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The Landis conjecture on exponential decay

A. Logunov[1] · E. Malinnikova[2] · N. Nadirashvili[3] · F. Nazarov[4]

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Abstract
Consider a solution u to $\Delta u + Vu = 0$ on $\mathbb{R}^2$, where V is real-valued, measurable and $|V| \leq 1$. If $|u(x)| \leq \exp(-C|x|\log^{1/2}|x|)$, $|x| > 2$, where C is a sufficiently large absolute constant, then $u \equiv 0$.

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1 The main result

Let u be a solution to

$$\Delta u + Vu = 0 \tag{1}$$

in $\mathbb{R}^n$, where V is a measurable function with $|V| \leq 1$ in the whole space. According to [13, 14], in the late 1960s Landis conjectured that if

$$|u(x)| \leq \exp(-C|x|),$$

where $C > 0$ is a sufficiently large constant, then $u \equiv 0$. The weaker statement, which was also conjectured by Landis according to [13], states that if $|u(x)|$ tends to 0 faster

✉ A. Logunov
log239@yandex.ru

E. Malinnikova
eugeniam@stanford.edu

N. Nadirashvili
nikolay.nadirashvili@univ-amu.fr

F. Nazarov
nazarov@math.kent.edu

1 Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

2 Department of Mathematics, Stanford University, Stanford, CA, USA

3 CNRS, Institut de Mathématique de Marseille, Marseille, France

4 Department of Mathematics, Kent State University, Kent, OH, USA

than exponentially at ∞, i.e.,

$$|u(x)| \leq \exp(-|x|^{1+\varepsilon}), \varepsilon > 0,$$

then $u \equiv 0$.

There are two versions of Landis' conjectures: real and complex. Meshkov [18] constructed a counter-example to the complex version of Landis' conjecture. He showed that there is a complex-valued potential V with $|V| \leq 1$ and a non-zero solution u to (1) on $\mathbb{R}^2$ such that $|u(x)| \leq \exp(-c|x|^{4/3})$. Meshkov also showed (in any dimension n) that if

$$\sup_{\mathbb{R}^n} |u(x)| e^{-\tau|x|^{4/3}} < \infty \text{ for all } \tau > 0,$$

then $u \equiv 0$. The question whether the Landis conjecture is true for real-valued V is open. The main result of this article confirms the weak version of the Landis conjecture in dimension two.

Theorem 1.1 *Suppose that* $\Delta u + Vu = 0$ *on* $\mathbb{R}^2$, *where* u *and* V *are real-valued and* $|V| \leq 1$. *If* $|u(x)| \leq \exp(-C|x|\log^{1/2}|x|)$, $|x| > 2$, *where* C *is a sufficiently large absolute constant*, *then* $u \equiv 0$.

A similar striking difference between the decay estimates for real and complex solutions has also been observed in [13], where a closely related equation $\Delta u + W \cdot \nabla u = 0$ with a bounded vector field $W : \mathbb{R}^2 \to \mathbb{R}^2$ was studied.

There is a simple example of a solution to (1) with bounded V that decays exponentially. Define $u = e^{-|x|}$ in $\{|x| > 1\}$ and extend it to a C^2 smooth positive function on the plane. Then $|\Delta u| \leq C|u|$ and by taking $u(\frac{1}{\sqrt{C}}\cdot)$ in place of u one can make $|V| \leq 1$ in this example.

The assumption that u is real-valued is redundant because in the case of real-valued V the real and imaginary parts of u also satisfy (1). But in the proof we will use that u is real-valued. The proof of Theorem 1.1 combines the technique of quasiconformal mappings with two tricks. The tricks involve nodal sets (zero sets) of u and holes that are made in nodal domains (connected components of the complement of the zero set). We describe the idea in Sect. 2. Some two-dimensional tools are used in the proof and the Landis conjecture in higher dimensions is still open.

Our second result is a local version of Landis' conjecture.

Theorem 1.2 *Let* u *be a real solution to* $\Delta u + Vu = 0$ *in* $B(0, 2R) \subset \mathbb{R}^2$, *where* V *is real-valued and* $|V| \leq 1$. *Suppose that* $|u(0)| = \sup_{B(0,2R)} |u| = 1$. *Then for any* x_0 *with* $|x_0| = R/2 > 2$, *we have*

$$\sup_{B(x_0,1)} |u| \geq \exp(-CR\log^{3/2} R)$$

with some absolute constant $C > 0$.

The previous best known bound $\sup_{B(x_0,1)} |u| \geq \exp(-CR^{4/3}\log R)$, was obtained in any dimension by Bourgain and Kenig [4] in their proof of Anderson localization for the Bernoulli model, see also [12].

Theorem 1.2 follows from the main local Theorem 2.2, where we don't assume that $|u(0)| = \sup_{B(0,2R)} |u| = 1$, and prove a version of the three balls inequality.

Landis' conjecture was a subject to an extensive study. Under additional assumptions on V, some versions of Landis' conjecture are known, see [3, 6, 7, 9, 12, 13, 15, 20] and references therein. A related problem in a cylinder was studied in [11].

Kévin Le Bal'ch and Diego A. Souza told us recently that they were able to extend the results of this article to the case of variable lower-order coefficients (see [17]).

Notation By $c, C, C', \ldots > 0$ we denote various constants. Typically small constants are denoted by small letters and we use capital letters for large constants. If a constant C depends on a domain (or some other parameter), we say it. Sometimes we state theorems without reminding that the functions are assumed to be real-valued and u is a solution to (1) on $\mathbb{R}^2$. A ball with center at x of radius r is denoted by $B(x,r)$ and the two-dimensional Lebesgue measure is denoted by m_2.

2 Strategy of the proof and local versions

The proof consists of three acts. First, we will explain the main ideas of each of them.

Description of act I We will use the following well-known fact about nodal sets, which is proved in the Appendix (Lemma A.14) for reader's convenience. There is an absolute constant $r_0 > 0$ such that if u is a solution to $\Delta u + Vu = 0$ in a neighborhood of a closed ball $\overline{B(z_0, r)}$ with $|V| \leq 1$, $u(z_0) = 0$ and $0 < r < r_0$, then the circle $C(z_0, r) = \{z : |z - z_0| = r\}$ is intersecting the zero set of u.

It is also true that the singular set

$$S = \{x : u(x) = 0 \text{ and } \nabla u(x) = 0\}$$

consists of isolated points and the nodal set

$$F_0 = \{x : u(x) = 0\}$$

is a union of smooth curves, see [5]. However the proof will not use it, but this structural result about nodal sets makes it easier to think about them.

Now, assume that u is a solution to (1) in $B(0, R)$, $R > 1$. Take $\varepsilon > 0$ (a small parameter to be chosen later) and add finitely many $C\varepsilon$ – separated closed disks of radius ε to F_0 so that the distance from each disk to F_0 is $\geq C\varepsilon$ and

$$F_0 \cup \text{ union of the disks } \cup \{z : |z| \geq R\}$$

is a $3C\varepsilon$ – net on the plane (assume $C > 2$). Let us denote by F_1 the union of the closed disks, see Fig. 1.

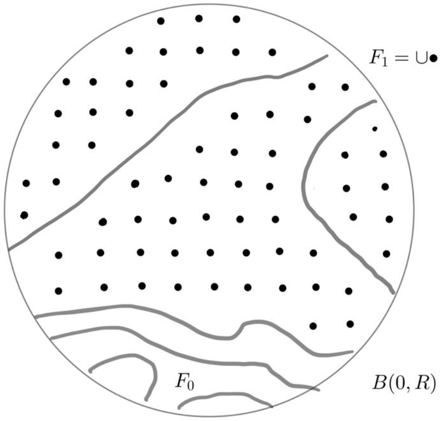


Fig. 1 Puncturing nodal domains

It can be shown that

$$\Omega = \{z : |z| < R, z \notin F_0 \cup F_1\}$$

is an open (possibly disconnected) set with the Poincare constant $\leq C'\varepsilon^2$, i.e., for every $u \in W_0^{1,2}(\Omega)$, we have

$$\int_\Omega u^2 \leq C'\varepsilon^2 \int_\Omega |\nabla u|^2.$$

It allows one to construct a function φ in $B(0, R)$ such that

- $\Delta\varphi + V\varphi = 0$ in Ω,
- $\varphi - 1 \in W_0^{1,2}(\Omega)$,
- $\|\varphi - 1\|_\infty \leq C''\varepsilon^2$.

The details are given in Sect. 3.

Description of act II Consider $f = \frac{u}{\varphi}$. Then f satisfies

$$\operatorname{div}(\varphi^2\nabla f) = 0$$

in Ω. The set Ω is usually not connected and the functions φ and f may not be smooth across F_0. However due to the fact that F_0 is the zero set of u it appears (after some work) that the equation $\operatorname{div}(\varphi^2\nabla f) = 0$ holds through F_0 in the whole $B(0, R) \setminus F_1$.

Here the theory of quasiconformal mappings joins the game. After noticing that $f \in W_{loc}^{1,2}$, we may use the Stoilow factorization theorem to make a $K-$ quasiconformal change of variables g mapping 0 to 0 and $B(0, R)$ onto $B(0, R)$ such that

$$f = h \circ g$$

where h is a harmonic function in $B(0, R) \setminus g(F_1)$. Moreover, K is very close to 1 when $\|\varphi - 1\|_\infty$ is small:

$$K \le \frac{1 + \left\| \frac{1-\varphi^2}{1+\varphi^2} \right\|_\infty}{1 - \left\| \frac{1-\varphi^2}{1+\varphi^2} \right\|_\infty} \le 1 + C\varepsilon^2.$$

Mori's theorem tells us how much the distances are distorted depending on K:

$$\frac{1}{16} \left| \frac{z_1 - z_2}{R} \right|^K \le \frac{|g(z_1) - g(z_2)|}{R} \le 16 \left| \frac{z_1 - z_2}{R} \right|^{1/K}. \tag{2}$$

We choose

$$\varepsilon \sim \frac{1}{\sqrt{\log R}}$$

so that the distortion on scales from $\frac{1}{R}$ to R is bounded and, moreover, the images of the disks in F_1 have size comparable to ε.

Then we get a harmonic function h in $B(0, R) \setminus g(F_1)$, where $g(F_1)$ is the union of sets of diameter $\sim \varepsilon$ and each set (the image of a single disk) is surrounded by an annulus of width $\sim C\varepsilon$ in which h does not change sign.

Description of act III By rescaling we get the following question: Let h be harmonic in a punctured domain $B(0, R') \setminus \cup_j D_j$ where $R' \sim \frac{R}{\varepsilon} \sim R\sqrt{\log R}$ and D_j are 1000–separated unit disks. Assume also that h does not change sign in $5D_j \setminus D_j$. What can be said about the decay of $|h|$?

Theorem 2.1 *Under the above assumptions, we have*

$$\sup_{B(0,R')\setminus \cup_j 3D_j} |h| \le \exp(CR') \sup_{\{z: R'/8 < |z| < R'\} \setminus \cup_j 3D_j} |h| \quad \text{for } R' > 2000$$

with some absolute constant $C > 0$.

Theorem 2.1 is an immediate consequence of a more general Theorem 5.3. The outcome is that $|u|$ cannot decay faster than $\exp(-CR\sqrt{\log R})$. A different proof of the estimate for harmonic functions in a punctured domain (with a slightly worse bound) is given in the Appendix. The second proof works in higher dimensions and uses the Carleman inequality with log linear weight.

Local versions Local versions of Theorem 1.1 (on the two dimensional plane) are also true. Here is the main local Theorem 2.2.

Theorem 2.2 *If u is a solution to $\Delta u + Vu = 0$ in $B(0, R)$, $R > 2$, V is real-valued, $|V| \le 1$, and*

$$\frac{\sup_{B(0,R)} |u|}{\sup_{B(0,R/2)} |u|} \le e^N,$$

then

$$\sup_{B(0,r)} |u| \geq (r/R)^{C(R\log^{1/2} R+N)} \sup_{B(0,R)} |u| \tag{3}$$

for any $r < R/4$, where C is an absolute positive constant.

Theorem 2.2 implies Theorems 1.1 and 1.2. In order to deduce Theorem 1.1, we may assume that $|u|$ attains its global maximum at some point on the plane, otherwise $|u|$ does not tend to 0 near infinity. Let

$$|u(z_{max})| = \max_{\mathbb{R}^2} |u| = 1.$$

Then for any $R > 6|z_{max}|$ and any x with $|x| = R/3$, we have

$$\sup_{B(x,R)} |u| = \sup_{B(x,R/2)} |u| = 1$$

and if additionally $R > 2$, then by Theorem 2.2 applied to $u(\cdot + x)$, we have

$$\sup_{B(x,R/4)} |u| \geq e^{-CR\log^{1/2} R}$$

and therefore

$$\sup_{|z|>R/12} |u| \geq e^{-CR\log^{1/2} R}.$$

In order to deduce Theorem 1.2 note that

$$\sup_{B(x,R/2)} |u| = \sup_{B(x,R)} |u| = 1$$

for any x with $|x| = R/2$ because $|u(0)| = \max_{B(0,2R)} |u|$. Applying Theorem 2.2 to $u(\cdot + x)$ we get

$$\sup_{B(x,1)} |u| \geq R^{-CR\log^{1/2} R} = e^{-CR\log^{3/2} R}.$$

Corollary 2.3 *Let $A > 4$. If u is a solution to $\Delta u + Vu = 0$ in $B(0,1)$, V is real-valued, $|V| \leq A$, and*

$$\frac{\sup_{B(0,1)} |u|}{\sup_{B(0,1/2)} |u|} \leq \exp(N),$$

then

$$\sup_{B(0,r)} |u| \geq r^{C(\sqrt{A\log A}+N)} \sup_{B(0,1)} |u| \quad \text{for } r \leq 1/4, \tag{4}$$

where C is an absolute positive constant.

For the proof, consider $u(\frac{1}{\sqrt{A}}\cdot)$ in place of u. We obtain a solution to $\Delta u + Vu = 0$ in $B(0, \sqrt{A})$ with $|V| \leq 1$ and

$$\frac{\sup_{B(0,\sqrt{A})} |u|}{\sup_{B(0,\sqrt{A}/2)} |u|} \leq e^N$$

and we can apply Theorem 2.2 to the new u and $R = \sqrt{A}$.

Remark 2.4 Inequality (4) implies that the vanishing order of u at 0 is bounded by $C(\sqrt{A \log A} + N)$. This question was previously studied in [3, 15, 20].

On any smooth two dimensional Riemannian manifold (M, g) every equation $\Delta_g u + Vu = 0$ can be simplified in local isothermal coordinates to $\Delta u + V'u = 0$ (with ordinary Euclidean Laplacian Δ). Corollary 2.3 gives information on the distribution of solutions to Schrodinger equations on compact manifolds of dimension 2.

Corollary 2.5 *Let (M, g) be a smooth closed (compact and without boundary) Riemannian manifold of dimension 2. Then for any function u satisfying $\Delta_g u + Vu = 0$ on M with $|V| \leq \lambda$, $\lambda > 2$, we have*

$$\sup_{B_r} |u| \geq r^{C\sqrt{\lambda \log \lambda}} \sup_M |u|$$

for any ball B_r of radius $r < 1/2$. The constant C depends on the manifold.

This result follows from Corollary 2.3 by an iteration argument similar to that in [8] on page 162, using isothermal coordinates. For the reader's convenience, we present the proof in Appendix A.6.

Remark 2.6 A slightly better bound was obtained in [8] by Donnelly and Fefferman for Laplace eigenfunctions on closed Riemannian manifolds of any dimension. If $\Delta_g u + \lambda u = 0$ on (M, g), then

$$\sup_{B_r} |u| \geq c r^{C\sqrt{\lambda}} \sup_M |u|, \quad r \leq \frac{1}{2}.$$

So the vanishing order at any point is at most $C\sqrt{\lambda}$.

In Act I and Act II we will reduce (with a logarithmic loss) the main local Theorem 2.2 to a general Theorem 5.3, which is a local statement about two dimensional harmonic functions.

3 Act I

3.1 Poincare constant for porous domains

Lemma 3.1 *Let F be a closed set in $B(0, R)$, $R > 1$, such that*

a) *For every* $z_0 \in F$, $r \in (0,1]$, *the circle* $C(z_0, r) = \{z : |z - z_0| = r\}$ *intersects* $F \cup \partial B(0, R)$.
b) $F \cup \partial B(0, R)$ *is* C*-dense in* $B(0, R)$, $C > 1$.

Then the Poincare constant of $\Omega = B(0, R) \setminus F$ *is bounded by some constant* $\widetilde{C}$ *that depends only on* C.

Proof Let $f \in C_0^\infty(\Omega)$. Extend f by zero outside Ω. First, we will show that if $z \in F \cup \partial B(0, R)$, then

$$\int_{B(z,3C)} |f|^2 \lesssim \int_{B(z,3C)} |\nabla f|^2.$$

Every circle $C_r = \partial B(z, r)$, $r \in (0, 1)$, has a zero of f, whence

$$\max_{C_r} |f| \le \int_{C_r} |\nabla f|$$

and

$$\int_{B(z,1)} |f|^2 = \int_0^1 \left(\int_{C_r} |f|^2 \right) dr \le \int_0^1 |C_r| \max_{C_r} |f|^2 dr \le \int_0^1 |C_r| \left(\int_{C_r} |\nabla f| \right)^2 dr \le$$

$$\le \int_0^1 |C_r|^2 \left(\int_{C_r} |\nabla f|^2 \right) dr \le (2\pi)^2 \int_0^1 \left(\int_{C_r} |\nabla f|^2 \right) dr = (2\pi)^2 \int_{B(z,1)} |\nabla f|^2.$$

We therefore can find $r \in (1/2, 1)$ such that

$$\int_{C_r} |f|^2 \le C_1 \int_{B(z,1)} |f|^2 \le C_2 \int_{B(z,1)} |\nabla f|^2.$$

Let Γ_ψ, $\psi \in [0, 2\pi)$, be a segment starting at the point

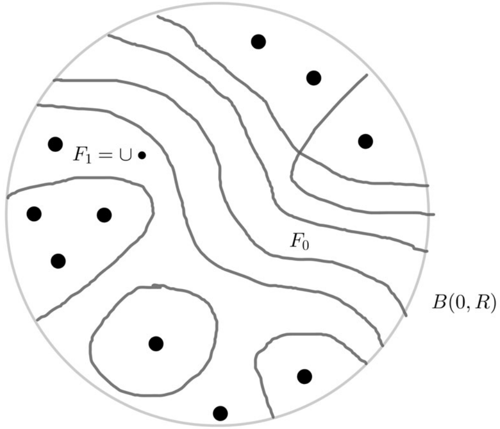
$$x_\psi := z + re^{i\psi}$$

and ending at the point $z + 3Ce^{i\psi}$. Note that

$$\max_{\Gamma_\psi} |f|^2 \le \left(|f(x_\psi)| + \int_{\Gamma_\psi} |\nabla f| \right)^2 \le 2|f(x_\psi)|^2 + 2 \left(\int_{\Gamma_\psi} |\nabla f| \right)^2 \le$$

$$\le 2|f(x_\psi)|^2 + 2|\Gamma_\psi| \int_{\Gamma_\psi} |\nabla f|^2 \le 2|f(x_\psi)|^2 + 6C \int_{\Gamma_\psi} |\nabla f|^2$$

Fig. 2 Puncturing nodal domains



and therefore

$$\int_{B(z,3C)\setminus B(z,1)} f^2 \leq (3C)^2 \int_0^{2\pi} \max_{\Gamma_\psi} |f|^2 d\psi \leq$$

$$\leq C_1(C)\left[\int_{C_r} |f|^2 + \int_0^{2\pi}\left(\int_{\Gamma_\psi} |\nabla f|^2\right) d\psi\right] \leq C_2(C)\int_{B(z,3C)} |\nabla f|^2.$$

Thus

$$\int_{B(z,3C)} |f|^2 \leq C_3(C)\int_{B(z,3C)} |\nabla f|^2.$$

We can choose a finite collection Z_* of points z in $F \cup \partial B(0, R)$ such that the balls $B(z, 3C)$ cover $B(0, R)$ and each point is covered a bounded number of times. Finally, we have

$$\int_{B(0,R)} f^2 \leq \sum_{z\in Z_*}\int_{B(z,3C)} |f|^2 \leq C_3(C)\sum_{z\in Z_*}\int_{B(z,3C)} |\nabla f|^2 \leq$$

$$\leq C_4(C)\int_{B(0,R)} |\nabla f|^2.$$

□

We start proving Theorem 2.2. Recall that $\Delta u + Vu = 0$ in the ball $B(0, R)$ (we may think that R is a large number) and F_0 is the zero set of u. We will use the fact that $u \in C^1(B(0, R))$, which is proved in the Appendix, see Fact A.5. Now, consider the following setting (see Figure 2):

Take $\varepsilon > 0$ (a small parameter to be chosen later). Choose finitely many $C\varepsilon$ – separated closed disks of radius ε, whose union will be denoted by F_1, so that the distance from each disk to F_0 and $\partial B(0,R)$ is $\geq C\varepsilon$ and

$$F_0 \cup F_1 \cup \partial B(0,R)$$

is a $3C\varepsilon$ – net in $B(0,R)$ (we assume $C>2$).

For instance, one can get F_1 by considering the maximal number of open non-intersecting disks of radius $(C+1)\varepsilon$ in $B(0,R)\setminus F_0$. The centers of the disks are $(2C+2)\varepsilon$ – separated. There is no point x in $B(0,R)\setminus F_0$ that is $(2C+2)\varepsilon$ far from the centers of the disks and from $F_0\cup\partial B(0,R)$, otherwise we could add one more disk of radius $(C+1)\varepsilon$ with center at this point. So we may choose the disks of radius $\varepsilon>0$ with the same centers, they will be $C\varepsilon$ – separated and $F_0\cup F_1\cup\partial B(0,R)$ will be a $2(C+1)\varepsilon$ – net.

Two points to avoid Now, let us remove from F_1 the disks that are $C\varepsilon$ close to 0 or to the point $z_{\max}\in\overline{B(0,R/2)}$ such that

$$|u(z_{\max})| = \sup_{B(0,R/2)} |u|.$$

The set $F_0\cup F_1\cup\partial B(0,R)$ will still be a $10C\varepsilon$ – net, but now all disks from F_1 are also $C\varepsilon$- separated from 0 and $z_{\max}$. The detail about avoiding those two points will be used only in the end of Act II.

Recall that F_0 has the property that for any $z_0\in F_0$, every circle $C(z_0,r)$ with $r<r_0$ intersects F_0 or ∂B_R. Taking $u(\varepsilon\cdot)$ in place of u (so the assumptions of Lemma 3.1 hold for $\varepsilon<r_0$) and applying Lemma 3.1 we arrive to the following conclusion.

Outcome The domain

$$\Omega = B(0,R)\setminus(F_0\cup F_1)$$

has Poincare constant $\leq C'\varepsilon^2$ and $B(0,R)\setminus F_1$ contains 0 and $z_{\max}$.

3.2 Solving $\Delta\varphi + V\varphi = 0$

The goal of this section is to construct an auxiliary solution to (1) in a domain with a small Poincare constant, so that the solution has boundary values 1 and is uniformly close to 1.

Lemma 3.2 *Let Ω be a bounded open set with the Poincare constant k^2. Let $V\in L^\infty(\Omega)$. Assume that*

$$k^2\|V\|_\infty \ll 1.$$

Then there exists $\varphi = 1+\tilde\varphi$ with

$$\tilde\varphi\in W_0^{1,2}(\Omega),\ \|\tilde\varphi\|_\infty \leq Ck^2\|V\|_\infty$$

such that φ is a weak solution to $\Delta\varphi + V\varphi = 0$ in Ω, where C is an absolute positive constant.

Proof We will use the following fact, which is proved in the Appendix, Lemma A.6 and Lemma A.10.

Fact *When the Poincare constant of* Ω *is* 1, $v \in L^\infty(\Omega)$, *there is a solution* φ *to* $\Delta\varphi = v$ *in* $W_0^{1,2}(\Omega)$ *with*

$$\|\varphi\|_\infty \leq C\|v\|_\infty$$

and

$$\|\varphi\|_{W_0^{1,2}} \leq C\|v\|_2.$$

Corollary (follows by rescaling) *If* Ω *has Poincare constant* k^2, *then we can find a solution* φ *to* $\Delta\varphi = v$ *with*

$$\|\varphi\|_\infty \leq Ck^2\|v\|_\infty$$

and

$$\|\varphi\|_{W_0^{1,2}} \leq C_1(k)\|v\|_2.$$

Now, let φ_1 solve $\Delta\varphi_1 = -V$ and for $n \geq 2$ let φ_n solve

$$\Delta\varphi_n = -V\varphi_{n-1}.$$

Note that this sequence is well defined since on each step the right-hand side is in L^∞. We have

$$\|\varphi_n\|_\infty \leq Ck^2\|V\|_\infty\|\varphi_{n-1}\|_\infty, \quad n \geq 2,$$

and $\|\varphi_1\|_\infty \leq Ck^2\|V\|_\infty$. We are assuming that $Ck^2\|V\|_\infty \leq 1/2$. Hence $\|\varphi_n\|_\infty \leq 2^{-n+1}Ck^2\|V\|_\infty$ and

$$\|\varphi_n\|_{W_0^{1,2}} \leq C_1(k)\|\varphi_{n-1}\|_2 \leq C_2(k)\|\varphi_{n-1}\|_\infty \leq C_3(k)2^{-n}.$$

Thus the series

$$\tilde{\varphi} = \varphi_1 + \varphi_2 + \cdots$$

converges both in L^∞ and in $W_0^{1,2}(\Omega)$ with

$$\|\tilde{\varphi}\|_\infty \leq C'k^2\|V\|_\infty.$$

Also for any $h \in W_0^{1,2}(\Omega)$, we have $\int \nabla\varphi_n \nabla h = \int V\varphi_{n-1}h$ for $n \geq 2$ and $\int \nabla\varphi_1 \nabla h = \int Vh$. Thus $\Delta\tilde{\varphi} = -V(1+\tilde{\varphi})$ and

$$\Delta(1+\tilde{\varphi}) + V(1+\tilde{\varphi}) = 0 \quad \text{in } \Omega$$

as required. $\square$

Outcome Since the Poincare constant of $\Omega = B(0,R) \setminus (F_0 \cup F_1)$ is $\leq \widetilde{C}\varepsilon^2$, using Lemma 3.2, we can find φ such that

- $\Delta\varphi + V\varphi = 0$ in Ω,
- $\varphi - 1 \in W_0^{1,2}(\Omega)$,
- $\|\varphi - 1\|_\infty \leq C'\varepsilon^2$.

4 Act II

4.1 Reduction to a divergence type equation in a domain with holes

Recall that u is a solution to $\Delta u + Vu = 0$ in $B(0,R)$ and F_0 is the zero set of u. Extend the function φ by 1 outside

$$\Omega = B(0,R) \setminus (F_0 \cup F_1).$$

Lemma 4.1 *The function* $\frac{u}{\varphi} \in W^{1,2}_{loc}(B(0,R))$ *and it is a solution to*

$$\operatorname{div}(\varphi^2 \nabla(\frac{u}{\varphi})) = 0$$

in $B(0,R) \setminus F_1$ *in the weak sense.*

Remark The lemma takes care of all "continuations through nodal lines" of u.

Proof First, we would like to notice that the extended functions $\frac{1}{\varphi}, \varphi \in W^{1,2}_{loc}(\mathbb{R}^2)$ and

$$\nabla \frac{1}{\varphi} = -\mathbb{1}_\Omega \frac{\nabla\varphi}{\varphi^2} \quad \text{and} \quad \nabla\varphi = \mathbb{1}_\Omega \nabla\varphi \tag{5}$$

in $\mathbb{R}^2$ in the sense of distributions:

$$\int_{\mathbb{R}^2} \frac{1}{\varphi}\nabla\xi = \int_\Omega \frac{\nabla\varphi}{\varphi^2}\xi \quad \text{and} \quad \int_{\mathbb{R}^2} \varphi\nabla\xi = -\int_\Omega \nabla\varphi\xi$$

for any $\xi \in C_0^\infty(\mathbb{R}^2)$. The formal check is performed in Fact A.12 in the Appendix.

Now, we would like to verify that $\frac{u}{\varphi} \in W^{1,2}_{loc}(B(0,R))$ and

$$\nabla\frac{u}{\varphi} = \frac{\varphi\nabla u}{\varphi^2} - \frac{u\nabla\varphi}{\varphi^2}\mathbb{1}_\Omega.$$

Fact 4.2 *Let* $u, v \in W^{1,2}_{loc}(B(0,R)) \cap L^\infty_{loc}(B(0,R))$. *Then* $uv \in W^{1,2}_{loc}(B(0,R))$ *and* $\nabla(uv) = u\nabla v + v\nabla u$.

Fact 4.2 is proved in the Appendix.

Recall that φ is extended by 1 outside Ω, $\frac{1}{\varphi} \in W^{1,2}_{loc}(\mathbb{R}^2)$ and u is C^1-smooth in $B(0, R)$ by Fact A.5. By Fact 4.2 we know that $\frac{u}{\varphi} \in W^{1,2}_{loc}(B(0, R))$ and, as expected,

$$\nabla \frac{u}{\varphi} = \frac{\varphi \nabla u}{\varphi^2} - \frac{u \nabla \varphi}{\varphi^2} \mathbb{1}_\Omega$$

in $B(0, R)$ in the sense of distributions. To establish the divergence-type equation for $\nabla \frac{u}{\varphi}$ we want to show that for every test function $h \in C_0^\infty(B(0, R) \setminus F_1)$, we have

$$\int_{B(0,R)\setminus F_1} \varphi^2 \nabla (\frac{u}{\varphi}) \nabla h = 0.$$

So we need to prove that

$$\int_{B(0,R)\setminus F_1} (\varphi \nabla u - u \nabla \varphi \mathbb{1}_\Omega) \cdot \nabla h = 0. \tag{6}$$

Since u is a solution to $\nabla u + V u = 0$ in $B(0, R)$, we have

$$\int_{B(0,R)\setminus F_1} \nabla u \cdot (\varphi \nabla h + h \nabla \varphi) = \int_{B(0,R)\setminus F_1} V \varphi u h \tag{7}$$

(we know the last equality under the assumption that φ is smooth, but it is also true for $\varphi \in W^{1,2}_{loc}(\mathbb{R}^2)$ by taking the norm limit). Consider a function $\xi \in C_0^\infty(B(0, R) \setminus F_0)$ that descends from 1 to 0 in the δ – neighborhood of $F_0 \cup \partial B(0, R)$ with $|\nabla \xi| < C/\delta$.

Since $\Delta \varphi + V \varphi = 0$ in

$$\Omega = B(0, R) \setminus (F_0 \cup F_1)$$

and $u h \xi \in C_0^1(\Omega)$, we have

$$\int_\Omega \nabla \varphi \cdot (h \nabla u \xi + u \nabla h \xi + u h \nabla \xi) = \int_\Omega V \varphi u h \xi. \tag{8}$$

Note that $\int_\Omega V \varphi u h \xi$ tends to $\int_\Omega V \varphi u h$ as $\delta \to 0$ (the functions V, φ, uh, ξ are uniformly bounded and the convergence holds pointwise in Ω because $\xi \to 1$ in $B(0, R) \setminus F_0$). Note that

$$h \nabla u \xi \to h \nabla u \quad \text{pointwise in } \Omega$$

and

$$u \nabla h \xi \to u \nabla h \quad \text{pointwise in } \Omega$$

because $\xi \to 1$ in Ω. Hence

$$\int_\Omega \nabla \varphi \cdot (h \nabla u \xi + u \nabla h \xi) \to \int_\Omega \nabla \varphi (h \nabla u + u \nabla h) \quad \text{as } \delta \to 0 \tag{9}$$

by the Lebesgue dominated convergence theorem with the majorant $|\nabla\varphi|(|h||\nabla u| + |u||\nabla h|)$.

In order to prove (6) we will show that

$$\int_\Omega \nabla\varphi \cdot (uh\nabla\xi) \to 0.$$

And here is the main place where we use that F_0 is the zero set of u! Note that $uh \in C_0^1(B(0,R))$ and vanishes on F_0, so $|uh| \le C_1(u,h)\varepsilon$ in the ε– neighborhood of the zero set of u. Thus $|uh\nabla\xi|$ is bounded by some constant $C(u,h)$ in $B(0,R)$. Also $m_2(\mathrm{supp}\nabla\xi)$ goes to 0. Hence

$$\int_\Omega \nabla\varphi \cdot (hu\nabla\xi) \le C(u,h)\sqrt{m_2(\mathrm{supp}\nabla\xi)}\sqrt{\int_\Omega |\nabla\varphi|^2} \to 0.$$

By (8), (9) we obtain

$$\int_\Omega \nabla\varphi \cdot (h\nabla u + u\nabla h) = \int_\Omega V\varphi uh = \int_{B(0,R)\setminus F_1} V\varphi uh$$

(the second equality is due to the fact that $u = 0$ on F_0). Using $\nabla\varphi = \nabla\varphi\mathbb{1}_\Omega$ in the sense of distributions, we have

$$\int_\Omega \nabla\varphi(h\nabla u + u\nabla h) = \int_{B(0,R)\setminus F_1} \nabla\varphi(h\nabla u + u\nabla h). \tag{10}$$

Thus

$$\int_{B(0,R)\setminus F_1} \nabla\varphi \cdot (h\nabla u + u\nabla h) = \int_{B(0,R)\setminus F_1} V\varphi uh$$

and, subtracting (7), we finish the proof of (6). □

4.2 Quasiconformal change of variables

We briefly describe some facts from the theory of quasiconformal mappings, which are used in the study of the solutions to equations in divergence form on the plane, and explain why the solutions behave like ordinary harmonic functions. We partially follow the exposition from [19], where the quasiconformal mappings are applied to quasi-symmetry of Laplace eigenfunctions.

Let B be a disk on the plane. Consider a real-valued function $f \in W_{loc}^{1,2}(B)$ satisfying

$$\mathrm{div}(\varphi^2\nabla f) = 0 \tag{11}$$

and assume that $0 < c < \varphi(x) < C < +\infty$ in B. One can find a function $\tilde{f} \in W_{loc}^{1,2}(B)$ such that

$$\varphi^2 f_x = \tilde{f}_y \text{ and } \varphi^2 f_y = -\tilde{f}_x$$

(see Sect. A.5) and f appears to be the real part of $w = f + i\tilde{f}$. A direct computation shows that w is a solution to the Beltrami equation:

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z} \tag{12}$$

with the Beltrami coefficient

$$\mu = \frac{1-\varphi^2}{1+\varphi^2} \cdot \frac{f_x + i f_y}{f_x - i f_y}. \tag{13}$$

When $\nabla f = 0$, we put $\mu = 0$.

We are going to apply the theory of quasiconformal mappings in a situation when $f = \frac{u}{\varphi}$ and the domain

$$\Omega_1 := B(0, R) \setminus F_1$$

is not simply connected. In this case w and $\tilde{f}$ can be defined only locally, but not in the whole Ω_1. However the Beltrami coefficient μ is well defined by (13) in Ω_1 and

$$|\mu| \leq \frac{1-\varphi^2}{1+\varphi^2} \leq C\varepsilon^2.$$

Let us extend μ by zero outside Ω_1 to the whole complex plane. Now μ has a compact support.

The existence Theorem 5.3.2 [2] claims that there is a K-quasiconformal homeomorphism ψ of the complex plane such that

- $\psi \in W^{1,2}_{loc}$,
- $\frac{\partial \psi}{\partial \overline{z}} = \mu \frac{\partial \psi}{\partial z}$,
- $K \leq \frac{1+\sup|\mu|}{1-\sup|\mu|}$.

In our case

$$K \leq 1 + C'\varepsilon^2.$$

Claim *The function $f \circ \psi^{-1}$ is harmonic in $\psi(\Omega_1)$.*

Indeed, for any ball $B \subset \Omega_1$, we can define $w \in W^{1,2}_{loc}(B)$ such that $f = \Re w$ and w, ψ solve the same Beltrami equation. Stoilow factorization theorem ([2], p.179, Theorem 5.5.1) claims that there is a holomorphic function W such that

$$w = W(\psi(z))$$

and therefore the harmonic function $\Re W$ satisfies

$$f(z) = \Re W(\psi(z)).$$

Clearly, the local observation shows that $f \circ \psi^{-1}$ is a harmonic function in $\psi(\Omega_1)$.

Note that $\psi(B(0,R))$ is a simply connected domain (and not the whole plane). Using the Riemann uniformisation theorem we can find a conformal map that sends $\psi(B(0,R))$ back to $B(0,R)$ and $\psi(0)$ to 0. The composition of this conformal map and the K-quasiconformal homeomorphism ψ will be a K-quasiconformal homeomorphism g of $B(0,R)$ onto itself with $g(0)=0$. Then the function $h=f\circ g^{-1}$ is harmonic in $g(\Omega_1)$.

Distortion of quasiconformal mappings Mori's theorem ([1], Chapter III, Section C) tells us that distances are changed by g in a controlled way:

$$\frac{1}{16}\left|\frac{z_1-z_2}{R}\right|^K \le \frac{|g(z_1)-g(z_2)|}{R} \le 16\left|\frac{z_1-z_2}{R}\right|^{1/K}. \tag{14}$$

We choose

$$\varepsilon=\frac{c}{\sqrt{\log R}}$$

so that

$$K\in[1,1+Cc^2/\log R),\qquad R^K\asymp R\asymp R^{1/K}$$

and the distortion on scales from $\frac{1}{R}$ to R is bounded. Namely, we may choose c so small that if $\frac{1}{R}\le|z_1-z_2|\le 2R$, then

$$\frac{1}{32}|z_1-z_2|\le|g(z_1)-g(z_2)|\le 32|z_1-z_2|.$$

Note that in the statement of Theorem 2.2 one can safely assume that R is sufficiently large ($R\gg 1$) by rescaling, which makes $\|V\|_\infty$ only smaller. It is needed to make $\varepsilon\ge 1/R$. Then we get a harmonic function h in $B(0,R)\setminus g(F_1)$, where $g(F_1)$ is the union of sets of diameter $\sim\varepsilon$. The image of a single disk of radius ε will be contained in a disk of radius 32ε. Let us denote these disks of radius 32ε by D_j. The images of disks from F_1 are $\frac{C}{32}\varepsilon$ – separated from each other and from the zero set of h. Hence D_j are $(\frac{C}{32}\varepsilon-128\varepsilon)=C_1 32\varepsilon$ – separated from each other and from the zero set of h, and h does not change sign in $C_1D_j\setminus D_j$. We have

$$C_1=\frac{C}{32^2}-4>100$$

if $C=10^6$.

We specifically asked that 0 and $z_{\max}$ (the point where $\sup_{B(0,R/2)}|u|$ is attained) are $C\varepsilon$ – separated from the disks. Recall that $g(0)=0$, so the disks C_1D_j do not contain 0 and $g(z_{\max})$. The distortion estimate implies that $g(z_{\max})\in\overline{B(0,R-R/64)}$. Since we had

$$\frac{\sup_{B(0,R)}|u|}{\sup_{B(0,R/2)}|u|}\le e^N,$$

we conclude that

$$\frac{\sup_{B(0,R)\setminus\cup 3D_j}|h|}{\sup_{B(0,R-R/64)\setminus\cup 3D_j}|h|}\le e^N.$$

If we make the rescaling by a factor of 32ε, then the disks D_j become 100-separated unit disks and R becomes

$$R' = R\cdot 32\varepsilon \sim R\sqrt{\log R}.$$

The goal of Theorem 2.2 is to estimate $\sup_{B(0,r)}|u|$ from below. If $r<1/R$, the image of $B(0,r)$ may have radius significantly smaller than r. However $g(B(0,r))$ contains a disk with center at 0 of radius

$$\frac{R}{16}\left(\frac{r}{R}\right)^K \ge \frac{R}{16}\left(\frac{r}{R}\right)^2.$$

Let $\tilde g = \frac{1}{32\varepsilon}g$. Then $\tilde g(B(0,r))$ contains a ball $B(0,r')$, where

$$r' \ge \frac{R'}{16}\left(\frac{r}{R}\right)^2.$$

So

$$\frac{R'}{r'}\le 16\frac{R^2}{r^2}.$$

In order to prove estimate (3), it is enough to show that

$$\sup_{B(0,r')\setminus\cup 3D_j}|h| \ge c(r'/R')^{C(R'+N)}\sup_{B(0,R')\setminus\cup 3D_j}|h|.$$

It will be proved in Theorem 5.3.

5 Act III

Before we formulate and prove the promised local Theorem 5.3 we will explain the main idea in the global case.

Theorem 5.1 (Toy problem) *Let $\{D_j\}$ be a collection of 100-separated disks with unit radius on the complex plane $\mathbb{C}$. Suppose that u is a harmonic function in $\mathbb{C}\setminus\cup_j D_j$ which preserves sign in each annulus $5D_j\setminus D_j$. If $|u(z)|\le e^{-L|z|}$ for all $z\in\mathbb{C}\setminus\cup_j D_j$ and L is sufficiently large, then $u\equiv 0$.*

Proof We start with a simple observation.

Claim 5.2 *Let $m_j = \min_{\partial 3D_j}|u|$. Then for some absolute constant $A>0$, we have*

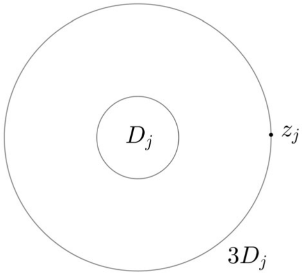
(1) $\max_{\partial 3D_j} |u| \le Am_j$,
(2) $\max_{\partial 3D_j} |\nabla u| \le Am_j$.

Proof By the Harnack inequality there exists a constant $A > 0$ such that

$$\sup_{4D_j \setminus 2D_j} |u| \le A \inf_{4D_j \setminus 2D_j} |u| \le Am_j,$$

which proves the first part of the claim. The second part follows from the Cauchy inequality. □

Let $k \in (0, L)$ and consider the numbers $m_j e^{k\Re z_j}$, where z_j is the rightmost point of $3D_j$.



Since

$$m_j \le |u(z_j)| \le e^{-L|z_j|},$$

there is j_0 such that

$$m_{j_0} e^{k\Re z_{j_0}} = \max_j m_j e^{k\Re z_j}.$$

Now, consider the analytic in $\mathbb{C} \setminus \cup(3D_j)$ function $f = (u_x - iu_y)e^{kz}$. If $|u(z)| \le e^{-L|z|}$ in $\mathbb{C} \setminus \cup(D_j)$, then

$$|\nabla u(z)| \le C \sup_{B(z,1)} |u| \le Ce^{-L(|z|-1)} \quad \text{for } z \in \mathbb{C} \setminus \cup(2D_j)$$

and $f(z) \to 0$ as $z \to \infty$, $z \in \mathbb{C} \setminus \cup(2D_j)$. So, by the maximum principle, there exists j_1 such that

$$\max_{\mathbb{C} \setminus \cup(3D_j)} |f| = \max_{\partial 3D_{j_1}} |f| \le Am_{j_1} e^{k\Re z_{j_1}} \le Am_{j_0} e^{k\Re z_{j_0}},$$

whence $|\nabla u| \le Am_{j_0} e^{-k\Re(z - z_{j_0})}$ in $\mathbb{C} \setminus \cup(3D_j)$. We may assume that $m_{j_0} \ne 0$, otherwise u is constant and therefore zero.

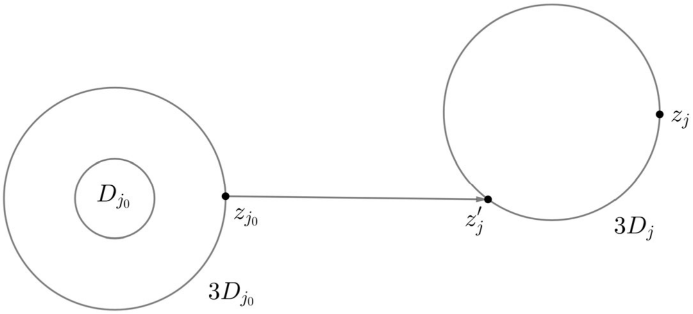
Now, consider the ray $\{z_{j_0} + y : y \in (0, +\infty)\}$. There are two possibilities:

(i) The ray goes to ∞ without hitting any other disks $(3D_j)$. Then for any $y > 0$,

$$|u(z_{j_0} + y) - u(z_{j_0})| \le \int_0^\infty |\nabla u(z_{j_0} + t)| dt \le \int_0^\infty A m_{j_0} e^{-kt} dt = \frac{A}{k} m_{j_0}.$$

Since $|u(z_{j_0})| \ge m_{j_0}$, wee see that $|u|$ stays bounded from below by $(1 - \frac{A}{k}) m_{j_0}$ on the ray. If $k > A$, this contradicts the decay assumption.

(ii) The ray hits another disk $3D_j$



at some point $z'_j = z_{j_0} + y$. Then we still have $|u(z'_j)| \ge (1 - \frac{A}{k}) m_{j_0}$ and, due to the fact that the disks are separated,

$$\Re(z'_j - z_{j_0}) = |z'_j - z_{j_0}| \ge 1.$$

Hence

$$m_j e^{k \Re z_j} \ge m_j e^{k \Re z'_j} \ge \frac{|u(z'_j)|}{A} e^{k(\Re z_{j_0} + 1)} \ge$$

$$\ge \frac{1}{A}(1 - \frac{A}{k}) e^k m_{j_0} e^{\Re z_{j_0}} > m_{j_0} e^{\Re z_{j_0}}$$

as soon as $k > 2A$, which contradicts the choice of j_0.

This proves the theorem with any $L > 2A$. □

Now we formulate and prove the harmonic counterpart of the main local theorem.

Theorem 5.3 *Let D_j be a collection of 100 – separated unit disks on $\mathbb{R}^2 = \mathbb{C}$ such that $0 \notin \cup 3D_j$. Let $R > 10^4$, $0 < r \le R/4$. Consider any harmonic function u in $B(0, R) \setminus \cup D_j$ such that u does not change sign in $(5D_j \setminus D_j) \cap B(0, R)$ for every j. Assume that*

$$\sup_{B(0, R - R/64) \setminus \cup 3D_j} |u| \ge e^{-N} \sup_{B(0,R) \setminus \cup 3D_j} |u|.$$

Then

$$\sup_{B(0,r) \setminus \cup 3D_j} |u| \ge \left(\frac{r}{R}\right)^{C(R+N)} \sup_{B(0,R) \setminus \cup 3D_j} |u|, \tag{15}$$

with some absolute constant $C > 0$.

Proof WLOG, $\sup_{B(0,R-R/64)\setminus\cup 3D_j} |u| = 1$. Fix $k = [C(N+R)]$ with sufficiently large $C > 0$ and assume that

$$\sup_{B(0,r)\setminus\cup 3D_j} |u| \le \left(\frac{r}{R}\right)^{3k}.$$

Consider the domain

$$\Omega := \{r/2 < |z| < R-1\} \setminus \cup(3D_j).$$

Let W_1 be the connected component of $\partial\Omega$ that intersects $\partial B(0, r/2)$. Note that each point of W_1 is either on $\partial B(0, r/2)$ or lies on some $\partial 3D_j$ that intersects $\partial B(0, r/2)$.

Estimate on W_1 Recall that if $5D_j \subset B(0, R)$, we have

(1) $\max_{\partial 3D_j} |u| \le A \min_{\partial 3D_j} |u|$,
(2) $\max_{\partial 3D_j} |\nabla u| \le A \min_{\partial 3D_j} |u|$.

Hence on $W_1 \setminus \partial B(0, r/2)$, we have

$$|u|, |\nabla u| \le A \sup_{B(0,r)\setminus\cup(3D_j)} |u| \le A \left(\frac{r}{R}\right)^{3k}.$$

If $x \in \partial B(0, r/2) \setminus \cup(3D_j)$, then either $x \in 4D_j$ for some j or $B(x, min(1, r/2)) \subset B(0, r) \setminus \cup(3D_j)$. In the first case u does not change sign in $B(x, 1)$ and

$$|\nabla u(x)| \le A|u(x)| \le A \sup_{B(0,r)\setminus\cup(3D_j)} |u| \le A \left(\frac{r}{R}\right)^{3k}.$$

In the second case, we have

$$|\nabla u(x)| \le \frac{A}{\min(1, r/2)} \sup_{B(0,r)\setminus\cup(3D_j)} |u| \le \frac{A}{\min(1, r/2)} \left(\frac{r}{R}\right)^{3k}.$$

Thus in all cases, if C in the definition of k is large enough, we have

$$\max_{W_1} |u|, \max_{W_1} |\nabla u| \le \frac{A}{\min(1, r/2)} \left(\frac{r}{R}\right)^{3k} \le \left(\frac{r}{R}\right)^{2k}$$

because

$$\left(\frac{R}{r}\right)^{k} \ge 4^k > A$$

and

$$\left(\frac{R}{r}\right)^{k} \ge 4^{k-1}\frac{R}{r} \ge \frac{2A}{r}.$$

Let W_2 be the connected component of $\partial\Omega$ that intersects $\partial B(0, R-1)$. Note that each point of W_2 is either on $\partial B(0, R-1)$ or lies on some $\partial 3D_j$ that intersects $\partial B(0, R-1)$.

Estimate on W_2 Any point $x \in \overline{B(0, R-1)} \setminus \cup(3D_j)$ is either in $4D_j$ for some j or $x \in \overline{B(0, R-1)} \setminus \cup(4D_j)$. In the first case u does not change sign in $B(x, 1)$ and therefore

$$|\nabla u(x)| \le A|u(x)| \le Ae^N.$$

In the second case $B(x, 1) \subset B(0, R) \setminus \cup(3D_j)$ and $|\nabla u(x)| \le Ae^N$. Thus

$$\max_{W_2} |u|, \max_{W_2} |\nabla u| \le Ae^N.$$

Note also that

$$W_2 \subset \overline{B(0, R-1)} \setminus B(0, R-7).$$

Now, consider the analytic in Ω function

$$f(z) = \frac{u_x - iu_y}{z^k}, \quad |f(z)| = \frac{|\nabla u(z)|}{|z|^k}.$$

Since

$$\sup_{B(0,R-R/64)\setminus\cup(3D_j)} |u| = 1 > \sup_{B(0,r/2)\setminus\cup 3D_j} |u|,$$

$$\max_{W_1} |u| \le \left(\frac{r}{R}\right)^{2k} < \frac{1}{2},$$

and since any point in

$$\Omega_1 = \{r/2 < |z| < R - R/64\} \setminus \cup(3D_j)$$

can be connected with W_1 by a curve of length at most $4R$ within Ω_1, we must have

$$\sup_{\Omega} |\nabla u| \ge \sup_{\Omega_1} |\nabla u| \ge \frac{1}{8R}$$

and

$$\sup_{\Omega_1} |f| \ge (R - R/64)^{-k} \sup_{\Omega_1} |\nabla u| \ge \frac{1}{8R} (R - R/64)^{-k}.$$

However

$$\max_{W_1} |f| \le \max_{W_1} |\nabla u| \left(\frac{2}{r}\right)^k \le \left(\frac{r}{R}\right)^{2k} \left(\frac{2}{r}\right)^k = \left(\frac{2r}{R}\right)^k R^{-k} \le$$

$$\leq 2^{-k} R^{-k} < \frac{1}{8R} R^{-k} < \sup_{\Omega_1} |f| \leq \sup_{\Omega} |f|$$

and

$$\max_{W_2} |f| \leq Ae^N \frac{1}{(R-7)^k} = Ae^N \left(\frac{R - R/64}{R-7} \right)^k (R - R/64)^{-k} \leq$$

$$\leq Ae^N \left(\frac{126}{127} \right)^k (R - R/64)^{-k} < \frac{1}{8R} (R - R/64)^{-k} \leq \sup_{\Omega} |f|$$

if $R - 7 > \frac{127}{128} R$ and C in the definition of k is large enough. By the maximum principle for holomorphic functions $\sup_\Omega |f|$ is achieved on $\partial 3D_j$ for some $3D_j \subset B(0, R-1) \setminus \overline{B(0, r/2)}$.

For every disk D_j with $3D_j \subset B(0, R-1) \setminus \overline{B(0, r/2)}$, consider the point z_j on $\partial 3D_j$ closest to the origin. All $3D_j$ that are not in the annulus $B(0, R-1) \setminus \overline{B(0, r/2)}$ will not be considered further. Put $m_j = \min_{\partial 3D_j} |u|$. Let j_0 be the index such that

$$\frac{m_{j_0}}{|z_{j_0}|^k} = \max_j \frac{m_j}{|z_j|^k}.$$

If $\sup_\Omega |f|$ is achieved on $\partial 3D_{j_1}$, then for $x \in \Omega$,

$$\frac{|\nabla u(x)|}{|x|^k} \leq \frac{1}{|z_{j_1}|^k} \max_{\partial 3D_{j_1}} |\nabla u| \leq A \frac{m_{j_1}}{|z_{j_1}|^k} \leq A \frac{m_{j_0}}{|z_{j_0}|^k}.$$

So we conclude that

$$|\nabla u(x)| \leq \left(\frac{|x|}{|z_{j_0}|} \right)^k A m_{j_0}.$$

Recalling that $\frac{1}{8R} \leq \sup_\Omega |\nabla u|$, we get

$$\frac{1}{8R} \leq \left(\frac{2R}{r} \right)^k A m_{j_0},$$

so

$$|u(z_{j_0})| \geq m_{j_0} \geq \frac{1}{8AR} \left(\frac{r}{2R} \right)^k.$$

Now, let (sz_{j_0}, z_{j_0}) be the longest subinterval of the radius $[0, z_{j_0}]$ starting at z_{j_0} that is contained in Ω. We have

$$|u(sz_{j_0})| \geq |u(z_{j_0})| - |z_{j_0}| \int_s^1 |\nabla u(tz_{j_0})| dt \geq |u(z_{j_0})| - |z_{j_0}| \int_0^1 A m_{j_0} t^k dt \geq$$

$$\geq m_{j_0} (1 - \frac{AR}{k+1}) \geq \frac{m_{j_0}}{2}$$

if the constant C in the definition of k is large enough. Note that

$$\left(\frac{R}{r}\right)^k \geq 4^k > 16AR \cdot 2^k.$$

Hence

$$\frac{m_{j_0}}{2} \geq \frac{1}{16AR}\left(\frac{r}{2R}\right)^k > \left(\frac{r}{R}\right)^{2k}$$

and the point sz_{j_0} cannot belong to W_1, whence it belongs to some $\partial 3D_j$ with $3D_j \subset B(0, R-1) \setminus \overline{B(0, r/2)}$. Then

$$\frac{m_j}{|z_j|^k} \geq \frac{|u(sz_{j_0})|}{A|sz_{j_0}|^k} \geq \frac{1}{2As^k}\frac{m_{j_0}}{|z_{j_0}|^k}.$$

It remains to notice that, since the distance from $3D_j$ to $3D_{j_0}$ is at least 96, we have

$$s^k \leq (1-96/R)^k < \frac{1}{2A}$$

if the constant C in the definition of k is large enough. But then $\frac{m_j}{|z_j|^k} > \frac{m_{j_0}}{|z_{j_0}|^k}$, which contradicts the choice of j_0. $\square$

Appendix

A.1 The toy problem for harmonic functions in higher dimensions: a proof with extra logarithm

Here we present another proof of a slightly worse bound for the toy problem for harmonic functions in a punctured domain. However this proof works in higher dimensions. We will denote by B_R the ball in $\mathbb{R}^n$ with center at 0 and of radius R.

Toy problem with extra logarithm Let D_j be a collection of unit, 100 – separated balls on the plane and let $R > 100$. Then for any harmonic function h in $B_R \setminus \cup D_j$ such that h does not change sign in each $B_R \cap 5D_j \setminus D_j$, we have

$$\int_{B_R \setminus (B_{R/2} \bigcup (\cup 3D_j))} h^2 \geq \exp(-CR\log R) \int_{B_{R/2} \setminus \cup 3D_j} h^2,$$

where C is an absolute positive constant.

This inequality implies that Theorem 5.1 holds in higher dimensions if we assume that $|u(z)| \leq e^{-L|z|\log|z|}$ for sufficiently large L.

Proof The proof is based on the Carleman inequality with log linear weight. Most of Carleman inequalities require strict log convexity-type properties of the weight. The next inequality is an exception:

$$\int_{B_R} |\Delta u|^2 e^{kx_1} \geq \frac{ck^2}{R^2} \int_{B_R} u^2 e^{kx_1} \tag{16}$$

for any $u \in C_0^2(B_R)$. The inequality is not difficult to prove. Let $v = ue^{kx_1/2}$, then

$$e^{kx_1/2} \Delta u = \Delta v - k v_{x_1} + \frac{k^2}{4} v$$

and

$$\int_{B_R} |\Delta u|^2 e^{kx_1} = \int_{B_R} |\Delta v + \frac{k^2}{4} v|^2 + \int_{B_R} |k v_{x_1}|^2 - \int_{B_R} 2(\Delta v + \frac{k^2}{4} v) k v_{x_1}.$$

Note that

$$\int_{B_R} 2 v v_{x_1} = \int_{B_R} \frac{\partial}{\partial x_1} v^2 = 0.$$

Integrating by parts, we see that

$$-\int_{B_R} \Delta v_{x_1} v = \int_{B_R} \Delta v v_{x_1} = \int_{B_R} v \Delta v_{x_1}$$

and therefore

$$\int_{B_R} \Delta v v_{x_1} = 0.$$

Hence

$$\int_{B_R} |\Delta u|^2 e^{kx_1} = \int_{B_R} |\Delta v + \frac{k^2}{4} v|^2 + \int_{B_R} |k v_{x_1}|^2 \geq \int_{B_R} |k v_{x_1}|^2 \geq$$

(by Poincare's inequality)

$$\geq \frac{\pi^2}{4} \frac{k^2}{R^2} \int_{B_R} v^2 = c \frac{k^2}{R^2} \int_{B_R} u^2 e^{kx_1}.$$

So we proved (16) and would like to apply it for the harmonic function h. However h is not in $C_0^2(B_R)$ and inequality (16) should be applied to

$$u = h\eta,$$

where η is a positive, C^2-smooth cut-off function:

- $\eta = 0$ in each $2D_j$ and in $\{x : |x| > R - 11\}$,
- $\eta = 1$ in $B_{\frac{3}{4}R \setminus \cup 3D_j}$,
- the function η, as well as its first and second derivatives are bounded by a numerical constant.

We will choose the parameter k later. For now we have

$$\int_{B_{\frac{3}{4}R}\setminus\cup 3D_j} |\Delta h|^2 e^{kx_1} + \text{“cut-off integrals”} \geq \frac{ck^2}{R^2}\int_{B_{\frac{3}{4}R}\setminus\cup 3D_j} h^2 e^{kx_1} =: \text{RHS}.$$

It is good that $\Delta h = 0$, so only the cut-off integrals are left on the left-hand side. There are two kinds of cut-off integrals:

$$\mathrm{I} = \sum_{5D_j \subset B_{\frac{3R}{4}}} \int_{3D_j\setminus 2D_j} \text{“cut-off terms”}$$

and

$$\mathrm{II} = \int_{B_{R-11}\setminus B_{(\frac{3R}{4}-10)}} \text{“cut-off terms”}$$

where

$$|\text{“cut-off terms”}| \lesssim (h^2 + |\nabla h|^2)e^{kx_1}$$

(recall that $\Delta h = 0$). Note that

$$\int_{3D_j\setminus 2D_j} e^{kx_1} \lesssim e^{-k/2}\int_{4D_j\setminus 3D_j} e^{kx_1}$$

because $4D_j \setminus 3D_j$ contains an open disk of radius $\frac{1}{4}$, where the function e^{kx_1} is pointwise bigger than $e^{k/2}\cdot \sup_{3D_j\setminus 2D_j} e^{kx_1}$. Now, assuming $5D_j \subset B_R$ we will use the sign condition in $5D_j \setminus D_j$. By the Harnack inequality and the Cauchy estimate we know that there is a constant $a_j \geq 0$ such that

$$|h| \asymp a_j \text{ and } |\nabla h| \lesssim a_j \text{ in } 4D_j \setminus 2D_j.$$

So

$$\int_{3D_j\setminus 2D_j} \text{“cut-off terms”} \lesssim a_j^2 \int_{3D_j\setminus 2D_j} e^{kx_1} \lesssim a_j^2 e^{-k/2}\int_{4D_j\setminus 3D_j} e^{kx_1} \lesssim$$

$$\lesssim e^{-k/2}\int_{4D_j\setminus 3D_j} h^2 e^{kx_1}.$$

Hence

$$\mathrm{I} \lesssim e^{-k/2}\int_{B_{\frac{3}{4}R}\setminus\cup 3D_j} h^2 e^{kx_1}.$$

Note that

$$\text{RHS} = \frac{ck^2}{R^2}\int_{B_{\frac{3}{4}R}\setminus\cup 3D_j} h^2 e^{kx_1} > 2\mathrm{I}$$

if

$$k^2/R^2 \gg e^{-k/2}.$$

We make the choice

$$k = C \log R$$

and it yields

$$\sup_{B_R} e^{kx_1} \le e^{CR\log R}.$$

Since

$$\mathrm{I} + \mathrm{II} \ge \mathrm{RHS} \quad \text{and} \quad I \le \frac{1}{2}\mathrm{RHS},$$

we have

$$\int_{B_{R-11}\setminus B_{(\frac{3R}{4}-10)}} \text{“cut-off terms”} = \mathrm{II} \ge \frac{1}{2}\mathrm{RHS} \asymp k^2/R^2 \int_{B_{\frac{3}{4}R}\setminus \cup 3D_j} h^2 e^{kx_1} \ge$$

$$\ge \exp(-CR\log R) \int_{B_{\frac{1}{2}R}\setminus \cup 3D_j} h^2.$$

If $5D_j \subset B_R$, then $\int_{3D_j\setminus 2D_j} h^2 \asymp \int_{4D_j\setminus 3D_j} h^2$, whence

$$\int_{B_R\setminus(B_{R/2}\bigcup(\cup 3D_j))} h^2 \ge c_1 \int_{B_{R-10}\setminus(B_{R/2}\bigcup(\cup 2D_j))} h^2 \ge$$

(by Cauchy estimate)

$$\ge c_2 \int_{B_{R-11}\setminus(B_{R/2}\bigcup(\cup 3D_j))} (h^2 + |\nabla h|^2)$$

and therefore

$$\sup_{B_R} e^{kx_1} \int_{B_R\setminus(B_{R/2}\bigcup(\cup 3D_j))} h^2 \ge c_2 \int_{B_{R-11}\setminus(B_{R/2}\bigcup(\cup 3D_j))} (h^2 + |\nabla h|^2) e^{kx_1} \ge c_3 \mathrm{II}.$$

Thus

$$\int_{B_R\setminus(B_{R/2}\bigcup(\cup 3D_j))} h^2 \ge \exp(-C' R\log R) \int_{B_{R/2}\setminus \cup 3D_j} h^2. \qquad \square$$

Deduction of Theorem 2.1 from Theorem 5.3 We may assume that

$$\sup_{B(0,R')\setminus\cup 3D_j} |h| = \sup_{B(0,R'/8)\setminus\cup 3D_j} |h|,$$

otherwise the statement is trivial. Consider any point x on $\partial B(0, R'/4) \setminus \cup 3D_j$.

Note that $B(0, R'/8) \subset B(x, 3R'/8)$ and $B(x, 3R'/4) \subset B(0, R')$. Hence

$$\sup_{B(x,3R'/8)\setminus\cup 3D_j} |h| = \sup_{B(x,3R'/4)\setminus\cup 3D_j} |h|.$$

Applying Theorem 5.3 for the disk with center at x (in place of 0) of radius $R = 3R'/4$ and $N = 0$, we obtain the bound

$$\sup_{\{R'/8<|z|<R'\}\setminus\cup 3D_j} |h| \geq \sup_{B(x,R'/8)\setminus\cup 3D_j} |h| \geq e^{-CR'} \sup_{B(0,R')\setminus\cup 3D_j} |h|.$$

A.2 Sketches of general elliptic theory

Fact A.1 *Denote by $E(z) = \frac{1}{2\pi}\log|z|$ the fundamental solution of the Laplace operator on the plane in the sense that for every C^∞ compactly supported function h, we have*

$$h = E * \Delta h.$$

Fact A.2 *Let Ω be a bounded open set and $g \in L^1(\Omega)$. Put $f = E * g$. Then*

(1) *$f \in L^p(\Omega)$ for all $p \geq 1$.*
(2) *$\Delta f = g$ in the sense that for every $h \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega f\Delta h = \int_\Omega gh.$$

Agreement Writing $E * g$ we assume that g is extended to $\mathbb{R}^2 \setminus \Omega$ by zero.

Proof (1) Let $D = \operatorname{diam}\Omega$. Let $E_D = \mathbb{1}_{B(0,D)}E$. Then in Ω we have $f = g * E_D$. Since $E_D \in L^p(\mathbb{R}^2)$ for all $p \geq 1$ and $g \in L^1$, the result follows from Young's convolution inequality.

(2) We have

$$\int f\Delta h = \int (E * g)\Delta h = \int_{\Omega\times\Omega} E(z-\zeta)g(\zeta)\Delta h(z)dm_2(z)dm_2(\zeta) =$$

$$= \int_\Omega g(\zeta)\left[\int_\Omega E(z-\zeta)\Delta h(z)dm_2(z)\right]dm_2(\zeta) = \int_\Omega gh.$$

□

Fact A.3 *Let $V \in L^\infty(\Omega)$, $u \in L^1_{loc}(\Omega)$ and $\Delta u + Vu = 0$ in Ω in the sense that for every $h \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u\Delta h + \int_\Omega Vuh = 0.$$

Then $u \in L^p_{loc}(\Omega)$ for every $p \geq 1$.

Proof Passing to a smaller bounded domain Ω', we may assume that $u \in L^1(\Omega)$, Ω is bounded. Consider $f = E * (Vu)$. By Fact A.2, $f \in L^p(\Omega)$ for all $p \geq 1$. Note that $u - f \in L^1(\Omega)$ and $\Delta(u - f) = 0$ in the sense of distributions. Hence, by Weyl's lemma, $u - f$ is harmonic in Ω, so $u - f \in L^p_{loc}(\Omega)$ and therefore $u \in L^p_{loc}(\Omega)$ too. □

Fact A.4 *Let $g \in L^p(\Omega)$ with $p > 2$ and let Ω be bounded. Then $g * E \in C^1(\Omega)$.*

Proof

$$E(z+t) - E(z) = \frac{1}{2\pi}(\log|z+t| - \log|z|) = \frac{1}{2\pi}\log\left|1 + \frac{t}{z}\right| =$$

$$= \frac{1}{2\pi}\Re\frac{t}{z} + O\left(\begin{cases} \frac{|t|^2}{|z|^2}, & \frac{1}{2} \geq \frac{|t|}{|z|} \\ \frac{|t|}{|z|} + |\log\left|1 + \frac{t}{z}\right||, & \frac{1}{2} \leq \frac{|t|}{|z|} \end{cases}\right).$$

Define

$$W(\zeta) = \begin{cases} \frac{1}{|\zeta|^2}, & \zeta > 2 \\ \frac{1}{|\zeta|} + \left|\log|1 + \frac{1}{\zeta}|\right|, & \zeta \leq 2. \end{cases}$$

Taking the convolution and applying Holder's inequality, we have

$$(g * E)(z+t) - (g * E)(z) = \left[g * \frac{1}{2\pi}\Re\frac{t}{\cdot}\right](z) + O\left(\|g\|_p \|W(\cdot/t)\|_q\right) =$$

$$= \Re\left(t\left[g * \frac{1}{2\pi\cdot}\right](z)\right) + O\left(\|W(\cdot/t)\|_q\right), \quad \text{where } 1/p + 1/q = 1.$$

Since $g * \frac{1}{2\pi\cdot} \in C(\Omega)$, the first term (as a function of $t \in \mathbb{C} = \mathbb{R}^2$) is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}$, which depends continuously on z. It is enough to show that $\|W\|_q < \infty$ because

$$\|W(\cdot/t)\|_q = \|W\|_q |t|^{2/q} = \|W\|_q\, o(t) \quad \text{as } t \to 0$$

($1 < q < 2$ if $p > 2$). Indeed,

$$\int |W|^q \lesssim \int_{|\zeta|>2} \frac{1}{|\zeta|^{2q}} + \int_{|\zeta|\leq 2}\left(\frac{1}{|\zeta|^q} + \left|\log\left|1 + \frac{1}{\zeta}\right|\right|^q\right) < \infty.$$ □

Fact A.5 *Let $V \in L^\infty(\Omega)$. If $\Delta u + Vu = 0$ in Ω in the sense of distributions and $u \in L^1_{loc}(\Omega)$, then $u \in C^1(\Omega)$.*

Proof By Fact A.3, $u \in L^p_{loc}(\Omega)$ with $p > 2$. Again passing to a subdomain, if necessary, we may assume that Ω is bounded and $u \in L^p(\Omega)$. Consider $f = E * (Vu)$. Since $Vu \in L^p(\Omega)$, $f \in C^1(\Omega)$. However $u - f$ is harmonic. Hence $u \in C^1(\Omega)$. $\square$

Lemma A.6 *Let Ω be a bounded domain with Poincare constant smaller than 1. Then for any $v \in L^\infty(\Omega)$, we can find a solution $u \in W_0^{1,2}(\Omega)$ to $\Delta u = v$ in the sense of distributions such that*

$$\|u\|_{W_0^{1,2}(\Omega)} \le 4\|v\|_2.$$

Remark A.7 Note that if $u \in W^{1,2}_{loc}(\Omega)$ and $h \in C_0^\infty(\Omega)$, then

$$\int_\Omega \nabla u \nabla h = -\int_\Omega u \Delta h.$$

So $u \in W^{1,2}_{loc}(\Omega)$ is a solution to $\Delta u = v$ in the sense of distributions if and only if

$$\int_\Omega \nabla u \nabla h = -\int_\Omega vh$$

for any $h \in C_0^\infty(\Omega)$.

Proof Consider the functional

$$\Phi(u) = \int |\nabla u|^2 + \int vu$$

for $u \in W_0^{1,2}(\Omega)$. Integrals in the next few lines will be over the domain Ω. Notice that by Poincare's inequality

$$\Phi(u) \ge \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |u|^2 - \|v\|_2\|u\|_2 \ge$$

$$\ge \frac{1}{2}\|u\|^2_{W_0^{1,2}} - \|v\|_2\|u\|_{W_0^{1,2}} \ge -\frac{1}{2}\|v\|_2.$$

Thus $\Phi(u)$ is bounded from below. Note that $\Phi(u) > 0 = \Phi(0)$ as soon as $\|u\|_{W_0^{1,2}} > 2\|v\|_2$. Let now $u_k \in C_0^\infty(\Omega)$ be any minimizing sequence for Φ. Note that

$$\frac{\Phi(u') + \Phi(u'')}{2} - \Phi\left(\frac{u' + u''}{2}\right) = \frac{1}{4}\int |\nabla(u' - u'')|^2 \ge \frac{1}{8}\|u' - u''\|^2_{W_0^{1,2}}.$$

Hence u_k is a Cauchy sequence in $W_0^{1,2}(\Omega)$, so the limit $u = \lim u_k$ exists and minimizes Φ. Now, take any test function $h \in C_0^\infty(\Omega)$ and consider

$$\Phi(u + th) = \Phi(u) + t\left(2\int \nabla u \nabla h + \int vh\right) + t^2\int |\nabla h|^2.$$

Since u is a minimizer, we must have

$$2\int \nabla u \nabla h + \int vh = 0,$$

i.e., $\Delta u = v/2$ in the sense of distributions. Taking $2u$ in place of u we get a solution to $\Delta u = v$ with

$$\|u\|_{W_0^{1,2}} \leq 4\|v\|_2. \qquad \square$$

The next step is to show that

$$\|u\|_\infty \leq C\|v\|_\infty$$

with some absolute constant $C > 0$. WLOG, we will assume $|v| \leq 1$.

A.3 Uniform bound via Di Giorgi method

Let Ω be any bounded open set in $\mathbb{R}^2$ with Poincare constant $k^2 \leq k_0^2$, i.e.,

$$\int u^2 \leq k^2 \int |\nabla u|^2 \text{ for all } u \in C_0^\infty(\Omega).$$

Claim I *Let k_0 be sufficiently small and consider any smooth ε-minimizer of*

$$\Phi(u) = \int_\Omega |\nabla u|^2 + \int_\Omega vu \quad (\|v\|_\infty \leq 1), \ \textit{i.e.},$$

$u \in C_0^\infty(\Omega)$ *and for any* $\tilde{u} \in C_0^\infty(\Omega)$, $\Phi(\tilde{u}) \geq \Phi(u) - \varepsilon$. *Then u satisfies*

$$\int_{B\cap\Omega} u^2 \leq Ck^2(k^2 + \varepsilon)$$

for every unit ball $B \subset \mathbb{R}^2$.

Proof Let $\varphi(x)$ be a smooth positive radial function such that

- $\varphi(x) = 1$ in $B(0, 1)$,
- $\varphi(x) \in (0, 1]$,
- $\varphi(x) \asymp e^{-|x|}$,
- $|\nabla \varphi| \leq \varphi$.

Let $\psi = \varphi^2$, so $|\nabla \psi| \leqslant 2|\psi|$. Applying the Poincare inequality to φu, we get

$$\int \varphi^2 u^2 \leq k^2 \int |\varphi \nabla u + u \nabla \varphi|^2 \leq 2k^2 \left(\int \varphi^2 |\nabla u|^2 + \int u^2 |\nabla \varphi|^2 \right) \leq$$

$$\leq 2k^2 \int \varphi^2 |\nabla u|^2 + 2k^2 \int u^2 \varphi^2, \text{ whence}$$

$$\int \varphi^2 u^2 \le \frac{2k^2}{1-2k^2}\int \varphi^2 |\nabla u|^2, \text{ i.e.,}$$

$$\int u^2\psi \le \frac{2k^2}{1-2k^2}\int |\nabla u|^2\psi \le 4k^2\int |\nabla u|^2\psi \quad \text{if} \quad k_0 \le \frac{1}{2}. \tag{17}$$

Now, consider the competitor $\tilde{u} = (1-\psi)u$. We have

$$\Phi(\tilde{u}) = \int |(1-\psi)\nabla u - u\nabla\psi|^2 + \int v(1-\psi)u \le$$

$$\le \int (1-\psi)^2|\nabla u|^2 + 2\int |\nabla u||u||\nabla\psi| + \int u^2|\nabla\psi|^2 + \int v(1-\psi)u$$

$$\le \int (\nabla u)^2 + \int vu - \int |\nabla u|^2\psi + 4\int |\nabla u||u|\psi + 4\int u^2\psi - \int v\psi u$$

(we used the inequalities $(1-\psi)^2 \le 1-\psi$, $|\nabla\psi| \leqslant 2\psi$, $\psi^2 \le \psi$).

Since $\Phi(\tilde{u}) \geqslant \Phi(u) - \varepsilon$, we must have

$$\int |\nabla u|^2\psi \le 4\left(\int |\nabla u||u|\psi + \int u^2\psi\right) + \varepsilon + \int |v\psi u|.$$

However

$$\int u^2\psi \leqslant 4k^2\int |\nabla u|^2\psi \quad \text{by (17)},$$

and

$$\int |\nabla u||u|\psi \le \sqrt{\int u^2\psi}\sqrt{\int |\nabla u|^2\psi} \leqslant 2k\int |\nabla u|^2\psi.$$

So for sufficiently small k_0,

$$4\left(\int |\nabla u||u|\psi + \int u^2\psi\right) \le \frac{1}{2}\int |\nabla u|^2\psi.$$

Hence

$$\int |\nabla u|^2\psi \le 2\varepsilon + 2\int |vu\psi|$$

$$\le 2\left(\varepsilon + \sqrt{\int \psi}\sqrt{\int u^2\psi}\right) \leqslant C\left(\varepsilon + k\cdot\sqrt{\int |\nabla u|^2\psi}\right).$$

If the first term dominates, then $\int |\nabla u|^2\psi \leqslant C\varepsilon$. Otherwise $\int |\nabla u|^2\psi \leqslant Ck^2$.

By (17) it follows that

$$\int u^2\psi \le Ck^2\left(k^2+\varepsilon\right).$$

□

Note that we did not care in Claim I where the Poincare constant came from and what was special about the geometry of Ω that made it small. The next lemma gives a simple bound for the Poincare constant of "thin" domains.

Lemma A.8 *Assume Ω is open and $m_2(\Omega \cap Q) \le c < 1$ for all unit squares $Q \subset \mathbb{R}^2$. Then the Poincare constant of Ω is at most $2 + \frac{2}{1-c}$.*

Proof Let $f \in C_0^\infty(\Omega)$. Extend f by zero outside Ω. It is sufficient to show that if Q is a unit square, then

$$\int_Q |f|^2 \le \left(2 + \frac{2}{1-c}\right) \int_Q |\nabla f|^2.$$

By tiling the plane with unit squares, it implies

$$\int_\Omega |f|^2 \le \left(2 + \frac{2}{1-c}\right) \int_\Omega |\nabla f|^2.$$

Let $Q = [0, 1]^2$,

$$I_x = ((x, y) : y \in [0, 1]) \quad \text{and} \quad I^y = ((x, y) : x \in [0, 1]).$$

Let X_0 be the set of $x \in [0, 1]$ such that I_x contains a zero of f. Then for every $x \in X_0$, we have

$$\max_{I_x} |f| \le \int_{I_x} |\nabla f|$$

and

$$\int_{I_x} f^2 \le \max_{I_x} f^2 \le \left(\int_{I_x} |\nabla f|\right)^2 \le \int_{I_x} |\nabla f|^2.$$

Hence

$$\int_{X_0 \times [0,1]} f^2 \le \int_Q |\nabla f|^2.$$

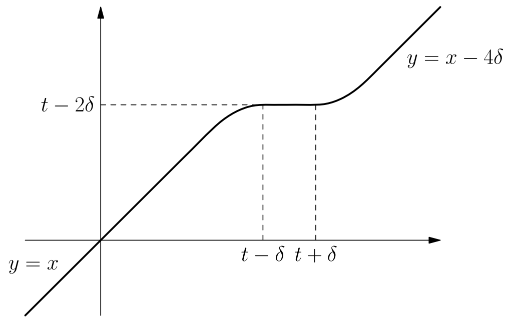
The set X_0 has Lebesgue measure at least $1 - c$, whence there is $x_0 \in X_0$ such that

$$\int_{I_{x_0}} f^2 \le \frac{1}{1-c} \int_Q |\nabla f|^2.$$

Claim *Let I be a unit interval and let z be any point in I. Then*

$$\int_I f^2 \le 2|f(z)|^2 + 2\int_I |\nabla f|^2.$$

Fig. 3 $y = \Theta(x)$



Indeed,

$$\int_I f^2 \leq \max_I f^2 \leq \left(|f(z)| + \int_I |\nabla f|\right)^2 \leq \left(|f(z)| + \sqrt{\int_I |\nabla f|^2}\right)^2 \leq$$

$$\leq 2|f(z)|^2 + 2\int_I |\nabla f|^2.$$

For every $y \in [0, 1]$, *it yields*

$$\int_{I^y} f^2 \leq 2|f(x_0, y)|^2 + 2\int_{I^y} |\nabla f|^2.$$

Thus

$$\int_Q f^2 = \int_0^1 \left(\int_{I^y} f^2 dx\right) dy \leq 2\int_{I_{x_0}} f^2 + 2\int_Q |\nabla f|^2 \leq \left(\frac{2}{1-c} + 2\right)\int_Q |\nabla f|^2.$$

□

Corollary A.9 *If*

$$m_2(\Omega \cap Q) \leq k^2 \ll 1$$

for any unit square Q, *then the Poincare consant of* Ω *is smaller than* Ck^2.

Proof For every square Q_{2k} of size $2k$, we have

$$m_2(\Omega \cap Q_{2k}) \leq \frac{1}{4} m_2(Q_{2k}).$$

By $2k$ rescaling we reduce the problem to Lemma A.8. □

Now we are almost ready to run the Di Georgi scheme. The only remaining preparatory part is smooth surgery. Let $u \in C_0^\infty(\Omega)$. Fix $t > 0$ (level) and $\delta > 0$ (extremely small number). Let Θ be a C^∞-smooth function on $\mathbb{R}$ described by Fig. 3.

The function Θ has the following properties:

- $0 \leq \Theta' \leq 1$,

- $\Theta(0) = 0$,
- $x - 4\delta \le \Theta(x) \le x$,
- $\Theta(x) = t - 2\delta$ on $(t - \delta, t + \delta)$.

Let $\tilde{u} = \Theta \circ u$. Then $|u - \tilde{u}| \le 4\delta$ and $|\nabla \tilde{u}| \le |\nabla u|$ pointwise. Thus, if u was an ε-minimizer, then $\tilde{u}$ is an $\varepsilon + A\delta$-minimizer (δ is purely qualitative and $A = 4\int_\Omega |v|$). Define

$$\Theta_-(x) = \begin{cases} \Theta(x), & x \le t + \delta \\ t - 2\delta, & x \ge t + \delta \end{cases} \qquad \text{and} \quad \Theta_+ = \Theta - \Theta_-.$$

The function $\tilde{u}$ naturally splits into two smooth compactly supported terms:

$$\tilde{u} = \tilde{u}_- + \tilde{u}_+, \text{ where } \tilde{u}_\pm = \Theta_\pm \circ u.$$

The function $\tilde{u}_+$ is compactly supported in $\{u > t\}$ ($\operatorname{supp} \tilde{u}_+ \subset \{u \ge t + \delta\}$) and $\nabla \tilde{u}_+$ and $\nabla \tilde{u}_-$ have disjoint supports. We may then try to replace $\tilde{u}_+$ by some smooth competitor $w \in C_0^\infty(\{u > t\})$ and see if the functional can drop. Note that

$$\Phi(\tilde{u}_- + w) = \int_\Omega |\nabla \tilde{u}_-|^2 + \int_{\{u>t\}} |\nabla w|^2 + \int_\Omega v\tilde{u}_- + \int_{\{u>t\}} vw,$$

so we just need to compare

$$\int_\Omega |\nabla \tilde{u}_+|^2 + \int_\Omega v\tilde{u}_+ \quad \text{with} \quad \int_{\{u>t\}} |\nabla w|^2 + \int_{\{u>t\}} vw.$$

Hence $\tilde{u}_+$ is an $(\varepsilon + A\delta)$-minimizer in the new domain $\{u > t\}$. We shall now fix the initial Poincare constant to be k_0 from Claim I. If u is an ε-minimizer, then $\int_B u^2 \le Ck_0^2(k_0^2 + \varepsilon)$ for every unit ball B, so

$$m_2(\{u > t_0\} \cap B) \le \frac{Ck_0^2(k_0^2 + \varepsilon)}{t_0^2}.$$

Choose $t_0 = C'\sqrt{k_0}$ with sufficiently large absolute constant C'. Then the domain $\Omega_1 = \{u > t_0\}$ satisfies $m_2(\Omega_1 \cap B) \le \frac{C}{C'}k_0\left(k_0^2 + \varepsilon\right)$ for any unit ball B and, by Corollary A.9, the Poincare constant of Ω_1 is at most $k_1^2 := k_0\frac{k_0^2+\varepsilon}{2}$. Also $u_1 = \tilde{u}_+ \in C_0^\infty(\Omega_1)$ will be an $\varepsilon_1 = \varepsilon + A\delta_0$-minimizer of Φ in $C_0^\infty(\Omega_1)$ where $\delta_0 > 0$ can be chosen arbitrary small. Finally, note that $u \le t_0 + u_1 + 4\delta_0$ everywhere in Ω. We can now repeat this construction with u_1, Ω_1, ε_1 instead of u, Ω, ε to get u_2, Ω_2, ε_2 and so on. We shall get a sequence of domains Ω_j, functions $u_j \in C_0^\infty(\Omega_j)$ and numbers $k_j, t_j, \varepsilon_j, \delta_j > 0$ such that

$$\Omega_1 \supset \Omega_2 \supset \dots \quad , \quad k_j^2 = k_{j-1}\frac{k_{j-1}^2 + \varepsilon_{j-1}}{2},$$

$$\varepsilon_j = \varepsilon_{j-1} + A\delta_{j-1}, \quad t_j = C'\sqrt{k_j},$$

$$m_2(\Omega_j \cap B) \le \frac{C}{C'} k_{j-1}(k_{j-1}^2 + \varepsilon_{j-1}) \quad \text{for any unit ball } B,$$

and

$$u \le t_0 + t_1 + \dots + t_l + u_{l+1} + 4\delta_0 + 4\delta_1 + \dots + 4\delta_l \quad \text{for any } l \ge 0.$$

In this construction, we can choose $\delta_j > 0$ as small as we want, so putting $\delta_j = \frac{c\varepsilon}{2^j}$ with sufficiently small absolute constant $c > 0$, we can guarantee that all $\varepsilon_j \le 2\varepsilon$.

Let l be the first index for which $k_l^2 < 2\varepsilon$. Then (provided that $C' > 8C$, $\varepsilon < 1/2$), we also have $m_2(\Omega_{l+1} \cap B) < \frac{\varepsilon}{2}$ for every unit ball B. For all $j \le l$, we have

$$k_j^2 \le k_{j-1}^3,$$

so, if k_0 was chosen less than $\frac{1}{4}$, it implies that $k_j^2 \le 2^{-j-2}$ for $j = 0, 1, \dots, l$, whence $t_j \le C' 2^{-\frac{j}{2}-1}$ and

$$u \le t_0 + t_1 + \dots + t_l + 4\delta_0 + 4\delta_1 + \dots + 4\delta_l \le C' \sum_{j\ge 0} 2^{-\frac{j}{2}-1} + \sum_{j \ge 0} 4\delta_j \le$$

$$\le 2C' + \varepsilon \le 2C' + 1 = C_0$$

in $\Omega \setminus \Omega_{l+1}$ because $u_{l+1} = 0$ in $\Omega \setminus \Omega_{l+1}$. Thus

$$m_2(\{u > C_0\} \cap B) < \varepsilon/2.$$

Considering $-u$ instead of u, we conclude that also $m_2(\{u < -C_0\} \cap B) < \varepsilon/2$ and therefore

$$m_2(\{|u| > C_0\} \cap B) < \varepsilon$$

for every ε-minimizer u. Since the true minimizer u is the limit of ε-minimizers in $L^2(\Omega)$, we get

$$m_2(\{|u| > C_0\} \cap B) = 0 \quad \text{for any unit ball } B,$$

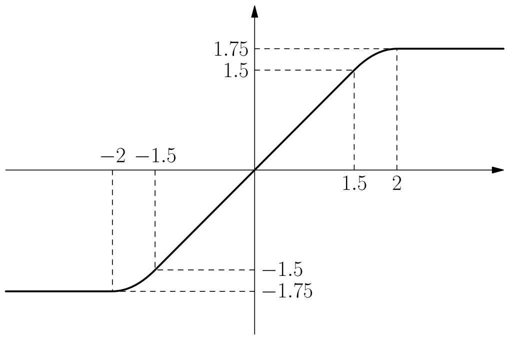
so $m_2(\{|u| > C_0\}) = 0$.

Conclusion If $|v| \le 1$ and the Poincare constant of Ω is not greater than $k_0^2 \ll 1$, then the minimizer of $\int_\Omega |\nabla u|^2 + \int_\Omega vu$ in $W_0^{1,2}(\Omega)$ satisfies

$$\|u\|_\infty \le C_0. \tag{18}$$

If the Poincare constant of Ω is k, we put $\widetilde{\Omega} = \frac{k_0}{k}\Omega$, $\widetilde{u} = \frac{k_0^2}{k^2} u(\frac{k}{k_0}\cdot)$, $\widetilde{v} = v(\frac{k}{k_0}\cdot)$, so $\widetilde{\Phi}(\widetilde{u}) = \int_{\widetilde{\Omega}} |\nabla \widetilde{u}|^2 + \int_{\widetilde{\Omega}} \widetilde{v}\widetilde{u} = \frac{k_0^4}{k^4}\Phi(u)$. Applying the result that was just obtained, we get the final observation.

Fig. 4 The graph of Θ



Lemma A.10 *If the Poincare constant of* Ω *is* $k > 0$, *then the minimizer of* $\Phi(u) = \int_\Omega |\nabla u|^2 + \int_\Omega vu$ *in* $W_0^{1,2}(\Omega)$ (*i.e., the solution to* $\Delta u = v/2$) *satisfies*:

$$\|u\|_\infty \le Ck^2 \|v\|_\infty, \tag{19}$$

where C *is an absolute positive constant.*

A.4 Other standard facts used in the proof

Fact A.11 *Let* Ω *be a bounded open set, let* $u \in W_0^{1,2}(\Omega)$ *satisfy* $\|u\|_\infty \le 1$. *Then there exists a sequence* $u_k \in C_0^\infty(\Omega)$ *with* $\|u_k\|_\infty \le 2$ *such that* $u_k \to u$, $\nabla u_k \to \nabla u$ *in* $L^2(\Omega)$ *and almost everywhere in* Ω.

Proof By the definition of $W_0^{1,2}(\Omega)$, we can find $\tilde{u}_k \in C_0^\infty(\Omega)$ with $\tilde{u}_k \to u$, $\nabla\tilde{u}_k \to \nabla u$ in L^2. Let Θ be defined by Fig. 4.

The function Θ has the following properties.

- $\Theta(x) = x$ for $x \in [-1.5, 1.5]$,
- Θ is C^∞-smooth and $|\Theta| \le 1.75$,
- $|\Theta'| \le 1$ and $|\Theta(x)| \le |x|$.

Put $u_k = \Theta(\tilde{u}_k)$. Note that $u_k = \tilde{u}_k$ and $\nabla u_k = \nabla\tilde{u}_k$ if $|\tilde{u}_k| \le 1.5$ and we always have $|u_k| \le |\tilde{u}_k|$, $|\nabla u_k| \le |\nabla\tilde{u}_k|$. We need to show that $\|u_k - u\|_2$, $\|\nabla u_k - \nabla u\|_2 \to 0$. Since $\tilde{u}_k$ converge in $W_0^{1,2}(\Omega)$, the functions $|\tilde{u}_k|^2$ and $|\nabla\tilde{u}_k|^2$ are uniformly integrable, i.e., for any $\varepsilon > 0$, there is $\delta > 0$ such that if $m_2(E) < \delta$, then

$$\int_E |\tilde{u}_k|^2, \int_E |\nabla\tilde{u}_k|^2 < \varepsilon.$$

In the following computation $\int$ will denote the integral over Ω:

$$\int |u_k - u|^2 \le 2\left[\int |u_k - \tilde{u}_k|^2 + \int |\tilde{u}_k - u|^2\right] \le$$

$$\le 4 \int\limits_{\{|\tilde{u}_k| > 1.5\}} (|u_k|^2 + |\tilde{u}_k|^2) + 2\int |\tilde{u}_k - u|^2 \le$$

$$\leq 8 \int\limits_{\{|\tilde{u}_k|>1.5\}} |\tilde{u}_k|^2 + 2\int |\tilde{u}_k - u|^2.$$

The second term tends to zero by the choice of $\tilde{u}_k$. Note that

$$m_2(\{|\tilde{u}_k| > 1.5\}) \leq m_2(|u - \tilde{u}_k| \geq 0.5) \leq 4\|u - \tilde{u}_k\|_2^2 \to 0.$$

So the first term tends to zero by the uniform integrability property. In a similar way one can show that $\int |\nabla u_k - \nabla u|^2 \to 0$.

Now we would like to choose a subsequence such that $u_k \to u$, $\nabla u_k \to \nabla u$ almost everywhere in Ω. It can be done by choosing any subsequence u_{k_j} with

$$\|u_{k_j} - u\|_2 \leq \frac{1}{4^j}, \quad \|\nabla u_{k_j} - \nabla u\|_2 \leq \frac{1}{4^j}.$$

Let E_j be the set of points $x \in \Omega$ such that $|u_{k_j}(x) - u(x)| \geq \frac{1}{2^j}$. Then

$$\sqrt{m_2(E_j)\frac{1}{4^j}} \leq \|u_{k_j} - u\|_2 \leq \frac{1}{4^j}, \quad \text{so} \quad m_2(E_j) \leq \frac{1}{4^j}.$$

Note that if $x \notin \cup_{j=n}^{\infty} E_j$, then $u_{k_j}(x)$ converge to $u(x)$. However

$$m_2(\cup_{j=n}^{\infty} E_j) \leq \sum_{j=n}^{\infty} m_2(E_j) \leq \frac{1}{2^n}.$$

Thus u_{k_j} converge to u almost everywhere in Ω. In a similar way one can show that ∇u_{k_j} also converge to ∇u almost everywhere. □

Fact 4.2 *Let Ω be an open set. Assume that $u, v \in W^{1,2}_{loc}(\Omega) \cap L^{\infty}_{loc}(\Omega)$. then $uv \in W^{1,2}_{loc}(\Omega)$ and $\nabla(uv) = u\nabla v + v\nabla u$.*

Proof Clearly, the fact is local. So we may assume $\Omega = B(0,r)$ and $u, v \in W^{1,2}(B(0,r)) \cap L^{\infty}(B(0,r))$.

Let us fix a small $\delta > 0$ and let

$$K_\varepsilon(z) = \frac{1}{\varepsilon^2}\kappa(|z|/\varepsilon)$$

be a C^∞-approximation to identity with $\operatorname{supp} K_\varepsilon(z) \subset B(0,\varepsilon)$, $\varepsilon < \delta$. Then $K_\varepsilon * u$ and $K_\varepsilon * v$ converge to u and v in $L^2(B(0, r-\delta))$ and a.e. in $B(0, r-\delta)$ as $\varepsilon \to 0$. Consider any test function $\eta \in C_0^\infty(B(0, r-\delta))$ and extend it by 0 outside $B(0, r-\delta)$. Then $\eta * K_\varepsilon \in C_0^\infty(B(0,r))$ and $K_\varepsilon * \nabla\eta = \nabla(K_\varepsilon * \eta)$. By Fubini's theorem we have

$$\int_{B(0,r-\delta)} (K_\varepsilon * u)\nabla\eta = \int_{B(0,r)} u(K_\varepsilon * \nabla\eta) =$$

$$= \int_{B(0,r)} u\nabla(K_\varepsilon * \eta) = -\int_{B(0,r)} \nabla u(K_\varepsilon * \eta) =$$

$$= -\int_{B(0,r)} \left(\int_{B(0,r-\delta)} \nabla u(x) K_\varepsilon(x-y)\eta(y)dy \right) dx =$$

$$= -\int_{B(0,r-\delta)} \left(\int_{B(0,r)} \nabla u(x) K_\varepsilon(y-x)\eta(y)dx \right) dy = -\int_{B(0,r-\delta)} (\nabla u * K_\varepsilon)\eta.$$

So

$$u_\varepsilon := K_\varepsilon * u \quad \text{and} \quad v_\varepsilon := \quad K_\varepsilon * v$$

are in $W^{1,2}(B(0,r-\delta)) \cap C^\infty(B(0,r-\delta))$ with

(1) $\nabla u_\varepsilon = \nabla u * K_\varepsilon$, $\nabla v_\varepsilon = \nabla v * K_\varepsilon$,
(2) $\nabla u * K_\varepsilon \to \nabla u$, $\nabla v * K_\varepsilon \to \nabla v$ in $L^2(B(0,r-\delta))$ and a.e. in $B(0,r-\delta)$ as $\varepsilon \to 0$,
(3) By Young's inequality for convolutions, we have $|u_\varepsilon| < \|u\|_{L^\infty(B(0,r))}$, $|v_\varepsilon| < \|v\|_{L^\infty(B(0,r))}$ in $B(0,r-\delta)$.

We know that the convergence of $u_\varepsilon v_\varepsilon$ holds a.e. in $B(0,r-\delta)$ and $|u_\varepsilon v_\varepsilon|$ are bounded by $\|u\|_\infty \|v\|_\infty$ a.e. in $B(0,r-\delta)$ if $\varepsilon < \delta$. So by the Lebesgue dominated convergence theorem $u_\varepsilon v_\varepsilon \to uv$ in $L^2(B(0,r-\delta))$.

We want to show that

$$\nabla(uv) = u\nabla v + v\nabla u$$

in the sense of $W^{1,2}(B(0,r-\delta))$. It is clear that $u\nabla v + v\nabla u$ is in $L^2(B(0,r-\delta))$ because u, v are bounded and their gradients are in $L^2(B(0,r-\delta))$.

Consider again a test function $\eta \in C_0^\infty(B(0,r-\delta))$. We have

$$\int uv\nabla\eta = \lim_{\varepsilon\to 0}\int u_\varepsilon v_\varepsilon \nabla\eta = -\lim_{\varepsilon\to 0}\int (\nabla u_\varepsilon v_\varepsilon + u_\varepsilon \nabla v_\varepsilon)\eta =$$

$$= -\lim_{\varepsilon\to 0}\left[\int ((\nabla u_\varepsilon - \nabla u)v_\varepsilon + u_\varepsilon(\nabla v_\varepsilon - \nabla v))\,\eta + \int (\nabla u v_\varepsilon + u_\varepsilon \nabla v)\eta \right].$$

Note that $\int(\nabla u_\varepsilon - \nabla u)v_\varepsilon\eta \to 0$ because $\nabla u_\varepsilon \to \nabla u$ in $L^2(B(0,r-\delta))$ and $|v_\varepsilon \eta| < \|v\|_\infty \|\eta\|_\infty$ in $B(0,r-\delta)$. Similarly, $\int u_\varepsilon(\nabla v_\varepsilon - \nabla v)\eta \to 0$. Finally, by the Lebesgue dominated convergence theorem

$$\int (\nabla u v_\varepsilon + u_\varepsilon \nabla v)\eta \to \int (\nabla u v + u\nabla v)\eta$$

because the convergence of the functions holds a.e. in $B(0,r-\delta)$ and there is the integrable majorant $(|\nabla u|\|v\|_\infty + \|u\|_\infty |\nabla v|)\|\eta\|_\infty$. Thus $\int uv\nabla\eta = -\int(\nabla uv + u\nabla v)\eta$. $\square$

Fact A.12 *Let Ω be a bounded open set. Let $\varphi = 1 + \psi$, where $\psi \in W_0^{1,2}(\Omega)$, $\|\psi\|_\infty \le \frac{1}{3}$. Then the functions*

$$\tilde{\varphi} = \begin{cases} \varphi \text{ in } \Omega \\ 1 \text{ outside } \Omega \end{cases} \qquad \text{and} \qquad \eta = \begin{cases} \frac{1}{\varphi} \text{ in } \Omega \\ 1 \text{ outside } \Omega \end{cases}$$

are in $W^{1,2}_{loc}(\mathbb{R}^2)$ *and*

$$\nabla\tilde{\varphi} = \nabla\varphi \mathbb{1}_\Omega, \quad \nabla\eta = -\frac{\nabla\varphi}{\varphi^2}\mathbb{1}_\Omega.$$

Proof Consider a sequence of functions $\psi_k \in C_0^\infty(\Omega)$ such that $\|\psi_k\|_\infty \leq \frac{2}{3}$ and $\psi_k \to \psi$, $\nabla\psi_k \to \nabla\psi$ in $L^2(\Omega)$ and a.e. in Ω. We can extend ψ_k by zero outside Ω and get a sequence of $C_0^\infty(\mathbb{R}^2)$ functions, which we will still denote by ψ_k, such that $\psi_k = 0$, $\nabla\psi_k = 0$ in $\mathbb{R}^2 \setminus \Omega$ while $\psi_k \to \psi\mathbb{1}_\Omega$ in $L^2(\mathbb{R}^2)$ and a.e., $\nabla\psi_k \to \nabla\psi\mathbb{1}_\Omega$ in $L^2(\mathbb{R}^2)$. This immediately implies that $1+\psi_k \to \tilde{\varphi}$, $\nabla(1+\psi_k) = \nabla\psi_k \to \nabla\psi\mathbb{1}_\Omega$ in $L^2_{loc}(\mathbb{R}^2)$, so $\tilde{\varphi} \in W^{1,2}_{loc}(\mathbb{R}^2)$ and $\nabla\tilde{\varphi} = \tilde{\varphi}\mathbb{1}_\Omega$. Note that

$$\left|\frac{1}{1+\psi_k} - \frac{1}{1+\psi}\right| = \left|\frac{\psi_k - \psi}{(1+\psi_k)(1+\psi)}\right| \leq 9|\psi_k - \psi|$$

and

$$\left|\nabla\frac{1}{1+\psi_k} + \frac{\nabla\psi}{(1+\psi)^2}\right| = \left|\frac{\nabla\psi_k}{(1+\psi_k)^2} - \frac{\nabla\psi}{(1+\psi)^2}\right| \leq$$

$$\leq |\nabla\psi_k - \nabla\psi|\frac{1}{(1+\psi_k)^2} + |\nabla\psi|\left|\frac{1}{(1+\psi_k)^2} - \frac{1}{(1+\psi)^2}\right| \quad \text{in } \Omega.$$

Also $\frac{1}{1+\psi_k} = 1$, $\nabla\frac{1}{1+\psi_k} = 0$ in $\mathbb{R}^2 \setminus \Omega$. Hence $\frac{1}{1+\psi_k} \to \eta$ in $L^2(\mathbb{R}^2)$ and we would like to show that

$$\nabla\frac{1}{1+\psi_k} \to -\frac{\nabla\psi}{(1+\psi)^2}\mathbb{1}_\Omega = -\frac{\nabla\varphi}{\varphi^2}\mathbb{1}_\Omega \quad \text{in } L^2(\mathbb{R}^2).$$

To see the latter, note that $\frac{1}{(1+\psi_k)^2} \leq 9$, so

$$\int_\Omega |\nabla\psi_k - \nabla\psi|^2 \frac{1}{(1+\psi_k)^4} \leq 81\int_\Omega |\nabla\psi_k - \nabla\psi|^2 \to 0,$$

and the functions $|\nabla\psi|^2\left[\frac{1}{(1+\psi_k)^2} - \frac{1}{(1+\psi)^2}\right]^2$ have the integrable majorant $81|\nabla\psi|^2$ and tend to 0 almost everywhere in Ω. Thus $\eta \in W^{1,2}_{loc}(\mathbb{R}^2)$ and $\nabla\eta = -\frac{\nabla\varphi}{\varphi^2}\mathbb{1}_\Omega$ as required. $\square$

Lemma A.13 *Let* Ω *be a bounded open set and let a function* $f \in C^1(\overline{\Omega})$ *be zero on* $\partial\Omega$. *Then* $f \in W^{1,2}_0(\Omega)$.

Proof Let $\varepsilon > 0$, denote by Ω_ε the set of points x in Ω with distance to the boundary of Ω at least ε. Let η be a function in $C_0^\infty(\Omega)$ with the following properties:

- $\eta(x) = 1$, if $x \in \Omega_\varepsilon$.
- $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq \frac{C}{\varepsilon}$ in Ω.

The function $f\eta$ is in $C_0^1(\Omega) \subset W_0^{1,2}(\Omega)$. We want to show that $f\eta$ converge to f in $W_0^{1,2}(\Omega)$ norm as $\varepsilon \to 0$. Observe that $|\nabla f|$ is uniformly bounded in Ω by some constant $A = A(f)$, so $|f(x)| \le A\varepsilon$ if the distance from x to $\partial\Omega$ is smaller than ε and therefore

$$|\nabla(f\eta)| \le |\nabla f||\eta| + |f||\nabla\eta| \le A + AC \quad \text{in } \Omega \setminus \Omega_\varepsilon.$$

Then

$$\int\limits_\Omega |f - f\eta|^2 = \int\limits_{\Omega\setminus\Omega_\varepsilon} |f - f\eta|^2 \le \int\limits_{\Omega\setminus\Omega_\varepsilon} |f|^2 \to 0$$

and

$$\int\limits_\Omega |\nabla f - \nabla(f\eta)|^2 = \int\limits_{\Omega\setminus\Omega_\varepsilon} |\nabla f - \nabla(f\eta)|^2 \le 2\int\limits_{\Omega\setminus\Omega_\varepsilon} (|\nabla f|^2 + |\nabla(f\eta)|^2) \le$$

$$\le m_2(\Omega \setminus \Omega_\varepsilon) C_1 A^2.$$

Since $m_2(\Omega \setminus \Omega_\varepsilon) \to 0$ as $\varepsilon \to 0$, we have verified that $f \in W_0^{1,2}(\Omega)$. ☐

Lemma A.14 *Let u be a solution to $\Delta u + Vu = 0$, $|V| \le 1$, in a ball $B(x,r)$, where $r < r_0$ and r_0 is a sufficiently small universal constant. If u is continuous up to $\partial B(x,r)$ and $u > 0$ on $\partial B(x,r)$, then $u > 0$ in $B(x,r)$.*

Proof We may assume that u is larger than a positive constant δ on $\partial B(x,r)$. Consider the set $\Omega = \{x \in B(x,r) : u(x) < \frac{\delta}{2}\}$. This is an open set strictly inside $B(x,r)$ and if u is not positive in $B(x,r)$, then Ω is not empty.

Since $u \in C^1(\overline{\Omega})$ by Fact A.5 and $u = \frac{\delta}{2}$ on $\partial\Omega$, we know by Lemma A.13 that $(u - \frac{\delta}{2}) \in W_0^{1,2}(\Omega)$.

Note that $\overline{\Omega} \subset B(0,r)$, so Ω has a Poincare constant smaller than Cr^2. By Lemma 3.2, if r is sufficiently small, we can find $\varphi = 1 + \tilde{\varphi}$ with $\tilde{\varphi} \in W_0^{1,2}(\Omega)$, $\|\tilde{\varphi}\|_\infty < \frac{1}{2}$ such that φ is a solution to $\Delta\varphi + V\varphi = 0$ in Ω. Then $(\frac{\delta}{2}\varphi - \frac{\delta}{2}) \in W_0^{1,2}(\Omega)$ and therefore the function $g = (\frac{\delta}{2}\varphi - u) \in W_0^{1,2}(\Omega)$. The function g is also a solution to $\Delta g + Vg = 0$. For any $\eta \in C_0^\infty(\Omega)$, we have $\int_\Omega \nabla g \nabla \eta = \int_\Omega Vg\eta$ and taking the limit in $W_0^{1,2}(\Omega)$, we get

$$\int_\Omega |\nabla g|^2 = \int_\Omega Vg^2 \le \int_\Omega g^2.$$

However Poincare's inequality implies

$$\int_\Omega g^2 \le Cr^2 \int_\Omega |\nabla g|^2.$$

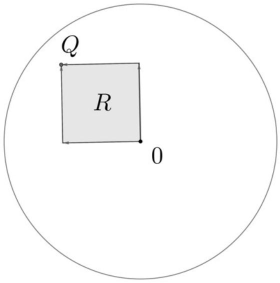
If r is sufficiently small, this could happen only if $g = 0$ in Ω. So $u = \frac{\delta}{2}\varphi$ in Ω, but $\varphi > \frac{1}{2}$ in Ω. So $u > \frac{\delta}{4}$ in Ω and in $B(x,r)$. ☐

A.5 Divergence free vector fields on the plane

If $F = (F_1, F_2) : B \to \mathbb{R}^2$ is a C^1- smooth vector field in a disk B on the plane such that F is divergence free: $\mathrm{div} F = 0$ in B, then there is a smooth function u such that

$$(F_1, F_2) = \nabla \times u := (u_{x_2}, -u_{x_1}).$$

Sometimes people refer to the statement above as to Poincare's lemma or the fundamental theorem of calculus, or the inverse gradient theorem. Here is the sketch of the standard proof. WLOG, $B = B(0, 1)$. Consider any point $Q \in B$ and the rectangle $R \subset B$ with opposite vertices 0 and Q, and sides parallel to x_1 and x_2 axes.



Note that the contour integral

$$\int_{\partial R} (-F_2, F_1) \cdot dx = \int_{\partial R} F \cdot n(x)|dx| = \int_R \mathrm{div}\ F$$

is zero. There are two simple paths that start at 0, go along the sides of R and end at Q. The integrals $\int (-F_2, F_1) dx$ over those two paths are the same and we define $u(Q)$ to be equal to both of them. The differentiation of $u(Q)$ in the horizontal and vertical directions shows that $(F_1, F_2) = (u_{x_2}, -u_{x_1})$.

We need the version with less regularity assumptions on the divergence free vector field F: if $F \in L^p_{\mathrm{loc}}(B(0, 1))$, $1 \le p < \infty$, and $\int_{B(0,1)} F \nabla h = 0$ for any $h \in C_0^\infty(B(0, 1))$ (the divergence free condition), then there is a function $u \in W^{1,p}_{loc}(B(0, 1))$ such that $(F_1, F_2) = \nabla \times u := (u_{x_2}, -u_{x_1})$.

Indeed, let

$$K_\varepsilon(z) = \frac{1}{\varepsilon^2} \kappa(|z|/\varepsilon)$$

be a C^∞-approximation to identity with $\mathrm{supp} K_\varepsilon(z) \subset B(0, \varepsilon)$. Define

$$F_\varepsilon = F * K_\varepsilon = (F_1 * K_\varepsilon, F_2 * K_\varepsilon)$$

in the smaller ball $B(0, 1 - \varepsilon)$. Then F_ε is divergence free in $B(0, 1 - \varepsilon)$. Indeed, if $f \in C_0^\infty(B(0, 1 - \varepsilon)$, then by Fubini's theorem

$$\int (F * K_\varepsilon) \nabla f = \int F (K_\varepsilon * \nabla f) = \int F \nabla (K_\varepsilon * f) = 0.$$

So there is a C^∞ function u_ε such that $F_\varepsilon = \nabla \times u_\varepsilon$ in $B(0, 1-\varepsilon)$. Fix $\delta \in (0, 1)$. By the Lebesgue theory $F * K_\varepsilon$ converge to F in $L^p(B(0, 1-\delta))$. Thus ∇u_ε is a Cauchy sequence in $L^p(B(0, 1-\delta))$. Let us add a constant to u_ε so that $\int_{B(0,1-\delta)} u_\varepsilon = 0$. By the Poincaré–Wirtinger inequality (see p.275, Theorem 1 in [10]) u_ε is a Cauchy sequence in $L^p(B(0, 1-\delta))$. Thus we can find a function $\tilde{u}_\delta$ such that u_ε converge to $\tilde{u}_\delta$ in $W^{1,p}(B(0, 1-\delta))$ and $\nabla \times \tilde{u}_\delta = \lim_{\varepsilon\to 0} \nabla \times u_\varepsilon = (F_1, F_2)$. For any $\delta_1, \delta_2 \in (0, 1)$, the gradients of $\tilde{u}_{\delta_1}$ and $\tilde{u}_{\delta_2}$ are the same in $B(0, 1 - \max(\delta_1, \delta_2))$ and therefore $\tilde{u}_{\delta_1} - \tilde{u}_{\delta_2}$ is constant almost everywhere in $B(0, 1 - \max(\delta_1, \delta_2))$. Finally, let us modify $\tilde{u}_\delta$ by subtracting a constant so that $\int_{B(0,1/2)} \tilde{u}_\delta = 0$ for all $\delta < 1/2$. Then u is well-defined by $u = \tilde{u}_\delta$ in $B(0, 1-\delta)$.

A.6 Local to global argument on compact manifolds

In this section we prove Corollary 2.5 using Corollary 2.3 by following the argument from [8], page 162.

Our 2-dimensional Riemannian manifold (M, g), which is assumed to be compact and without boundary, can be covered by a finite number of charts U with isothermal coordinates, where the Laplace operator has a simplified form $\Delta_g u = \frac{\Delta u}{q}$ with some positive bounded function q and the Euclidean distance in each chart is comparable to the geodesic distance on the manifold up to some factor $K = K(M, g) \geq 1$. In each chart, the equation $\Delta_g u + Vu = 0$ is then simplified to $\Delta u + Vqu = 0$, where $\|qV\|_\infty \leq \|q\|_\infty \|V\|_\infty \leq C\lambda$. From now on, we fix this finite collection of charts.

Given a chart U, we will denote by $B_U(x, r)$ the Euclidean ball in the isothermal coordinates with center x and radius r. Due to the compactness of M there is $c > 0$ such that for any $x \in M$, there is a chart U such that $B_U(x, c) \subset U$. Since the Euclidean distance in each chart is comparable to the manifold distance, there is a constant

$$\rho = \rho(M) \in (0, c/4)$$

such that if a point y satisfies $B_{U'}(y, c) \subset U'$ and

$$y \in B_U(x, c/4) \subset B_U(x, c) \subset U,$$

then

$$B_{U'}(y, \rho) \subset B_U(x, c/2).$$

For any two points x and y on M, we can find a finite sequence of points x_1, $x_2, \dots, x_k$ such that $x = x_1$, $y = x_k$ and the manifold distance between x_i and x_{i+1} is smaller than $c/(4K)$. Note that the constant k can be chosen to be bounded by some constant depending only on (M, g). For each i, choose a chart U_i such that $B_i(x_i, c) := B_{U_i}(x_i, c) \subset U_i$. Then

- both x_i and x_{i+1} lie in both U_i and U_{i+1},
- $x_i \in B_{i+1}(x_{i+1}, c/4)$.

We will call such a sequence the Harnack sequence for x and y.

Without loss of generality, we may assume that $\max_M |u| = 1$ and it is achieved at some point $x_{\max} \in M$. For any other point $y \in M$, consider the Harnack sequence x_i for $x_{\max} = x_1$ and $y = x_k$. We will show by induction on i that

$$\sup_{B_i(x_i,\rho)} |u| \geq e^{-C_i\sqrt{\lambda \log \lambda}}$$

for some C_i depending on i and (M, g).

For $i = 1$ the ball $B_i(x_i, \rho)$ contains the maximum point and there is nothing to prove. Now, assume that $\sup\limits_{B_i(x_i,\rho)} |u| \geq e^{-C_i\sqrt{\lambda \log \lambda}}$. We would like to show the same for x_{i+1}.

Indeed, we have $B_i(x_i, \rho) \subset B_{i+1}(x_{i+1}, c/2)$ and therefore

$$\sup_{B_{i+1}(x_{i+1},c)} |u| \leq 1 \leq e^{C_i\sqrt{\lambda \log \lambda}} \sup_{B_{i+1}(x_{i+1},c/2)} |u|.$$

By rescaled Corollary 2.3 it yields the lower bound for $\sup\limits_{B_{i+1}(x_{i+1},\rho)} |u|$, which finishes the proof by induction on i.

Finally, the estimate

$$\sup_{B_k(y,r)} |u| \geq r^{C\sqrt{\lambda \log \lambda}}$$

for $r < \min(1/2, \rho/2)$ follows from

$$\sup_{B_k(y,2\rho)} |u| \leq 1 \leq e^{C_k\sqrt{\lambda \log \lambda}} \sup_{B_k(y,\rho)} |u|$$

and rescaled Corollary 2.3.

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