PLASTIC ANISOTROPY OF BODY-CENTERED CUBIC METALS

by

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ABSTRACT

Plastic anisotropy in body-centered cubic (BCC) metals was studied analytically and experimentally with reference to crystallographically derived anisotropic yield loci. Two separate two-dimensional loci, denoted planar-stress and planar-strain, were chosen to bound the yield locus for any textured BCC metal. These loci were chosen such that planar-stress deformation occurs in the absence of shear stresses, planar-strain, in the absence of shear strains.

Planar-stress and planar-strain loci were calculated for both restricted glide, or (110) <111> slip, and pencil glide, or slip with equal ease on any plane containing the <111> slip direction. Restricted-glide planar-stress and planar-strain loci were calculated from existing procedures: the generalized Schmid's law and the procedure of Bishop and Hill, respectively. To calculate pencil-glide planar-stress and planar-strain loci, the pencil-glide equivalent of the generalized Schmid's law was derived, and an approximate equivalent of the Bishop-Hill procedure was found.

Component and rotationally symmetric yield loci were calculated for the "cube-on-face", "cube-on-edge", and "cube-on-corner" textures. The limiting drawing ratio (LDR) in deep drawing and earing behavior were treated with rotationally symmetric and component loci, respectively. The results of these calculations were insensitive to slipping mode.
The analytically and experimentally determined anisotropy in a textured low-carbon steel sheet were compared. Attention was focussed on the ratio of plane-strain to uniaxial flow stress. The crystallographic texture was determined from (200) and (110) pole figures, and the results were used to calculate upper and lower bounds to the yield loci for both restricted and pencil glide. An increase in the ratio of plane-strain to uniaxial flow stress was calculated and traced to two textural origins: the presence of a cube-on-corner texture and the absence of a cube-on-face, with the former providing the larger increase. Uniaxial-tensile and plane-strain-tension and compression tests were performed. The ratio of plane-strain to uniaxial flow stress determined from these tests was in good agreement with average ratios calculated from the crystallographically derived loci for both slipping modes. The plane-strain to uniaxial stress ratio calculated from the Hill rotationally symmetric anisotropic continuum theory was also in good agreement with the experimental results.

Plane-strain compression tests were performed on corner orientation α-iron single crystals. A pronounced anisotropy was noticed among the various orientations. On the basis of a shear-stress:shear-strain comparison, deformation appeared to occur more nearly under conditions of planar stress.

Thesis Supervisor: Walter A. Backofen
Title: Professor of Metallurgy
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Preface

Plastic anisotropy, long a concern in both deformation processing and structural applications, has only recently been recognized as a means of property control. "Texture hardening" has been shown to provide dramatic improvements in the plastic properties of HCP metals, which commonly develop very sharp textures and can deform by slip on only a limited number of systems. Body-centered cubic metals, which have a larger number of slip systems than HCP metals, would still be expected to develop plastic anisotropy as a result of crystallographic texture, although the magnitude of the plastic anisotropy would not be expected to be as pronounced.

The plastic anisotropy of BCC metals, particularly a iron and low carbon steels, is the subject of this investigation. Special emphasis is placed on the determination, both theoretically and experimentally, of appropriate anisotropic yield loci, since with these loci the most nearly complete descriptions can be made of the plastic behavior of textured BCC metals. Yield loci for textured BCC metals are derived for both "restricted glide", or \{110\} \{211\} slip, and "pencil glide", or slip with equal ease on any plane containing the \{211\} slip direction.

An experimental program has been carried out to measure the extremes of texture hardening that might be present in ideally-textured BCC metals, as well as to measure the plastic anisotropy
in a sheet which contains several diffuse or "smeared" textural components. The results of these experiments provide a valuable test of the theoretical procedures for predicting plastic anisotropy.

With this overall concern, it was decided that the thesis would be presented most conveniently in two parts:

I - A Theoretical Description of the Plastic Anisotropy of Body-centered Cubic Metals

II - An Experimental Investigation of the Plastic Anisotropy of Body-centered Cubic Metals.
PART I

A THEORETICAL DESCRIPTION OF THE PLASTIC ANISOTROPY
OF BODY-CENTERED CUBIC METALS

I. Introduction

Thus far, theoretical attempts to describe plastic anisotropy have frequently made use of the ratio of width to thickness strains (R) as measured in a conventional tension test. Backofen, Hosford and Burke\(^1\), in first introducing "texture hardening", used R as a parameter to describe the yielding and plastic flow of a material with varying degrees of "normal" anisotropy. This was done by making use of a modified form of the anisotropic continuum locus suggested by Hill\(^2\). To arrive at their end result, it was also necessary to assume that the uniaxial yield strength did not change as R changed and that the principal axes of stress and strain coincided, both conditions that are often absent in actual textured metals.

A more direct theoretical description of plastic anisotropy is possible through a crystallographic analysis which follows directly from a knowledge of the constituent textural components and a description of the crystallographic slipping process. By deriving anisotropic yield loci for both restricted and pencil glide directly from crystallographic analyses, it was thought that the plastic behavior of textured BCC metals would be more realistically described. Using these crystallographically derived
anisotropic yield loci, it should then be possible to evaluate the relative structural and processing characteristics of different textural components which can be present alone or in combination in a textured BCC metal.

The theoretical attempts to relate the plastic properties of polycrystalline aggregates to those of single crystals have been primarily concerned with randomly oriented metals, particularly of FCC structure. These descriptions are also valid for BCC metals deforming by restricted glide, since the \{111\} <110> systems of FCC are equivalent to \{110\} <111> restricted-glide systems of BCC. Sachs\(^3\) first attempted to find the uniaxial stresses necessary to cause yielding through operation of the most highly stressed \{111\} <110> system in crystals of different orientations. Similar calculations were made by Cox and Sopwith\(^4\) and Kochendörfer.\(^5\) Since all of these above procedures satisfy only the conditions of stress equilibrium and stress boundary conditions in an average sense, but violate some or all of the strain boundary conditions, the results will all be in the nature of average lower bounds.\(^6\)

Taylor\(^7\) provided for the maintenance of cohesion among grains of various orientations by assuming that each grain undergoes the same uniform plastic strain as the aggregate. Recognizing the requirement, first pointed out by v. Mises\(^8\), that at least five independent slip systems must operate simultaneously in order to accommodate an arbitrary shape change in a FCC crystal, he used his intuitively derived minimum shear principle to find the ratio of uniaxial yield stress to critical
resolved shear stress. Taylor's minimum shear principle was put on a firmer theoretical basis by Bishop and Hill$^9$ and Bishop$^{10}$, who showed that those five (or more) slip systems which Taylor chose to accommodate the imposed strain are indeed the most highly stressed slip systems. In addition, these latter authors showed that there are only 56 distinct states of stress which are capable of operating at least five slip systems, and that, in fact, these states of stress always cause the critical resolved shear stress to be reached simultaneously on 6 or 8 slip systems. These latter authors also suggested the use of the mathematically equivalent, but computationally less cumbersome, maximum work principle. The details of the Bishop-Hill procedure are contained in Appendix I. As Taylor$^6$ subsequently pointed out, all these procedures which satisfy only strain compatibility lead to upper bound solutions.

The maximum work principle, which can conveniently be applied to conditions other than uniaxial loading, was used by Bishop and Hill$^9$ to calculate the entire yield locus for a randomly oriented FCC polycrystalline aggregate. This was accomplished by subjecting crystals of various orientations to the same strain state, and, through the use of the maximum work principle, finding the appropriate yield stresses and averaging the results. Hutchinson$^{11}$ developed a similar procedure to find approximately the yield locus for a randomly oriented BCC metal deforming by pencil glide. Slip was allowed to occur on 40 slip systems (10 slip planes for each $\langle 111 \rangle$ direction) and the state of stress that simultaneously operated the five most
highly stressed systems was found. This latter stress-based procedure was shown to correspond to the condition that all grains deform by the same strain as the aggregate. Some steps were taken by Hosford and Backofen\textsuperscript{13} to predict anisotropic yield loci for FCC metals (or BCC metals deforming by restricted glide); their procedure was based on a hybrid minimal principle and involved the macroscopic parameter, $R$, but it did provide the first results dealing explicitly with the textured state.

Lin\textsuperscript{14,15}, Kröner\textsuperscript{16}, Budiansky and Wu\textsuperscript{17}, and Hutchinson\textsuperscript{11,12} have proposed several sophisticated models to account for the elastic interactions among the deforming grains. An important and convenient consequence of the calculations of Budiansky and Wu is that the stress-strain curves, both in tension and shear, rapidly become asymptotic to the rigid-plastic isostrain solution of Taylor, reaching 99 percent of the Taylor value by the time the total strain is 10 times the elastic yield strain. Similar results were found by Hutchinson from his pencil-glide model. In this latter case, 99 percent of the rigid-plastic isostrain yield stress was reached by the time the total strain was only 6 times the elastic yield strain. In a textured metal, where grain boundary restraints would be expected to be less severe than in a metal with randomly oriented grains, the isostrain assumption seems even more justified. The lessening of the grain boundary restraints in a textured metal also leads one to expect that the grain size
dependence of the yield stress in a textured metal would be weaker than in a metal with randomly oriented grains.

In addition to the lack of a grain size dependence of the yield stress, the isostrain assumption has several other shortcomings. Implicit in this model is the assumption that the critical resolved shear stress is constant for all slip systems and that latent hardening is equal to active hardening, conditions which do not exist in many metals. Deformation is also assumed to occur only by slip, which is taken to occur uniformly. However, there are many cases of practical interest in which the isostrain assumption can lead to useful conclusions about plastic anisotropy.
II. Derivation of Anisotropic Yield Loci for Textured BCC Metals Deforming by Restricted Glide

A difficulty arises in the description in two dimensions of an anisotropic yield locus because of the frequent lack of coincidence of the principal axes of stress and strain. For isotropic materials, in which principal axes of stress and strain coincide, the complete yield locus can be described by one two-dimensional section taken normal to the hydrostatic line in principal stress space. Since this simple two-dimensional description is not always possible for anisotropic metals, two separate mathematically and physically meaningful two-dimensional sections are suggested.

Of the two-dimensional yield loci chosen to describe anisotropic plastic behavior, one will be called the "planar-stress" and the other the "planar-strain" yield locus. Both will have as variables two normal stresses acting in the plane of a sheet and will be chosen such that the third normal stress, which acts perpendicular to the plane of the sheet, is zero. The distinction between the two types of loci is that, in the planar-stress locus, the normal stresses are chosen to be principal stresses and the resulting strain state may contain both normal and shear strains, while, in the planar strain locus, the strain state is chosen to contain only principal strains and the requisite stress state may contain both normal and shear components. Hence, the planar strain locus cannot always show the complete state of stress necessary to cause the material to experience only normal strains along
the locus coordinate directions. The planar stress locus, on the other hand, will often apply when there are components of shear strain which cannot be represented but which act normal to the locus coordinate directions in the chosen principal stress space. When the principal directions of stress and strain coincide, the planar-stress and planar-strain loci are identical. An example of the properties of the planar-stress and planar-strain yield loci for the pure "cube-on-corner" textural component is contained in Appendix II. The planar-stress and planar-strain yield loci for textured BCC metals deforming by restricted glide can be derived using existing analytical procedures.

Planar-stress restricted-glide loci can be found directly from the generalized Schmid's law (Appendix I). This is accomplished by expressing the stresses referred to the cubic axes in terms of the stress \( \sigma_{rr} \) and \( \sigma_{tt} \) acting along the rolling and transverse directions in a sheet containing the particular texture of interest. The appropriate stress relationships can be found from the transformation

\[
\sigma_{ij} = l_ir_j \sigma_{rr} + l_it_j \sigma_{tt} \quad (1)
\]

Substituting equation (1) into the generalized Schmid's law equations gives the planar-stress yield loci for restricted glide.
Planar-strain yield loci for restricted glide can be found from the maximum work of Bishop and Hill. Now the first step is to express the strains $d\varepsilon_{ij}$ referred to the cubic axes in terms of the imposed principal strains $d\varepsilon_{rr}$, $d\varepsilon_{tt}$, and $d\varepsilon_{nn}$ acting in the rolling, transverse, and sheet normal directions for the particular textural component. This is done by using the transformation equation

$$d\varepsilon_{ij} = l_{ir} l_{jr} d\varepsilon_{rr} + l_{it} l_{jt} d\varepsilon_{tt} + l_{in} l_{jn} d\varepsilon_{nn}$$

(2)

The values of $d\varepsilon_{ij}$ are then substituted into the work expression (equation 1d), which is then evaluated for the 56 permissible polyslip stress states, and the operative stress state (or states) which maximizes the work is selected. The normal stresses referred to the textural axes can be found from the Bishop-Hill stress state $\sigma_{ij}$ referred to the cubic axes by using the transformations

$$\sigma_{rr} = l_{ri} l_{rj} \sigma_{ij}$$
$$\sigma_{tt} = l_{ti} l_{tj} \sigma_{ij}$$
$$\sigma_{nn} = l_{ni} l_{nj} \sigma_{ij}$$

(3)

The additional requirement that $\sigma_{nn}$ vanish amounts to adjusting the hydrostatic pressure and allows $\sigma_{rr}$ and $\sigma_{tt}$ to be determined uniquely. By repeating this procedure for various strain states, a complete planar-strain yield locus can be generated for a
particular single crystal orientation or, equivalently, a particular textural component. The derivations of the planar stress and planar strain yield loci for the pure "cube-on-corner" textural component are also included in Appendix II.

Mathematically, yield loci determined under conditions of planar-strain and planar-stress are upper and lower bounds to the yield locus whose principal axes of stress and strain are assumed to coincide. The upper bound character of the planar-strain locus is readily apparent, since enforcing the same strain state in all textural components immediately satisfies in each component the displacement boundary conditions imposed on the aggregate. The planar-stress locus, however, must be considered a lower bound in an average sense.

Although there are no shear stresses acting in any of the textural components, as for the case in which the principal axes of stress and strain coincide, the principal stresses which enforce the same normal strain in different textural components need not be the same. Hence the average of the principal stresses necessary to enforce the same normal strain state in different textural components satisfies the principal stress boundary conditions only in an average sense. This is similar to the condition used by Sachs\textsuperscript{3} to find an average lower bound to the uniaxial yield strength of a randomly oriented FCC aggregate.

In addition to the significance as upper and lower bounds to the locus in which the principal axes of stress and strain
coincide, the two types of loci have physical and practical significance individually. This is most apparent for cases involving a single crystal or only one ideal textural component. If, for example, a single crystal were deformed in plane strain, use of the planar-strain plane-strain strength would be appropriate if the deformation were fully constrained to avoid all shear strains. If the crystal were unconstrained in shear, the planar-stress locus would be appropriate. In a textured metal with more than one textural component the situation is not as simple because of the complicated interactions among the various components. In actuality, the deformation in this latter case cannot occur purely under conditions of planar stress or planar strain, since either stress or strain boundary conditions are violated at the grain boundaries. The actual behavior of a textured polycrystalline aggregate would be expected to be some compromise between the two types of behavior, although the details of the external shear restraints imposed during a test would again be expected to influence the plastic behavior.

In order to find the yield locus for a textured metal containing more than one textural component, it is necessary to average in some manner the contributions from the various components. The choice of averaging procedure is most obvious in the case of planar-strain loci. The values of the average normal stresses acting in the plane of the sheet are found by averaging the stresses necessary to impose the same strain state in all the textural components. The average values of the two
normal stresses for a given strain state determine the coordinates of a point on the average planar-strain yield locus, and the orientation of the strain vector, which must be normal to the yield locus, determines the slope at that point. The normality of the strain vector to the yield locus can be derived thermodynamically and can also be shown to follow from the critical resolved shear stress law for single crystals. Again, additional shear stresses are often required to enforce the imposed principal strain state, and these shear stresses may vary from component to component subjected to the same strain state, but this information is not contained in planar-strain yield loci.

An analogous averaging procedure will be used to calculate composite planar-stress yield loci. The values of the operative normal stresses, which are principal stresses in this case, will again be found by averaging the stresses necessary to produce the same normal strains in the various textural components. However, the normal strains will, in general, not be principal strains. Additional shear strains will be present and the magnitude of these shear strains may be different in the various textural components that are being forced to undergo the same state of normal strain. Shear stresses, however, are always absent.

Restricted-glide loci will now be calculated for the three common textures in iron, which have been described as the cube-on-face, cube-on-edge, and cube-on-corner textures because of the orientation of the unit BCC cube relative to
the plane of the sheet. These textures are commonly smeared somewhat about the ideal orientations. Very often the smearing occurs about the sheet normal, and, in some cases the textures are rotationally symmetric. Rotationally symmetric yield loci for these textures are of interest not only because they describe the plastic behavior when the texture is so characterized, but also because they are useful in determining the average plastic response in rotationally symmetric deformations, such as occur in deep drawing.

The averaging procedures described previously can best be visualized with reference to the individual loci of components that are rotated by various amounts about the sheet normal. Representative planar-stress and planar-strain component loci for the cube-on-face, cube-on-edge, and cube-on-corner textures chosen at three symmetric positions are shown in Figure 1. The rotationally planar-stress and planar-strain symmetric loci are calculated by first subjecting all components to the same normal strain ratio \( \frac{\varepsilon_{yy}}{\varepsilon_{xx}} \) and finding the corresponding stress components for yielding, \( \sigma_{xx} \) and \( \sigma_{yy} \). Textural components every 2.5° about the sheet normal were used in forming the average, which was calculated from the trapezoidal rule

\[
\bar{\sigma}_{xx} = \frac{1}{n} \left[ \frac{1}{2} (\sigma_{xx}^0 + \sigma_{xx}^n) + \sigma_{xx}^1 + \ldots + \sigma_{xx}^{n-1} \right]
\]

\[
\bar{\sigma}_{yy} = \frac{1}{n} \left[ \frac{1}{2} (\sigma_{yy}^0 + \sigma_{yy}^n) + \sigma_{yy}^1 + \ldots + \sigma_{yy}^{n-1} \right]
\]
Figure 1. Representative Planar-stress (inner) and Planar-strain (outer) Component Loci for the Cube-on-face, Cube-on-edge, and Cube-on-corner Textures Deforming by Restricted Glide. The orientations of the components are specified by the rotations indicated.
where 0 and n both refer to the ideal textural component and hence must only be counted once. Again, the values of $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ determine a point on the rotationally symmetric locus and the strain ratio $d\varepsilon_{yy}/d\varepsilon_{xx}$ determines the slope at that point. The details of these calculations are given in Appendix III. The rotationally symmetric loci for the cube-on-face, cube-on-edge, and cube-on-corner textures are shown in Figures 2, 3, and 4, respectively. The isotropic restricted-glide locus calculated by Bishop and Hill is shown in Figure 5.
Figure 2. Rotationally Symmetric Planar-stress and Planar-strain Yield Loci for the Cube-on-face Texture Deforming by Restricted Glide.
Figure 3. Rotationally Symmetric Planar-stress and Planar-strain Yield Loci for the Cube-on-edge Texture Deforming by Restricted Glide.
Figure 4. Rotationally Symmetric Planar-stress and Planar-strain Yield Loci for the Cube-on-corner Texture Deforming by Restricted Glide.
Figure 5. Isotropic Restricted-glide Yield Locus Calculated by Bishop and Hill (ref. 9).
III. The Prediction of Anisotropic Yield Loci for Textured BCC Metals Deforming by Pencil Glide

Thus far, plastic flow in BCC metals under combined stresses has been described only for restricted glide. Taylor\textsuperscript{6} derived a minimum-shear principle for finding the axisymmetric flow strengths of BCC crystals deforming by pencil glide, but made no calculations. Chin and Mammel\textsuperscript{20}, assuming that slip occurs only on \{110\}, \{112\}, and \{123\} planes, calculated the distribution of axisymmetric flow strengths over the stereographic triangle by using the principle of minimum shear. Only the approximate procedure of Hutchinson\textsuperscript{11} is now available for describing pencil-glide deformation under more complicated states of stress. This pencil-glide behavior under more complicated stress states will be investigated here.

The pencil-glide equivalent of the generalized Schmid's law yield conditions will be derived and polyslip stress states which simultaneously operate slip in four and three <111> directions will be found. It will then be shown that the stress states found which simultaneously operate three <111> directions do not provide the minimum of five independent degrees of freedom necessary to accommodate an arbitrary shape change, and that stress states which simultaneously operate four <111> slip directions, though they can provide five degrees of freedom, are insufficient to accommodate any arbitrary shape change. An approximate pencil-glide equivalent of the Bishop-Hill procedure will be derived and used to describe yielding under prescribed states of strain.
A. The Generalized Yield Conditions for BCC Crystals Deforming by Pencil Glide

The pencil-glide yield conditions cannot be given as directly as for restricted-glide, since the operative slip planes are not specifically designated. Slip by pencil glide can occur when the critical resolved shear stress is reached on any slip plane containing a <111> slip direction (Figure 6).

The first step now is to recognize that slip with equal ease on any plane containing a <111> slip direction is equivalent to slip with equal ease in any direction which lies in a {111} plane. This transposition was suggested by Kocks\textsuperscript{21} and amounts to switching the identity of the two direction cosines in the stress transformation equation. This equivalence can also be demonstrated geometrically (Figure 7). The geometric argument proceeds as follows: examine any plane containing a <111> slip direction on which the critical resolved shear stress $k$ is reached. The shear stress normal to the shear stress in question must also equal $k$ to ensure moment equilibrium. The latter shear stress can lie in any direction in the {111} slip plane, since the former can act on any plane containing the slip direction. Hence, a crystal which slips with equal ease on any plane containing a <111> slip direction can be viewed as slipping with equal ease in any direction in a {111} slip plane. Since it is easier to visualize slip as being equally favored along any direction in a {111} plane, this viewpoint will be used to describe the yielding of BCC crystals deforming by pencil glide.
Figure 6. Permissible Orientations of Pencil-glide Slip Planes. The dotted line in the stereogram indicates the positions of slip planes which can operate in conjunction with the [111] slip direction.
Figure 7. Geometric Representation of the Equivalence of Slip with Equal Ease on Any Plane Containing the $\{111\}$ Slip Directions and Slip with Equal Ease in Any Direction in the $\{111\}$ Slip Planes. At yielding, the orthogonal shears shown must both be equal to $k$ to endure moment equilibrium.
Let $\tau_{np}$ and $\tau_{nq}$ be shear stresses acting on a $\{111\}$ plane in a $<110>$ and $<112>$ direction, respectively. If slip is to occur with equal ease in any direction in the $\{111\}$ plane, the vector formed by adding $\tau_{np}$ and $\tau_{nq}$ must lie somewhere on a circle of radius $k$. The generalized pencil-glide yield criteria can be expressed in polar and quadratic form as follows:

Polar form: \[ \tau_{np} \cos\theta + \tau_{nq} \sin\theta = \pm k \] (5)

Quadratic form: \[ \tau_{np}^2 + \tau_{nq}^2 = k^2 \]

This pencil-glide yield locus is shown in Figure 8. The yield locus for restricted glide, which is described by 6 lines representing the attainment of the critical resolved shear stress $k$ in the $<110>$ directions, is also shown.

The shear stresses $\tau_{np}$ and $\tau_{nq}$ are found from the general state of stress $\sigma_{ij}$ acting along the cubic axes by using the stress transformations

\[ \tau_{np} = \lambda_{ni} \lambda_{pj} \sigma_{ij} \] (7)

\[ \tau_{nq} = \lambda_{ni} \lambda_{qj} \sigma_{ij} \] (8)

The results for the four $\{111\}$ slip planes, written in Bishop-Hill notation (Appendix I), are shown in Table I.
Figure 8. Pencil-glide and Restricted-glide Yield Loci for Slip on \{111\} Planes. Slip by pencil glide will occur when \( k \) is reached in any direction in the \{111\} planes (circular locus), while slip by restricted glide will occur when \( k \) is reached in one or two \(<110>\) directions (hexagonal locus).
Table I

Shear Stresses Acting in \{111\} Planes

<table>
<thead>
<tr>
<th>System</th>
<th>n</th>
<th>p</th>
<th>q</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(111)</td>
<td>[1\bar{1}0]</td>
<td>[\bar{1}1\bar{2}]</td>
<td>( \tau_{np}^{(a)} = \frac{1}{\sqrt{6}} ) (-C+F-G)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \tau_{nq}^{(a)} = \frac{1}{3\sqrt{2}} ) (-A+B+F+G-2H)</td>
</tr>
<tr>
<td>(b)</td>
<td>(\bar{1}1\bar{1})</td>
<td>[110]</td>
<td>[\bar{1}2]</td>
<td>( \tau_{np}^{(b)} = \frac{1}{\sqrt{6}} ) (C+F+G)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \tau_{nq}^{(c)} = \frac{1}{3\sqrt{2}} ) (-A+B-F+G+2H)</td>
</tr>
<tr>
<td>(c)</td>
<td>(\bar{1}1\bar{1})</td>
<td>[\bar{1}00]</td>
<td>[12]</td>
<td>( \tau_{np}^{(c)} = \frac{1}{\sqrt{6}} ) (-C-F+G)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \tau_{nq}^{(c)} = \frac{1}{3\sqrt{2}} ) (-A+B-F-G-2H)</td>
</tr>
<tr>
<td>(d)</td>
<td>(\bar{1}1\bar{1})</td>
<td>[\bar{1}00]</td>
<td>[\bar{1}2]</td>
<td>( \tau_{np}^{(d)} = \frac{1}{\sqrt{6}} ) (C-F-G)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \tau_{nq}^{(d)} = \frac{1}{3\sqrt{2}} ) (-A+B+F-G+2H)</td>
</tr>
</tbody>
</table>

To find the polar and quadratic forms of the generalized pencil-glide yield conditions, the expressions for \( \tau_{np} \) and \( \tau_{nq} \) are substituted into equations (5) and (6). The polar form of the resulting yield conditions can specify the orientations of the operative slip planes, once the stress state is known. The quadratic forms, on the other hand, can predict yielding under an arbitrary state of stress directly. The quadratic forms are, for slip with equal ease in the
[\{111\}] slip direction: \((C-F+G)^2 + (A-G+H)^2 + (B-H+F)^2 = 9k^2\)  \((9)\)

[\{1\bar{1}1\}] slip direction: \((C+F+G)^2 + (A-G-H)^2 + (B+H-F)^2 = 9k^2\)  \((10)\)

[\{\bar{1}1\bar{1}\}] slip direction: \((C+F-G)^2 + (A+G+H)^2 + (B-H-F)^2 = 9k^2\)  \((11)\)

[\{\bar{1}1\bar{1}\}] slip direction: \((C-F-G)^2 + (A+G-H)^2 + (B+H+F)^2 = 9k^2\)  \((12)\)

For the special case of principal stresses acting along the cubic axes, equations \((9) - (12)\) reduce to the v. Mises yield locus. This result is physically sensible, since one interpretation of the v. Mises yield criterion is that it represents yielding at a critical value of the shear stress acting on a macroscopic octahedral plane. The \{111\} planes bear an octahedral relationship to the cubic axes: therefore, if slip is allowed to occur with equal ease along any direction in these planes, the pencil-glide locus must reduce to the octahedral shear-stress criterion.

B. Simultaneous Slip in Three or Four <\{111\}> Directions

If an arbitrary imposed strain is to be accommodated in a BCC crystal deforming by pencil glide, slip must occur simultaneously in at least three <\{111\}> slip directions. This can be demonstrated geometrically with the aid of Figure 9, in which a shear strain acting in a \{111\} plane is imposed for both restricted-glide and pencil-glide behavior. The imposed strain, which requires the operation of two restricted-glide slip systems, can be accommodated by operating only one pencil-glide system. Hence, since arbitrary imposed strains can be accommodated by the operation of 6 or 8 restricted-glide systems, one would expect that the same strains could be accommodated by three or four pencil-glide systems.
Figure 9. Accommodation by Pencil- and Restricted-glide Slip of a Shear Strain Acting on the (111) Plane. The strain $d\varepsilon$ must be accommodated by restricted-glide slip on the $(a_1)$ and $-(a_3)$ systems. The same strain is shown being accommodated by a single pencil-glide system.
The pencil-glide strain-slip equations derived by Taylor\textsuperscript{6} can also be used to show that the accommodation of an arbitrary imposed strain requires the operation of at least three $<111>$ slip directions. The five independent components of strain are expressed by Taylor in terms of eight variables: four of the variables are the amounts of shear in the $<111>$ directions; the other four identify the orientation of the operative slip planes. In order to provide the minimum of five degrees of freedom necessary to accommodate the five independent components of strain, slip must occur in at least three $<111>$ directions. The simultaneous operation of three $<111>$ slip directions provides six degrees of freedom (three amounts of shear and three slip-plane designations), while the operation of only two $<111>$ directions would provide only four.

Unlike the generalized Schmid's law for restricted glide, the pencil-glide yield conditions cannot be viewed geometrically in three separate 3-dimensional spaces. Each pencil-glide condition contains not three, but all six of the Bishop-Hill stresses, and furthermore, coupling exists among all these stresses. However, if the four yield expressions (equations 9 - 12) are added, the coupling cancels, and the following relationship results.

$$A^2 + B^2 + C^2 + 2F^2 + 2G^2 + 2H^2 = 9k^2$$  \hspace{1cm} (13)

If equation (13) is subtracted from equations (9)-(12), the following relationships, which must hold for any stress system
which simultaneously operates all four $\langle 111 \rangle$ slip directions, are obtained.

\begin{align}
-CF + CG - FG - AG + AH - GH - BH + BF - HF &= 0 \quad (14) \\
CF + CG + FG - AG - AH + GH + BH - BF - HF &= 0 \quad (15) \\
CF - CG - FG + AG + AH + GH - BH - BF + HF &= 0 \quad (16) \\
-CF - CG + FG + AG - AH - GH + BH + BF + HF &= 0 \quad (17)
\end{align}

Only three of the four relationships are independent, however, since the sum of the first three is equal to the negative of the fourth. By rearranging equations (14)-(17), the following independent relationships are obtained.

\begin{align}
\text{equation 15 + equation 16}/2 & \quad F(B-C) = GH \quad (18) \\
\text{equation 16 + equation 17}/2 & \quad G(C-A) = HF \quad (19) \\
\text{equation 14 + equation 15} \\
\text{equation 19}/2 & \quad H(A-B) = FG \quad (20)
\end{align}

Multiplying equations (18), (19), and (20) by $F$, $G$, and $H$, respectively, gives

\begin{align}
F^2(B - C) &= FGH \quad (21) \\
G^2(C - A) &= FGH \quad (22) \\
H^2(A - B) &= FGH \quad (23)
\end{align}

or

\begin{equation}
F^2(B - C) = G^2(C - A) = H^2(A - B) \quad (24)
\end{equation}

It will next be shown that equation (24) must equal zero. The first step is to recognize that the sum of the coefficients of $F^2$, $G^2$, and $H^2$ is zero identically, since

\begin{equation}
(B - C) + (C - A) + (A - B) = 0 \quad (25)
\end{equation}
If one of the coefficients \((B - C), (C - A),\) or \((A - B)\) differs from zero, at least one of the remaining two coefficients must be of opposite sign for equation (25) to be satisfied. If this is the case, equation (24) will be violated, since \(F^2, G^2,\) and \(H^2\) are always positive for real values of \(F, G,\) and \(H,\) but one of their coefficients is of opposite sign from the other two. Hence, admissible states of stress than can simultaneously operate slip in all four \(<\text{lll}\>\) slip directions must satisfy the relationship

\[
F^2(B - C) = G^2(C - A) = H^2(A - B) = 0
\]  

(26)

The solutions to equation (26) are

Group I: \(F = G = H = 0\)

Group II: \(A = B = C (= 0 \text{ since } A + B + C \equiv 0)\)

Group IIIa: \(B = C, G = H = 0\)

b: \(C = A, H = F = 0\)

c: \(A = B, F = G = 0\)

The states of stress that correspond to the above solutions to equation (26) can be found by substituting the solutions into equation (13). The conditions imposed by the solutions in Group I lead to the locus of permissible stress states

Group I: \(A^2 + B^2 + C^2 = 9k^2\)

The conditions imposed by the Group II solutions further imply that only one of the shear stresses \(F, G,\) or \(H\) can be present at a time. This can be shown by referring back to equations (18)-(20), the original independent requirements for operating four
systems simultaneously. If $A = B = C = 0$, equations (18), (19), and (20) state that $GH = HF = FG = 0$. Hence two of the three shear stresses must vanish. Group II must then be further subdivided into

Group IIa: $A = B = C = G = H = 0, F \neq 0$

b: $A = B = C = H = F = 0, G \neq 0$

c: $A = B = C = F = G = 0, H \neq 0$

The following stress states which satisfy the above conditions can be found from equation (13).

Group IIa: $F^2 = \frac{3}{2} k^2$ or $F = \pm \frac{3\sqrt{2}}{2}$

b: $G^2 = \frac{3}{2} k^2$ or $G = \pm \frac{3\sqrt{2}}{2}$ (28)

c: $H^2 = \frac{3}{2} k^2$ or $H = \pm \frac{3\sqrt{2}}{2}$

The stress states that satisfy the conditions of Group III are, from equation (13)

Group IIIa: $A^2 + B^2 + C^2 + 2F^2 = 9k^2, B = C = \frac{1}{2} A$

b: $A^2 + B^2 + C^2 + 2G^2 = 9k^2, C = A = \frac{1}{2} B$

c: $A^2 + B^2 + C^2 + 2H^2 = 9k^2, A = B = \frac{1}{2} C$

..... (29)

It is interesting to note that the pencil-glide stress states in Groups I to III which simultaneously operate four slip directions have the same character as the Bishop-Hill restricted-glide stress states in Groups I to III which simultaneously operate eight slip systems (Appendix I). In both cases, Group I contains
stress states that involve only A, B, and C. Group II involves stress states that contain only one of the stresses, F, G, or H. Group III contains stress states that involve the stresses A, B, and C and, in addition, one of the stresses F, G, or H.

The procedure used to find stress states for simultaneous slip in three <111> slip directions is analogous to that used to describe the simultaneous operation of four <111> directions. Now, however, there are four separate cases to consider, which correspond to the absence of slip in one of the four <111> slip directions.

If slip does not occur in the [111] slip direction, the shear stress on that system must be below the yield stress by some quantity \( \varepsilon \). Physically this means that the radius vectors formed by adding \( \tau_{np} \) and \( \tau_{nq} \) must equal \( k \) for the three other slip directions and must be less than \( k \) for the [111]. This fourth requirement can be expressed, with the aid of equation (9) as

\[
(C - F + G)^2 + (A - G + H)^2 + (B - H + F)^2 = 9k^2 - \varepsilon \quad (37)
\]

where \( 9k^2 > \varepsilon > 0 \) for the stress on the inoperative system to be real and below \( k \). Equations (10), (11), and (12) must still be satisfied, however, since the critical resolved shear stress must be reached in the three remaining slip directions, and hence, in all, four independent conditions must be met.

Adding Equations (10), (11), (12), and (37) will again remove the coupling among the stresses, but the result, the equivalent of equation (13), will now be
\[ A^2 + B^2 + C^2 + 2F^2 + 2G^2 + 2H^2 = 9k^2 - \frac{\varepsilon}{4} \quad (38) \]

The same sum will be obtained if slip does not occur in one of the other three slip directions. However, the remaining three independent conditions, the equivalents of equations (18)-(20), will be different for the four different cases.

Case 1: \([\text{\overline{1}1\overline{1}}]\) Inoperative

Subtracting equation (38) from equations (37), (10), (11), and (12) yields the following relationships which must hold if slip occurs only in the \([\text{\overline{1}1\overline{1}}], \text{[\overline{1}1\overline{1}}],\text{ and \text{[\overline{1}1\overline{1}}] directions.}

\[
-CF + CG - FG - AG + AH - GH - BH + BF - HF = -\frac{3}{8} \quad (39)
\]
\[
CF + CG + FG - AG - AH + GH + BH - BF - HF = \frac{\varepsilon}{8} \quad (40)
\]
\[
CF - CG - FG + AG + AH - GH - BH + BF + HF = \frac{\varepsilon}{8} \quad (41)
\]
\[
-CF - CG + FG + AG - AH - GH + BH + BF + HF = \frac{\varepsilon}{8} \quad (42)
\]

Again, only three of equations (39)-(42) are independent, since equation (39) is the negative sum of the other three.

Rearranging equations (39)-(42) results in the following three independent relationships

\[
\text{(equation 39 + equation 40)/2, } C - A = \frac{1}{G} (HF - \frac{\varepsilon}{8}) \quad (43)
\]
\[
\text{(equation 39 + equation 41)/2, } A - B = \frac{1}{H} (FG - \frac{\varepsilon}{8}) \quad (44)
\]
\[
\text{(equation 39 + equation 42)/2, } B - C = \frac{1}{F} (GH - \frac{\varepsilon}{8}) \quad (45)
\]

Since the sum of equations (43)-(45) is zero, \(\varepsilon\) can be expressed in terms of the three shear stresses \(F, G,\) and \(H.\)
\[
\frac{\varepsilon}{8} = \frac{(FG)^2 + (GH) + (HF)^2}{FG + GH + HG}
\]  

(46)

Only three of equations (43)-(46) are independent, since equation (46) was found from the sum of equations (43)-(45). The fourth independent condition is equation (38).

Case 2: \([1\bar{1}1]\) Inoperative

Similar results are obtained if the \([1\bar{1}1]\) slip direction is inoperative. The yield expression for slip in the \([1\bar{1}1]\) direction (equation 10) is set equal to \(9k^2 - \varepsilon\), and, after making rearrangements similar to case 1, the following relationships result:

\[
C - A = \frac{1}{G} \left( HF - \frac{\varepsilon}{8} \right)
\]  

(47)

\[
B - C = \frac{1}{F} \left( GH + \frac{\varepsilon}{8} \right)
\]  

(48)

\[
A - B = \frac{1}{H} \left( FG + \frac{\varepsilon}{8} \right)
\]  

(49)

\[
\frac{\varepsilon}{8} = \frac{(FG)^2 + (GH)^2 + (HF)^2}{-FG + GH + HF}
\]  

(50)

Again, only three of equations (47)-(50) are independent, since equation (50) is found from the sum of equations (47)-(49). The fourth independent condition will once more be equation (38).

Analogous procedures for the remaining two cases lead to the two groups of four relationships, three of which are independent.
Case 3: \textbf{Inoperative}

\[ A - B = \frac{1}{H} \left( FG - \frac{\epsilon}{8} \right) \]  
\[ B - C = \frac{1}{F} \left( GH + \frac{\epsilon}{8} \right) \]  
\[ C - A = \frac{1}{G} \left( HF + \frac{\epsilon}{8} \right) \]  
\[ \frac{\epsilon}{8} = \frac{(FG)^2 + (GH)^2 + (HF)^2}{FG - GH - HF} \]  

Case 4: \textbf{Inoperative}

\[ B - C = \frac{1}{F} \left( GH - \frac{\epsilon}{8} \right) \]  
\[ A - B = \frac{1}{H} \left( FG + \frac{\epsilon}{8} \right) \]  
\[ C - A = \frac{1}{G} \left( HF + \frac{\epsilon}{8} \right) \]  
\[ \frac{\epsilon}{8} = \frac{(FG)^2 + (GH)^2 + (HF)^2}{-FG + GH - HF} \]  

Again, these latter cases require that equation (38) be satisfied.

The relationships governing the simultaneous operation of three slip directions are complicated by the fact that the additional parameter \( \epsilon \), which determines the stress level on the inoperative system, must be introduced. Furthermore, unlike for restricted glide, the stress level on the inoperative system, \( 9k^2 - \epsilon \), influences the yield expression for the operative systems.
Stress states that simultaneously operate three slip directions can be found by analogy from the stress states found by Bishop and Hill which simultaneously operate six restricted-glide slip systems. (Bishop-Hill Groups IV and V, Appendix I) Since the stress states that operate eight restricted-glide slip systems have the same character as those which operate four pencil-glide slip directions, a parallel would also be anticipated between six restricted-glide slip systems and three pencil-glide slip directions.

In Bishop-Hill Group IV, the stress states are of the form \( A = -B, C = 0, \) etc., with one of the shear stresses, \( F, G, \) or \( H \) also vanishing.

If only \( F, C, \) or \( H \) is set equal to zero, three pencil-glide slip directions can be made operative. The following 12 cases can be distinguished.

Group IVa: \( F = 0, [111] \) Inoperative

Since \( F = 0, \) \( \frac{C}{8} = GH \) from equation (46), and, from equations (43)-(45) and equation (38)

\[
A = \frac{1}{3} (-G + H)
\]

\[
B = \frac{1}{3} (2G + H)
\]

\[
C = -\frac{1}{3} (G + 2H)
\]

\[
G^2 + GH + H^2 = \frac{27}{8} k^2
\]
Any stress state which satisfies equation (59) will operate only the [111], [1$ar{1}$1], and [1$ar{1}$1] directions as long as $0 < \varepsilon = 8GH \leq 9k^2$. Similar results are obtained for the remaining 11 cases, which are characterized by

Group IVb:  
F = 0, [111] Inoperative  
c: F = 0, [1$ar{1}$1] Inoperative  
d: F = 0, [1$ar{1}$1] Inoperative  
e: G = 0, [111] Inoperative  
f: G = 0, [1$ar{1}$1] Inoperative  
g: G = 0, [1$ar{1}$1] Inoperative  
h: G = 0, [1$ar{1}$1] Inoperative  
i: H = 0, [111] Inoperative  
j: H = 0, [1$ar{1}$1] Inoperative  
k: H = 0, [1$ar{1}$1] Inoperative  
l: H = 0, [1$ar{1}$1] Inoperative

The detailed stress relationships for these last 11 cases are given in Appendix IV.

Equation (60) or (61) can be used to write A or -B in terms of F and H, yielding

$$A = -B = \frac{F(F - H)}{F + 2H} \quad (64)$$

If the equality $F = G$ is substituted into equation (46), $\frac{\varepsilon}{8}$ becomes

$$\frac{\varepsilon}{8} = \frac{F(F^2 + 2H^2)}{F + 2H} \quad (65)$$
The final relationship between $F$ and $H$ which must be satisfied is found by substituting $F = G$ and equations (64) and (65) into equation (38), giving

$$\left( \frac{F(F - H)}{F + 2H} \right)^2 + 2F^2 + H^2 + \frac{F(F^2 + 2H^2)}{F + 2H} = \frac{9}{2} k^2 \quad (66)$$

Analogous results are obtained for the remaining 11 cases, which are characterized by the relationships

**Group IVn:** $A = -B, C = 0, F = -G, [1\bar{1}1]$ Inoperative

- **o:** $A = -B, C = 0, F = G, [\bar{1}11]$ Inoperative
- **p:** $A = -B, C = 0, F = -G, [\bar{1}11]$ Inoperative
- **q:** $B = -C, A = 0, G = H, [\bar{1}11]$ Inoperative
- **r:** $B = -C, A = 0, G = -H, [\bar{1}11]$ Inoperative
- **s:** $B = -C, A = 0, G = -H, [\bar{1}11]$ Inoperative
- **t:** $B = -C, A = 0, G = H, [\bar{1}11]$ Inoperative
- **u:** $C = -A, B = 0, H = F, [111]$ Inoperative
- **v:** $C = -A, B = 0, H = F, [\bar{1}11]$ Inoperative
- **w:** $C = -A, B = 0, H = -F, [\bar{1}11]$ Inoperative
- **x:** $C = -A, B = 0, H = -F, [\bar{1}11]$ Inoperative

The detailed stress relationships for these last 11 cases are again contained in Appendix IV.

**Group IVy:** $A = -B, C = 0, A = H, [111]$ Inoperative

If $A = H$, equation (60) or (62) states that

$$\frac{e}{b} = HF + GH \quad (67)$$
Substituting equation (67) into equation (61) leads to the relationship among \( F, G, \) and \( H \)

\[
2H^2 + H(F + G) - FG = 0
\]  
\[
(68)
\]

A quadratic relationship between the stresses \( F \) and \( G \) alone can be obtained by substituting equations (67) and (68) into equation (38), yielding

\[
F^2 + FG + G^2 = \frac{9}{2} k^2
\]  
\[
(69)
\]

Similar relationships can be obtained for the analogous cases characterized by

Group IVz: \( A = -B, C = 0, A = -H, [\overline{1}1\overline{1}] \) Inoperative

aa: \( A = -B, C = 0, A = H, [\overline{1}1\overline{1}] \) Inoperative

bb: \( A = -B, C = 0, A = -H, [\overline{1}1\overline{1}] \) Inoperative

c: \( B = -C, A = 0, B = F, [111] \) Inoperative

d: \( B = -C, A = 0, B = -F, [\overline{1}1\overline{1}] \) Inoperative

ee: \( B = -C, A = 0, B = -F, [\overline{1}1\overline{1}] \) Inoperative

ff: \( B = -C, A = 0, B = F, [\overline{1}1\overline{1}] \) Inoperative

gg: \( C = -A, B = 0, C = G, [111] \) Inoperative

hh: \( C = -A, B = 0, C = G, [\overline{1}1\overline{1}] \) Inoperative

ii: \( C = -A, B = 0, C = -G, [\overline{1}1\overline{1}] \) Inoperative

jj: \( C = -A, B = 0, C = -G, [\overline{1}1\overline{1}] \) Inoperative

Appendix IV also contains the details of these latter 11 cases.
Bishop-Hill Group V contains stress states in which only F, G, or H appear. If A, B, and C are set equal to 0 in equations (43)-(58), distinct values of F, G, and H which simultaneously operate three slip directions can be found directly. Again, four different stress states, depending on which slip system is inoperative, will result.

Group Va: \( A = B = C = 0, \quad [\overline{111}] \) Inoperative

Since \( A = B = C = 0 \), equations (43)-(45) will be zero, which requires that \( HF = FG = GH = 0 \), or \( F = G = H \). Substituting this equality into equations (38) and (46) and solving for F, G, and H gives

\[
F = G = H = \pm \frac{3\sqrt{2}}{4} \ k
\]

Group Vb: \( A = B = C = 0, \quad [\overline{\overline{11}}] \) Inoperative

In this case, \( HF = -GH = -FG \) from equations (47)-(49) and hence

\[
F = -G = H = \pm \frac{3\sqrt{2}}{4} \ k
\]

Similar results are obtained in the \([\overline{\overline{11}}]\) or \([\overline{\overline{11}}]\) direction is inoperative

Group Vc: \( A = B = C = 0, \quad [\overline{1\overline{1}}] \) Inoperative

\[
F = G = -H = \pm \frac{3\sqrt{2}}{4} \ k
\]

Group Vd: \( A = B = C = 0, \quad [\overline{11}] \) Inoperative

\[
-F = G = H = \pm \frac{3\sqrt{2}}{4} \ k
\]
If there is an exact parallel between the polyslip stress states for restricted glide and the corresponding stress states for pencil glide, the stress states in Groups IV and V are the only stress states which can simultaneously operate three pencil-glide slip directions. The proof of this is not as straightforward as for the case of four slip directions operating simultaneously.

The general requirements for the states of stress which simultaneously operate three slip directions will be derived for case 1, [111] inoperative. The requirements for cases 2, 3, and 4 can be found from the requirements for case 1 by cyclically permuting the signs of F, G, and H. Equations (43)-(45) can be used to find A, B, and C in terms of F, G, H, and \( \varepsilon \). If the expressions for A, B, and C are substituted into equation (38), a polynomial in F, G, H, and \( \varepsilon \) results. If equation (46) is used to write \( \varepsilon \) in terms of F, G, and H, a fourth order polynomial relationship, which must be obeyed by any stress state which can simultaneously operate the [111], [111], and [111] slip directions, is obtained. This polynomial, written in descending powers of F, is

\[
\psi F^4 + wF^3 + (\psi^2 - \frac{27}{8}k^2w^2)F^2 + w(w^2 - \psi)(\psi - \frac{27}{4}k^2)F + (w^2 - \psi)^2(\psi - \frac{27}{8}k^2) = 0 \tag{70}
\]

where \( \psi = G^2 + GH + H^2 \) and \( w = G + H \).

In addition to satisfying equation (70), values of F, G, and H must obey the requirement

\[
0 < \frac{\varepsilon}{8} = \frac{(FG)^2 + (GH)^2 + (HF)^2}{FG + GH + HF} \leq \frac{9}{8}k^2
\]

in order to insure that the stress in the [111] direction is real.
and below the yield stress. Roots to equation (75) were sought by selecting two values for G and H within the permissible range \(-\frac{3\sqrt{2}}{2} < G, H < \frac{3\sqrt{2}}{2}\) k, varying the value of F by one percent within the same interval, and computing the polynomial to look for changes of sign. No additional permissible stress states were found.

The stress states found which simultaneously operate three slip directions can be shown to lack the five degree of freedom necessary to accommodate an arbitrary shape change. This can be demonstrated by substituting the stress conditions of Groups IV and V into the expression for the shear stresses \(\tau_{np}\) and \(\tau_{nq}\) contained in Table I. Groups IVa, IVm, IVr, and IVy, which are typical of the remaining members in Group IV, will be examined in detail, along with Group Va.

**Group IVa:** \(F = 0, [\text{111}]\) Inoperative

If the expressions for A, B, and C in terms of G and H are substituted into the expressions for \(\tau_{np}^{(b)}, \tau_{nq}^{(b)}, \tau_{np}^{(c)}, \tau_{nq}^{(c)}, \tau_{np}^{(d)},\) and \(\tau_{nq}^{(d)},\) the following results are obtained.

\[
\begin{align*}
\tau_{np}^{(b)} &= \frac{2}{3\sqrt{6}} (G - H) \\
\tau_{nq}^{(b)} &= \frac{2}{3\sqrt{2}} (G + H)
\end{align*}
\] (71)
\[ \tau_{np} = \frac{2}{3\sqrt{6}} (2G + H) \]
\[ \tau_{nq} = \frac{-2}{3\sqrt{2}} (H) \]
\[ \tau_{np} = \frac{-2}{3\sqrt{6}} (2G + H) \]
\[ \tau_{nq} = \frac{2}{3\sqrt{2}} (H) \] (71)

If one of the three slip planes is chosen, the other two are uniquely determined and no further choice of slip planes is available. If, for example, the orientation of the (b) is chosen, \( \tau_{np}^{(b)} \) and \( \tau_{nq}^{(b)} \) are fixed. However, the remaining shear stresses \( \tau_{np}^{(c)}, \tau_{nq}^{(c)}, \tau_{np}^{(d)}, \) and \( \tau_{nq}^{(d)} \), and hence the orientations of the \( (c) \) and \( (d) \) slip planes, are also fixed, since these latter shear stresses can all be expressed in terms of \( \tau_{np}^{(b)} \) and \( \tau_{nq}^{(b)} \).

\[ \tau_{np}^{(c)} = -\tau_{np}^{(d)} = \frac{1}{2} \tau_{np}^{(b)} + \frac{\sqrt{3}}{2} \tau_{nq}^{(b)} \] (72)

\[ \tau_{nq}^{(d)} = -\tau_{nq}^{(c)} = -\frac{\sqrt{3}}{2} \tau_{np}^{(b)} + \frac{1}{2} \tau_{nq}^{(b)} \]

Similar relationships for the remaining two pairs of shear stresses can be found if the freedom of choice is used for the \( (c) \) or \( (d) \) system.

Group IVm: \( A = -B, C = 0, F = G, [\text{111}] \) Inoperative

Substituting \( A = -B, C = 0, \) and \( F = G \) into the appropriate shear stress expressions of Table I gives
\[ \tau_{np}^{(b)} = \frac{1}{\sqrt{6}} \quad (2F) \]

\[ \tau_{nq}^{(b)} = \frac{2}{3\sqrt{2}} \quad (-A + H) \]

\[ \tau_{np}^{(c)} = 0 \]

\[ \tau_{nq}^{(c)} = -\frac{2}{3\sqrt{2}} \quad (A + F + H) \]

\[ \tau_{np}^{(d)} = -\frac{1}{\sqrt{6}} \quad (2F) \]

\[ \tau_{nq}^{(d)} = \frac{2}{3\sqrt{2}} \quad (-A + H) \]

Since \( \tau_{np}^{(c)} \) vanishes, the (c) system must be either [\( \overline{1}11 \)] (112) or [\( \overline{1}11 \)] (\( \overline{1} \overline{1} \overline{2} \)) from equation (5) and Table I. These two slip systems are not independent, however, because one slip plane is the negative of the other.

If freedom of choice is exerted to pick the orientation of the (b) system, \( \tau_{np}^{(b)} \) and \( \tau_{nq}^{(b)} \) are fixed. This also restricts the orientation of the (d) system, since

\[ \tau_{np}^{(d)} = -\tau_{np}^{(b)} \]

\[ \tau_{nq}^{(d)} = \tau_{nq}^{(b)} \]

Again, only four degrees of freedom, one choice of slip-plane orientation and three amounts of shear, are available.

Group IVr: \( B = -C, A = 0, G = -H, [\overline{1}11] \) Inoperative

The shear stresses from Table I appropriate to this case are
\[\tau_{np}^{(a)} = \frac{1}{\sqrt{6}} (B + F - G)\]
\[\tau_{nq}^{(a)} = \frac{1}{3\sqrt{2}} (B + F + 3G)\]
\[\tau_{np}^{(c)} = \frac{1}{\sqrt{6}} (B - F + G)\]
\[\tau_{nq}^{(c)} = \frac{1}{3\sqrt{2}} (B - F + G)\]
\[\tau_{np}^{(d)} = -\frac{1}{\sqrt{6}} (B + F + G)\]
\[\tau_{nq}^{(d)} = \frac{1}{3\sqrt{2}} (B + F - 3G)\]  

(75)

The (c) system is determined in this case, since \(\tau_{np}^{(c)}\) and \(\tau_{nq}^{(c)}\) are related from equation (75) by

\[\tau_{np}^{(c)} = \sqrt{3} \tau_{nq}^{(c)}\]  

(76)

and by the yield requirement

\[\left[\tau_{np}^{(c)}\right]^2 + \left[\tau_{nq}^{(c)}\right]^2 = \kappa^2\]  

(77)

The choices available for \(\tau_{np}^{(c)}\) and \(\tau_{nq}^{(c)}\) are, from equations (76) and (77)

\[\tau_{np}^{(c)} = \frac{\sqrt{3}}{2}, \quad \tau_{nq}^{(c)} = \frac{1}{2}\]

(78)

If the results of equation (78) are substituted into equation (5), the values of the angle \(\theta\) measured from the p or [110] direction toward the [112] direction are found to be either \(\theta = 30^\circ\) or \(\theta = 210^\circ\). This fixes the orientation of the slip plane as either (211) or (211).
Again, if the orientation of one of the other slip planes is chosen, the other is also fixed. If the (a) system and hence \( \tau_{np}^{(a)} \) and \( \tau_{nq}^{(a)} \) are fixed, \( \tau_{np}^{(d)} \) and \( \tau_{nq}^{(d)} \) must be

\[
\begin{align*}
\tau_{np}^{(d)} &= -\frac{1}{2} \tau_{np}^{(a)} - \frac{\sqrt{3}}{2} \tau_{nq}^{(a)} \\
\tau_{nq}^{(d)} &= \frac{\sqrt{3}}{2} \tau_{np}^{(a)} - \frac{1}{2} \tau_{nq}^{(a)}
\end{align*}
\] (79)

Group IVy: A = -B, C = 0, A = H, [111] Inoperative

The above restrictions require that

\[
\begin{align*}
\tau_{np}^{(b)} &= \frac{1}{\sqrt{6}} (F + G) \\
\tau_{nq}^{(b)} &= \frac{1}{3\sqrt{2}} (-F + G) \\
\tau_{np}^{(c)} &= \frac{1}{\sqrt{6}} (-F + G) \\
\tau_{nq}^{(c)} &= -\frac{1}{3\sqrt{2}} (F + G + 4H) \\
\tau_{np}^{(d)} &= -\frac{1}{\sqrt{6}} (F + G) \\
\tau_{nq}^{(d)} &= \frac{1}{3\sqrt{2}} (F - G)
\end{align*}
\] (80)

Again, only one choice of slip-plane orientation is available. If, for example the (b) system is fixed and \( \tau_{np}^{(b)} \) and \( \tau_{nq}^{(b)} \) are determined, the shear stresses on the other two systems, and hence the orientations of these systems, are also determined.
It must first be shown that the stresses $F$, $G$, and $H$ must all be of the same sign. The quantities $\frac{\epsilon}{\delta}$ and $H^2$ must both be positive, which implies from equations (46), (67), and (68) that

$$\begin{align*}
FG + GH + HF & > 0 \\
HF + GH & = H(F + G) > 0 \\
FG - GH - HF & > 0
\end{align*}$$

The sum of equations (81) and (83) implies that

$$FG > 0$$

or that $F$ and $G$ must be of the same sign. Equation (82) then requires that $H$ must be of the same sign as $F + G$, and, since $F$ and $G$ must have similar signs, $F$, $G$, and $H$ must all be of the same sign.

Having fixed the $(b)$ system, one can immediately find the orientation of the $(d)$ system from stress relationships

$$\begin{align*}
\tau_{np}^{(d)} &= - \tau_{np}^{(b)} \\
\tau_{nq}^{(d)} &= - \tau_{nq}^{(b)}
\end{align*}$$

The stress $\tau_{np}^{(c)}$ can also be found from the relationship

$$\tau_{np}^{(c)} = \sqrt{3} \tau_{nq}^{(b)}$$

This then determines the magnitude of the remaining shear stress, $\tau_{nq}^{(c)}$ which can be found from the quadratic form of the yield locus (equation 6). However, since $F$, $G$, and $H$ must be
of the same sign, \( F + G + 4H \) will have the same sign as \( F + G \) or \( \tau_{np}^{(b)} \). Hence the sign of \( \tau_{nq}^{(c)} \) will be opposite that of \( \tau_{np}^{(b)} \) (equation 80) and the (c) system will be specified.

Similar conclusions will be reached if the freedom of choice is exerted to fix the (c) or (d) systems.

Group Va: \( A = B = C = 0, F = G = H = \pm \frac{3\sqrt{2}}{4} k \),

[111] Inoperative

These distinct values of \( F, G, \) and \( H \) require that

\[
\begin{align*}
\tau_{np}^{(b)} &= \pm \frac{\sqrt{3}}{2} k \\
\tau_{nq}^{(b)} &= \pm \frac{1}{2} k \\
\tau_{np}^{(c)} &= 0 \\
\tau_{nq}^{(c)} &= \mp k \\
\tau_{np}^{(d)} &= \mp \frac{\sqrt{3}}{2} k \\
\tau_{nq}^{(d)} &= \pm \frac{1}{2} k
\end{align*}
\]

Substituting the values in equation (87) into equation (5) and solving for \( \theta \) requires the three slip systems to be

\[
\begin{align*}
[111] & \text{ (121) or [111] (120)} \\
[111] & \text{ (112) or [111] (112)} \\
[111] & \text{ (211) or [111] (211)}
\end{align*}
\]
The stress states in Groups Vb, c, and d will also operate three \{1\bar{1}1\} \{1\bar{1}2\} type systems, and hence Group V offers only three degrees of freedom, the three amounts of shear on the three \{1\bar{1}1\} \{1\bar{1}2\} systems.

The stress states in Group II, which operate four systems simultaneously, also impose \{1\bar{1}2\} slip-plane restrictions. Group IIa, for example imposes the shear stress conditions

\[
\begin{align*}
\tau_{np}^{(a)} &= \pm \frac{\sqrt{3}}{2} k \\
\tau_{nq}^{(a)} &= \pm \frac{1}{2} k \\
\tau_{np}^{(b)} &= \pm \frac{\sqrt{3}}{2} k \\
\tau_{nq}^{(b)} &= \pm \frac{1}{2} k \\
\tau_{nq}^{(c)} &= \pm \frac{\sqrt{3}}{2} k \\
\tau_{np}^{(c)} &= \pm \frac{1}{2} k \\
\tau_{nq}^{(d)} &= \pm \frac{\sqrt{3}}{2} k \\
\tau_{np}^{(d)} &= \pm \frac{1}{2} k
\end{align*}
\]

(88)

and requires the four slip systems to be

[1\bar{1}1] \ (\bar{2}1\bar{1}) \ or \ [1\bar{1}1] \ (2\bar{1}\bar{1})

[1\bar{1}1] \ (2\bar{1}\bar{1}) \ or \ [1\bar{1}1] \ (\bar{2}\bar{1}\bar{1})

[\bar{1}\bar{1}1] \ (\bar{2}1\bar{1}) \ or \ [\bar{1}\bar{1}1] \ (2\bar{1}1)

[\bar{1}\bar{1}1] \ (\bar{2}\bar{1}\bar{1}) \ or \ [\bar{1}\bar{1}1] \ (211)
Thus Group II, which operates four slip directions simultaneously, also fails to provide the required minimum of five degrees of freedom necessary to accommodate an arbitrary shape change, since only the four amounts of shear on the four \( <111> \{112\} \) systems can be varied.

If four slip directions are operated simultaneously under the stress restraints of Groups I or III, however, five degrees of freedom are available. This conclusion can be reached by appreciating that, after imposing the four yield requirements (equations 9-12) on the prevailing stress state, there will still remain one undetermined component of stress, which provides one degree of freedom. This independent stress component, together with the four amounts of slip on the operative systems, constitute the required five degrees of freedom. This same conclusion can also be reached by examining the consequences of the restrictions of Groups I and III on the orientations of the operative slip planes.

Group I imposes the shear stress restraints.

Group I: \( F = G = H = 0 \), Four Systems Operative

\[
\begin{align*}
\tau_{np}^{(a)} &= \frac{1}{\sqrt{6}} (A + B) \\
\tau_{nq}^{(a)} &= \frac{1}{3\sqrt{2}} (-A + B) \\
\tau_{np}^{(b)} &= -\frac{1}{\sqrt{6}} (A + B) \\
\tau_{nq}^{(b)} &= \frac{1}{3\sqrt{2}} (-A + B)
\end{align*}
\]
\[ \tau_{\text{np}}^{(c)} = \frac{1}{\sqrt{6}} (A + B) \]
\[ \tau_{\text{nq}}^{(c)} = \frac{1}{3\sqrt{2}} (-A + B) \]  

(89)
\[ \tau_{\text{np}}^{(d)} = -\frac{1}{\sqrt{6}} (A + B) \]
\[ \tau_{\text{nq}}^{(d)} = \frac{1}{3\sqrt{2}} (-A + B) \]

There remains available now one choice of slip-plane orientation. If system (a) is fixed, the others are also, since

\[ \tau_{\text{np}}^{(b)} = -\tau_{\text{np}}^{(c)} = \tau_{\text{np}}^{(d)} = -\tau_{\text{np}}^{(a)} \]
\[ \tau_{\text{nq}}^{(b)} = \tau_{\text{nq}}^{(c)} = \tau_{\text{nq}}^{(d)} = \tau_{\text{nq}}^{(a)} \]  

(90)

The four amounts of shear and the one choice of slip-plane orientation provide the requisite five degrees of freedom.

Group III contains similar degrees of freedom. For example, the restraints of Group IIIa impose the shear stress conditions:

Group IIIa: \( B = C, \ A = -2B, \ G = H = 0, \)

Four Systems Operative

\[ \tau_{\text{np}}^{(a)} = \frac{1}{\sqrt{2}} (-B + F) \]
\[ \tau_{\text{nq}}^{(a)} = \frac{1}{3\sqrt{2}} (3B + F) \]  

(90)
\[ \tau_{\text{np}}^{(b)} = \frac{1}{\sqrt{6}} (B + F) \]
\[ \tau_{\text{nq}}^{(b)} = \frac{1}{3\sqrt{2}} (3B - F) \]
\[
\tau_{np} = -\frac{1}{\sqrt{6}} (B + F)
\]
\[
\tau_{np} = \frac{1}{3\sqrt{2}} (3B - F)
\]
\[
\tau_{nq} = \frac{1}{\sqrt{6}} (B - F)
\]
\[
\tau_{nq} = \frac{1}{3\sqrt{2}} (3B + F)
\]  

(90)

If the freedom of slip-plane choice is exercised for the (a) system, the shear stresses on the (b), (c), and (d) systems are fixed by the relationships

\[
\tau_{np} = -\tau_{np} = \frac{1}{2} \tau_{nq} + \frac{\sqrt{3}}{2} \tau_{nq}
\]
\[
\tau_{nq} = \tau_{nq} = -\frac{\sqrt{3}}{2} \tau_{np} + \frac{1}{2} \tau_{np}
\]
\[
\tau_{nq} = -\tau_{nq}
\]
\[
\tau_{nq} = \tau_{nq}
\]  

(91)

Similar conclusions can be drawn for the other members of Group III.

The stress states in Group I and III, even though they have available five degrees of freedom, cannot by themselves accommodate any arbitrary imposed strain. This can be shown with reference to the stress-strain relationships obtained by differentiating the individual yield loci (equations 9-12) which operate each of the \langle 111 \rangle slip directions with respect to the stresses \( \sigma_{ij} \) referred to the cubic axes. This can be expressed as
\[ d\varepsilon_{ij} = \sum_{\alpha=1}^{4} \frac{\partial f_{\alpha}}{\partial \sigma_{ij}} \ d\lambda_{\alpha} \]  

(93)

where the \( f_{\alpha} \) are the four loci of equations (9)-(12) written in terms of the stresses \( \sigma_{ij} \) and the \( d\lambda_{\alpha} \) are proportional to the amounts of slip on the four slip systems. \( d\lambda_{\alpha} \) must be positive if the strain increments are in the directions of the applied stresses, or, stated alternatively, the strain increments are in the direction of the outward normals to the yield loci.

If the following changes of variables are made

\[
\begin{align*}
d\lambda_1 &= a, \ d\lambda_2 = b, \ d\lambda_3 = c, \text{ and } d\lambda_4 = d \\
k_1 &= \frac{2}{9} (a + b + c + d) \\
k_2 &= \frac{2}{9} (-a - b + c + d) \\
k_3 &= \frac{2}{9} (a - b + c - d) \\
k_4 &= \frac{2}{9} (a - b - c + d)
\end{align*}
\]

(94)

the stress-strain relationships, written in Bishop-Hill notation, become

\[
\begin{align*}
d\varepsilon_{11} &= k_1 (-B + C) + k_3 H - 2k_4 F - k_2 G \\
2d\varepsilon_{12} &= k_3 (A - B) + 2k_1 H + k_2 F - k_4 G \\
2d\varepsilon_{23} &= k_4 (B - C) + k_2 H + 2k_1 F - k_3 G \\
2d\varepsilon_{31} &= k_2 (A - C) - k_4 H - k_3 F + 2k_1 G
\end{align*}
\]

(95)
An arbitrary state of strain must not only be accommodated by operating four slip directions under the stress conditions of Groups I and III, but must also obey equation (95). The stress conditions of Groups I and II can be used to write five of the Bishop-Hill stresses in terms of sixth. If these stress expressions are substituted into equation (95), there will remain five unknowns, four $k$'s and the one undetermined Bishop-Hill stress. One must then be able to invert equation (95) to find the undetermined stress and the four $k$'s in terms of the five independent components of applied strain $d\varepsilon_{ij}$. However, the condition that the strain increments be in the direction of the outward normal to the yield loci further requires that

$$a = \frac{9}{8} (k_1 - k_2 + k_3 + k_4) \geq 0$$

$$b = \frac{9}{8} (k_1 - k_2 - k_3 - k_4) \geq 0$$

$$c = \frac{9}{8} (k_1 + k_2 + k_3 - k_4) \geq 0$$

$$d = \frac{9}{8} (k_1 + k_2 - k_3 + k_4) \geq 0$$

(96)

conditions that are not met for any arbitrary strain state.

Hence, the pencil-glide polyslip stress states in Groups I and III, like the equivalent restricted-glide stress states in Bishop-Hill Groups I and III, cannot alone operate slip systems that will accommodate any arbitrary state of strain. Since the stress states in Groups II, IV, and V do not make available the required five degrees of freedom, these latter stress states, in
general, cannot supplement the stress states in Group I and III such that a stress state in Groups I through V can always be found which will accommodate any arbitrary strain state. This deficiency can be remedied in two possible ways. The first is that additional undetected stress states that are solutions to equation (70) may exist in the regions very near the stress states of Group IV. These additional stress states may retain the requisite five degrees of freedom and may be capable of supplementing the stress states of Groups I and III such that any arbitrary strain state can be accommodated. The second is that the critical resolved shear stress is simultaneously reached on two planes in at least one ⟨111⟩ direction (Figure 10). Then, however, the critical resolved shear stress will be exceeded on any plane whose normal lies between the normals of the two operative slip planes. However, neither addition is expected to lead to stress predictions that are significantly different from those found using only the stress states in Groups I to V.

C. An Approximate Procedure for Describing BCC Pencil-Glide Yielding Under Prescribed States of Strain

A computational procedure will now be developed to approximate yielding under prescribed states of strain when the slip plane is not specifically designated. States of stress which simultaneously cause yielding in four or three ⟨111⟩ slip directions on the most highly stressed planes will be considered.
Figure 10. Simultaneous Operation of Two Slip Directions in a \{111\} Plane. If the critical resolved shear stress $k$ is simultaneously reached in directions 1 and 2, the shear stress $\mathcal{T}_3$ will be greater than $k$. 
Since these polyslip stress states cannot always operate slip systems which can accommodate an arbitrary state of strain, the imposed strain states will, in general, not be completely enforced.

For computational purposes, only the stress states in Groups I, II, III, V, and IVa-l will be included and used in conjunction with the maximum-work principle. For a prescribed strain state, the work done by the stresses in Groups II and V can be found directly, since these stress states are distinct. The remaining groups all contain permissible stress states which are functionally related. However, for a given strain state, distinct states of stress can also be found for these latter groups.

The first step in establishing distinct stress states for the remaining groups is to make use of the functional relationships describing the permissible stress states to write the entire stress state in terms of only one stress. The work expression (Appendix I)

\[ dW = -B\varepsilon_{11} + A\varepsilon_{22} + 2F\varepsilon_{23} + 2G\varepsilon_{31} + 2H\varepsilon_{12} \]  (97)

is then given in terms of the imposed strain state and only one of the six Bishop-Hill stresses. The level of this one remaining stress can be found by differentiating the work expression with respect to this stress, setting the result equal to zero, and requiring also that the second derivative be negative. The particular stress state which maximizes the work done can then be determined.
To illustrate this maximization procedure, the distinct state of stress that simultaneously operates four slip directions under the restraints of Group IIIc will be determined. In Group IIIc, the four Bishop-Hill stresses $A$, $B$, $C$, and $H$ are involved. However, the functional restraint among these four stresses can be written in terms of just two stresses by making use of the equality $A = B = -\frac{1}{2}C$. Equation (29c) can then be rewritten as

$$6A^2 + 2H^2 = 9k^2$$

(98)

Solving equation (98) for $H$ gives

$$H = \pm \left( \frac{9}{2} - 3A^2 \right)^{1/2}$$

(99)

The work expression now becomes

$$dW = A(d\varepsilon_{22} - d\varepsilon_{11}) + 2\left( \frac{9}{2} - 3A^2 \right)^{1/2}$$

(100)

Differentiating equation (100) with respect to $A$, setting the result equal to zero, and solving for $A$ gives

$$A = \pm \frac{\sqrt{6} (d\varepsilon_{11} - d\varepsilon_{22})}{2[12d\varepsilon_{12}^2 + (d\varepsilon_{11} - d\varepsilon_{22})^2]^{1/2}}$$

(101)

Substituting for $A$ in the expressions for $B$, $C$, and $H$ (equation 29c) yields

$$A = B = -\frac{1}{2}C = \pm \frac{\sqrt{6} (d\varepsilon_{11} - d\varepsilon_{22})}{2[12d\varepsilon_{12}^2 + (d\varepsilon_{11} - d\varepsilon_{22})^2]^{1/2}}$$

(102)
\[ H = \pm \frac{\sqrt{6} \ (d\varepsilon_{11} - d\varepsilon_{22})}{\left[ 12d\varepsilon_{12}^2 + (d\varepsilon_{11} - d\varepsilon_{22})^2 \right]^{1/2}} \]  

(102)

\[ F = G = 0 \]

Requiring that \( dW \) be a maximum gives

\[ A = B = -\frac{1}{2} \quad C = \frac{\sqrt{6} \ (d\varepsilon_{11} - d\varepsilon_{22})}{2 \left[ 12d\varepsilon_{12}^2 + (d\varepsilon_{11} - d\varepsilon_{22})^2 \right]^{1/2}} \]

(103)

\[ H = \frac{3\sqrt{6} \ d\varepsilon_{12}}{\left[ 12d\varepsilon_{12}^2 + (d\varepsilon_{11} - d\varepsilon_{22})^2 \right]^{1/2}} \]

\[ F = G = 0 \]

By following a similar procedure for the remaining cases, the corresponding distinct values of the Bishop-Hill stresses can be found as a function of the applied strain. A summary of all the stress states to be used in approximating BCC pencil-glide yielding under prescribed states of strain is contained in Table II.

D. **Pencil-Glide Planar-Stress and Planar-Strain Yield Loci**

Having established the generalized pencil-glide yield criteria (equations 9-12) and an approximate procedure for enforcing arbitrary shape changes (Table II), rotationally symmetric yield loci for the cube-on-face, cube-on-edge, and cube-on-corner textures can now be calculated. Using these tools and the definitions of planar-stress and planar-strain, the rotationally symmetric yield loci were calculated using an averaging procedure similar to that used for restricted glide (Appendix III).
<table>
<thead>
<tr>
<th>GROUP</th>
<th>H</th>
<th>P</th>
<th>G</th>
<th>F</th>
<th>S</th>
<th>D</th>
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<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
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<td>0</td>
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<td>0</td>
<td>α</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>α</td>
<td>β</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>α</td>
<td>β</td>
</tr>
</tbody>
</table>

Table II
Stress States Which Simultaneously Operate Three or Four Slip Directions
Representative planar-stress and planar-strain loci are shown in Figure 11. The inadequacy of the approximate procedure for enforcing an arbitrary shape change is evidenced most prominently in the component loci for the 45° orientation of the cube-on-edge texture, where the yield stress in the second quadrant for the planar-stress or lower-bound locus actually exceeds that of the planar-strain or upper-bound locus. The rotationally symmetric loci for the three textures are shown in Figures 12-14, the isotropic locus calculated by Hutchinson is shown in Figure 15.

The pencil-glide rotationally symmetric loci shown in Figures 12-14 are remarkably similar to their restricted-glide counterparts shown in Figures 2-4. In each case, the loci for the cube-on-edge texture deviate least from the isotropic (Figures 5 and 15). The loci for the cube-on-face texture are contracted in the first and third quadrants, while those for the cube-on-corner are extended. The Hill rotationally symmetric anisotropic theory associates a contraction in the first and third quadrants with an R value less than one, an extension, with an R greater than one. Hence, on this basis, R would be expected to be less than one for the cube-on-face texture, approximately equal to one for the cube-on-edge, and greater than one for the cube-on-corner, independent of the assumed slipping mode.
Figure 11. Representative Planar-stress and Planar-strain (approximate) Component Loci for the Cube-on-face, Cube-on-edge, and Cube-on-corner Textures Deforming by Pencil Glide.
Figure 12. Rotationally Symmetric Planar-stress and Planar-strain (approximate) Yield Loci for the Cube-on-face Texture Deforming by Pencil Glide.
Figure 13. Rotationally Symmetric Planar-stress and Planar-strain (approximate) Yield Loci for the Cube-on-edge Texture Deforming by Pencil Glide.
Figure 14. Rotationally Symmetric Planar-stress and Planar-strain (approximate) Yield Loci for the Cube-on-corner Texture Deforming by Pencil Glide.
Figure 15. Isotropic Pencil-glide Yield Locus Calculated by Hutchinson (ref. 11).
IV. The Influence of Plastic Anisotropy on the Limiting Drawing Ratio and Earing Behavior

The analytical procedures for describing plastic anisotropy can also be used for predicting certain consequences of crystallographic texture in deformation processing operations. In particular, the limiting drawing ratio and earing behavior can be treated with rotationally symmetric yield loci and component loci, respectively.

A. The Limiting Drawing Ratio

Deformation in deep drawing, which is inherently non-steady-state, is often idealized by pure radial drawing, which is illustrated in Figure 16a by a frictionless wedge-drawing test. The relationship of the basic shape change in radial drawing to that encountered in deep drawing is shown in Figure 16b, in which the additional bending required at the flange-to-wall and wall-to-bottom transitions has been included. The definition of the limiting drawing ratio (LDR) as the maximum ratio of initial to final cup diameter is also indicated.

The strain patterns in the flange and cup-wall are schematically represented in Figure 17. The coordinates, x, y, and z are located in both regions along the radial, circumferential, and through-thickness directions, respectively. Deformation in the flange is imagined to occur in plane strain without change in thickness (dε_{zz} = 0) under radial tension and circumferential compression as shown.
Figure 16. (a) - Pure Radial Drawing, Illustrated by a Frictionless Wedge-drawing Test.

(b) - Sector in a Partially Drawn Cup.
Figure 17. A Schematic Representation of Plastic Strains in the Flange, and, in the Event of Failure, in the Lower Cup Wall. Plane-strain deformation is indicated in both locations, with $\varepsilon_{zz} = 0$ in the flange and $\varepsilon_{yy} = 0$ in the lower wall.
A flange reference element before (unshaded) and after (shaded) some deformation is drawn accordingly. In the lower cup wall, straining in the circumferential direction \((d\varepsilon_{yy} = 0)\) is effectively precluded by the rigid punch. The reference element for this latter element before and after deformation is again shown. The stresses \(\sigma_{xx}\) and \(\sigma_{yy}\) are both tensile in this case, a positive \(\sigma_{yy}\) being necessary to preserve \(d\varepsilon_{yy} = 0\).

In pure radial drawing (Figure 16a) the load requirements originate in the deformation of the flange region. This load must be transmitted through the end of the sector (in the lower cup wall) which is nominally unstrained and therefore is the potential failure site in the system. Since deformation in the cup wall occurs in the absence of circumferential strain \((d\varepsilon_{yy} = 0)\), necking down before failure will be accompanied by a reduction in thickness in order to keep volume constant. With these considerations in mind, it would be expected that a material anisotropy which acts to increase the resistance to plastic thinning without strengthening the flange would increase the limiting drawing ratio.

Whiteley\(^{22}\), using a wide variety of materials, first showed that the limiting drawing ratio correlates best with a normal anisotropy in which thinning resistance is enhanced. As a measure of normal anisotropy, \(\bar{R}^*\), the planar average of the strain ratio, was used. Similar correlations of the drawing limit with

* \(\bar{R}\) is conventionally defined as \((R_0 + 2R_{45} + R_{90})/4\) where \(R_0\) and \(R_{90}\) are measured along the rolling and transverse directions, respectively, and \(R_{45}\) is taken at 45° to these axes.
\( \bar{R} \) were obtained by Lloyd\textsuperscript{23}, Wilson and Butler\textsuperscript{24}, Atkinson and Maclean\textsuperscript{25}, and Liliet and Wybo\textsuperscript{26}.

The dependence of the limiting drawing ratio on \( \bar{R} \) can be interpreted with the aid of the yield loci shown in Figure 18. Two loci are shown, both being constructed from the Hill rotationally symmetric anisotropic theory. With \( \bar{R} = 1 \), the v. Mises locus (solid line) results. With \( \bar{R} > 1 \), the locus (dotted line) is distorted as shown. Cup-wall and flange reference elements are also shown in the first and fourth quadrants, respectively.

With \( \bar{R} = 1 \), loading in the cup wall occurs along path 1, with \( \bar{R} > 1 \), along path 1'. The locus distortion for \( \bar{R} > 1 \) is seen to increase the wall strength from \( \sigma_{xx}(\text{max}) \) to \( \sigma'_{xx}(\text{max}) \). Loading in the flange occurs along path 2 in both cases. However, the stress necessary to deform the flange in the material with \( \bar{R} > 1 \) is seen to be less than that for \( \bar{R} = 1 \). Hence, increasing \( R \) would be expected to increase the limiting drawing ratio both by enhancing wall strength and decreasing flange resistance.

Hosford and Backofen\textsuperscript{13} have suggested an alternate parameter, \( \beta \), to express the anisotropy conducive to increasing the limiting drawing ratio. The \( \beta \) parameter is defined directly as the ratio of wall strength to the strength of the deforming flange, or

\[
\beta = \frac{\sigma_{xx}(\text{wall, where } d\epsilon_{yy} = 0)}{c_{yy}(\text{flange, where } d\epsilon_{zz} = 0)}
\]  

(104)
Figure 18. A Schematic Representation of the Stress States which Enforce Plane-strain Deformations in the Flange and Cup Wall. Loading in the cup wall occurs along path 1 in the isotropic metal, along path 1' in the anisotropic metal. Flange loading occurs along path 2 in both metals.
Written in terms of the notation shown in Figure 18, $\beta$ becomes

$$\beta = \frac{\sigma_{xx}(\text{max})}{-2\sigma_{IV}}$$  \hspace{1cm} (105)

The $\beta$ parameter is defined such that $\sigma_{xx} = 0$ in the flange, which is strictly true only at the outer edge. In order to make $\sigma_{xx} = 0$, it is necessary to add to the stress state determined along path 2 a hydrostatic compression of $-\sigma_{IV}$, which accounts for the factor of two in the denominator of equation (105). This additional factor makes $\beta$ equal to one for an isotropic material. As defined, $\beta$ will always be negative, since its denominator is negative, but only the magnitude of the stress ratio need be considered.

Although $\beta$ is more difficult to measure than $R^2$, it can be more reliably predicted from crystallographically derived yield loci. The strain ratio $R$ can be found from the slope of the yield locus at its intersection with the coordinate axes. For example, the strain ratio $R_x = \frac{d\varepsilon_{yy}}{d\varepsilon_{zz}}$ can be found by making use of the fact that the vector whose components are $d\varepsilon_{xx}$ and $d\varepsilon_{yy}$ is normal to the yield locus at the point $\sigma_{yy} = 0$ (uniaxial tension in the $x$ direction) and that the volume does not change during plastic deformation ($d\varepsilon_{xx} + d\varepsilon_{yy} + d\varepsilon_{zz} = 0$). If $\alpha$ is the angle between the tangent to the yield locus at the point $\sigma_{yy} = 0$ and the $\sigma_{xx}$ axis, the strain ratio $R_x$ can be found from the relationship

$$\frac{1}{R_x} + 1 = \tan \alpha$$  \hspace{1cm} (106)
However, whenever the yield locus has a corner along one of the coordinate axes, R will be indeterminate. This occurs, for example, in the Tresca locus, in which a corner is formed by a vertical branch and another oriented at 45° to both coordinate axes. The R value predicted from the vertical branch will be zero, since $d\varepsilon_{yy}$ must be zero to satisfy the normality condition; on the other hand, R for the 45° branch will be infinite, since $d\varepsilon_{zz} = 0$. Hence R for the Tresca locus can vary from zero to infinity.

Several of the planar-stress and planar-strain rotationally symmetric yield loci shown in Figures 2-4 and 12-14 also cross the $\sigma_{xx}$ axis at 45°, and hence the R values calculated from these individual loci would be infinite. An added indeterminacy in predicting R for a particular texture arises from the fact that the uniaxial stress itself is not uniquely determined, i.e., there is a gap between the upper (planar-strain) and lower (planar-stress) bound loci at the point $\sigma_{yy} = 0$. Hence one can imagine a variety of slopes (and R values) which are formed by permissible yield loci that lie between the upper and lower bounds.

The $\beta$ parameter, since it involves the ratio of two plane-strain strengths, can always be calculated unambiguously from a particular locus. These calculations were made from the rotationally symmetric loci for the three common textures under conditions of both planar stress and planar strain for both restricted and pencil glide. In addition, upper and lower bounds for $\beta$ were calculated by dividing the stress $\sigma_{xx}(\text{max})$ for planar strain by the stress $2\sigma_{IV}$ for planar stress, and vice-versa.
The results are contained in Table III. Restricted glide is denoted by RG and pencil glide by PG.

Table III

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<tr>
<th>Component</th>
<th>Planar Stress</th>
<th>Planar Strain</th>
<th>Maximum</th>
<th>Minimum</th>
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<td>RG</td>
<td>PG</td>
<td>RG</td>
<td>PG</td>
</tr>
<tr>
<td>Cube-on-face</td>
<td>0.737</td>
<td>0.700</td>
<td>0.651</td>
<td>0.654</td>
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<tr>
<td>Cube-on-edge</td>
<td>1.159</td>
<td>1.200</td>
<td>1.178</td>
<td>1.178</td>
</tr>
<tr>
<td>Cube-on-corner</td>
<td>1.649</td>
<td>1.446</td>
<td>1.239</td>
<td>1.226</td>
</tr>
</tbody>
</table>

The β values for restricted-glide planar-strain conditions are identical to those predicted previously by Hosford and Backofen. However, use of the planar-stress shear conditions produces no relative changes in β for the three components. More strikingly, the percentage difference between the two deformation extremes is extremely small.

The procedure for calculating β from rotationally symmetric loci need not be restricted to the single ideal components considered here. Any texture or combination of textures can be considered, making the calculation of β from crystallographically derived rotationally symmetric yield loci a powerful tool for analyzing the limiting drawing ratio for any existing or hypothetical texture.
B. **Earing Behavior**

Earing during deep drawing, though, in general, unrelated to cup failure\textsuperscript{24,28}, often necessitates an extra trimming operation and thus produces unwanted scrap.

Unlike the limiting drawing ratio, earing behavior is related to planar rather than normal anisotropy. Theories which relate earing to planar anisotropy all aim to predict ear position and height and can be divided into three categories, depending on their basis: an anisotropic continuum locus, the planar variation of $R$, and the plastic properties of single crystals.

A general theory of earing was first proposed by Hill\textsuperscript{2}. The observed symmetry about the ear axis, which suggests an absence of shear strains, led Hill to assume that ears will form on the rim of the blank at positions where the radial direction is a principal axis of strain. A maximum of four ears could be predicted, along the rolling and transverse directions or at 45° to these directions, depending on the relative magnitudes of the continuum anisotropy parameters. This upper limit makes it impossible to deal with the six ears that are frequently observed, reported in copper, for example, by Baldwin, Howard, and Ross\textsuperscript{29}, and the occasional examples of eight ears. An attempt by Hill to account for these additional ears by generalizing the continuum locus to a polynomial of higher degree resulted in a form which is difficult to test experimentally.
The planar variation of R has been used by Baldwin et. al., Whiteley and Wise\textsuperscript{30}, and Wilson and Butler\textsuperscript{24} to predict both ear position and height. The ear axes were always found to lie in directions along which R is a maximum. Ear height was correlated by Baldwin et. al. with the factor $R_{0,90} - R_{45}$, where $R_{0,90}$ is the average of the R values for the rolling and transverse directions and $R_{45}$ is the value found at 45° to these directions. Wilson and Butler, also using copper, found that ear height correlated best with $(R_{0,90} - R_{45})/R_a$, where $R_a$ is the average R value in the plane of the sheet. Whiteley and Wise correlated ear height in low carbon steel with the variation in the strain ratio $\Delta R$, which is identical to the factor introduced by Baldwin et. al., if the minimum in R is at the 45° position.

If there are two different maxima in R for the rolling and transverse directions, the ear heights in these directions would also be expected to be different. Bourne and Hill\textsuperscript{31} suggested that the relationship between ear position and the maxima in R could be interpreted physically by examining the deformation at the rim of the blank. At this point the radial stress vanishes and the stress state is predominantly circumferential compression. Ears would be expected to form where a pure circumferential compression would produce a maximum radial extension. If the sign of the stress state were reversed, this would imply that ears form at points where a circumferential tension would produce a maximum radial contraction. This latter condition is equivalent to the criterion that ears form at positions where a tensile specimen taken in the circumferential direction produces
a maximum R. Furthermore, this viewpoint suggests that, if there are unequal maxima in R in the rolling and transverse directions, the axis of the larger ear should be perpendicular to the direction of the higher R value. This was observed to be true by Wilson and Butler, and Wright,\textsuperscript{28} but the opposite was found by Baldwin et al., Bourne, and Hill, and Whiteley.

The first attempt to relate earing directly to the properties of textures was made by McEvily\textsuperscript{32}. Like Bourne and Hill, he assumed that earing was determined by conditions at the rim. Ear positions were predicted at points where maxima in the radial strain component resulted from slip on \{111\} <110> systems. However, in addition to assuming that the stress state is pure circumferential compression, he also assumed that the circumferential strain is the same in each element of the flange, an assumption which is contrary to the findings of Baldwin et al. and Zaat.\textsuperscript{33}

Tucker\textsuperscript{34} made a more general crystallographic analysis without requiring that the circumferential strain be the same in each element of the flange. His stress state was one of equal circumferential compression and radial tension, which is shown along path (2) in Figure 18. Assuming that slip occurs on \{111\} <110> systems and that work hardening is parabolic, the angular distribution of radial strains resulting from slip on the most highly stressed systems was found. Maxima in the radial strain component would correspond to ears, minima to troughs. A complication was introduced by radial strain discontinuities that
occurred at the transitions from one slip system to another, but the earing predictions were in excellent agreement with the detailed earing observations made on aluminum single crystals.

The appearance of radial strain discontinuities might cause more serious difficulties when several texture components are present and their earing contributions are added. To avoid this difficulty, as well as to provide a more general framework for incorporating different slipping mode and different imposed stress or strain states, a stress-based earing theory will be proposed.

If the radial direction is again chosen to lie along the x axis (Figure 17), the flange stress state will be located somewhere in quadrant IV, varying from rim to die radius. Loading will again be assumed to occur along path 2 ($\sigma_{xx} = \sigma_{IV}$, and $\sigma_{yy} = -\sigma_{IV}$) and the shear stresses $\sigma_{xy}$, $\sigma_{yz}$, and $\sigma_{zx}$ will be taken to be zero (conditions of planar stress). The variation in the stress $\sigma_{IV}$ necessary to caused yielding in various textural components oriented at different angles in the plane of the sheet is shown in Figures 1 and 11. The component corresponding to the smallest value of $\sigma_{IV}$ would be expected to yield first, producing a local extension in the x direction that leads to an ear. As hardening occurs in this weakest component, other components will yield, but the radial strain would be expected to be largest in the weakest component. Hence ear axes would be expected to coincide with the x axes of the weakest textural components.
If $\sigma_{IV}$ varies significantly in the plane of the sheet, large ear heights would be expected, since the weakest textural components would be required to undergo more deformation in order to harden to the stress level necessary to initiate yielding in the strongest components. The ratio $\sigma_{IV}/\bar{\sigma}_{IV}$, where $\bar{\sigma}_{IV}$ is the average level of $\sigma_{IV}$ computed from the yield locus for the rotationally symmetric texture, would be expected to be a measure of the strength of the local component relative to that of the rotationally symmetric aggregate, and hence should be an indicator of ear height.

Ear position and ear height will hence be predicted as follows. The ratios $\sigma_{IV}/\bar{\sigma}_{IV}$ will be calculated for all textural components in the plane of the sheet. The location of the individual components will be specified by noting the angle $\theta$ between transverse direction of the ideal texture and the $x$ axis of the individual textural component. Ears will be taken to form along the $x$ axes of those components for which $\sigma_{IV}/\bar{\sigma}_{IV}$ is less than one and troughs, along the $x$ axes of components for which $\sigma_{IV}/\bar{\sigma}_{IV}$ is greater than one. The height of the ear formed will be taken to be proportional to the amount by which $\sigma_{IV}/\bar{\sigma}_{IV}$ is less than one at the earing site.

The earing patterns so calculated for the ideal cube-on-face, cube-on-edge, and cube-on-corner textures are shown in Figure 19 for both restricted- and pencil-glide deformation. The patterns are strikingly similar for both deformation modes. Four severe 45° ears are anticipated for the cube-on-face texture. The
Figure 19. Earing Curves Computed for the Ideal Cube-on-face, Cube-on-edge, and Cube-on-corner Textures. The angle $\theta$ locates the local x (radial) direction relative to the transverse direction of the ideal texture. The solid lines represent restricted-glide deformation, the dotted lines, pencil-glide deformation.
cube-on-edge texture is expected to form 8 ears, but these occur in pairs near the rolling and transverse directions and almost reduce to four ears. The cube-on-corner texture shows the smallest earing tendency and 12 ears are anticipated to form. The position and severity of the ears calculated for the three common textures agree well with Tucker's observations on aluminum single crystals of the corresponding orientations.

Earring patterns for mixed textures can be calculated from a series of yield loci formed by locating x and y axes at various angles in the plane of the aggregate sheet and applying principal stresses along these axes. Local values of $\sigma_{IV}/\sigma_{IV}$ can again be compared and earing patterns calculated for the mixed texture.

Earring patterns similar to those calculated under conditions of planar stress can also be calculated for conditions of planar strain. However, the symmetries for the various textural components shown in Figures 1 and 11 are similar for both planar-stress and planar-strain restraints, suggesting that the crystallographically derived earing calculations are not very sensitive to the details of shear restraint.

Hence, generalizing the Tucker earing theory to polyslip situations leads to calculations of ear positions and ear heights that are remarkably similar for extremes of both deformation mode and shear restraint.
PART II

An Experimental Investigation of the Plastic Anisotropy of Body-centered Cubic Metals

V. Introduction

Previous studies of the textural origins of plastic anisotropy have been made largely with steel sheet. The experimental emphasis in that work was on the effects of various textural components on the ratio of width to thickness strains measured in a conventional tension test. The relationship of \( R \) to the shape of the yield locus for the sheet has been pointed out by Backofen, Hosford, and Burke.\(^1\) (Chapter III) High \( R \) (\( R > 1 \)) was associated with an elongation of the \( v \). Mises ellipse in the first and third quadrant and a contraction in the second and fourth. Low \( R \) (\( R < 1 \)) has the opposite effect, with \( R = 0 \) reducing the \( v \). Mises ellipse to a circle.

Another sensitive measure of locus distortion is the ratio of the flow stresses determined under conditions of plane-strain and uniaxial tension (Figure 20). Uniaxial tension is represented by loading paths 1 and 2, plane-strain tension (or compression) by paths 3 and 4. As shown in Figure 20, the plane-strain stresses \( \sigma_{xx}(\text{max}) \) and \( \sigma_{yy}(\text{max}) \) determine the maximum spread of the yield locus in the \( x \) and \( y \) directions, respectively. The ratios \( \sigma_{xx}(\text{max})/x_t \) and \( \sigma_{yy}(\text{max})/y_t \) indicate the degree of first-quadrant distortion. These stress ratios are more useful than the strain ratio \( R \) in comparing experimentally determined and analytically
Figure 20. Schematic Yield Locus Indicating Loadings Imposed on the Low-carbon Steel Sheet. Uniaxial loading in the transverse and radial directions occurs along paths 1 and 2, respectively. Plane-strain loading in the transverse and radial directions occurs along paths 3 and 4. The relationship of the strain ratios $R_x$ and $R_y$ to the slopes of the yield locus determined along paths 1 and 2 is also indicated.
calculated locus distortion, since the analytically calculated R values are often indeterminate (Chapter IV).

In work being reported here, the analytically and experimentally determined stress ratios $\sigma_{xx}(\text{max})/X_t$ and $\sigma_{yy}(\text{max})/Y_t$ were compared for a textured 0.05 percent carbon steel sheet. The character and proportions of the textural components present were estimated from (200) and (110) pole figures. Using this textural information, composite yield loci were calculated and the ratios $\sigma_{xx}(\text{max})/X_t$ and $\sigma_{yy}(\text{max})/Y_t$ were computed. The stress $\sigma_{xx}(\text{max})$ and $\sigma_{yy}(\text{max})$ were determined from both plane-strain tension and plane-strain compression tests. The stresses $X_t$ and $Y_t$ and the strain ratios $R_x$ and $R_y$ were also measured in more conventional tension tests. The experimentally determined stress ratios were formed and compared to the same ratios from two other sources: the crystallographically derived yield loci, and the Hill anisotropic continuum locus which was calculated using the average of the measured values of $R_x$ and $R_y$.

To examine the extremes of plastic anisotropy which might be anticipated in ideally textured rods and sheets, wire drawing\textsuperscript{37,38} and plane-strain compression\textsuperscript{39,40} tests have been performed on single crystals. Similar plane-strain compression tests were performed here on corner-orientation $\alpha$-iron single crystals. The measured anisotropy was compared to that to be anticipated if deformation occurred under conditions of planar stress and planar strain for both restricted- and pencil-glide behavior.
VI. The Plastic Anisotropy of a Low-carbon Steel Sheet

A. Material

The material for this investigation was an Al-killed, hot-rolled and box-annealed 0.05 percent carbon steel sheet. The grain size was ASTM 7 and the thickness 0.034 in. The details of chemical composition and processing history are contained in Appendix IV.

B. Texture

(200) and (110) poles figures were prepared using the Lopata-Kula modification of the Schultz method. Additional corrections for the variation of intensity with angle to the reflecting surface were made with a factor determined for each reflection from a random powder sample of pure iron. The measured diffracted intensities were normalized so that the average pole density \( \bar{\rho} \) projected onto a spherical surface is unity. This was done by dividing the pole figure into segments of equal spherical area, noting the average pole density in these areas, and averaging the results to find the intensity that corresponds to a random distribution of crystals. This procedure can be expressed as

\[
\sum \rho_n A_n = \bar{\rho} A_t = A_t
\]

or

\[
\frac{1}{A_t} \sum \rho_n A_n = 1
\]

(107)
where $\rho_n$ and $A_n$ are the local pole densities and area elements and $A_t$ is the total spherical area. The random intensities so determined were about 20 percent higher than those measured for the corresponding peaks in the random powder sample of pure iron. The normalized pole figures are shown in Figure 21.

The intense peaks in both pole figures correspond to a combined (111) [\overline{1}10] and (111) [1\overline{1}2] texture which has mirror symmetry about the planes normal to the rolling and transverse directions. The areas in which pole density is $< 1$, in both figures, indicate that there are fewer crystals with {100} planes in the plane of the sheet than would be present in an isotropic sheet, or, stated alternatively, the sheet has a "negative" {100} texture. The pole densities in the remaining areas are approximately equal to 1.

C. **Textural Idealization and Yield Locus Construction**

The intensified and depleted regions are more crisply delineated in the (200) pole figure, and, for this reason, the texture will be idealized from the information from that figure. The final result is shown in Figure 22. The first step in the idealization was to account for the areas of density ranging from 2 to 4. This was done by assigning them an average pole density of 3 and assuming that their extent could be represented by the two rotations shown. The intensified areas are thus modeled as a cube-on-corner texture that is rotationally symmetric about the sheet normal except for the 6° gaps shown and is smeared by 10° about axes in the plane of the sheet.
Figure 21. (200) and (110) Pole Figures for the Low-carbon Steel Sheet. The intensified areas in both figures correspond to the presence of a cube-on-corner texture, the depleted areas, to the absence of a cube-on-face.
Figure 22. Idealized (200) Pole Figure and its Decomposition into Constituent Components.
(a) - Idealized (200) Pole Figures.
(b) - Isotropic Component (component 1).
(c) - Cube-on-corner Component (component 2).
(d) - Isotropic Component minus Cube-on-face Contribution (component 3).
Next, the two depleted areas in the center and at the periphery were assumed to contain an average pole density of 0.3 and were taken to represent rotational symmetry about the sheet normal. The geometry of the depleted area in the center of the pole figure was then approximated by providing an additional rotation of 25° about axes lying in the plane of the sheet. To account for the number of poles in the central area, the additional rotation required in the peripheral area is 12°. To maintain an average pole density of 1 over the entire figure, the remaining area was assigned an average density of 0.9.

This idealized result was finally modeled by a superposition of three separate pole figures, or components, as shown in Figure 22. The three textural components represented by these pole figures are identified as follows: an isotropic component with average pole density of 0.3 over the entire area (component 1, Figure 22b), a component with average pole density of 2.1 in the intensified areas and zero elsewhere (component 2, Figure 22c), and a component with average pole density of 0.6 everywhere except in the areas occupied by the negative {100} or cube-on-face texture, where the pole density is zero (component 3, Figure 22d).

The fraction $f_n$ of the total crystals representing each of these components was then estimated by finding the fraction of the total number of poles contained in the area occupied by each component. The fraction $f_n$ was found by multiplying the average pole density in the $n^{th}$ area by the proportion of the total area occupied by the $n^{th}$ component, or
\[ f_n = \rho_n \frac{A_n}{A_t} \]  \hspace{1cm} (107)

If this calculation is made for the three idealized components, the fractions estimated to the nearest 5 percent are

\[ f_1 = 0.30 \]
\[ f_2 = 0.30 \]  \hspace{1cm} (108)
\[ f_3 = 0.40 \]

It was next assumed that the yield loci for crystals in the intensified areas can be represented by the yield loci for the pure \{111\} or cube-on-corner texture rotated only about the sheet normal. This assumption should not lead to any serious computational errors with a smearing of only 10°. The yield loci for crystals that have, in effect, been removed from the zones depleted to zero (Figure 213) were assumed to be represented by loci for the rotationally symmetric cube-on-face or \{100\} texture. Based on the extent of the assumed central smearing, the axis of an average crystal in this area would be 12.5° from the \{100\} sheet normal, while the peripheral smearing suggests that an average crystal is only 6° off-axis.

The average stresses \( \bar{\sigma}_{xx} \) and \( \bar{\sigma}_{yy} \) necessary to enforce a prescribed strain state on the combination of the three textural components were found from the stresses necessary to enforce the same strain state on each component by use of the relationship

\[ \bar{\sigma}_{xx} = f_1 \sigma_{xx_1} + f_2 \sigma_{xx_2} + f_3 \sigma_{xx_3} \]
\[ \bar{\sigma}_{yy} = f_1 \sigma_{yy_1} + f_2 \sigma_{yy_2} + f_3 \sigma_{yy_3} \]  \hspace{1cm} (109)
Having established yield loci for components 1 and 2, their contributions to the average can be found directly. The yield loci for component 3, however, were found by a subtraction process.

Consider the pole figure for a random (or isotropic) distribution of crystals which is decomposed into two components as shown in Figure 23. The first component is taken to be component 3 (Figure 22d), the second is a smeared cube-on-face component. The stresses necessary to enforce a prescribed strain state in the isotropic aggregate, \( \sigma_{xx_i} \) and \( \sigma_{yy_i} \), can then be viewed as originating from contributions of the two new constituent components as follows:

\[
\sigma_{xx_i} = (1 - A_f) \sigma_{xx_3} + A_f \sigma_{xx_f}
\]

\[
\sigma_{yy_i} = (1 - A_f) \sigma_{yy_3} + A_f \sigma_{yy_f}
\]

(110)

where the subscript \( f \) refers to the cube-on-face component of Figure 23c. Rearranging equation (110) yields the desired stress levels for component 3 in Figure 22d.

\[
\sigma_{xx_3} = \frac{\sigma_{xx_i} - A_f \sigma_{xx_f}}{1 - A_f}
\]

(111)

\[
\sigma_{yy_3} = \frac{\sigma_{yy_i} - A_f \sigma_{yy_f}}{1 - A_f}
\]
Figure 23. Decomposition of (200) Pole Figure for an Isotropic Metal.
(a) - Isotropic (200) Pole Figure.
(b) - Isotropic Component minus Cube-on-face Contribution (component 3).
(c) - Cube-on-face Contribution.
or
\[
\sigma_{xx_3} = \sigma_{xx_i} + \frac{A_f}{1 - A_f} (\sigma_{xx_i} - \sigma_{xx_f})
\]

(111)

\[
\sigma_{yy_3} = \sigma_{yy_i} + \frac{A_f}{1 - A_f} (\sigma_{yy_i} - \sigma_{yy_f})
\]

Two interesting observations can be made directly from equation (111). The first is that, if the stress necessary for yielding in the negative cube-on-face texture is less than that required for an isotropic metal, a strengthening will occur, and vice-versa. The second is that the contribution of the negative cube-on-face texture is proportional to \( A_f/(1 - A_f) \). This indicates that the effects of the negative texture depend not only on removing crystals in the ideal orientation, but also on the removal of immediate neighbors, which have nearly the same yielding characteristics. However, as the area removed becomes too large, the average locus for the material that would be in the site of the negative texture site approaches isotropy and the factors \( \sigma_{xx_i} - \sigma_{xx_f} \) and \( \sigma_{yy_i} - \sigma_{yy_f} \) diminish. Hence the influence of the negative texture should reach a maximum when enough smearing occurs to remove adjacent crystals of nearly similar properties, but not enough to greatly reduce the factors \( \sigma_{xx_i} - \sigma_{xx_f} \) and \( \sigma_{yy_i} - \sigma_{yy_f} \).

To consolidate notation, let \( \sigma_{xx_i} \) and \( \sigma_{yy_i} \) apply to both component 1 (Figure 22b) and the random part of component 3 (Figure 23a). Also, let the cube-on-corner component 2 (Figure 21c) be represented by \( \sigma_{xx_c} \) and \( \sigma_{yy_c} \). Therefore, after substituting equation (111) into (109), the final relationships for the average stresses \( \bar{\sigma}_{xx} \) and \( \bar{\sigma}_{yy} \) become
\[
\sigma_{xx} = f_1 \sigma_{xx_i} + f_2 \sigma_{xx_C} + f_3 \left[ \sigma_{xx_i} + \frac{A_f}{1 - A_f} \left( \sigma_{xx_i} - \sigma_{xx_f} \right) \right]
\]

\[
\sigma_{yy} = f_1 \sigma_{yy_i} + f_2 \sigma_{yy_C} + f_3 \left[ \sigma_{yy_i} + \frac{A_f}{1 - A_f} \left( \sigma_{yy_i} - \sigma_{yy_f} \right) \right]
\]

\[
\ldots\ldots (112)
\]

At this point, upper and lower bounds to the yield loci for the steel sheet were calculated for both restricted glide and pencil glide behavior. The calculation of these bounds proceeds from equation (112) as follows:

1. All crystals in the sheet are imagined to deform by the same normal strain state.

2. The stresses \( \sigma_{xx_i} \) and \( \sigma_{yy_i} \) necessary to enforce this strain state in an isotropic metal are found from the calculations of Bishop and Hill\(^9\) for restricted glide and Hutchinson\(^11\) for pencil glide. These stresses are then weighted by the factor \( f_1 \), the fraction of the purely isotropic crystals present. These contributions to the sheet locus will be identical for both upper and lower bounds, since the upper and lower bounds to the isotropic loci coincide.

3. The stresses \( \sigma_{xx_C} \) and \( \sigma_{yy_C} \) which enforce the assumed strain state on the smeared cube-on-corner component are calculated and multiplied by the factor \( f_2 \). These stresses are calculated from planar-strain (upper bound) cube-on-corner loci for the upper bound to the sheet locus and planar-stress loci for the lower bound.
4. The factors $\sigma_{xx_i} - \sigma_{xx_f}$ and $\sigma_{yy_i} - \sigma_{yy_f}$ are computed by subtracting the stresses necessary to enforce the assumed strain state in the rotationally symmetric cube-on-face component that would be present in an isotropic metal from the stresses necessary to enforce the assumed strain state in an isotropic metal, which were found in step 2. A choice of yield loci for the cube-on-face component is now available depending on whether these crystals which would be present in an isotropic metal are taken to deform under conditions of planar stress or planar strain. An examination of the yield loci for the rotationally symmetric cube-on-face texture (Figures 2 and 12) shows a marked contraction in the first and third quadrants. Removing these crystals will hence lead to an extension in the first and third quadrants, which is reflected in a positive value of $\sigma_{xx_i} - \sigma_{xx_f}$ $\sigma_{yy_i} - \sigma_{yy_f}$ in these quadrants. The upper bound is then formed so as to maximize this extension and hence the planar-stress (lower-bound) loci for the cube-on-face component are used to calculate the upper bound to the sheet locus, and vice-versa. Since the planar-stress and planar-strain loci for the rotationally symmetric cube-on-face texture are almost identical, this distinction will actually have little effect on the final calculation.
5. The stress differences found in step 4 are multiplied by $A_f/(1 - A_f)$, the product added to the corresponding isotropic stress, and this sum multiplied by $f_3$.

6. The contributions found in steps 2, 3, and 5 are added to find the average stresses $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$.

The upper and lower bound loci calculated through the use of equation (112) are shown in Figures 24 and 25 for restricted-glide and pencil-glide behavior, respectively. Upper and lower bounds to the ratio $\sigma_{xx(\max)}/x_t$, which, because of the assumed textural symmetry, are identical to those for $\sigma_{yy(\max)}/y_t$, were computed from these loci by dividing the largest plane-strain value by the smallest uniaxial value, and vice-versa. The ratios obtained in this manner, along with the isotropic ratios calculated by Bishop and Hill$^9$ for restricted glide and Hutchinson$^{11}$ for pencil-glide (Figures 5 and 15) are shown in Table IV.

<table>
<thead>
<tr>
<th></th>
<th>Maximum</th>
<th>Minimum</th>
<th>Average</th>
<th>Isotropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Restricted Glide</td>
<td>1.28</td>
<td>1.16</td>
<td>1.22</td>
<td>1.09</td>
</tr>
<tr>
<td>Pencil Glide</td>
<td>1.30</td>
<td>1.20</td>
<td>1.25</td>
<td>1.12</td>
</tr>
</tbody>
</table>

The ratios calculated for the textured sheet are all larger than the isotropic values, which are reflected in first- and third-quadrant locus elongations. The ratios in Table IV are strikingly insensitive to slipping mode.
Figure 24. Restricted-glide Upper- and Lower-bound Yield Loci Calculated for Low-carbon Steel Sheet.
Figure 25. Approximate Pencil-glide Upper- and Lower-bound Yield Loci Calculated for Low-carbon Steel Sheet.
To trace the origins of this locus distortion, the averages of the upper and lower bounds to the uniaxial and plane-strain flow stresses will be examined for each component. These average stresses are contained in Table V. The notation RG denotes restricted glide and PG pencil glide.

Table V

<table>
<thead>
<tr>
<th>Component</th>
<th>Uniaxial: $\bar{x}_t/k$ or $\bar{y}_t/k$</th>
<th>Plane Strain: $\sigma_{xx}(\text{max})/k$ or $\sigma_{yy}(\text{max})/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RG</td>
<td>PG</td>
</tr>
<tr>
<td>1</td>
<td>3.06</td>
<td>2.75</td>
</tr>
<tr>
<td>2</td>
<td>2.72</td>
<td>2.47</td>
</tr>
<tr>
<td>3</td>
<td>3.10</td>
<td>2.79</td>
</tr>
</tbody>
</table>

The loci for component 2 are distorted from the isotropic both as a result of an increase in the plane-strain stresses and a decrease in the uniaxial stresses. The average uniaxial stresses for component 3 are almost identical to the isotropic, but a locus distortion still results because of an increase in the plane-strain stresses.

The ratios of plane-strain to uniaxial stresses computed from the values in Table V are summarized in Table VI on the following page.
Table VI

$\bar{\sigma}_{xx}(\text{max})/\bar{x}_t$ and $\bar{\sigma}_{yy}(\text{max})/\bar{y}_t$ for Textural Components

<table>
<thead>
<tr>
<th>Component</th>
<th>RG</th>
<th>PG</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.09</td>
<td>1.12</td>
</tr>
<tr>
<td>2</td>
<td>1.35</td>
<td>1.33</td>
</tr>
<tr>
<td>3</td>
<td>1.21</td>
<td>1.25</td>
</tr>
</tbody>
</table>

The combination of an increase above the isotropic in the plane-strain stress and a decrease in the uniaxial stress for component 2 has resulted in a ratio of these stresses that is larger than that for component 3. Hence, even though the plane-strain stresses themselves are larger for component 3 than for component 2, the distortion of the yield locus is more pronounced in component 2.

In the textured sheet as idealized, the presence of the cube-on-corner component (component 3) would be more effective in increasing the ratio of plane-strain to uniaxial stress than the absence of the cube-on-face, even though there is 10 percent more of this latter component present. However, both components contribute significantly to increase this stress ratio above that of the isotropic. If the sheet here is assumed to be composed only of 30 percent cube-on-corner and 70 percent isotropic (0.70 component 1 and 0.3 component 2), a ratio of plane-strain to uniaxial stress lower than that for the three-component idealization would result. Similarly, if only the
absence of cube-on-face is included (0.6 component 1 and 0.4 component 3), this stress ratio would again be lower than that for the three component idealization and would be even lower than that for 0.7 component 1 and 0.3 component 2.

D. The Experiments

The experiments performed on the steel sheet were designed to measure the flow stresses for loading along paths 1-4 (Figure 20). The flow stresses for loading along paths 1 and 2 were determined from conventional tensile tests. Those for paths 3 and 4, from plane-strain tension and compression tests.

E. Uniaxial Tension

Sheet tensile specimens with a gage section 2 in. long and 0.025 in. wide were tested at a strain rate 0.01/min. Specimens were taken along both the rolling and transverse directions.

Plastic strains were determined using composite strain gages that simultaneously measured length and width strains. Thickness strains were determined using the constancy of volume condition. The procedure used was to load to certain level, unload and measure plastic length and width strains. The true-stress:strain curves so determined in the rolling and transverse directions are shown in Figures 26 and 27, respectively. The results for the two different tests performed are seen to be in close agreement. The plastic-strain ratios, which were calculated from the slopes of the width strain versus thickness
Figure 26. Rolling-direction Stress-strain Curves. Uniaxial tension occurs along path 2, Fig. 20. Plane-strain tension and compression both occur along path 4.
Figure 27. Transverse-direction Stress-strain Curves. Uniaxial tension occurs along path 1, Fig. 20. Plane-strain tension and compression both occur along path 3.
strain curves, were $R_y = 1.81 \pm 0.09$ in the rolling direction and $R_x = 1.83 \pm 0.07$ in the transverse direction for the range of strains investigated. Values of $R$ in this range, though not uncommon, are high for Al-killed steels.

F. Plane-strain Tension

Plane-strain tension experiments were performed on face-notched tensile specimens similar to those used by Corrigan et al.\textsuperscript{43} and by Lee and Backofen\textsuperscript{44}. The specimens were of 0.50 in. width ($w$), with a reduced section height ($b_o$) of 0.042 in. and a thickness ($t_o$) of 0.020 in. The ratios $b_o/t_o$ and $w/b_o$ are 2.1 and 12, respectively, for these specimens. These values were chosen to ensure that deformation occurs by plane strain. From the experiences of Corrigan et al. with essentially isotropic materials the condition for plane strain has been reported as $w/b_o > 10$ and $b_o/t_o < 3$. Lee and Backofen have used specimens of similar geometry to enforce plane strain in highly anisotropic titanium alloys.

A slitting saw was used to machine the reduced sections in the samples, which were held in a rigid fixture. This fixture had two pins which attached to the specimen through the grip holes for accurate location and also had two clamps very near the machining site. To eliminate material damaged by machining, about 0.001 in. was removed from the reduced section by etching lightly in 5 percent nital. A microhardness traverse across the reduced section was performed, and a detectable increase in
hardness was noted only in the outer 10 percent of the section, indicating that no serious machining damage remained.

Reduced section height and thickness were measured prior to straining using the traveling microscope section of a Leitz microhardness tester.

With a reduced section thickness of 0.020 in., there still remained at least 30 crystals in the thickness direction, a sufficient number to avoid serious contributions to the overall flow stress from the weaker, less confined surface crystals. This conclusion is based on the observations of Fleischer and Hosford\textsuperscript{45}, who found that weakening from the "number of crystals per cross section effect" decreased rapidly when more than 8 crystals were present in the cross section.

Loading proceeded at a rate of about 0.05/min. Plastic extension strains were measuring using strain gages of 0.020 in. gage height which were attached to the center of the reduced section. The load-unload technique was again employed. The plane-strain tension-stress:strain curves are also shown in Figures 26 and 27. Repeat experiments produced more scatter than in the uniaxial tests, but the agreement between the pairs of tests is still good, especially at higher strains.

G. Plane-strain Compression

The method used to impose plane-strain compression was essentially that described by Lee and Backofen. Testing was
done in a precisely assembled subpress mounted in an Instron testing machine. A 0.0625 in. indentor was used to apply load to a 0.5 in. wide strip. This choice of geometry results in the deformation-zone ratios $b/t_o = 1.5$ and $w/b = 12$. The former should be large enough to ensure nearly homogeneous compression and the latter to provide the constraint required for plane strain. Lubrication was provided using a thin Teflon film which was sprayed on the surface. A series of indentations was made and the reduced thickness measured with a micrometer. Straining proceeded at a rate of about 0.06/min.

The plane-strain compression tests were performed primarily to substantiate the results of the more accurate plane-strain tension tests, and hence only one test was performed for each direction. The plane-strain compression-stress:strain curves are shown in Figures 26 and 27. Good agreement is shown between the plane-strain tension and compression results.

In order to compare flow stresses measured along different stress paths, it is necessary to make the comparison at a constant level of effective strain, which is defined as

$$d\bar{e} = \frac{dW}{\bar{\sigma}} \quad (113)$$

where $W$ is the plastic work (independent of loading path) and $\bar{\sigma}$ is the corresponding stress.\textsuperscript{46} The plastic work was taken as the area under the stress-strain curves for both uniaxial and plane-strain tension. The plane-strain and uniaxial flow
stresses were then compared for identical work on each loading path. Upper and lower limits to the ratio of plane-strain to uniaxial stress were computed by comparing the highest plane-strain tension curve with the lowest uniaxial tension curve, and vice-versa. The results of these comparisons are shown in Figures 28 and 29.

The relatively low initial ratios are attributed to the lack of a complete absence of width strains in the plane-strain tension specimens at low plastic strains. When this restraint has developed, however, the ratios of plane-strain to uniaxial stresses all approach constant values which are no longer strain dependent. These values lie in the range between 1.23 and 1.33.

Also shown in Figures 28 and 29 are the analytical bounds calculated for the idealized sheet (solid lines). The average of the bounds for both restricted- and pencil-glide behavior ($R_G = 1.22$ and $P_G = 1.25$) are in good agreement with the experimental results. The ratios of plane-strain to uniaxial stress calculated from the various isotropic loci (dotted lines 1-4) fall significantly below the experimental results. Excellent agreement with the experimental ratios is also achieved if the ratio of plane-strain stress is calculated from the Hill rotationally symmetric anisotropic theory (dotted line 5) using the average of the $R$ values measured in the rolling and transverse directions.
Figure 28. The Ratio of Rolling-direction Plane-strain ($\bar{\gamma}_{yy(\text{max})}$) to Uniaxial ($Y_e$) Flow Stresses Compared as a Function of Effective Strain. The upper-bound and lower-bound ratios computed from Fig. 24. and 25. are joined by the cross-hatching shown. $\overline{RG}$ and $\overline{PG}$ designate the average of the bounds for restricted and pencil glide, respectively. Isotropic ratios (1-4) and the ratio computed from the Hill rotationally symmetric anisotropic theory (5) are indicated.
Figure 29. The Ratio of Transverse-direction Plane-strain ($\nabla_{xx}^{\text{max}}$) to Uniaxial ($X_{\tau}$) Flow Stresses Compared as a Function of Effective Strain. The upper-bound and lower-bound ratios computed from Fig. 24 and 25. are joined by the cross-hatching shown. RG and PG designate the average of the bounds for restricted glide and pencil glide respectively. Isotropic ratios (1-4) and the ratio computed from the Hill rotationally symmetric anisotropic theory (5) are indicated.
The observed anisotropy measured by a comparison of plane-strain and uniaxial tension tests can be accounted for equally well through the use of the crystallographically derived yield loci or the Hill rotationally symmetric anisotropic locus calculated from the average R values. However, the former method, though more tedious, provides a more detailed insight into the specific textural origins of the plastic anisotropy. The observed increase above the isotropic in the ratio of plane-strain to uniaxial stress is attributed to two sources: the presence of a measured cube-on-corner texture and the absence of a cube-on-face, with the former contribution providing the larger increase. The results are remarkably similar for both pencil and restricted glide.
VII. Plane-strain Compression of α-Iron Single Crystals

To investigate the extremes of plastic anisotropy which might be present in ideally textured BCC metals, plane-strain compression tests were made on corner-orientation α-iron single crystals. Four tests on crystals whose plane of compression and direction of extension were located near the (001)[010], (111)[112], (110)[001], and (110)[1110] directions were performed.

A. Crystal Growth

Sheet crystals were grown from Ferrovac-E iron (Appendix VI) by the strain-anneal technique. This procedure has been described by Dunn and Nonken, Stein and Low, and in detail by Mayer.

Crystals near the corner orientations were selected from partially grown sheets, eliminating the need for a bending reorientation step.

The crystals grown were 0.050 in. thick and 0.80 in. wide.

The divergences from the nominal orientations, described by rotations about the sheet normal, direction of extension, and with direction (which suffers no strain) are 2°, 6°, and 5° for the (001)[010] crystal; 3°, 6°, and 1° for the (110)[001] crystal which was tested in two orthogonal directions; and 1°, 5°, and 0° for the (111)[112] crystal. The (110)[1110] specimen was cut from the (110)[001] crystal with a jeweler’s saw and the cut surface was subsequently etched in 30 percent HNO₃ in H₂O.
B. Plane-strain Compression

Plane-strain compression tests were performed using the same apparatus described in Chapter VI, except that a 0.125 in. indentor was employed. The deformation geometry was hence characterized by the deformation-zone ratios $b/t_o = 2.5$ and $w/b = 6.4$

A Teflon spray lubricant was again used.

Deflections were measured using a DC-LVDT which was attached to the side of the subpress. The final thickness after straining was measured with a micrometer and the deflection so determined was used to establish the zero strain position.

The plane-strain compression-stress:strain curves are shown in Figure 30.

The trends shown among the various orientations are in accord with the results anticipated by examining the component loci for these orientations (Figures 1 and 11). The most pronounced anisotropy is expected in the (110) crystals, for which a 2:1 ratio of plane-strain strengths is anticipated for the ideal orientations. The plane-strain strength of the (111) crystal would be expected to lie between these extremes and be higher than that for the (001).

If the stress-strain curves in Figure 30 are compared on a resolved shear-stress resolved shear-strain basis, however, the scatter is considerable. The details of this comparison were outlined by Hosford\textsuperscript{39}. Shear stress-strain ($\tau$-$\gamma$) curves
Figure 30. α-Iron Single-crystal Plane-strain Compression True-stress: strain Curves.
constructed from compressive stress-strain ($\sigma$-$\varepsilon$) data by taking

$$\tau = \frac{\sigma}{M}$$

and

$$\gamma = M\varepsilon$$

where $M$, the Taylor factor, reduces to the ratio of the plane-strain strength to the critical resolved shear stress for plane-strain deformation.

If these comparisons are made using the planar-strain plane-strain strengths calculated of the different crystals, a scatter of $^{+28}_-$ percent in $\tau$ exists at a shear strain ($\gamma$) of 0.30. This scatter is of the same order as that found for aluminum crystals by Hosford and permalloy crystals by Chin, Nesbitt, and Williams.$^{40}$

Comparing the compressive stress-strain data using the approximate planar-strain plane-strain strengths for pencil-deformation in the different components produces no significant improvement. The scatter in $\tau$ at $\gamma = 0.30$ for this case is $^{+29}_-$ percent.

The primary source of error in plane-strain compression tests on single crystals arises from the crystal rotations that occur during straining. The (111)[112] crystal tested here, for example, rotated by 12° in the interval of imposed strain. This rotation would significantly change the M factor for the crystal. The effect of this change in M is magnified when the
compressive stress-strain data are used to calculate shear stress-strain curves, since M is involved in both the calculation of the shear stress and the shear strain.

If planar-stress plane-strain strengths are used to construct $\tau - \gamma$ curves, however, some improvement results. The scatter in $\tau$ at $\gamma = 0.30$ is now $\pm 17$ percent for restricted glide and $\pm 19$ percent for pencil glide, suggesting that the plane-strain deformation of single crystals is more realistically described as occurring under conditions of planar-stress.
VIII. Conclusions

1. The plastic anisotropy calculated from crystallographically derived yield loci for BCC metals is insensitive to the choice of slipping mode. The shapes of the restricted-glide and pencil-glide loci for a given texture are remarkably similar. Making available additional pencil-glide slip planes generally shrinks the yield locus, but the shape remains similar to the locus for restricted glide.

2. Rotationally symmetric yield loci and component loci can be used to study the limiting drawing ratio (LDR) and earing behavior respectively. Calculations of the $\beta$ parameter from the rotationally symmetric yield loci for the three textures indicate a slight increase above the iostropic in the LDR as a result of presence of the cube-on-edge texture and a pronounced improvement with the presence of the cube-on-corner. The presence of cube-on-face texture decreases the LDR. Ear heights are anticipated to be largest for the ideal cube-on-face texture and smallest for the cube-on-corner. Four ears are anticipated for the cube-on-face texture, 8 for the cube-on-edge, and 12 for the cube-on-corner. These results are again insensitive to slipping mode. These earing calculations are in good agreement with Tucker's observations on deep-drawn aluminum single crystals.
3. The average ratios of plane-strain to uniaxial flow stresses calculated from the upper- and lower-bound loci for the textured low-carbon steel sheet are in good agreement with the measured ratios. The primary increases in this stress ratio above the isotropic is due to the presence of a cube-on-corner texture, but the absence of a cube-on-face also contributes to an increase. The effects of this "negative" texture depend not only on the absence of crystals in ideal orientation, but also on the absence of their immediate neighbors. These conclusions are similar for both restricted and pencil glide.

4. The ratio of plane-strain to uniaxial flow stress calculated from the Hill rotationally symmetric anisotropic theory using the R values measured in the rolling and transverse directions is in good agreement with the measured stress ratios.

5. A pronounced anisotropy similar to that anticipated from the component yield loci is observed in the plane-strain compression of α-iron single crystals. The plane-strain stress-strain data can best be correlated on a shear-stress: shear-strain basis by considering that deformation occurs under conditions of planar stress.
Suggestions for Future Work

The following areas are suggested as extensions of research related to this thesis.

1. A further investigation of the stress states which simultaneously operate three pencil-glide slip directions would be of interest to supplement the present work. In particular, a search for additional stress states near those contained in Group IV and an analysis of their ability to accommodate an arbitrary strain state would be fruitful.

2. The development of a method for calculating yield loci for mixed textures which follows directly from a knowledge of the distribution of the crystals present, rather than a positive and negative deviation from a random array, would be useful.

3. A third area is the extension of the earing calculations to mixed textures and a comparison of the results with the ear positions and ear heights observed in these textures.

4. An investigation of the limitations of the assumptions made in describing plastic anisotropy is in order. This investigation should include a study of the effects of changes in grain size on the properties of textured BCC metals.
References


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42. L. G. Schulz, J. Appl. Phys. 20, 1030 (1949).


47. C. G. Dunn and G. C. Nonken, Met. Prog. 64(6), 71 (1953).

APPENDIX I

The Bishop-Hill Procedure

The procedure used by Bishop and Hill\(^9\) and Bishop\(^10\) to find the stress states that can enforce an arbitrary change on a FCC crystal begins by recognizing the requirement of v. Mises\(^8\) that at least five independent slip systems must act simultaneously. The slip system notation used by these authors, which will also be adopted here, is shown schematically in Figure Ia and is summarized in Table Ia.

Table Ia

<table>
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<td>[011][101]</td>
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<td>((c_1)(c_2))</td>
<td>((d_1)(d_2))</td>
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</table>

Introducing the stress notation

\[
\sigma_{22} - \sigma_{33} = A \\
\sigma_{33} - \sigma_{11} = B \\
\sigma_{11} - \sigma_{22} = C \\
\sigma_{23} = F \\
\sigma_{31} = G \\
\sigma_{12} = H
\]
Figure Ia. Schematic Representation of Bishop-Hill Slip-system Notation.
one can write the generalized Schmid's law as

\[
\begin{array}{c|c}
\text{Slip System} & \text{Yield Expression} \\
\hline
\pm (a_1) & A - G + H = \pm \sqrt{6} k \\
\pm (a_2) & B - H + F = \pm \sqrt{6} k \\
\pm (a_3) & C - F + G = \pm \sqrt{6} k \\
\mp (b_1) & A + G + H = \pm \sqrt{6} k \\
\pm (b_2) & B - H - F = \pm \sqrt{6} k \\
\pm (b_3) & C + F - G = \pm \sqrt{6} k \\
\mp (c_1) & A + G - H = \pm \sqrt{6} k \\
\mp (c_2) & B + H + F = \pm \sqrt{6} k \\
\mp (c_3) & C - F - G = \pm \sqrt{6} k \\
\pm (d_1) & A - G - H = \pm \sqrt{6} k \\
\pm (d_2) & B + H - F = \pm \sqrt{6} k \\
\pm (d_3) & C + F + G = \pm \sqrt{6} k \\
\end{array}
\]

where \( k \) is the critical resolved shear stress.

The 24 yield expression can be plotted in three separate three-dimensional stress spaces which have as coordinates \( A, G, \) and \( H; B, H, \) and \( F; \) and \( C, F, \) and \( G. \) The yield condition for a particular slip system will be represented by a plane in one of the three three-dimensional spaces. These planes are oriented at equal angles to the three coordinate axes, similar to \{111\} crystallographic planes, and hence the intersections of the planes, which represent polyslip stress states, will form three octahedra (Figure Ib).
Figure Ib. Generalized Schmid's Law Plotted in Three Separate Coordinate Systems. The yield expressions can all be plotted in one of the three-dimensional spaces with coordinates A, G, and H; B, H, and F; or C, F, and G.
One then observes that there are two types of polyslip stress states: one which is situated at the vertex of an octahedron (points $x_1$ and $x_2$ in Figure Ib), and the other which lies on a line oriented at 45° to two coordinate axes ($y_1$, $y_2$, and $y_3$). The former will simultaneously operate four slip systems, the latter two. Hence any polyslip stress state which operates at least five slip systems will actually activate six or eight. Slip will be activated on eight slip systems if the stress state is located at two vertices ($x_1$, $x_2$) or at one vertex and on two 45° lines ($x_1$, $y_1$, $y_2$); six systems will be activated if the stress state is located on three 45° lines ($y_1$, $y_3$, $y_4$). (One vertex and one 45° line would violate the condition that $A + B + C = 0$.) A summary of all permissible polyslip stress states is given in Table Ib. Stress states 29-56 are the negative of 1-28.

The three cases considered above correspond to stress states -2, -12, and -16. As can be shown with reference to Figure Ib, stress state -2 will operate the eight slip systems -$(a_2)$, $(a_3)$, -$(b_2)$, $(b_3)$, -$(c_2)$, $(c_3)$, -$(d_2)$, and $(d_3)$; stress state -12 will operate systems -$(a_2)$, $(a_3)$, -$(b_2)$, $(b_3)$, -$(c_1)$, $(c_3)$, -$(d_1)$, and $(d_3)$; while stress state -16 will operate systems -$(b_2)$, $(b_3)$, -$(c_1)$, $(c_2)$, -$(d_1)$, and $(d_3)$. A similar examination of the remaining polyslip stress states will show that the stress states in Groups I, II, and III simultaneously operate eight slip systems, while those in Groups IV and V operate six.
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**Group I**

**Group II**

**Group III**

**Group IV**

**Group V**
The particular state of stress which acts to accommodate an imposed state of strain is found by selecting from the 56 permissible stress states that particular state which maximizes the external work done. The increment of work done, with reference to the cubic axes, is

\[
d\mathcal{W} = \sigma_{11} \varepsilon_{11} + \sigma_{22} \varepsilon_{22} + \sigma_{33} \varepsilon_{33} + 2\sigma_{12} \varepsilon_{12} + 2\sigma_{23} \varepsilon_{23} + 2\sigma_{31} \varepsilon_{31}
\]

............... (Ic)

or, in Bishop-Hill notation

\[
d\mathcal{W} = -B\varepsilon_{11} + A\varepsilon_{22} + 2F\varepsilon_{23} + 2G\varepsilon_{31} + 2H\varepsilon_{12}
\]

(Id)

The strains \( \varepsilon_{ij} \) are then written in terms of the imposed strains and the result is substituted into equation Id. The values of the 56 permissible states of stress are then substituted sequentially into equation Id. That particular state of stress, or those states of stress, since the stress state need not be unique, which maximizes \( d\mathcal{W} \) for the given strain state will prevail.
APPENDIX II

Planar-Stress and Planar-Strain Yield Loci

The properties of planar-stress and planar-strain yield loci will be illustrated by a detailed examination of the restricted-glide yield loci for the pure cube-on-corner textural component, the cube-on-corner component whose axes are rotated by 15° about the sheet normal, and the combination of these two components. Specific attention will be focused on the plane-strain deformation of these textures.

The coordinates r, t, and n are chosen for the [11\text{	ext{\text{-}}}2] (rolling), [1\text{\text{\text{-}}}1\text{\text{\text{-}}}0] transverse, and [111] (sheet normal) directions of the pure cube-on-corner component; x, y, and z are chosen for the rotated cube-on-corner component (Figures IIa and IIc).

In order to examine the properties of the restricted-glide loci for the pure cube-on-corner component, it is necessary to write the generalized Schmid's law in terms of the stresses acting in the r, t, n system and to express the strains in the t, t, n system in terms of the amounts of shear on the 12 slip systems. These expressions are contained in equations (IIa) and (IIb), respectively.
\[
- \frac{1}{2} \sigma_{tn} + \frac{\sqrt{3}}{2} \sigma_{nr} = \pm \sqrt{6} k \pm (a_1) \\
- \frac{1}{2} \sigma_{tn} - \frac{\sqrt{3}}{2} \sigma_{nr} = \pm \sqrt{6} k \pm (a_2) \\
\sigma_{tn} = \pm \sqrt{6} k \pm (a_3)
\]

\[
- \frac{2}{3\sqrt{6}} \sigma_{rr} + \frac{2}{3\sqrt{6}} \sigma_{nn} - \frac{2}{3\sqrt{2}} \sigma_{rt} - \frac{1}{6} \sigma_{tn} + \frac{7}{6\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (b_1)
\]

\[
\frac{2}{3\sqrt{6}} \sigma_{rr} - \frac{2}{3\sqrt{6}} \sigma_{nn} - \frac{2}{3\sqrt{2}} \sigma_{rt} - \frac{1}{6} \sigma_{tn} - \frac{7}{6\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (b_2)
\]

\[
\frac{4}{3\sqrt{2}} \sigma_{rt} + \frac{1}{3} \sigma_{tn} = \pm \sqrt{6} k \pm (b_3)
\]

\[
- \frac{1}{\sqrt{6}} \sigma_{rr} + \frac{1}{\sqrt{6}} \sigma_{tt} - \frac{2}{3\sqrt{2}} \sigma_{rt} - \frac{1}{6} \sigma_{tn} + \frac{1}{2\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (c_1)
\]

\[
\frac{1}{3\sqrt{6}} \sigma_{rr} - \frac{1}{\sqrt{6}} \sigma_{tt} + \frac{2}{3\sqrt{6}} \sigma_{nn} - \frac{1}{2} \sigma_{tn} - \frac{5}{6\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (c_2)
\]

\[
\frac{2}{3\sqrt{6}} \sigma_{rr} - \frac{2}{3\sqrt{6}} \sigma_{tt} + \frac{2}{3\sqrt{2}} \sigma_{rt} + \frac{2}{3} \sigma_{tn} + \frac{1}{3\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (c_3)
\]

\[
- \frac{1}{3\sqrt{6}} \sigma_{rr} + \frac{1}{\sqrt{6}} \sigma_{tt} - \frac{2}{3\sqrt{6}} \sigma_{nn} - \frac{1}{2} \sigma_{tn} + \frac{5}{6\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (d_1)
\]

\[
\frac{1}{\sqrt{6}} \sigma_{rr} - \frac{1}{\sqrt{6}} \sigma_{tt} - \frac{2}{3\sqrt{2}} \sigma_{rt} - \frac{1}{6} \sigma_{tn} - \frac{1}{2\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (d_2)
\]

\[
- \frac{2}{3\sqrt{6}} \sigma_{rr} + \frac{2}{3\sqrt{6}} \sigma_{nn} + \frac{2}{3\sqrt{2}} \sigma_{rt} + \frac{2}{3} \sigma_{tn} - \frac{1}{3\sqrt{3}} \sigma_{nr} = \pm \sqrt{6} k \pm (d_3)
\]

\[\text{………………… (IIa)}\]
\[ \varepsilon_{rr} = \frac{1}{3\sqrt{6}} (-2b_1 + 2b_2 - 3c_1 + c_2 + 2c_3 - d_1 + 3d_2 - 2d_3) \]

\[ \varepsilon_{tt} = \frac{1}{\sqrt{6}} (c_1 - c_2 + d_1 - d_2) \]

\[ \varepsilon_{nn} = \frac{2}{3\sqrt{6}} (b_1 - b_2 + c_2 - c_3 - d_1 + d_3) \]

\[ \varepsilon_{nt} = \frac{1}{3\sqrt{2}} (-b_1 - b_2 + 2b_3 - c_1 + c_3 - d_2 + d_3) \]

\[ \varepsilon_{tn} = \frac{1}{12} (-3a_1 - 3a_2 + 6a_3 - b_1 - b_2 + 2b_3 - c_1 - 3c_2 + 4c_3 - d_2 + 4d_3) \]

\[ \varepsilon_{nr} = \frac{1}{12\sqrt{3}} (9a_1 - 9a_2 + 7b_1 - 7b_2 + 3c_1 - 5c_2 + 2c_3 + 5d_1 - 3d_2 - 2d_3) \]

\[ \varepsilon_{rn} = \varepsilon_{nr} = 0 \]

The slip-system and shear-strain notation is summarized in Appendix I.

The planar-stress yield locus for the pure cube-on-corner component (Figure IIa) can be found directly from equation (IIa) by requiring that only \( \sigma_{rr} \) and \( \sigma_{tt} \) act, i.e., \( \sigma_{nn} = \sigma_{nt} = \sigma_{tn} = \sigma_{nr} = 0 \). The planar-strain locus (Figure IIb) is found from the procedure of Bishop and Hill (Appendix I). The stress states which make up the planar-strain locus are stress states No. 6, 24, 25, and 28.

Under the conditions of planar stress, a state of plane-strain in the transverse direction of the pure cube-on-corner component is imposed by the stress state \( \sigma_{rr} = \sigma_{tt} = \frac{3}{2} \sqrt{6} \), which simultaneously operates two branches of the yield locus; the first
Figure IIa. Restricted glide Planar-stress Yield Locus for the Ideal Cube-on-corner Texture. The slip systems which act to accommodate the imposed plane strain under planar-stress conditions are indicated.
Figure IIb. Restricted-glide Planar-strain Yield Locus for the Ideal Cube-on-corner Texture. The slip systems which act to accommodate the imposed plane strain under planar-strain conditions are indicated.
(horizontal) branch of the yield locus represents yielding on the \(-(b_1), (b_2), (c_3),\) and \(-(d_3)\) systems, the second on the \(-(c_2)\) and \((d_1)\) systems.

It should first be noted that the strains \(d\varepsilon_{rr}\) and \(d\varepsilon_{tt}\) that result from slip on each particular system are normal to their respective branch of the yield locus in the \(\sigma_{rr}', \sigma_{tt}'\) space. This can be shown with reference to equation (IIb). If slip occurs by a unit amount on the \(-(b_1), (b_2), (c_3),\) or \(-(d_3)\) system, the resulting normal strain state will be \(d\varepsilon_{rr} = \frac{2}{3\sqrt{6}},\) \(d\varepsilon_{tt} = 0,\) and \(d\varepsilon_{nn} = -\frac{2}{3\sqrt{6}},\) which is perpendicular to the horizontal branch of the yield locus. Similarly, if slip occurs on the \(-(c_2)\) or \((d_1)\) system, the resulting normal strain state will be \(d\varepsilon_{rr} = -\frac{\sqrt{6}}{3},\) \(d\varepsilon_{tt} = \sqrt{6},\) and \(d\varepsilon_{nn} = -\frac{2\sqrt{6}}{3},\) which again is perpendicular to its branch of the yield locus. This illustrates the general property of planar-stress yield loci that, even though the normal strains resulting from slip on a single system are not principal strains, these normal strain components are perpendicular to the two-dimensional principal stress (planar-stress) yield locus.\(^{19}\)

Plane-strain in the transverse direction is achieved by combining the strain vectors normal to the two branches of the yield locus to get the normal-strain state \(d\varepsilon_{rr} = 0, d\varepsilon_{tt} = 1,\) and \(d\varepsilon_{nn} = -1.\) If active hardening is assumed to be equal to latent hardening, it is possible to allow different amounts of shear to occur on the various operative systems. By adjusting the amounts of shear on the operative systems to be \(b_1 = d_3 = -\frac{\sqrt{6}}{8},\)
\[ b_2 = c_3 = \frac{\sqrt{6}}{8}, \quad c_2 = -\frac{\sqrt{6}}{2}, \quad \text{and} \quad d_1 = \frac{\sqrt{6}}{2} \] and substituting these expressions into the strain-slip relationships (equation IIb), a state of transverse plane strain is achieved, since \( \delta \varepsilon_{rr} = 0, \delta \varepsilon_{tt} = 1, \) and \( \delta \varepsilon_{nn} = -1. \) However, only two of the shear strains, \( \delta \sigma_{rt} \) and \( \delta \sigma_{tn}, \) vanish; the remaining shear strain \( \delta \varepsilon_{nr} \) is equal to \( \frac{15\sqrt{2}}{8}. \) The appearance of the additional shear strain \( \delta \sigma_{nr} \) occurs not only for the particular amounts of shear chosen, but is completely general. This general behavior can be shown by using the first five expressions of equation (IIb) \( (\delta \varepsilon_{rr} = 0, \delta \varepsilon_{tt} = 1, \delta \varepsilon_{nn} = -1, \delta \varepsilon_{rt} = 0, \delta \varepsilon_{tn} = 0) \) to write the expression for \( \delta \varepsilon_{nr} \) in terms of the amounts of shear \( c_3 \) and \( d_3. \) If, in addition, \( \delta \varepsilon_{nr} \) is to be equal to zero, the following condition must be met:

\[ c_3 - d_3 + \frac{\sqrt{6}}{6} = 0 \]  

(IIc)

Since \( c_3 \) must be positive and \( d_3 \) must be negative, equation (IIc) can never be satisfied and the strain \( \delta \varepsilon_{nr} \) will always be present.

In order to impose a state of transverse plane strain in the absence of shear strains, i.e. under conditions of planar strain, it is necessary both to alter the level of \( \sigma_{rr} \) and \( \sigma_{tt} \) and to apply an additional shear stress \( \sigma_{nr}. \) Specifically, if
\[ \sigma_{rr} = \frac{\sqrt{6}}{3} k, \quad \sigma_{tt} = \frac{5\sqrt{6}}{3} k, \quad \sigma_{nr} = -\frac{2\sqrt{3}}{3} k, \quad \text{and} \quad \sigma_{nn} = \sigma_{rt} = \sigma_{tn} = 0 \]
(Bishop-Hill stress state No. 6, \( \sigma_{12} = \sqrt{6} k, \) when referred to the cubic axes), transverse plane strain in the absence of shear strains is achieved. By substituting the stresses in the \( r, t, n \) system into equation (IIa), one can show that the critical resolved shear stress is simultaneously reached on the eight systems \(- (a_+)\),
\((a_2), -(b_1), (b_2), (c_1), -(c_2), \text{ and } -(d_2)\). Plane strain under the conditions of planar strain can be achieved, for example, if the amounts of shear on the operative systems are
\[
a_1 = \frac{\sqrt{6}}{6}, \quad a_2 = -\frac{\sqrt{6}}{2}, \quad b_1 = -\frac{\sqrt{6}}{2}, \quad b_2 = \frac{\sqrt{6}}{2}, \quad c_1 = d_1 = \frac{\sqrt{6}}{4},
\]
and \(c_2 = d_2 = -\frac{\sqrt{6}}{4}\).

Hence, a state of plane strain in the transverse direction of the pure cube-on-corner texture is accompanied by the additional shear strain \(d_{nr}\) when imposed under conditions of planar stress. When planar-strain conditions are imposed, the additional shear stress \(\sigma_{nr}\) must be applied.

The restricted-glide yield loci for the cube-on-corner component that is rotated by 15° about the sheet normal have properties similar to those found for the pure cube-on-corner component. The planar-stress and planar-strain yield loci for this latter case are shown in Figures IIc and d.

Under conditions of planar stress, plane strain in the x direction of the rotated cube-on-corner component is accomplished by the principal stress state \(\sigma_{xx} = \sigma_{yy} = \frac{3}{2} \sqrt{6} k\) (Figure IIc). Since, in both components, plane strain is accomplished by the same principal stresses \(\sigma_{rr} = \sigma_{tt} = \frac{3}{2} \sqrt{6} k, \sigma_{xx} = \sigma_{yy} = \frac{3}{2} \sqrt{6} k\), this principal-stress state will also operate plane strain in the combination of the two components \(\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \frac{3}{2} \sqrt{6} k\).

In the plane-strain deformation of the rotated cube-on-corner component, the same six slip systems are activated as in the pure
Figure IIc. Restricted-glide Planar-stress Yield Locus for the Cube-on-corner Component Rotated about the Sheet Normal as Shown. The slip systems which act to accommodate the imposed plane strain under planar-stress conditions are indicated.
Figure IIId. Restricted-glide Planar-strain Yield Locus for the Cube-on-corner Component Rotated about the Sheet Normal as Shown. The slip systems which act to accommodate the imposed plane strain under planar-strain conditions are indicated.
cube-on-corner component. The horizontal branch of the yield locus has split into two branches, one which remained horizontal and one oriented at 45° to the coordinate axes. Although the principal-stress state necessary to activate planar-stress plane strain is the same for both textural components, and the same six slip systems are activated, the amounts of slip on the six systems will differ in the two components, since the two plane-strain states are different when referred to the cubic axes. Hence, the additional shear strains accompanying the planar-stress plane-strain deformation will also be different for the two components.

Under conditions of planar-strain, plane strain in the x direction is enforced by the same Bishop-Hill stress state (No. 6, \( \sigma_{12} = \sqrt{6} \) k) that enforced plane strain in the transverse direction of the pure cube-on-corner component. When referred to the x, y, z coordinates of the rotated cube-on-corner texture, the required normal stresses are \( \sigma_{xx} = \frac{1}{2} (\sqrt{6} + \sqrt{2})k \) and \( \sigma_{yy} = \frac{1}{2} (\sqrt{6} - \sqrt{2})k \).

In addition, the additional shear stresses in the x, y, z system are \( \sigma_{xy} = -\frac{1}{3} \sqrt{6} \) k, \( \sigma_{yz} = -\frac{1}{3} \sqrt{3} + 2\sqrt{3} \) k, and \( \sigma_{zx} = -\frac{1}{3} \sqrt{-3 + 2\sqrt{3}} \) k. Hence, while only the one additional shear stress (\( \sigma_{np} \)) was required to enforce plane strain in the pure cube-on-corner component, all three shear stresses (\( \sigma_{xy}, \sigma_{yz}, \) and \( \sigma_{zx} \)) must act to enforce plane strain in the rotated cube-on-corner component.

The composite average normal stresses necessary to enforce planar-strain plane strain in the combination of the two cube-on-corner components are \( \bar{\sigma}_{xx} = \frac{1}{2} (\sigma_{tt} + \sigma_{xx}) = (\frac{4}{3} \sqrt{6} + \frac{1}{2} \sqrt{2})k \) acting in the direction of extension and \( \bar{\sigma}_{yy} = \frac{1}{2} (\sigma_{rr} + \sigma_{yy}) = (\frac{2}{3} \sqrt{6} - \frac{1}{2} \sqrt{2})k \) acting in the direction which suffers no strain. The
accompanying average shear stresses are $\bar{\sigma}_{xy} = \frac{1}{2} \left( \sigma_{tr} + \sigma_{xy} \right) = \frac{1}{6} \sqrt{3} k$, $\bar{\sigma}_{yz} = \frac{1}{2} \left( \sigma_{nr} + \sigma_{yz} \right) = \frac{1}{6} \left( 2\sqrt{3} + \sqrt{3} + 2\sqrt{3} \right) k$, and $\bar{\sigma}_{zx} = \frac{1}{2} \left( \sigma_{nt} + \sigma_{zx} \right) = \frac{1}{6} \sqrt{-3 + 2\sqrt{3}} k$. 
APPENDIX III

The Averaging Process Used to Compute Composite Anisotropic Yield Loci

The averaging process used to compute composite anisotropic yield loci which was outlined in Chapter II will be illustrated here by the calculation of the restricted-glide planar-stress yield locus for the rotationally symmetric cube-on-face texture. Specific attention will be focused on plane-strain and axisymmetric deformations.

The coordinate notation for this texture and representative component loci whose axes are rotated by 0°, 22.5°, and 45° about the [001] sheet normal are shown in Figure IIIa. Also shown acting normal to the component loci and the rotationally symmetric locus are the two strain vectors which represent plane strain \( \varepsilon_{xx} = 1, \varepsilon_{yy} = 0 \) and axisymmetric reduction \( \varepsilon_{xx} = 1, \varepsilon_{yy} = -\frac{1}{2} \).

The stress states which enforce these strain states in the rotationally symmetric composite are found by averaging the stresses which enforce these same strain states in the individual textural components. This is accomplished through use of the trapezoidal rule (equation 4)

\[
\bar{\sigma}_{xx} = \frac{1}{n} \left[ \frac{1}{2}(\sigma^{o}_{xx} + \sigma^{n}_{xx}) + \sigma^{1}_{xx} + \ldots + \sigma^{n-1}_{xx} \right]
\]

(IIIa)

\[
\bar{\sigma}_{yy} = \frac{1}{n} \left[ \frac{1}{2}(\sigma^{o}_{yy} + \sigma^{n}_{yy}) + \sigma_{yy} + \ldots + \sigma^{n-1}_{yy} \right]
\]
Figure IIIa. Schematic Representation of the Averaging Process Used to Calculate the Restricted-glide Planar-stress Yield Locus for the Rotationally Symmetric Cube-on-face Texture. The stress states necessary to impose axisymmetric and plane-strain deformation in the three components and the rotationally symmetric texture are indicated.
In order to find rotationally symmetric loci for the cube-on-face texture, it is necessary only to include components whose $x$ axis lies somewhere in the $45^\circ$ interval between the [110] and [100] directions. This results because of the four-fold rotational symmetry about the [001] axis and the mirror symmetry about the {110} planes. The rotational symmetry reduces the interval to $90^\circ$; the mirror symmetry, to $45^\circ$.

Component loci were computed in $2.5^\circ$ intervals about the [001] sheet normal, giving 19 components which were used in the average. The stresses $\sigma_{xx}$ and $\sigma_{yy}$ for plane-strain reduction are the same for all the components ($\sigma_{xx} = \sigma_{yy} = \sqrt{6} \, k$) and hence the average plane-strain strength of the rotationally symmetric texture will also be $\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \sqrt{6} \, k$. The stresses $\sigma_{xx}$ and $\sigma_{yy}$ necessary to enforce axisymmetric flow in the 19 cube-on-face components are summarized in Table IIIa. The angle $\theta$ is measured about the [001] direction from the [110] direction toward the [100].

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\frac{\sigma_{xx}}{k}$</th>
<th>$\frac{\sigma_{yy}}{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.449</td>
<td>-2.449</td>
</tr>
<tr>
<td>2.5</td>
<td>2.261</td>
<td>-2.261</td>
</tr>
<tr>
<td>5</td>
<td>2.114</td>
<td>-2.114</td>
</tr>
<tr>
<td>7.5</td>
<td>2.000</td>
<td>-2.000</td>
</tr>
<tr>
<td>10</td>
<td>1.911</td>
<td>-1.911</td>
</tr>
<tr>
<td>12.5</td>
<td>1.843</td>
<td>-1.843</td>
</tr>
<tr>
<td>15</td>
<td>1.793</td>
<td>-1.793</td>
</tr>
<tr>
<td>17.5</td>
<td>1.759</td>
<td>-1.759</td>
</tr>
</tbody>
</table>
### Table IIIa (cont'd)

<table>
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<tr>
<th></th>
<th>$\bar{\sigma}_{xx}$</th>
<th>$\bar{\sigma}_{yy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.739</td>
<td>-1.739</td>
</tr>
<tr>
<td>22.5</td>
<td>1.732</td>
<td>-1.732</td>
</tr>
<tr>
<td>25</td>
<td>1.796</td>
<td>-1.402</td>
</tr>
<tr>
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</tr>
<tr>
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<td>-0.112</td>
</tr>
<tr>
<td>45</td>
<td>2.449</td>
<td>0</td>
</tr>
</tbody>
</table>

The average stresses $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ necessary to enforce planar stress axisymmetric flow in the rotationally symmetric cube-on-face texture are found from equation (IIIa) to be $\bar{\sigma}_{xx} = 2.002k$ and $\bar{\sigma}_{yy} = -1.319k$.

A general normal-strain state in the plane of a sheet can be characterized by the strain ratio $\rho = \frac{d\varepsilon_{yy}}{d\varepsilon_{xx}}$. The strain ratio $\rho$ can be related to the angle $\phi$, the angular orientation of the strain vector from the $\sigma_{xx}$ axis, by the relation

$$\phi = \tan^{-1}{\rho} \quad \text{(IIIb)}$$

In the calculation of the rotationally symmetric cube-on-face locus, $\phi$, and hence $\tan^{-1}{\rho}$, was varied in $1^\circ$ increments because the slope of the yield locus changed rapidly in this interval.
In the calculation of the rotationally symmetric loci for the cube-on-edge and cube-on-corner textures, $\phi$ was varied in $2.5^\circ$ increments.

For composite yield loci which contain an isotropic component, only 12 strain ratios, $\rho = \frac{d \varepsilon_{yy}}{d \varepsilon_{xx}}$, can be included in the average, since these are the only strain ratios for which the isotropic yield strengths were calculated. These strain ratios, transferred from the reduced stress space used by Bishop and Hill and Hutchinson to the $\sigma_{xx}$, $\sigma_{yy}$ axes are $\rho = -1, -\frac{7}{8}, -\frac{3}{4}, -\frac{5}{8}, -\frac{1}{2}, -\frac{3}{8}, -\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{7}, \frac{1}{3}, \frac{3}{5}$, and 1.

Averaging techniques similar to those illustrated for planar-stress restricted-glide loci were used to calculate planar-strain and pencil-glide loci.
APPENDIX IV

A Summary of Group IV Stress States

The complete stress relationships for Groups IVa, IVm, and IVy were derived in Chapter III. The stress relationships for the remaining Group IV stress states are found using procedures similar to those used for Groups IVa, IVm, and IVy. The entire set of Group IV stress states is summarized below.

Group IVa: \( F = 0, [\overline{111}] \) Inoperative

\[
A = \frac{1}{3} (-G + H)
\]
\[
B = \frac{1}{3} (2G + H)
\]
\[
C = \frac{1}{3} (G + 2H)
\]
\[
G^2 + GH + H^2 = \frac{27}{8} k^2
\]
\[
0 < \varepsilon = 8GH \leq 9k^2
\]

Group IVb: \( F = 0, [\overline{1\overline{1}}1] \) Inoperative

\[
A = \frac{1}{3} (G + H)
\]
\[
B = \frac{1}{3} (2G - H)
\]
\[
C = \frac{1}{3} (-G + 2H)
\]
\[
G^2 - GH + H^2 = \frac{27}{8} k^2
\]
\[
0 < \varepsilon = -8GH \leq 9k^2
\]
Group IVc: \( F = 0, [\bar{\text{III}}] \) Inoperative

\[
A = \frac{1}{3} (G + H) \\
B = \frac{1}{3} (-2G + H) \\
C = \frac{1}{3} (G - 2H) \\
G^2 - GH + H^2 = \frac{27}{8} k^2 \\
0 < \varepsilon = -8GH < 9k^2
\]

Group IVd: \( F = 0, [\text{III}] \) Inoperative

\[
A = \frac{1}{3} (G - H) \\
B = \frac{1}{3} (2G + H) \\
C = \frac{1}{3} (G + 2H) \\
G^2 + GH + H^2 = \frac{27}{8} k^2 \\
0 < \varepsilon = 8GH < 9k^2
\]

Group IVe: \( G = 0, [\text{III}] \) Inoperative

\[
A = -\frac{1}{3} (H + 2F) \\
B = \frac{1}{3} (-H + F) \\
C = \frac{1}{3} (2H + F) \\
H^2 + HF + F^2 = \frac{27}{8} k^2 \\
0 < \varepsilon = 8HF < 9k^2
\]
Group IVf: \( G = 0, [\bar{1}1\bar{1}] \) Inoperative

\[
A = \frac{1}{3} (H + 2F) \\
B = \frac{1}{3} (H - F) \\
C = \frac{1}{3} (2H + F) \\
H^2 + HG + F^2 = \frac{27}{8} k^2 \\
0 < \epsilon = 8HF < 9k^2
\]

Group IVg: \( G = 0, [\bar{1}1\bar{1}] \) Inoperative

\[
A = \frac{1}{3} (-H + 2F) \\
B = \frac{1}{3} (H + F) \\
C = \frac{1}{3} (2H - F) \\
H^2 - HF + F^2 = \frac{27}{8} k^2 \\
0 < \epsilon = -8HF < 9k^2
\]

Group IVh: \( G = 0, [\bar{1}1\bar{1}] \) Inoperative

\[
A = \frac{1}{3} (H - 2F) \\
B = \frac{1}{3} (H + F) \\
C = \frac{1}{3} (-2H + F) \\
0 < \epsilon = -8HF < 9k^2
\]
Group IVi: \( H = 0, \{111\} \) Inoperative

\[
A = \frac{1}{3} (2F + G) \\
B = \frac{1}{3} (F + 2G) \\
C = \frac{1}{3} (-F + G) \\
F^2 + FG + G^2 = \frac{27}{8} k^2 \\
0 < \epsilon = 8FG < 9k^2
\]

Group IVj: \( H = 0, \{1\bar{1}1\} \) Inoperative

\[
A = \frac{1}{3} (-2F + G) \\
B = \frac{1}{3} (F - 2G) \\
C = \frac{1}{3} (F + G) \\
F^2 - FG + G^2 = \frac{27}{8} k^2 \\
0 < \epsilon = -8FG < 9k^2
\]

Group IVk: \( H = 0, \{\bar{1}\bar{1}1\} \) Inoperative

\[
A = \frac{1}{3} (2F + G) \\
B = \frac{1}{3} (F + 2G) \\
C = \frac{1}{3} (F - G) \\
F^2 + FG + G^2 = \frac{27}{8} k^2 \\
0 < \epsilon = 8FG < 9k^2
\]
Group IVl: $H = 0$, [111] Inoperative

$$A = \frac{1}{3} (2F - G)$$

$$B = \frac{1}{3} (-F + 2G)$$

$$C = \frac{1}{3} (F + G)$$

$$F^2 - FG + G^2 = \frac{27}{8} k^2$$

$$0 < \varepsilon = -8FG \leq 9k^2$$

Group IVm: $A = -B$, $C = 0$, $F = G$, [111] Inoperative

$$A = -B = \frac{F(F - H)}{F + 2H}$$

$$\left[ \frac{F(F - H)}{F + 2H} \right]^2 + 2F^2 + H^2 + \frac{F(F^2 + 2H^2)}{F + 2H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8F(F^2 + 2H^2)}{F + 2H} \leq 9k^2$$

Group IVn: $A = -B$, $C = 0$, $F = -G$, [111] Inoperative

$$A = -B = \frac{F(-F + H)}{F + 2H}$$

$$\left[ \frac{F(-F + H)}{F + 2H} \right]^2 + 2F^2 + H^2 + \frac{F(F^2 + 2H^2)}{F + 2H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8F(F^2 + 2H^2)}{F + 2H} \leq 9k^2$$

Group IVo: $A = -B$, $C = 0$, $F = G$, [111] Inoperative

$$A = -B = \frac{F(F + H)}{-F + 2H}$$

$$\left[ \frac{F(F + H)}{-F + 2H} \right]^2 + 2F^2 + H^2 + \frac{F(F^2 + 2H^2)}{F - 2H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8F(F^2 + 2H^2)}{F - 2H} \leq 9k^2$$
Group IVp: \( A = -B, C = 0, F = -G, [\bar{III}] \) Inoperative

\[
A = -B = \frac{F(F - H)}{-F + 2H} \\
\left(\frac{F(F - H)}{-F + 2H}\right)^2 + \frac{2F^2}{-F + 2H} \leq \frac{k^2}{2} \\
0 < \varepsilon = \frac{8F(2F^2 + 2H^2)}{-F + 2H} \leq 9k^2
\]

Group IVq: \( B = -C, A = 0, G = H, [\bar{III}] \) Inoperative

\[
B = -C = \frac{G(-F + G)}{2F + G} \\
\left(\frac{G(-F + G)}{2F + G}\right)^2 + \frac{2G^2}{2F + G} + \frac{G(2F^2 + G^2)}{2F + G} = \frac{9}{2} k^2 \\
0 < \varepsilon = \frac{8G(2F^2 + G^2)}{2F + G} \leq 9k^2
\]

Group IVr: \( B = -C, A = 0, G = -H, [\bar{III}] \) Inoperative

\[
B = -C = \frac{G(F + G)}{-2F + G} \\
\left(\frac{G(F + G)}{-2F + G}\right)^2 + \frac{2G^2}{-2F + G} + \frac{G(2F^2 + G^2)}{-2F + G} = \frac{9}{2} k^2 \\
0 < \varepsilon = \frac{8G(2F^2 + G^2)}{-2F + G} \leq 9k^2
\]

Group IVs: \( B = -C, A = 0, G = -H, [\bar{III}] \) Inoperative

\[
B = -C = \frac{G(F - G)}{2F + G} \\
\left(\frac{G(F - G)}{2F + G}\right)^2 + \frac{2G^2}{2F + G} + \frac{G(2F^2 + G^2)}{2F + G} = \frac{9}{2} k^2 \\
0 < \varepsilon = \frac{8G(2F^2 + G^2)}{2F + G} \leq 9k^2
\]
Group IVt: $B = -C, A = 0, G = H, [\bar{1}l1]$ Inoperative

$$B = -C = \frac{G(F + G)}{2F - G}$$

$$\left[\frac{G(F + G)}{2F - G}\right]^2 + F^2 + 2G^2 + \frac{G(2F^2 + G^2)}{-2F + G} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8G(2F^2 + G^2)}{-2F + G} \leq 9k^2$$

Group IVu: $C = -A, B = 0, H = F, [l1l]$ Inoperative

$$C = -A = \frac{H(-G + H)}{2G + H}$$

$$\left[\frac{H(-G + H)}{2G + H}\right]^2 + G^2 + 2H^2 + \frac{H(2G^2 + H^2)}{2G + H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8H(2G^2 + H^2)}{2G + H} \leq 9k^2$$

Group IVv: $C = -A, B = 0, H = F, [l\bar{1}l]$ Inoperative

$$C = -A = \frac{H(G + H)}{2G - H}$$

$$\left[\frac{H(G + H)}{2G - H}\right]^2 + G^2 + 2H^2 + \frac{H(2G^2 + H^2)}{-2G + H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8H(2G^2 + H^2)}{-2G + H} \leq 9k^2$$

Group IVw: $C = -A, B = 0, H = -F, [\bar{1}l1]$ Inoperative

$$C = -A = \frac{H(G + H)}{-2G + H}$$

$$\left[\frac{H(G + H)}{-2G + H}\right]^2 + G^2 + 2H^2 + \frac{H(2G^2 + H^2)}{-2G + H} = \frac{9}{2} k^2$$

$$0 < \varepsilon = \frac{8H(2G^2 + H^2)}{-2G + H} \leq 9k^2$$
Group IVx: \( C = -A, B = 0, H = -F, [\overline{1}11] \) Inoperative

\[
C = -A = \frac{H(G - H)}{2G + H}
\]

\[
\left[ \frac{H(G - H)}{2G + H} \right]^2 + G^2 + 2H^2 + \frac{H(2G^2 + H^2)}{2G + H} = \frac{9}{2} k^2
\]

\[0 < \varepsilon = \frac{8H(2G^2 + H^2)}{2G + H} < 9k^2\]

Group IVy: \( A = -B, C = 0, A = H, [\overline{1}11] \) Inoperative

\[2H^2 + H(F + G) - FG = 0\]

\[F^2 + FG + G^2 = \frac{9}{2} k^2\]

\[0 < \varepsilon = 8H(F + G) < 9k^2\]

\[FG + GH + HF > 0\]

Group IVz: \( A = -B, C = 0, A = -H, [\overline{1}11] \) Inoperative

\[2H^2 + H(F - G) + FG = 0\]

\[F^2 - FG + G^2 = \frac{9}{2} k^2\]

\[0 < \varepsilon = 8H(F - G) < 9k^2\]

\[-FG - GH + HF > 0\]

Group IVaa: \( A = -B, C = 0, A = H, [\overline{1}11] \) Inoperative

\[2H^2 - H(F + G) - FG = 0\]

\[F^2 + FG + G^2 = \frac{9}{2} k^2\]

\[0 < \varepsilon = -8H(F + G) < 9k^2\]

\[FG - GH - HF > 0\]
Group IVbb: $A = -B, C = 0, A = -H, [\overline{111}]$ Inoperative

\[ 2H^2 + H(-F + G) + FG = 0 \]
\[ F^2 - FG + G^2 = \frac{9}{2} k^2 \]
\[ 0 < \varepsilon = 8H(-F + G) < 9k^2 \]
\[ -FG + GH - HF > 0 \]

Group IVcc: $B = -C, A = 0, B = F, [\overline{111}]$ Inoperative

\[ 2F^2 + F(G + H) - GH = 0 \]
\[ G^2 + GH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \varepsilon = 8F(G + H) < 9k^2 \]
\[ FG + GH + HF > 0 \]

Group IVdd: $B = -C, A = 0, B = -F, [\overline{11}1]$ Inoperative

\[ 2F^2 + F(-G + H) + GH = 0 \]
\[ G^2 - GH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \varepsilon = 8F(-G + H) < 9k^2 \]
\[ -FG - GH + HF > 0 \]

Group IVee: $B = -C, A = 0, B = -F, [\overline{111}]$ Inoperative

\[ 2F^2 + F(G - H) + GH = 0 \]
\[ G^2 - GH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \varepsilon = 8F(G - H) < 9k^2 \]
\[ FG - GH - HF > 0 \]
Group IVff: \( B = -C, A = 0, B = F, [\bar{\bar{1}}11] \) Inoperative

\[ 2F^2 - F(G + H) - GH = 0 \]
\[ G^2 + GH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \epsilon = -8F(G + H) \leq 9k^2 \]
\[-FG + GH - HF > 0 \]

Group IVgg: \( C = -A, B = 0, C = G, [1\bar{1}1] \) Inoperative

\[ 2G^2 + G(F + H) - HF = 0 \]
\[ F^2 + FH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \epsilon = 8G(F + H) \leq 9k^2 \]
\[ FG + GH + HF > 0 \]

Group IVhh: \( C = -A, B = 0, C = G, [1\bar{1}1] \) Inoperative

\[ 2G^2 - G(F + H) - HF = 0 \]
\[ F^2 + FH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \epsilon = -8G(F + H) \leq 9k^2 \]
\[-FG - GH + HF > 0 \]

Group IVii: \( C = -A, B = 0, C = -G, [\bar{1}\bar{1}1] \) Inoperative

\[ 2G^2 + G(F - H) + HF = 0 \]
\[ F^2 - FH + H^2 = \frac{9}{2} k^2 \]
\[ 0 < \epsilon = 8G(F - H) \leq 9k^2 \]
\[ FG - GH - HF > 0 \]
Group IVjj:  \( C = -A, \ B = 0, \ C = -G, \ [\overline{111}] \) Inoperative

\[
2G^2 + G(-F + H) + HF = 0
\]

\[
F^2 - FH + H^2 = \frac{9}{2} k^2
\]

\[
0 < \varepsilon = 8G(-F + H) \leq 9k^2
\]

\[-FG + GH - HF > 0\]
APPENDIX V

Processing History and Chemical Composition of Low-Carbon Steel Sheet

The Al-killed low-carbon sheet was made in a basic oxygen furnace and ladle-killed. Finish rolling was done at about 1625°F, and the sheet was spray quenched to below 1100°F before coiling. Using a heating rate of less than 30°F per hour, the material was box annealed for 30 hours at 1290-1310°F. The sheet was given a slight skin pass of about 1/2 percent reduction after annealing, which resulted in a final thickness of 0.034 in. The ASTM grain size was 7. The chemical composition is described below.

<table>
<thead>
<tr>
<th>Element</th>
<th>C</th>
<th>Mn</th>
<th>P</th>
<th>S</th>
<th>Cu</th>
<th>Al</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>0.05</td>
<td>0.37</td>
<td>0.005</td>
<td>0.025</td>
<td>0.030</td>
<td>0.040</td>
<td>0.014</td>
</tr>
</tbody>
</table>
APPENDIX VI

Growth of α-Iron Single Crystals

The composition of the Ferrovac E used in the strain anneal technique is shown below in Table VIa.

**Table VIa**

<table>
<thead>
<tr>
<th>Element</th>
<th>C</th>
<th>Mn</th>
<th>P</th>
<th>S</th>
<th>Si</th>
<th>Ni</th>
<th>Cr</th>
<th>Mo</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.006</td>
<td>0.03</td>
<td>0.006</td>
<td>0.001</td>
</tr>
<tr>
<td>V</td>
<td>0.004</td>
<td>0.01</td>
<td>0.01</td>
<td>0.001</td>
<td>0.0010</td>
<td>0.0042</td>
<td>0.000015</td>
<td></td>
</tr>
</tbody>
</table>

The 1 in. starting stock was given a sequence of fifty-percent reductions by rolling followed by 2 hr. anneals at 825°C. The final anneal was for 4 hr. at 825°C.

Critical straining was done at a rate of about 0.0002/min., which limited the Luder's strain to 1.9-2.4%. Straining was continued for 0.5% beyond the Luder's strain.

Crystals were grown in hydrogen in a differentially wound horizontal gradient furnace. A sharp gradient was achieved by inserting a slotted copper block in the hot zone and butting a water cooled tube against the copper block with only a mica sheet between the two. The samples were pulled into the hot
zone at a rate of 1 cm./hr. After about 3 cm. had grown, the crystal was removed, etched in 30% HNO₃ in H₂O, and X-rayed using Laue back-reflection patterns. When a suitable crystal was found, all others present were cut away and the sheet was again introduced into the furnace to complete growth.
Biographical Note

Henry R. Piehler was born in East Paterson, New Jersey, on April 16, 1938. He attended elementary school in East Paterson, and was graduated from Lodi High School, Lodi, New Jersey, in 1956. In 1960 he received the S.B. degree in Aeronautics and Astronautics from M.I.T. As a student in M.I.T.'s cooperative program, he spent six months of his junior year working as an engineering assistant at Northrop Aircraft Corporation, Hawthorne, California. During the summers of 1960 and 1961, he was employed as a consultant at the RAND Corporation, Santa Monica, California. He received the S.M. degree in Aeronautics and Astronautics from M.I.T. in 1962. After spending that summer as a research engineer in the M.I.T. Aeroelastic and Structures Research Laboratory, he became a graduate student in the Department of Metallurgy in the fall of 1962.

In 1965, he married the former Margaret M. Forbes, of Montreal, Quebec.