CAUSAL STRUCTURES IN LIE GROUPS AND APPLICATIONS TO STABILITY OF DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

According to results of Kostant, a real simple Lie algebra $\mathfrak{g}$ has an invariant convex cone if and only if $\mathfrak{g}$ is Hermitian symmetric. We classify and explicitly describe such cones in the classical algebras, specifically, all such cones in $sp(n,\mathbb{R})$ and all open or closed invariant convex cones in $su(p,q)$, $so^*(2n)$, and $so(2,n)$.

The universal covering group $\tilde{G}$ receives a left- and right-invariant causal structure from any such cone. In the tube-type cases $\tilde{G}$ is shown to be globally causal under all such structures, and likewise for certain of the structures in the other cases.

The open cones are found to consist of elliptic elements, each within a unique maximal compact subalgebra. This mapping of positive elliptics to maximal compacts (Kählerization) is shown to be real-analytic. The same results hold for infinite-dimensional analogues of $sp(n,\mathbb{R})$ and $so^*(2n)$. In these two cases the Cayley transform is a causal mapping of an open subset of $\mathfrak{g}$ into the group. Some of the mapping's other properties are collected.

Let $\mathcal{S} = \{g \in Sp(n,\mathbb{R})$: g within a unique maximal compact subgroup$\}$, an open set having 2^n connected components $\mathcal{S}_j$. The inverse images of the $\mathcal{S}_j$ in the universal covering $\widetilde{Sp(n,\mathbb{R})}$ correspond naturally to the well-known regions of strong stability of linear periodic canonical differential equations. Two stability criteria are found for certain such equations of positive type, applicable in finite or infinite dimensions. One involves the operator bounds of the coefficients, the other an invariant Hilbert-Schmidt bound.

Thesis Supervisor: Irving E. Segal
Title: Professor of Mathematics

To the memory of

my grandfather,

Ernest W. Engelkemier

(1895 - 1978)

Let us then provisionally adopt the unispace cosmos, and seek to analyze free propagation over very long times, and its effect on the measurement of frequency of light.

I. E. Segal

Let us go then, you and I,
When the evening is spread out against the sky...

T. S. Eliot

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Marge Zabierek deserves praise for a beautiful typing job and such a cheerful disposition. I'm glad she enjoyed doing this thesis so much, especially the tables and graphs.

Most of all I thank my parents, Joanne and Marvin Paneitz, for their love and constancy and everything else. I thank David Goering and Holly and Warren Schoming for the friendship and "spiritualizing" of nearly ten true years, and not only summers. My Watertown housemates, Daniel "Jackson" Friedman and John "Johnson" Lutostanski, deserve public recognition for skillfully instructing an erstwhile raving liberal. May they always be free to choose.

The MIT math dept. has been a friendly and intriguing place to work and converse, and it will not be easy to leave. And I have appreciated the Pleasant Folks of Cambridge.

Do I dare
Disturb the universe?...
I have measured out my life
with coffee spoons.
-Prufrock again

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INTRODUCTORY REMARKS

The main ideas of chapters one to five on the classification of invariant convex cones in the classical Hermitian symmetric Lie algebras, seem to be the following.

1) We use the orbit classification of Burgoyne and Cushman [2] from time to time, particularly to deal with the boundaries of the cones and $so(2,n)$. These are the only really technical results we quote, and the above authors claim they "only need quote one result, Sylvester's theorem on the signature."! Also, the form of the orbits in the open maximal cones on $su(p,q)$ could have been determined by the idea of Theorem 14.2, just as was done for the corresponding open maximal cones in $so^*(2n)$ (Theorem 17.3), independently of classification.

2) The quadratic mapping

$$v \in V \rightarrow \mathfrak{a}(\cdot v, v) \in \mathfrak{g}^*$$

where V is a fundamental representation space and $\mathfrak{a}$ an invariant symplectic form (and a rather different expression, $\tau(\cdot e, f)$, for $so(2,n)$; see section 21) is shown to single out orbits with particular positivity properties (Theorems 6.1, 10.1, and 15.1), connecting the geometries of the Lie algebra and the space V , and is essential to establishing the maximality of certain

explicitly described cones (Theorems 8.2, 12.3, 18.2).
3) The argument of sections 7, 11, and 16 seems well-known but appears in other context, such as the Klein-Gordon equation in Minkowski space

$$\Box\varphi + m^2\varphi = 0 \ , \quad m > 0 \ .$$

In section 7 the form $\mathfrak{s}(\cdot,\cdot)$ is the energy norm

$$\left\langle\!\left\langle \binom{\varphi}{\dot\varphi}, \binom{\varphi}{\dot\varphi} \right\rangle\!\right\rangle_E = \int_{\mathbb{R}^3} \left\{ |\dot\varphi|^2 + m^2|\varphi|^2 + |\vec\nabla\varphi|^2 \right\} d_3\vec{x}$$

and $\mathfrak{s}_1(\cdot,\cdot)$ is the Lorentz-invariant norm

$$\left\langle\!\left\langle \binom{\varphi}{\dot\varphi}, \binom{\varphi}{\dot\varphi} \right\rangle\!\right\rangle_L = \int_{\mathbb{R}^3} \left\{ |C\varphi|^2 + |C^{-1}\dot\varphi|^2 \right\} d_3\vec{x}$$

where $C = (m^2-\Delta)^{1/4}$ and say $\varphi, \dot\varphi \in C_0^\infty(\mathbb{R}^3)$.
4) The "noncompact convexity" theorems 14.2, 17.3, 21.3, and possibly 9.2 for $sp(n,\mathbb{R})$ (a "discrete" version), are in the direction of generalizing to noncompact groups Horn's theorem for Hermitian matrices (see section 14) and Kostant's generalizations to maximal compact subgroups of semisimple Lie groups, within [16]. The classification of the open and closed invariant convex cones and <u>all</u> invariant convex cones for $sp(n,\mathbb{R})$ (Theorems 8.1, 14.4, 19.3, and 22.2) is a fairly immediate consequence, just as Horn's theorem easily classifies the invariant convex cones in $u(n)$.

Chapter six is a fairly direct application of the preceeding chapters. The uniqueness and analyticity of

Kählerization (for $sp(\mathcal{H})$, at least), shown in chapter seven, 25.2, 25.4, and 27.2, are valid in infinite dimensions and, like chapter nine, suggested by problems in quantum field theory. Chapter eight is a rather non-systematic collection of properties of the mapping

$$\mathcal{C}: X \rightarrow \frac{I+X}{I-X}$$

of the Lie algebra to the Lie group. One remark that should have been made there, is that all group elements in the range of the Cayley transform restricted to the cones $\pm C_1$, described in Proposition 30.3, commute with a unique complex structure, by Theorem 25.2. The proof of the latter applies equally to $SO^*(\mathcal{H})$, defined in section 16.

One comment on notation: $A \subset B$ means A is contained in B and $A \neq B$.

CHAPTER I: Preliminaries.

1. Preliminaries on Convex Cones.

We need some standard facts about convex cones as, for example, in [6], to which we refer for proofs. In this section all vector spaces are finite-dimensional.

Definition. Let E be a real vector space. $C \subseteq E$ is convex if $\lambda x + (1-\lambda) \in C$ whenever $x, y \in C$ and $\lambda \in [0,1]$. $C \subseteq E$ is a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. A non-trivial cone is a nonempty cone C satisfying $\{0\} \neq C \neq E$.

Elementary properties of convex cones are, for example: the closure or interior of a convex cone is also a convex cone. Less intuitive is the following

Lemma 1.1. A convex cone which is dense in E is equal to E.

Let $\hat{E}$ be the vector space dual to E. There is a "separating hyperplane theorem" for convex cones.

Lemma 1.2. If C is a closed convex cone in E and $x \notin C$, then there exists $f \in \hat{E}$ such that $f(x) > 0$ and $f(y) \leq 0$ for all $y \in C$.

Corollary 1.3. A convex cone, not all of E, is in some closed half-space.

It is convenient to introduce a real positive-definite scalar product $\langle \cdot , \cdot \rangle$ in E to state the duality theorem for cones, but this is not strictly necessary as $\hat{\hat{E}} \cong E$ canonically.

Definition. Given any cone C in E, let $C^* = \{y : \langle y,x \rangle \le 0$ for all $x \in C\}$. C^* is a closed convex cone, called the dual cone of C.

Theorem 1.4. For any cone C in E, $(C^*)^*$ is the closed convex hull of C. In particular, if C is closed and convex, $(C^*)^* = C$.

Proof. It follows from the definitions and Lemma 1.2.

2. Existence of the Minimal Cone.

In this section we recall the results of Kostant on the existence of invariant convex cones in semisimple Lie algebras.

Definition. A causal cone in a real Lie algebra $\mathfrak{g}$ is a non-zero closed convex cone C in $\mathfrak{g}$ invariant under the adjoint group of $\mathfrak{g}$ and satisfying $C \cap -C = \{0\}$.

Proposition 2.1. Any non-trivial invariant convex cone C in a simple real Lie algebra satisfies $C \cap -C \subseteq \{0\}$.

Proof. $\bar{C} \cap -\bar{C}$ is an ideal in $\mathfrak{g}$ by the invariance.

<u>Theorem 2.2</u> (Kostant). Let G be a semisimple Lie group acting on a real finite-dimensional space V. Assume that K is a maximal compact subgroup of G. Then there exists a non-trivial G-invariant convex cone C in V satisfying $C \cap -C \subseteq \{0\}$ iff V has a non-zero K-invariant vector.

<u>Proof</u>. The proof is given in [19], but the ideas are useful later so we repeat them here. Let C be a cone with the given properties. By Lemma 1.2, $\exists\, f \in \hat{V}$ such that $f(x) \geq 0 \quad x \in C$ and $f(z) > 0$ for some $z \in C$. Then

$$w = \int_K k(z)dk$$

is K-invariant and $f(w) > 0$ so $w \neq 0$.

Conversely, let $w \neq 0$ be K-invariant. Let $\underline{k}$ be the Lie algebra of K and $\mathfrak{g} = \underline{k} + \underline{p}$ the Cartan decomposition. As G is a matrix group, its complexification G_c acts on $V_c = V + iV$, and any subgroup G_u corresponding to $\underline{k} + i\underline{p}$ is compact. We can assume $K \subset G_u$, and G_u leaves invariant some complex Hilbert structure $\langle \cdot , \cdot \rangle$. All $X \in \underline{k} + i\underline{p}$ are skew-Hermitian on V_c, so all $X \in \underline{p}$ are Hermitian and $g \in \exp \underline{p}$ positive-definite Hermitian.

Any $g \in G$ can be written uniquely as $(\exp X)k$ for $X \in \underline{p}$, $k \in K$. Thus $\langle gw,w \rangle = \langle (\exp X)w,w \rangle > 0$. Now $\langle gu,v \rangle = \langle u, \Theta(g^{-1})v \rangle \ \forall\, g \in G$, $u,v \in V$, where $\Theta: G \to G: (\exp X)k \to (\exp -X)k$ is the Cartan involution corresponding to $\theta: X + Y \to X - Y$ for $X \in \underline{k}$, $Y \in \underline{p}$. Letting C_o denote the convex cone generated by the gw for $g \in G$, it is clear that C_o is a non-trivial G-invariant convex cone in V, and $\langle u,v\rangle > 0 \ \forall u,v \in C_o$.

<u>Corollary 2.3</u>. Let $\mathfrak{g}$ be a semisimple Lie algebra and G the adjoint group of $\mathfrak{g}$. Let K be any maximal compact subgroup of G, with Lie algebra $\underline{k}$. Then there is a non-trivial G-invariant convex cone C in $\mathfrak{g}$ satisfying $C \cap -C = \{0\}$ if $\underline{k}$ has a non-trivial center.

<u>Proof</u>. Take $V = \mathfrak{g}$ in Theorem 2.2. As K acts irreducibly on the $\underline{p}$'s of the simple components of $\mathfrak{g}$, any non-zero K-fixed vector must be in $\underline{k}$.

It is clear that any invariant convex cone in a semi-simple $\mathfrak{g}$ is contained in the direct sum of invariant cones in the simple factors, and we restrict to the simple case from now on. It is well known that if $\mathfrak{g}$ is a real simple Lie algebra the dimension of the center $Z(\underline{k})$ is either 0 or 1. Thus by Proposition 2.1, a simple $\mathfrak{g}$ admits a non-trivial invariant convex cone iff $\dim Z(\underline{k}) = 1$. The integration argument above shows that in this case there are unique minimal causal cones $\pm\overline{C}_o$. By the simplicity, the positive-definite K-invariant form used in the proof of Theorem 2.2 was on $\mathfrak{g}\, B_\theta(\cdot,\cdot) = -B(\cdot,\theta\cdot)$ up to a scalar, where B is the killing form on $\mathfrak{g}$. We use later the fact that $B_\theta(X,Y) \geq 0 \ \forall X,Y \in \overline{C}_o$.

We see that no compact or complex simple Lie algebra admits a causal cone. By the classification the only algebras that do are $sp(n,\mathbb{R})$ $(n \geq 1)$, $su(p,q)$ $(p \geq q \geq 1)$, $so^*(2n)$ $(n \geq 3)$, $so(2,n)$ $(n \geq 3)$, and two exceptional algebras. We list information about the dimensions of these algebras below, using the notation of [9]. Recall that $\text{char}\,\mathfrak{g} = \dim \underline{p} - \dim \underline{k}$ where $\mathfrak{g} = \underline{k} + \underline{p}$ is any Cartan decomposition, and $\text{rank}\,\mathfrak{g}$ is the dimension of any maximal abelian subalgebra of $\underline{p}$.

$\underline{sp(n,\mathbb{R})}$ $\dim \underline{k} = n^2$, $\dim \underline{p} = n^2 + n$; $\text{rank}\,\mathfrak{g} = n$; $\text{char}\,\mathfrak{g} > 0\ \forall n$.

$\underline{su(p,q)}$ $\dim \underline{k} = p^2 + q^2 - 1$, $\dim \underline{p} = 2pq$; $\text{rank}\,\mathfrak{g} = q$; $\text{char}\,\mathfrak{g} > 0$ when $p = q$, $\text{char}\,\mathfrak{g} = 0$ when $p = q + 1$, and $\text{char}\,\mathfrak{g} < 0$ otherwise.

$\underline{so^*(2n)}$ $\dim \underline{k} = n^2$, $\dim \underline{p} = n^2 - n$; $\text{rank}\,\mathfrak{g} = [n/2]$; $\text{char}\,\mathfrak{g} < 0\ \forall n$.

$\underline{so(2,n)}$ $\dim \underline{k} = \frac{1}{2}n(n-1) + 1$, $\dim \underline{p} = 2n$; $\text{rank}\,\mathfrak{g} = 2$; $\text{char}\,\mathfrak{g} > 0$ for $n = 3,4$ and $\text{char}\,\mathfrak{g} < 0$ otherwise.

$\underline{e\ III}$ (a real form of e_6) $\dim \underline{k} = 46$, $\dim \underline{p} = 32$; $\text{rank}\,\mathfrak{g} = 2$.

$\underline{e\ VII}$ (a real form of e_7) $\dim \underline{k} = 79$, $\dim \underline{p} = 54$; $\text{rank}\,\mathfrak{g} = 3$.

A glance at the list on p. 346 of [9] proves the following

Corollary. A complex simple Lie algebra $\mathfrak{g}$ admits a non-compact real form having a causal cone iff the adjoint group corresponding to a compact real form of $\mathfrak{g}$ is not simply connected.

3. Structure of Hermitian Lie algebras.

Let $\mathfrak{g}$ be one of the Lie algebras in the list in Section 2, and $\mathfrak{g} = \underline{k} + \underline{p}$ a fixed Cartan decomposition. Let $\mathfrak{g}^c$ be its complexification, and set $\underline{k}^c = \underline{k} + i\underline{k}$, $\underline{p}^c = \underline{p} + i\underline{p}$. Let c be the one-dimensional center of $\underline{k}$, $\underline{h}$ a maximal abelian subalgebra of $\underline{k}$, and set $\underline{h}^c = \underline{h} + i\underline{h}$. Then $c \subseteq \underline{h} \subset \underline{h}^c$. We have $\underline{k} = \underline{k}_s \oplus c$, where $\underline{k}_s = [\underline{k},\underline{k}]$ is a compact semisimple Lie algebra.

We recount the notation and analysis in [9], p. 312-16. It is shown there that $\underline{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. Let Δ denote the set of roots of $\mathfrak{g}^c$ with respect to $\underline{h}^c$. (A root is always nonzero.) If

$$\mathfrak{g}^c = \underline{h}^c + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

is the root space decomposition, then each $\mathfrak{g}_\alpha$ satisfies either $\mathfrak{g}_\alpha \subset \underline{k}^c$, in which case the root α is called <u>compact</u>, or $\mathfrak{g}_\alpha \subset \underline{p}^c$, in which case α is called <u>noncompact</u>. We have the decompositions

$$\underline{k}^c = \underline{h}^c + \sum_{\alpha} \mathfrak{g}_\alpha \qquad\qquad \underline{p}^c = \sum_{\beta} \mathfrak{g}_\beta$$

α running over the compact roots and β over the noncompact roots.

All roots are real-valued on $i\underline{h}$, and a root is compact iff it vanishes on c . Let B be the restriction of the killing form to $i\underline{h}$, a real positive-definite inner product. (For our purposes, any positive multiple of B would serve equally as well.)

Choose compatible orderings in the real duals of ic and $i\underline{h}$. $\forall \alpha \in \Delta$ take $H_\alpha \in i\underline{h}$ such that $\alpha(H_\alpha) = 2$ and $B(H_\alpha,\cdot)$ proportional to α . Also take $E_\alpha \in \mathfrak{g}_\alpha$ such that $i(E_\alpha + E_{-\alpha})$, $E_\alpha - E_{-\alpha} \in \underline{k} + i\underline{p}$, and $[E_\alpha, E_{-\alpha}] = H_\alpha$ $\forall \alpha \in \Delta$. Let Q_+ be the set of positive and noncompact roots. As in [15] $\forall \alpha \in Q_+$ set

$$X_\alpha = E_\alpha + E_{-\alpha}$$

$$Y_\alpha = -i(E_\alpha - E_{-\alpha}) \quad .$$

These vectors are a basis for $\underline{p}$. We have

$$(1) \qquad \begin{aligned} [H_\alpha, X_\alpha] &= 2iY_\alpha \\ [H_\alpha, Y_\alpha] &= -2iX_\alpha \\ [X_\alpha, Y_\alpha] &= 2iH_\alpha \end{aligned}$$

for all $\alpha \in Q_+$.

Let $\Delta^+ \subset \Delta$ be the positive roots with respect to the ordering, and let $\Delta_o^+ = \{\alpha_1,\ldots,\alpha_r\}$ be the simple positive roots. The α_i are a basis for the dual of $i\underline{h}$, and any $\alpha \in \Delta^+$ is a nonnegative integral linear combination of the α_i . By the compatibility of the ordering, $\exists Z \in c$ such that $\alpha(iZ) > 0$ iff α is noncompact and positive.

The following is well known.

Proposition 3.1. $\alpha(iZ) = \beta(iZ)\ \forall\, \alpha,\beta \in Q_+$, and all but exactly one of the $\alpha_i \in \Delta_o^+$ are compact.

Proof. As $[\underline{p},\underline{p}] \subseteq \underline{k}$, if α,β are noncompact roots,

$$0 = [Z,[E_\alpha,E_\beta]] = (\alpha+\beta)(Z)[E_\alpha,E_\beta] \quad .$$

Thus $\alpha+\beta \in \Delta$ implies $(\alpha+\beta)(Z) = 0$. We recall the lemma on root systems ([13, p. 45]): $B(\alpha,\beta) < 0$ implies $\alpha+\beta$ is a root (so $B(\alpha,\beta) > 0$ implies $\alpha-\beta$ is a root). Therefore $|\alpha(Z)| \neq |\beta(Z)|$ implies $\alpha \perp \beta$.

Thus one can put the noncompact roots into mutually perpendicular subspaces, each singled out by a particular value of $|\alpha(Z)|$. As shown above, root spaces corresponding to different values must commute (and recall $[E_\alpha,E_\beta] = 0$ for $\alpha,\beta \in Q_+$).

We wish to show that each compact root α is in one of these subspaces. Now there must be some noncompact β_1 such that $\alpha+\beta_1$ is a root; otherwise $[E_\alpha,\underline{p}] = 0$ and $\{X \in \underline{k}\colon [X,\underline{p}] = 0\}$ would be a non-trivial ideal in $\mathfrak{g}$, contradicting simplicity. Thus $\alpha + \beta_1 = \beta_2$ for noncompact β_1, β_2 , $\alpha = \beta_2 - \beta_1$ is a root, and β_1 and β_2 must be in the same subspace.

The root system is then decomposed into non-trivial mutually orthogonal systems, contradicting simplicity, unless all $|\alpha(Z)| = |\beta(Z)|$ for α, β noncompact. Assume then that $\alpha(iZ) = 1 \ \forall \alpha \in Q_+$. Let $\beta_1, \ldots, \beta_n$ be the noncompact roots in Δ_o^+ . Any noncompact root involves one β_i but no others, as $(\beta_i + \beta_j)(iZ) = 2$. If γ, δ are noncompact and involve different β_i's , then $[E_\gamma, E_\delta] = 0$, and the noncompact roots are again partitioned non-trivially unless $n = 1$.

If $\alpha(iZ) = 1 \ \forall \alpha \in Q_+$, we have $[Z, E_\alpha] = -iE_\alpha$ and $[Z, E_{-\alpha}] = iE_{-\alpha}$, so $[Z, X_\alpha] = Y_\alpha$ and $[Z, Y_\alpha] = -X_\alpha$ $\forall \alpha \in Q_+$. Thus ad Z: $\underline{p} \to \underline{p}$ is a complex structure on $\underline{p}$.

In [9] it is shown that there exists a set $\Sigma_o \subseteq Q_+$ such that

$$\mathfrak{a} = \sum_{\delta \in \Sigma_o} \mathbb{R} X_\delta$$

is maximal abelian in $\underline{p}$, and $\gamma \pm \delta$ are not roots $\forall \gamma, \delta \in \Sigma_o$. Let $\underline{h}^-$ be the real span of iH_α , $\alpha \in \Sigma_o$. Then $\underline{h}^- \subseteq \underline{h}$ and these iH_α are orthogonal, so $\dim \underline{h}^- = \dim \mathfrak{a}$. Let $\underline{h}^+$ be the orthocomplement of $\underline{h}^-$ in $\underline{h}$ with respect to the killing form. In [15] it is shown that $\underline{h}^+$ is the centralizer of $\mathfrak{a}$ in $\underline{h}$, and that the complexification of $\underline{h}^+ \oplus \mathfrak{a}$ is also a Cartan subalgebra of $\mathfrak{g}^c$. It may be shown that the centralizer of $\mathfrak{a}$ in $\underline{k}$, usually denoted $\mathfrak{m}$, is spanned by $\underline{h}^+$ and all $i(E_\alpha + E_{-\alpha})$, $E_\alpha - E_{-\alpha}$ as above, where α is compact and $iH_\alpha \in \underline{h}^+$.

We introduce one more bit of notation; as is standard, let $Z_0 = -(i/2) \sum_{\alpha \in \Sigma_0} H_\alpha \in \underline{h}^-$. Then

$$\begin{aligned}[Z - Z_0, X_\alpha] &= Y_\alpha - [Z_0, X_\alpha] \\ &= Y_\alpha - Y_\alpha = 0 ,\end{aligned}$$

so $Z - Z_0 \in \underline{h}^+$, and we have decomposed

$$Z = (Z - Z_0) + Z_0 \in \underline{h}^+ \oplus \underline{h}^- .$$

Only in the symplectic case is $\underline{h}^+ = 0$, but often $Z = Z_0$ even if $\underline{h}^+ \neq 0$. Whether $Z = Z_0$ or not depends only on $\mathfrak{g}$ and not on the choices we have made, as is proven in [15]:

Proposition 3.8. $Z = Z_0$ iff Z is contained in a three-dimensional simple subalgebra of $\mathfrak{g}$.

Finally, we compute the action of $A = \exp \mathfrak{a}$ on Z. (The action of A on $\underline{h}$ is used a great deal in the next several sections.) Let $X = \sum_{\alpha \in \Sigma_0} t^{\alpha} X_{\alpha} \in \mathfrak{a}$. The relations (1) imply

$$\begin{aligned} \mathrm{Ad}(\exp X)(Z) &= Z - Z_0 + \mathrm{Ad}(\exp X)\Big(-(i/2) \sum_{\alpha \in \Sigma_0} H_{\alpha}\Big) \\ &= Z - Z_0 - (i/2) \sum_{\alpha \in \Sigma_0} (\cosh 2t^{\alpha}) H_{\alpha} \\ &\qquad - (1/2) \sum_{\alpha \in \Sigma_i} (\sinh 2t^{\alpha}) Y_{\alpha} . \end{aligned}$$

Corollary 3.2. Any invariant convex cone in $\mathfrak{g}$ containing Z contains all

$$c_0(Z - Z_0) + \sum_{\alpha \in \Sigma_0} c_{\alpha}(-iH_{\alpha}) \in \underline{h}$$

for c_0, c_{α} satisfying $0 < c_0 \leq 2 \min_{\alpha \in \Sigma_0} c_{\alpha}$.

Proof. Average over positive and negative t^{α} in the above.

4. Certain Classical Linear Groups.

To classify some of the invariant convex cones in the classical Hermitian Lie algebras we need information about their orbits, and the work [2] of Burgoyne and Cushman, from the point of view of the lowest-dimensional representation, is ideal for our purposes. The exceptional algebras will probably require a more abstract approach, which is currently being suggested by the similarities and analogies seen in the following treatments of the four classical series.

In this section we review their description of the classical linear groups, and in the next describe their classification of the orbits. It would not simplify matters at this stage to restrict to the Hermitian case, and we refrain from doing so until the end of section five.

Let V be a finite-dimensional complex vector space, always non-zero. If $A: V \to V$ is linear, $\det A$ denotes the usual complex determiant. Let $G = GL(V)$. Let σ be any conjugate-linear operator $\sigma: V \to V$ such that $\sigma^2 = \pm I$. σ induces an automorphism of G of order two.

We consider the real groups $H_\sigma = \{g \in H: g\sigma = \sigma g\}$ for $H = G$ and certain special proper subgroups H of G. Such σ and σ' are called <u>equivalent with respect to</u> H if $\sigma = g\sigma' g^{-1}$ for some $g \in H$. If this happens H_σ and $H_{\sigma'}$ are isomorphic.

The two cases $\sigma^2 = \pm I$ induces quite different structures on V.

(a) $\underline{\sigma^2 = I}$. $V_\sigma^+ = \{v \in V: \sigma v = v\}$ is a real form of V, and any $g \in H_\sigma$ leaves V_σ^+ invariant. $\det g \in \mathbb{R}$ for any $g \in H_\sigma$.

(b) $\underline{\sigma^2 = -I}$. In this case V must have even dimension, and V receives the structure of a quaternionic vector space as follows. Let

$\mathbb{Q} = \{a+bi+cj+dk\colon a,b,c,d \in \mathbb{R}\}$ denote the quaternions as usual. If $\alpha+\beta j \in \mathbb{Q}$ for $\alpha,\beta \in \mathbb{C}$ and $v \in V$, define $(\alpha+\beta j)v = \alpha v+\beta\sigma(v)$. (Note $\alpha j = j\bar{\alpha}$ for $\alpha \in \mathbb{C}$.) We use the anti-automorphism

$$a+bi+cj+dk \in \mathbb{Q} \to a+bi-cj+dk \in \mathbb{Q} ,$$

denoted $\lambda \to \lambda^q$; it satisfies $(\lambda\mu)^q = \mu^q\lambda^q$ $\forall\, \lambda,\mu \in \mathbb{Q}$. Note $\lambda\lambda^q$ is not necessarily real, and usually $\lambda\lambda^q \neq \lambda^q\lambda$.

Again, $\det g \in \mathbb{R}$ for any $g \in H_\sigma$.

Real forms of $GL(V)$.

The first case involves no σ; we assume only a nondegenerate Hermitian form $\tau_*(\cdot,\cdot)$ on V, and define $G_{\tau_*} = \{g \in G\colon \tau_*(g\cdot,g\cdot) = \tau_*(\cdot,\cdot)\}$. $|\det g| = 1\ \forall g \in G_{\tau_*}$, and G_{τ_*} is connected. The subgroup of $\det g = 1$ is connected and isomorphic to an $SU(p,q)$.

(a) $\sigma^2 = I$. All σ are equivalent with respect to G. $G_\sigma \cong GL(n,\mathbb{R})$ where $n = \dim V$. G_σ has two components, and $SL(n,\mathbb{R})$ is connected.

(b) $\sigma^2 = -I$. All σ are equivalent with respect to G. $G_\sigma \cong U^*(2n)$, where $2n = \dim V$. G_σ has two components, and is isomorphic to the general linear group on a quaternionic vector space. $SU^*(2n)$ is connected.

Real Forms of $O(V,\tau)$.

Let τ be a nondegenerate symmetric complex-bilinear form on V . We consider only those σ as above such that $\tau(\sigma\cdot,\sigma\cdot) = \overline{\tau(\cdot,\cdot)}$. Let $H = \{g \in G: \tau(g\cdot,g\cdot) = \tau(\cdot,\cdot)\}$.

(a) $\sigma^2 = I$. There are several equivalence classes of the σ's with respect to H , depending on the (real) signature of τ restricted to V_σ^+ , denoted τ_+. H_σ has either two or four components, and $\det g = \pm 1$ for each $g \in H_\sigma$. Each H_σ is isomorphic to some $O(p,q)$.

(b) $\sigma^2 = -I$. Let $\dim V = 2n$. V becomes a quaternionic vector space as above, and defining $\tau_-(u,v) = \tau(u,v) + \tau(u,\sigma v)j$ for $u,v \in V$, τ_- is a nondegenerate $\mathbb{Q}$-valued form on V . It safisfies

(2) $$\tau_-(\lambda u,\mu v) = \lambda\tau_-(u,v)\mu^q$$

and $\tau_-(u,v) = \tau_-(v,u)^q$ for all $u,v \in V$, $\lambda,\mu \in \mathbb{Q}$. There is one equivalence class for σ , and $\mathbb{Q}$-bases $\{e_i\}$ exist for V such that $\tau_-(e_i,e_j) = \delta_{ij}$ [5]. Each $g \in H_\sigma$ automatically has determinant 1, just as for the symplectic case, and $H_\sigma \cong SO^*(2n)$ is connected.

Real Forms of $Sp(V,\tau)$.

Let τ be a nondegenerate anti-symmetric complex bilinear form on V . $\dim V$ is necessarily even. We consider only those σ such that $\tau(\sigma\cdot,\sigma\cdot) = \overline{\tau(\cdot,\cdot)}$. Let $H = \{g \in G: \tau(g\cdot,g\cdot) = \tau(\cdot,\cdot)\}$.

(a) $\underline{\sigma^2 = I}$. There is a unique equivalence class for σ with respect to H , and $H_\sigma \cong Sp(n,\mathbb{R})$ is connected. Each $g \in H_\sigma$ has determinant 1.

(b) $\underline{\sigma^2 = -I}$. Again V becomes a quaternionic vector space, and τ_- is defined on V by the same expression as above. (2) still holds, but now $\tau_-(u,v) = -\tau(v,u)^q$, and there are various equivalence classes for σ with respect to H , classified by the signature of τ_- . That is, there are $\mathbb{Q}$-bases $\{e_r\}$ such that $\tau_-(e_r,e_s) = \pm\delta_{rs}j$. Each $g \in H_\sigma$ has determinant 1, and $H_\sigma \cong Sp(p,q)$ is connected.

5. <u>Classification of the Orbits</u>. (Burgoyne and Cushman)

One may designate any one of the groups (real or complex) of the previous section by $G(V,\sigma,\tau)$, using the notation there. The Lie algebra of $G(V,\sigma,\tau)$, written $L(V,\sigma,\tau)$, is just those linear endomorphisms of V commuting with σ and skew with respect to τ . However, as in [2], either τ or σ may not actually occur in the definition of the group or algebra. Moreover, τ may denote τ_* , in which case σ is absent.

There is the notion of a <u>type</u> Δ, an equivalence class of pairs (A,V) where $A \in L(V,\sigma,\tau)$, under the obvious notation of equivalence. $\dim \Delta$ ($= \dim V$ for all $(A,V) \in \Delta$) is well defined. The connection between types and orbits is the following

<u>Proposition 5.1</u>. Let $A, B \in L(V,\sigma,\tau)$. Then there exists $g \in G(V,\sigma,\tau)$ such that $g^{-1}Ag = B$ iff (A,V) and (B,V) belong to the same type.

The <u>sum</u> of two types is defined to be just the obvious τ-orthogonal, σ-invariant direct sum. A type Δ is <u>indecomposable</u> if it cannot be written as the sum of two other types.

Given any $A \in L(V,\sigma,\tau)$, one can write it uniquely as $A = S + N$, where $S, N \in L(V,\sigma,\tau)$, S is semisimple, N is nilpotent, and $SN = NS$.

<u>Definitions</u>. 1) Let $m \geq 0$ be the unique integer such that $N^m \neq 0$ and $N^{m+1} = 0$ in the above. m is called the <u>height</u> of (A,V), and the notation <u>$\text{ht}\,\Delta$</u> for any type Δ is well defined.

2) Let $K = \ker N^m$; then $K \supseteq NV$. If $K = NV$, we say (A,V) is <u>uniform</u>, and clearly one can speak of uniform types.

3) If $\text{ht}\,\Delta = 0$ call Δ a <u>semisimple type</u>. A semisimple type is uniform.

There is a natural mapping of uniform types to semisimple types, defined as follows. Let Δ be uniform and $m = \text{ht}\,\Delta$. If $(A,V) \in \Delta$ put $\overline{V} = V/NV$, and for $v \in V$ put $\bar{v} = v + NV$. Define $\overline{A}$, $\bar{\sigma}$, and $\bar{\tau}$ on $\overline{V}$ by $\overline{A}\bar{v} = \overline{Av}$, $\bar{\sigma}\bar{v} = \overline{\sigma v}$, and $\bar{\tau}(\bar{u},\bar{v}) = \tau(u,N^m v)$. Since τ is nondegenerate on V and (A,V) is uniform, $\bar{\tau}$ is nondegenerate on $\overline{V}$. $G(\overline{V},\bar{\sigma},\bar{\tau})$ and $L(\overline{V},\bar{\sigma},\bar{\tau})$ are well defined, and $A \in L(\overline{U},\bar{\sigma},\bar{\tau})$. Let $\overline{\Delta}$ denote the type containing $(\overline{A},\overline{V})$; $\overline{\Delta}$ is semisimple. Note $\overline{\Delta}$ has nothing to do with complex conjugation in $\mathbb{C}$.

<u>Remark</u>. $G(\overline{V},\bar{\sigma},\bar{\tau})$ may be a group in a different class than $G(V,\sigma,\tau)$: if τ is complex bilinear and $\tau(u,v) = \lambda\tau(v,u)$, $\lambda = \pm 1$, then $\bar{\tau}$ satisfies $\bar{\tau}(\bar{u},\bar{v}) = \lambda(-1)^m\bar{\tau}(\bar{v},\bar{u})$. If τ denotes τ_* one can assume that $\bar{\tau}$ is again an Hermitian form by replacing $\bar{\tau}$ by $i\bar{\tau}$ if necessary.

They prove the following results.

<u>Theorem 5.2</u>. The decomposition of a type $\Delta = \Delta_1 + \dots + \Delta_s$ into indecomposable types (which clearly exists) is unique.

<u>Proposition 5.3</u>. If Δ is uniform it is uniquely determined by $\text{ht}\,\Delta$ and $\overline{\Delta}$.

<u>Proposition 5.4</u>. If Δ is indecomposable then Δ is uniform and $\overline{\Delta}$ is indecomposable.

We next give the simple construction recovering a uniform type Δ from ht Δ and the semisimple type $\bar{\Delta}$, as in Proposition 5.3. Let $(S,E) \in \bar{\Delta}$ and $S \in L(E,\sigma,\tilde{\tau})$. Let V be the direct sum of $m+1$ copies of E, written conveniently as $E + NE + \ldots + N^m E$. Extend σ and S to V, acting componentwise on this decomposition, and let N act on V in the obvious way. Then $N^m \neq 0$, $N^{m+1} = 0$, and $\ker N^m = NV$. To define τ on V, let $\tau(N^r E, N^s E) = 0$ unless $r+s = m$, and in that case let $\tau(N^r e_1, N^s e_2) = (-1)^r \tilde{\tau}(e_1,e_2) \ \forall\ e_1,e_2 \in E$. Then clearly $G(V,\sigma,\tau)$ is one of the above groups, $A = S + N \in L(V,\sigma,\tau)$, and $(A,V) \in \Delta$.

By the above, the problem of the classification of the orbits under the adjoint group of any of these algebras reduces to the determination of the semisimple indecomposable types, at least in the case where G is connected. This description is straightforward for the complex groups and the $U(p,q)$. The results are as follows. Let Δ be a semisimple indecomposable type for either a complex G or $U(p,q)$ and $(S,W) \in \Delta$.

$G = GL(W)$. W is one-dimensional and Δ is determined by a single eigenvalue $\zeta \in \mathbb{C}$. Denote this type by $\Delta(\zeta)$.

$G = O(W,\tau)$. If $S = 0$ W is one-dimensional. There is a basis element e such that $\tau(e,e) = 1$. Denote this type by $\Delta(0)$.

If $S \neq 0$ the set of eigenvalues for S is $\{\zeta, -\zeta\}$ for some $0 \neq \zeta \in C$. W is two-dimensional and there is a basis $\{e,f\}$ such that $Se = \zeta e$, $Sf = -\zeta f$, $\tau(e,e) = \tau(f,f) = 0$ and $\tau(e,f) = 1$. Denote this type by $\Delta(\zeta,-\zeta)$.

$\underline{G = Sp(W,\tau)}$. Whether $S \neq 0$ or not, $\dim W = 2$, and there is a basis $\{e,f\}$ and $\zeta \in C$ such that $Se = \zeta e$, $Sf = -\zeta f$, and $\tau(e,f) = 1$. Denote this type also by $\Delta(\zeta,-\zeta)$.

$\underline{G = U(p,q)}$. $\dim W$ is either 2 or 1. When $\dim W = 2$ there is a basis $\{e,f\}$ and $\zeta \in C$ such that $\zeta \neq -\bar{\zeta}$ where $\tau_*(e,e) = \tau_*(f,f) = 0$, $\tau_*(e,f) = 1$, and $Se = \zeta e$, $Sf = -\bar{\zeta} f$. Denote this type by $\Delta(\zeta,-\bar{\zeta})$.

When $\dim W = 1$ there is $\zeta \in C$ such that $\bar{\zeta} = -\zeta$ and a basis element e such that $Se = \zeta e$ and $\tau_*(e,e) = \pm 1$. The two signs give different types. Denote these by $\Delta^{\pm}(\zeta)$.

For the real groups $G(V,\sigma,\tau)$ one proceeds as follows. An indecomposable semisimple type Δ for $G(V,\sigma,\tau)$ clearly gives rise to a type Δ^c for the corresponding complex group $G(V,\tau)$: just omit σ. Let $(S,V) \in \Delta$. Δ^c is a sum of semisimple indecomposable types for $G(V,\tau)$, so let Δ_1^c be an indecomposable component of Δ^c. Let $(S,W) \in \Delta_1^c$ where $W \subseteq V$. Clearly $(S,\sigma W)$

is also a type for $G(V,\tau)$, and we have either $\sigma W = W$ or or $\sigma W \cap W = \{0\}$. Since $\sigma^2 = \pm I$ and Δ is indecomposable we have three possible decompositions for Δ^c :

(a) $\Delta^c = \Delta_1^c + \sigma\Delta_1^c$ and $\Delta_1^c \neq \sigma\Delta_1^c$;

(b) $\Delta^c = \Delta_1^c + \sigma\Delta_1^c$ and $\Delta_1^c = \sigma\Delta_1^c$;

(c) $\Delta^c = \Delta_1^c$ and $\Delta_1^c = \sigma\Delta_1^c$.

Let eig Δ_1^c denote the set of eigenvalues of S on W . They prove the following

<u>Lemma A.1</u>. (a) occurs iff $\overline{\text{eig } \Delta_1^c} \neq \text{eig } \Delta_1^c$.

<u>Lemma A.2</u>. Suppose $S \neq 0$. Then (b) occurs iff $\sigma^2 = -I$ and all elements of eig Δ_1^c are real.

We now consider each of the six classes of fixed-point groups and indicate the classification of and notation for the semisimple indecomposable types, based mostly on Lemmas A.1 and A.2.

<u>$G = GL(V)$</u>

<u>$\sigma^2 = I$</u> . Let $\Delta_1^c = \Delta(\zeta)$.

1) $\zeta \neq \bar{\zeta}$; (a) by A.1; type $\Delta(\zeta,\bar{\zeta})$.

2) $\zeta = \bar{\zeta} \neq 0$; (c) by A.1 and A.2; type $\Delta(\zeta)$.

3) $\zeta = 0$; (c) because if (b) and $W = \mathbb{C}e$, $\mathbb{C}(e+\sigma e)$ is σ-invariant, contrary to indecomposability; type $\Delta(0)$.

<u>$\sigma^2 = -I$</u> . Let $\Delta_1^c = \Delta(\zeta)$.

1) $\zeta \neq \bar{\zeta}$; (a) by A.1; type $\Delta(\zeta,\bar{\zeta})$.

2) $\zeta = \bar{\zeta} \neq 0$; (b) by A.2; type $\Delta(\zeta,\zeta)$.

3) $\zeta = 0$; (b) as $\dim V$ must be even; type $\Delta(0,0)$.

$\underline{G = O(V,\tau)}$

$\underline{\sigma^2 = I}$. Let $\Delta_1^c = \Delta(0)$.

1) Clearly (c) ; $\operatorname{sig} \tau_+ = (1,0)$ or $(0,1)$; two types $\Delta^{\pm}(0)$.

Let $\Delta_1^c = \Delta(\zeta,-\zeta)$, $\zeta \neq 0$.

2) $\zeta \neq \pm\bar{\zeta}$; (a) by A.1; $\operatorname{sig} \tau_+ = (2,2)$; type $\Delta(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$.

3) $\zeta = \bar{\zeta}$; (c) by A.1, A.2; $\operatorname{sig} \tau_+ = (1,1)$; type $\Delta(\zeta,-\zeta)$.

4) $\zeta = -\bar{\zeta}$; again (c); $\operatorname{sig} \tau_+ = (2,0)$ or $(0,2)$; two types $\Delta^{\pm}(\zeta,-\zeta)$.

$\underline{\sigma^2 = -I}$. Let $\Delta_1^c = \Delta(0)$.

1) (b) as V must be even-dimensional; type $\Delta(0,0)$.

Let $\Delta_1^c = \Delta(\zeta,-\zeta)$, $\zeta \neq 0$.

2) $\zeta \neq \pm\bar{\zeta}$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$.

3) $\zeta = \bar{\zeta}$; (b) by A.2; type $\Delta(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$.

4) $\zeta = -\bar{\zeta}$; (c) by A.1, A.2; two types $\Delta^{\pm}(\zeta,-\zeta)$ depending on the signature of $\tau_-(\cdot,S\cdot)$ (V is one-dimensional over $\mathbb{Q}$.)

$\underline{G = Sp(V,\tau)}$

$\underline{\sigma^2 = I}$. Let $\Delta_1^c = \Delta(\zeta,-\zeta)$.

1) $\zeta \neq \pm\bar{\zeta}$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$.

2) $\zeta = \bar{\zeta} \neq 0$; (c) by A.1, A.2; type $\Delta(\zeta,-\zeta)$.

3) $\zeta = -\bar{\zeta} \neq 0$; (c) by A.1, A.2; $\tau(S\cdot,\cdot)$ either positive or negative definite on V_σ^+ ; two types $\Delta^\pm(\zeta,-\zeta)$.

4) $\zeta = 0$; clearly (c); type $\Delta(0,0)$.

$\underline{\sigma^2 = -I}$. Let $\Delta_1^c = \Delta(\zeta,-\zeta)$.

1) $\zeta \neq \pm\zeta$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$.

2) $\zeta = \bar{\zeta} \neq 0$; (b) by A.2; type $\Delta(\zeta,-\zeta,\zeta,-\zeta)$.

3) $\zeta = -\bar{\zeta} \neq 0$; (c) by A.1, A.2; two types $\Delta^\pm(\zeta,-\zeta)$ depending on sig $\tau_-(\cdot,\cdot)$.

4) $\zeta = 0$; clearly (c); again two types $\Delta^\pm(0,0)$.

This information is summarized in the following table taken from [2].

The Semisimple Indecomposable Types

σ^2	τ	Type	Conditions
		$\Delta(\zeta)$	
+I		$\Delta(\zeta,\bar\zeta)$	$\zeta \neq \bar\zeta$
		$\Delta(\zeta)$	$\zeta = \bar\zeta$
-I		$\Delta(\zeta,\bar\zeta)$	
	*	$\Delta(\zeta,-\bar\zeta)$	$\zeta \neq -\bar\zeta$
		$\Delta^{\pm}(\zeta)$	$\zeta = -\bar\zeta$
	sym	$\Delta(\zeta,-\zeta)$	$\zeta \neq 0$
		$\Delta(0)$	
+I	sym	$\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq \pm\bar\zeta$
		$\Delta(\zeta,-\zeta)$	$\zeta = \bar\zeta \neq 0$
		$\Delta^{\pm}(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$
		$\Delta^{\pm}(0)$	
-I	sym	$\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq -\bar\zeta$
		$\Delta^{\pm}(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$
		$\Delta(0,0)$	
	alt	$\Delta(\zeta,-\zeta)$	
+I	alt	$\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq \pm\bar\zeta$
		$\Delta(\zeta,-\zeta)$	$\zeta = \bar\zeta$
		$\Delta^{\pm}(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$
-I	alt	$\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq -\bar\zeta$
		$\Delta^{\pm}(\zeta,-\zeta)$	$\zeta = -\bar\zeta$

Using the construction outlined before, it is now a simple matter to list the indecomposable types. We do so for the Hermitian symmetric Lie algebras, and in fact for all the $o(p,q)$. Write $\sigma = \sigma_{\pm}$ according to whether $\sigma^2 = \pm I$, and set $\delta = (-1)^{m/2}\epsilon$ for m even where $\epsilon = \pm 1$.

Indecomposable Types for Hermitian Lie Algebras

Group	Type	Conditions	Signature
$GL(V,\tau_*)$	$\Delta_m(\zeta,-\bar\zeta)$	$\zeta \neq -\bar\zeta$	$(m+1,m+1)$
$[\supset SU(p,q)]$	$\Delta_m^\epsilon(\zeta)$	$\zeta = -\bar\zeta$	m even $(\frac{1}{2}(m+1+\delta),\frac{1}{2}(m+1-\delta))$ m odd $(\frac{m+1}{2},\frac{m+1}{2})$
$O(V,\sigma_+,\tau)$	$\Delta_m(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq \pm\bar\zeta$	$(2(m+1),2(m+1))$
$[= o(p,q)]$	$\Delta_m(\zeta,-\zeta)$	$\zeta = \bar\zeta \neq 0$	$(m+1,m+1)$
	$\Delta_m^\epsilon(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$	m even $(m+1+\delta,m+1-\delta)$ m odd $(m+1,m+1)$
	$\Delta_m^\epsilon(0)$	m even	$(\frac{1}{2}(m+1+\delta),\frac{1}{2}(m+1-\delta))$
	$\Delta_m(0,0)$	m odd	$(m+1,m+1)$
$O(V,\sigma_-,\tau)$	$\Delta_m(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq -\bar\zeta$	
$[= SO^*(2n)]$	$\Delta_m^\epsilon(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$	
	$\Delta_m(0,0)$	m even	
	$\Delta_m^\epsilon(0,0)$	m odd	
$Sp(V,\sigma_+,\tau)$	$\Delta_m(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$	$\zeta \neq \pm\bar\zeta$	
$[=Sp(n,\mathbb{R})]$	$\Delta_m^\epsilon(\zeta,-\zeta)$	$\zeta = \bar\zeta \neq 0$	
	$\Delta_m^\epsilon(\zeta,-\zeta)$	$\zeta = -\bar\zeta \neq 0$	
	$\Delta_m(0,0)$	m even	
	$\Delta_m^\epsilon(0)$	m odd	

The indecomposable types which can contribute to a type for $o(2,n)$ are quite limited, and listed below.

$\underline{\Delta_m(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})}$ $\quad m = 0$

$\underline{\Delta_m(\zeta,-\zeta)}$ $\quad m = 0,1$

$\underline{\Delta_m^{\epsilon}(\zeta,-\zeta)}$ $\quad m = 0,\ \epsilon = \pm 1;\ m = 2$ with $\epsilon = 1$;
$m = 1,\ \epsilon = \pm 1$

$\underline{\Delta_m^{\epsilon}(0)}$ $\quad m = 0,\ \epsilon = \pm 1;\ m = 2,\ \epsilon = \pm 1;\ m = 4$ with
$\epsilon = -1$

$\underline{\Delta_m(0,0)}$ $\quad m = 1$

CHAPTER II. Invariant Convex Cones in $sp(n,\mathbb{R})$.

6. Conventions for $sp(n,\mathbb{R})$.

It will be convenient to work with fixed models for the classical Hermitian Lie algebras, fixed Cartan decompositions, etc. The notation is as in section 3.

We take

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : \begin{array}{l} A,B,C \text{ real } n\times n \text{ matrices,} \\ B,C \text{ symmetric, } A \text{ arbitrary} \end{array}\right\}$$

$$\underline{k} = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{g} : A \text{ skew, } B \text{ symmetric}\right\}$$

$$\underline{p} = \left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{g} : A,B \text{ symmetric}\right\}$$

$$\underline{h} = \left\{\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in \mathfrak{g} : D \text{ diagonal}\right\}$$

$$c = \mathbb{R}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Let d_j $(j = 1,\dots,n)$ be linear functionals spanning the dual of $\underline{h}$:

$$d_j : \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \longrightarrow D_{jj} \quad .$$

Then

$$\{\text{noncompact roots}\} = \{\pm i(d_j+d_k) : 1\le j<k\le n;\ \pm i2d_j : 1\le j\le n\}$$

$$\{\text{compact roots}\} = \{\pm i(d_j-d_k) : 1\le j<k\le n\}$$

and the positive roots we take to be the above with "+" only. Also,

$$\{\text{simple positive roots}\} = \Delta_o^+ = \{i(d_1-d_2),\ldots,i(d_{n-1}-d_n),i2d_n\} .$$

We give the H_α, X_α, Y_α for $\alpha \in Q_+$. For $\alpha = i(d_j+d_k)$ $(1 \le j < k \le n)$

$$H_\alpha = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \qquad X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \qquad Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$$

where $D = -iE_{jj} - iE_{kk}$, $A = B = E_{jk} + E_{kj}$.

For $\alpha = i2d_j$ $(1 \le j \le n)$

$$H_\alpha = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \qquad X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \qquad Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$$

where $H_\alpha = -iE_{jj}$, $A = B = E_{jj}$.

We have then $Z = \begin{pmatrix} 0 & -\frac{1}{2}I \\ \frac{1}{2}I & 0 \end{pmatrix} = Z_o$, $\underline{h} = \underline{h}^-$, and take

$$\Sigma_o = \{i2d_1,\ldots,i2d_n\} \quad , \quad \mathfrak{a} = \left\{\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \in \mathfrak{g}: \Lambda \text{ diagonal}\right\} .$$

The cone in $\underline{h}$ obtained from Corollary 3.2 is

$$\left\{\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \in \underline{h}: D \text{ has positive diagonal entries}\right\} .$$

Define $B(X,Y) = \mathrm{Tr}(XY)$ for $X,Y \in \mathfrak{g}$. If

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \quad , \quad Y = \begin{pmatrix} D & G \\ H & -D^t \end{pmatrix} \in \mathfrak{g} ,$$

we have

$$B(X,Y) = 2\sum_{i,j} A_{ij}D_{ji} + \sum_{i,j} B_{ij}H_{ji} + \sum_{i,j} C_{ij}G_{ji} .$$

The Cartan involution is $\theta: X \to -X^t$ and $B_\theta(X,Y) = -B(X,\theta Y)$, so

$$B_\theta\left(\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \begin{pmatrix} D & G \\ H & -D^t \end{pmatrix}\right) = 2\sum_{i,j} A_{ij}D_{ij} + \sum_{i,j} B_{ij}G_{ij} + \sum_{i,j} C_{ij}H_{ij} .$$

Define the symplectic form $\mathcal{a}(\cdot,\cdot)$ on $\mathbb{R}^{2n}$ by

$$\mathcal{a}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = y\cdot u - x\cdot v$$

for $x,y,u,v \in \mathbb{R}^n$.

<u>Theorem 6.1</u>. Let $u \in \mathbb{R}^{2n}$ and $X \in \mathfrak{g}$. Then $\mathcal{a}(Xu,u) = B_\theta(X,Y)$ where $\mathcal{a}(Yv,v) \geq 0$ for all $v \in \mathbb{R}^{2n}$.

<u>Proof</u>. Let $u = \begin{pmatrix} v_i \\ w_i \end{pmatrix}$ and $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$. Then $\mathcal{a}(Xu,u) = (Cv)\cdot v - 2(Av)\cdot w - (Bw)\cdot w = \sum_{i,j} C_{ij}v_iv_j - \sum_{i,j} B_{ij}w_iw_j - 2\sum_{i,j} A_{ij}w_iv_j = B_\theta\left(\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \begin{pmatrix} D & G \\ H & -D^t \end{pmatrix}\right) = B_\theta(X,Y)$, where $D_{ij} = -w_iv_j$, $G_{ij} = -w_iw_j$, and $H_{ij} = v_iv_j$. Clearly $Y \in \mathfrak{g}$.

Secondly, let $\begin{pmatrix} x \\ y \end{pmatrix} = v \in \mathbb{R}^{2n}$. Then

$$\begin{aligned} \mathcal{a}(Yv,v) &= (Hx)\cdot x - 2(Dx)\cdot y - (Gy)\cdot y \\ &= \sum_{i,j} x_iH_{ij}x_j - 2\sum_{i,j} x_jD_{ij}y_i - \sum_{i,j} y_iG_{ij}y_j \\ &= (x\cdot v)(x\cdot v) + 2(x\cdot v)(w\cdot y) + (y\cdot w)(y\cdot w) \\ &= (x\cdot v + w\cdot y)^2 \geq 0 . \end{aligned}$$

7. Conjugacy of Positive Elliptics in $sp(\mathcal{H})$.

Let $\mathcal{H}$ be a real Hilbert space, possibly infinite dimensional, with inner product $g_0(\cdot,\cdot)$. We assume that $\mathcal{H}$ has a complex structure J , i.e., an orthogonal with square $-I$. Define the symplectic form $\mathcal{a}(x,y) = g_0(J^{-1}X,Y)$ for $x,y \in \mathcal{H}$. $\mathcal{a}$ is skew and nondegenerate. $(\mathcal{H}, \langle\ ,\ \rangle)$ is then a complex Hilbert space, where $\langle\cdot,\cdot\rangle = g_0(\cdot,\cdot) + i\mathcal{a}(\cdot,\cdot)$; note $g_0(\cdot,\cdot) = \mathcal{a}(J\cdot,\cdot)$. $\mathcal{a}$ is really the primary bilinear form here, but we need some g_0 to define the topology.

Write $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$ or sometimes $\|\cdot\|^2_{g_0}$ for clarity. In this section all operators are bounded. Given a (real-linear) operator $B\colon \mathcal{H} \to \mathcal{H}$, $\|B\|$ denotes the norm of B . Let $Sp(\mathcal{H})$ be the group of invertible linear operators $\mathcal{H} \to \mathcal{H}$ preserving $\mathcal{a}$. Call $B\colon \mathcal{H} \to \mathcal{H}$ <u>infinitesimally symplectic</u> if B is skew with respect to $\mathcal{a}$. The set of infinitesimally symplectic operators we denote $\mathfrak{g}$, or $sp(\mathcal{H})$.

<u>Theorem 7.1</u>. Let $B \in \mathfrak{g}$ satisfy $\mathcal{a}(Bv,v) \geq k(v,v)\ \forall\ v \in \mathcal{H}$, for some $k > 0$. Then there exists $S \in Sp(\mathcal{H})$ such that $SBS^{-1} = JH$ where H is a positive self-adjoint operator on $(\mathcal{H}, \langle\ ,\ \rangle)$ (i.e., H commutes with J). Furthermore, the spectrum of H is bounded as follows:

$$(\|B\|^{3/2}/k^{1/2})\|v\|^2 \geq \langle Hv,v\rangle \geq (k^{3/2}/\|B\|^{1/2})\|v\|^2$$

for all $v \in \mathcal{H}$.

Proof. Clearly $\|B\| \geq \sup_{\|v\|\leq 1} \mathcal{A}(Bv,v)$, and in fact by polarization one can prove equality, but we do not need this. It follows from the identity $\mathcal{S}_o(Bv,w) = \mathcal{A}(Bu_-,u_-) - \mathcal{A}(Bu_+,u_+)$ where $u_\pm = (v \pm Jw)/2$ and the positivity of $\mathcal{A}(B\cdot,\cdot)$.

Define $\mathcal{S}(v,w) = \mathcal{A}(Bv,w)$ for all $v,w \in \mathcal{H}$. $\mathcal{S}$ is symmetric as $B \in \mathfrak{g}$ and $\mathcal{A}$ is skew. We have

(3) $$\|B\|\,\|v\|^2 \geq \mathcal{S}(v,v) \geq k\|v\|^2 \;\forall\; v \in \mathcal{H}$$

so $\mathcal{S}_o$ and $\mathcal{S}$ are real Hilbert structures defining equivalent norms on $\mathcal{H}$. We have $\mathcal{S}(\cdot,\cdot) = \mathcal{S}_o(J^{-1}B\cdot,\cdot)$, and J is orthogonal for $\mathcal{S}$ iff B commutes with J . $J^{-1}B$ is a symmetric operator with respect to $\mathcal{S}_o$, bounded below by k and above by $\|B\|$, so B is bijective. Hence it has a bounded inverse, and $\|B^{-1}\| \leq 1/k$.

Now B is skew with respect to $\mathcal{S}$, so if $B = K\sqrt{B^tB} = K\sqrt{-B^2}$ is the polar decomposition in $(\mathcal{H},\mathcal{S})$, all three operators B,K, and $\sqrt{-B^2}$ commute. We have $K^2 = -I$, and $\sqrt{-B^2}$ is invertible and positive-definite symmetric in $(H,\mathcal{S})$.

Furthermore, $K \in Sp(\mathcal{H})$ as

$$\begin{aligned}\mathcal{A}(Kv,Kw) &= \mathcal{A}((B/\sqrt{-B^2})v,\ (B/\sqrt{-B^2})w)\\ &= \mathcal{S}((1/\sqrt{-B^2})v,\ (B/\sqrt{-B^2})w)\\ &= \mathcal{S}(v,-B^{-1},w) = \mathcal{A}(v,w) \quad .\end{aligned}$$

Setting $\mathcal{S}_1(x,y) = \mathcal{A}(Kx,y) \ \forall \ x,y \in \mathcal{H}$, we have

$$(4) \qquad \mathcal{S}_1(\cdot,\cdot) = \mathcal{S}((-B^2)^{-\frac{1}{2}}\cdot,\cdot) \quad ,$$

and $\mathcal{S}_1$ defines yet another equivalent norm on $\mathcal{H}$. Its advantage over $\mathcal{S}$ is that

$$\langle\cdot,\cdot\rangle_1 = \mathcal{S}_1(\cdot,\cdot) + i\mathcal{A}(\cdot,\cdot)$$

defines a complex Hilbert structure on $\mathcal{H}$ (with complex structure K) with the same imaginary part as that of $\langle \ , \ \rangle$.

It is not difficult to see that the topological equivalence of $(\mathcal{H}, \langle \ , \ \rangle)$ and $(\mathcal{H}, \langle \ , \ \rangle_1)$ implies there exists a unitary equivalence

$$S: (\mathcal{H},\langle \ , \ \rangle_1) \longrightarrow (\mathcal{H},\langle \ , \ \rangle) \quad .$$

We have $JS = SK$, $S \in Sp(\mathcal{H})$, and $SBS^{-1} = J(S\sqrt{-B^2}\,S^{-1})$. So, we set $H = S\sqrt{-B^2}\,S^{-1}$, and $HJ = JH$ because K commutes with $\sqrt{-B^2}$.

By (4) we have orthogonal equivalences

$$(\mathcal{H},\mathcal{S}) \xrightarrow{(-B^2)^{1/4}} (\mathcal{H},\mathcal{S}_1) \xrightarrow{\quad S \quad} (\mathcal{H},\mathcal{S}_0)$$

of real Hilbert spaces. As $(-B^2)^{1/4}$ commutes with $\sqrt{-B^2}$, $\sqrt{-B^2}$ is positive-definite symmetric in $(\mathcal{H},\mathcal{S}_1)$ also. By the equivalence, H has the same property in $(\mathcal{H},\mathcal{S}_0)$, and is self-adjoint in $(\mathcal{H},\langle \ , \ \rangle)$ as $HJ = JH$.

It remains to prove the bounds on H . They are equivalent to

$$(\|B\|^{3/2}/k^{1/2})\|\cdot\|_{\mathfrak{s}}^{2} \;\geq\; \mathfrak{s}(\sqrt{-B^{2}}\cdot,\cdot) \;\geq\; (k^{3/2}/\|B\|^{1/2})\|\cdot\|_{\mathfrak{s}}^{2}$$

or

$$(5) \qquad (\|B\|^{3}/k)\|\cdot\|_{\mathfrak{s}}^{2} \;\geq\; \mathfrak{s}(-B^{2}\cdot,\cdot) \;\geq\; (k^{3}/\|B\|)\|\cdot\|_{\mathfrak{s}}^{2} \;.$$

For the upper bound of (5),

$$\begin{aligned} \mathfrak{s}(-B^{2}w,w) &= \mathfrak{s}(Bw,Bw) \\ &\leq \|B\|_{\mathfrak{s}_o}\mathfrak{s}_o(Bw,Bw) \\ &\leq \|B\|_{\mathfrak{s}_o}^{3}\|w\|_{\mathfrak{s}_o}^{2} \;\leq\; (\|B\|_{\mathfrak{s}_o}^{3}/k)\|w\|_{\mathfrak{s}}^{2} \;, \end{aligned}$$

the first and third inequalities following from (3). For the lower bound,

$$\begin{aligned} \mathfrak{s}(Bw,Bw) &\geq k\mathfrak{s}_o(Bw,Bw) \\ &\geq k\|B^{-1}\|_{\mathfrak{s}_o}^{-2}\|w\|_{\mathfrak{s}_o}^{2} \\ &\geq k^{3}(1/\|B\|_{\mathfrak{s}_o})\|w\|_{\mathfrak{s}}^{2} \end{aligned}$$

by (3) and $\|B^{-1}\|_{\mathfrak{s}_o} \leq 1/k$ shown earlier.

8. Uniqueness of Causal Cones in $sp(n,\mathbb{R})$.

The goal of this section is to prove Theorem 8.1. We use the notation of section 6, and in order to apply Theorem 7.1 we indicate the connection between the notations of sections 6 and 7. Our real Hilbert space $\mathcal{H}$ is $\mathbb{R}^{2n}$ with the standard inner product. The complex structure J is $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$; note that $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -w \\ v \end{pmatrix}$ corresponds to $i(v+iw) = -w + iv$ for $v,w \in \mathbb{R}^n$. Also $\langle v+iw, x+iy\rangle = v\cdot w + w\cdot y + i\{w\cdot x - v\cdot y\}$, so the two definitions of the symplectic form α agree.

A short computation shows that $g = \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in Sp(n,\mathbb{R})$ iff $P^tM = M^tP$, $N^tQ = Q^tN$, and $Q^tM - N^tP = I$, or equivalently $MN^t = NM^t$, $QP^t = PQ^t$, and $MQ^t - NP^t = I$. Thus $g^{-1} = \begin{pmatrix} Q^t & -N^t \\ -P^t & M^t \end{pmatrix}$. Also $X \in gl(2n,\mathbb{R})$ commutes with J iff X has the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ for $A,B \in gl(n,\mathbb{R})$. We find $g \in Sp(n,\mathbb{R})$ in the subgroup K corresponding to <u>k</u> iff $g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where $AA^t + BB^t = I$ and $AB^t = BA^t$, or $A^tA + B^tB = I$ and $A^tB = B^tA$. As shown on p. 343 of [9], K is algebraically isomorphic to $U(n)$ acting on $\mathbb{C}^n$, i.e. $I = (A+iB)(A+iB)^*$ gives the same equations.

Let $\mathcal{C}$ be the set of all non-trivial invariant convex cones in $\mathfrak{g}$, and let $C_o = \{X \in \mathfrak{g}: \alpha(Xv,v) > 0 \ \forall\, v \neq 0$ in $\mathbb{R}^{2n}\}$.

<u>Theorem 8.1</u>. C_o is nonempty and in $\mathcal{C}$. Any $C \in \mathcal{C}$ contains either C_o or $-C_o$ but not both. $\pm C_o$ are the unique open cones in $\mathcal{C}$. $\overline{C}_o = \{X \in \ \ : \alpha(Xv,v) \geq 0 \ \forall\, v \in \mathbb{R}^n\}$, and $\pm\overline{C}_o$ are the unique closed cones in $\mathcal{C}$.

<u>Proof</u>. Clearly $J \in \mathcal{C}_o$ and $C_o \in \mathcal{C}$. Let $C \in \mathcal{C}$. By Lemma 1.1 $\overline{C} \in \mathcal{C}$ also, and by the proof of Theorem 2.2 $\overline{C}$ contains either J or $-J$. Assume $J \in \overline{C}$. Now

$\mathcal{A}(X\cdot,\cdot)$ is a symmetric form on $\mathbb{R}^{2n}$ for all $X \in \mathfrak{g}$. Since all norms are equivalent in a finite-dimensional space C_o is open. Thus $J \in \overline{C} \cap C_o$ implies $\exists W \in C \cap C_o$.

Let W be the B in Theorem 7.1. The invariance of C implies $JH \in C$ as in the theorem, and the diagonalization of JH by a unitary implies that C also contains

$$J\tilde{D} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$$

for some diagonal matrix D with positive entries.

Now conjugation by $a, a^{-1} \in A$ and averaging (C is convex) shows that C also contains any $\begin{pmatrix} 0 & -D' \\ D' & 0 \end{pmatrix} \in \underline{h}$ where $D'_{ii} \geq D_{ii}$ for $i = 1,\ldots,n$. As C is a cone it must contain all $\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, D positive diagonal. The argument can now be repeated to show $C_o \subseteq C$. But C cannot contain C_o and $-C_o$ by Proposition 2.1.

Of the remaining three statements we show how the first follows from the last two. Let C be an open cone in $\mathcal{C}$, and assume $\overline{C} = \overline{C}_o$. As $\overline{C}_o \cap -\overline{C}_o = \{0\}$ we must have $C_o \subseteq C$. If $X \in C - C_o \subseteq \overline{C}_o - C_o$ $\exists v \in \mathbb{R}^{2n}$ such that $\mathcal{A}(Xv,v) = 0$ and $v \neq 0$. C being open implies $\exists \varepsilon > 0$ such that $Y = X - \varepsilon J \in C \subseteq \overline{C}_o$ but $\mathcal{A}(Yv,v) < 0$, a contradiction.

To prove $\overline{C}_o = \{X \in \mathfrak{g} : \mathcal{A}(Xv,v) \geq 0 \ \forall\, v \in \mathbb{R}^{2n}\} \equiv S$, note first that clearly $\overline{C}_o \subseteq S$. Conversely, if $X \in S$ then $X + \varepsilon J \in C_o \ \forall\ \varepsilon > 0$, so $X \in \overline{C}_o$.

Finally, let C_1 be closed and in $\mathcal{C}$. We may assume

$C_0 \subset C_1$ so that $\overline{C}_0 \subseteq \overline{C}_1 = C_1$. If $\overline{C}_0 \subset C_1$ then $C_1^* \subset \overline{C}_0^* = -\overline{C}_0$ by Theorem 8.2 below. But we have proven that then $-C_0 \subset C_1^*$ must occur, hence $-\overline{C}_0 \subseteq C_1^*$, a contradiction.

<u>Theorem 8.2</u>. $\overline{C}_0^* = -\overline{C}_0$, i.e., the closed cone $\overline{C}_0$ is self-dual with respect to any B_θ form.

<u>Proof</u>. We saw in section 2 that $B_\theta(X,Y) \geq 0$ $\forall X,Y \in \overline{C}_0$, which is just $-\overline{C}_0 \subseteq \overline{C}_0^*$. If now $X \notin -\overline{C}_0$ $\exists$ $v \in \mathbb{R}^{2n}$ such that $\mathcal{Q}(Xv,v) > 0$. By Theorem 6.1 $B_\theta(X,Y) > 0$ for some $Y \in \overline{C}_0$, so $X \notin \overline{C}_0^*$.

<u>Corollary 8.3</u>. C_0 is the topological interior of $\overline{C}_0$.

<u>Proof</u>. The interior of $\overline{C}_0$ contains C_0 and is an invariant convex cone, and so equals C_0 by the uniqueness.

9. <u>Classification of Cones in $sp(n,\mathbb{R})$</u> .

We have seen that any invariant convex cone C (or its negative) in $sp(n,\mathbb{R})$ must sit between C_0 and $\overline{C}_0$. Let $\mathcal{C}_+ = \{C \in \mathcal{C}: C_0 \subseteq C \subseteq \overline{C}_0\}$. It turns out that $\mathcal{C}_+$ is a finite collection, and we will be able to classify its elements. For example, in $sp(2,\mathbb{R})$ there are 12 such cones in $\mathcal{C}_+$.

Any $C \in \mathcal{C}_+$ can be considered a collection of types Δ , each of which is a sum of indecomposable types $\Delta = \Delta_1 + \ldots + \Delta_m$. By Theorem 8.1 it is clear that Δ

is in $\overline{C}_0$ iff each Δ_j is in the $\overline{C}_0$'s for the (possibly) lower-dimensional $\mathcal{a}$-orthogonal subspaces. Thus to determine the types on the boundary of C_0 it suffices to consider the indecomposable types.

Using the notation and analysis in Section 5, it is clear that $X \in C_0$ determines a type whose indecomposable components are all of the form $\Delta_0^+(\zeta,-\zeta)$ for $0 \neq \zeta \in i\mathbb{R}$. $\pm\zeta$ are the eigenvalues of this type, so the eigenvalues of any $X \in \overline{C}_0$ must also lie on the imaginary axis. Thus it suffices to consider only $\Delta_m^{\pm}(0)$ (m odd), $\Delta_m(0,0)$ (m even), and $\Delta_m^{\pm}(\zeta,-\zeta)$ ($\zeta = -\bar{\zeta} \neq 0$).

Theorem 9.1. Only the types $\Delta_1^+(0)$, $\Delta_0(0,0)$, and $\Delta_0^+(\zeta,-\zeta)$ are in $\overline{C}_0$.

Proof. Let $A = S + N$ be a representative of one of the above types of height m. The representation space is a direct sum $E + NE + \ldots + N^m E$, where $\mathcal{a}(\cdot, N^m\cdot)$ is non-degenerate on E.

Assume first $m \geq 1$, take $v,w \in E$, and consider

$$\mathcal{a}(A(N^{m-1}v+w), N^{m-1}v+w)$$
$$= \mathcal{a}(Aw,w) + (-1)^{m-1}\mathcal{a}(AN^{2m-2}v,v) + 2\mathcal{a}(N^m v,w)$$

$2m-2 \geq m+1$ iff $m \geq 3$, in which case the first and second terms vanish, and the expression is negative for the proper choice of v and w.

If $m = 2$ the expression is $-\mathcal{a}(SN^2v,v) + 2\mathcal{a}(N^2v,w)$, and this is negative if w is sufficiently large and of the proper sign.

Finally, if $m = 1$ take $v, w \in E$ and consider

$$\mathcal{a}(A(Nv+w), Nv+w) = -2\mathcal{a}(NSw, v) + \mathcal{a}(Nw, w) \quad .$$

If $S \neq 0$ it is injective, and again this is not positive-definite.

The remaining cases are just those stated in the theorem. All are orbits in $sp(1,\mathbb{R}) \approx sl(2,\mathbb{R})$. $\Delta_0(0,0)$ is just $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is in the orbit of a $\Delta_0^+(\zeta,-\zeta)$ iff iff $-a^2 - bc > 0$, $b < 0$, and $c > 0$. $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \neq 0$ is in $\Delta_0^+(0)$ iff $-a^2 - bc = 0$, $b \leq 0$, and $c \geq 0$. This completes the proof.

To classify the cones we need some further machinery and a convexity lemma. Let $E_n = \{[(x_1,y_1),\ldots,(x_n,y_n)]:$ all $x_i, y_i \geq 0\}$. We consider it a cone in $\mathbb{R}^{2n}$. Define a semigroup $\mathcal{G}$ acting on $\mathbf{E}_n$ generated by

1) $[(x_1,y_1),\ldots] \to [(y_1,x_1),\ldots]$,
2) $[(x_1,y_1),\ldots,(x_i,y_i),\ldots,(x_n,y_n)] \to$
   $[(x_i,y_i),\ldots,(x_1,y_1),\ldots,(x_n,y_n)]$ for $i=2,\ldots,n$,
3) $[(x_1,y_1),\ldots] \to [(\lambda x_1,(1/\lambda)y_1),\ldots]$ for $\lambda > 0$,

and

4) $[(x_1,y_1),(x_2,y_2),\ldots] \to [(x_1,y_1+y_2),(x_1+x_2,y_2),\ldots]$.

Let Φ be the collection of all Schrödinger bases in $\mathbb{R}^{2n}$ on which G acts, i.e.

$$\Phi = \{(a_i,b_i)_{i=1}^n : \mathcal{A}(a_i,a_j) = \mathcal{A}(b_i,b_j) = 0,\ \mathcal{A}(a_i,b_j) = \delta_{ij}\} ,$$

and let $\widetilde{\mathcal{C}}_+$ be the collection of all convex cones in E_n which are invariant under Θ and contain $\mathcal{O}_n = \{[(x_1,y_1),\ldots,(x_n,y_n)]: \text{all } x_i,y_i > 0\}$. We will put $\mathcal{C}_+$ in 1-1 correspondence with $\widetilde{\mathcal{C}}_+$ by means of the following maps. Given $X \in \overline{C}_0$ and $\varphi = (a_i,b_i)_{i=1}^n \in \Phi$, define

$$M_\varphi(X) = [(\mathcal{A}(Xa_1,a_1),\mathcal{A}(Xb_1,b_1)),\ldots,(\mathcal{A}(Xa_n,a_n),\mathcal{A}(Xb_n,b_n))] \in E_n .$$

For any $C \in \mathcal{C}_+$ let E_C be the convex cone generated by all $M_\varphi(X)$ for $X \in C$, $\varphi \in \Phi$.

<u>Lemma 9.2</u>. Let $X \in \overline{C}_0$ and let the decomposition of the type of X be P summands of the form $\Delta^+(\zeta,-\zeta)$ $(\zeta = -\bar{\zeta} \neq 0)$, N summands of the form $\Delta_1^+(0)$, and Z of the form $\Delta(0,0)$. Then $n = P + N + Z$. Let $\varphi = (a_i,b_i) \in \Phi$ exhibit this decomposition, so that $M_\varphi(X)$ has P pairs of the form (e,f) , $e,f > 0$, N of the form $(\lambda,0)$ or $(0,\lambda)$, $\lambda > 0$, and Z of the form $(0,0)$.

Let $\psi = (c_i,d_i) \in \Phi$ be arbitrary. We can assume ψ is ordered so that $\mathcal{A}(Xc_i,c_i),\mathcal{A}(Xd_i,d_i) > 0$ for $i = 1,\ldots,P'$, $\mathcal{A}(Xc_i,c_i) > 0$, $\mathcal{A}(Xd_i,d_i) = 0$ for $i = P'+1,\ldots,P'+N'$, and $\mathcal{A}(Xc_i,c_i) = \mathcal{A}(Xd_i,d_i) = 0$ for $i = P'+N'+1,\ldots,n$.

Then $Z' \le Z$ and $N'+Z' \le N+Z$.

<u>Proof</u>. Assume that the diagonalizing basis φ has been ordered so that the sentence above is true with (a_i,b_i) replacing (c_i,d_i) , and P,N,Z replacing P',N',Z'

respectively. Now each c_j and d_j is a linear combination of the a_i and b_i. If $j = P'+N'+1,\dots,n$, c_j and d_j can involve no a_i or b_i with $i = 1,\dots,P$ or a_i with $i = p+1,\dots,P+N$. The $2Z' \times 2Z'$ matrix

$$\begin{pmatrix} \mathcal{A}(c_k,c_\ell) & \mathcal{A}(c_k,d_\ell) \\ \mathcal{A}(d_k,c_\ell) & \mathcal{A}(d_k,d_\ell) \end{pmatrix}_{k,\ell=P'+N'+1,\dots,n}$$

(which depends only on the combinations of the a_i, b_i for $i = P+N+1,\dots,n$) is of rank $2Z'$ as $\psi \in \Phi$, but is equal to $A^t\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}A$ where A is of order $2Z \times 2Z'$, hence $Z' \leq Z$.

Express likewise d_j for $j = P'+1,\dots,n$. These d_j can involve no a_i, b_i for $i = 1,\dots,P$. Now the maximal dimension of an isotropic subspace in a $2(N+Z)$-dimensional symplectic space is $N+Z$, and $N'+Z' \leq N+Z$ follows.

<u>Definition</u>. If $e \in E_n$ has exactly N pairs of the form $(0,\lambda)$ or $(\lambda,0)$, $\lambda > 0$, and Z pairs of the form $(0,0)$, say <u>e gives (N,Z)</u>.

<u>Theorem 9.3</u>. $E_C \in \tilde{\mathcal{C}}_+$ for each $C \in \mathcal{C}_+$, and the mapping

$$E\colon \mathcal{C}_+ \to \tilde{\mathcal{C}}_+ \colon C \to E_C$$

is a 1-1 correspondence.

<u>Proof</u>. E_C clearly contains $\mathcal{O}_n$ since each C contains C_o. Invariance under the generators 1) - 3) follows from

the trivial basis changes $a_1 \to b_1$, $b_1 \to -a_1$ and $a_1 \to \lambda a_1$, $b_1 \to (1/\lambda)b_1$.

Invariance under 4) follows from an $Sp(2,\mathbb{R})$ symmetry: given (a_1,b_1,a_2,b_2) , make the change of basis

$$(6) \qquad \begin{array}{ll} a_1 \to c_1 = a_1 & a_2 \to c_2 = a_2 + ka_1 \\ b_1 \to d_1 = b_1 - kb_2 & b_2 \to d_2 = b_2 \end{array}$$

for $k \in \mathbb{R}$, which is symplectic. If (a_1,b_1,a_2,b_2) is part of a $\varphi \in \Phi$ and $X \in C$ so $M_\varphi(X) \in E_C$, do (6) for $k = \pm 1$ to obtain φ_+,φ_- , and $\frac{1}{2}(M_{\varphi_+}(X) + M_{\varphi_-}(X)) \in E_C$ exhibits the invariance under 4). Thus $E_C \in \widetilde{\mathcal{C}}_+$.

Before proving the 1-1 correspondence, we first make a simplifying remark about the $D \in \widetilde{\mathcal{C}}_+$. If $e \in D$, then D also contains any other e' obtained from e by changing any positive x_i or y_i in e to any other positive x_i' or y_i' . This is trivial for the $(\lambda,0)$ and $(0,\lambda)$ $(\lambda \neq 0)$ pairs by (3), and accomplished for the (x,y) $(x,y > 0)$ pairs by initially multiplying by a small constant, and then applying (3) twice so

$$(x,y) \to (\lambda x,(1/\lambda)y) \quad \text{and} \quad (x,y) \to ((1/\lambda)x,\lambda y)$$

and averaging, to "raise" (x,y) by any desired amount.

Now let $D \in \widetilde{\mathcal{C}}_+$. By the above D is characterized by the set of pairs (N,Z) which the $e \in D$ give. (Of course, the (N,Z) cannot be specified arbitrarily.)

Each (N,Z) determines many types Δ, as in the statement of Lemma 9.2; let C be the convex cone in $\overline{C}_0$ generated by the representatives of these types determined by D. That $E_C = D$ follows from the fact that D is convex, Lemma 9.2, and the observation that if D contains e giving (N,Z) where $Z < n$ $(e \neq 0)$, then it also contains all points giving (N',Z') where $Z' \leq Z$ and $N'+Z' \leq N+Z$, by invariance under 1) to 4). Thus $C \to E_C$ is surjective.

Finally, suppose $E_{C_1} = E_{C_2}$ for $C_1, C_2 \in \mathcal{C}_+$. We must show that each type in C_1 is in C_2 also. Let Δ be a type in C_1 and $X \in \mathfrak{g}$ representing Δ. If $\varphi_1 \in \Phi$ is a diagonalizing basis as in Lemma 9.2, $e = M_{\varphi_1}(X) \in E_{C_1}$ and gives a pair (N,Z). By hypothesis $e \in E_{C_2}$, so e is a convex combination of $M_{\varphi_i}(Y_i)$ for $Y_i \in C_2$, $\varphi_i \in \Phi$.

Observe that if $x,y \in E_n$, and x,y, and $x+y$ give $(N_1,Z_1),(N_2,Z_2)$, and (N_3,Z_3) respectively, then $Z_3 \leq \min(Z_1,Z_2)$, and $N_3+Z_3 \leq \min(N_1+Z_1, N_2+Z_2)$. Thus for some $Y \in C_2$ and $\varphi \in \Phi$, $M_\varphi(Y)$ gives (N_1,Z_1) where $Z \leq Z_1$ and $N+Z \leq N_1+Z_1$. Now Y may be diagonalized by some $\psi \in \Phi$, and $M_\psi(Y)$ gives (N_2,Z_2). By the lemma, $N_2+Z_2 \geq N_1+Z_1$ and $Z_2 \geq Z_1$, so $N_2+Z_2 \geq N+Z$ and $Z_2 \geq Z$. It remains to see that these inequalities and $Y \in C_2$ imply $X \in C_2$.

Via the basis ψ, Y acts on n copies of $\mathbb{R}^2$; on Z_2 of them Y acts as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, on N_2 like $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and on $n - Z_2 - N_2$ copies like a representative of a $\Delta^+(\zeta,-\zeta)$, $\zeta = -\bar{\zeta} \neq 0$. $Sp(n,\mathbb{R})$ includes $SL(2,\mathbb{R})$ on each $\mathbb{R}^2$ and all permutations of the $\mathbb{R}^2$'s . The average of, say $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ with its transform $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so a $\Delta_1^{+}(0)$ can be changed to a $\Delta^+(\zeta,-\zeta)$ easily. Also, the average of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with its transform $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ gives $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, etc. In this way one can see that X is in the invariant convex cone generated by Y , <u>provided</u> $N_2 > 0$.

If $N_2 = 0$, $Z_2 < n$ (otherwise $Y = 0$) , and we must show how $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ can give rise to an $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This is done by the $Sp(2,\mathbb{R})$ symmetry (6). If $Ya_1 = -b_1$, $Yb_1 = a_1$, $Ya_2 = Yb_2 = 0$, we consider the action on the basis (c_1,d_1,c_2,d_2) from (6):

$$Yc_1 = -d_1 - kd_2 \qquad\qquad Yc_2 = -kd_1 - k^2 d_2$$
$$Yd_1 = c_1 \qquad\qquad Yd_2 = 0 \quad .$$

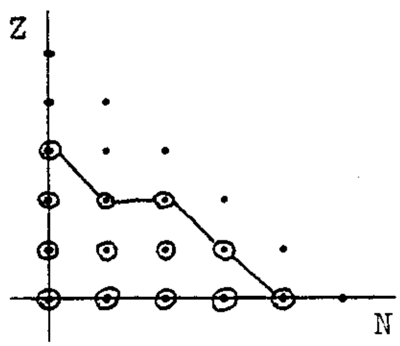
The average for $k = \pm 1$ is

$$Yc_1 = -d_1 \qquad\qquad Yc_2 = -d_2$$
$$Yd_1 = c_1 \qquad\qquad Yd_2 = 0 \quad ,$$

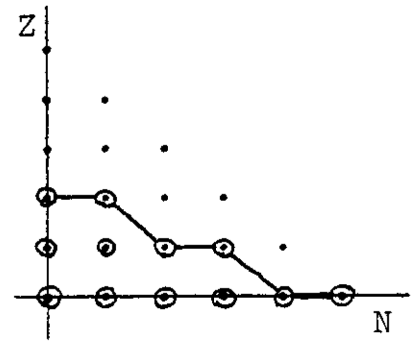
as desired.

Thus $X \in C_2$ also, so $C_1 = C_2$ and $E \to E_C$ is injective. This completes the proof.

We see from the proof that any cone in $\mathcal{C}_+$ may be represented graphically by a set of lattice points $\{(N,Z)\}$ in $\{(x,y) \in \mathbb{Z} \oplus \mathbb{Z} : x,y \geq 0,\ x+y \leq n\}$, having the general form



or



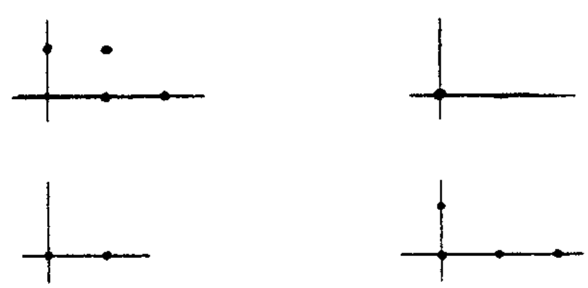
(in $sp(5,\mathbb{R})$), for example.

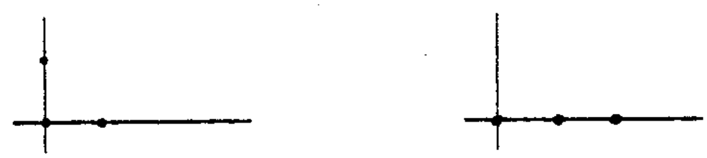
The segments bounding the region containing the (N,Z) are either horizontal or have slope -1. $(0,0)$ is always included ($C_o \subseteq C$ always), and $(0,n)$ is included optionally according to whether $0 \in C$ or not.

It is clear anyway that $sl(2,\mathbb{R})$ has 4 cones in $\mathcal{C}_+$:

(closed) (open)

and there are 6 cones in $sp(2,\mathbb{R})$ which do not contain 0 :





They are listed this way to exhibit a duality. Letting $\mathcal{C}_+^o$ denote the cones in $\mathcal{C}_+$ not containing 0, we have the mapping

$$C \in \mathcal{C}_+^o \to \hat{C} = \{X \in \mathfrak{g}: B_\theta(X,Y) > 0 \ \forall Y \in C\} \in \mathcal{C}_+^o ,$$

and perhaps there is a corresponding explicit duality in these graphs. Are there any "self-dual" cones?

Since the previous was written, Bob Proctor pointed out that the enumeration of the above graphs involves no complicated combinatorics. The positive cones not containing 0, in 1-1 correspondence with the above graphs not containing $(0,n)$, are also in 1-1 correspondence with the set of nonempty and proper subsets of $S_n = \{1,2,\ldots,n+1\}$, as follows. Notice that each row in a graph contains strictly fewer points than any rows below it, and the graph is faithfully represented by the collection of positive integers each the number of points in some row of the graph. For example, in the $sp(5,\mathbb{R})$ examples above, the subsets are $\{5,4,3,1\}$ and $\{6,4,2\}$, respectively.

Thus there are $2(2^n-1)$ positive invariant convex cones not containing 0 in $sp(n,\mathbb{R})$. And it appears that the geometrical duality above corresponds simply to complementation in S_n of these subsets, in which case there are evidently no cones C satisfying $\hat{C} = C$.

CHAPTER III. Invariant Convex Cones in su(p,q).

10. Conventions for su(p,q).

We take $p \geq q \geq 1$ and $n = p+q$. As in section 3,

$$\mathfrak{g} = \left\{ \begin{pmatrix} B & A \\ A^* & C \end{pmatrix} : \begin{array}{l} A,B,C \text{ complex matrices; } B \text{ and } C \text{ skew-} \\ \text{Hermitian of order } p \text{ and } q\text{, resp.;} \\ A \text{ arbitrary; } \operatorname{Tr} B + \operatorname{Tr} C = 0 \end{array} \right\}$$

$$\underline{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{g} \right\}$$

$$\underline{p} = \left\{ \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \in \mathfrak{g} \right\}$$

$$\underline{h} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \underline{k} : B \text{ and } C \text{ diagonal} \right\}$$

and if $Z = \begin{pmatrix} -(q/(p+q))iI & 0 \\ 0 & (p/(p+q))iI \end{pmatrix}$, $c = \mathbb{R}Z$.

Let d_j $(j = 1,\ldots,n)$ be linear functionals spanning the dual of $\underline{h}$:

$$d_j : \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \to \begin{cases} B_{jj} & \text{for } j \leq p \\ C_{j-p,j-p} & \text{for } j > p \end{cases}.$$

Then $\{\text{noncompact roots}\} = \{\pm(d_j - d_k) : 1 \leq j \leq p,\ p+1 \leq k \leq n\}$

$\{\text{compact roots}\} = \{\pm(d_j - d_k) : 1 \le j < k \le p \text{ and } p+1 \le j < k \le n\}$

and the positive roots we take to be the above with "+" only.

$\Delta_o^+ = \{d_1-d_2, \ldots, d_{p-1}-d_p, d_p-d_{p+1}, d_{p+1}-d_{p+2}, \ldots, d_{n-1}-d_n\}$

and $d_p - d_{p+1}$ is the only noncompact simple positive root. For $1 \le j \le p$ and $p+1 \le k \le n$, $\alpha = d_j - d_k \in Q_+$, and $H_\alpha = E_{jj} - E_{kk}$, $X_\alpha = E_{jk} + E_{kj}$, and $Y_\alpha = -iE_{jk} + iE_{kj}$.

We take $\Sigma_o = \{d_i - d_{p+i} : 1 \le i \le q\}$ so

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ D & 0 & 0 \end{pmatrix} : D \text{ real and diagonal} \right\}$$

$$Z_o = -\frac{i}{2} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} ,$$

$$\underline{h}^- = \left\{ \begin{pmatrix} Di & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -Di \end{pmatrix} \in \underline{h} \right\} ,$$

and $$\underline{h}^+ = \left\{ \begin{pmatrix} Di & 0 & 0 \\ 0 & Fi & 0 \\ 0 & 0 & Di \end{pmatrix} \in \underline{h} \right\} .$$

Thus $Z = Z_o$ iff $p = q$, and $\underline{h}^+ = 0$ iff $p = q = 1$.

$$Z - Z_o = \begin{pmatrix} (-q/(p+q)+\frac{1}{2})iI & 0 & 0 \\ 0 & -(q/p+q))iI & 0 \\ 0 & 0 & (p/(p+q)-\frac{1}{2})iI \end{pmatrix}$$

and the cone from Corollary 3.2 is the set of rays containing

$$Z - Z_o + \begin{pmatrix} -\frac{i}{2}D & & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2}D \end{pmatrix} \in \underline{h}$$

where all $D_{jj} \geq 1$.

Define $B(X,Y) = \mathrm{Tr}(XY)$ for $X,Y \in \mathfrak{g}$. If $X = \begin{pmatrix} B & A \\ A^* & C \end{pmatrix}$, $Y = \begin{pmatrix} G & D \\ D^* & H \end{pmatrix} \in \mathfrak{g}$, we have

$$B(X,Y) = -\sum_{i,j} B_{ij}\overline{G_{ij}} + 2\,\mathrm{Re}\sum_{i,j} A_{ij}\overline{D_{ij}} - \sum_{i,j} C_{ij}\overline{H_{ij}} \;.$$

The Cartan involution is $\theta: X \to \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} X \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, and

$$B_\theta\left(\begin{pmatrix} B & A \\ A^* & C \end{pmatrix}, \begin{pmatrix} G & D \\ D^* & H \end{pmatrix}\right) = \sum_{i,j} B_{ij}\overline{G_{ij}} + 2\,\mathrm{Re}\sum_{i,j} A_{ij}\overline{D_{ij}} + \sum_{i,j} C_{ij}\overline{H_{ij}} \;.$$

Define the Hermitian form $H(\cdot,\cdot)$ on $\mathbb{C}^n$ by

$$H\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{i=1}^{p} x_i\overline{u_i} - \sum_{j=1}^{q} y_j\overline{v_j} \;,$$

for $x,y \in \mathbb{C}^p$, $y,v \in \mathbb{C}^q$. $H(Xv,v)$ is purely imaginary $\forall\, X \in \mathfrak{g}$, $v \in \mathbb{C}^n$.

<u>Theorem 10.1</u>. Let $u \in \mathbb{C}^n$ and $X \in \mathfrak{g}$. Then $iH(Xu,u) = B_\theta(X,Y)$ where $Y \in u(p,q)$ (using the same formula for B_θ). $\mathrm{Tr}\, Y = -iH(u,u)$, and $iH(Yv,v) \geq 0 \;\forall\, v \in \mathbb{C}^n$.

<u>Proof</u>. Let $u = \begin{pmatrix} v \\ w \end{pmatrix}$ and $X = \begin{pmatrix} B & A \\ A^* & C \end{pmatrix}$. Then

$$iH(Xu,u) = i\Big\{\sum_{i,j} v_i B_{ji}\overline{v_j} + \sum_{i,j} w_i A_{ji}\overline{v_j} - \sum_{i,j} \overline{w_j}\,\overline{A_{ij}} v_i - \sum_{i,j} w_i C_{ji}\overline{w_j}\Big\}$$

$$= B_\theta\left(\begin{pmatrix} B & A \\ A^* & C \end{pmatrix}, \begin{pmatrix} G & D \\ D^* & H \end{pmatrix}\right) = B_\theta(X,Y)$$

where $G_{ij} = -iv_i\overline{v_j}$, $H_{ij} = iw_i\overline{w_j}$, and $D_{ij} = -iv_i\overline{w_j}$. Clearly $Y \in u(p,q)$ and $\mathrm{Tr}\, Y = -iH(u,u)$.

If $v = \binom{x}{y} \in \mathbb{C}^{p+q}$,

$$\begin{aligned} iH(Yv,v) &= |\langle x,v\rangle|^2 + 2\,\mathrm{Re}\,(\langle y,w\rangle\langle v,x\rangle) + |\langle y,w\rangle|^2 \\ &= |\langle x,v\rangle + \langle y,w\rangle|^2 \geq 0 \ . \end{aligned}$$

11. Conjugacy of Positive Elliptics in $u(p,q)$.

Let $\mathcal{H}$ be a complex Hilbert space with complex structure i as usual and complex inner product $\langle\ ,\ \rangle_0$. "Complex-linear" (or i-linear) in this section means commuting with i . Let $\mathcal{J}$ be an Hermitian operator on $\mathcal{H}$ with spectrum $\{+1,-1\}$ and define $H(\cdot,\cdot) = \langle \mathcal{J}\cdot,\cdot\rangle_0$. Let $\mathcal{Q}(\cdot,\cdot) = -\mathrm{Im}\, H(\cdot,\cdot)$, a (real) symplectic form on $\mathcal{H}$. If we set

$$\langle\cdot,\cdot\rangle = \mathrm{Re}\langle\cdot,\cdot\rangle_0 + i\mathcal{Q}(\cdot,\cdot)$$

and $J = -\mathcal{J}i = -i\mathcal{J}$, then $(\mathcal{H}, \langle\ ,\ \rangle)$ is a complex Hilbert space with complex structure J .

Our notation here is chosen carefully to correspond with section 7; in particular, the J and $\langle\ ,\ \rangle$ correspond, and the earlier remarks on $\|\cdot\|$ apply. All operators are bounded.

Let $U(\mathcal{H},\mathcal{J}) = \{S\colon \mathcal{H} \to \mathcal{H} \mid S$ an i-linear invertible operator preserving H (or $\mathcal{Q}$)$\}$, and $\mathfrak{g} = \{B\colon \mathcal{H} \to \mathcal{H} \mid$ B i-linear and skew for $H\}$. The following lemma is clear.

Lemma 11.1. For all $A \in \mathcal{G}$, $\mathcal{A}(A\cdot,\cdot) = \mathrm{Re}(iH(A\cdot,\cdot))$ is a symmetric real bilinear form on $\mathcal{X}$ for which i is orthogonal. $iH(Ax,x)$ is real for all $A \in \mathcal{G}$.

Theorem 11.2. Let $B \in \mathcal{G}$ satisfy $\mathcal{A}(Bv,v) \geq k\langle v,v\rangle \;\forall\; v \in \mathcal{X}$, for some $k > 0$. Then there exists $S \in U(\mathcal{X},\mathcal{J})$ such that $SBS^{-1} = JH_1$ where H_1 satisfies $H_1\mathcal{J} = \mathcal{J}H_1$ and is a positive self-adjoint operator on $(\mathcal{X}, \langle\ ,\ \rangle)$ (and $(\mathcal{X}, \langle\ ,\ \rangle_o)$). Furthermore, H_1 is bounded as follows:

$$(\|B\|^{3/2}/k^{1/2})\|v\|^2 \geq \langle H_1 v,v\rangle \geq (k^{3/2}/\|B\|^{1/2})\|v\|^2$$

for all $v \in \mathcal{X}$.

Proof. The situation is the same as in section 7 except that we have included an i which is intended to commute with all operators we consider and be orthogonal for every bilinear form. The proof is virtually the same.

We define $\mathcal{S}(\cdot,\cdot) = \mathcal{A}(B\cdot,\cdot)$ and (3) still holds. We decompose $B = K\sqrt{-B^2}$ in the real space $(\mathcal{X},\mathcal{S})$ in which i is orthogonal. $Bi = iB$ so K and $\sqrt{-B^2}$ also commute with i, and $K \in \mathcal{G} \cap U(\mathcal{X},\mathcal{J})$. Letting $\mathcal{S}_1(\cdot,\cdot) = \mathcal{A}(K\cdot,\cdot) = \mathcal{S}((-B^2)^{-\frac{1}{2}}\cdot,\cdot)$, $\langle\cdot,\cdot\rangle_1 = \mathcal{S}_1(\cdot,\cdot) + i\mathcal{A}(\cdot,\cdot)$ is a complex Hilbert structure on $\mathcal{X}$ with complex structure K.

Any unitary equivalence

$$S\colon (\mathcal{X}, \langle\ ,\ \rangle_1) \to (\mathcal{X}, \langle\ ,\ \rangle)$$

satisfies $SKS^{-1} = J = -i\mathscr{J}$ and preserves $\operatorname{Im} H$, but we want S to be also i-linear. This is clearly equivalent to $S^{-1}\mathscr{J}S = iK$ if S is a unitary equivalence, which we now assume. Now $\mathscr{J}$ and iK are Hermitian with spectra $\{+1,-1\}$ in the complex spaces $(\mathscr{H}, \langle\ ,\ \rangle)$ and $(\mathscr{H}, \langle\ ,\ \rangle_1)$ respectively. Thus we need an S taking the ± 1 eigenspaces of iK to the ± 1 eigenspaces of $\mathscr{J}$. Clearly such an S exists iff the ℓ_2-dimensions of these subspaces match.

Observe

$$\langle\cdot,\cdot\rangle = \operatorname{Re} H(\mathscr{J}\cdot,\cdot) - i \operatorname{Im} H(\cdot,\cdot)$$

and

$$\langle\cdot,\cdot\rangle_1 = \operatorname{Re} H(iK\cdot,\cdot) - i \operatorname{Im} H(\cdot,\cdot) .$$

Thus if $L \subset \mathscr{H}$ is in the positive eigenspace of iK, $H(\cdot,\cdot)$ is strictly positive-definite on L. We need an infinite-dimensional "Sylvester's law of inertia," and the lemma below is sufficient for this.

The rest of the theorem is proven as in section 7.

__Lemma 11.3__. Let $(\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2, \langle\ ,\ \rangle)$ be a nontrivial direct sum of complex Hilbert spaces. Let $\mathscr{J}$ be the Hermitian operator on $\mathscr{H}$ equal to $+I$ on $\mathscr{H}_1$ and $-I$ on $\mathscr{H}_2$, and set $H(\cdot,\cdot) = \langle\mathscr{J}\cdot,\cdot\rangle$. Let α be the cardinality of the dimension of $\mathscr{H}_1$. If $S \subset \mathscr{H}$ is a closed complex-linear subspace such that $H|_S$ is strictly positive-definite on S, then $\operatorname{card}(\dim S) \leq \alpha$.

Proof. Let $\{e_\lambda\}_{\lambda\in\Gamma}$ be an H-basis of S, i.e. $H(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. By the open mapping theorem

$$M: \ell_2(\Gamma) \to S: 1_\lambda \to e_\lambda$$

is an embedding. Clearly $v_\lambda = H(M(v), e_\lambda)$ $\forall v \in \ell_2(\Gamma)$, $\lambda \in \Gamma$.

If $\mathrm{Card}(\dim S) > \alpha$, then $\exists v \in \ell_2(\Gamma)$ where $v \neq 0$ and $(Pr_{\mathscr{K}_1} \circ M)(v) = 0$. (Otherwise the adjoint of $Pr_{\mathscr{K}_1} \circ M$ has dense range from $\mathscr{K}_1$ to $\ell_2(\Gamma)$, and the Gram-Schmidt process gives a contradiction.) Then

$$0 < |v|^2 = \sum_{\mu\in\Gamma} H(Pr_{\mathscr{K}_2} \circ M(v), e_\mu)\overline{v_\mu}$$

$$= \sum_{\mu\in\Gamma} H(Pr_{\mathscr{K}_2} \circ M(v), Pr_{\mathscr{K}_2} \circ M(v)) \leq 0$$

as $H \leq 0$ on $\mathscr{K}$.

12. Minimal and Maximal Causal Cones in $su(p,q)$.

Let $\mathcal{C}$ be the set of all nontrivial invariant convex cones in $su(p,q)$. For convenience we now exclude the case $p = q = 1$. We use the notation of section 10, and now indicate the connection with the notation of section 11. The complex Hilbert space $(\mathscr{K}, \langle\, , \rangle_o)$ is $\mathbb{C}^n$ $(n = p+q)$ with the usual Hilbert structure. Writing $x \in \mathbb{C}^n$ as $\binom{y}{z}$ where $y \in \mathbb{C}^p$, $z \in \mathbb{C}^q$, $\mathscr{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $H(\cdot,\cdot)$ is an Hermitian form on $\mathscr{K}$ with signature (p,q). $J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $J \in \mathfrak{g}$ iff $p = q$, but this is not a problem; recall the closely related $Z \in \mathfrak{g}$. Let $G = SU(p,q)$.

One computes that $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,q)$ iff $AA^* - BB^* = I$, $DD^* - CC^* = I$, and $AC^* = BD^*$, or equivalently $A^*A - C^*C = I$, $D^*D - B^*B = I$, and $A^*B = C^*D$. It follows that $g^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}$.

Now $\sigma \in gl(n,C)$ commutes with J iff $\sigma Z = Z\sigma$ iff σ has the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Our maximal compact K is all $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$, U_1, U_2 unitary with $(\det U_1)(\det U_2) = 1$.

<u>Definition</u>. Let

$$C_0 = \{X \in \mathfrak{g}\colon iH(Xv,v) > 0 \;\forall \text{ nonzero } v \in C^n\}$$

(Note Lemma 11.1) and

$$C_1 = \{X \in \mathfrak{g}\colon iH(Xv,v) > 0 \;\forall \text{ nonzero } v \in C^n \text{ such that } \mathcal{H}(v,v) = 0\} .$$

<u>Theorem 12.1</u>. C_0 and C_1 are nonempty, open, and in $\mathcal{C}$. Any $C \in \mathcal{C}$ contains either C_0 or $-C_0$ but not both. $\overline{C}_0$ and $\overline{C}_1$ are obtained by changing $>$ to $\geq$ in the definitions of C_0 and C_1, and $C_0 \subset C_1$, $\overline{C}_0 \subset \overline{C}_1$. $\overline{C}_0, \overline{C}_1 \in \mathcal{C}$, and any closed (open) cone in $\mathcal{C}$ which contains C_0 is contained in $\overline{C}_1$ (resp. C_1).

<u>Proof</u>. It is clear that $Z \in C_0 \subseteq C_1$ and $C_0, C_1 \in \mathcal{C}$. $\mathcal{C}_0$ is open by the same argument as in Theorem 8.1; we will return to $\mathcal{C}_1$ later.

Let $C \in \mathcal{C}$. As before, we may assume $Z \in \overline{C}$, so again $\exists\, B \in C \cap C_0$. Apply Theorem 11.2 and note that a conjugation by $S \in U(p,q)$ can be achieved by an element of G . Diagonalizing further by means of unitaries, it follows that C contains $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ where λ and σ are positive diagonal matrices. Averaging over permutations of the λ_i and σ_j (obtained from conjugations from K), we see $Z \in C$.

Now reverse the argument to show $C_0 \subseteq C$. The unclear step is that any $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ as above can be obtained from Z by conjugations, convex combinations, and scalar multiplication. Now the effect of conjugating $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ as above, where all $\lambda_i + \sigma_j > 0$, by $a, a^{-1} \in A$ and averaging, is as follows. Any $\begin{pmatrix} -i\tilde{\lambda} & 0 \\ 0 & i\tilde{\sigma} \end{pmatrix}$ can be so obtained from $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$, where $\tilde{\lambda}$ and $\tilde{\sigma}$ are related to λ and σ as follows. Take i,j such that $1 \le i \le p$, $1 \le j \le q$, and $c > 0$. Then $\tilde{\lambda}_\ell = \lambda_\ell$, $\tilde{\sigma}_k = \sigma_k$ for $\ell \ne i$ and $k \ne j$, and $\tilde{\lambda}_i = \lambda_i + c$, $\tilde{\sigma}_j = \sigma_j + c$. (We emphasize this only depends on $\lambda_i + \sigma_j > 0$.) It is now clear how any $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$, $\lambda_i, \sigma_j > 0$, can be obtained from Z , so $C_0 \subseteq C$.

That $\overline{C_0}$ and $\overline{C}_1$ have the asserted forms follows exactly in the symplectic case.

$C_0 \ne C_1$ follows from the exclusion of the case $p = q = 1$ earlier. The situation is clarified by the following

<u>Lemma 12.2</u>. If $X = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix} \in \underline{h}$, $X \in C_1$ iff $\forall\, i,j$ $\lambda_i + \sigma_j > 0$. This is clearly equivalent to: $\exists c \in \mathbb{R}$

such that $\lambda_i > c > -\sigma_j \ \forall\ i,j$. Thus either all $\lambda_i > 0$ or all $\sigma_j > 0$, but not necessarily both. $X \in C_0$ if $c = 0$ may be taken.

Proof. Let $y = \binom{u}{v} \in C^n$ satisfy $y \neq 0$ and $\mathcal{H}(y,y) = 0$. Then

$$iH(Xy,y) = \sum_{i=1}^{p} \lambda_i |u_i|^2 + \sum_{j=1}^{q} \sigma_j |v_j|^2$$
$$> c\left(\sum_{i=1}^{p} |u_i|^2 - \sum_{j=1}^{q} |v_j|^2 \right) = 0 ,$$

the inequality being strict as $y \neq 0$.

Thus $C_0 \neq C_1$ as, for example, in $su(2,1)$ $\begin{pmatrix} -3i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{pmatrix} \in C_1 - C_0$, and other examples show $\overline{C}_0 \neq \overline{C}_1$. $\overline{C}_0, \overline{C}_1 \in \mathcal{C}$ is clear from Lemma 1.1.

To show C_1 is open we need the fact, proven by classification of the orbits in the next section, that each $X \in C_1$ is conjugate to a $Y = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix} \in \underline{h}$ where all $\lambda_i + \sigma_j > 0$. Then $\exists \epsilon > 0$ such that $Y - \epsilon Z \in C_1$ also, so $Y \in (Y - \epsilon Z) + C_0 \subseteq C_1$, so Y , hence X , is in the interior of C_1 .

The last assertion about open cones follows just as in the symplectic case from the maximality of $\overline{C}_1$, which we now prove. Let $C \in \mathcal{C}$ be closed and contain C_0 . Clearly then $-C_0 \subset C^*$ so $-\overline{C}_0 \subseteq C^*$, hence $C \subseteq -\overline{C}_0^* = C_1$ by Theorem 12.3 below and Theorem 1.4.

Theorem 12.3. $\overline{C}_0^* = -\overline{C}_1$.

Proof. $\overline{C}_0^* \supseteq -\overline{C}_1$ follows from the minimality of C_0 : $\overline{C}_1^* \in \mathcal{C}$ and $-C_0 \subset \overline{C}_1^*$ so $-\overline{C}_0 \subseteq \overline{C}_1^*$, hence $-\overline{C}_1 \subseteq \overline{C}_0^*$ by duality.

If now $X \not\subseteq -\overline{C}_1$ $\exists$ $v \neq 0$ such that $\mathcal{H}(v,v) = 0$ and $iH(Xv,v) > 0$. By Theorem 10.1 $B_\theta(X,Y) > 0$ for some $Y \in \overline{C}_0$, so $X \not\subseteq \overline{C}_0^*$ also.

Corollary 12.4. C_1 (resp. C_0) is the topological interior of $\overline{C}_1$ (resp. $\overline{C}_0$) .

Proof. The assertion for C_1 follows as before (Corollary 8.3) from the maximality of C_1 . We cannot use this argument for $(\overline{C}_0)^\circ$, however. But $(\overline{C}_0)^\circ \subset C_1$, and if $X \in (\overline{C}_0)^\circ \subseteq \overline{C}_0$, X is conjugate to a $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\sigma \end{pmatrix}$ such that all $\lambda_i, \sigma_j \geq 0$, again by the fact we prove in the next section. But in fact all λ_i, σ_j must be positive as $(\overline{C}_0)^\circ$ is open and in $\overline{C}_0$.

Remark. One can see already that $\overline{C}_0$ and C_1 intersect nontrivially by examining particular diagonal matrices.

13. Orbits in $\overline{C}_1$ of $su(p,q)$.

It is clear from our description of $\overline{C}_1$ in the last section that, in examining which types Δ can lie in $\overline{C}_1$, we obtain only necessary conditions on Δ by considering the possible indecomposable types in $\overline{C}_1$. Also, we must examine all the orbits as it is not yet clear that all elements of C_1 are elliptic. Still, the necessary conditions are quite restrictive on the possible indecomposable types.

Theorem 13.1. Among the indecomposable types for $u(p,q)$, only $\Delta_0^{\pm}(\zeta)$ and $\Delta_1^{\pm}(\zeta)$ $(\zeta=-\bar{\zeta})$ are in $\pm\,\overline{C}_1$. (We define $\overline{C}_1$, C_0, etc. for $u(p,q)$ the same as for $su(p,q)$.)

Proof. Consider first a type $\Delta_m(\zeta,-\bar{\zeta})$ for $\zeta \neq -\bar{\zeta}$, containing $(A = S+N, V)$ where $V = E + NE + \ldots + N^m E$. E is two-dimensional, and there is a basis $\{e,f\}$ for E such that $H(f,N^m f) = H(e,N^m e) = 0$ and $H(e,N^m f) = 1$. Let $a,b \in \mathbb{C}$. If $m \geq 1$, then only the cross-terms in the following expression contribute:

$$H(A(aN^{m-1}f + be), aN^{m-1}f + be) \quad = \quad -2i\,\mathrm{Im}\, b\bar{a}$$

and this is of indeterminant sign.

If $m = 0$, let $Ae = \zeta e$, $Af = -\bar{\zeta} f$, $H(e,e) = H(f,f) = 0$, and $H(e,f) = 1$. Then $H(ae+bf,ae+bf) = 2\,\mathrm{Re}\, a\bar{b}$, so let $a\bar{b} = ci$, $c \in \mathbb{R}$. Then $H(A(ae+bf),ae+bf) = 2ci\,\mathrm{Re}\,\zeta$, $\mathrm{Re}\,\zeta \neq 0$, and $c > 0$ or $c < 0$ are possible.

Finally, consider $\Delta_m^{\epsilon}(\zeta)$ for $\zeta = -\bar{\zeta}$, V as above. E is one-dimensional, $A = S+N$, and S has eigenvalue ζ. The case $m \geq 3$ is quickly ruled out just as in the symplectic case. Next let $m = 2$. If $u,v,w \in E$ set $y = N^2 v + Nw + u$, and note $H(y,y) = 2\,\mathrm{Re}\, H(u,N^2 v) - H(w,N^2 w)$

and $H(Ay,y) = \zeta(2\,\mathrm{Re}\,H(u,N^2v) - H(w,N^2w)) + 2i\,\mathrm{Im}\,H(N^2w,u)$. Take u,v,w all nonzero such that $H(y,y) = 0$. As the phase of w is rotated this is maintained but $iH(Ay,y)$ takes both positive and negative values.

The remaining cases are indeed in $\pm\overline{C}_1$, and lie in $u(1,1)$ or $u(1)$.

To aid in understanding these types $\Delta^{\pm}_{0,1}(\zeta)$, we record a convenient isomorphism between $sp(1,\mathbb{R})$ and $su(1,1)$. It is given by conjugating by a particular 2x2 unitary, coming from a "Cayley transform". In fact, we lose nothing in letting our 2x2 matrices have square matrix entries.

Let A , B , and C be real nxn matrices, B and C symmetric, and let $D = A + \lambda iI$, $\lambda \in \mathbb{R}$. Then

$$\begin{pmatrix} D & B \\ C & -D^* \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} + \lambda i \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \equiv \mathcal{O} .$$

Let $U = (1/\sqrt{2})\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$, so $U^{-1} = U^* = (1/\sqrt{2})\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$.
Then

$$U\mathcal{O}U^{-1} = \begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} A-A^t+i(B-C) & -A-A^t+i(B+C) \\ -A-A^t-i(B+C) & A-A^t+i(C-B) \end{pmatrix}$$

which is in $u(n,n)$. Letting $\lambda = 0$, note this embeds $sp(n,\mathbb{R})$ in $su(n,n)$.

If $n = 1$ then $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} \in sl(2,\mathbb{R}) + \mathbb{R}iI$ corresponds, under a Lie algebra isomorphism, to

$$\begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i(b-c) & -2a+i(b+c) \\ -2a-i(b+c) & -i(b-c) \end{pmatrix} = \begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} + \begin{pmatrix} -di & \beta \\ \bar{\beta} & di \end{pmatrix} .$$

Recall $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ represents a positive elliptic type iff $b < 0$, $c > 0$, and $-a^2 - bc > 0$, and a positive nilpotent type iff $b \le 0$, $c \ge 0$, $c-b > 0$, and $-a^2 - bc = 0$. These conditions correspond for $\begin{pmatrix} -di & \beta \\ \bar{\beta} & di \end{pmatrix} \in su(1,1)$ to $d > 0$, $d > |\beta|$, and $d > 0$, $d = |\beta|$. Note $\begin{pmatrix} -di & \beta \\ \bar{\beta} & di \end{pmatrix}$ is conjugate under $SU(1,1)$ to $\begin{pmatrix} -d'i & 0 \\ 0 & d'i \end{pmatrix}$ iff $|d| > |\beta|$.

We relate the types $\Delta^{\pm}_{0,1}(i\lambda)$ $(\lambda \in \mathbb{R})$ to particular matrix representatives. Recall the notation $\Delta^{\pm}(\zeta)$ means E is spanned by e with $H(e,e) = \pm 1$.

<u>In</u> C_0 . $\Delta^{+}(i\lambda)$ for $\lambda < 0$

and $\Delta^{-}(i\lambda)$ for $\lambda > 0$.

Both correspond to the 1×1 $(i\lambda)$.

<u>In</u> $\bar{C}_0 - C_0$ <u>and</u> C_1 <u>(vacuously)</u>. $\Delta\ (0)$; (0) .

<u>In</u> $\bar{C}_0 - C_1$. $\Delta^{+}_{0}(0)$; corresponds to, say $\begin{pmatrix} -i & i \\ -i & i \end{pmatrix}$.

<u>In</u> $\bar{C}_1 - (C_1 \cup \bar{C}_0)$. $\Delta^{+}_{1}(i\lambda)$ for $\lambda \neq 0$ corresponds to

$$\begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} + \begin{pmatrix} -i & i \\ -i & i \end{pmatrix} .$$

The above facts follows from the lemma below, proved by computation.

Lemma 13.2. Let $X = \begin{pmatrix} \lambda i & 0 \\ 0 & \lambda i \end{pmatrix} + \begin{pmatrix} -di & \beta \\ \bar{\beta} & di \end{pmatrix} \in u(1,1)$. Then $X \in C_0$ iff $d > 0$ and $d^2 > \lambda^2 + |\beta|^2$ (a consequence is $d-\lambda, d+\lambda > 0$), $X \in \overline{C}_0$ iff $d \geq 0$ and $d^2 \geq \lambda^2 + |\beta|^2$, $X \in C_1$ iff $d > 0$ and $d > |\beta|$, and $X \in \overline{C}_1$ iff $d \geq 0$ and $d \geq |\beta|$.

Finally, we list the types in $\overline{C}_1$ for $su(p,q)$.

Proposition 13.3. The types Δ in C_1 are of the form

$$(7) \qquad \Delta = \Delta^+(-i\lambda_1) + \ldots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_1) + \ldots + \Delta^-(i\sigma_q)$$

where all $\lambda_i + \sigma_j > 0$ and $\Sigma_i \lambda_i = \Sigma_j \sigma_j$. Δ is furthermore in C_0 iff all $\lambda_i, \sigma_j > 0$.

The types Δ in $\overline{C}_0$ are of either of two forms:

1) (7) with all $\lambda_i, \sigma_j \geq 0$ (and such a Δ may also be in C_1), or

2)

$$(8) \quad \Delta = \Delta^+(-i\lambda_1) + \ldots + \Delta^+(-i\lambda_{p-\ell}) + \underbrace{\Delta_1^+(-i\lambda) + \ldots + \Delta_1^+(-i\lambda)}_{\ell \text{ times}} + \Delta^-(i\sigma_1) + \ldots + \Delta^-(i\sigma_{q-\ell})$$

for $\lambda = 0$, some ℓ with $1 \leq \ell \leq q$, all $\lambda_i, \sigma_j \geq 0$ and $\Sigma_i \lambda_i = \Sigma_j \sigma_j$. Such a type is on the boundaries of $\overline{C}_0$ and $\overline{C}_1$.

The remaining types in $\overline{C}_1$ are those either of the form (7) with all $\lambda_i + \sigma_j \geq 0$, $\Sigma_i \lambda_i = \Sigma_j \sigma_j$, and some $\lambda_i = -\sigma_j \neq 0$, or (8) for $\lambda \neq 0$, some ℓ with $q \geq \ell \geq 1$,

λ_i and σ_j satisfying $\lambda_i \geq \lambda \geq -\sigma_j \ \forall i,j$, and

$$2\ell\lambda + \sum_{i=1}^{p-\ell} \lambda_i = \sum_{j=1}^{q-\ell} \sigma_j .$$

Proof. It is clear from the classification of indecomposable types in $\overline{C}_1$ and the computation in the proof of Lemma 12.2.

14. Classification of Open and Closed Cones in $su(p,q)$.

To classify some of the invariant convex cones in $\mathfrak{g}$ we need a convexity result (Theorem 14.2), the proof of which depends on the

Lemma 14.1. All orbits in C_1 are closed.

Proof. Let $Y \in C_1$, and suppose

$$X = \begin{pmatrix} -i\lambda & \beta \\ \beta^* & i\sigma \end{pmatrix}$$

is in the K-orbit of Y , for Hermitian λ,σ . We need some measure of the finite distance of the K-orbit of Y from the boundary of C_1 : clearly $\exists k > 0$ such that $iH(Yv,v) \geq k\|v\|^2 \ \forall \ v \in \mathbb{C}^n$ with $H(v,v) = 0$, $\|\cdot\|$ the usual (K-invariant) norm on $\mathbb{C}^n$. The same holds for X , and a simple computation shows that a consequence is

$$(\lambda_{ii}+\sigma_{jj})/2 - |\beta_{ij}| \geq k$$

for i,j satisfying $1 \leq i \leq p, 1 \leq j \leq q$.

Now let $Ad(g_m)Y \to Q \in \mathfrak{g}$ as $m \to \infty$. We can write

g_m according to $G = K\overline{A}^+K$. Let

$$a = \begin{pmatrix} \cosh\theta_j & \sinh\theta_j \\ \sinh\theta_j & \cosh\theta_j \end{pmatrix} \in \overline{A^+} , \quad \theta_j \geq 0 .$$

Letting $Ad(a)X = \begin{pmatrix} L & * \\ * & M \end{pmatrix}$, then for $1 \leq j \leq q$,

$$L_{jj} = -i\lambda_{jj}\cosh^2\theta_j - i\sigma_{jj}\sinh^2\theta_j - 2i\sinh\theta_j\cosh\theta_j \operatorname{Im}\beta_{jj}$$

and

$$M_{jj} = i\sigma_{jj}\cosh^2\theta_j + i\lambda_{jj}\sinh^2\theta_j + 2i\sinh\theta_j\cosh\theta_j \operatorname{Im}\beta_{jj} .$$

Also,

$$\begin{aligned} iL_{jj} - iM_{jj} &\geq \cosh 2\theta_j(\lambda_{jj} + \sigma_{jj}) - 2|\beta_{jj}|\sinh 2\theta_j \\ &\geq (\sinh 2\theta_j)2k , \end{aligned}$$

and note $|x|^2 + |y|^2 \geq \frac{1}{2}|x \pm y|^2 \ \forall \ x,y \in C$.

Now return to the $Y_m = Ad(g_m)Y$. The B_θ-norms of the Y_m remain bounded by some fixed number as $m \to \infty$, and B_θ is K-invariant. The above estimate shows that the allowable $\overline{A}^+$ components in g_m must lie in some compact set. As K is compact the g_m have some convergent subsequence, proving the lemma.

We recall a result of Horn [11]: given an Hermitian matrix A with eigenvalues $\lambda_1,\ldots,\lambda_n$, the set of diagonal entries of all conjugates UAU^{-1} , U unitary, is precisely the convex hull of the points $(\lambda_{p(1)},\ldots,\lambda_{p(n)})$, p a permutation of $(1,\ldots,n)$. We will only need the (presumably) easier half of this theorem, where "precisely" is replaced by "contained in".

Definitions. 1) Let $V,W = (\lambda_1,\dots,\lambda_p,\sigma_1,\dots,\sigma_q) \in \mathbb{R}^{p+q}$. say V is in the noncompact convex hull of W if V is equal to a vector in the convex hull of the set

$$\{(\lambda_{r(1)},\dots,\lambda_{r(p)},\sigma_{s(1)},\dots,\sigma_{s(q)}) : r,s \text{ permutations of } (1,\dots,p),(1,\dots q) \text{, respectively}\}$$

plus a vector in the closed positive orthant of $\mathbb{R}^{p+q}$. Clearly this relation is transitive, and the noncompact convex hull of a point is closed.

2) If $X = \begin{pmatrix} * & \beta \\ \beta^* & * \end{pmatrix} \in \mathfrak{P}$, call $R = \sqrt{\sum_{i,j} |\beta_{ij}|^2}$ the $\underline{p}$-norm of X (although it is not a norm on $\mathfrak{P}$).

Theorem 14.2*. Let $X \in C_1$, and let X represent the type

$$\Delta = \Delta^+(-i\lambda_1) + \dots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_1) + \dots + \Delta^-(i\sigma_q)$$

as in Proposition 13.3. Let the diagonal elements of X be $(-ia_1,\dots,-ia_p,ib_1,\dots,ib_q)$. Then $(a_1,\dots,a_p,b_1,\dots,b_q)$ is in the noncompact convex hull of $(\lambda_1,\dots,\lambda_p,\sigma_1,\dots,\sigma_q)$.

The proof is dependent on the following

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*I thank Reneé Chipman for an accidental wish for a "good flow" which helped to get this theorem and its lemma unstuck.

<u>Lemma 14.3</u>. Let $X \in C_1 - \underline{k}$. Then there exists Y in the orbit of X with strictly smaller $\underline{p}$-norm such that the diagonal of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} X$ is in the non-compact convex hull of the diagonal of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Y$.

<u>Proof</u>. By Horn's theorem we may assume

$$X = \begin{pmatrix} -i\lambda & \beta \\ \beta^* & i\sigma \end{pmatrix}$$

where λ and σ are diagonal. (The $\underline{p}$-norm is K-invariant.) As in the proof of Lemma 14.1, $\lambda_i + \sigma_j > 2|\beta_{ij}|\ \forall\, i,j$. Set $\gamma_{ij} = i\beta_{ij}/(\lambda_i+\sigma_j)$, and consider

$$\mathrm{Ad}(\exp\ \varepsilon P)X = Y(\varepsilon) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & \gamma \\ \gamma^* & 0 \end{pmatrix} \in \mathcal{P}$$

We have

$$A = -i\lambda + \varepsilon(\gamma\beta^* - \beta\gamma^*) + O(\varepsilon^2) \quad ,$$

$$C = i\sigma + \varepsilon(\gamma^*\beta - \beta^*\gamma) + O(\varepsilon^2) \quad , \text{ and}$$

$$B = \beta + i\varepsilon(\gamma\sigma + \lambda\gamma) + O(\varepsilon^2) \quad .$$

The diagonal entries of A and C are

$$-i\lambda_i + 2i\varepsilon \sum_{j=1}^{q} (|\beta_{ij}|^2/(\lambda_i+\sigma_j)) + O(\varepsilon^2)$$

and $$i\sigma_j - 2i\varepsilon \sum_{i=1}^{p} (|\beta_{ij}|^2/(\lambda_i+\sigma_j)) + O(\varepsilon^2) \quad ,$$

and we have $B_{ij} = (1-\epsilon)\beta_{ij} + O(\epsilon^2)$.

Note that if $\sum_{j=1}^{q} |\beta_{ij}|^2 = 0$ for some i , then the ith diagonal element of A is equal to $-i\lambda_i$ for all ϵ , and likewise for the jth diagonal element of C if $\sum_{i=1}^{p} |\beta_{ij}|^2 = 0$. Thus if $\epsilon > 0$ is small enough $Y(\epsilon)$ satisfies the requirements of the lemma.

Proof of Theorem 14.2. It is not clear that the lemma above can be strengthened to produce a sequence converging to an element of $\underline{k}$, but Zorn's lemma can be used.

First define some notation. If $Y,Z \in \mathfrak{g}$ write $Y \geq Z$ if the $\underline{p}$-norm of Y is $\leq$ the $\underline{p}$-norm of Z , and the diagonal of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Z$ is in the noncompact convex hull of the diagonal of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Y$. Let $\Omega = \{Y \in Ad(G)X : Y \geq X\}$. We apply Zorn's lemma to $(\Omega, \geq)$. By Lemma 14.3 it is clear that a $\geq$-maximal element Y of Ω must be in $\underline{k}$, and if such an element exists the K-diagonalization of Y exhibits the type of X , proving the theorem.

To prove the hypothesis of Zorn's lemma, first note that Ω is a bounded subset of $\mathfrak{g}$, as the Killing form is constant on the orbit of X . Let $\Phi_{\lambda \in \Gamma}$ be a simply ordered subset of Ω , considered as a net indexed by itself. The closure of Φ is compact, so Φ has a subnet $\tilde{\Phi}$ converging to Y . Y is in the orbit of X by

Lemma 14.1, and clearly the $\underline{p}$-norm of Y is $\leq$ the $\underline{p}$-norm of any $Z \in \tilde{\Phi}$, hence any $Z \in \Phi$ by the cofinality of $\tilde{\Phi}$.

Finally, if $Z \in \tilde{\Phi}$ express the diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Z$ as a vector in the noncompact convex hull of the diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} W$, where W is any element of $\tilde{\Phi}$ where $W \geq Z$. The set of possible nonnegative vectors in these expressions is bounded as Ω is bounded, so the expressions must have a convergent subnet. The limit expresses the diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Z$ as a vector in the noncompact convex hull of the diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Y$. Thus $Y \in \Omega$ and $Y \geq Z \ \forall\, Z \in \Phi$, proving the hypothesis of Zorn's lemma.

It is now not difficult to classify the open and closed invariant convex cones in $\mathfrak{g}$. We need some notation.

Let

$$E_1 = \{(\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_q) \in \mathbb{R}^{p+q} \colon \ \Sigma_i \lambda_i = \Sigma_j \sigma_j \ \text{ and } \ \lambda_i + \sigma_j \geq 0 \ \forall\, i, j\}$$

and $E_0 = \{e \in E_1 \colon$ all components of $e \geq 0\}$. Let $\mathcal{P}$ be the group of permutations of the λ's and permutations of the σ's acting on E_1. Let $\mathcal{C}_+ = \{C \in \mathcal{C} \colon C_0 \subseteq C \subseteq \overline{C}_1\}$, $\mathcal{C}_+^o = \{C \in \mathcal{C}_+ \colon C$ is open$\}$, $\mathcal{C}_+^c = \{C \in \mathcal{C}_+ \colon C$ is closed$\}$,

$\mathcal{S}^o = \{$all $\mathcal{P}$-invariant open convex cones in E_1 containing $(E_0)^o\}$, and let

$\mathcal{S}^c = \{$all $\mathcal{P}$-invariant closed convex cones in E_1 containing $E_0\}$.

Let Φ be the collection of all bases $\{(e_i, f_j): i = 1,\ldots,p,\ j = 1,\ldots,q\}$ of $\mathbb{C}^n$ such that $H(e_i, e_\ell) = \delta_{i\ell}$, $H(f_j, f_k) = -\delta_{jk}$ and $H(e_i, f_j) = 0\ \forall\ i,j$. Given $\varphi = (e_i, f_j) \in \Phi$ define

$$M_\varphi: \mathfrak{g} \to \mathbb{R}^{p+q}:$$
$$X \to (iH(Xe_1,e_1),\ldots,iH(Xe_p,e_p),iH(Xf_1,f_1),\ldots,iH(Xf_q,f_q)) .$$

As noted previously, $M_\varphi(X) \in E_1$ if $X \in \overline{C}_1$, and $M_\varphi(X) \in (E_1)^o$ if $X \in C_1$.

<u>Theorem 14.4</u>. Let $C_3 \in \mathcal{C}_+^o$ and $C_4 \in \mathcal{C}_+^c$. Let E_{C_i} be the convex cone generated by $\{M_\varphi(X): X \in C_i,\ \varphi \in \Phi\}$ for $i = 3,4$. Then $E_{C_3} \in \mathcal{E}^o$, $E_{C_4} \in \mathcal{E}^c$, and these mappings

$$E: \mathcal{C}_+^o \to \mathcal{E}^o$$
$$E: \mathcal{C}_+^c \to \mathcal{E}^c$$

are bijections.

<u>Proof</u>. We deal with the open cones first. To see that E_{C_3} is open we show that E_{C_3} contains a neighborhood of each $M_\varphi(X)$, $X \in C_3$, $\varphi \in \Phi$. By Theorem 14.2 $M_\varphi(X)$ is a nonnegative vector plus a convex combination of other vectors

$$S = \{(\lambda_{r(1)},\ldots,\lambda_{r(p)},\sigma_{s(1)},\ldots,\sigma_{s(q)})\}$$

in E_{C_3} coming from the type of X. E_{C_3} must contain a neighborhood of each element of S as the intersection of C_3 with $\underline{h}$ must be open in $\underline{h}$. E_{C_3} also contains all positive vectors as $C_0 \subseteq C_3$. A simple picture now shows $M_\varphi(X)$ to be in the interior of E_{C_3}. Thus E_{C_3} is open and clearly then $E_{C_3} \in \mathcal{S}^o$.

To show surjectivity, let $D \in \mathcal{S}^o$, and let $C_3 = \{X \in C_1 : M_\varphi(X) \in D \ \forall \varphi \in \Phi\}$. Clearly C_3 is invariant, convex, and contains C_0. To show C_3 is open, we may take $X \in \underline{h} \cap C_3$. $\exists \epsilon > 0$ such that the diagonal of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(X - \epsilon Z)$ is in D. By Theorem 14.2 $M_\varphi(X - \epsilon Z) \in D$ $\forall \varphi \in \Phi$, so $X - \epsilon Z \in C_3$, and $X \in (X - \epsilon Z) + C_0 \subseteq C_3$ proves $X \in (C_3)^o$. Thus $C_3 \in \mathcal{C}_+^o$. That $E_{C_3} = D$ follows from Theorem 14.2.

If now $C_2 \in \mathcal{C}_+^o$ and $E_{C_2} = D$ then clearly $C_2 \subseteq C_3$. By Theorem 14.3 E_C is simply the set of diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(C \cap \underline{h})$ for any open cone C, so C_2 and C_3 clearly contain the same elliptic orbits and are equal. Thus E is a bijection for the open cones.

Now let $C_4 \in \mathcal{C}_+^c$. The set S of diagonals of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(C_4 \cap \underline{h})$ is closed and in E_{C_4}; we will see that it is all of E_{C_4}. If $M_\varphi(X) \not\subseteq S$ for some $X \in C_4$, $\varphi \in \Phi$, the same is true for $X + \epsilon Z \in C_4 \cap C_1$ for some $\epsilon > 0$, and Theorem 14.2 may be applied to reach a contradiction. Thus $E_{C_4} \in \mathcal{S}^c$.

To show surjectivity, take $D \in \mathcal{S}^c$ and let $C_4 = \{X \in \overline{C}_1 : M_\varphi(X) \in D \ \forall \varphi \in \Phi\}$. Then $C_4 \in \mathcal{C}_+^c$ is automatic, and clearly $E_{C_4} \subseteq D$. Further, equality holds again by an application of 14.2 to $X + \epsilon Z$.

Finally, suppose $E_{C_2} = D$ for $C_2 \in \mathcal{C}_+^c$. $C_2 \subseteq C_4$ again, and $E_{C_2} = E_{C_4}$ implies $C_2 \cap \underline{h} = C_4 \cap \underline{h}$, so C_2 and C_4 have the same elliptic orbits. Suppose then

$$\Delta = \Delta_1^+(-i\lambda) + \dots + \Delta_1^+(-i\lambda) + \Delta_e$$

where

$$\Delta_e = \Delta^+(-i\lambda_{\ell+1}) + \dots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_{\ell+1}) + \dots + \Delta^-(i\sigma_q)$$

as in Proposition 13.3, is a nonelliptic type in C_4. Then

$$\tilde{\Delta} = \Delta^+(-i\lambda) + \dots + \Delta^-(-i\lambda) + \Delta_e$$

is an elliptic type in C_4 as C_4 is closed, so $\tilde{\Delta}$ is in C_2, too. But then Δ is obtained back from $\tilde{\Delta}$ by adding on $\Delta_1^+(0) + \Delta^+(0) + \dots + \Delta^-(0)$ (which is in $\overline{C}_0 \subseteq C_2$) the appropriate number of times. Thus $C_2 = C_4$, and the proof is complete.

The classification of <u>all</u> invariant convex cones in $\mathfrak{g}$ does not seem hopeless, and probably involves ideas similiar to those successful for $sp(n,\mathbb{R})$. Just as for those groups, the set of elliptic orbits in the closed maximal cone do not form a convex cone: we saw that a convex combination of $Sp(2,\mathbb{R})$ transforms of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. There is a similar symmetry here, existing in any $SU(2,2)$ subgroup of $SU(p,q)$. If $(e_1,e_2,f_1,f_2) \in \Phi$, then one may check that

$$E_1 = e_1 - k(e_2 - f_2) \qquad E_2 = e_2 + k(e_1 + f_1)$$
$$F_1 = f_1 + k(e_2 - f_2) \qquad F_2 = f_2 + k(e_1 + f_1)$$

for k real, is also orthonormal in $(\mathbb{C}_4, H)$. It comes from the nilpotent

$$\begin{pmatrix} 0 & k & 0 & k \\ -k & 0 & k & 0 \\ 0 & k & 0 & k \\ k & 0 & -k & 0 \end{pmatrix}$$

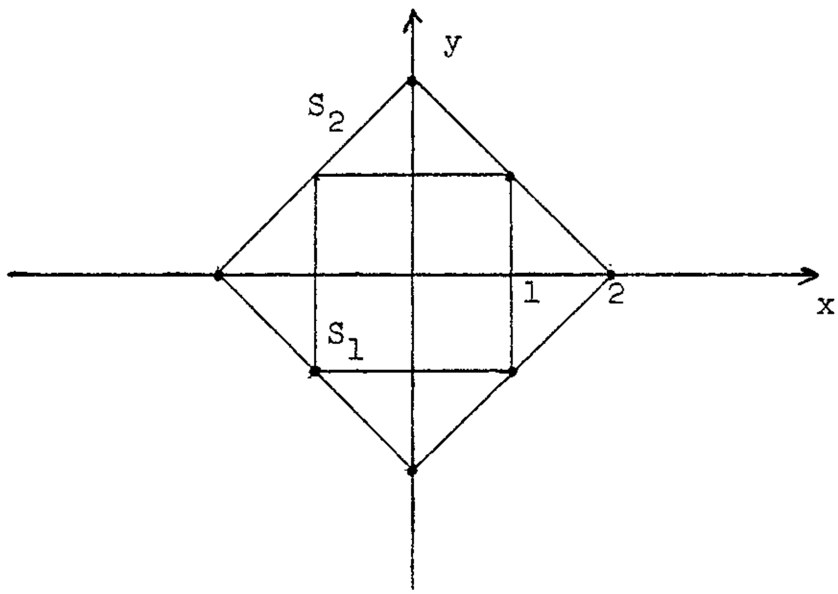
in $su(2,2)$, which has square zero.

Furthermore, one may calculate that averaging the transforms of the elliptic type $\Delta^+(-i\lambda) + \Delta^+(-i\lambda) + \Delta^-(-i\lambda) + \Delta^-(id)$ $(\lambda + d > 0)$ for k positive and negative gives $\Delta_1^+(-i\lambda) + \Delta^+(-i\lambda) + \Delta^-(id)$. There may be other $SU(p,q)$ symmetries involved in classifying the invariant cones in $su(p,q)$, but based on the analogy with $sp(n,\mathbb{R})$ I suspect not.

Finally, there is a simple geometric picture that one can use to see how the various closed and open invariant cones in $su(2,2)$ overlap. We consider the slice $\lambda_1 + \lambda_2 = \sigma_1 + \sigma_2 = 1$ in $E_1 = \{(\lambda_1, \lambda_2, \sigma_1, \sigma_2): \lambda_i + \sigma_j \geq 0, \lambda_1 + \lambda_2 = \sigma_1 + \sigma_2\}$, perpendicular to $(1,1,1,1)$, regarded as a z-axis.

Define $x = \lambda_1 - \lambda_2$ and $y = \sigma_1 - \sigma_2$. As closed convex cones, $E_{\bar{C}_0}$ is generated by $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, and $(0,1,0,1)$, and $E_{\bar{C}_1}$ is generated by

$\frac{1}{2}(3,-1,1,1)$, $\frac{1}{2}(-1,3,1,1)$, $\frac{1}{2}(1,1,3,-1)$, and $\frac{1}{2}(1,1,-1,3)$. (In general, $\overline{C}_0$ has pq generators and $\overline{C}_1$ has $p+q$ generators.) They correspond in the (x,y) plane to $\{(x,y)\colon x = \pm 1,\ y = \pm 1\}$ and $\{(0,\pm 2),(\pm 2,0)\}$, respectively. We have



The minimal cone corresponds to the inner square S_1 , the maximal to the outer S_2 . Closed invariant convex cones correspond to closed convex sets containing S_1 and contained in S_2 which are invariant under reflections about the x and y axes.

The outer automorphisms of $su(2,2)$ are generated by the inner as well as "time reversal" $X \to \overline{X}$ and "space reversal" $X \to \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \overline{X} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The former takes C_0 to $-C_0$, and the latter has the effect in the diagram of a reflection about $x = y$, or, up to inner automorphisms, a $90°$ rotation. Thus there are clearly lots of invariant convex cones also invariant under that second outer automorphism.

CHAPTER IV. Invariant Convex Cones in so*(2n) .

15. Conventions for so*(2n) .

We take $n \geq 3$, and let $\ell = [n/2]$, the greatest integer $\leq n/2$. As in section 3,

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} : \begin{matrix} A,B \text{ complex } n\times n \text{ matrices;} \\ A \text{ skew, } B \text{ Hermitian} \end{matrix} \right\}$$

$$\underline{k} = \left\{ X \in \mathfrak{g} : X \text{ real} \right\} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{g} : \; A \text{ skew, } B \text{ symmetric} \right\}$$

$$\underline{p} = \left\{ X \in \mathfrak{g} : iX \text{ real} \right\} = \left\{ i\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{g} : \; A,B \text{ skew} \right\}$$

$$\underline{h} = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in \underline{k} : \; D \text{ diagonal} \right\}$$

$$c = \mathbb{R}Z \quad \text{where} \quad Z = \begin{pmatrix} 0 & -\tfrac{1}{2}I \\ \tfrac{1}{2}I & 0 \end{pmatrix} .$$

Let d_j $(j = 1,\ldots,n)$ be linear functionals spanning the dual of $\underline{h}$:

$$d_j : \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \to D_{jj} .$$

Then $\{\text{noncompact roots}\} = \{\pm i(d_j + d_k) : 1 \leq j < k \leq n\}$

$\{\text{compact roots}\} = \{\pm i(d_j - d_k) : 1 \leq j < k \leq n\}$

and the positive roots we take to be the above with "+" only.

$$\Delta_0^+ = \{i(d_1 + d_2), i(d_1 - d_2), \ldots, i(d_{n-1} - d_n)\} .$$

Let $1 \le j < k \le n$. For $\alpha = i(d_j + d_k)$,
$H_\alpha = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, $X_\alpha = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$, and $Y_\alpha = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$, where
where $D = -iE_{jj} - iE_{kk}$, $A = -iE_{jk} + iE_{kj}$, and
$B = -iE_{jk} + iE_{kj}$.

We take $\Sigma_o = \{i(d_1 + d_2), i(d_3 + d_4), \ldots, i(d_{2\ell-1} + d_{2\ell})\}$,
so

$$\underline{h}^- = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in \underline{h}: \begin{array}{l} D_{11} = D_{22}, \ldots, D_{2\ell-1,2\ell-1} = d_{2\ell,2\ell}, \\ D_{2\ell+1} = 0 \quad \text{(if it exists)} \end{array} \right\}$$

$$\underline{h}^+ = \left\{ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in \underline{h}: D_{11} = -D_{22}, \ldots, D_{2\ell-1,2\ell-1} = -D_{2\ell,2\ell} \right\}$$

$$Z_o = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in \underline{h} \quad \begin{array}{l} \text{where all } D_{jj} = -\tfrac{1}{2} \text{ if } n \text{ even;} \\ D_{jj} = -\tfrac{1}{2} \text{ for } j \le 2\ell,\ D_{nn} = 0, \\ \text{if } n \text{ odd,} \end{array}$$

so

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \in \mathfrak{g}: A = \begin{pmatrix} \begin{pmatrix} 0 & -c_1 i \\ c_1 i & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & -c_\ell i \\ c_\ell i & 0 \end{pmatrix} \end{pmatrix}, c_j \in \mathbb{R} \right\}.$$

Thus $Z = Z_o$ iff n even, and $\underline{h}^+ \neq 0$ always.

The cone in $\underline{h}$ obtained from Corollary 3.2 is all $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ in $\underline{h}$ where $\operatorname{diag} D = (-d_1, -d_1, \ldots, -d_\ell, -d_\ell)$ for $d_j > 0$ if n is even, and $\operatorname{diag} D = (-d_1, -d_1, \ldots, -d_\ell, -d_\ell, -e)$ where $0 < e \le d_j$ if n is odd.

Define $B(X,Y) = \frac{1}{2}\operatorname{Tr}(XY)$ for $X,Y \in \mathfrak{g}$. If

$$X = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \quad Y = \begin{pmatrix} C & D \\ -\overline{D} & \overline{C} \end{pmatrix} \in \mathfrak{g},$$

$$B(X,Y) = -\operatorname{Re} \sum_{i,j} A_{ij}C_{ij} - \operatorname{Re} \sum_{i,j} B_{ij}D_{ij} .$$

The Cartan involution is $X \to \overline{X}$, and

$$B_\theta\left(\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \begin{pmatrix} C & D \\ -\overline{D} & \overline{C} \end{pmatrix}\right) = \operatorname{Re}\left(\sum_{i,j} A_{ij}\overline{C_{ij}}\right) + \operatorname{Re}\left(\sum_{i,j} B_{ij}\overline{D_{ij}}\right) .$$

Define the complex symmetric form $\tau(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ by

$$\tau(x,y) = \sum_{j=1}^{2n} x_i y_i ,$$

and the symplectic form $\mathcal{A}(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ by

$$\mathcal{A}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \operatorname{Re}\left(\sum_{j=1}^{n} (y_j\bar{u}_j - x_j\bar{v}_j)\right)$$

for $x,y,u,v \in \mathbb{C}^n$. In fact $\mathfrak{g}$ is precisely the complex-linear endomorphisms of $\mathbb{C}^{2n}$ skew with respect to τ and $\mathcal{A}$. Also, $\mathcal{A}(\cdot,\cdot) = \operatorname{Re}\tau(\cdot,\sigma\cdot)$, where $\sigma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \circ \mathcal{C}$ where $\mathcal{C}$ is complex conjugation on $\mathbb{C}^{2n}$. All elements of $\mathfrak{g}$ commute with the anti-linear σ.

<u>Theorem 15.1</u>. Let $w \in \mathbb{C}^{2n}$ and $X \in \mathfrak{g}$. Then $\mathcal{A}(Xw,w) = B_\theta(X,Y)$ where $Y \in \mathfrak{g}$, and $\mathcal{A}(Yv,v) \geq 0$ for all $v \in \mathbb{C}^{2n}$.

<u>Proof</u>. Note $\tau(Xu,\sigma u)$ is automatically real for $X \in \mathfrak{g}$, $u \in \mathbb{C}^{2n}$. Let $w = \begin{pmatrix} u \\ v \end{pmatrix}$ and $X = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$. Then

$$\mathcal{a}(Xw,w) = \sum_i \left\{-(\bar{B}u)_i\bar{u}_i + (\bar{A}v)_i\bar{u}_i - (Au)_i\bar{v}_i - (Bv)_i\bar{v}_i\right\}$$

$$= \sum_{i,j} B_{ij}\overline{(-\bar{u}_i u_j - v_i\bar{v}_j)} + \mathrm{Re}\left\{\sum_{i,j} A_{ij}\overline{(\bar{u}_i v_j - \bar{u}_j v_i)}\right\}$$

$$= B_\theta\left(\begin{pmatrix} A & B \\ -\bar{B} & \bar{A}\end{pmatrix}, \begin{pmatrix} C & D \\ -\bar{D} & \bar{C}\end{pmatrix}\right) = B_\theta(X,Y)$$

where $C_{ij} = \bar{u}_i v_j - \bar{u}_j v_i$, $D_{ij} = -v_i\bar{v}_j - \bar{u}_i u_j$.

Clearly $Y \in \mathfrak{g}$.

If $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^{2n}$,

$$\mathcal{a}(Yv,v) = -\sum_{i,j} x_i D_{ij}\bar{x}_j - \sum_{i,j} y_i\overline{C_{ij}}\bar{x}_j + \sum_{i,j} x_i C_{ij}\bar{y}_j - y_i\overline{D_{ij}}\bar{y}_j$$

$$= |\langle x,\bar{v}\rangle|^2 + |\langle x,u\rangle|^2 + |\langle v,y\rangle|^2 + |\langle y,\bar{u}\rangle|^2 + 2\,\mathrm{Re}\,(\langle y,v\rangle\langle u,x\rangle) - 2\,\mathrm{Re}\,(\langle x,\bar{v}\rangle\langle\bar{u},y\rangle)$$

$$= |\langle x,\bar{v}\rangle - \langle u,\bar{y}\rangle|^2 + |\langle y,v\rangle + \langle x,u\rangle|^2 \geq 0 \quad ,$$

$\langle\cdot,\cdot\rangle$ being the usual complex Hilbert structure on $\mathbb{C}^n$.

16. Conjugacy of Positive Elliptics in $so^*(\mathcal{H})$.

Let $(H, \langle\ ,\ \rangle_0)$ be a real Hilbert space and $J: H \to H$ an orthogonal with square $-I$. Let $\mathcal{H} = H \oplus iH$, and extend J to $\mathcal{H}$ the obvious way so that $iJ = Ji$. Extend $\langle\ ,\ \rangle_0$ to $\mathcal{H}$ so that $(\mathcal{H}, \langle\ ,\ \rangle_0)$ is a complex Hilbert space with complex structure i . We write $\|x\|^2 = \langle x,x\rangle_0$ for $x \in \mathcal{H}$, and $\|B\|$ for the usual norm with respect to $\langle\ ,\ \rangle_0$ for any linear operator B . In this section all operators are bounded and commute with i . Finally, extend $\langle\ ,\ \rangle_0$ on H to $\tau(\cdot,\cdot)$ on $\mathscr{K}$ by <u>complex bilinearity</u> so that τ is a complex symmetric form on $\mathscr{K}$. If $\mathcal{C}$ is the anti-linear conjugation of $\mathscr{K}$ with respect to the decomposition $H \oplus iH$ ($\mathcal{C} = +I$ on H , $-I$ on iH) , then $\tau(\cdot,\cdot) = \langle\cdot,\mathcal{C}\cdot\rangle_0$.

Letting $\mathcal{J} = iJ$, $\mathcal{J}$ is an Hermitian operator on $(\mathscr{K}, \langle\ ,\ \rangle_0)$ with spectrum $\{+1,-1\}$. In fact, the $+1$ eigenspace of $\mathcal{J}$ is $\mathscr{K}_+ = \{x + iJx\colon x \in H\}$ and the -1 eigenspace is $\mathscr{K}_- = \{x - iJx\colon x \in H\}$. One can check that τ vanishes when restricted to either $\mathscr{K}_+$ or $\mathscr{K}_-$.

Our notation is chosen to correspond with sections 7 and 11, and this section is a further restriction on the situation in section 11. In particular, the J , $\mathcal{J}$, and $\langle\ ,\ \rangle_0$ correspond.

Let $\sigma = J\mathcal{C} = \mathcal{C}J$, an anti-linear operator with square $-I$. We have $\tau(\sigma\cdot,\sigma\cdot) = \overline{\tau(\cdot,\cdot)}$, and note that σ switches $\mathscr{K}_+$ and $\mathscr{K}_-$. As before, we define

$$\mathcal{A}(\cdot,\cdot) = -\operatorname{Im}\langle\mathcal{J}\cdot,\cdot\rangle_0 ,$$

a real symplectic form on $\mathscr{K}$. Furthermore,

$$\mathcal{A}(\cdot,\cdot) = \operatorname{Re}\,\tau(\cdot,\sigma\cdot) . \tag{9}$$

Defining $\langle\cdot,\cdot\rangle = \operatorname{Re}\langle\cdot,\cdot\rangle_0 + i\mathcal{A}(\cdot,\cdot)$, $(\mathscr{K}, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space with complex structure J .

Define

$SO^*(\mathcal{H}) = \{g\colon \mathcal{H} \to \mathcal{H} \mid g$ invertible, g commutes with i and σ, and $\tau(g\cdot,g\cdot) = \tau(\cdot,\cdot)\}$

and

$\mathfrak{g} = \{B\colon\mathcal{H} \to \mathcal{H} \mid B$ commutes with i and σ and is skew for $\tau\} = so^*(\mathcal{H})$.

Note τ could be replaced by the symplectic form $\mathcal{A}$ in the above two definitions by (9). Also note $J \in \mathfrak{g} \cap SO^*(\mathcal{H})$.

The condition defining the "maximal compact subgroup" K of $SO^*(\mathcal{H})$ is the same as in the previous sections, i.e. $g \in K$ iff $gJ = Jg$, which here is equivalent to g leaving the space H invariant. (Not the "K" in 16.2.)

<u>Lemma 16.1</u>. For all $X \in \mathfrak{g}$, $\mathcal{A}(A\cdot,\cdot) = \mathrm{Re}\ \tau(A\cdot,\sigma\cdot)$ is a real symmetric form on $\mathcal{H}$ for which i and σ are orthogonal. $\tau(Ax,\sigma x) \in \mathbb{R}\ \forall\, x \in \mathcal{H}$.

<u>Theorem 16.2</u>. Let $B \in \mathfrak{g}$ satisfy $\mathcal{A}(Bx,x) \geq k\langle x,x\rangle\ \forall\, x \in \mathcal{H}$, for some $k > 0$. Then there exists $S \in SO^*(\mathcal{H})$ such that $SBS^{-1} = JH_1 = H_1J$, where H_1 is a positive self-adjoint operator in $(\mathcal{H},\langle\ ,\ \rangle)$ (and $(\mathcal{H},\langle\ ,\ \rangle_0)$). Also,

$$\left(\|B\|^{3/2}/k^{1/2}\right)\|x\|^2 \geq \langle H_1x,x\rangle \geq \left(k^{3/2}/\|B\|^{1/2}\right)\|x\|^2 \quad \forall x \in \mathcal{H}.$$

<u>Proof</u>. Again we follow the proof in section 7. Defining $\mathcal{S}(\cdot,\cdot) = \mathcal{A}(B\cdot,\cdot)$, we polar-decompose $B = K\sqrt{-B^2}$ in $(\mathcal{H},\mathcal{S})$. As i and σ are orthogonal for $\mathcal{S}$ and commute with B,

K and $\sqrt{-B^2}$ also commute with i and σ, and again $K \in \mathfrak{g} \cap SO^*(\mathcal{H})$.

Defining $s_1(\cdot,\cdot) = \mathcal{A}(K\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_1 = s_1(\cdot,\cdot) + i\mathcal{A}(\cdot,\cdot)$, $(\mathcal{H}, \langle\ ,\ \rangle_1)$ is a complex Hilbert space with complex structure K. It suffices to find a unitary equivalence

$$S: (\mathcal{H}, \langle\ ,\ \rangle_1) \to (\mathcal{H}, \langle\ ,\ \rangle)$$

which commutes with both i and σ. As before, i-linearity follows if S takes the +1 eigenspace $\mathcal{L}_+$ (resp. -1 eigenspace $\mathcal{L}_-$) of iK to $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Note σ switches $\mathcal{L}_+$ and $\mathcal{L}_-$ also as σ is anti-linear and commutes with K. Thus clearly all of $\mathcal{H}_\pm$ and $\mathcal{L}_\pm$ have the same dimension, and the desired S is obtained by taking any unitary equivalence $(\mathcal{L}_+, \langle\ ,\ \rangle_1) \to (\mathcal{H}_+, \langle\ ,\ \rangle)$ and extending it to $\mathcal{H}$ by requiring it to commute with σ.

The rest of the theorem follows as before.

17. Noncompact Convexity in so*(2n).

Just as the orbits of interest in su(p,q) were associated with the non-simple u(1,1) and u(1), we will have occasion to examine the orbits of so*(4) ≈ $sl(2,\mathbb{R}) \oplus su(2)$ and so*(2) ≈ so(2) in classifying invariant convex cones in so*(2n). However, $n \geq 3$ is assumed in what follows unless otherwise stated.

To relate the notations of sections 15 and 16, we let $H = \mathbb{R}^{2n}$ with the usual inner product, $\mathcal{H} = \mathbb{C}^{2n}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ (regarding $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$), $\mathcal{C}$ complex conjugation in $\mathbb{C}^{2n}$, and $\sigma = J\mathcal{C}$. There are basically three forms on $\mathbb{C}^{2n}$ invariant under $SO^*(2n)$, all closely related:

1) the complex symmetric form τ,
2) the symplectic form $\mathcal{A}(\cdot,\cdot) = \mathrm{Re}\,\tau(\cdot,\sigma\cdot)$ (both defined in section 15), and
3) the Hermitian form $H_o(\cdot,\cdot) = -i\tau(\cdot,\sigma\cdot)$ with signature (n,n) alluded to in section 16.

As with the earlier groups, we define a class of bases that are transformed by $G = SO^*(2n)$. Call

$$\Phi = \{(e_j,f_j)_{j=1}^n : \tau(e_k,e_\ell) = \tau(f_k,f_\ell) = 0 \ \forall\, k,\ell\,, \quad \tau(e_j,f_k) = \delta_{jk}\,, \text{ and } \sigma e_j = if_j\,, \quad \sigma f_j = -ie_j\}$$

the set of G-<u>bases</u>. Our <u>standard basis</u> is $e_j = (\tilde{e}_j + i\tilde{e}_{n+j})/\sqrt{2}$, $f_j = (\tilde{e}_j - ie_{n+j})/\sqrt{2}$ where $\tilde{e}_1,\ldots,\tilde{e}_{2n}$ are the standard unit vectors in $\mathbb{C}^{2n}$. For $\varphi = (e_j,f_j) \in \Phi$, note φ is a basis for H_o in the sense of section 14 for $su(p,q)$, and our notation is consistent.

We will need several observations about "matrix elements" $\mathcal{A}(X\cdot,\cdot)$. Note that $\mathcal{A}(\sigma\cdot,\sigma\cdot) = \mathcal{A}(\cdot,\cdot)$, so $\forall\, X \in \mathfrak{g}$ and $\forall (e_j,f_j) \in \Phi$ we have $\mathcal{A}(Xe_i,e_i) = \mathcal{A}(Xf_i,f_i)$.

Also, if $a_i = (e_i + \gamma f_i)/\sqrt{2}$ for $|\gamma| = 1$, then $\mathcal{a}(Xa_i,a_i) = \mathcal{a}(Xe_i,e_i)$. (Such $\{a_i\}$ are quaternionic bases in the sense of section 4.)

In particular, if $X = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{g}$ and a_i is the unit vector $\tilde{e}_i$, then $\mathcal{a}(Xa_i,a_i) = -B_{ii}$. (Also, $\mathcal{a}\left(\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) = \sum_{j=1}^{n} D_j(|v_j|^2 + |w_j|^2)$ where D is diagonal.) If $e = \tilde{e}_1 \pm i\tilde{e}_2$, then $\tau(e,e) = \tau(e,\sigma e) = 0$, and $\mathcal{a}(Xe,e) = -B_{11} - B_{22} \mp 2\,\mathrm{Im}\,B_{12}$.

Any $g \in SO^*(2n)$ has the form $g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ where $A^tB = \bar{B}^t\bar{A}$ and $A^tA + \bar{B}^t\bar{B} = I$, or equivalently $AA^t + BB^t = I$ and $A\bar{B}^t = B\bar{A}^t$. We have $g^{-1} = \begin{pmatrix} A^t & -\bar{B}^t \\ B^t & \bar{A}^t \end{pmatrix}$, and g is in the maximal compact $K \approx U(n)$ iff g is a real matrix.

<u>Definitions</u>. 1) Let

$$C_2 = \{X \in \mathfrak{g}: \mathcal{a}(Xe,e) > 0 \ \forall\, e \neq 0 \text{ such that } \tau(e,e) = \tau(e,\sigma e) = 0\} .$$

2) If $X = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{g}$, call $(-B_{11},\ldots,-B_{nn}) \in \mathbb{R}^n$ the <u>G-diagonal of</u> X. (It is essentially the projection of X on the maximal torus $\underline{h}$.

3) Let

$$E_0 = \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n: 0 \le \lambda_i \le \sum_{\substack{j=1 \\ j\neq i}} \lambda_j,\ i = 1,\ldots,n\}$$

and $E_2 = \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n: \lambda_i + \lambda_j \ge 0 \ \forall\, i \neq j\}$.

4) If $V, W = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, say V is in the noncompact convex hull of W if V is equal to an element of E_o plus a vector in the convex hull of $\{(\lambda_{p(1)}, \ldots, \lambda_{p(n)}) : p \text{ permutation of } (1, \ldots, n)\}$.

5) If $X = Y + Z$, $Y \in \underline{k}$, $Z \in \underline{p}$, call $\sqrt{B(Z,Z)}$ the p-norm of X.

Lemma 17.1. $X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \in C_2$ if D is diagonal and $D_i + D_j > 0 \ \forall i \neq j$.

Proof. Let $e = \begin{pmatrix} c + id \\ f + ig \end{pmatrix} \neq 0$ satisfy $\tau(e,e) = \tau(e,\sigma e) = 0$, where $c,d,f,g \in \mathbb{R}^n$. The conditions are equivalent to $c \cdot g = f \cdot d$, $c \cdot c + f \cdot f = d \cdot d + g \cdot g$, and $c \cdot d + f \cdot g = 0$. In terms of vectors in $\mathbb{C}^n$, they become $\|c+if\|^2 = \|d+ig\|^2$ and $\langle c+if, d+ig\rangle = 0$ ($\langle\ ,\ \rangle$ the sesquilinear form on C^n). Recalling the isomorphism $\mathbb{R}^{2n} \approx \mathbb{C}^n$ and $K \approx U(n)$, there clearly exists $g \in K$ and $s > 0$ such that

$$g\left(s\binom{c}{f} + is\binom{d}{g}\right) = \tilde{e}_1 + i\tilde{e}_2 = f \ .$$

Then $\mathcal{A}(Xe,e) = s^{-2}\mathcal{A}((gXg^{-1})f,f)$. gXg^{-1} is a real matrix, so $\mathcal{A}(Xe,e) > 0$ follows from the remark above (on the $\mathcal{A}(Xe,e)$) and Horn's theorem.

Lemma 17.2. The G-orbit of any $Y \in C_2$ is closed.

Proof. $\exists k > 0$ such that $\mathcal{A}(Ye,e) \geq k\|e\|^2 \ \forall e$ such that $\tau(e,e) = \tau(e,\sigma e) = 0$. (Note $-i\tau(e,\sigma e) = H_o(e,e)$, and the proof is the same as for $su(p,q)$.) The inequality is also true for any X in the K-orbit of Y. We compute $\mathrm{Ad}(a)X$ for $a \in \bar{A}^+$. As

$$a = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix},$$

where

$$S = \begin{pmatrix} \begin{pmatrix} \cosh\theta_1 & -i\sinh\theta_1 \\ i\sinh\theta_1 & \cosh\theta_2 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \cosh\theta_\ell & -i\sinh\theta_\ell \\ i\sinh\theta_\ell & \cosh\theta_\ell \end{pmatrix} \end{pmatrix}$$

$\theta_j \geq 0$, it suffices to multiply

$$\begin{pmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} -c & e+ir \\ e-ir & -d \end{pmatrix} \begin{pmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{pmatrix}$$

where c,d,e,r are real. We have $c+d-2|r| \geq 2k$ by our choice of k and a remark earlier. The diagonal elements of this matrix are $-c\cosh^2\theta - d\sinh^2\theta - r\sinh 2\theta$ and $-d\cosh^2\theta - c\sinh^2\theta - r\sinh 2\theta$, and their sum is

$$\begin{aligned} -(c+d)\cosh 2\theta - 2r\sinh 2\theta &\leq (-(c+d)+2|r|)\sinh 2\theta \\ &\leq -2k\sinh 2\theta \quad . \end{aligned}$$

Now argue as in the proof of Lemma 14.1, using the estimate above.

<u>Theorem 17.3</u>. Let $Y \in C_2$. Then Y is conjugate to an $X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$ as in the statement of 17.1, where the G-diagonal of Y is in the noncompact convex hull of the G-diagonal of X.

The proof runs exactly as the proof of Theorem 14.2, and is dependent only on Lemma 17.2 and the following

Lemma 17.4. Let $X \in C_2 - \underline{k}$. Then there exists Y in the orbit of X with strictly smaller $\underline{p}$-norm such that the G-diagonal of X is in the noncompact convex hull of the G-diagonal of Y.

Proof. By Horn's theorem we may assume $X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} + i\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ where A and B are real and skew, and D is diagonal and satisfies the hypothesis of Lemma 17.1.

If G and H are nxn real and skew,

$$\mathrm{Ad}\left(\exp\left(\varepsilon\begin{pmatrix} iG & iH \\ iH & -iG \end{pmatrix}\right)\right)X = \begin{pmatrix} iA+\varepsilon(S+iM) & iB-D+\varepsilon(T+iL) \\ iB+D+\varepsilon(-T+iL) & iA+\varepsilon(S-iM) \end{pmatrix} + O(\varepsilon^2)$$

where S,T,L,M are real and nxn :

$$S = AG - GA + BH - HB$$
$$T = HA + AH - BG - GB$$
$$L = -GD - DG$$
$$M = HD + DH \quad .$$

Therefore we set

$$G_{ij} = \left(B_{ij}/(D_i+D_j)\right) \quad \text{and} \quad H_{ij} = \left(-Aij/(D_i+D_j)\right)$$

so that

$$(B+\varepsilon L)_{ij} = (1-\varepsilon)B_{ij}$$

$$(A+\epsilon M)_{ij} = (1-\epsilon)A_{ij} \ , \quad \text{and}$$

$$(-D+\epsilon T)_{ii} = -D_i + 2\epsilon \sum_{j=1}^{n} \left(((A_{ij})^2 + (B_{ij})^2)/(D_i + D_j) \right) \ .$$

As before, if $T_{ii} = 0$ the corresponding $O(\epsilon^2)$ terms are zero $\forall\ \epsilon$. It is clear that $\{T_{ii}\}_{i=1}^{n} \in E_o$ as A and B are skew. Thus the desired Y is obtained if ϵ is positive and sufficiently small.

18. Minimal and Maximal Causal Cones in so*(2n).

In addition to the definition of C_2 in the last section, let

$$C_1 = \{X \in \ : \mathcal{A}(Xv,v) > 0 \ \forall\ v \neq 0 \ \text{ in } \ \mathbb{C}^{2n}\} \ ,$$

$$C_o = \{X \in C_1 : \sum_{j=2}^{n} \mathcal{A}(Xe_j,e_j) > \mathcal{A}(Xe_1,e_1) \ \forall\ (e_j,f_j) \in \Phi\} \ ,$$

and now let $\mathcal{C}$ be the collection of all nontrivial invariant convex cones in $\mathfrak{g}$.

<u>Theorem 18.1</u>. C_i $(i = 0,1,2)$ are nonempty, open, and in $\mathcal{C}$. Any $C \in \mathcal{C}$ contains either C_0 or $-C_0$ but not both. $\bar{C}_0$, $\bar{C}_1$, and $\bar{C}_2$ are obtained by changing ">" to "≥" in the definitions. Any closed (open) cone in $\mathcal{C}$ containing C_o is contained in $\bar{C}_2$ (resp. C_2).

<u>Proof</u>. Clearly $C_0 \subseteq C_1 \subseteq C_2$ and that the C_i are in $\mathcal{C}$ provided they are nonempty. In fact the inclusions are proper: if $X = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, D diagonal, and $\operatorname{diag} D$ is

$(n,1,\ldots,1)$ then $X \in C_1 - C_0$, and if it is $(-1,2,\ldots,2)$ then $X \in C_2 - C_1$ by Lemma 17.1.

We show $J \in C_0$ to prove $C_0 \neq \emptyset$. Using $G = K\bar{A}^+K$, it suffices to see that

$$\sum_{j=2}^{n} \mathcal{A}(Ja\beta_i, a\beta_i) > \mathcal{A}(Ja\beta_1, a\beta_1)$$

where $a \in \bar{A}^+$ and $\beta_i = \begin{pmatrix} c_i^\alpha \\ d_i^\alpha \end{pmatrix} \in \mathbb{R}^{2n}$ $(i,\alpha = 1,\ldots,n)$, the β_i corresponding under $\mathbb{R}^{2n} \approx \mathbb{C}^n$ to an orthonormal basis of $\mathbb{C}^n$. Setting $N_i^\alpha = |c_i^\alpha|^2 + |d_i^\alpha|^2$, we have $\sum_{i=1}^{n} N_i^\alpha = 1$ and $\sum_{\alpha=1}^{n} N_i^\alpha = 1 \ \forall\, i,\alpha$.

Taking an a as in the proof of 17.2, we have

$$\mathcal{A}(Ja\beta_i, a\beta_i) = \sum_{\alpha=1}^{n} (\cosh 2\theta_{[\alpha+1\,/\,2]})N_i^\alpha$$

(where $\theta_{\ell+1} \equiv 0$), and, dropping the factor of 2 from $\theta_1,\ldots,\theta_\ell$, we must show

$$\sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})N_1^\alpha < \sum_{i=2}^{n}\sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})N_i^\alpha$$

$$= \sum_{\alpha=1}^{n} (\cosh \theta_{[\alpha+1/2]})(1 - N_1^\alpha)$$

or

$$2\cosh\theta_1 + \ldots + 2\cosh\theta_\ell + 1$$

$$> 2(N_1^1 + N_1^2)\cosh\theta_1 + \ldots + 2(N_1^{2\ell-1} + N_1^{2\ell})\cosh\theta_\ell + 2N_1^{2\ell+1}$$

if n is odd. (If n is even the same with the 1 and $2N_1^{2\ell+1}$ removed must be shown, but this is now clear from

$\sum_{\alpha=1}^{n} N_1^{\alpha} = 1$ and $n \geq 3$.) In that case it suffices to show $2\cosh\theta + 1 > 2(N_1 + N_2)\cosh\theta + 2N_3$ where $0 \leq N_1, N_2, N_3$ and $N_1 + N_2 + N_3 \leq 1$. But the r.h.s. is $\leq 2(1 - N_3)\cosh\theta + 2N_3$ and $1 > 2(1 - \cosh\theta)N_3$ is clear.

Thus C_0 is nonempty. C_1 is open by the same proof as that for C_0 in $sp(n,\mathbb{R})$. By Theorem 16.2, any $X \in C_1$ is conjugate to a $\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, D diagonal, where all $D_{jj} > 0$. D is clearly unique up to a permutation as D essentially gives the eigenvalues of X acting on $\mathbb{C}^{2n}$. By 17.3 $X \in C_0$ iff it is conjugate to a $Y = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, D diagonal, where diag D is in $(E_0)^{\circ}$. C_0 is then open by the continuity of eigenvalues. We have already seen that any $Y \in C_2$ is conjugate to an X as in 17.1, so C_2 is open by the same argument as for C_1 in $su(p,q)$.

Now let $C \in \mathcal{C}$; we may assume $J \in \overline{C}$. As before C_0 being open implies $\exists W \in C \cap C_0$, so $J \in C$ by the above conjugacy in C_0. (Average over permutations of the G-diagonal of $\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$.) Now $\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$ (where diag $D = (1,1,\epsilon,\ldots,\epsilon)$ for $1 > \epsilon > 0$) can be obtained from averaging the transforms of J by $a, a^{-1} \in A$, as in Corollary 3.2. It may be seen without much difficulty that the convex cone generated by permutations of $(1,1,0,\ldots,0)$ is precisely E_0, and likewise the convex cone generated by permutations of the above $(1,1,\epsilon,\ldots,\epsilon)$ is just $(E_0)^{\circ}$. Thus C_0 is the invariant convex cone generated by J, and $C_0 \subseteq C$.

That $\bar{C}_0$, $\bar{C}_1$, and $\bar{C}_2$ have the asserted forms follows exactly as in the symplectic case.

Finally, the last two statements follow, as in the proof for $su(p,q)$, from

Theorem 18.2. $\bar{C}_1^* = -C_1$ and $\bar{C}_0^* = -\bar{C}_2$.

Proof. $B_\theta(X,Y) > 0 \ \forall\, X,Y \in C_1$ follows from Theorem 16.2 and the definitions of C_1 and B_θ, so $-\bar{C}_1 \subseteq \bar{C}_1^*$. The converse proceeds in the by now familiar way from Theorem 15.1.

$-\bar{C}_2 \subseteq C_0^*$ follows from the minimality of $\pm C_0$. Conversely, if $X \notin \bar{C}_2$ then, as in the proof of 17.1, there is some K-conjugate of X of the form $Y = \begin{pmatrix} * & B \\ -\bar{B} & * \end{pmatrix}$ where B has $\begin{pmatrix} -\lambda_1 & \alpha \\ \bar{\alpha} & -\lambda_2 \end{pmatrix}$ in the upper left corner, and $\lambda_1 + \lambda_2 - 2\,\mathrm{Im}\,\alpha < 0$. But now $W = \begin{pmatrix} 0 & L \\ -\bar{L} & 0 \end{pmatrix}$ where L is all zeros except for $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}$ in the upper left corner, is in $\bar{C}_0$, and $B_\theta(Y,W) = \lambda_1 + \lambda_2 - 2\,\mathrm{Im}\,\alpha < 0$. Thus Y, and hence X, is not in $-\bar{C}_0^*$.

Corollary 18.3. C_0, C_1, and C_2 are the interiors of their closures.

Proof. As we have shown that all $X \in C_2$ are conjugate to a $Y \in \underline{h}$, the proof follows that of Corollary 12.4.

19. Classification of Orbits and Cones in so*(2n) .

We shall see that if Δ is a type in some $\bar{C}_2$ and $\Delta = \Delta_1 + \ldots + \Delta_m$ its decomposition into indecomposable types, then each Δ_j lies in an so*(4) or so*(2) . It will be useful to examine the orbits in so*(4) , as the same definitions of C_i $(i = 0,1,2)$ make sense there (and only C_1 for so*(2)), and the C_i can be explicitly described.

The $sl(2,\mathbb{R})$ component of so*(4) is spanned by

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \begin{pmatrix} \begin{matrix} 0 & i \\ -i & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & \begin{matrix} 0 & i \\ -i & 0 \end{matrix} \\ \begin{matrix} 0 & i \\ -i & 0 \end{matrix} & 0 \end{pmatrix}$$

and the $su(2)$ component is spanned by

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}, \quad \begin{pmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & 0 \end{pmatrix}.$$

We let B_1 and B_2 be $-\mathrm{Tr}(\cdot\cdot)$ restricted to the $sl(2,\mathbb{R})$ and $su(2)$ components, respectively.

By the uniqueness of the decomposition it is clear that if $Y \in sl(2,\mathbb{R})$, $Z \in su(2)$, and $(e_1,e_2,f_1,f_2) \in \Phi$, $\mathcal{A}(Ye_1,e_1) = \mathcal{A}(Ye_2,e_2)$ and $\mathcal{A}(Ze_1,e_1) = -\mathcal{A}(Ze_2,e_2)$. Lemma 17.1 shows that $\forall e \neq 0$ such that $0 = \tau(e,e) = \tau(e,\sigma e)$, $\mathcal{A}(Ze,e) = 0$ and $\mathcal{A}(Ye,e) > 0 \ \forall Y$ in the positive cone of $sl(2,\mathbb{R})$. Note also

$$\begin{pmatrix} 0 & & -1 & i \\ & & -i & -1 \\ 1 & i & & \\ -i & 1 & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = 0 \quad ,$$

the 4x4 matrix being in the positive nilpotent orbit.

We see that C_0 is empty and "$\bar{C}_0$", so to speak, is the closed positive cone in $sl(2,\mathbb{R})$. C_2 is the direct sum of the open positive cone in $sl(2,\mathbb{R})$ and $su(2)$.

The cone C_1 with self-dual closure (and 18.2 is valid here, too) is

$$C_1 = \left\{(Y,Z): \begin{array}{l} Y \in \text{open positive cone in } sl(2,\mathbb{R}), \\ Z \in su(2), \text{ and } B_1(Y,Y) > B_2(Z,Z) \end{array}\right\} .$$

That C_1 is convex follows from an elementary estimate.

<u>Theorem 19.1</u>. Only the indecomposables $\Delta_0(0,0)$, $\Delta_1^{\pm}(0,0)$, $\Delta_0^{\pm}(\zeta,-\zeta)$, and $\Delta_1^{\pm}(\zeta,-\zeta)$ where $\zeta = -\bar{\zeta} \neq 0$, lie in $\pm \bar{C}_2$.

<u>Proof</u>. We consider first any of the types $\Delta_m^{\pm}(\cdots)$ or $\Delta_m(\cdots)$ with $m \geq 3$. As in the earlier proofs we considered $\mathcal{Q}(Aa,a)$ where $a \in N^{m-1}E + E$, and found it to be of indeterminant sign. τ vanishes on the σ-invariant $N^{m-1}E + E$ so clearly no such type can be in $\pm\bar{C}_2$.

For the other orbits we take $A = S + N$ as usual. To rule out the remaining types excluded by the statement of the theorem, it is sufficient to let $S = 0$, as that orbit is in any closed invariant cone containing the original type with $S \neq 0$.

For $\Delta_2(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$ $(\zeta \neq -\bar\zeta)$, $\alpha(N(Ne+w),Ne+w) = -2\alpha(W,N^2e)$. NE has dimension 4, so $e \neq 0$ can be found with $\tau(Ne,Ne) = \tau(Ne,\sigma Ne) = 0$. $w + Ne$ has the same properties, and $\alpha(w,N^2e)$ can take any value.

For $\Delta_2^{\pm}(\zeta,-\zeta)$ $(\zeta = -\bar\zeta \neq 0)$ or $\Delta_2(0,0)$ again let $S \to 0$. $V = N^2E + NE + E$ so let $T = +Id.$ on $N^2E + E$ and $T = -Id.$ on NE . Then $T \in SO^*(6)$ and $TNT = -N$, showing that $N \notin \bar C_2$ as $\bar C_2 \cap -\bar C_2 = \{0\}$.

For $\Delta_1(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$ $(\zeta \neq -\bar\zeta)$ again let $S \to 0$. E has a basis $e,f,\sigma e,\sigma f$ where $\tau(e,Nf) = 1$, $\tau(e,Ne) = \tau(f,Nf) = 0$, and $\mathbb{C}e + \mathbb{C}f$ is τ-orthogonal to $N\{\mathbb{C}\sigma e + \mathbb{C}\sigma f\}$. Let $Te = f$, $Tf = e$, $T(Ne) = -Nf$, $T(Nf) = -Ne$, and extend it to V by requiring it to commute with σ . Then $T \in SO^*(8)$ and $TNT = -N$, excluding this type.

The remaining types are in $so^*(2)$ or $so^*(4)$. $\Delta_0(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$ has a hyperbolic $sl(2,\mathbb{R})$ component so is never in a $\bar C_2$. $\Delta_1^{\pm}(\zeta,-\zeta)$ $(\Delta_1^{\pm}(0,0))$ correspond to a nilpotent $sl(2,\mathbb{R})$ component and a nonzero (resp. zero) $su(2)$ component, so are always on the boundary of $\bar C_2$.

From now on reserve the notation $\Delta_1^{\pm}(\zeta,-\zeta)$ for the case $\zeta = -\bar\zeta \neq 0$ only; this is the only ambiguity in the notation for the indecomposables left. We note that $\Delta_1^{+}(0,0)$ is on the boundary of $\bar C_0$ and $\bar C_2$, but $\Delta_1^{+}(\zeta,-\zeta)$ is on the boundary of $\bar C_2$ but not that of $\bar C_1$ or $\bar C_0$.

This is quite analogous to the types $\Delta_1^+(0)$ and $\Delta_1^+(i\lambda)$ $(\lambda \neq 0)$ for su(p,q) . Finally, just as for su(p,q) , representatives of $\Delta_1^+(\zeta,-\zeta)$ are a sum within so*(4) of a representative of $\Delta_1^+(0,0)$ and a representative of the elliptic type $\Delta^+(\zeta,-\zeta) + \Delta^-(\zeta,-\zeta)$, which is on the boundary of $\bar{C}_2$ but not that of $\bar{C}_1$ or $\bar{C}_0$.

Finally, as a consequence of the results of the last three sections, we list the types in $\bar{C}_2$ for so*(2n) , $n \geq 3$.

Proposition 19.2. Types in C_1 are those of the form

$$(10) \qquad \Delta = \Delta^+(\zeta_1,-\zeta_1) + \ldots + \Delta^+(\zeta_n,-\zeta_n)$$

Δ is furthermore in C_0 if $(|\zeta_1|,\ldots,|\zeta_n|) \in (E_o)^o$. Other types in C_2 have either the form

$$(11) \qquad \Delta = \Delta(0,0) + \Delta^+(\zeta_2,-\zeta_2) + \ldots + \Delta^+(\zeta_n-\zeta_n)$$

or

$$(12) \qquad \Delta = \Delta^-(\zeta_1,-\zeta_1) + \Delta^+(\zeta_2,-\zeta_2) + \ldots + \Delta^+(\zeta_n,-\zeta_n)$$

for $(-|\zeta_1|,|\zeta_2|,\ldots,|\zeta_n|) \in (E_2)^o$.

Types in $\bar{C}_0$ have either the form (10) with $(|\zeta_1|,\ldots,|\zeta_n|) \in E_o$ or

$$(13) \qquad \Delta = \underbrace{\Delta_1^+(0,0) + \ldots + \Delta_1^+(0,0)}_{\ell \text{ times}} + \underbrace{\Delta(0,0) + \ldots + \Delta(0,0)}_{m \text{ times}}$$
$$+ \Delta^+(\zeta_{2\ell+m+1},-\zeta_{2\ell+m+1}) + \ldots + \Delta^+(\zeta_n-\zeta_n)$$

with $(0,\ldots,0,|\zeta_{2\ell+m+1}|,\ldots,|\zeta_n|) \in E_o$ and $2\ell + m \geq 1$.

Types in $\partial\bar{C}_1$ have the form (13) with $2\ell + m \geq 1$.

Finally, types in $\partial\bar{C}_2$ have either the form (12) with $(-|\zeta_1|,|\zeta_2|,\ldots,|\zeta_n|) \in \partial E_2$, (13) with $\ell \geq 1$ or $m \geq 2$, or

$$(14) \qquad \Delta = \Delta_1^+(\zeta_1,-\zeta_1) + \Delta^+(\zeta_3,-\zeta_3) + \ldots + \Delta^+(\zeta_n,-\zeta_n)$$

where $(|\zeta_1|,-|\zeta_1|,|\zeta_3|,\ldots,|\zeta_n|) \in \partial E_2$.

It is now easy to classify the open and closed invariant convex cones in $\mathfrak{g}$. The proof involves no new ideas over those for su(p,q) , but one must use the remarks about the nilpotents above for the closed cones. The proof uses the maps

$$M_\varphi\colon \mathfrak{g} \to \mathbb{R}^n\colon \ X \to (\mathcal{A}(Xe_1,e_1),\ldots,\mathcal{A}(Xe_n,e_n))$$

for $\varphi = (e_j,f_j) \in \Phi$. We have seen that $M_\varphi(X) \in E_2$ if $X \in \bar{C}_2$, $M_\varphi(X) \in (E_2)^o$ if $X \in C_2$, etc.

<u>Theorem 19.3</u>. The open (closed) non-trivial invariant convex cones in so*(2n) $(n \geq 3)$ containing C_o are in 1-1 correspondence with the open (resp. closed) cones in $(E_2)^o$ (resp. E_2) containing $(E_o)^o$ (resp. E_o) and invariant under the finite group of permutations of the Euclidean coordinates of $\mathbb{R}^n$.

CHAPTER V. Invariant Convex Cones in so(2,n) .

20. Conventions and Preliminaries for so(2,n) .

so(2,n) is simple if $n \geq 3$. Let $\ell = [n/2]$. We take

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} : A,B,C \text{ real; } B \ 2\times n; \ A,C \text{ skew} \right\}$$

$$\underline{k} = \{X \in \mathfrak{g} : X \text{ skew}\} \quad , \quad \underline{p} = \{X \in \mathfrak{g} : X \text{ symmetric}\}$$

$$\underline{h} = \left\{ \begin{bmatrix} \begin{pmatrix} 0 & d_0 \\ -d_0 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & d_1 \\ -d_1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & d_\ell \\ -d_\ell & 0 \end{pmatrix} \end{bmatrix} \in \mathfrak{g} \right\}$$

$$c = \mathbb{R}Z \quad \text{where} \quad Z = \begin{bmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix} .$$

Let d_j $(j = 0,\ldots,\ell)$ be the linear functionals spanning the dual of $\underline{h}$, whose definition is clear from the above notation for $\underline{h}$. Then

$$\{\text{noncompact roots}\} = \{\pm i(d_0+d_j),\ \pm i(d_0-d_j) : j = 1,\ldots,\ell\}$$

if n is even; if n is odd one has the additional noncompact roots $\pm id_0$.

$$\{\text{compact roots}\} = \{\pm i(d_j+d_k),\ \pm i(d_j-d_k) : 1 \le j < k \le \ell\}$$

if n is even; if n is odd one has the additional compact roots $\pm id_1,\ldots,\pm id_\ell$. The positive roots we take to be the above with "+" only, and

$$\Delta_o^+ = \{i(d_0-d_1), i(d_1-d_2),\ldots, i(d_{\ell-1}-d_\ell), i(d_{\ell-1}+d_\ell)\}$$

if n is even, and

$$\Delta_o^+ = \{i(d_0-d_1), i(d_1-d_2),\ldots, i(d_{\ell-1}-d_\ell), id_\ell\}$$

if n is odd. We forego listing the H_α, X_α, Y_α for $\alpha \in Q_+$ as we will not really need them.

We let $\Sigma_o = \{i(d_0+d_1), i(d_0-d_1)\}$, so $Z = Z_o$ always, and

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & 0 & a & 0 & 0 & \cdots \\ 0 & 0 & 0 & b & 0 & \cdots \\ a & 0 & & & & \\ 0 & b & & & & \\ 0 & 0 & & & 0 & \\ \vdots & & & & & \end{pmatrix} : a,b \in \mathbb{R} \right\}.$$

Then $\underline{h}^- = \{X \in \underline{h} : d_j(X) = 0 \text{ for } 2 \le j \le \ell\}$ and $\underline{h}^+ = \{X \in \underline{h} : d_o(X) = d_1(X) = 0\}$, so $\underline{h}^+ = 0$ iff $n = 3$.

The cone in $\underline{h}^-$ given by Corollary 3.2 is (in $\mathfrak{o}(2,2)$) generated by

$$-\frac{i}{2}\sum_{\alpha\in\Sigma_o} d_\alpha H_\alpha = \begin{pmatrix} 0 & -\frac{1}{2}(d_1+d_2) & 0 & 0 \\ \frac{1}{2}(d_1+d_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(d_1-d_2) \\ 0 & 0 & \frac{1}{2}(d_1-d_2) & 0 \end{pmatrix}$$

for $d_\alpha \geq 1$, or all

$$(15) \qquad \begin{pmatrix} \begin{matrix} 0 & -d \\ d & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -g \\ g & 0 \end{matrix} \end{pmatrix}$$

for $|g| \leq d-1$, $d \geq 1$. The cone generated by this is all of the form (15) where $d > 0$, $|g| < d$.

We take $B(X,Y) = \mathrm{Tr}(XY)$ for $X,Y \in \mathfrak{g}$, so if $X = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, $Y = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix} \in \mathfrak{g}$

$$B(X,Y) = -\sum_{i,j} A_{ij}P_{ij} - \sum_{i,j} C_{ij}R_{ij} + 2\sum_{i,j} B_{ij}Q_{ij}$$

and $B_\theta(X,Y)$ is the same with "+" replacing "-" in the above. $(\theta(X) = -X^t)$.

Given $V = (d_0, d_1, \ldots, d_\ell) \in \mathbb{R}^{\ell+1}$, for concreteness we associate

$$(16) \qquad X = \begin{pmatrix} \begin{pmatrix} 0 & -d_0 \\ d_0 & 0 \end{pmatrix} & & \bigcirc \\ & \begin{pmatrix} 0 & -d_1 \\ d_1 & 0 \end{pmatrix} & \\ \bigcirc & & \ddots \end{pmatrix} \in \underline{h}$$

In what follows we replace Z by the more comfortable J , so $J \in \underline{h}$ corresponds to $(1,0,\ldots,0)$ as in (16).

Note $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(2,n)$ iff $A^tA - C^tC = I$, $D^tD - B^tB = I$, and $A^tB = C^tD$, or equivalently $AA^t - BB^t = I$, $DD^t - CC^t = I$, and $AC^t = BD^t$, so

$g^{-1} = \begin{pmatrix} A^t & -C^t \\ -B^t & D^t \end{pmatrix}$. g is in the maximal compact K (of $SO_o(2,n)$) iff $B = 0$, $C = 0$, $\det A = \det D = 1$. $O(2,n)$ preserves the symmetric form τ on $\mathbb{R}^{2+n}$

$$\tau\left(\binom{x}{y},\binom{z}{w}\right) = x_1 z_1 + x_2 z_2 - \sum_{j=1}^{n} y_j w_j \quad ,$$

where $x,z \in \mathbb{R}^2$, $y,w \in \mathbb{R}^n$.

We characterize the identity component $G = SO_o(2,n)$ as follows. Let $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0) \in \mathbb{R}^{2+n}$, and let $p\colon \mathbb{R}^{2+n} \to \mathbb{R}^2$ be the projection. If $g \in O(2,n)$, then $g \in G$ iff $\det g = 1$ and the linearly independent vectors $p(g(e_1))$ and $p(g(e_2))$ have the same orientation as e_1 and e_2 .

Recall that the reflections of $\underline{h}$ coming from conjugations by K (or G) are, for n odd, all permutations and sign changes of the $d_1,\ldots,d_\ell$, and all permutations and even numbers of sign changes of the $d_1,\ldots,d_\ell$ for n even. This finite group we call W .

There is an analogue of Horn's theorem (see section 14) for any semisimple Lie group, one side of which we state here for K acting on its Lie algebra $\underline{k}$ via Ad . Let $\Gamma\colon \underline{k} \to \underline{h}$ be the orthogonal projection.

Proposition 20.1. Let $X \in \underline{h}$. Then $\Gamma(\mathrm{Ad}(K)X)$ is contained in the convex hull of the points $\sigma(X)$ for $\sigma \in W$.

Proof. The proof is essentially word for word that of

Kostant's proof on p. 452 in [16], and is the standard maximization of a continuous function over the compact K argument. In fact, equality holds in the above, but we do not need this.

We record how A acts on $\underline{h}$ by conjugation. If

$$a = \begin{pmatrix} 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \\ \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \end{pmatrix}, \text{ then } \quad \mathrm{Ad}(e^a)\begin{pmatrix} 0 & -d_0 & & 0 \\ d_0 & 0 & & \\ & & 0 & -d_1 \\ 0 & & d_1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & -f_0 & & * \\ f_0 & 0 & & \\ & * & 0 & -f_1 \\ & & f_1 & 0 \end{bmatrix}$$

where $f_0 = d_0 \cosh\theta_1 \cosh\theta_2 - d_1 \sinh\theta_1 \sinh\theta_2$, $f_1 = d_1 \cosh\theta_1 \cosh\theta_2 - d_0 \sinh\theta_1 \sinh\theta_2$. Note the $*$ changes sign if $-a$ is used instead.

Define Φ to be the collection of all ordered bases of $\mathbb{R}^{2+n}$ which are transforms by a $g \in G$ of the standard basis $e_1,\ldots,e_{2+n}$. Let Ω be the set of all pairs (e,f) of linearly independent vectors in $\mathbb{R}^{2+n}$ satisfying $\tau(e,e) = \tau(f,f) = \tau(e,f) = 0$, and such that the $\mathbb{R}^2$ components of e and f are oriented the same as e_1 and e_2 . It is clear that G acts on Ω , and given any $(e,f) \in \Omega$ $\exists$ $r > 0$ and $\underline{g \in K}$ such that $g(re) = (1,0,1,0,\ldots,0)$ and $g(f) = (a,b,a,b,0,\ldots,0)$ for $a,b \in \mathbb{R}$ with $b > 0$.

Finally, we define two closed cones in $\mathbb{R}^{1+\ell}$:

$$E_1 = \{(d_0,\ldots,d_\ell):\ d_0 \geq |d_j|\ \forall\ j\}\ ,\quad \text{and}$$

$$E_0 = \{(d_0,\ldots,d_\ell):\ d_0 \geq \sum_{j=1}^{\ell} |d_j|\}\ .$$

21. Noncompact Convexity in so(2,n) .

The representation of $\mathfrak{g}$ on $\mathbb{R}^{2+n}$ which we use here is quite different from the defining representations of the three previous series of simple causal groups. (Their analogue for so(2,n) is the spin representation.) Here we do not have available Hilbert space arguments to prove ellipticity of the orbits in the interior of the minimal cone, so we rely more heavily on the classification of the orbits. In this section and the next $n \geq 4$ for convenience.

Let $\mathcal{C}$ be the collection of all nontrivial invariant convex cones in $\mathfrak{g}$. We define

$$C_1 = \{X \in \mathfrak{g}:\ \tau(Xe,f) > 0\ \forall\ (e,f) \in \Omega\}$$

and

$$C_0 = \{X \in C_1:\ \tau(Xv_1,v_2) > \sum_{j=1}^{\ell} \left|\tau(Xv_{2j+1},V_{2j+2})\right| \quad \forall\ (v_1,\ldots,v_{2+n}) \in \Phi\}$$

We will see in the next section that C_0 and C_1 are, respectively minimal and maximal open invariant convex cones in $\mathfrak{g}$.

Proposition 21.1. Let $(d_0,\dots,d_\ell) \in \mathbb{R}^{1+\ell}$ such that $d_o > |d_j|$ for $j = 1,\dots,\ell$. Then X as in (16) is in C_1 . If also $d_o > \sum_{j=1}^{\ell} |d_j|$, then X is in C_0 .

Proof. By 20.1 and the remark following the definition of Ω , it suffices to consider a K-conjugate Y of X whose $o(2,2)$ part is $\begin{pmatrix} 0 & -d_0 & & \\ d_0 & 0 & & \\ & & 0 & -d \\ & & d & 0 \end{pmatrix}$ where $|d| \le \max_{j>0} |d_j|$, $\tilde{e} = (1,0,1,0,\dots,0)$, and $\tilde{f} = (a,b,a,b,0,\dots,0)$ where $b > 0$. One computes $\tau(Y\tilde{e},\tilde{f}) = b(d_o-d) > 0$, so $X \in C_1$.

To prove the second statement, we first note that $J \in C_0$. This is clear from $\mathrm{Ad}(G)J = \mathrm{Ad}(K)\mathrm{Ad}(A)J$, the computation of the A-action earlier, and Proposition 20.1. But the X above is clearly in the invariant convex cone generated by G-transforms of J , as $\exists\ \epsilon > 0$ such that $d_o = \sum_{j=1}^{\ell} (|d_j|+\epsilon)$, and $(d_0,d_1,\dots,d_\ell) = (|d_1|+\epsilon,d_1,0,\dots,0) + \dots + (|d_\ell|+\epsilon,0,\dots,0,d_\ell)$. Thus $X \in C_0$ as C_0 is G-invariant.

Lemma 21.2. The G-orbit of an X as in (16) where $d_o > |d_j|$ $(j = 1,\dots,\ell)$ is closed.

Proof. Again we follow the proofs in Lemmas 14.1 and 17.2, and give the necessary estimate. $\exists\ k > 0$ such that $d_o - |d_j| \ge k\ \forall\ j > 0$, and this inequality also holds for the $(d_o,h_1,\dots,h_\ell)$ corresponding to any Y in the

K-orbit of X. We may assume $\theta_1, \theta_2 \geq 0$, and using the earlier computation and the notation there,

$$\begin{aligned} f_0 - f_1 &= d_0 \cosh(\theta_1 + \theta_2) - h_1 \cosh(\theta_1 + \theta_2) \\ &\geq k \cosh(\theta_1 + \theta_2) . \end{aligned}$$

Thus $\theta_1 + \theta_2$ cannot grow too large, and likewise θ_1 and θ_2.

Remark. As $n \geq 4$, $(v_1, \ldots, v_{2+n}) \in \Phi$ implies $(v_1 \pm v_3, v_2 \pm v_4) \in \Omega$, and so if $X \in C_1$ we have $\tau(Xv_1, v_2) > |\tau(Xv_3, v_4)|$. It will turn out by classification that C_1 consists of conjugates of the X considered in the previous lemma.

Definitions. 1) Given $X \in \mathfrak{g}$, let $X = Y + Z$, $Y \in \underline{k}$, $Z \in \underline{p}$. Call $\sqrt{B(Z,Z)}$ the p-norm of X. Call the vector $(d_0, \ldots, d_\ell)$ obtained from $\Gamma(Y) \in \underline{h}$ as in (16) the G-diagonal of X.
2) If $V, U \in \mathbb{R}^{1+\ell}$, say V is in the noncompact convex hull of U if V is equal to a vector in E_0 (see section 20) plus a vector in the convex hull of $\{\sigma(U) : \sigma \in W\}$.

Theorem 21.3. Let $Y \in C_1$ be in the orbit of an X as in the statement of Lemma 21.2. Then the G-diagonal of Y is in the noncompact convex hull of the $(d_0, \ldots, d_\ell)$ there.

Proof. It is exactly the same argument as for $su(p,q)$

and so*(2n) , using the following.

Lemma 21.4. Let X,Y be as in the statement of 21.3, and assume $Y \not\in \underline{k}$. Then there exists Z in the orbit of Y such that $\underline{p}$-norm of Z is strictly less than the $\underline{p}$-norm of Y , and the G-diagonal of Y is in the noncompact convex hull of the G-diagonal of Z .

Proof. The proof is computationally more difficult because the noncompact root spaces are not as easily expressed in terms of our representation. By Proposition 20.1 we may assume the $\underline{k}$-component of Y is in $\underline{h}$. Let $Y = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, $B \neq 0$. For D 2xn we compute

$$(17)\quad \begin{pmatrix} I & \varepsilon D \\ \varepsilon D^t & I \end{pmatrix}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}\begin{pmatrix} I & -\varepsilon D \\ -\varepsilon D^t & I \end{pmatrix} =$$

$$\begin{pmatrix} A+\varepsilon(DB^t-BD^t) & B+\varepsilon(DC-AD) \\ B^t+\varepsilon(D^tA-CD^t) & C+\varepsilon(D^tB-B^tD) \end{pmatrix} + O(\varepsilon^2) \quad .$$

Assume $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ gives rise to $(c_0, c_1, \ldots, c_\ell)$ as in (16). Break B and D up into 2x2 blocks plus possibly (n odd) a 2x1 vector:

$$B = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdots \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right] \qquad D = \left[\begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdots \begin{pmatrix} u \\ v \end{pmatrix}\right] .$$

Then

$$DC - AD = \left[\begin{pmatrix} yc_1+zc_0 & -xc_1+c_0w \\ -c_0x+wc_1 & -c_0y-zc_1 \end{pmatrix} \cdots \begin{pmatrix} c_0v \\ -c_0u \end{pmatrix}\right] .$$

Now $c_0 > |c_j|$ $(j = 1, \ldots, \ell)$ by the Remark in this section, so x,y,z,w , etc. and u,v are determined uniquely by

$$\Big((c_0+c_1)/2\Big)(z+y) + \Big((c_0-c_1)/2\Big)(z-y) = -a$$

$$\Big((c_0+c_1)/2\Big)(w-x) + \Big((c_0-c_1)/2\Big)(w+x) = -b$$

$$-\Big((c_0+c_1)/2\Big)(w-x) + \Big((c_0-c_1)/2\Big)(w+x) = c$$

$$\Big((c_0+c_1)/2\Big)(z+y) - \Big((c_0-c_1)/2\Big)(z-y) = d \text{ , etc.,}$$

and $\alpha = -c_0 v$, $\beta = c_0 u$. Then

$$DC - AD = \left[\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \cdots \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}\right] .$$

We will take ε small and positive.

Then

$$DB^t - BD^t = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}$$

where $P = -za - wb + cx + dy + \ldots -\alpha v + \beta u =$ $\Big((c_0+c_1)/2\Big)\Big\{(z+y)^2 + (x-w)^2\Big\} + \Big((c_0-c_1)/2\Big)\Big\{(y-z)^2 + (x+w)^2\Big\}$ $+ \ldots + c_0(u^2+v^2) > 0$, and

$$D^tB - B^tD = \begin{bmatrix} \begin{pmatrix} 0 & -Q \\ Q & 0 \end{pmatrix} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

where $Q = ya + wc - bx - dz = -\Big((c_0+c_1)/2\Big)\Big\{(z+y)^2 + (w-x)^2\Big\}$ $+ \Big((c_0-c_1)/2\Big)\Big\{(z-y)^2 + (w+x)^2\Big\}$, which is of indeterminant sign, but it is clear that the G-diagonal of the ε terms of (17) are in $-E_0$.

22. Classification of Orbits and Cones in so(2,n) .

Just as the types in the maximal cone of so*(2n) $(n \le 3)$ were composed of indecomposable types coming from so*(4) , we will see that the corresponding orbits for so(2,n) come from the orbits of $o(2,2) \approx sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. In fact, all but two of the indecomposable types for the series o(2,n) , $n \ge 1$ (for the entire algebra), are in O(2,2) or a subalgebra. (Of these two, one is in o(2,3) and one in o(2,4) .)

Temporarily let C_{-1} be the minimal closed convex invariant cone containing J , and let C_2 be the negative of the cone dual to C_{-1} under any $B_\theta(\cdot,\cdot)$. (By the G-invariance C_2 is clearly independent of the particular θ .) We have already seen that C_2 contains $\overline{C}_1$, which in turn contains all X as in (16) with $d_0 \ge |d_j|$ for $j > 0$. Also, we know already that $C_{-1} \subseteq \overline{C}_0$, and that C_{-1} contains all X as in (16) with $d_0 \ge \sum_{j=1}^{\ell} |d_j|$. As $-(E_0)^* = E_1$ (Euclidean metric), a necessary condition for $X \in C_2$ is that all G-diagonals of all conjugates of X lie in E_1 .

It is appropriate to first consider o(2,2) . One $sl(2,\mathbb{R})$ component is spanned by

$$J_1 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & 0 \\ & & 0 & -1 \\ 0 & & 1 & 0 \end{pmatrix} \quad \text{and all} \quad \begin{pmatrix} & 0 & x & y \\ & & y & -x \\ x & y & & \\ y & -x & & 0 \end{pmatrix} ,$$

the other is spanned by

$$J_2 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & 0 \\ & & 0 & 1 \\ 0 & & -1 & 0 \end{pmatrix} \quad \text{and all} \quad \begin{pmatrix} & 0 & x & y \\ & & -y & x \\ x & -y & & \\ y & x & & 0 \end{pmatrix}$$

Of the outer automorphisms from $O(2,2)$, conjugation by $\begin{pmatrix} I & & 0 & \\ & & -1 & 0 \\ 0 & & 0 & 1 \end{pmatrix}$ exchanges the two subalgebras (and exchanges J_1 and J_2), whereas conjugation by $\begin{pmatrix} -1 & 0 & & 0 \\ 0 & 1 & & \\ & & -1 & 0 \\ 0 & & 0 & 1 \end{pmatrix}$ "inverts" each $sl(2,\mathbb{R})$ separately, exchanging the positive and negative cones. In $Ad(G)$ the latter does not exist, but the former does in the sense that $o(2,2) \subset o(2,n)$ and $n \geq 4$. Note $J = \frac{1}{2}(J_1 + J_2)$ with this inclusion.

Recall each indecomposable type for $so(2,n)$ carries a certain signature (p,q) , and we find $(p+q)\times(p+q)$ matrices representing such types.

$\underline{\Delta_m^\varepsilon(0) \ , \quad m \quad \text{even}}$.

A) $\underline{m = 0}$

1) $\varepsilon = +1$; $sig(1,0)$; (0) .

2) $\varepsilon = -1$; $sig(0,1)$; (0) .

B) $\underline{m = 2}$

1) $\varepsilon = +1$; $sig(1,2)$; $\left(\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & -1 \\ 1 & 1 & 0 \end{array}\right)$.

In o(2,2) it is the sum of two nonzero nilpotents in the sl(2,ℝ)'s , one with positive J_1 component, the other with negative such. It cannot contribute to a type in C_2 for so(2,n) .

2) $\varepsilon = -1$; sig(2.1) ; $\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)$; the other sum of two nilpotents; it can contribute to C_2 .

C) m = 4

Only $\varepsilon = -1$ gives a type for so(2,n) ; sig(2,3) ;

$$\left(\begin{array}{cc|ccc} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \sqrt{2} \\ \hline 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 & \sqrt{2} & 0 \end{array}\right) .$$

It is conjugate under K to an element with o(3) component $\begin{pmatrix} 0 & -\sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so cannot contribute.

$\Delta_m(0,0)$, m odd

Only m = 1 is relevant; sig (2,2) ; $\left(\begin{array}{cc|cc} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{array}\right)$.

It is a nonzero nilpotent in one sl(2,ℝ) and 0 in the other; it can contribute.

$\Delta_m^{\varepsilon}(\zeta,-\zeta)$ $(\zeta = -\bar{\zeta} \neq 0)$

A) m = 0

1) $\varepsilon = 1$; sig (2,0) ; $\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$.

2) $\varepsilon = -1$; sig(0,2) ; the same matrix.

B) <u>m = 2</u>

Only $\varepsilon = 1$ is relevant; sig(2,4) ; ($\zeta = i\lambda$)

$$\left(\begin{array}{cc|cccc} 0 & -\lambda & 0 & 0 & 1/\sqrt{2} & 0 \\ \lambda & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\ \hline 0 & 0 & 0 & -\lambda & -1/\sqrt{2} & 0 \\ 0 & 0 & \lambda & 0 & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 & -\lambda \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & \lambda & 0 \end{array}\right)$$

It is conjugate under $SO_o(2,4)$ to a matrix with unchanged λ-entries and non- λ entries multiplied by an arbitrarily large factor, so clearly cannot contribute to C_2 .

C) <u>m = 1</u>

sig(2,2) . The $\varepsilon = \pm 1$ cases are the two possibilities for an elliptic element in one $sl(2,\mathbb{R})$ and a nonzero nilpotent in the other. One can contribute to $\pm C_2$, the other cannot. Let us say $\Delta_1^1(\zeta,-\zeta)$ can contribute. A typical representative is

$$X = \begin{pmatrix} 0 & -\lambda & & \\ \lambda & 0 & & 0 \\ & & 0 & -\lambda \\ 0 & & \lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} .$$

The remaining cases $\Delta_{0,1}(\zeta,-\zeta)$ ($0 \neq \zeta$ real) and $\Delta_o(\zeta,-\zeta,\bar{\zeta},-\bar{\zeta})$ ($\zeta \neq \pm\bar{\zeta}$) are in $o(2,2)$, and correspond to hyp. $\oplus$ (0 or hyp.) , hyp. $\oplus$ nilp. , and hyp. $\oplus$ elliptic.

None can contribute to C_2 .

Now a type involving $\Delta_1^1(\zeta,-\zeta)$ above cannot contribute to C_1 , as $(e,f) = ((1,0,-1,0),(0,1,0,-1)) \in \Omega$, and $Xe = (0,\lambda,0,-\lambda)$, which is τ-orthogonal to f . It follows that $\Delta_2^-(0)$ (nilp. $\oplus$ nilp.) and $\Delta_1(0,0)$ (nilp. $\oplus$ 0) cannot contribute to C_1 , as $\Delta_1^+(\zeta,-\zeta)$ is in the invariant convex cone generated by $\Delta_2^{-1}(0)$, and $\Delta_1(0,0)$ is just $\Delta_1^1(\zeta,-\zeta)$ for $\zeta = 0$, so to speak; the same (e,f) rules out $\Delta_1(0,0)$.

By this analysis and Proposition 21.1, we see that C_1 consists of conjugates of X as in (16) with $(d_0,\ldots,d_\ell) \in (E_1)^\circ$, and $C_2 = \bar{C}_1$. Also, we see that C_0 contains nothing else besides what we already knew it contained, namely, conjugates of X as in (16) with $(d_0,\ldots,d_\ell) \in (E_0)^\circ$. Thus $C_{-1} = \bar{C}_0$.

<u>Theorem 22.1</u>. C_0 and C_1 are open cones in $\mathcal{C}$, and any $C \in \mathcal{C}$ contains either C_0 or $-C_0$ but not both. $\bar{C}_0$ and $\bar{C}_1$ are obtained by changing "$>$" to "$\geq$" in the definitions. Any closed (open) cone in $\mathcal{C}$ containing C_0 is contained in $\bar{C}_1$ (resp. C_1) .

<u>Proof</u>. From the form of C_1 and C_0 exhibited in the paragraph above, to prove C_0 and C_1 open it suffices to show there is a neighborhood of J in C_0 . By the continuity of the eigenvalues there is a neighborhood N

about J such that $\forall X \in N$ X has one eigenvalue near i, one near $-i$, and the rest in a small disc about 0. By the classification of types in $\mathfrak{g}$ the ones near i must be on the imaginary axis and be negatives of each other. If then $X(v_1 + iv_2) = \lambda i(v_1 + iv_2)$, $\lambda > 0$, then $Xv_1 = -\lambda v_2$, $Xv_2 = \lambda v_1$, $\tau(v_1, v_2) = 0$, and $\tau(v_1, v_1), \tau(v_2, v_2) > 0$ are evident if N is small enough. X leaves the τ-complement of $\mathbb{R}v_1 + \mathbb{R}v_2$ invariant, and by the classification of types in $o(n)$ and the continuity of eigenvalues it is clear that $X \in C_0$.

Thus C_0 is open, and any $C \in \mathcal{C}$ with $J \in \bar{C}$ must contain $U \in C_0$, so $J \in C$. (The sum of all W-conjugates of $X \in \underline{h}$ is a multiple of J.) Thus $C_0 \subseteq C$ as before.

The statements about the form of $\bar{C}_0$ and $\bar{C}_1$ follow as before. Finally, we have already noted $\bar{C}_0^* = -\bar{C}_1$, so the remaining statements follow as before.

It is clear from our classification and Theorem 21.3 that types in $\bar{C}_1$ are composed of $\Delta_0^{\pm}(0)$ (a 1x1 zero), $\Delta_2^-(0)$ (nilp. $\oplus$ nilp.), $\Delta_1(0,0)$ (nilp. $\oplus$ zero), $\Delta_1^+(\zeta,-\zeta)$ (nilp. $\oplus$ elliptic), and $\Delta_0^{\pm}(\zeta,-\zeta)$ (the $\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$'s). Furthermore, representatives of $\Delta_2^-(0)$ and $\Delta_1(0,0)$ are conjugate to arbitrarily small multiples of themselves, so they can appear in a type in $\bar{C}_1$ only if they are followed by all $\Delta_0^-(0)$'s. These types are in the minimal closed cone $\bar{C}_0$. Also, representatives of

$\Delta_1^+(\zeta,-\zeta)$ are a sum of representatives of the elliptic types $\Delta_0^{\pm}(\zeta,-\zeta)$ and those of $\Delta_1(0,0)$ (or $\Delta_2^-(0)$) .

Using the maps

$$X \in \mathfrak{g} \to (\tau(Xv_1,v_2),\tau(Xv_3,v_4),\ldots,\tau(Xv_{2\ell+1},v_{2\ell+2}))$$

for $(v_1,\ldots,v_{2+n}) \in \Phi$, the proof of the following parallels the earlier proofs.

Theorem 22.2. The closed (open) non-trivial invariant convex cones in $so(2,n)$ $(n \geq 4)$ containing C_0 are in 1-1 correspondence with the closed (resp. open) convex cones in E_1 (resp. $(E_1)^o$) containing E_0 (resp. $(E_0)^o$) and invariant under the finite permutation and sign-changing group W acting on E_1 .

CHAPTER VI. Global Causality of the Covering Groups.

23. Definitions and Isomorphisms.

If a Lie group G acts transitively on a manifold M, then it is well known that the universal covering group $\tilde{G}$ acts on the universal cover $\tilde{M}$. In fact, if $M = G/H$ for some subgroup H, and if $0 \to D \to \tilde{G} \xrightarrow{\pi} G \to 0$ is the exact sequence of groups, then there is the sequence of covering maps

$$\tilde{G}/\tilde{H}' \to \tilde{G}/\tilde{H} \xrightarrow{\approx} G/H_0 \to G/H$$

where H_0 is the identity component of H, $\tilde{H} = \pi^{-1}(H_0)$, and $\tilde{H}'$ is the identity component of $\tilde{H}$. $\tilde{G}/\tilde{H}'$ is simply connected, so $\tilde{M} \cong \tilde{G}/\tilde{H}'$. (See for example [19, p. 33].) Furthermore, the composition $\tilde{H}' \to \pi^{-1}(H_0) \xrightarrow{\pi} H_0$ is also a covering map.

Recall that G is said to act _effectively_ on M if the identity element $e \in G$ is the only element of G acting trivially on M. It is easily seen that if $M = G/H$, then the subgroup of G acting trivially on M is precisely the largest normal subgroup of G contained in H.

The proof of the following is not difficult, and we will not use it in this chapter.

Proposition 23.1. With the above notation, if G acts effectively on M, then $\widetilde{G}$ acts effectively on $\widetilde{M}$ if and only if $\pi: \widetilde{H}' \to H_0$ is an isomorphism. In any case, the subgroup of $\widetilde{G}$ acting trivially on $\widetilde{M}$ is $D \cap \widetilde{H}'$.

We recall now some of the notions and definitions in Chapter 2 of [19]. A causal structure on a manifold M is a smooth assignment to each $p \in M$ of a non-trivial closed convex cone in the tangent space $T_p(M)$, defined in a neighborhood of each point by a finite number of functions on points and the components of tangent vectors. One say that a Lie group acts causally on M if the group transformations preserve this structure. A causal structure on a Lie group is said to be invariant if it is invariant under both left and right translations. Clearly such a structure is completely determined by a causal cone in the Lie algebra.

In [19] causal structures on groups acting causally on causally oriented manifolds were defined implicitly (Scholium 2.4). As we now have more detailed information on the possibilities for causal structures on simple Lie groups, it seems worthwhile to make the following

Definition 23.2. Given a Lie group G with a causal cone C in its Lie algebra, and a manifold M with causal structure $\{C_p\}_{p\in M}$ on which G acts, say the group action is compatible with the causal orientations, or briefly, G <u>acts temporally</u>, if G acts causally on M and if

$$\Big((d/dt)\exp tX\cdot y\Big)_{t=0} \in C_y$$

for all $X \in C$, $y \in M$. If $\Phi\colon G\times M \to M$ is the group action, these conditions are equivalent to requiring that the differential $d\Phi_{(g,p)}$ map the direct sum of cones $C_g \times C_p$ into $C_{g(p)}$ for all $g \in G$, $p \in M$. (Of course $C_g = dL_gC = dR_gC$.)

<u>Remark</u>. In [19] the cones C_g were essentially defined by projecting the inverse images $(d\Phi_{(g,p)})^{-1}(C_{g(p)})$ onto $T_g(G)$.

We now specialize to the cases of the simple groups considered thus far. We have sometimes regarded $su(p,q)$ as a subalgebra of $sp(p+q,\mathbb{R})$ (section 11) and $so^*(2n)$ as a further restriction of $su(n,n)$. Here we keep the second embedding $so^*(2n) \hookrightarrow su(n,n)$ (detailed below) and consider it in parallel with another map $sp(n,\mathbb{R}) \hookrightarrow su(n,n)$, which was in fact already defined in section 13. This last was accomplished by a linear operator that we now call $\mathcal{C}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, originally denoted U . Regarding $sp(n,\mathbb{R})$ acting as complex-linear operators on $\mathbb{C}^{2n}$, we have

$$\mathcal{C}\, sp(n,\mathbb{R})\, \mathcal{C}^{-1} \equiv sp(n,\mathbb{R})_1 \subseteq su(n,n)$$

with equality only if $n = 1$.

We extend the original symplectic form $\mathcal{A}(\cdot,\cdot)$ on $\mathbb{R}^{2n}$ to a real symplectic form $\mathcal{A}(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ by $\mathcal{A}(ix,y) = 0$, $\mathcal{A}(ix,iy) = \mathcal{A}(x,y)$ $\forall\, x,y \in \mathbb{R}^{2n}$. This form is to be regarded as the $\mathcal{A}$ appearing in the treatment of $so^*(2n)$ (section 17): $\mathcal{A}(x,y) = \tau(x,\sigma y) = \mathrm{Re}\langle x,Jy\rangle_0$, $\langle\cdot,\cdot\rangle_0$ the standard complex Hilbert structure on $\mathbb{C}^{2n}$. One checks that $\mathcal{A}(v,w) = \mathrm{Re}\{iH(\mathcal{C}v,\mathcal{C}w)\}$ for $v,w \in \mathbb{C}^{2n}$, where $H(\cdot,\cdot)$ is the Hermitian form on $\mathbb{C}^{2n}$ defined in section 12, and in fact

$$\mathcal{C}\, so^*(2n)\, \mathcal{C}^{-1} \equiv so^*(2n)_1 \subset su(n,n)$$

for all $n \geq 1$. Thus we can compare directly the matrix elements $\mathcal{A}(Xv,v)$ for X in $sp(n,\mathbb{R})$ and $so^*(2n)$ with the corresponding $\mathrm{Re}\{iH(Yu,u)\}$ for Y in $sp(n,\mathbb{R})_1$, $so^*(2n)_1$.

It is useful to have the explicit forms of these new algebras and their corresponding analytic groups in $SU(n,n)$. We have

$$sp(n,\mathbb{R})_1 = \left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \begin{array}{l} \alpha,\beta \ \ n\times n \ \ \text{complex matrices,} \\ \alpha \ \text{skew Hermitian,} \ \ \beta \ \text{symmetric} \end{array}\right\}$$

and $$\mathcal{C}\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}\mathcal{C}^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where $$\alpha = \tfrac{1}{2}(A-A^t + i(B-C))$$

and $$\beta = \tfrac{1}{2}(-A-A^t+i(B-C)) \quad .$$

Also, $$Sp(n,\mathbb{R})_1 = \left\{\begin{pmatrix} F & G \\ \bar{G} & \bar{F} \end{pmatrix} : G^*F \ \text{symmetric,} \ \ F^*F-G^t\bar{G} = I\right\} ,$$

the two conditions being equivalent to FG^t symmetric and $FF^* - GG^* = I$.

Analogously,

$$so^*(2n)_1 = \left\{\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} : \begin{array}{l} \alpha,\beta \;\; n\times n \;\; \text{complex matrices,} \\ \alpha \;\; \text{skew Hermitian,} \;\; \beta \;\; \text{skew} \end{array}\right\}$$

and

$$\mathcal{C}\begin{pmatrix} A & B \\ -\bar B & \bar A \end{pmatrix}\mathcal{C}^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$$

where $\alpha = \frac{1}{2}(A + \bar A + i(B + \bar B))$ and $\beta = \frac{1}{2}(-A + \bar A + i(B - \bar B))$.
Finally,

$$SO^*(2n)_1 = \left\{\begin{pmatrix} F & G \\ -\bar G & \bar F \end{pmatrix} : \begin{array}{l} G^*F \;\; \text{skew,} \\ F^*F - G^t\bar G = I \end{array}\right\} ,$$

the two conditions equivalent to FG^t skew and $FF^* - GG^* = I$.

<u>Remark</u>. By examining the maximal tori $\underline{h}_1$ in $sp(n,\mathbb{R})_1$ and $so^*(2n)_1$ corresponding to the $\underline{h}$ for $sp(n,\mathbb{R})$ and $so^*(2n)$, and comparing them with the $\underline{h}$ for $su(n,n)$, we have the following: the invariant positive cone in $sp(n,\mathbb{R})_1$ and the "middle" (self-dual) cone C_1 in $so^*(2n)_1$ are contained in the minimal cone C_0 for $su(n,n)$. However, the maximal cone C_2 of $so^*(2n)_1$ extends outside the maximal cone C_1 of $su(n,n)$.

24. <u>Causal Actions on Šhilov Boundaries</u>.

We recall the standard action of $SU(n,n)$ on the group of unitaries $U(n)$ by fractional linear transformations [19, p. 35]. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,n)$ and $U \in U(n)$, define

$$(\rho(g))U = (AU + B)(CU + D)^{-1} \in U(n) .$$

ρ defines a (left) group action, and the action is causal. ($U(n)$ has the standard invariant causal structure determined by positive-definite Hermitians.)

Let

$$U_s(n) = \{U \in U(n): U \text{ symmetric}\}$$

and (only for n even!)

$$U_k(n) = \{U \in U(n): U \text{ skew}\} .$$

One checks that $Sp(n,\mathbb{R})_1$ and $SO^*(2n)_1$ transform $U_s(n)$ $U_k(n)$, respectively. Restricting the causal structure of $U(n)$ to these closed submanifolds, one finds that the resulting cone fields on $U_s(n)$ and $U_k(2n)$ are non-trivial for all $n \geq 1$. Thus $Sp(n,\mathbb{R})_1$ and $SO^*(4n)_1$ act causally on $U_s(n)$ and $U_k(2n)$, respectively. (These two actions and the action of $SU(n,n)$ on $U(n)$ are individually isomorphic to the actions of each group G on the Shilov boundary of its associated Hermitian symmetric domain G/K .)

<u>Theorem 24.1</u>. $SU(n,n)$ acts temporally on $U(n)$, with the causal structure on $SU(n,n)$ obtained from the maximal cone $\bar{C}_1$ in $su(n,n)$ described earlier.

Proof. As $SU(n,n)$ acts causally on $U(n)$ and its causal structure is invariant, it suffices to check the second condition in Definition 23.2 for $g = e$; we compute the differential of ρ at this point. For $X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ in the Lie algebra, $(U + \epsilon AU + \epsilon B)(\epsilon B^*U + I + \epsilon C)^{-1} = U(I + \epsilon L) + o(\epsilon^2)$, where $L = U^{-1}AU - C - B^*U + U^{-1}B$. We check that iL is positive-definite Hermitian if $X \in \bar{C}_1$.

Given $y \in \mathbb{C}^n$ set $x = Uy$, and note

$$\langle iLy,y\rangle = i\langle Ax,x\rangle - i\langle Cy,y\rangle \\ -i\langle B^*x,y\rangle + i\langle By,x\rangle \quad .$$

In fact, this expression is just $iH(Xw,w)$ for $w = \begin{pmatrix} x \\ y \end{pmatrix}$ and $H(w,w) = 0$, so $\langle iLy,y\rangle \geq 0 \;\forall\; y \in \mathbb{C}^n$ by the definition of $\bar{C}_1$.

In particular, $SU(n,n)$ acts temporally with the causal structure obtained from the minimal cone $\bar{C}_0$, and by the last remark in Section 23 we have the

Corollary 24.2. For all $n \geq 1$, $Sp(n,\mathbb{R})_1$ acts temporally on $U_s(n)$ (with the essentially unique causal structure on $Sp(n,\mathbb{R})_1$) . For the causal structures on $SO^*(2n)_1$ obtained from the self-dual cones $\bar{C}_1$, $SO^*(2n)_1$ acts temporally on $U_k(n)$ (for $n \geq 4$ even) and on $U(n)$ (for all $n \geq 3$) .

The case above of $SO^*(2n)$, n even, can be strengthened.

Theorem 24.3. For $n \geq 2$, $SO^*(4n)$ acts temporally on $U_k(2n)$, with the causal structure on $SO^*(4n)$ obtained from the maximal cone $\bar{C}_2$.

Proof. As in the previous proof, we must show that $iL = i(U^{-1}\alpha U - \bar{\alpha} + \bar{\beta}U + U^{-1}\beta)$ is positive-definite Hermitian if U is unitary and skew and $\mathcal{C}^{-1}\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\mathcal{C} \in \bar{C}_2$ of $so^*(2n)$.

To simplify the argument we use the fact that any $U \in U_k(2n)$ may be written as $U_1 D U_1^t$ where

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(as in [12, p. 6]) and U_1 is unitary. (This can be proven by the usual diagonalization arguments, but abstractly one knows that K acts transitively on the Shilov boundary [15, p. 270].) Then

$$(18) \quad \langle iLv,v\rangle = i\langle \alpha_1 Dw,Dw\rangle + i\langle \bar{\beta}_1 Dw,w\rangle - i\langle \bar{\alpha}_1 w,w\rangle + i\langle \beta_1 w,Dw\rangle$$

where $w = U_1^t v$, $\alpha_1 = U_1^{-1}\alpha U_1$, $\beta_1 = U_1^{-1}\beta U_1^{t-1}$. Then $\mathcal{C}^{-1}\begin{pmatrix} \alpha_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix}\mathcal{C} \in \bar{C}_2$ also, and we drop the subscript.

Given any $w \in \mathbb{C}^n$ let $e = \begin{pmatrix} w \\ -iDw \end{pmatrix}$, and check that $\tau(e,e) = \tau(e,\sigma e) = 0$ (as in Section 17), so $\mathcal{A}(Xe,e) \geq 0$ $\forall X \in \bar{C}_2$. By the identity of the last section

$$(19) \quad iH\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\mathcal{C}e,\mathcal{C}e\right) \geq 0 .$$

Now $e = \binom{Dv}{v}$ for some v, and in fact (19) equals (18) with α,β,v replacing α_1,β_1,w respectively, which completes the proof.

Before stating the consequences of the above for global causality we consider $SO_o(2,n)$ and its action on the projective quadric

$$\underset{\sim}{Q} = \{[x \in \mathbb{R}^{2+n}: \tau(x,x) = 0]\} ,$$

$[x]$ meaning the line determined by x. (See [19], Section II.4.) The tangent space to $[x] \in \underset{\sim}{Q}$ can be identified with the projective space of vectors $y \in \mathbb{R}^{2+n}$ such that $\tau(y,x) = 0$, modulo $\mathbb{R}x$. τ factors to a linear conformal structure τ_1 on $\underset{\sim}{Q}$ with signature $(1,n)$. As $S^1 \times S^n$ is a double cover of $\underset{\sim}{Q}$ a system of forward cones can easily be chosen.

<u>Theorem 24.4</u>. $SO_o(2,n)$ acts temporally on $\underset{\sim}{Q}$, with the causal structure on the group coming from a closed maximal invariant convex cone in the Lie algebra.

<u>Proof</u>. Let $\overline{C}_1$ be the cone defined earlier. (See Sections 20 and 21 for notation.) Then $X \in \overline{C}_1$ iff $\tau(Xe,f) \geq 0 \ \forall \ (e,f) \in \Omega$.

This condition has a pleasant geometric interpretation. Given any $[e] \in \underset{\sim}{Q}$ and $X \in \mathfrak{g}$, $Xe \in T_e(\mathbb{R}^{2+n})$ as usual, and with care we can write $[Xe] \in T_{[e]}(\underset{\sim}{Q})$, as $\tau(Xe,e) = 0$ always. Now the projective space of $f \in \mathbb{R}^{2+n}$ such that

$\tau(f,f) = \tau(f,e) = 0$ and $(e,f) \in \Omega$, modulo $\mathbb{R}e$, gives one of the cones $\tilde{C}$ (not convex) in $T_{[e]}(\underline{Q})$ of τ_1-isotropic vectors. Then $\tau(Xe,f) \geq 0 \; \forall \; (e,f) \in \Omega$ implies $\tau_1([Xe],w) \geq 0 \; \forall \; w \in \tilde{C}$, which implies (as τ_1 has signature $(1,n)$) that $[Xe]$ is a tangent vector in the convex cone generated by $\tilde{C}$. Thus $[Xe]$ is a forward-pointing tangent vector at $[e]$ for all $[e] \in \underline{Q}$.

Definition. A manifold M with causal structure is called globally causal if there exists no closed non-trivial piecewise C^1 curve in M the differential of which lies in the causal cone at each point.

Theorem 24.5. For all $n \geq 1$, $\widetilde{Sp(n,\mathbb{R})}$, $\widetilde{SU(n,n)}$, $\widetilde{SO^*(4n)}$, and $\widetilde{SO_o(2,n+2)}$ are globally causal with respect to their causal structures coming from a maximal causal cone in their Lie algebras. $\widetilde{SU(p,q)}$ $(p > q \geq 1)$ is globally causal with respect to the minimal cone $(\overline{C}_0)$ causal structure, and $\widetilde{SO^*(4n+2)}$ $(n \geq 1)$ is globally causal with respect to the causal structure coming from the self-dual cone $\overline{C}_1$.

Proof. The first four series of groups act temporally on $\widetilde{U_s(n)}$, $\widetilde{U(n)}$, $\widetilde{U_k(2n)}$, and $\mathbb{R} \times S^n$, respectively, by the initial remarks in section 23 and Theorems 24.1-4. $\widetilde{U(n)}$ and $\mathbb{R} \times S^n$ are globally causal by Corollary 2.3.1 and Scholium 2.11 in [19].

To identify $\widetilde{U_s(n)}$ and $\widetilde{U_k(2n)}$, we set

$$M_s(n) = \{U \in U_s(n) : U \in SU(n)\} = \{UU^t : U \in SU(n)\}$$

and

$$M_k(2n) = \{UDU^t : U \in SU(2n)\} \quad ,$$

D as in 24.3. They are coset spaces, isomorphic to $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$, respectively, and are simply connected because $SU(n)$ is simply connected and the factor spaces are connected.* Thus

$$\widetilde{U_s(n)} \cong M_s(n) \times \mathbb{R} \quad \text{and} \quad \widetilde{U_k(2n)} \cong M_k(2n) \times \mathbb{R} ,$$

and the spaces on the r.h.s. are contained component-wise in $SU(n) \times \mathbb{R} \cong \widetilde{U(n)}$. Clearly then $\widetilde{U_s(n)}$ and $\widetilde{U_k(2n)}$ are also globally causal.

We can assume that any closed time-like curve $g(t)$ $(0 \leq t \leq 1)$ in any of the covering groups $\widetilde{G}$ starts and ends at $e \in \widetilde{G}$. As the action of $\widetilde{G}$ on the relevant manifold $\widetilde{S}$ is temporal, $g(t)p$ $(\forall p \in \widetilde{S})$ is a closed time-like curve in $\widetilde{S}$, contradicting the global causality unless $g(t)p = p \; \forall \; t$, so $g(t) = e \; \forall \; t$.

The last statement follows in the same way by Corollary 24.2, and the next-to-last by the embedding $su(p,q) \hookrightarrow su(p,p)$, under which the minimal cone of $su(p,q)$ goes into the minimal cone of $su(p,p)$.

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*I thank Peter Greenberg for these observations.

Corollary 24.6. If $\widetilde{G}$ is any of the above (24.5) groups with the causal cone C indicated, then there exists a semigroup S in $\widetilde{G}$ such that $S \cap S^{-1} = \{e\}$ and $gSg^{-1} = S$ for all $g \in \widetilde{G}$, generated by $\{\exp X: X \in C\}$.

Proof. If $\exp X_1 \ldots \exp X_n = e$, $X_j \in C$, the expression gives a piecewise C^1 time-like curve in $\widetilde{G}$, so all X_j must be 0. This suffices to get $S \cap S^{-1} = \{e\}$, and $gSg^{-1} = S$ is clear.

Remarks. In the theory of Hermitian symmetric spaces the first four series of groups are referred to as of "tube type", the latter two of "non-tube type". (The e_7 domain is also of tube type, the e_6 domain not.) Only in the case of tube type domains does the Šhilov boundary unravel and have a globally causal universal covering. (For example, the Šhilov boundary for $SU(2,1)$ is isomorphic to S^3.) Thus proving the global causality with the maximal causal structure may require some new ideas in the non-tube type case.

In the former cases one can ask whether the semigroup of Corollary 24.6 is closed and in fact the entire set of forward displacements of the universal covering of the Šhilov boundary coming from the group action. I believe that it is, but this question may require a consideration of the classification of the conjugacy classes in the groups. [2] also gives a treatment of this, a simple extension of the Lie algebra classification. It is interesting that here the Cayley transform (see section 29) appears naturally.

Based on the analogy with $\widetilde{SL(2,\mathbb{R})}$ and some other cases, I would conjecture that any forward displacement is a product of <u>two</u> exponentials as in 24.6, one of which is some $\operatorname{Exp} tZ$, $Z \in Z(k)$, $t \geq 0$, the other in the range of the Cayley transform restricted to the forward maximal cone C . (Any such transform is also an exponential of an $X \in C$.)

CHAPTER VII. Kählerization in Causal Lie Algebras.

25. Uniqueness of the Complex Structure.

We have seen that each X in the interior of a maximal open invariant convex cone in the Lie algebras thus far considered, is conjugate to an element in an open cone in the maximal torus h of k . In particular, such X are elliptic, i.e., in some maximal compact subalgebra. In fact, this maximal compact is uniquely determined by X . Such uniqueness does not hold in general for non-positive elliptics, but indications are that it holds for at least a "generic" set (Proposition 25.3). We call this mapping of elliptics to maximal compacts Kählerization*.

Uniqueness in the symplectic case, from the standpoint of linear quantum fields, was first shown by Weinless in his thesis [20]. His proof involved holomorphic functions in the upper half-plane and particularly some results of Goodman [8] on such functions. Here we give two fairly straightforward proofs of the above claim. The first is an argument in terms of real Hilbert spaces, finite- or infinite-dimensional, applicable to the open invariant cone in $sp(n,\mathbb{R})$, the minimal open cone in $su(p,q)$, and the

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*Although, this is a term perhaps best reserved for only the symplectic groups.

"middle" cone C_1 in $so^*(2n)$. The second proof is more non-linear, involving the action of G on the symmetric space G/K , but covers all the maximal open cones.

Consider first $sp(\mathcal{H})$, where $(\mathcal{H}, \mathcal{S}_0)$ is a real Hilbert space possessing an orthogonal J with square $-I$. (We use the notation in section 7, and again consider only bounded operators.) The extension appropriate for our purposes of the notion of "maximal compact subalgebra" of $sp(\mathcal{H})$ to infinite dimensions* is as follows. Such a subalgebra is all those elements skew with respect to a complex Hilbert structure on $\mathcal{H}$, topologically equivalent to and having the same imaginary part $\mathcal{A}(\cdot,\cdot)$ as the original one. A complex structure in $sp(\mathcal{H})$ is a $J_1 \in sp(\mathcal{H})$ such that $J_1^2 = -I$ and $\mathcal{A}(J_1\cdot,\cdot)$ is a positive-definite symmetric form on $\mathcal{H}$ defining an equivalent norm. Denote the set of complex structures by Γ . One checks that $\Gamma \subset Sp(\mathcal{H})$ also, and that Γ is in 1-1 correspondence with the afore-mentioned set of complex Hilbert structures. Furthermore, the correspondence extends to the set of maximal compacts: given $J_1 \in \Gamma$, the corresponding maximal compact $\underline{k}$ is all $X \in sp(\mathcal{H})$ commuting with J_1 . That $\underline{k}$ in fact determines J_1 is

*"Quasi-compact" in [18], but we shall refrain from introducing this term as the compactness of the corresponding group in the finite-dimensional case is not used here.

clear from the observation that no two distinct complex structures commute: if say J_1 commuted with the original J, then $J^{-1}J_1 = D$ is positive-definite symmetric in $(\mathcal{H}, \mathfrak{s}_o)$ and has square I, hence equals I and $J_1 = J$. (This argument is a special case of 25.2.)

By the unitary equivalence of Hilbert spaces with the same dimension, all maximal compacts in $sp(\mathcal{H})$ are conjugate under $Sp(\mathcal{H})$, and Γ is a single orbit in the positive cone in $sp(\mathcal{H})$.

Now $U(\mathcal{H}, \mathcal{J})$ and $SO^*(\mathcal{H})$ (see sections 11 and 16) are further restrictions of this situation. To pass to $u(\mathcal{H}, \mathcal{J})$ from $sp(\mathcal{H})$, one takes an $i \in sp(\mathcal{H})$ such that $i^2 = -I$, $iJ = Ji \equiv \mathcal{J}$, and $\mathcal{A}(i\cdot,\cdot)$ <u>not</u> positive- or negative-definite (in finite dimensions the (real) signature being $(2p,2q)$, $p,q \geq 1$), and defines $u(\mathcal{H},\mathcal{J}) = \{X \in sp(\mathcal{H}) : Xi = iX\}$. (All $i \in sp(n,\mathbb{R})$ such that $i^2 = -I$ and $\mathcal{A}(i\cdot,\cdot)$ has a particular signature, also form a single $sp(n,\mathbb{R})$-orbit.) This embeds $u(p,q) \subset sp(p+q,\mathbb{R})$, and one sees that maximal compact subalgebras of $su(p,q)$ are in 1-1 correspondence with $\Gamma \cap u(p,q)$, a single $u(p,q)$-orbit, which lies in a minimal open invariant cone in $su(p,q)$ iff $p = q$. If $p \neq q$, $\Gamma \cap u(p,q)$ projects injectively onto an $su(p,q)$-orbit, each element of which spans the center of some maximal compact.

As shown in section 16, $so^*(\mathcal{H})$ is a further restriction on $u(\mathcal{H},\mathcal{J})$ in the case when the dimensions of the eigenspaces of $\mathcal{J}$ are equal. In finite dimensions one has $so^*(2n) \subset su(n,n) \subset sp(2n,\mathbb{R})$, as follows. $so^*(2n)$ is defined by taking $i,\sigma \in sp(2n,\mathbb{R})$ which arise as in section 16, and letting

$$so^*(2n) = \{X \in sp(2n,\mathbb{R}) : Xi = iX,\ X\sigma = \sigma X\} \quad .$$

The intersection of the open positive invariant cone in $sp(2n,\mathbb{R})$ with $so^*(2n)$ is the cone C_1 above, and furthermore maximal compacts of $so^*(2n)$ are in 1-1 correspondence with $\Gamma \cap so^*(2n)$, a single $SO^*(2n)$-orbit.

The purpose of the preceeding two paragraphs is to make the uniqueness argument of 25.2 applicable to the indicated cones in $su(p,q)$ and $so^*(2n)$. We see that maximal compacts in these two algebras are contained in unique maximal compacts in a symplectic algebra, so uniqueness for $sp(n,\mathbb{R})$'s open cone C_0 implies uniqueness for the cones obtained by intersecting C_0 with $su(p\ q)$ and $so^*(2n)$.

There is a simple parametrization of the set of positive complex structures Γ for $sp(\mathcal{H})$. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$, and A^t the transpose, with respect to $s_0(\cdot,\cdot) = \mathcal{A}(J\cdot,\cdot)$, of $A \in \mathcal{L}(\mathcal{H})$. We have

$$Sp(\mathcal{H}) = \{g \in \mathcal{L}(\mathcal{H}) : g^tJg = J\} \quad \text{and}$$
$$sp(\mathcal{H}) = \{X \in \mathcal{L}(\mathcal{H}) : X^tJ + JX = 0\} = \{JH : H \in \mathcal{L}(\mathcal{H}),\ H^t = H\} \ .$$

If $K \in \Gamma$ then $K = JD$ where $D^t = D$ is positive-definite symmetric (with respect to $\mathcal{S}_o$) . $K^2 = -I$ implies $J = DJD$, so $D \in Sp(\mathcal{H})$.

In the finite-dimensional case we have in fact $D \in \exp \underline{p}$. D is the exponential of a unique symmetrix X , and $D^{-1} = JDJ^{-1}$ is equivalent to $-X = J^{-1}XJ$; together with $X^t = X$ this implies $X \in \underline{p}$. Reversing the steps, we have

$$\Gamma = \{J \exp X \colon X \in \underline{p}\} \quad .$$

Returning to the general case, note that (positive-definite) symmetric forms $\mathcal{S}(\cdot,\cdot)$ on $\mathcal{H}$ are in linear 1-1 correspondence with (positive-definite, resp.) symmetric operators $\widetilde{\mathcal{S}}$ on $(\mathcal{H},\mathcal{S}_o)$, via $\mathcal{S}(\cdot,\cdot) = \mathcal{S}_o(\widetilde{\mathcal{S}}\cdot,\cdot)$.

<u>Proposition 25.1</u>. A positive-definite symmetric form $\mathcal{S}(\cdot,\cdot)$ can be the real part of a topologically equivalent complex Hilbert structure on $\mathcal{H}$ with imaginary part $\mathcal{A}(\cdot,\cdot)$, if and only if $\widetilde{\mathcal{S}} \in Sp(\mathcal{H})$.

<u>Proof</u>. If $R \in \Gamma$ and $\mathcal{S}(\cdot,\cdot) = \mathcal{A}(R\cdot,\cdot)$, then $\mathcal{A}(R\cdot,\cdot) = \mathcal{S}(\widetilde{\mathcal{S}}\cdot,\cdot) = \mathcal{A}(J\widetilde{\mathcal{S}}\cdot,\cdot)$, so $\widetilde{\mathcal{S}} = J^{-1}R \in Sp(\mathcal{H})$. Conversely, $\widetilde{\mathcal{S}} \in Sp(\mathcal{H})$ implies $R = J\widetilde{\mathcal{S}} \in Sp(\mathcal{H})$ and $R^2 = J(\widetilde{\mathcal{S}}J\widetilde{\mathcal{S}}) = J^2 = -I$.

<u>Theorem 25.2</u>. Suppose $X \in sp(\mathcal{H})$ and $\mathcal{A}(Xv,v) \geq k\mathcal{S}_o(v,v)$ $\forall\, v \in \mathcal{H}$, for some $k > 0$. Then X commutes with a unique complex structure.

Proof. The existence of one such complex structure is clear from Theorem 7.1, and so it suffices to assume $JX = XJ$. If now $XK = KX$ for some $K \in \Gamma$, write $K = JD$ and $X = JH$ as above. The hypothesis on X implies that H is positive-definite symmetric in $(\mathcal{X}, s_o)$. Further, we have $(JH)D = D(JH)$ as $JH = HJ$, so H^2 commutes with D. Thus H commutes with D by the functional calculus, and $D(JH) = (JH)D = J(HD) = J(DH)$. As H is surjective, $JD = DJ$ also. Now $J = DJD$ and D is positive-definite, so $D^2 = I$, hence $D = I$ and $K = J$.

Mostly for the sake of convenience, let us restrict to the finite-dimensional case for the rest of this section. It is not difficult to see that $X \in sp(n, \mathbb{R})$ is elliptic iff X is diagonalizable over $\mathbb{C}$ with purely imaginary eigenvalues. The condition that X is in a unique maximal compact subalgebra is fairly simple.

Proposition 25.3. Let $X = JH = HJ \in sp(n, \mathbb{R})$. Then H is in a unique maximal compact subalgebra iff H and $-H$ have no eigenvalues in common.

Proof. Let $\mathbb{R}^{2n} = V_1 \oplus \ldots \oplus V_m$ be the eigenspace decomposition of H, where $H = \alpha_j$ on V_j, $\alpha_j \in \mathbb{R}$. The V_j are s_o- and $\mathcal{A}$-orthogonal and J-invariant, so all $\dim V_j$ are even. Let JD, D positive-definite symmetric and symplectic, be a candidate for an additional complex structure commuting with JH. As above, D

commutes with H^2 , hence with $|H|$, so the sufficiency is clear as in the proof of 25.2. Conversely, if say $\alpha_1 = 0$, then any $D \in \exp \underline{p}$ on V_1 is possible, and there is no uniqueness. Secondly, if say $\alpha_1 = -\alpha_2 \neq 0$, it suffices to see that there are lots of complex structures in $sp(2,\mathbb{R})$ commuting with the elliptic

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

These can be determined, and are a two-parameter family: L commutes with all

$$JD = \begin{bmatrix} 0 & a & x & b \\ a & 0 & b & x \\ -x & b & 0 & -a \\ b & -x & -a & 0 \end{bmatrix}$$

where $x > 0$ and $1 + a^2 + b^2 = x^2$. (The eigenvalues of D are $x \pm \sqrt{x^2-1}$, each with multiplicity two.)

We next give the second proof of uniqueness mentioned at the start. The idea is to note that each maximal compact subgroup leaves fixed a unique point in G/K , and also that one-parameter groups generated by an X in an open maximal cone also have this property. The first statement is a general fact about symmetric spaces, and is sharpened in the Hermitian symmetric case as follows (Theorem 3.2, Chapter IX [9]). Let G be a simple Lie group and K a maximal compact subgroup, such that G/K is Hermitian symmetric, and $\mathfrak{g},\underline{k},\underline{p}$ as usual. Let x_o be the identity coset in G/K and $J \in Z(\underline{k})$ such that $\mathrm{ad}(J)\colon \underline{p} \to \underline{p}$ is a complex structure (see section 3). Then $\mathrm{Ad}(\exp \pi J)$ is the Cartan involution for $\mathfrak{g} = \underline{k} + \underline{p}$, and the geodesic symmetry (reflection) about x_o is the action of $\exp(\pi J)$ on G/K. x_o is the unique fixed point of the one-parameter group generated by J.

Now consider the standard actions of $SU(p,q)$, $Sp(n,\mathbb{R})_1$, and $SO^*(2n)_1$ (see section 23) on the appropriate bounded domain in some $\mathbb{C}^m$ by fractional linear transformations [14, p. 215]. To prove the second assertion above, it suffices to consider X in the maximal torus $\underline{h}$.

<u>SU(p,q)</u> Take $(\lambda_1,\ldots,\lambda_p,\sigma_1,\ldots,\sigma_q) \in \mathbb{R}^{p+q}$ such that $\lambda_i + \sigma_j > 0$ i,j. If Z is any $p\times q$ complex matrix then clearly

$$\begin{pmatrix} e^{-ti\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{-ti\lambda_p} \end{pmatrix} Z \begin{pmatrix} e^{-ti\sigma_1} & & 0 \\ & \ddots & \\ 0 & & e^{-ti\sigma_q} \end{pmatrix} = Z \ \forall\, t \in \mathbb{R}$$

iff $Z = 0$.

<u>Sp(n,ℝ)₁</u> Let $(d_1,\ldots,d_n) \in \mathbb{R}^n$ such that all $d_j > 0$. If Z is any $n\times n$ symmetric complex matrix then

$$(20) \qquad \begin{pmatrix} e^{-tid_1} & & 0 \\ & \ddots & \\ 0 & & e^{-tid_n} \end{pmatrix} Z \begin{pmatrix} e^{-tid_1} & & 0 \\ & \ddots & \\ 0 & & e^{-tid_n} \end{pmatrix} = Z \ \forall\, t \in \mathbb{R}$$

iff $Z = 0$.

$\underline{SO^*(2n)_1}$ Let $(d_1,\ldots,d_n) \in \mathbb{R}^n$ such that $d_j + d_k > 0$ $\forall j \neq k$. If Z is any nxn skew-symmetric complex matrix, then clearly (20) holds $\forall t \in \mathbb{R}$ iff $Z = 0$.

Finally, consider $so(2,n)$. The action of $SO_o(2,n)$ on its bounded domain is a bit of a mess [21, p.168], but the $K = SO(2) \times SO(n)$ -action is simple in a particular representation. In fact it is equivariant with the linear adjoint action of K on $\underline{p}$ [15]. So, let $d_o > |d_j|$ $(j = 1,\ldots,[n/2])$ give an element in a maximal open cone as in (16), set $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and let B be $2xn$. Then

$$\begin{pmatrix} R_{d_o} & 0 \\ 0 & R_{d_j} \end{pmatrix}\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}\begin{pmatrix} R_{-d_o} & 0 \\ 0 & R_{-d_j} \end{pmatrix} = \begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}$$

where $C = R_{d_o} B R_{-d_j}$. One checks easily that

$$\frac{d}{dt}(BR_{td_j})_{t=0} = \frac{d}{dt}(R_{td_o}B)_{t=0}$$

iff $B = 0$.

As a result, we have proven

Theorem 25.4. Let $\mathfrak{g}$ be one of the simple causal Lie algebras $sp(n,\mathbb{R})$, $su(p,q)$, $so^*(2n)$, $so(2,n)$. Then each $X \in \mathfrak{g}$ in an open maximal invariant convex cone is contained in a unique maximal compact subalgebra.

26. Generalizations to Unbounded Operators.

In this section we extend the discussion in section 7 to some unbounded infinitesimally symplectic operators. Using the notation there, we consider real-linear, possibly unbounded $S: \mathcal{D}_S \to \mathcal{H}$ in $(\mathcal{H}, \langle\ ,\ \rangle)$ having dense domain $\mathcal{D}_S \subseteq \mathcal{H}$ and satisfying $\mathcal{A}(Sx,y) + \mathcal{A}(x,Sy) = 0 \ \forall\ x,y \in \mathcal{D}_S$. We raise the question: when can we regard S as a skew-adjoint operator on some complex Hilbert space $(\tilde{\mathcal{H}}, \langle\ ,\ \rangle_1)$ having a "dense intersection" with $\mathcal{H}$ and such that $\mathrm{Im}\langle\ ,\ \rangle = \mathrm{Im}\langle\ ,\ \rangle_1$ on that intersection? On this question mention should be made of [3] and [18].

When S satisfies a positivity condition on its domain, we find a canonical $\tilde{\mathcal{H}}$, and $\tilde{\mathcal{H}} \cap \mathcal{H}$ contains a certain "finite-energy" subspace, in complete analogy with the positive-mass Klein-Gordan wave equation in Minkowski space. However, the formulation of a uniqueness statement like Theorem 25.2 is not yet clear.

We first recall the Friedrichs extension. Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be a real or complex Hilbert space and H an Hermitian operator on $\mathcal{H}$ with dense domain $\mathcal{D}_H$, bounded from below by $k > 0$:

$$(21) \qquad \langle Hv,v\rangle \ \geq\ k\langle v,v\rangle \ \forall\ v \in \mathcal{D}_H \ .$$

(If $\mathcal{H}$ is complex we assume H is complex-linear.) H extends to a surjective self-adjoint $A: \mathcal{D}_A \to \mathcal{H}$, also bounded below by k, as follows. Let $[\mathcal{D}_H]$ be the completion of $\mathcal{D}_H$ in $\langle\cdot,\cdot\rangle_\varepsilon \equiv \langle H\cdot,\cdot\rangle$, and extend $\langle\cdot,\cdot\rangle_\varepsilon$ to a real Hilbert structure on $[\mathcal{D}_H]$. $([\mathcal{D}_H],\langle\cdot,\cdot\rangle_\varepsilon)$ is the so-called "finite-energy" subspace. Let $\mathcal{D}_A = \mathcal{D}_{H*} \cap [\mathcal{D}_H]$ and $A = H^*|_{\mathcal{D}_A}$. A consequence of the proof is $\langle x,y\rangle_\varepsilon = \langle Ax,y\rangle \ \forall\ x \in \mathcal{D}_A$, $y \in [\mathcal{D}_H]$, and it is not hard to see that $[\mathcal{D}_H] = [\mathcal{D}_A]$.

Typically, the inclusion $\mathcal{D}_A \subsetneq [\mathcal{D}_A]$ is proper.

<u>Lemma 26.1</u>. $([\mathcal{D}_A],\langle\cdot,\cdot\rangle_\varepsilon)$ is unitarily equivalent to $(\mathcal{H},\langle\cdot,\cdot\rangle)$ via the densely defined unitary map

$$U = (A^{-1})^{\frac{1}{2}} \circ A: \mathcal{D}_A \to \mathcal{H} \to \mathcal{R}_{(A^{-1})^{\frac{1}{2}}}$$

where $\mathcal{R}_{(A^{-1})^{\frac{1}{2}}} \supseteq \mathcal{D}_A$ is the range of $(A^{-1})^{\frac{1}{2}}$. Thus $\mathcal{D}_A = [\mathcal{D}_A]$ iff A is bounded.

<u>Proof</u>. By a remark above, if $x,y \in \mathcal{D}_A$

$$\langle x,y\rangle_\varepsilon = \langle Ax,y\rangle = \langle Ax,A^{-1}Ay\rangle = \langle Ux,Uy\rangle .$$

Let $\mathcal{D}_{A_1} = A^{-1}([\mathcal{D}_A])$ and $A_1 = A|_{\mathcal{D}A_1}$. Then $A_1 : \mathcal{D}_{A_1} \to [\mathcal{D}_A]$ is surjective.

<u>Lemma 26.2</u>. In $([\mathcal{D}_A],\langle\cdot,\cdot\rangle_\varepsilon)$, $\mathcal{D}_{A_1}$ is dense (-also in $(\mathcal{H},\langle\cdot,\cdot\rangle)$) and A_1 is self-adjoint and bounded from below also by k .

<u>Proof</u>. That $\mathcal{D}_{A_1}$ is dense and A_1 Hermitian is easily shown. To see that A_1 is self-adjoint, let $y \in \mathcal{D}_{(A_1)^*} \subseteq [\mathcal{D}_A]$, i.e. $\exists\ y^* \in [\mathcal{D}_A]$ such that $\langle x,y^*\rangle_\varepsilon = \langle A_1x,y\rangle_\varepsilon$

$\forall\ x \in \mathcal{D}_{A_1}$. Let $A_1 x = x^* \in [\mathcal{D}_A]$, and obtain $\langle x^*,y^*\rangle = \langle x^*,y\rangle_\varepsilon$ for all $x^* \in [\mathcal{D}_A]$. If we assume only $x^* \in \mathcal{D}_A$, then $\langle x^*,y^*\rangle = \langle Ax^*,y\rangle$, so $y \in \mathcal{D}_{A^*} = \mathcal{D}_A$, and $Ay = y^* \in [\mathcal{D}_A]$, implying $y \in \mathcal{D}_{A_1}$ as desired.

Finally, $\langle A_1 x,x\rangle_\varepsilon \geq k\langle x,x\rangle_\varepsilon\ \forall\ x \in \mathcal{D}_A$ iff $\langle x^*,x^*\rangle \geq k\langle x^*,A^{-1}x^*\rangle\ \forall\ x^* \in [\mathcal{D}_A]$, which is clear as $\|A^{-1}\| \leq 1/k$.

Now take $S\colon \mathcal{D}_S \to \mathcal{H}$ as in the first paragraph of this section, where $(\mathcal{H},\langle\cdot,\cdot\rangle = \mathcal{S}_o(\cdot,\cdot) + i\mathcal{Q}(\cdot,\cdot))$ is a complex Hilbert space with complex structure J . Assume that $\mathcal{Q}(Sv,v) \geq k\langle v,v\rangle\ \forall\ v \in \mathcal{D}_S$, for some $k > 0$. We define $H = J^{-1}S$ so that $\mathcal{D}_H = \mathcal{D}_S$ and note that H is symmetric in $(\mathcal{H},\mathcal{S}_o)$ and bounded from below by k .

Now extend H to a surjective (real) self-adjoint operator $A\colon \mathcal{D}_A \to \mathcal{H}$ à la Friedrichs as above, and S likewise, so that $JA = S\colon \mathcal{D}_A \to \mathcal{H}$. As previously, define the surjective restriction $A_1\colon \mathcal{D}_{A_1} \to [\mathcal{D}_A]$, and likewise let $\mathcal{D}_{S_1} = S^{-1}([\mathcal{D}_A])$, $S_1 = S|_{\mathcal{D}_{S_1}}\colon \mathcal{D}_{S_1} \to [\mathcal{D}_A]$. Then Lemma 26.2 applies, and the following is the analogue for S_1 , proven the same way.

<u>Lemma 26.3</u>. In $([\mathcal{D}_A],\langle\cdot,\cdot\rangle_\varepsilon)$, $\mathcal{D}_{S_1}$ is dense (-also in $(\mathcal{H},\langle\cdot,\cdot\rangle)$), and S_1 is skew-adjoint.

Note that we have $\mathcal{D}_{A_1},\mathcal{D}_{S_1} \subseteq \mathcal{D}_A \subseteq [\mathcal{D}_A] \subseteq \mathcal{H}$; in general $\mathcal{D}_{A_1}$ and $\mathcal{D}_{S_1}$ overlap non-trivially, and equality holds in any of these inclusions iff A is bounded.

We apply the polar decomposition in $([\mathcal{D}_A],\langle\cdot,\cdot\rangle_\varepsilon)$ for closed operators in a real space, writing $S_1 = RT$, where R is orthogonal, T is non-negative self-adjoint, and R and T commute with all orthogonals commuting with S_1. (R is orthogonal as $\ker S_1 = \{0\}$ and range $S_1 = [\mathcal{D}_A]$.) We have $S_1 = -S_1^* = -TR^{-1} = (-R^{-1})(RTR^{-1})$ by 26.3, so by uniqueness $R^2 = -I$, $R(\mathcal{D}_{S_1}) = \mathcal{D}_{S_1} = \mathcal{D}_T$, and $RT = TR$.

T is also surjective, so $T^{-1}: [\mathcal{D}_A] \to \mathcal{D}_{S_1}$ is bounded and positive-semidefinite. $\mathcal{A}(Rx,Ry) = \mathcal{A}(x,y)$ $\forall\, x,y \in [\mathcal{D}_A]$ is shown using the equation $JAx = RTx$ $\forall\, x \in \mathcal{D}_{S_1}$.

Now define

$$\mathcal{S}_1(x,y) = \mathcal{A}(Rx,y) \;\forall\, x,y \in [\mathcal{D}_A] \;.$$

We have

$$\begin{aligned} \mathcal{S}_1(x,y) &= \mathcal{S}_0(J^{-1}Rx,y) \\ &= \mathcal{S}_0(AT^{-1}x,y) \qquad (21) \\ &= \langle T^{-1}x,y\rangle_\varepsilon \;\forall\, x,y \in [\mathcal{D}_A] \;, \end{aligned}$$

so $\mathcal{S}_1$ is a positive-definite form on $[\mathcal{D}_A]$. Let $\widetilde{\mathcal{H}}$ be the completion of $[\mathcal{D}_A]$ with respect to $\mathcal{S}_1$. R, $\mathcal{S}_1$, and $\mathcal{A}$ extend continuously to $\widetilde{\mathcal{H}}$ so that $(\widetilde{\mathcal{H}}, \mathcal{S}_1(\cdot,\cdot) + i\mathcal{A}(\cdot,\cdot) = \langle\cdot,\cdot\rangle_1)$ is a complex Hilbert space with complex structure R.

<u>Proposition 26.4</u>. 1) $\|x\|_1^2 \le \|T^{-1}\|_\varepsilon \|x\|_\varepsilon^2 \;\forall\, x \in [\mathcal{D}_A]$.
2) In $(\widetilde{\mathcal{H}}, \langle\ ,\ \rangle_1)$, $\mathcal{D}_{S_1}$ is dense and T is Hermitian (complex-linear) and bounded from below by $1/\|T^{-1}\|_\varepsilon$.

3) Making the Friedrichs extension (in the complex case) as earlier, T and S_1 extend to surjective self-adjoint and skew-adjoint operators $\widetilde{T}$, $\widetilde{S}$ respectively, so that $R\widetilde{T} = \widetilde{T}R = \widetilde{S}$.

<u>Proof</u>. 1) follows from (21) and 3) follows from 2). The density of $\mathcal{D}_{S_1}$ is not difficult, and the bound on T follows so: $\forall\, x \in \mathcal{D}_{S_1}$,

$$\mathfrak{s}_1(Tx,x) = \langle x,x\rangle_\varepsilon \geq (1/\|T^{-1}\|_\varepsilon)\langle T^{-1}x,x\rangle_\varepsilon$$
$$= (1/\|T^{-1}\|_\varepsilon)\mathfrak{s}_1(x,x) \quad .$$

27. <u>Analyticity of Kählerization in</u> $sp(\mathcal{H})$.

As in section 7 let $(\mathcal{H},\mathfrak{s}_0(\cdot,\cdot)+i\mathcal{A}(\cdot,\cdot) = \langle\cdot,\cdot\rangle)$ be a complex Hilbert space with complex structure J . We return to bounded operators only, and recall the definitions of Γ and $\mathcal{L}(\mathcal{H})$ from section 25, using only the $\mathfrak{s}_0$-norm topology on $\mathcal{H}$.

Let

$$C_0 = \{X \in sp(\mathcal{H}):\ \mathcal{A}(Xv,v) \geq k\langle v,v\rangle\ \forall\ v \in \mathcal{H} \text{ for some } k > 0\} ,$$

an open convex cone in $sp(\mathcal{H})$, and $\Gamma \subset C_0$. To define an analytic structure in Γ we use the following.

<u>Lemma 27.1</u>. Γ is closed in $\mathcal{L}(\mathcal{H})$.

<u>Proof</u>. We saw in section 25 that

$\Gamma = J\{D \in \mathcal{L}(\mathcal{V}): D^t = D,\ DJD = J$, and D positive-definite$\}$.

The first two conditions are obviously preserved on passing to limits, and the third follows from $D^{-1} = JDJ^{-1}$.

If $X \in C_0$ determines a complex structure J_1 as in 25.2, we have $X = J_1H = HJ_1$, where H is symmetric and positive-definite with respect to $s_X(\cdot,\cdot) = \sigma(X\cdot,\cdot)$. Then clearly

$$(22) \qquad J_1 = X(-X^2)^{-\frac{1}{2}} \equiv \mathcal{K}(X) ,$$

the square root being well-defined in $(\mathcal{V}, s_X)$.

<u>Theorem 27.2</u>. The mapping $\mathcal{K}: C_0 \to \Gamma$ defined by (22) is real-analytic.

<u>Proof</u>. This is a local property, and as $\mathcal{K}$ is constant on rays it suffices to check analyticity at $X \in C_0$ such that $0 < I + X^2 < I$ with respect to s_X . ($\|Y\|_{s_{tX}}$ $(Y \in \mathcal{L}(\mathcal{V}), t > 0)$ is independent of t .) Now if Y is in a small neighborhood N of 0 in $sp(\mathcal{V})$, $I + (X+Y)^2 \equiv M$ is longer symmetric in $(\mathcal{V}, s_X)$, but $\|M\|_{s_X} \leq 1$ may be assumed, and the power series

$$\sqrt{-(X+Y)^2} = \sqrt{I-M} = I - \frac{(1/2)}{1!}M - \frac{(1/2)(3/2)}{2!}M^2 - \dots$$

is valid for $\|M\| \leq 1$. M is also analytic in Y , and thus $\mathcal{K}(X+Y)$ is a power series in Y for $Y \in N$.

The theorem is no doubt true for the maximal open cones in the other algebras, too. It would follow if one could show that the unique fixed point in G/K of $\{e^{tX}\}$ (see section 25, second proof) depended analytically on X.

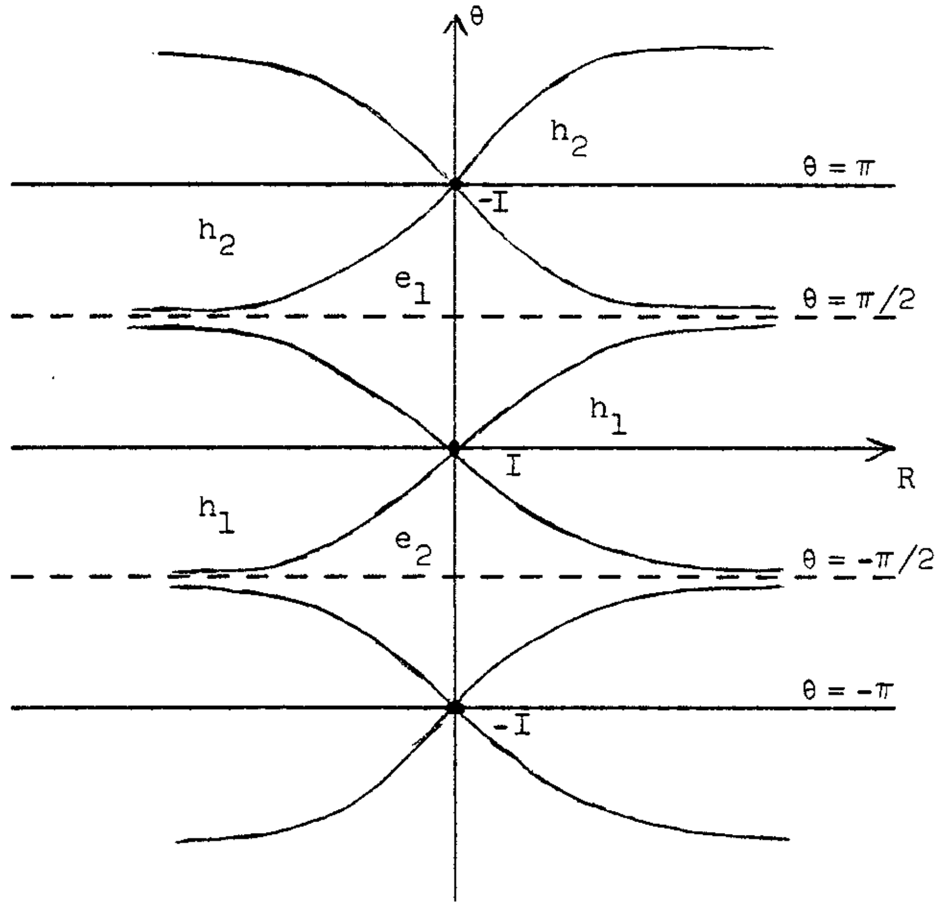
28. An Example and Analytic Continuation: SU(1,1).

It is not only customary but also usually quite worthwhile to illustrate general theory in Lie groups by considering in detail the case of $SL(2,\mathbb{R})$ [10]. For this group we can analytically continue the map κ in the last section to the whole Lie algebra minus zero (the range space no longer being merely complex structures), give an interpretation in the symmetric space, and complex-analytically extend κ to $SL(2,\mathbb{C}) - \{\pm I\}$. Some indications for $SU(2,1)$ will also be given, but an overall picture of Kählerization there has not yet emerged.

It will be useful to give $SL(2,\mathbb{R})$ some global coordinates in order to visualize the various regions of ellipticity, etc. (Compare with the picture in [7].) This we do by the Cartan decomposition $G = K \exp \underline{p}$, where

$$K = \left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}, \quad \underline{p} = \left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\right\}.$$

Conjugation by K is merely orthogonal rotations in the A,B-planes, with respect to the coordinates θ, A, B. Setting $R = \sqrt{A^2+B^2}$, the coordinates for the subgroup $\left\{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right\}$ satisfy $\cosh R \cos\theta = 1$, and a two-dimensional slice of the group is as follows.



The four conjugacy classes containing $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$, and $\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}$ are each homeomorphic to $\mathbb{R}^2-(0)$, and form boundaries between the simply connected elliptic regions e_1, e_2 and the two hyperbolic regions h_1, h_2 homeomorphic to $S^1 \times \mathbb{R}^2$. The orbit of complex structures $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \exp \underline{p}$ is clearly just the plane $\theta = \pi/2$.

Turning to the Lie algebra, in Theorem 9.1 we saw that $X = \begin{pmatrix}a & b\\ c & -a\end{pmatrix}$ is elliptic iff $\det X = -a^2 - bc > 0$, in which case b and c must be non-zero and of opposite sign.

The positive cone C_0 is where $-b,c > 0$, and with our coordinates C_0 exponentiates into e_1 .

One can see that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad - bc = 1)$ is in $e_1 \cup e_2$ iff $|\mathrm{Tr}\, g| = |a+d| < 2$, and we have

$$e_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \colon \ |a+d| < 2,\ -b,c > 0 \right\} ,$$

$$e_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \colon \ |a+d| < 2,\ -b,c < 0 \right\} .$$

These restrictions are closely related to formulas for $\varkappa$, which can be computed. $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in C_0$ commutes with the unique positive complex structure

$$\varkappa(X) = \left(1/\sqrt{-a^2-bc}\right)\begin{pmatrix} a & b \\ c & -a \end{pmatrix} ,$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in e_1$ is in the unique maximal compact determined by

$$\left(1/\sqrt{4-(a+d)^2}\right)\begin{pmatrix} a-d & 2b \\ 2c & -a+d \end{pmatrix} .$$

Now $\varkappa(X)$ $(X \in C_0)$ is just the unique complex structure in the one-parameter group generated by X , so clearly $\varkappa$ could be analytically extended to $sl(2,\mathbb{R}) - 0$ with range S^2 , by mapping each X to the ray in $sl(2,\mathbb{R})$ containing X . However, this prescription does not give must insight into all the other cases, where $\dim K > 1$.

We pass next to the bounded domain, where $G = SU(1,1)$ acts on $D' = \{|z| < 1\}$ and nonsingularly on the entire Šhilov boundary $\check{S} = S^1 = \{|z| = 1\}$. The complex structures go over to

$$\Gamma_1 = \left\{\begin{pmatrix} ic & D \\ \bar{D} & -ic \end{pmatrix} : c < 0,\ D \in \mathbb{C},\ c^2 = |D|^2 + 1\right\}$$

which act as reflections about points in D'. Let $\mathfrak{g} = su(1,1)$, recall

$$C_0 = \left\{\begin{pmatrix} ia & B \\ \bar{B} & -ia \end{pmatrix} \in \mathfrak{g} : a^2 - |B|^2 > 0,\ a < 0\right\} ,$$

and note that the corresponding elliptic regions are

$$E_1 = \left\{\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in G : |\mathrm{Re}\ A| < 1,\ \mathrm{Im}\ B > \mathrm{Im}\ A\right\}$$

and

$$E_2 = \left\{\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in G : |\mathrm{Re}\ A| < 1,\ \mathrm{Im}\ B < \mathrm{Im}\ A\right\} ,$$

the two latter conditions being equivalent to $\mathrm{Im}\,A + \mathrm{Im}\,B < 0$, $\mathrm{Im}\,A + \mathrm{Im}\,B > 0$ respectively.

In what follows we define Kählerization $\mathcal{K}_1$ on the group; this is a bit more general as the corresponding map on the Lie algebra can be recovered by evaluating $\mathcal{K}_1$ on exponentials.

One computes that $T = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in E_1$ determines

$$J_1 = \begin{pmatrix} ic & D \\ \bar{D} & -ic \end{pmatrix} \in \Gamma_1 \tag{23}$$

where $c = (\mathrm{Im}\ A)/L$, $D = B/L$, $L = \sqrt{1-(\mathrm{Re}\ A)^2}$, and the unique fixed points of J_1 are

$$z_T = i\,\frac{c+1}{D} \quad \text{(interior)}$$

and

$$w_T = i\,\frac{c-1}{D} \quad \text{(exterior)} .$$

But z_T, w_T are not analytic functions of T near a nilpotent boundary. To remedy this, we assign to z_T two points ℓ_T, m_T on $\check{S}$ which will be analytic functions of T as T crosses a boundary, and seem to have a chance to generalize to higher dimensions. We will see that these points are closely related to the ordinary polar coordinates on S^2. At least in the tube-type cases $\dim \check{S} = \frac{1}{2} \dim \underline{p}$, so two points on $\check{S}$ would seem appropriate to determine a point in the bounded domain.

Of course this assignment is not made completely invariantly, but relative to prior choices of a maximal compact K, a maximal abelian $\mathfrak{a} \subset \underline{p}$, and a Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$. We take

$$\mathfrak{a}_+ = \left\{ \begin{pmatrix} 0 & -s \\ -s & 0 \end{pmatrix} : s \leq 0 \right\}$$

and then $A^+ = \exp \mathfrak{a}_+$ moves the real line in D' to the left with the unique fixed points ± 1.

Given now $T \in E_1$, $z_T = S(0)$ for a unique transvection S from 0. S is conjugate under K to a unique $a \in A^+$, so let $m_T = a^2(i)$. As T moves to a boundary m_T heads toward -1; only with the square power does m_T extend analytically! If $T \in K$, $z_T = 0$ and ℓ_T is undefined; if $T \notin K$, let ℓ_T be the unique point on S^1 on the same ray from 0 as z_T. ("Ray from 0" is a K-invariant notion.)

With $T = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \in E_1$,

$$\ell_T = -i(B/|B|) \qquad (\text{if} \quad T \not\in K)$$

(24)

$$m_T = \frac{2|B|\,\mathrm{Im}\,A + i((\mathrm{Im}\,A)^2 - |B|^2)}{(\mathrm{Im}\,A)^2 + |B|^2} \quad .$$

The $a \in A^+$ above is $\begin{pmatrix} r & s \\ s & r \end{pmatrix}$ where $r = |D|/M$, $s = (c+1)/M$, $M = \sqrt{-2-2c}$, c, D as in (23).

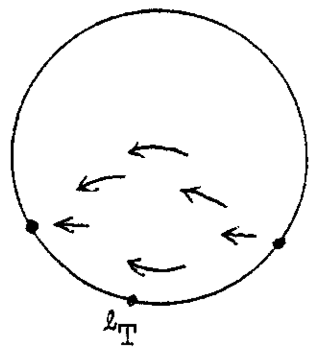
m_T now extends to all of $G - (\pm I)$, and ℓ_T to $G - K$. We interpret ℓ_T, m_T for T ± hyperbolic, ± unipotent below, but first indicate the connection with polar coordinates (φ, θ) on S^2. Set $\ell_T = e^{i\varphi}$ and

$$\cos\theta = (-\mathrm{Im}\ A)/(\sqrt{(\mathrm{Im}\ A)^2 + |B|^2})\ ,\ \sin\theta = |B|\ /\ (\sqrt{(\mathrm{Im}\,A)^2 + |B|^2})$$

for $0 \le \theta \le \pi$. Then $m_T = ie^{2i\theta}$. Clearly the mapping $\mathcal{K}_1 \colon G - (\pm I) \to S^2$ factors to $\widetilde{\mathcal{K}}_1 \colon (G/\pm I) - \{\pm I\} \to \mathbb{R}P^2$ in the obvious way. $\mathcal{K}_1(E_1)$ is the set of all points with latitude $\theta > +45°$.

If $T \in G$ is on the boundary of E_1, ℓ_T is the unique fixed point of T; if on the boundary of E_2, the negative of that point. $m_T = -1, +1$ respectively in the two cases.

Any ± hyperbolic $T \in G$ has two unique fixed points on S, and is a sort of finite "flow" from one to the other. Then ℓ_T is one of the two points midway between these two fixed points: the midpoint on the "left" side of the flow if T is in the hyperbolic region with $-I$ in its closure (example below), and the opposite midpoint if T is in the other region. m_T measures the



relative distance of the two fixed points from each other, but we will not make this precise.

With our polar coordinates, it is clear that

$$\kappa_1\begin{pmatrix} x+iy & z+iw \\ z-iw & x-iy \end{pmatrix} = (y,z,w)/\sqrt{y^2+z^2+w^2} \in S^2$$

for $x,y,z,w \in \mathbb{R}$ satisfying

$$(25) \qquad x^2 + y^2 - z^2 - w^2 = 1\ , \quad |y|^2 + |z|^2 + |w|^2 \neq 0\ ,$$

and it is clear why $\pm I \in G$ is excluded. The mapping extends complex-analytically to

$$\kappa_1^c\colon SL(2,C) - (\pm I) \to \mathbb{C}P^2\colon \begin{pmatrix} x+iy & z+iw \\ z-iw & x-iy \end{pmatrix} \to [y,z,w]$$

for $x,y,z,w \in \mathbb{C}$ satisfying (25). At least for $SU(1,1) - (\pm I)$, κ_1 is clearly equivariant with respect to conjugation in G and some representation of G by analytic transformations of S^2 leaving the "polar regions" $\theta > 45°$, $\theta < -45°$ invariant (hence did not really depend on the original choice of K and α_+) , but this has not been worked out.

We conclude this section with a few remarks for Kählerization in $su(2,1)$, where most of the real work related to this chapter has gone! From [2] one sees that all orbits in $su(2,1)$ come from elements of $u(1) \oplus u(1,1)$, except for the six-dimensional "principal nilpotent" N orbit, where $N^2 \neq 0$, $N^3 = 0$ as in [2]. It seems that $\mathcal{K}$ should extend to all of $su(2,1)$, minus the union of all the four-dimensional elliptic orbits of real multiples of $X_e = \left(\begin{array}{cc|c} -2i & 0 & 0 \\ 0 & i & 0 \\ \hline 0 & 0 & i \end{array}\right)$, which we recognize as lying on the boundary of a closed maximal invariant cone. If this is the case, $\mathcal{K}_1$ would be singular on a closed five-dimensional submanifold of $SU(2,1)$. Note that orbits of elements like $X_e + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & -i \\ 0 & i & i \end{pmatrix}$ are six-dimensional, and their multiples make up a seven-dimensional union of orbits composing the rest of the boundary of the maximal cones.

CHAPTER VIII. The Cayley Transform.

29. General Properties.

If H is a self-adjoint operator on some complex Hilbert space, the Cayley transform $\mathcal{C}(iH) = (I+iH)(I-iH)^{-1} = U$ is a unitary operator, and H can be recovered by the equation

$$H = -i\,\frac{U-I}{U+I} .$$

We have seen that, among the algebras enumerated in section 2, only $sp(n,\mathbb{R})$, $so^*(2n)$, and $su(n,1)$ have maximal compact subalgebras isomorphic to all skew-Hermitian operators on some complex Hilbert space. However, $sp(n,\mathbb{R})$ and $so^*(2n)$ are distinguished as there this complex space can be naturally taken to be a complexification of the representation space of the algebra's fundamental (defining) representation. We will see that in the former two cases (but not in the case of $su(n,1)$) $\mathcal{C}$ extends to an open subset of the noncompact algebra, mapping into the corresponding group, and has several interesting algebraic and "causal" properties. We discuss features specific to $sp(\mathcal{H})$ and $so^*(\mathcal{H})$ in the next section, but in this section consider the mapping

$$\mathcal{C}(X) = \frac{I+X}{I-X} \tag{26}$$

in the general context of a Lie group acting on a finite-dimensional vector space.

So, let the Lie group G and its Lie algebra $\mathfrak{g}$ act on V .

<u>Proposition 29.1</u>. $\mathcal{C}$ maps a neighborhood of 0 in $\mathfrak{g}$ into G iff tanh (given by the power series) maps a neighborhood of 0 in $\mathfrak{g}$ into $\mathfrak{g}$.

<u>Proof</u>. As $\mathcal{C}$ is nonsingular near 0, the condition is equivalent to $\mathcal{C}^{-1}(e^Y) \in \mathfrak{g}$ for all Y in a neighborhood of 0, and the equivalence follows from the identity

$$\frac{e^Y - I}{e^Y + I} = \frac{e^{Y/2} - e^{-Y/2}}{e^{Y/2} + e^{-Y/2}} = \tanh Y/2 \quad .$$

As the series for tanh has only odd powers, it is clear that $\mathcal{C}$ locally maps $\mathfrak{g}$ into G if G is a subgroup of $GL(V)$ defined by conditions of commuting with a family of operators and/or preserving some bilinear forms.

Let us now consider only those G satisfying 29.1, a fairly wide class. Nevertheless, $\{\mathcal{C}(tX) : t \in \mathbb{R}\}$ is rarely a one-parameter subgroup of G (even with a different parameter than t).

<u>Proposition 29.2</u>. The mapping $t \to \mathcal{C}(tX)$ is (locally) a a homomorphism iff $X^3 = 0$. If $X \neq 0$, $\{\mathcal{C}(tX)\}$ is a (local) group iff X^3 is a multiple of X .

<u>Proof</u>. If s,t,u are small, one checks that

$\mathcal{C}(tX)\mathcal{C}(sX) = \mathcal{C}(uX)$ iff $(t+s-u)X = stuX^3$. If $X^3 = cX$, u is uniquely determined by

$$u = \frac{t+s}{1+cts} .$$

$\mathcal{C}$ is defined on an open subset of $\mathfrak{g}$, and the following is easily shown.

Proposition 29.3. Let $(d\mathcal{C})_X$ be the differential of $\mathcal{C}$ at $X \in \mathfrak{g}$. If $\mathcal{C}$ is defined at X ,

$$[dL_{\mathcal{C}(X)^{-1}} \circ (d\mathcal{C})_X](y) = 2(I+X)^{-1}Y(I-X)^{-1}$$

for all $Y \in \mathfrak{g}$, where L_g $(g \in G)$ is left translation by g .

Remark. Let $R_X = \frac{1}{2}dL_{\mathcal{C}(X)^{-1}} \circ (d\mathcal{C})_X: \mathfrak{g} \to \mathfrak{g}$ be the linear map above. Then

$$\mathrm{Ad}(\mathcal{C}(X)) = (R_X)^{-1} R_{-X} ,$$

so $R_{(\cdot)}$ is something of a "square root" of the adjoint action.

We record some easy identities, in the context of $\mathcal{C}$ as a partially defined map from $gl(V)$ to $gl(V)$:

1) $\mathcal{C}(-X) = \mathcal{C}(X)^{-1}$
2) $\mathcal{C}(X^{-1}) = -\mathcal{C}(X)$
3) $\mathcal{C}^2 = -I$
4) $\mathcal{C}(\tanh(X+Y)) = \mathcal{C}(\tanh X)\mathcal{C}(\tanh Y)$

hold whenever all $\mathcal{C}$-transforms make sense, and in 4), when X and Y commute.

30. Cayley Transforms for Causal Groups.

We consider simultaneously the algebras of bounded operators $sp(\mathcal{H})$ and $so^*(\mathcal{H})$ and their corresponding groups $Sp(\mathcal{H})$ and $SO^*(\mathcal{H})$ as defined in sections 7 and 16. In each case the group acts as continuous automorphisms of the symplectic space $(\mathcal{H},\mathcal{A}(\cdot,\cdot))$, and additionally, elements of $SO^*(\mathcal{H})$ commute with the i and σ which anti-commute, have squares $-I$, and preserve $\mathcal{A}$. The propositions in this section are valid for both classes of groups; we let G denote the group, $\mathfrak{g}$ the algebra, and τ the topology on $\mathcal{H}$.

Definitions. Let

$$C_1 = \{X \in \mathfrak{g}: \mathcal{A}(X\cdot,\cdot) \text{ is a positive-definite form defining the topology } \tau\} ,$$

and $\Gamma = \{X \in C_1: X^2 = -I\}$.

Proposition 30.1. $\mathcal{C}(X) \in G$ for $X \in \mathfrak{g}$ whenever it is defined, and $\mathcal{C}$ is causal with respect to the cone C_1 .

Proof. Let $v_i \in \mathcal{H}$ $(i = 1,2)$ and $X \in \mathfrak{g}$ such that $(I-X)^{-1}$ exists. Thus there exists unique w_i such that $w_i - Xw_i = v_i$. Then

$$\begin{aligned} \mathcal{A}(\mathcal{C}(X)v_1,\mathcal{C}(X)v_2) &= \mathcal{A}(w_1 + Xw_2, w_2 + Xw_2) \\ &= \mathcal{A}(w_1 - Xw_1, w_2 - Xw_2) \\ &= \mathcal{A}(v_1,v_2) \end{aligned}$$

as X is skew with respect to $\mathcal{A}$, so the first statement is clear. The second follows from the computation of the differential of $\mathcal{C}$ in the last section and the identity

$$\mathcal{A}((I+X)^{-1}Y(I-X)^{-1}v,v) = \mathcal{A}(Y(I-X)^{-1}v,(I-X)^{-1}v) ,$$

$X,Y \in \mathfrak{g}$, $v \in \mathcal{V}$.

<u>Remark</u>. (Refer to the remark in the last section.) The proof of 30.1 shows that the R_X, although not automorphisms of $\mathfrak{g}$, nevertheless preserve the causal cone. In the case of $sl(2,\mathbb{R})$, the R_X turn out to be "conformal automorphisms", that is, positive multiples of automorphisms; this is a low-dimensional "accident". In fact, if $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ and $B(\cdot,\cdot)$ is Killing form, a lengthy computation shows that

$$B(R_X(Y),R_X(Z)) = (1/(1-a^2-bc)^2)B(Y,Z)$$

for all $Y,Z \in sl(2,\mathbb{R})$; of course a^2+bc is a multiple of $B(X,X)$.

It is clear that $\mathcal{C}(J) = J$ and $\exp(\pi/2)J = J$ for all $J \in \Gamma$. A generalization of this relationship between $\mathcal{C}$ and exp holds for all of C_1. (By Theorems 7.1 and 16.2 $\mathcal{C}$ is defined on all of C_1.)

Proposition 30.2. Let $X \in C_1$ so that $X = JH = HJ$ for a unique $J \in \Gamma$ and H positive-definite with respect to $\mathcal{S}_J(\cdot,\cdot) = \mathcal{A}(J\cdot,\cdot)$. Then

$$\mathcal{C}(JH) = \exp(JT)$$

where $JT = TJ$, $H = \tan \frac{1}{2}T$, and $\pi > T > 0$ in $(\mathcal{H},\mathcal{S}_J)$.

Proof. $I \pm X$ are invertible, and

$$\begin{aligned}\mathcal{C}(X) = (I+X)/(I-X) &= (I+X^2)/(I-X^2) + 2X/(I-X^2) \\ &= (I-H^2)/(I+H^2) + J(2H/(I+H^2)) \quad .\end{aligned}$$

Take T positive and symmetric in $(\mathcal{H},\mathcal{S}_J)$ such that $\pi > T > 0$ and $(I-H^2)/(I+H^2) = \cos T$. Then $\sin T = 2H/(I+H^2)$ and $\mathcal{C}(X) = \exp JT$. Also,

$$\begin{aligned}\tan^2 \tfrac{1}{2}T &= \Big(I-(I-2\sin^2 \tfrac{1}{2}T)\Big)/\Big(I+(2\cos^2 \tfrac{1}{2}T-I)\Big) \\ &= (I - \cos T)/(I + \cos T) = H^2\end{aligned}$$

so that $H = \tan \frac{1}{2}T$.

Proposition 30.3. Let $J \in \Gamma$ be arbitrary. The image of C_1 under $\mathcal{C}$ is

$$\{g \in G\colon \mathcal{A}(gv,v) \geq k\mathcal{A}(Jv,v)\ \forall\ v \in \mathcal{H} \text{ , for some } k > 0\} \quad .$$

Proof. Use the identity

$$\mathcal{A}((g-I)(g+I)^{-1}v,v) = 2\mathcal{A}(gw,w)$$

where $w = (g+I)^{-1}v$. The inequality above for a $g \in G$

implies first that $g+I$ has dense range, and that $(g+I)^{-1}$ exists quickly follows.

Naturally many mathematical questions arise concerning the applicability and causality of $\mathcal{C}$ for the other causal groups and structures. For the $su(p,q)$ series $\mathcal{C}$ just does not map into $SU(p,q)$, only $U(p,q)$; $tr(\cdot) = 0$ implies $\det \mathcal{C}(\cdot) = 1$ holds in general only for 2×2 matrices. $\mathcal{C}$ does map $so(2,n)$ into $SO_o(2,n)$ but is not causal; it suffices to examine $(d\mathcal{C})_X(Y)$ for X,Y in the two-dimensional maximal torus $\underline{h}$ of $o(2,2)$. Finally, an example in $so^*(4)$ shows that $\mathcal{C}$ is not causal with respect to the minimal invariant cone, and causality seems doubtful for the maximal cone, too. However, it does not yet seem unlikely that causal mappings from algebra to group exist. (The exponential mapping is not causal, but note [19, p. 31].) Hopefully the isomorphism $sp(2,\mathbb{R}) \approx so(2,3)$ can suggest a definition for the $so(2,n)$, and the failure for $su(2,1)$ should be examined more closely.

Finally, our $\mathcal{C}$ seems to have but loose ties with the Cayley transform for Hermitian symmetric spaces [15]. However, the conformal mapping $z \to \frac{1+z}{1-z}$ in $\mathbb{C}$, which takes the unit disc to the right half-plane and fixes $\pm i$, <u>is</u> a Cayley transform for $Sl(2,\mathbb{R})/SO(2)$.

CHAPTER IX. Applications to Differential Equations.

31. Preliminaries.

Let $(H_o, \langle\cdot,\cdot\rangle)$ be a real Hilbert space possessing an orthogonal J with square $-I$. Define the symplectic form $\mathcal{A}(\cdot,\cdot) = \langle J^{-1}\cdot,\cdot\rangle$ as usual. The norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ gives the only topology on H_o that we use. In this final chapter we consider linear differential equations of the form

$$(27) \qquad \frac{d}{dt}X(t) = A(t)X(t)$$

with periodic coefficients, i.e. for some $T > 0$

$$A(t+T) = A(t) \ , \quad t \in [0,\infty)$$

where $X(t) \in H_o$, and $A(t)$ is a strongly continuous curve in $sp(H_o)$. Recall that the latter means that for all $v \in H_o$, $t \to A(t)v$ is a continuous curve in H_o .

The usual existence and uniqueness statements hold for such equations. However in the infinite-dimensional case a solution to (27) is not necessarily (Fréchet) differentiable; we define the solutions of (27) to be those $X(t)$ satisfying the integrated form of the equation

$$(28) \qquad X(t) = X(0) + \int_0^t A(s)X(s)ds \quad .$$

Recall (section 25) that $u(H_o,\mathcal{J})$ and $so^*(H_o)$ can be regarded as subalgebras of $sp(H_o)$.

(27) is equivalent in the usual way to

(29) $$\frac{d}{dt}U(t) = A(t)U(t) \ , \quad U(0) = I \ ,$$

where $t \to U(t)$ is a curve in $Sp(H_o)$, as the coefficients are in $sp(H_o)$. As in Floquet theory, $U(\cdot)$ satisfies $U(t+T) = U(t)U(T)$, and as in [4], our basic reference for this chapter, we call $U(T)$ the <u>monodromy operator</u>.

One advantage of the form (29) in the finite-dimensional case is that (29) lifts to

(30) $$\frac{d}{dt}\widetilde{U(t)} = A(t)\widetilde{U(t)}$$

where $t \to \widetilde{U(t)}$ is a curve in the universal covering group $\widetilde{Sp(H_o)}$ of $Sp(H_o)$. (We identify the two Lie algebras.) Of course the solution of (30) covers that of (29). We shall see in the next section that classical stability "bands" and regions correspond to certain regions of ellipticity in $\widetilde{Sp(H_o)}$.

In accordance with general convention, we say (27) is <u>stable</u> if each of its solutions (on $[0,\infty)$) is bounded, and <u>strongly stable</u> [4, p. 200] if for some $\varepsilon > 0$, all equations $X' = A_1(t)X$, where $\int_0^T \|A(t) - A_1(t)\| dt < \varepsilon$, are stable. It remains true in the infinite-dimensional case that stability and strong stability are equivalent to corresponding properties of the monodromy operator $U(T)$, described presently. (The uniform-boundedness principle is crucial in this reduction.) We say $U \in \mathcal{L}(H_o) \equiv$

$\{A: H_o \to H_o \mid A$ real-linear, bounded$\}$ is <u>stable</u> if U^{-1} exists and $\exists\, M > 0$ such that $\|U^n\| \le M$ for all integers n. $U \in Sp(H_o)$ is called <u>strongly stable</u> if all $U_1 \in Sp(H_o)$ in some norm-neighborhood of U are stable.

<u>Lemma 31.1.</u> ([4, pp. 219-20]) (27) is stable (strongly stable) if and only if the corresponding $U(T)$ is stable (strongly stable, respectively).

32. <u>Regions of Stability</u>.

We continue with the notation of the previous section, and in this section restrict H_o to be finite-dimensional. For concreteness take $H_o = \mathbb{R}^{2n}$, and define the symplectic form as in section six.

Now the spectrum of any $U \in Sp(H_o)$ is symmetric with respect to the unit circle (the Poincaré-Lyapunov theorem), so U stable implies $\sigma(U)$ is a subset of $S^1 = \{|z| = 1\}$, invariant under complex conjugation.

<u>Lemma 32.1.</u> $U \in Sp(H_o)$ is stable iff U is in some maximal compact subgroup, i.e., is elliptic.

<u>Proof</u>. The sufficiency is obvious. If U is stable, U clearly must be diagonalizable over $\mathbb{C}$. Extend $\mathcal{A}$ to $\mathbb{C}^{2n}$ by complex bilinearity. By the invariance of $\mathcal{A}$ under U, one sees that a U-invariant subspace on which $U = e^{i\theta}$ is $\mathcal{A}$-orthogonal to one where $U = e^{i\varphi}$, unless $e^{i\theta} = e^{-i\varphi}$.

As the eigenvalues come in conjugate pairs, it is easy to find the proper basis of $\mathbb{R}^{2n}$ so that U is conjugate under $Sp(n,\mathbb{R})$ to some

$$\exp\begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}, \quad D \text{ diagonal.} \tag{31}$$

Given a stable U, the diagonal elements of D in (31), say d_j where $-\pi < d_j \leq \pi$, $j = 1,\ldots,n$, are unique up to a permutation. The eigenvalues of U (with multiplicity) are $e^{\pm id_1},\ldots,e^{\pm id_n}$. The classical terminology is that $e^{id_1},\ldots,e^{id_n}$ are the <u>multipliers of the first kind</u>, $e^{-id_1},\ldots,e^{-id_n}$ the <u>multipliers of the second kind</u>.

<u>Definition</u>. Let the systems of coefficients $A(t),A_1(t)$ give strongly stable equations. We say $A(t)$ and $A_1(t)$ belong to the same <u>stability region</u> if and only if there is a continuous map

$$A_2\colon [0,T] \times [0,1] \to sp(H_o)\colon (t,s) \to A_2(t,s)$$

such that $A_2(t,0) = A(t)$, $A_2(t,1) = A_1(t)$, $A_2(0,s) = A_2(T,s)$, and that the equation

$$\frac{d}{dt}X(t) = A_2(t,s)X(t)$$

is strongly stable $\forall s \in [0,1]$. (One could require all these functions A, A_1, A_2 to be merely piece-wise continuous in t, but the present definition suffices here.)

The stability regions were determined by I.M. Gelfand and V.B. Lidskiĭ in [7]. They are determined by an integer m, called the <u>index</u>, corresponding to the number of times (in one direction or the other) $U(t)$ $(0 \le t \le T)$ "winds around the group", and a pattern of the eigenvalues of $U(T)$, out of a total number of 2^n such patterns, described as follows. $U(T)$ can be the monodromy operator of a strongly stable equation (equivalently, $U(T)$ itself is strongly stable (Lemma 31.1), i.e., has a neighborhood in $Sp(H_o)$ consisting of elliptic elements) iff the $e^{\pm id_j}$ obtained above have the property that no $\gamma \in S^1$ is a multiplier both of the first and second kinds, that is, there are no "repeated multipliers of unlike type". Equivalently, no $e^{\pm id_j}$ can be real, and, with the above restriction on the range of the d_j, $\forall\, i,j = 1,\ldots,n$ $d_i = -d_j \neq 0$ does not occur.

<u>Corollary 32.2</u>. An elliptic $U \in Sp(n,\mathbb{R})$ is strongly stable iff U is contained in a unique maximal compact subgroup.

<u>Proof</u>. It follows from Proposition 25.3, the use of the Cayley transform $\mathcal{C}$, and the obvious fact that $[J,X] = 0$ iff $J\mathcal{C}(X) = \mathcal{C}(X)J$ for J a complex structure and X elliptic. Note that each strongly stable U is in the range of $\mathcal{C}$, as -1 is not an eigenvalue.

Arranging the d_j is order of ascending absolute value, the "pattern of eigenvalues" above is just the sequence $d_1/|d_1|, \ldots, d_n/|d_n|$. Clearly there are 2^n such patterns. We say that there are 2^n <u>elliptic regions</u> in $Sp(n,\mathbb{R})$. As the mapping

$$i\mathbb{R} \to S^1 - (-1) \quad : \quad ir \to \frac{1+ir}{1-ir}$$

preserves the linear orders of $i\mathbb{R}$ and $S^1 - (-1)$, it is clear that these regions are diffeomorphic under $\mathcal{C}$ to the various regions in $sp(n,\mathbb{R})$ each containing elements uniquely Kählerizable.

In [7] it is proven that each elliptic region is connected and simply connected. The region of conjugates of (31) where all $d_j > 0$, corresponds under $\mathcal{C}$ to the open positive cone in $sp(n,\mathbb{R})$, hence being convex, is contractible. One may ask, are the other elliptic regions contractible?

Note that the $X \in sp(n,\mathbb{R})$ such that $X^2 = -I$ (see the remarks in section 25; there are $n+1$ such orbits) are stable, but also strongly stable only in the two cases where X or $-X$ is a complex structure, i.e. $\sigma(X\cdot,\cdot)$ is positive- or negative-definite.

As the elliptic regions in $Sp(n,\mathbb{R})$ are simply connected, the components of their inverse images in $\widetilde{Sp(n,\mathbb{R})}$ project back to $Sp(n,\mathbb{R})$ diffeomorphicly. The components of the inverse image of an elliptic region are parametrized by the index $m \in \mathbb{Z}$ above. The following is clear from the theorems of [7] and Corollary 32.2.

Proposition 32.3. Stability regions for equations of the form (29) are in natural 1-1 correspondence with the open regions in $\widetilde{Sp(n,\mathbb{R})}$ consisting of those group elements uniquely Kählerizable, i.e., those which are each contained in a unique (essentially) maximal compact subgroup, as follows. Lift (29) to (30), and associate a strongly stable $A(t)$ with the region in $Sp(n,\mathbb{R})$ containing $\widetilde{U(T)}$.

It becomes clear why there should be stability "bands" for one-dimensional equations like the Mathieu equation

$$(32) \qquad Y''(t) + (\varepsilon + \delta \cos t)y(t) = 0 \; , \quad T = 2\pi \; ;$$

they just reflect the geometry of $\widetilde{SL(2,\mathbb{R})}$. One has a mapping of (ε,δ)-pairs to $U(T) \in SL(2,\mathbb{R})$, or rather $\widetilde{U(T)} \in \widetilde{SL(2,\mathbb{R})}$, by solving the equation. The inverse image of the picture in section 28, under this mapping, is just the diagram appearing in textbooks containing sections on the Mathieu equation, for example, [1 , p. 562].

33. Two Stability Criteria.

We return to the notation of section 31, and consider the equation (27) with $t \to A(t)$ strongly continuous and norm-bounded: $\|A(t)\| \leq M \;\forall t$. Further, let $\mathcal{H} = H_0 \oplus iH_0$ be the complexification of H_0 with complex structure i , and extend $\langle\cdot,\cdot\rangle$ on H_0 to a complex Hilbert structure

$\langle\cdot,\cdot\rangle$ on $\mathcal{H}$. We extend all operators on H_o to i-linear operators on $\mathcal{H}$. Our goal in this last section is to prove Theorems 33.4 and 33.6. The latter is a generalization to infinite dimensions of a theorem of M. G. Krein [17].

In the applications to quantum fields that we have in mind $U(T)$ becomes the "scattering operator", and we are interested in finding easily expressed conditions on $A(t)$ sufficient to guarantee the stability, i.e., unitarizability, of $U(T)$. Also, in these applications $t \to A(t)$ will not be continuous in the norm topology.

We restrict now to "positive-energy" equations, that is, coefficients $A(t)$ where

(33) $$\alpha(A(t)v,v) \geq 0 \quad \forall v \in H_o, \ \forall t \in [0,T]$$

and for some $k > 0$

(34) $$\alpha(A_{av}v,v) \geq k\langle v,v\rangle \ \forall v \in H_o, \text{ where } A_{av}v = \int_0^T A(t)v\,dt,$$

the integral of a continuous function in H_o. (The Bochner integrals we use are all integrals over an interval of $\mathcal{H}$-valued functions.)

Under somewhat more general hypotheses than these, one can prove

Lemma 33.1. ([4, p. 222]) Let $[a,b]$ be a closed interval containing 0, and for $\lambda \in [a,b]$ let $U(T;\lambda)$ be the monodromy operator for the equation

(35) $$(d/dt)X(t) = \lambda A(t)X(t) \quad .$$

If

(36) $$(U(T;\lambda) + I)^{-1} \text{ exists}$$

as a bounded operator $\forall \lambda \in [a,b]$, (35) is strongly stable $\forall \lambda \in [a,b]$.

We recognize that (36) is equivalent to $U(T;\lambda)$ being a Cayley transform of some $Y_\lambda \in sp(H_o)$, here necessarily in the cone C_o of section 27, as the proof shows.

Now let $\mathcal{B}$ be the Banach space of continuous functions from $[0,T]$ to $\mathcal{H}$, with the norm

$$|||f||| = \sup_{t\in[0,T]} \|f(t)\| \quad (f \in \mathcal{B}) \ ,$$

and define

$$\langle f,g\rangle_c = \int_0^T \langle f(t),g(t)\rangle dt \quad \text{for} \quad f,g \in \mathcal{B} \quad .$$

Let $(\mathcal{H}_c, \langle\cdot,\cdot\rangle_c)$ be the completion of $\mathcal{B}$ in $\langle\cdot,\cdot\rangle_c$. ('c' stands for curve.)

Lemma 33.2. ([4, p. 224]) As in Lemma 33.1, $(U(T;\lambda) + I)^{-1}$ exists if and only if for any $f \in \mathcal{H}$ there exists a unique solution $X(t)$ in $\mathcal{B}$ depending continuously on f (in the norm $|||\cdot|||$) of the boundary value problem

(37) $$(d/dt)X(t) = \lambda A(t)X(t) \ , \quad X(0) + X(T) = f \ .$$

We will solve the integrated form of this equation ((39) below), but first need some more notation.

Let $A(t) = JH(t)$; then $H(t)$ is a non-negative Hermitian operator in $(\mathcal{H}_c, \langle\cdot,\cdot\rangle_c)$. Define $\langle\cdot,\cdot\rangle_{\varepsilon_0}$ on $\mathcal{B}\times\mathcal{B}$ by

$$\langle f,g\rangle_{\varepsilon_0} = \int_0^T \langle H(t)f(t),g(t)\rangle dt \quad ,$$

and let $\mathcal{R} = \{f \in \mathcal{B}\colon H(t)f(t) = 0 \;\; \forall t\}$. Then clearly $\mathcal{R} = \{f \in \mathcal{B}\colon \langle f,f\rangle_{\varepsilon_0} = 0\}$, and $\mathcal{R}$ is $|||\cdot|||$-closed in $\mathcal{B}$. Also if $f \in \mathcal{B}$

$$\text{(38)} \qquad \langle f,f\rangle_{\varepsilon_0} \le \left(\int_0^T \|H(t)\|dt\right)|||f|||^2 \quad .$$

Define the function h on $[-T,T]$ by $h(t) = \begin{cases} \frac{1}{2} & (0 \le t \le T) \\ -\frac{1}{2} & (-T \le t < 0) \end{cases}$. Let $\omega = \pi/T$. Then $\left\{e^{i\omega(2k+1)t}\right\}_{k\in\mathbb{Z}}$ is an orthogonal basis of $L_2([0,T])$, and on $[-T,T]$ it is a basis for that subspace of functions satisfying $f(t+T) = -f(t)$ a.e. We have the L_2-expansion

$$h(t) = (1/\pi i)\sum_k (1/2k+1)e^{i\omega(2k+1)t} \quad .$$

($\sum_k$, $\sum_\ell$, etc. always denote a sum over all integers.)

For $f \in \mathcal{B}$ define

$$(Kf)(t) = \int_0^T h(t-s)A(s)f(s)ds \quad .$$

Clearly $K\colon \mathcal{B} \to \mathcal{B}$ is continuous and vanishes on $\mathcal{R}$. In fact, Lemma 33.3. $\ker K = \mathcal{R}$.

Proof. $Kf = 0$ implies $\int_0^T e^{-i\omega(2k+1)s} H(s)f(s)ds = 0$ for all $k \in \mathbb{Z}$. Now $s \to H(s)f(s)$ is a continuous function under our assumptions ($t \to H(t)$ strongly continuous and norm-bounded), so $H(s)f(s) = 0$ and $f \in \mathcal{R}$.

We have introduced K because if $\lambda = 1$, (37) is equivalent to

$$(39) \qquad ((I-K)X)(t) = \tfrac{1}{2}f \ .$$

Thus it suffices to find conditions under which $|||K||| < 1$, or just invert $I-K$ in the algebra of bounded operators in $\mathcal{B}$, in order to deduce strong stability of (27).

In fact K is continuous with respect to all the other norms, too. Let $f \in \mathcal{B}$ and $g \in \mathcal{H}$. Then for all t

$$\begin{aligned} |\langle (Kf)(t),g\rangle|^2 &= \left|\int_0^T h(t-s)\langle JH(s)f(s),g\rangle ds\right|^2 \\ &\leq \tfrac{1}{4}\left\{\int_0^T \|H^{1/2}(s)f(s)\|\,\|H^{1/2}(s)Jg\|\,ds\right\}^2 \\ &\leq \tfrac{1}{4}\left(\int_0^T \|H^{1/2}(s)f(s)\|^2 ds\right)\left(\int_0^T \|H^{1/2}(s)Jg\|^2 ds\right) \\ &\leq \tfrac{1}{4}\langle f,f\rangle_{\epsilon_0}\|g\|^2\left(\int_0^T \|H(s)\|ds\right) , \quad \text{so} \end{aligned}$$

$$|||Kf||| \leq \tfrac{1}{2}\|f\|_{\epsilon_0} \left(\int_0^T \|H(s)\|ds\right)^{1/2} .$$

Now $\mathcal{B}/\mathcal{R}$ is a Banach space with the quotient norm $|||\cdot|||_q$. By an earlier remark, $\langle\cdot,\cdot\rangle_{\epsilon_0}$ factors to a positive-definite form $\langle\cdot,\cdot\rangle_\epsilon$ on $\mathcal{B}/\mathcal{R}$, a kind of "average-energy" norm. Let $(\widetilde{\mathcal{V}}, \langle\cdot,\cdot\rangle_\varepsilon)$ be the completion of $\mathcal{B}/\mathcal{R}$ in this norm. Then the inclusion $j: \mathcal{B}/\mathcal{R} \hookrightarrow \widetilde{\mathcal{V}}$ is continuous with norm

$$(41) \qquad \|j\| \;\le\; \left(\int_0^T \|H(t)\|dt\right)^{1/2}$$

by (38).

By (40) and (41), it follows that $|||K||| \le \frac{1}{2}\int_0^T \|H(s)\|ds$. By this estimate and the remark following 33.3, the following is evident.

<u>Theorem 33.4</u>. Suppose $t \to A(t) \in sp(H_0)$ is strongly continuous, norm-bounded, periodic with period T , and satisfies (33) and (34). Then the equation

$$(d/dt)X(t) \;=\; A(t)X(t)$$

is strongly stable provided

$$\int_0^T \|A(t)\|dt \;<\; 2 \quad .$$

We need additional notation in order to state our second stability criterion. There are several other continuous operators closely related to K that one can define. By 33.3, K factors to

$$\overline{K}: (\mathcal{B}/\mathcal{R}, |||\cdot|||_q) \;\to\; (\mathcal{B}, |||\cdot|||)$$

and this extends to

$$\widetilde{K}: (\widetilde{\mathcal{V}}, \langle\cdot,\cdot\rangle_\varepsilon) \;\to\; (\mathcal{B}, |||\cdot|||) \quad .$$

Denote by K_1 the composition

$$K_1 : \widetilde{\mathcal{H}} \xrightarrow{\widetilde{K}} \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{R} \xhookrightarrow{j} \widetilde{\mathcal{H}} \ .$$

Then $\|\widetilde{K}\| \le \frac{1}{2}(\int_0^T \|H(s)\|ds)^{1/2}$, $\|K_1\| \le \frac{1}{2}\int_0^T \|H(s)\|ds$, and of course range $K_1 \subseteq \mathcal{B}/\mathcal{R}$.

Lemma 33.5. K_1 is self-adjoint.

Proof. To show $\langle f, K_1 g\rangle_\epsilon = \langle K_1 f, g\rangle_\epsilon$ $(f, g \in \mathcal{B}/\mathcal{R})$, one uses $h(-t) = -h(t)$, $A(t) \in sp(H_o)$, and does an interchange of order of integration.

Now $\|K_1\| < 1$ is sufficient to force $I - K$ to be invertible as a bounded operator in $(\mathcal{B}, |||\cdot|||)$, as then $(I-K)^{-1} = \sum_{\ell=0}^{\infty} K^\ell = I + K + K\sum_{\ell=1}^{\infty} K^\ell$, where the sum converges in the $\mathcal{B}$-norm as the terms $K \cdot K^\ell$ can be factored

$$\mathcal{B} \longrightarrow \mathcal{B}/\mathcal{R} \xhookrightarrow{j} \widetilde{\mathcal{H}} \xrightarrow{K_1^\ell} \widetilde{\mathcal{H}} \xrightarrow{\widetilde{K}} \mathcal{B} \ .$$

Recall the definition (34) of A_{av} . We are assuming $H_{av} > 0$ where $A_{av} = JH_{av}$, and

$$(A_{av})^2 = (JH_{av})(JH_{av})$$

$$= -H_{av}^{-1/2}(H_{av}^{1/2}J^*H_{av}^{1/2})(H_{av}^{1/2}JH_{av}^{1/2})H_{av}^{1/2} \quad ,$$

so $(A_{av})^2$ has nonpositive trace whenever it is trace class. (Compare with the Killing form in $sp(n,\mathbb{R})$.)

Unlike the criterion of Theorem 33.4, the following test is invariant under $Sp(H_o)$.

<u>Theorem 33.6</u>. Suppose that $A(t)$ has an absolutely convergent Fourier expansion on $[0,T]$, and $(A_{av})^2$ is trace-class, in addition to the assumptions of Theorem 33.4. Then (note 33.5)

$$\mathrm{tr}(K_1^2) = -(T^2/4)\,\mathrm{tr}\,(A_{av})^2 , \tag{42}$$

and $(d/dt)X = A(t)X$ is strongly stable if

$$-T^2\,\mathrm{tr}\,(A_{av})^2 < 4 .$$

<u>Remark</u>. It is hoped that the assumption of an absolutely convergent expansion for $A(t)$ can be weakened substantially or eliminated, as it is only used to justify a certain rearrangement of a series, as shall be seen.

<u>Proof</u>. We first show that $\widetilde{\mathcal{V}}$ is unitarily equivalent to a closed subspace of $\mathcal{V}_c$ (proper if $\mathcal{R} \neq 0$), and deal with the transformed K_1. Define

$$T_o\colon \mathcal{B}/\mathcal{R} \to \mathcal{B}\colon f(t) + \mathcal{R} \to H^{1/2}(t)f(t) .$$

Then T_o extends to a unitary equivalence

$$T\colon (\widetilde{\mathcal{V}}, \langle\cdot,\cdot\rangle_\varepsilon) \to (V, \langle\cdot,\cdot\rangle_c)$$

where $V \subseteq \mathcal{V}$ is closed. One sees that $f \in V^\perp$ iff $H^{1/2}(t)f(t) = 0$ a.e., and

$$\begin{aligned}(K_2f)(t) &\equiv ((TK_1T^{-1})f)(t) \\ &= \int_0^T H^{1/2}(t)h(t-s)JH^{1/2}(s)f(s)ds\end{aligned}$$

for $f \in V$. K_2 extends to $\mathcal{H}_c$ by this expression and vanishes on $V^{\perp}$; denote this extension also by K_2 . Then $tr(K_1^2) = tr(K_2^2)$.

Now $H^{1/2}(t)$ can also be expanded absolutely, so let $H^{1/2}(t) = \sum_k e^{i2\omega kt} L_k$, $L_k : \mathcal{H} \to \mathcal{H}$, satisfy

$$(43) \qquad \sum_k \|L_k\| < \infty .$$

Let $z \in \mathcal{H}$ have norm 1; then $w_\ell(t) = (1/\sqrt{T}) e^{i\omega(2\ell+1)t} z$ has norm 1 in $\mathcal{H}_c$. We have

$$(K_2 w_\ell)(t) = (\sqrt{T}/\pi i) \sum_{k,m} (1/2m+1)(L_{k-m} J L_{m-\ell} z) e^{i\omega(2k+1)t} ,$$

an absolutely convergent sum. Then $\sum_\ell \langle K_2 w_\ell, K_2 w_\ell \rangle$ is equal to

$$(T^2/\pi^2) \sum_{k,\ell} \langle \sum_m (1/2m+1) L_{k-m} J L_{m-\ell} z, \sum_n (1/2n+1) L_{k-n} J L_{n-\ell} z \rangle .$$

We wish to rearrange the summation in n, m, k, ℓ in order to faciliate massive cancellation. To show absolute convergence, note

$$\sum_{k,\ell,m,n} (1/|2m+1||2n+1|) \|L_{k-m}\| \|L_{m-\ell}\| \|L_{k-n}\| \|L_{n-\ell}\|$$

$$= \sum_{k,\ell,n,r} (1/|2n+1||2(n+r)+1| \|L_{k-r}\| \|L_{\ell+r}\| \|L_k\| \|L_\ell\|$$

is less than a constant times

$$(45) \qquad \sum_r (\sum_k \|L_k\| \|L_{k-r}\|)^2 .$$

Now $\sum_r (\sum_r \|L_k\| \|L_{k-r}\|) < \infty$ by (43), so clearly the sum of squares (45) also converges.

Thus we can rearrange (44) arbitrarily. It becomes

$$(T^2/\pi^2)\sum_r\Big(\sum_n (1/(2n+1)(2(n+r)+1)\Big)\sum_{k,\ell} \langle L_{k-r}JL_{-\ell+r}z, L_kJL_{-\ell}z\rangle \ .$$

However,

$$\sum_n (1/(2n+1)(2(n+r)+1)) = 0$$

if $r \neq 0$! and $\sum_n (1/(2n+1)^2 = \pi^2/4$. To see this note $h(t)\overline{h(t)} = 1/4$, and multiply out the earlier expansion for $h(t)$.

The above then reduces to

$$(T^2/4)\sum_{k,\ell} \langle L_kJL_{-\ell}z, L_kJL_{-\ell}\rangle$$

Now $L_k^* = L_{-k}$ and $\sum_k (L_kL_{-k}) = H_{av}$, so

$$\sum_\ell \langle K_2w_\ell, K_2w_\ell\rangle = -(T^2/4)\sum_\ell \langle L_\ell JH_{av}JL_{-\ell}z, z\rangle$$
$$= (T^2/4)\sum_\ell \langle (H_{av}^{1/2}JL_{-\ell})^*(H_{av}^{1/2}JL_{-\ell})z, z\rangle \ .$$

Letting z run over an orthonormal basis $\{z_\alpha\}$ of $\mathcal{K}$, we have

$$\mathrm{tr}(K_1^2) = (T^2/4)\sum_\alpha\sum_\ell \langle (H_{av}^{1/2}JL_{-\ell})(H_{av}^{1/2}JL_{-\ell})^* z_\alpha, z_\alpha\rangle$$
$$= -(T^2/4)\sum_\alpha \langle H_{av}^{1/2}JH_{av}JH_{av}^{1/2}z_\alpha, z_\alpha\rangle$$
$$= -(T^2/4)\ \mathrm{tr}\,((JH_{av})^2) = -(T^2/4)\ \mathrm{tr}\,(A_{av})^2$$

as claimed.

By Lemma 33.5, $\|K_1\|^2 \le tr(K_1^2)$, and so the last statement follows from the remarks after that lemma. This completes the proof.

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BIOGRAPHICAL NOTE - AFTERWORD

Steve Paneitz was born in Lawton, Oklahoma on March 24, 1955, and lived in Red Oak, Iowa for about ten years until his family moved to McPherson, Kansas, where he attended the public high school. He attended the University of Kansas from 1973 to 1976, and subsequently moved to Boston to study at MIT. For the next two years he has a postdoctoral fellowship from the Miller Institute for Basic Research in Science at the University of California at Berkeley.

"The startling nearness of so many galaxies in the astronomy of the Other Men could be explained on the theory of the 'expanding universe'. Well I knew that this dramatic theory was but tentative and very far from satisfactory...

"I, too, now sought to capture the infinite spirit, the Star Maker, in an image spun by my own finite though cosmical nature. For now it seemed to me, it seemed, that I suddenly outgrew the three-dimensional vision proper to all creatures, and that I saw with physical sight the Star Maker. I saw, though nowhere is cosmical space, the blazing source of the hypercosmical light, as though it were an overwhelmingly brilliant point, a star, a sun more powerful than all suns together. It seemed to me that this effulgent star was the centre of a four-dimensional sphere whose curved surface was the three-dimensional cosmos..."

Olaf Stapldon, Star Maker (1937)

Would it have been worth while,
To have bitten off the matter with a smile,
To have squeezed the universe into a ball,
To roll it toward some overwhelming question...

T.S. Eliot, "The Love Song of
J. Alfred Prufrock"