

Topology and Combinatorics of Ordered Sets

by

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(1977)

SUBMITTED IN PARTIAL FULFILLMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1981

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Submitted to the Department of Mathematics on May 1, 1981  
in partial fulfillment of the requirements for the degree  
of Doctor of Philosophy in Mathematics.

## ABSTRACT

This thesis is about applications of topology to the combinatorics of partially ordered sets (posets). We start out with Möbius inversion and another generalization of the principle of inclusion and exclusion. This leads to the study of Möbius functions and Möbius numbers of posets. Chain-counting techniques are used to prove two new theorems about Möbius numbers, which generalize Rota's Galois connection theorem and Crapo's complementation theorem.

The set of ideals of a poset form a topology, called the ideal topology. We discuss results about the ideal topology which are largely due to R. E. Stong. Stong's work was overlooked by combinatorialists, probably because it was disguised as topology.

The primary method for turning posets into topological spaces is called geometric realization. We begin with several canonical homeomorphisms, some of them new, between certain poset constructions.

The next best thing to a homeomorphism is a homotopy equivalence. Quillen proved a theorem which is a powerful and convenient tool for proving homotopy equivalences, and which is closely related to extensions of inclusion-exclusion and to the Galois connection theorem. Various applications of Quillen's theorem are given, some of them new.

The theory of Cohen-Macaulay complexes and Cohen-Macaulay posets provides connections between combinatorics,

ring theory, and algebraic topology. We begin by reviewing and extending various known results about the Cohen-Macaulay property. We then answer a question of D. Eisenbud and C. Huneke about when a certain poset construction preserves the Cohen-Macaulay property. We also prove that a certain stronger version of the Cohen-Macaulay property is a topological invariant, as conjectured by K. Baclawski.

L. Lovász used algebraic topology to prove a long-standing conjecture of Kneser about families of finite sets. This result can be restated in terms of the chromatic numbers of certain graphs. In fact, Lovász gave a general technique for using algebraic topology to find a lower bound on the chromatic number of a graph. We reformulate and generalize these results in such a way as to involve the theory of Cohen-Macaulay posets.

Thesis Supervisor: Richard P. Stanley

Title: Professor of Applied Mathematics

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## ACKNOWLEDGEMENTS

I thank Professor David P. Roselle of Virginia Tech for introducing me to combinatorics, and I thank Professor James R. Munkres of M.I.T. for introducing me to algebraic topology. Of course, this thesis could not have been written without the help of the combinatorics faculty at M.I.T.--particularly my thesis advisor, Richard P. Stanley. Correspondence with Professor Anders Björner of the University of Stockholm has also been very helpful.

My fellow graduate students have been another source of mathematical information, encouragement, and camaraderie. I would particularly like to mention Paul H. Edelman, Ranee Kathryn Gupta, and Jeffrey H. Smith.

Finally, I thank Orville Redenbacher for growing the popcorn that sustained me through years of toil.

## INTRODUCTION

If this thesis has a theme, then it has two. The first theme is applications of topology to the combinatorics of partially ordered sets (posets). We study two ways of making posets into topological spaces. Then topological tools can be applied to learn about such things as Möbius inversion, fixed points in posets, Cohen-Macaulay rings, and even chromatic numbers of graphs.

The second theme deals with attempts to generalize theorems about lattices to arbitrary posets. Sometimes a theorem about lattices can be generalized to posets either by basic combinatorics (i.e. counting things) or by topology. In the case of Rota's cross-cut theorem, we have a combinatorial generalization (Theorem 1.12) and a topological generalization (Theorem 5.9, due to Björner), either of which implies the original result. In the case of Crapo's complementation theorem, we have a combinatorial theorem for finite posets (Theorem 2.5) and a topological theorem (Theorem 6.1) which together imply Crapo's theorem.

Consideration of the principle of inclusion and exclusion leads to another sense in which one might want to do away with lattice assumptions. The principle of inclusion and exclusion deals with a collection of finite sets and their intersections. What if you'd rather not deal

with the intersections--are you stuck? The Möbius Inversion Theorem 1.3 is a way of avoiding intersections, and Theorem 1.5 is a more general answer.

In topology, there is a similar problem with intersections. If a geometric simplicial complex is covered by contractible subcomplexes, and if every finite intersection of members of the cover is contractible, then the underlying space is homotopy equivalent to the nerve of the cover. But again, the question arises, what if we don't want to deal with the intersections. An answer to this question is given by two theorems of Quillen, 5.5 and 5.7, which are quite reminiscent of 1.3 and 1.5. Quillen's theorems also form our most powerful tool for proving theorems about homotopy type of posets.

A few words on prerequisites: I have tried to define all of the poset terminology that I use. The most important terms are collected in Chapter 0, and a few other terms are defined when the need arises. The books by Aigner [Ai] and Birkhoff [Bi] are good references for posets.

I feel that it would be impractical to review all of the material from algebraic topology that I use herein. Most of it is quite standard. In a few cases, I give references to the books by Spanier [Sp] or Whitehead [Wh].



## Chapter 0. Definitions from Order Theory

A quasi-order is a relation, usually denoted  $\leq$ , which is reflexive and transitive. If the relation is antisymmetric as well, it is called a partial order. A set equipped with such a relation is called a quasi-ordered set or a partially ordered set, respectively. Henceforth, "partially ordered set" will always be abbreviated as poset. Note that if you start with a quasi-ordered set and identify every pair of elements  $x, y$  such that  $x \leq y$  and  $y \leq x$ , the result is a poset. The symbols  $\geq$ ,  $<$ , and  $>$  are defined from  $\leq$  in the obvious ways. Two elements  $x, y$  are comparable if  $x \leq y$  or  $y \leq x$ .

A finite poset is often indicated by its Hasse diagram: Say that  $x$  covers  $y$  if  $x > y$  and there is no  $z$  such that  $x > z > y$ . Draw a small circle or dot to represent each element of the poset, placing  $x$  higher than  $y$  if  $x > y$ . Draw a line segment from  $x$  to  $y$  whenever  $x$  covers  $y$ . For example, the poset whose elements are  $\{a, b, c, d\}$  and whose relations are  $a < c$ ,  $a < d$ ,  $b < c$ , and  $b < d$  has the Hasse diagram shown in figure 1 (a). The poset of subsets of a 3-element set, ordered by inclusion, is shown in figure 1 (b).

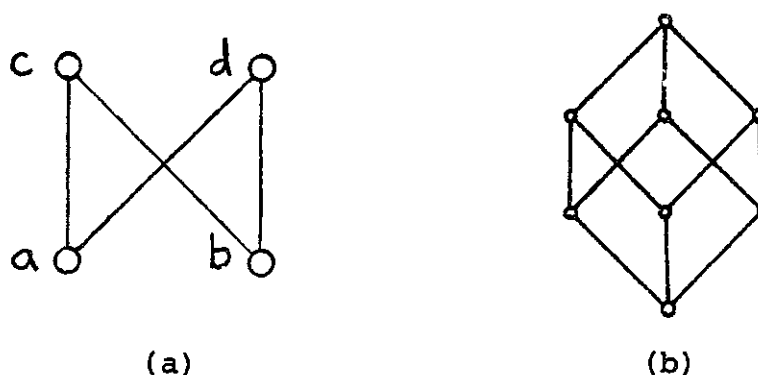


Figure 1.

A subset of a poset, given the induced partial order, is a subposet. A subposet  $I$  of a poset  $P$  is an ideal of  $P$  if, whenever  $x \leq y$  in  $P$  and  $y \in I$ , it follows that  $x \in I$ . In particular, the subposet  $P_{\leq x} = \{y \in P : y \leq x\}$  is called the principal ideal generated by  $x$ . (The notations  $P_{\geq x}$ ,  $P_{< x}$ , and  $P_{> x}$  are defined analogously.) If  $x \leq y$ , the subposet  $[x, y] = \{z \in P : x \leq z \leq y\}$  is called a closed interval of  $P$ . A subposet of  $P$  of the form  $P$ ,  $P_{> x}$ ,  $P_{< y}$ , or  $(x, y) = \{z \in P : x < z < y\}$  is called an open interval of  $P$ . (To avoid confusion, I will use the notation  $\langle x, y \rangle$  for ordered pairs.)

If  $P$  and  $Q$  are posets, the direct product poset  $P \times Q$  is the poset whose underlying set is the Cartesian product of the underlying sets of  $P$  and  $Q$ , and whose

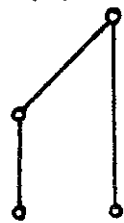
ordering is defined by  $\langle a, b \rangle \leq \langle c, d \rangle$  if and only if  $a \leq c$  and  $b \leq d$ .

A poset is bounded if it has a greatest element and a least element. The greatest element is usually denoted by  $\hat{1}$ , and the least element is usually denoted by  $\hat{0}$ . Given any poset  $P$ , there is a bounded poset  $\hat{P}$  formed by simply adjoining a new greatest element  $\hat{1}$  and a new least element  $\hat{0}$ . (These new elements are adjoined whether or not  $P$  had a least element or a greatest element to begin with.) Conversely, if  $P$  is a bounded poset, the proper part of  $P$  is the subposet  $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ .

If two elements  $x, y$  of a poset have a least upper bound, it is called the join of  $x$  and  $y$ , and is written  $x \vee y$ . Similarly, the greatest lower bound, if it exists, is called the meet of  $x$  and  $y$ , and is written  $x \wedge y$ . (Sometimes the terms meet and join are used for greatest lower bounds and least upper bounds of arbitrary subsets.) A lattice is a bounded poset in which every pair of elements has a meet and a join. Figure 1 (b) depicts a lattice. In fact, the poset  $\mathcal{P}_m^S$  of subsets of a set  $S$  is always a lattice, in which meet means intersection and join means union. Even more generally, any direct product of lattices is a lattice. (One can view  $\mathcal{P}_m^S$  as the direct product of  $(\text{card } S)$ -many copies of  $\mathcal{P}_m = \begin{matrix} \circ \\ | \\ \circ \end{matrix}$ .) The poset of figure 1 (a) is not a lattice, for

several reasons: it is not bounded,  $a \vee b$  does not exist, and  $c \wedge d$  does not exist.

An antichain is a poset in which no two elements are comparable. A chain is a poset in which every two elements are comparable. The length of a finite chain is one less than its cardinality. The length of a poset is the greatest length of a chain contained in the poset. The length of  $P_{\leq x}$  is called the height of  $x$  in  $P$ .

A poset is said to be ranked if every element has finite height, and if  $\text{height}(x) = \text{height}(y) + 1$  whenever  $x$  covers  $y$ . In that case, the height of  $x$  is usually called the rank of  $x$ , and is denoted by  $r(x)$ . The posets of figure 1 are ranked, whereas the poset  is not ranked.

The dual  $P^*$  of a poset  $P$  is the poset obtained by reversing the order of  $P$ . To find a Hasse diagram for  $P^*$ , take a diagram for  $P$  and turn it upside down. Frequently, a result about posets can be reformulated by reversing some orders. The new result is then said to follow by duality, or by standing on your head.

If  $P$  and  $Q$  are posets, a function  $f : P \rightarrow Q$  is said to be isotone if it is order-preserving, i.e. if  $x \leq y$  implies  $f(x) \leq f(y)$ . Similarly,  $f$  is said to be antitone if  $x \leq y$  implies  $f(x) \geq f(y)$ . Note that an

antitone map  $f : P \longrightarrow Q$  corresponds to an isotone map from  $P$  to  $Q^*$  or from  $P^*$  to  $Q$ , in an obvious way.

An isomorphism of posets is an isotone map which has an isotone two-sided inverse function. An automorphism of a poset  $P$  is an isomorphism of  $P$  with itself.

The barycentric subdivision of  $P$ , denoted  $sd(P)$ , is the poset of finite nonempty chains of  $P$ , ordered by inclusion. The interval poset of  $P$ , denoted  $Int(P)$ , is the set of closed intervals of  $P$ , ordered by inclusion.

## Chapter 1: Möbius Inversion

One of the most fundamental and well known principles of combinatorics is the principle of inclusion and exclusion. Its basic form is this: Suppose  $X$  is a finite set, and  $\mathcal{A}$  is a collection of subsets of  $X$  which cover  $X$ . Then

$$\text{card } X = \sum_{\mathcal{C} \subset \mathcal{A}} (-1)^{1+\text{card } \mathcal{C}} \text{card } (\cap \mathcal{C}). \quad (1)$$

Actually, it is the elements of  $X$ , rather than just their cardinalities, which are being included and excluded. So we could write

$$X = \sum_{\mathcal{C} \subset \mathcal{A}} (-1)^{1+\text{card } \mathcal{C}} \cap \mathcal{C}. \quad (2)$$

To make this precise, we can identify a subset of  $X$  with its indicator function, and view (2) as taking place in the abelian group  $\text{Maps}(X, \mathbb{Z})$ . Then it is no longer important for  $X$  to be finite, although  $\mathcal{A}$  should still be finite. Any homomorphism  $\phi$  from  $\text{Maps}(X, \mathbb{Z})$  to an abelian group  $G$  yields an identity

$$\phi(X) = \sum_{\mathcal{C} \subset \mathcal{A}} (-1)^{1+\text{card } \mathcal{C}} \phi(\cap \mathcal{C}). \quad (3)$$

If  $X$  is finite, and if  $\phi$  is the homomorphism from  $\text{Maps}(X, \mathbb{Z})$  to  $\mathbb{Z}$  which just adds up the values of a map, then (3) reduces to (1).

The trouble with the inclusion-exclusion principle (2) is that in some situations, we may not have good descriptions of the intersections of subcollections of  $\mathcal{A}$ . For example, suppose  $X$  is a poset, and  $\mathcal{A}$  is the collection  $\{X_{\leq c} : c \in X\}$  of principal ideals of  $X$ . An intersection of principal ideals is not a principal ideal unless  $X$  happens to be a meet-semilattice, that is, unless every pair of elements of  $X$  has a meet (greatest lower bound).

This brings us to the following general problem: We are given a set  $X$  and a collection  $\mathcal{A}$  of subsets of  $X$ . Let  $\tilde{\mathcal{A}}$  denote the subgroup of  $\text{Maps}(X, \mathbb{Z})$  generated by indicator functions of members of  $\mathcal{A}$ . The question is, does  $X$  belong to  $\tilde{\mathcal{A}}$ ? In other words, do there exist integers  $\lambda_A$  such that

$$X = \sum_{A \in \mathcal{A}} \lambda_A A?$$

If so, we will say that  $\mathcal{A}$  is an additive cover of  $X$ .

Define an equivalence relation on  $X$  by saying that two members  $x$  and  $y$  are equivalent if every member of  $\mathcal{A}$  contains both or neither of  $x$  and  $y$ . Call the

equivalence classes atoms of  $\mathcal{A}$ , and let  $Y$  denote the set of atoms. Obviously no linear combination of members of  $\mathcal{A}$  can have different coefficients on members of the same atom. So the best possible situation is that every atom belongs to  $\tilde{\mathcal{A}}$ , in which case we say that  $\mathcal{A}$  is an atomic cover. Note that  $Y$  is finite if  $\mathcal{A}$  is finite. Since  $X$  is partitioned into atoms, every finite atomic cover is an additive cover.

In general, the question of whether a given finite cover is additive or atomic amounts to determining whether a certain system of linear equations has an integer solution. But we are aiming toward convenient sufficient conditions.

These questions have applications in measure theory as well as combinatorics. Define a measure as a function  $m$  which is defined on the closure of  $\mathcal{A}$  with respect to finite unions and intersections, which has values in an abelian group  $G$ , and which satisfies the modular equality

$$m(A) + m(B) = m(A \cup B) + m(A \cap B).$$

Such a measure extends by linearity to a homomorphism  $\bar{m} : \tilde{\mathcal{A}} \rightarrow G$ . To show that  $\bar{m}$  is well-defined, suppose that



$$\sum_{i \in I} A_i = \sum_{j \in J} B_j,$$

where  $A_i, B_j$  are (not necessarily distinct) elements of  $\mathcal{A}$ , and the indexing sets  $I, J$  are finite, and show that

$$\sum_{i \in I} m(A_i) = \sum_{j \in J} m(B_j).$$

Consider the collection  $\{A_i : i \in I\}$ , quasi-ordered by inclusion. Suppose  $A_i$  and  $A_j$  are not comparable. By the modular equality, it is no loss of generality to replace  $A_i$  and  $A_j$  by  $A_i \cup A_j$  and  $A_i \cap A_j$ , and that operation must increase the amount of order in  $\{A_i : i \in I\}$ . Thus, we may assume that  $\{A_i : i \in I\}$  and  $\{B_j : j \in J\}$  are chains. It follows easily that the collections are actually identical. Therefore  $\bar{m}$  is well-defined. Of course,  $\bar{m}$  can be applied to any equation in  $\tilde{\mathcal{A}}$ .

Given  $x$  in  $X$ , let  $\mathcal{A}_x$  denote the subposet of  $\mathcal{A}$  consisting of those members which contain  $x$ .

**1.1 Proposition:** If  $\mathcal{A}$  is a finite cover such that  $\mathcal{A}_x$  has a greatest element for every  $x$  in  $X$ , then  $\mathcal{A}$  is an additive cover.

Proof: Let  $T_x$  denote the greatest member of  $\mathcal{A}_x$ . It is easy to see that for any  $x, y$  in  $X$ , either  $T_x = T_y$  or

else  $T_x$  is disjoint from  $T_y$ . Thus  $X$  is partitioned into the sets  $T_x$ , so  $X$  is the sum of the distinct values of  $T_x$ .  $\square$

To convince yourself that "additive" cannot be replaced by "atomic" in the proposition above, let  $X = \{1,2,3,4\}$ ,  $A = \{2,3\}$ ,  $B = \{3,4\}$ , and  $\mathcal{A} = \{X,A,B\}$ .

1.2 Proposition: If each  $\mathcal{A}_x$  has a least element  $S_x$ , and if each  $S_x$  contains finitely many atoms, then  $\mathcal{A}$  is an atomic cover.

Proof: Define a quasi-order on  $X$  by  $x \leq y \Leftrightarrow x \in S_y$ . This is a partial order on the set  $Y$  of atoms. If  $y$  is the atom containing  $x$ , then  $S_x$  corresponds to  $Y_{\leq y}$ . Therefore  $y = S_x - \sum_{z < y} z$ . And if  $y$  is minimal in  $Y$ , then  $S_x = y$ , so  $y$  belongs to  $\mathcal{A}$ . The result follows by induction.  $\square$

In the case of the proposition above, we can actually derive a formula for the coefficients which make  $\mathcal{A}$  into an atomic cover. For simplicity, we may replace  $X$  by  $Y$ , and thus assume that every atom is a singleton. Suppose that  $\mu(x,y)$  are integers such that

$$\{y\} = \sum_{x \ll y} \mu(x, y) S_x \quad (4)$$

for each  $y$  in  $X$ . By the proof of 1.2,

$$\begin{aligned} \{y\} &= S_y - \sum_{z < y} \{z\} \\ &= S_y - \sum_{z < y} \sum_{x \ll z} \mu(x, z) S_x \\ &= S_y - \sum_{x \ll y} S_x \sum_{z: x \ll z < y} \mu(x, z) \\ &= \sum_{x \ll y} S_x (\delta(x, y) - \sum_{z: x \ll z < y} \mu(x, z)), \end{aligned} \quad (5)$$

where  $\delta$  is the Kronecker delta. The most obvious way to reconcile (4) and (5) is to let

$$\mu(x, y) = \delta(x, y) - \sum_{z: x \ll z < y} \mu(x, z). \quad (6)$$

In fact, (6) is a perfectly good recurrence for  $\mu$ .

Proposition 1.2 can now be rephrased as follows.

**1.3 Möbius Inversion Theorem:** If  $X$  is a poset such that every principal ideal is finite, and if  $\mu$  is the function defined recursively by  $\mu(x, y) = \delta(x, y) - \sum_{z: x \ll z < y} \mu(x, z)$ ,

then  $\{y\} = \sum_{x \ll y} \mu(x, y) X_{\ll x}$  for each  $y$  in  $X$ .  $\square$

The function  $\mu$  above is called the Möbius function of the poset  $X$ , and was first discussed in full generality in [Ro]. Note that the value of  $\mu(x,y)$  depends only on the closed interval  $[x,y]$ . Therefore  $\mu$  is defined so long as the poset is locally finite, i.e., so long as all of the closed intervals are finite. Also,  $\mu(x,y) = 0$  if  $x \not\leq y$ , by the convention that an empty sum is zero.

The zeta function of a poset is defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

The recurrence (6) can be rewritten as

$$\sum_z \mu(x,z)\zeta(z,y) = \delta(x,y),$$

so long as we stipulate that  $\mu(x,y) = 0$  if  $x \not\leq y$ . Thus, in a certain sense,  $\mu$  and  $\zeta$  are inverses of each other. See [Ro] for more information on this point of view. In this thesis, the zeta function will only be used as a notational convenience.

If  $P$  is a finite poset, define an integer  $\mu(P)$ , called the Möbius number of  $P$ , by the recurrence

$$\mu(P) = -1 - \sum_{x \in P} \mu(P_{>x}). \quad (7)$$

Note that  $\mu(\emptyset) = -1$ , by the convention that an empty sum is zero.

1.4 Theorem: If  $\mathcal{A}$  is a finite cover of  $X$  such that  $\mu(\mathcal{A}_x) = 0$  for all  $x$  in  $X$ , then  $\mathcal{A}$  is an additive cover:

$$X = - \sum_{A \in \mathcal{A}} \mu(\mathcal{A}_{>A}) A.$$

Proof: Let  $x$  be an arbitrary element of  $X$ . By hypothesis,

$$0 = \mu(\mathcal{A}_x) = -1 - \sum_{A \in \mathcal{A}_x} \mu((\mathcal{A}_x)_{>A}).$$

If  $A \in \mathcal{A}_x$ , then  $(\mathcal{A}_x)_{>A} = \mathcal{A}_{>A}$ , so

$$0 = -1 - \sum_{A \in \mathcal{A}_x} \mu(\mathcal{A}_{>A}), \quad \text{or}$$

$$1 = - \sum_{A \in \mathcal{A}_x} \mu(\mathcal{A}_{>A}).$$

This last equation is just the coefficient of  $x$  in

$$X = - \sum_{A \in \mathcal{A}} \mu(\mathcal{A}_{>A}) A. \quad \square$$

Theorem 1.4 extends easily to the case of an indexed cover:

1.5 Theorem: If  $P$  is a finite poset,  $g : P \rightarrow \mathcal{A}$  is an isotone map, and  $\mu(g^{-1}(\mathcal{A}_x)) = 0$  for all  $x$  in  $X$ , then

$$X = - \sum_{t \in P} \mu(P_{>t}) g(t). \quad \square$$

Before giving an application of 1.5, we will need to discuss some general properties of Möbius numbers and Möbius functions.

1.6 Theorem (P. Hall): For any finite poset  $P$ ,  $\mu(P)$  equals the number of odd chains in  $P$  minus the number of even chains in  $P$ , where the empty chain is counted as an even chain.

Proof: If  $c$  is a nonempty chain of  $P$ , let  $x$  be the least element of  $c$ . The rest of  $c$  is a chain of the opposite parity in  $P_{>x}$ . Therefore the chain-counting formula above satisfies the recurrence (7).  $\square$

1.7 Corollary: If  $P$  has an element which is comparable with every other element, then  $\mu(P) = 0$ .

Proof: If  $z$  is comparable with everything else, then the chains which contain  $z$  are in bijective correspondence with the chains which do not contain  $z$ . Therefore  $P$  has the same number of odd and even chains, so  $\mu(P) = 0$  by 1.6. □

In particular,  $\mu(P) = 0$  if  $P$  has a least element or a greatest element. Therefore, Theorem 1.4 implies 1.1, and partially implies 1.2.

1.8 Corollary:  $\mu(P^*) = \mu(P)$ .

Proof:  $P^*$  has the same set of chains as  $P$ . □

1.9 Corollary:  $\mu(P) = -1 - \sum_{x \in P} \mu(P_{<x})$ . (8)

Proof: Corollary 1.8 and recurrence (7). □

It is no accident that the Möbius function and Möbius number have similar names and symbols:

1.10 Proposition:  $\mu(P) = \mu_{\hat{P}}(\hat{0}, \hat{1})$ , and  $\mu(x, y) = \mu((x, y))$  if  $x < y$ .

Proof: Compare the recurrences (6) and (8). □

1.11 Proposition: The Möbius function of the direct product poset  $P \times Q$  is given by

$$\mu_{P \times Q}(\langle a, b \rangle, \langle c, d \rangle) = \mu_P(a, c) \mu_Q(b, d).$$

Proof: Check that the right-hand side obeys the recurrence (6) for the left-hand side. □

1.12 Proposition: If  $S$  is finite, the Möbius function of the power set  $\underline{2}^S$  is given by  $\mu(A, B) = (-1)^{\text{card}(B \setminus A)}$  if  $A \subset B$ .

Proof: Compute the Möbius function of the two-element chain  $\underline{2}$ , and apply 1.11. □

Given a subset  $A$  of a poset  $P$ , let  $P(A)$  denote the subposet consisting of elements which are comparable to every element of  $A$ . We say that a subset  $C$  of  $P$  is a cutset of  $P$  if  $C \cap P(\sigma)$  is nonempty for every finite chain  $\sigma$  of  $P$ , or equivalently if  $\text{sd}(P)$  is covered by



$\{\text{sd}(P(A)) : A \subset C, A \neq \emptyset\}$ . We will see that it is an additive cover.

1.13 Cross-Cut Theorem: If  $C$  is a finite cutset of a poset  $P$ , then

$$\sum_{\text{ACC}} (-1)^{\text{card } A} \text{sd}(P(A)) = 0.$$

Proof: Let  $X = \text{sd}(P)$  and  $\mathcal{A} = \{\text{sd}(P(A)) : A \subset C\}$ . Define an isotone map  $g : (2^C \setminus \{\emptyset\})^* \rightarrow \mathcal{A}$  by  $A \rightarrow \text{sd}(P(A))$ . If  $\sigma \in \text{sd}(P)$ , note that

$$\begin{aligned} g^{-1}(\mathcal{A}_\sigma) &= \{A \subset C : \sigma \subset P(A), A \neq \emptyset\} \\ &= \{A \subset C : A \subset P(\sigma), A \neq \emptyset\} \\ &= \{A \subset C \cap P(\sigma), A \neq \emptyset\}. \end{aligned}$$

Since  $C$  is a cutset,  $C \cap P(\sigma)$  is nonempty, and is thus the least element of  $g^{-1}(\mathcal{A}_\sigma)$ . Therefore  $\mu(g^{-1}(\mathcal{A}_\sigma)) = 0$  by 1.7, so 1.5 says that

$$\text{sd}(P) = - \sum_{\substack{\text{ACC} \\ A \neq \emptyset}} \mu((2^C \setminus \{\emptyset\})_{>A}^*) \text{sd}(P(A)).$$

Now  $\mu((2_m^C \setminus \{\emptyset\})_{>A}^*) = \mu((2_m^C \setminus \{\emptyset\})_{<A})$  by 1.8,

$$= \mu_{2_m^C}(\emptyset, A) \quad \text{by 1.10,}$$

$$= (-1)^{\text{card } A} \quad \text{by 1.12.}$$

Thus

$$\text{sd}(P) = - \sum_{ACC, A \neq \emptyset} (-1)^{\text{card } A} \text{sd}(P(A)).$$

Since  $P(\emptyset) = P$ , the result follows.  $\square$

The theorem above is called the cross-cut theorem because it was originally proved [Ro, p. 352] in the case where  $C$  is a cross-cut (i.e. a cutset which is also an antichain) and where  $\hat{P}$  is a lattice.

As mentioned earlier, one can obtain new equations by applying a measure to additive covering equations. In case  $X$  is of the form  $\text{sd}(P)$ , where  $P$  is finite, an important measure is the measure  $\chi$  which has the value  $(-1)^n$  on a chain of length  $n$ . Using Theorem 1.6, we see that

$$\mu(P) = \chi(\text{sd}(P)) - 1,$$

where the  $-1$  comes from the fact that  $sd(P)$  does not include the empty chain. Apply  $\chi$  to 1.13 and note that

$$\sum_{ACC} (-1)^{\text{card } A} = 0$$

to obtain

$$\sum_{ACC} (-1)^{\text{card } A} \mu(P(A)) = 0.$$

If  $\hat{P}$  is a finite lattice, one can show that  $\mu(P(A)) = 0$  if  $A$  and  $P(A)$  are nonempty, so

$$\begin{aligned} \mu(P) &= \sum \{(-1)^{\text{card } A} : A \subset C, P(A) = \emptyset\} \\ &= \sum \{(-1)^{\text{card } A} : A \subset C, P(A) \neq \emptyset\}. \end{aligned}$$

This fact will be generalized in Chapter 5.

As a generalization of  $\chi$ , one can let each chain of length  $k$  have the measure  $\binom{n-2}{k}$ , which can be regarded as a polynomial in  $n$ . Then the measure of  $sd(P)$  is denoted  $Z(P;n)$ , and is called the zeta polynomial of  $P$ . See [Ed 1] or [Ed 2] for further information.

## Chapter 2: Proving Möbius Identities by Chain Counting

We saw in Theorem 1.6 that  $\mu(P)$ , the Möbius number of a finite poset  $P$ , equals the number of odd chains minus the number of even chains. ("Odd" and "even" refer to the cardinality, not the length.) Therefore, one can prove things about Möbius numbers by counting chains, as we did in Proposition 1.7. First, we will consider the Möbius numbers of two finite posets connected by a relation.

We will say that a relation  $R$  between two posets  $P$  and  $Q$  is an ideal relation if  $R$  is an ideal in the direct product poset  $P \times Q$ . We will need the following construction, which is a slight modification of one given by Baclawski in [Ba 3]. Given  $P, Q$ , and  $R$ , construct a poset whose underlying set is the disjoint union of  $P$  and  $Q$ , and whose ordering is given by  $x \leq y$  if and only if one of the following holds:

- (1)  $x \leq y$  in  $P$ ,
- (2)  $x \geq y$  in  $Q$ ,
- (3)  $x \in P, y \in Q$ , and  $\langle x, y \rangle \in R$ .

This definition gives a partial order precisely when  $R$  is an ideal relation. Following Baclawski, we denote this new poset by  $P +_R Q$ , and call it the R-join of  $P$  and  $Q$ .

2.1 Theorem. Let  $R$  be an ideal relation between two finite posets,  $P$  and  $Q$ . Then

$$\begin{aligned} \mu(Q) + \sum_{x \in P} \mu(P_{<x}) \mu(R(x)) &= \mu(P +_R Q) \\ &= \mu(P) + \sum_{y \in Q} \mu(Q_{<y}) \mu(R^{-1}(y)). \end{aligned}$$

Proof: We will show that

$$\mu(P +_R Q) = \mu(Q) + \sum_{x \in P} \mu(P_{<x}) \mu(R(x));$$

the other half is analogous.

If  $C$  is a chain in  $P +_R Q$ , let  $x$  be the largest member of  $C$  which belongs to  $P$ . If no such  $x$  exists, then  $C$  is counted in the term  $\mu(Q)$ . Otherwise,  $C$  can be split into a chain in  $P_{<x}$ , the singleton  $\{x\}$ , and a chain in  $R(x)$ . It is easy to check that the signs work out, so the result follows.  $\square$

A very simple sort of ideal relation is  $R = P \times Q$ . Then  $P +_R Q$  is denoted  $P * Q$ , and is called the join (or ordinal sum) of  $P$  and  $Q$ .

2.2. Corollary:  $\mu(P * Q) = -\mu(P)\mu(Q)$ .

Proof: Theorem 2.1 says that

$$\begin{aligned}\mu(P*Q) &= \mu(Q) + \sum_{x \in P} \mu(P_{<x})\mu(Q) \\ &= \mu(Q) \left(1 + \sum_{x \in P} \mu(P_{<x})\right).\end{aligned}$$

By Corollary 1.9, or by 2.1 with  $Q = \emptyset$ , we know that

$$\mu(P) = -1 - \sum_{x \in P} \mu(P_{<x}).$$

Substitution yields the result. □

Suppose that  $g : P \rightarrow Q$  is an isotone map. If we identify  $g$  with the set of ordered pairs  $\langle x, g(x) \rangle$ , then the ideal in  $P \times Q^*$  generated by  $g$  is

$$R_g = \{\langle x, y \rangle \in P \times Q^* : g(x) \leq y\}.$$

For this relation, we have  $R(x) = Q_{>g(x)}$  and  $R^{-1}(y) = g^{-1}(Q_{<y})$ . Recall that by 1.7,  $\mu(Q_{>g(x)}) = 0$ . So 2.1 yields:

**2.3 Corollary** [Ba 3, Theorem 5.5]: If  $g : P \rightarrow Q$  is an isotone map of finite posets, then

$$\mu(Q) = \mu(P) + \sum_{y \in Q} \mu(Q_{\leq y}) \mu(g^{-1}(Q_{\leq y})).$$

□

The ideals  $g^{-1}(Q_{\leq y})$  are called the fibers of  $g$ . Corollary 2.3 implies that if all of the fibers of  $g$  have Möbius number zero, then  $\mu(P) = \mu(Q)$ .

If every fiber of  $g$  has a greatest element, then  $g$  is called a (lower) Galois map. This is an especially important case in which the fibers have Möbius number zero. The function  $f$  from  $Q$  to  $P$  which sends each element  $y$  to the greatest element of the fiber  $g^{-1}(Q_{\leq y})$  is isotone. One can check that  $g(f(y)) \leq y$  and  $f(g(x)) \geq x$  for all  $x$  in  $P$  and  $y$  in  $Q$ . Also, every fiber of  $f$  has a least element. Fans of category theory may be amused to note that  $f$  and  $g$  can be viewed as a pair of adjoint functors.

The situation gains symmetry if we phrase it in terms of antitone maps. A Galois connection between  $P$  and  $Q$  is a pair of antitone maps,  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$ , such that  $g(f(x)) \geq x$  and  $f(g(y)) \geq y$  for all  $x$  in  $P$  and  $y$  in  $Q$ . By the reasoning above, either of these maps completely determines the Galois connection. We saw that if two posets are related by a Galois connection, then their Möbius numbers are equal.

One can also use a Galois connection in a more "local" fashion, to relate the Möbius functions of two posets.

2.4 Theorem [Ro, p. 347]: Let  $f$  and  $g$  form a Galois connection between  $P$  and  $Q$ . For each  $x$  in  $P$  and  $y$  in  $Q$ ,

$$\sum_{s:f(s)=y} \mu_P(x,s) = \sum_{t:g(t)=x} \mu_Q(y,t).$$

Proof: It is a simple consequence of the definitions that for every  $s$  in  $P$  and  $t$  in  $Q$ ,

$$(*) \quad f(s) \geq t \Leftrightarrow g(t) \geq s.$$

Let  $R$  be the ideal relation consisting of pairs  $\langle s,t \rangle$  which satisfy  $(*)$ . We may assume that  $\langle x,y \rangle$  is in  $R$ ; otherwise both sides of the equation are zero, by the convention that  $\mu(a,b) = 0$  if  $a \not\leq b$ .

We will apply 2.1 to the half-open intervals  $(x,g(y)]$  and  $(y,f(x)]$ . Each of these posets is either empty or has a largest element, so we find that

$$\mu((x,g(y)]) = -\delta(x,g(y)), \quad \text{and}$$

$$\mu((y,f(x)]) = -\delta(y,f(x)).$$



Since  $R(s) = (y, f(s))$  and  $R^{-1}(t) = (x, g(t))$ , we similarly have

$$\mu(R(s)) = -\delta(y, f(s)),$$

$$\mu(R^{-1}(t)) = -\delta(x, g(t)).$$

Finally, using 1.10,

$$\mu((x, g(y))|_{<s}) = \mu_P(x, s) \quad \text{and}$$

$$\mu((y, f(x))|_{<t}) = \mu_Q(y, t).$$

So 2.1 says

$$\begin{aligned} & \delta(y, f(x)) + \sum_{s: x < s \leq g(y)} \mu_P(x, s) \delta(y, f(s)) \\ &= \delta(x, g(y)) + \sum_{t: y < t \leq f(x)} \mu_Q(y, t) \delta(x, g(t)). \end{aligned}$$

If  $f(s) = y$ , then  $s \leq g(y)$ ; and if  $\mu_P(x, s) \neq 0$ , then  $x < s$ . Therefore the constraint  $x < s \leq g(y)$  on the first summation can be replaced by  $s \neq x$ . Similarly, the second summation can be taken over  $t \neq y$ . The result follows. □

Crapo's complementation theorem [Cr] is a theorem about the Mobius function of a finite lattice. The next theorem is a similar statement which holds for any finite poset. In Chapter 6, we will say more about how the two results are related.

A subset  $C$  of a poset  $P$  is called convex if  $x, y \in C$  implies that the closed interval  $[x, y]$  is contained in  $C$ . For example, any ideal, interval, or anti-chain in  $P$  is convex.

**2.5 Theorem:** If  $C$  is a convex subset of a finite poset  $P$ , then

$$\mu(P) = \mu(P \setminus C) + \sum_{x, y \in C} \mu(P_{<x}) \zeta(x, y) \mu(P_{>y}).$$

Proof: We know that  $\mu(P)$  counts the chains in  $P$ , and  $\mu(P \setminus C)$  counts the chains in  $P$  which do not meet  $C$ . It remains to be shown that the expression

$$(*) \quad \sum_{x, y \in C} \mu(P_{<x}) \zeta(x, y) \mu(P_{>y})$$

counts the chains which do meet  $C$  (and counts them with the right coefficients).

The formula (\*) counts quadruples of the form  $\langle A, x, y, B \rangle$ , where  $x, y \in C$ ,  $x < y$ ,  $A$  is a chain in  $P_{<x}$ ,

and  $B$  is a chain in  $P_{>y}$ . Any such quadruple corresponds uniquely to a chain which meets  $C$ . On the other hand, a chain which meets  $C$  corresponds to such a quadruple, but not necessarily uniquely.

Let  $T$  be a chain which meets  $C$ . Let  $n = \text{card}(T)$  and  $m = \text{card}(T \cap C)$ . If  $\langle A, x, y, B \rangle$  corresponds to  $T$ , then either  $x$  is covered by  $y$  in  $T$  (i.e.  $x$  and  $y$  are consecutive in  $T$ ) or else  $x = y$ . Of course, there are  $n$  such quadruples with  $x = y$ , and each of these is counted with the coefficient  $(-1)^{n+1}$ . On the other hand, since  $C$  is convex, there are  $m-1$  quadruples in which  $x$  is covered by  $y$  in  $T$ . This kind of quadruple is counted with the coefficient  $(-1)^n$ . Therefore  $T$  is counted by (\*) with a coefficient of

$$m(-1)^{n+1} + (m-1)(-1)^n = (-1)^{n+1}.$$

That is the same coefficient as is used for  $T$  in  $\mu(P)$ , so we are done. □

## Chapter 3: The Ideal Topology

In this chapter, we discuss the first of two ways of viewing posets as topological spaces. The most important results of this theory appeared in an article called "Finite Topological Spaces" by R. E. Stong [Sto]. (To be exact, Propositions 3.1 and 3.3 through 3.10 are all essentially due to Stong, except insofar as they apply to infinite posets.) That paper seems to have escaped the notice of some researchers interested in partially ordered sets, perhaps because of its title and point of view.

It is easy to check that the ideals of a poset  $P$  form the open sets of a topology on  $P$ , which we call the ideal topology. An element  $x$  of  $P$  has a smallest neighborhood, the principal ideal  $P_{\leq x}$ . The principal ideals form a basis for the topology. Although the ideal topology is  $T_0$ , it is not Hausdorff or even  $T_1$  unless  $P$  is an antichain. (Recall that a space is  $T_0$  if, for any two points, there exists an open set containing one but not both of the points. In a  $T_1$  space, one can specify which of the points is to be contained in the open set. Equivalently, a space is  $T_1$  if and only if every singleton is a closed set.)

Incidentally, one could actually discuss the ideal topology on quasi-ordered sets without much complication.

It is interesting that every finite topological space corresponds to a quasi-ordered set in this way: Every point in a finite topological space has a minimum neighborhood. Define  $x \leq y$  if  $x$  belongs to the minimum neighborhood of  $y$ . Alexandroff [A1] may have been the first to notice this bijection between finite topological spaces and finite quasi-ordered sets.

A poset  $P$  is said to be connected if every pair of points  $x, y$  in  $P$  can be connected by a finite sequence

$$x = a_0 \leq a_1 \geq a_2 \leq a_3 \geq \dots \leq a_n = y.$$

It is easy to show that  $P$  is connected if and only if  $P$  cannot be written as a disjoint union of two ideals. But this is precisely the condition that  $P$  be connected as a space.

**3.1 Proposition:** A function  $f : P \rightarrow Q$  between two posets is continuous in the ideal topology if and only if it is isotone.

Proof: It is easy to see that if  $f$  is isotone, then the inverse image of an ideal is an ideal. Conversely, suppose that the inverse image of each ideal is an ideal. In particular, for each  $y$  in  $P$ ,  $f^{-1}(Q_{\leq f(y)})$  is an ideal

containing  $y$ . So if  $x < y$ , then  $x \in f^{-1}(Q_{\leq f(y)})$ , so  $f(x) \leq f(y)$ .  $\square$

Proposition 3.1 shows that the study of posets and isotone maps belongs, in some sense, to topology. Or, if you like, the category of posets and isotone maps is a full subcategory of the category of topological spaces and continuous maps.

We should point out that there are not many maps from posets into "familiar" spaces.

**3.2 Proposition:** Suppose  $f : P \rightarrow X$  is a continuous map from a connected poset into a  $T_1$  space. Then  $f$  is a constant map.

Proof: Suppose  $x < y$  but  $f(x) \neq f(y)$ . Since  $X$  is  $T_1$ , there is an open set  $V$  which contains  $f(y)$  but not  $f(x)$ . Then  $f^{-1}(V)$  is an open set of  $P$  which contains  $y$  but not  $x$ , which is impossible.  $\square$

The situation for maps into a poset is more interesting. If  $X$  is a space and  $P$  is a poset, the set  $\text{Hom}(X, P)$  of continuous maps from  $X$  into  $P$  can be partially ordered componentwise:  $f \leq g$  if  $f(x) \leq g(x)$  for all  $x$  in  $X$ .

Before stating a result about  $\text{Hom}(X,P)$ , let us recall some definitions from topology. If  $f, g : X \rightarrow Y$  are two continuous maps, then a homotopy from  $f$  to  $g$  is a continuous map  $H : X \times I \rightarrow Y$  such that  $H(x,0) = f(x)$  and  $H(x,1) = g(x)$  for all  $x$  in  $X$ . If there exists a homotopy from  $f$  to  $g$ , then we say that  $f$  and  $g$  are homotopic. It is an easy exercise that "homotopic" is an equivalence relation on maps from  $X$  to  $Y$ .

3.3 Proposition: Let  $X$  be a space and let  $P$  be a poset. If  $f, g \in \text{Hom}(X,P)$  and  $f \leq g$ , then  $f$  and  $g$  are homotopic.

Proof: Define  $H : X \times I \rightarrow P$  by

$$H(x,t) = \begin{cases} f(x) & \text{if } t < 1 \\ g(x) & \text{if } t = 1. \end{cases}$$

If  $V$  is open in  $P$ , then

$$H^{-1}(V) = f^{-1}(V) \times [0,1) \cup g^{-1}(V) \times \{1\}.$$

But  $f \leq g$ , so  $g^{-1}(V) \subset f^{-1}(V)$ ; therefore

$$H^{-1}(V) = f^{-1}(V) \times [0,1) \cup g^{-1}(V) \times [0,1].$$

Thus  $H$  is continuous, so  $H$  is the desired homotopy.

□

We need a few more definitions: If  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are maps such that  $g \circ f$  is homotopic to  $\text{id}_X$  (the identity map of  $X$ ) and  $f \circ g$  is homotopic to  $\text{id}_Y$ , then  $f$  and  $g$  are homotopy inverses of each other. A map which has a homotopy inverse is a homotopy equivalence. If there is a homotopy equivalence from  $X$  to  $Y$ , then we say that  $X$  and  $Y$  are homotopy equivalent, or that  $X$  and  $Y$  have the same homotopy type. Homotopy equivalence is an equivalence relation on spaces. A space with the homotopy type of a point is contractible. Equivalently,  $X$  is contractible if and only if  $\text{id}_X$  is homotopic to a constant map. A homotopy from  $\text{id}_X$  to a constant is called a contraction of  $X$ .

3.4 Corollary: A poset with a greatest element is contractible.

Proof: Suppose the poset  $P$  has the greatest element  $y$ . Let  $g : P \rightarrow P$  be the constant map with value  $y$ . Then  $\text{id}_P \leq g$ , so by Proposition 3.3, the identity map of  $P$  is homotopic to a constant map. □



3.5 Corollary: Every poset is locally contractible.

Proof: The smallest neighborhood of any point is a principal ideal, which is contractible by 3.4.  $\square$

3.6 Corollary: The path components of a poset are the same as the connected components.

Proof: Any contractible space is path connected, so 3.5 implies that posets are locally path connected. In any locally path connected space, the path components are the same as the components [Mu 1, p. 162].  $\square$

Proposition 3.3 implies that if two maps belong to the same component of  $\text{Hom}(X, P)$ , then they are homotopic. We will see that there is a partial converse.

If  $\{P_\alpha : \alpha \in J\}$  is a family of posets, then the ideal topology on the direct product poset  $\prod P_\alpha$  is unfortunately not the same as the product topology determined by the ideal topology on the factors. It is the box topology, which coincides with the product topology for finite products. (The box topology on a product  $\prod X_\alpha$  is the coarsest topology such that  $\prod V_\alpha$  is open whenever each  $V_\alpha$  is open in  $X_\alpha$ . See [Mu 1, §2-8].)

The poset of isotone maps from  $P$  to  $Q$ ,  $\text{Hom}(P, Q)$ , can be viewed as a subposet of the product  $\prod_P Q$ . That is,  $\text{Hom}(P, Q)$  has the box topology. So if  $P$  is finite, then  $\text{Hom}(P, Q)$  has the product topology. (In the context of function spaces, the product topology is sometimes called the topology of pointwise convergence.)

**3.7 Proposition:** If  $P$  and  $Q$  are posets,  $P$  is finite, and  $f$  and  $g$  are homotopic maps in  $\text{Hom}(P, Q)$ , then  $f$  and  $g$  belong to the same component of  $\text{Hom}(P, Q)$ .

Proof: Let  $H : P \times I \rightarrow Q$  be a homotopy from  $f$  to  $g$ . Then there is a map  $\hat{H} : I \rightarrow \text{Hom}(P, Q)$  defined by  $\hat{H}(t)(x) = H(x, t)$ . Note that  $\hat{H}(t)$  belongs to  $\text{Hom}(P, Q)$  because a continuous map of two variables is continuous in the first variable. Also,  $\hat{H}(0) = f$  and  $\hat{H}(1) = g$ . If we can show that  $\hat{H}$  is continuous, then it will be a path from  $f$  to  $g$  in  $\text{Hom}(P, Q)$ , and we will be done.

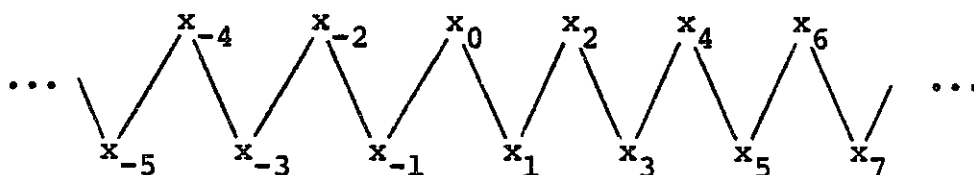
Since  $\text{Hom}(P, Q)$  has the product topology, it suffices to check that  $\hat{H}$  is continuous in each coordinate. This follows from the fact that  $H(x, t)$  is continuous in the second variable. □

**3.8 Theorem:** If  $P$  and  $Q$  are posets and  $P$  is finite, then the components of  $\text{Hom}(P, Q)$  are the same as the

homotopy classes of maps from  $P$  to  $Q$ .

Proof: Combine 3.3 and 3.7. □

Example: The finiteness assumption cannot be removed from Theorem 3.8. Consider the poset  $P$  shown below.



It is easy to see that the identity map of  $P$  is not comparable to any other member of  $\text{Hom}(P,P)$ , therefore  $\text{id}_P$  is not connected to any other member of  $\text{Hom}(P,P)$ . However, we now show that  $\text{id}_P$  is homotopic to the constant map with value  $x_0$ .

Define maps  $\phi_n$  in  $\text{Hom}(P,P)$  by

$$\phi_n(x_i) = \begin{cases} x_{-n} & i \leq -n \\ x_i & -n \leq i \leq n \\ x_n & n \leq i, \end{cases}$$

for each  $n = 0, 1, 2, 3, \dots$ . Then define  $H : P \times I \rightarrow P$  by  $H(x, 0) = x$ ,  $H(x, t) = \phi_{2n-1}(x)$  if  $1/(2n+1) < t < 1/(2n-1)$ , and  $H(x, 1/(2n+1)) = \phi_{2n}(x)$ . Note that

$H(x,1) = \phi_0(x) = x_0$  and  $H(x,0) = x$ . It remains to check continuity. It suffices to check that the inverse images of basic open sets, namely principal ideals, are open. Thus, for example, one finds that for  $n \geq 1$ ,

$$H^{-1}(\{x_{2n-1}\}) = \{x_{2n-1}\} \times [0, 1/(2n-1)) \\ \cup \{x_{2n-1}, x_{2n}, x_{2n+1}, \dots\} \times (1/(2n+1), 1/(2n-1)).$$

The other cases are left for the reader to check.  $\square$

Recall from Chapter 2 that an isotone map  $f : P \rightarrow Q$  is (lower) Galois if, for each  $y$  in  $Q$ , the fiber  $f^{-1}(Q_{\leq y})$  has a greatest element. Then there is a uniquely defined isotone map  $g : Q \rightarrow P$  such that  $g \circ f \geq \text{id}_P$  and  $f \circ g \leq \text{id}_Q$ . The map  $g$  satisfies the dual condition that  $g^{-1}(P_{> x})$  always has a least element, and is thus said to be an upper Galois map. From 3.3, we see that an (upper or lower) Galois map is a homotopy equivalence.

An element  $x$  of a poset  $P$  is said to be an irreducible of  $P$  if either  $P_{< x}$  has a greatest element or  $P_{> x}$  has a least element. (This is not to be confused with the concept of "join-irreducible", which is only defined in a lattice.)

3.9 Proposition: If  $x$  is an irreducible of  $P$ , then the inclusion  $P \setminus \{x\} \rightarrow P$  is a homotopy equivalence.

Proof: If  $P_{<x}$  has a greatest element, then the inclusion is lower Galois, and if  $P_{>x}$  has a least element, then the inclusion is upper Galois.  $\square$

By removing irreducibles, we can make a poset smaller without changing its homotopy type. But if we started with a finite poset, then of course we'll have to stop after a finite number of steps. What we have left is called a core of the poset we started with. (The core is not a uniquely defined subposet. For example, either element of a two-element chain is a core of the chain.) Thus the problem of determining whether two finite posets are homotopy equivalent reduces to the problem of determining whether they have homotopy equivalent cores. That is quite a simplification, as the next result shows.

3.10 Theorem [Sto]: Two finite posets  $P, Q$  are homotopy equivalent if and only if each core of  $P$  is isomorphic to each core of  $Q$ .

Proof: Suppose  $A$  is a core of  $P$  and  $B$  is a core of  $Q$ . If  $A$  is isomorphic to  $B$ , then by transitivity  $P$  is

homotopy equivalent to  $Q$ .

On the other hand, suppose  $P$  is homotopy equivalent to  $Q$ . By transitivity,  $A$  is homotopy equivalent to  $B$ . Choose  $f$  in  $\text{Hom}(A,B)$  and  $g$  in  $\text{Hom}(B,A)$  such that  $f \circ g$  is homotopic to  $\text{id}_B$  and  $g \circ f$  is homotopic to  $\text{id}_A$ .

Suppose  $f \circ g \neq \text{id}_B$ . Then by 3.7, there is some map  $h$  in  $\text{Hom}(B,B)$  which is comparable, but not equal, to  $\text{id}_B$ . Without loss of generality, assume  $h > \text{id}_B$ . Choose  $b$  maximal such that  $h(b) > b$ . If  $x > b$ , then by choice of  $b$ ,  $h(x) = x$ ; but since  $h$  is isotone,  $h(x) \geq h(b)$ , so  $x \geq h(b)$ . Thus  $b$  is an irreducible, because  $B_{>b}$  has the least element  $h(b)$ . That is a contradiction, since  $B$  is a core. Therefore  $f \circ g = \text{id}_B$ . By the same argument,  $g \circ f = \text{id}_A$ , so  $A$  is isomorphic to  $B$ .  $\square$

Note that the theorem above shows that the core of a poset is unique up to isomorphism.

By Theorem 3.10, a finite poset is contractible in the ideal topology if and only if its core is a single point. In the literature (e.g. [Ri]) such a poset is said to be dismantlable by irreducibles. We will shorten the term to dismantlable, since we will not discuss any other form of dismantlability. Due to the equivalence of dismantlability and contractibility, results about dismantlable posets are sometimes immediate consequences of

easy topological results, such as Propositions 3.11 and 3.13 below.

**3.11 Proposition:** The product of two spaces is contractible if and only if both of the factors are contractible.

Proof: ( $\Rightarrow$ ) If  $X \times Y$  is contractible, there is a map  $C : X \times Y \times I \rightarrow X \times Y$  such that  $C(x,y,0) = \langle x,y \rangle$  and  $C(x,y,1) = \langle x_0,y_0 \rangle$ , for some  $x_0$  in  $X$  and  $y_0$  in  $Y$ . We show that  $X$  is contractible; the other part is analogous. Define  $H : X \times I \rightarrow X$  by  $H(x,t) = \text{pr}_1(C(x,y_0,t))$ , where  $\text{pr}_1$  is projection onto the first coordinate. Check that  $H$  is continuous,  $H(x,0) = x$ , and  $H(x,1) = x_0$ .

( $\Leftarrow$ ) Suppose  $C : X \times I \rightarrow X$  and  $D : Y \times I \rightarrow Y$  are contractions of  $X$  and  $Y$ , respectively. Define  $H : X \times Y \times I \rightarrow X \times Y$  by  $H(x,y,t) = \langle C(x,t), D(y,t) \rangle$ , and check that  $H$  is a contraction of  $X \times Y$ .  $\square$

**3.12 Corollary** [D-R, Lemma 5]: The direct product of two finite posets is dismantlable if and only if both of the factors are dismantlable.  $\square$

Suppose that  $A$  is a subspace of  $X$  and  $j : A \rightarrow X$  is the inclusion map. If there exists a map

$r : X \rightarrow A$  such that  $r \circ j = \text{id}_A$ , then we say that  $A$  is a retract of  $X$ , and  $r$  is a retraction.

**3.13 Proposition:** A retract of a contractible space is contractible.

Proof: Suppose that  $C : X \times I \rightarrow X$  is a contraction,  $r : X \rightarrow A$  is a retraction, and  $j : A \rightarrow X$  is the inclusion. Then the composite  $r \circ C \circ (j \times \text{id}_I)$  is a contraction of  $A$ . □

**3.14 Corollary** [D-P-R, Lemma 5]: A retract of a dismantlable poset is dismantlable. □

A poset  $P$  has the fixed point property (FPP) if every isotone map  $f : P \rightarrow P$  has a fixed point.

**3.15 Proposition** [Ri, Prop. 1]: If  $P$  is a finite poset and  $a$  is an irreducible of  $P$ , then  $P$  has FPP if and only if  $P \setminus \{a\}$  has FPP.

Proof: Without loss of generality, assume that  $P_{<a}$  has the least element  $b$ .

( $\Rightarrow$ ) Let  $f : P \setminus \{a\} \rightarrow P \setminus \{a\}$  be any isotone map. Extend  $f$  to a map  $f' : P \rightarrow P$  by letting  $f'(a) = f(b)$ . Check



that  $f'$  is isotone. By hypothesis,  $f'$  has a fixed point  $x$ , which is unequal to  $a$  and must therefore be a fixed point of  $f$ .

( $\Leftarrow$ ) Let  $g : P \rightarrow P$  be an isotone map. Define  $g' : P \setminus \{a\} \rightarrow P \setminus \{a\}$  by  $g'(x) = \begin{cases} b & \text{if } g(x) = a, \\ g(x) & \text{otherwise.} \end{cases}$

Check that  $g'$  is isotone. By hypothesis,  $g'$  has a fixed point. Now either we have found a fixed point for  $g$ , or else  $g(b) = a$ . In the latter case, we can iterate  $g$  to find a fixed point, since  $g(b) > b$  and  $P$  is finite.  $\square$

3.16 Corollary: For finite posets, FPP depends only on ideal homotopy type.  $\square$

## Chapter 4. Geometric Realization

Although the ideal topology is theoretically appealing, its usefulness is limited. For the very reason that no information is lost in passing from the poset to its space, no simplification is achieved. Furthermore, many theorems and techniques in topology apply only to "nice" spaces, where "nice" does not include the ideal topology of a poset. Finally, one sometimes wishes to consider antitone maps of posets, which are (in general) discontinuous in the ideal topology. Therefore, in this chapter, we consider another way of associating a topological space to a poset, which does involve "nice" spaces.

For each poset  $P$ , let  $\Delta(P)$  denote the set of finite nonempty chains of  $P$ , called the order complex of  $P$ . This is a simplicial complex. Then for each simplicial complex  $K$ , there is a well known construction [Sp, §3-1] of a topological space  $|K|$  called the geometric realization. Furthermore, these constructions are functorial. That is, an isotone map induces a simplicial map, which in turn induces a continuous map, in a way which commutes with composition of maps.

Since geometric simplicial complexes (i.e. geometric realizations of abstract simplicial complexes) have been a favorite object of study in algebraic topology, it is

generally much more fruitful to study  $|\Delta(P)|$  than to study the ideal topology of  $P$ . Therefore, from now on,  $|\Delta(P)|$  will be the space of choice to associate to a poset  $P$ . For example, if we say that  $P$  is contractible, we mean that  $|\Delta(P)|$  is contractible.

Note that  $\Delta(P)$  depends only upon the comparability relation of  $P$ . The most important consequence is that  $\Delta(P^*) = \Delta(P)$ . Also note that antitone maps induce simplicial maps of order complexes, just as isotone maps do. See [G, Chapter 5] for more information on comparability relations.

The Euler characteristic  $\chi(K)$  of a simplicial complex  $K$  is the number of even dimensional (odd cardinality) simplices minus the number of odd dimensional simplices. Since  $\Delta(P)$  does not include the empty chain, we have

$$\mu(P) = \chi(\Delta(P)) - 1, \quad (1)$$

for any finite poset  $P$ . There is a standard result of algebraic topology [Sp, 4.4.15] that the Euler characteristic of a complex can be computed from its homology groups.

Thus

$$\mu(P) = \sum_n (-1)^n \text{rank } \tilde{H}_n(\Delta(P)) \quad (2)$$

where  $\tilde{H}_n(\cdot)$  represents reduced simplicial homology with integer or field coefficients. This relationship between Möbius numbers and homology is one of the main reasons for interest in the geometric realizations of posets.

A simplicial complex is a cone if there is some vertex  $v$  such that for every simplex  $\sigma$ ,  $\{v\} \cup \sigma$  is also a simplex. In particular, if  $P$  has some element which is comparable to every other element, then  $\Delta(P)$  is a cone. It is well known that any realization of a cone is contractible. Since a homotopy equivalence induces homology isomorphism, any contractible space is acyclic (has trivial homology groups). It follows from equation (2) that if  $P$  is acyclic, then  $\mu(P) = 0$ . Thus we have another way of seeing Proposition 1.7.

The order complex is also useful in the study of the fixed point problem for finite posets. Baclawski and Björner [B-B 1] have adapted the Lefschetz fixed point theorem of simplicial homology to isotone and antitone maps from a poset to itself. For instance, their results imply that a finite acyclic poset has the fixed point property.

There is a surprising relationship between geometric realization and the ideal topology. McCord [Mc] has shown that there is a natural continuous map from  $|\Delta(P)|$  to  $P$  in the ideal topology, and this map is a weak homotopy equivalence. That means that the two spaces have the same

homotopy groups, homology groups, and cohomology groups.

The rest of this chapter will involve constructing homeomorphisms between the realizations of various posets.

Therefore, we will give an explicit construction for

$|\Delta(P)|$ : Let  $|\Delta(P)|$  be the set of formal linear combinations  $\sum \{t_a a : a \in P\}$  such that each  $t_a$  is nonnegative,  $\sum \{t_a : a \in P\} = 1$ , and the subposet  $\{a : t_a \neq 0\}$  is a finite nonempty chain. For each simplex  $\sigma$  of  $\Delta(P)$ , there is a corresponding closed simplex  $|\sigma|$  consisting of the linear combinations  $\sum \{t_a a : a \in \sigma\}$  such that each  $t_a$  is nonnegative and  $\sum \{t_a : a \in \sigma\} = 1$ . Note that  $|\sigma|$  can be regarded as a compact, convex subset of some finite-dimensional Euclidean space. We give  $|\Delta(P)|$  the finest topology such that every closed simplex has the usual Euclidean topology. One consequence of this definition is that a function on  $|\Delta(P)|$  is continuous if and only if it is continuous when restricted to each closed simplex. An isotone or antitone map  $f : P \rightarrow Q$  induces a continuous map  $f : |\Delta(P)| \rightarrow |\Delta(Q)|$  which sends  $\sum t_a a$  to  $\sum t_a f(a)$ .

The join of posets, defined in Chapter 2, bears an obvious relation to the join of simplicial complexes:

$$\Delta(P*Q) = \Delta(P) * \Delta(Q). \quad (3)$$

There is also a join operation on topological spaces. If  $X$  and  $Y$  are spaces, then  $X * Y$  denotes the quotient space of  $X \times Y \times I$  determined by the equivalence relation which identifies  $\langle x, y_1, 0 \rangle$  with  $\langle x, y_2, 0 \rangle$  and  $\langle x_1, y, 1 \rangle$  with  $\langle x_2, y, 1 \rangle$  for all  $x, x_1, x_2$  in  $X$  and all  $y, y_1, y_2$  in  $Y$ . (By convention,  $\emptyset * X = X * \emptyset = X$ .) Then there is a natural homeomorphism

$$|K * L| \cong |K| * |L| \quad (4)$$

for simplicial complexes  $K$  and  $L$ .

Actually, in order that (4) be a homeomorphism, we have to be careful about the topology on  $|K| * |L|$ . For any Hausdorff space, the associated compactly generated topology is the finest topology which determines the same compact subspaces as the original topology. The realization of a simplicial complex is automatically compactly generated. But in the definition of  $|K| * |L|$ , we need to use the compactly generated topology on  $|K| \times |L| \times I$ , which may be strictly finer than the product topology if  $K$  and  $L$  are both infinite. See [Wh, §I-4] for more information on compactly generated spaces.

From now on, we will denote the realization of a poset  $P$  by simply  $|P|$ , rather than  $|\Delta(P)|$ . Then by combining (3) and (4), we obtain a natural homeomorphism

$$|P*Q| \cong |P| * |Q|. \quad (5)$$

Now we consider the realization of a direct product. The product  $|P| \times |Q|$  can be regarded as a CW complex, with cells of the form  $|\sigma| \times |\tau|$ , where  $\sigma \in \Delta(P)$  and  $\tau \in \Delta(Q)$ . In order that  $|P| \times |Q|$  be a CW complex in the usual sense [Sp, §7-6], we again have to use the compactly generated topology instead of the usual product topology. With that proviso, we have the following result, which was stated but not proved in [Q 1].

**4.1 Proposition:** For posets  $P, Q$ , there is a natural homeomorphism

$$|P \times Q| \cong |P| \times |Q|.$$

Proof: Define a function  $\phi : |P \times Q| \rightarrow |P| \times |Q|$  by  $\sum t_{a,b} \langle a, b \rangle \rightarrow \langle \sum t_{a,b} a, \sum t_{a,b} b \rangle$ . Naturality is clear. If  $\sigma \in \Delta(P \times Q)$ , then  $\phi(|\sigma|) \subset |\text{pr}_1 \sigma| \times |\text{pr}_2 \sigma|$ , where  $\text{pr}_1$  and  $\text{pr}_2$  are the coordinate projections. Since  $|\sigma|$  and  $|\text{pr}_1 \sigma| \times |\text{pr}_2 \sigma|$  have the usual Euclidean topology, it is easy to see that the restriction of  $\phi$  to  $|\sigma|$  is continuous. It follows that  $\phi$  is continuous.

Most of the work lies in showing that  $\phi$  is bijective. Let  $x \in |P|$  and  $y \in |Q|$ ; we will show that there

is a unique  $z \in |P \times Q|$  such that  $\phi(z) = \langle x, y \rangle$ . We can write  $x$  as  $r_1x_1 + r_2x_2 + \dots + r_mx_m$ , for some chain  $x_1 < x_2 < \dots < x_m$  of  $P$  and some nonnegative coefficients  $r_i$  with  $\sum r_i = 1$ . Similarly,  $y = s_1y_1 + s_2y_2 + \dots + s_ny_n$ , where  $y_1 < \dots < y_n$ ,  $s_i \geq 0$ , and  $\sum s_i = 1$ . Suppose  $z$  has the form  $\sum t_{a,b} \langle a, b \rangle$ . Whenever  $a \notin \{x_1, x_2, \dots, x_m\}$  or  $b \notin \{y_1, y_2, \dots, y_n\}$ , we must have  $t_{a,b} = 0$  if it is to be possible that  $\phi(z) = \langle x, y \rangle$ . Therefore we can write

$$z = \sum_{i=1}^m \sum_{j=1}^n t_{i,j} \langle x_i, y_j \rangle,$$

so  $\phi(z) = \langle \sum t_{i,j}x_i, \sum t_{i,j}y_j \rangle$ . Thus we want  $\sum t_{i,j}x_i = \sum r_ix_i$  and  $\sum t_{i,j}y_j = \sum s_jy_j$ . It is necessary that for each  $i$ ,

$$\sum_j t_{i,j} = r_i,$$

and for each  $j$ ,

$$\sum_i t_{i,j} = s_j.$$

We also require that the subposet  $\{\langle x_i, y_j \rangle : t_{i,j} \neq 0\}$  be a chain in  $P \times Q$ .



We proceed inductively. If  $r_m = 0$ , then we must have  $t_{m,j} = 0$  for all  $j$ ; then we have reduced to a smaller case. We make a similar reduction if  $s_n = 0$ . On the other hand, if  $r_m$  and  $s_n$  are both nonzero, then we must have  $t_{m,n} \neq 0$ . In fact, since the chain  $\{\langle x_i, y_j \rangle : t_{i,j} \neq 0\}$  cannot have more than one member in both the last row and the last column,  $t_{m,n}$  must equal the smaller of  $r_m$  and  $s_n$ . Say  $r_m$  is smaller. Then set  $t_{m,n} = r_m$ ,  $t_{m,j} = 0$  for  $j < n$ , and replace  $s_n$  by  $s_n - r_m$ ; then we have reduced to an  $(m-1)$  by  $n$  matrix. This procedure will yield one and only one solution.

The inversion algorithm above is continuous on each cell  $|\sigma| \times |\tau|$  of  $|P| \times |Q|$ . Since we use the compactly generated topology on  $|P| \times |Q|$ , it follows that the inversion algorithm is continuous on  $|P| \times |Q|$ . Therefore  $\phi$  is a homeomorphism.  $\square$

Example: Given a point

$w = \langle .1x_1 + .4x_2 + .2x_3 + .3x_4, .5y_1 + .3y_2 + .2y_3 \rangle$  in  $|\{x_1 < x_2 < x_3 < x_4\}| \times |\{y_1 < y_2 < y_3\}|$ , find the unique  $z$  in  $|\{x_1 < x_2 < x_3 < x_4\}| \times |\{y_1 < y_2 < y_3\}|$  such that  $\phi(z) = w$ .

Write a matrix to hold the coefficients for  $z$ , with the desired row and column sums written along the sides.

.3			
.2			
.4			
.1			
	.5	.3	.2

Since  $.2 < .3$ , put  $.2$  in the upper right corner, and zeroes in the rest of the last column. Also, subtract  $.2$  from the last row sum.

.1			.2
.2			0
.4			0
.1			0
	.5	.3	

Now put  $.1$  in the upper right empty slot, and zeroes in the rest of the last row. Also, subtract  $.1$  from the last column sum.

	0	.1	.2
.2			0
.4			0
.1			0
	.5	.2	

Here the last row sum equals the last column sum. Whichever way we break the tie, the result is the same: put  $.2$  in the upper right

empty slot, and zeroes in the rest of the last row and column.

	0	.1	.2
	0	.2	0
.4		0	0
.1		0	0
	.5		

With only one empty column, the solution is obvious:

	0	.1	.2
	0	.2	0
.4		0	0
.1		0	0

□

Recall that  $sd(P)$  denotes the poset of finite chains of  $P$ . The order complex of  $sd(P)$  is the first barycentric subdivision of the order complex of  $P$ . It is well known that any complex has the same space as its barycentric subdivision [Sp, 3.3.9], so in particular  $|sd(P)|$  is homeomorphic to  $|P|$ . However, it has not been previously observed that  $|Int(P)|$  is also homeomorphic to  $|P|$ .

4.2 Proposition: There are natural homeomorphisms

$$|sd(P)| \cong |P| \cong |Int(P)|.$$

Proof: Define a map  $|sd(P)| \longrightarrow |P|$  by defining the vertex map

$$(x_1 < x_2 < \cdots < x_n) \longrightarrow \sum (1/n)x_i$$

and extending linearly. Similarly define a map

$$|\text{Int}(P)| \longrightarrow |P| \quad \text{using the vertex map } [a,b] \longmapsto \frac{1}{2}a + \frac{1}{2}b.$$

Check that these maps are homeomorphisms, using an elimination algorithm similar to the one used in 4.1.  $\square$

We are now in a position to derive homeomorphisms describing all of the open intervals of  $P \times Q$  and  $\text{Int}(P)$ . This will prove useful in Chapter 9.

For any poset  $P$ , let  $P_{\hat{0}}$  denote the poset formed by adjoining a new least element  $\hat{0}$ . Note that the complex  $\Delta(P_{\hat{0}}) = \{\hat{0}\} * \Delta(P)$  is a cone. The suspension  $S(X)$  of a space or complex  $X$  is formed by joining  $X$  with a two-point discrete space.

Quillen [Q 1] gives the homeomorphism

$$|P_{\hat{0}} \times Q_{\hat{0}} \setminus \{\hat{0} \times \hat{0}\}| \cong |P * Q|, \quad (6)$$

which can be justified as follows. Using Proposition 4.1 and formula (5),

$$\begin{aligned}
|P_0^\wedge \times Q_0^\wedge \setminus \hat{0} \times \hat{0}| &= |P \times Q_0^\wedge \cup P_0^\wedge \times Q| \\
&= |P| \times |Q_0^\wedge| \cup |P_0^\wedge| \times |Q| \\
&= |P| \times (\hat{0} * |Q|) \cup (\hat{0} * |P|) \times |Q|.
\end{aligned}$$

This is homeomorphic to  $|P| * |Q|$ , as the following diagram indicates.

$$\begin{array}{ccc}
|P| \times \hat{0} & \xrightarrow{|P| \times (\hat{0} * |Q|)} & |P| \times |Q| \xrightarrow{(\hat{0} * |P|) \times |Q|} \hat{0} \times |Q| \\
|P| & \xrightarrow{\hspace{10em}} & |P| * |Q| \xrightarrow{\hspace{10em}} |Q|
\end{array}$$

Note that  $|P| * |Q|$  was defined as a quotient of  $|P| \times |Q| \times I$ , which does have the central cross section

$$|P| \times |Q| \times \{\frac{1}{2}\} \cong |P| \times |Q|.$$

There is a further, apparently new, canonical homeomorphism along these lines:

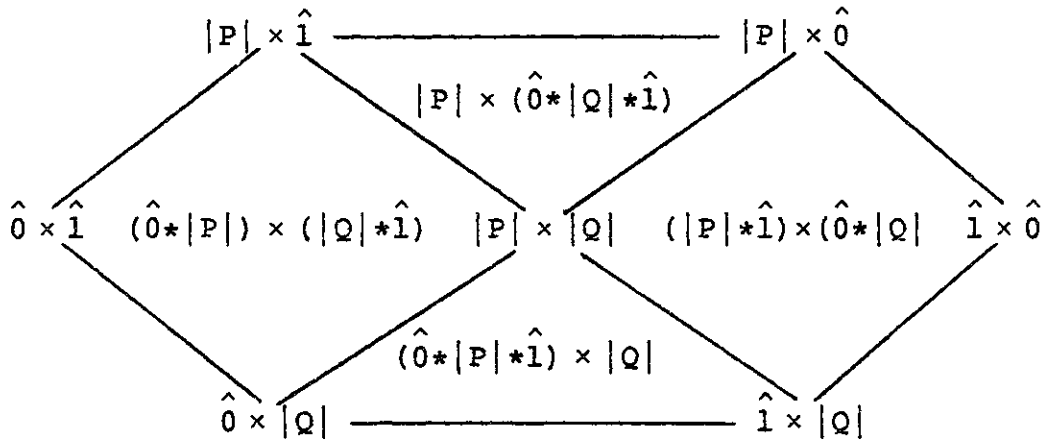
$$|\hat{P} \times \hat{Q} \setminus \{\hat{0} \times \hat{0}, \hat{1} \times \hat{1}\}| \cong S(|P| * |Q|). \quad (7)$$

To justify (7), we again start out using Proposition 4.1

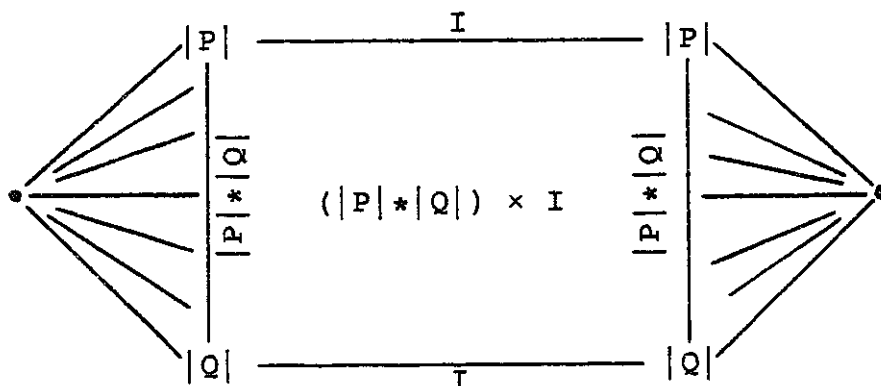
and formula (5). Let  $P^{\hat{1}}$  denote the poset formed by adjoining a new greatest element  $\hat{1}$  to  $P$ .

$$\begin{aligned}
 & |\hat{P} \times \hat{Q} \setminus \{\hat{0} \times \hat{0}, \hat{1} \times \hat{1}\}| \\
 &= |\hat{P} \times Q \cup P \times \hat{Q} \cup P_0^{\hat{1}} \cup Q^{\hat{1}} \cup P^{\hat{1}} \times Q_0| \\
 &= |\hat{P}| \times |Q| \cup |P| \times |\hat{Q}| \cup |P_0^{\hat{1}}| \times |Q^{\hat{1}}| \cup |P^{\hat{1}}| \times |Q_0| \\
 &= (\hat{0} * |P| * \hat{1}) \times |Q| \cup |P| \times (\hat{0} * |Q| * \hat{1}) \\
 &\quad \cup (\hat{0} * |P|) \times (|Q| * \hat{1}) \cup (|P| * \hat{1}) \times (\hat{0} * |Q|).
 \end{aligned}$$

Now we draw a diagram representing this space.



The picture above is canonically homeomorphic to the picture below, which is the suspension of  $|P| * |Q|$ .



Now we can describe the open intervals in a product poset.

4.3 Theorem: There are canonical homeomorphisms

- (a)  $|P \times Q| \cong |P| \times |Q|$ ,
- (b)  $|P \times Q_{>\langle a, b \rangle}| \cong |P_{>a}| * |Q_{>b}|$ ,
- (c)  $|P \times Q_{<\langle a, b \rangle}| \cong |P_{<a}| * |Q_{<b}|$ ,
- (d)  $(\langle a, b \rangle, \langle c, d \rangle) \cong \begin{cases} S(|(a, c)| * |(b, d)|) & \text{if } a \neq c, b \neq d, \\ |(b, d)| & \text{if } a = c, \\ |(a, c)| & \text{if } b = d, \end{cases}$

where all products and joins of spaces have the compactly generated topology.

Proof: (a) is Proposition 4.1. Part (b) is formula (6), and then (c) follows by standing on your head. Part (d) follows from equation (7), where we note that in order to write the closed interval  $[\langle a, b \rangle, \langle c, d \rangle] = [a, c] \times [b, d]$

in the form  $\hat{P} \times \hat{Q}$ , we must have  $a \neq c$  and  $b \neq d$ .  $\square$

Having analyzed the open intervals of  $P \times Q$ , we move on to the open intervals of  $\text{Int}(P)$ . There is an obvious way of regarding  $\text{Int}(P)$  as a filter (dual ideal) of  $P^* \times P$ , so the open interval  $([a,b],[c,d])$  of  $\text{Int}(P)$  is isomorphic to the interval  $\langle a,b \rangle, \langle c,d \rangle$  of  $P^* \times P$ . So by 4.3 (d), we have

$$|([a,b],[c,d])| \cong \begin{cases} S(|(c,a)| * |(b,d)|) & \text{if } a \neq c, b \neq d, \\ |(b,d)| & \text{if } a = c, \\ |(c,a)| & \text{if } b = d. \end{cases}$$

Also, the interval  $\text{Int}(P)_{>[a,b]}$  is isomorphic to  $(P^* \times P)_{<\langle a,b \rangle}$ , so by 4.3 (b)

$$|\text{Int}(P)_{>[a,b]}| \cong |P_{<a}| * |P_{>b}|.$$

Finally we consider  $\text{Int}(P)_{<[a,b]}$ , which is isomorphic to  $\text{Int}([a,b]) \setminus \{[a,b]\}$ . This is empty if  $a = b$ , so assume  $a \neq b$ . Note that



$$\begin{aligned}
 & |\text{Int}([a,b]) \setminus \{[a,b]\}| \\
 &= |\text{Int}([a,b))| \cup |\text{Int}((a,b])|.
 \end{aligned}$$

By Proposition 4.2 and formula (5),

$$|\text{Int}([a,b))| \cong a * |(a,b)| \quad \text{and}$$

$$|\text{Int}((a,b])| \cong b * |(a,b)|.$$

Furthermore,

$$\begin{aligned}
 & |\text{Int}([a,b))| \cap |\text{Int}((a,b])| \\
 &= |\text{Int}((a,b))|,
 \end{aligned}$$

which is homeomorphic to  $|(a,b)|$ . When two cones over a space  $X$  have  $X$  as their intersection, then their union is the suspension of  $X$ :

$$|\text{Int}([a,b]) \setminus \{[a,b]\}| \cong S(|(a,b)|).$$

We collect these results for easy reference:

4.4 Theorem: There are canonical homeomorphisms

$$\begin{aligned}
 \text{(a)} \quad & |\text{Int}(P)| \cong |P|, \\
 \text{(b)} \quad & |\text{Int}(P)_{>[a,b]}| \cong |P_{<a}| * |P_{>b}|, \\
 \text{(c)} \quad & |\text{Int}(P)_{<[a,b]}| \cong \begin{cases} S(|(a,b)|) & \text{if } a \neq b \\ \emptyset & \text{if } a = b, \end{cases} \\
 \text{(d)} \quad & |([a,b], [c,d])| \cong \begin{cases} S(|(c,a)| * |(b,d)|) & \text{if } a \neq c, b \neq d, \\ |(c,a)| & \text{if } b = d, \\ |(b,d)| & \text{if } a = c, \end{cases}
 \end{aligned}$$

where joins of spaces have the compactly generated topology.

□

## Chapter 5: Homology and Homotopy Type of Posets

In this chapter, we will introduce some techniques for proving theorems about the homology and homotopy type of posets. As a first application, we will prove another generalization of the cross-cut theorem, which we first met in Chapter 1.

A nonempty space  $X$  is defined to be  $n$ -connected if, for every  $m \leq n$ , every continuous map of the  $m$ -sphere  $S^m$  into  $X$  is homotopic to a constant map. Then 0-connected means path connected, and 1-connected means simply connected. If we think of  $S^m$  as the boundary of an  $(m+1)$ -cell  $B^{m+1}$ , then a map  $f : S^m \rightarrow X$  is homotopic to a constant map if and only if  $f$  can be continuously extended across  $B^{m+1}$ .

For any simplicial complex  $K$ , the  $n$ -skeleton  $K^{(n)}$  is the subcomplex consisting of all simplices of dimension at most  $n$ .

Suppose  $K$  is a simplicial complex and  $X$  is a space. A carrier from  $K$  to  $X$  is a function  $B$  which sends simplices of  $K$  to subspaces of  $X$ , such that  $\tau \subset \sigma$  implies  $B(\tau) \subset B(\sigma)$ . An  $n$ -connected carrier is a carrier  $B$  such that  $B(\sigma)$  is  $n$ -connected for every simplex  $\sigma$  of  $K$ . A continuous function  $f : |K| \rightarrow X$  is carried by  $B$  if, for each simplex  $\sigma$  of  $K$ ,  $f(|\sigma|) \subset B(\sigma)$ .

**5.1 Homotopy Carrier Theorem:** If  $B$  is an  $n$ -connected carrier from  $K$  to  $X$ , then

- (1) there exists a map  $|K^{(n+1)}| \rightarrow X$  carried by  $B$ , and
- (2) [La] any two maps  $|K^{(n)}| \rightarrow X$  carried by  $B$  are homotopic.

Proof (1): To construct a map  $g : |K^{(n+1)}| \rightarrow X$  carried by  $B$ , we proceed by induction on dimensions.

For each vertex  $v$  in  $K^{(0)}$ , let  $g(v)$  be an arbitrarily chosen point of  $B(v)$ . Now we have a map  $g : |K^{(0)}| \rightarrow X$  carried by  $B$ .

Suppose we have a continuous map  $g : |K^{(i)}| \rightarrow X$  carried by  $B$ , where  $i \leq n$ . Suppose  $\tau$  is an  $(i+1)$ -simplex of  $K$ . For each proper face  $\sigma$  of  $\tau$ , we know that  $g(|\sigma|) \subset B(\sigma) \subset B(\tau)$ , so  $g(|\partial\tau|) \subset B(\tau)$ . Since  $|\tau|$  is an  $(i+1)$ -cell and  $B(\tau)$  is  $n$ -connected, we can extend  $g$  over  $|\tau|$  in such a way that  $g(|\tau|) \subset B(\tau)$ . All simplices have disjoint interiors, so we can do this for all of the  $(i+1)$ -simplices. And recall that a map on a simplicial complex is continuous when it is continuous on each closed simplex. Therefore we have constructed a continuous map  $g : |K^{(i+1)}| \rightarrow X$  carried by  $B$ , which completes the induction.

(2): Suppose  $f, g : |K^{(n)}| \longrightarrow X$  are both carried by  $B$ . We will inductively construct a homotopy  $H : |K^{(n)}| \times I \longrightarrow X$  from  $f$  to  $g$ . Note that although  $|K^{(n)}| \times I$  has no natural simplicial structure, it is a CW complex with cells of the form  $|\sigma| \times I$ ,  $|\sigma| \times \{0\}$ , and  $|\sigma| \times \{1\}$ .

Define  $H : |K^{(n)}| \times \{0,1\} \longrightarrow X$  by  $H(x,0) = f(x)$  and  $H(x,1) = g(x)$ . If  $v$  is a vertex of  $K$ , then  $f(v) \in B(v)$  and  $g(v) \in B(v)$ , so since  $B(v)$  is path connected,  $H$  can be continuously extended across  $\{v\} \times I$ .

The inductive hypothesis is that  $H$  is defined continuously on  $|K^{(n)}| \times \{0,1\} \cup |K^{(i)}| \times I$ , and  $H(|\sigma| \times I) \subset B(\sigma)$  for each  $\sigma$  in  $K^{(i)}$ . Continue as in part (1). □

If  $R$  is a commutative ring with identity, let  $\tilde{\mathcal{C}}(K;R)$  denote the augmented simplicial chain complex of  $K$  with coefficients in  $R$ , and let  $\tilde{\mathcal{P}}(X;R)$  denote the augmented singular chain complex of  $X$  with coefficients in  $R$ . Then Theorem 5.1 has the following homology analog, which is essentially the familiar acyclic carrier theorem [E-S, Theorem VI-5.7].

**5.2 Homology Carrier Theorem:** If  $R$  is a commutative ring with identity and  $B$  is a carrier from  $K$  to  $X$  such that

$\tilde{H}_i(B(\sigma); R) = 0$  for all  $i \leq n$  and all  $\sigma$  in  $K$ , then

- (1) there exists a chain map  $\tilde{\mathcal{C}}(K^{(n+1)}; R) \rightarrow \tilde{\mathcal{F}}(X; R)$  carried by  $B$ , and
- (2) any two chain maps  $\tilde{\mathcal{C}}(K^{(n)}; R) \rightarrow \tilde{\mathcal{F}}(X; R)$  carried by  $B$  are chain-homotopic.  $\square$

Note that the carrier theorems are valid for  $n = \infty$ . Indeed, that is the only case we will need until Chapter 9!

A contractible carrier is, of course, a carrier  $B$  such that  $B(\sigma)$  is contractible for every  $\sigma$ . Note that a contractible space is  $\infty$ -connected ( $n$ -connected for all  $n$ ): If  $C$  is a contraction of  $X$  and  $f : S^n \rightarrow X$  is a continuous map, then  $C \circ (f \times \text{id}_I)$  is a homotopy of  $f$  to a constant map. In particular, a contractible carrier is an  $\infty$ -connected carrier. (Conversely, it follows easily from 5.1 that any  $\infty$ -connected simplicial complex is contractible.)

Recall that if a poset has a greatest element or a least element, then its order complex is a cone, hence its realization is contractible.

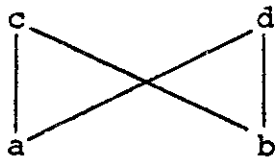
Proposition 3.3 said that comparable isotone maps are homotopic with respect to the ideal topology. The following is an analog for geometric realization of posets.

**5.3 Proposition [Q 1, Proposition 1.3]:** If  $P$  and  $Q$  are posets,  $f, g \in \text{Hom}(P, Q)$ , and  $f \leq g$ , then  $f$  and  $g$

induce homotopic maps from  $|P|$  to  $|Q|$ .

Proof [A. Björner, private communication]: Define a carrier  $B$  from  $\Delta(P)$  to  $|Q|$  by  $B(\sigma) = |f(\sigma) \cup g(\sigma)|$ . Since  $f \leq g$ ,  $f(\sigma) \cup g(\sigma)$  always has a least element, so  $B$  is a contractible carrier. Since  $f$  and  $g$  are both carried by  $B$ , Theorem 5.1 says that  $f$  and  $g$  are homotopic.  $\square$

In contrast to the situation for the ideal topology, 5.3 has no converse, even for finite posets. Consider the poset  $P$  below.



Let  $f : P \rightarrow P$  be the map which exchanges  $a$  and  $b$ , and exchanges  $c$  and  $d$ . The realization of  $P$  is a circle. The realization of  $f$  is a 180-degree rotation, which is homotopic to the identity map. However, it is easy to see that the identity map of  $P$  is not comparable to any other map.

It is immediate from 5.3 that Galois maps induce homotopy equivalence. Here is a particularly useful special case of Galois maps: An isotone map  $\phi : P \rightarrow P$

is called a closure map if  $\phi \geq \text{id}$  and  $\phi \circ \phi = \phi$ . Elements in the range of  $\phi$  are said to be closed. If  $K$  is the subposet of closed elements of  $\phi$ , if  $j : K \rightarrow P$  is the inclusion, and if  $\phi_0 : P \rightarrow K$  is obtained from  $\phi$  by restriction, then we see that  $j \circ \phi_0 = \phi \geq \text{id}_P$  and  $\phi_0 \circ j = \text{id}_K$ . So by 5.3,  $\phi_0$  induces homotopy equivalence between  $P$  and  $K$ .

A poset will be called join-contractible (via  $y$ ) if there is some element  $y$  such that the join  $x \vee y$  exists for every  $x$ .

**5.4 Corollary:** Every join-contractible poset is contractible.

Proof: The map  $x \mapsto x \vee y$  is a closure map, and the set of closed elements has a least element,  $y$ .  $\square$

Recall that if  $f : P \rightarrow Q$  is an isotone map, the fibers of  $f$  are the subposets of the form  $f^{-1}(Q_{\leq y})$ . We saw in Chapter 2 that if all of the fibers have Möbius number zero, then  $\mu(P) = \mu(Q)$ . Here are two analogous results.

**5.5 Theorem** [Q 2, Theorem A; Q 1, Proposition 1.6]: If  $f : P \rightarrow Q$  is an isotone map, all of whose fibers are



contractible, then  $f$  induces homotopy equivalence.

Proof: The function  $\sigma \mapsto |f^{-1}(Q_{\leftarrow \max \sigma})|$  is a contractible carrier from  $\Delta(Q)$  to  $|P|$ . So by part (1) of 5.1, there exists a continuous map  $g : |Q| \rightarrow |P|$  such that  $g(|\sigma|) \subset |f^{-1}(Q_{\leftarrow \max \sigma})|$  for every  $\sigma$  in  $\Delta(Q)$ . We will show that  $g$  is a homotopy inverse for  $f$ .

If  $\sigma \in \Delta(Q)$ , then  $g(|\sigma|) \subset |f^{-1}(Q_{\leftarrow \max \sigma})| = f^{-1}(|Q_{\leftarrow \max \sigma}|)$ , so  $f \circ g(|\sigma|) \subset |Q_{\leftarrow \max \sigma}|$ . But also  $|\sigma| \subset |Q_{\leftarrow \max \sigma}|$ , so the function  $\sigma \mapsto |Q_{\leftarrow \max \sigma}|$  is a contractible carrier which carries both  $f \circ g$  and  $\text{id}_Q$ . Then by part (2) of 5.1,  $f \circ g$  is homotopic to  $\text{id}_Q$ .

If  $\sigma \in \Delta(P)$ , then  $f(\sigma) \in \Delta(Q)$ , so  $g \circ f(|\sigma|) = g(|f(\sigma)|) \subset |f^{-1}(Q_{\leftarrow \max f(\sigma)})|$ . But also  $|\sigma| \subset |f^{-1}(Q_{\leftarrow \max f(\sigma)})|$ , so the function  $\sigma \mapsto |f^{-1}(Q_{\leftarrow \max f(\sigma)})|$  is a contractible carrier which carries  $g \circ f$  and  $\text{id}_P$ . By part (2) of 5.1,  $g \circ f$  is homotopic to  $\text{id}_P$ . □

**5.6 Theorem:** If  $f : P \rightarrow Q$  is an isotone map, all of whose fibers are acyclic with respect to  $R$ , then  $f_* : H_*(P; R) \rightarrow H_*(Q; R)$  is an isomorphism.

Proof: Proceed as in the proof of 5.5, using 5.2 instead of 5.1. □

Note that if  $f : P \rightarrow Q$  is an isotone map such that all of the subposets  $f^{-1}(Q_{\geq y})$  are contractible, then  $f$  is a homotopy equivalence. This follows from 5.5 by standing on your head.

Theorems such as 5.5 can be rephrased to resemble the additive covering results of Chapter 1. Let  $K$  be a simplicial complex. Let  $\mathcal{A}$  be a collection of contractible subcomplexes which cover  $K$ . We order  $\mathcal{A}$  by inclusion. The question is: When is  $\mathcal{A}$  homotopy equivalent to  $K$ ? Here's an answer which is reminiscent of Proposition 1.2:

If  $\mathcal{A}_\sigma$  has a least element for each simplex  $\sigma$  of  $K$ , then  $\mathcal{A}$  is homotopy equivalent to  $K$ .

To see why the statement above follows from Theorem 5.5, consider the isotone map which sends each  $\sigma$  to the least element of  $\mathcal{A}_\sigma$ . The fibers of this map are precisely the members of  $\mathcal{A}$ , which are given to be contractible.

Now we generalize Theorem 5.5 to the case of an ideal relation between two posets, thus obtaining a topological analog to Theorem 2.1.

**5.7 Theorem** [Q 1, Corollary 1.8]: If  $R$  is an ideal relation between  $P$  and  $Q$ , and if the subposets  $R(x)$  and  $R^{-1}(y)$  are contractible for each  $x$  in  $P$  and  $y$  in  $Q$ , then  $P$  is homotopy equivalent to  $Q$ .

Proof: Let  $i : P \rightarrow P +_R Q$  and  $j : Q \rightarrow (P +_R Q)^*$  be the inclusion maps, which are isotone. The fibers of  $i$  are the subposets  $R^{-1}(y)$  and  $P_{<x}$ . By hypothesis,  $R^{-1}(y)$  is contractible, and  $P_{<x}$  is a cone. Therefore, by 5.5,  $i$  is a homotopy equivalence. We can similarly apply 5.5 to  $j$ .  $\square$

There is a covering version of 5.7, which is obtained by considering the membership relation between  $K$  and  $\mathcal{A}^*$ :

If  $\mathcal{A}_\sigma$  is a contractible poset for each  $\sigma$  in  $K$ , then  $\mathcal{A}$  is homotopy equivalent to  $K$ .

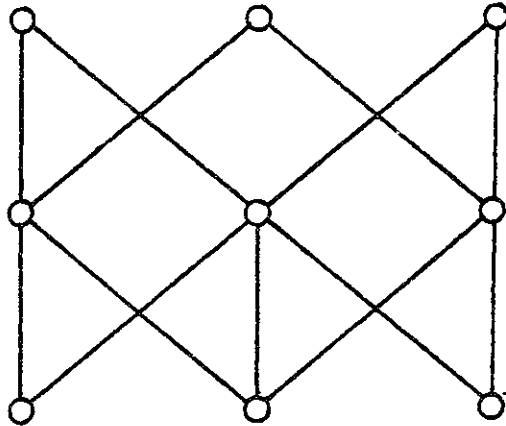
Of course, there is also a homology version of 5.7, which follows from 5.6. I do not bother to state it.

Recall from Chapter 3 that an element  $x$  of a poset  $P$  is an irreducible of  $P$  if  $P_{<x}$  has a greatest element or if  $P_{>x}$  has a least element. Björner showed [Bj 1] that if  $x$  is an irreducible of  $P$ , then the inclusion of  $P \setminus \{x\}$  into  $P$  is a homotopy equivalence. This fact has an obvious generalization:

**5.8 Proposition:** If  $P_{<x}$  or  $P_{>x}$  is contractible, then the inclusion of  $P \setminus \{x\}$  into  $P$  is a homotopy equivalence.

Proof: If  $P_{<x}$  is contractible, apply 5.5. The other case follows by standing on your head. □

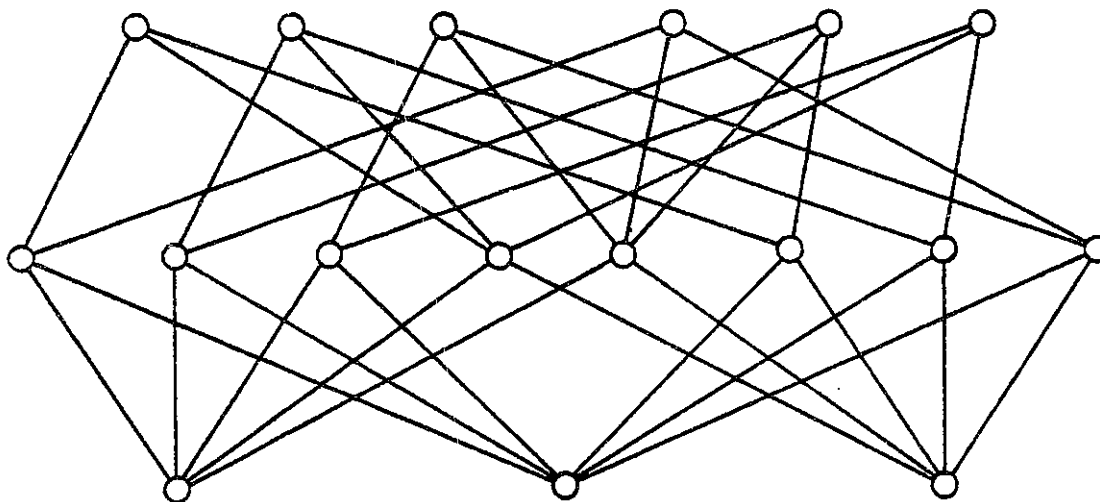
Repeated applications of Proposition 5.8 can sometimes be used to determine the homotopy type of a given poset. For example, consider the poset below, which was given by Rival [Ri] as an example of a poset which has no irreducibles but which has the fixed point property.



Proposition 5.8 can be used to remove the lower left and lower right elements. What remains has a least element, so the poset above is contractible.

Unfortunately, there is no analog of Stong's theorem 3.10 for geometric realization. The figure below shows a poset which is contractible, but such that none of the subposets  $P_{<x}$  or  $P_{>x}$  is contractible. This example was obtained by subdividing the "dunce hat", which is a familiar example of a CW complex which is contractible but not

collapsible [Z].



By the way, a finite complex is "collapsible" if it can be contracted by repeatedly collapsing free faces of maximal cells. I won't define those terms precisely, but here's the general idea: Picture a hollow tetrahedron with three solid, rigid faces and one face which is a rubber membrane. This model represents a 3-cell which is attached to the rest of some complex along the solid faces, and the membrane represents a free face. Now pump out the air from the interior of the tetrahedron, and watch what happens.

We now discuss another version of the cross-cut theorem, which we first encountered in Chapter 1. Recall that if  $A$  is a subset of  $P$ , then  $P(A)$  denotes the set of all elements of  $P$  comparable with every element of  $A$ . A subset  $C$  of  $P$  is a cutset of  $P$  if, for every  $\sigma$  in  $\Delta(P)$ ,  $P(\sigma) \cap C \neq \emptyset$ .

Given a subset  $C$  of  $P$ , let  $\Gamma(P,C)$  denote the poset of those finite nonempty subsets  $A$  of  $C$  such that  $P(A) \neq \emptyset$ . Rota showed [Ro, p. 352] that if  $\hat{P}$  is a finite lattice and if  $C$  is a cutset which is also an antichain (a cross-cut), then  $\mu(P) = \mu(\Gamma(P,C))$ . Here is a generalization due to Björner [Bj 1].

We say that a cutset  $C$  of  $P$  is coherent if, for every  $A$  in  $\Gamma(P,C)$ ,  $P(A)$  is a cone. It is easy to check that if  $\hat{P}$  is a lattice, then every cutset of  $C$  is coherent. More generally, it is enough to require that every finite nonempty subset of  $C$  which is bounded above or below in  $P$  has a meet or a join in  $P$ .

5.9 Theorem [Bj 1, Theorem 2.3]: If  $C$  is a coherent cutset of  $P$ , then  $P$  is homotopy equivalent to  $\Gamma(P,C)$ .

Proof: Consider the ideal relation

$$R = \{ \langle x, A \rangle \in \text{sd}(P) \times \Gamma(P,C) : x \subset P(A) \}.$$

For any  $x$  in  $\text{sd}(P)$ , we have

$$\begin{aligned} R(x) &= \{ A \in \Gamma(P,C) : x \subset P(A) \} \\ &= \{ A \in \Gamma(P,C) : A \subset P(x) \}. \end{aligned}$$

Note that if  $x \subset P(A)$ , then  $P(A) \neq \emptyset$ . Thus  $R(x)$  consists of all finite nonempty subsets of  $P(x) \cap C$ . This is a simplicial complex which is a cone on any of its vertices, so  $R(x)$  is contractible.

For any  $A$  in  $\Gamma(P,C)$ .

$$\begin{aligned} R^{-1}(A) &= \{x \in \text{sd}(P) : x \subset P(A)\} \\ &= \text{sd}(P(A)). \end{aligned}$$

Since  $C$  is coherent,  $P(A)$  is a cone, so  $R^{-1}(A)$  is contractible by Proposition 4.2.

Now by Theorem 5.7,  $\Gamma(P,C)$  is homotopy equivalent to  $\text{sd}(P)$ , and hence (by 4.2) to  $P$ . □

## Chapter 6: Complementation

If  $x$  and  $y$  are elements of a poset  $P$ , we say that  $x$  and  $y$  are complements in  $P$  if the set  $\{x,y\}$  has no upper bound or lower bound in  $P$ . If  $\{x,y\}$  has no upper bound in  $P$ , we say that  $x$  and  $y$  are upper semicomplements. If  $P$  is bounded, we will always define complements with respect to the proper part  $\bar{P}$ .

A poset in which every pair of elements has a join is called a join-semilattice.

The next theorem strengthens homological results in [Ba 3] and [B-B 2], and verifies an unpublished conjecture of Björner. See also [B-W] for related results.

**6.1 Theorem:** Let  $L$  be a bounded join-semilattice. If  $s \in \bar{L}$ , and if  $B$  is a set of upper semicomplements of  $s$ , including all of the complements of  $s$ , then the subposet  $\bar{L} \setminus B$  is contractible.

Proof: Let  $N = \bar{L} \setminus B$  for convenience, and let  $G = \{x \in \bar{L} : x \vee s < \hat{1}\}$ . Note that  $G$  is an ideal in  $N$ . Also,  $s \in G$ , and if  $x \in G$ , then  $x \vee s \in G$ ; so  $G$  is join-contractible via  $s$ .

Consider the inclusion map of  $G$  into  $N$ . Its fibers are  $\{G \cap N_{\leq x} : x \in N\}$ . If  $x \in G$ , then



$G \cap N_{\leq x} = N_{\leq x}$ , which is a cone. Suppose  $x \in N \setminus G$ . Then  $x \vee s = \hat{1}$ , but  $x$  is not a complement of  $s$ , so  $\{x, s\}$  has a lower bound  $t$  in  $\bar{L}$ . In fact,  $t \in G \cap N_{\leq x}$ . For any  $y$  in  $G \cap N_{\leq x}$ , we have

$$(y \vee t) \vee s = y \vee (t \vee s) = y \vee s < 1,$$

so  $y \vee t \in G$ . Also  $y \vee t \leq x$ , so  $y \vee t \in G \cap N_{\leq x}$ . Thus  $G \cap N_{\leq x}$  is join-contractible. Therefore by Quillen's Theorem 5.5,  $N$  is homotopy equivalent to  $G$ , which we know to be contractible.  $\square$

Theorem 6.1 leads to a new proof of Crapo's complementation theorem.

6.2 Theorem [Cr, Theorem 3]. If  $L$  is a finite lattice,  $s \in L$ , and  $C$  is the set of complements of  $s$ , then

$$\mu(\hat{0}, \hat{1}) = \sum_{x, y \in C} \mu(\hat{0}, x) \zeta(x, y) \mu(y, \hat{1}).$$

Proof: By 6.1,  $\bar{L} \setminus C$  is contractible, so  $\mu(\bar{L} \setminus C) = 0$ .

The result follows from Theorem 2.5.  $\square$

A subset  $A$  of a poset  $P$  is initial if, for every element of  $P$ , there is some element of  $A$  below it.

(If  $P$  has no infinite chains, then  $A$  is initial if and only if  $A$  contains all of the minimal elements of  $P$ .)

An initial subset  $A$  is join-coherent if every finite non-empty subset of  $A$  which is bounded above has a join.

The result below generalizes [Bj, Theorem 3.3] and [B-B 1, Prop. 3.1].

**6.3 Theorem:** Suppose  $P$  is a poset,  $A$  is a join-coherent initial subset of  $P$ , and  $s \in A$ . If  $s$  has no complement which is a join of finitely many elements of  $A$ , then  $P$  is contractible.

Proof: Let  $Q$  be the subposet of  $P$  consisting of joins of finite subsets of  $A$ . The fibers of the inclusion map of  $Q$  into  $P$  are of the form  $Q \cap P_{\leq y}$ . This is nonempty since  $A$  is initial. Every pair of elements of  $Q \cap P_{\leq y}$  has a join, since  $A$  is join-coherent, so  $Q \cap P_{\leq y}$  is join-contractible. By Theorem 5.5, it follows that  $P$  and  $Q$  are homotopy equivalent. So it suffices to show that  $Q$  is contractible.

Suppose  $x \in Q$ . By hypothesis,  $\{x, s\}$  must be bounded above or below in  $P$ . If  $\{x, s\}$  is bounded above in  $P$ , then by join-coherence,  $x \vee s$  exists and belongs to  $Q$ . If  $\{x, s\}$  is bounded below in  $P$ , then  $\{x, s\}$  is bounded below by an element of  $A$ , since  $A$  is initial.

Thus  $s$  has no complement in  $Q$ .

Note that  $\hat{Q}$  is a bounded join-semilattice. By 6.1, with  $B = \emptyset$ ,  $Q$  is contractible.  $\square$

## Chapter 7. The Higher Order Complexes

The width (or first Dilworth number) of a poset  $P$  is defined to be the maximum size of an antichain in  $P$ , and is denoted by  $d(P)$ . The  $k$ th order complex  $\Delta_k(P)$  of a poset  $P$  is the set of finite nonempty subsets of  $P$  with width at most  $k$ , ordered by inclusion. These simplicial complexes (which we are thinking of here as posets) were studied by Bjorner in [Bj 2].

Observe that  $\Delta_1(P) = \text{sd}(P)$ . We saw in Chapter 4 that  $\text{sd}(P)$  is homeomorphic to  $P$ . There is no isotone map which induces that homeomorphism, but there is an isotone map from  $\text{sd}(P)$  to  $P$ , defined by sending a chain to its greatest element, which has contractible fibers. That suggests the following generalization: Define the  $k$ th ideal poset  $J_k(P)$  to be the set of nonempty ideals of  $P$  which are generated by at most  $k$  elements, ordered by inclusion. (Notice that  $J_1(P)$  is isomorphic to  $P$ .) There is a natural isotone map  $\sigma_k : \Delta_k(P) \rightarrow J_k(P)$  which sends a subset of  $P$  to the ideal that it generates. Then we expect the following to hold:

**7.1 Theorem:**  $\sigma_k$  is a homotopy equivalence.

Before proceeding to the proof, we will need a few facts from matroid theory.

Suppose  $S$  is a finite set. A function  $\lambda : 2^S \rightarrow \mathbb{N}$  is called submodular if, for all  $A, B$  in  $2^S$ ,

$$\lambda(A \cap B) + \lambda(A \cup B) \leq \lambda(A) + \lambda(B).$$

If  $\lambda$  is isotone as well as submodular, then the collection

$$\{A \subset S : (\forall B \subset A) \lambda(B) \geq \text{card } B\}$$

forms the set of independent sets of a matroid  $M(\lambda)$ . See [We, chapter 8] for details.

**7.2 Lemma:** If  $\lambda : 2^S \rightarrow \mathbb{N}$  is isotone and submodular, and if  $\lambda(S) > \text{card } S$ , then  $M(\lambda)$  has a coloop (isthmus).

Proof of Lemma: If  $S$  is independent, we are done. Otherwise, there exists  $I \subset S$ ,  $I \neq S$ , such that  $\lambda(I) < \text{card } I$ . Choose  $I$  maximal with that property, and let  $B = S \setminus I$ . We will show that  $M(\lambda)$  is the direct sum of its restrictions to  $I$  and  $B$ , and that the restriction to  $B$  is the free matroid.

Let  $D$  be an independent subset of  $I$ . We need to show that  $D \cup B$  is independent. So suppose that  $D \cup B$  is dependent. Then there exists  $J \subset D \cup B$  such that  $\lambda(J) < \text{card } J$ . Note that  $\text{card } J = \text{card}(J \cap D) + \text{card}(J \cap B)$ . By submodularity,

$$\begin{aligned} \lambda(I \cap J) + \lambda(I \cup J) &\leq \lambda(I) + \lambda(J) \\ &\leq \text{card } I + \text{card}(J \cap B) + \text{card}(J \cap D) - 2 \\ &= \text{card}(I \cup J) + \text{card}(J \cap D) - 2. \end{aligned}$$

Since  $\lambda$  is isotone and  $D$  is independent,

$$\lambda(I \cap J) \geq \lambda(I \cap J \cap D) \geq \text{card}(I \cap J \cap D) = \text{card}(J \cap D).$$

Therefore

$$\text{card}(J \cap D) + \lambda(I \cup J) \leq \text{card}(I \cup J) + \text{card}(J \cap D) - 2$$

or

$$\lambda(I \cup J) \leq \text{card}(I \cup J) - 2 < \text{card}(I \cup J).$$

By choice of  $I$ , this implies that  $J \subset I$ . Since

$J \subset D \cup B$ , it follows that  $J \subset D$ . But that is absurd, since  $D$  is independent and  $J$  is dependent.  $\square$

Proof of Theorem: By Theorem 5.5, it suffices to show that  $\sigma_k$  has contractible fibers. What it boils down to is this: Let  $P$  be a poset which has exactly  $m$  maximal elements,  $m \leq k$ , and such that every element of  $P$  is below some maximal element. We need to show that  $\Delta_k(P)$  is contractible.

Let  $t_1, t_2, \dots, t_m$  be the maximal elements of  $P$ . Given  $A \subset [m] = \{1, 2, 3, \dots, m\}$ , define

$$t(A) = \{t_i : i \in A\},$$

$$E(A) = \{x \in \Delta_k(P) : x \cup t(A) \in \Delta_k(P)\}, \quad \text{and}$$

$$N(A) = \{y \in P : (\forall i \in A) y \not\leq t_i\}.$$

Note that  $N(A)$  is a filter (dual ideal) in  $P$ . Define the relation

$$R = \{(x, A) \in \Delta_k(P) \times (2^{[m]} \setminus \{\emptyset\}) : x \in E(A)\},$$

which is an ideal relation.

For  $A \in 2^{[m]} \setminus \{\emptyset\}$ , we have  $R^{-1}(A) = E(A)$ . Since  $m \leq k$ ,  $t(A) \in E(A)$ , and in particular  $E(A)$  is nonempty. Now it is clear from the definition that  $E(A)$  is join-contractible via  $t(A)$ . If we can show that every  $R(x)$  is contractible, then Theorem 5.7 will imply that  $\Delta_k(P)$  is homotopy equivalent to  $2^{[m]} \setminus \{\emptyset\}$ , which has a greatest element, so we will be done.

From now on, let  $x$  be a fixed member of  $\Delta_k(P)$ .

The  $N$  notation has the properties that  $X \subset Y$  implies  $N(Y) \subset N(X)$ , hence  $N(X) \cup N(Y) \subset N(X \cap Y)$ , and also  $N(X) \cap N(Y) = N(X \cup Y)$ . Greene and Kleitman showed [G-K, Lemma 4.3] that if  $F$  and  $G$  are filters in a finite poset, then

$$d(F \cap G) + d(F \cup G) \geq d(F) + d(G).$$

Combining these facts, we find that the function  $\lambda : 2^{[m]} \rightarrow \mathbb{N}$  defined by  $\lambda(A) = k - d(x \cap N(A))$  is isotone and submodular. Furthermore,  $\lambda([m]) = k \geq m$ , so lemma 7.2 says that the matroid  $M(\lambda)$  has a coloop.

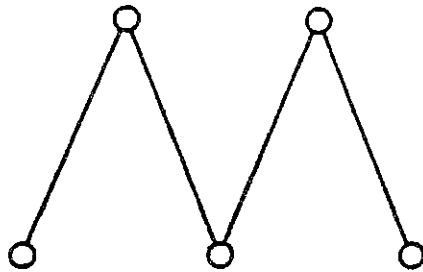
A nonempty set  $A \subset [m]$  does not belong to  $R(x)$  precisely when  $d(x \cup t(A)) > k$ . This happens just in case there is some  $B \subset A$  such that  $\text{card } B + d(x \cap N(B)) > k$ , or  $\text{card } B > \lambda(B)$ . Therefore,  $R(x)$  consists precisely of the nonempty independent sets of  $M(\lambda)$ . Saying that  $M(\lambda)$



has a coloop is the same as saying that  $R(x)$  is join-contractible. Thus  $R(x)$  is contractible, which completes the proof.  $\square$

Remark 1: If  $x$  is disjoint from  $t([m])$  and  $d(x) = k$ , then  $R(x)$  is the matroid  $\Gamma^{(1)}$  of [G-K, p. 67].

Remark 2: Although  $\Delta_k(P)$  is homeomorphic to  $J_k(P)$  for  $k = 1$ , they need not be homeomorphic for  $k > 1$ . For example, if  $P$  is the poset of the figure below, then the realizations of  $\Delta_2(P)$  and  $J_2(P)$  are not of the same dimension.



## Chapter 8. Cohen-Macaulay Complexes: Background

The theory of Cohen-Macaulay complexes provides connections between ring theory, algebraic topology, and combinatorics. We will begin with the ring theory, but soon leave it.

Let  $R$  be a commutative ring with identity, and let  $K$  be a simplicial complex with vertices  $x_1, x_2, \dots, x_n$ . By thinking of the vertices as indeterminates, we can form the polynomial ring  $R[x_1, x_2, \dots, x_n]$ . Let  $I(K)$  be the ideal generated by the monomials which do not correspond to simplices of  $K$ . Then we say that  $K$  is Cohen-Macaulay over  $R$  if the quotient ring  $R[x_1, \dots, x_n]/I(K)$  (sometimes called the Stanley-Reisner ring of  $K$ ) is a Cohen-Macaulay ring. See [Ma, p. 103] for a definition of Cohen-Macaulay rings. See also [H] and [S] for information on the ring-theoretic approach to Cohen-Macaulay complexes.

If  $\sigma$  is a simplex of a simplicial complex  $K$ , then the link of  $\sigma$  in  $K$ , written  $Lk(\sigma, K)$  or  $Lk(\sigma)$ , is the subcomplex consisting of simplices  $\tau$  such that  $\sigma \cap \tau = \emptyset$  but  $\sigma \cup \tau \in K$ . By convention,  $Lk(\emptyset, K) = K$ . The connection between Cohen-Macaulay complexes and algebraic topology is given by the following theorem of G. Reisner:

8.1 Theorem [Re]: Suppose  $K$  is a finite simplicial complex and  $R$  is a field or  $\mathbb{Z}$ . Then  $K$  is Cohen-Macaulay over  $R$  if and only if  $\tilde{H}_i(\text{Lk}(\sigma, K); R) = 0$  for all simplices  $\sigma$  (including  $\emptyset$ ) and for all  $i < \dim \text{Lk}(\sigma, K)$ . □

With Reisner's theorem as motivation, we can redefine the Cohen-Macaulay property in terms of algebraic topology. We will use the topological approach from now onward.

A finite-dimensional simplicial complex  $K$  is said to be a homology bouquet (over  $R$ ) if  $\tilde{H}_i(K; R) = 0$  for all  $i < \dim K$ . By convention,  $\dim \emptyset = -1$ , so the empty complex is a bouquet.

Let  $R$  be a field over  $\mathbb{Z}$ . A finite-dimensional simplicial complex  $K$  is said to be almost Cohen-Macaulay over  $R$  (ACM, for short) if the link of nonempty simplex of  $K$  is a homology bouquet over  $R$ . We say that  $K$  is Cohen-Macaulay over  $R$  (CM) if  $K$  is ACM over  $R$  and if  $\text{Lk}(\emptyset, K) = K$  is also a homology bouquet over  $R$ .

Remark. Most of the results in this chapter and the next chapter are true with coefficient rings more general than fields or the integers. However, there seems to be no single level of generality which is appropriate for everything. In any case, more general coefficients seem to be of

limited usefulness, so I will resist the temptation to do everything in the utmost generality. Throughout,  $R$  will be a field or  $\mathbb{Z}$ .

A simplicial complex is pure if every maximal simplex has the same dimension. The next theorem implies that all CM complexes, and all connected ACM complexes, are pure.

8.2 Theorem [Ba 2, Prop. 3.1]: Let  $K$  be a finite-dimensional simplicial complex. If the link of each simplex of  $K$  (including  $\emptyset$ ) is empty, discrete, or connected, then  $K$  is pure.

Proof: Suppose  $K$  is not pure. Let  $L$  equal the subcomplex generated by the maximal simplices of maximum dimension, and let  $S$  equal the subcomplex generated by the other maximal simplices. Then  $L$  and  $S$  are nonempty, and  $L \cup S = K$ .

If  $K$  were empty or discrete, then  $K$  would be pure; so  $K$  is connected. In particular,  $L \cap S$  is nonempty. Choose a simplex  $\sigma$  which is maximal in  $L \cap S$ . Since  $\sigma$  is contained in maximal simplices of two different dimensions,  $Lk(\sigma, K)$  cannot be empty or discrete, so  $Lk(\sigma, K)$  is connected.

By maximality of  $\sigma$ , none of the vertices of  $Lk(\sigma, K)$  belong to  $L \cap S$ . But  $\sigma$  is not maximal in  $K$ , so  $Lk(\sigma, K)$  must contain vertices of both  $L$  and  $S$ . Since  $Lk(\sigma, K)$  is connected, there exists a 1-simplex  $\tau$  in  $Lk(\sigma, K)$ , one of whose ends is in  $L$  and one of whose ends is in  $S$ . Then  $\tau$  cannot belong to either  $L$  or  $S$ , which is impossible.  $\square$

A poset of finite length is said to be ACM (resp. CM) whenever its order complex is ACM (CM). (Note that a poset has finite length if and only if its order complex is finite-dimensional.) By 8.2, all maximal chains in a CM or connected ACM poset have the same length. It follows that such a poset is ranked.

The closed star of a simplex  $\sigma$ ,  $\overline{St}(\sigma, K) = \overline{St}(\sigma)$ , is the set of simplices  $\tau$  of  $K$  such that  $\sigma \cup \tau$  is also a simplex of  $K$ . It follows that  $|\overline{St} \sigma| = |\sigma| * |Lk \sigma|$ . The boundary of  $\sigma$ ,  $Bd \sigma$ , is the complex of its proper faces. If  $p$  is an interior point of  $|\sigma|$ , then  $|\sigma| = p * |Bd \sigma|$ , so  $|\overline{St} \sigma| = p * |Bd \sigma| * |Lk \sigma|$ .

The following theorem shows that ACM and CM are topological properties. That is, they depend only upon the geometric realization of the simplicial complex.

**8.3 Topological Invariance Theorem [Mu 2]:** A finite-dimensional simplicial complex  $K$  is ACM over  $R$  if and only if  $H_i(|K|, |K| \setminus p; R) = 0$  for all points  $p$  of  $|K|$  and all integers  $i$  less than the dimension of the component of  $K$  which contains  $p$ . Furthermore,  $K$  is CM over  $R$  if and only if  $K$  is ACM over  $R$  and  $\tilde{H}_i(|K|; R) = 0$  for all  $i < \dim K$ .

Proof: Let  $\sigma$  be a simplex of  $K$ , and let  $p$  be an interior point of  $|\sigma|$ . Since  $|\text{Bd } \sigma|$  is a sphere of dimension  $\dim(\sigma) - 1$ , the suspension isomorphism says that

$$\tilde{H}_i(\text{Lk } \sigma; R) \cong \tilde{H}_{i+\dim \sigma}(\text{Bd } \sigma * \text{Lk } \sigma; R).$$

Now  $\overline{\text{St } \sigma}$  is a cone, hence contractible, so the long exact sequence of a pair [Sp, §4.5] tells us that

$$H_{i+\dim \sigma}(\text{Bd } \sigma * \text{Lk } \sigma; R) \cong H_{i+\dim \sigma+1}(\overline{\text{St } \sigma}, \text{Bd } \sigma * \text{Lk } \sigma; R).$$

Since  $|\overline{\text{St } \sigma}| = p * |\text{Bd } \sigma| * |\text{Lk } \sigma|$ , there is a deformation retraction of  $|\overline{\text{St } \sigma}| \setminus p$  onto  $|\text{Bd } \sigma * \text{Lk } \sigma|$ . Therefore

$$\begin{aligned} & H_{i+\dim \sigma+1}(\overline{\text{St } \sigma}, \text{Bd } \sigma * \text{Lk } \sigma; R) \\ & \cong H_{i+\dim \sigma+1}(|\overline{\text{St } \sigma}|, |\overline{\text{St } \sigma}| \setminus p; R). \end{aligned}$$

Finally, by the excision property [Sp, 4.6.1],

$$\begin{aligned} H_{i+\dim \sigma+1}(|\overline{St} \sigma|, |\overline{St} \sigma| \setminus p; R) \\ \cong H_{i+\dim \sigma+1}(|K|, |K| \setminus p; R). \end{aligned}$$

So we have shown that

$$(*) \quad \tilde{H}_i(\text{Lk } \sigma; R) \cong H_{i+\dim \sigma+1}(|K|, |K| \setminus p; R).$$

We assume, without loss of generality, that  $K$  is connected. We may also assume that  $K$  is pure of dimension  $n = \dim K$ , by the following reasoning. If we know that  $K$  is ACM and connected, then  $K$  is pure by 8.2. On the other hand, if  $\sigma$  is a maximal simplex and  $i = -1$ , then  $(*)$  reduces to

$$R \cong \tilde{H}_{-1}(\emptyset; R) \cong H_{\dim \sigma}(|K|, |K| \setminus p; R).$$

So if we know that  $H_j(|K|, |K| \setminus p; R)$  is only allowed to be nontrivial when  $j = n$ , then it also follows that  $K$  is pure.

Given that  $K$  is pure of dimension  $n$ , we have  $\dim(\text{Lk } \sigma) = n - \dim \sigma - 1$ . Therefore  $i < \dim(\text{Lk } \sigma)$  if

and only if  $i + \dim \sigma + 1 < n$ . Now the ACM part of the theorem follows from (\*), and the remainder of the theorem is immediate.  $\square$

The open intervals in a poset  $P$  of finite length correspond to certain links in the order complex of  $P$ . For example, given an interval of the form  $(x, y)$ , choose a maximal chain  $\sigma$  in  $P_{<x} \cup P_{>y}$ . Then  $\Delta((x, y))$  is the link of  $\sigma$  in  $\Delta(P)$ . On the other hand, any link can be expressed as a join of open intervals. Explicitly, the link of the chain  $x_1 < x_2 < \dots < x_n$  in  $P$  is  $\Delta(P_{<x_1}) * \Delta((x_1, x_2)) * \dots * \Delta((x_{n-1}, x_n)) * \Delta(P_{>x_n})$ . This leads to a nicer characterization of ACM and CM for posets.

**8.4 Theorem:** If  $K$  and  $L$  are homology bouquets over  $R$ , then the complex  $K * L$  is a homology bouquet over  $R$ .

Proof: If we allow empty simplices, then the simplices of  $K * L$  are in bijective correspondence with ordered pairs of simplices from  $K$  and  $L$ . This gives rise to an isomorphism of augmented simplicial chain complexes with coefficients in  $R$ :

$$\tilde{\mathcal{C}}(K * L; R) \cong S(\tilde{\mathcal{C}}(K; R) \otimes_R \tilde{\mathcal{C}}(L; R)).$$



Here  $S$  is suspension of a chain complex, which merely shifts the indexing. Then the Künneth formula [Sp, 5.3.4] says that there is a short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=m} \tilde{H}_p(K;R) \otimes_R \tilde{H}_q(L;R) \longrightarrow \tilde{H}_{m+1}(K*L;R) \\ \longrightarrow \bigoplus_{p+q=m-1} \tilde{H}_p(K;R) *_R \tilde{H}_q(L;R) \longrightarrow 0.$$

The result follows. □

**8.5 Corollary:** A poset  $P$  is CM over  $R$  if and only if every open interval of  $P$  is a homology bouquet over  $R$ . In order that  $P$  be ACM, we no longer require that  $P$  itself be a homology bouquet. □

We will now define a stronger form of the Cohen-Macaulay property, first studied by Quillen [Q 1]. A finite-dimensional simplicial complex  $K$  is a homotopy bouquet if  $K$  is  $(\dim K - 1)$ -connected or empty. (By convention,  $(-1)$ -connected means nonempty, so any complex of dimension zero is a homotopy bouquet.) A complex  $K$  is homotopy ACM if the link of every nonempty simplex of  $K$  is a homotopy bouquet. We say that  $K$  is homotopy CM if  $K$  is homotopy ACM and if  $K$  is also a homotopy bouquet.

Remark. The proof of the next result uses the most advanced algebraic topology in this thesis, although it is still well known to algebraic topologists. I regret that I do not know a more elementary proof.

Given a collection of disjoint spaces  $\{X_\alpha\}$ , each  $X_\alpha$  having a specified base point  $x_\alpha$ , construct a space called the wedge of  $\{X_\alpha\}$  by identifying all of the base points with each other. By convention, the empty wedge is a point.

8.6 Lemma: A nonempty homotopy bouquet  $K$  is homotopy equivalent to a wedge of spheres of dimension  $\dim K$ .

Proof: Let  $n = \dim K$ . If  $n = 0$ , the theorem is trivial.

Suppose  $n = 1$ . With the aid of Zorn's lemma, choose a maximal contractible subcomplex  $T$ . Argue that  $T$  contains all of the vertices of  $K$ . (If  $K$  is finite, we have simply chosen a spanning tree for a graph.) See [Sp, 3.7.2] for details.

Since  $T$  is a contractible subcomplex of  $K$ ,  $|K|/|T|$  is homotopy equivalent to  $|K|$  [Wh, I-5.13 or B-W Lemma 2.2]. Since  $T$  contains all of the vertices of  $K$ ,  $|K|/|T|$  is a 1-dimensional CW complex having precisely one vertex. Therefore  $|K|/|T|$  is a wedge of 1-spheres.

Suppose  $n > 1$ . The Hurewicz isomorphism theorem [Sp, 7.5.5] tells us that  $\tilde{H}_i(K) = 0$  for  $i \leq n-1$ , and  $H_n(K) \cong \pi_n(K)$ . Since  $K$  is  $n$ -dimensional,  $H_n(K)$  is free abelian. Therefore  $\pi_n(K)$  is free abelian.

Choose maps  $\{g_\alpha : S^n \rightarrow |K|\}$  corresponding to a basis of  $\pi_n(K)$ . Then there is a map

$$\text{wedge}_\alpha g_\alpha : \text{wedge}_\alpha S^n \rightarrow |K|$$

which induces isomorphism in homology. Since  $n > 1$ , these spaces are simply connected, so the Whitehead theorem [Sp, 7.5.9] says that the map  $\text{wedge}_\alpha g_\alpha$  induces isomorphism of homotopy groups. It follows [Sp, 7.6.24] that  $\text{wedge}_\alpha g_\alpha$  is a homotopy equivalence.  $\square$

**8.7 Theorem:** If  $K$  is homotopy CM (respectively homotopy ACM), then  $K$  is homology CM (resp. homology ACM) over any  $R$ .

Proof: Lemma 8.6, and the fact that a wedge of spheres is a homology bouquet.  $\square$

R. D. Edwards [Edw] gave an example of a compact 3-manifold  $M$  which has the homology of a 3-sphere but is not simply connected, and whose double suspension  $S(S(M))$

is homeomorphic to a 5-sphere. Any triangulation of  $M$  gives rise to a triangulation of  $S^5$  which is not homotopy ACM, because certain 1-simplices have  $M$  as their link. (See also [G-S] for information on such examples.) But of course  $S^5$  does have triangulations which are homotopy CM, such as the boundary of a 6-simplex. Therefore, unlike the homology versions, homotopy ACM and CM are not topological properties. However, we will see that they are preserved by many common operations on complexes and posets.

**8.8 Theorem:** If  $K$  and  $L$  are homotopy bouquets, then  $K * L$  is a homotopy bouquet.

Proof: We saw in Lemma 8.6 that a homotopy bouquet is homotopy equivalent to a wedge of spheres. The join operation is well defined on homotopy classes of spaces, so it is enough to show that the join of two wedges of spheres is homotopy equivalent to a wedge of spheres.

Suppose  $|K| = \text{wedge}_{\alpha} S_{\alpha}^m$  and  $|L| = \text{wedge}_{\beta} S_{\beta}^n$ . If  $m = n = 0$ , then clearly  $K * L$  is connected, hence a bouquet. Otherwise, suppose  $m > 0$ . Then we can write  $S_{\alpha}^m$  as a suspension:  $S_{\alpha}^m = S(S_{\alpha}^{m-1})$ , so  $|K| = \text{wedge} S(S_{\alpha}^{m-1})$ .

Now observe that suspension commutes with wedge, up to homotopy type. (That can be justified using the fact that a contractible subcomplex of a simplicial complex can

be smashed without changing the homotopy type of the complex.) Therefore  $|K|$  is homotopy equivalent to  $S(\text{wedge } S_{\alpha}^{m-1}) = S^0 * \text{wedge } S_{\alpha}^{m-1}$ , so  $|K*L|$  is homotopy equivalent to

$$S^0 * (\text{wedge } S_{\alpha}^{m-1}) * (\text{wedge } S_{\beta}^n).$$

By induction,  $(\text{wedge } S_{\alpha}^{m-1}) * (\text{wedge } S_{\beta}^n)$  is a homotopy bouquet, say  $\text{wedge } S_{\gamma}^{m+n}$ . Therefore  $|K*L|$  is homotopy equivalent to  $S(\text{wedge } S_{\gamma}^{m+n})$ . By commuting suspension with wedge again, we are done.  $\square$

**8.9 Corollary:** A poset  $P$  of finite length is homotopy CM if and only if every open interval of  $P$  is a homotopy bouquet. In order that  $P$  be ACM, we no longer require that  $P$  itself be a homotopy bouquet.  $\square$

Recall from Chapter 5 that  $K^{(n)}$  denotes the  $n$ -skeleton of the simplicial complex  $K$ .

**8.10 Lemma:** For any simplicial complex  $K$ ,  $|K|$  is  $n$ -connected if and only if  $|K^{(n+1)}|$  is  $n$ -connected.

Sketch of Proof: Each continuous map of  $S^m$  into  $|K|$  has a simplicial approximation [Sp, 3.4.8]. Since  $S^m$  is

$m$ -dimensional, a simplicial map sends  $S^m$  into  $|K^{(m)}|$ . Similarly, homotopies between maps from  $S^m$  into  $|K|$  are given by maps of  $S^m \times I$  into  $|K|$ , and a simplicial approximation to such a map has its range within  $|K^{(m+1)}|$ .

□

A lattice of finite length is semimodular if and only if it is ranked, and its rank function satisfies the submodular inequality

$$r(x \vee y) + r(x \wedge y) \leq r(x) + r(y).$$

Recall that  $r(\hat{0}) = 0$ . Note that semimodularity is a local property, i.e. if a lattice of finite length is semimodular, then so are all of its closed intervals.

8.11 Theorem [F]: Every semimodular lattice of finite length is homotopy CM.

Proof: Let  $L$  be a semimodular lattice of finite length. By 8.9, it is enough to show that every interval of  $L$  is a homotopy bouquet. An interval which contains  $\hat{0}$  or  $\hat{1}$  is a cone, so it is enough to consider intervals of the form  $(x, y)$ . And since semimodularity is a local property, it suffices to show that  $\bar{L} = (\hat{0}, \hat{1})$  is a bouquet.

Let  $n = r(\hat{1})$ . Let  $C$  be the cross-cut of all minimal elements of  $\bar{L}$  (atoms of  $L$ ). By Theorem 5.9,  $\bar{L}$  is homotopy equivalent to  $\Gamma(\bar{L}, C)$ . We know that  $\bar{L}$  has dimension  $n-2$ , so it suffices to show that  $\Gamma(\bar{L}, C)$  is  $(n-3)$ -connected.

Let  $B$  be the simplicial complex of all finite non-empty subsets of  $C$ . Since  $B$  is a cone on any of its vertices,  $B$  is contractible. If  $c_1, c_2, \dots, c_j$  are elements of  $C$ , then by the submodular inequality and induction,

$$r(c_1 \vee c_2 \vee \dots \vee c_j) \leq r(c_1) + \dots + r(c_j) = j,$$

hence  $c_1 \vee c_2 \vee \dots \vee c_j < \hat{1}$  if  $j < n$ . That means that every subset of  $C$  of cardinality  $n-1$  or less belongs to  $\Gamma(\bar{L}, C)$ . In other words,  $\Gamma(\bar{L}, C)$  and  $B$  have the same  $(n-2)$ -skeleton. By Lemma 8.10, it follows that  $\Gamma(\bar{L}, C)$  is  $(n-3)$ -connected.  $\square$

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In the case of finite lattices, stronger results have been obtained; see for example [Ba 2] or [Bj 2]. See also [B-W] for a more constructive proof of 8.11.

## Chapter 9. Cohen-Macaulay Complexes: Further Properties

When studying CM posets, it is natural to ask which operations on posets preserve CM. Probably the easiest such question involves joins. Since a link in a join is a join of links, Theorems 8.4 and 8.8 immediately imply:

9.1 Proposition:  $P * Q$  is CM  $\Leftrightarrow P$  and  $Q$  are CM.  $\square$

We will proceed to answer several more such questions. The results were known, at least for homology CM (see [Ba 2]), but the proofs are new.

Next, we will consider a less familiar operation, which might be called "twinning on ideal," and answer a question of D. Eisenbud and C. Huneke about when that construction yields a CM poset.

Finally, we will prove a conjecture of K. Baclawski, showing that a property called "2-Cohen-Macaulay connectivity" is a topological property.

From the homeomorphism theorem 4.2 and the Topological Invariance Theorem 8.3, it is immediate that  $sd$  and  $Int$  preserve homology ACM and CM. We can extend this fact to homotopy CM by a different argument.

9.2 Proposition:  $Int(P)$  is ACM  $\Leftrightarrow P$  is ACM  $\Leftrightarrow sd(P)$  is



ACM.

$$\text{Int}(P) \text{ is CM} \Leftrightarrow P \text{ is CM}$$

$$\Leftrightarrow \text{sd}(P) \text{ is CM.}$$

Proof: By the homeomorphisms of 4.2, it suffices to consider the ACM property.

From our analysis of the intervals of  $\text{Int}(P)$ , Theorem 4.4, we see that if  $P$  is ACM, then  $\text{Int}(P)$  is ACM. On the other hand, suppose  $\text{Int}(P)$  is ACM. An interval of the form  $(a,b)$  in  $P$  is isomorphic to the interval  $([a,a],[a,b])$  in  $\text{Int}(P)$ , hence is a bouquet. By definition of ACM,  $\text{Int}(P)$  has no infinite chains, so  $P$  has no infinite chains. Therefore, given an element  $a$  of  $P$ , we can choose a minimal element  $x$  of  $P_{<a}$  and a maximal element  $y$  of  $P_{>a}$ . There are poset isomorphisms

$$\text{Int}(P)_{>[x,a]} = P_{>a} \quad \text{and}$$

$$\text{Int}(P)_{>[a,y]} = P_{<a};$$

therefore  $P_{>a}$  and  $P_{<a}$  are bouquets.

Suppose  $\text{sd}(P)$  is ACM. If  $I$  is an open interval of  $P$ , other than  $P$  itself, then there is an element  $\sigma$

of  $sd(P)$  such that  $sd(I)$  is isomorphic as a poset to  $sd(P)_{>\sigma}$ . Since  $sd(I)$  is homeomorphic to  $I$ , it follows that  $I$  is a bouquet.

Now suppose  $P$  is ACM. An open interval of  $sd(P)$  of the form  $(a,b)$  or  $sd(P)_{<\sigma}$  is isomorphic to  $2^S \setminus \{\emptyset, S\}$  for some finite  $S$ . We can see that such a poset is a bouquet by noting that it is the boundary of a simplex, or by using the product theorem 4.3 (d). An interval of the form  $sd(P)_{>\sigma}$  is isomorphic as a poset to  $sd(I_1 * I_2 * \dots * I_n)$  for some open intervals  $I_1, \dots, I_n$  in  $P$ . This is homeomorphic to  $I_1 * I_2 * \dots * I_n$ , which is a join of bouquets.  $\square$

**9.3 Theorem:** Suppose  $P$  and  $Q$  are posets having at least two elements. Then  $P \times Q$  is ACM  $\Leftrightarrow P$  and  $Q$  are ACM, and  $P \times Q$  is CM  $\Leftrightarrow P$  and  $Q$  are CM, and either  $P$  and  $Q$  are both antichains or  $P$  and  $Q$  are both acyclic.

**Proof:** It follows easily from the product theorem 4.3 that  $P \times Q$  is ACM if and only if  $P$  and  $Q$  are ACM. To finish the proof, we have to consider when  $P \times Q$  is a bouquet.

If  $P$  and  $Q$  are both antichains, then  $P \times Q$  is an antichain, hence a bouquet. If  $P$  and  $Q$  are acyclic,

then so is  $P \times Q$ . In the homotopy case, note that if  $P$  and  $Q$  are acyclic homotopy bouquets, then  $P$  and  $Q$  are contractible, hence  $P \times Q$  is contractible by 3.11.

Suppose  $P \times Q$  is a bouquet. By the Künneth formula, this is only possible if  $P$ ,  $Q$ , and  $P \times Q$  have nontrivial ordinary homology in only one dimension, namely dimension zero. Therefore  $P \times Q$  is an antichain or acyclic. If  $P \times Q$  is an antichain, then  $P$  and  $Q$  are antichains. If  $P \times Q$  is acyclic, then  $P$  and  $Q$  are acyclic. In the homotopy case, we note that if  $P \times Q$  is an acyclic homotopy bouquet, then  $P \times Q$  is contractible, hence (by 4.1 and 3.11)  $P$  and  $Q$  are contractible. Therefore  $P$  and  $Q$  are bouquets.  $\square$

If  $X$  is a topological space, let  $[S^n, X]$  denote the set of homotopy classes of maps from  $S^n$  into  $X$ . (If  $X$  is path connected, then the set  $[S^n, X]$  is in bijective correspondence with the homotopy group  $\pi_n(X)$ .) A map  $g : X \rightarrow Y$  induces a function  $g_{\#} : [S^n, X] \rightarrow [S^n, Y]$  by the rule  $g_{\#}([f]) = [g \circ f]$ . The correspondences  $X \mapsto [S^n, X]$  and  $g \mapsto g_{\#}$  define a functor. Mainly, that means that the correspondence  $g \mapsto g_{\#}$  commutes with composition and sends identity maps to identity maps.

Recall that each CM poset  $P$  has a rank function  $r$ . A level in  $P$  is the set of all elements having a given

fixed rank. A subposet of  $P$  which is obtained by deleting several levels of  $P$  is called a rank-selected subposet of  $P$ .

9.4 Rank Selection Theorem: Any rank-selected subposet of a CM poset is CM.

Proof: We will discuss in detail the case of homotopy CM. For homology with coefficients in  $R$ , one uses Theorem 5.2 in place of 5.1, and the functor  $H_1(\cdot; R)$  in place of the functor  $[S^1; \cdot]$ . See [Mu 2] for a more general homological result.

By induction, it is enough to show that CM is preserved by deleting one level.

If  $K$  is a simplicial complex,  $\sigma \in K$ , and  $\tau \in \text{Lk}(\sigma)$ , then the link of  $\tau$  in  $\text{Lk}(\sigma)$  equals the link of  $\sigma \cup \tau$  in  $K$ . It follows easily that a link in a CM complex is CM. Also, a link in a rank-selected subposet is a rank-selected subposet of a link. Therefore, it suffices to show that if  $P$  is a CM poset, and  $Q$  is obtained by deleting one level from  $P$ , then  $Q$  is a bouquet. To be specific, assume that  $P$  is a homotopy CM poset of length  $n+2$ , and  $Q$  is obtained by deleting level  $k$ . (If  $P$  has length less than 2, the result is immediate.)

Let  $j : Q \rightarrow P$  be the inclusion map. Note that there is an induced map  $j : |Q| \rightarrow |\Delta^{(n+1)}(P)|$ .

Given a chain  $\sigma$  of  $P$ , define the subposet  $C(\sigma)$  as follows: let

$$a = \min\{x \in \sigma : r(x) \geq k\},$$

$$b = \max\{x \in \sigma : r(x) \leq k\}, \text{ and}$$

$$C(\sigma) = Q_{\leq b} \cup Q_{\geq a}.$$

Note that  $\tau \subset \sigma$  implies  $C(\tau) \subset C(\sigma)$ .

Now  $j^{-1}(C(\sigma))$  is a cone if  $\sigma$  has no element of rank  $k$ ; otherwise it is a link of a singleton chain of  $P$ , hence it is a bouquet of dimension  $n+1$  by hypothesis. Thus  $\sigma \mapsto |j^{-1}(C(\sigma))|$  is an  $n$ -connected carrier from  $\Delta(P)$  to  $|Q|$ . By part (1) of the Homotopy Carrier Theorem 5.1, there is a map  $g : |\Delta^{(n+1)}(P)| \rightarrow |Q|$  such that  $g(|\sigma|) \subset |j^{-1}(C(\sigma))|$  for all  $\sigma$  in  $\Delta^{(n+1)}(P)$ .

Since  $j$  is isotone, we have

$$g \circ j(|\sigma|) = g(|j(\sigma)|) \subset |j^{-1}(C(j(\sigma)))|$$

for all  $\sigma$  in  $\Delta(Q)$ . Also,

$|\sigma| \subset |j^{-1}(C(j(\sigma)))|$ , so the function  $\sigma \mapsto |j^{-1}(C(j(\sigma)))|$  is a contractible carrier from  $\Delta(Q)$  to  $|Q|$  which carries  $g \circ j$  and  $\text{id}_Q$ . By part (2) of 5.1,  $g \circ j$  is homotopic to  $\text{id}_Q$ . It follows that  $(\text{id}_Q)_\# = (g \circ j)_\# = g_\# \circ j_\#$ , hence the map

$$j_\# : [S^i, |Q|] \longrightarrow [S^i, |\Delta^{(n+1)}(P)|]$$

is an injection for all  $i$ . From Lemma 8.10 and the assumption that  $P$  is a bouquet,  $\Delta^{(n+1)}(P)$  is a bouquet. Thus for all  $i \leq n$ ,  $[S^i, |\Delta^{(n+1)}(P)|]$  has just one element, so  $[S^i, Q]$  has just one element.  $\square$

Suppose  $P$  is a poset and  $I$  is an ideal of  $P$ . We define a new poset  $P \times I$  to be the subposet  $I \times \{\hat{0}\} \cup P \times \{\hat{1}\}$  of  $P \times \underline{2}$ . (Alternatively, one could define  $P \times I$  as a special case of the R-join construction introduced in Chapter 2.) In connection with their studies of Rees algebras, Eisenbud and Huneke [E-H] have asked when  $P \times I$  is CM. We will give an answer to that question.

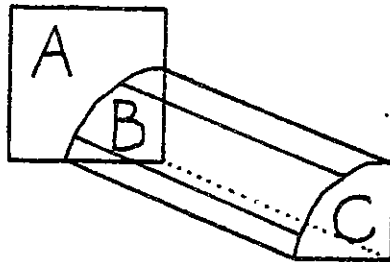
**9.5 Lemma:** If  $K$  and  $L$  are two simplicial complexes,  $K$  is a bouquet,  $K$  and  $L$  are homotopy equivalent to each other, and  $K$  and  $L$  have different dimensions, then  $L$  is a bouquet if and only if  $K$  is acyclic.

Proof: The homology case is immediate from the definition of a homology bouquet. In the homotopy case, we use the fact that a space with the homotopy type of a wedge of spheres is contractible if and only if it is acyclic over some ring  $R$ . □

9.6 Theorem: Suppose  $P$  is a CM poset and  $I$  is an ideal of  $P$ . Then  $P \times I$  is CM if and only if

- (i) if  $I \neq \emptyset$ , then  $P$  is acyclic, and
- (ii) for all  $x$  in  $P \setminus I$ , if  $I \cap P_{<x} \neq \emptyset$ , then  $P_{<x}$  is acyclic.

Proof: For convenience, let  $A = (P \setminus I) \times \{\hat{1}\}$ ,  $B = I \times \{\hat{1}\}$ , and  $C = I \times \{\hat{0}\}$ . Let  $h : P \times I \rightarrow P$  be the projection onto the first coordinate. We will proceed by analyzing the open intervals of  $P \times I$ .



First we consider  $P \times I$  as a whole. If  $I$  is empty, then of course  $P \times I$  is the same as  $P$ , which is a bouquet. Otherwise, since  $P$  is ranked (by 8.2) it is

easy to see that the dimension (length) of  $P \times I$  is greater by 1 than the dimension of  $P$ . Also,  $P \times I$  is homotopy equivalent to  $P$ , via the closure map which sends  $\langle x, \hat{0} \rangle$  to  $\langle x, \hat{1} \rangle$ . So by Lemma 9.5,  $P \times I$  is a bouquet if and only if  $P$  is acyclic. This yields constraint (i).

Next we consider open intervals of the form

$(P \times I)_{>x}$ . If  $x \in A \cup B$ , then  $(P \times I)_{>x}$  is isomorphic to an open interval of  $P$ , which we know to be a bouquet. So suppose  $x \in C$ . Then when we apply the closure map which sends each  $\langle y, \hat{0} \rangle$  to  $\langle y, \hat{1} \rangle$ , the set of closed elements has a least element, the closure of  $x$ . So in that case,  $(P \times I)_{>x}$  is contractible. Thus, the intervals  $(P \times I)_{>x}$  are bouquets without any extra constraints.

Thirdly, we consider intervals of the form  $(P \times I)_{<x}$ . If  $x \in C$ , or if  $x \in A$  but  $I \cap P_{<h(x)} = \emptyset$ , then  $(P \times I)_{<x}$  is isomorphic to the bouquet  $P_{<h(x)}$ . If  $x \in B$ , then when we apply the dual closure map which sends  $\langle y, \hat{1} \rangle$  to  $\langle y, \hat{0} \rangle$ , there is a greatest closed element, so  $(P \times I)_{<x}$  is contractible. Suppose  $x \in A$  and  $I \cap P_{<h(x)} \neq \emptyset$ . We can again apply Lemma 9.5, and conclude that  $(P \times I)_{<x}$  is a bouquet if and only if  $P_{<h(x)}$  is acyclic. This yields constraint (ii).

Finally, we consider intervals of  $P \times I$  of the form  $(x, y)$ . If  $x$  and  $y$  are both in  $A \cup B$  or both in  $C$ , then of course  $(x, y)$  is isomorphic to the interval



$(h(x), h(y))$  of  $P$ , which is a bouquet. If  $x \in C$  and  $y \in B$ , then the product theorem 4.3 (d) shows that the realization of  $(x, y)$  is either empty or homeomorphic to the suspension of  $|(h(x), h(y))|$ . A suspension of a bouquet is a bouquet, so  $(x, y)$  is a bouquet. If  $x \in C$  and  $y \in A$ , we can apply a closure map which sends each  $\langle z, \hat{0} \rangle$  to  $\langle z, \hat{1} \rangle$ , and the set of closed elements of  $(x, y)$  will have a least element. Thus, in any case, intervals of the form  $(x, y)$  lead to no constraints.  $\square$

We will now discuss a stronger version of the Cohen-Macaulay property, invented by K. Baclawski [Ba 1]. If  $K$  is a simplicial complex and  $\sigma$  is a simplex of  $K$ , then the open star of  $\sigma$ , denoted  $\text{St}(\sigma, K)$  or  $\text{St}(\sigma)$ , is the set of simplices which contain  $\sigma$ . (Note that  $\text{St}(\sigma)$  is not a subcomplex, but  $K \setminus \text{St}(\sigma)$  is a subcomplex.) We say that  $K$  is 2-Cohen-Macaulay connected over  $R$  (2-CM, for short) if  $K$  is CM over  $R$  and, for each vertex  $v$  of  $K$ ,  $K \setminus \text{St}(v)$  is CM over  $R$  and has the same dimension as  $K$ . If  $K = \Delta(P)$ , note that removing the open star of a vertex of  $K$  is equivalent to removing an element from  $P$ .

We have observed earlier that "a link in a link is a link", hence a link in a CM complex is CM. Similarly, Baclawski observed that if  $\sigma \in K$  and  $v$  is a vertex of  $\text{Lk}(\sigma, K)$ , then

$$\begin{aligned} & \text{Lk}(\sigma, K) \setminus \text{St}(v, \text{Lk}(\sigma, K)) \\ &= \text{Lk}(\sigma, K \setminus \text{St}(v, K)). \end{aligned}$$

Therefore 2-CM is also a "local" property:

9.7 Proposition: If  $K$  is 2-CM (respectively CM) of dimension  $n$ , and if  $\sigma \in K$ , then  $\text{Lk}(\sigma, K)$  is 2-CM (resp. CM) of dimension  $n - \dim \sigma - 1$ . □

Baclawski showed [Ba 1], among other things, that the proper part of a finite semimodular lattice is 2-CM if and only if the lattice is geometric. He also showed that 2-CM is invariant with respect to barycentric subdivision, and suggested that 2-CM is a topological property. We will see that his conjecture is correct.

In the proof, we will deal with both posets and simplicial complexes. It is important to keep track of the distinction, so that we will not accidentally do any barycentric subdivisions. We will use the operator  $P(\cdot)$  to denote the poset of simplices of a simplicial complex. Note that  $\Delta(P(\cdot))$  is the barycentric subdivision operator on simplicial complexes, and  $P(\Delta(\cdot)) = \text{sd}(\cdot)$ .

9.8 Theorem: Suppose  $K$  is CM over  $R$  and  $n = \dim K$ . Then  $K$  is 2-CM over  $R \Leftrightarrow$  for all  $p$  in  $|K|$ ,  $\tilde{H}_{n-1}(|K| \setminus p; R) = 0$ .

Proof: We will draw heavily on the Topological Invariance Theorem 8.3. All homology groups have coefficients in  $R$ , which is suppressed from the notation.

( $\Rightarrow$ )

Case I:  $p$  is a vertex of  $K$ .

Since  $K$  is 2-CM,  $K \setminus \text{St}(p, K)$  is CM of dimension  $n$ , so  $\tilde{H}_{n-1}(|K \setminus \text{St } p|) = 0$ . But  $|K \setminus \text{St } p|$  is a deformation retract of  $|K| \setminus p$ , so  $\tilde{H}_{n-1}(|K| \setminus p) = 0$ .

Case II:  $p$  is not a vertex of  $K$ .

Let  $\sigma$  be the unique simplex of  $K$  which has  $p$  in its interior. Choose a vertex  $v$  in  $\sigma$ . Since  $\sigma$  is not a vertex,  $\sigma_0 = \sigma \setminus \{v\}$  is another simplex.

By Proposition 9.7,  $\text{Lk}(v, K)$  is 2-CM of dimension  $n-1$ . By induction on dimension,  $\tilde{H}_{n-2}(|\text{Lk } v| \setminus \hat{\sigma}_0) = 0$ , where  $\hat{\sigma}_0$  is the barycenter of  $\sigma_0$ . There is a deformation retraction of  $|\text{Lk } v| \setminus \hat{\sigma}_0$  onto  $|P(\text{Lk } v) \setminus \{\sigma_0\}|$ , so  $\tilde{H}_{n-2}(P(\text{Lk } v) \setminus \{\sigma_0\}) = 0$ .

Let  $P = P(K)$ ,  $\Delta = \Delta(P)$ , and  $Q = P(\text{St } v) = P_{\geq v}$ . The poset  $P(\text{Lk } v) \setminus \{\sigma_0\}$  is isomorphic to  $Q \setminus \{\sigma\} \setminus \{v\}$ ; just

add  $v$  to everything. Therefore  $\tilde{H}_{n-2}(Q \setminus \{\sigma\} \setminus \{v\}) = 0$ . Since  $Q \setminus \{\sigma\}$  has a least element, it follows from a long exact sequence that  $H_{n-1}(Q \setminus \{\sigma\}, Q \setminus \{\sigma\} \setminus \{v\}) = 0$ . Then by excision,  $H_{n-1}(P \setminus \{\sigma\}, P \setminus \{\sigma\} \setminus \{v\}) = 0$ .

Note that  $P \setminus Q = P(K \setminus \text{St } v)$ , and since  $K$  is 2-CM of dimension  $n$ ,  $K \setminus \text{St } v$  is CM of dimension  $n$ . In particular, we know that  $\tilde{H}_{n-1}(P \setminus Q) = 0$ . Now we observe that there is a dual closure map on  $P \setminus \{\sigma\} \setminus \{v\}$ , defined by removing  $v$  from simplices, whose set of closed elements is  $P \setminus Q$ . Therefore  $\tilde{H}_{n-1}(P \setminus \{\sigma\} \setminus \{v\}) = 0$ .

In the two paragraphs above, we showed that  $H_{n-1}(P \setminus \{\sigma\}, P \setminus \{\sigma\} \setminus \{v\}) = 0$  and that  $\tilde{H}_{n-1}(P \setminus \{\sigma\} \setminus \{v\}) = 0$ . By a long exact sequence, it follows that  $\tilde{H}_{n-1}(P \setminus \{\sigma\}) = 0$ . Finally, if we use  $p$  as the "barycenter" of  $\sigma$ , we have  $\Delta(P \setminus \{\sigma\}) \simeq \Delta \setminus \text{St}(p, \Delta)$ , which is a deformation retract of  $|K| \setminus p$ . Therefore  $\tilde{H}_{n-1}(|K| \setminus p) = 0$ .

( $\Leftarrow$ ) Let  $v$  be a vertex of  $K$ . We will show that  $K \setminus \text{St } v$  is CM of dimension  $n$ .

Step 1. We show that  $\dim(K \setminus \text{St } v) = n$ .

Assume not. That is,  $v$  belongs to every  $n$ -simplex of  $K$ . Let  $\sigma$  be an  $n$ -simplex, and choose a point  $p$  in the interior of  $\sigma$ . Since  $\sigma$  is maximal,  $\text{St } \sigma = \{\sigma\}$ . Therefore there is a deformation retraction of  $|K| \setminus p$  onto

$|K \setminus \{\sigma\}$ . By hypothesis,  $\tilde{H}_{n-1}(|K \setminus p|) = 0$ , so  $\tilde{H}_{n-1}(|K \setminus \{\sigma\}|) = 0$ .

The case  $n = 0$  is trivial, so assume  $n > 0$ . The boundary of  $\sigma$  is an  $(n-1)$ -cycle in the complex  $K \setminus \{\sigma\}$ . However, it is not a boundary, since the face  $\sigma \setminus \{v\}$  is not a face of any  $n$ -simplex other than  $\sigma$ . Therefore  $\tilde{H}_{n-1}(|K \setminus \{\sigma\}|) \neq 0$ , which is a contradiction.

Step 2: For all  $j < n$ ,  $\tilde{H}_j(K \setminus \text{St } v) = 0$ .

Since  $|K \setminus \text{St } v|$  is a deformation retract of  $|K \setminus v|$ , it is enough to show that  $\tilde{H}_j(|K \setminus v|) = 0$  for  $j < n$ . The case  $j = n-1$  is by hypothesis. The other cases follow from the assumption that  $K$  is CM and from the long exact sequence of the pair  $(|K|, |K \setminus v|)$ .

Step 3: It remains to be shown that for all  $p$  in  $|K \setminus \text{St } v|$  and all  $j < n$ ,

$$H_j(|K \setminus \text{St } v|, |K \setminus \text{St } v \setminus p|) = 0.$$

Case I:  $p \notin \overline{\text{St}}(v, K)$ .

By excision,  $H_j(|K \setminus \text{St } v|, |K \setminus \text{St } v \setminus p|) \cong H_j(|K|, |K \setminus p|)$ , which is zero since  $K$  is CM.

Case II:  $p \in \overline{\text{St}} v$ , so  $p \in |\text{Lk}(v, K)|$ .

By Step 2 and a long exact sequence, it is equivalent to show that  $\tilde{H}_j(|K \setminus \text{St } v| \setminus p) = 0$  for all  $j \leq n-2$ .

Let  $\sigma$  be the simplex of  $\text{Lk}(v)$  which contains  $p$  in its interior. Let  $\Delta$  be the barycentric subdivision of  $K$ , where we take  $p$  as the barycenter of  $\sigma$ . Let  $m$  denote the barycenter of  $\sigma \cup \{v\}$ . The 1-simplices  $vm$  and  $mp$  determine a full subcomplex  $A$ , which we can think of as a straight line from  $v$  to  $p$ .

The complex

$$\Delta \setminus [\text{St}(v, \Delta) \cup \text{St}(m, \Delta) \cup \text{St}(p, \Delta)]$$

is a deformation retract of  $|K| \setminus |A|$ . On the other hand, the deformation retraction of  $|K| \setminus v$  onto  $|\bar{K} \setminus \text{St } v|$  restricts to a deformation retraction of  $|K| \setminus |A|$  onto  $|K \setminus \text{St } v| \setminus p$ . Thus it is enough to show that

$$\tilde{H}_j(\Delta \setminus (\text{St}(v, \Delta) \cup \text{St } m \cup \text{St } p)) = 0$$

for all  $j \leq n-2$ .

Note that  $\text{St}(v, \Delta) \cap \text{St } m = \text{St}(vm)$ . By the same reasoning as in Step 2,  $\tilde{H}_j(\Delta \setminus \text{St}(v, \Delta))$ ,  $\tilde{H}_j(\Delta \setminus \text{St } m)$ , and  $\tilde{H}_j(\Delta \setminus \text{St}(vm))$  are all zero for all  $j \leq n-1$ . So by a Mayer-Vietoris sequence [Sp, §4.6],  $\tilde{H}_j(\Delta \setminus (\text{St}(v, \Delta) \cup \text{St}(m))) = 0$  for all  $j \leq n-2$ . Since  $\text{St}(v, \Delta) \cap \text{St } p = \emptyset$ ,

$$\begin{aligned}
& [\Delta \setminus (\text{St}(v, \Delta) \cup \text{St } m)] \cup (\Delta \setminus \text{St } p) \\
& = \Delta \setminus (\text{St } m \cap \text{St } p) = \Delta \setminus \text{St}(mp).
\end{aligned}$$

As above,  $\tilde{H}_j(\Delta \setminus \text{St}(mp)) = 0$  for all  $j \leq n-1$ . So by another Mayer-Vietoris sequence,

$$\tilde{H}_j(\Delta \setminus (\text{St}(v, \Delta) \cup \text{St } m \cup \text{St } p)) = 0$$

for  $j \leq n-2$ , as desired. □

## Chapter 10. Involutions and Chromatic Number

M. Kneser made the following conjecture, which was proved about twenty-two years later by L. Lovász:

10.1 Theorem [L]: If we split the  $n$ -subsets of a  $(2n+k)$ -element set into  $k+1$  classes, then one of the classes will contain two disjoint  $n$ -subsets.

This result can be rephrased in terms of chromatic number, as follows: Construct a graph  $KG_{n,k}$  (called a Kneser graph) whose vertices are the  $n$ -subsets of a  $(2n+k)$ -set, and whose edges connect disjoint  $n$ -subsets. Then 10.1 says that the chromatic number of  $KG_{n,k}$  is at least  $k+2$ .

Here is the approach that Lovász used to prove 10.1. Let  $G$  be a (possibly infinite) simple graph. Two vertices are neighbors if they are incident to a common edge. (No vertex is its own neighbor.) Let  $\mathcal{N}(G)$  denote the set of nonempty subsets of  $V(G)$  which have a common neighbor. If  $G$  has finite neighborhoods, then  $\mathcal{N}(G)$  is a simplicial complex; therefore  $\mathcal{N}(G)$  is called the neighborhood complex of  $G$ . Theorem 2 of [L] says that

If  $|\mathcal{N}(G)|$  is  $(k-2)$ -connected, then  $G$  is not  $k$ -colorable,



and Theorem 3 of [L] implies that

$$|\mathcal{N}(KG_{n,k})| \text{ is } (k-1)\text{-connected.}$$

From these two facts, 10.1 is immediate.

The purpose of this chapter is to reformulate and generalize Lovász's work. In particular, Lovász made two suggestions for further study:

"... could Theorem 2 be strengthened by considering homology instead of homotopy, or as follows? If the  $(k-2)$ -dimensional homotopy group of  $\mathcal{N}(G)$  is trivial, then the chromatic number of  $G$  differs from  $k$ ."

We will see that the answer to the first question is yes, and the answer to the second question is no.

As above, let  $G$  be a simple graph. Given a set  $A$  of vertices of  $G$ , let  $v(A)$  denote the set of common neighbors of  $A$ . Then  $v$  is an antitone map from  $2^{V(G)}$  to  $2^{V(G)}$ , and  $v \circ v \geq \text{id}$ . It follows that  $v \circ v$  is a closure map, and  $v \circ v \circ v = v$ . Now define  $\mathcal{L}(G)$  to be the subset of closed elements of  $2^{V(G)}$  with respect to the closure map  $v \circ v$ .  $\mathcal{L}(G)$  is bounded, with  $\hat{0} = \emptyset$  and  $\hat{1} = V(G)$ . It is a general fact about closure maps that any

meet of closed elements is closed:

$$(\forall i) \quad \bigwedge \{x_i\} \leq x_i$$

$$\Rightarrow (\forall i) \quad \overline{\bigwedge \{x_i\}} \leq \bar{x}_i = x_i$$

$$\Rightarrow \overline{\bigwedge \{x_i\}} \leq \bigwedge \{x_i\}.$$

In particular, an arbitrary subset of  $\mathcal{L}(G)$  has a meet in  $\mathcal{L}(G)$ . It follows that an arbitrary subset  $A$  of  $\mathcal{L}(G)$  has a join in  $\mathcal{L}(G)$ : take the meet of all upper bounds of  $A$ . Therefore,  $\mathcal{L}(G)$  is a complete lattice. (Complete means that arbitrary subsets, not just finite subsets, have meets and joins.) We call  $\mathcal{L}(G)$  the neighborhood lattice of  $G$ . Since  $v \circ v \circ v = v$ ,  $\mathcal{L}(G)$  is precisely the image of  $v$ . Therefore  $v$  restricts to an antitone map from  $\mathcal{L}(G)$  to  $\mathcal{L}(G)$ , such that  $v \circ v = \text{id}$ .

It is easy to check that  $v \circ v$  restricts to a closure map of  $\mathcal{N}(G)$ , and that the set of closed elements is precisely the proper part of  $\mathcal{L}(G)$ . Therefore,  $|\bar{\mathcal{L}}(G)|$  is homotopy equivalent to  $|\mathcal{N}(G)|$ . So any statement about the homology or homotopy of  $|\bar{\mathcal{L}}(G)|$  applies equally well to  $|\mathcal{N}(G)|$ .

An ortholattice is a (bounded) lattice equipped with a unary operation  $a \rightarrow a'$  satisfying the properties:

- (i)  $a \leq a' \Rightarrow a = \hat{0}$ ,
- (ii)  $a \leq b \Rightarrow b' \leq a'$ , and
- (iii)  $a'' = a$ ,

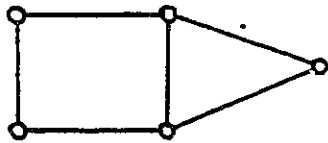
for all  $a$  and  $b$ . (An equivalent set of axioms is given in [Bi, §II-14].) We have already seen that the map  $v$  on  $\mathcal{L}(G)$  satisfies properties (ii) and (iii). Since no vertex is its own neighbor,  $v$  also satisfies (i). Therefore the pair  $\langle \mathcal{L}(G), v \rangle$  is an example of a complete ortholattice.

A 6-cycle has the same neighborhood lattice as two disjoint 3-cycles. However, the map  $v$  behaves differently in those two examples. In other words,  $\mathcal{L}(C_6)$  and  $\mathcal{L}(C_3 + C_3)$  are not isomorphic as ortholattices. So the question arises: Does the ortholattice  $\mathcal{L}(G)$  uniquely determine the graph  $G$ ? Strictly speaking, the answer is no--for example,  $\mathcal{L}(G)$  ignores isolated vertices. However,  $\mathcal{L}(G)$  comes close to determining  $G$ , in a sense that we now discuss.

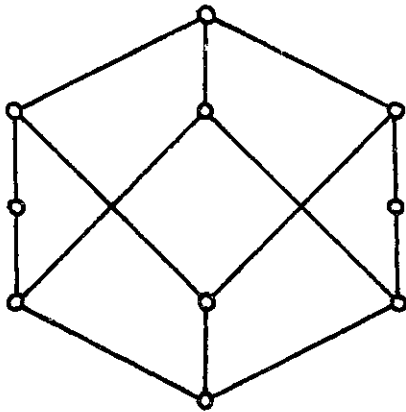
Recall that the atoms of a lattice  $L$  are the minimal elements of  $\bar{L}$ . Given an ortholattice  $\langle L, ' \rangle$ , construct a graph  $\mathcal{G}(L)$  whose vertices are the atoms of  $L$ , and such that there is an edge from  $a$  to  $b$  if  $a \leq b'$ . A lattice is atomic if each element is a join of atoms. If  $L$  is atomic, then the neighborhoods of  $\mathcal{G}(L)$  form an antichain. Then any single vertex of  $\mathcal{G}(L)$  is the set of common neighbors of its neighborhood, so the atoms of

$\mathcal{L}(\mathcal{G}(L))$  correspond precisely to the atoms of  $L$ .  
 Conversely, if the neighborhoods of  $G$  form an antichain,  
 then  $\mathcal{L}(G)$  is atomic. In fact, it turns out that  $\mathcal{L}$  and  
 $\mathcal{G}$  determine a bijective correspondence between graphs  
 whose neighborhoods form an antichain, and complete atomic  
 ortholattices.

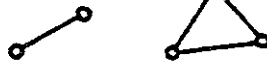
Lest the reader suspect that a neighborhood lattice  
 is always ranked, we observe that the graph



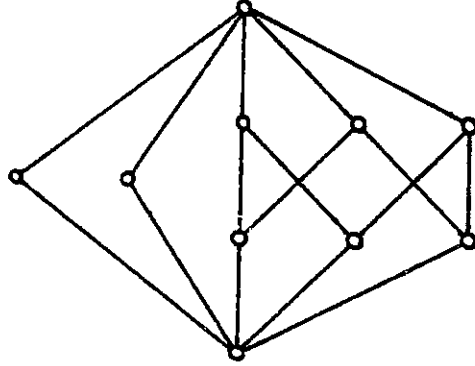
has the neighborhood lattice



and the graph



has the neighborhood lattice



Let us consider the Kneser graphs  $KG_{n,k}$ . We have a  $(2n+k)$ -element set, call it  $S$ , and the vertices are  $n$ -subsets. Given a set of vertices  $\{A_1, A_2, \dots, A_m\}$ , the set of common neighbors is the set of  $n$ -subsets of  $S \setminus \cup\{A_i\}$ . Conversely, if  $B$  is a subset of  $S$  with at most  $n+k$  elements, then the set of  $n$ -subsets of  $B$  is the set of common neighbors of some nonempty set of  $n$ -subsets. By this correspondence between  $B$  and its set of  $n$ -subsets, we see that  $\bar{\mathcal{L}}(KG_{n,k})$  is isomorphic to the set of subsets of  $S$  of cardinality at least  $n$  and at most  $n+k$ . Thus  $\bar{\mathcal{L}}(KG_{n,k})$  and  $\mathcal{L}(KG_{n,k})$  are rank-selected subposets of  $2^S$ , and the involution  $v$  is simply set complementation. This leads to our version of Lovász's Theorem 3:

10.2 Theorem:  $\bar{\mathcal{L}}(KG_{n,k})$  is homotopy CM of dimension  $k$ .

Proof: We can see that a finite Boolean algebra is homotopy CM by using the product theorem 9.3, the semimodularity theorem 8.11, or by just looking at the intervals. So the result follows from the Rank Selection Theorem 9.4. □

Now we need to translate colorability into a property of neighborhood lattices.

10.3 Lemma:  $G$  can be properly  $k$ -colored if and only if  $\bar{\mathcal{P}}(G)$  can be written as a union of ideals  $J_1, J_2, \dots, J_k$  such that  $J_i \cap v(J_i) = \emptyset$  for each  $i$ .

Proof: ( $\Rightarrow$ ) Suppose the vertices of  $G$  have been properly colored with colors  $1, 2, \dots, k$ . For each  $i$ , let  $J_i$  denote the set of all  $A$  in  $\bar{\mathcal{P}}(G)$  such that  $v(A)$  contains a vertex of color  $i$ . Clearly  $J_i$  is an ideal and  $\bar{\mathcal{P}}(G) = J_1 \cup J_2 \cup \dots \cup J_k$ . Suppose  $A \in J_i$  and  $v(A) \in J_i$ . Then  $A = v(v(A))$  and  $v(A)$  both contain a vertex of color  $i$ . But every vertex of  $A$  is a neighbor of every vertex of  $v(A)$ , so this contradicts the assumption that  $G$  is properly colored. Therefore  $J_i \cap v(J_i) = \emptyset$ .

( $\Leftarrow$ ) Suppose  $\bar{\mathcal{P}}(G) = J_1 \cup \dots \cup J_k$ , where each  $J_i$  is an ideal such that  $J_i \cap v(J_i) = \emptyset$ . Color each vertex  $s$  of  $G$

with some color  $i$  such that  $v(\{s\}) \in J_i$ . (If  $s$  is an isolated vertex, color it arbitrarily.) Now suppose two vertices  $s, t$  are neighbors and have the same color. Then there is some  $i$  such that  $v(\{s\}) \in J_i$  and  $v(\{t\}) \in J_i$ . Since  $t$  is a neighbor of  $s$ ,  $\{t\} \subset v(\{s\})$ . Since  $v$  is antitone, it follows that  $v(v(\{s\})) \subset v(\{t\})$ . But  $v(\{t\}) \in J_i$  and  $J_i$  is an ideal, so  $v(v(\{s\})) \in J_i$ . We have shown that  $v(\{s\}) \in J_i$  and  $v(v(\{s\})) \in J_i$ , which contradicts the assumption that  $J_i \cap v(J_i) = \emptyset$ .  $\square$

If  $X$  is a topological space, a map  $f : X \rightarrow X$  is called an involution if  $f \circ f = \text{id}_X$ . For example, the realization of  $v$  is an involution of  $|\mathcal{Z}(G)|$ .

Suppose  $X$  and  $Y$  are spaces with involutions  $f, g$  respectively. A continuous map  $\phi : X \rightarrow Y$  is said to be equivariant if it respects the involutions, i.e.

$g \circ \phi = \phi \circ f$ . We will be particularly concerned with equivariant maps into the sphere  $S^n$ , where the involution  $a : S^n \rightarrow S^n$  is always taken to be the antipodal map.

The following fact is known (see [C-F]) but we reproduce it here for completeness.

10.4 Lemma: Suppose  $X$  is a normal space with an involution  $g$ . There exists an equivariant map  $\phi : X \rightarrow S^n \rtimes X$  can be written as a union of closed sets  $A_1, A_2, \dots, A_{n+2}$ ,

such that  $A_i \cap g(A_i) = \emptyset$  for each  $i$ .

Proof: ( $\Rightarrow$ ) Assume  $\phi : X \rightarrow S^n$  is equivariant. We can triangulate  $S^n$  as the boundary of an  $(n+1)$ -simplex; let  $B_1, B_2, \dots, B_{n+2}$  be the closed maximal faces. Then  $B_i \cap a(B_i) = \emptyset$  for each  $i$ , so the closed sets  $A_i = \phi^{-1}(B_i)$  have the desired property.

( $\Leftarrow$ ) Assume  $X = A_1 \cup A_2 \cup \dots \cup A_{n+2}$ , where each  $A_i$  is a closed set with the property that  $A_i \cap g(A_i) = \emptyset$ . Since  $A_i$  and  $g(A_i)$  are disjoint closed sets in a normal space, Urysohn's lemma says that there is a continuous map  $\beta_i : X \rightarrow I$  such that  $\beta_i(A_i) = \{0\}$  and  $\beta_i(g(A_i)) = \{1\}$ . Define a map  $f : X \rightarrow \mathbb{R}^{n+1}$  by

$$f(x) = \langle \beta_1(x), \beta_2(x), \dots, \beta_{n+1}(x) \rangle.$$

I claim that  $f(x)$  never equals  $f(g(x))$ . If  $f(x) = f(g(x))$ , then by definition of the  $\beta_i$ 's,  $x$  cannot belong to any of  $A_1, A_2, \dots, A_{n+1}$ . Therefore  $x$  must belong to  $A_{n+2}$ . By the same argument,  $g(x)$  must belong to  $A_{n+2}$ . This violates the assumption that  $A_{n+2} \cap g(A_{n+2}) = \emptyset$ .

Having shown that  $f(x) \neq f(g(x))$  for all  $x$ , we can define a map  $\phi : X \rightarrow S^n$  by



$$\phi(x) = \frac{f(x) - f(g(x))}{|f(x) - f(g(x))|}.$$

It is easy to check that  $\phi$  is equivariant. □

10.5 Theorem: If  $G$  can be properly  $k$ -colored, then there exists an equivariant map  $\phi : |\bar{\mathcal{Z}}(G)| \rightarrow S^{k-2}$ .

Proof: Assume  $G$  can be properly  $k$ -colored. By Lemma 10.3,  $\bar{\mathcal{Z}}(G)$  can be written as a union of ideals  $J_1, J_2, \dots, J_k$  such that  $J_i \cap v(J_i) = \emptyset$ , for each  $i$ . Each finite chain of  $\bar{\mathcal{Z}}(G)$  belongs to one of these ideals, so

$$|\bar{\mathcal{Z}}(G)| = |J_1| \cup |J_2| \cup \dots \cup |J_k|.$$

If  $J_i \cap v(J_i) = \emptyset$ , then  $|J_i| \cap |v(J_i)| = \emptyset$ . Since  $v(|J_i|) \subset |v(J_i)|$ , it follows that  $|J_i| \cap v(|J_i|) = \emptyset$ . The sets  $|J_i|$  are closed in  $|\bar{\mathcal{Z}}(G)|$ , and any realization of a simplicial complex is normal [Sp, 3.1.17], so the result follows by Lemma 10.4. □

Unfortunately, the converse of Theorem 10.5 is false. To prove that, we need the following fact.

10.6 Proposition: If  $G$  contains no 4-cycles, then  $\dim |\bar{\mathcal{Z}}(G)| \leq 1$ .

Proof: We prove the contrapositive. Suppose  $\bar{\mathcal{P}}(G)$  has a chain of length 2:  $v(A) > v(B) > v(C)$ . Then  $v^2(C) > v^2(B) > v^2(A)$ . Choose vertices of  $G$ :  $r \in v(B) \setminus v(C)$ ,  $s \in v(C)$ ,  $t \in v^2(B) \setminus v^2(A)$ ,  $u \in v^2(A)$ . Since  $v(B) \cap v^2(B) = \emptyset$ , these vertices are all distinct. Now  $r$  and  $s$  are neighbors of  $t$  and  $u$ , so  $G$  contains a 4-cycle.  $\square$

There is a theorem [C-F, 3.7] which implies that if  $P$  is a poset of length  $n$  with an involution without fixed points, then there is an equivariant map  $|P| \rightarrow S^n$ . So by 10.6, if  $G$  has no 4-cycles, then there is an equivariant map  $|\bar{\mathcal{P}}(G)| \rightarrow S^1$ . If the converse of 10.5 were true, we could conclude that any graph without 4-cycles can be properly 3-colored. But that is false, since there exist finite graphs with arbitrarily high girth and chromatic number [Er].

The fact that if  $G$  has no 4-cycles then  $\dim |\bar{\mathcal{P}}(G)| \leq 1$  also allows us to answer Lovasz's second question, in the negative. If  $\dim |\bar{\mathcal{P}}(G)| \leq 1$ , then the higher homotopy groups of  $|\bar{\mathcal{P}}(G)|$  are all trivial. That is because each 1-dimensional simplicial complex (which is essentially the same as a graph) has a universal covering space which is a tree, hence contractible. Therefore, the chromatic number of  $G$  is not always indicated by a non-trivial homotopy group.

Now we need another general fact about equivariant maps, which seems to be new.

10.7 Theorem: Suppose  $X$  is a space with involution  $v$ , and  $g : X \rightarrow S^n$  is an equivariant map.

Then there exists  $j \leq n$  and  $\beta \in \tilde{H}_j(X; \mathbb{Z}/2)$  such that  $\beta \neq 0$  and  $v_*(\beta) = \beta$ .

Furthermore, if no such  $\beta$  exists for  $j < n$ , then  $\beta$  can be chosen so that  $g_*(\beta)$  is the nonzero element of  $\tilde{H}_n(S^n; \mathbb{Z}/2)$ .

Proof: The case  $n = 0$  is straightforward, so we assume  $n > 0$ . The proof will proceed by inductively constructing singular chains. Bear in mind that signs can be ignored, since we are using  $\mathbb{Z}/2$  coefficients.

We will assume that there do not exist  $j < n$  and  $\beta \in \tilde{H}_j(X; \mathbb{Z}/2)$  such that  $\beta \neq 0$  and  $v_*(\beta) = \beta$ . So the goal is to produce a nontrivial element  $\beta$  of  $H_n(X; \mathbb{Z}/2)$  such that  $v_*(\beta) = \beta$  and  $g_*(\beta)$  is nontrivial.

It is convenient to define a "symmetrizer" chain map  $\theta = \text{id}_\# + v_\#$  on  $X$ . We use the same notation for the chain map  $\text{id}_\# + a_\#$  on  $S^n$ , where  $a$  is the antipodal map. These operators can be easily verified to satisfy  $\theta\theta = 0$  and  $\theta g_\# = g_\# \theta$ .

If  $j < n$ , we have assumed that if  $\beta \in \tilde{H}_j(X; \mathbb{Z}/2)$  and  $v_*(\beta) = \beta$ , then  $\beta = 0$ . Consequently, if a  $j$ -cycle  $x_j$  satisfies  $\theta x_j = 0$ , then  $x_j$  must be a boundary.

Step 1: We observe that there are singular  $j$ -chains  $h_j$  in  $S^n$ ,  $0 < j < n$ , such that

$h_0$  is an elementary 0-chain,  $\partial h_j = \theta h_{j-1}$   
for  $1 \leq j \leq n$ , and  $\theta h_n$  generates  
 $H_n(S^n; \mathbb{Z}/2)$ .

One should think of  $h_j$  as a hemisphere of dimension  $j$ .

Step 2. We construct singular  $j$ -chains  $c_j$  in  $X$ ,  
 $0 < j \leq n$ , such that

$c_0$  is an elementary 0-chain, and  $\partial c_j = \theta c_{j-1}$   
for  $1 \leq j \leq n$ .

We have assumed that there is no nontrivial  $\beta \in \tilde{H}_{-1}(X; \mathbb{Z}/2)$  such that  $v_*(\beta) = \beta$ , so  $X$  is nonempty. Pick a point in  $X$ , and let  $c_0$  be the corresponding elementary 0-chain. Now  $\theta c_0$  is a 0-cycle (with respect to reduced homology.) Since  $\theta \theta c_0 = 0$ , there is a 1-chain  $c_1$  such that  
 $\partial c_1 = \theta c_0$ .

Suppose that  $\partial c_j = \theta c_{j-1}$  for some  $j$  such that  $1 < j < n$ . We note that  $\partial \theta c_j = \theta \partial c_j = \theta \theta c_{j-1} = 0$ , so  $\theta c_j$  is a cycle. Since  $\theta \theta c_j = 0$ , there exists a  $(j+1)$ -chain  $c_{j+1}$  such that  $\partial c_{j+1} = \theta c_j$ . This completes the induction.

Step 3. We inductively construct  $j$ -chains  $e_j$  in  $S^n$ ,  $0 \leq j \leq n$ , such that

$$h_j - g_{\#} c_j - \theta e_j \quad \text{is a cycle.}$$

To begin, note that  $h_0 - g_{\#} c_0$  is a 0-cycle, since  $h_0$  and  $c_0$  were chosen to be elementary 0-chains, so we can take  $e_0 = 0$ .

Now suppose that  $e_j$  is a  $j$ -chain,  $j < n$ , such that  $h_j - g_{\#} c_j - \theta e_j$  is a cycle. Since  $\tilde{H}_j(S^n; \mathbb{Z}/2) = 0$ , there is a  $(j+1)$ -chain  $e_{j+1}$  such that

$$\partial e_{j+1} = h_j - g_{\#} c_j - \theta e_j.$$

When we apply  $\theta$ , we obtain

$$\partial \theta e_{j+1} = \theta h_j - g_{\#} \theta c_j.$$

By Step 1 and Step 2, this is the same as

$$\partial\theta e_{j+1} = \partial h_{j+1} - \partial g_{\#} c_{j+1},$$

which implies that  $h_{j+1} - g_{\#} c_{j+1} - \theta e_{j+1}$  is a cycle, as desired.

Step 4: We complete the proof.

By Step 1 and Step 3,  $h_n - g_{\#} c_n - \theta e_n$  is a cycle which is homologous to either zero or  $\theta h_n$ . In either case, when we apply  $\theta$ , we find that  $\theta h_n - g_{\#} \theta c_n$  is homologous to zero. That is,  $\theta h_n$  and  $g_{\#} \theta c_n$  belong to the same homology class. Note that  $\partial\theta c_n = \theta\partial c_n = \theta\theta c_{n-1} = 0$ , so  $\theta c_n$  is a cycle. Therefore, if  $\beta$  is the homology class of  $\theta c_n$ , then  $g_{\#}(\beta)$  is the nonzero element of  $H_n(S^n; \mathbb{Z}/2)$ . Finally, since  $\theta\theta c_n = 0$ ,  $v_{\#} \theta c_n = \theta c_n$ , so  $v_{\#}(\beta) = \beta$ .  $\square$

Remark: Theorem 10.7 implies the well known fact that there is no equivariant map  $S^m \rightarrow S^n$  for  $m > n$ , which in turn implies such familiar facts as the Borsuk-Ulam theorem, the hairy ball theorem, and the ham sandwich theorem. With the aid of the Universal Coefficient Theorem, one can also deduce that any equivariant map from  $S^n$  to  $S^n$  has odd degree.

Finally we have our homology version of Lovász's Theorem 2:

10.8 Theorem: If  $\tilde{H}_j(\mathcal{P}(G); \mathbb{Z}/2) = 0$  for  $j \leq k-2$ , then  $G$  is not  $k$ -colorable.

Proof: By 10.5 and 10.7. □

Proof of 10.1: By 10.2 and 10.8. □

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