INTERACTIONS BETWEEN COMBINATORICS, LIE THEORY AND ALGEBRAIC GEOMETRY VIA THE BRUHAT ORDERS

by

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ABSTRACT

Bruhat orders are partially ordered sets which arise in algebraic geometry during the study of the geometry of semisimple algebraic groups. As a preliminary step, any Bruhat order arising from a Weyl group is depicted with certain weights of a representation of a semisimple Lie algebra.

Classical Bruhat orders are Bruhat orders which arise from the classical semisimple algebraic groups. We first describe the classical Bruhat orders with tableaux of integers, allowing any two elements of an order to be directly compared. These descriptions are then used to show that the classical Bruhat orders are lexicographically shellable, a property concerning the simplicial complexes of chains in the orders. Recent work of C. Deconcini and V. Lakshmibai which applies this lexicographic shellability result to algebraic geometry is briefly discussed. Two other applications of the lexicographic shellability of the classical orders are also described: a new means of computing the Möbius function of the full classical orders, and a proof that the simplicial complexes of chains in the classical orders are triangulations of double suspensions of either spheres or balls. A second application of the tableaux description is the confirmation of a conjecture of Lusztig concerning the description of the Bruhat order on the symmetric group with arrays of dimensions of intersections of pairs of flags of subspaces in specified relative positions. Finally, one of the families of tableaux obtained here is related to the tableaux employed by Young in his description of the representations of the special linear group.

Bruhat lattices are Bruhat orders which are lattices. First, the Bruhat lattices are classified. We then employ a recent algebraic geometric result of C.S. Seshadri to show that certain combinatorial generating functions associated to these lattices can be expressed as the quotients of certain products. In particular, the same methods provide new proofs for two plane partition generating function identities as well as identifying two new exceptional irreducible Gaussian posets. We also use closely related Lie algebraic techniques to provide a new proof of the fact that the Bruhat lattices possess the strong Sperner property, an extremal combinatorial property concerning the sizes of antichains in ranked partially ordered sets.

Part of the Lie algebraic proof of the strong Sperner property for Bruhat lattices is abstracted to the context of arbitrary ranked partially ordered sets and then translated into the language of elementary linear algebra. A special case of this abstraction, stated for distributive lattices, is a priori applicable to the Bruhat lattices. Surprisingly, it is possible to prove that this special case can be applied to no other distributive lattices. Dynkin diagrams arise naturally in the proof of this classification theorem. We present a total of five proven or potential ways of characterizing or describing the Bruhat lattices.

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(continued)

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Mathematically speaking, I would like to thank Paul Edelman for generously showing me his proof of lexicographic shellability for the Bruhat order on the full symmetric group. I also had several helpful conversations with Corrado DeConcini on the subject of shellability and its consequences in algebraic geometry. And I thank George Lusztig for suggesting a problem which led to the third section of Chapter IV. Most of all, however, I thank my adviser Richard P. Stanley for his mathematical guidance and encouragement. His paper, "Weyl Groups, the Hard Lefschetz Theorem, and the Sperner Property", was the principal source of mathematical motivation for this thesis.

Chapter I

Introduction and an Example

1. Introduction

This thesis presents several results obtained by exploiting some of the connections between combinatorics, Lie theory and algebraic geometry which arise in the study of semisimple algebraic groups. The portion of Lie theory we shall be concerned with is the theory of representations of semisimple Lie algebras. The relevant area of algebraic geometry concerns the projective varieties, or flag manifolds, G/P, where G is a semisimple algebraic group and P is a parabolic subgroup. The mathematical objects of central concern to us are the Bruhat orders, which are partially ordered sets arising in both of these subjects. These orders have played a central role in recent work of Kazhdan and Lusztig [KLu] in representation theory and of Seshadri, et. al. in algebraic geometry [LM4]. Bruhat orders are usually defined in terms of the elements of Weyl groups, which are finite groups whose structures resemble the structures of the symmetric groups. These orders are therefore of interest to combinatorialists. Some of the most combinatorially interesting Bruhat orders were in fact defined independently by combinatorialists who were unaware of the algebraic definitions of the orders.

Which of the six possible logical relationships between combinatorics, Lie theory and algebraic geometry arise in this thesis? Lie theory and algebraic geometry overlap heavily in the area we are concerned with, since semisimple algebraic groups are essentially semisimple Lie groups with algebraic geometric rather than differential geometric structures. This area has been extensively studied and is beyond the scope of this thesis. We shall be concerned with applications to and from combinatorics on the one hand, and the algebraic subjects on the other hand. More of the applications are from the representation theory of semisimple Lie algebras to combinatorics, although each of the other three possibilities is also represented.

We now describe the two most interesting results of this thesis. Lexicographic shellability for a partially ordered set is a property invented by A. Björner [Bjö] as a condition sufficient to insure that the simplicial complex of chains in the partially ordered set is a "shellable" simplicial complex. It is known that this in turn implies that a certain commutative ring associated to the partially ordered set has the Cohen-Macaulay property. Utilizing explicit combinatorial descriptions, we show (Theorem III.4) that all "classical" Bruhat orders are lexicographically shellable. This extends a theorem of Edelman [Ede] to many more cases. C. DeConcini and V. Lakshmibai [DeL] have used our result to show that the canonical embeddings of certain projective varieties are arithmetically Cohen-Macaulay and arithmetically normal, a result of current interest in algebraic geometry.

The description of the other most interesting result of this thesis begins with a result of R. Stanley. In [StW], Stanley used the hard Lefschetz theorem of algebraic geometry to show that the Bruhat orders possess the strong Sperner property, an extremal combinatorial property concerning the sizes of antichains in a ranked partially ordered set. We simplify and generalize Stanley's methods for the special case of distributive lattices, obtaining a new sufficient condition for a distributive lattice to have the strong Sperner property. This condition consists of certain linear equations which are specified in terms of the combinatorial structure of the lattice. Surprisingly, it is possible to list exactly which distributive lattices satisfy this sufficient condition. Dynkin diagrams arise naturally during the classification procedure (Theorem VI.3.2). These diagrams, or certain subsets of them, classify many different kinds of mathematical objects, including semisimple Lie algebras and Lie groups, point crystallographic groups, and critical points of functions of several complex variables having no moduli [HHS].

We now quickly introduce some basic terminology and facts necessary for the remainder of the introduction; the formal definitions will be given in Chapter II. Henceforth the terms "partially ordered set" and "partial order" will often be replaced with the word poset. Weyl groups are finite groups whose presentations have a certain specified form. They play an important role in the structure theory of semisimple Lie algebras and semisimple algebraic groups. An irreducible Weyl group is one which cannot be expressed as the direct product of two smaller Weyl groups. The irreducible Weyl groups have been completely classified. There are three infinite families of them, the members of which are respectively denoted by A_{n-1}, BC_n, and D_n. These groups are called the classical Weyl groups. (This is because they arise from the classical semisimple algebraic groups SL_n, SO_{2n+1}, Sp_{2n}, SO_{2n}.) In addition, there are five *exceptional* irreducible Weyl groups, denoted E_6, E_7, E_8, F_4, and G_2. The Weyl group A_{n-1} is just the *nth* symmetric group.

It was mentioned earlier that Bruhat orders are defined on the elements of Weyl groups. However, this can be generalized in two ways. First, it is possible to extend the definition to Coxeter groups, a class of groups containing the Weyl groups. Second, an analog of the Bruhat order can be defined on the elements of certain coset spaces of Weyl or Coxeter groups obtained by dividing by certain subgroups. In this thesis, the term "*Bruhat order*" shall refer *only* to Bruhat orders defined on *Weyl groups or their appropriate coset spaces*. Note that all Weyl groups in this thesis are finite; we shall not consider affine Weyl groups, which are a special kind of infinite Coxeter group. These restrictions arise because the Bruhat orders are being studied in their original context of complex semisimple algebraic groups, where finite Weyl groups play a central role. A *Bruhat order of type X* is a Bruhat order defined on an irreducible Weyl group of type X or an appropriate coset space of a group of type X. A *classical Bruhat order* is a Bruhat order of type A, BC, or D.

The remainder of this introduction is an overview of the entire thesis. In broad terms, this thesis consists of three parts. The first part, Chapters I and II, contains introductory material and a preliminary proposition which applies to all Bruhat orders. The second part, Chapters III and IV, studies the classical Bruhat orders. The third part, Chapters V and VI, is largely (but not solely) concerned with the Bruhat orders which are lattices (*Bruhat lattices*). It could be said that the classical Bruhat orders are the nicest Bruhat orders from a combinatorial viewpoint, but that the Bruhat lattices are the most combinatorially interesting. The classical Weyl groups are very closely related to the symmetric groups. As a consequence of this, the classical Bruhat orders have nice combinatorial descriptions. On the other hand, lattices are generally more combinatorially interesting than arbitrary posets. Also, the two families of Bruhat orders previously studied by combinatorialists for purely combinatorial reasons are in fact Bruhat lattices. Finally, the Bruhat lattices appear in Lie representation theory in a particularly advantageous manner, allowing combinatorial conclusions to be drawn from representation theoretic facts.

For the sake of an example, the most famous kind of Bruhat order is described in the second section of this chapter. The formal definitions of Weyl groups and Bruhat orders appear in the first section of Chapter II. The second section of Chapter II presents a preliminary result, Proposition II.2, which is used throughout the thesis. This proposition depicts the Bruhat orders with certain weights of representations of semisimple Lie algebras, thus producing the connection with Lie representation theory.

Chapter III begins the study of the classical Bruhat orders by employing the aforementioned Proposition II.2 to obtain descriptions of the classical Bruhat orders in terms of n-tuples of integers. Except for this, all of the techniques used in Chapter III are combinatorial. (However, Chapter III is in some sense entirely combinatorial in content, since even the n-tuple descriptions may be obtained combinatorially, albeit more slowly.) Both the Weyl group definitions and these n-tuple descriptions do not permit the direct comparison of an arbitrary pair of elements in one of these orders. This situation is rectified in the third section of Chapter III with the derivation of tableau descriptions for the classical orders. The tableau description for orders of type D is new, whereas the tableau descriptions for orders of types A and BC turn out to have been known already in Indian algebraic geometry folklore. In the last section of Chapter III, the previously mentioned lexicographic shellability of the classical orders is deduced as virtually a corollary to the tableau descriptions and their proofs.

Chapter IV describes five applications of the two main results of Chapter III. The applications described in the first two sections of the chapter are almost entirely due to other people, but are described in this thesis for the sake of completeness. The first section is a summary of DeConcini's and Lakshmibai's application of the lexicographic shellability for classical orders to algebraic geometry that was mentioned earlier. Section 2 of Chapter IV briefly describes how a new derivation of the Möbius function for the full classical orders can be obtained from the proof of their lexicographic shellability. Section 2 also describes how triangulations of spheres and balls can be produced from the classical Bruhat orders with lexicographic shellability and knowledge of the Möbius function. The third section of Chapter IV uses the tableau descriptions for orders of type A to confirm a conjecture of Lusztig's concerning arrays of dimensions of intersections of pairs of flags of subspaces in specified relative positions. As a consequence, a more direct description is obtained for the Bruhat orders of type A in their original contexts, that of Schubert varieties in flag manifolds. The last section of Chapter IV describes the relationship between the tableaux obtained in Chapter III for orders of type A and the tableaux employed by Young in his description of representations of the special linear group.

Chapter V studies the Bruhat lattices with the representation theory of complex semisimple Lie algebras. In the second section of Chapter V, Proposition II.2 is used to identify which Bruhat orders are in fact lattices. The third section presents the tools from representation theory which are needed in the last two sections of the chapter. In Section 4 (which represents joint work with R.P. Stanley), Weyl's character formula is combined with recent algebraic geometric work of Seshadri [LM3] to show that the rank weighted generating functions for multichains in Bruhat lattices can be expressed as the quotients of certain products. In Section 5, principal three dimensional subalgebras are used to provide a new proof that the Bruhat lattices possess the strong Sperner property. Both of these results have consequences in more traditional combinatorics. New proofs of certain plane partition generating function identities can be obtained as special cases of the first result. Also, Lindström and Stanley have shown that a conjecture of Erdös and Moser in extremal number theory [Erd] can be proved using a particular case of the second result [Lin] [StW].

Each Bruhat lattice is actually a distributive lattice. It is well known that distributive lattices are in one-to-one correspondence with the subposets of their join irreducible elements. We shall call any poset that arises from a Bruhat lattice in this manner a miniscule poset. Years before combinatorialists were aware of the Lie theoretic notions utilized here, Stanley defined the notion of "Gaussian poset" for purely combinatorial reasons [StO]. This definition concerns the form of a certain family of generating functions associated to a partially ordered set. The main result of the fourth section of Chapter V can be rephrased as: All miniscule posets are Gaussian posets. It is interesting to note that all known Gaussian posets are miniscule posets, and it seems plausible that these are all possible Gaussian posets. This is just one of the five proven or potential ways of describing or characterizing the miniscule posets that are presented in Chapters V and VI.

The first main topic of Chapter VI is the abstraction of part of the Lie algebraic proof of the strong Spernerity of Bruhat lattices. Sufficient conditions for strong Spernerity are stated in the context of arbitrary ranked posets, and the relevant part of the Lie algebraic proof is translated into elementary linear algebra for the benefit of readers unfamiliar with Lie representation theory. Unfortunately, the most general statement of this abstracted sufficient condition is fairly difficult to work with. A special case of this condition, expressed only in the context of distributive lattices, is much easier to use. It is known from the Lie algebraic proof in Chapter V that it is possible to apply this special case to the Bruhat lattices. This leads to the second topic of Chapter VI, the classification of the distributive lattices which satisfy this special sufficient condition. The third section of the chapter presents a proof that no other lattices beside the Bruhat lattices can satisfy the condition in question. As was indicated earlier in this introduction, we consider this result to be one of the two most interesting results of this thesis, partly because Dynkin diagrams arise naturally in the course of the proof.

Since distributive lattices are in one-to-one correspondence with the subposets of their join irreducible elements, the classification theorem just described is also a characterization of the miniscule posets. The last section of Chapter VI summarizes the four ways in which the miniscule posets arise up to that point, and also describes a fifth (empirical, but interesting) method by which these posets can be described.

2. Example

The most famous Bruhat order can be described as the partially ordered set of j-tuples $(a_1, a_2, \cdots, a_j)$ satisfying $1 \leq a_1 < a_2 < \cdots < a_j \leq n$, with order given by $a \leq b$ if and only if $a_1 \leq b_1$, $a_2 \leq b_2$, $\cdots$, $a_j \leq b_j$. This partial order has $\binom{n}{j}$ elements, and is in fact a distributive lattice. It is denoted $A_{n-1}(j)$ in this thesis. In algebraic geometry, this lattice describes the inclusion relationships of the Schubert subvarieties of the Grassmannian projective variety of j-dimensional subspaces of an n-dimensional space [StW]. In Lie representation theory, $A_{n-1}(j)$ is the partially ordered set of weights of the jth exterior power of the natural representation of $sl(n, \mathbb{C})$. There are two ways by which this lattice is often described in combinatorics. The first is as the partially ordered set of all partitions of integers into j or fewer parts, with each part no larger than n-j. The second way is as the lattice of order ideals of the poset which is the product of a j-element chain with an (n-j)-element chain. Stanley [StW] denotes this poset by L(j,n-j).

Chapter II

Definitions and a Preliminary Result

1. Definitions and Notation

The term "poset" stands for "partially ordered set". If x and y are elements of a poset P such that $x \leq z < y$ implies that z equals x, then we say that y covers x in P. The Hasse diagram of a finite poset P is the directed graph whose vertices are the elements of P, and whose edges are the covering relations of P. Namely, (y,x) is an edge of the graph if y covers x. An order ideal I of P is a subset $I \subseteq P$ such that $y \in I$ and $x \leq y$ imply $x \in I$. An order filter is an analogously defined subset of P, with $\geq$ replacing $\leq$.

Notation: $[n] := \{1, 2, \cdots, n\}$

$\pm[n] := \{-n, -n+1, \cdots, -1, 1, 2, \cdots, n\}$

Bruhat partial orders are defined on the elements of Weyl groups. It is possible to characterize a Weyl group as a finite group with n designated generators s_i, $1 \leq i \leq n$, whose presentation with respect to these generators has the form:

$$< s_i : s_i^2 = e, (s_i s_j)^{m_{i,j}} = e \text{ where } m_{i,j} \in \{2, 3, 4, 6\} >.$$

The designated generators s_i are called simple reflections. An irreducible Weyl group is one which cannot be expressed as the direct product of two smaller Weyl groups. The irreducible Weyl groups have been completely classified. There are three infinite families of *classical* irreducible Weyl groups, denoted with the letters A, BC, and D. Ignoring designated generators, these three infinite families can be simply described. The Weyl group of type A_{n-1} can be depicted with $n \times n$ permutation matrices (symmetric group, order $n!$), of type BC_n with "signed" permutation matrices (hyperoctahedral group, order $2^n n!$), and of type D_n with "signed" permutation matrices which have an even number of negative ones (order $2^{n-1}n!$). There are also five *exceptional* irreducible Weyl groups. They are denoted E_6, E_7, E_8, F_4, and G_2.

For any Weyl group W of rank n and any subset $J \subseteq [n]$, the parabolic subgroup W_J is defined to be $< s_j : j \in J >$. The set of left cosets, or coset space, W/W_J is denoted W^J.

Bruhat orders are defined on the coset spaces W^J as well as on Weyl groups W, but we must first define the Bruhat orders on Weyl groups. Any element $w \in W$ can be expressed $w = s_{i_k} \cdots s_{i_2} s_{i_1}$. Define the *length of w*, *l(w)*, to be the smallest such k possible. Any conjugate t of a designated generator, $t = ws_i w^{-1}$, is called a *reflection*.

Definition. The Bruhat partial order on a Weyl group W is the partial order defined by:

(i) The unique maximal element is the identity e.

(ii) For two elements w, w' of W, the relation $w \leq w'$ holds if and only if there exist reflections $t_1, \cdots, t_k$ such that $w = t_k \cdots t_2 t_1 w'$ and $l(t_{i+1} \cdots t_1 w) > l(t_i \cdots t_1 w)$ for $1 \leq i < k$.

This definition is the order dual of the usual one, e.g. normally e is the unique minimal element. We have reversed this convention for the sake of much nicer notation in the future. Note that as a result, $w \leq w'$ implies $l(w) \geq l(w')$. All of this relatively harmless, since the Bruhat orders are self-dual (Corollary 2).

Given $J \subseteq [n]$, it is known that each element $w \in W$ has a unique expression $w = w^J w_J$ where $l(w) = l(w^J) + l(w_J)$, $w_J \in W_J$, and w^J is the unique element of wW_J of minimal length. Thus by ignoring the w_J part of each element in a coset in W^J, we can identify each coset in W^J with an element of W in a natural way. The Bruhat order on W^J is defined to be the induced order under this identification. We will use this subset of W to depict W^J rather than the cosets themselves. Henceforth the term Bruhat poset shall refer to a Bruhat order defined on any Weyl group W or coset space W^J. The term irreducible Bruhat poset shall refer to a Bruhat poset defined on any W or $W^J = W/W_J$ for which W is irreducible.

Notation. Let W be an irreducible Weyl group of rank n and of type X, $X \in \{A, BC, D, E, F, G\}$. If $J \subseteq [n]$, set $J^c = [n] - J$. The statements of our results always require the set J^c rather than the set J. Hence we shall let $X_n(J^c)$ denote the irreducible Bruhat poset W^J. If $J^c = \{j\}$, then $X_n(j)$ shall denote the poset W^J.

2. Depiction of Bruhat Orders with Weights of Representations

In this section we present a useful preliminary proposition which depicts the Bruhat orders with certain weights of representations of semisimple Lie algebras. In addition to giving the connection of Bruhat orders to representation theory, this depiction facilitates some computations. It will be used several times in this thesis. Although it is possible that some researchers may already know this result, some are unaware of it, and it does not explicitly appear in the literature.

We assume familiarity with the theory of weights of representations of semisimple Lie algebras [Hu1]. Let H be the Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ of rank n and fix a set of positive roots Φ^+ in H_R^*. Denote the inner product on H_R^* with $(\cdot,\cdot)$ and let $\langle\cdot,\alpha\rangle = 2(\cdot,\alpha)/(\alpha,\alpha)$. Denote the action of an element w of the Weyl group on a weight $\lambda \in H_R^*$ by $w\lambda$. Denote the fundamental weights by λ_i, where $1 \leq i \leq n$.

<u>Proposition 2.</u> Let W be the Weyl group of a complex semisimple Lie algebra $\mathfrak{g}$ of rank n. Let $\lambda = \sum m_i\lambda_i$, $m_i \geq 0$, be a dominant weight for $\mathfrak{g}$ in H_R^*, and let $J^c = \{i: m_i > 0\}$. Define P to be the poset consisting of the weights $w\lambda$, $w \in W$, with order generated by the relations $u\lambda < v\lambda$ if $v\lambda - u\lambda = k\alpha$, where α is a positive root and $k > 0$. Then P is isomorphic to the Bruhat order W^J. The unique maximal element of P is λ.

<u>Remark.</u> The lattice of weights in H_R^* is often endowed with an order given by $\mu < \omega$ if and only if $\omega - \mu$ is a non-zero sum of positive roots. Thus the theorem almost states that the set of weights in the orbit $W\lambda$ ordered by the usual ordering of weights is just W^J. However, for $u\lambda < v\lambda$ to imply $u < v$, we must also require that $u\lambda$ and $v\lambda$ be related by a sequence of weights $w\lambda$ whose successive differences are positive multiples of positive roots. The example A_3, $\lambda = \lambda_1 + \lambda_3$ shows that this additional requirement is necessary in general. (There is a family of representations for which the additional assumption is not needed. See Section V.3.)

The first of the following lemmas is equivalent to Lemma 8.10 of Berstein, Gelfand and Gelfand [BGG], and the second is Lemma 3.5 of Deodhar [Deo].

Lemma 2.1. Let α be any positive root and let t be the corresponding reflection. Then $\alpha \in w\Phi^+$ if and only if $l(tw) > l(w)$.

Lemma 2.2. Let $w \in W$ and $w' \in W^J$. Then $w \leq w'$ if and only if the W^J part of w is $\leq w'$.

Proof of Proposition 2. The stabilizing subgroup of W at λ is exactly W_J, so the map $w \mapsto w\lambda$ is a bijection between W^J and the orbit of λ under W.

The order relations $u < v$ if $u = tv$ and $l(u) > l(v)$ generate the order on W^J. Let α be the positive root corresponding to t. Lemma 2.1 implies $\alpha \in v\Phi^+$. Therefore $\langle v\lambda,\alpha\rangle \geq 0$. Now $u\lambda = v\lambda - \langle v\lambda,\alpha\rangle$ implies $\langle v\lambda,\alpha\rangle > 0$. Hence $u\lambda < v\lambda$ in the partial order P defined on the orbit $W\lambda$.

Conversely, suppose that $v\lambda - u\lambda = k\alpha$ with α a positive root and $k > 0$. Consider the line $v + a\alpha$ where a is real. At most two points on this line have norm $\|v\lambda\| = \|u\lambda\| = \|tv\lambda\|$. This implies that $u\lambda = tv\lambda = v\lambda - \langle v\lambda,\alpha\rangle$, where t corresponds to α. Therefore $\langle v\lambda,\alpha\rangle > 0$. So $\alpha \in v\Phi^+$, and Lemma 2.1 implies $l(tv) > l(v)$. Hence $tv < v$. Now $v \in W^J$, and u is the W^J part of tv. Hence Lemma 2.2 implies that $u < v$ in W^J.

The following corollary actually holds for all Bruhat orders arising from finite Coxeter groups. The proof in the (slightly) more general case is the same in spirit [Bou, Ex. IV.1.22].

<u>Corollary 2.</u> Bruhat orders are self-dual.

<u>Proof.</u> Let w_0 denote the unique element of the Weyl group which takes positive roots to negative roots [Hu1, Ex. 10.9]. Apply w_0 to the orbit W^J.

Chapter III

Combinatorial Descriptions and Lexicographic Shellability of the Classical Orders

1. Introduction

The classical Bruhat orders are the nicest Bruhat orders from a combinatorial viewpoint, because the groups upon which they are defined are very closely related to the symmetric groups. (See Section II.1) In the second section of this chapter, Proposition II.2 is used to obtain descriptions of the classical orders in terms of signed multipermutations.

The rest of the chapter is entirely combinatorial in methods and content. Neither the original definition of Bruhat order nor the n-tuple descriptions permit the direct comparison of an arbitrary pair of elements from one of the classical orders. This is the purpose of the tableau descriptions which are derived in Section 3. These descriptions are used at the end of Section 3 to help specify which of the order generating relations given in Section 2 are actually covering relations.

The last section of the chapter uses the proofs of the tableau descriptions to help prove that the classical Bruhat orders are lexicographically shellable (Theorem 4). Given a poset P, let C(P) denote the set of its covering relations, i.e. $C(P) = \{(x,y): \ x \text{ covers } y\}$. Let Ω be any partially ordered set.

Definition. A poset P is said to be *lexicographically shellable* if there exists a map ω: $C(P) \longrightarrow \Omega$ such that:

(i) For every pair $x \geq y$ in P there exists a unique unrefineable chain $x = z_0 > z_1 > \cdots > z_r = y$ with $\omega(z_{t-1}, z_t) \geq \omega(z_t, z_{t+1})$ for $1 \leq t < r$.

(ii) If x covers w and $w \geq y$, then $\omega(x, z_1) > \omega(x, w)$, where z_1 is defined by (i).

Given any poset P, the *order complex* of P is defined to be the simplicial complex whose vertices are the elements of P and whose faces are the chains in P. Björner has shown [Bjö] that if a poset is lexicographically shellable, then the order complex of the poset is a "shellable" simplicial complex. Roughly speaking, a simplicial complex is "shellable" if it can be assembled from its maximal faces in a certain nice sequential fashion. Stanley [StC] and Reisner [Rei] have shown that if the order complex of a poset is shellable, then a certain commutative ring associated to the poset, the "Stanley-Reisner ring", has the Cohen-Macaulay property. This is the consequence of Theorem 4 that is used by DeConcini and Lakshmibai in their algebraic geometric application of lexicographic shellability. This application will be briefly described in Section VI.1.

Just as this thesis was being written, Björner and Wachs obtained a result [BjW] which essentially supersedes the lexicographic shellability result proved in this chapter. By considering a property slightly weaker than lexicographic shellability, they have shown that the order complex of any interval of any Bruhat order on any Coxeter group is shellable.

2. n-tuple and 2n-tuple Descriptions

Here Proposition II.2 is used to obtain descriptions of the classical Bruhat orders in terms of n-tuples and 2n-tuples of integers. The following is a list of the positive roots and fundamental weights for Lie algebras of types A, C, and D. Stanley [StW] obtains these descriptions directly from the presentations of the classical Weyl groups, but the method used here is faster and more precise. This way also has the advantage of explicitly retaining the connection with semisimple Lie algebras, which will be exploited in Chapter V.

A_{n-1}. Positive roots: $-e_i + e_j$, $1 \leq i < j \leq n$.

Fundamental weights: $\lambda_i = (-i/n)[e_1 + \cdots + e_{n-i}] + [(n-i)/n][e_{n-i+1} + \cdots + e_n]$, $1 \leq i \leq n-1$.

C . Positive roots: $-e_i + e_j$, $1 \leq i < j \leq n$; $e_i + e_j$, $1 \leq i \leq j \leq n$.

Fundamental weights: $\lambda_i = e_{n-i+1} + e_{n-i+2} + \cdots + e_n$, $1 \leq i \leq n$.

D . Positive roots: $-e_i + e_j$, $1 \leq i < j \leq n$; $e_i + e_j$, $1 \leq i < j \leq n$.

Fundamental weights: $\lambda_i = e_{n-i+1} + e_{n-i+2} + \cdots + e_n$, $3 \leq i \leq n$.

$$\lambda_{n-1} = \tfrac{1}{2}(-e_1 + e_2 + \cdots + e_n)$$

$$\lambda_n = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_n)$$

The action of a Weyl group on a weight space is generated by reflections with respect to the positive roots. Both of the root systems B_n and C_n generate the same Weyl group BC_n; we will use the fundamental weights of type C because they have nicer coordinates than those of type B. For the classical root systems, the possible effects on the coordinates $(a_1, a_2, \cdots, a_n)$ of a vector in the weight space from reflecting with respect to a positive root are:

Root $-e_i + e_j$, $i < j$ (Switch): $S_{ij}(\underset{\sim}{a}) = \underset{\sim}{b}$, $b_k = \begin{cases} a_j & \text{if } k = i, \\ a_i & \text{if } k = j, \\ a_k & \text{otherwise.} \end{cases}$

Root e_i (Negate): $N_i(\underset{\sim}{a}) = \underset{\sim}{b}$, $b_k = \begin{cases} -a_i & \text{if } k = i, \\ a_k & \text{otherwise.} \end{cases}$

Root $e_i + e_j$, $i < j$ (Switch-Negate): $SN_{ij}(\underset{\sim}{a}) = \underset{\sim}{b}$, $b_k = \begin{cases} -a_j & \text{if } k = i, \\ -a_i & \text{if } k = j, \\ a_k & \text{otherwise.} \end{cases}$

A "permutation of an n-tuple" is an n-tuple obtained by rearranging the components of the original n-tuple. "Signed permutation of an n-tuple" shall mean the same thing, except that the signs of the components may be changed as well. The notation $p^i q^j \cdots r^k$ denotes the n-tuple $(p, \cdots, p, q, \cdots, r, \cdots, r)$, where $n = i + j + \cdots + k$.

<u>Proposition 2A.</u> Let $J^c = \{j_1, j_2, \cdots, j_m\}$ with $n - 1 \geq j_1 > j_2 > \cdots > j_m \geq 1$. The Bruhat order $A_{n-1}(J^c)$ is isomorphic to the poset of all permutations of the n-tuple $e = 0^{n-j_1} 1^{j_1 - j_2} \cdots m^{j_m}$ with order generating relations $S_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i < j$, $a_i < a_j$. The maximal element is the n-tuple e.

<u>Proposition 2BC.</u> Let $J^c = \{j_1, j_2, \cdots, j_m\}$ with $n \geq j_1 > j_2 > \cdots > j_m \geq 1$. The Bruhat order $BC_n(J^c)$ is isomorphic to the poset of all

signed permutations of $e = 0^{n-j_1} 1^{j_1-j_2} \cdots m^{j_m}$ with order generating relations $S_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i < j$, $a_i < a_j$; $SN_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i + a_j > 0$; and $N_i(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i > 0$. The maximal element is the n-tuple e.

__Proposition 2D.__ Let $J^c = \{j_1, j_2, \cdots, j_m\}$ with $n \geq j_1 > j_2 > \cdots > j_m \geq 1$. The Bruhat order $D_n(J^c)$ is isomorphic to a poset of certain n-tuples as described below for various J^c. The order generating relations for all cases are $S_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i < j$, $a_i < a_j$ and $SN_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i + a_j > 0$.

__J^c__	__e__	__Set of n-tuples__
$n-1 \notin J^c$, $n \notin J^c$	$0^{n-j_1} 1^{j_1-j_2} \cdots m^{j_m}$	All signed permutations of e.
$n-1 \notin J^c$, $n \in J^c$	$1^{j_1-j_2} \cdots m^{j_m}$	All signed permutations of e with an even number of negative components.
$n-1 \in J^c$, $n \notin J^c$	$(-1)^1 1^{j_1-j_2} \cdots m^{j_m}$	All signed permutations of e with an odd number of negative components.
$n-1 \in J^c$, $n \in J^c$	$0^1 1^{j_2-j_3} \cdots (m-1)^{j_m}$	All signed permutations of e.

The maximal element in each case is the n-tuple e.

Figure 2 shows $D_3(1,3)$. The 3-tuples are parsed with commas, and underlines denote negative numbers.

__Proof of Proposition 2D.__ Choose λ as follows and apply Proposition II.2.

(i) $n-1 \notin J^c$, $n \notin J^c$: $\lambda = \Sigma_{j \in J^c} \lambda_j$,

(ii) $n-1 \notin J^c$, $n \in J^c$: $\lambda = \lambda_n + \Sigma_{j \in J^c} \lambda_j$,

(iii) $n-1 \in J^c$, $n \notin J^c$: $\lambda = \lambda_{n-1} + \Sigma_{j \in J^c} \lambda_j$,

(iv) $n-1 \in J^c$, $n \in J^c$: $\lambda = \Sigma_{j \in J^c} \lambda_j$.

The n-tuple of coordinates of λ with respect to the standard basis is e, and the orbit of λ under W is the set indicated in the table. The difference of two weights is a multiple of a positive root if and only if the respective n-tuples are related by S_{ij} or SN_{ij}, $i < j$. The conditions $a_i < a_j$ and $a_i + a_j > 0$ respectively hold if and only if $\underset{\sim}{a} - S_{ij}(\underset{\sim}{a})$ and $\underset{\sim}{a} - SN_{ij}(\underset{\sim}{a})$, are positive multiples of positive roots.

The proofs of Propositions 2A and 2BC are similar, except for type A one must verify that it is alright to avoid fractional and negative coordinates by using $\lambda_i = e_{n-i+1} + e_{n-i+2} + \cdots + e_n$ for $1 \leq i \leq n-1$, rather than the value originally given.

There is an alternative way to describe the Bruhat orders of types BC_n and D_n which will be needed in the next section. To each n-tuple $(a_i)_{i \in [n]}$ of integers associate a 2n-tuple $(a_i)_{i \in \pm[n]}$ of non-negative integers according to:

$$j = i > 0: \qquad a_j = \begin{cases} a_i & \text{if } a_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$j = -i < 0: \qquad a_j = \begin{cases} -a_i & \text{if } a_i < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $a_i > 0$ implies $a_{-i} = 0$. The 6-tuples in Figure 2 appear directly beneath the 3-tuples. The following two operations describe the possible effects of reflecting with respect to a positive root:

$$\text{(i)} \quad \text{(Switch)} \qquad S_{ij}(\underset{\sim}{a}) = \underset{\sim}{b},\ i < j, \quad b_k = \begin{cases} a_j & \text{if } k = i, \\ a_i & \text{if } k = j, \\ a_k & \text{otherwise.} \end{cases}$$

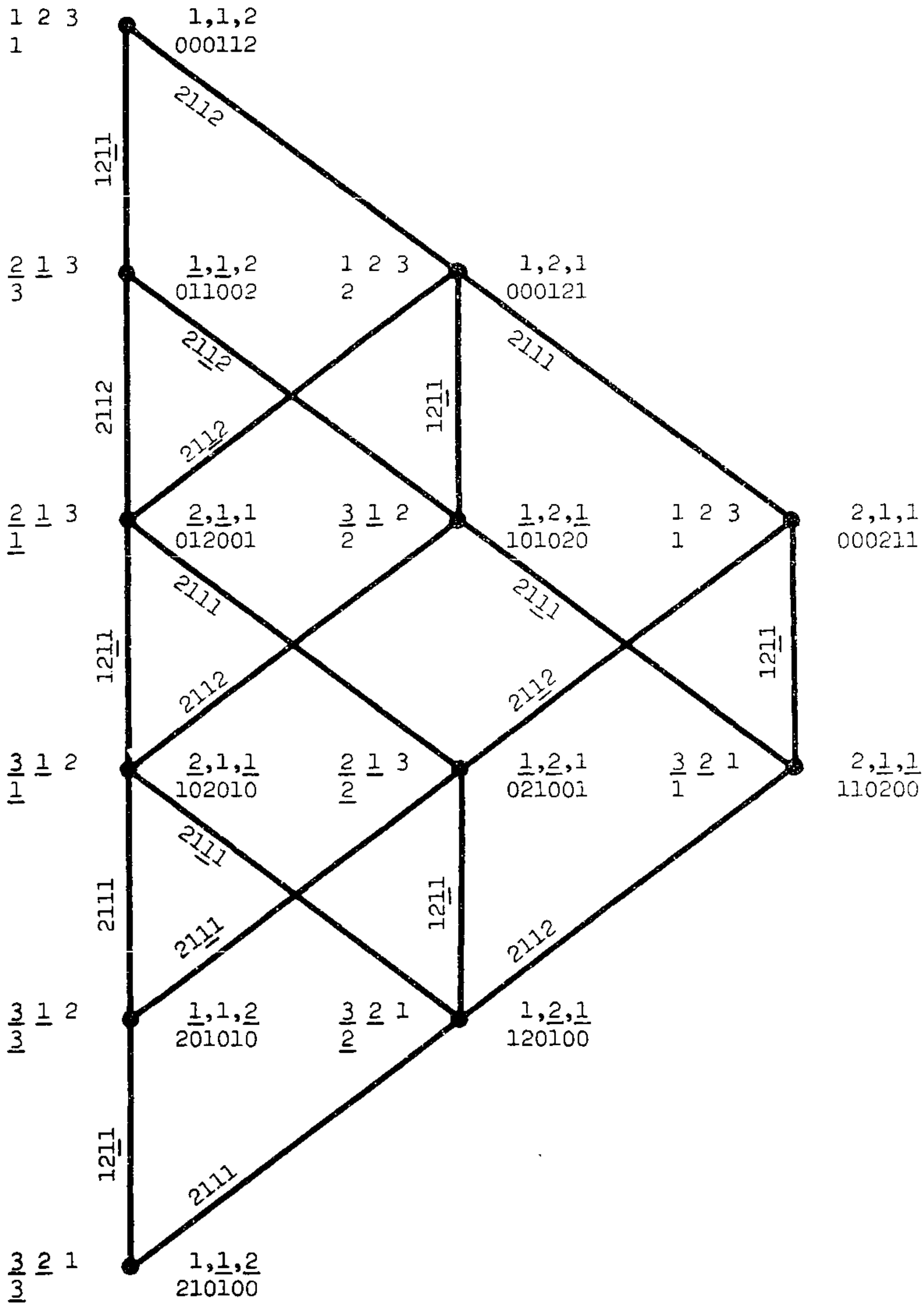


Figure 2

(ii) (Double Jump) $DJ_{ij}(\underset{\sim}{a}) = \underset{\sim}{b}$, $i < j$, $a_i = a_{-j} = 0$,

$$b_k = \begin{cases} a_j & \text{if } k = i, \\ a_{-i} & \text{if } k = -j, \\ 0 & \text{if } k = -i \text{ or } j, \\ a_k & \text{otherwise.} \end{cases}$$

It is easy to find the appropriate sets of 2n-tuples to describe the posets $BC_n(J^c)$ and $D_n(J^c)$. The following corollaries describe the translation of the generating relations into 2n-tuple notation.

Corollaries 2BCD. If the elements of $D_n(J^c)$ are portrayed with 2n-tuples as described above, then the relations $S_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i < j$, $i \neq -j$, $a_{-i} = 0$, $a_i < a_j$, and $DJ_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i < j$, $i \neq -j$, $a_i = a_{-j} = 0$, $a_{-i} + a_j > 0$ generate the desired partial order. Similarly, these relations together with $S_{-i,i}(\underset{\sim}{a}) < \underset{\sim}{a}$, $i > 0$, $a_i > 0$ generate the orders $BC_n(J^c)$ when their elements are portrayed with 2n-tuples.

3. Tableau Descriptions

We now describe the classical Bruhat orders with tableaux of integers. The tableau description for type D orders is new; the tableau descriptions for orders of types A and BC have apparently existed before only in the folklore of Indian algebraic geometers.

Given $J^c = \{j_1 \geq j_2 \geq \cdots \geq j_m\}$, a tableau of shape J^c is an array of non-zero integers of the form $(T_{p,d})_{1 \leq p \leq m,\ 1 \leq d \leq j_p}$. A standard tableau is one in which the entries in any row are strictly increasing and the entries in any column are non-increasing. The symbol T_p will denote either the set $\{T_{p,d}\}_{1\le d\le j_p}$ or the row vector $(T_{p,d})_{1\le d\le j_p}$. An <u>extreme tableau</u> is a tableau such that $T_{p+1} \subseteq T_p$ for $1 \leq p < m$. Given any set of tableau of the same shape J^c, one can define a partial order on them by <u>entrywise comparison</u>; namely, $U \leq V$ iff $U_{p,d} \leq V_{p,d}$ for $1 \leq p \leq m$, $1 \leq d \leq j_p$. This order will also be used to compare respective rows of two tableau.

<u>Definition.</u> Define a map ξ_n (ξ_{2n}) from the set of n-tuples (2n-tuples) of non-negative integers to the set of extreme standard tableau by:

$$\xi(\underset{\sim}{c}) = T, \text{ where for } p > 1,$$

$$T_p = \{i \in [n]: \ a_i \geq p\} \quad (\{i \in \pm[n]: \ a_i \geq p\}),$$

$$\text{with } T_{p,1} < T_{p,2} < \cdot \cdot \cdot \cdot$$

<u>Notation.</u> Fix $U = \xi(a)$, $V = \xi(b)$, $T = \xi(c)$ throughout this section.

<u>Theorem 3A.</u> Let $J^c = \{j_1, j_2, \ldots, j_m\}$ with $n-1 \geq j_1 > j_2 > \cdots > j_m \geq 1$. Let P be the poset of all extreme standard tableau of shape J^c with entries from [n], with partial order defined by entrywise comparison. Then P is isomorphic to the Bruhat poset $A_{n-1}(J^c)$.

<u>Theorem 3BC.</u> Let $J^c = \{j_1, j_2, \cdots, j_m\}$ with $n \geq j_1 > j_2 > \cdots > j_m \geq 1$. Let P be the poset of all extreme standard tableau of shape J^c with entries from $\pm[n]$ such that both i and -i never occur in the same tableau, with partial order defined by entrywise comparison. Then P is isomorphic to the Bruhat poset $BC_n(J^c)$.

<u>Definitions.</u> A <u>segment</u> of a row vector T_p is a row vector $T_p[f,g] = (T_{p,f}, T_{p,f+1}, \cdots, T_{p,g})$ for some f and g such that $i \leq f \leq g \leq j$. Segments which occupy the same positions in two tableaux of the same shape, e.g. $U_p[f,g]$ and $V_p[f,g]$, are said to be <u>analogous segments</u>. Two such analogous segments are said to be <u>D-incompatible</u> if $\{|U_{p,d}|: f \leq d \leq g\} = \{|V_{p,d}|: f \leq d \leq g\} = \{1, 2, \cdots, g-f+1\}$ and one of $|\{U_{p,d}: U_{p,d} < 0, f \leq d \leq g\}|$, $|\{V_{p,d}: V_{p,d} < 0, f \leq d \leq g\}|$ is an odd number while the other one is an even number. Two tableaux are said to be <u>D-compatible</u> if they have no analogous D-incompatible segments. The <u>D-compatible entrywise comparison partial order</u> on a set of tableau of the same shape is the usual entrywise comparison partial order together with the additional stipulation that any two tableaux must be D-compatible in order to be comparable.

<u>Theorem 3D.</u> Let $J^c = \{j_1, j_2, \cdots, j_m\}$ with $n \geq j_1 > j_2 > \cdots > j_m \geq 1$. Define $J^{c'}$ by:

$$j'_1 = \begin{cases} j_2 & n-1,\ n \in J^c \\ j_1 + 1 & n-1 \in J^c,\ n \notin J^c \\ j_1 & \text{otherwise.} \end{cases} \qquad j'_{1<i\leq m} = \begin{cases} j_{i+1} & n-1, n \in J^c, \\ j_i & \text{otherwise.} \end{cases}$$

Let P be the poset of all extreme standard tableau of shape $J^{c'}$ with entries such that both i and -i never occur in the same tableau, with partial order defined by D-compatible entrywise comparison. Then P is isomorphic to the Bruhat poset $D_n(J^c)$.

The isomorphism for Theorem 3A is the map ξ_n; the isomorphism for Theorems 3BCD is the map ξ_{2n}. These maps are clearly bijective. The tableaux for $D_3(1,3)$ (of shape $\{3,1\}$) appear to the left of the points in Figure 2. The proof of Theorem 3A is contained in the proof of Theorem 3B, since $A_{n-1}(J^c)$ can be identified in an obvious manner with an interval of $BC_n(J^c)$. All ordered tuples in the proofs of Theorems 3BCD will be 2n-tuples.

<u>Proof of Theorem 3BC.</u> Suppose that $\underset{\sim}{b} < \underset{\sim}{a}$ in $BC_n(J^c)$ by one of the generating relations of Corollary 2BC. It is straightforward to show that V is less than U by componentwise comparison. Conversely, assume that $V < U$ in P. We shall construct $\underset{\sim}{c}$ such that $\underset{\sim}{c} < \underset{\sim}{a}$ and $V \leq T < U$. Applying induction on the sum of the differences of the respective tableau entries will complete the proof.

Set $l_i(\underset{\sim}{a}) = \{j: \ j \leq i, \ a_j = a_i\}$.

Let $x = \max\{a_h: \ a_h \neq b_h, \ -n \leq h \leq n\}$.

Note: Expressions such as $j \leq h < i$ refer only to non-zero h.

Let $i = \min\{ h: \ a_h = x, \ (a_h \neq b_h) \text{ or } (a_h = b_h \text{ and } l_h(\underset{\sim}{a}) \neq l_h(\underset{\sim}{b})), \ -n \leq h \leq n\}$.

In the following arguments, we can assume $x = m$ and $l_i(\underset{\sim}{a}) = 1$. (If not, the locations $\pm h$ such that $a_h \geq x$ and $l_h(\underset{\sim}{a}) = l_h(\underset{\sim}{b})$ can be ignored using the reduction $V_p \leq U_p$ iff $V_p \cup \{h\} \leq U_p \cup \{h\}$.

Let j be such that $b_j = x$ and $l_j(\underset{\sim}{b}) = l_i(\underset{\sim}{a})$.

Let $y = \max\{a_h: \ j \leq h < i\}$.

If $y > 0$, let $k = \max\{h: \ a_h = y, \ j \leq h < i\}$.

If $y = 0$, let $z = \min\{a_{-h}: \ j \leq h < i\}$,

$$\text{and let } k = \begin{cases} -1 \text{ if } i = 1, \\ \max\{h: \ a_{-h} = z, \ j \leq h < i\} \text{ otherwise.} \end{cases}$$

(i) $y > 0$; or $y = 0$, $i \neq 1$, $z = 0$; or $y = 0$, $i = 1$.

Set $\underset{\sim}{c} = S_{ki}(\underset{\sim}{a})$. Clearly $\underset{\sim}{c} < \underset{\sim}{a}$ and $T < U$. Fix p with $y < p \leq m$. Then

$$T_p = U_p - \{i\} \cup \{k\}.$$

The choice of k implies $h \notin U_p$, $j \leq h < i$. Hence if $U_{p,d} = i$, then $T_{p,d} = k$. Thus T is obtained from U by replacing i with k, with no shifting of other entries. The various choices made also imply $U_{p,d-1} < j$. Hence $V_{p,d-1} < j$. But $j \in V_p$. Thus $V_{p,d} \leq j \leq k = T_{p,d}$. We conclude that $V \leq T < U$.

(ii) $y = 0$, $i \neq 1$, $z > 0$. This can only occur when $j < k < i < 0$ or $0 < j < k < i$. Set $\underset{\sim}{c} = DJ_{ki}(\underset{\sim}{a})$. Note that $a_i > 0$ and $k \neq -i$. Therefore $\underset{\sim}{c} < \underset{\sim}{a}$.

For $z < p \leq x$, $\quad T_p = U_p - \{i\} \cup \{k\}$.

For $0 < p \leq z$, $\quad T_p = U_p - \{i\} \cup \{k\} - \{-k\} \cup \{-i\}$.

Now $a_h = 0$ for $j \leq h < i$, so the replacement of i with k works as in (i), and the corresponding entry in V is less than k. Suppose $0 < j < k < i$; the case $j < k < i < 0$ is similar. The first p rows of U contain $-i+1$, $-i+2, \cdots, -j$. Since $j \in V$, at worst the corresponding elements in V_p for $0 < p \leq z$ are $-i, -i+1, \cdots, -j-1$. Thus $T_p \geq V_p$ for $0 < p \leq z$, after T_p has been obtained from U_p by removing $-k$ and inserting $-i$. Again we conclude $V \leq T < U$.

<u>Proof of Theorem 3D.</u> As in case BC, $\underset{\sim}{b} < \underset{\sim}{a}$ in $D_n(J^c)$ by one of the generating relations of Corollary 2D implies that U is entrywise less than V. We verify that U_p and V_p are D-compatible in one such situation. Let $\underset{\sim}{b} = DJ_{ij}(\underset{\sim}{a})$ with $i < 0 < j$, $-i < j$, $a_i = a_{-j} = 0$, $0 < a_{-i} < a_j$. Fix p such

that $0 < p \leq a_{-i}$. Suppose there is a segment of length t in V_p which is D-incompatible with the analogous segment of U_p. This implies both of these segments must contain one each of $\pm 1, \pm 2, \cdots, \pm t$. If $-i$ or j are in U_p, they are replaced by $-j$ and i respectively when passing to V_p. (Some shifting of entries may occur.) This forces $t > j$. But then this segment of V_p has exactly two more negative entries than the analogous segment of U_p. Therefore V_p and U_p are in fact D-compatible. The other cases are easier. Hence $\underset{\sim}{b} < \underset{\sim}{a}$ implies $V < U$ in P.

Conversely, suppose that U and V are D-compatible and that $V < U$ by entrywise comparison. Define $l_i(\underset{\sim}{a})$, x, i, j, y, z, and k as in the proof of Theorem 3BC. Proceed as before, unless:

(i) $y = 0$, $i = 1$. Let $k = \max\{h: a_{-h} = z, j \leq h < i\}$.

Set $\underset{\sim}{c} = DJ_{ki}(\underset{\sim}{a})$.

(ii) $j = -1$, $k = 1$. Redefine $y = \max\{a_h: 2 \leq h < i\}$.

If $y > 0$, proceed as before. Otherwise, redefine

$$z = \min\{a_{-h}: -1 \leq h < i, h \neq 1\}$$

$$\text{and } k = \max\{h : -1 \leq h < i, h \neq 1, a_{-h} = z\}.$$

In both cases, the proofs that T is entrywise greater than or equal to V are similar to those used for Theorem 3BC.

We verify that the tableaux T and V are D-compatible for one case. Suppose $0 < j < k < i$, $y = 0$ and $z > 0$. Then $\underset{\sim}{c} = DJ_{ki}(\underset{\sim}{a})$. The reduction of the proof of Theorem 3BC which assumes $x = m$ and $l_i(\underset{\sim}{a}) = 1$ is still valid. To obtain T from U, the entry $i > 0$ is replaced by $k > 0$ in a fixed position in each row, and the entries $-(i-1), -(i-2), \cdots, -k$ are replaced by $-i, -(i-1), \cdots, -(k+1)$ respectively in each of the first a_{-k} rows. Fix p, and suppose that $T_p[f,g]$ and $V_p[f,g]$ are D-incompatible. Set $t = g - f + 1$. If $t < j$ or $t \geq i$, it is clear that $U_p[f,g]$ and $T_p[f,g]$ have the same number of negative entries. Since U and V are D-compatible by assumption, no D-incompatibilities will arise between T_p and V_p for these values of t. Thus $j \leq t < i$, implying $\pm j$ occurs in any D-incompatible segment. Let $T_{p,d} = k$, $V_{p,e} = j$. By the proof of Theorem 3BC, $d \leq e$. By the choice of k, $-j \in T_1$. Let $T_{p,c} = -j$. Then $e < d$. Now $f \leq c$ and $e \leq g$, implying $d \in [f,g]$. But $T_{p,d} = k$. Thus each segment must contain one each of $\pm 1, \pm 2, \cdots, \pm k$, and $k > j$. By the choice of k, the entries $-i, -(i-1), \cdots, -k, \cdots, -j$ occur in T_1. Therefore no h such that $k < h < i$ occur in T_p, implying $g = d$. (Recall that $t < i$.) We conclude that any D-incompatible segment must be of the form $T_p[f,d] = (-h, -h+1, \cdots, -k, \cdots, -j, T_p[f+h-j, d-1], k)$, where $k < h < i$ and $\{|T_{p,c}|: f+h-j \leq c \leq d-1\} = \{1, 2, \cdots, j-1\}$. Recall $V_{p,e} = j$ and $e \geq d$. But $\pm j$ must appear in $V_p[f,d]$, since $j < h$. This forces $V_{p,d} = j$. We must have $V_{p,c} \leq T_{p,c}$ for $f \leq c \leq f+h-j-1$ and $-j$ is not available. Thus $V_p[f, f+h-j-1] = (-h, -h+1, \cdots, -j-1)$. Now $T_p[f+h-j, d-1] = U_p[f+h-j, d-1]$, and both U_p and V_p have one each of $\pm 1, \pm 2, \cdots, \pm(j-1)$ in this segment. The D-compatiblity of U and V therefore implies the D-compatibility of T_p and V_p along the segment $[f+h-j, d-1]$. The remaining entries in the segments $T_p[f,d]$ and $V_p[f,d]$ have identical signs. Hence these larger segments are actually D-compatible. Thus no D-incompatible segments exist between T and V. The other cases are easier. Therefore $V \leq T < U$ in P, and the proof is complete.

For the statements of the next results, we revert to n-tuple notation for cases BC_n and D_n. However, retain the correspondence between a and U, b and V, c and T via the map ξ_{2n} composed with the equivalence of n-tuples and 2n-tuples.

Corollaries 3ABCD. The order generating relations $S_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i < a_j$, of Propositions 2ABCD are covering relations iff $i < k < j$ implies either $a_k < a_i$ or $a_k > a_j$. The generating relations $SN_{ij}(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i + a_j > 0$, of Propositions 2BCD are covering relations iff $i < k < j$ implies $a_k < -a_i$ or $a_k > a_j$, $k < i$ implies $|a_k| < -\min(a_i, a_j)$ or $|a_k| > \max(a_i, a_j)$, and (for case BC, not case D) $a_i a_j < 0$. The relations $N_i(\underset{\sim}{a}) < \underset{\sim}{a}$, $a_i > 0$, of Proposition 2BC are covers iff $k < i$ implies $|a_k| > a_i$.

Proof. All of the "only if" parts can be easily proved by finding counterexamples to weakenings of the conditions. In the case of type BC posets, let us prove that $\underset{\sim}{b} = SN_{ij}(\underset{\sim}{a}) < a$, $a_i + a_j > 0$ is a covering relation for the case $a_i < 0$, $a_j > 0$. This implies $a_j > -a_i$. Let $K = \{k\colon a_k > a_j,\ i < k < j\} \cup \{k\colon a_k > a_j,\ k < i\} \cup \{-k\colon a_k < -a_j,\ k < i\} = \{k_1 < k_2 < \cdot \cdot \cdot < k_r\}$. To obtain the tableau V from the tableau U, replace entries $k_1, k_2, \cdot \cdot \cdot, k_r, j$ with $-i, k_1, k_2, \cdot \cdot \cdot, k_r$ in rows $-a_i + 1$ to a_j. Fix p such that $-a_i < p \leq a_j$ and let $V_{p,e} = -i$, $U_{p,f} = j$. Suppose there is an extreme standard tableau S such that $V < S < U$ by entrywise comparison. Let $H = \{S_{p,g}\colon e \leq g \leq f\}$. Clearly $H \subset \{-i, -i+1, \cdot \cdot \cdot, j\}$. If $k \in K$, then $k \in U_q$ for some $q > a$. But $V_q = U_q$ forces $S_q = U_q$. Thus $k \in S_q \subseteq S_p$, since S is extreme. Then $k \in H$, because $S_{p,f+1} = U_{p,f+1} > j$. Hence $H = K \cup \{h\}$, with $-i \leq h \leq j$. Note that $U_{-a_i} = V_{-a_i}$ forces $U_{-a_i} = S_{-a_i}$, and $-i \leq U_{-a_i,g} \leq j$ implies $U_{-a_i} \in K \cup \{-i,j\}$. But $h \in S_{-a_i}$ since S is extreme. Therefore $h = -i$ or $h = j$, implying $T = V$ or $S = U$. Proofs of the other cases are similar in spirit.

4. Lexicographic Shellability

Lexicographic shellability was defined in the introductory section to this chapter. In [Ede], P. Edelman showed that the Bruhat partial order $A_{n-1}([n-1])$ is lexicographically shellable. We extend this result in two ways: to the other two classical Bruhat orders $BC_n([n])$ and $D_n([n])$, and also to the coset space Bruhat orders $A_n(J^c)$, $BC_n(J^c)$, and $D_n(J^c)$, where $J^c \subseteq [n]$. Much of the proof of the following theorem comes from the proofs of the tableau descriptions.

Theorem 4. All classical Bruhat orders are lexicographically shellable.

Proof. Orders of types BC and D will described with 2n-tuples indexed by $\pm[n]$. Orders of type A will not be treated separately because $A_{n-1}(J^c)$ is an interval of $BC_n(J^c)$.

Let Ω be the lexicographic total order on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. For example, $(2,3,7,1) > (2,3,6,9)$. Note that the word "lexicographic" is being used in two entirely different contexts. Given any 2n-tuple $(a_i)_{i\in\pm[n]}$, define

$$r_i(\underset{\sim}{a}) = \{j: \ j \geq i, \ a_j = a_i\} \ .$$

(Recall the similar definition of $l_i(\underset{\sim}{a})$ in Section 3.) Define a labelling ω of the cover relations of the classical orders, i.e.

$$\omega: \ C(X_n(J^c)) \longrightarrow \Omega.$$

If $\underset{\sim}{a}$ covers $\underset{\sim}{b}$, set $\omega(\underset{\sim}{a},\underset{\sim}{b})$ equal to:

$(a_j,\ r_j(\underset{\sim}{a}),\quad a_i,\ \ l_i(\underset{\sim}{a}))$ if $\underset{\sim}{b} = S_{ij}(\underset{\sim}{a})$,

$(a_j,\ r_j(\underset{\sim}{a}),\ -a_{-i},\ r_{-i}(\underset{\sim}{a}))$ if $\underset{\sim}{b} = DJ_{ij}(\underset{\sim}{a})$, $a_i = a_{-j} = 0$, $a_j > a_{-i}$,

$(a_j,\ r_j(\underset{\sim}{a}),\ -a_{-i},\ r_{-i}(\underset{\sim}{a}))$ if $\underset{\sim}{b} = DJ_{ij}(\underset{\sim}{a})$, $a_i = a_{-j} = 0$, $a_j = a_{-i}$, $j < -i$,

$(a_{-i},\ r_{-i}(\underset{\sim}{a}),\ -a_j,\ r_j(\underset{\sim}{a}))$ if $\underset{\sim}{b} = DJ_{ij}(\underset{\sim}{a})$, $a_i = a_{-j} = 0$, $a_j < a_{-i}$,

where $i < j$. These labels are shown in Figure 2 for $D_3(1,3)$; minus signs are denoted with underscores for typographical convenience.

Most of the content of this proof resides in the manner in which the definition of the labels above complements the proof of Theorem 3. We will only outline the rest of the proof, which consists of trivial to easy (but sometimes tedious) verifications. The tedious aspect arises because one must consider the various covering relations or combinations of covering relations for orders of types BC and D at each step.

Given $\underset{\sim}{a} > \underset{\sim}{b}$ in a classical Bruhat order, the proof of Theorem 3 recursively constructed a particular chain of elements $\underset{\sim}{a} = \underset{\sim}{c}_0 > \underset{\sim}{c}_1 > \cdots > \underset{\sim}{c}_r = \underset{\sim}{b}$ from $\underset{\sim}{a}$ to $\underset{\sim}{b}$. In each case, Corollary 3 can be used to verify that each of the relations $\underset{\sim}{c}_t > \underset{\sim}{c}_{t+1}$ is a covering relation. Again considering various cases, one can confirm that the choice of two consecutive elements is always such that $\omega(\underset{\sim}{c}_{t-1},\underset{\sim}{c}_t) > \omega(\underset{\sim}{c}_t,\underset{\sim}{c}_{t+1})$. The third consequence of the construction is that if $\underset{\sim}{a}$ covers $\underset{\sim}{d}$ and $\underset{\sim}{d} > \underset{\sim}{b}$, then $\omega(\underset{\sim}{a},\underset{\sim}{c}_1) > \omega(\underset{\sim}{a},\underset{\sim}{d})$. To see this, note that the tableaux for $\underset{\sim}{a}$ and $\underset{\sim}{b}$ imply that the first two entries of the label quadruples are as large as possible with the choice of $\underset{\sim}{c}_1 = \underset{\sim}{c}$. The tableaux also imply that the locations searched in the process of defining y, z, and k are the only locations to which the entry x at location i can be moved such that the resulting 2n-tuple is greater than $\underset{\sim}{b}$. Given these restrictions on the movement of the entry x at i, the choice $\underset{\sim}{c}_1 = \underset{\sim}{c}$ from the proof of Theorem 3 is exactly the choice of $\underset{\sim}{c}_1$ that maximizes the third and fourth entries of the label quadruple.

To complete the proof, one must show that no other chain from $\underset{\sim}{a}$ to $\underset{\sim}{b}$ has non-increasing covering relation labels. Let $\underset{\sim}{a} = \underset{\sim}{d}_0 > \underset{\sim}{d}_1 > \cdots > \underset{\sim}{d}_r = \underset{\sim}{b}$ be some other chain from $\underset{\sim}{a}$ to $\underset{\sim}{b}$. As above, the tableaux indicate that the entry x at i is the largest entry (in terms of the first two label entries) which can be moved in any of the 2n-tuples between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Since it must be moved sometime, and since the labels must never increase, the entry x at i must be moved first. The tableaux again restrict the locations to which this entry can be moved. At this point, various cases for each type of order must be considered in order to rule out any other "first moves" beside $\underset{\sim}{c}_1 = \underset{\sim}{c}$. To treat these cases for orders of types BC and D, it is helpful to occasionally return to n-tuple notation and use Corollary 3. Each of these situations essentially follows the same pattern: Moving the entry x at i to location $h < k$ with either a "switch" or "double jump" is either impossible or not a covering relation. And if the entry x at i is moved to a location $h > k$ to produce a 2n-tuple greater than $\underset{\sim}{b}$, one can show that eventually this entry x must "hop" over location k, which again is not a cover, or move to location k. This eventual forced move to location k produces an increase in the labels of the covering relations of the alternative chain. The proof is complete once all of the apparent alternative "first moves" for each case have been eliminated.

Chapter IV

Applications of Lexicographic Shellability and Tableau Descriptions

1. An Application to Algebraic Geometry due to DeConcini and Lakshmibai

In this section we will briefly describe how DeConcini and Lakshmibai have used Theorem III.4 to show that certain embeddings of certain projective varieties arising in algebraic geometry are arithmetically Cohen-Macaulay and arithmetically normal [DeL]. Let G be a classical semisimple algebraic group over an algebraically closed field, and let P_j be the jth maximal parabolic subgroup of G. Then G/P_j is a projective variety. Let R_τ denote the homogeneous coordinate ring (for the canonical projective embedding of G/P_j) of a Schubert subvariety $S(\tau)$ of G/P_j. The main result of DeConcini and Lakshmibai is:

Theorem. The ring R_τ is Cohen-Macaulay and normal.

For certain choices of G and P_j, there is a straightforward proof of this result utilizing the theory of algebras with straightening laws [Bac], [DEP]. For example, if G is of type A_{n-1} and P_j is the jth maximal parabolic subgroup of G, then one can use the work of Rota, et. al. [DKR] (or other authors) to show that the ring R_τ is a ring with straightening law over a prinicipal ideal of the poset $A_{n-1}(j)$, which is in fact a distributive lattice. A consequence of the theory of algebras with straightening laws is that R_τ is Cohen-Macaulay if the

Stanley-Reisner ring of this principal ideal is Cohen-Macaulay. But the Cohen-Macaulayness of this ring follows from the shellability of the order complex of the principal ideal, which can in turn be deduced from S. Provan's theorem [Prv] that the order complex of a distributive lattice is shellable.

The more general work of DeConcini and Lakshmibai follows the same pattern. Two difficulties arise. First, the connection between the Bruhat poset $X_n(j)$ and the ring R_τ for a Schubert variety in G/P_j, where G is of any classical type X_n, is not in general as straightforward as when G is of type A_{n-1}. DeConcini and Lakshmibai introduce an intermediate object, called a doset, which is a subset of $X_n(j) \times X_n(j)$. They then define the concept of an algebra with straightening law over a doset and show that any ring related by this mechanism to a doset defined on a poset is Cohen-Macaulay if the Stanley-Reisner ring of the poset is Cohen-Macaulay. Work of Seshadri, et. al. [LM4] is used to confirm that the ring R_τ is an algebra with straightening law over a doset defined on a principal order ideal of the Bruhat order $X_n(j)$. Hence the problem is reduced to the question of whether the order complexes of principal order ideals in the posets $X_n(j)$ are shellable. The second aspect of difficulty for the more general case now arises because the posets $X_n(j)$ are not distributive lattices in general. However, Björner's proposition [Bjö] that the lexicographic shellability of the poset implies the shellability of the order complex can be combined with Theorem III.4 to complete the proof.

In a personal communication, DeConcini has indicated that the methods of [DeL] together with Theorem III.4 can also be applied to the Schubert varieties of the flag manifolds G/P which correspond to the Bruhat posets of the form $A_{n-1}(j_1, j_2)$. Furthermore, DeConcini and Lakshmibai point out in their paper that their methods also apply to the Schubert varieties of G/P_j, with G an exceptional semisimple algebraic group and P_j a classical maximal parabolic subgroup, if the order complexes of the corresponding Bruhat posets are lexicographically shellable. These order complexes are now known to be shellable by the recent work of Björner and Wachs [BjW] which was described at the end of Section III.1.

2. Computation of the Möbius Function and Triangulations of Balls and Spheres

The Möbius function of a partially ordered set P is a certain integer valued function on the set $P \times P$ which played a central role in G.-C. Rota's theory of enumeration with respect the poset P [Rot]. By careful use of the Coxeter group axioms, D.-N. Verma has shown [VeM] that the Möbius function for the Bruhat order on any full Coxeter group W has the following expression: $\mu(u,v) = (-1)^{l(u)-l(v)}$. A theorem of Stanley and Björner [Bjö, Theorem 2.7] for arbitrary posets provides a more concrete way to obtain this result for the classical Weyl groups by using the labelling of the covering relations specified above. A particular case of their theorem states that if a labelling of the covers satisfies the requirements for lexicographic shellability, then $(-1)^{r(x)-r(y)}\mu(y,x)$ is the number of chains from x to y which have strictly increasing labels, where $x > y$ and $r(x)-r(y)$ is the length of any unrefineable chain from x to y. With our labelling, it is easy to verify that there is always exactly one such chain for any pair of comparable elements in a classical Weyl group. With a little more work, one could probably also use the same methods to obtain (for the classical cases) a more concrete derivation of Deodhar's expression [Deo] for the Möbius function of the Bruhat orders defined on the coset spaces W^J.

Stanley and Edelman have noted that Theorem III.4 can be combined with the corollary just described to produce triangulations of spheres [Ede]. The particular form of the Möbius function for the Bruhat order $X_n([n])$ on a classical Weyl group combined with the lexicographic shellability of $X_n([n])$ implies the following: If one deletes the minimal and maximal elements from $X_n([n])$ and forms the order complex of the resulting poset, then the simplicial complex so obtained is a triangulation of a sphere. The interested reader should consult Edelman's proof for the case $A_n([n])$. (This proof applies immediately to $BC_n([n])$ and $D_n([n])$.) In addition, Stanley has pointed out (personal communication) that Deodhar's computation of the Möbius function for the Bruhat orders on the coset spaces W^J can be combined with Theorem III.4 to produce triangulations of balls by the same procedure. In summary, this procedure of forming the order complex after deleting the minimal and maximal elements from a classical Bruhat order W^J yields a triangulation of a sphere when $J = \emptyset$ and a triangulation of a ball when $J \neq \emptyset$. DeConcini conjectured (personal communication) that this procedure always yields triangulations of spheres or balls when it is applied to any interval of a Bruhat order defined on a Coxeter group or an appropriate coset space of a Coxeter group. This conjecture was recently confirmed by Björner and Wachs [BjW]. (See Section III.1.)

3. Relationship with the Original Definition of Bruhat Order

In this section we use one of the tableau descriptions (Theorem III.3A) to prove a conjecture of G. Lusztig's (Proposition 3.1) concerning the Bruhat order on the symmetric group and arrays of dimensions of intersections of pairs of flags of subspaces in specified relative positions. As a consequence, we obtain a more direct description of the Bruhat orders of type A in their original contexts.

<u>Definition.</u> Let $\mathcal{W}$ be an n-dimensional vector space. A <u>maximal flag of subspaces</u> $\{\mathcal{W}_i\}$ in $\mathcal{W}$ is a strictly increasing sequence of subpaces $0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_n = \mathcal{W}$ in $\mathcal{W}$.

<u>Notation.</u> Throughout this section W will denote the <u>nth</u> symmetric group, i.e. the Weyl group of type A_{n-1}. Its elements shall be denoted with the small Greek letters σ, τ.

<u>Definition.</u> A maximal flag $\{\mathcal{U}_i\}$ is said to be in <u>relative position σ</u> with respect to a fixed maximal flag $\{\mathcal{W}_i\}$ if and only if $\mathcal{W}_i \cap \mathcal{U}_{\sigma^{-1}(i)-1} \subset \mathcal{W}_i \cap \mathcal{U}_{\sigma^{-1}(i)}$ for $1 \leq i \leq n$.

It is a fact that any two flags are in exactly one relative position σ with respect to each other.

<u>Proposition 3.1.</u> Let $\{\mathcal{W}_i\}$ be a fixed maximal flag in $\mathcal{W}$. Let $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ be in relative positions σ and τ with respect to $\{\mathcal{W}_i\}$. Then $\tau \leq \sigma$ in the Bruhat order on W if and only if $\dim(\mathcal{V}_i \cap \mathcal{W}_j) \leq \dim(\mathcal{U}_i \cap \mathcal{W}_j)$ for $1 \leq i,j \leq n$.

We defer the proof of Proposition 3.1.

Consult [Hu2] as a reference for the following material. Let G = SL(n,$\mathbb{C}$) act on an n-dimensional complex vector space $\mathcal{W}$. Fix a maximal torus T and a Borel subgroup containing T. Let $\{\mathcal{W}_i\}$ be the maximal flag stabilized by B. The points of the manifold G/B correspond to maximal flags in $\mathcal{W}$. Let $N_G(T)$ be the normalizer of T in G. Then $N_G(T)/T = W$, the Weyl group of G, which is the <u>nth</u> symmetric group. Let W' be a set of representatives of $N_G(T)/T$ in G. The Bruhat decomposition of the flag manifold is described with these representatives: $G/B = \dot{\bigcup}_{\sigma \in W'} B\sigma B/B$. The Bruhat order on W was originally defined by inclusion (reverse inclusion for this paper) of the subsets $\overline{B\sigma B/B}$ (bar denotes topological closure) of the flag manifold. Hence the following proposition uses Theorem III.3A to obtain a more direct description of the Bruhat order of type A_{n-1} in its original context.

<u>Proposition 3.2.</u> Let G = SL(n, $\mathbb{C}$) act on $\mathcal{W}$ and let G/B be the manifold of maximal flags in $\mathcal{W}$. Denote the flag stabilized by B with $\{\mathcal{W}_i\}$. Let $\sigma, \tau \in W'$ as above. If $\{\mathcal{U}_i\} \in B\sigma B/B$ and $\{\mathcal{V}_i\} \in B\tau B/B$ are two flags in two Bruhat cells, then $\overline{B\tau B/B} \supseteq \overline{B\sigma B/B}$ if and only if $\dim(\mathcal{V}_i \cap \mathcal{W}_j) \leq \dim(\mathcal{U}_i \cap \mathcal{W}_j)$ for $1 \leq i,j \leq n$.

Proof. It can be shown that $\{\mathcal{U}_i\} \in B\sigma B/B$ if and only if $\{\mathcal{U}_i\}$ is in relative position σ with respect to $\{\mathcal{W}_i\}$. The proposition then follows from the original definition of Bruhat order and Proposition 3.1.

The following definition of relative position is equivalent to the one given above.

Definition. A maximal flag $\{\mathcal{U}_i\}$ is said to be in relative position σ with respect to a fixed maximal flag $\{\mathcal{W}_i\}$ if and only if there exists a basis $\{w_i\}$ of $\mathcal{W}$ such that $\mathcal{W}_i = [w_1, w_2, \cdots, w_i]$ and $\mathcal{U}_i = [w_{\sigma(1)}, w_{\sigma(2)}, \cdots, w_{\sigma(i)}]$.

Proof of Proposition 3.1. Pick a basis $\{w_i\}$ for $\mathcal{W}$ such that the flags $\{\mathcal{W}_i\}$ and $\{\mathcal{U}_i\}$ can be described as above. Then $\dim(\mathcal{U}_i \cap \mathcal{W}_j) = |\{\sigma(k): k \leq i, \sigma(k) \leq j\}|$. A similar expression computes $\dim(\mathcal{V}_i \cap \mathcal{W}_j)$ in terms of τ. Apply Proposition 3.3.

Proposition 3.3. Let σ, τ be permutations on $\{1, 2, \cdots, n\}$. Then $\tau \leq \sigma$ in the Bruhat order on the nth symmetric group if and only if $|\{\tau(k): k \leq i, \tau(k) \leq j\}| \leq |\{\sigma(k): k \leq i, \sigma(k) \leq j\}|$ for $1 \leq i,j \leq n$.

Proof. Modify Theorem III.3A to handle permutations of $\{1, 2, \cdots, n\}$ rather than $\{0, 1, \cdots, n-1\}$. The resulting tableaux have n rather than n-1 rows. Let the tableau U correspond to σ, i.e. $U_p = \{ i: \sigma(i) \geq p\}$. Set $m_{i,j} = |\{\sigma(k): k \leq i, \sigma(k) \leq j\}|$, $m_{0,j} = m_{i,0} = m_{0,0} = 0$. Define V and $n_{i,j}$ similarly with respect to τ. Note that $U_p = \{ i: m_{i,p-1} = m_{i-1,p-1}\}$, and similarly for V_p. The numbers $m_{i,p-1}$ progress from 0 to p-1 as i runs from 0 to n. There are n-p+1 locations i such that $m_{i,p-1} = m_{i-1,p-1}$. Since U_p and V_p are increasing row vectors, it is easy to see that $V_p \le U_p$ by entrywise comparison if and only if $n_{i,p-1} \le m_{i,p-1}$ for $0 \le i \le n$. Apply Theorem III.3A to finish the proof.

Choose m integers $J^c = \{j_1, \ldots, j_m\}$ such that $n-1 \ge j_1 \ge \ldots > j_m \ge 1$. A flag of type J^c in an n-dimensional vector space $\mathcal{W}$ is a strictly increasing sequence of subspaces $0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \cdots \subset \mathcal{U}_m = \mathcal{W}$ such that $\dim \mathcal{U}_k = j_{m-k+1}$ for $1 \le k \le m$. Fix one such flag and let $P \subseteq G$ denote its stabilizer. If one gives a definition of relative position between a flag of type J^c and a fixed maximal flag using elements of W^J, then there are appropriate W^J analogs to each of the results above. The objects involved are: flags of type J^c, relative positions from W^J, manifold G/P of flags of type J^c with Bruhat cells $B\sigma P/P$, and multi-permutations (shuffles). Perhaps these results can also be extended in some fashion to Bruhat orders of types B, C, and D, if the appropriate definitions of flags are used.

4. Relationship with Young's Tableaux

Alfred Young utilized standard tableau with entries from {1, 2, . . . , n} in his construction of finite dimensional irreducible representations of $sl(n, \mathbb{C})$ [Boe, Theorem 5.3]. The extreme standard tableaux used in Section III.3 to describe the Bruhat orders of type A_{n-1} are a subset of the tableaux employed by Young. Since it was shown in Section II.2 that Bruhat orders arise in the context of representations, one might ask whether the tableaux used in this paper can be identified in a natural manner with a subset of the tableaux used by Young. Proposition 4 gives an affirmative answer to this question.

Refer to Section II.2 and [Hu1] for representation notation and definitions. Let us briefly describe Young's construction of a finite dimensional irreducible representation ρ of $sl(n, \mathbb{C})$ with highest weight $\lambda = \Sigma_1^{n-1} m_i \lambda_i$. Set $N = m_1 + m_2 + \cdots + m_{n-1}$. Let ν be the natural representation of $sl(n, \mathbb{C})$ on $\mathbb{C}^n$, and let $\mathcal{W} = \otimes^N \mathbb{C}^n$. Young explicitly constructed a certain projection P on $\mathcal{W}$. Set $\mathcal{V} = P(\mathcal{W})$. He then showed that the desired representation ρ is the map from $sl(n, \mathbb{C})$ to $gl(\mathcal{V}, \mathbb{C})$ given by $\rho(x)v = P\{[\otimes^N \nu](x)w\} = [\otimes^N \nu](x)(Pw)$ where $w \in \mathcal{V}$, $v = Pw \in \mathcal{V}$, $x \in sl(n, \mathbb{C})$. Let w_T denote the element $e_{t_1} \otimes e_{t_2} \otimes \cdots \otimes e_{t_N}$ of the usual basis for $\mathcal{W}$, where T is a tableau with m_{n-1} rows of length n-1, m_{n-2} rows of length n-2, $\cdots$, and whose entries are $t_1, t_2, \cdots, t_N$ when the tableaux is read like a page of English text. Whenever T is a standard tableau, let $v_T = P(w_T)$. Young proved that the set of the vectors v_T forms a basis for the representation ρ of $sl(n, \mathbb{C})$ on $\mathcal{V}$.

Given a basis vector v_T, let $q_i = |\{t_r: \ t_r = i\}|$, $1 \leq i \leq n$. Then v_T is a weight vector for ρ with weight $\Sigma_{j=1}^{n-1}(q_{j+1}-q_j)\lambda_j$. Since the weight of v_T can be computed in terms of T, and since each $w\lambda$ weight space has dimension 1, Young's techniques assign to each weight $w\lambda$ exactly one standard tableau.

Any rows of the same length in an extreme standard tableau must have identical entries. If an extreme standard tableau has more than one row of a given length, then we shall call the second, third, $\cdots$ rows of that length <u>repeated rows</u>. Essentially no information is lost if these rows are deleted from the tableau.

<u>Proposition 4.</u> Let ρ be the finite dimensional irreducible representation of $sl(n, \mathbb{C})$ with highest weight $\lambda = \Sigma_{j=1}^{n-1} m_j \lambda_j$ and set $J^c = \{j: m_j > 0\}$. Let $w \in W$, the Weyl group of type A_{n-1}. Then the standard tableau $T_{w\lambda}$ assigned by Young to the weight $w\lambda$ is an extreme tableau. Furthermore, if any repeated rows in $T_{w\lambda}$ are deleted, then the resulting tableau is equal to the tableau T_w assigned to the coset of w in W^J by the constructions of Proposition III.2A and Theorem III.3A.

<u>Proof.</u> The tableau T_λ corresponding to the highest weight of the representation has as many entries as possible equal to n, then as many entries equal to n-1 as possible, etc. It is easy to see that T_λ is an extreme tableau, and that the tableau T_e corresponding to the identity $e \in W$ via the work in Chapter III is obtained when the redundant rows are deleted from T_λ. The effect in tableau terms of operating on a weight μ with a simple reflection s_i is to replace the entry n-i+1 with the entry n-i in every row of the tableau T_μ where n-i does not already appear. (If $U = T_\mu$, this corresponds to finding the largest k such that $P[\otimes^N Y_i^k(\omega_U)] \neq 0$, where $Y_i(e_{n-i+1}) = e_{n-i}$.) In the context of Chapter III, it is easy to show that $T_{s_i u}$ is obtained from T_u by exactly the same procedure. The proof is complete with the observation that every element of W^J can be expressed as a product of simple reflections.

Chapter V

Application of Lie Representation Theory to Bruhat Lattices

1. Introduction and Combinatorial Definitions

In the introductory chapter to this thesis, we asserted that the Bruhat lattices were the most interesting Bruhat orders from a combinatorial point of view. One of the reasons given was that their structures were particularly susceptible to analysis with Lie algebraic methods. This chapter will utilize the Lie algebraic notions of miniscule representation and principal three dimensional subalgebra to obtain combinatorial information about the Bruhat lattices.

Section 2 classifies the Bruhat lattices using Proposition II.2 as a computational aid. Section 3 presents the Lie algebraic machinery which will be needed, including the definition of a miniscule representation of a complex semisimple Lie algebra. The miniscule representations have been classified, and the list of them is given in Section 3. We define a "miniscule lattice" to be the poset of weights of a miniscule representation. It turns out that the set of miniscule lattices is the same as the set of Bruhat lattices, and the correspondence is almost "natural" in a certain sense. (See Section 2.) Because of the methods used in this chapter, these lattices will be referred to as miniscule lattices. This terminology has the added advantage that certain posets associated to these lattices can be referred to as "miniscule posets", rather than as "Bruhat posets".

__Definitions.__ Let L be a distributive lattice. The __poset of join irreducibles__ j(L) of L is defined to be the subposet of all elements of L which cover exactly one element. Let P be a poset. The (distributive) __lattice of order ideals__ J(P) of P is the set of order ideals in P ordered by inclusion.

It is well known that L = J(j(L)) and P = j(J(P)). A "miniscule poset" will turn out to be any poset P such that P = j(L), where L is a Bruhat (miniscule) lattice.

The following sequence of combinatorial definitions are necessary to explain the content of Section 4.

__Definition.__ Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Let P be a tableau of shape λ with non-negative integer entries no larger than m. If $P_{p,d} \geq P_{p+1,d}$ and $P_{p,d} \geq P_{p,d+1}$ for all possible p and d, and if the sum of the entries of P is N, then P is a __plane partition of N contained in λ with part size bounded by m__.

__Definition.__ The generating function for plane partitions of shape contained in λ with part size bounded by m, $G(\lambda,m,x)$, is defined to be:

$$G(\lambda,m,x) = \sum_{P} x^{|P|} ,$$

where the sum is over all such possible plane partitions P and where $|P| = N$ if P is a plane partition of N.

In his thesis [StO], Stanley abstracted this generating function to arbitrary posets as follows. A plane partition with part size bounded by m can be thought of as a non-decreasing sequence of (possibly empty) order ideals of the fixed order ideal of "shape λ" of the poset $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the natural numbers under their usual ordering. E.g., the non-negative integer in the upper left hand corner of the tableau counts the number of non-empty ideals in the sequence, since every non-empty ideal must contain the unique minimal element (0,0) of the fixed order ideal.

Definition. Given any finite poset P, the m-nested ideal generating function of P is defined to be:

$$F(P,m,x) \quad = \quad \sum_{I_1 \leq I_2 \leq \cdots \leq I_m} x^{|I_1|+|I_2|+\cdots+|I_m|}$$

where the I_i are order ideals in P.

Definition. A poset P is said to be Gaussian if its m-nested ideal generating functions have the following form for every non-negative m:

$$F(P,m,x) \quad = \quad \frac{(1-x^{h_1+m})(1-x^{h_2+m}) \quad \cdots \quad (1-x^{h_r+m})}{(1-x^{h_1}) \quad (1-x^{h_2}) \quad \cdots \quad (1-x^{h_r})} \quad .$$

where r and the h_i are non-negative integers independent of m.

It is easy to see that the direct sum of any two Gaussian posets is a Gaussian poset. (The converse is also true: Stanley has shown (personal communication) that if a direct sum is Gaussian, then each of the summands is Gaussian.) The concept of Gaussian poset was introduced by Stanley for purely combinatorial reasons years before combinatorialists studied combinatorial aspects of Lie algebras [St0]. Certain plane partition generating function identities motivated the definition of

Gaussian poset. Propositions 4.1 and 4.2 of this chapter give new proofs for two of these identities. These proofs are special cases of the main result of Section 4, which can be succintly stated as: All miniscule posets are Gaussian posets. The two exceptional irreducible miniscule posets $e_6(6)$ and $e_7(7)$ are new irreducible Gaussian posets. There are no known Gaussian posets beside the miniscule posets. (See Section VI.4.) The m-nested ideal generating functions for the miniscule posets can be interpreted in an obvious manner as "rank weighted m-multichain" generating functions for the Bruhat lattices.

We now describe the content of the fifth and last section of this chapter.

Definition. A ranked poset is a partial order on a set L together with a partition $\{L_0, L_1, \cdots, L_r\}$ of L into ranks L_i such that the elements of L_{i+1} cover only elements of L_i. If $x \in L_i$, then we say x has rank i and set $r(x) = i$.

Definitions. Let a ranked poset L have ranks $L_0, L_1, \cdots, L_r$. If the sizes of the ranks are such that $|L_i| = |L_{r-i}|$ for $0 \leq i \leq r$, then L is rank symmetric. If there is some k such that $0 \leq k \leq r$ and $|L_0| \leq |L_1| \leq \cdots \leq |L_k| \geq \cdots \geq |L_{r-1}| \geq |L_r|$, then L is rank unimodal.

Definitions. A ranked poset L is said to have the Sperner property if no antichain in L has more elements than the largest rank of L does. The poset L is k-Sperner if no union of k antichains in L exceeds the union of the k largest ranks of L in size. If L has r ranks, then it is said to be strongly Sperner if it is k-Sperner for $k = 1, 2, \cdots, r$.

The last section of this chapter presents a new proof that miniscule (Bruhat) lattices are rank symmetric, rank unimodal, and strongly Sperner. Although this proof is more limited in scope than Stanley's original proof [StW], it leads to a better understanding of how Stanley's proof works. As noted in the introduction to this thesis, this better understanding leads to a result in Chapter VI which is more general in some sense than Stanley's. Another reason for presenting the new proof is that the Lie algebraic techniques used in Section 5 are very closely related to the techniques used in Section 4.

This chapter makes heavy use of the representation theory of complex semisimple Lie algebras as described in [Hu1]. We should note, however, that the main result of Section 5 is superseded by Theorem 2.1 of Chapter VI. The proof of this theorem is presented in purely linear algebraic terms, so it should be accessible to all readers.

2. Classification of Bruhat Lattices

A *Bruhat lattice* is a Bruhat poset which is a lattice. Let W be a Weyl group with simple reflections $s_1, s_2, \cdots, s_n$, and let $\mathfrak{g}$ be the corresponding complex semisimple Lie algebra with fixed Cartan subalgebra H, positive simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$, and fundamental weights $\lambda_1, \lambda_2, \cdots, \lambda_n$. If $J^c \subseteq [n]$, set $\lambda = \sum_{i \in J^c} \lambda_i$. Recall that Proposition II.2 uses the weights $w\lambda$, $w \in W$, to portray the Bruhat order W^J. Also recall that nodes j and k are connected in the Dynkin diagram for $\mathfrak{g}$ if and only if $(\alpha_j, \alpha_k) < 0$.

__Lemma 2.__ If there is an element u of W^J such that $u\lambda = \Sigma_{i=1}^{n} r_i\lambda_i$ with $r_j = p > 0$, $r_k = q > 0$ and $(\alpha_j, \alpha_k) < 0$, then W^J is not a lattice.

__Proof.__ For convenient (albeit imprecise) notation, refer to an element w of W^J with the __jth__ and __kth__ coordinates of $w\lambda$ with respect to the basis of fundamental weights, e.q. u = (p,q). Suppose that $(\alpha_k, \alpha_k) = 2(\alpha_j, \alpha_j)$. Then $w\lambda$ covers $s_jw\lambda = (-p, p+q)$ and $s_kw\lambda = (p+2q, -q)$. In turn, $s_jw\lambda$ covers $s_ks_jw\lambda = (p+2q, -p-q)$ and $s_kw\lambda$ covers $s_js_kw\lambda = (-p-2q, p+q)$. Now $l(s_ks_jw) = l(s_js_kw) = l(s_jw) + 1 = l(s_kw) + 1$. But $(s_ks_js_ks_js_k)s_jw = s_js_kw$ and $(s_js_ks_js_ks_j)s_kw = s_ks_jw$. Hence both s_jw and s_kw cover both s_ks_jw and s_js_kw, implying that W^J is not a lattice. The cases $(\alpha_k, \alpha_k) = (\alpha_j, \alpha_j)$ and $(\alpha_k, \alpha_k) = 3(\alpha_j, \alpha_j)$ are similar.

__Proposition 2.__ The following is a list of all irreducible Bruhat posets which are lattices: $A_{n-1}(j)$, $1 \leq j \leq n-1$, $BC_n(1)$, $BC_n(n)$, $D_n(1)$, $D_n(n-1)$, $D_n(n)$, $E_6(1)$, $E_6(6)$, $E_7(7)$, $G_2(1)$, $G_2(2)$. (Simple reflections numbered as in [Hu1, p. 58].) All of these lattices are distributive lattices.

__Proof.__ Using tables [Bou, pp. 250-275], one may eliminate all other finite irreducible Bruhat posets in less than an hour with the following method. Take an X_n and J^c which do not appear in the list. Set $\lambda = \Sigma_{j\in J^c}\lambda_j$. Operate on λ with simple reflections until the situation of Lemma 2 is produced. The poset $A_{n-1}(j)$ was described in Section 2 of Chapter I and is easily seen to be a distributive lattice. The posets $BC_n(n)$, $D_{n+1}(n)$, $D_{n+1}(n+1)$ were shown to be isomorphic distributive lattices by Stanley [StW]. The other posets listed are distributive lattices by inspection.

Stanley denotes the lattice $BC_n(n)$ by $M(n)$. We will describe it explicitly in the proof of Proposition 4.2. The lattices $BC_n(1)$, $G_2(1)$ and $G_2(2)$ are chains of lengths 2n, 6 and 6 respectively. The lattice $D_n(1)$ has 2n elements, with two elements in the middle rank and one element in every other rank. The lattices $E_6(1)$ and $E_6(6)$ are isomorphic, and each has 27 elements. The lattice $E_7(7)$ has 56 elements. The Hasse diagrams for $E_6(6)$ and $E_7(7)$ appear in Figure 2.2 of Section VI.2.

3. Miniscule Representations and Principal Three Dimensional Subalgebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with fixed Cartan subalgebra H and Weyl group W. A finite dimensional irreducible representation of $\mathfrak{g}$ is completely determined by its "highest weight".

Definition. Let ρ be a finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ. The representation ρ is a miniscule representation if every one of its weights is of the form $w\lambda$ for some $w \in W$.

Fact 3.1. [Hu1, Ex. 13.13] If $X_n(\lambda)$ denotes the finite dimensional irreducible representation of the complex simple Lie algebra of type X_n with highest weight λ, then the miniscule representations of complex simple Lie algebras are: $A_{n-1}(\lambda_j)$, $1 \leq j \leq n-1$, $B_n(\lambda_n)$, $C_n(\lambda_1)$, $D_n(\lambda_1)$, $D_n(\lambda_{n-1})$, $D_n(\lambda_n)$, $E_6(\lambda_1)$, $E_6(\lambda_6)$, $E_7(\lambda_7)$.

Recall that a partial order is defined on the weights of any representation by: $\mu \leq \omega$ if and only if $\omega - \mu$ is a sum of positive roots.

Definitions. A miniscule lattice is the set of weights of some miniscule representation ordered by the usual partial order on weights. An irreducible miniscule lattice is one which arises from a miniscule representation of a simple Lie algebra. An (irreducible) miniscule poset is the partially ordered set of join irreducible elements of some (irreducible) miniscule lattice.

Remark. As a consequence of the manner in which representations of semisimple Lie algebras can be decomposed into representations of simple Lie algebras, every miniscule lattice can be expressed as a product of irreducible miniscule lattices.

The use of the word "lattice" for these posets is justified by the following lemma.

Lemma 3. Each miniscule lattice is just the Bruhat poset which corresponds to the miniscule representation as in Proposition II.2. Hence each miniscule lattice is in fact a distributive lattice.

Proof. Let $W_J \subseteq W$ be the stabilizer of the highest weight λ. Suppose that $u\lambda \leq v\lambda$ in $W\lambda$ with $u, v \in W^J$. Then $u\lambda = v\lambda - \Sigma k_i\alpha_i$ with $k_i \geq 0$, $1 \leq i \leq n$. Now $\|v\lambda\| = \|u\lambda\|$ implies that $\langle v\lambda, \alpha_j\rangle > 0$ for some j. Lemma II.2.1 then implies that $s_jv < v$ in W. Exercise 13.13 of [Hu1] states that $\langle v\lambda, \alpha_j\rangle = +1, 0$, or -1, since λ corresponds to a miniscule representation. Thus $s_jv\lambda = v\lambda - \alpha_j$. Apply induction and Lemma II.2.2 to conclude that $u < v$ in W^J. Thus each miniscule lattice is a Bruhat poset. Comparing the list of miniscule representations of simple Lie algebras with the list of irreducible Bruhat lattices in Proposition 2, it is evident that every irreducible miniscule lattice is a distributive lattice. Taking products implies that every miniscule lattice is distributive.

Note that $G_2(1)$ and $G_2(2)$ are Bruhat lattices, but $G_2(\lambda_1)$ and $G_2(\lambda_2)$ are not miniscule representations. However, $G_2(1)$ and $G_2(2)$ are both six element chains, and $A_5(1)$ is a miniscule lattice which is a six element chain. Therefore, the set of miniscule lattices is the same as the set of Bruhat lattices, but the correspondence is not quite compatible with the Lie representation indexing of the lattices. We shall use essentially the same notation to describe irreducible miniscule lattices as was used for irreducible Bruhat lattices. To emphasize the representation dependent nature of this chapter, a slight change will be made for type BC orders.

<u>Fact 3.2.</u> The irreducible miniscule lattices are: $A_{n-1}(j)$, $1 \leq j \leq n-1$, $B_n(n)$, $C_n(1)$, $D_n(1)$, $D_n(n-1)$, $D_n(n)$, $E_6(1)$, $E_6(6)$, $E_7(7)$.

<u>Notation.</u> The irreducible miniscule poset corresponding to $X_n(j)$ shall be denoted $x_n(j)$.

<u>Fact 3.3.</u> The irreducible miniscule posets are: $a_{n-1}(j)$, $1 \leq j \leq n-1$, $b_n(n)$, $c_n(1)$, $d_n(1)$, $d_n(n-1)$, $d_n(n)$, $e_6(1)$, $e_6(6)$, $e_7(7)$.

The Hasse diagrams for the irreducible miniscule posets are given in Figure VI.3.8.

A review of the proof of Lemma 3 reveals that if $v\lambda$ covers $u\lambda$ in a miniscule lattice W^J, then $v\lambda - u\lambda = \alpha_j$, a particular positive simple root. Hence if $u\lambda = \Sigma k_i\alpha_i$, then the poset rank of $u\lambda$ in $W\lambda$ is given by Σk_i up to an additive constant. Let $\delta^\vee$ denote the unique element of $H_{\mathbb{R}}^*$ (the Euclidean space where roots and weights live) such that $(\alpha_i,\delta^\vee) = 1$ for $1 \leq i \leq n$. It is easy to show that $\delta^\vee = \Sigma 2\lambda_i/(\alpha_i,\alpha_i)$. If w_0 is the unique element of the Weyl group which takes every positive root to a negative root [Hu1, Ex. 10.9], then by Propostion II.2 $w_0\lambda$ is the unique minimal element of $W\lambda$. Since positive simple roots must pass to negative simple roots, $(w_0\lambda,\delta^\vee) = -(\lambda,\delta^\vee)$. Hence $W\lambda$ has $2(\lambda,\delta^\vee)+1$ ranks as a ranked poset.

Let ρ denote the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ. There is a unique element h of the fixed Cartan subalgebra H of $\mathfrak{g}$ such that $\rho(h)v = (\mu,\delta^\vee)v$ if v is a vector in the representation space of weight μ. If ρ is miniscule, a basis for the representation space can be chosen which is in one-to-one correspondence with the weights of the representation or the elements of the miniscule lattice. To determine which rank a lattice element is in, multiply the corresponding weight basis vector by $\rho(h)$ and add $(\lambda,\delta^\vee)$ to the observed eigenvalue. (Recall that the ranks of a poset are numbered 0, 1, 2, · · ·, r.)

This leads to the second topic of this section, principal three dimensional subalgebras. It will be useful to define two other elements x and y in $\mathfrak{g}$ such that x, y, and h span a subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{C})$. The definition below is from B. Kostant [Kos, p. 996], but the idea originated with E.B. Dynkin [Dyn, p. 168] and J. de Siebenthal

[Sie]. First recall that one may find 3n elements $\{x_i, y_i, h_i\}_{i=1}^{n}$ in $\mathfrak{g}$ such that any triple $\{x_i, y_i, h_i\}$ spans a subalgebra isomorphic to $sl(2,\mathbb{C})$, where $h_j \in H$ is defined by $\lambda_i(h_j) = \delta_{ij}$ [Hu1, pp. 37, 112]. Using $\alpha_i(h) = 1$, it is easy to show that $h = \Sigma\, 2h_i/(\alpha_i,\alpha_i)$.

<u>Definition.</u> Let $x = \Sigma\, c_i x_i$, where $c_1, c_2, \cdots, c_n$ are any n non-zero complex numbers. Set $y = \Sigma\, 2y_i/[c_i(\alpha_i,\alpha_i)]$. Then any subalgebra of conjugate to the subalgebra spanned by x, y, and h is called a *principal three dimensional subalgebra*.

It is easy to check that x, y, and h span a subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{C})$. Any representation of $\mathfrak{g}$ induces a representation of $sl(2,\mathbb{C})$ via this embedding. Principal three dimensional subalgebras have been employed previously in combinatorics in [Hug], [Lep], and [StU].

4. Plane Partition Generating Function Identities

This section represents joint work with R. Stanley.

The Weyl character formula is a multivariate generating function for the dimensions of the weight spaces of a finite dimensional irreducible representation of a complex semisimple Lie algebra. There is a particularly nice one variable specialization of this formula for principal three dimensional subalgebras. Recall that a finite dimensional representation of $sl(2,\mathbb{C})$ has weights $-r/2, (-r/2)+1, \cdots, (r/2)-1, r/2$ for some non-negative integer r.

Lemma 4.1. Let ρ be a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ. Let $\Pi(\lambda)$ denote the set of weights of ρ, and let d_μ be the dimension of the weight space of weight μ. Let d_i denote the dimension of the weight space of weight i for the induced representation of a principal three dimensional subalgebra of $\mathfrak{g}$. Then

$$\sum_{i=-r/2}^{i=r/2} d_i x^i = \sum_{\mu\in\Pi(\lambda)} d_\mu x^{(\mu,\delta^\vee)} = x^{-(\lambda,\delta^\vee)} \frac{\prod_{\alpha\in\Phi^+}(1-x^{\langle\lambda+\delta,\alpha\rangle})}{\prod_{\alpha\in\Phi^+}(1-x^{\langle\delta,\alpha\rangle})} .$$

where $\langle\cdot,\alpha\rangle = 2(\cdot,\alpha)/(\alpha,\alpha)$, Φ^+ is the set of positive roots for $\mathfrak{g}$, and δ is the sum of the fundamental weights (or half the sum of the positive roots).

Proof. The first equality holds because a weight vector for ρ of weight μ is a weight vector for the induced representation of weight $(\mu,\delta^\vee)$, by the definition of h. For the second equality, use Jacobson's derivation of Weyl's total degree formula [Jac, p. 256] with $\delta^\vee$ rather than δ. No problem arises for Jacobson since he eventually sets $x = 1$. Using Jacobson's original proof has in the past [StU] caused certain generating functions normally associated with Lie algebras of type B to be labelled type C, and conversely. (The root systems of types B and C are dual to each other.)

Lepowsky first resurrected this form of the character formula for combinatorial identities [Lep]. He calls this identity the "principal specialization of Weyl's character formula". It has also been used by Stanley [StU]. When applied to a miniscule representation, this identity produces an expression for the "rank generating polynomial" of the corresponding miniscule lattice: Bring the factor $x^{(\lambda,\delta^\vee)}$ to the left hand side of the equation, and then the coefficient of x^j is the number of elements in the jth rank of the lattice.

The following non-trivial lemma is one of the main results of the recent algebraic geometric paper entitled "G/P-I" by C.S. Seshadri. We refer instead to "G/P-III" [LM3] (with coauthors V. Lakshmibai and C. Musili) because it is more readily available.

Lemma 4.2. Let λ be the highest weight of a miniscule representation of a complex semisimple Lie algebra $\mathfrak{g}$. Then the dimension of the weight space of weight μ of the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight $m\lambda$ is equal to the number of multichains $u_1\lambda \leq u_2\lambda \leq \cdot\cdot\cdot \leq u_m\lambda$ in the corresponding miniscule lattice such that $\mu = u_1\lambda + u_2\lambda + \cdot\cdot\cdot + u_m\lambda$.

We are now ready to prove the main result of this section.

Theorem 4. Every miniscule poset is a Gaussian poset.

Proof. Apply Lemmas 4.1 and 4.2 to the representation of $\mathfrak{g}$ of highest weight $m\lambda$, where λ is the highest weight of the corresponding miniscule representation. After bringing the factor $x^{(m\lambda,\delta^\vee)}$ to the left hand side of the character formula, the left hand side counts the number of m-multichains in the corresponding miniscule lattice, weighted by ranks. Therefore the m-nested ideal generating function for the miniscule poset is the quotient of two products. However, it must be verified that there are fixed non-negative integers r and $h_1, h_2, \cdot\cdot\cdot, h_r$ such that the right hand sides for various values of m have exactly the required form. Since δ is a weight, $h_\alpha = \langle\delta,\alpha\rangle$ is an integer. Cancel the terms in the product where $\langle\lambda,\alpha\rangle = 0$. For each irreducible miniscule representation, one can verify by hand that $\langle\lambda,\alpha\rangle = 1$ whenever $\langle\lambda,\alpha\rangle \neq 0$. (This is easy; the worst case $E_7(\lambda_7)$ has only 27 positive roots α such that $\langle\lambda_7,\alpha\rangle \neq 0$.) Hence if $\langle\lambda,\alpha\rangle \neq 0$, then $\langle m\lambda+\delta,\alpha\rangle = m + h_\alpha$. Thus each irreducible miniscule poset is Gaussian. But every miniscule poset can be expressed as the direct sum of irreducible miniscule posets, since products of lattices pass to sums of posets when forming posets of join irreducibles. The proof is complete with the observation that the direct sum of two Gaussian posets is Gaussian.

The irreducible miniscule posets $a_{n-1}(j)$, $1 \leq j \leq n-1$, and $b_n(n)$ ($\cong d_{n+1}(n+1) \cong d_{n+1}(n)$)) were first shown to be Gaussian with intricate generating function manipulations. We shall work out the details of these cases below. It is trivial to directly prove that $c_n(1)$ and $d_n(1)$ are Gaussian. The exceptional irreducible miniscule posets $e_6(6)$ ($\cong e_6(1)$) and $e_7(7)$ are new Gaussian posets. The miniscule posets are all known examples of Gaussian posets. They will be shown to be remarkable in other respects in Chapter VI. We formally pose the following question.

<u>Problem 4</u>. Is every Gaussian poset a miniscule poset?

New proofs of two plane generating function identities can be obtained by working out the two cases of Theorem 4 mentioned above in detail. The identity for $a_{n-1}(j)$, $1 \leq j \leq n-1$, is originally due to

MacMahon [McM, p. 243]. For a modern treatment of a more general result, see [StT, Theorem 15.3]. The identity for $b_n(n)$ was first conjectured by Bender and Knuth [BKn] and later proved by Gordon [unpublished], Andrews [And], and MacDonald [McD, Ex. I.5.19]. Plane partitions contained in λ with part size bounded by m were defined in Section 1, as were their associated generating functions. Such a plane partition P is called a <u>column strict plane partition contained in λ with part size bounded by m</u> if it satisfies $P_{p,d} > P_{p,d+1}$ rather than $P_{p,d} \geq P_{p,d+1}$.

<u>Proposition 4.1.</u> Let $\lambda = (j, j, \cdots, j)$ be an (n-j)-tuple. Then the generating function $G(\lambda,m,x)$ for plane partitions contained in λ with part size bounded by m is:

$$G(\lambda,m,x) = \frac{\prod_{p=1}^{j}\prod_{q=1}^{n-j}(1 - x^{m+p+q-1})}{\prod_{p=1}^{j}\prod_{q=1}^{n-j}(1 - x^{p+q-1})} .$$

<u>Proof.</u> It is easy to see that $G(\lambda,m,x)$ counts m-nested ideals in $a_{n-1}(j)$, which is the product of a j-chain with an (n-j)-chain. The right hand side is found using by using Lemma 4.1 for $A_{n-1}(m\lambda_j)$ and computing the values $\langle\delta,\alpha\rangle$ for all α such that $\langle\lambda_j,\alpha\rangle \neq 0$. The equality of the two sides follows from Lemma 4.2 and the fact that appropiately weighted m-multichains in $A_{n-1}(j)$ correspond to m-nested ideals in $a_{n-1}(j)$.

<u>Proposition 4.2.</u> Let $\lambda = (n, n, \cdots, n)$ be an m-tuple. Then the generating function $H(\lambda,n,x)$ for column strict plane partitions contained in λ with part size bounded by n is:

$$H(\lambda,n,x) = \frac{\prod_{p=1}^{n}\prod_{q=1}^{p}(1 - x^{m+p+q-1})}{\prod_{p=1}^{n}\prod_{q=1}^{p}(1 - x^{p+q-1})} .$$

Proof. Using Theorem 5BC, it is easy to show that $B_n(n)$ is the lattice of all n-tuples $\underset{\sim}{a} = (a_i)$ with $0 = a_1 = a_2 = \cdots = a_r < a_{r+1} < a_{r+2} < \cdots < a_n \leq n$, $0 \leq r \leq n$, ordered by $\underset{\sim}{a} \leq \underset{\sim}{b}$ iff $a_1 \leq b_1, \cdots, a_n \leq b_n$. Each column of one of these plane partitions is an element of $B_n(n)$ by this description and thus each such plane partition is an m-multichain in $B_n(n)$. (Note that the plane partitions are being "sliced" differently in this proof when compared to Proposition 4.1.) Apply Lemmas 4.1 and 4.2 to $B_n(m\lambda_n)$.

The previously known proofs of these two plane partition identities are somewhat unsatisfactory from a combinatorial viewpoint because they involve evaluations of determinants and/or manipulation of symmetric function identities. The proofs presented here are not "combinatorial". However, they have a strong algebraic combinatorial flavor. The basis result of Seshadri can be described in terms of algebras with straightening laws, which have been studied by Rota, Garsia, Eisenbud, Baclawski, DeConcini and Procesi [Bac] [DEP] [DKR]. Rota, et. al. in fact provide a predominantly combinatorial proof of Seshadri's result for the case A_{n-1}. Furthermore, the Weyl character formula has a very combinatorial flavor when viewed in the proper light, as illustrated by Verma's derivation utilizing Möbius inversion on the Bruhat order [VeS]. Hence it may be possible to present a proof of Proposition 4.1 which develops the necessary Lie theoretic results along the way in a nice algebraic combinatorial fashion, with linear independence essentially being the only algebraic notion used. It is apparently very hard to give purely combinatorial proofs of Propositions 4.1 and 4.2. The cases $m = 1$ are the only cases done to date [Sag, p. 31].

5. Bruhat Lattices are Strongly Sperner

In this section we prove that every miniscule (Bruhat) lattice is rank symmetric, rank unimodal, and strongly Sperner.

The following lemma is due to Griggs [Gri]:

Lemma 5.1. Let L be a ranked poset. the following two conditions are equivalent:

(i) L is rank unimodal and has the strong Sperner property.

(ii) If $0 \leq i \leq j \leq r$, then there exists $\min\{|L_i|,|L_j|\}$ disjoint chains each containing one element from each of the ranks $L_i, L_{i+1}, \cdot \cdot \cdot, L_j$.

Different versions of (ii) have in the past been referred to as Property T. We shall refer to it by a more descriptive name, rank matching property.

Definitions. If L is a ranked poset with r ranks, let $\tilde{L}$ denote the complex vector space with basis $\{\tilde{a}, \tilde{b}, \cdot \cdot \cdot\}$ where $\{a, b, \cdot \cdot \cdot\}$ are the elements of L. This vector space can be graded by $L = \oplus_{i=0}^{r}\tilde{L}_i$, where $\tilde{L}_i$, the ith rank subspace, is the span of the elements in L_i. A linear operator X on $\tilde{L}$ of degree +1 will be called a raising operator if $X(\tilde{b}) =$

$\Sigma_{c \in L_{i+1}} x(b,c)\tilde{c}$, $b \in L_i$, implies $x(b,c) = 0$ unless c covers b in L. A <u>lowering operator</u> Y for $\tilde{L}$ is an analogously restricted linear operator of degree -1.

The following lemma is due to Stanley [StW]:

<u>Lemma 5.2.</u> Let L be a ranked poset. The following two conditions are equivalent:

(i) L is rank symmetric and has the rank matching property.

(ii) If $0 \leq i < r/2$, then there exists a raising operator X such that X^{r-2i}: $\tilde{L}_i \longrightarrow \tilde{L}_{r-i}$ is a vector space isomorphism.

Both the proof of the following theorem and Section VI.3 will reveal that the rank matching property is more relevant to this thesis than the strong Sperner property. However, the Sperner property is more well-known.

<u>Theorem 5.</u> Miniscule lattices are rank symmetric, rank unimodal, and strongly Sperner.

<u>Proof.</u> Let L be a miniscule lattice. Consider the corresponding miniscule representation ρ with highest weight λ of a semisimple Lie algebra $\mathfrak{g}$. The representation space has a basis $\{v_{w\lambda}\}_{w\ W}$ indexed by the elements of the miniscule lattice, and thus may be denoted $\tilde{L}$. Recall that any representation of $\mathfrak{g}$ induces a representation of $sl(2,\mathbb{C})$ on $\tilde{L}$ via the embedding of the principal three dimensional subalgebra spanned by x, y, and h. From [Hu1, p. 107], it is possible to deduce that if $\rho(x)v_{u\lambda} = \Sigma_{w \in W} x(u,w)v_{w\lambda}$, then $x(u,w) = 0$ unless $w\lambda = u\lambda + \alpha_i$ for some

positive simple root α_i. Hence $\rho(x)$ is a raising operator on $\tilde{L}$ for the miniscule lattice.

Set $r = 2(\lambda, \delta^{\vee})$. With respect to h, the induced representation has weight spaces $\tilde{L}_0, \tilde{L}_1, \cdots, \tilde{L}_r$ with weights $-r/2, (-r/2)+1, \cdots, (r/2)-1, r/2$. Since $sl(2,\mathbb{C})$ is semisimple, the induced representation on $\tilde{L}$ can be expressed as a direct sum of irreducible representations of $sl(2,\mathbb{C})$. It is well known that the restriction of $\rho(x)$ to one of these irreducible subrepresentations composed with itself 2j times (where j is not too large) is an isomorphism from the one dimensional weight space of the subrepresentation of weight $-j$ to the one dimensional weight space of weight $+j$. These isomorphisms can be combined to show that $\rho(x)^{r-2i}$ is an isomorphism from $\tilde{L}_i$ to $\tilde{L}_{r-i}$. The proof is complete with the application of Lemmas 4.2 and 4.1.

In his original proof of this theorem for all Bruhat orders, Stanley used the hard Lefschetz theorem from algebraic geometry to produce a raising operator with the desired properties. Initially the relationship of the proof given here to Stanley's proof was not understood, but it turns out that the hard Lefschetz theorem is sometimes proved with the same technique used in the proof above: decomposition of a representation of $sl(2,\mathbb{C})$ [CGr, p. 44]. Thus, in a certain sense, it may seem that the hard Lefschetz theorem is superfluous. However, that this is not the case is illustrated by the fact that Stanley's methods apply to any Bruhat order arising from a Weyl group, whereas Theorem 5 applies only to Bruhat lattices. The necessary representations of $sl(2,\mathbb{C})$ apparently arise readily in the context of Lie algebras only with miniscule representations, whereas the hard Lefschetz theorem of algebraic geometry supplies the desired representations of $sl(2,\mathbb{C})$ for all Bruhat orders after some preliminary algebraic geometric work has been done. As mentioned in the introduction, Section VI.2 will abstract the essential aspects of the proof of Theorem 5 with arbitrary posets in mind.

For the sake of mathematical culture, we now mention an application of a particular case of Theorem 5. The truth of a fact closely related to the following proposition was conjectured by P. Erdős and L. Moser in 1965 [Erd]. The original conjecture and this proposition were recently proved with the combined efforts of B. Lindström [Lin] and R. Stanley [StW]. The <u>element sum</u> of a set of real numbers is the sum of the real numbers in the set.

<u>Proposition.</u> No set of n positive real numbers has more distinct subsets with equal element sums than does the set $\{1, 2, \cdots, n\}$.

<u>Proof.</u> Let $a_1 < a_2 < \cdots < a_n$ be the real numbers. Associate to any subset $(a_{i_1}, \cdots, a_{i_k})$ the element $(0, \cdots, 0, i_1, \cdots, i_k)$ of $B_n(n)$. (See the proof of Proposition 4.2.) If $\underset{\sim}{b} < \underset{\sim}{c}$ in $B_n(n)$, then the element sum of the subset corresponding to $\underset{\sim}{b}$ is strictly less than the element sum of the subset corresponding to $\underset{\sim}{c}$. Therefore, in order to have equal element sums, a collection of subsets of real numbers must correspond to a collection of incomparable elements in $B_n(n)$. The Sperner property puts an upper bound on the number of simueltaneously incomparable elements in $B_n(n)$. This bound is attained by the numbers specified in the statement of the proposition.

Chapter VI

A Dynkin Diagram Classification of Certain Partially Ordered Sets

1. Introduction

Theorem V.5 proved that the Bruhat (miniscule) lattices were rank symmetric, rank unimodal, and strongly Sperner. (These combinatorial terms were defined in Section V.1.) In the second section of this chapter we will abstract part of the proof of this theorem to obtain a new sufficient condition for an arbitrary ranked poset to have these combinatorial properties. No Lie representation theory will be used in this chapter, but linear algebra will play a central role via Lemma V.5.2. This lemma, due to Stanley, was the first step in the proof of Theorem V.5. It translated the question of whether a ranked poset L was rank symmetric, rank unimodal, and strongly Sperner into a question concerning the existence of a certain kind of linear operator on the vector space $\tilde{L}$ associated to the ranked poset. Lie algebraic techniques were then used to construct an appropriate linear operator for each Bruhat lattice. A closer look at this proof reveals that the latter part of it is independent of whether the ranked poset is a Bruhat lattice or not. Theorem 2.1 is the promised abstraction of this part of the proof of Theorem V.5. Its proof uses only elementary linear algebra and hence should be readily accessible to all readers.

Theorem 2.1 is fairly difficult to use in its full generality. Its hypothesis can be restricted in various ways to make it easier to use. One restricted version, Theorem 2.2, applies only to distributive lattices. The hypothesis of this theorem requires that each edge of the Hasse diagram of the distributive lattice be assigned a rational number in such a way that certain simple linear conditions are satisfied. This condition is not difficult to work with and Theorem 2.2 can in fact be readily applied to any of the Bruhat lattices without any knowledge of Lie representation theory.

Attempting to apply Theorem 2.2 to distributive lattices other than the Bruhat lattices leads to the next topic of this chapter. It is possible to express the hypothesis of this theorem in terms of the poset of join irreducibles of the distributive lattice. The analogous condition involves assigning a rational number to each element of the poset of join irreducibles in such a way that certain linear conditions specified by the combinatorial structure of the poset are satisfied. Recall that the posets of join irreducibles of Bruhat lattices are called miniscule posets. The question at hand becomes: Are there any posets beside the miniscule posets which satisfy this condition? Surprisingly, it is possible to prove that there are no other such posets. (And thus the Bruhat lattices are the only lattices satisfying the hypothesis of Theorem 2.2.) One interesting aspect of the proof is that the key step utilizes a combinatorial consequence of Theorem 2.2 itself. Even more interesting is the natural appearance of a form of Dynkin diagram. Dynkin diagrams mysteriously arise in several different branches of mathematics [HHS].

The classification of V-labellable posets can also be considered a characterization of the miniscule posets. This characterization is the fourth description of the miniscule posets presented in this thesis. The last section of this chapter summarizes these four ways, and additionally specifies a fifth (empirical, but interesting) description.

2. Sufficient Conditions for the Strong Sperner Property

Suppose that L is a ranked poset with ranks $L_0, L_1, \cdots, L_r$. To apply Lemma V.5.2, one must find a raising linear operator X on the vector space L such that $X^{r-2i}: \tilde{L}_i \to \tilde{L}_{r-i}$ is a vector space isomorphism for $0 \leq i < r/2$. The proof of Theorem V.5 showed that the raising operator at hand satisfied this requirement by utilizing two additional operators: a lowering operator Y and an operator H which multiplied each vector in $\tilde{L}_i$ by (2i-r). The three operators obeyed the commutation relations XY - YX = H, HX - XH = 2X, and HY - YH = -2Y. The following theorem assumes that three such linear operators have been defined on an arbitrary ranked poset.

__Theorem 2.1.__ Let L be a ranked poset with r+1 ranks. Suppose that complex numbers x(b,e) and y(e,b) can be assigned to each covering relationship b < e such that the following equations hold:

For every $b \in L_i$,

$$\sum_{b \text{ covers } d} y(b,d)x(d,b) - \sum_{e \text{ covers } b} x(b,e)y(e,b) = 2i - r .$$

For every $b, c \in L_i$,

$$\sum_{\substack{b \text{ and } c \\ \text{cover } d}} y(b,d)x(d,c) \quad - \quad \sum_{\substack{e \text{ covers} \\ b \text{ and } c}} x(b,e)y(e,c) \;=\; 0 \;.$$

Then L is rank symmetric, rank unimodal, and strongly Sperner.

Proof. This proof is given in greater detail in [Prc]. Define three linear operators on $\tilde{L}$:

$$\text{For } b \in L_i, \quad X\tilde{b} \;=\; \sum_{e \text{ covers } b} x(b,e)\tilde{e} \;,$$

$$Y\tilde{b} \;=\; \sum_{b \text{ covers } d} y(b,d)\tilde{d} \;,$$

$$\text{and} \quad H\tilde{b} \;=\; (2i-r)\tilde{b} \;.$$

Then $XY - YX = H$, $HX - XH = 2X$, and $HY - YH = -2Y$. (Thus X, Y, and H span a representation of $sl(2,\mathbb{C})$ on $\tilde{L}$.) Choose any minimal element u_0 of L, and let U be the subspace of $\tilde{L}$ spanned by all vectors obtained by acting on u_0 with various compositions of X, Y, and H. The commutation relations can be used together with $Yu_0 = 0$ to show that u_0, Xu_0, X^2u_0, form a basis for U. Let X_U, Y_U, and H_U denote the restrictions of X, Y, and H to U. Now trace $H_U = -r + (-r+2) + \cdot\cdot\cdot\cdot$. But $H_U = X_UY_U - Y_UX_U$, implying trace $H_U = 0$. Thus $X^r u_0$ must be the last non-zero vector in the sequence. Let j be the smallest integer such that $\tilde{L}_j \cap U \neq \emptyset$. Pick $v_j \in \tilde{L}_j$ such that $v_j \notin U$. Let V be the subspace generated by all actions of X, Y, and H on u_0 and v_j. By the same reasoning as before, u_0, Xu_0, $\cdot\cdot\cdot$, $X^r u_0$, v_j, Xv_j, $\cdot\cdot\cdot$, $X^{r-2j}v_j$ form a basis for V. Continue this process until a basis is obtained for all of $\tilde{L}$. It is now easy to see that the order operator X satisfies the requirements of Lemma V.5.2. Apply Lemma V.5.1 to finish the proof.

Note that the proof remains valid if either the condition { x(b,e) = 0 when e does not cover b } or the condition { y(b,d) = 0 when b does not cover d } is dropped. Both were required so that the theorem could be more simply stated.

Because of the large number of quadratic equations, Theorem 2.1 is difficult to apply. However, as noted in Section V.5, Stanley's work [StW] and a proof of the hard Lefschetz theorem [CGr] combine to guarantee that the more general version of Theorem 2.1 just noted can be applied to all Bruhat orders. Figure 2.1 shows one way that the covering

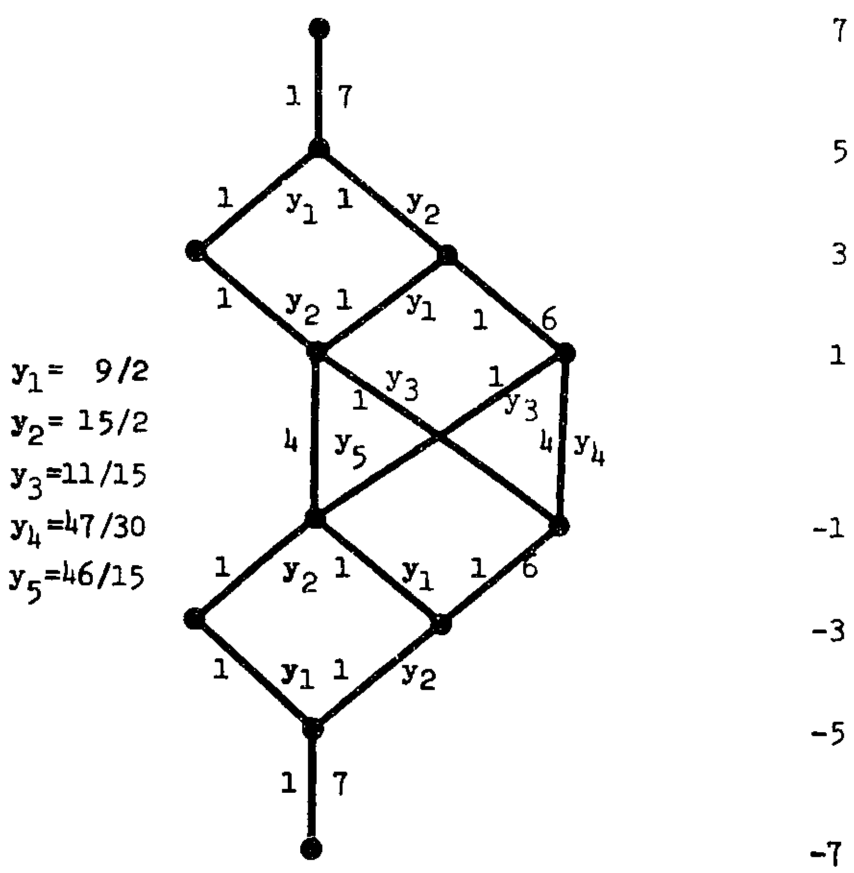


Figure 2.1

relations of the Bruhat order $BC_3(2)$ can be assigned appropriate complex numbers. (The coefficient $x(d,b)$ appears to the left of the edge (b,d), and the coefficient $y(b,d)$ appears to the right.)

The equations in the statement of Theorem 2.1 become linear equations if one sets $x(b,e) = 1$ whenever e covers b in L. Further suppose that the ranked poset L is "uniquely modular", namely: If b and c both cover d, then there exists a unique element e which covers both b and c, and similarly for e covering both b and c. Then requiring $y(e,b) = y(c,d)$ in such a poset L eliminates the need for the second set of equations altogether. Finally, we require that the ranked poset actually be a distributive lattice. Not only does this make the statement of the definition below simpler, but it also sets the stage for a smooth transition to the next section of this chapter.

Definition. A distributive lattice L with r+1 ranks is E-labellable (edge labellable) if each covering relationship $b < e$ can be assigned a rational number $y(e,b)$ such that:

(i) If e covers both b and c, and both b and c cover d, then $y(e,b) = y(c,d)$, and

(ii) If $b \in L_i$, then $\sum_{b \text{ covers } d} y(b,d) - \sum_{e \text{ covers } b} y(e,b) = 2i-r.$

Pictorially, each edge of the Hasse diagram of L is to be labelled with a rational number such that opposite edges in any square must have equal labels and such that that the sum of the labels of edges emanating below an element b minus the sum of the labels of edges emanating above b must equal $2i-r$, if $b \in L_i$. Figure 2.2 shows the Hasse diagrams for the

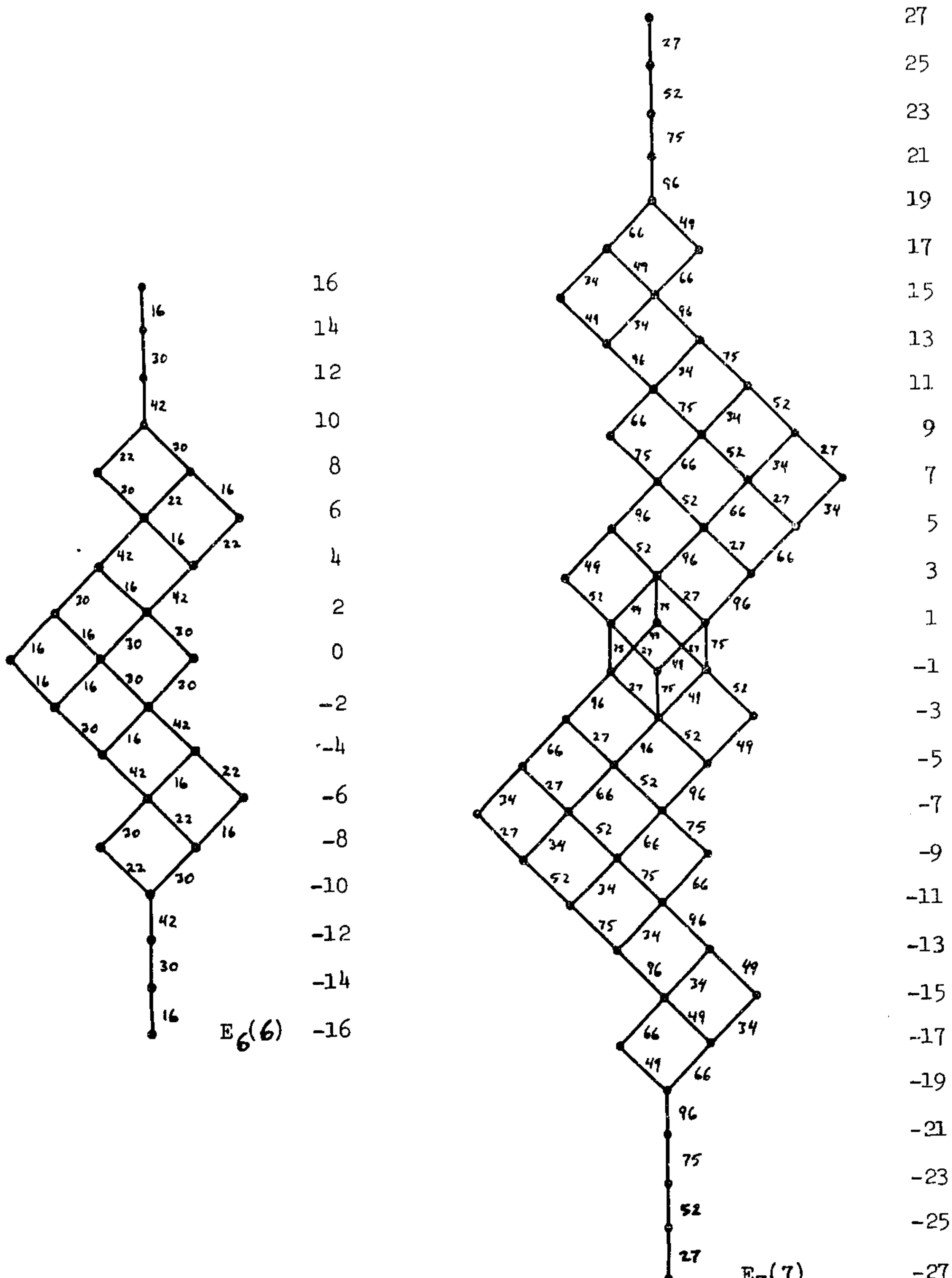

16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
E6(6)
27
25
23
21
19
17
15
13
11
9
7
5
3
1
-1
-3
-5
-7
-9
-11
-13
-15
-17
-19
-21
-23
-25
-27
E7(7)

Figure 2.2

exceptional irreducible Bruhat lattices $E_6(6)$ and $E_7(7)$ with valid E-labellings. The most interesting special case of Theorem 2.1 can now be stated.

Theorem 2.2. If a distributive lattice is E-labellable, then it is rank symmetric, rank unimodal, and strongly Sperner.

Unlike Theorem 2.1, it is not difficult to attempt to apply Theorem 2.2 to an arbitrary distributive lattice. The proof of Theorem V.5 implies that the edge labels necessary to apply Theorem 2.2 to the miniscule (Bruhat) lattices can be found by the explicit computation of certain representations of Lie algebras. But it is actually far easier to compute the necessary edge labels for the irreducible miniscule lattices by directly solving the required equations. This is how the edge lables shown in Figure 2.2 were obtained. However, the classification theorem of the next section proves that Theorem 2.2 cannot be applied to any distributive lattices other than the miniscule lattices!

3. Classification of V-Labellable Posets

In this section we prove that Theorem 2.2 can be applied only to miniscule lattices. The only possible edge labelling for each irreducible miniscule lattice will be computed as part of the proof. The main result of this section can also be viewed as a Dynkin diagram type classification of the E-labellable distributive lattices, since Dynkin diagrams arise in the course of the proof as a natural means of indexing the possible E-labellable lattices. The objects directly under consideration will actually be the posets of join irreducibles of the distributive lattices rather than the distributive lattices themselves. We will thus obtain a characterization of the miniscule posets.

The techniques used in this section are almost entirely elementary combinatorics and linear algebra. However, it is interesting to note that Theorem 2.2 itself will be the key non-trivial fact. Despite the fact that it was possible to express its proof entirely in the language of elementary linear algebra, it is probably best to view Theorem 2.2 as an application of elementary facts about representations of $sl(2,\mathbb{C})$.

Not only is it useful to recast the concept of E-labellability of a distributive lattice L in terms of the poset $P = j(L)$ of join irreducibles, but the resulting condition is also more elegant. The terms "order ideal" and "order filter" were defined in Section II.1.

<u>Definition.</u> A finite poset P is <u>V-labellable</u> if there exists a function $\pi: P \longrightarrow \mathbb{Q}$ such that for every antichain $A \subseteq P$,

$$\sum_{x \in A} \pi(x) - |I_A| = \sum_{y \in B} \pi(y) - |I_B| ,$$

where I_A is the order ideal in P with maximal elements A, the number of in I_A is denoted by $|I_A|$, the order filter $P - I_A$ is denoted by I_B, and B is the set of minimal elements of I_B.

It is easy to verify that P is V-labellable if and only if $L = J(P)$ is E-labellable. Thus, the following theorem is a restatement of Theorem 2.2.

__Theorem 3.1.__ If a poset P is V-labellable, then the latti[illegible] = J(P) of order ideals of P is rank symmetric, rank unimodal, and strongly Sperner.

A few definitions must be made before the classification theorem can be stated. Roughly speaking, "Dynkin diagrams" are graphs which classify complex semisimple Lie algebras (among other things).

__Definition.__ A __connected rooted Dynkin diagram__ $X_n[j]$ is a connected Dynkin diagram X_n in which the __jth__ node has been designated as a special node. A __rooted Dynkin diagram__ is a finite disjoint union of connected

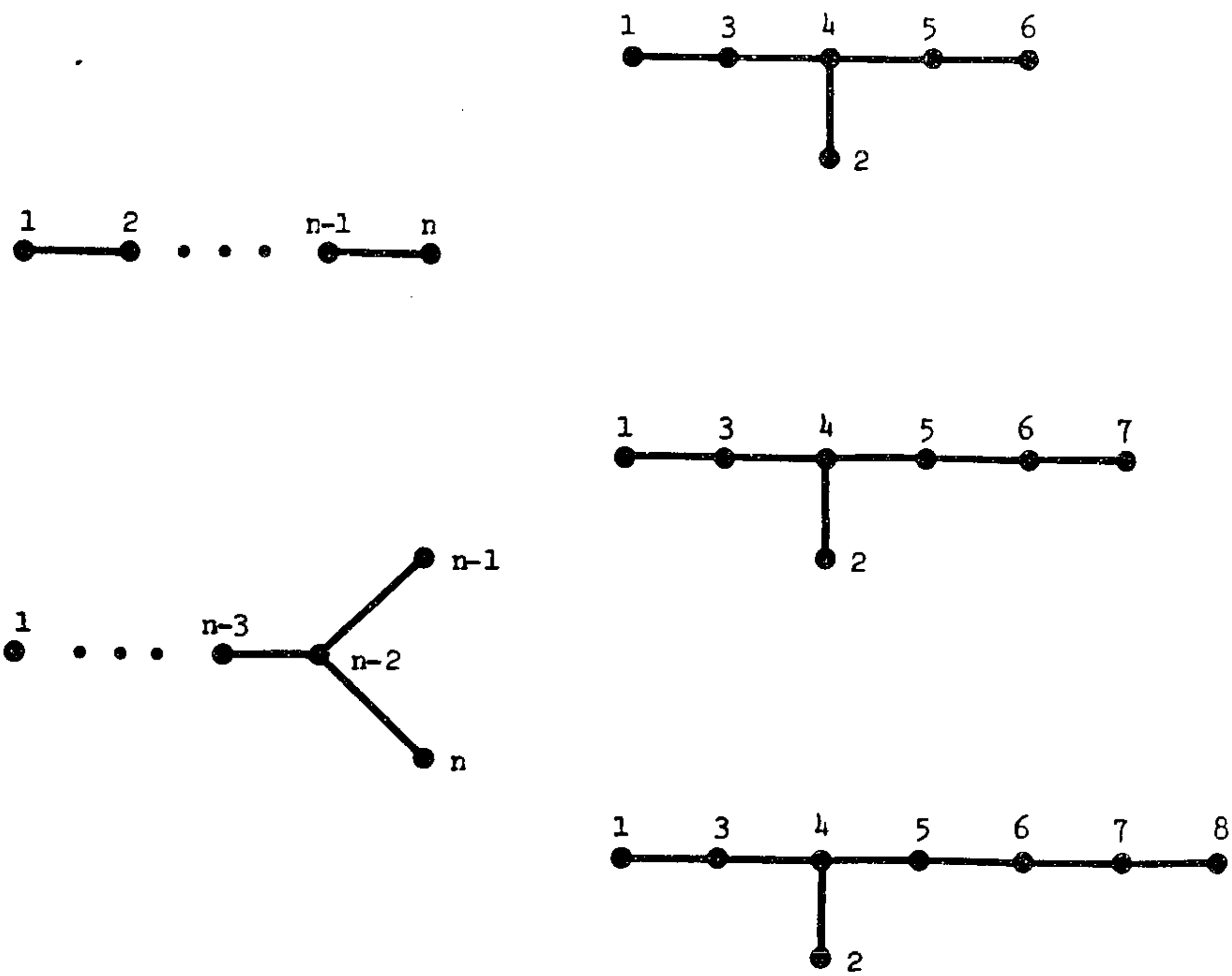


Figure 3.1

rooted Dynkin diagrams.

The Dynkin diagrams A_n, D_n, E_6, E_7, and E_8 are pictured in Figure 3.1. We shall be concerned with only these Dynkin diagrams (i.e. not B_n, C_n, F_4, or G_2); this is not unusual [HHS].

Definitions. A poset is irreducible if it cannot be expressed as the direct sum (disjoint union) of two non-empty posets. An irreducible component of a poset is a maximal irreducible subposet.

Definitions. The basic tree of an irreducible poset P is the multi-rooted tree (acyclic graph with special vertices) whose vertices are the elements x in P such that $\{y: y \leq x\}$ is a chain, whose edges are the covering relations between these vertices, and whose roots (special vertices) are the minimal elements of P.

It will be shown in the course of the proof that the basic trees of irreducible V-labellable posets are rooted trees in the usual sense; namely, they have exactly one special vertex apiece.

The main result of this section can now be stated.

Theorem 3.2. The basic tree of each irreducible component of a V-labellable poset is one of the following rooted Dynkin diagrams: $A_n[j]$, $1 \leq j \leq n$, $D_n[1]$, $D_n[n-1]$, $D_n[n]$, $E_6[1]$, $E_6[6]$, or $E_7[7]$. The miniscule poset $x_n(j)$ is the unique V-labellable poset with basic tree $X_n[j]$. Hence the direct sums of the miniscule posets $a_n(j)$, $d_n(1)$, $d_n(n-1)$, $d_n(n)$, $e_6(1)$, $e_6(6)$, and $e_7(7)$ exhaust all possible V-labellable posets.

In words, each irreducible V-labellable poset has one of a few possible connected rooted Dynkin diagrams embedded in the "lower" part of its Hasse diagram. See Figure 3.8, where the vertices of the basic trees are denoted with circles rather than dots.

The following theorem is a restatement of Theorem 3.2.

Theorem 3.3. The only E-labellable distributive lattices are the miniscule lattices.

Proof of Theorem 3.2. Unlike some other Dynkin diagram type classification procedures, it will not be possible here to immediately reduce to the case of an irreducible V-labellable poset. (See Corollary 3.) Our attention will, however, eventually focus on one irreducible component of a V-labellable poset. After some work which restricts the possible local structure of a V-labellable poset, a simpler object, the basic tree, is associated to each irreducible component of the V-labellable poset. Systems of linear equations closely related to the Cartan matrices of simple Lie algebras are then used to eliminate all but a handful of rooted trees as possible basic trees of irreducible components of V-labellable posets. Finally, by direct construction, almost all of these potential basic trees are shown to uniquely determine an irreducible V-labellable poset. It is interesting to note that the six potential basic trees which do not lead to irreducible components of V-labellable posets are $E_6[2]$, $E_7[1]$, $E_7[2]$, $E_8[8]$, $E_8[1]$, and $E_8[2]$, all of which correspond to fundamental representations of semisimple Lie algebras which are not quite miniscule.

The proof of Theorem 3.2 is now presented as a series of lemmas. Throughout the proof, P will denote a V-labellable poset with p elements and labelling function $\mathcal{N}$. For simplicity of notation, the same symbols x, y, · · · will sometimes be used to refer both to elements of P and to the vertex labels $\mathcal{N}(x)$, $\mathcal{N}(y)$, · · · . Similarly, an upper case latin letter can refer to either a subset of P or to the sum of the vertex labels of the elements in the subset.

The following crucial lemma is the only part of the proof which uses something more (Theorem 3.1) than straightforward combinatorial reasoning and linear algebra. This lemma will be used in five distinct steps later in the proof.

<u>Lemma 3.1.</u> All vertex labels are positive.

<u>Proof.</u> Consider L = J(P). This distributive lattice has p+1 ranks. The Hasse diagram of L can be viewed as a network, where a vertex in the <u>ith</u> rank of L is a source or sink of (2i-r) units of flow, and an edge corresponding to an element x in P carries $\mathcal{N}(x)$ units of flow downward. Since L is E-labellable, Kirchhoff's first law (conservation of mass) is satisfied at every vertex of L. Let F $\subseteq$ L be any order filter of L. By conservation of mass, the sum of the flows on edges descending from the minimal vertices of F must equal the sum of the sources and sinks which are members of F. Sinks are vertices in ranks 0, 1, · · ·, (p-1)/2 (p odd) or 0, 1, · · ·, (p-2)/2 (p even). By Theorem 3.1, L has the rank matching property. (The strong Sperner property is irrelevant. Ignore the application of Lemma V.5.1 at the end of the proof of Theorem 2.2.) Therefore each sink of size (2i-r) in F can be matched with a source of size -(2i-r) <u>which lies in F</u>. Thus the sum of the sources and sinks in F is non-negative. In particular, let F be the filter in L consisting of

all order ideals of P which contain the element x. Every edge descending from a minimal element of F has flow $\pi(x)$, and thus the sum of the sources and sinks in F is a positive integral multiple of $\pi(x)$. The sum of sources and sinks in F is zero only when F = L, and this F does not correspond to any poset element x under the construction above. Therefore $\pi(x)$ must be positive.

The following lemma follows immediately from the definition of V-labellable.

Lemma 3.2. The poset P is V-labellable if and only if its order dual P^* is V-labellable.

Notation. The order ideal with maximal elements $\{b, c, \cdots\}$ shall be denoted by $(b, c, \cdots)$.

Lemma 3.3. The poset P is modular; i.e., if elements b and c both cover d, then there exists at least one element e which covers both b and c, and order dually. Hence P is ranked.

Proof. Let $D = \{$ d covered by b and c $\}$,

$E = \{$ e which cover b and c $\}$,

$F = \{$ f covered by c but not b $\}$,

$G = \{$ g covered by b but not c $\}$,

$S = \{$ s which cover b but not c $\}$,

$T = \{$ t which cover c but not b $\}$.

Finally, let $m = 2|(b,c)| - p$. Four equations in nine unknowns are obtained by considering the ideals (b,c), $(b,c) - \{b\}$, $(b,c) - \{c\}$, and (b,c) - {b,c}:

$$\begin{array}{rcl} b + c - E - S - T &=& m \, , \\ b - c + F - S &=& m - 2 \, , \\ -b + c + G - T &=& m - 2 \, , \\ -b - c + D + F + G &=& m - 4 \, . \end{array}$$

Solving these equations yields $E = D$. Lemma 3.1 implies $D > 0$. Hence E is non-empty. Use Lemma 3.2 to obtain the dual result. Apply Theorem II.16 of [Bir] to conclude that P is ranked.

<u>Lemma 3.4.</u> No element ever covers or is covered by three or more other elements.

<u>Proof.</u> Proceed by induction on the ranks of P. Let q be an element of minimal rank which covers at least three elements b, c, and d. Let K be the set of other elements covered by q. Figure 3.2 shows the four possible situations for the highest three ranks of the ideal (q). It will be-

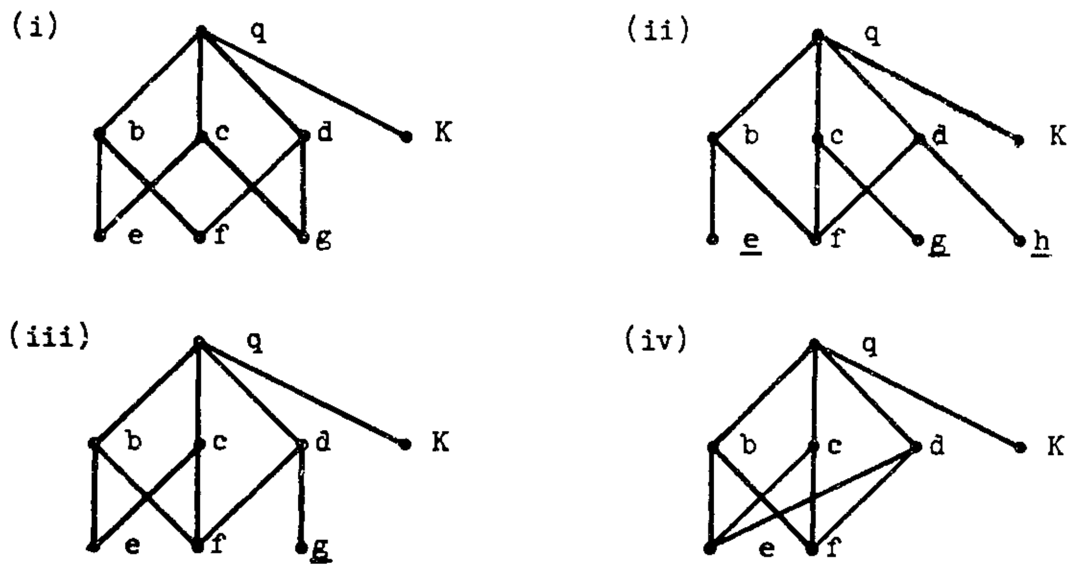


Figure 3.2

come clear that the existence of the underscored elements is irrelevant. Assume for now that they exist. It will also become clear that it does not matter whether any elements in K cover any of the elements shown in the lowest rank. Assume for now that elements in K do not cover any of the named elements.

For each case, consider the 8 equations in 17 or 18 unknowns generated by the ideals $(q) - \{q\}$, $(q) - \{q,b\}$, $(q) - \{q,c\}$, $(q) - \{q,d\}$, $(q) - \{q,b,c\}$, $(q) - \{q,b,d\}$, $(q) - \{q,c,d\}$, $(q) - \{q,b,c,d\}$. We shall write out the equations only for case (i); the other cases are similar. Let Y denote the minimal elements of $(q) - \{q,b,c,d\}$, let X denote the elements which cover b but not c or d, let U denote the elements which cover b and c but not d, etc. Finally, let R denote the elements other than q which cover b, c, and d, and let $m = 2|(v)| - p$. Then

$$
\begin{aligned}
b + c + d + K - q - R - S - T - U - V - W - X - Y &= m-2, \\
-b + c + d + K - S - V - W - Y &= m-4, \\
b - c + d + K - T - V - X - Y &= m-4, \\
b + c - d + K - U - W - X - Y &= m-4, \\
-b - c + d + e + K - V - Y &= m-6, \\
-b - c - d + f + K - W - Y &= m-6, \\
b - c - d + g + K - X - Y &= m-6, \\
-b - c - d + e + f + g + K - Y &= m-8.
\end{aligned}
$$

Add the 2nd, 3rd, 4th, and 8th equations, and then subtract the 1st, 5th, 6th, and 7th equations. The resulting equation is $q + R = 0$. For cases (ii) and (iii), the resulting equation is $f + q + R = 0$. In case (iv), it is $e + f + q + R = 0$. Apply Lemma 3.1 to obtain contradictions in all cases. Q.E.D.

The next lemma completes the analysis of the local structure of P.

Lemma 3.5. No two elements both cover each of two other elements. Therefore P is "uniquely modular", i.e. if b and c both cover d, then there exists a unique element e which covers both b and c, and order dually.

Proof. (See Figure 3.3.) Suppose that d and e both cover b and c. Let G denote the elements in the rank of d and e beside these two elements, and similarly for F. Let S (T) be the set of elements covered only by b (c), and let U (V) be the set of elements covering only d (e). Finally, let m = 2k-p, where k is the number of elements of P of rank less than or equal to the rank of b and c. Lemma 3.4 guarantees that the situation described in Figure 3.3 is sufficiently general. Consider the ideals (d,F), (e,F), (b,F), and (c,F). Then

$$d - e + F - G \qquad - U \qquad = m + 2 ,$$

$$d + e + F - G \qquad - V = m + 2 ,$$

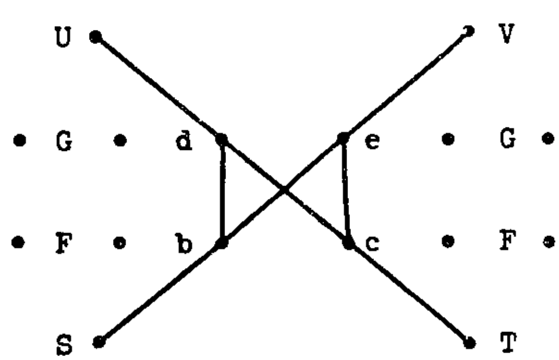


Figure 3.3

$$\begin{array}{lllll} b - c & + F - G & + T & = & m - 2 , \\ - b + c & + F - G + S & & = & m - 2 . \end{array}$$

Then $- S - T - U - V = 8$ contradicts Lemma 3.1. This proves the first statement. Combine it with Lemma 3.3 to obtain the second statement.

We now study the global structure of an irreducible component Q of the V-labellable poset P. Let q denote the number of elements of Q, let T denote the basic tree of Q, and let n denote the number of elements of T. The number n could be called the rank of Q, since it will be seen to be analogous to the rank of a Weyl group or the rank of a semisimple Lie algebra.

Lemma 3.6. The basic tree of Q has exactly one root and is either a chain or "Y-shaped", i.e. it has at most one vertex with three or more adjacent vertices.

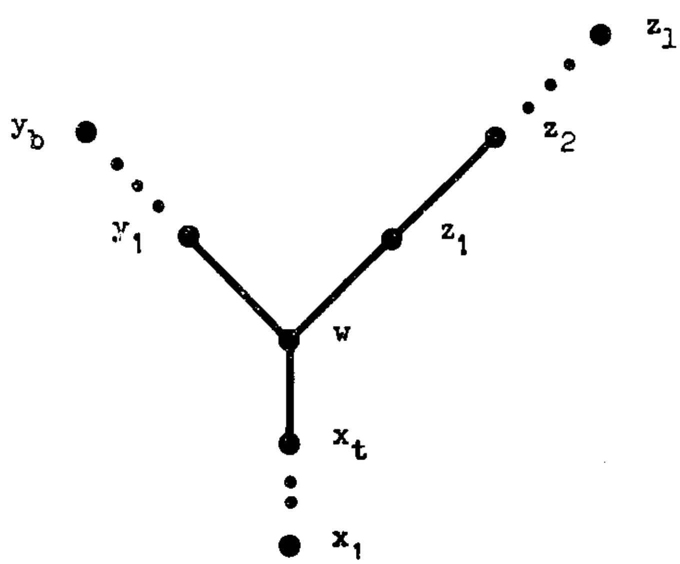


Figure 3.4

Proof. Lemma 3.5 precludes the existence of more than one minimal element of Q. If there is more than one "branching" in T, use Lemma 3.5 to produce a vertex in the basic tree which is covered by three or more element, contradicting Lemma 3.4.

Notation. (See Figure 3.4.) Set $n = b + l + t + 1$, where t (trunk) is the number of vertices in the branch of the basic tree T containing the root ($t = 0$ if the root is covered by two elements), and b (branch) and l (limb) are the numbers of elements in the other two branches of T. Refer to the elements of T with the letters shown in Figure 3.4.

Lemma 3.7. The following connected rooted Dynkin diagrams are the only possibilities for the basic tree of the irreducible component Q: $A_n[j]$, $1 \leq j \leq n$, $D_n[1]$, $D_n[n-1]$, $D_n[n]$, $E_6[1]$, $E_6[2]$, $E_6[6]$, $E_7[1]$, $E_7[2]$, $E_7[7]$, $E_8[1]$, $E_8[2]$, and $E_8[8]$.

Proof. Let s equal p minus the sum of the labels of the minimal elements of P lying outside Q. Consider the empty ideal of P together with the n ideals of P each generated by one element of the basic tree T of Q. The following system of n+1 equations in n+1 unknowns is obtained:

$$
\begin{array}{lr}
-x_1 \quad + s = 0 \\
x_1 - x_2 \quad + s = 2 \\
\quad \vdots & \vdots \\
x_t - w \quad + s = 2t \\
w - y_1 \qquad - z_1 \quad + s = 2(t+1) \\
y_1 - y_2 \qquad - z_1 \quad + s = 2(t+2) \\
\quad \vdots & \vdots
\end{array}
$$

$$\begin{array}{rcrcl} & y_b - z_1 & & + s & = 2(t+b+1) \\ - y_1 & & z_1 - z_2 & + s & = 2(t+2) \\ \ddots & & & \vdots & \\ - y_1 & & & z_l + s & = 2(t+l+1) \end{array}$$

The unique solution is:

For $1 \leq i \leq t$, $x_i = i(s-i+1)$,

$w = (t+1)(s-t)$,

for $1 \leq j \leq b$, $y_j = (t+j+1)(s-t-j) - jz_1$,

for $1 \leq k \leq l$, $z_k = (t+k+1)(s-t-k) - ky_1$,

and

$$s = \frac{-blt^2+blt+bl^2+b^2l+4bl+2bt+2lt+b^2+l^2+t^2+3b+3l+3t+2}{-blt+b+l+t+2} .$$

Since $w = (t+1)(s-t)$, this vertex label will be negative if $s - t < 0$:

$$s - t = \frac{blt+bl^2+b^2l+4bl+bt+lt+b^2+l^2+3b+3l+t+2}{-blt+b+l+t+2} .$$

It is easy to check that the denominator of this expression is positive only for the following unordered values of b, l, and t: $\{ \{0,j,k\}: 0 \leq j < \infty, 0 \leq k < \infty\} \cup \{ \{1,1,k\}: 1 \leq k < \infty\} \cup \{ \{1,2,2\}, \{1,2,3\}, \{1,2,4\} \}$. Lemma 3.1 thus implies that no other values are permissible. Consulting Figure 3.1 reveals that the rooted trees in this list are exactly the rooted Dynkin diagrams listed in the statement of the theorem.

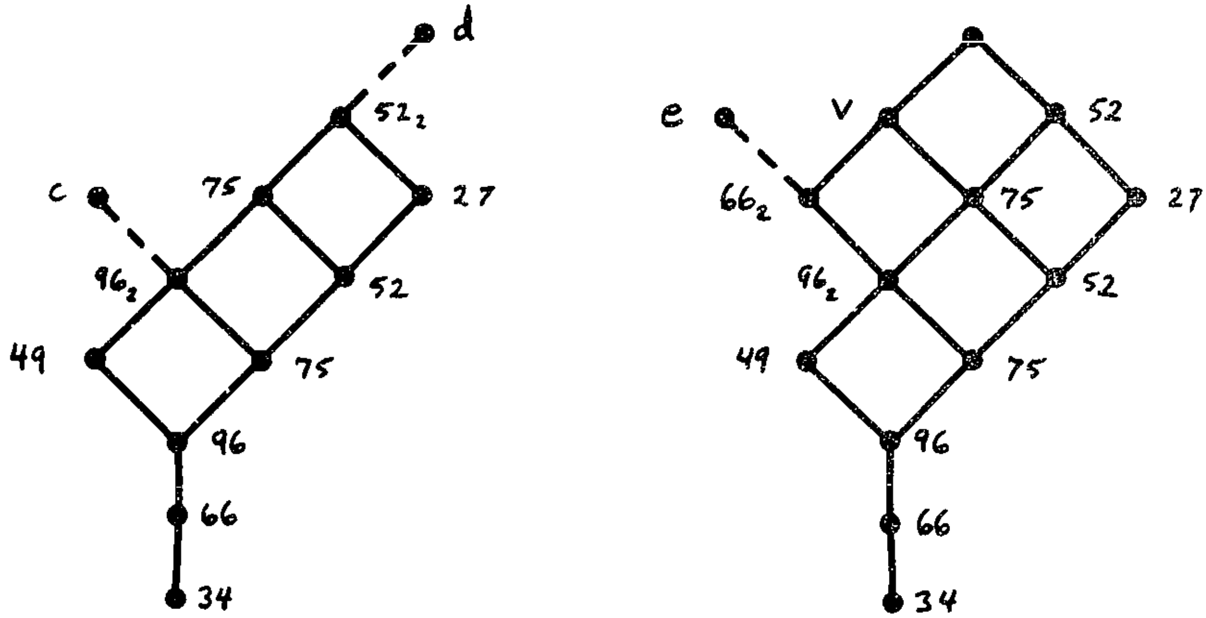
The following lemma is the last step in the proof.

<u>Lemma 3.6.</u> Each of the basic trees $A_n[j]$, $1 \leq j \leq n$, $D_n[1]$, $D_n[n-1]$, $D_n[n]$, $E_6[1]$, $E_6[6]$, and $E_7[7]$ determines one irreducible V-labellable poset, which is the miniscule poset $x_n(j)$ if the basic tree is $X_n[j]$. None of the rooted Dynkin diagrams $E_6[2]$, $E_7[1]$, $E_7[2]$, $E_8[8]$, $E_8[1]$, or $E_8[2]$ is a basic tree for an irreducible component of a V-labellable poset.

<u>Proof.</u> If elements b and c both cover d, and e is the unique element required by Lemma 3.5 which covers both b and c, then the proof of Lemma 3.3 implies that $\pi(e) = \pi(d)$. This fact, Lemma 3.4, and Lemma 3.5 will be collectively referred to with the phrase "local structure".

First consider $E_6[2]$, $E_7[2]$, and $E_8[2]$. Let v be the unique element covering both y_1 and z_1. By considering the ideals (v) and (y_1, z_1), one obtains $v = (y_1 + z_1)/2 + 1$. Computing v for these three cases yields the numbers v = 31, 143/2, and 202. But local structure implies that v = w = 42, 96, and 270. Thus these three rooted Dynkin diagrams cannot be basic trees of irreducible components.

Now consider $E_7[1]$. After computing the values for the basic tree and applying local structure, one can immediately construct as much of the irreducible component Q as is shown in Figure 3.5(a). Using the ideal (96_2), one finds c = 66. Then the ideal (52_2) leads to d = 0, implying that 52_2 is not covered by such an element. Figure 3.5(b) now depicts the situation. Using (66_2), one computes e = 34. Considering the ideal (v) leads to v = 47. But v = 96 by local structure. Similar arguments lead to inconsistencies in the 6<u>th</u> and 12<u>th</u> ranks of the irre-



(a) Figure 3.5 (b)

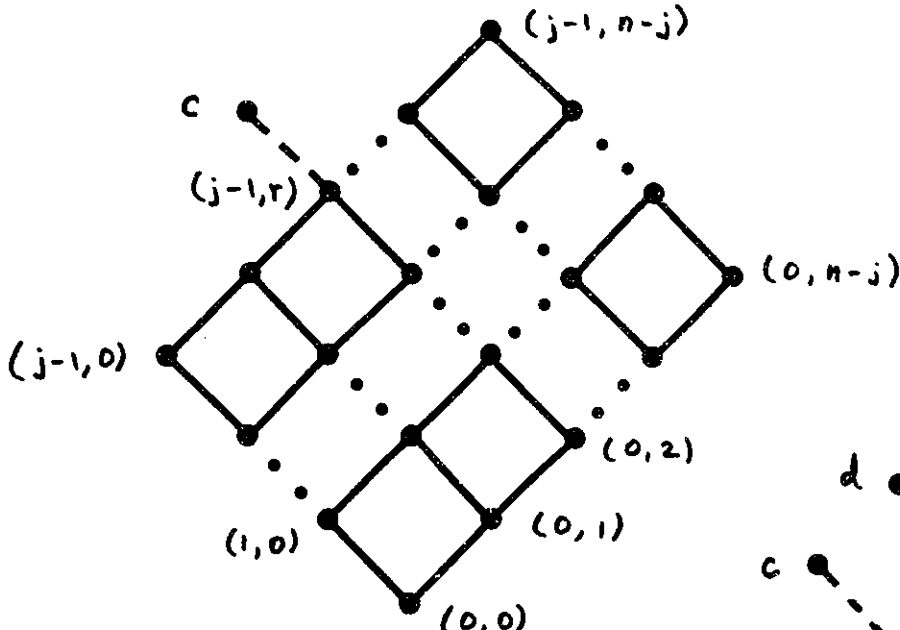


Figure 3.6

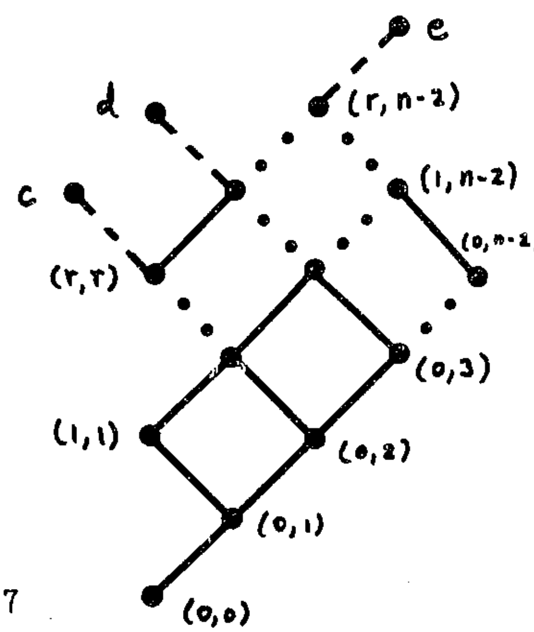


Figure 3.7

ducible components of $E_8[1]$ and $E_8[8]$.

Next consider $A_n[j] = A_n[n-j]$. Local structure implies that Q has at least the elements which are shown in Figure 3.6. Successively consider the principal ideals $((j-1,r))$ for $r = 1, 2, \cdots, n-j-1$:

$$(r+1)(n-r) - c - (r+j+1)(n-r-j) + j(n-j+1) = 2j(r+1) .$$

In each case, this equation implies $c = 0$. A similar result holds for the ideals $((r,n-j))$ for $r = 1, 2, \cdots, j-2$. The equation for the ideal $((j-1,n-j))$ is different in form but leads to the same conclusion. Thus there are no other elements in Q and $q = j(n-j+1) = s$.

Now consider $D_n[n] = D_n[n-1]$ with $n \geq 4$. Denote the elements of Q as shown in Figure 3.7, and proceed by induction on r. Assume that $\pi(i,i) = x_1$ for $i \leq r$. First consider the ideal $((r,r))$:

$$\begin{aligned} x_1 - c - z_r + s &= (r+1)(r+2) , \\ c &= 2x_1 - z_r - (r+1)(r+2) , \\ c &= 0 . \end{aligned}$$

Next consider the ideal $((r,r+1))$:

$$\begin{aligned} w - d - z_{r+1} + s &= (r+1)(r+4) , \\ d &= w - z_{r+1} + x_1 - (r+1)(r+4) , \\ d &= x_1 . \end{aligned}$$

And consider the ideal $((r,n-2))$ for $r \leq n-4$:

$$\begin{aligned} z_{n-r-3} - e - x_1 + s &= r(r+1) - 2(r+1)(n-1) , \\ e &= z_{n-r-3} + 2(r+1)(n-1) - r(r+1) , \end{aligned}$$

$$e = 0 .$$

After consideration of the ideals ((n-3,n-2)) and ((n-2,n-2)), one can conclude that Q has $q = n(n-1)/2 = s$ elements. The structure of Q can be described as the lattice of order ideals of the poset which is a product of a 2-element chain with an (n-2)-element chain.

The constructions of Q for $D_n[1]$, $n \geq 5$, $E_6[1] = E_6[6]$, and $E_7[7]$ are similar and will be omitted. In each case $s = q$, the number of elements in Q. This implies that Q is by itself a V-labellable poset. (See Corollary 3.) The poset Q for $D_n[1]$ has 2n-2 elements on 2n-3 ranks, with 2 elements on the middle rank. The posets Q for $E_6[6]$ and $E_7[7]$ have 16 and 27 elements respectively. The proofs of Lemma 3.8 and Theorem 3.2 are now complete.

The proof of Lemma 3.8 showed that each irreducible V-labellable poset has a unique V-labelling. The possible irreducible V-labellable posets are shown in Figure 3.8. Vertices of the basic trees are denoted with circles rather than dots.

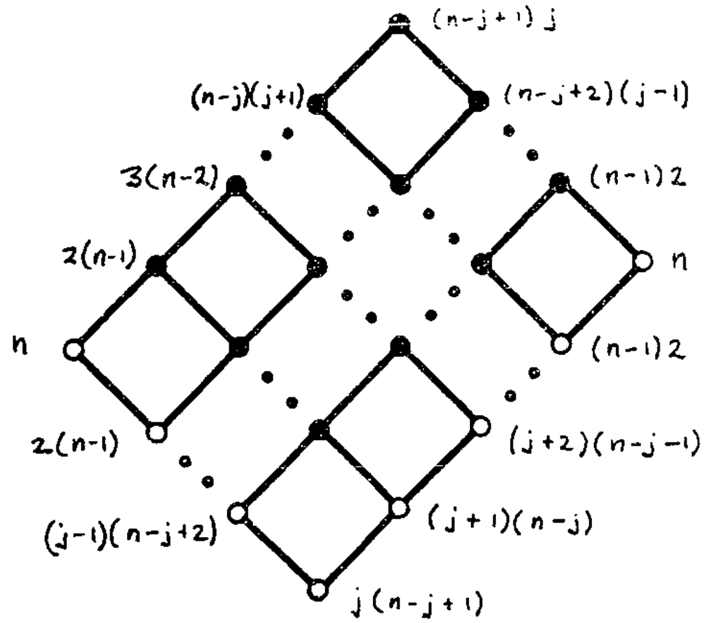
The following fact is a consequence of the proof of Theorem 3.2.

<u>Corollary 3.</u> A poset is V-labellable if and only if each of its irreducible components is V-labellable.

<u>Proof.</u> It is conceivable that $P = Q_1 \oplus Q_2$, with the following equation holding for every antichain $A_1 \subseteq Q_1$:

$$\Sigma_{x \in A_1} (x) - |I_{A_1}| - \Sigma_{y \in B_1} (y) + |I_{B_1}| = \alpha_1 ,$$

where $\alpha_1 \neq 0$, and with a similar equation holding for every antichain



$a_n(j)$

$d_n(n)$

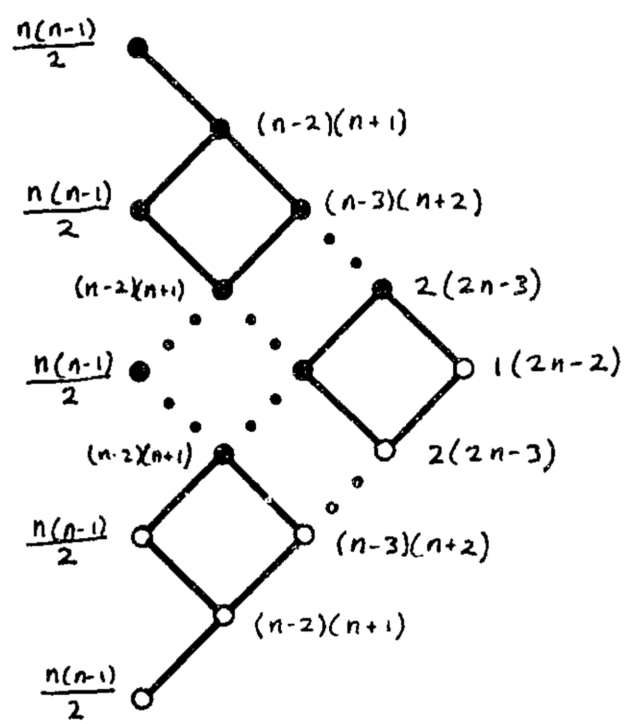
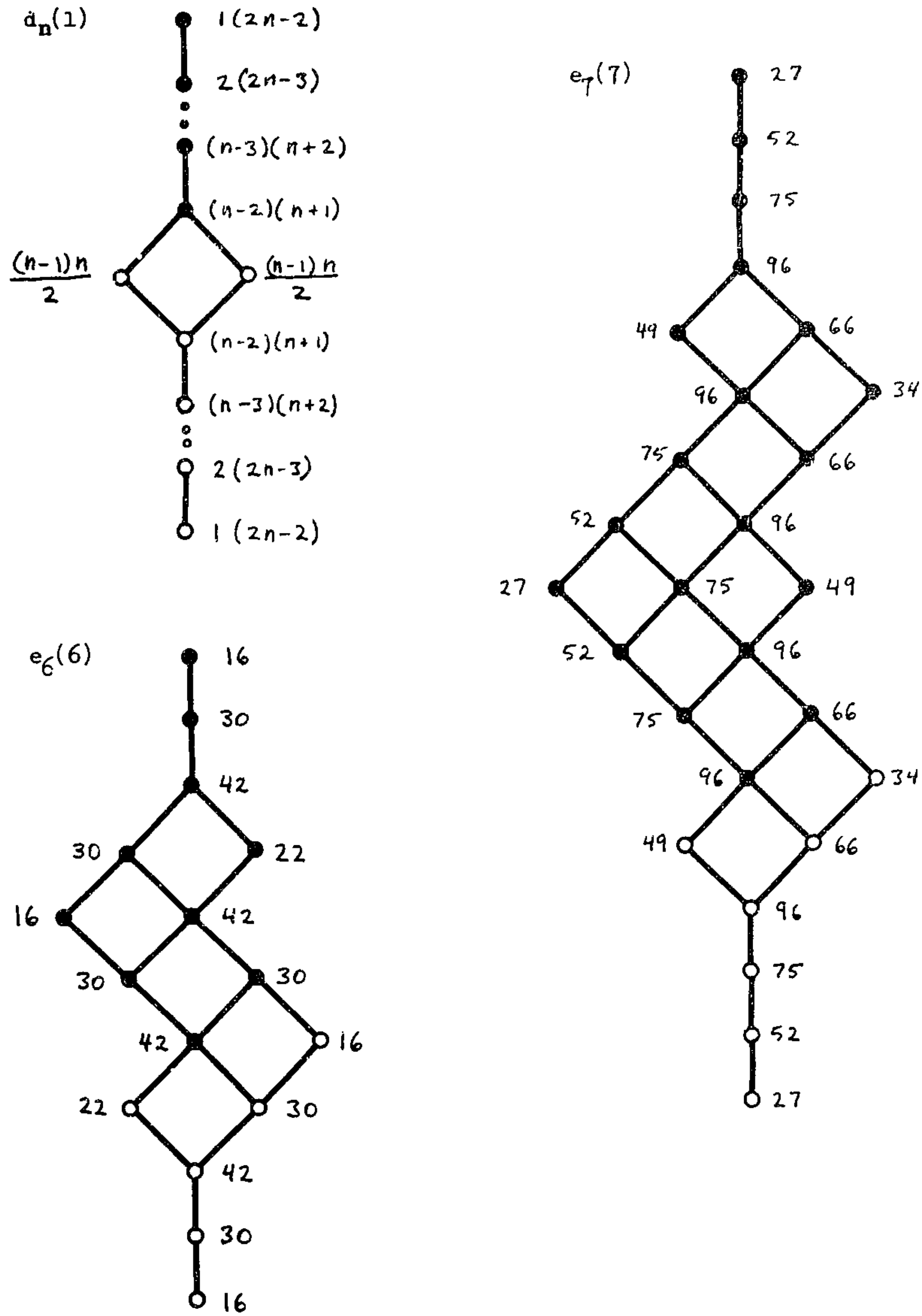


Figure 3.8a



$d_n(1)$
1(2n-2)
2(2n-3)
(n-3)(n+2)
(n-2)(n+1)
(n-1)n / 2
(n-1)n / 2
(n-2)(n+1)
(n-3)(n+2)
2(2n-3)
1(2n-2)
$e_6(6)$
16
30
42
30
22
16
42
30
30
42
16
22
30
42
30
16
$e_7(7)$
27
52
75
96
49
66
96
34
75
66
52
96
27
75
49
52
96
75
66
96
34
49
66
96
75
52
27

Figure 3.8b

$A_2 \subseteq Q_2$. If $\alpha_1 = -\alpha_1$, then P is V-labellable. This kind of situation is ruled out by the proof of Lemma 3.8, which shows that $\alpha_i = 0$ for every possible irreducible component Q_i of a V-labellable poset.

4. The Ubiquity of the Miniscule Posets

The miniscule posets have been described in four distinct ways in this thesis. In this section we summarize these four ways and present an additional, empirical way to describe them. The miniscule posets are:

(a) (Definition) All posets of join irreducibles of the distributive lattices defined by the weights of miniscule representations of semisimple Lie algebras.

(b) All posets of join irreducibles of the Bruhat lattices.

(c) All known Gaussian posets.

(d) All V-labellable posets.

Notation. Let $\mathbf{p}$ denote the total order with p elements. Let $\oplus$ denote the operation of direct sum (disjoint union) for posets. Recall that if P is a poset, then J(P) denotes the lattice of order ideals in P.

The fifth description is:

(e) All direct sums of the posets shown in Table 4, each of which is of the form $J^r(\mathbf{p} \oplus \mathbf{q})$, where $r \geq 1$ and $0 \leq p \leq q$.

Although there is no known "theoretical" significance to this description, it is interesting to note how closely the irreducible miniscule posets are related by the operation $J(\cdot)$. Except for the chains, each irreducible miniscule poset appears exactly once in the list. Also note that $p + q + r = n$, the rank of the miniscule poset.

Three of the five descriptions of the miniscule posets are empirical identifications, one is a non-trivial result (Theorem VI.3), and one is an open problem (Problem V.4). Of the twenty possible *a priori* "theoretical" implications, only two are known: (a) $\Rightarrow$ (c) (Theorem V.4) and (a) $\Rightarrow$ (d) (Theorem V.5). Of course many of these are fairly unlikely, but some would be nice to see, e.g. (a) $\Rightarrow$ (b) (The set of weights of a miniscule representation necessarily forms a distributive lattice $\cdot\ \cdot\ \cdot$), or useful to have, e.g. (c) $\Rightarrow$ (d), which would solve Problem V.4.

$J(\underset{\sim}{0} \oplus \underset{\sim}{0})$ $a_1(1)$	$J(J(\underset{\sim}{0} \oplus \underset{\sim}{0}))$ $a_2(1)$	$J(J(J(\underset{\sim}{0} \oplus \underset{\sim}{0})))$ $a_3(1)$	$J(J(J(J(\underset{\sim}{0} \oplus \underset{\sim}{0}))))$ $a_4(1)$	. . .
⋮	⋮	⋮	⋮	
$J(\underset{\sim}{0} \oplus \underset{\sim}{q})$ $a_{q+1}(1)$	$J(J(\underset{\sim}{0} \oplus \underset{\sim}{q}))$ $a_{q+2}(1)$	$J(J(J(\underset{\sim}{0} \oplus \underset{\sim}{q})))$ $a_{q+3}(1)$	$J(J(J(J(\underset{\sim}{0} \oplus \underset{\sim}{q}))))$ $a_{q+4}(1)$	. . .
⋮	⋮	⋮	⋮	
$J(\underset{\sim}{1} \oplus \underset{\sim}{1})$ $a_3(2)=d_3(1)$	$J(J(\underset{\sim}{1} \oplus \underset{\sim}{1}))$ $d_4(1)$	$J(J(J(\underset{\sim}{1} \oplus \underset{\sim}{1})))$ $d_5(1)$	$J(J(J(J(\underset{\sim}{1} \oplus \underset{\sim}{1}))))$ $d_6(1)$	. . .
$J(\underset{\sim}{1} \oplus \underset{\sim}{2})$ $a_4(2)$	$J(J(\underset{\sim}{1} \oplus \underset{\sim}{2}))$ $d_5(5)$	$J(J(J(\underset{\sim}{1} \oplus \underset{\sim}{2})))$ $e_6(6)$	$J(J(J(J(\underset{\sim}{1} \oplus \underset{\sim}{2}))))$ $e_7(7)$	
$J(\underset{\sim}{1} \oplus \underset{\sim}{3})$ $a_5(2)$	$J(J(\underset{\sim}{1} \oplus \underset{\sim}{3}))$ $d_6(6)$			
⋮	⋮			
$J(\underset{\sim}{1} \oplus \underset{\sim}{q})$ $a_{q+2}(2)$	$J(J(\underset{\sim}{1} \oplus \underset{\sim}{q}))$ $d_{q+3}(q+3)$			
⋮	⋮			
$J(\underset{\sim}{2} \oplus \underset{\sim}{2})$ $a_5(3)$				
⋮				
$J(\underset{\sim}{p} \oplus \underset{\sim}{q})$ $a_{p+q+1}(p+1)$				
⋮				

Table 4

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