) is equivalent in the usual way to

\[ \frac{d}{dt} U(t) = A(t)U(t) , \quad U(0) = I , \]

where \( t \rightarrow U(t) \) is a curve in \( \text{Sp}(H_0) \), as the coefficients are in \( \text{sp}(H_0) \). As in Floquet theory, \( U(\cdot) \) satisfies \( U(t+T) = U(t)U(T) \), and as in \([4]\), our basic reference for this chapter, we call \( U(T) \) the monodromy operator.

One advantage of the form (29) in the finite-dimensional case is that (29) lifts to

\[ \frac{d}{dt} \widetilde{U}(t) = A(t)\widetilde{U}(t) \]

where \( t \rightarrow \widetilde{U}(t) \) is a curve in the universal covering group \( \widetilde{\text{Sp}}(H_0) \) of \( \text{Sp}(H_0) \). (We identify the two Lie algebras.) Of course the solution of (30) covers that of (29). We shall see in the next section that classical stability "bands" and regions correspond to certain regions of ellipticity in \( \widetilde{\text{Sp}}(H_0) \).

In accordance with general convention, we say (27) is stable if each of its solutions (on \([0, \infty)\)) is bounded, and strongly stable \([4, p. 200]\) if for some \( \epsilon > 0 \), all equations \( X' = A_1(t)X \), where \( \int_0^T ||A(t) - A_1(t)||dt < \epsilon \), are stable. It remains true in the infinite-dimensional case that stability and strong stability are equivalent to corresponding properties of the monodromy operator \( U(T) \), described presently. (The uniform-boundedness principle is crucial in this reduction.) We say \( U \in \mathcal{L}(H_0) \)
{A: H₀ → H₀ | A real-linear, bounded} is stable if U⁻¹ exists and \( \exists M > 0 \) such that \( ||U^n|| \leq M \) for all integers n. \( U \in Sp(H₀) \) is called strongly stable if all \( U \in Sp(H₀) \) in some norm-neighborhood of \( U \) are stable.

Lemma 31.1. ([4, pp. 219-20]) (27) is stable (strongly stable) if and only if the corresponding \( U(T) \) is stable (strongly stable, respectively).

32. Regions of Stability.

We continue with the notation of the previous section, and in this section restrict \( H₀ \) to be finite-dimensional. For concreteness take \( H₀ = \mathbb{R}^{2n} \), and define the symplectic form as in section six.

Now the spectrum of any \( U \in Sp(H₀) \) is symmetric with respect to the unit circle (the Poincaré-Lyapunov theorem), so \( U \) stable implies \( \sigma(U) \) is a subset of \( S^1 = \{ |z| = 1 \} \), invariant under complex conjugation.

Lemma 32.1. \( U \in Sp(H₀) \) is stable iff \( U \) is in some maximal compact subgroup, i.e., is elliptic.

Proof. The sufficiency is obvious. If \( U \) is stable, \( U \) clearly must be diagonalizable over \( \mathbb{C} \). Extend \( \sigma \) to \( \mathbb{C}^{2n} \) by complex bilinearity. By the invariance of \( \sigma \) under \( U \), one sees that a \( U \)-invariant subspace on which \( U = e^{i\theta} \) is \( \sigma \)-orthogonal to one where \( U = e^{i\phi} \), unless \( e^{i\theta} = e^{-i\phi} \).
As the eigenvalues come in conjugate pairs, it is easy to find the proper basis of $\mathbb{R}^{2n}$ so that $U$ is conjugate under $\text{Sp}(n,\mathbb{R})$ to some

$$ (31) \quad \exp \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}, \ D \ \text{diagonal}. $$

Given a stable $U$, the diagonal elements of $D$ in (31), say $d_j$ where $-\pi < d_j \leq \pi$, $j = 1, \ldots, n$, are unique up to a permutation. The eigenvalues of $U$ (with multiplicity) are $e^{i d_1}, \ldots, e^{i d_n}$. The classical terminology is that $e^{i d_1}, \ldots, e^{i d_n}$ are the multipliers of the first kind, $e^{-i d_1}, \ldots, e^{-i d_n}$ the multipliers of the second kind.

**Definition.** Let the systems of coefficients $A(t), A_1(t)$ give strongly stable equations. We say $A(t)$ and $A_1(t)$ belong to the same stability region if and only if there is a continuous map

$$ A_2: [0,T] \times [0,1] \to \text{sp}(H_0): (t,s) \to A_2(t,s) $$

such that $A_2(t,0) = A(t), A_2(t,1) = A_1(t), A_2(0,s) = A_2(T,s)$, and that the equation

$$ \frac{d}{dt} X(t) = A_2(t,s)X(t) $$

is strongly stable $\forall s \in [0,1]$. (One could require all these functions $A, A_1, A_2$ to be merely piece-wise continuous in $t$, but the present definition suffices here.)
The stability regions were determined by I.M. Gelfand and V.B. Lidskii in [7]. They are determined by an integer \( m \), called the index, corresponding to the number of times (in one direction or the other) \( U(t) (0 \leq t \leq T) \) "winds around the group", and a pattern of the eigenvalues of \( U(T) \), out of a total number of \( 2^n \) such patterns, described as follows. \( U(T) \) can be the monodromy operator of a strongly stable equation (equivalently, \( U(T) \) itself is strongly stable (Lemma 31.1), i.e., has a neighborhood in \( \text{Sp}(H_0) \) consisting of elliptic elements) iff the \( e^{\pm id_j} \) obtained above have the property that no \( \gamma \in S^1 \) is a multiplier both of the first and second kinds, that is, there are no "repeated multipliers of unlike type". Equivalently, no \( e^{\pm id_j} \) can be real, and, with the above restriction on the range of the \( d_j, \forall i, j = 1, \ldots, n \), \( d_i = -d_j \neq 0 \) does not occur.

**Corollary 32.2.** An elliptic \( U \in \text{Sp}(n, \mathbb{R}) \) is strongly stable iff \( U \) is contained in a unique maximal compact subgroup.

**Proof.** It follows from Proposition 25.3, the use of the Cayley transform \( C \), and the obvious fact that \( [J, X] = 0 \) iff \( JC(X) = C(X)J \) for \( J \) a complex structure and \( X \) elliptic. Note that each strongly stable \( U \) is in the range of \( C \), as \(-1\) is not an eigenvalue.

Arranging the \( d_j \) is order of ascending absolute
value, the "pattern of eigenvalues" above is just the sequence $d_1/|d_1|, \ldots, d_n/|d_n|$. Clearly there are $2^n$ such patterns. We say that there are $2^n$ **elliptic regions** in $\text{Sp}(n, \mathbb{R})$. As the mapping

$$i\mathbb{R} \rightarrow S^1(-1) : ir \rightarrow \frac{1+ir}{1-ir}$$

preserves the linear orders of $i\mathbb{R}$ and $S^1(-1)$, it is clear that these regions are diffeomorphic under $C$ to the various regions in $\text{sp}(n, \mathbb{R})$ each containing elements uniquely Kählerizable.

In [7] it is proven that each elliptic region is connected and simply connected. The region of conjugates of (31) where all $d_j > 0$, corresponds under $C$ to the open positive cone in $\text{sp}(n, \mathbb{R})$, hence being convex, is contractible. One may ask, are the other elliptic regions contractible?

Note that the $X \in \text{sp}(n, \mathbb{R})$ such that $X^2 = -I$ (see the remarks in section 25; there are $n+1$ such orbits) are stable, but also strongly stable only in the two cases where $X$ or $-X$ is a complex structure, i.e. $\mathcal{S}(X^*, \cdot)$ is positive- or negative-definite.

As the elliptic regions in $\text{Sp}(n, \mathbb{R})$ are simply connected, the components of their inverse images in $\tilde{\text{Sp}}(n, \mathbb{R})$ project back to $\text{Sp}(n, \mathbb{R})$ diffeomorphically. The components of the inverse image of an elliptic region are parametrized by the index $m \in \mathbb{Z}$ above. The following is
clear from the theorems of [7] and Corollary 32.2.

Proposition 32.3. Stability regions for equations of the form (29) are in natural 1-1 correspondence with the open regions in \( \text{Sp}(n,\mathbb{R}) \) consisting of those group elements uniquely \( K \)-\( \text{ählerizable} \), i.e., those which are each contained in a unique (essentially) maximal compact subgroup, as follows. Lift (29) to (30), and associate a strongly stable \( A(t) \) with the region in \( \text{Sp}(n,\mathbb{R}) \) containing \( \overline{U(T)} \).

It becomes clear why there should be stability "bands" for one-dimensional equations like the Mathieu equation

\[
Y''(t) + (\varepsilon + 5 \cos t)y(t) = 0, \quad T = 2\pi,
\]
they just reflect the geometry of \( \text{SL}(2,\mathbb{R}) \). One has a mapping of \((\varepsilon,s)\)-pairs to \( U(T) \in \text{SL}(2,\mathbb{R}) \), or rather \( \overline{U(T)} \in \text{SL}(2,\mathbb{R}) \), by solving the equation. The inverse image of the picture in section 28, under this mapping, is just the diagram appearing in textbooks containing sections on the Mathieu equation, for example, [1, p. 562].

33. Two Stability Criteria.

We return to the notation of section 31, and consider the equation (27) with \( t \to A(t) \) strongly continuous and norm-bounded: \( \|A(t)\| \leq M \forall t \). Further, let \( \mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \) be the complexification of \( \mathbb{R} \) with complex structure \( i \), and extend \( \langle \cdot,\cdot \rangle \) on \( \mathbb{R} \) to a complex Hilbert structure
<\cdot,\cdot> on \mathcal{H}. We extend all operators on \mathcal{H}_0 to i-linear operators on \mathcal{H}. Our goal in this last section is to prove Theorems 33.4 and 33.6. The latter is a generalization to infinite dimensions of a theorem of M. G. Krein [17].

In the applications to quantum fields that we have in mind U(T) becomes the "scattering operator", and we are interested in finding easily expressed conditions on \lambda(t) sufficient to guarantee the stability, i.e., unitarizability, of U(T). Also, in these applications \lambda \to A(t) will not be continuous in the norm topology.

We restrict now to "positive-energy" equations, that is, coefficients \lambda(t) where

\begin{equation}
\lambda(A(t)v,v) \geq 0 \quad \forall v \in \mathcal{H}_0, \forall t \in [0,T]
\end{equation}

and for some \( k > 0 \)

\begin{equation}
\lambda(A_{av}v,v) \geq k\langle v,v \rangle \quad \forall v \in \mathcal{H}_0, \quad \text{where} \quad A_{av}v = \int_0^T A(t)v dt,
\end{equation}

the integral of a continuous function in \mathcal{H}_0. (The Bochner integrals we use are all integrals over an interval of \mathbb{H}-valued functions.)

Under somewhat more general hypotheses than these, one can prove

**Lemma 33.1.** ([4, p. 222]) Let \([a,b]\) be a closed interval containing 0, and for \( \lambda \in [a,b] \) let \( U(T;\lambda) \) be the monodromy operator for the equation
If

$$ (U(T;\lambda) + I)^{-1} $$

exists as a bounded operator \( \forall \lambda \in [a,b] \), (35) is strongly stable \( \forall \lambda \in [a,b] \).

We recognize that (36) is equivalent to \( U(T;\lambda) \) being a Cayley transform of some \( Y_\lambda \in \text{sp}(H_0) \), here necessarily in the cone \( C_\circ \) of section 27, as the proof shows.

Now let \( \mathcal{B} \) be the Banach space of continuous functions from \([0,T]\) to \( \mathcal{A} \), with the norm

$$ ||f|| = \sup_{t\in[0,T]} ||f(t)|| \quad (f \in \mathcal{B}), $$

and define

$$ <f,g>_c = \int_0^T <f(t),g(t)>dt \quad \text{for} \quad f,g \in \mathcal{B}. $$

Let \( (\mathcal{A}_c,\langle \cdot,\cdot \rangle_c) \) be the completion of \( \mathcal{B} \) in \( \langle \cdot,\cdot \rangle_c \).

('c' stands for curve.)

**Lemma 33.2.** ([4, p. 224]) As in Lemma 33.1, \( (U(T;\lambda) + I)^{-1} \) exists if and only if for any \( f \in \mathcal{A} \) there exists a unique solution \( X(t) \) in \( \mathcal{B} \) depending continuously on \( f \) (in the norm \( ||\cdot|| \)) of the boundary value problem

$$ (d/dt)X(t) = \lambda A(t)X(t), \quad X(0) + X(T) = f. $$
We will solve the integrated form of this equation ((39) below), but first need some more notation.

Let \( A(t) = JH(t) \); then \( H(t) \) is a non-negative Hermitian operator in \( (\mathcal{H}, \langle \cdot, \cdot \rangle_c) \). Define \( \langle \cdot, \cdot \rangle_{\mathcal{C}} \) on \( \mathcal{B} \times \mathcal{B} \) by

\[
\langle f, g \rangle_{\mathcal{C}} = \int_0^T \langle H(t)f(t), g(t) \rangle dt,
\]

and let \( \mathcal{R} = \{ f \in \mathcal{B}: H(t)f(t) = 0 \ \forall t \} \). Then clearly \( \mathcal{R} = \{ f \in \mathcal{B}: \langle f, f \rangle_{\mathcal{C}} = 0 \} \), and \( \mathcal{R} \) is \( \| \cdot \| \) -closed in \( \mathcal{B} \). Also if \( f \in \mathcal{B} \).

(38) \[
\langle f, f \rangle_{\mathcal{C}} \leq \left( \int_0^T \| H(t) \| \| dt \right) \| f \| ^2.
\]

Define the function \( h \) on \([-T, T]\) by \( h(t) = \begin{cases} \frac{1}{2} & (0 \leq t \leq T) \\ -\frac{1}{2} & (-T \leq t < 0) \end{cases} \). Let \( \omega = \pi / T \). Then \( \{ e^{i\omega(2k+1)t} \}_{k \in \mathbb{Z}} \) is an orthogonal basis of \( L_2([0, T]) \), and on \([-T, T]\) it is a basis for that subspace of functions satisfying \( f(t + T) = -f(t) \) a.e. We have the \( L_2 \)-expansion

\[
h(t) = \frac{1}{\pi i} \sum_{k} \frac{e^{i\omega(2k+1)t}}{1/2k+1}.
\]

(\( \sum \), etc. always denote a sum over all integers.)

For \( f \in \mathcal{B} \) define

\[
(Kf)(t) = \int_0^T h(t-s)A(s)f(s)ds.
\]

Clearly \( K: \mathcal{B} \to \mathcal{B} \) is continuous and vanishes on \( \mathcal{R} \). In fact,
Lemma 33.3. \( \text{ker } K = \mathcal{R} \).

Proof. \( Kf = 0 \) implies \( \int_0^T e^{-i\omega(2k+1)s} H(s)f(s)ds = 0 \) for all \( k \in \mathbb{Z} \). Now \( s \to H(s)f(s) \) is a continuous function under our assumptions \( (t \to H(t) \) strongly continuous and norm-bounded), so \( H(s)f(s) = 0 \) and \( f \in \mathcal{R} \).

We have introduced \( K \) because if \( \lambda = 1 \), \( (37) \) is equivalent to

\[
(39) \quad ((I-K)X)(t) = \frac{1}{2}f.
\]

Thus it suffices to find conditions under which \( |||K||| < 1 \), or just invert \( I-K \) in the algebra of bounded operators in \( S \), in order to deduce strong stability of \( (27) \).

In fact \( K \) is continuous with respect to all the other norms, too. Let \( f \in S \) and \( g \in \mathcal{V} \). Then for all \( t \)

\[
|\langle (Kf)(t), g \rangle|^2 = \left| \int_0^T h(t-s) \langle JH(s)f(s), g \rangle ds \right|^2
\]

\[
\leq \frac{1}{2} \left( \int_0^T ||H^{1/2}(s)f(s)|| \cdot ||H^{1/2}(s)Jg|| ds \right)^2
\]

\[
\leq \frac{1}{2} \left( \int_0^T ||H^{1/2}(s)f(s)||^2 ds \right) \left( \int_0^T ||H^{1/2}(s)Jg||^2 ds \right)
\]

\[
\leq \frac{1}{2} \langle f, f \rangle_{\mathcal{V}} ||g||^2 \left( \int_0^T ||H(s)|| ds \right), \text{ so}
\]

\[
|||Kf||| \leq \frac{1}{2} ||f||_{\mathcal{V}} \left( \int_0^T ||H(s)|| ds \right)^{1/2}.
\]

Now \( S/\mathcal{R} \) is a Banach space with the quotient norm \( ||| \cdot |||_q \). By an earlier remark, \( \langle \cdot, \cdot \rangle_{\mathcal{V}} \) factors to a positive-definite form \( \langle \cdot, \cdot \rangle_{\mathcal{E}} \) on \( S/\mathcal{R} \), a kind of
"average-energy" norm. Let $(\tilde{V},<\cdot,\cdot>_{\varepsilon})$ be the completion of $\mathcal{B}/\mathcal{R}$ in this norm. Then the inclusion $j: \mathcal{B}/\mathcal{R} \hookrightarrow \tilde{V}$ is continuous with norm

$$\|j\| \leq \left( \int_0^T \|H(t)\|dt \right)^{1/2}$$

by (38).

By (40) and (41), it follows that $\|\|K\|\| \leq \frac{1}{T} \int_0^T \|H(s)\|ds$. By this estimate and the remark following 33.3, the following is evident.

**Theorem 33.4.** Suppose $t \rightarrow A(t) \in \text{sp}(H_0)$ is strongly continuous, norm-bounded, periodic with period $T$, and satisfies (33) and (34). Then the equation

$$(d/dt)X(t) = A(t)X(t)$$

is strongly stable provided

$$\int_0^T \|A(t)\|dt < 2.$$

We need additional notation in order to state our second stability criterion. There are several other continuous operators closely related to $K$ that one can define. By 33.3, $K$ factors to

$$\tilde{K}: (\mathcal{B}/\mathcal{R},\|\cdot\|_q) \rightarrow (\mathcal{B},\|\cdot\|)$$

and this extends to

$$\tilde{K}: (\tilde{V},<\cdot,\cdot>_{\varepsilon}) \rightarrow (\mathcal{B},\|\cdot\|).$$
Denote by $K_1$ the composition

$$K_1: \tilde{\mathcal{H}} \xrightarrow{\sim} \mathcal{V} \longrightarrow \mathcal{E}/\mathcal{R} \xrightarrow{\iota} \tilde{\mathcal{H}}.$$ 

Then $\|\tilde{K}\| \leq \frac{1}{2}(\int_0^T \|H(s)\|ds)^{1/2}$, $\|K_1\| \leq \frac{1}{2}(\int_0^T \|H(s)\|ds)$, and of course range $K_1 \subseteq \mathcal{E}/\mathcal{R}$.

**Lemma 33.5.** $K_1$ is self-adjoint.

**Proof.** To show $\langle f, K_1 g \rangle_\epsilon = \langle K_1^* f, g \rangle_\epsilon$ $(f, g \in \mathcal{E}/\mathcal{R})$, one uses $h(-t) = -h(t)$, $A(t) \in \text{sp}(H_0)$, and does an interchange of order of integration.

Now $\|K_1\| < 1$ is sufficient to force $I - K$ to be invertible as a bounded operator in $(\mathcal{E}, \|\cdot\|)$, as then

$$(I - K)^{-1} = \sum_{l=0}^\infty K^l = I + K + K^2 + \cdots,$$

where the sum converges in the $\mathcal{E}$-norm as the terms $K \cdot K^l$ can be factored

$$\mathcal{E} \longrightarrow \mathcal{E}/\mathcal{R} \xrightarrow{\iota} \tilde{\mathcal{H}} \xrightarrow{\iota} \tilde{\mathcal{V}}$$

Recall the definition (34) of $A_{av}$. We are assuming $H_{av} > 0$ where $A_{av} = JH_{av}$, and

$$A_{av}^{-2} = (JH_{av})(JH_{av})$$

$$= -H_{av}^{-1/2}(H_{av}^{-1/2}JH_{av}^{-1/2})(H_{av}^{-1/2}JH_{av}^{-1/2})H_{av}^{-1/2},$$

so $(A_{av})^2$ has nonpositive trace whenever it is trace class. (Compare with the Killing form in $\text{sp}(n,\mathbb{R})$.)

Unlike the criterion of Theorem 33.4, the following test is invariant under $\text{Sp}(H_0)$. 

Theorem 33.6. Suppose that $A(t)$ has an absolutely convergent Fourier expansion on $[0,T]$, and $(A_{av})^2$ is trace-class, in addition to the assumptions of Theorem 33.4. Then (note 33.5)

$$(42) \quad \text{tr}(K_2^2) = -(T^2/4) \text{tr} (A_{av})^2,$$

and $(d/dt)X = A(t)X$ is strongly stable if

$$-T^2 \text{tr} (A_{av})^2 < 4.$$ 

**Remark.** It is hoped that the assumption of an absolutely convergent expansion for $A(t)$ can be weakened substantially or eliminated, as it is only used to justify a certain rearrangement of a series, as shall be seen.

**Proof.** We first show that $\tilde{Y}$ is unitarily equivalent to a closed subspace of $\mathcal{H}_c$ (proper if $R \neq 0$), and deal with the transformed $K_1$. Define

$$T_0 : \mathcal{B}/\mathcal{R} \to \mathcal{B} : f(t) + \mathcal{R} \to H^{1/2}(t)f(t).$$

Then $T_0$ extends to a unitary equivalence

$$T : (\tilde{V}, \langle \cdot, \cdot \rangle_0) \to (V, \langle \cdot, \cdot \rangle_c)$$

where $V \subset \mathcal{V}$ is closed. One sees that $f \in V^\perp$ iff $H^{1/2}(t)f(t) = 0$ a.e., and

$$(K_2 f)(t) = ((TK_1 T^{-1}) f)(t) = \int_0^T H^{1/2}(t) h(t-s) J H^{1/2}(s) f(s) ds$$
for \( f \in V \). \( K_2 \) extends to \( V_c \) by this expression and vanishes on \( V_1 \); denote this extension also by \( K_2 \). Then \( \text{tr}(K_2^2) = \text{tr}(K_2^2) \).

Now \( H^{1/2}(t) \) can also be expanded absolutely, so let \( H^{1/2}(t) = \sum_k e^{i2\pi kt} L_k \), \( L_k: V \to V \), satisfy

\[
\sum_k \|L_k\| < \infty .
\]

Let \( z \in V \) have norm 1; then \( w(t) = (1/\sqrt{T}) e^{i\omega (2k+l)t} t_z \) has norm 1 in \( V_c \). We have

\[
(K_2 w_k)(t) = (\sqrt{T}/\tau i) \sum_{k,m} (1/2^m) (L_{k-m}^n J_{m-\ell}^n z) e^{i\omega (2k+l)t},
\]

an absolutely convergent sum. Then \( \sum_k \langle K_2 w_k, K_2 w_k \rangle \) is equal to

\[
(\tau^2/\tau^2) \sum_k \langle \sum_{k,m} (1/2^m) L_{k-m}^n J_{m-\ell}^n z, \sum_{n} (1/2^m) L_{k-n}^n J_{n-\ell}^n z \rangle .
\]

We wish to rearrange the summation in \( n, m, k, \ell \) in order to facilitate massive cancellation. To show absolute convergence, note

\[
\sum_{k, \ell, m, n} (1/|2m+1| |2n+1|) \|L_{k-m}\| \|L_{m-\ell}\| \|L_{k-n}\| \|L_{n-\ell}\|
\]

\[
\sum_{k, \ell, n, r} (1/|2n+1| |2(n+r)+1|) \|L_{k-r}\| \|L_{n+r}\| \|L_{k}\| \|L_{\ell}\|
\]

is less than a constant times

\[
\sum \left( \sum_k \|L_k\| \|L_{k-r}\| \right)^2 .
\]

Now \( \sum \left( \sum_k \|L_k\| \|L_{k-r}\| \right) < \infty \) by (43), so clearly the sum of
squares (45) also converges.

Thus we can rearrange (44) arbitrarily. It becomes

\[(T^2/\pi^2) \sum_{r \ n} \left( \sum_{l} (1/(2n+1)(2(n+r)+1)) \right) \sum_{k, l} <L_{k-r}JL_{-l}^*, z, L_{k}JL_{-l} z> .\]

However,

\[\sum_{n} (1/(2n+1)(2(n+r)+1)) = 0\]

if \(r \neq 0\) and \(\sum_{n} (1/(2n+1)^2 = \pi^2/4\) . To see this note \(h(t)h(t) = 1/4\), and multiply out the earlier expansion for \(h(t)\).

The above then reduces to

\[(T^2/4) \sum_{k, l} <L_{k}JL_{-l}^*, z, L_{k}JL_{-l} z> .\]

Now \(L_{k}^* = L_{-k}\) and \(\sum_{k} (L_{k}L_{-k}) = H_{av}\), so

\[\sum_{k} <K_{2w_{k}}, K_{2w_{l}} > = -(T^2/4) \sum_{k} <L_{k}JH_{av}JL_{-l}^*, z, z> \]

\[= (T^2/4) \sum_{l} <(H_{av}^{1/2}JL_{-l}^*)^*(H_{av}^{1/2}JL_{-l}) z, z> .\]

Letting \(z\) run over an orthonormal basis \(\{z_{\alpha}\}\) of \(\mathcal{N}\), we have

\[\text{tr}(K_{1}^2) = (T^2/4) \sum_{\alpha} \sum_{l} <(H_{av}^{1/2}JL_{-l}^*)^*(H_{av}^{1/2}JL_{-l}), z_{\alpha}, z_{\alpha}> \]

\[= -(T^2/4) \sum_{\alpha} <H_{av}^{1/2}JH_{av}JH_{av}^{1/2}z_{\alpha}, z_{\alpha}> \]

\[= -(T^2/4) \text{tr} ((JH_{av})^2) = -(T^2/4) \text{tr} (A_{av})^2 \]

as claimed.
By Lemma 33.5, \[ \| K_1 \|^2 \leq \text{tr}(K_1^2) \], and so the last statement follows from the remarks after that lemma. This completes the proof.
REFERENCES


18. I. E. Segal, "Conjugacy to Unitary Groups within the Infinite-dimensional Symplectic Group", Argonne Nat.


BIOGRAPHICAL NOTE - AFTERWORD

Steve Paneitz was born in Lawton, Oklahoma on March 24, 1955, and lived in Red Oak, Iowa for about ten years until his family moved to McPherson, Kansas, where he attended the public high school. He attended the University of Kansas from 1973 to 1976, and subsequently moved to Boston to study at MIT. For the next two years he has a postdoctoral fellowship from the Miller Institute for Basic Research in Science at the University of California at Berkeley.

"The startling nearness of so many galaxies in the astronomy of the Other Men could be explained on the theory of the 'expanding universe'. Well I knew that this dramatic theory was but tentative and very far from satisfactory...

"I, too, now sought to capture the infinite spirit, the Star Maker, in an image spun by my own finite though cosmical nature. For now it seemed to me, it seemed, that I suddenly outgrew the three-dimensional vision proper to all creatures, and that I saw with physical sight the Star Maker. I saw, though nowhere is cosmical space, the blazing source of the hypercosmical light, as though it were an overwhelmingly brilliant point, a star, a sun more powerful than all suns together. It seemed to me that this effulgent star was the centre of a four-dimensional sphere whose curved surface was the three-dimensional cosmos..."

Olaf Stapledon, Star Maker (1937)

Would it have been worth while,
To have bitten off the matter with a smile,
To have squeezed the universe into a ball,
To roll it toward some overwhelming question...

T.S. Eliot, "The Love Song of J. Alfred Prufrock"