HIERARCHICAL PRODUCTION PLANNING

by

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ABSTRACT

This thesis develops and tests a heuristic production planning approach designed for tactical and operational planning in a two-stage manufacturing environment. The methodology developed incorporates the hierarchical philosophy in the planning framework for both finished product assembly and component fabrication. The assumptions incorporated in the approach are tested, and the heuristic procedure is compared to Material Requirements Planning.

The thesis concludes that the Hierarchcial approach is based on sound assumptions and that there are situations in which a corporation which currently uses M.R.P. would benefit from using such an approach. For the cases herein tested, the only situations in which the M.R.P. approach outperformed the hierarchical method were those for which part setup costs were primary and part manufacturing was not capacity constrained.

Thesis Supervisor: Arnoldo C. Hax

Title: Professor of Management Science

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CHAPTER 1 - INTRODUCTION

The hierarchical approach to production planning was originally designed for determining production schedules in single-stage manufacturing environments [18]. Since the original concept was developed, extensive work has centered on improving the method [4], [5], [12], [13]. The idea of using the hierarchical framework for a multistage environment, one in which costs from several stages, like part fabrication and finished product assembly, are important, was first suggested by Candea [8]. This thesis develops a practical hierarchical approach designed for two-stage production scheduling. This method is a heuristic and whenever possible an intuitive justification for chosen proceedures will be given. This chapter presents an overview of the issues addressed by production planning and the content of the remaining chapters.

1.1 Production Planning

Production planning is a term which refers to a wide spectrum of issues, and addresses questions ranging from plant expansion to scheduling employees' vacations. This spectrum can be partitioned into three layers - strategic, tactical, and operational. This classification follows Anthony's general structure for managerial decisions [2]. Anthony's framework has proven to be extremely helpful in generating insights into the decision-making process, and in defining the characteristics of a sound planning method.

These three echelons - strategic, tactical, and operational, have different scopes and planning horizons, require different degrees of data aggregation, and are subject to varying intensities of risk and uncertainties.

Strategic decisions fix the location, size and number of plants and warehouses, set the choice of appropriate technology, the means of distribution, contingency plans, the selection of product-markets, and product mix. Tactical plans determine the size of the workforce, number of shifts, amount of overtime and subcontracting, inventory targets, and levels of production for each period of the planning horizon. The allocation of resources determined at the tactical level must fit in the world defined by strategic plans.

Both strategic and tactical decisions deal with aggregate quantities. For example, manpower is not determined for individual products in each hour, but rather for groups of products over a period of time. The existence of an aggregation structure for products, customers, time periods and resources is necessary for cohesive tactical planning.

At the operational level, management is concerned with day-to-day issues such as, which workmen to assign to which machines, which orders to expedite or de-expedite, and in what order products should be processed on differing machines. These decisions are constrained by aggregate tactical plans.

Production planning involves complex choices from a very large number of alternatives. Decisions are made considering conflicting objectives while satisfying financial, technological, and marketing policies. Since managerial interaction potentially affects the cost of goods sold and hence the efficiency of an organization, it is natural and necessary that several managerial echelons be active in the planning process. Hierarchical production planning, as designed by Hax and Meal [18] facilitates this critical interaction.

1.2 Outline of the Thesis

The next chapter of this thesis reviews single-stage production planning with emphasis on the theory and contributions of the hierarchical process. The multistage environment and the methodology most frequently used for production scheduling, Material Requirements Planning, is described in the third chapter of this thesis. The fourth chapter conceptually develops a two-stage hierarchical production planning model. This chapter discusses the special problems the hierarchical approach faces when placed in a multistage environment, and the solution procedures adopted for our modeling efforts. The fifth chapter tests some of the heuristic procedures described in the fourth chapter. A limited comparison between the hierarchical approach and Material Requirements Planning is presented in chapter 6. Finally conclusions and topics for future research are discussed in the last chapter of the thesis.

CHAPTER 2 - APPROACHES TO SINGLE-STAGE PRODUCTION PLANNING

In a single-stage batch processing environment, distinct items are manufactured with identical equipment. Production planning addresses the issue of determining the number of units of each good to manufacture and the sequence in which items are to be produced. A simple lot sizing approach works well when the process being managed is not seasonal in nature, or capacity constraints are nonexistant. A global optimization method is appropriate for systems with primary setup costs, accurate detailed forecasts, and accurate parameters. The hierarchical approach was designed to fit the needs of systems which are capacity constrained and in which setup costs are secondary in nature. The traditional lot sizing approach and the optimization method are both briefly reviewed in this chapter, and then the hierarchical framework is described.

2.1 Lot Sizing

The lot sizing method was designed to exploit the economic order quantity concept. This approach was developed for pure inventory systems to minimize total inventory holding and machine changeover costs. It deals with each order independently, and therefore does not address capacity constraints, overtime or subcontracting. An illustration of the cost tradeoffs considered by this method is shown in figure 2.1.

The economic order quantity is manufactured whenever the inventory of an item reaches a critical point. If demand is

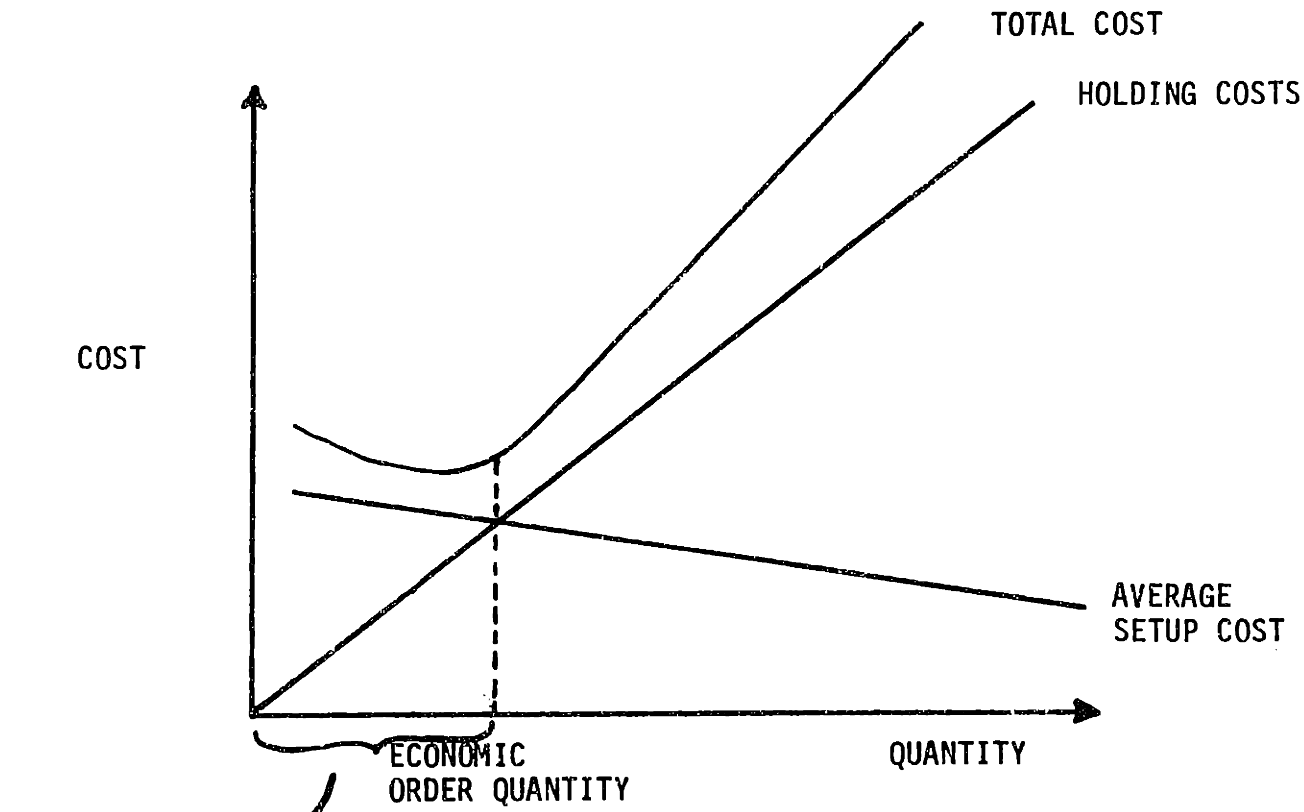


FIGURE 2.1 EOQ

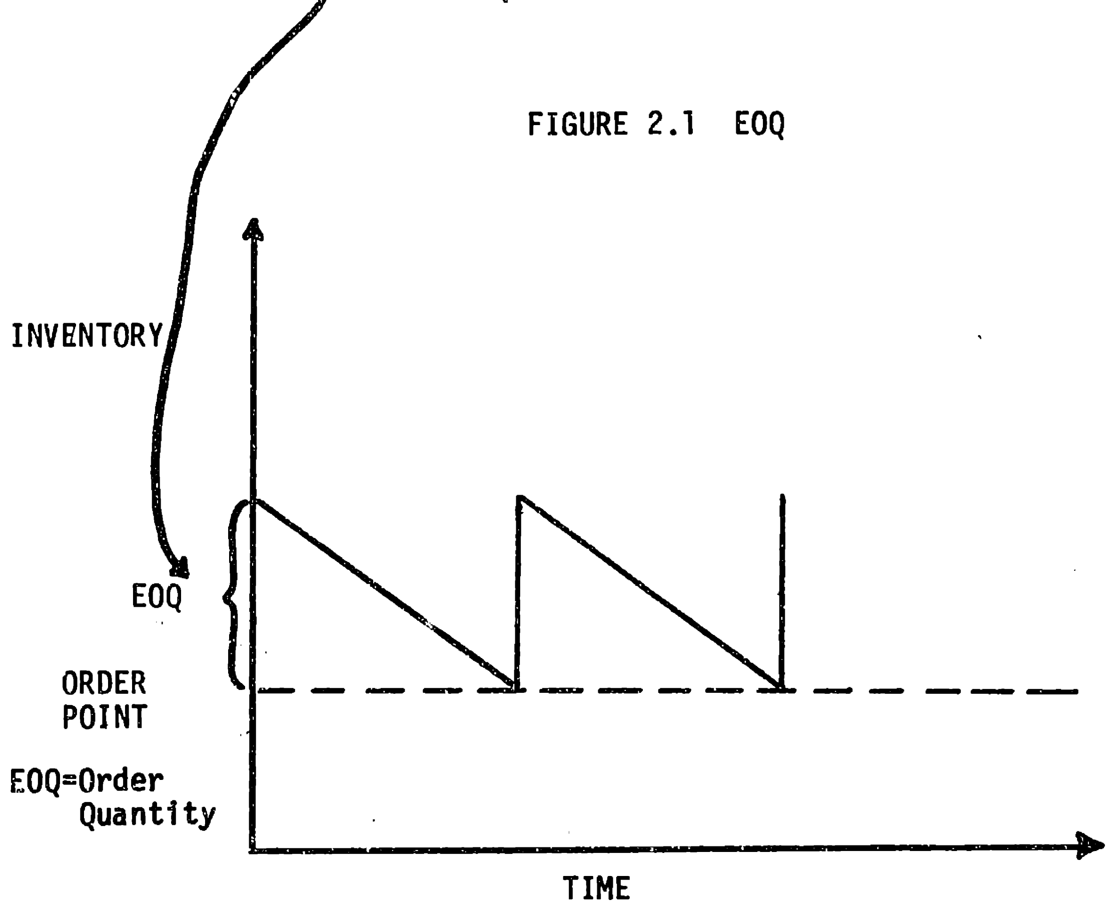


Figure 2.2 Inventory Profile

smooth and capacity is available, the product's inventory will resemble Figure 2.2.

The application of this approach to a manufacturing environment may be very disappointing, since it does not consider either the costs associated with fluctuating work loads, or capacity constraints. For example, when several items have similar seasonalities and capacity is limited lot sizing procedures may not be appropriate. In these situations, many items must be produced simultaneously to satisfy the corresponding demands. If the run quantities of each of these items are substantial in size and capacity is limited, the system necessarily is in trouble, necessitating small production quantities. This creates high manufacturing costs and excessive downtime due to machine changeovers. Basically, this approach is designed for controlling product inventories in the absence of seasonalities and capacity limitations. It has been used extensively for general manufacturing activities. Unfortunately, it is frequently applied in improper settings.

2.2 The Monolithic Optimization Approach

Since the development of linear programming, operations researchers have been attracted to modeling production planning and scheduling systems. Monolithic or global models attempt to minimize all production costs while satisfying demand and without exceeding capacity limitations.

The essence of the direct optimization approach is to capture the indivisibilities of run lengths and economies of scale. This necessitates planning at a level of detail where setup costs are incurred. Unfortunately, the resulting mixed-integer programming formulation is too large to be solved by existing computational algorithms. Thus there has been major emphasis on transforming the problem into an equivalent linear programming model which can be solved using large scale programming methods. Pioneering work in this area was done by Manne in 1958 [23]. This work was continued by Dzielinski, Baker and Manne [9], Dzielinski and Gomory [10], and later by Lasdon and Terjung [20]. Theoretically, these approaches guarantee an optimal solution and are a great contribution to the operations management literature. However, there are two principal drawbacks to the implementation of monolithic models - the data required by this approach are inappropriate for supporting aggregate production decisions, and the models do not facilitate managerial interaction with the solution process.

Tactical decisions are based on long run production plans for aggregate products. However, monolithic models are forced to deal with products at the most detailed level, in order to include all relevant costs. To support tactical planning, detailed product data must be aggregated. For example, a critical input for direct optimization is forecasted demand for each item over the entire planning horizon, typically one year. The use of detailed data in these situations presents two major problems.

Detailed forecasting requires significantly more computation than aggregate forecasting, thus not allowing complex and more accurate techniques to be used. Forecasting at the item level and then aggregating the estimates results in greater errors than aggregate forecasting.

The second practical disadvantage of direct optimization, the inability to coordinate managerial interaction with the model's solution, is a severe handicap in real world situations. There are many details warranting managerial consideration which are difficult if not impossible to include in the modeling efforts. For example, managment may prefer spurts of overtime to continual small amounts, so that machine repairs and tuneups are easier.

2.3 The Hierarchical Approach

Both lot sizing and direct or global optimization share a common characteristic, despite their extensive differences. These models do not distinguish between different echelons in the decision-making process. The hierarchical production planning approach is characterized by its recognition of the need to separate tactical from operational decisions and by its ability to deal with individual decisions at each level while using linking mechanisms for transferring higher level results to lower levels. Hierarchical production planning is designed to encourage managerial interactions at all levels.

Hierarchical production planning addresses the continuous issues involved with tactical and operational planning done in an environment defined by strategic plans. Long-range planning defines parameters for hierarchical production planning. For example, service level and the products being produced are determined by strategic plans.

The first stage of the hierarchy addresses tactical planning and determines an aggregate production plan: the timing of inventory buildups, manpower decisions, capacity utilization, and overtime schedules for the entire planning horizon, typically a full year.

Given an aggregate production plan, a variety of methods have been proposed for disaggregating the production schedule for the current time period. Basically, the second level of the model attempts to minimize setup costs subject to constraints imposed by the aggregate production plan.

The third and final level in the hierarchy schedules the production quantities of each item in order to maximize the time until the next setup of similar items is required and maintain the constraints imposed by previous levels. A conceptual overview of the hierarchical framework is illustrated in Figure 2.3. The design of a hierarchical system is a complex task. An excellent framework is provided by Hax [15]. The framework needs to both fit the organizational structure it is designed for and be analytically sound.

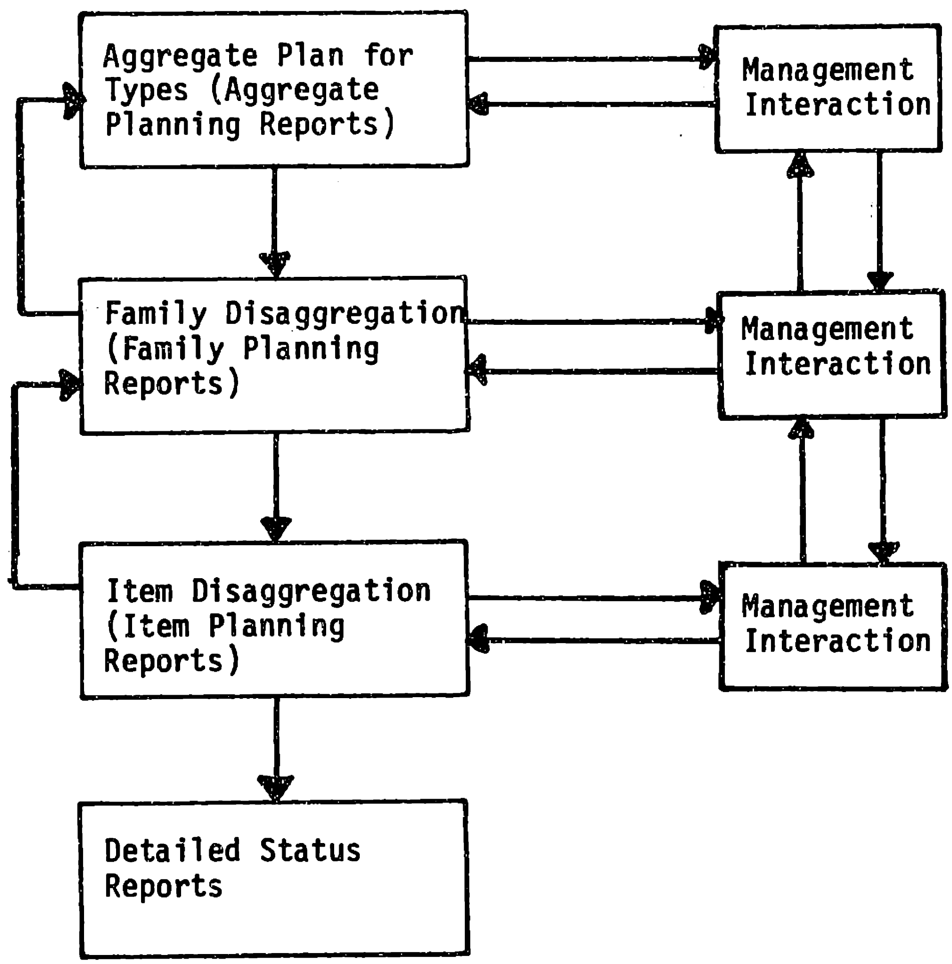


FIGURE 2.3 Conceptual Overview of Hierarchical Planning System

For the hierarchical framework three levels of product aggregation have been identified [18]. At the highest level of aggregation, items with similar costs and productivity characteristics are grouped into product types. This grouping is used to determine the optimum aggregate levels of production, manpower, and inventory. Within a product type, items sharing major setup costs are grouped into a family. This is done in order to allow all items in a familiy to be produced jointly, thereby avoiding unnecessary setup costs. Figure 2.4 illustrates an example of item partitioning.

The Aggregate Model

The first stage in hierarchical production planning is the aggregate level. A model frequently used is a linear program designed to minimize the total costs involved in accumulating seasonal stock, regular and overtime production, and holding inventory. Other costs can easily be added to the formulation. Any aggregate production planning model can be used in the first stage of hierarchical production planning as long as it adequately represents the practical problem under consideration. For extensive discussions of possible models, see [7], [15].

As tactical planning is concerned with complete seasonal cycles, typically a year, the model finds the optimal solution for a complete cycle. Tactical managerial decisions may affect the solution of the model at this stage. This aggregate model is run every period with a rolling horizon, so

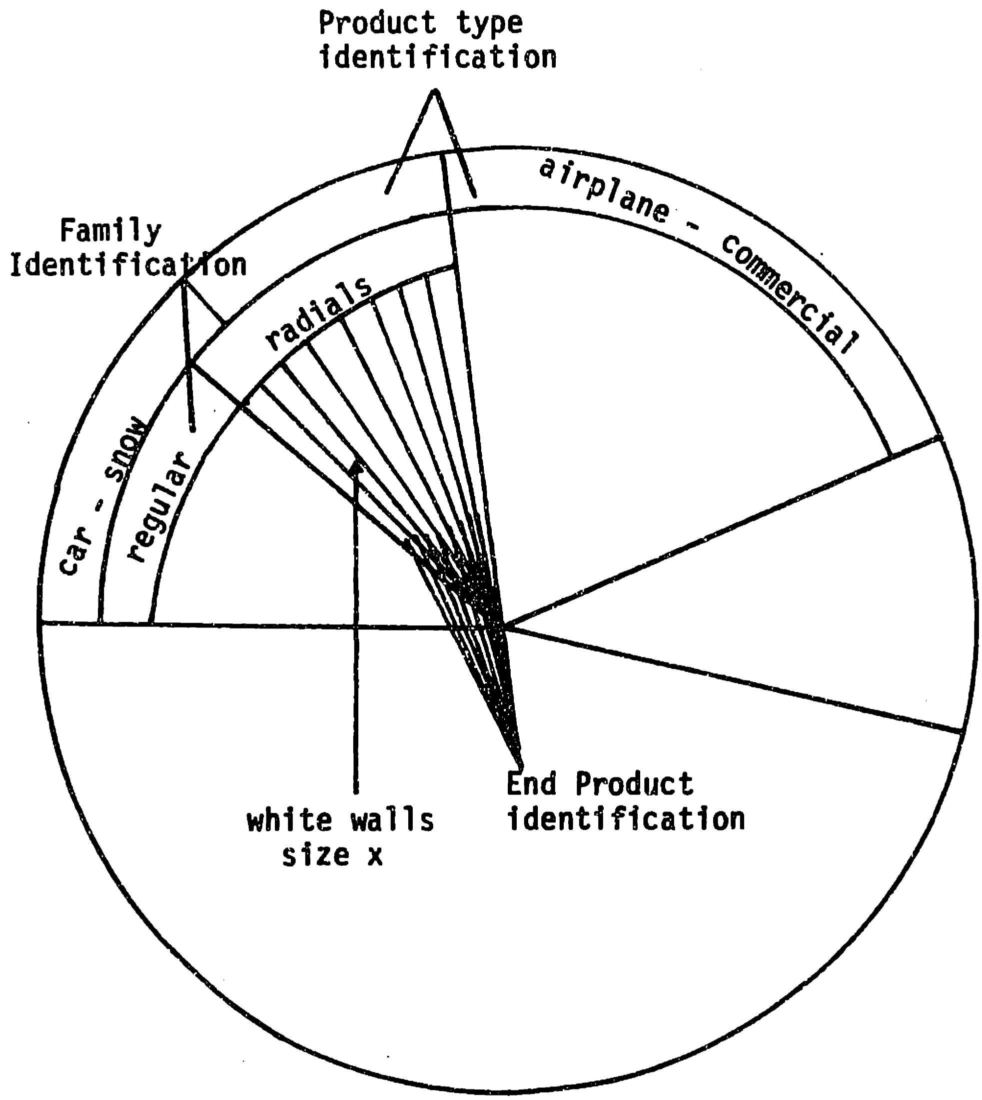


FIGURE 2.4 Example of Item Classification

that a complete cycle is always considered. A period in this context is situation dependent. In practice, it has been found that four weeks in a period and 13 periods in a year are a good choice. Virtually all disaggregation methodologies use as a primary input the aggregate production levels specified by the aggregate model. An aggregate model is illustrated in Figure 2.5.

The aggregate model works with annual aggregate forecasts. The forecasting techniques can therefore be sophisticated, and in general will be more accurate than the forecasts required by the approaches discussed earlier. Decisions on regular time, overtime, and hiring and firing are based on total demand. Therefore, the forecasts associated with hierarchical production planning should result in better decision-making ability than the approaches associated with less accurate forecasts. Figure 2.6 depicts the flow of information at the aggregate level in hierarchical production planning.

The Disaggregation Methodologies

A critical step in the hierarchical scheme is to determine how aggregate production quantities should be allocated among the families belonging to each product type. It is at this level in the hierarchy where setup costs are considered. To insure feasibility and consistency in the system, the sum of the production of the families in each product type must not be greater than the amount dictated by

FIGURE 2.5 Determination of an Aggregate Production Plan

MINIMIZE PRIMARY COSTS

$$\text{MIN} \sum_t \left[\sum_i \left(h(i)*I(i,t) + cr*R(i,t) + co*O(i,t) \right) \right]$$

such that:

$$p(i)*[R(i,t)+O(i,t)] + I(i,t-1) - I(i,t) = d(i,t)$$

$$R(i,t) \geq 0 \qquad \text{for all } i,t$$

$$O(i,t) \geq 0$$

$$I(i,t) \geq ss(i,t)$$

$$\sum_i R(i,t) \leq r(t) \qquad \text{for all } t$$

$$\Sigma O(i,t) \leq o(t)$$

---

where:

h(i)	= the holding cost per unit of product type i
I(i,t)	= the planned aggregate inventory of product type i in period t
cr	= the cost of regular time associated with the production of finished products
R(i,t)	= the regular time allotted to the production of product type i in period t
ss(i,t)	= the safety stock of product type i required in period t and is determined based on required service levels and forecast errors
r(t)	= the available regular time for the production of finished goods in period t
o(t)	= the available overtime for the production of finished goods in period t
d(i,t)	= the effective demand for product type i in period t
p(i)	= the productivity rate - units of product type i which can be aasembled per hour of production time

INPUT*	STAGE	OUTPUT**
1. Forecast by Product Type 2. Service Level for each Product Type 3. Production Costs: Inventory Holding Costs, Labor Costs, Backorder Costs 4. Production Rates 5. Policy Variables: determined by Strategic Plans, e.g. available capacity	→ Aggregate Level Planning Planning Horizon is one year Review once a month ←	1. Production Schedules*** 2. Capacity Utilization 3. Inventory Levels*** 4. Planned changes in the workforce 5. Required Overtime 6. Marginal costs of Strategic Decisions

* These variables must be available for each product type over every period in the planning horizon.

** These variables are specified for each product over the entire planning horizon.

*** These variables are passed to Figure 2.9.

FIGURE 2.6 Flow of Information at the Aggregate Stage the aggregate model for this product type. Four disaggregation methods proposed in the literature will be presented: Equalization-of-Run-Out-Times (EROT) [19], Winters [35], Hax and Meal [18], and Knapsack [4]. Figure 2.7 displays a flow chart comparing the algorithms.

The Winters Approach

The Winters approach was first proposed in 1962 [35]. Its central idea is to exploit the Economic Quantity (EOQ) concept. The methodology allocates $X^*(I,t)$, the aggregate production quantity for product type I in the immediate period. The allocation process is conducted through the following steps:

1. Compute the run-out-time for each item, k, in product type I:

$$ROT(k) = [AI(k) - ss(k)]/d(k)$$

where:

AI(k) represents the available inventory of item k,

ss(k) represents the safety stock of item k, and

d(k) represents the demand for item k in the current period.

2. Compute the run-out-time for each family, j in product type I -

$$ROT(j) = \underset{k \in k(j)}{\text{Minimum}}\ ROT(k)$$

where:

k(j)represents the set of indices of all items in family j.

3. Rank all families within a product type by their associated run-out-time so that those families with smaller run-out-times are at the top of the list. An index, Index(j) indicates what position family j occupies on the list.

4. The run quantitiy of the first family on the list is scheduled for production, and that family is moved to the bottom of the list. The run quantity for a family, RQ(j), is defined as:

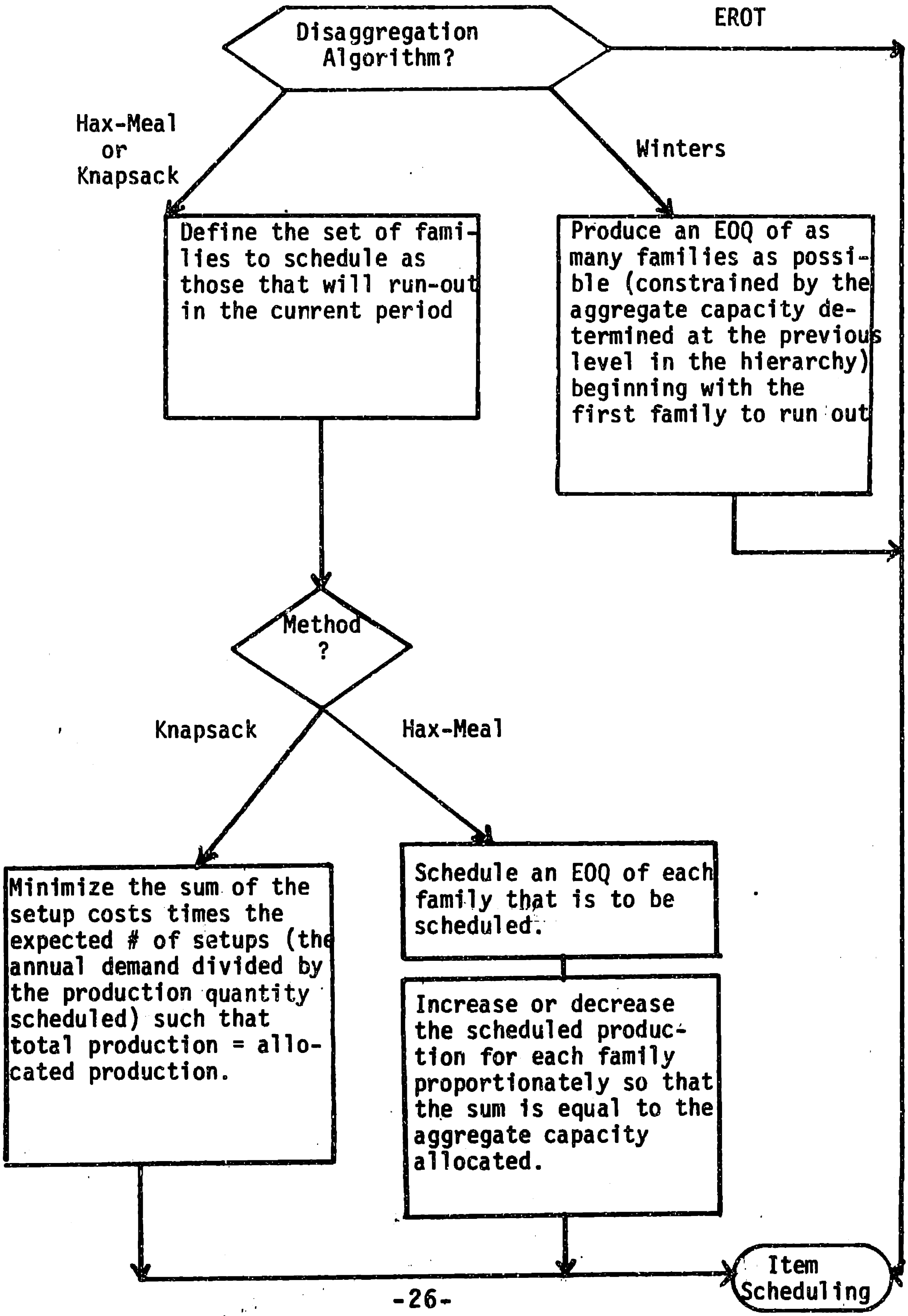
$$RQ(j) = MIN\{EOQ(j), OS(j)-AI(j)\}$$

where OS(j) represents the overstock limit of family j. When family j has a terminal demand at the end of its season, OS(j) can be calculated by means of a newsboy model (see Zimmermann and Sovereign [36]). If demand for the product is continuous, storage space or annual demand can dictate the level overstock limit is set at.

5. If a run quantity of the second family on the list, now moved up to the top position,can be produced without the total planned family production exceeding X(I,t) it will be planned.

6. This process continues until a family is reached that cannot be produced without the total running over the planned aggregate total.

FIGURE 2.7 The Disaggregation Algorithms



There are two drawbacks associated with this method of family scheduling which will briefly be mentioned. If capacity is tight, it may be important to allocate all of $X^*(I,t)$, while this approach usually will not do so. It would be a rare occurence if the sum of the run quantities of the families produced exactly matched the aggregately planned number. Secondly, it is possible for a run-quantity of every family on the list to be allocated, while a significant quantity remains to be allocated. In this case two EOQ's of a family may be planned for production. The economic order quantity conecpt was designed to offset setup and holding costs. The idea of a complete second EOQ or not to produce the remaining quantity is meaningless, as a second setup cost does not occur when more than one EOQ is produced. After disaggregating to the family level the approach disaggregates family quantities to determine item production. An approach for doing this is described later.

<u>The Equalization-Of-Run-Out-Time Approach (EROT)</u>

The production levels determined at the aggregate planning stage for a given type are disaggregated using as a criterion the equalization of the run-out-times of all items belonging to that type. This is equivalent to maximizing the time until any item within the product type will run out. If the type run quantity is set at $X^*(I,t)$, for the immediate period, the run quantities, $Z(k)$, are initially set for each item k -

$$Z(k) = d(k) * [(X^*(I,t) + \sum_{k \in I} (AI(k) - ss(k))/ \sum_{k \in I} d(k)] + ss(k) - AI(k).$$

where:
d(k) is the forecasted demand for item k in the immediate period,
AI(k) is the available inventory of item k, and
ss(k) is the safety stock for item k.

All the Z(k)'s are required to lie between zero and the overstock limits less the available inventory plus the safety stock.
If some Z(k) violates its upper or lower bound, or production is less than zero, it is defined as being equal to that bound it violated , $X^*(I,t)$ is decreased by the quantity allocated to that item and the process is repeated with the remaining items - the Z's are redefined, until a production quantity of every item is determined.

This approach ignores the family level and setup costs and typical solutions are such that whenever the aggregate problem allocates a non - zero sum in a given period for production of a product type all of the items in that type end up being produced in that period. Little is lost by using such an approach when setup costs are very low or holding costs are so high every product will be setup in every period, regardless of the disaggregation scheme.

The Hax-Meal Approach

The Hax-Meal algorithm was developed in 1973 [18]. The concept behind this algorithm is to base intial allocations on the classical EOQ formula and then to adjust these levels to better fit the aggregate model. The Hax-Meal approach recognizes that holding costs over future periods have been determined by the aggregate model, and that the primary concern should be minimization of setup costs. This methodology allocates production capacity only to those families that will run-out in the immediate period in lot sizes proportional to their EOQ's. If all families scheduled for production hit their overstock limits other families are considered. Formally the algorithm works as follows:

1. A list composed of all families within a product type which will run-out in the immediate period is constructed. This set of families will be called J^*.

2. For each family in J^*, an initial run quantity is scheduled:

   $$\hat{Y}(j) = \text{Minimum}\ [EOQ(j),\ OS(j) - AI(j)\ ] \quad \text{for all } j \varepsilon J^*.$$

3. If the sum of $\hat{Y}(j)$ for all j in J^* does not add to $X(I)^*$, they are adjusted in the following manner -

   A. If $\sum_{j \varepsilon J^*} \hat{Y}(j) < X(I)^*$,

      new run quantities are defined as:

      $$\hat{Y}^*(j) = \text{Minimum}\ [\ OS(j) - AI(j),\ \hat{Y}(j) + (X(I)^* - \sum_{j\varepsilon J^*} \hat{Y}(j))*R(j)]$$

      where:

      $$R(j) = (OS(j) - AI(j))/[\ \sum_{j\varepsilon J^*} (OS(j) - AI(j))];$$

      until all families triggering hit their overstock limits in which case new families are added to the trigger list.

   B. If $\sum_{j \varepsilon J^*} \hat{Y}(j) > X(I)^*$, the run quantities are decreased proportionally to their initial assignments :

      $$\hat{Y}^*(j) = X(I)^* * \hat{Y}(j)/[\ \sum_{j\varepsilon J^*} \hat{Y}(j)].$$

After these family production quantities are determined, they are then disaggregated to set item production quantities. A routine for this second level of disaggregation is described later in this chapter.

The Knapsack Approach

The Knapsack approach was formalized in 1977 by Bitran and Hax [4]. It was an attempt to build the Hax-Meal concepts into a sub-optimization model. This approach minimizes setup costs at the family level. The minimization disaggregates the total production allocated to a product type among its families imposing feasibility constraints. The algorithm is illustrated in Figure 2.8. The structure which emerges is that of a bounded convex knapsack program with continuous variables, hence the name knapsack. The approach then passes the family production levels to the item disaggregation routine.

The Item Model

The final stage in the hierarchical production planning process is the scheduling of items within a family. For this task, overstock limits and service requirements must be observed. The general approach to accomplish this task is to equalize the expected run-out times of the items in a family. The expected time until it is necessary to set up the family again is thereby maximized. This item approach parallels the EROT routine explained earlier.

FIGURE 2.8 The Knapsack Routine

Minimize secondary costs

$$\text{MIN} \sum_{j \in i} s(j)*d(j)/Q(j)$$

such that:

$$\sum_{j} Q(j) = Y(i)$$

$$lb(j) \leq Q(j) \leq ub(j)$$

where:

s(j) = the setup cost necessary for family j

d(j) = the demand for family j (annual or other specified time interval)

Q(j) = the production of family j in the immediate time period

Y(j) = the total production of type i designated in the first level of the hierarchy

lb(j) = the lower bound on production of family j

ub(j) = the upper bound on production of family j

An alternate approach has been proposed by Bitran and Hax [5]. This approach differs from EROT when overstock limits or understock limits become binding.

<u>Applicability of the Hierarchical Approach</u>

Hierarchical production planning was designed to be used in situations with the following characteristics:

1. production is done in batch sizes,
2. setup costs are secondary in nature while other costs, like inventory holding costs, overtime, and hiring and firing are primary, and
3. the planning desired is for a single-stage process. This occurs whenever one stage in a process is truly dominant because of costs or capacity constraints. The environment need not be single-stage, just single-stage dominated. The constraints considered are from this unique stage. This plan may then be used as the driving force in an MRP system.

The initial Hax-Meal approach was designed for a major corporation. Seasonal inventory needed to be planned for differing products. An example of such products are the snow-tires. The inventory was to be built up such that:

1. The company did not have to carry large inventories from one season until the next,
2. peak demands would be met with a minimal amount of overtime,

3. the costs of necessarily changing the production equipment to be used from one family to another were minimal, and

4. the chance of running out of a product demanded coincides with the corporation's service level policy.

This is a batch processing environment. The primary costs included regular and overtime and inventory holding costs. Setup costs were classified as secondary. Only the final stage of production is considered - the availability and production of materials is assumed to be of secondary importance to the model and to management, principally for cost reasons.

This type of production process is handled well by the Hax-Meal or Knapsack algorithms. This claim is supported with data in a paper by Haas, Hax and Welsch [9]. The general flow of information while disaggregating an aggregate schedule is illustrated in Figure 2.9.

FIGURE 2.9 Flow of Information at the Family and Item Levels

INPUT*

1. Service Levels
2. Time and Cost Requirements of Machine Changeovers
3. Planning Distribution for the immediate month
4. Demand forecasts
5. Policy variables
6. Total production and inventory levels***

STAGE

Family Level

Planning Horizon is one month

Review Period is one week

Item Level

Planning Horizon is one week

Review Period is one day

OUTPUT**

1. Frequency of Changeovers
2. Structure of Inventory
3. Structure of Production Schedule
4. Accuracy of Parameters used in higher levels
5. Part Orders for future periods

---

* These variables must be available for each family or item over the indicated period - one month or one week.

** These variables are specified for each family and item over the indicated planning horizon.

*** These numbers are determined in Figure 2.6.

CHAPTER 3 - THE TWO-STAGE ENVIRONMENT

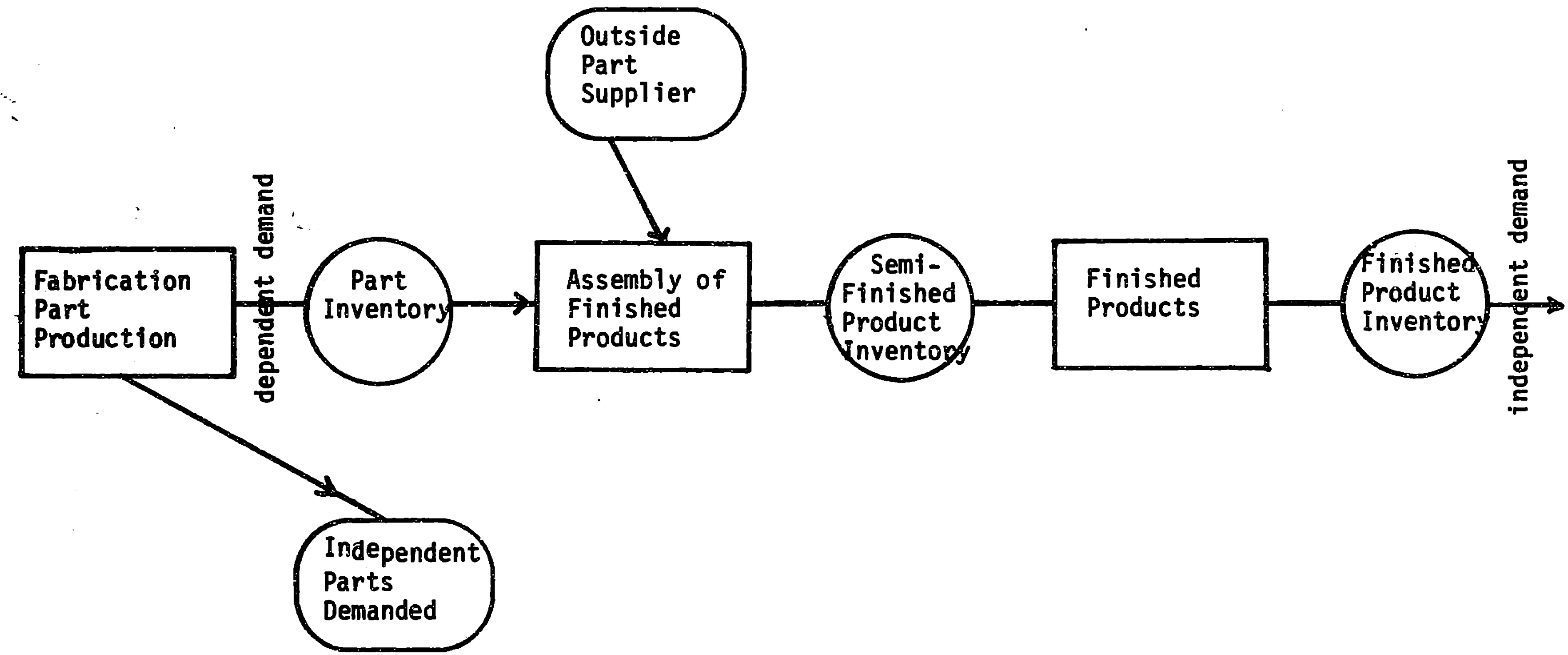
This chapter first briefly describes the two-stage production process and how its planning differs from the single-stage case. It then presents a general overview of Material Requirements Planning - what it does and how it fits into a manufacturing system.

3.1 The Two-Stage Process

In this research, a two-stage production system is defined as a manufacturing process in which component parts need to be fabricated or purchased, and then assembled into finished goods. These two phases, fabrication and assembly, occur at separate work centers. The first phase may consist of distinct components being fabricated at or purchased from separate localities. For example, prior to manufacturing a pencil, the lead, the wood, the eraser, and the fennel, the metal rim, need to be purchased or fabricated, generally all from different work centers. We will use this manufacturing environment in subsequent chapters to test the performance of the proposed systems at supporting production decisions in two-stage production process.

A typical two-stage production process is illustrated in Figure 3.1. Three distinct elements exist in the schematic representation -

1. Production operations - the fabrication and assembly processes,
2. Stocking points, where inventory of parts and finished products are accumulated, and



The arrows indicate the flow of goods

FIGURE 3.1 The Two-Stage Setting

3. Flow of material between stocking points, into the system and out of the system.

This definition of the two-stage production process is not all inclusive, there are a large number of two-stage processes which will not be treated here, such as distribution systems, and job shop scheduling and dispatching problems.

Production planning in the two-stage environment has the objectives described in the first section of the thesis, cost minimization while satisfying demand. The pursuit of that objective, in a two stage situation, naturally involves more complex considerations and constraints than the single-stage process does. The principal difference between one and two-stage production is that in the two-stage case the assembly of finished products is dependent on the fabrication and availability of components.

In this setting, it is important to distinguish between "dependent" and "independent" demand. "Independent" demand for an item comes from customers outside of the manufacturing process. In the two-stage setting depicted in Figure 3.1, the demand for finished products is "independent". "Dependent" demand of a particular item, is generated inside the planning model and is determined by internal needs of the final assembly process. For example, the demand for components necessary to assemble finished products is determined by the demand for finished products. Items may be subject to both sorts of demands.

3.2 An Overview of Material Requirements Planning, M.R.P.

M.R.P. is directed towards coordinating the manufacturing of all items required for the production of other items. The technique is designed for any number of stages in the manufacturing process. The description here is for the two-stage process and the concepts can be readily extended to any number of stages.

Most commercial codes begin with a master schedule indicating the planned production of all goods which have independent demand and require the production of other items for at least L periods. L represents the maximum lead time required for parts to be available. The development of such a schedule is generally external to the M.R.P. process.

The technique explodes the scheduled production of finished products to determine the resulting component requirements. Included in these explosions may be a normal shrinkage or spoilage factor.

Once the time-phased requirements for components are calculated, the on hand and on order part quantities are netted out to establish effective requirements. The "dependent" demands for components are calculated, and component ordering quantities are set from these effective requirements. There are many approaches for setting ordering quantities, examples are - fixed order quantity, lot for lot, fixed period requirements, part-period balancing [31], Wagner-Whitin [28], Silver-Meal [30], and Economic Order

Quantity [7]. Given the planned order quantites, the proper timing of orders is set to ensure that the scheduled assembly of finished products is feasible.

The overall flow of the M.R.P. process and how it fits with assembly planning and control is illustrated in Figure 3.2. For an M.R.P. system to work it requires the existence of an accurate product-component requirement file, on-hand and on-order component status file and a master schedule.

M.R.P. has five direct benefits -

1. Multiple component availability is coordinated through the use of detailed forecasting and planning;

2. Inventory investment is reduced because the increased information flows needed to support an MRP system facilitate more timely purchasing and fabrication decisions;

3. Management is provided with feedback which allows them to fine tune component orders.

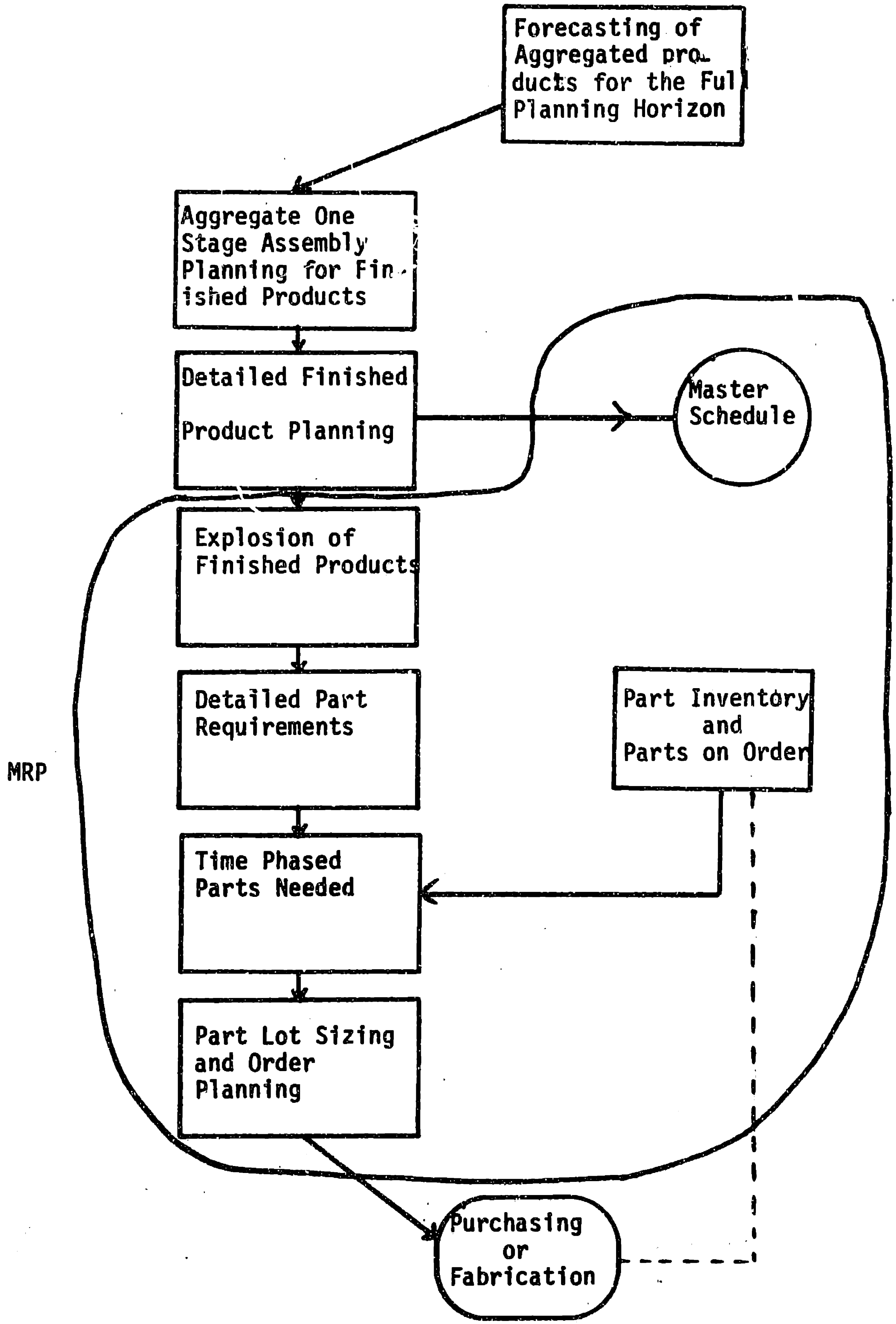
4. Due to the planning of component production, part setup costs may be reduced; and

5. Managment can do a better job of capacity planning with the aid of an M.R.P. system.

M.R.P. has two principal disadvantages and several potential drawbacks if the system is not properly constructed. The principal drawbacks are -

1. The need for information processing and accurate

FIGURE 3.2 Material Requirements Planning (MRP)



files of bill of material requirements and inventory status are expensive

and

2. The failure to consider component fabrication costs in the master schedule may lead to non-optimal planning decisions.

Examples of potential disadvantages include -

1. M.R.P. systems generally assume that part capacity is unlimited, and lead times are independent of work loads; these assumptions can result in component backorders.

2. Systems may have nervous responses to temporary changes in shrinkage factors or lead times causing unnecessarily large part inventory,

3. Unrealistic and inaccurate shrinkage factors may result in component shortages.

4. M.R.P. systems may treat indenpendent component demand inappropriately, thereby component inventory may not be properly matched.

5 The inadequate provision for uncertainty, particularly when lead times are long, may result in component shortages.

Overall, all of these potential disadvantages can be handled with numerous tricks or special techniques which have been developed, while the principal disadvantages cannot. For a more thourough discussion of M.R.P. see Orlicky [27] or

Smith [31].

The MRP model used for Comparative Purposes

As stated earlier the determination of a master schedule is a moot issue in MRP literature. However, the existance of such a schedule is a critical input to an MRP system. It was felt that the determination of such a plan would reasonably be done in a manner to minimize all primary costs associated with assembling the finished products. The level of detail required in the master schedule is such that part requirements can be specified. The master schedule used for our comparison of algorithms uses the aggregate model from a single-stage hierarchical scheme and disaggregates the resulting aggregate schedule for every period using an equalization-of-run-out-time approach.

Given the master schedule, the part requirements for an entire year will be analyzed. For each part, a Silver-Meal algorithm will be used to plan part production quantities over the year. The total requirements of men and machines will be examined to determine the feasiblity of the desired part production schedule. If the schedule is not feasible, due to part capacity, two further schedules will be developed and the one with minimal cost chosen. The two schedules will alter the original schedule in the following manner:

Schedule one

Move parts scheduled in the periods of trouble back one period if possible. If the schedule is still not feasible, delay the production of finished products requiring the infeasible parts up one period.

Schedule two

Move the production of finished products requiring the infeasible parts up one period, whenever possible. If the part schedule is still not feasible, move the production of parts where necessary and possible back one or more periods.

The master schedule will be updated and exploded every period. This is the MRP system we will be comparing with the two-stage hierarchical approach, later to be developed.

It was felt that we were giving the MRP system a fair chance in the comparison of algorithms. As the Silver-Meal approach has been rated very favorably as a heuristic for scheduling parts [28]; the adjustment procedures gives the MRP two shots at every simulation; and the manner of determining the master schedule optimizes all primary costs over the feasible region for scheduling, as will be proved in Proposition 3.1.

Proposition 3.1

If there is an optimal solution to the aggregated model used in single-stage hierarchial planning, illustrated in Figure 3.4, and that solution is disaggregated using the EROT approach for every period, that solution is optimal and feasible in the detailed LP illustrated in Figure 3.3 which may be used in MRP to determine the master schedule.

FIGURE 3.3 The Detailed Linear Program Supporting M.R.P.

P1

Objective:

1) $$\text{Min} \sum_{I}\Big[\sum_{i\in I}\sum_{t=1}^{T} h(i,t)*I(i,t)+cr(t)*R(i,t)+co(t)*O(i,t)\Big]$$

s.t.

2) $$\sum_{I}\sum_{i\in I} R(i,t)\leq r(t) \text{ for all } t$$

3) $$\sum_{I}\sum_{i\in I} O(i,t)\leq o(t) \text{ for all } t$$

4) $$R(i,t),\ O(i,t)\geq 0 \text{ for all } i,t$$

5) $$I(i,t)\geq ss(i,t)$$

6) $$I(i,t-1)+p(I)*[R(i,t)+O(i,t)]-I(i,t)=d(i,t) \quad \text{for all } i,t$$

Where:

h(i,t) = the holding cost per unit of item i in period t

I(i,t) = the inventory of item i planned for period t

cr(t) = the cost of regular time associated with the assembly of finished products in period t

R(i,t) = the regular time allotted to the production of family i in period t

co(t) = the cost of overtime associated with the assembly of finished products in period t

O(i,t) = the overtime allotted to the productions of item i in period t

ss(i,t) = the safety stock of item i required in period t

$r(t)$ = the available regular time in period t

$o(t)$ = the available overtime in period t

$d(i,t)$ = the demand for item i in period t

$p(I)$ = the productivity rate associated with type I.

FIGURE 3.4 The Aggregated Linear Program Supporting the First Stage of Hierarchical Production Planning

P2

Objective:

7) $\text{Min} \sum_{I} \sum_{t} (h(I,t)*I(i,t)+cr(t)*R(I,t) +co(t)*O(i,t)$

s.t.

8) $\sum_{I} R(I,t) \leq r(t)$ for all t

9) $\sum_{I} O(I,t) \leq o(t)$ for all t

10) $R(I,t),\ O(I,t) \geq 0$ for all I,t

11) $I(I,t) \geq ss(I,t)$

12) $p(I)*(R(I,t)+O(I,t))+I(I,t-1) -I(I,t)=ED(I,t)$

Where:

ED(I,t) represents the effective demand of product type I in period t.

Proof of 3.1

1. The objective functions, equations 1 and 7 of the two problems are identical as -

   a. every product has 3 components in the objective function, an inventory, regular time and overtime term,

   b. if two products have an identical set of components, the values of those two components can be aggregated in the objective function.

   c. by definition of product type, all families within a product type have identical components in the objective function

   d. therefore the aggregation of families into types in the objective function, does not change the solution.

2. Any solution to the detailed problem, P1 is feasible in the aggregate problem, P2 -

   a. if the constraints in the detailed problem are satisfied, the sums will be satisfied so constraints 2 to 5 summed match equations 8 to 11.

   b. by summing all of the families equalities in equation 6 in a given product type, equality would be assured in equation 12. by the definition of effective demand. The ED's in the aggregate model are effective demand, and the I's are adjusted properly.

3. This implies that if an optimal solution in the aggregate model is feasible in the detailed problem, it is also optimal.

4. All feasible solutions to the aggregate problem, disaggregated using EROT[1] are feasible in the detailed problem as -

   a. constraints 2,3 are clearly satisfied,
   b. constraints 4 are satisfied, as a positive number is broken into pieces, all the pieces remain positive.
   c. constraint 5 is true, as the definition of safety stock gurantees that:

1. This version of the EROT pays attention to safety stock.

$$\sum_{i \in I} ss(i,t) \leq ss(I,t)$$

d. To show that constraint 6 in the detailed model is true, we will add to each side of equation 12 the difference between demand and effective demand. Use of EROT gurantees constraints 6 of the detialed algorithm.

note - this would not necessarily be true, if effective demand were not used.

5. This implies an optimal solution to the aggregated model, disaggregated with the EROT routine is feasible and optimal in the detailed problem.

Q.E.D.

3.3 Overall Remarks

In general, M.R.P. can add a lot to a production system that currently is operating without any form of coordination. In the next chapter an alternative approach for two-stage production planning is developed.

CHAPTER 4 - A CONCEPTUAL PRESENTATION OF THE TWO-STAGE HIERARCHICAL METHODOLOGY

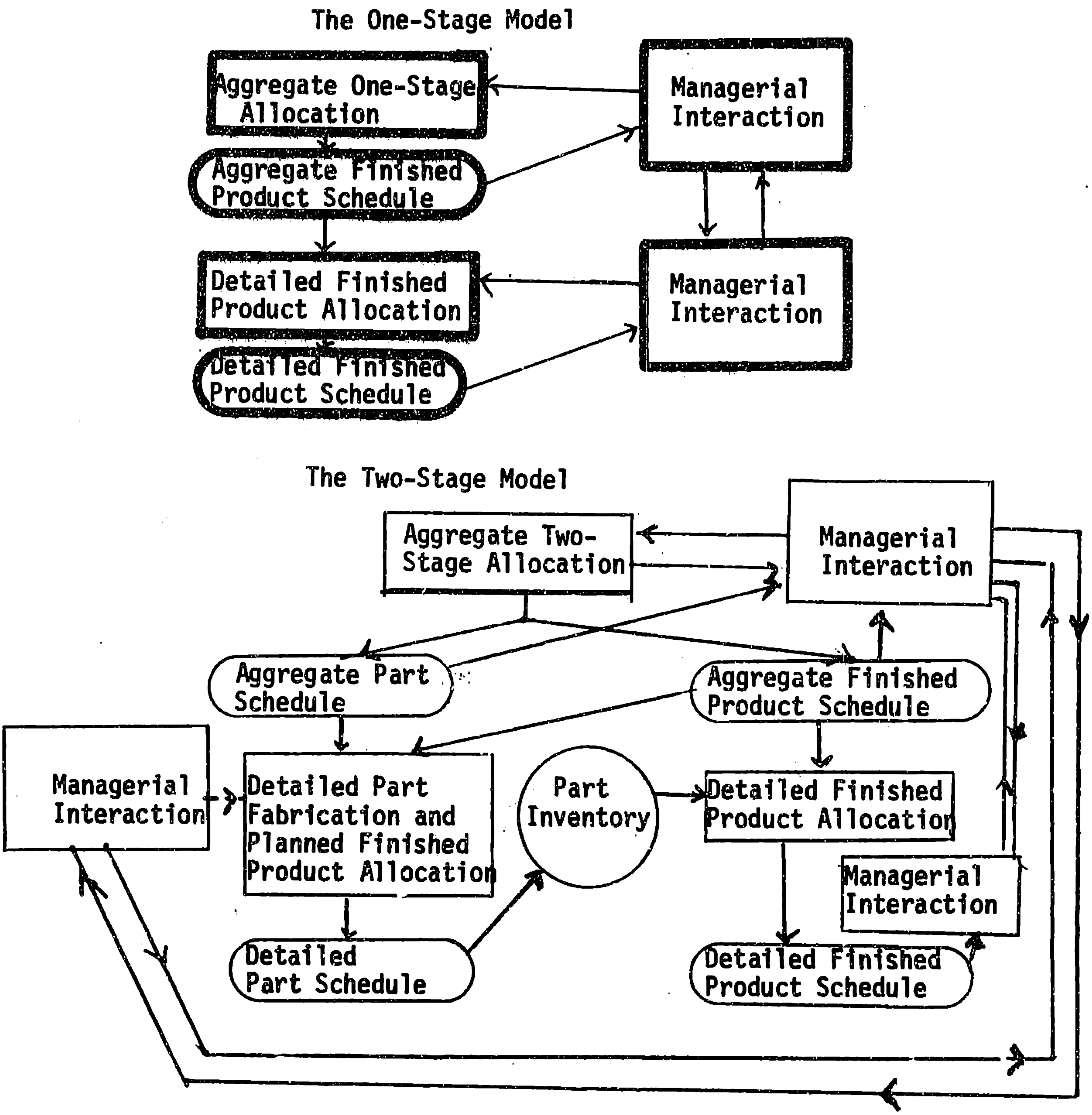
4.1 An Overview

It is the purpose of this thesis to develop and test a hierarchical planning framework suitable for a two-stage production environment. This methodology has extensive potential utilization as an approach for coordinating the production of finished products and their associated components, while minimizing all primary costs and allowing flexibility in the scheduling process. Throughout the development of this heuristic approach the ability of corporations to actually use the process will influence modeling decisions.

With this planning process, a company which fabricates some components used in assembly can appropriately account for all primary factors when designing the assembly schedule for finished products, and the associated plan for component fabrication. This list of primary factors includes whatever limitations exist on component fabrication. It was our desire to offer an alternative to M.R.P. which originally motivated the use of this hierarchical framework in a two-stage setting.

As we develop the two-stage model we will strongly rely on our knowledge of the one-stage hierarchical process. The general structure and the flow of information associated with the two-stage hierarchical approach is illustrated and contrasted with the single-stage structure in Figure 4.1. For the two-stage case, the first level in the hierarchy determines

Figure 4.1 The one and two-stage Hierarchical Flow



the production of aggregate parts and finished products period by period over the entire planning horizon. This schedule is concerned with feasibility and resource utilization, while minimizing primary costs. As in the single-stage model, tactical planning and high level managerial interaction are readily facilitated and encouraged by this portion of the model.

The second level in the hierarchical framework determines a detailed production schedule which coordinates the scheduled assembly of finished products for L periods, with the fabrication of detailed components in the immediate period. In this context, L represents the maximum lead-time associated with component fabrication or orders. Simultaneously, this schedule is feasible and in aggreement with the aggregate schedule and the tactical plans previously developed. From these disaggregated schedules assembly and fabrication or orders in the immediate period are determined. Supporting the determination of the parts to fabricate and the associated quantities is the planned production of finished products for L periods. The planned production is not frozen. However, the parts that will be available in L periods is frozen. This level, as in the single-stage case, addresses operational plans and facilitates low level managerial interaction. However, the degree of operational managerial interaction possible in the single-stage hierarchical approach may be greater than in multistage cases. This is because the production plan in the multi-stage case tends to be constrained by the interaction of the different stages involved. Therefore, principal operational deviations may best be done by managerial insights being fed back to higher levels in the process. In this manner changes in the operational schedule can be done in a coordinated fashion. It is worth pointing out that the ability for management to change the planned production in a coordinated fashion with full understanding of potential repercussions is not readily supported with an M.R.P. system.

To support this planning process, a method for aggregating finished products and parts must be defined. The rationale for this need is identical to that for the aggregation of products in the single-stage situation: by working with aggregated products and parts, the tactical planning process is built on more accurate forecasts; managerial interaction in an imprecise environment is more readily facilitated and can be done with a greater understanding of its consequences; and potential problems are more easily identified in the aggregate schedule than in a detailed schedule.

For the purpose of this study, parts were aggregated in two levels, parts and part types, and finished products were aggregated in three, items, families and types. We based the method of aggregation on an actual practical setting: the production of pencils. The number of levels and manner of aggregation are critical, as will be demonstrated by the detailed development of the model which follows. The grouping of products and parts herein described should be viewed as a possibility that worked for the particular situation considered but not necessarily correct for another product structure.

Finished product items with similar production parameters, (e.g. productivity, and seasonality) were grouped into types. Items within a product type sharing major setup costs were grouped into families.

Parts were aggregated in a somewhat different fashion. Each part required seperate setups and the grouping into families was felt to be unnecessary. Parts produced by the same capacity or purchased from the same supplier were grouped into types.

This chapter develops in detail each stage of the hierarchical process. The next section addresses the issues involved with the development of the aggregate model. The problems associated with disaggregation and suggested approaches are then presented. The last section reviews the overall hierarchical framework.

4.2 The Aggregate Model

The aggregate model is the first step in the hierarchical process. It guides management in tactical decision-making in an environment defined by strategic plans. Its overall purpose is to determine an effective feasible schedule, for aggregate finished product assembly and aggregate component fabrication over the entire planning horizon. This schedule is to minimize the primary assembly and fabrication costs while coordinating the production of finished products with the availability of associated components and satisfying the demand for finished products. This model is run as frequently as tactical planning requires. To simplify the exposition, we assume throughout this thesis that the model is run every period with updated forecasts and data. A period is defined as four weeks. This definition of period is used instead of a month so that each period is of equal length - the capacity available in every period is identical and the part lead time can be expressed in terms of periods.

The aggregate model used for two-stage planning is of the same format as that recommended in the single-stage case, a linear program. This particular method is not required, but has the advantages earlier discribed. For other possible models see [7]. Each part of the two-stage aggregate model will be built and explained in the this section.

The Objective Function

The objective function in the two-stage aggregate model mirrors the one used in the single-stage model. The two-stage objective function minimizes both part and finished product production costs, those associated with holding products and parts in inventory and the overtime used for their fabrication or assembly. The single-stage model was solely concerned with the expenses associated with finished product assembly. A comparison of the two objective functions follows:

The Single-Stage Model

(1) $$\text{Minimize } \sum_t \Big[\sum_i \{ h(i)*I(i,t) + cr*R(i,t) + co*O(i,t) \} \Big]$$

The Two-Stage Model

(2) $$\text{Minimize } \sum_t \Big[\sum_i \{ h(i)*I(i,t) + cr*R(i,t) + co*O(i,t) \} + \sum_j \{ \hat{h}(j)*\hat{I}(j,t) + \hat{cr}(j)*\hat{R}(j,t) + \hat{co}(j)*\hat{O}(j,t) \} \Big].$$

where:

t represents the time period
i represents the finished product type, and
j represents the part type;

h(i) represents the holding cost for product type i for one period,
$\hat{h}(j)$ represents the holding cost for part type j for one period,
I(i,t) represents the inventory of product type i at the end of period t,
$\hat{I}(j,t)$ represents the inventory of part type j at the end of period t
cr represents the cost of regular time used for finished product assembly,
$\hat{cr}(j)$ represents the cost of regular time used for the fabrication of part type j.

R(i,t)	represents the regular time allocated for the production of product type i in period t,
$\hat{R}(j,t)$	represents the regular time allocated for the fabrication of part type j in period t,
co	represents the cost of overtime for finished product assembly,
$\hat{co}(j)$	represents the cost of overtime for the fabrication of component j,
O(i,t)	represents the overtime allocated to the assembly of finished product type i in period t, and
$\hat{O}(j,t)$	represents the overtime allocated for the fabrication of part type j in period t.

It is worth noting that the overtime and regular time costs for finished products have all been assigned the same value, while those same parameters associated with part types have been assigned differing values depending on the specific part involved. This is due to our definition of part types. In this model, part types may originate from separate sources, and within those sources the overtime costs may differ. For example, the assembly of a pencil requires an eraser and a lead, each of these components may be manufactured at separate locations or distinct plants. One of the components may require more skilled labor or more machine time than the other and therefore be more expensive.

Other primary linear costs, such as hiring and firing, or separate finished product labor costs may be added without loss of generality of the results in either case - single or multi-stage. However, for the purposes of discussion and development in this thesis, the model is kept as simple as possible.

There are three types of constraints: those solely concerned with aggregated finished products, those requiring feasibility of the planned production of parts, and those coordinating the process of product assembly and component fabrication.

Constraints on Finished Products

In the single-stage hierarchical model, the production of finished products is constrained so that:

1. The expected demand for all products is satisfied -

(3) $$p(i)*[R(i,t)+O(i,t)] + I(i,t-1) - I(i,t) = D(i,t) \qquad \text{for all } i,t$$

where:
p(i) represents the productivity associated with finished product type i, the units of product type i produced per hour, and
D(i,t) represents the demand of finished product type i in period t;

2. The planned usage of regular and overtime in any period is not greater than that available -

(4) $$\sum_i R(i,t) \leq r(t) \qquad \sum_i O(i,t) \leq o(t) \qquad \text{for all } t$$

where:
r(t) represents the regular time available in period t, and
o(t) represents the overtime available in period t;

3. The expected inventory levels for all products remain within bounds -

(5) $$I(i,t) \geq ss(i,t) \qquad I(i,t) \leq os(i,t) \qquad \text{for all } i,t$$

where:
ss(i,t) represents the safety stock of product type i in period t, and
os(i,t) represents the overstock limit of product type i in period t; and

4. The regular and overtime allocated to any product type must be nonnegative -

$$(6) \qquad \begin{matrix} R(i,t) \geq 0 \\ O(i,t) \geq 0 \end{matrix} \qquad \text{for all } i,t$$

The consideration of component fabrication, while introducing new limits on feasibility does not affect any of these sets of constraints, and simultaneously does not generate any new constraints which only affect finished product assembly. All of the constraints on fabrication or purchase of parts do not limit finished products without explicit reference to the availability of components. The only coupling constraints are mass balance equations. The constraints on finished products in the two-stage model are identical to those just presented for the sngle-stage model.

Constraints on Components

The constraints which only address the fabrication of parts parallel those solely dealing with finished product production. The following equations set bounds on inventory and the use of regular and overtime for parts and in part fabrication -

$$(7) \qquad \begin{matrix} \hat{I}(j,t) \geq \hat{ss}(j,t) \\ \hat{I}(j,t) \leq \hat{os}(j,t) \end{matrix} \qquad \text{for all } j,t$$

$$(8) \qquad \begin{matrix} \hat{R}(j,t) \geq 0 \\ \hat{O}(j,t) \geq 0 \end{matrix} \qquad \text{for all } j,t$$

and

$$(9) \qquad \sum_j \hat{R}(j,t) \leq \hat{r}(t) \qquad \sum_j \hat{O}(j,t) \leq \hat{o}(t) \qquad \text{for all } t$$

However, all parts may not originate at the same source, as indicated in our discussion of the objective function. In these situations, equation 9 gets broken down by source. This is exemplified for the case of two sources in equations 9A -

$$(9A) \qquad \sum_{j \varepsilon J1} \hat{R}(j,t) \leq \hat{r}(J1,t), \quad \sum_{j \varepsilon J2} \hat{R}(j,t) \leq \hat{r}(J2,t)$$

$$\sum_{j \varepsilon J1} \hat{O}(j,t) \leq \hat{o}(J1,t), \quad \sum_{j \varepsilon J2} \hat{O}(j,t) \leq \hat{o}(J2,t)$$

where:

- ss(j,t) represents the safety stock of part type j required in period t,
- os(j,t) represents the overstock limit of part type j in period t,
- r(t) represents the total available regular time for part fabrication in period t, when all parts originate at the same source,
- o(t) represents the total available overtime for part fabrication in period t, when all parts originate at the same source.
- r(J1,t) represents the total available regular time for parts originating at source 1 in period t,
- r(J2,t) represents the total available regular time for parts originating at source 2 in period t,
- o(J1,t) represent the total available overtime for parts originating at source 1 in period t, and
- o(J2,t) represents the total availabe overtime for parts originating at source 2 in period t.

The single-stage model does not consider part or component availability, and therefore has no constraints dealing with part fabrication.

It has been assumed implicitly that part and finished product production are separate. This assumption is not necessary, but built in for simplicity, and it resembles the real cases we have been exposed to. If the assembly and fabrication were done with the same labor force, distinct constraints relating to available regular and overtime would collapse into one.

The Interacting Constraints

There is a single type of limitation connecting the fabrication of parts with the assembly of finished products. This sort of constraint is the part production - part demand balance equation. The demand for parts in any given period is not computed from outside parameters, like the demand for finished products, but rather is determined by the model itself and the planned production of finished products. Component demand is "dependent". Simultaneously component availability limits the production of finished products. The model determines component availability for all periods, considering initial part inventory and orders outstanding. This interaction between factors, part fabrication and finished product assembly, is reflected in equations of the form -

(10) $\hat{p}(j)[\hat{R}(j,t_a) + \hat{O}(j,t_a)] + \hat{I}(j,t_b-1) - \hat{I}(j,t_b)$

$$= \sum_i f(i,j)*p(i)*[R(i,t_b) + O(i,t_b)] \quad \text{for every } j,\ t_b$$

where:
$\hat{p}(j)$ represents the production factor associated with part type j,
f(i,j) represents the number of part type j required per unit of finished product type i,
t_a and t_b are time indices, where the difference represents the lead time associated with part j.

Given a lead time of L periods, the equation for the first L-1 periods, strictly constrains the production of finished products based on the parts fabricated or ordered previously. In this context t_a may be negative, and $\hat{R}(j,t_a)$ may be historical data which cannot be influenced by the model. However, if t_b is L or larger $\hat{R}(j,t_a)$ and $\hat{O}(j,t_a)$ are not historical data, but decision variables of the model. This equation equates part production and the change in part inventory with parts necessary for the finished products scheduled to be assembled in L periods. These constraints resemble the finished product demand balance equation (equation 3). That equation equates the assembly of finished products and the change in the product inventory with the expected demand for those products.

The entire two-stage aggregate model is illustrated in Figure 4.2.

Figure 4.2 The Aggregate Model

The objective funtion:

$$\text{Minimize } \sum_t \Big[\sum_i \{ h(i)*I(i,t) + cr*R(i,t) + co*O(i,t) \} + \sum_j \{ \hat{h}(j)*\hat{I}(j,t) + \hat{cr}(j)*\hat{R}(j,t) + \hat{co}(j)*\hat{O}(j,t) \} \Big]$$

subject to:

$$p(i)*[R(i,t) + O(i,t)] + I(i,t-1) - I(i,t) = D(i,t) \quad \text{for all } i,t$$

$$\sum_i R(i,t) \le r(t), \quad \sum_i O(i,t) \le o(t) \quad \text{for all } t$$

$$ss(i,t) \le I(i,t) \le os(i,t) \quad \text{for all } i,t$$

$$R(i,t),\ O(i,t) \ge 0 \quad \text{for all } i,t$$

$$\sum_j \hat{R}(j,t) \le \hat{r}(t) \qquad \hat{O}(i,t) \le \hat{o}(t) \quad \text{for all } j,t$$

$$\hat{ss}(j,t) \le \hat{I}(j,t) \le \hat{os}(j,t) \quad \text{for all } j,t$$

$$\hat{R}(j,t),\ \hat{O}(j,t) \ge 0 \quad \text{for all } j,t$$

and

$$\hat{p}(j)*[\hat{R}(j,t_a) + \hat{O}(j,t_a)] + \hat{I}(j,t_b-1) - \hat{I}(j,t_b)$$
$$= \sum_i f(i,j)*p(i)*[R(i,t_b) + O(i,t_b)] \quad \text{for all } j,t_b$$

Special Needs of the Aggregate Model

There are two special requirements of the interacting constraints (equation 10). These are the only unusual needs of the aggregate two-stage model. All finished products within a given product type may not have identical part type requirements. In which case the finished product-part factor used in the part demand - production balance equation may be difficult to define.

The definition of effective part demand is the second difficulty of this constraint. Effective demand for finished product types was defined by Golovin[13], to ensure feasibility of tactical plans. He defined effective product type demand as :

$$(11) \quad ED(i,t) = \text{maximum}\left[0, \sum_{r=1}^{t} D(i,r) - AI(i,1)\right]$$

$$ED(I,t) = \sum_{i \in I} ED(i,t)$$

where:

- $ED(i,t)$ is the effective demand of product family i in period t
- $ED(I,t)$ is the effective demand of product type I in period t
- $i \in I$ represents all items in product type I
- $D(i,t)$ represents the demand for item i in period t, and
- $AI(i,1)$ represents the available inventory of item i in period 1, based on the initial inventory less safety stock.

The aggregate model then constrains the production of every product type to be not smaller than that product type's effective demand. In all previous equations, related to the aggregate model, whenever demand was used it was referring to effective product demand.

Unfortunately, the demand for specific parts within a part type is not known until after the aggregate model has been solved, and in most cases disaggregated. For this two-stage aggregate model to be operable, the planned usage of wrong parts and inappropriate schedules must be avoided. Each of these needs will be addressed and solution procedures suggested.

The Effective Part Demand

Effective part demand ideally is defined in a manner similar to that used in the definition of effective product demand -

$$(12) \qquad \hat{ED}(J,t) = \sum_{j \varepsilon J} \text{maximum } [0, \hat{D}(j,t) - \hat{AI}(j,t)]$$

where:

$\hat{ED}(j,t)$ represents effective part demand in period t,

$\hat{D}(j,t)$ represents the demand for part j in period t,

j represents all items in part type J

$\hat{AI}(j,t)$ represents the available inventory of part j in period t, based solely on the initial inventory and those parts already on order.

This equation is equivalent to -

$$(13) \qquad \hat{ED}(J,t) = \sum_{j \varepsilon J} \text{maximum } [0, \sum_{I} \hat{f}(I,j)*[R(I,t+L) + O(I,t+L)] - \hat{AI}(j,t+L)]$$

where:

$\hat{f}(I,j)$ represents the requirements of product type I for part j,

since the demand for parts is a function of the planned assembly of finished products. However, this cannot be determined without knowing the schedule for finished product production.

Instead of attempting to define effective demand for parts, we define effective part inventory, $\hat{EI}(j,t)$. Effective part inventory attempts to measure the maximum combination of parts in inventory for which planned part type utilization can be assured. Effective Demand for a part type could then be defined as:

$$(14) \qquad \hat{ED}(J,t) = \text{Maximum}\ [0,\ \hat{D}(J,t) - \hat{EI}(J,t)]$$

where:

$\hat{D}(J,t)$ represents the sum of the demand of all parts in part type J in period t.

Note: This equation differs from equation 13 in that here the terms are aggregated and refer to part types rather than specific parts.

The measure of effective inventory is particularly important for the first L-1 periods, when finished product assembly is constrained by available parts (available parts are reflected in the effective part inventory). The interacting constraint for period one is equivalent to the following -

$$(15) \qquad \sum_i f(i,j)*p(i)*[R(i,1) + O(i,1)] \leq \hat{EI}(j,1)$$

This equivalence assumes that the lead time for part type j is one or more periods. It is felt that effective part inventory must be used in this constraint rather than actual part type inventory to assure feasibility in the aggregate schedule. If actual inventory were used, the production of a finished product requiring part B may be scheduled, while only part A is available. Basically, the smaller the effective inventory measure used, the better is the guarantee that the aggregate schedule can be feasibly disaggregated. Simultaneously, the smaller the effective inventory is in any period, the more constrained is the aggregate schedule. These two pulls in the direction that effective inventory is set at, the one for minimal costs in the aggregate schedule, and the other for guaranteeing feasibility in the detailed schedule, are both satisfied at the actual level. However, the true value is unknown prior to the determination of an aggregate schedule.

The Measurement of Effective Inventory

As a measurement of effective part inventory in any period, we attempt to provide an estimate. This estimate is obtained by summing over all parts in a part type the minimum of that which we expect to be demanded in the indicated period and the actual availability of that part.

$$\hat{E}I(J,t) = \sum_{j \in J} \text{minimum}\,(\hat{E}U(j,t),\ \hat{A}I(j,t))$$

where:

$\hat{E}I(J,t)$ reflects the effective inventory of part type J in period t,

$\hat{E}U(j,t)$ reflects the expected utilization of part j in period t, where part j is in part type J, and

$\hat{A}I(j,t)$ reflects the available inventory of part j in period t less the safety stock associated with part j.

The use of expected demand or utilization tries to prohibit the planned utilization of part B when only part A is available at the aggregate level.

In the first period, if all parts in a part type have inventory levels greater than that which we expect to be demanded, the effective inventory for period one can be increased to add to the model's flexibility, while guaranteeing a feasible disaggregation scheme. This quantity that the effective inventory can be increased by will be definined in analytic terms later in this chapter when the estimation procedure is described. The manner that this increase in expected inventory builds flexibility into the model can be witnessed by examining equation 15. The feasible region for finished product assembly in the immediate period is increased. The rationale for only including this aspect of effective inventory in the first period will be explained when the details of the calculation are presented.

We begin describing the details of calculations involved with the estimation of effective part inventory, by explaining the process for the immediate period. Effective part inventory for any period is broken into two components - one component which reflects the amount that is expected to be utilized in that period, and one component aimed at increasing the aggregate model's flexibility if the parts for such an increase are available. Each of these components and the associated estimation process will be discussed.

Estimation of Expected Part Utilization

The part inventory that is expected to be used in the immediate period is estimated using previous aggregate schedules and the latest demand forecasts. Aggregate schedules are generated every period in the hierarchical framework. We assume that these aggregate schedules, once updated for changes in forecasts, do not change substantially from period to period. This assumption was tested and the results of the experiments are presented in the first section of the next chapter. Our methodology to determine expected part utilization proceeds as follows -

1. Begin with the previous period's aggregately planned production of finished products for the immediate period, X(I,Current Period); (where X(I,Current Period) represents the planned aggregate production of product type I for the current period.)

2. Determine how current inventory is different from what the previous aggregate schedule had planned inventory to be - or the aggregate forecast errors for the past period, ΔI(I,Current Period). (where ΔI(I, Current Period) represents the planned aggregate inventory less the actual aggregate inventory.)

3. Determine how aggregate forecasts for the current period have changed, ΔF(I,Current Period). (where ΔF(I,Current Period) represents the previous period's forecasts for the current period less the current period's forecasts.)

4. Determine an adjusted aggregate level of production for each product type,

(16) $X^*(I,\text{Current Period})$

$$= X(I,\text{Current Period}) + \Delta I(I,\text{Current Period}) + \Delta F(I,\text{Current Period})$$

(where $X^*(I,\text{Current Period})$ represents the adjusted planned aggregate production of product type I in the current period.).

5. The feasibility of the aggregate schedule of all products is checked against finished product capacity. If it is infeasible, all the product types with planned ending inventory are scaled down proportionately until the schedule is feasible, or no ending inventory remains. If the schedule remains infeasible, all product types scheduled for production are scaled down in proportion to their scheduled quantities until the plan is feasible. X^* will represent this feasible production plan.

6. The aggregate levels $X^*(I,\text{Current Period})$ are then disaggregated, via an algorithm described later in this chapter, and part requirements are determined by exploding the detailed production plan.

7. If the part requirements of all parts within a part type are less than the part availability, the aggregate part type inventory representing that planned for utilization is calculated as -

(17) $$UI(J,\text{Current Period}) = \sum_I f(I,J)*X^*(I,\text{Current Period})$$

for all J.

(where UI(J,Current Period) represents the expected utilization of part type J in the current period.) However, if the disaggregation algorithm finds that part availability does not permit $X^*(I,\text{Current Period})$ to be produced, but only $X^{**}(I,\text{Current Period})$, the planned inventory available for utilization is set at -

(18) $$UI(J,\text{Current Period}) = \sum_I f(I,J)*X^{**}(I,\text{Current Period})$$

Basically, this routine attempts to pre-guess the aggregate production levels which the tactical plans will decide upon and then determines if the parts are available for those levels.

If parts are available, the expected part utilization is determined from disaggregating the aggregate schedule. If the parts are not available, adjustments are made and the expected part utilization is determined based on the adjusted aggregate schedule.

Estimation of Flexibility Providing Part Inventory

This component of effective part inventory attempts to measure the quantitiy of any product that could be produced from the part inventory remaining after those parts expected to be utilized are netted out. In the estimation procedure described for this term, it is assumed that all items in a family of finished products have identical part requirements. This term is calculated in the following manner -

1. The parts expected to be used, and allocated in the first component, are subtracted from each part's actual inventory to determine expected excess part inventory:

$$\widehat{EPI}(j,\text{Current Period}) = \widehat{API}(j,\text{Current Period}) - \widehat{EUI}(j,\text{Current Period}) \quad (19)$$

for all j

where:

$\widehat{EPI}$(j,Current Period) represents the expected excess part inventory for part j,

$\widehat{API}$(j,Current Period) represents the actual inventory of part j in the Current Period, and

$\widehat{EUI}$(j,Current Period) represents the expected utilization of part j as calculated from the expected detailed production schedule.

It is worth noting that in this calculation, j refers to specific parts and not part types.

2. A dummy finished product is designed which requires rd(j) units of part j. Where -

(20) $$rd(j) = \underset{I,\ i\varepsilon I}{\text{Maximum}} [f^*(I,i,j)]$$

where $f^*(I,i,j)$ represents the requirements of family i in product type I for part j.

In this manner rd(j) represents the maximum requirement of any finished product for part j.

3. For each part type, the number of units of the dummy finished product which can be produced from the expected excess inventory is calculated as -

(21) $$N(J) = \underset{j\varepsilon J}{\text{Minimum}} [\ \hat{E}PI(j,\text{Current Period})/rd(j)\ ]$$

This number will be zero if the finished product disaggregation was constrained by any part in the given part type, as that part's EPI will be zero.

4. The flexibility providing part inventory for each part type is then calculated as -

(22) $$\hat{F}PPI(I,\text{Current Period}) = N(J) * \underset{I,\ i\varepsilon I}{\text{Minimum}} [f^*(I,i,j)]$$

where:

$\hat{F}PPI(I,\text{Current Period})$ represents the flexibility providing part inventory.

The rationale for using the minimum in this context is based on the guarantee that N(J) units of any product can be produced. If a number larger than the minimal factor is used, then the model will expect that more than N(J) units of the product with the minimal factor can be produced.

This flexibility providing part inventory is necessarily zero for all periods beyond the current one. This is because part inventory that has been allocated is netted out prior to computing future period's effective inventory. It is impossible for all parts in a part type to remain in inventory given the definition of flexibility providing part inventory. If parts are on order and will arrive in future periods, such a term may be necessary. The aggregate model allows the part inventory available in the current period and not used to be used in future periods, so that the aggregate model can distribute this flexibility providing inventory for use throughout the planning horizon as it chooses.

<u>The Final Calculations of Effective Part Inventory</u>

We have defined the two components used in the calculation of effective part inventory for the current period. These two components are summed -

(23) $\hat{EI}$(J,Current Period) = $\hat{UI}$(J,Current Period) + $\hat{FPPI}$(J,Current Period)

where:

$\hat{EI}$(J,Current Period) represents the effective part inventory in the current period.

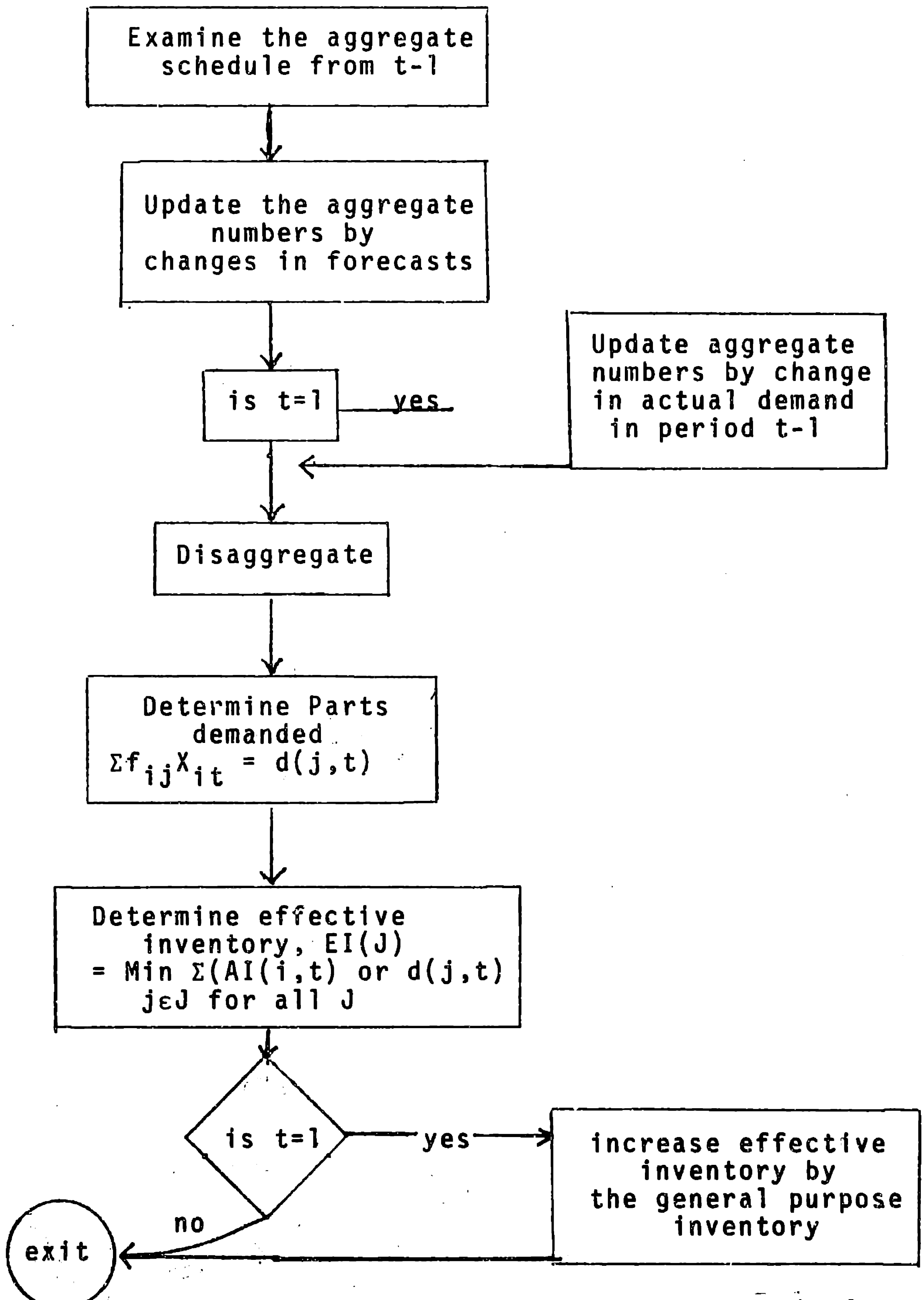
For all periods beyond the current period, effective inventory is calculated by simply estimating the expected part utilization since the flexibility providing part inventory is zero. As each period's effective inventory is calculated, those parts associated with the calculation are netted out of the available inventory. The estimation of expected part utilization, $\hat{UI}$(J,t) for all periods beyond the current one is done in the manner just described for the current period skipping the second step. The overall flow of the calcuation for each period, including the current period is illustrated in Figure 4.3.

The cumulative finished product production for the first L-1 periods, where L is the maximum part lead time, is constrained by the available parts. Simultaneously, the effective demand for parts in the immediate period is dependent upon planned finished product production in L periods and the estimated effective part inventory in L period.

It is necessary to determine effective part inventory for at least L periods in order to be able to estimate effective part demand for the immediate period. However, as indicated, to calculate effective part inventory for k periods requires disaggregating an aggregate schedule for the same number of periods. When disaggregating some forecast accuracy is lost. Given the imprecise nature of the estimations involved, it is the author's opinion that in general effective inventory should be estimated only for L periods.

It should be brought to the reader's attention that there is an advantage and a disadvantage in including this flexibility providing part inventory in the calculation of effective inventory. The advantage of increasing the flexibility in the aggregate model has been discussed. The disadvantage is that the existance of a positive flexibility providing part inventory in the first period creates problems in the calculation of the second period's effective inventory. This problem will be illustrated with the following example -

Figure 4.3 The Calculation of Effective Part Inventory



Where: J is a part type and j is a part in J.

Assume that every finished product that a company assembles requires one unit of one of two parts in a part type. That part type has the following initial inventory and expected utilization of each of its two parts in the current period -

100	50	initial inventory
30	30	expected utilization
70	20	remaining inventory.

It is clear that 20 units of any product, beyond those expected to be produced can be manufactured. The disadvantage of flexibility providing part inventory is the problem of netting it out. Since the planner does not know which part the twenty units will require, the manner in which they should be netted out is not clear.

The use of flexibility providing part inventory should therefore be viewed with caution and there are situations in which it may not be advisable to use it at all. For example, when part capacity is very tight and the aggregate schedule is very robust, the added flexibility may not be as powerful as the more exact estimate of effective part demand.

Overall, our method of estimating effective part inventory will not be 100% accurate. If small miscalculations are expensive, some adjustment procedure or feedback should be available. In the next chapter we will test how miscalculations in effective part inventory affect manufacturing costs.

Critical errors in calculations are revealed when the aggregate schedule is disaggregated. If the true level of effective part inventory is binding in a given period and that level has been overestimated, it will show up as an infeasibility in the disaggregation process. If the part inventory used in the aggregate schedule for a particular part type had been binding, and after disaggregating, all parts in that part type have a positive inventory the effective part inventory had been underestimated. If the effective inventory had been misestimated, and was not and should not be binding, little adjustment is necessary. This information can be fed back to the aggregate level and reflected in an adjusted aggregate schedule in the following manner -

1. Determine if the part availability was binding in the aggregate schedule. Excluding dual degeneracy, this can be done by examining the shadow prices associated with the interactive constraints for the immediate period. If any shadow prices are nonzero, the associated part availability was binding, otherwise no adjustments are necessary.

2. Given that the constraint was binding, determine if the effective inventory was estimated correctly. If disaggregation is impossible, the estimate was high, and if each part in the part type has a positive ending inventory, the estimate was low. If the estimate was accurate no adjustment is necessary.

3. If the estimate was high, disaggregation was infeasible, adjust the effective inventory by the number of units of finished products that the disaggregation process was short times their associated product-part factor -

$$\hat{EPI}(J,t) = \hat{EPI}(J,t) - \sum_{I} f(I,J) * \Delta S(I,t) \quad (24)$$

where:

$\Delta S(I,t)$ represents the number of units of product type I, the number of units of product type I the aggregate schedule planned less the number of units the operational schedule planned.

4. If the estimate was low, inventory remains of each part in the part type, the quantity it was low by is calculated by increasing the FPPI(J,t). The number of dummy products that could be produced from the remaining inventory of part type J is calculated, M(J) and then -

(25) $\hat{I}EI(J,t) = M(J)* \text{Minimum } f^{*}(I,i,j)$

where this minimum is calculated over all families i in all finished product types I and for all parts j in part type J, and
$\hat{I}EI(J,t)$ represents the increase in effective inventory.

Effective inventory is then set as follows -

(26) $\hat{E}PI(J,t) = \hat{E}PI(J,t) + \hat{I}EI(J,t)$

5. The new aggregate solution can be calculated based on the old solution and the new information as -

(27) $X^{new} = B^{-1}*(b + \Delta b)$

where:
X represents the aggregate solution,
B represents the basis for the existing aggregate solution,
b represents the original right hand side of the linear programming problem used to solve the aggregate model, including the constraints on part availability, and
Δb represents a vector of changes in the right hand side - or changes in effective inventory levels.

6. With this new aggregate solution, the disaggregation process may begun a second time. There are two considerations which may prevent this simple proceedure from solving at the exact effective inventory:

   a. The correction for low estimates is not precise, as the dummy product was used rather than the exact finished products that the disaggregation scheme would have chosen. If the process is repeated continually - eventually full adjustment will occur or the constraint will no longer be binding; and

   b. It may be impossible for the aggregate solution to change by the full adjustment in the indicated manner, if one of the basic variables becomes negative. This binds the degree of feasible

> adjustments. In this case, the aggregate solution could be resolved using the dual simplex method, or the modeler could stop adjusting when this occurs, settling for a sub-optimal solution. The dual in this case provides the modeler with a bound, so that the maximum degree of sub-optimality can be evaluated.

A limited part of this adjustment process was tested, the testing was done on a single adjustment to each effective inventory and was only done as long as the basic variables remained basic.

The Product-Part Requirements

The second special need of the two-stage hierarchical methodology deals with the definition of f(i,j), where this term represents the requirement of finished product type i for part type j. This term may not be constant, as the amount of part type j required per unit of product type i may depend on which familes within the product type are produced. For example, not all pencils in a product type require an eraser. The requirement for erasers of that product type will be dependent on the relative quantities of the varying families being produced in any period. The method for aggregating finished products and parts into types needs to consider how f(i,j) will be defined, and the hierarchical model needs to be flexible enough to adapt to a changing product-part factor.

If the part requirements for every family in every type are identical, no adjustment will be needed. In the case of the pencils, each pencil requires one wood piece. There is no problem in setting the factors associated with this part. In one sense, this problem is not distinct from the previous problem of defining effective inventory. When the interactive constraint associated with the immediate period is examined -

$$(29) \quad \sum_{I} f(i,j) * X(I,1) \leq \hat{E}PI(J,1)$$

it appears as though, if all factors are low and effective inventory is low, the actual constraint is not altered, but merely scaled. Therefore, effective inventory could be adjusted to account for misestimating f. However, if the errors in the part-product factors are not all proportional, the relative quantities of differing product types scheduled for production may be distorted and nonoptimal, where optimal is defined as the true solution to the aggregate production plan using the real factors.

Estimation of the Part-Product Factors

We will use the following procedure to estimate the part-product factors -

1. Determine the annual demand forecasts for every family of finished goods. (note - it is assummed that every item in a family of finished goods has identical part requirements)

2. For every product type and every part type, define -

$$(30) \quad f(I,J) = [\sum_{i \in I} A(i) * f^{**}(i,I,J)]/[\sum_{i \in I} A(i)]$$

where:

f(I,J) represents the part-product factor associated with product type I and part type J,

i represents all families of finished products in product type I,

A(i) represents the annual forecasted demand of family i, and

$f^{**}(i,I,J)$ represents the requirements of family i in product type I for part type J.

As is the case with effective part inventory, this number is only an estimate. The estimate based on relative annual demands of finished products may be in error, if the disaggregation scheme is not designed to produce all product families in proportion to their annual demand in every period, or if forecast error exists.

Adjusting for Misestimated Part-Product Factors

One option is to feed back to the aggregate level, after disaggregating, the actual factors for the first L periods, and resolve the aggregate schedule using the dual simplex method. This approach was not tested.

Alternatively, a heuristic adjustment routine might be used when disaggregating parts for the current period and finished products for L periods ahead. Such a routine is described here and tested in the next chapter.

The following fact will be used throughout the development of the heuristic adjustment procedure, and is proved in appendix 1 -

> Given, zero forecast error, zero initial inventory of finished products and parts, zero lead time, identical seasonality of all families within a product type, no aggregately planned backorders, and no ending inventory at the end of a year, at the end of 13 periods for each product type I -

$$\sum_t \sum_{i \in I} f^{**}(i,I,J)*Y(i,t) = \sum_t f(I,J)*X(I,t) \qquad (31)$$

where:

Y(i,t) can represent either - the demand for family i in period t, or the production of family i in period t, and

X(I,t) represents the production of part type I tactically planned for period t.

The fact that at any point in time the product-part factor reflects relative annual demands of products in a product type implies that if in a given period the actual factor, $\hat{f}(I,J)$, is less than the average number used in the model, at some later time the difference will be made up. One procedure for dealing with this low factor is to produce the average number of parts. The result will be greater part holding costs than considered in the aggregate model. This situation corresponds to the case in which one family of a finished product with part requirements smaller than the average is built up in inventory to a greater extent than other families.

The heuristic procedure that is described recognizes that forecast errors exist, but assumes that at any point in time the expected forecast error is zero. Our procedure for dealing with high estimates attempts to heuristically delay the extra part production as long as possible. A routine for doing this is illustrated in Figure 4.4. and is tested in the next chapter.

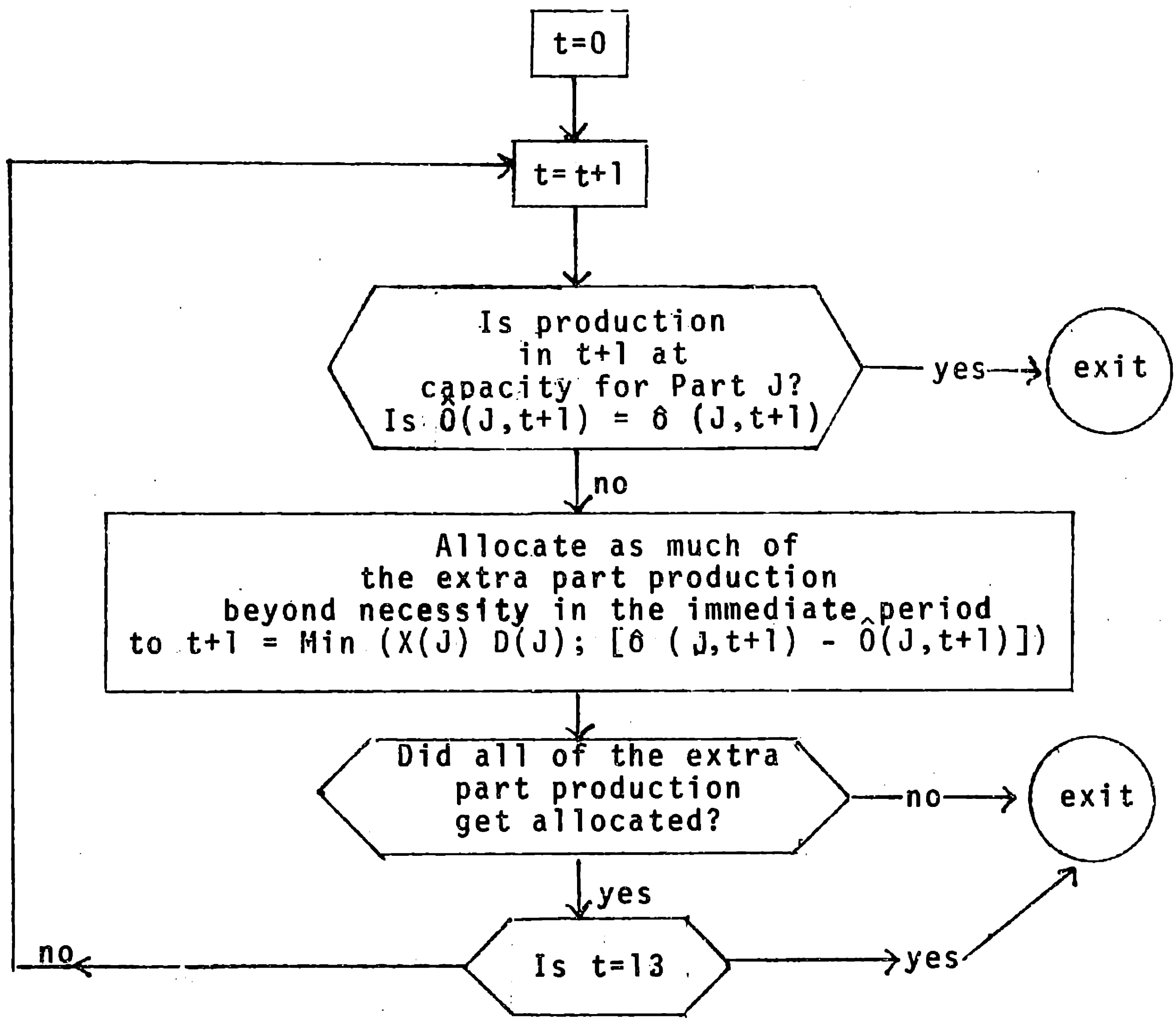
If the actual factor is greater than the average, the f used in the aggregate model, a number of options are available. It may be that in a previous period the actual was lower than the average and that overall the cumulative part requirements have averaged out, or inventory of a family of finished products, which need more parts than the average, are being built up. The difference in part requirements and average part production is not due to parts necessary for finished products to satisfy demand, but for inventory buildup. One option earlier mentioned is to resolve the aggregate schedule. The heuristic procedure we have developed and tested in the next chapter is the following -

1. If previously parts have been built up, obtain the necessary difference from inventory, and
2. If the parts are not available, constrain the finished product disaggregation algorithm to feasible levels.

The adjustments necessary with this procedure will tend to be less than expected, since while the factor for one product is low another may be high, and the buildup of parts does not need to be held in inventory. This trading of parts between product types must be done with care. At any point in time, given the assumptions stated prior to equation 31, there will be a cumulative feasible disaggregation schedule which does not require any backorders of finished products. This is proved in appendix 3. However, if the disaggregation scheme is not careful, backorders may result from using parts intended for later use in the production of one product type for assembly of another product type in an earlier period.

Figure 4.4 Adjusting the f's

The part need is less than the average - can we produce less than the average?



4.3 The Two-Stage Disaggregation Process

Once the total aggregate production of product and part types is determined, as explained in the previous section, the model must address the need to break these total quantities up between families and parts for the immediate period. The single-stage hierarchical methodology recognizes that at the operational level, when disaggregating, each product type can be handled separately. A viable objective when disaggregating suggested by Hax and Meal [18], is the minimization of setup costs which were not considered in the aggregate model. A process by which this is accomplished is the knapsack routine [4]. Unfortunately, disaggregating in the two-stage case is more complicated.

Proposition 4.1
Although the aggregate model recognizes that for a production schedule of finished products to be feasible the necessary components for their assembly must be available, each product and part type can be disaggregated separately in a manner which minimizes setup costs and is guaranteed to be feasible, if the following fact is true:

- All products in each product type have an identical profile of part requirements.

(32) all $f^*(I,i,j)$ are identical for all i in a given I, $\forall$ I,j

The proof of this proposition will begin by stating the full scale mathematical program involved with such a disaggregation -

The Mathematical Program -

(33) $$\text{Minimize} \sum_{t=1}^{L} \sum_{I} \sum_{i} [s(i)*d(i)/Q(i,t)] + \sum_{J}\sum_{j}[\hat{s}(j)*\hat{d}(j)/\hat{Q}(j,1)]$$

subject to the following constraints -

(34) $$\sum_{I} \sum_{i} f^*(I,i,j)*Q(i,t) \leq \hat{A}I(j,t) + \hat{O}I(j,t) \quad \text{for all } j, \text{ for all } t < L$$

(35) $$\sum_{I} \sum_{i} f^*(I,i,j)*Q(i,L) \leq \hat{A}I(j,L) + \hat{Q}(j,1) \quad \text{for all } j$$

(36) $$\sum_{i \in I} Q(i,t) = X(I,t) \quad \text{for all } I,t$$

(37) $$\sum_{j \in J} \hat{Q}(j,1) = \hat{X}(J,1) \quad \text{for all } J$$

(38) $$lb(i,t) \leq Q(i,t) \leq ub(i,t) \quad \text{for all } i,t$$

(39) $$\hat{l}b(j,1) \leq \hat{Q}(j,1) \leq \hat{u}b(j,1) \quad \text{for all } j$$

---

where:

$\hat{O}I(j,t)$ represents the quantity of part j on order or being fabricated which will become available to be used in assembly operations in period t.

Proof:

1. since all $f^*(I,i,j)$ are equivalent over all $i \epsilon I$, equation 34 is equivalent to -

$$\sum_I f(I,j) * \sum_{i \epsilon I} Q(i,t) \leq \hat{A}I(j,t) + \hat{O}I(j,t) \tag{40}$$

for all j and all $t < L$

2. by use of the constraint requiring the sum of family production to equal the scheduled type production, equation 36, this is equivalent to -

$$\sum_I f(I,j) * X(I,t) \leq \hat{A}I(j,t) + \hat{O}I(j,t) \tag{41}$$

3. Since all terms on the left hand side of this equation are known prior to disaggregating, the available inventory of part j in period t+1 can be calculated as

$$\hat{A}I(j,t+1) = \hat{A}I(j,t) + \hat{O}I(j,t) - \sum_I f(I,j)*X(I,t) \tag{42}$$

4. We are assuming that the aggregate schedule is feasible, this implies that equation 41 is satisfied by the aggregate schedule for all j,t.

5. The entire program can now be restated as -

$$\text{Minimize} \sum_{t=1}^{L} \sum_I \sum_i [s(i)*d(i)/Q(i,t)] + \sum_J \sum_j [\hat{s}(j)*\hat{d}(j)/\hat{Q}(j,1)]$$

subect to the following constraints -

$$\sum_I f(I,j)*\sum_i Q(i,t) \leq \hat{A}I(j,t) + \hat{O}I(j,t) \text{ for all } j,t < L$$

$$\sum_I f(I,j)*\cancel{\sum_i Q(i,L)}\ X(I,t) \leq \hat{A}I(j,L) + \hat{Q}(j,1) \quad \text{for all } j$$

$$\sum_i Q(i,t) = X(I,t) \quad \text{for all } I,t$$

$$\sum_j \hat{Q}(j,1) = \hat{X}(J,1)$$

$$lb(i,t) \leq Q(i,t) \leq ub(i,t) \quad \text{for all } i,t$$

$$\hat{l}b(j,1) \leq \hat{Q}(j,1) \leq \hat{u}b(j,1) \quad \text{for all } j$$

6. Since the $AI(j,L)$ is definable without disaggregating finished products, by simply using equation 42 L times and transferring the part inventory across time; the finished product minimization and the minimization over part types can be split up, as there are no linking equations dependent upon the disaggregated schedules - the two separate mathematical programs are -

I. For Finished Products

$$\text{Min.} \sum_{t=1}^{L} \sum_{I} \sum_{i} [s(i)*d(i)/Q(i,t)]$$

$$\text{st.} \sum_I f(I,j)*\sum_i Q(i,t) \leq \hat{A}I(j,t) + \hat{O}I(j,t) \quad \text{for all } j, t<L$$

$$\sum_i Q(i,t) = X(I,t) \quad \text{for all } I$$

$$lb(i,t) \leq Q(i,t) \leq ub(i,t) \quad \text{for all } i,t$$

II. For Parts

$$\text{Min} \sum_J \sum_j [\hat{s}(j)*\hat{d}(j)/\hat{Q}(j,t)]$$

$$\text{st.} \sum_I f(I,j)*X(I,L) \leq \hat{Q}(j,1) + \hat{A}I(j,L) \quad \text{for all } j$$

$$\sum_j \hat{Q}(j,1) = \hat{X}(J,1) \quad \text{for all } J$$

$$\hat{l}b(j,1) \leq \hat{Q}(j,1) \leq \hat{u}b(j,1) \quad \text{for all } j.$$

7. The minimization relating to finished products is separable by type, as the first constraint is guaranteed by aggregate feasibility as stated in point 4, and given this guarantee, the problem is identical to that in the single stage state for each t.

It is worth noting, that here there is no point in disaggregating the finished product schedule for any more than one period as nothing is gained in terms of what parts will be needed and the future part available inventory.

8. The minimization over part types is also separable by type, as the first constraint is merely a lower bound for each j. The lower bound used in the third constraint

can be redefined as -

the maximum $(lb(j,l), \sum_{I} f(I,j)*X(I,L) \leq \hat{A}I(j,L))$

and then the first constraint on parts can be eliminated.

9. The mathematical problem relating to part types for the immediate period can be separated by part type for the same reason that single-stage finished product types can be disaggregated separably - no constraint cuts across any two types.

This proposition implies that in general the simple one-stage disaggregation routines which disaggregate type by type may be neither feasible nor optimal in a multi-stage setting.

However, for the simple structure, where finished products in a type have identical part profiles and each product type requires proportionally identical numbers of parts within each part type, a knapsack like algorithm can be used to disaggregate finished products in the immediate period. Unfortunately, this will not be the usual situation. We built this hierarchical model with the intention of a more general applicability. For the more typical situation, where the assembly of families within product types requires differing components, we present two distinct approaches for disaggregating ; an equalization-of-run-out-time algorithm and an optimization methodology. The characteristics of all three of these possible disaggregation schemes will be described.

<u>The Knapsack Routine</u>

This approach to disaggregating in the single-stage situation was proposed by Bitran and Hax [4], and briefly described in the chapter on single-stage production planning. Given the conditions required for this approach to be useful, no part disaggregation will generally be necessary. This is because to a large degree the aggregate schedule dictates the number of each part in a part type to be produced, either explicitly as there is only one part in the part type, or implicitly by the required proportions of each part.

There are two modifications in the knapsack method for disaggregating finished products that will be described in this section and briefly tested in the next chapter. These modifications are applicable when using the knapsack routine in any hierarchical framework, either single-stage or multistage.

The first modification addresses the implicit planning horizon incorporated in the disaggregation scheme. The objective when using the original knapsack approach to disaggregate a given product type is -

(39) $$\text{Minimize} \sum_{i \in I} S(i)*A(i)/Q(i)$$

where:

S(i) represents the setup cost associated with family i in product type I,
A(i) represents the annual forecasted demand of family i, and
Q(i) represents the quantity of family i to be produced.

A comparison of various disaggregation algorithms [14] used in single-stage hierarchical production planning, indicates that there are circumstances in which a shorter planning horizon, less than annual demand in the objective function, does a better job in terms of costs. The basic concept in this modification is that when disaggregating from type to family, the planning horizon should be short enough for seasonality and existing inventory to play a significant role and long enough so that the plans made at higher levels in the hierarchy not be violated.

Operational planning schedules the assembly of finished products for one period. This adjustment defines the appropriate planning horizon to be as long as the current production will influence product availability. In an attempt to capture this concept in the model, we modified the objective function to the following format -

(40) $$\text{Minimize } \sum_{i} [S(i) * \sum_{t=1}^{ph} d(i,t)]/Q(i)$$

where:

d(i,t) represents the demand for family i in period t, and

ph is the desirable planning horizon.

The number ,ph ,is calculated for each product type each time the disaggregation routine is called. It was felt that as the aggregate schedule was concerned with longer periods of time and building inventory so should the disaggregation routine. Moreover, if the aggregate schedule was only concerned with the immediate period, the disaggregation routine also should be.

We define the planning horizon, ph, as the planned quantity of the product type plus the product type's available inventory divided by the immediate demand for that product type. A flowchart of this calculation is illustrated in Figure 4.5. The algorithm proceeds as follows -

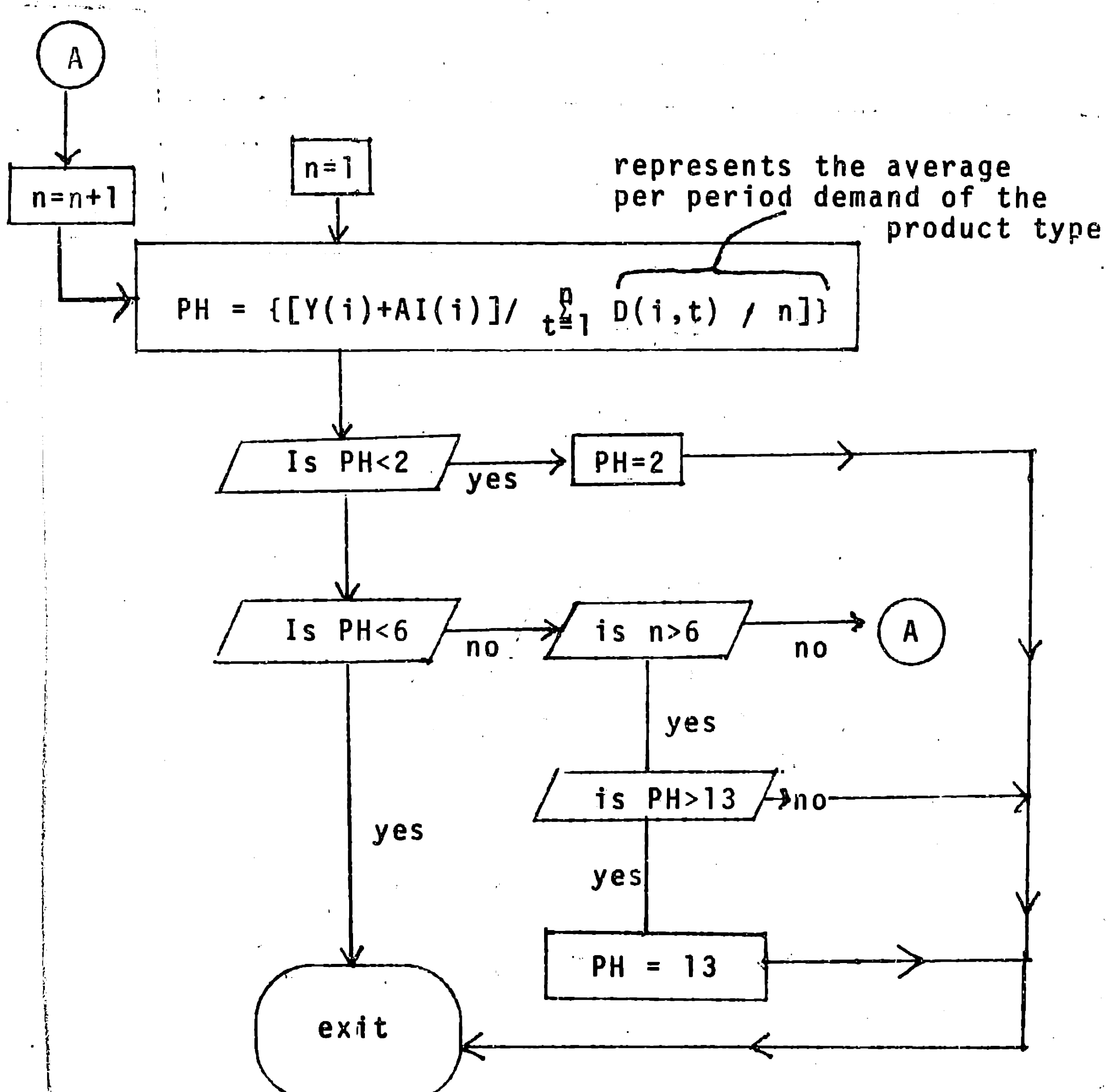
1. The planning horizon must be at least one complete period; due to forecast errors and to avoid a nervous system we choose the value 2 as a minimum.
2. Next, the algorithm checks if the planning horizon is greater than 6 periods. If it is, the algorithm attempts to check that it isn't due to unusual seasonalities, like zero demand in a particular period. If this is the case, the planning horizon is recalculated using an average demand of a greater number of periods.

Figure 4.5 The Time Horizon Used in Level 2 of The Modified Knapsack

The Modified Algorithm

$$\text{MIN} \left\{ \sum_{j \in i} [S(j)] \left[\sum_{t=1}^{PH} d(j,t) \right] \right\}$$

Where the planning horizon P H is determined as follows:



3. A maximum of one year is used, as it is felt that in a batch processing environment, the probability that the disaggregation of finished products within a single period could have influence on manufacturing costs for anything beyond a year is close to zero.

This myopic planning horizon concept was tested and the results will be presented in chapter 5.

The second modification aims to ensure future feasibility of the aggregate schedule. The need for this modification will be illustrated with an example -

Suppose an aggregate schedule had production of a particular product type planned in at the following levels, with the indicated planned ending inventories -

	Period 1	Period 2
Production	25 units	5 units
Inventory	15 units	0 units

In the first period the following situation existed for the families in the product type -

	Family 1	Family 2
Initial* Inventory	0 units	10 units
Expected Demand in Period 1	10 units	10 units
Expected Demand in Period 2	10 units	10 units

* = initial inventory - safety stock

Given this situation, with either the Hax-Meal algorithm or the Knapsack routine, only Family 1 would trigger in the immediate period, as this is the only family with a positive effective demand. However, if all 25 units planned for production in period 1 were made of Family 1, and demand was exactly as forecasted, the following would result -

	Family 1	Family 2	Total
Ending Inventory	15 units	0 units	15 units
Expected Demand in Period 2	10 units	10 units	20 units

However, the production of only five units of this product type in period 2 would not be sufficient to satisfy the demand for family 2.

The traditional knapsack approach schedules only those families with a positive effective demand in any given period, as long as the total to be produced can be produced using only those families without violating the upper bounding constraints. A disaggregation scheme should attempt to the maximum possible degree to be consistent with the aggregate schedule. It is desirable that if forecast error would be zero, that the disaggregation scheme permits all future aggregate plans to remain feasible. For a full discussion of this consistency property see Golovin [13].

To avoid the problem illustrated in the example, a simple test is required. When only Family 1 triggers, the production in the current period plus the current available inventory of that family less two periods' demand for the family should be compared with the planned inventory of the product type in two periods -

$$(41) \quad X(I,\text{current period}) + \sum_{i \in I^*} (AI(i) - d(i,1) - d(i,2)) \overset{?}{\leq} AI(I,2)$$

where: X(I,current period) represents the type production to be disaggregated,

$AI(i)$ represents the available inventory of family i

I^* represents the set of families in product type I that have positive effective demands for the immediate period,

$d(i,1)$, $d(i,2)$ represent the forecasted demand for family i in periods 1 and 2 respecitively, and

$AI(I,2)$ represents the aggregately planned type inventory for the end of period 2.

If the first number is larger than the second, as is the case in the example given earlier, another family is scheduled to trigger in the current period. A flowchart of this check is illustrated in Figure 4.6. It is worth noting that when disaggregating, a lower bound on families that are forced to trigger to guarantee feasibility of the aggregate schedule can be set at:

$$lb(i^*,1) = \text{minimum}\left[\, X(I,1) + \sum_{i \in I^*} (AI(i) - d(i,1) - d(i,2) - AI(I,2);\ ub(i^*,1)\right].$$

where:

i^* indicates the family that was forced to trigger, and

$lb(i^*,1)$, $ub(i^*,1)$ indicate the lower and upper bounds associated with family i^* in period 1

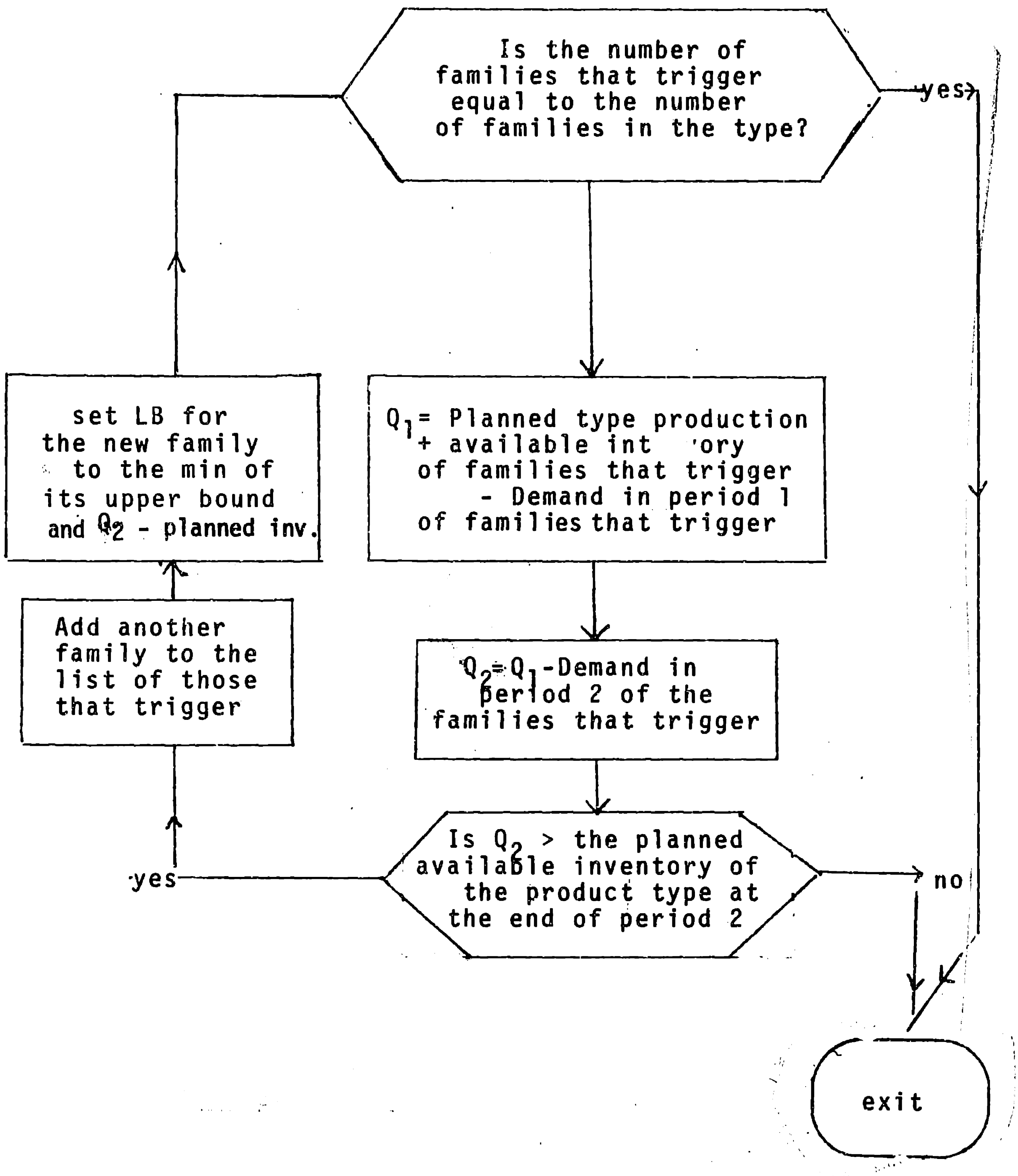
To be truly certain of a compatibility of the schedules, this should be checked for all 13 periods. However, there is a cost associated with checking. To check detailed demand forecasts of future periods are required. One of the problems identified with the monolithic optimization approach was the need for complete detailed forecasts. Simultaneously, the probability of violating future planned aggregate inventories decreases as the time into the future increases.

When we are working with a two-stage system, which needs L periods of detailed data for part planning, the availability of that data should be used to further ensure consistency. For the data we tested, checking for L periods was sufficient. The author would recommend using the maximum of L or 1 periods of checking in any hierarchical system. The exception would be when the user has knowledge that allows him or her to feel it necessary to check further into the future.

Golovin [13] noted the possibility of forced backorders because of these situations, and identified a point where Hax-Meal created unnecessary backorders. In the next chapter we compare the adjusted algorithm with the unadjusted Hax-Meal for that particular point. This potential problem with the Hax-Meal and knapsack routines has also been identified by Gabbay [12], and he suggests an alternate approach for dealing with it. However, his modifications break the knapsack structure of the model, which creates significant computational problems.

These two modifications will be included in any future reference to the knapsack routine, and in the knapsack routine we use when comparing the two-stage approach to M.R.P.

Figure 4.6 A Consistency Safeguard for Disaggregation Routines



The Equalization-of-Run-Out-Times, EROT

A relatively simple process, suggested for use in the single-stage environment, EROT[13], offers an alternative to disaggregating the tactical plans.

The EROT procedure in the two-stage environment would disaggregate finished products and part types in a manner so that inventory of all items or parts within a part type will last an equal expected amount of time. This methodology disregards finished product families and setup costs. For a more complete description of this approach see [13]. When the method for aggregating items into product types was presented, items grouped into a product type were defined to have similar seasonalities. When using the EROT approach the product-part factors for every product are known precisely for the immediate period; and known within a forecast error range for future periods.

If forecast error is zero, for all future periods, all product-part factors are known in advance. The feasibility of this disaggregation scheme is dependent on the accuracy of the effective part type inventories. Although feasibility cannot be guaranteed, as is the general case, with small forecast errors, part safety stocks can be used to increase the feasibility probability.

A General Two-Stage Disaggregation Approach

This methodology disaggregates all product and part types simultaneously using the Frank-Wolfe algorithm. When optimizing at this level in the single-stage process, the objective of minimizing family setup costs is equivalent to the knapsack approach [4], in which each finished product type is disaggregated separately and the fabrication or purchase of parts is not considered. The two-stage model has this same style objective, with part setup costs included as illustrated below.

The Single-Stage Objective -

(42) $$\text{Minimize} \sum_{I} \sum_{i \in I} [S(i)*d(i)/Q(i)]$$

The Two-Stage Objective -

(43) $$\text{Minimize} \sum_{I} \sum_{i \in I} [S(i)*d(i)/Q(i)] + \sum_{J} \sum_{j \in J} [\hat{S}(j)*\hat{d}(j)/\hat{Q}(j)]$$

where:

$S(i),\hat{S}(j)$ represent the setup costs associated with the production of family i or part j,

$d(i),\hat{d}(j)$ represents the demand for finished product family i, or part j, and

$Q(i),\hat{Q}(j)$ represents the quantity of the associated family or part to be manufactured.

The following constraints are associated with the single-stage disaggregation for any given period -

(44) $\sum_{i \in I} Q(i) = X(I,t)$ for all I

(45) $lb(i,t) \leq Q(i) \leq ub(i,t)$ for all i

where:
X(I,t) represents the aggregate production quantity of product type I scheduled to be produced in the period being disaggregated - t,
Q(i) is the quantity of family i scheduled for production in period t,
lb(i,t) represents a lowerbound on the quantity to be produced of family i, and
ub(i,t) represents a upper bound on the quantity to be produced of family i.

The lower bound for a family is defined to be the maximum of zero and that family's effective demand, while the upper bound for a family is defined to be that family's overstock limit less the available inventory of that family. When the family has a terminal demand at the end of its season, OS(i), can be calculated by means of a newsboy model (see Zimmermann and Sovereign [36]).

In the two-stage case with lead time, the disaggregation is subject to many more constraints. With lead time, more than one period of planned production needs to be disaggregated, therefore the Q(i)s need a time dimension attached to them. Equations (44) and (45) need to be true for every period being disaggregated. Simultaneously, similar constraints exist associated with the part types -

(46) $\sum_{j \in J} \hat{Q}(j,t) = \hat{X}(J,t)$ for all J and all t being disaggregated over.

(47) $\hat{lb}(j,t) \leq \hat{Q}(j,t) \leq \hat{ub}(j,t)$ for all j and all t being disaggregated over.

where:

$\hat{Q}(j,t)$ represents the quantity of part j planned for production in period t,
$\hat{X}(J,t)$ represents the quantity of part type J planned by the aggregate schedule for period t,
$\hat{lb}(j,t)$ represents the lower bound for production of part j in period t, and
$\hat{ub}(j,t)$ represents the upper bound for production of part j in period t.

There exists two sorts of interacting constraints which the disaggregation process needs to be concerned with. The first type addresses periods in which the part availabilty is frozen - the first L-1 periods. These constraints are of the following form -

$$(48) \quad \sum_{I} \sum_{i \in I} f^{*}(I,i,j) * Q(i,t) \leq \hat{A}I(j,t) \quad \text{for all } j$$

where:

$\hat{A}I(j,t)$ indicates the part availability of part j in period t.

The second sort of interacting constraints addresses the relationship between parts demanded and parts produced L periods prior -

$$(49) \quad \sum_{I} \sum_{i \in I} f^{*}(I,i,j) * Q(i,t+L) \leq \hat{Q}(j,t) + \hat{A}I(j,t)$$

These two equations prevent the general separability that the single-stage model has. Fortunately, the constraints on finished product production for the immediate period are totally free from constraints relating to part production in the immediate period and finished product production in L periods. The interaction between periods in the disaggregation scheme may be in the carrying of part inventory from one period to the next. However, we chose to break the disaggregation problem up in a block manner. The reason we chose this was so that a phantom inventory reconcilation could be done between periods and the families that were to trigger in L periods could be determined. With a lead time of one period two seperate disaggregation routines were done, one for the immediate period, and one done for one period in the future. The second disaggregation is done after planned inventories at the part and family levels are established for the next period. The two disaggregations necessary are illustrated below.

The two disaggregation pieces
for a one period lead time case

For the Immediate Period -

$$\text{Minimize} \sum_{I}\sum_{i \in I} [S(i) * d(i)/Q(i,1)]$$

subject to -

$$\sum_{i \in I} Q(i,1) = X(I,1) \qquad \text{for all } I$$

$$lb(i,1) \leq Q(i,1) \leq ub(i,1)$$

$$\sum_{I}\sum_{i \in I} f(I,i,j) * Q(i,1) \leq \hat{A}I(j,1) \text{ for all } j$$

Parts for the Immediate Period and Finished Products for one period in the future -

$$\text{Minimize} \sum_{I}\sum_{i \in I} [S(i) * d(i)/Q(i,2)] + \sum_{J}\sum_{j \in J} [\hat{S}(j)*\hat{d}(j)/\hat{Q}(j,1)]$$

subject to -

$$\sum_{i \in I} Q(i,2) = X(I,2) \quad \text{for all } I$$

$$lb(i,2) \leq Q(i,2) \leq ub(i,2) \quad \text{for all } i$$

$$\sum_{j \in J} \hat{Q}(j,1) = \hat{X}(J,1) \quad \text{for all } J$$

$$\hat{l}b(j,1) \leq \hat{Q}(j,1) \leq \hat{u}b(j,1) \quad \text{for all } j$$

$$\sum_{I}\sum_{i \in I} f(I,i,j) * Q(i,2) \leq \hat{Q}(j,1) + \hat{A}I(j,2)$$

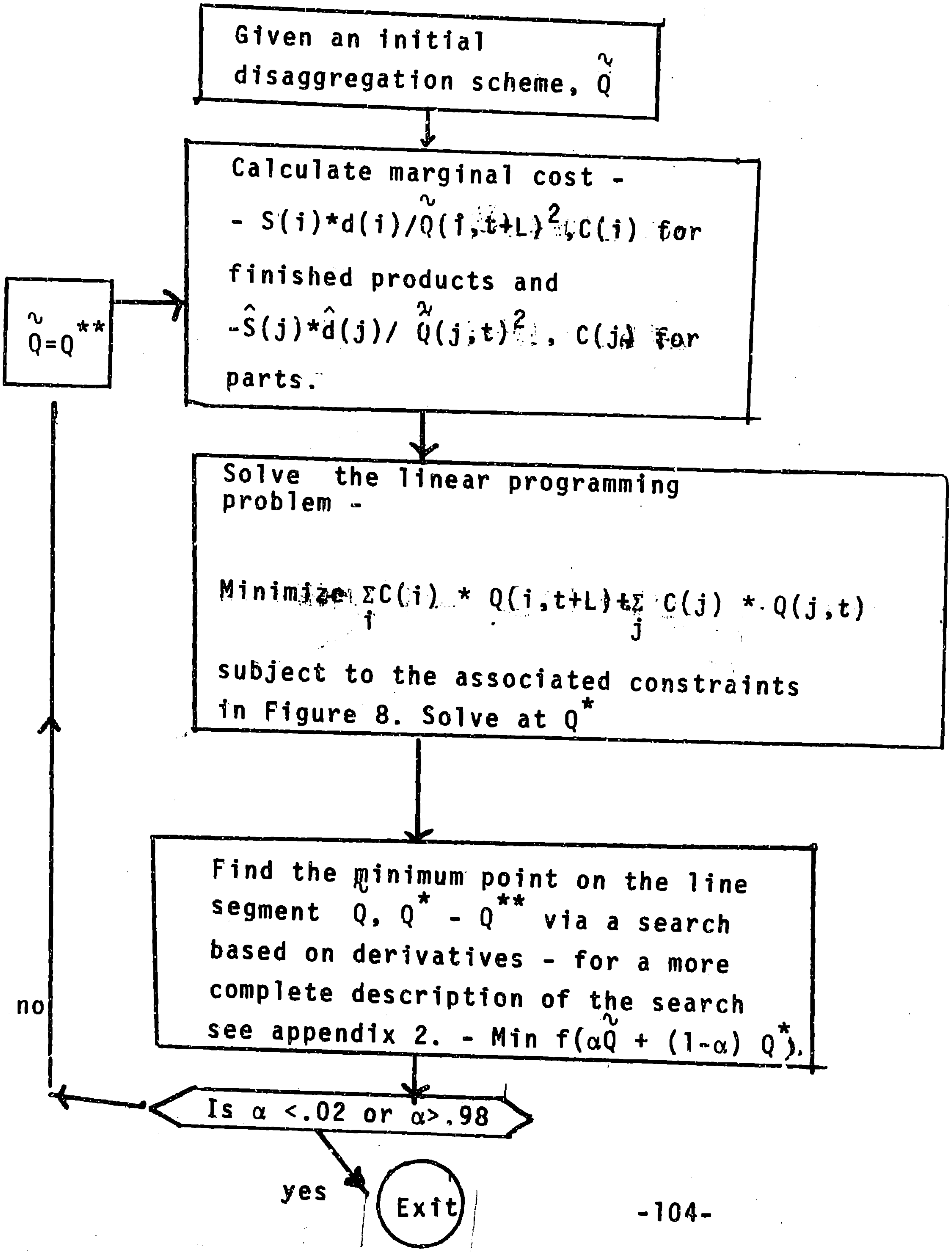
For these disaggregation problems, the disaggregation of finished products can be done simultaneously using the Frank-Wolfe decomposition. This approach solves the math programs previously illustrated in the following manner ;

1. An initial solution is found;
2. The derivative of the objective function with respect to each variable is evaluated at that point;
3. A linear objective function is defined with the partial derivatives previously evaluated as the cost coefficients;
4. A new solution is found;
5. The actual objective function is minimized over all weighted combinations of the two points. The minimum is treated as an initial solution and step 2 is returned to until a true minimum is found. This is identified when the minimum of a weighted average is entirely at one of the two points.

This algorithm applies nicely to our problem, as all the constraints are linear, and the objective function is convex. For a more complete description of the characteristics of the objective function see Appendix 2.

A flowchart indicating the steps used in the disaggregation scheme based on Frank-Wolfe are illustrated in Figure 4.7. When this is solved for the current period, the accuracy in the estimated effective inventories and part -product factors can be examine, and if necessary fed back to the aggregate

Figure 4.7 - The Flow of the Frank-Wolfe Algorithm with Our Disaggregation Needs



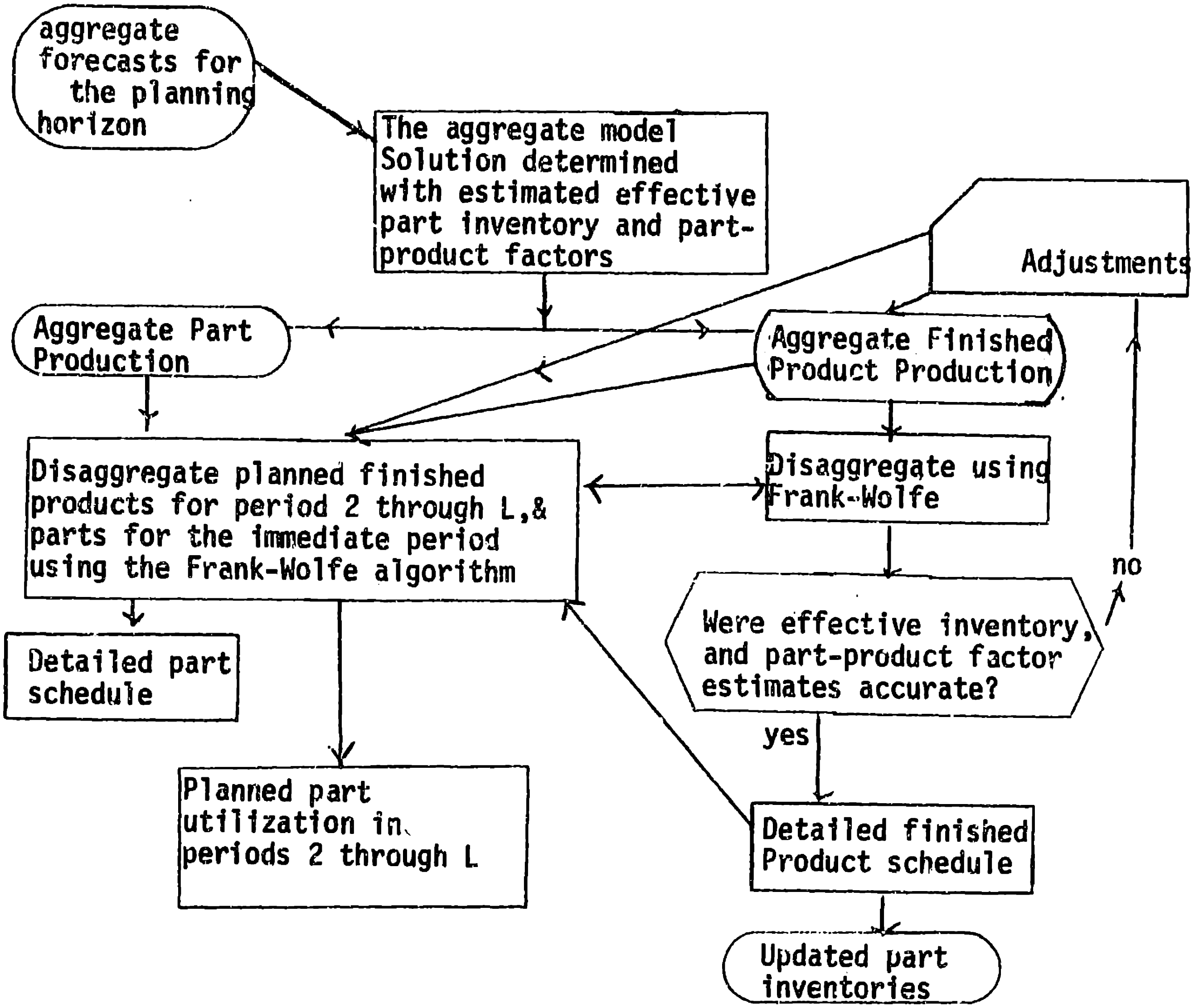
solution. Simultaneously, after the current period is over - planned production levels can be adjusted for changed forecasts and given the disaggregated solution for the next period, effective inventory and part-product factors to be used in the next aggregate schedule can be estimated.

4.4 The Overall Model

The hierarchical model is based on the concept of planning production at the tactical level so as to minimize both finished product assembly, and component fabrication costs. In a general sense, the hierarchical process has difficulties defining effective part inventory and product-part factors. However, there are reasonable methods to handle inaccuracies, and with simple corrective procedures the overall cost levels are relatively robust, as will be shown in the next chapter.

For disaggregating products and parts, the solution procedure is not as simple as in the single-stage case, yet it can be done with mathematical programming or foregoing the minimization of setup costs and using an EROT algorithm The flow of the two-stage approach is illustrated in Figure 4.8. This general scheme for planning production will be compared, in limited situations to the MRP approach in chapter six.

Figure 4.8 The Overall Flow of the Two-Stage Hierarchical Methodology



CHAPTER 5 - TESTING THE TWO-STAGE HIERARCHICAL MODEL

Throughout the process of testing varying aspects of the model, a scaled down version of an actual situation is used and several parameters are varied. All data was measured in a standard manner and shared the structural characteristics of a pencil company. The overall framework and measurements will be presented first then, a description of the tests performed and the data used will be specified.

5.1 The Framework

The company assembles a variety of pencils requiring a number of distinct components, as illustrated in Figure 5.1. The pencil product-part requirements are illustrated in Figure 5.2. All items of the same size were grouped into product types. The result is that all items within a type require identical assembly time. Within types, items sharing common setup costs, and having the same inscriptions were grouped into families - exactly as would be done when grouping products for sinlge-stage hierarchical systems. In this case, all items within a family had identical part requirements. If this were not the case, perhaps a fourth level of product aggregation would have been desired.

Parts sharing common production facilities were classified into types. In this case, the two different leads were one type, and the erasers and wood were each separate types. Fortunately, the grouping of leads into a type was identical to developing a part-type so that each item required one part in that type and never more than one. The classification of

Figure 5.1

The Product: Pencils

Variations → 2 Sizes
3 insignias on the side
2 lead types
some without erasers

Figure 5.2 Product Part Requirements

Pencil			Part			
Size 1			Wood	Eraser	Lead 1	Lead 2
	insignia 1	color 1	1	1	1	0
		color 2	1	1	1	0
	insignia 2	color 1	1	1	0	1
		color 2	1	1	0	1
Size 2						
	insignia 1	color 1	1	1	1	0
		color 2	1	1	1	0
	insignia 2	color 1	1	1	1	0
		color 2	1	1	1	0
	insignia 3	color 1	1	0	0	1
		color 2	1	0	0	1

A 1 denotes the requirement of a given part in a given pencil;
A 0 denotes the absence of that requirement.

product types was checked against part requirements, in an attempt to avoid problems resulting from variability in product-part factors within a product type. An illustration of this classification and family requirements is given in Figure 5.3.

Figure 5.3 A Tree Diagram of Product-Part Requirements

	Product type 1		Product type 2		
family	1	2	1	2	3
Part Requirements					
Part type 1 family 1	1	1	1	1	1
Part type 2 family 1	1	1	1	1	0
Part type 3 family 1	1	0	1	1	0
family 2	0	1	0	0	1

Ideally, all finished products within a product type should have identical part type requirements. The desire-ability of this characteristic; and the potential problems that occur if it does not hold were discussed in the previous chapter when the aggregate interacting constraints were pre - sented. Again, we would like to emphasize that the aggrega-tion process should be custom designed for each production problem .

The Parameters and Measurements Associated With the Data

Given the large number of possible input parameters, we varied those we felt most important to test, as the computational time was expensive. The product structure described in the previous section, as well as the annual demand, finished product holding costs, finished product overtime costs and productivity were held fixed in all tests. The values of the fixed parameters are indicated in Figure 5.4.

We believe that changing the productivity would principally affect the aggregate model in a manner similar to changing capacity. We did not feel it was necessary to vary both part and finished product holding costs, as altering either one alters the relative magnitudes which are the important factors in testing the two-stage model.

Finished Product

	type 1	type 2
Annual Demand	100,000	120,000
Holding Cost$/unit period	$.30	$.40
Productivity Hours/unit	.1	.12
Overtime cost	$9.5/hour	$9.5/hour

Part

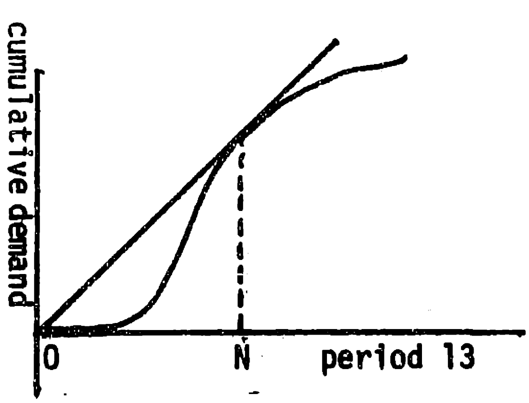
	type 1	type 2	type 3
Productivity Hrs/unit	.5	.1	.6

The planning horizon of the aggregate model = 1yr.

Figure 5.4 Parameters Held Constant

Both part fabrication capacity and finished product assembly capacity were measured between 0 and 100 in the following manner;

(a) Determine at which of the thirteen periods there exists the highest average demand per period (based on cumulative demand - see Figure 5.5). Let us call this period N.



Note: the point where the tangent from the origin intersects the cumulative demand curve is the point of highest average demand per period.

Figure 5. 5

(b) At period N compute the average demand per period, or:

$$\text{Avg Demand}_N = (\text{Cumulative demand up to period N})/N.$$

(c) If capacity with no overtime is equal to to Avg. Demand_N, then capacity is set to 100. If capacity with the maximum allowable overtime is equal to Avg. Demand_N, the capacity is set to 0. At all points between these two extremes capacity is scaled appropriately.

Forcast error was generated by a function of the form:

$$\text{f.e.} = (A+Bt^C)\cdot(d(t))$$

where A, B, and C are parameters

t is the time period and

d(t) is the actual demand in period t.

A fourth parameter of the error function is the probability that the error is positive or negative. The probability that it is positive is referred to as the positive bias. We felt that such a function would be an accurate description of how forecast errors are. The size of the errors grow with the length of the forecast horizon.

Each demand pattern was treated as two connected half sine waves (see Figure 5.6) with peaks and valleys. The waves could meet at any of the 13 periods. It was required that the half waves directions be opposite one another. With this construct any seasonal pattern of demand could be created, where the X-axis is viewed as the month's mean demand. The seasonal factors were normalized, so that the average was equal to 1.0

As a measure of seasonality, the coefficient of variation was calculated for each demand pattern.

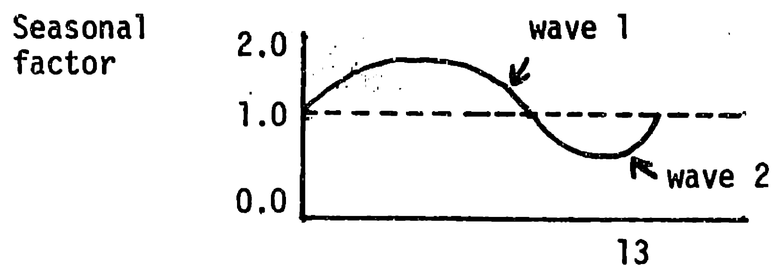
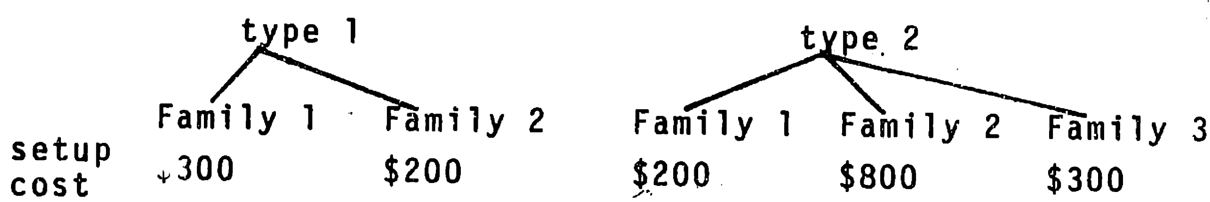


Figure 5.6 An Example of Seasonal Factors

The setup costs for families within a finished product type were not taken equal in order to test the model under less favorable conditions. Simultaneously, we felt that the exact levels of the setup costs of finished products were not key for most tests performed. The setup cost structure used for finished products in all tests (unless otherwise mentioned)is illustrated in Figure 5.7.

Figure 5.7 Family Setup Costs



5.2 Testing the Heuristics Built into the Model

In this section the assumptions and heuristics built into the two-stage methodology developed in the previous chapter are tested. This is done prior to comparing the overall algorithm with M.R.P. to ensure that the base upon which the two-stage hierarchical approach is set is solid. The first test presented checks the assumption that the aggregate schedule does not substantially change from period to period. The second and third tests examine the manufacturing cost of misestimating effective part inventory with and without an adjustment procedure. The fourth and fifth tests examine the expense associated with misestimating product-part factors with and without an adjustment process. The sixth test compares the single-stage knapsack algorithm and the Hax-Meal algorithm with the adjusted knapsack routine, that with a shorter planning horizon when disaggregating. The seventh test examines how Golovin's point [13] responds to a knapsack routine that checks aggregate schedule's feasibility one period ahead.

Testing the Robustness of the Aggregate Schedule

When estimating effective part inventory, a previous aggregate schedule updated for changes in forecasts is used. The first test performed was to determine how close to actual values these aggregate estimates were.

The data used to test this hypothesis had the following characteristics -

1. The only part required was part type 1, the wood piece. In this manner, changes in the aggregate schedule could not be attributed to effective part inventory.

2. Two patterns of seasonality were used:

period	Pattern 1 Type 1	Pattern 1 Type 2	Pattern 2 Type 1	Pattern 2 Type 2
1	1.35	2.11	0.43	0.83
2	2.52	1.62	0.61	1.25
3	2.52	1.16	0.89	1.55
4	1.35	0.76	1.21	1.67
5	0.93	0.46	1.21	1.55
6	0.57	0.28	1.65	1.25
7	0.30	0.21	1.82	0.83
8	0.15	0.28	1.65	0.71
9	0.15	0.46	1.21	0.62
10	0.30	0.76	0.89	0.55
11	0.57	1.16	0.61	0.62
12	0.93	1.62	0.43	0.71
13	1.35	2.11	0.36	0.83

Figure 5.8 Seasonality Factors

3. Finished product capacity was set at 10%,35%, and 40%.

4. Part capacity was set at 20%, 100%.

5. Part holding costs were set at $.15/unit period and $.25/unit period.

6. Setup costs were held at standard levels.

7. The planning horizon of the aggregate problem was 13 periods, or one year.

8. Forecast error was set by functions with the following coefficients -
   A=B=C=0 or A=.05, B=.02, C=1.1 with a positive bias of .6.

The testing was limited by computational expense. For each of the 48 points, the planned aggregate production for period 3 in period 2 updated for changed forecasts and bound by capacity and part availability was compared with the aggregate production for period 3 planned in period 3.

A statistic that was examined is the absolute percent deviation for each of the finished product types. These numbers are the mean absolute deviation divided by the true value times 100. The average indicates that the estimates were quite close to the actual, as the mean absolute deviation for both product types was less than 1%.

The maximum deviation for either product type on all 48 points was 2.9%, indicating a robustness in the aggregate schedule. All points that were generated had an upper bound on the forecast error for the next period of 9.7%. Our basic conclusion is that within the range of forecast errors that were tested, the aggregate schedule is robust. The hypothesis that the actual level of the estimated aggregate production comes from a distribution centered at the true value and with a standard deviation less than 2% cannot be rejected.

Testing the Cost of Misestimating Effective Inventory

The second test performed was to determine how misestimating effective part inventory, without any adjustment routine, affected manufacturing costs. For this test all the data had the following characteristics:

1. High seasonality - identical to pattern 1 in Figure 5.7;
2. The only part requirements were for part type 3. In this manner, the chosen disaggregation scheme would affect the level of effective inventory;
3. The inital part inventory was set to exactly match that which would be required in the first period, if the single-stage aggregate schedule applied to this data was disaggregated using the knapsack algorithm.

5. Finished product capacity was set at 40%;

6. Part capacity was set at one of two levels, 100% or 25%;

7. Part holding costs took on two values, one approximately 50% that of finished product holding cost and one approximately 75% of finished product holding cost;

8. The planning horizon was set at 13 periods, or one year; and

9. We varied the error in our estimates of effective inventory from minus 15% to plus 15%.

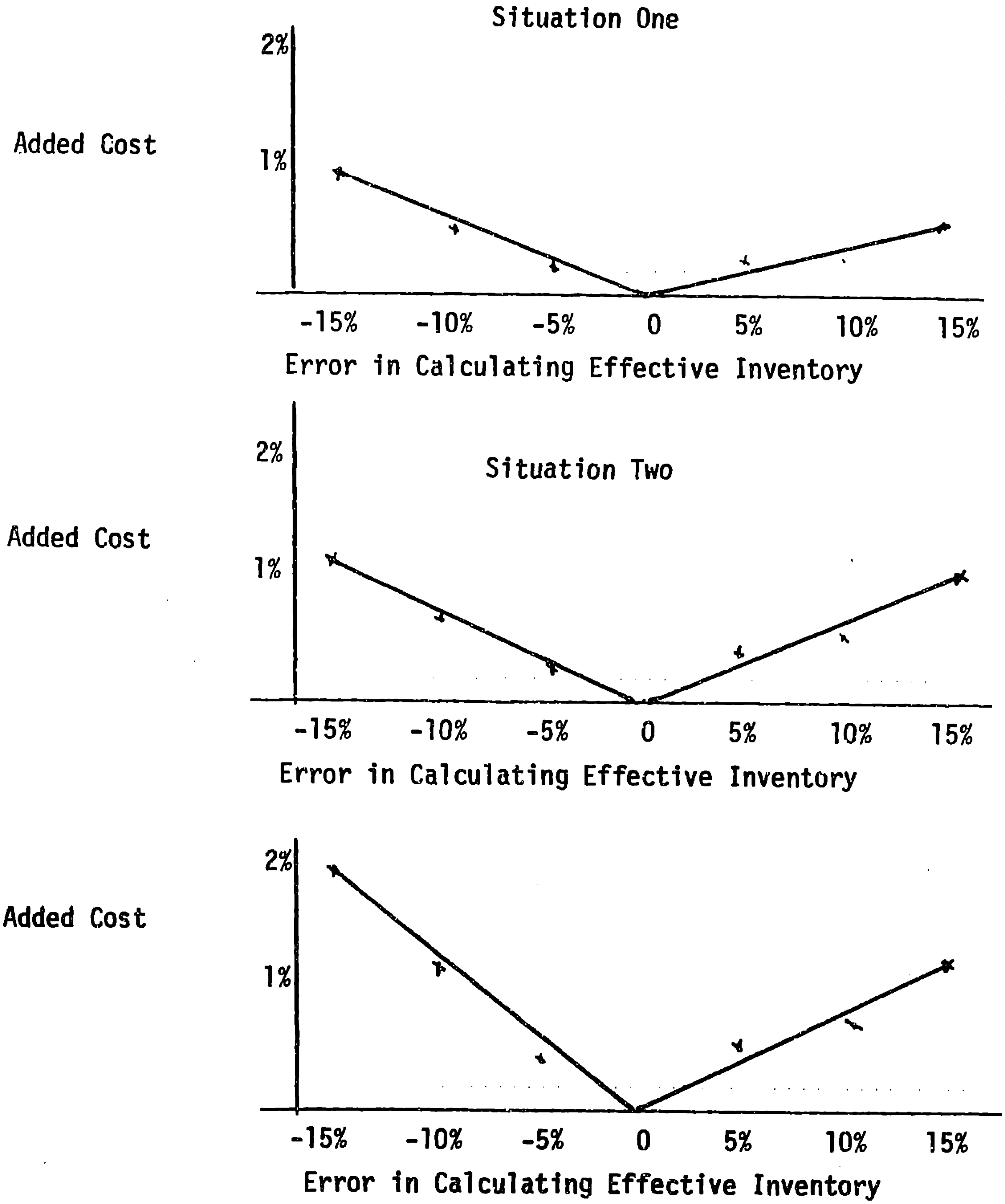
Overall three situations were tested, all with high seasonality and relatively tight finished product capacity. In the first situation part capacity was loose and part holding costs were 50% that of finished products. The second situation tightened part capacity beyond the tightness of finished product capacity, and the third situation raised the part holding costs to 75% that of finished products.

The cost associated with misestimating effective inventory for period one using the two-stage hierarchical model developed in the previous chapter with no correction routine or feedback for these three situations are illustrated in Figure 5.9.

It was felt that these situations represented those in which costs would be most affected by the misestimates. If effective inventory is greater than needed, misestimating the level will have little effect on costs.

The results of this test indicate that misestimating effective part inventory results in a small cost. With estimates within 5% of the actual numbers, the costs were all less than .3% of the total manufacturing costs.

Figure 5.9 Testing the Cost Associated with Misestimating the Effective Inventory



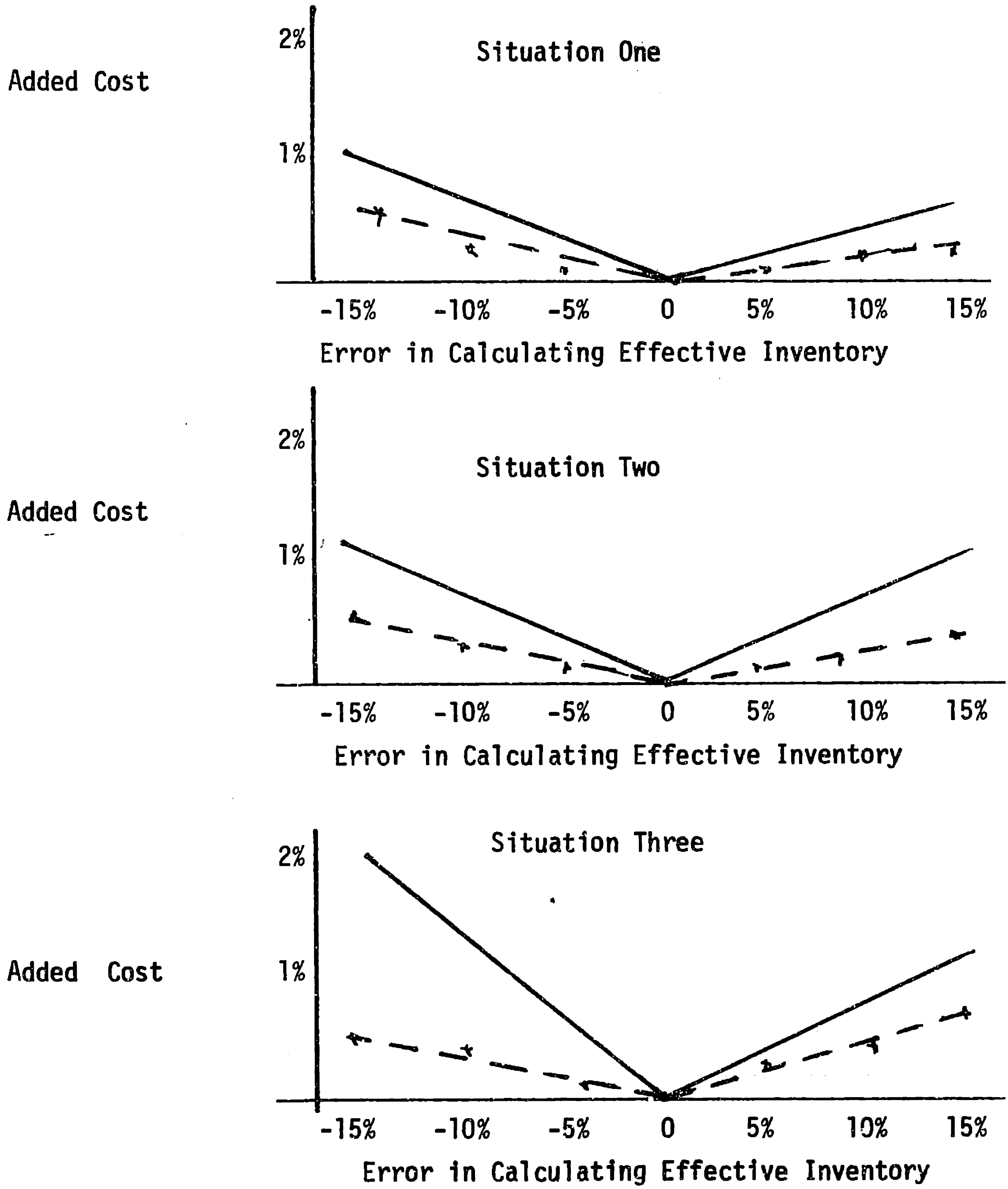
The third test performed examined how adding the simple one-step feeback routine to correct for poor effective inventory calculations affected the cost of bad estimates. Each point previously tested was rerun with the correction routine. The results of these runs are illustrated in Figure 5.10.

The simple adjustment routine generally cuts the small cost of misestimating the effective inventory at least in half. It is worth noting that high estimates appear to be less costly with no adjustment routine than low estimates. The reason for this skewness, is that high estimates are necessarily partially corrected for when disaggregating. However, if more than an optimal quantity of a family with low part requirements can be assembled, it will be so that the disaggregation keeps in line with the aggregate schedule to the maximum possible degree. However, if it is impossible to produce the aggregate production levels due to part availability, all production will be scaled down. This may not,and usually does not result in an identical solution to that found when accurate estimates for part available inventory are used.

<u>Testing the Expense Associated with Misestimating the Product-Part Factors</u>

The fourth test performed on the aggregate model was designed to determine the impact of misestimating the product-part factors on the cost of the solution. For this test we used the points described for the second test. However, we worked with part type one as opposed to part type three. This was done so that the calculations of effective inventory

Figure 5.10 Testing the Cost Associated with Misestimating Effective Inventory and Using a One-Step Adjustment



Broken lines indicate cost of error when one-step adjustment procedure is used, and
Solid lines indicate cost of error when no adjustment is used.

could be as accurate as possible. It was felt that these points would truely test the model's robustness, as in period one the part factors and part availability constrain the aggregate schedule. The results of this test are illustrated in Figure 5.11 . The errors used are in the product-part factors associated with both product types. It is worth emphasizing that small miscalculations do not appear to be expensive.

When the simple adjustment routine suggested in the previous chapter's section on product-part factors is used, the costs assoicated with misestimates are decreased. These costs are illustrated in Figure 5.11 ,where they are compared with the costs associated with no adjustment.

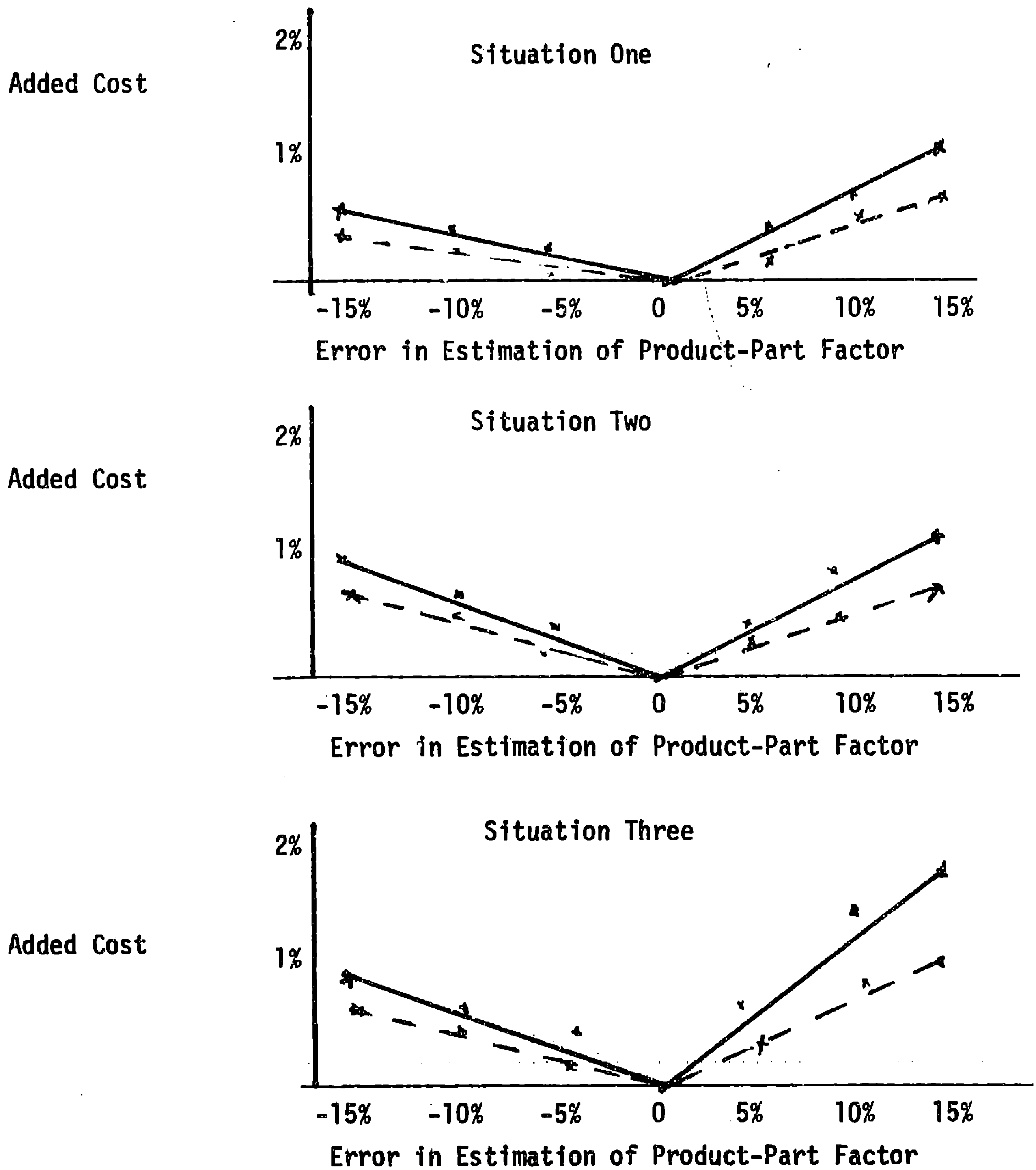
The basic results of our tests regarding the cost to the solution of misestimating effective inventory or product-part factors by small amounts indicate the expense is particularly small, and the procedure is relatively robust with respect to these two parameters.

Testing the Planned Alterations in the Knapsack Routine

The next two tests were performed on the single-stage hierarchical model presented in Chapter 2 to determine if the changes incorporated in the knapsack routine were justifiable. The data used to test the knapsack algorithm with a myopic planning horizon had the following characteristics:

1. No part requirements, therefore the two-stage hierarchical approach was identical to a one-stage model;

Figure 5.11 Testing the Cost of Misestimating the Product-Part Factors with and without Adjustment Routines



Broken lines indicate costs of misestimates when an adjustment routine is used, and
Solid lines indicate costs of misestimates when no adjustement routine is used.

2. Capacity was set at 20% - relatively tight;

3. The seasonality of demand took on the two levels illustrated in Figure 5.8;

4. We felt it was important to vary relative setup costs as these have a significant affect on the disaggregation scheme when the knapsack approach is used. The values the setup costs took on are illustrated in Figure 5.12.

Figure 5.12 The Setup Cost Structure

		type 1		type 2		
family		1	2	1	2	3
setup cost pattern	A	$200.	$300.	$300.	$800.	$200.
	B	$200.	$200.	$200.	$200.	$200.
	C	$1000	$300.	$1000.	$1000.	$200.
	D	$700.	$200.	$700.	$1000.	$200.

5. The forecast error was determined by one of the following three sets of coefficients -

	A	B	C	Positive bias
set 1	0.	0.	0.	0.
set 2	.01	.02	1.3	.5
set 3	.01	.02	1.3	.9

With the two sets of seasonality, four sets of setup costs, and three groups of forecast error parameters, 24 points were tested.

The results of the comparison of the modified knapsack, the unmodified knapsack, and the Hax-Meal approach are summarized with the Wilcoxon statistic and presented in Figure 5.13.

Figure 5.13 A Comparison of Disaggregation Methodologies -

Decision Criterion	Knapsack Adjusted vs. Knapsack Unadjusted	Knapsack Adjusted vs. Hax-Meal
Total Costs	1.42	.58
Backorders	1.63	.16

The numbers in the boxes indicate the number of standard deviations by which the Wilcoxon statistic favors the adjusted knapsack routine.

The hypothesis that the adjusted knapsack routine is superior to the unadjusted routine cannot be rejected at the 95% significance level. No strong statements relating it to the Hax-Meal algorithm can be made. It is worth noting that the adjusted knapsack routine was never dominated by either the unadjusted knapsack routine or the Hax-Meal approach. It is clear that this shorter planning horizon is worth implementing at the operational level in the hierarchy.

To test the routine designed to check the disaggregation schedule with the planned aggregate inventory one period in advance, we chose to use the point identified by Golovin [13]. The data associated with that point and the result of running that point with the modified knapsack are presented in Figure 5.14.

Figure 5.14 - Golovin's Point

INPUT

Product Structure and Setup Costs

	Type I		Type II			
family	1	2	1	2	3	4
setup cost	$6000.	$4500.	$400.	$5000.	$3000.	$2000.
items	1	1	1	1	1	1

Capacity was set at 2000 hours of regular time and 1200 hours of overtime

The demand schedule for the two product types is shown below -

period	Product Type 1	Product Type II
1	20549.	8029.
2	0.	4023.
3	0.	4860.
4	0.	7132.
5	0.	9664.
6	1545.	17603.
7	7895.	14276.
8	10982.	11706.
9	15782.	15056.
10	16870.	8232.
11	15870.	7880.
12	9878.	10762.
13	12736.	6174.
total	112107.	125395.

Forecast error was set at 0.
The production time required for type 1 was set at .10 Hours/Unit, and for type 2 was set at .12 Hours/Unit

OUTPUT

The Approach	Total Cost	Total Backorders
Mixed Integer Programming	$291,234.	0.units
Hax-Meal	$344,391.	1065 units
Adjusted Knapsack	$305,078.	0. units

It is worth noting that 72% of the difference in cost and 100% of the difference in backorders between the Hax-Meal routine and the solution found with a Mixed Integer Program by Golovin are avoided when the adjusted knapsack routine is used. The costs do not include back-order costs. The difference in costs between the adjusted knapsack and the Hax-Meal approaches can principally be attributed to less use of overtime. This savings in cost is an added benefit from the adjustment aimed at preventing backorders.

Overall our tests indicate that:

1. The heuristic method suggested for measuring effective part inventory is based on a sound assumption regarding the robustness of an aggregate schedule, as long as forecast error for the following period is less than 9%;

2. Errors less than 5% in estimates of effective part inventory and product-part factors, have mild affects on total manufacturing costs; and

3. Both adjustments suggested for the knapsack routine appear to strengthen the disaggregation approach.

CHAPTER 6 - THE COMPARISON OF THE TWO-STAGE ALGORITHM WITH MATERIAL REQUIREMENTS PLANNING

This comparison was done in two steps. The first step examines data which could be disaggregated using the knapsack routine, and attempts to compare the simple structured two-stage hierarchical algorithm with M.R.P. The second step uses the results of the first tests to identify the situations in which one approach outperforms the other, and presents limited testing of similar situations requiring a complicated array of parts. This test compares the two-stage hierarchical approach using Frank-Wolfe in disaggregating the two-stage model with M.R.P.

The set of experiments was limited by computational time and budget. This chapter will initially present the data base tested in the first step, and the associated results and implications. It will then present the data and results corresponding to the second step of experiments. The chapter will conclude with an overview of the results.

6.1 Comparison of the Hierarchical Approach using Knapsack with M.R.P.

The data base for this comparison was centered on the same data structure used in the previous chapter in the tests on the two-stage model (i.e., that resembling an actual pencil company).

The following parameters were considered critical in comparing M.R.P. with the hierarchical approach:

1. Relative capacities - finished products to parts;

2. Part capacity - is it unlimited;

3. Relative holding costs - the difference in holding a finished product and holding all or a part of its components;

4. Forecast error;

5. Part setup costs, in absolute numbers and in relation to part holding costs;

6. Seasonality of demand; and

7. Initial part inventory.

We began the testing procedure with two sets of points both based on the requirement of all finished products for exactly one part, like the pencil requirements for part type one. The parameters that were fixed for the two sets of points are illustrated in Figure 6.1. The parameters that were allowed to vary, and the values they were given were identical for both sets of points and are illustrated in Figure 6.2.

Results of the Comparison

In total 54 situations were simulated. There was only one type of situation in which M.R.P. cost less than the two-stage hierarchical algorithm :

When part capacity was high - 250%;

Part setup costs were significant - more than 20% of total manufacturing costs, and

Either -
 a. forecast errors were small, or
 b. part safety stock was large and part holding costs were small.

Figure 6.1 - The Parameters of the points tested -

All points in set A had the following characteristics -

The seasonality factors for each period and each product type were -

period	Type 1	Type 2
1	1.35	2.11
2	2.52	1.62
3	2.52	1.16
4	1.35	0.76
5	0.93	0.46
6	0.57	0.28
7	0.30	0.21
8	0.15	0.28
9	0.15	0.46
10	0.30	0.76
11	0.57	1.16
12	0.93	1.62
13	1.35	2.11

the coefficients of variation were -

type 1	type 2
5.605	5.064

Finished product capacity was set at 30%

The setup costs associated with finished product families were the standard numbers illustrated in Figure 5.6.

The holding costs were -

type 1	type 2	
$.30	$.40	per unit period

The planning horizon was set at 13 periods covering one year.

All points in set B had the following characteristics -

The seasonality factors for each period and each product type were -

period	Type 1	Type 2
1	0.43	0.83
2	0.61	1.25
3	0.89	1.55
4	1.21	1.67
5	1.21	1.55
6	1.65	1.25
7	1.82	0.83
8	1.65	0.71
9	1.21	0.62
10	0.89	0.55
11	0.61	0.62
12	0.43	0.71
13	0.36	0.83

the coefficients of variation were -

type 1	type 2
3.037	1.926

Figure 6.1 continued

Finished product capacity was set at 50%

The setup costs associated with finished product families were the standard numbers illustrated in Figure 5.6

The holding costs were -

	type 1	type 2	
	$.30	$.40	per unit period

The planning horizon was set at 13 periods - one year

Figure 6.2 The Multi-Valued Parameters Associated with both sets of data -

The part capacity was set at one of three levels -

250%
110%
and 25%

The part holding cost was set at one of two levels -

$.15/unit period or
$.25/unit period

The forecast error was set at one of two levels -

set 1. A=B=C=0. or zero forecast error, and
set 2. A=.02, B=.01, C=1.3 and the positive bias = .5 so forecast error as a percent of actual demand is

$.02 + .01 * t^{1.3}$.

The part setup cost was set at one of two levels -

$1500. per setup or
$2500. per setup

The initial part inventory was set at two distinct levels -

1. A tight level was defined as those parts which would exactly match the requirements for period one, of a finished product schedule determined by a single-stage hierarchical run; and
2. A loose level of initial inventory was defined as twice a tight level.

In general the following characteristics were noted:

1. If any forecast error existed, and part capacity was tighter than finished product capacity, M.R.P. tended to have greater backorders than the knapsack routine. This can be explained by the structure of the aggregate model. Once the parts are being held, the cost of assembling and holding the finished product is only the difference in the holding cost - not the full finished product holding cost. It then may be worth avoiding future overtime by building inventory of finished products sooner. In effect, the two-stage model, recognizing the existance of part in inventory will occassionally have a larger stock of finished products than M.R.P. will and be more protected against forecast error.

2. Regardless of the forecast error, if part capacity was not set at 250%, the knapsack approach had lower total costs than the MRP approach. The MRP costs associated with finished products was lower, while the difference was more than made up in the part costs. This difference in part costs was a combination of greater use of overtime and larger inventory levels.

3. When part capacity was tighter than finished product capacity and there was no forecast error, the difference in costs were the largest in favor of the two-stage model. This makes sense, as the MRP approach is blind to component overtime costs, and component holding costs are considered at a secondary level. The differences in cost were as high as 48% of total cost. The tighter the part capacity, and the higher the overtime and holding costs of parts - the greater the difference.

4. M.R.P. outperformed the two-stage approach when part capacity was 250% and forecast error was zero.

An example of the costs and backorders of a test point and the parameters associated with that point are illustrated in Figure 6.3.

The results of this test indicate, that if part types

Figure 6.3 An Example Point

Approach :

M.R.P.

finished product costs	=	52,612.
part costs	=	68,637.
total costs		121, 249.
total backorders		1923 - 2% of the annual demand.

Knapsack

finished product costs	=	69,581.
part costs	=	49.646.
total costs		119, 227.
total backorders		0 - 0% of the annual demand.

This point belonged to set A.
The part capacity was 25%
the part holding costs were $.15/unit period
the forecast error was at level 2,
the part setup costs were 1500. per setup and
the initial part inventory was at the higher of the two levels earlier stated.

can be as well defined as it was for our tests, and if the overtime associated with component production and part holding costs are of greater concern than part setup costs, the two-stage approach is more appropriate than M.R.P. Simultaneously, if the two-stage approach could be more flexible on aggregate part production levels when disaggregating or provide feedback to the aggregate schedule concerning part setup costs, an improved solution may be found, and the approach may always be superior to M.R.P.

To further tests these results we added a second part type to the product requirement structure. The part type, 2A, had two families - each finished product required one of the first family and two of the second family for production. This part type was defined in such a way, so that the knapsack product disaggregation remained appropriate. Every finished product in a product type had an identical profile of part requirements, thus facilitating the use of the knapsack routine. Effective part inventory for this part type was defined as -

(1) Minimum (3 * $\hat{A}I(1)$, 1.5*$\hat{A}I(2)$)

for the first period. Where $\hat{A}I(1)$ indicates the available inventory of family one and $\hat{A}I(2)$ indicates the available inventory of family two. This definition is equivalent to our earlier definition, given the relative requirements for the parts in the part type.

The parts were also disaggregated using a knapsack approach and relying on the aggregate schedule to remain stable. For this test we used the features of set A illustrated in Figure 6.1. The other characteristics of the points tested are listed in Figure 6.4.

The result of this comparison further enhanced our belief that the two-stage model can outperform M.R.P. Of the 16 points generated, there was only one in which M.R.P. had actual costs less than the two-stage hierarchical model; the one with high setup costs, large part capacity and zero forecast error. If part safety stock were set at levels designed for forecast error and M.R.P. systems, the same situation with some forecast error and low part holding costs may have credited M.R.P.

Overall, when part capacity is limited, and part setup costs are not primary, the two-stage hierarchical approach dominates the M.R.P. planning process.

It is worth noting that the differences in cost are not insignificant. On the average[1], for all 70 cases tested the knapsack approach's total manufacturing costs were 4% less than those associated with M.R.P. and the average difference in backorders was small. A bar graph illustrating the distribution of the difference in costs and backorders is shown in Figure 6.5.

1. The average represents the median, as the distribution is skewed.

Figure 6.4 - Characteristics of the third set of points tested -

The part capacity for the two different part types took on one of the following two sets of values -

20%	and 25%	or
200%	and 250%.	

Part holding costs were either -

Part type 1		Part type 2	
\$.10/unit period	and	\$.05/unit period	or
\$.09/unit period	and	\$.02/unit period.	

Part setup costs were set at either -

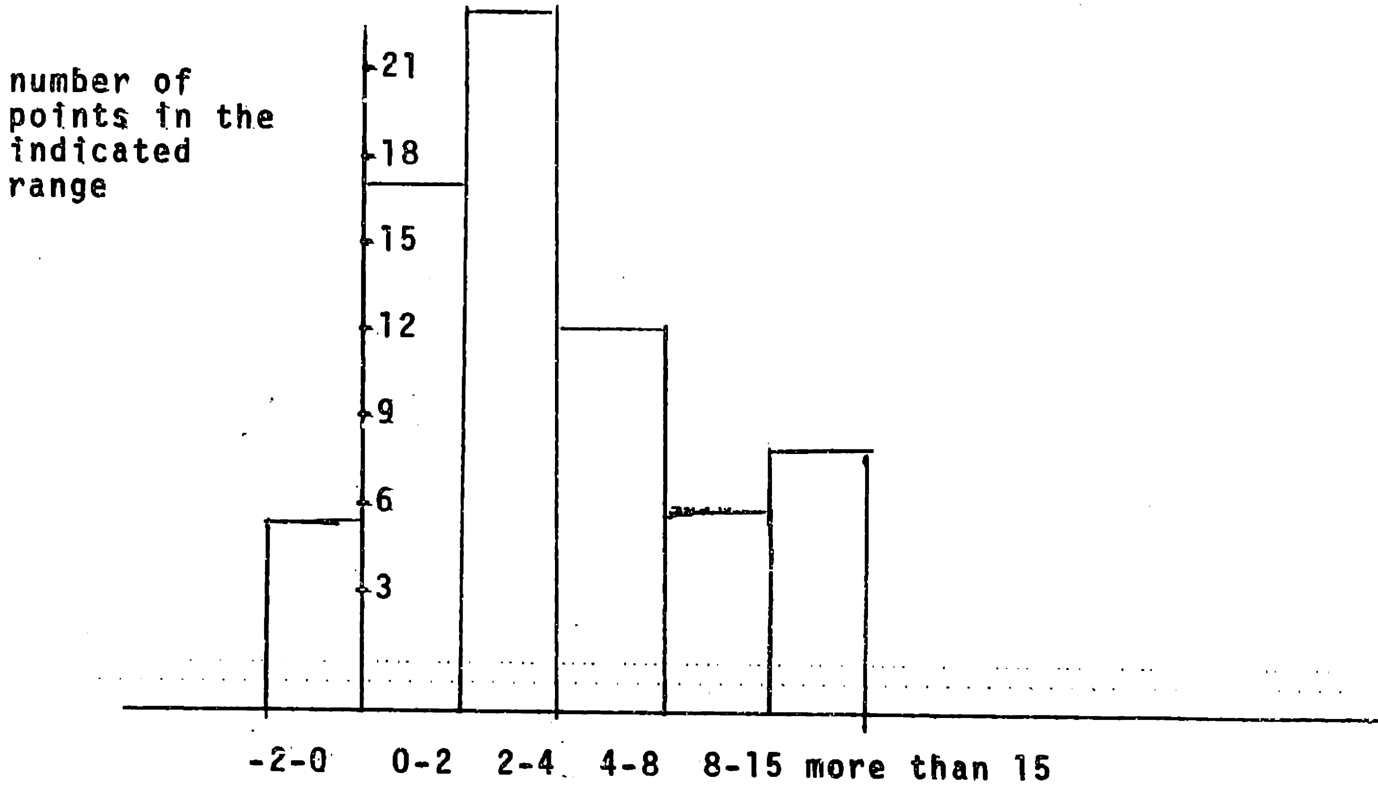
	Part type 1 family 1	Part type 2 family 1		Part type 2 family 2		
	2000.	2000.	and	2000.	\$/setup	
or	1200.	200.	and	1200.	\$/setup,	and

The parameters used in determining the forecast error took on the following values -

A=B=C=0.
or
A= .01 B=.02 C=1.3 and positive bias = .5

where the forecast error as a % of the true value of demand in any period is defined as $(A + B*t^C)$.

Figure 6.5 - A Summary of the Results Comparing
The Two-Stage Model with M.R.P.



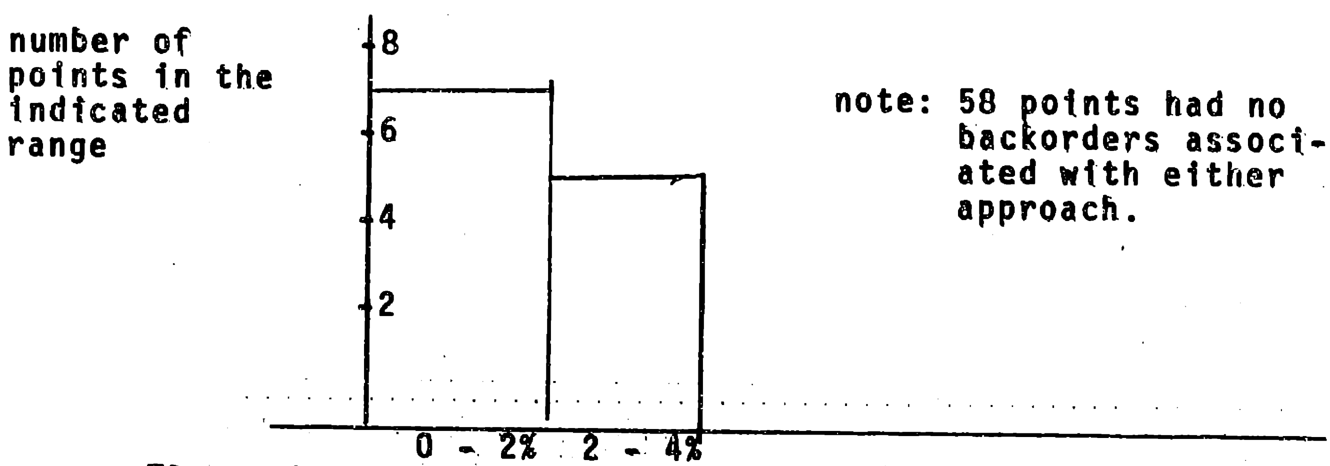
100 * [C(MRP) - C(TSH)]/AC

where:

C(MRP) represents the cost associated with the MRP run,

C(TSH) represents the cost associated with the two-stage hierarchical run, and

AC represents the average cost of the two runs.



The number of backorders associated with the MRP approach less the number associated with the hierarchical approach as a percentage of annual demand.

6.2 Comparison of the General Structure Two-Stage Hierarchical Algorithm with M.R.P.

For testing the validity of the model in situations in which the more general Frank-Wolfe based disaggregation scheme is required, we identified three facets of the model to test:

1. How the use of average product-part factors affects the comparison of the 2-stage methodology with MRP;
2. How using or not using the flexibility providing part inventory affects the comparison of the 2-stage methodology with MRP: and
3. How the total model compares with MRP in situations previously identified as strongly favoring one approach over the other.

In this section we will present the limited data and results from each of these tests.

The first test consisted of three data points all with identical product-part requirements as illustrated in Figure 6.6.

In these situations, three of the six product-part factors used in the aggregate model would probably be incorrect. Simultaneoulsy, part effective inventory is cleanly defined as actual part type inventory for the first period. The critical parameter in this test was part capacity. The levels at which we set all other parameters that were identical in all three points are illustrated in Figure 6.7.

Figure 6.6 - The Finished Product Family Requirements for Part Families -

	Part type 1 family 1	Part type 2 family 1	Part type 3 family 1
Finished Product Type 1 family 1	1	1	1
Finished Product Type 1 family 2	1	1	0
Finished Product Type 2 family 1	1	1	1
Finished Product Type 2 family 2	1	1	1
Finished Product Type 2 family 3	1	0	0

Figure 6.7 - Parameters of the First Three Points Used in the Comparison of the General Two-Stage Hierarchical Model with M.R.P.

The seasonality factors for each period and each product type were -

period	Type 1	Type 2
1	1.35	2.11
2	2.52	1.62
3	2.52	1.16
4	1.35	0.76
5	0.93	0.46
6	0.57	0.28
7	0.30	0.21
8	0.15	0.28
9	0.15	0.46
10	0.30	0.76
11	0.57	1.16
12	0.93	1.62
13	1.35	2.11

overtime availability for finished product assembly was set at 20% of regular time availability

the coefficients of variation were -

type 1	type 2
5.605	5.064

Finished Product capacity was set at 30%

Finished Product setup costs were set at the standard levels

The planning horizon was set at 13 periods - one year

The holding costs of finished products were -

Type 1	Type 2
$.30/unit period	$.40/unit period

The holding costs of Parts were set at

Type 1	Type 2	Type 3
$.1/unit period	$.08/unit period	$.08/unit period

The forecast error was set at zero.

The part setup costs were

Type 1	Type 2	Type 3
$1200./setup	$2500./setup	$1000./setup

The initial part inventory was set at 1.2 times that which would exactly match the requirements for period one, of a finished product schedule determined by a single-stage hierarchical run.

The three points were associated with the capacities illustrated in Figure 6.8.

	Part type 1	Part type 2	Part type 3
point 1	100%	100%	100%
point 2	20%	40%	25%
point 3	200%	200%	200%.

Figure 6.8 Part Type Capacity

The results of simulating these three situations and planning with both the M.R.P. approach and the two-stage hierarchical method are illustrated in Figure 6.9.

Point number	MRP results	Two-Stage results
point 1	total costs 216,883.0	total costs 149,815.0
	total backorders 0.	total backorders 1.1% of annual demand
point 2	total costs 390,715.0	total costs 160,184.0
	total backorders 0	total backorders 1.9% of annual demand
point 3	total costs 163,807.0	total costs 134,967.0
	total backorders 0	total backorders 1.15% of annual demand

Figure 6.9 The Results Associated with the First three points.

The differnece in costs in points one and two can be attributed primarily to the use of overtime in part fabrication. However, when part capacity was loose - 200%, point 3, the principal difference is due to part holding costs.

The results of small numbers of backorders and large cost savings indicate that with a small part safety stock to be used for errors in the definition of part-product factors, the two-stage approach could be superior to the M.R.P. appproach.

For the second test, determining if flexibility providing part inventory is useful, the family part requirements used were the full structure illustrated in Figure 5.2. The parameter we felt was key to vary was forecast error. The parameters held constant for this test are illustrated in Figure 6.10. The coefficients assoiciated with the forecast error took on the three sets of values illustrated in Figure 6.11.

point number	A	B	C	PPE[1]
point 4	0.	0.	0.	0.
point 5	.02	.01	1.3	.5
point 6	.02	.01	1.3	1.

Figure 6.11 The Coefficients Associated with the Forecast Error for points 4 through 6

1. PPE represents the probability that the error is positive.

Figure 6.10 - Parameters Associated with Points 4 through 6 Used in the Comparison of the General Two-Stage Hierarchical Model with M.R.P.

The seasonality factors for each period and each product type were -

period	Type 1	Type 2
1	0.43	0.83
2	0.61	1.25
3	0.89	1.55
4	1.21	1.67
5	1.21	1.55
6	1.65	1.25
7	1.82	0.83
8	1.65	0.71
9	1.21	0.62
10	0.89	0.55
11	0.61	0.62
12	0.43	0.71
13	0.36	0.83

the coefficients of variation were -

type 1	type 2
3.037	1.926

Finished Product Capacity was set at 50%, and the available overtime was set at 20% of the available regular time

Finished product setup costs were set at the standard levels

Part setup costs took on the following values -

Part type 1 family 1	Part type 2 family 1	Part type 3 family 1	family 2
1500.	1500.	800.	2000.

The planning horizon was set at 13 periods - one year

The holding costs of finished products were -

type 1	type 2
\$.30/unit period	\$.40/unit period

The holding costs of part types were -

type 1	type 2	type 3
\$.11/unit period	\$.08/unit period	\$.08/unit period

The initial part inventory was set at 1.3 times that which would exactly match the requirements for period one, of a finished product schedule determined by a single-stage hierarchical run.

The part capacity for all part types was set at 110%.

The results of the simulations associated with these three situations are illustrated in Figure 6.12.

Figure 6.12 The Results of Situations 4 - 6

Point 4	Point 5	Point 6
MRP	MRP	MRP
total costs	total costs	total costs
169,706	178,977	215,856
total backorders	total backorders	total backorders
0	1.3% annual demand	0
Two-Stage Hierarchical Production Planning with FPPI		
total costs	total costs	total costs
157,387	160,222	220,817
total backorders	total backorders	total backorders
0	.073% annual demand	0
Two-Stage Hierarchical Production Planning without FPPI		
total costs	total costs	total costs
157,223	173,990	242,095
total backorders	total backorders	total backorders
0	.2% annual demand	0

The difference in costs in point 6 are primarily finished product holding costs. This situation is due to the consideration of part availability and the gain in producing finished products earlier. However, due to the positive forecast error, the quantity of finished products planned for production is already high. The implication of this is that if it is known that forecast errors are always positive, M.R.P. might be a better approach than two-stage hierarchical planning. However, if it is known that forecast errors are always high there is indication that a new approach to forecasting would be beneficial.

The data indicates that with forecast error the flexibility providing part inventory has a positive effect on the two-stage approach. It is felt that with tighter part capacity the positive effect would be magnified.

For the final test on the general two-stage hierarchical model, the performance of the methodology was compared to M.R.P. in five situations. In all five of these situations:

1. The finished product part requirements were identical to the requirements in runs 4 through 6;
2. The seasonality of demand was identical to that used in runs 1 through 3;
3. Finished product capacity was set at 30%;
4. The holding costs and setup costs associated with finished products were the standard numbers;

The remainder of the parameters for each of the five points, and the associated results are given in Figure 6.13.

The difference in costs associated with point 7 can primarily be attributed to holding finished products, again resulting from the biased forecasts. Point 9 was designed to favor the M.R.P. approach-plenty of part capacity, high setup

Figure 6.13 - The Characteristics and Results Assoicated with Runs 7 through 11

Point 7
Characteristics:
Part Capacity = 100% for each part type
Forecast error coefficients -
A=.02, B=.01, C=1.3 PPE=1.

Part Setup Costs

type 1	type 2	type 3	type 3
family 1	family 1	family 1	family 2
$1200./setup	$2500./setup	$1000/setup	$350/setup

Results

	MRP	Hierarchical
total costs	185,813	187001
total backorders	0	0

Point 8
Characteristics:
Part Capacity

type 1	type 2	type 3
50%	100%	50%

Forecast error coefficients -
A=.01, B=.03, C=1.3 PPE=1.
Part Setup Costs

type 1	type 2	type 3	type 3
family 1	family 1	family 1	family 2
$1200/setup	$500/setup	$1000/setup	$2500/setup

Results

	MRP	Hierarchical
total costs	260,595	260,676
total backorders	0	0

Point 9
Characteristics:
Part Capacity = 200% for each part type
Forecast error coefficients -
A=.01, B=.03, C=1.3, PPE=1.
Part Setup Costs

type 1	type 2	type 3	type 3
family 1	family 1	family 1	family 2
$2500/setup	$2500/setup	$3000/setup	$2000/setup

Results

	MRP	Hierarchical
total costs	384,500	346,502
total backorders	0	0

Figure 6.13 continued

Point 10
Characteristics:
Part Capacity = 25% for all parts with 30% of regular time used in part fabrication available for overtime.
Forecast error coefficients -
A=.01, B=.03, C=1.3, PPE=1.

Part Setup Costs

type 1 family 1	type 2 family 1	type 3 family 1	type 3 family 2
$1000./setup	$500/setup	$1400/setup	$700/setup

Results

	MRP	Hierarchical
total costs	278,844	203,634
total backorders	0	0

Point 11
Characteristics:
Part Capacity = 25% for all parts with 30% of regular time used in part fabrication available for overtime.
Forecast error coefficients -
A=.01, B=.01, C=1.6 PPE=.8

Part Setup Costs

type 1 family 1	type 2 family 1	type 3 family 1	type 3 family 2
$1000/setup	$500/setup	$1000/setup	$700/setup

Results

	MRP	Hierarchical
total costs	189,565	133,250
total backorders	0	0

costs, and a bias on the forecasts. However, despite the design, the hierarchical approach outperformed MRP. Despite the high setup costs, capacity was not set at a high enough level for parts and resulted in extensive use of part overtime by the MRP approach. Points 10 and 11 were designed to be favorable for the hierarchical approach - part capacity was tight and part setup costs were not steep. However, in point 10 the full positive bias in forecast error remained, yet the hierarchical approach did substantially better than MRP in both of these simulations.

6.3 An Overview of the Results

In total, we compared the Hierarchical approach to the version of M.R.P. earlier described in 81 distinct situations. An elegant and real two-stage production process, a pencil company,was found with all of the characteristics needted to test the two-stage hierarchical model. The number of comparisons performed was limited by computational time and expense.

The first seventy cases tested were done with part requirements permiting the knapsack disaggregation scheme to be used. The last eleven runs were done on data with full part requirements in situations identified as strongly favoring one approach over the other.

The parameters we varied, and the values those parameters took on included:

1. Finished Product Capacity - 20 to 100 %,
2. Part Capacity - 20 to 250 %,

3. Part Holding Costs in relation to finished product holding costs - 33 to 75%,

4. Forecast Errors and biases from no error and no bias to a growing error, starting at 9% in the first period with full positive bias,

5. The Part Setup Costs from - $200 to $3000,

6. The Seasonality of demand from steep to moderate and,

7. The Initial Part Inventory from tight to loose levels.

We felt the ranges chosen for the variables encompassed a fairly broad and realistic spectrum. For further justification for the ranges chosen refer to earlier sections of this chapter.

The results of the comparisons imply that the hierarchical heuristic herein developed can be a better planning tool than M.R.P. in most situations where setup costs are secondary and the parts are internally fabricated. In almost every case tested the hierarchical approach outperformed the M.R.P. approach in terms of costs and backorders. The only exceptions were when part capacity was extremely high - 250%, part setup costs were substantial, and forecast errors were small; or when a strong positive bias in the forecasts existed and part capacity was relatively loose.

Basically, the hierarchical approach is strongly favored for situations in which part capacity is limited (less than 250%) and forecast error does not have an unnaturally high positive bias.

CHAPTER 7 - CONCLUSIONS

7.1 Basic Conclusions

The value of the method developed in the thesis, in the writer's opinion, is that the approach is practical and the solutions are robust for the kinds of problems they address. In the development of the hierarchical model; existing disaggregation schemes for single-stage hierarchical production planning are improved, a definition of effective component demand is developed, and a general disaggregation methodology is provided. Each of these aspects of the model are contributions to the area of hierarchical production planning.

The thesis addresses the question as to whether or not the two-stage hierarchical model outperforms M.R.P. The question is answered favorably for the two-stage hierarchical model in almost all situations tested. The results indicate that if a corporation fabricates its own parts or components and then assembles the finished products, if part capacity is constraining, or part holding costs and overtime costs are principal while part setup costs are secondary in nature, a two-stage hierarchical approach will result in superior production planning to an M.R.P. system. The concept of setup costs being secondary in nature, is in line with the conditions defined for successful implementation of the single-stage hierarchical approach [18].

7.2 Areas for Future Research

The value of the work is not only that it develops a new method for production planning and that method works, but also that areas for future research are exposed. In future research, the structure of products and parts can be enlarged, and complicated; the number of stages can be extended; feedback algorithms can be incorporated; the impact of part safety stocks can be examined; the approach can be used to generate a master schedule for an M.R.P. system; the impact of a changing component cost structure can be simulated; and the needed interfaces with strategic planning can be developed. All of these steps would enlarge the sphere of tactical questions to which the work can be applied and contribute value to methods used today.

Feedback algorithms would enlarge the range of situations for which a hierarchical approach is worthwhile. The feedback could be related to setup costs, misestimated parameters, or managerial insights which involve many separate areas. The feedback algorithms and use are not limited to multi-stage situations, but also could add to the single-stage approach, particularly regarding the need for setup costs to be secondary in design.

The knapsack disaggregation algorithm was proved to be applicable whenever all products in a product type have identical part requirement profiles. However, the disaggregation may be appropriate when small differences in profiles exist.

This needs to be tested to determine the relationship between the differences in profiles and the necessary levels of part safety stocks. The use of such a disaggregation approach simplifies the required disaggregation process immensely, and the required safety stocks may be a small price to pay for the extra simplification.

Some sort of combination approach which uses the two-stage hierarchical algorithm to determine the aggregate finished product levels and an M.R.P. type of part scheduling spinoff might be developed. This algorithm could benefit from the best of both approaches. The master schedule would be determined in a manner so that part availability, part holding costs and part overtime costs would have an impact on the schedule; simultaneously when disaggregating finished products to determine a master schedule parts would not play a role, but while parts were being scheduled the part setup costs would be appropriately considered. Alternatively the combination approach could schedule those parts internally fabricated and the master schedule could be determined using the hierarcical approach, while the parts purchased could be scheduled in an M.R.P. fashion.

Further simulations comparing the hierarcical approach to M.R.P. may be done in an attempt to improve the hierarchical algorithm and further identify situations for which one algorithm is better than the other. In particular, the approach might be tested with more complex finished product part requirement structures or with a greater number of parts sharing capacity than examined in this thesis.

Alternatively, one might like to compare other M.R.P. setups with the hierarchical approach. It is important to realize that what we defined and simulated as M.R.P. is not all of M.R.P. but rather an example we felt favorably reflected the M.R.P. approach.

The present research regards costs as constant, cost components may be changing. The hierarchical approach would allow this changing structure of part costs to be incorporated in the planning process, while M.R.P. does not allow the impact of changing costs to be reflected in any part of the planning process. The hierarchical approach appears to be better able to interface with the strategic planning associated with changing costs. This hypothesis needs to be tested and interface developed.

Overall, there remains substantial research to be done, in the tactical stages of production planning. Simultaneously, the potential utilization of the hierarchical approach is promising as evidenced by data presented in this thesis.

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Appendix 1 The Use of Average Product-Part Factors

This appendix is based on the following claim:

Given:

1. Product types composed of items with identical seasonality
2. No forecast error
3. No initial inventory of parts or finished products
4. No lead time associated with the fabrication of parts, &
5. No aggregately planned backorders.

There will always be a feasible disagregation, if the product-part factors used in the aggregate schedule are defined as follows:

$$\hat{F}(I,J) = \frac{\sum_{i \in I} \sum_{j \in J} A(i) * f^{*}(I,i,j)}{\sum_{i \in I} A(i)} \qquad (1)$$

where:

$A(i)$ represents the annual demand of family i, and

$f^{*}(I,i,j)$ represent the requirement of part j for for finished product family i in type I.

Proof:

1. Let us call:

$$f^{*}(I,i,J) = \sum_{j \in J} f^{*}(I,i,j) \qquad \text{for all } J$$

2. The total planned production of part type J for the first t periods is:

(3) $\sum_I \sum_t \hat{F}(I,J) * X(I,t)$

3. For disaggregation to be feasible for the first t periods -

(4) $\sum_t \sum_I \sum_{i \in I} f^*(I,i,J) * d(i,t)$

of part type J must be available and disaggregated properly.

4. To show:

(5) $\sum_I \sum_t \sum_{i \in I} f^*(I,i,J) * d(i,t) \leq \sum_I \sum_t \hat{F}(I,J) * X(I,t)$

We will show this separately for each finished product type which will imply that it is true in the aggregate.

5. for each i, I

(6) $$\frac{\sum_t d(i,t)}{\sum_t \sum_{i \in I} d(i,t)} = \frac{A(i)}{\Sigma A(i,t)}$$

by the original assumption of identical seasonality for all items in a given product type.

6. This implies

(7) $$\frac{\sum_t \sum_{i \in I} f^*(I,i,J) * d(i,t)}{\sum_t \sum_{i \in I} d(i,t)} = \hat{F}(I,J)$$

7. Multiplying both sides of this equation by $\sum_t \sum_{j \in I} d(i,t)$

(8) $\sum_t \sum_{i \in I} f^*(I,i,J) * d(i,t) = \hat{F}(I,J) * \sum_t \sum_{i \in I} d(i,t)$

8. With no aggregately planned backorders,

(9) $\sum_t \sum_{i \in I} d(i,t)$ must be $\leq \sum_t X(I,t)$.

9. This implies -

$$(10) \quad \sum_{t}\sum_{i\epsilon I} f^*(I,i,J) * d(i,t) \leq \sum_{t}\hat{F}(I,J)*X(I,t) \text{ for all } J.$$

10. We must now show that there is a feasible method of disaggregating the part types so that a sufficient quantity of each part in that part type is available.

11. Equation (10) is equivalent to -

$$(11) \quad \sum_{t}\sum_{i\epsilon I}\sum_{j\epsilon J} f^*(I,i,j)*d(i,t) \leq \sum_{t}\hat{F}(I,J)*X(I,t) \text{ for all } J.$$

12. This implies that the total on the right can be broken up into k pieces, where k is the number of parts in part type J, such that there is a sufficient quantity of each part -

at least $\sum_{t}\sum_{i\epsilon I} f^*(I,i,j)*d(I,t)$ of part j can be produced in the first t periods, for each j.

Q.E.D.

Appendix 2 - Characteristics of the Objective Function used When Disaggregating

The objective function used when disaggregating, i.e. minimizing setup costs, has the following format:

$$(1)\quad \text{Minimize} \sum_{t}\sum_{I}\sum_{i\varepsilon I} [s(i)*\ d(i)/Q(i,t)] + \sum_{J}\sum_{j\varepsilon J} [\hat{s}(j)*\hat{d}(j)/\hat{Q}(j)]$$

where: I represents the finished product types,
i represents the products in product type I,

d(i) represents the effective demand for family i, associated with the calculated planning horizon,
s(i) represents the setup cost associated with family i, and
Q(i,t) represents the quantity to be produced of family i in period t,
J represents the part types,
j represents the parts in part type J,
$\hat{s}(j)$ represents the setup cost associated with part j,
$\hat{d}(j)$ represents the effective demand for part j, and
$\hat{Q}(j)$ represents the quantity of part j to be fabricated in the immediate period.

Originally when working with this objective function, we hoped it would be quasiconcave - so that when doing the Frank-Wolfe procedure it would be unnecessary to search between solutions.

Proposition 1: Although, equation 1 is a sum of quasiconcave functions, it is not quasiconcave.

Proof:

1. Each term s(i)*d(i)/Q(i,t) or $\hat{s}(j)*\hat{d}(j)/\hat{Q}(j)$ as

Definition 1 - "A numerical function Θ defined on a set $\Gamma \subset R^n$ is said to be quasiconcate at $\bar{X} \varepsilon \Gamma$ (w.r.t. Γ) if for each $x \varepsilon \Gamma$ such that $\Theta(x) \geq \Theta(\bar{X})$, the function Θ assumes a value no smaller than $\Theta(\bar{X})$ on each point on the intersection of the closed line segment $[\bar{X};x]$ and Γ" [2].

Let $Q(1), Q(2) \in \Gamma \in R^n$

$$f(Q(1)) = \alpha(1)$$

$$f(Q(2)) = \alpha(2)$$

$$Q^* = \lambda Q(1) + \hat{\lambda} Q(2) \quad \text{where} \quad \hat{\lambda} = 1-\lambda \text{ , } 0 \le \lambda \le 1$$

$$f(Q^*) = d(i)*s(i)/[\lambda Q(1) + \hat{\lambda} Q(2)]$$

but $Q^* \le$ maximum $(Q(1),Q(2))$, Hence,

$$d(i)*s(i)/Q^* \ge d(i)*s(i)/[\text{maximum}(Q(1),Q(2)]$$

this is equivalent to -

$$f(Q^*) \ge d(i)*s(i)/[\text{maximum}(Q(1),Q(2)] \ge \min(\alpha 1,\alpha 2)$$

therefore f(Q) is quasiconcave.

2. The sum of these quasiconcave functions is not quasiconcave -

proof: an example

$d(1)=d(2)$, $s(1)=s(2)$, $i=2, j=0, I=1$

$X_1 = (51,1)$, $X_2 = (1,51)$

$$\Theta(51,1) = d(1)*s(1)/51 + d(1)*s(1)/1 = d(1)*s(1) *(1+1/51)$$

$$\Theta(1,51) = d(1)*s(1)/1 + d(1)*s(1)/51 = d(1)*s(1) *(1+1/51)$$

$$Q(\tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \Theta(26,26) = d(1)*s(1)/26 + d(1)*s(1)/26 = 1/13 *(s(1)*d(1))$$

If $d(1)*s(1) > 0$

$\Theta(\tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) < \Theta(X_1)$, and $\Theta(X_2)$

Therefore the objective function, equation 1 is not quasiconcave and minimum $\Theta(x)$ does not necessarily solve at an extreme point. Search procedures between Frank-Wolfe solutions to the objective function are necessary.

Proposition 2: The function we are minimizing, $\Theta(x)$ is convex

Proof:

1. $\Theta'(Q(i,t)) = \dfrac{-s(i)*d(i)}{Q(i,t)^2}$ for all i,t or j

2. $$\Theta''(Q) = \begin{bmatrix} \dfrac{2d(1)s(1)}{Q(1,1)^3} & & 0 \\ & \searrow & \\ 0 & & \dfrac{2s(j)d(j)}{Q(j)^3} \end{bmatrix}$$

3. since each s,d,q are positive, $\Theta(Q)$'s second derivative is positive definite and Θ is concave.

Q.E.D.

The Search Between Solutions

The concavity of the objective function is used when searching between points generated by the Frank-Wolfe algorithm for a minimum. Given two points, and because Θ is concave, the following will be used -

We are seeking the minimum value of Θ along the line segment between Q_1 and Q_2:

$$\alpha Q_1 + (1-\alpha)Q_2 = Q_2 + \alpha(Q_1 - Q_2)$$

At the minimum the derivative of Θ with respect to α will be zero. Or -

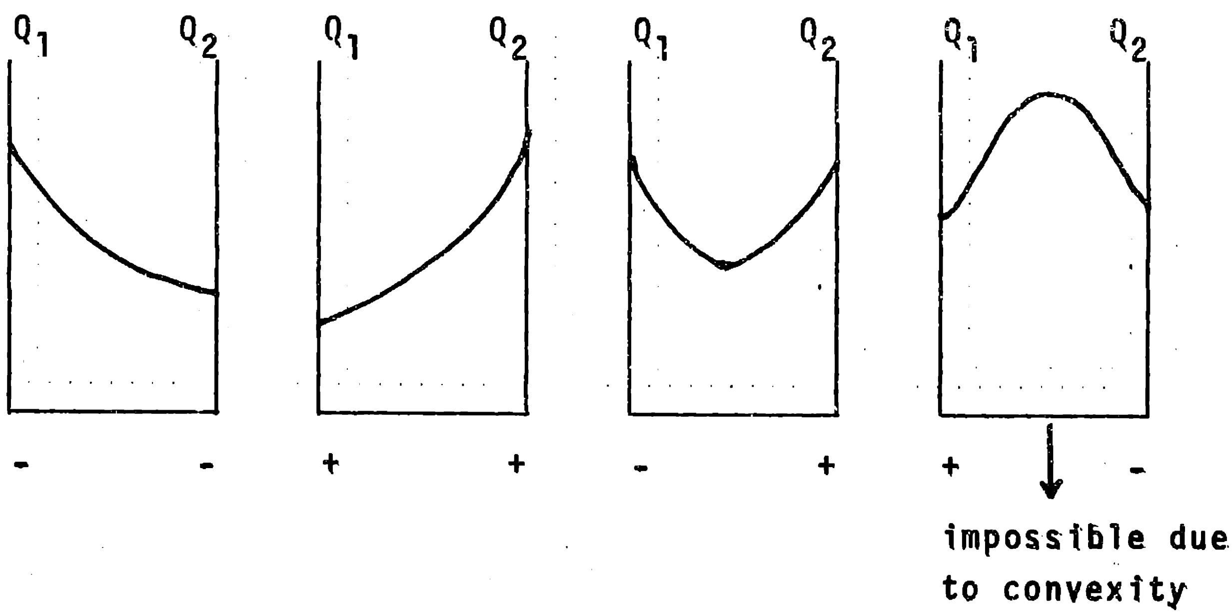
$$\frac{\partial\Theta\ (\alpha Q_1 + (1-\alpha)Q_2)}{\partial\alpha} = 0$$ is the minimum along the line of search.

$$\frac{\partial\Theta\ [Q_2 + \alpha(Q_1 - Q_2)]}{\partial\alpha}$$

$$= \frac{\partial}{\partial\alpha}\ \Sigma\ \left(d(i)*s(i)/(Q_2^i + \alpha(Q_1^i - Q_2^i)\right)$$

$$= \Sigma\ [s(i)d(i)(Q_1^i - Q_2^i)]/[Q_2^i + \alpha(Q_1^i - Q_2^i)^2] = 0, \quad \alpha\varepsilon[0,1].$$

Possible signs of the derivative at Q_1 and Q_2 are as follows:



The search algorithm procees as follows:

1. If $\partial(\alpha)$ is negative at both values of Q, Q_1 and Q_2, Q_2 is the minimum.
2. If $\partial(\alpha)$ is positive at both values of Q, the minimum is at Q_1.
3. Otherwise a search procedure is required. This is done by computing the derivative at $(Q_1+Q_2)/2$ and determining which half of the line segment that the minimum is on. This is continued until the derivative is less than epsilon in absolute value.