SIMPLE FLOW-REVERSIBLE MODELS FOR DYNAMICS AND CONTROL OF HEAT EXCHANGERS

by

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ABSTRACT

Simple models are required to handle nonlinear effects in heat exchangers due to flow rate changes, and especially to flow reversal transients, such as occur in loss-of-coolant accidents. This thesis presents very simple models for the dynamic behavior of such systems. Exact linear distributed models have been presented by Takahashi(2), Paynter and Takahashi (1), and Hsu and Gilbert (9).Approximate methods for simplifying these models include Friedly (4) who presented an asymptotic approximation which satisfies dynamic response of such systems at both low and high frequencies; however, this technique involves an infinite order model using a distributed system.

The simple models of the present paper employ finite state models or "Lumped Models"; and two, three, and four lump heat exchanger models are discussed. Both dynamic and static behaviors of these models are compared with exact results. In addition, some results for flow reversals are shown.

Studying the results for these simple models shows that to get the best agreement with exact solutions, a linear combination of the intermediate out puts should be used.The benefits of such an improvement are shown.

Finally, the monotonic parameters are calculated directly and they are compared with graphical results which are obtained by technique due to Paynter (8).

Thesis Supervisor: Henry M. Paynter
Title: Professor of Mechanical Engineering

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I wish to express my sincere appreciation and gratitude to my advisor Professor Henry M. Paynter, a source of never ending knowledge, for his stimulating guidance, and counsel through out the course of this thesis. I am grateful to him for discussions which helped greatly to clarify my thoughts.

I would like to express my appreciation to my wife, Azar, for her encouragement and for typing the manuscript.

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NOMENCLATURE

A	Heat transfer area (m^2)
A_s	Cross sectional area for each fluid (m^2)
a_1	$\frac{U.A}{\dot{M}_s.C}$ (dimensionless)
a_2	$\frac{U.A}{\dot{M}_t.C}$ (dimensionless)
C	Specific heat transfer at constant pressure ($\frac{cal}{kg.sec}$)
$G(s)$	Transfer function (dimensionless)
$\|G\|$	Gain (dimensionless)
$\|G\|_N$	Normalized gain (dimensionless)
j	$\sqrt{-1}$
K	Gain, used for temperature in improved case (dimensionless)
L	Total length of heat exchanger's tubes or shell (m)
l_n	Length of nth section (m)
M	Amount of mass accumulated in each section (kgm)
$\dot{M}$	Mass flow rate (kgm/sec)
N	Number of Lumped-Models (dimensionless)
$\dot{Q}$	Total heat transfer per unit time (cal/sec)
r	Ratio of velocities of two fluids (dimensionless)
s	Laplace transform variable with respect to θ_0 (dimensionless)
T_i	Temperature of ith fluid (°C), (i=1,2,...)

T^n	Temperature of nth section (°C)
t	Time (sec)
τ_n	Time during that fluid goes through nth section (sec)
U	Overall heat transfer coefficient ($cal/m^2.sec.C$)
V	Velocity of fluid (m/sec)
$\dot{W}_{sh}$	Shear work per unit time (joul/sec)
α	Temperature factor, in stirred tank assumption (dimensionless)
θ	Phase angle (deg.)
θ_o	Dimensionless time
ρ	Density of fluid (kg/m^3)
φ	Distribution coefficient for heat transfer (dimensionless)
γ	$\varphi.a_1$ or $\varphi.a_2$
Ω	Frequency (dimensionless)

Subscripts:

1	Supply flow (control agent), tube side
2	Demand flow (controlled medium), shell side
c	Cold fluid
H	Hot fluid
i	Inlet
n	nth section
o	Outlet

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SECTION I

INTRODUCTION

One of the problems which has recently arisen is accounting for thermal lag in fluid systems; in the general case, such systems contain heat capacitance, which produce significant effects in many situations such as mechanical, chemical, and aeronautical applications where precise temperature control is very important. Temperature-control systems in air-craft are subject to extreme environmental variation. The controls must be designed to adjust quickly to these changes in ambient conditions so as to deliver an air stream without excessive temperature fluctuation. The choice among proposed control systems which achieve a required steady state is based on their transient operation. It is then desirable that the analysis of systems be carried quite far with paper and pencil alone leaving a minimum of adjustment to be made on a working model. However, at present, temperature control problems are solved mainly through costly experimentation on models of proposed systems.

To obtain design information without experiment, equations describing the transient operation of the separate parts of a proposed system must be solved. A major difficulty is that the equations for many of these parts are so complicated that only very rough approximate methods have been available for their solutions. One such general system which is used in almost all areas is the heat exchanger. Upon changing physical or chemical conditions of the working fluids flows in such a system, it is then important to know what is happening in different parts of industrial or power plants, etc., containing such heat exchangers. Therefore, it would be very useful to determine sensitivity of heat exchangers response upon changing different parameters.

Finally, it should be mentioned that the engineer frequently must simplify the basic scientific picture to make it more useful for practical application. Calculations, which render approximate results but at the same time allow a rapid survey over a wide range of conditions and assumptions, are important in the approach to an engineering problem. They serve as timesavers in that they confine the more detailed investigations to a smaller numerical range. The purpose of this thesis is to explore the task of finding simpler forms of the exact equations for dynamic response of heat-exchangers.

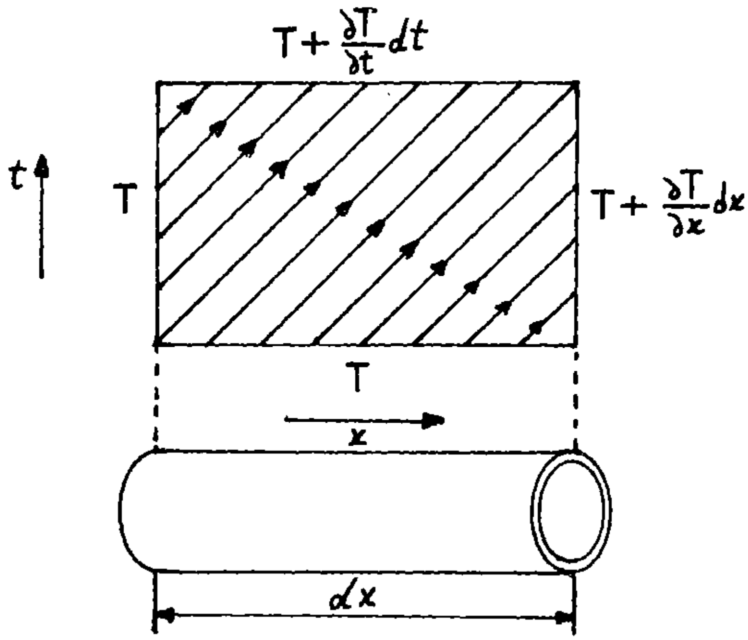
A PICTURE OF PRECEDING ATTEMPTS FOR THIS PROBLEM:

In the literature, dynamic performance of many heat exchangers of various configurations have been explored in very great detail . Takahashi (2)presented transfer function analysis of heat exchangers processes in 1952, then Paynter and Takahashi (1) gave a new method of evaluating dynamic response of heat exchangers, Hsu and Gilbert represented the same results of Takahashi in 1966. Wen-Jei Yang (17) has produced an analysis of transient heat transfer in a vapor-heated heat exchanger with arbitrary time-wise variant flow perturbation. Myers and others (16),(18) analysed the transient response of cross-flow heat exchangers, evaporators and condensers. Rea and Ablow (13) presented a model for transient air temperatures in a duct. They investigated experimentally and theoretically a thin-walled duct carrying heated air, and they found the duct wall is shown to be an important heat reservoir. Rizika (14) produced a method to find the thermal lag in systems such as heat exchanger and pipes. Dusinbere (19) showed a numerical methods for calculation of transient temperatures in pipes and heat exchangers. Finally Friedly (4) presented an asymptotic approximation for exact solution of heat exchangers. His method is useful both at high and low frequencies. All of these researches and evaluations are based on Profos(7) method of evaluating such systems, and his operator is still used for this problem.

SECTION II

EXACT SOLUTION AND FRIEDLY'S METHOD

Takahashi (2) used the Profos(7) operator to solve the heat exchanger problem as follow:



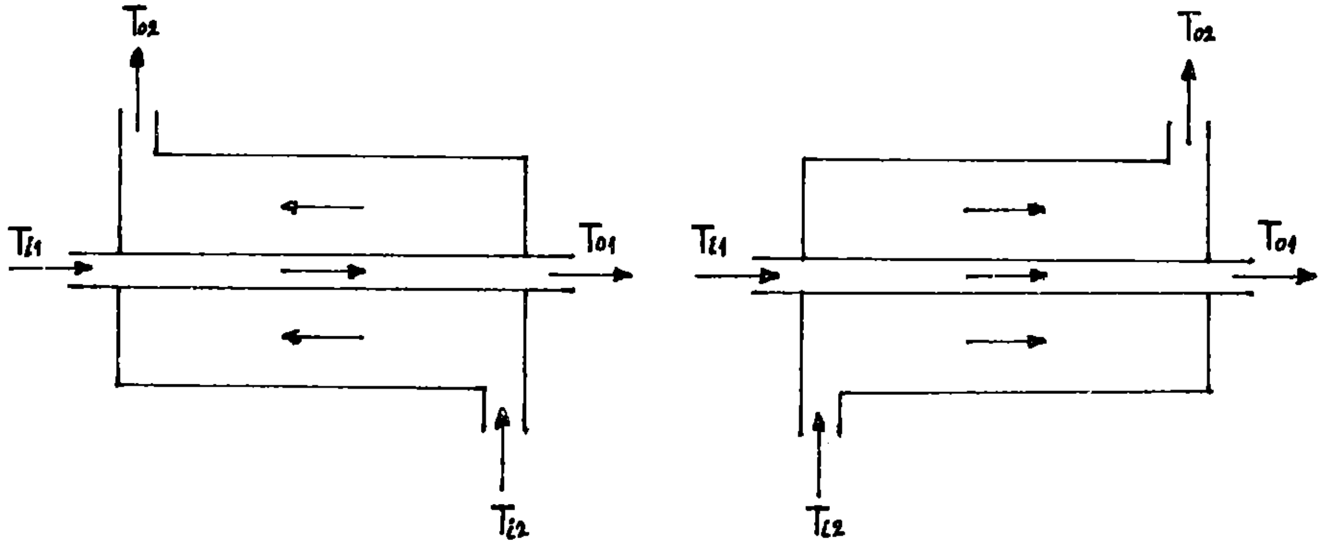
FIGURE(2-1): Profos Operator

From energy balance(applied for above element) following formula is gained:

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}=A(\varphi-T)$$

Where φ is the surface temperature of solid.

By applying this result for the heat exchanger of Fig.(2-2), the following results are produced:



FIGURE(2-2):Counter-Flow and Parallel-Flow Heat-Exchanger

$$\begin{cases} \dfrac{\partial T_1}{\partial \tau} + \dfrac{\partial T_1}{\partial x} = a_1(T_2 - T_1) \\ r\dfrac{\partial T_2}{\partial \tau} \pm^{*} \dfrac{\partial T_2}{\partial x} = a_2(T_1 - T_2) \end{cases}$$

[*Plus sign is for parallel flow, minus sign is for counter-flow.]

These results are based on the assumption that both fluids are unmixed, solid capacities are neglected, and system parameters are constant.

By assuming that the "hot fluid" (tube side) of temperature T_1 is the control agent (supply side), and the "cold fluid" (shell side), fluid of temperature T_2 is the controlled medium (demand side), the transfer function is:

$$G(j\Omega)=\frac{T_{o2}}{T_{i1}}$$

Then the transfer functions are in the following form:

i- For the parallel flow:

$$G(j\Omega)=\frac{g_2}{P_1-P_2}\left(e^{P_1}-e^{P_2}\right) \qquad (2\text{-}1)$$

Where $$P_1\,,P_2=\left[-(f_1+f_2)\pm\sqrt{(f_1-f_2)^2+4g_1g_2}\,\right]/2 \qquad (2\text{-}2)$$

ii- For the counter flow:

$$G(j\Omega)=\frac{g_2\left(1-e^{P_1-P_2}\right)}{-\left(P_1-P_2e^{P_1-P_2}\right)+f_2\left(1-e^{P_1-P_2}\right)} \qquad (2\text{-}3)$$

Where $$P_1,P_2=\left[-(f_1-f_2)\pm\sqrt{(f_1+f_2)^2-4g_1g_2}\,\right]/2$$

The parameters are defined as follow:

$f_1=a_1+j\Omega$

$f_2=a_2+jr\Omega$

$g_1=a_1$

$g_2=a_2$

Friedly's approximate method is based on Schöe's(4) approach to the dynamics of double-pipe heat exchangers, which is in the general form of:

$$\bar{G}(S)=\frac{m+TS}{1+TS}\times\frac{1-e^{-(\alpha_1+\beta_1 s)}}{\alpha_2+\beta_2 s} \qquad (2\text{-}5)$$

Where α_i , β_i are constants which result from the high-frequency limit of the exact transfer function and m results from the low frequency limit. The time constant T is arbitrary and adjusted to match the exact frequency response as well as possible. Friedly retained the delay time but eliminated the adjustable time constant. Then his approximate form is:

$$\bar{G}(S) = \frac{K}{1+TS}\left[1 - e^{-(\alpha_1+\beta_1 S)}\right] \qquad (2\text{-}6)$$

Where:

$$\frac{K}{T} = \frac{1}{\beta_2} \quad , \quad K = \frac{m}{a_1}$$

In other words, he expanded the denominator of exact solution and neglected the terms of order two and more; therefore he got:

$$G(x, S\rightarrow\infty) \longrightarrow \frac{a_2 \, e^{a_2(x-1)}}{(1+r)S}\left[e^{-r(1-x)S} - e^{-(x+r)S-(a_1+a_2)x} + \cdots\right]$$

$$G(x,0) = a_2 \frac{e^{(a_1-a_2)x} - 1}{a_2 e^{(a_2-a_1)} - a_1}$$

Therefore his approximate transfer function is:

$$\bar{G}(x,S) = \frac{K}{1+TS}\left[e^{-r(1-x)S} - e^{-(a_1+a_2)x} \cdot e^{-(x+r)S}\right] \qquad (2\text{-}7)$$

Where:

$$\frac{K}{T} = \frac{a_2}{1+r}\, e^{a_2(x-1)}$$

$$K = \frac{a_2}{1-e^{-(a_1+a_2)x}} \times \frac{e^{(a_2-a_1)x} - 1}{a_2 e^{a_2-a_1} - a_1}$$

But as the results show , this method is infinite order and Friedly used a distributed system model.

It should be mentioned that after presenting the exact solution, Paynter and Takahashi (1) created another method for evaluating which based on monotonic systems that Paynter (3) presented. As Paynter says: The Laplace transform solution of a monotone process can be written as:

$$G(S)=e^{\zeta - T_m S + \frac{T_s^2}{2} S^2 - \frac{T_a^3}{6} S^3 + \cdots\cdots}$$

Where the parameters ζ , T_m , T_s , T_a are given in terms of system constants, the symbol S is the complex variable of the Laplace transformation. In summary, ζ measures the steady state amplitude ratio between response and disturbance, T_m measures the mean time delay between response and disturbance, T_s defines the dispersion or att uation, and T_a the assymmetry or phase nonlinearity. This characterization is very efficient for any physical process where the step response is monotonic and nondecreasing in time.

If the Laplace transformation of system is in the form of

$$G(S)=\frac{1}{1+a_1 S+a_2 S^2+a_3 S^3+\cdots\cdots} = \frac{1}{\sum_k a_k S^k} \qquad (2\text{-}8)$$

Where the denominator polynomial may be either finite or infinite, then the monotone parameters can be obtained directly by following relationships:

$T_m = a_1$

$T_s^2 = a_1^2 - 2a_2$

$T_a^3 = 2a_1^3 - 6a_1a_2 + 6a_3$

......etc........

These monotonic parameters are obtained by this method for simple model of this present paper.

SECTION III

ANALYSIS AND FORMATION OF HEAT EXCHANGER PROBLEM:

The dynamic performance of a cocenteric pipe heat exchanger could be characterized by sol ution of four simultaneous nonlinear partial differential equations, the assumptions for this analysis are:

1-The heat flow and temperature distribution are functions of time and axial distance from tube inlet.

2-Both the inner radius and the outer radius of the tube are assumed constant.

3-The tube material is homogeneous and isentropic, the density and the specific heat are constant.

4-The thermal conductivity of tube material is zero in the axial direction; the thermal conductivity of the tube material is considered infinite in the radial direction, this condition valid for thin metal walls.

5-There is no energy source within the tube material itself.

6-The thermal conductivity in the outer wall in the longitudinal direction is zero and in the transverse direction is finite, a condition valid for thick insulated walls.

7-The film coefficients of heat transfer between the fluid and tube material h are uniform and constant over the inner and outer tube surfaces.

8- The specific heats at constant pressure of both the inner and outer fluids are assumed to be constant.

9- The fluid pressure at any section is indepentent of time for both inner and outer fluids.

Nonlinear partial differential equations are:

Heat balance, for inner fluid:

$$\frac{\partial T_1}{\partial \tau} + \frac{\partial T_1}{\partial x} = \frac{U_1 A_1}{M_1 C_1}(T_H - T_1)$$

Heat balance, inner wall:

$$\frac{\partial T_H}{\partial \tau} = \frac{U_1 A_1}{M_1 C_1}(T_1 - T_H) + \frac{U_2 A_2}{M_2 C_2}(T_2 - T_H)$$

Heat balance, outer fluid:

$$\frac{M_1}{M_2} \times \frac{\partial T_2}{\partial \tau} \pm^{*} \frac{\partial T_2}{\partial x} = \frac{U_2 A_2}{M_2 C_2}(T_H - T_2) + \frac{U_s A_s}{M_2 C_2}(T_{so} - T_2)$$

Heat balance, outer wall:

$$\frac{\partial T_s}{\partial \tau} = \frac{a_s \theta_1}{r^2} \times \frac{\partial^2 T_s}{\partial y^2}$$

[*Positive sign is for parallel-flow, negative sign is for counter-flow.]

These results can be gotten by Profos(7) operator, in appendix(a) there are the manners of getting these formulas.. These formulas are for general cases that the thermal storage of walls aren't negligible.

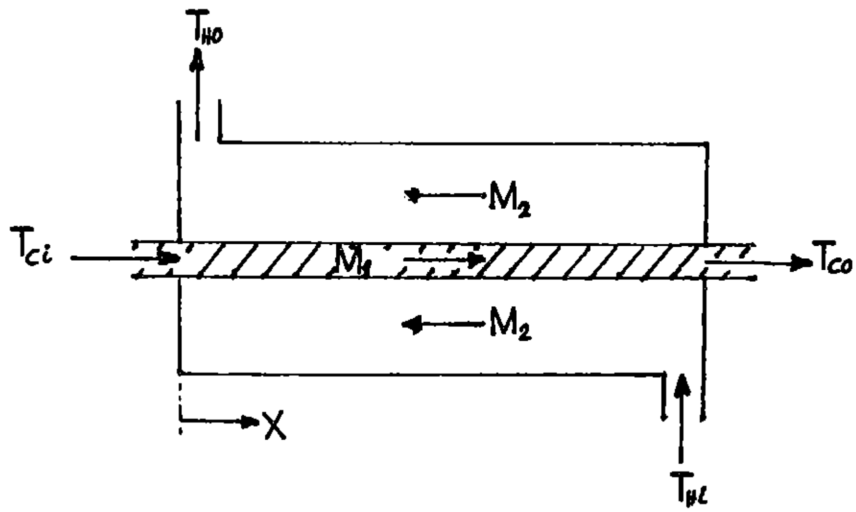
Analysis the problem consists of two parts: Static case, and Dynamic response, as follow:

STATIC CASE OF HEAT EXCHANGERS:

Based on basic assumptions and neglecting the thermal storage of walls, for steady state case, from energy balance (appendix-a), the results are:

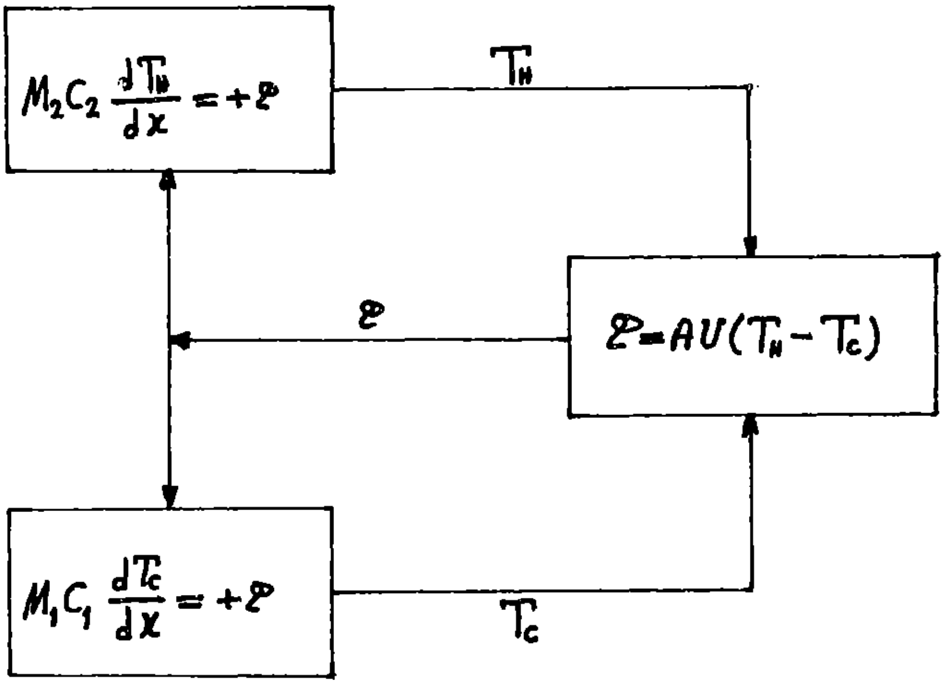
for counter-flow

$$\begin{cases} M_1 C_1 \dfrac{dT_c}{dx} = +\mathcal{P} \\ M_2 C_2 \dfrac{dT_H}{dx} = +\mathcal{P} \end{cases} \qquad (3-1)$$



FIGURE(3-1): Counter Flow Heat Exchanger

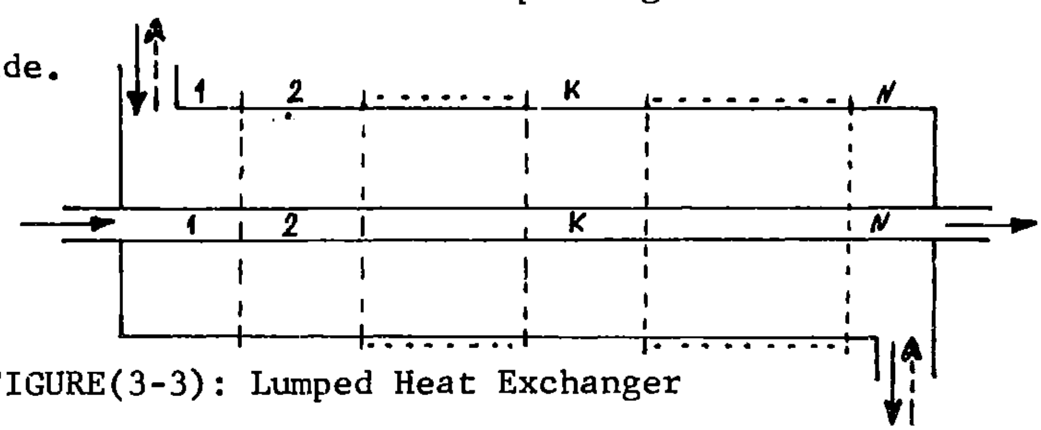
A model for this counter-flow case is shown in figure (3-2):



FIGURE(3-2): Model for Counter-Flow Heat Exchanger

Since the objective of this thesis is to reduce the mathematical model to a set of simultaneous ordinary differential equations, the system will be "finite differenced" or "lumped"

The exchanger is divided into N sections (Fig.3-3) with each section on **tube** side corresponding to a section on the shell side.



FIGURE(3-3): Lumped Heat Exchanger

For each lump the Profos(7) operator is used then:

$$\tau_k s \bar{T}_{Hk} = \bar{T}_{Hk-1} - \bar{T}_{Hk} - \gamma_{Hk}(\bar{T}_{Hk} - \bar{T}_{cj}) \qquad \text{Hot Side}$$

$$\tau_j s \bar{T}_{cj} = \bar{T}_{cj-1} - \bar{T}_{cj} + \gamma_{cj}(\bar{T}_{Hk} - \bar{T}_{cj}) \qquad \text{Cold Side}$$

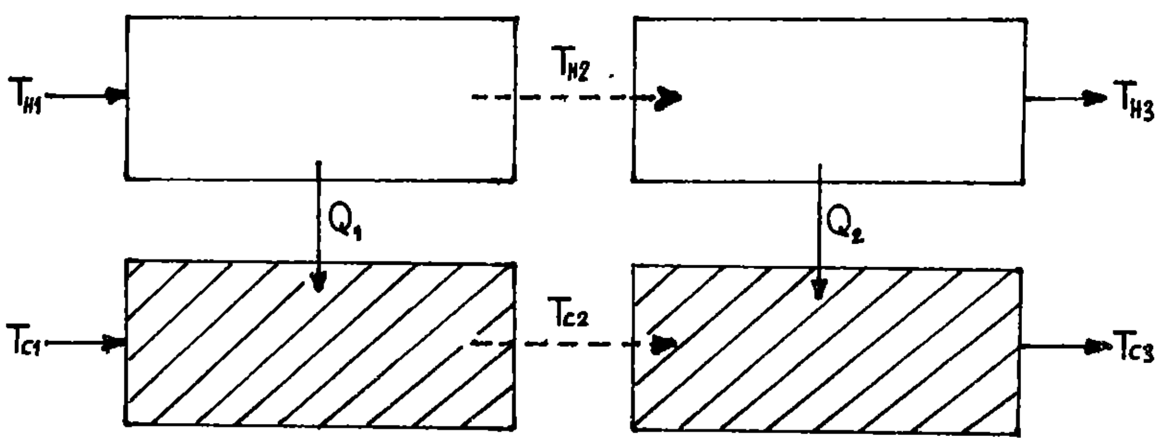
Calculations to get these results are in appendix(c). For static case(steady-state) they are:

$$\begin{cases} T_{Hk-1} - T_{Hk} - \gamma_{Hk}(T_{Hk} - T_{cj}) = 0 \\ T_{cj-1} - T_{cj} + \gamma_{cj}(T_{Hk} - T_{cj}) = 0 \end{cases} \qquad (3\text{-}2)$$

Therefore, it depends on number of lumps, and it is clear that by increasing this number better results will be gotten.

For N=2:

i- Parallel Flow Heat Exchanger:



FIGURE(3-4): Two Lumped Models Heat Exchanger(Parallel)

Calculations are in appendix(c) which give following results by disturbing only inlet temperature of hot side fluid:

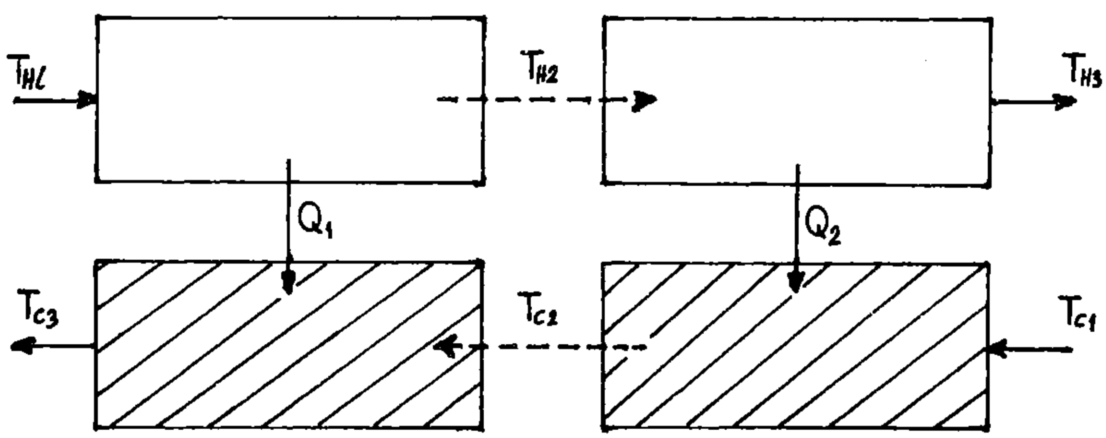
$$\frac{T_{C3}}{T_{H1}} = \frac{\varphi_2 a_2(1+\varphi_1 a_2) + \varphi_1 a_2(1+\varphi_2 a_1)}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{(1+\varphi_2 a_2)(1+\varphi_1 a_2) + \varphi_1\varphi_2 a_1 a_2}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{1+\varphi_1 a_2}{(1+\varphi_1 a_1)(1+\varphi_1 a_2) - \varphi_1^2 a_1 a_2}$$

$$\frac{T_{C2}}{T_{H1}} = \frac{\varphi_1 a_2}{(1+\varphi_1 a_1)(1+\varphi_1 a_2) - \varphi_1^2 a_1 a_2}$$

ii- Counter Flow Heat Exchanger:



FIGURE(3-5): Tow Lumped Models Heat Exchanger(counter-flow)

The beauty of this model is that for counter-flow only one of the fluids is reversed and everything is the same:

$$\frac{T_{C3}}{T_{H1}} = \frac{\varphi_1 a_2(1+\varphi_2 a_1+\varphi_2 a_2) + \varphi_2 a_2}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2) - \varphi_1\varphi_2 a_1 a_2}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_2)}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)-\varphi_1\varphi_2 a_1 a_2}$$

$$\frac{T_{C2}}{T_{H1}} = \frac{\varphi_2 a_2(1+\varphi_1 a_2)}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)-\varphi_1\varphi_2 a_1 a_2}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)}{(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_1+\varphi_2 a_2)-\varphi_1\varphi_2 a_1 a_2}$$

Now, for different values of a_1 , a_2 we can compare the results of this model and results of exact solution, but by changing φ_1 and φ_2 the optimum results can be gained.Table(3-1) through(3-4) contain the results of different values of a_1 , a_2 φ_1 , and φ_2 .

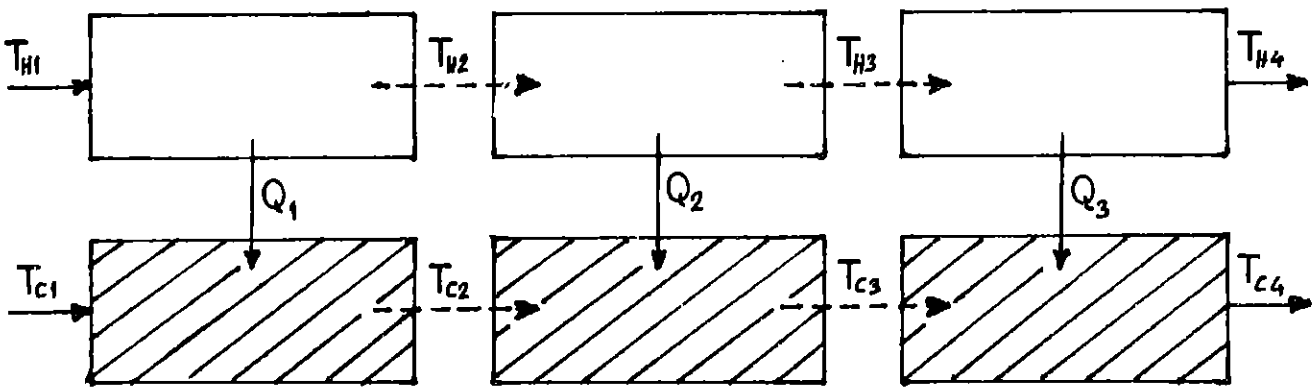
For N=3:

i-Parallel Flow: As shown following , for this case there are six equations for six outlet temperatures as shown in Fig. (3-6):

Equations are:

$$\begin{cases} T_{H1} - (1+\varphi_1 a_1)T_{H2} + \varphi_1 a_1 T_{C2} = 0 \\ \varphi_1 a_2 T_{H2} + T_{C1} - (1+\varphi_1 a_2)T_{C2} = 0 \\ T_{H2} - (1+\varphi_2 a_1)T_{H3} + \varphi_2 a_1 T_{C3} = 0 \end{cases}$$

$$\begin{cases} \varphi_2 a_2 T_{H3} + T_{C2} - (1+\varphi_2 a_2) T_{C3} = 0 \\ T_{H3} - (1+\varphi_3 a_1) T_{H4} + \varphi_3 a_1 T_{C4} = 0 \\ \varphi_3 a_2 T_{H4} + T_{C3} - (1+\varphi_3 a_2) T_{C4} = 0 \end{cases}$$



FIGURE(3-6): Three Lumped Models Heat Exchanger(parallel-flow)

Therefore, results for disturbing only inlet temperature of hot fluid(T_{H1}) are:

$$\frac{T_{C4}}{T_{H1}} = \frac{\varphi_3 a_2(1+\varphi_1 a_2)(1+\varphi_2 a_2) + a_2(1+\varphi_3 a_1)(\varphi_1+\varphi_2+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2) + \varphi_1\varphi_2\varphi_3 a_1 a_2^2}{\prod_{i=1}^{3}(1+\varphi_i a_1+\varphi_i a_2)}$$

$$\frac{T_{H4}}{T_{H1}} = \frac{\prod_{i=1}^{3}(1+\varphi_i a_2) + \varphi_3 a_1 a_2(\varphi_2+\varphi_1+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2) + \varphi_1\varphi_2 a_1 a_2(1+\varphi_3 a_2)}{\prod_{i=1}^{3}(1+\varphi_i a_1+\varphi_i a_2)}$$

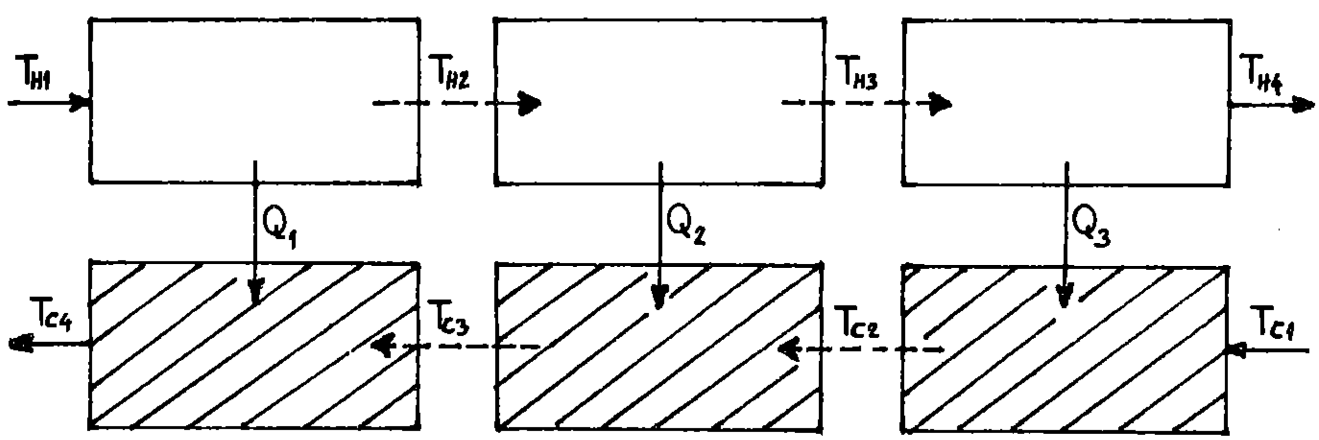
$$\frac{T_{C3}}{T_{H1}} = \frac{a_2(\varphi_1+\varphi_2+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2)}{(1+\varphi_2 a_1+\varphi_2 a_2)(1+\varphi_1 a_1+\varphi_1 a_2)}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_2)+\varphi_1\varphi_2 a_1 a_2}{(1+\varphi_2 a_1+\varphi_2 a_2)(1+\varphi_1 a_1+\varphi_1 a_2)}$$

$$\frac{T_{C2}}{T_{H1}} = \frac{\varphi_1 a_2}{1+\varphi_1 a_1+\varphi_1 a_2}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{1+\varphi_1 a_2}{1+\varphi_1 a_1+\varphi_1 a_2}$$

ii- For Counter-Flow(N=3): As shown in following, these six equations are for this case(see Fig.3-7):



FIGURE(3-7): Three Lumped Models Heat Exchanger(counter-flow)

$$\begin{cases} T_{H1} - (1+\varphi_1 a_1) T_{H2} + \varphi_1 a_1 T_{C4} = 0 \\ \varphi_1 a_2 T_{H2} + T_{C3} - (1+\varphi_1 a_2) T_{C4} = 0 \\ T_{H2} - (1+\varphi_2 a_1) T_{H3} + \varphi_2 a_1 T_{C3} = 0 \\ \varphi_2 a_2 T_{H3} + T_{C2} - (1+\varphi_2 a_2) T_{C3} = 0 \\ T_{H3} - (1+\varphi_3 a_1) T_{H4} + \varphi_3 a_1 T_{C2} = 0 \\ \varphi_3 a_2 T_{H4} + T_{C1} - (1+\varphi_3 a_2) T_{C1} = 0 \end{cases}$$

Results for disturbing T_{H1} are:

$$\frac{T_{C4}}{T_{H1}} = \frac{\varphi_1 a_2 (1+\varphi_2 a_2)(1+\varphi_2 a_1)(1+\varphi_3 a_2+\varphi_3 a_1) + (1-\varphi_1\varphi_2 a_1 a_2)(\varphi_2 a_2+\varphi_2\varphi_3 a_2^2+\varphi_2\varphi_3 a_1 a_2+\varphi_3 a_2)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2} (\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i\varphi_{i+1} a_2)}$$

$$\frac{T_{H4}}{T_{H1}} = \frac{(1+\varphi_1 a_2-\varphi_2 a_2) \prod_{i=1}^{3} (1+\varphi_i a_2)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2} (\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i\varphi_{i+1} a_2)}$$

$$\frac{T_{C3}}{T_{H1}} = \frac{a_2 (1+\varphi_1 a_2)(\varphi_2+\varphi_3+\varphi_2\varphi_3 a_2+\varphi_2\varphi_3 a_1)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2} (\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i\varphi_{i+1} a_2)}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_3 a_2+\varphi_3 a_1)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2} (\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i\varphi_{i+1} a_2)}$$

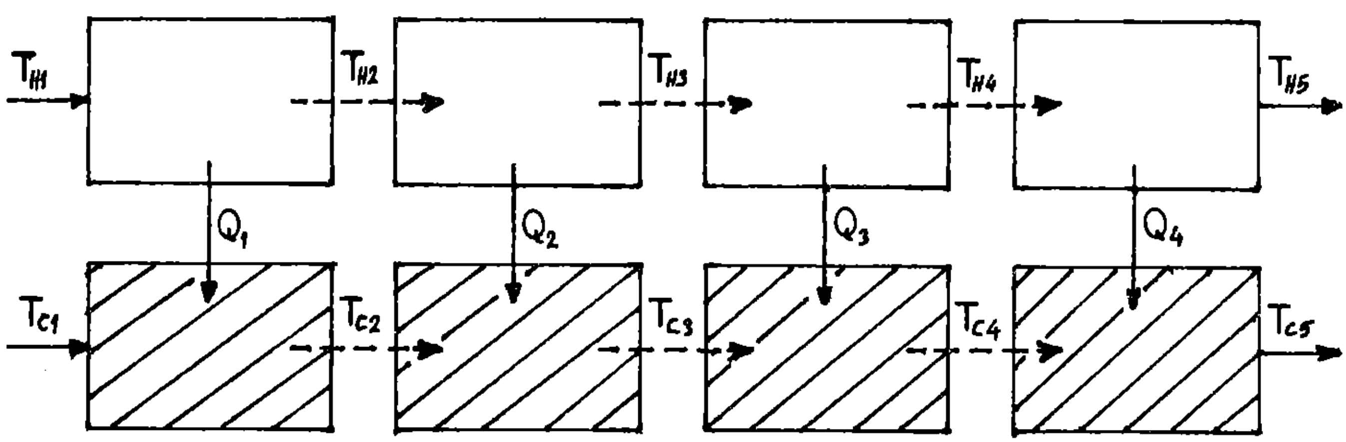
$$\frac{T_{C2}}{T_{H1}} = \frac{\varphi_3 a_2 (1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_1 a_2-\varphi_2 a_2)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_2 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2}(\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i \varphi_{i+1} a_2)}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1)(1+\varphi_3 a_1+\varphi_3 a_2) - a_1 a_2 \varphi_2 (\varphi_2+\varphi_3+\varphi_2\varphi_3 a_2+\varphi_2\varphi_3 a_1)}{(1+\varphi_3 a_2+\varphi_3 a_1)(1+\varphi_1 a_1+\varphi_2 a_2)(1+\varphi_2 a_2)(1+\varphi_2 a_1) - a_1 a_2 \prod_{i=1}^{2}(\varphi_i+\varphi_{i+1}+\varphi_i\varphi_{i+1} a_1+\varphi_i \varphi_{i+1} a_2)}$$

The results for different values of a_1, a_2, φ_1, φ_2, and φ_3 are in tables (3-1) through (3-4)

For N=4:

i- Parallel flow heat exchanger, for eight outlet temperatures as shown in Fig.(3-8) there are eight equations which can be obtained by using the energy balance for each section.



FIGURE(3-8): Four Lumped Model Heat-Exhanger(parallel-flow)

$$\begin{cases} T_{H1} - (1+\varphi_1 a_1) T_{H2} + \varphi_1 a_1 T_{C2} = 0 \\ \varphi_1 a_2 T_{H2} + T_{C1} - (1+\varphi_1 a_2) T_{C2} = 0 \end{cases}$$

$$
\begin{cases}
\varphi_2 a_2 T_{H3} + T_{C2} - (1+\varphi_2 a_2) T_{C3} = 0 \\
T_{H3} - (1+\varphi_3 a_1) T_{H4} + \varphi_3 a_1 T_{C4} = 0 \\
\varphi_3 a_2 T_{H4} + T_{C3} - (1+\varphi_3 a_2) T_{C4} = 0 \\
T_{H2} - (1+\varphi_2 a_1) T_{H3} + \varphi_2 a_1 T_{C3} = 0 \\
T_{H4} - (1+\varphi_4 a_1) T_{H5} + \varphi_4 a_1 T_{C5} = 0 \\
\varphi_4 a_2 T_{H5} + T_{C4} - (1+\varphi_4 a_2) T_{C5} = 0
\end{cases}
$$

The results for disturbing T_{H1} are:

$$
\frac{T_{C5}}{T_{H1}} = \frac{a_2 \prod_{i=1}^{2} \left[\varphi_{2i-1}+\varphi_{2i}+\varphi_{2i-1}\varphi_{2i}(a_1+a_2)\right]\left[1+\varphi_{5-2i} a_1+\varphi_{6-2i} a_2+a_1 \varphi_{5-2i}\varphi_{6-2i}(a_1+a_2)\right]}{\prod_{i=1}^{4}(1+\varphi_i a_1+\varphi_i a_2)}
$$

$$
\frac{T_{H5}}{T_{H1}} = \frac{\prod_{i=1}^{2}\left[1+\varphi_{2i-1} a_2+\varphi_{2i} a_2+a_1\varphi_{2i}\varphi_{2i-1}(a_1+a_2)\right]+a_1 a_2 \prod_{i=1}^{2}(\varphi_{2i-1}+\varphi_{2i}+\varphi_{2i}\varphi_{2i-1} a_1+\varphi_{2i}\varphi_{2i-1} a_2)}{\prod_{i=1}^{4}(1+\varphi_i a_1+\varphi_i a_2)}
$$

$$
\frac{T_{C4}}{T_{H1}} = \frac{\varphi_3 a_2(1+\varphi_1 a_2)(1+\varphi_2 a_2)+a_2(1+\varphi_3 a_1)(\varphi_1+\varphi_2+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2)+\varphi_1\varphi_2\varphi_3 a_1 a_2^2}{\prod_{i=1}^{3}(1+\varphi_i a_1+\varphi_i a_2)}
$$

$$
\frac{T_{H4}}{T_{H1}} = \frac{\prod_{i=1}^{3}(1+\varphi_i a_2)+\varphi_3 a_1 a_2(\varphi_2+\varphi_1+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2)+\varphi_1\varphi_2 a_1 a_2(1+\varphi_3 a_2)}{\prod_{i=1}^{3}(1+\varphi_i a_1+\varphi_i a_2)}
$$

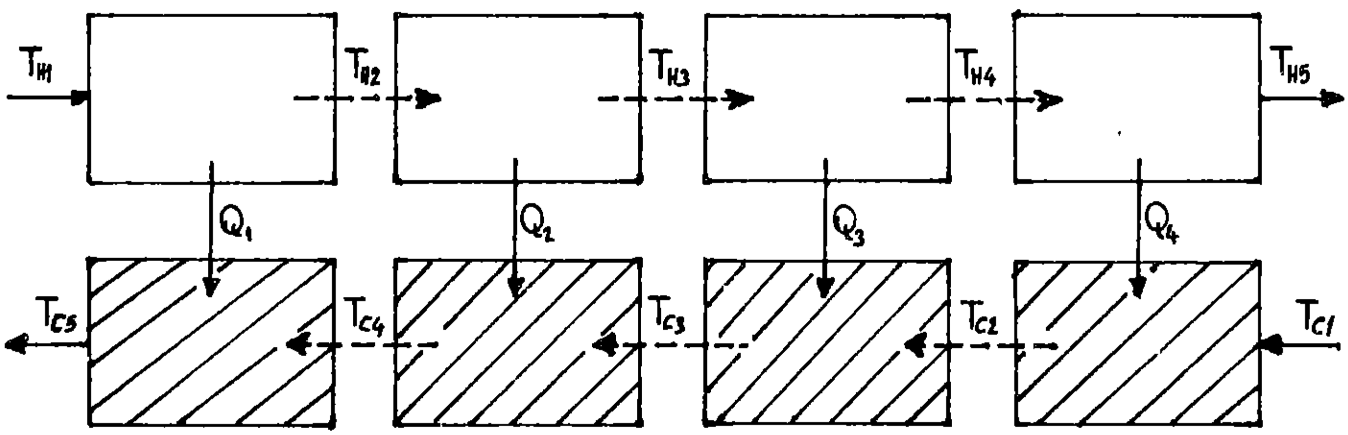
$$
\frac{T_{C3}}{T_{H1}} = \frac{a_2(\varphi_2+\varphi_1+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2)}{(1+\varphi_2 a_1+\varphi_2 a_2)(1+\varphi_1 a_1+\varphi_1 a_2)}
$$

$$
\frac{T_{H3}}{T_{H1}} = \frac{(1+\varphi_1 a_2)(1+\varphi_2 a_2)+\varphi_1\varphi_2 a_1 a_2}{(1+\varphi_2 a_1+\varphi_2 a_2)(1+\varphi_1 a_1+\varphi_1 a_2)}
$$

$$\frac{T_{C2}}{T_{H1}} = \frac{\varphi_1 a_2}{1+\varphi_1 a_1 + \varphi_1 a_2}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{1+\varphi_1 a_2}{1+\varphi_1 a_1 + \varphi_1 a_2}$$

ii- For Counter-Flow(N=4): from appendix(e) which shows how to get the equations, and Fig.(3-9) the following results obtained:



FIGURE(3-9): Four Lumped Models Heat-Exchanger(Counter-Flow)

$$\begin{cases} T_{H1} - (1+\varphi_1 a_1) T_{H2} + \varphi_1 a_1 T_{C5} = 0 \\ \varphi_1 a_2 T_{H2} + T_{C4} - (1+\varphi_1 a_2) T_{C5} = 0 \\ T_{H2} - (1+\varphi_2 a_1) T_{H3} + \varphi_2 a_1 T_{C4} = 0 \\ \varphi_2 T_{H3} a_2 + T_{C3} - (1+\varphi_2 a_2) T_{C4} = 0 \\ T_{H3} - (1+\varphi_3 a_1) T_{H4} + \varphi_3 a_1 T_{C3} = 0 \\ \varphi_3 a_2 T_{H4} + T_{C2} - (1+\varphi_3 a_2) T_{C3} = 0 \end{cases}$$

$$\begin{cases} T_{H4} - (1+\varphi_4 a_1) T_{H5} + \varphi_4 a_1 T_{C2} = 0 \\ \varphi_4 a_2 T_{H5} + T_{C1} - (1+\varphi_4 a_2) T_{C2} = 0 \end{cases}$$

Therefore, the outlet temperatures by disturbing T are:

$$\frac{T_{C5}}{T_{H1}} = \frac{\varphi_1 a_2 (1+\varphi_2 a_1+\varphi_1 a_2) \prod_{i=3}^{4}[1+\varphi_i(a_1+a_2)] + a_2 \sum_{i=1}^{2} \varphi_{i+1}[1+\varphi_{2i}(a_1+a_2)](1-\varphi_i\varphi_{5-i} a_1 a_2) + \varphi_4 a_2 (1-\varphi_1\varphi_2 a_1 a_2 - \varphi_2\varphi_3 a_1 a_2)}{\prod_{i=1}^{4}[1+\varphi_i(a_1+a_2)] + \varphi_1\varphi_2\varphi_3\varphi_4 a_1^2 a_2^2 - \varphi_3 a_1 a_2 (1+\varphi_4 a_1+\varphi_4 a_2)(\varphi_1+\varphi_2+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2) - \varphi_1\varphi_2 a_1 a_2 \prod_{i=3}^{4}(1+\varphi_i a_1+\varphi_i a_2) - \varphi_4 a_1 a_2 [\ldots]}$$

$$\frac{T_{H5}}{T_{H1}} = \frac{1+\varphi_4 a_2}{\varphi_4 a_2}\left[\frac{\varphi_1 a_2 \prod_{i=1}^{2}[1+\varphi_{2i-1}(a_1+a_2)] - \varphi_1\varphi_2\varphi_3 a_1 a_2^2 + a_2(\varphi_2+\varphi_3+\varphi_1\varphi_3 a_1+\varphi_2\varphi_3 a_1)}{\prod_{i=1}^{2}(1+\varphi_i a_1)} + \frac{\prod_{i=1}^{3}[1+\varphi_i(a_1+a_2)] - a_1 a_2[\varphi_1\varphi_2+\varphi_2\varphi_3+\varphi_3\varphi_1+2\varphi_1\varphi_2\varphi_3(a_1+a_2)]}{\prod_{i=1}^{3}(1+\varphi_i a_1)} \frac{T_{C5}}{T_{H1}}\right]$$

$$\frac{T_{C4}}{T_{H1}} = -\frac{\varphi_1 a_2}{1+\varphi_1 a_1} + \frac{1+\varphi_1 a_1+\varphi_1 a_2}{1+\varphi_1 a_1} \times \frac{T_{C5}}{T_{H1}}$$

$$\frac{T_{H4}}{T_{H1}} = \frac{1-a_1 a_2(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3) - \varphi_1\varphi_2\varphi_3 a_1 a_2 (a_1+a_2)}{\prod_{i=1}^{3}(1+\varphi_i a_1)} + \frac{a_1(1+\varphi_1 a_1+\varphi_1 a_2)(\varphi_2+\varphi_3+\varphi_2\varphi_3 a_1+\varphi_2\varphi_3 a_2)+a_1\varphi_1(1-\varphi_2\varphi_3 a_1 a_2)}{\prod_{i=1}^{3}(1+\varphi_i a_1)} \times \frac{T_{C5}}{T_{H1}}$$

$$\frac{T_{C3}}{T_{H1}} = \frac{-\varphi_2 a_2 - \varphi_1 a_2(1+\varphi_2 a_1+\varphi_2 a_2)}{(1+\varphi_2 a_1)(1+\varphi_1 a_1)} + \frac{(1+\varphi_2 a_2+\varphi_2 a_1)(1+\varphi_1 a_1+\varphi_1 a_2) - \varphi_1\varphi_2 a_1 a_2}{(1+\varphi_2 a_1)(1+\varphi_1 a_1)} \times \frac{T_{C5}}{T_{H1}}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{1-\varphi_1\varphi_2 a_1 a_2}{(1+\varphi_2 a_1)(1+\varphi_1 a_1)} + \frac{a_1(\varphi_1+\varphi_2+\varphi_1\varphi_2 a_1+\varphi_1\varphi_2 a_2)}{(1+\varphi_2 a_1)(1+\varphi_1 a_1)} \times \frac{T_{C5}}{T_{H1}}$$

$$\frac{T_{C2}}{T_{H1}} = \frac{-\varphi_4 a_2}{1+\varphi_4 a_2} \times \frac{T_{H5}}{T_{H1}} + 2\,\frac{\prod_{i=1}^{3}[1+\varphi_i(a_1+a_2)] - a_1 a_2[\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3+2\varphi_1\varphi_2\varphi_3(a_1+a_2)]}{\prod_{i=1}^{3}(1+\varphi_i a_1)} \times \frac{T_{C5}}{T_{H1}}$$

$$\frac{T_{H2}}{T_{H1}} = \frac{1}{1+\varphi_1 a_1} + \frac{\varphi_1 a_1}{1+\varphi_1 a_1} \times \frac{T_{C5}}{T_{H1}}$$

Adding the number of Lumped elements gives a better solution compare with exact results, but if only a simple model is desired then only two, three, and four Lumped elements need to be used, as shown later, by choosing good values for the distribution coefficients φ_1 & φ_2 results will be very close to exact results.

The restrictive assumptions which are shown at the beginning of this analysis may be omitted and the solution will appear for the more general case. This solution can be realized by virtue of the Lumped Model approach via appropriate choice of parameters. Quantitive effects of variable heat transfer coefficients, existance of heat capacitors, heat interaction with ambient or any other cases can be practically obtained only by getting the results of a few equations in a very simple form.

The results of steady state(static) case are tabulated and plotted. In appendix(a) the exact solutions are presented, for steady state case, and in Fig.(3-12a) through(3-13b)the results of exact method and several Lumped Models are shown, for the static case.

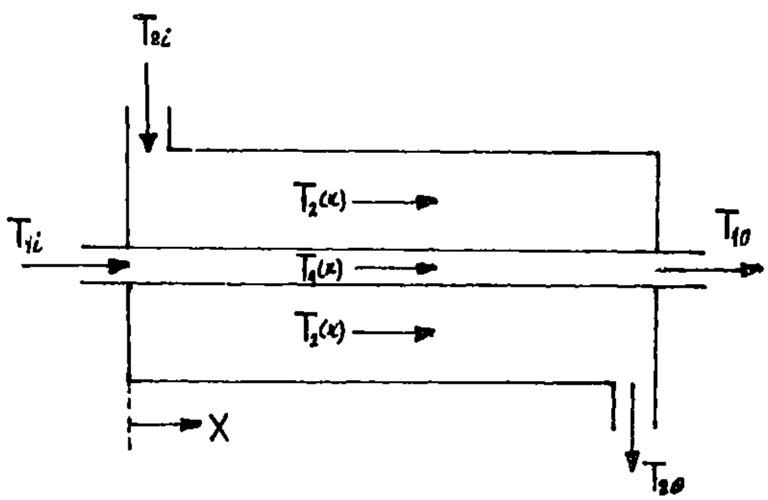
Because of differences between exact results and the Lumped Models, fitting the Lumped Model parameters via the exact solution yields substantial improvement. First the parallel flow case is fitted and its effects on counter flow are shown, then the results of fitting counter flow and its effects on parallel flow are discussed.

The exact equations for static cases (as they are presented in appendix a) are:

i- For Parallel Flow:

$$\frac{T_2(x)}{T_{1i}} = \frac{a_2\left[1 - e^{-(a_1+a_2)x}\right]}{a_1 + a_2} \tag{3-3}$$

$$\frac{T_1(x)}{T_{1i}} = \frac{a_2 + a_1 e^{-(a_1+a_2)x}}{a_1 + a_2} \tag{3-4}$$

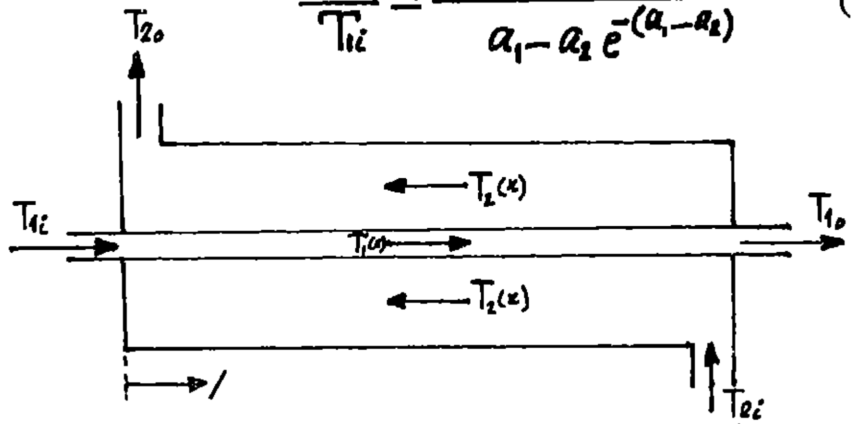


FIGURE(3-10): Parallel-Flow Heat Exchanger

ii- For Counter Flow:

$$\frac{T_2(x)}{T_{1i}} = \frac{a_1 e^{-(a_1-a_2)x} - a_2 e^{-(a_1-a_2)}}{a_1 - a_2 e^{-(a_1-a_2)}} \tag{3-5}$$

$$\frac{T_1(x)}{T_{1i}} = \frac{a_2\left[e^{-(a_1-a_2)x} - e^{-(a_1-a_2)}\right]}{a_1 - a_2 e^{-(a_1-a_2)}} \quad (3\text{-}6)$$



FIGURE(3-11): Counter-Flow Heat-Exchanger

Now, for two special cases, the results of static case are computed and plotted:

1)- For case: $a_1 = 4$, $a_2 = 1$

$T_2 \equiv T_c$ (cold fluid) $T_1 \equiv T_H$ (hot fluid)

Parallel Flow

$$\begin{cases} \dfrac{T_c(x)}{T_{Hi}} = \dfrac{1 - e^{-5x}}{5} \\[2ex] \dfrac{T_H(x)}{T_{Hi}} = \dfrac{1 + 4e^{-5x}}{5} \end{cases} \quad (3\text{-}7)$$

Counter Flow

$$\begin{cases} \dfrac{T_H(x)}{T_{Hi}} = \dfrac{4e^{-3x} - e^{-3}}{4 - e^{-3}} \\[2ex] \dfrac{T_c(x)}{T_{Hi}} = \dfrac{e^{-3x} - e^{-3}}{4 - e^{-3}} \end{cases} \quad (3\text{-}8)$$

2)- For case: $a_1 = 1$, $a_2 = 1$

Parallel Flow
$$\begin{cases} \dfrac{T_c(x)}{T_{Hi}} = \frac{1}{2}\left(1 - e^{-2x}\right) \\ \\ \dfrac{T_H(x)}{T_{Hi}} = \frac{1}{2}\left(1 + e^{-2x}\right) \end{cases} \qquad (3\text{-}9)$$

Counter Flow
$$\begin{cases} \dfrac{T_c(x)}{T_{Hi}} = \frac{1}{2}(1 - x) \\ \\ \dfrac{T_H(x)}{T_{Hi}} = 1 - \frac{1}{2}x \end{cases} \qquad (3\text{-}10)$$

The results of these two special cases are plotted in Fig. (3-12a) through(3-13b), as they indicated these results bear a very good agreement with Lumped Model results.

For making clear and easier comparison, the results of static cases for outlet temperatures are plotted respect to number of Lumped sections. For determining the approximate slopes of these results, they are plotted respect to N^m, that N is number of Lumped elements, and m is an exponent value which makes the relationship linear, and Fig. (3-14) through(3-17) show these results,indicating for each case the minimum number of Lumped elements which gives the best results, and they are shown on the figures (3-14) through(3-17)

For preceding special cases, the outlet temperatures are fitted as follows:

a)- Fitting Parallel Flow: Tables (3-1) through (3-4) contain the results of Lumped Models-outlet temperatures for each value of φ , by calling those data, and putting x=0 or x=1 in equations (3-7) through (3-10), the results are:

For case $a_1=4$, $a_2=1$

Parallel Flow: $\left.\frac{T_{oc}}{T_{iH}}\right|_{EXACT} = .1986$, $\left.\frac{T_{oH}}{T_{iH}}\right|_{EXACT} = .2054$

Lumped Models:

$$\varphi=\tfrac{1}{2} \Rightarrow \frac{T_{oc}}{T_{iH}} = .1837 \quad , \quad \frac{T_{oH}}{T_{iH}} = .2653$$

Counter Flow:

$$\varphi=\tfrac{1}{2} \Rightarrow \frac{T_{oc}}{T_{iH}} = .2 \quad , \quad \frac{T_{oH}}{T_{iH}} = .2$$

$$EXACT \Rightarrow \frac{T_{oc}}{T_{iH}} = .24 \quad , \quad \frac{T_{oH}}{T_{iH}} = .04$$

$$\left.\frac{T_{oc})_L}{T_{oc})_E}\right|_{Parallel} = \frac{1}{1.0811} \Longrightarrow \left.T_{oc})_L\right|_{Fitted-CounterFlow} = .2\times1.0811 = .2162 \Longrightarrow \left.\frac{T_{oc})_{F.L}}{T_{oc})_E}\right|_{CounterFlow} = .90$$

Therefore, by fitting the parallel flow with exact results, the counter flow gets the 90% value of the exact results, in appendix(d) calculations for other cases(different φ) are presented and their results are in tables (3-5)&(3-6).

b)- Fitting Counter-Flow: The results of Lumped Models and exact results make following calculations, they are for whole values of φ's, in appendix(d):

For $a_1 = 4$, $a_2 = 1$

Counter Flow:

$$\varphi = \frac{1}{2} \Longrightarrow \begin{cases} \frac{T_{oc}}{T_{iH}} = .2 \\ \frac{T_{oH}}{T_{iH}} = .2 \end{cases} \Longrightarrow \left.\frac{T_{oc})_E}{T_{oc})_L}\right|_{Counter} = 1.2$$

Parallel Flow:

$$\varphi = \frac{1}{2} \Longrightarrow \begin{cases} \frac{T_{oc}}{T_{iH}} = .1837 \\ \frac{T_{oH}}{T_{iH}} = .2653 \end{cases}$$

Therefore,

$$\left.T_{oc})_L\right|_{Fitted\text{-}Parallel} = .1837 \times 1.2 = .220 \Longrightarrow \left.\frac{T_{oc})_{F\text{-}L}}{T_{oc})_E}\right|_{Parallel} = 1.108$$

Then by fitting the counter flow with exact results, the parallel flow gets the 10.8% higher of exact value. Therefore, as the results show it is better to fit the parallel flow and improve the counter flow. Figures (3-14) through (3-17) show the fitting results for different flow, and different cases.

This improvement of static case, by fitting the outlet temperature, is useful for dynamic and transient responses, because the Lumped Model matches the exact solution at low frequency(by fitting) and it improves the results for high frequency or in early transient time, as they are shown later.

SECTION IV

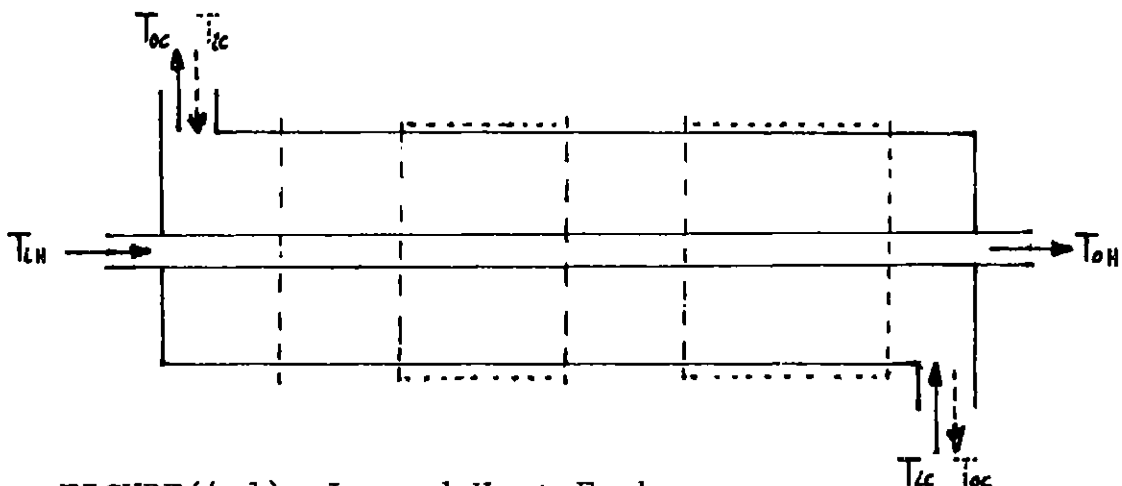
DYNAMIC AND TRANSIENT RESPONSE OF HEAT-EXCHANGERS:

In the first part of this thesis, the steady state case was discussed and equations were obtained showing the temperature gradient as a function of location of each Lumped section in the heat-exchanger. The more general case of unsteady-state operation can now be assumed so that the dependent temperature variables are functions of the independent variables. Since there is more than one independent variable, the relationship between temperature, time, and distance can be stated as a partial differential equation, as it is shown in(1), (4), (6). However, in keeping with the objective of this thesis, this partial differential equation can in turn be written as a set of simultaneous ordinary differential equations. For this staged system it is always possible to use the canonical transformation to uncouple a large number of simultaneous equations and write them in terms of single first and second order systems. However, there are cases where it is both simpler and faster to make a direct attack on the problem. A direct approach is particulary advantageous when the coefficient matrix is bi-ortri-diagonal, since for these cases the eigenvalues, or characteristic roots, can be obtained analytically.

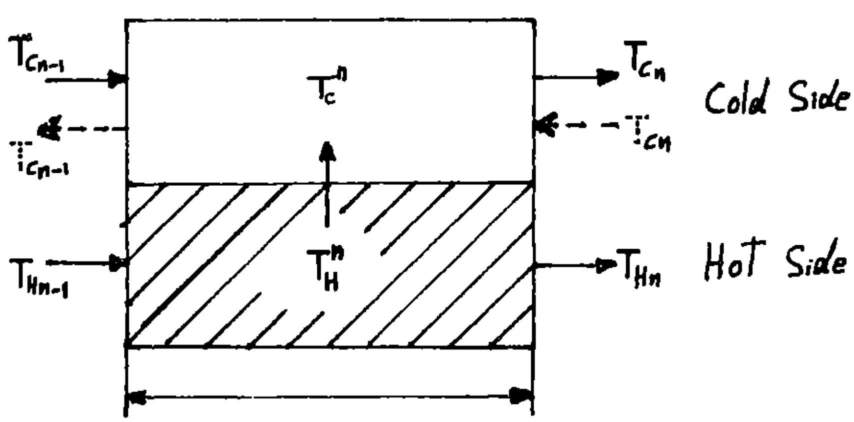
Lumped Formulation

As Fig.(4-1) shows, a heat exchanger is divided by N Lumped, from first law of thermodynamics(energy equation), there is following relationship for each Lumped:

Rate of Heat Accumulation=Heat Flow in- Heat Flow out± Heat Transferred



FIGURE(4-1): Lumped Heat Exchanger



FIGURE(4-2): Element of Heat Exchanger

For analysis the problem one of the sections should be formulated for energy balance and then it can be developed for other sections, that section analysis can be the same as Profos operator(7). By using above relationship for each section(as shown in fig.4-2) the results are:

For hot side:

Rate of heat accumulation = $M_H \cdot \frac{du_H}{dt} = M_H C_H \frac{dT_H^n}{dt}$

Heat flow in = $\dot{M}_H h_{H_{n-1}} = \dot{M}_H C_H T_{H_{n-1}}$

Heat flow out = $\dot{M}_H h_{Hn} = \dot{M}_H C_H T_{Hn}$

Heat transferred = $\dot{Q} = UA(T_H^n - T_C^n)$

Therefore, energy balance is:

$$M_H C_H \frac{dT_H^n}{dt} = \dot{M}_H C_H T_{H_{n-1}} - \dot{M}_H C_H T_{Hn} - UA_n(T_H^n - T_C^n)$$

or

$$\frac{dT_H^n}{dt} = \frac{\dot{M}_H}{M_H}(T_{H_{n-1}} - T_{Hn}) - \frac{UA_n}{M_H C_H}(T_H^n - T_C^n) \qquad (4\text{-}1)$$

An used for area because all of the sections don't need to be equal, only hot and cold sections in the same number should be the same.

$M_H = \rho_H A_s l_n$ $\qquad$ $A_s l_n$ = volume of section n

$\dot{M}_H = \rho_H A_s V_h$

By assumption of constant density and constant geometry for both sides:

continuty (mass conservation) - $\rho_1 A_s V_1 = \rho_2 A_s V_2$

or V =const.

If τ_n defined as the time which fluid going through section n then:

$$V = \frac{\ell_n}{\tau_n}$$

Therefore, $\dot{M}_H = \rho_H A_s \frac{\ell_n}{\tau_n}$

and $\frac{\dot{M}_H}{M_H} = \frac{1}{\tau_n}$

If the total length of tube is L and φ_n is the fraction or ratio of the length section n and total length, and defined as total time which hot fluid takes to flow through tube, then:

$$\varphi_n L = \ell_n \implies \tau_n = \varphi_n \tau \qquad (4\text{-}2)$$

Defined a_1 as:

$$a_1 \equiv \frac{U \cdot A}{\dot{M}_H \cdot C_H} = \frac{U \cdot P \cdot L}{\frac{M_H}{\tau_n} \cdot C_H} = \frac{U \cdot P \, \ell_n}{\varphi_n \cdot \frac{M_H}{\tau_n} \cdot C_H}$$

Where $P \cdot \ell_n = A_n$

Therefore: $$\frac{U \cdot A_n}{M_H \cdot C_H} = \frac{\varphi_n a_1}{\tau_n} \qquad (4\text{-}3)$$

By substituting (4-2)&(4-3) in(4-1):

$$\frac{dT_H^n}{dt} = \frac{1}{\tau_n}\left(T_{H_{n-1}} - T_{H_n}\right) - \frac{\varphi_n a_1}{\tau_n}\left(T_H^n - T_C^n\right) \qquad (4\text{-}4)$$

If $\theta_0 \equiv \frac{t}{\tau} \implies dt = \frac{\tau_n}{\varphi_n} d\theta_0$

"using equation(4-2)"

Therefore, equation (4-5) becomes as:

Hot Side $$\boxed{\varphi_n \frac{dT_H^n}{d\theta_0} = \left(T_{H_{n-1}} - T_{H_n}\right) - \varphi_n a_1 \left(T_H^n - T_C^n\right)}$$

The same equation can be derived for cold side flow, only r which is velocity ratio of two fluids is added to equation:

Cold Side
Parallel Flow

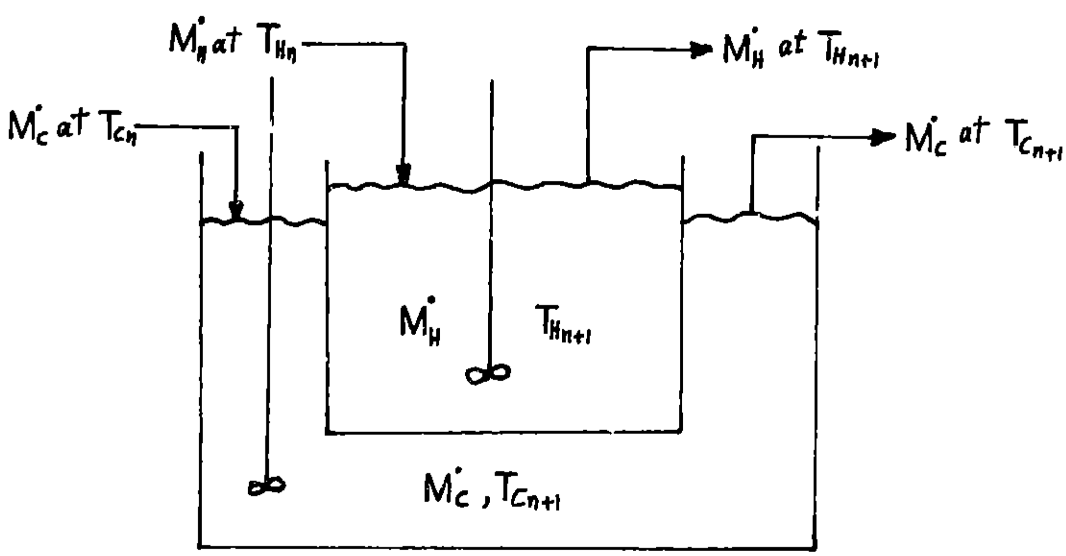
$$\varphi_n r \frac{dT_c^n}{d\theta_0} = (T_{c_{n-1}} - T_{c_n}) + \varphi_n a_2 (T_H^n - T_c^n) \qquad (4\text{-}7)$$

There are some changes because of counter flow, and for such case the equation is:

Cold Side
Counter Flow

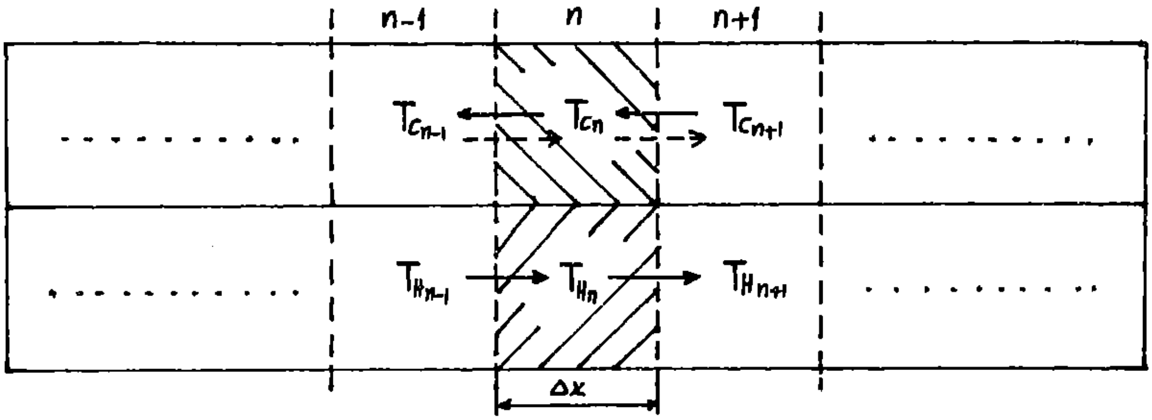
$$\varphi_n r \frac{dT_c^n}{d\theta_0} = (T_{c_n} - T_{c_{n+1}}) + \varphi_n a_2 (T_H^n - T_c^n) \qquad (4\text{-}8)$$

Before any further step, the temperature of each stage should be defined. As it was used in steady state case and also it is shown in appendix(c), the assumption of stirred tank is used for each lumped as in following figure:



FIGURE(4-3): A Model of Lumped Section

Therefore, the exit temperature of each lumped(section) is used for the temperature of whole fluid in that section, then as one of the sections picked up as an element for evaluating, it looks like Fig.(4-4):



FIGURE(4-4): nth Lumped Model

Therefore, by using this assumption and substitute the outlet temperature for section temperature, equations (4-6) through (4-8) become:

Parallel Flow
$$\begin{cases} \varphi_n \dfrac{dT_{Hn}}{d\theta_0} = (T_{Hn-1} - T_{Hn}) - \varphi_n a_1 (T_{Hn} - T_{Cn}) \\ r\varphi_n \dfrac{dT_{Cn}}{d\theta_0} = (T_{Cn-1} - T_{Cn}) - \varphi_n a_2 (T_{Hn} - T_{Cn}) \end{cases} \tag{4-9}$$

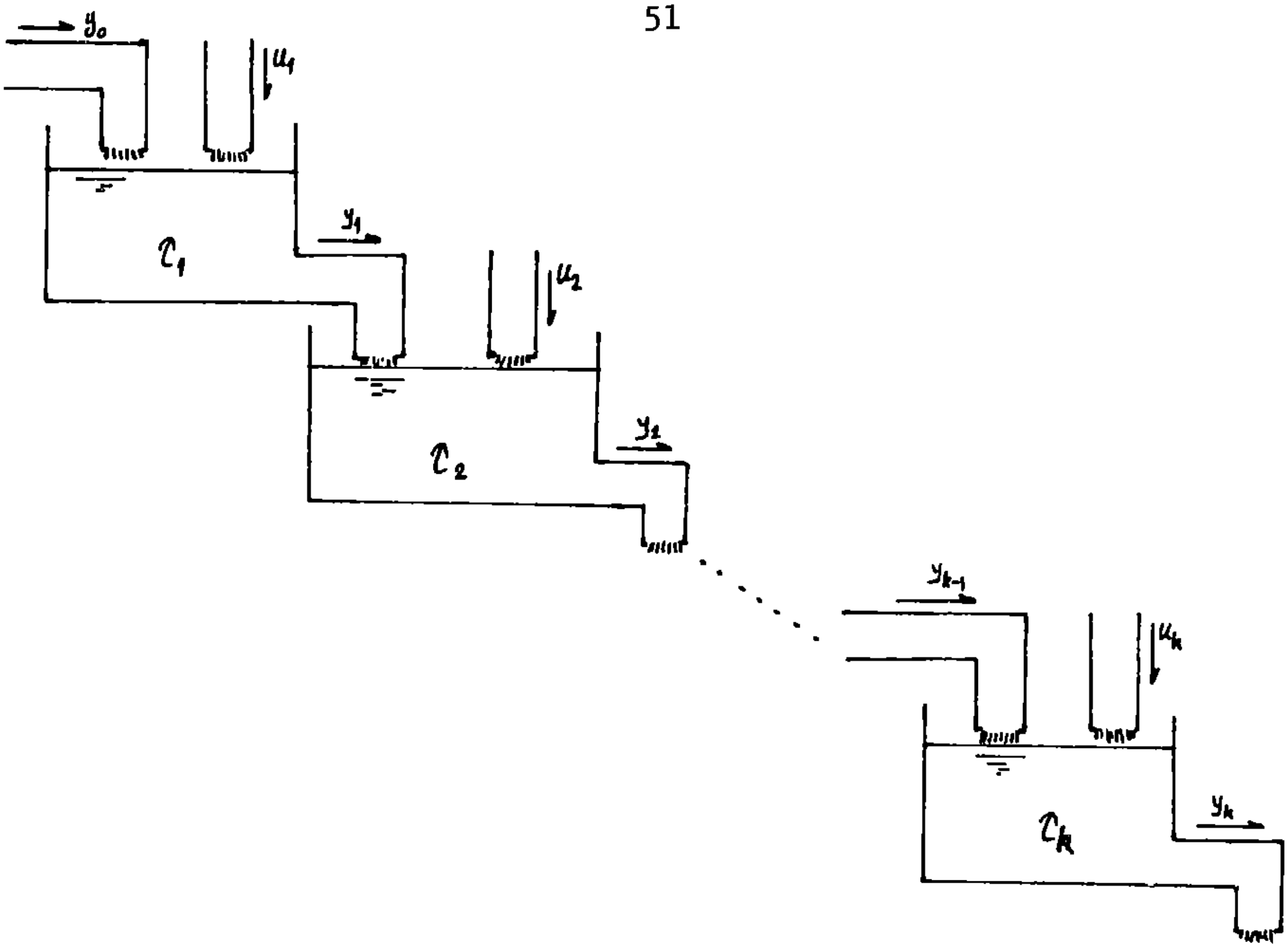
Counter Flow
$$\begin{cases} \varphi_n \dfrac{dT_{Hn}}{d\theta_0} = (T_{Hn-1} - T_{Hn}) - \varphi_n (T_{Hn} - T_{Cn+1}) \alpha_1 \\ r\varphi_n \dfrac{dT_{Cn+1}}{d\theta_0} = (T_{Cn} - T_{Cn+1}) - \varphi_n a_2 (T_{Hn} - T_{Cn+1}) \end{cases} \tag{4-10}$$

Because solving the equations by Laplace transform is easier; therefore, equations (4-9)&(4-10), by using zero condition at the inlet or t =0 are:

Parallel Flow
$$\begin{cases} (\varphi_n S + \varphi_n a_1 + 1)\bar{T}_{Hn} = \bar{T}_{Hn-1} + \varphi_n a_1 \bar{T}_{cn} \\ (r\varphi_n S + \varphi_n a_2 + 1)\bar{T}_{cn} = \bar{T}_{cn-1} + \varphi_n a_2 \bar{T}_{Hn} \end{cases} \tag{4-11}$$

Counter Flow
$$\begin{cases} (\varphi_n S + \varphi_n a_1 + 1)\bar{T}_{Hn} = \bar{T}_{Hn-1} + \varphi_n a_1 \bar{T}_{cn+1} \\ (r\varphi_n S + \varphi_n a_2 + 1)\bar{T}_{cn+1} = \bar{T}_{cn-1} + \varphi_n a_2 \bar{T}_{Hn} \end{cases} \tag{4-12}$$

As equations (4-11)&(4-12) show, if the number of Lumped sections are given, these equations can be extended and their solutions give the temperature at any section. The same as steady state case, here for N equals to 2,3, and 4 it will be discussed, also we can see by putting a large value for time ($t\rightarrow\infty$) the results of static case can be obtained. Following is a schematic of this evaluation:

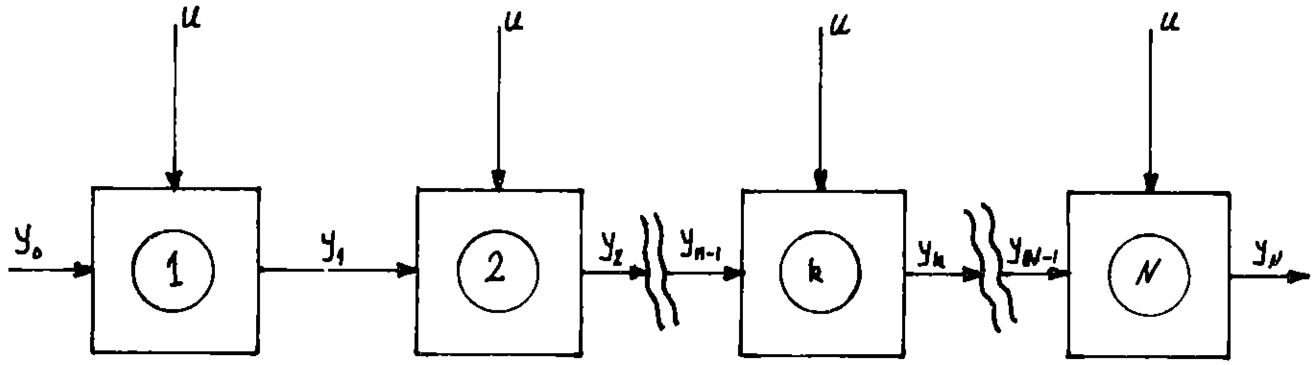


FIGURE(4-5): Schematic of Lumped Model for Heat-Exchanger

A simplest problem which is close to heat exchanger problem of dynamic response is one where there are a number of first-order systems arranged in series. Letting the time constant of the nth stage be τ_n, denoting the dependent variable leaving stage by the stage number, and including the possibility of being able to control each stage, a balance on the nth stage gives the equation:

$$\tau_n \frac{dy_n}{dt} + y_n = \alpha_n y_{n-1} + \beta_n u \qquad (4\text{-}13)$$

and the same equation will apply to every stage. Fig.(4-6) shows this series of first order systems.



FIGURE(4-6): Series of First-Order Systems

If we assume that the system is originally operating at some steady state conditions, (as it discussed for heat exchanger is last part), and that the dependent variable represents deviations from steady state, the Laplace transform of equation (4-13) gives the result:

$$(\tau_n s+1)\bar{y}_n=\alpha_n\bar{y}_{n-1}+\beta_n\bar{u} \qquad (4\text{-}14)$$

Thus the complete set of equations can be written as:

$$(\tau_1 s+1)\bar{y}_1=\alpha_1\bar{y}_0+\beta_1\bar{u}$$

$$(\tau_2 s+1)\bar{y}_2=\alpha_2\bar{y}_1+\beta_2\bar{u}$$

$$\vdots$$

$$(\tau_n s+1)\bar{y}_n=\alpha_n\bar{y}_{n-1}+\beta_n\bar{u}$$

$$\vdots$$

Now, the first equation can be solved in the set for $\bar{y}_1$ in terms of $\bar{y}_0$ and $\bar{u}$ and then use this result to eliminate $\bar{y}_1$

from the second equation. This procedure can be repeated until we obtain an explicit expression for $\bar{y}_n$ in terms of $\bar{y}_0$ and $\bar{u}$, the system inputs. Hence

$$\bar{y}_n = \frac{(\alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1)\bar{y}_0}{(\tau_n s+1)(\tau_{n-1} s+1)\cdots\cdots(\tau_1 s+1)} + \left[\frac{\beta_n(\tau_{n-1}s+1)(\tau_{n-2}s+1)\cdots\cdots(\tau_1 s+1)}{(\tau_n s+1)(\tau_{n-1}s+1)\cdots\cdots(\tau_1 s+1)} + \right.$$

$$+ \alpha_n \frac{\beta_{n-1}(\tau_{n-2}s+1)(\tau_{n-3}s+1)\cdots\cdots(\tau_1 s+1)}{(\tau_n s+1)(\tau_{n-1}s+1)\cdots\cdots(\tau_1 s+1)} + \cdots\cdots$$

$$\left. + \frac{\beta_1 \alpha_1 \alpha_2 \cdots\cdots \alpha_{n-1}}{(\tau_n s+1)(\tau_{n-1}s+1)\cdots\cdots(\tau_1 s+1)}\right]\bar{u}$$

or, for a case where all of the time constants τ and system gain α are equal, the transfer function for inlet disturbances is

$$\frac{\bar{y}_n}{\bar{y}_0} = \frac{\alpha^n}{(\tau s+1)^n}$$

and the transfer function for control variable changes is:

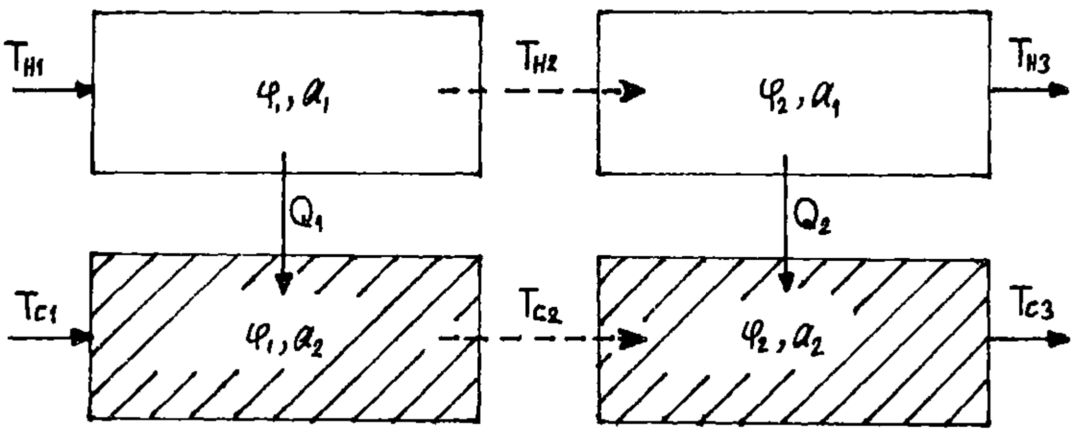
$$\frac{\bar{y}_n}{\bar{u}} = \frac{\beta\alpha^{n-1}\sum_{r=0}^{n-1}\alpha^{-r}(\tau s+1)^r}{(\tau s+1)^n}$$

SPECIAL CASES FOR DYNAMIC RESPONSE:

For two lumped models(N=2):

i-Parallel Flow:

From equation(4-11), for N=2 the results are:



FIGURE(4-7): Two Lumped Models Heat-Exchanger(parallel-flow)

$$\begin{cases}
(\varphi_1 S + \varphi_1 a_1 + 1)\,\bar{T}_{H2} = \bar{T}_{H1} + \varphi_1 a_1 \bar{T}_{C2} \\
(\varphi_1 r S + \varphi_1 a_2 + 1)\,\bar{T}_{C2} = \bar{T}_{C1} + \varphi_1 a_2 \bar{T}_{H2} \\
(\varphi_2 S + \varphi_2 a_1 + 1)\,\bar{T}_{H3} = \bar{T}_{H2} + \varphi_2 a_1 \bar{T}_{C3} \\
(\varphi_2 r S + \varphi_2 a_2 + 1)\,\bar{T}_{C3} = \bar{T}_{C2} + \varphi_2 a_2 \bar{T}_{H3}
\end{cases}$$

The results of these equations, by disturbing only inlet temperature of hot fluid, are:

$$\frac{\bar{T}_{C2}}{\bar{T}_{H1}} = \frac{\varphi_1 a_2}{(\varphi_1 S + \varphi_1 a_1 + 1)(\varphi_1 r S + \varphi_1 a_2 + 1) - \varphi_1^2 a_1 a_2}$$

$$\frac{\bar{T}_{H2}}{\bar{T}_{H1}} = \frac{\varphi_1 r S + \varphi_1 a_2 + 1}{(\varphi_1 S + \varphi_1 a_1 + 1)(\varphi_1 r S + \varphi_1 a_2 + 1) - \varphi_1^2 a_1 a_2}$$

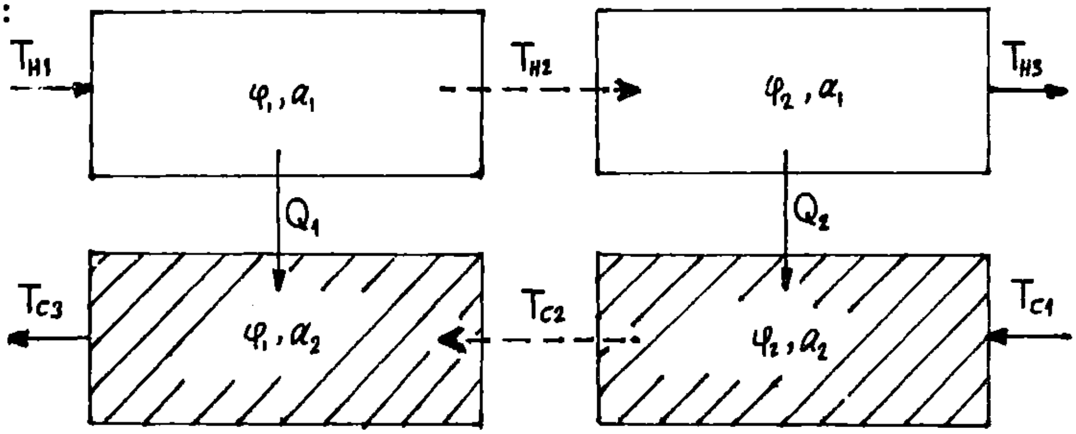
(4-15a-0)

$$\frac{\overline{\overline{T}}_{H3}}{\overline{\overline{T}}_{H1}} = \frac{(\varphi_1 r s + \varphi_1 a_2 + 1)(\varphi_2 r s + \varphi_2 a_2 + 1) + \varphi_1 \varphi_2 a_1 a_2}{\prod_{i=1}^{2}\left[(\varphi_i s + \varphi_i a_1 + 1)(\varphi_i r s + \varphi_i a_2 + 1) - \varphi_i^2 a_1 a_2\right]} \qquad (4\text{-}15a)$$

$$\frac{\overline{\overline{T}}_{C3}}{\overline{\overline{T}}_{H1}} = \frac{\varphi_2 a_2(\varphi_1 r s + \varphi_1 a_2 + 1) + \varphi_1 a_2(\varphi_2 s + \varphi_2 a_1 + 1)}{\prod_{i=1}^{2}\left[(\varphi_i s + \varphi_i a_1 + 1)(\varphi_i r s + \varphi_i a_2 + 1) - \varphi_i^2 a_1 a_2\right]} \qquad (4\text{-}15b)$$

ii- Counter Flow:

From equation(4-12), and as Fig.(4-8)shows the results are:



FIGURE(4-8): Two Lumped Models of Counter Flow Heat-Exchanger

$$\begin{cases} (\varphi_1 s + \varphi_1 a_1 + 1)\overline{\overline{T}}_{H2} = \overline{\overline{T}}_{H1} + \varphi_1 a_1 \overline{\overline{T}}_{C3} \\ (r\varphi_1 s + \varphi_1 a_2 + 1)\overline{\overline{T}}_{C3} = \overline{\overline{T}}_{C2} + \varphi_1 a_2 \overline{\overline{T}}_{H2} \\ (\varphi_2 s + \varphi_2 a_1 + 1)\overline{\overline{T}}_{H3} = \overline{\overline{T}}_{H2} + \varphi_2 a_1 \overline{\overline{T}}_{C2} \\ (r\varphi_2 s + \varphi_2 a_2 + 1)\overline{\overline{T}}_{C2} = \overline{\overline{T}}_{C1} + \varphi_2 a_2 \overline{\overline{T}}_{H3} \end{cases}$$

For disturbing only inlet temperature of hot fluid, the solutions of above equations are:

$$\frac{\bar{\bar{T}}_{C3}}{\bar{\bar{T}}_{H1}}=\frac{\varphi_1 a_2\left[(\varphi_2 s+\varphi_2 a_1+1)(r\varphi_2 s+\varphi_2 a_2+1)-\varphi_2^2 a_1 a_2\right]+\varphi_2 a_2}{\prod_{i=1}^{2}\left[(\varphi_i s+\varphi_i a_1+1)(r\varphi_i s+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]-\varphi_1\varphi_2 a_1 a_2} \qquad (4\text{-}16a)$$

$$\frac{\bar{\bar{T}}_{C2}}{\bar{\bar{T}}_{H1}}=\frac{\varphi_2 a_2(r\varphi_1 s+\varphi_1 a_2+1)}{\prod_{i=1}^{2}\left[(\varphi_i s+\varphi_i a_1+1)(r\varphi_i s+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]-\varphi_1\varphi_2 a_1 a_2} \qquad (4\text{-}16b)$$

$$\frac{\bar{\bar{T}}_{H3}}{\bar{\bar{T}}_{H1}}=\frac{(r\varphi_2 s+\varphi_2 a_2+1)(r\varphi_1 s+\varphi_1 a_2+1)}{\prod_{i=1}^{2}\left[(\varphi_i s+\varphi_i a_1+1)(r\varphi_i s+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]-\varphi_1\varphi_2 a_1 a_2}$$

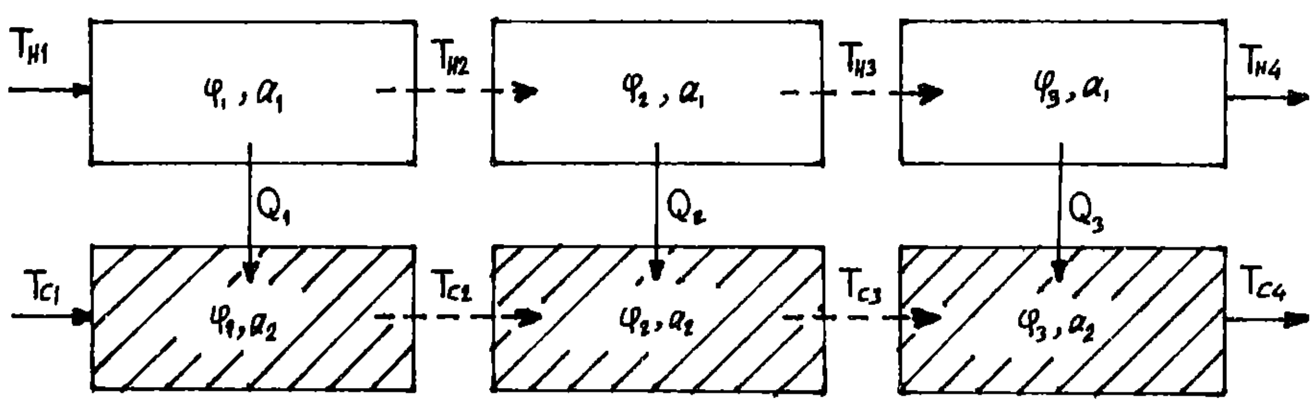
$$\frac{\bar{\bar{T}}_{H2}}{\bar{\bar{T}}_{H1}}=\frac{(r\varphi_1 s+\varphi_1 a_2+1)\left[(\varphi_2 s+\varphi_2 a_1+1)(r\varphi_2 s+\varphi_2 a_2+1)-\varphi_2^2 a_1 a_2\right]}{\prod_{i=1}^{2}\left[(\varphi_i s+\varphi_i a_1+1)(r\varphi_i s+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]-\varphi_1\varphi_2 a_1 a_2}$$

It is clear that by putting $s=0$, the result of steady-state case can be gotten.

For three lumped models(N=3):

i- Parallel Flow:

From equation(4-11) for N=3 the results for temperature labeled on fig.(4-9)are:



FIGURE(4-9): Three Lumped Models for Parallel Flow Heat-Exchanger(parallel-flow)

There are three φ's, which can be equal or different, but the sum of them should be unit.

$$\begin{cases} (\varphi_1 S + \varphi_1 a_1 + 1)\,\bar{T}_{H2} = \bar{T}_{H1} + \varphi_1 a_1 \bar{T}_{C2} \\ (r\varphi_1 S + \varphi_1 a_2 + 1)\,\bar{T}_{C2} = \bar{T}_{C1} + \varphi_1 a_2 \bar{T}_{H2} \\ (\varphi_2 S + \varphi_2 a_1 + 1)\,\bar{T}_{H3} = \bar{T}_{H2} + \varphi_2 a_1 \bar{T}_{C3} \\ (\varphi_2 r S + \varphi_2 a_2 + 1)\,\bar{T}_{C3} = \bar{T}_{C2} + \varphi_2 a_2 \bar{T}_{H3} \\ (\varphi_3 S + \varphi_3 a_1 + 1)\,\bar{T}_{H4} = \bar{T}_{H3} + \varphi_3 a_1 \bar{T}_{C4} \\ (\varphi_3 r S + \varphi_3 a_2 + 1)\,\bar{T}_{C4} = \bar{T}_{C3} + \varphi_3 a_2 \bar{T}_{H4} \end{cases}$$

And the solutions for changing only inlet hot temperature fluid are:

$$\frac{\bar{T}_{C2}}{\bar{T}_{H1}} = \frac{\varphi_1 a_2}{(\varphi_1 S + \varphi_1 a_1 + 1)(\varphi_1 r S + \varphi_1 a_2 + 1) - \varphi_1^2 a_1 a_2}$$

$$\frac{\bar{T}_{H2}}{\bar{T}_{H1}} = \frac{\varphi_1 r S + \varphi_1 a_2 + 1}{(\varphi_1 S + \varphi_1 a_1 + 1)(\varphi_1 r S + \varphi_1 a_2 + 1) - \varphi_1^2 a_1 a_2}$$

$$\frac{\bar{T}_{C3}}{\bar{T}_{H1}} = \frac{\varphi_2 a_2 (\varphi_1 r S + \varphi_1 a_2 + 1) + \varphi_1 a_2 (\varphi_2 S + \varphi_2 a_1 + 1)}{\prod_{i=1}^{2} \left[(\varphi_i S + \varphi_i a_1 + 1)(\varphi_i r S + \varphi_i a_2 + 1) - \varphi_i^2 a_1 a_2\right]}$$

$$\frac{\bar{T}_{H3}}{\bar{T}_{H1}} = \frac{(\varphi_1 r S + \varphi_1 a_2 + 1)(\varphi_2 r S + \varphi_2 a_2 + 1) + \varphi_1 \varphi_2 a_1 a_2}{\prod_{i=1}^{2} \left[(\varphi_i S + \varphi_i a_1 + 1)(\varphi_i r S + \varphi_i a_2 + 1) - \varphi_i^2 a_1 a_2\right]}$$

$$\frac{\overline{T}_{C4}}{\overline{T}_{H1}}=\frac{\varphi_3 a_2\prod_{i=1}^{2}(\varphi_i rs+\varphi_i a_2+1)+\varphi_1\varphi_2\varphi_3 a_1 a_2^2+a_2(\varphi_3 s+\varphi_3 a_1+1)\sum_{i=1}^{2}\varphi_{3-i}(\varphi_i rs+\varphi_i a_2+1)(\varphi_i s+\varphi_i a_1+1)}{\prod_{i=1}^{3}\left[(\varphi_i s+\varphi_i a_1+1)(\varphi_i rs+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]}$$

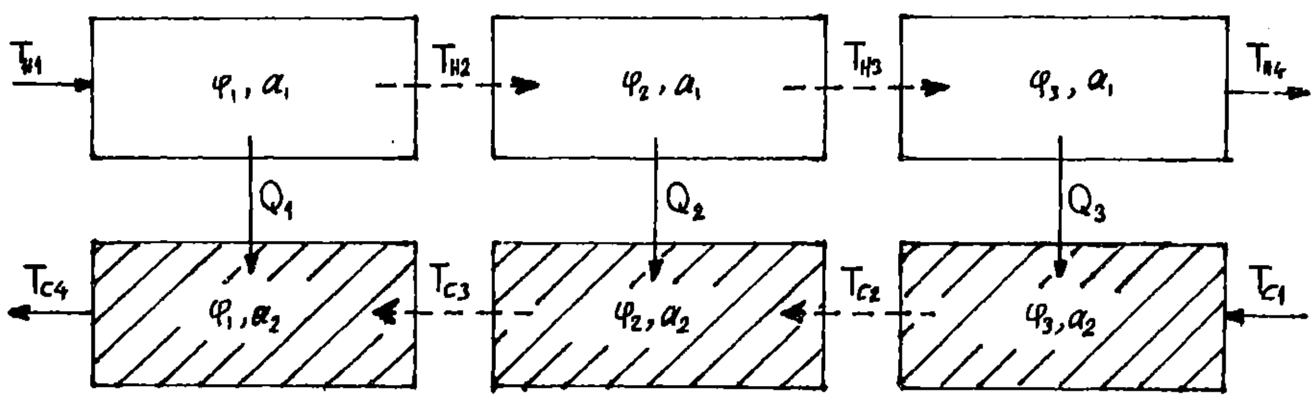
(4-17a)

$$\frac{\overline{T}_{H4}}{\overline{T}_{H1}}=\frac{\prod_{i=1}^{3}(\varphi_i rs+\varphi_i a_2+1)+\varphi_1\varphi_3 a_1 a_2(\varphi_2 s+\varphi_2 a_1+1)+\varphi_2 a_1 a_2\sum_{i=1}^{2}\varphi_{5-2i}(\varphi_{2i-1} rs+\varphi_{2i-1} a_2+1)}{\prod_{i=1}^{3}\left[(\varphi_i s+\varphi_i a_1+1)(\varphi_i rs+\varphi_i a_2+1)-\varphi_i^2 a_1 a_2\right]}$$

(4-17b)

ii- Counter Flow:

Using equation(4-12) for temperature labeled on Fig. (4-10), they give six equations and six unknown temperatures as follows:



FIGURE(4-10): Three Lumped Models of Counter Flow Heat-Exchanger

$$\begin{cases}(\varphi_1 s+\varphi_1 a_1+1)\,\overline{T}_{H2}=\overline{T}_{H1}+\varphi_1 a_1\,\overline{T}_{C4}\\(\varphi_1 rs+\varphi_1 a_2+1)\,\overline{T}_{C4}=\overline{T}_{C3}+\varphi_1 a_2\,\overline{T}_{H2}\\(\varphi_2 s+\varphi_2 a_1+1)\,\overline{T}_{H3}=\overline{T}_{H2}+\varphi_2 a_1\,\overline{T}_{C3}\\(\varphi_2 rs+\varphi_2 a_2+1)\,\overline{T}_{C3}=\overline{T}_{C2}+\varphi_2 a_2\,\overline{T}_{H3}\\(\varphi_3 s+\varphi_3 a_1+1)\,\overline{T}_{H4}=\overline{T}_{H3}+\varphi_3 a_1\,\overline{T}_{C2}\\(\varphi_3 rs+\varphi_3 a_2+1)\,\overline{T}_{C2}=\overline{T}_{C1}+\varphi_3 a_2\,\overline{T}_{H4}\end{cases}$$

Solution for getting outlet temperatures of each section based on disturbing only hot inlet fluid temperature are in the next page but instead of these long answers by putting the numerical values of parameters (namely φ_1 , φ_2 , φ_3 ,and a_1 ,a_2 ,a_3 ,r) the results will be very simple as they are shown for special values of these parameters.

$$\frac{\bar{\bar{T}}_{C2}}{\bar{\bar{T}}_{H1}}=\frac{\varphi_3 a_2(\varphi_1 rs+\varphi_1 a_2+1)(\varphi_2 rs+\varphi_2 a_2+1)}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]}$$

$$\frac{\bar{\bar{T}}_{H2}}{\bar{\bar{T}}_{H1}}=\frac{(\varphi_1 rs+\varphi_1 a_2+1)(\varphi_2 rs+\varphi_2 a_2+1)(\varphi_2 s+\varphi_2 a_1+1)\left[(\varphi_3 s+\varphi_3 a_1+1)(\varphi_3 rs+\varphi_3 a_2+1)-\varphi_3^2 a_1 a_2\right]-\varphi_2 a_1 a_2(\varphi_1 rs+\varphi_1 a_2+1)\left[\varphi_3+\varphi_2(1+\varphi_3 s+\varphi_3 a_1)(1+\varphi_3 rs+\varphi_3 a_2)-\varphi_2\varphi_3^2 a_1 a_2\right]}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]}$$

$$\frac{\bar{\bar{T}}_{C3}}{\bar{\bar{T}}_{H1}}=\frac{a_2(\varphi_1 rs+\varphi_1 a_2+1)\left[\varphi_3+\varphi_2(\varphi_3 s+\varphi_3 a_1+1)(\varphi_3 rs+\varphi_3 a_2+1)-\varphi_2\varphi_3^2 a_1 a_2\right]}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]}$$

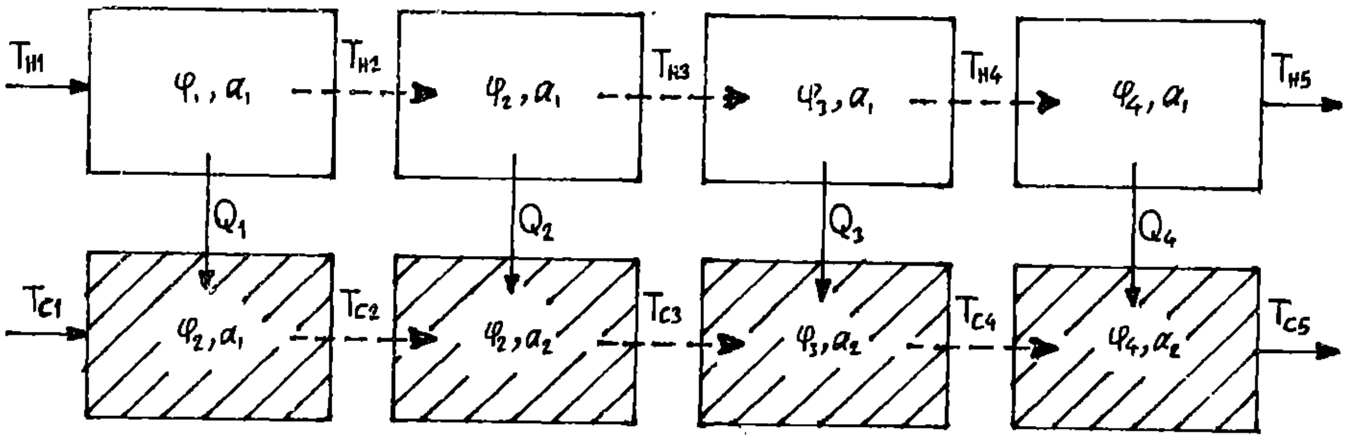
$$\frac{\bar{\bar{T}}_{C4}}{\bar{\bar{T}}_{H1}}=\frac{a_2(1-\varphi_1\varphi_2 a_1 a_2)\left[\varphi_3+\varphi_2(\varphi_3 s+\varphi_3 a_1+1)(\varphi_3 rs+\varphi_3 a_2+1)-\varphi_2\varphi_3^2 a_1 a_2\right]+\varphi_1 a_2(\varphi_2 rs+\varphi_2 a_2+1)(\varphi_2 s+\varphi_2 a_1+1)\left[(\varphi_3 s+\varphi_3 a_1+1)(\varphi_3 rs+\varphi_3 a_2+1)-\varphi_3^2 a_1 a_2\right]}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]} \quad (4\text{-}18a)$$

$$\frac{\bar{\bar{T}}_{H3}}{\bar{\bar{T}}_{H1}}=\frac{(\varphi_1 rs+\varphi_1 a_2+1)(\varphi_2 rs+\varphi_2 a_2+1)\left[(\varphi_3 s+\varphi_3 a_1+1)(\varphi_3 rs+\varphi_3 a_2+1)-\varphi_3^2 a_1 a_2\right]}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]}$$

$$\frac{\bar{\bar{T}}_{H4}}{\bar{\bar{T}}_{H1}}=\frac{\prod_{i=1}^{3}(\varphi_i rs+\varphi_i a_2+1)}{(\varphi_2 s+\varphi_2 a_1+1)(\varphi_2 rs+\varphi_2 a_2+1)\prod_{i=1}^{2}\left[(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_{2i-1}^2 a_1 a_2\right]-\prod_{i=1}^{2}\left[\varphi_{2i-1}a_i+\varphi_2 a_i(\varphi_{2i-1}s+\varphi_{2i-1}a_1+1)(\varphi_{2i-1}rs+\varphi_{2i-1}a_2+1)-\varphi_2\varphi_{2i-1}^2 a_1 a_2 a_i\right]} \quad (4\text{-}18b)$$

For four lumped models(N=4):

i- Parallel Flow:



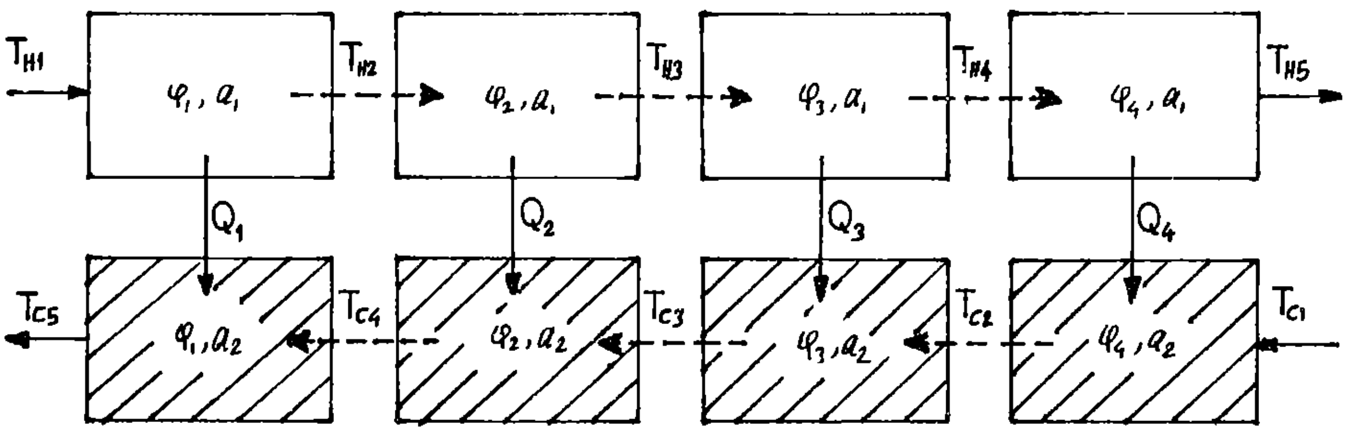
FIGURE(4-11): Four Lumped Models for Parallel Flow Heat-exchanger.

By using N=4, there are four φ's, and equation(4-11)is extend to following equations:

$$
\begin{cases}
(\varphi_1 s + \varphi_1 a_1 + 1)\bar{T}_{H2} = \bar{T}_{H1} + \varphi_1 a_1 \bar{T}_{C2} \\
(\varphi_1 r s + \varphi_1 a_2 + 1)\bar{T}_{C2} = \bar{T}_{C1} + \varphi_1 a_2 \bar{T}_{H2} \\
(\varphi_2 s + \varphi_2 a_1 + 1)\bar{T}_{H3} = \bar{T}_{H2} + \varphi_2 a_1 \bar{T}_{C3} \\
(\varphi_2 r s + \varphi_2 a_2 + 1)\bar{T}_{C3} = \bar{T}_{C2} + \varphi_2 a_2 \bar{T}_{H3} \\
(\varphi_3 s + \varphi_3 a_1 + 1)\bar{T}_{H4} = \bar{T}_{H3} + \varphi_3 a_1 \bar{T}_{C4} \\
(\varphi_3 r s + \varphi_3 a_2 + 1)\bar{T}_{C4} = \bar{T}_{C3} + \varphi_3 a_2 \bar{T}_{H4} \\
(\varphi_4 s + \varphi_4 a_1 + 1)\bar{T}_{H5} = \bar{T}_{H4} + \varphi_4 a_1 \bar{T}_{C5} \\
(\varphi_4 r s + \varphi_4 a_2 + 1)\bar{T}_{C5} = \bar{T}_{C4} + \varphi_4 a_2 \bar{T}_{H5}
\end{cases}
$$

The solutions can be obtained in the same manner of other cases, but they obtained only for special cases that are in the next part.

ii- Counter Flow:



FIGURE(4-12):Four Lumped Models for Counter Flow Heat-Exchanger

Using equation(4-12) for N=4 and four different φ's, it is extended to the following equations:

$$\begin{cases}
(\varphi_1 s+\varphi_1 a_1+1)\bar{T}_{H2}=\bar{T}_{H1}+\varphi_1 a_1\bar{T}_{C5}\\
(\varphi_1 rs+\varphi_1 a_2+1)\bar{T}_{C5}=\bar{T}_{C4}+\varphi_1 a_2\bar{T}_{H2}\\
(\varphi_2 s+\varphi_2 a_1+1)\bar{T}_{H3}=\bar{T}_{H2}+\varphi_2 a_1\bar{T}_{C4}\\
(\varphi_2 rs+\varphi_2 a_2+1)\bar{T}_{C4}=\bar{T}_{C3}+\varphi_2 a_2\bar{T}_{H3}\\
(\varphi_3 s+\varphi_3 a_1+1)\bar{T}_{H4}=\bar{T}_{H3}+\varphi_3 a_1\bar{T}_{C3}\\
(\varphi_3 rs+\varphi_3 a_2+1)\bar{T}_{C3}=\bar{T}_{C2}+\varphi_3 a_2\bar{T}_{H4}\\
(\varphi_4 s+\varphi_4 a_1+1)\bar{T}_{H5}=\bar{T}_{H4}+\varphi_4 a_1\bar{T}_{C2}\\
(\varphi_4 rs+\varphi_4 a_2+1)\bar{T}_{C2}=\bar{T}_{C1}+\varphi_4 a_2\bar{T}_{H5}
\end{cases}$$

Solutions for cases which are discussed in next part are presented. We can develop and get better results by putting more or larger N.

RESULTS OF TRANSIENT RESPONSE FOR THREE CASES:

In the following cases, solutions for outlet temperatures of last section for each fluid are found and compared with the exact results(2) and results of Friedly's model(4).

The exact solution and Friedly's results, as shown in appendix(a), can be obtained by putting $x=1$ in them:

Exact solution for counter flow:

$$G(j\Omega)=\frac{T_{oc}}{T_{iH}}=\frac{g_2\left(1-e^{P_1-P_2}\right)}{-\left(P_1-P_2\,e^{P_1-P_2}\right)+f_2\left(1-e^{P_1-P_2}\right)}$$

Where,

$$P_1\,,P_2=\left[-(f_1-f_2)\pm\sqrt{(f_1+f_2)^2-4\,g_1\,g_2}\right]/2$$

Exact solution for parallel flow:

$$G(j\Omega)=\frac{T_{oc}}{T_{iH}}=\frac{g_2}{P_1-P_2}\left(e^{P_1}-e^{P_2}\right)$$

Where, $$P_1\,,P_2=\left[-(f_1+f_2)\pm\sqrt{(f_1-f_2)^2+4\,g_1\,g_2}\right]/2$$

$f_1=a_1+j\Omega$, $f_2=a_2+rj\Omega$, $g_1=a_1$, $g_2=a_2$

Friedly's solution for counter flow:

$$\bar{G}(s)=\frac{K}{1+TS}\left[1-e^{-(a_1+a_2)}\cdot e^{-(r+1)s}\right]$$

Where, $\frac{K}{T}=\frac{a_2}{1+r}$ and, $$K=\frac{a_2}{1-e^{-(a_1+a_2)}}\times\frac{e^{a_2-a_1}-1}{a_2 e^{a_2-a_1}-a_1}$$

Then for cases that a_1 , a_2 ,and r are specified the above equations appear as follows:

i- Exact:

For case $a_1=4$, $a_2=1$, $r=1$

-Counter Flow

$$G(j\Omega)=\frac{1-e^{2\sqrt{2.25-\Omega^2+5j\Omega}}}{\left(-.5+j\Omega-\sqrt{2.25-\Omega^2+5j\Omega}\right)-\left(2.5+j\Omega+\sqrt{2.25-\Omega^2+j\Omega}\right)e^{2\sqrt{2.25-\Omega^2+5j\Omega}}} \tag{4-19}$$

-Parallel Flow

$$G(j\Omega)=\frac{1-e^{-5}}{5}e^{-j\Omega} \tag{4-20}$$

For case: $a_1=1$, $a_2=1$, $r=1$

-Counter Flow

$$G(j\Omega)=\frac{1-e^{2\sqrt{2j\Omega-\Omega^2}}}{\left(1+j\Omega-\sqrt{2j\Omega-\Omega^2}\right)-\left(1+j\Omega+\sqrt{2j\Omega-\Omega^2}\right)e^{2\sqrt{2j\Omega-\Omega^2}}} \tag{4-21}$$

-Parallel Flow $$G(j\Omega)=\frac{1-e^{-2}}{2}e^{-j\Omega} \tag{4-22}$$

For case: $a_1=2$, $a_2=1$, $r=2$

-Counter Flow

$$G(j\Omega)=\frac{1-e^{2\sqrt{.25-\Omega^2 \times 2.25+4.5j\Omega}}}{\left(1.5+1.5j\Omega-\sqrt{.25-2.25\Omega^2+4.5j\Omega}\right)-\left(1.5+1.5j\Omega+\sqrt{.25-2.25\Omega^2+4.5j\Omega}\right)e^{2\sqrt{.25-2.25\Omega^2+4.5j\Omega}}} \tag{4-23}$$

-Parallel Flow

$$G(j\Omega)=\frac{e^{-1.5(1+j\Omega)}}{\sqrt{9-\Omega^2-2j\Omega}}\left(e^{\sqrt{2.25-.25\Omega^2-.5j\Omega}}-e^{-\sqrt{2.25-.25\Omega^2-.5j\Omega}}\right) \tag{4-24}$$

As it is shown in reference(4), Friedly discussed only counter-flow heat exchanger, and its results for preceding cases are:

ii- Friedly's Results:

For $a_1=4$, $a_2=1$, $r=1$

-Counter Flow

$$\bar{G}(S)=\frac{K}{1+2KS}\left[1-e^{-5-2S}\right] \tag{4-25}$$

Where, $K=\frac{1-e^{3}}{(4-e^{3})(1-e^{5})}$

For $a_1=1$, $a_2=1$ $r=1$

-Counter Flow

$$\bar{G}(S)=\frac{K}{1+2KS}\left[1-e^{-2-2S}\right] \tag{4-26}$$

Where, $K=\frac{1}{2(1-e^{2})}$

For $a_1=2$, $a_2=1$, $r=2$

-Counter Flow

$$\bar{G}(S)=\frac{K}{1+3KS}\left[1-e^{-3-3S}\right] \tag{4-27}$$

Where, $K=\frac{e-1}{(2e-1)(1-e^{3})}$

iii- Lumped Model Results:

First, for simplify the calculations all of φ's assumed to be equal for all cases.

For $a_1=4$, $a_2=1$, $r=1$

a-Two Lumped Models:

$\varphi_1=\varphi_2=1/2$ from equations (4-15)&(4-16):

Counter Flow

(4-22a)
$$\begin{cases} G_1(S) = \dfrac{\bar{T}_{oc}}{\bar{T}_{iH}} = \dfrac{.5}{.25S^2 + 2.25S + 2.5} \\[2ex] G_2(S) = \dfrac{\bar{T}_{oH}}{\bar{T}_{iH}} = \dfrac{.5S + 1.5}{(.5S+3)(.25S^2+2.25S+2.5)} \end{cases}$$

Parallel Flow

(4-22b)
$$\begin{cases} G_1(S) = \dfrac{\bar{T}_{oc}}{\bar{T}_{iH}} = \dfrac{.5(S+4.5)}{(.5S+3.5)^2(.5S+1)^2} \\[2ex] G_2(S) = \dfrac{\bar{T}_{oc}}{\bar{T}_{iH}} = \dfrac{.25S^2 + 1.5S + 3.25}{(.5S+3.5)^2(.5S+1)^2} \end{cases}$$

b- Three Lumped Models:

$\varphi_1 = \varphi_2 = \varphi_3 = 1/3$ using equations (4-17)&(4-18)

Counter Flow

(4-23a)
$$\begin{cases} G_1(S) = \dfrac{\bar{T}_{oc}}{\bar{T}_{iH}} = \dfrac{3S^4 + 66S^3 + 534S^2 + 1881S + 2511}{S^6 + 33S^5 + 435S^4 + 2915S^3 + 10368S^2 + 18216S + 11772} \\[2ex] G_2(S) = \dfrac{\bar{T}_{oH}}{\bar{T}_{iH}} = \dfrac{27(S+4)^3}{S^6 + 33S^5 + 435S^4 + 2915S^3 + 10368S^2 + 18216S + 11772} \end{cases}$$

Parallel Flow

(4-23b)
$$\begin{cases} G_1(S) = \dfrac{\bar{T}_{oc}}{\bar{T}_{iH}} = \dfrac{27\left[(S+7)^2 + (S+4)^2 + (S+7)(S+4) + 4\right]}{(S^2+11S+24)^3} \\[2ex] G_2(S) = \dfrac{\bar{T}_{oH}}{\bar{T}_{iH}} = \dfrac{27\left[(S+4)^3 + 12S + 60\right]}{(S^2+11S+24)^3} \end{cases}$$

c- Four Lumped Models:

$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 1/4$ using equations(A-15)&(A-16),(A-21)&(A-22) in the appendix(e), solution of four lumped model are presented for special cases only.

Parallel Flow

$$\begin{cases} G_1(s) = \dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \dfrac{256(4s^3+78s^2+532s+1261)}{(s+9)^4(s+4)^4} \\ \\ G_2(s) = \dfrac{\bar{\bar{T}}_{oH}}{\bar{\bar{T}}_{iH}} = \dfrac{256\left[(s+5)^4+12(s+5)^2+8(s+8)(s+5)+4(s+8)^2+16\right]}{(s+9)^4(s+4)^4} \end{cases}$$

There are following results for counter flow case:

Counter Flow

$$\begin{cases} G_1(s) = \dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \dfrac{4(s^6+39s^5+631s^4+5421s^3+26124s^2+67184s+74048)}{s^8+52s^7+1158s^6+14604s^5+109153s^4+513552s^3+1452448s^2+2219776s+1344768} \\ \\ G_2(s) = \dfrac{\bar{\bar{T}}_{oH}}{\bar{\bar{T}}_{iH}} = \dfrac{256(s+5)(s^6+39s^5+627s^4+5317s^3+25080s^2+62400s+64000)}{(s+8)^3(s^8+52s^7+1158s^6+14404s^5+109153s^4+513552s^3+1452448s^2+2819776s+1344768} \end{cases}$$

For $a_1 = 1$, $a_2 = 1$, $r = 1$

a- Two Lumped Models:

Using $\varphi_1 = \varphi_2 = 1/2$ in equations(4-15)&(4-16), gives:

Counter Flow

$$\begin{cases} G_1(s) = \dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \dfrac{2(s^2+6s+12)}{(s^2+6s+6)(s^2+6s+10)} \\ \\ G_2(s) = \dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \dfrac{4(s+3)^2}{(s^2+6s+6)(s^2+6s+10)} \end{cases}$$

Parallel Flow

$$\begin{cases} G_1(s)=\dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}}=\dfrac{8(s+3)}{(s^2+6s+8)^2} \\[2ex] G_2(s)=\dfrac{\bar{\bar{T}}_{oH}}{\bar{\bar{T}}_{iH}}=\dfrac{4(s+3)^2+4}{(s^2+6s+8)^2} \end{cases}$$

b- Three Lumped Models:

Using three equal φ's $\varphi_1=\varphi_2=\varphi_3=1/3$, it means the heat exchanger has been divided into three equal sections; then,by applying equations (4-17)&(4-18), the results are:

Counter Flow

$$\begin{cases} G_1(s)=\dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}}=\dfrac{3(s+4)^4+21(s+4)^2+192}{(s^3+13s^2+55s+84)(s^3+11s^2+39s+36)} \\[2ex] G_2(s)=\dfrac{\bar{\bar{T}}_{oH}}{\bar{\bar{T}}_{iH}}=\dfrac{27(s+4)^3}{(s^3+13s^2+55s+84)(s^3+11s^2+39s+36)} \end{cases}$$

Parallel Flow

$$\begin{cases} G_1(s)=\dfrac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}}=\dfrac{81(s+4)^2+27}{(s+3)^3(s+5)^3} \\[2ex] G_2(s)=\dfrac{\bar{\bar{T}}_{oH}}{\bar{\bar{T}}_{iH}}=\dfrac{27(s+4)^3+81(s+4)}{(s+3)^3(s+5)^3} \end{cases}$$

c- Four Lumped Models:

Applying $\varphi_1=\varphi_2=\varphi_3=\varphi_4=1/4$ in the results which are presented in appendix(e), give following equations:

Parallel Flow

$$G_1(S)=\frac{\bar{T}_{oc}}{\bar{T}_{iH}}=\frac{1024(S+5)(S^2+10S+26)}{(S+6)^4(S+4)^4}$$

$$G_2(S)=\frac{\bar{T}_{oH}}{\bar{T}_{iH}}=\frac{256(S+5)^4+1536(S+5)^2+256}{(S+6)^4(S+4)^4}$$

Counter Flow

$$G_1(S)=\frac{\bar{T}_{oc}}{\bar{T}_{iH}}=\frac{4(S+4)(S+6)(S+5)^2(S^2+10S+39)+836S^2+8936S+34400}{S^8+40S^7+696S^6+6880S^5+42208S^4+164000S^3+391680S^2+520960S+288000}$$

$$G_2(S)=\frac{\bar{T}_{oH}}{\bar{T}_{iH}}=\frac{64(16S^8+640S^7+11184S^6+57539S^5+694380S^4+2710716S^3+6038160S^2+9662800S+6\times10^6)}{(S+4)(S+6)(S+5)^2[S^8+40S^7+696S^6+6880S^5+42208S^4+164000S^3+391680S^2+520960S+288000]}$$

For $a_1=2$, $a_2=1$, $r=2$

a- Two Lumped Models:

Using $\varphi_1=\varphi_2=1/2$ in equations(4-15)&(4-16), gives following results:

Counter Flow

$$G_1(S)=\frac{\bar{T}_{oc}}{\bar{T}_{iH}}=\frac{S^2+5.5S+7}{(S^2+5.5S+5)^2-2}$$

$$G_2(S)=\frac{\bar{T}_{oH}}{\bar{T}_{iH}}=\frac{(S+1.5)^2}{(S^2+5.5S+5)^2-2}$$

Parallel Flow

$$\begin{cases} G(s)=\dfrac{\overline{T}_{oc}}{\overline{T}_{iH}}=\dfrac{3s+7}{(s^2+5.5s+5)^2} \\ G_2(s)=\dfrac{\overline{T}_{oH}}{\overline{T}_{iH}}=\dfrac{4s^2+12s+11}{(s^2+5.5s+5)^2} \end{cases}$$

b- Three Lumped Models:

By using $\varphi_1=\varphi_2=\varphi_3=1/3$ and equations(4-17)&(4-18),

there are:

Counter Flow

$$\begin{cases} G(s)=\dfrac{\overline{T}_{oc}}{\overline{T}_{iH}}=\dfrac{12(s+5)^2(s+2)^2+30(s+5)(s+2)+147}{8(s+5)^3(s+2)^3-24(s+5)^2(s+2)^2-48(s+5)(s+2)-98} \\ G_2(s)=\dfrac{\overline{T}_{oH}}{\overline{T}_{iH}}=\dfrac{216(s+2)^3}{8(s+5)^3(s+2)^3-24(s+5)^2(s+2)^2-48(s+5)(s+2)-98} \end{cases}$$

$$G(s)=\frac{\overline{T}_{oc}}{\overline{T}_{iH}}=\frac{108(s+2)^2+54(s+2)(s+5)+27(s+5)^2+54}{8(s^2+7s+9)^3}$$

Parallel Flow

$$G_2(s)=\frac{\overline{T}_{oH}}{\overline{T}_{iH}}=\frac{27(s+2)(2s^2+8s+11)}{2(s^2+7s+9)^3}$$

c- Four Lumped Models:

By using $\varphi_1=\varphi_2=\varphi_3=\varphi_4=1/4$ and the result of solution for four lumped in appendix(e), following results are obtained:

Parallel Flow

$$G(s)=\frac{\bar{T}_{oc}}{\bar{T}_{iH}}=\frac{16(2s+5)^3+16(s+6)^3+16(2s+5)(s+6)(3s+11)+64(3s+11)}{(s^2+8.5s+14)^4}$$

$$G_2(s)=\frac{\bar{T}_{oH}}{\bar{T}_{iH}}=\frac{16(2s+5)^3+32(s+6)^2+96(2s+5)^2+64(s+6)(2s+5)+64}{(s^2+8.5s+14)^4}$$

Counter Flow

$$G_1(s)=\frac{2(s^6+25.5s^5+266.75s^4+1464s^3+4472.5s^2+7310s+5368)}{s^8+34s^7+433.5s^6+38845s^5+18510s^4+53975s^3+93408s^2+86496s+31472}$$

$$G_2(s)=\frac{(2s+5)\left[(s^2+8.5s+14)^3(s^2+8.5s+23.8)+32(s^2+8.5s+14)(s^2+8.5s+22)(s^2+8.5s+30)-192(17s^2+144.5s+239)\right]}{2(s+6)^3(s^8+34s^7+433.5s^6+38845s^5+18510s^4+53975s^3+93408s^2+86496s+31472)}$$

All of above cases are normal situations of heat-exchangers, for comparing the results of Lumped Models, exact solution, and Friedly's approximate method all of above cases are plotted in frequency domain, and the results of step responses for two Lumped Models(the simplest form) are plotted.

One of the beautiesof the Lumped Models is finding the Monotone parameters directly by the method which H.M.Paynter presented in(8), and these coefficients can be compared with the same parameters which are obtained by simple technique that again Paynter presented which are gained by using the probability paper. All of these evaluations are in next parts.

RESULTS OF FREQUENCY RESPONSE:

I- Counter Flow Heat-Exchanger

i-Exact Solution:

In preceding section, transfer functions for transient response were discussed and equations for special cases were derived, now by recalling those equations the numerical values of gains and phase angles can be calculated and plotted, here is for exact solution; recall equation(4-19):

For case: $a_1=4$, $a_2=1$, $r=1$

$$G(j\Omega)=\frac{1-e^{2\sqrt{2.25-\Omega^2+5j\Omega}}}{\left(-.5+j\Omega-\sqrt{2.25-\Omega^2+5j\Omega}\right)-\left(2.5+j\Omega+\sqrt{2.25-\Omega^2+5j\Omega}\right)e^{2\sqrt{2.25-\Omega^2+5j\Omega}}}$$

For $\Omega=0$ $\quad |G|=\frac{1-e^3}{1-4e^3}=.2405$, $\theta=\angle G(j\Omega)=0°$

For $\Omega\rightarrow\infty$ $\quad |G|\rightarrow 0$, $\theta\rightarrow -90°$

Table(4-1) contains the value of gain and phase angle for different frequencies.

For case: $a_1=1$, $a_2=1$, $r=1$

$$G(j\Omega)=\frac{1-e^{2\sqrt{2j\Omega-\Omega^2}}}{\left(1+j\Omega-\sqrt{2j\Omega-\Omega^2}\right)-\left(1+j\Omega+\sqrt{2j\Omega-\Omega^2}\right)e^{2\sqrt{2j\Omega-\Omega^2}}}$$

For $\Omega=0$ $\quad |G|=0.5$, $\theta=0°$

For $\Omega\rightarrow\infty$ $\quad |G|\rightarrow 0$, $\theta\rightarrow -90°$

In this case $|G|$ is obtained by using Hopital's rule.In table (4-5) there are numerical values of $|G|$ and θ° for different frequencies.

For case: $a_1=2$, $a_2=1$, $r=2$

$$G(j\Omega)=\frac{1-e^{2\sqrt{.25-2.25\Omega^2+4.5j\Omega}}}{\left(1.5+1.5j\Omega-\sqrt{.25-2.25\Omega^2+4.5j\Omega}\right)-\left(1.5+1.5j\Omega+\sqrt{.25-2.25\Omega^2+4.5j\Omega}\right)e^{2\sqrt{.25-2.25\Omega^2+4.5j\Omega}}}$$

For $\Omega=0$ $|G|=.3873$, $\theta=0^{\circ}$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$, $\theta\longrightarrow -90^{\circ}$

Results for other values of Ω are presented in table(4-9).

ii- Friedly's Solution:

For case $a_1=4$, $a_2=1$, $r=1$

$$G(j\Omega)=\frac{K}{1+2Kj\Omega}\left[1-e^{-5-2j\Omega}\right] \quad \text{where} \quad K=.2422$$

For $\Omega=0$ $|G|=.2405$, $\theta=0^{\circ}$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$, $\theta\longrightarrow -90^{\circ}$

Table(4-1) contains all numerical values for $|G|$ and θ° , for different frequencies..

For case $a_1=1$, $a_2=1$, $r=1$

Recall equation(4-26):

$$G(j\Omega)=\frac{K}{1+2Kj\Omega}\left[1-e^{-2-2j\Omega}\right] \quad \text{where} \quad K=.578$$

For $\Omega=0$ $|G|=.5$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$, $\theta\longrightarrow -90^\circ$

For other values of Ω the results are presented in table (4-5).

For case $a_1=2$, $a_2=1$, $r=2$

By using equation(4-27) from preceding section, following results can be gotten:

$$G(j\Omega)=\frac{K}{1+3j\Omega K}\left[1-e^{-3-3j\Omega}\right] \quad \text{where} \quad K=.4076$$

For $\Omega=0$ $|G|=.3873$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$, $\theta\longrightarrow -90^\circ$

Results for other values of frequency are in table(4-9).

iii- Lumped Models Results:

The more interested outlet temperature is usually the outlet temperature of cold fluid;therefore, the Lumped Models cold fluid results are discussed.

For case $a_1=4$, $a_2=1$, $r=1$

a- Two Lumped Models:

$$G(j\Omega)=\frac{.5}{2.5-.25\Omega^2+2.25j\Omega}$$

For $\Omega=0$ $|G|=.2$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$, $\theta\longrightarrow -180^\circ$

As in static case discussed, for getting better results we can fit the Lumped Models results with the exact results at low frequency;therefore, for two Lumped Models all of results are multiplied by a factor:

$$Factor = \frac{T_{oc})_{Exact}}{T_{oc})_{Lumped}} = \frac{.2405}{.2} = 1.202$$

As it's shown in table(4-2) in addition for absolute results of Lumped Models there is a column for fitted results.

b- Three Lumped Models:

$$G(j\Omega) = \frac{(2511 + 3\Omega^4 - 584\Omega^2) + j(1881\Omega - 66\Omega^3)}{(11972 + 435\Omega^4 - 10368\Omega^2 - \Omega^6) + j(18216\Omega + 33\Omega^5 - 2915\Omega^3)}$$

For $\Omega = 0$ $\quad |G| = .2133 \quad , \quad \theta = 0^\circ$

For $\Omega \longrightarrow \infty$ $\quad |G| \longrightarrow 0 \quad , \quad \theta \longrightarrow -180^\circ$

factor for fitting $= \frac{.2405}{.2133} = 1.1275$

The results for other values of frequency are shown in table (4-3).

c- Four Lumped Models:

$$G(j\Omega) = \frac{4(74048 + 631\Omega^4 - 26112\Omega^2 - \Omega^6) + 4j(67184\Omega + 39\Omega^5 - 5421\Omega^3)}{(1344768 + 109153\Omega^4 + \Omega^8 - 158\Omega^6 - 1452448\Omega^2) + j(2219776\Omega + 14404\Omega^5 - 62\Omega^7 - 513552\Omega^3)}$$

For $\Omega = 0$ $\quad |G| = .22025 \quad , \quad \theta = 0^\circ$

For $\Omega \longrightarrow \infty$ $\quad |G| \longrightarrow 0 \quad \theta \longrightarrow -180^\circ$

factor for fitting $= \frac{.2405}{.22025} = 1.092$

Results for other values of frequency arepresented in table (4-4).

For case $a_1 = 1$, $a_2 = 1$, $r = 1$

a-Two Lumped Models:

$$G(j\Omega) = \frac{24 - 2\Omega^2 + 12j\Omega}{(60 + \Omega^4 - 52\Omega^2) + j(96\Omega - 12\Omega^3)}$$

For $\Omega=0$ $|G|=0.4$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -180^\circ$

fitting factor $=\dfrac{.5}{.4}=1.25$

Table(4-6) shows these results for other values of Ω .

b- Three Lumped Models:

$$G(j\Omega)=\frac{3(432+\Omega^4-103\Omega^2)+3j(312\Omega-16\Omega^3)}{\left[(84-13\Omega^2)+j(55\Omega-\Omega^3)\right]\left[(36-11\Omega^2)+j(39\Omega-\Omega^3)\right]}$$

For $\Omega=0$ $|G|=.4286$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -180^\circ$

fitting factor $=\dfrac{.5}{.4286}=1.166$

The whole results are shown in table(4-7).

c- Four Lumped Models

$$G(j\Omega)=\frac{4(32000+388\Omega^4-11520\Omega^2-\Omega^6)+4j(27344\Omega+30\Omega^5-2760\Omega^3)}{(288000+42208\Omega^4+\Omega^8-696\Omega^6-391680\Omega^2)+j(520960\Omega+6080\Omega^5-40\Omega^7-164000\Omega^3)}$$

For $\Omega=0$ $|G|=.4444$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -180^\circ$

fitting factor $=\dfrac{.5000}{.4444}=1.125$

Results for other values of frequency are shown in table(4-8).

For case: $a_1=2$, $a_2=1$, $r=2$

a- Two Lumped Models:

$$G(j\Omega)=\frac{(7-\Omega^2)+5.5j\Omega}{(23+\Omega^4-14.25\Omega^2)+j(55\Omega-11\Omega^3)}$$

For $\Omega=0$ $|G|=.3043$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -180^\circ$

fitting factor $= \frac{.3873}{.3043} = 1.2725$

Other results for different values of frequency are in table (4-10).

b- Three Lumped Models:

$$G(j\Omega)=\frac{3(274.5+2\Omega^4-243\Omega^2)+3j(315\Omega-28\Omega^3)}{(2511+696\Omega^4-4\Omega^6-1228\Omega^2)+j(6552\Omega+84\Omega^5-2884\Omega^3)}$$

For $\Omega=0$ $\quad |G| = .328$, $\theta = 0^\circ$

For $\Omega \longrightarrow \infty$ $\quad |G| \longrightarrow 0$ $\quad \theta \longrightarrow -180^\circ$

fitting factor $= \frac{.3873}{.328} = 1.18$

Results are presented in table (4-11)

c- Four Lumped Models:

$$G(j\Omega)=\frac{2(-44725\Omega^2+5368+2675\Omega^4-\Omega^6)+2j(7310\Omega+25.5\Omega^5-1464\Omega^3)}{(31472+\Omega^8+18510\Omega^4-435\Omega^6-93408\Omega^2)+j(86496\Omega+3884.5\Omega^5-34\Omega^7-53975\Omega^3)}$$

For $\Omega=0$ $\quad |G| = .3411$, $\theta = 0^\circ$

For $\Omega \longrightarrow \infty$ $\quad |G| \longrightarrow 0$, $\theta \longrightarrow -180^\circ$

fitting factor $= \frac{.3873}{.3411} = 1.135$

Table (4-12) shows the numerical values of gains and phase angle for other amount of frequency.

In all tables there is a column for normalized gain which means ratio of gain for every frequency and exact gain for zero frequency.

$$|G|_N = \frac{|G|_\Omega}{|G|_0}$$

II- PARALLEL FLOW HEAT EXCHANGER

i- The Exact Solution:

From equation (4-20) we have following transfer function for transient response in parallel flow heat exchanger:

For case: $a_1=4$, $a_2=1$, $r=1$

$$G(j\Omega)=\frac{1-e^{-5}}{5}e^{-j\Omega}$$

As it shows, it is a complete circle with radius R=.1986 and the phase angle is $\theta=\Omega$. Table(4-13) contains the value of phase angles for different value of Ω .

For case: $a_1=2$, $a_2=1$, $r=2$

$$G(j\Omega)=\frac{e^{-1.5(1+j\Omega)}}{2\sqrt{2.25-.25\Omega^2-.5j\Omega}}\left[e^{\sqrt{2.25-.25\Omega^2-.5j\Omega}}-e^{-\sqrt{2.25-.25\Omega^2-.5j\Omega}}\right]$$

For $\Omega=0$ $\quad |G|=.3167$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $\quad |G|\longrightarrow 0$ $\quad \theta\longrightarrow-\infty$

Table(4-17) contains the results for other values of Ω .

ii- The Lumped Model Results:

Because the interested outlet temperatureis the outlet temperature of cold side, then the following results are for that temperature, also in following there is a fitting factor for each case which defined as:

$$\text{fitting factor}=\frac{|G(0)|_{Exact}}{|G(0)|_{Lumped}}$$

For case: $a_1=4$, $a_2=1$, $r=1$

a- Two Lumped Models:

$$G(j\Omega)=\frac{.5(4.5+j\Omega)}{(3.5+.5j\Omega)^2(1+.5j\Omega)^2}$$

For $\Omega=0$ $\quad |G|=.1837$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $\quad |G|\longrightarrow 0$ $\quad \theta\longrightarrow -270^\circ$

fitting factor $=\frac{.1986}{.1837}=1.081$

Table(4-14) contains the other values of $|G|$ and θ for different amount of Ω , also it has a column for $|G|_N$,which is the ratio of gain for each Ω and gain of exact solution at zero frequency.

b- Three Lumped Models:

$$G(j\Omega)=\frac{27(97-3\omega^2+33j\omega)}{(24-\omega^2+11j\omega)^3}$$

For $\Omega=0$ $\quad |G|=.1894$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $\quad |G|\longrightarrow 0$ $\quad \theta\longrightarrow -360^\circ$

fitting factor $=\frac{.1986}{.1894}=1.0483$

The results for other values of frequency are in table (4-15).

c- Four Lumped Models:

$$G(j\Omega)=\frac{256(1261-78\Omega^2)+256j(522\Omega-4\Omega^3)}{(36-\Omega^2+13j\Omega)^4}$$

For $\Omega=0$ $\quad |G|=.1922$, $\theta=0^\circ$

For $\Omega\longrightarrow\infty$ $\quad |G|\longrightarrow 0$ $\quad \theta\longrightarrow -450^\circ$

fitting factor $=\frac{.1986}{.1922}=1.0333$

Results for other values of frequency are presented in table (4-16).

For case: $a_1=2$, $a_2=1$, $r=2$

a- Two Lumped Models:

$$G(j\Omega)=\frac{7+3j\Omega}{(5-\Omega^2+5.5j\Omega)^2}$$

For $\Omega=0$ $|G|=.28$, $\theta=0°$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -270°$

fitting factor $=\frac{.3167}{.28}=1.131$

Table(4-18) contains the value of gain and phase angle for different amount of frequency.

b- Three Lumped Models:

$$G(j\Omega)=\frac{1701-189\Omega^2+1080j\Omega}{8(9-\Omega^2+7j\Omega)^3}$$

For $\Omega=0$ $|G|=.292$, $\theta=0°$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -360°$

fitting factor $=\frac{.3167}{.292}=1.086$

Table(4-19) contains all of the results.

c- Four Lumped Models:

$$G(j\Omega)=\frac{11440-2416\Omega^2+j(8752\Omega-240\Omega^3)}{(14-\Omega^2+8.5j\Omega)^4}$$

For $\Omega=0$ $|G|=.298$, $\theta=0°$

For $\Omega\longrightarrow\infty$ $|G|\longrightarrow 0$ $\theta\longrightarrow -450°$

fitting factor $=\frac{.3167}{.298}=1.0635$

Results for other values of gain and phase angle for different amount of frequency are presented in table(4-20).

To compare the exact solution and simple Lumped Models results, all of results are plotted in Fig.(4-19)&(4-20).

SECTION V

EFFECT OF USING NONEQUAL φ's:

As it was mentioned before, existance of φ's in all of results of Lumped Model systems makes it flexible to adjust the results with exact values. All of preceding calculations and results are based on using equal φ 's in the whole Lumped Models, here it is shown that for one case(a_1=4 , a_2=1 , r=1) in two Lumped Models what is the effect of using different φ 's.

Counter Flow Heat Exchanger, two Lumped Models:

Recalling equation(4-16a) and substitute the a_1 =4 ,a_2 =1 ,r=1, and using φ_1 =1/3 ,φ_2=2/3, following transfer function is gotten:

$$G(j\Omega) \equiv \frac{\bar{T}_{oc}}{\bar{T}_{iH}} = \frac{3(59-4\Omega^2+32j\Omega)}{(864+4\Omega^4-487\Omega^2)+j(1197\Omega-76\Omega^3)}$$

For $\Omega=0$ $\quad |G|=.1979$, $\theta=0^\circ$

For $\Omega \longrightarrow \infty$ $\quad |G| \longrightarrow 0$ $\quad \theta \longrightarrow -180^\circ$

fitting factor $= \frac{.2405}{.1979} = 1.2152$

Table(5-1) shows the results of this case for different values of frequency, also Fig.(5-1) shows this results comparing to the exact value and results of using equal φ's.

IMPROVEMENT OF LUMPED MODELS TO GET THE BEST RESULTS:

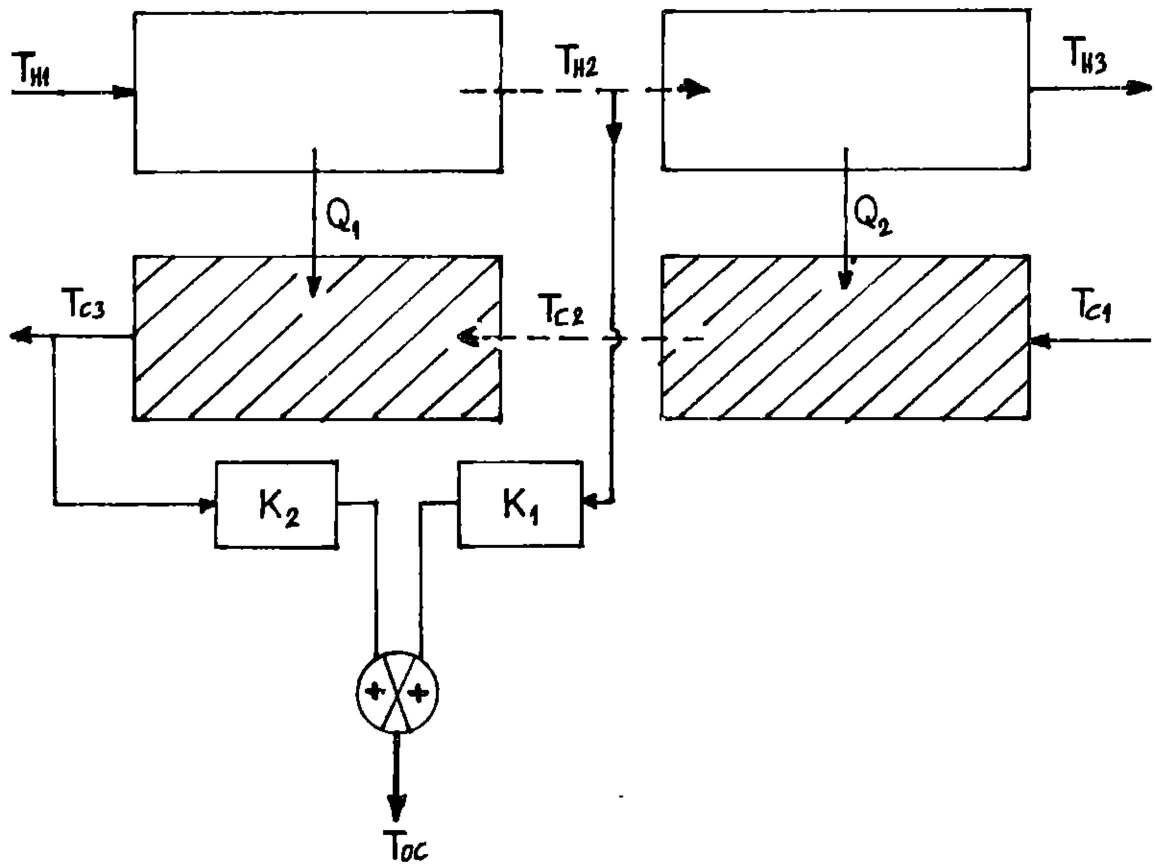
In the preceding section the results of various Lumped-Models and exact solutions were compared, via tabulations and plots. The results for counter-flow heat exchangers, Fig. (4-3) through (4-18), show that a Lumped Model can yield a good agreement with exact results at low frequency, but at high frequency both gains and phase angles are characteristically different. If one looks very carefully at these results, one finds in all cases the discrepancy between the Lumped-Model and the exact solution is always the same, i.e. the phase angles always have 90 degree difference and gains are about half of the exact values; therefore, one may conclude that some feature is characteristically the same for all of them. It is clear that thus 90 degree difference between phase angles means if a zero is added to the transfer function of the Lumped Model, then the phase angles at high frequency will match the exact results very well.

Since the purpose of this thesis is to find simple models which well match the exact results, evaluating the transfer functions which were found for different Lumped Models showed that two-lump models have a very simple form to handle; therefore, one should look for a way to add a zero to results of two Lump Models to get better agreement.

Recalling the outlet temperature results of two-lump models for counter-flow heat exchangers, they show that the transfer function of the first lump has one zero more than the transfer function of outlet temperature of the second lump; therefore, if the outlet temperature of the second lump of the cold fluid is combined with outlet temperature of the first lump of hot fluid, then the overall result has one zero more than the zeroes of the outlet temperature of the cold fluid.

Therefore, from above discussion we conclude that to get the best agreement for lumped model approximations,we should employ in general a linear combination of the outlet temperatures of the various lump, and in particular we have the combination for two lump models as shown in Fig.(5-2).

Now, it is very interesting to determine whether there is any value for K_1 and K_2 which yields a good match between Lumped Model results and exact solutions for both gain and phase angle. The positive answer to this question is explored for the various cases already discussed. The results are expressed in terms of the values of K_1 and K_2 for different cases. Finally one may ask whether these values are the same or different for different cases and if the parameters are different how great is the range of their deviation?



FIGURE(5-2): Linear Combination for Two Lumped Models Counter-Flow Heat Exchanger

K_1 and K_2 for case: $a_1=4$, $a_2=1$, $r=1$

Counter-Flow Heat Exchanger-

By substituting the parameters in equations(4-15a)& (4-15a-0) and using equal φ's i.e. $\varphi=1/2$, following transfer functions are gotten:

$$\frac{\bar{T}_{C3}}{\bar{T}_{H1}} = \frac{2}{S^2+9S+10}$$

$$\frac{\bar{T}_{H2}}{\bar{T}_{H1}} = \frac{2(S^2+9S+14)}{(S^2+9S+10)(S+6)}$$

Therefore, the new outlet temperature of cold fluid is:

$$\frac{\bar{T}_{oc}}{\bar{T}_{iH}} = \frac{2K_2}{s^2+9s+10} + \frac{2(s^2+9s+14)K_1}{(s^2+9s+10)(s+6)}$$

or,

$$G(s) = \frac{2(K_1 s^2+9K_1 s+K_2 s+14K_1+6K_2)}{(s^2+9s+10)(s+6)}$$

$$G(j\Omega) = \frac{2(14K_1+6K_2-K_1\Omega^2)+2j(9K_1\Omega+K_2\Omega)}{(10-\Omega^2+9j\Omega)(6+j\Omega)}$$

To decide how should the gains K_1 and K_2 be determined, we have to see the effect of high values of frequency, also we can't adjust K_1 and K_2 exactly for high frequency because we thereby preclude the adjustment at low frequency. Therefore, it is clear that adjustments must begin at middle frequencies. For value of Ω =2, the amount of gain and phase angle are set equal to the exact values and the necessary K_1 and K_2 calculated. Then the results for other values of frequency are compared.

The results show a very satisfactory agreement at most values of frequency except at very lowest frequencies;therefore, by optimizing K_1 and K_2 around the values that was gotten for Ω =2, two values for K's are gotten which they are:

$$K_1 = 1.10 \qquad , \qquad K_2 = 2.44$$

Therefore, transfer function will be:

$$G(j\Omega)=\frac{2(30-1.1\Omega^2+12.34j\Omega)}{(10-\Omega^2+9j\Omega)(6+j\Omega)}$$

For $\Omega=0$ $|G|_N=1$ $\theta=0^\circ$

For $\Omega\rightarrow\infty$ $|G|_N\rightarrow 0$ $\theta\rightarrow -90^\circ$

This transfer function is for normalized gain, and the results for other values of frequency are in table (5-2).

As we compare these results with those for the exact solution, we conclude that this is the best agreement for whole range of frequency in a very simple closed form.

Fig.(5-3) shows the plotted results and it is shown that the results of the simple Lumped Model is more close to exact results than Friedly's model, with advantage of having a very easy a simple form which is very useful for control of heat-exchangers.

K_1 and K_2 for case: $a_1=2$, $a_2=1$, $r=2$

Counter-Flow Heat Exchanger-

Recalling equations (4-15a)&(4-15a-0) and substituting given numbers, the results are:

$$\frac{\bar{T}_{C3}}{\bar{T}_{H1}}=\frac{s^2+5.5s+7}{(s^2+5.5s+5)^2-2}$$

$$\frac{\bar{T}_{H2}}{\bar{T}_{H1}}=\frac{2(s+1.5)(s^2+5.5s+5)}{(s^2+5.5s+5)^2-2}$$

Then, the new outlet temperature for cold fluid is:

$$\frac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \frac{K_1(2S^3+14S^2+21.5S+15)+K_2(S^2+5.5S+7)}{(S^2+5.5S+5)^2-2}$$

or,

$$G(j\Omega) \equiv \frac{\bar{\bar{T}}_{oc}}{\bar{\bar{T}}_{iH}} = \frac{(7K_2+15K_1-K_2\Omega^2-14K_1\Omega^2)+j(5.5K_2\Omega+21.5K_1\Omega-2K_1\Omega^3)}{(23+\Omega^4-40.25\Omega^2)+(55\Omega-11\Omega^3)j}$$

To get the proper values for K_1, K_2, the same manner which used for first case is used here and the results are:

$K_1 = .4$, $K_2 = 2.55$

Table (5-3) shows the results of improvement case and it compares them with the exact results. In fig.(5-4) the exact solution and improvement of Lumped Model results are plotted, and it is clear that how this simple model predicts the transient response of heat exchanger so close to exact solution.

In third case which $a_1 = a_2 = r = 1$, we do the same as what we have done so forth, but for flactuation part we can use average values for them and adjust the K's with those values.

STEP AND IMPULSE RESPONSES:

One of the advantages of Lumped Models system is that it describes the transfer functions in a very simple form which it is easy to find the response of heat exchanger respect to any kind of disturbance like step function or impulse disturbance or periodic perturbation and so forth. Specially the expression of the improved case has a very simple form because it works only with two stirred tank. Here the results of step and impulse responses are presented for improved case and is compared with exact responses.The method is the same for every case;therefore, only for one case the results are presented.

a)- Step response for case: $a_1=4$, $a_2=1$ $r=1$

In reference(4), the results of step response for exact solution is presented, and they are used here and they are tabulated in table(5-4).

For Friedly's Method:

$$\bar{G}(x,s)=\frac{K}{1+Ts}\left[e^{-rs(1-x)}-e^{-(a_1+a_2)x}\cdot e^{-s(x+r)}\right]$$

Where: $\frac{K}{T}=\frac{a_2 e^{a_2(x-1)}}{1+r}$, $K=\frac{a_2}{a_1\left[1-e^{-(a_1+a_2)x}\right]}\times\frac{1-e^{-(a_1-a_2)x}}{1-\frac{a_2}{a_1}e^{-(a_1-a_2)}}$

For step response it will be:

$$\frac{\bar{T}_{oc}}{\bar{T}_{in}}=\frac{K}{s(1+Ts)}\left[e^{-rs(1-x)}-e^{-(a_1+a_2)x}\cdot e^{-s(x+r)}\right]$$

$$\frac{T_{oc}}{T_{iH}} = \frac{K}{T} x \begin{cases} 0 & t \leqq r(1-x) \\ T\left(1-e^{-\frac{t-r(1-x)}{T}}\right) & t \geqq r(1-x) \end{cases} - \frac{K}{T} x\, e^{-(a_1+a_2)x} \begin{cases} 0 & t \leqq x+r \\ T\left(1-e^{-\frac{t-(x+r)}{T}}\right) & t \geqq x+r \end{cases}$$

Therefore, for $a_1=4$, $a_2=r=1$ and x=1:

$$\frac{K}{T} = 2/4 \;,\; K = .242 \longrightarrow T = .484$$

$$\frac{T_{oc}}{T_{iH}} = \begin{cases} 0 & t \leqq 0 \\ .242\left(1-e^{-2.065t}\right) & 0 \leqq t \leqq 2.0 \\ .242\left(.993-.581\, e^{-2.065t}\right) & t \geqq 2.0 \end{cases} \qquad (5\text{-}1)$$

For Lumped Model:

$$\bar{G}(s) = \frac{\bar{T}_{oc}}{\bar{T}_{iH}} = \frac{2(30+1.1s^2+12.34s)}{(s^2+9s+10)(s+6)}$$

For Step Response:

$$\frac{\bar{T}_{oc}}{\bar{T}_{iH}} = \frac{2(1.1s^2+12.34s+30)}{s(s+6)(s+4.5+\sqrt{10.25})(s+4.5-\sqrt{10.25})}$$

$$\frac{T_{oc}}{T_{iH}} = \mathcal{L}^{-1}\left[\frac{2(1.1s^2+12.34s+30)}{s(s+6)(s+4.5+\sqrt{10.25})(s+4.5-\sqrt{10.25})}\right]$$

Then the result is:

$$\frac{T_{oc}}{T_{iH}} = .2405\left(1-.185\, e^{-6t}-.005\, e^{-7.7t}-.81\, e^{-1.298t}\right) \qquad (5\text{-}2)$$

Table(5-4) shows the result of step response for three methods which are Lumped Models, Exact and Friedly's approximation method. Fig.(5-5) presents the plotted comparison between these three methods. It shows clearly that the Lumped-Model results are closer to exact results than Friedly's method.

b)- Impulse Response for Case: a =4 , a =1 , r=1

The exact results for impulse response were presented in reference(10), they also are shown in table (5-5).

Friedly's Method:

Because the impulse response is derivative of step response therefore, by taking the derivative of equation (5-1) following results are obtained:

$$\frac{T_{oc}}{T_{iH}} = \begin{cases} 0 & t \leq 0 \\ 2.065\, e^{-2.065t} & 0 \leq t \leq 2 \\ 1.199\, e^{-2.065t} & t \geq 2 \end{cases}$$

Results for different values of time (normalized time) are presented in table (5-5).

Lumped Model:

To get the impulse response we take only the derivative of equation which is for step response;therefore, its results are:

$$\frac{T_{\alpha}}{T_{iH}} = 1.11\, e^{-6t} + .038\, e^{-7.7t} + 1.051\, e^{-1.298t}$$

Table (5-5) contains the results of impulse response for exact solution and Friedly's method and Lumped Model system.

In Fig.(5-6) these results are plotted for comparing the accuracy of two approximate methods, and it is clear that in Lumped-Model method the error is almost zero for all values of time but we can't say for Friedly's method the error is near zero.

CALCULATIONS OF MONOTONIC PARAMETERS:

H.M.Paynter (3) presented a new evaluation method for dynamic response including that for counter-flow and parallel-flow heat exchangers. He showed that the Laplace transformation solutions for heat exchangers can be written as:

$$G(s) = e^{\delta - T_m s + \frac{T_s^2}{2} s^2 - \frac{T_a^3}{6} s^3 + \cdots}$$

Where, δ measures the steady state amplitude ratio between response and disturbance, T_m measures the mean time delay between response and disturbance, T_s defines the dispersion or atten uation, and T_a is the asymmetry or phase nonlinearity. As it is shown in his paper, he concluded that for a given Laplace transformation solution of the form:

$$G(s) = \frac{N(s)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots\cdots}$$

and for special case N(s) =1 the monotonic parameters can be written directly by following relationships:

$$T_m = a_1$$

$$T_s^2 = a_1^2 - 2a_2$$

$$T_a^3 = 2a_1^3 - 6a_1 a_2 + 6a_3$$

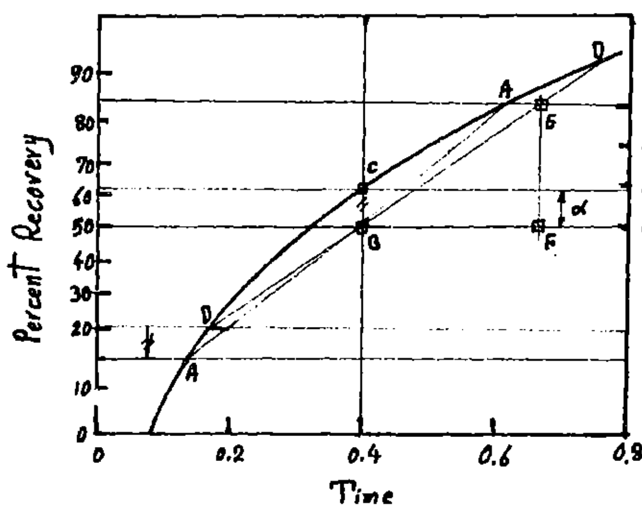
If N(s) is a function of (s) then the results of parameters should be diminished by the same amount which are obtained from N(s).

Therefore, since the results of Lumped Model method are in the form of $G(s)=\frac{N(s)}{D(s)}$, then the monotonic parameters can be calculated very easily.

To compare the calculated parameter with exact monotonic parameters H.M. Paynter presented following technique(8):

Simple Technique for determination of Monotonic Parameters:

By using a commercially available "Probability Paper", a Gaussian distribution can be plotted linearly. If a step response of any dynamic behavior is plotted on such paper (as shown in Fig.5-6)by following method the monotonic parameters are obtained.



FIGURE(5-6): Use of Probability Paper for Monotone Parameters

PROCEDURE:

1- Points A-A are determined where the step response curve crosses the 16% and 84% recovery levels.

2- Point C is on the response curve vertically above point B .

3- The vertical distance BC set off upwards from the 16% and 84% levels determines the points D-D.

4- The intersection of line DD with the 84% level fixes point E.

5- Point F is vertically below E at the 50% level.

Then for determine the parameters we have:

T_m: - The time interval from the origin to point B gives the meandelay, T_m.

T_s: - The time interval from point B to point F gives the dispersion time, T_s.

α : - The ratio of distance BC to distance FE gives one-sixth the skew coefficient, α .

T_a: - Then by determination of α we have: $T_a^3 = \alpha T_s^3$

Now, by plotting the results of step responses of exact solution, Lumped Model, and Friedly's method, on probability paper we can get the Monotone parameters,also these parameters can be obtained directly from Laplace transformation solution for Lumped Model Method.

Fig. (5-7) through (5-9) show results of plotting the step responses to exact solution, Lumped Model, and Friedly's method, the above technique is used and from those plotting the following results are gained:

For Exact Solution $\begin{cases} T_m = .65 \\ T_s = .81 \\ T_a = 1.08 \end{cases}$

For Lumped Model $\begin{cases} T_m = .66 \\ T_s = .87 \\ T_a = 1.19 \end{cases}$

For Friedly's Method $\begin{cases} T_m = .48 \\ T_s = .51 \\ T_a = .62 \end{cases}$

As the results of Paynter's method for determination of Monotonic parameters show, the values which obtained by Lumped Model are very close to exact values, but we can't see this accuracy from Friedly's method, and this is one of the other advantage of Lumped Model.

Monotone parameters are calculated by the direct method for Lumped Model and written as follow:

$$G(s)=\frac{2.2s^2+24.68s+60}{s^3+15s^2+64s+60}=\frac{1+.411s+.0367s^2}{1+1.06s+.25s^2+.0167s^3}$$

Therefore:

$$T_{m_1}=a_{11}=1.06 \quad , \quad T_{m_2}=a_{12}=.411 \implies T_m=1.06-.41=.65$$

$$T_{s_1}=\sqrt{a_{11}^2-2a_{21}}=.80 \; , \; T_{s_2}=\sqrt{a_{12}^2-2a_{22}}=.3 \implies T_s=\sqrt{.8^2-.3^2}=.74$$

$$T_{a_1}=\sqrt[3]{2a_{11}^3-6a_{11}a_{21}+6a_{31}}=1.0 \; , \; T_{a_2}=\sqrt[3]{2a_{12}^3-6a_{12}a_{22}+6a_{32}}=.3 \implies T_a=\sqrt[3]{1^3-.3^3}=.99$$

The difference between these results and graphical results are because of approximation in graphical method.

SECTION VI

DISCUSSION, CONCLUSION, AND SUGGESTIONS FOR FUTURE WORKS:

The essential part of this thesis has been devoted to determination of very simple conceptual and physical models that would describe heat-exchangers in some of the most important practical cases that one expects to encounter, particularly under conditions where the flow rates are varied. The low frequency behavior for shell-and-tube exchangers has been matched as carefully as possible by use of a method due to Professor H.M. Paynter. It has appeared reasonable to take into account at least qualitatively the very high frequency behavior to complement the low frequency information. To this end, Friedly presented an approximate method which works for low and high frequencies, but which is infinite order;therefore, a simple low-order model should be found. For certain applications (e.g. optimal control) since lump models constitute the most linear way to get Simple results, such Lumped Model have been used in present work.

Analysis of Lumped Models has illustrated that it is possible to approximate the dynamics of a variety of plug flow processes with very simple and basic transfer functions.

As the discussed examples show, the simple Lumped Models even without improvement demonstrate very good agreement at low frequency or for long-duration fluctuations. As those results show, there is no sensible difference between two lump-Models and four-lumps ones;in other words, the rate of convergence of the Lumped Model to the exact solution is quite small with respect to changes in the number of sections. Therefore, from the stand-point of significant improvement only two-lump models were discussed and results show that they sensibly agree with exact solution over the whole range of frequency.

Since by this technique the outlet temperature of each section is determined by a very simple transfer function, this model gives not only the exit temperature of the heat-exchanger but also the temperatures for all intermediate points at each section along the heat-exchanger, the special cases, which are included, show that both the frequency and time response are well approximated over their entire range by a transfer function derived only from a two-lump model.

There is no evidence limitation for this model, especially in the improved case, because it includes some adjustable parameters which make it flexible for all situations;therefore, many assumptions can be removed by changing those particular parameters to achieve a satisfactory solution for most cases.

Another important aspect of this thesis is that the Lumped Model is capable of directly predicting certain Flow Reversal effects. The results for steady state behavior show, thus the outlet temperature of Lumped-Models is very close to exact solution. To obtain the exact values the Lumped-Models results can be adjusted by calibration. Thus, it is important to know that if the parallel-flow case (or alternatively, the counter-flow case) is so fitted, what will then happen if flow is reversed. Fitting the parallel-flow situation with exact value is shown to yield better results in such flow-reversal cases.

Comparison of Friedly's approximate method and the technique of Lumped-Models shows that the use of Lumped-Models has many advantages with respect to Friedly's model which some of them are:

1- Basically Friedly's method is an infinite order model which is complicated from the control point of view while Lumped-Models are of finite order which are easy to handle and work with.

2- Friedly's method doesn't yield good agreement in all cases (as it is shown in reference 4) but the Lumped-Models can be used in a variety of cases with overall better results than Friedly's model.

3- Friedly's model has the limitations indicated in reference (4) but as previously mentioned, Lumped-Models provide flexibility which can be expanded for many cases.

Since the present work gives very useful results for the discussed cases it would appear to be worthwhile to further develop the method for more general cases. Therefore, following efforts are recommended:

I- The improvement technique is physically acceptable because Two Lumped-Models for whole heat exchanger yield too low an exit temperature. Therefore, a combination of an intermediate temperature and an exit temperature gives the best results, but it should be proved mathematically why such a compensatory technique is true and why a linear combination is sufficient.

II- It is interesting to know the effectiveness and sensitivity of the parameters α and φ , where α is used in temperature of each section-$[T=\alpha T_i+(1-\alpha)T_0]$ -because in the present work α was assumed to be zero which clearly isn't precisely correct for any real case. Also for one special case different φ's are used to improve the results, Therefore, this parameter has been shown to have significant influence.

III- Lumped-Model could be used (especially the improvement case) to eliminate all of the common simplifying assumption such as: including the wall heat capacities, changing overall heat transfer coefficient between two fluids along the heat-exchanger, boundary layer effects, and eliminating the plug flow assumption.

IV- Heat-exchangers involving gases and vapors are more complicated to deal with, since the momentum and continuity equations are now coupled with the thermal equations. Therefore, it will be interesting to consider Lumped-Models for such two phase situations.

REFERENCES:

1. H.M. Paynter, Y. Takahashi, "A New Method of Evaluating Dynamic Response of Counter Flow and Parallel Flow Heat Exchangers." Trans. ASME,78,1956, pp 749-758.

2. Y. Takahashi,"Automatic and Manual Control.", A. Tustin, ed. Butterworths London,1952, pp 235-248.

3. H.M. Paynter, "On an Analogy Between Stochatic Processes and Monotone Dynamic Systems."

4. J.C. Friedly, "Asymptotic Approximation to Plug Flow Process Dynamics." Joint Automatic Control Conference Preprints of Papers, June, 1967, pp. 216-228.

5. H.M. Paynter, Class Notes on"Dynamics and Control of Thermofluid Processes and Systems." M.I.T. 1977.

6. R.G.E. Franks, "Modeling and Simulation in Chemical Engineering." Wiley-Interscience, New York 1972.

7. P. Profos, Die Behandlung von Regelproblemen vermittels des Frequenzganges des Regelkreises Zürich,1943, pp.48.

8. H.M. Paynter, "Regelungstechnik, Moderne Theorien und ihre Verwend brakeit." Verlag R. Oldenbourg Munchen,1957 ,pp.135.

9. J.P. Hsu, N. Gilbert, "Transfer Functions of Heat Exchangers." A.I.Ch.E. Journal, Vol. 8.No.5, November 1962, pp.593-598.

10. J.M. Douglas, "Process Dynamic and Control" VolumeI, Analysis of Dynamic Systems, Prentice-Hall, Inc., Englwood Cliffs, New Jersey 1972.

11. J.M. Mozley, "Predicting Dynamics of Concentric Pipe Heat Exchangers." Ind. Eng. Chem., 48, 1956, pp.1035-1051.

12. M. Masubuchi, "Dynamic Response and Control of Multipass Heat Exchanger." ASME, Journal of Basic Engineering, March 1960, pp.51-59.

13. S.E. Rea, C.M. Ablow, "Transient Air Temperatures in a Duct." Trans. ASME, October 1957, pp.1536-1544.

14. J.W. Rizika, "Thermal Lags in Flowing Systems Containing Heat Capacitors." Trans. ASME, April 1954, pp.411-420.

15. P.D. Hansen, "The Dynamics of Heat Exchange Processes." Sc.D. Thesis, Dept. of Mechanical Eng., M.I.T. 1960.

16. G.E. Myer, J.W. Mitchell, C.F.Jr.Lindeman, "The Transient Response of Heat Exchanger Having an Infinite Capacitance Rate Fluid." ASME, Journal of Heat Transfer, May 1970, pp 267-275.

17. W.J. Yang, "Transient Heat Transfer in a Vapor Heated Exchanger With Arbitrary Time-Wise Perturbation." ASME Journal of Heat Transfer, May 1964, pp.133-142.

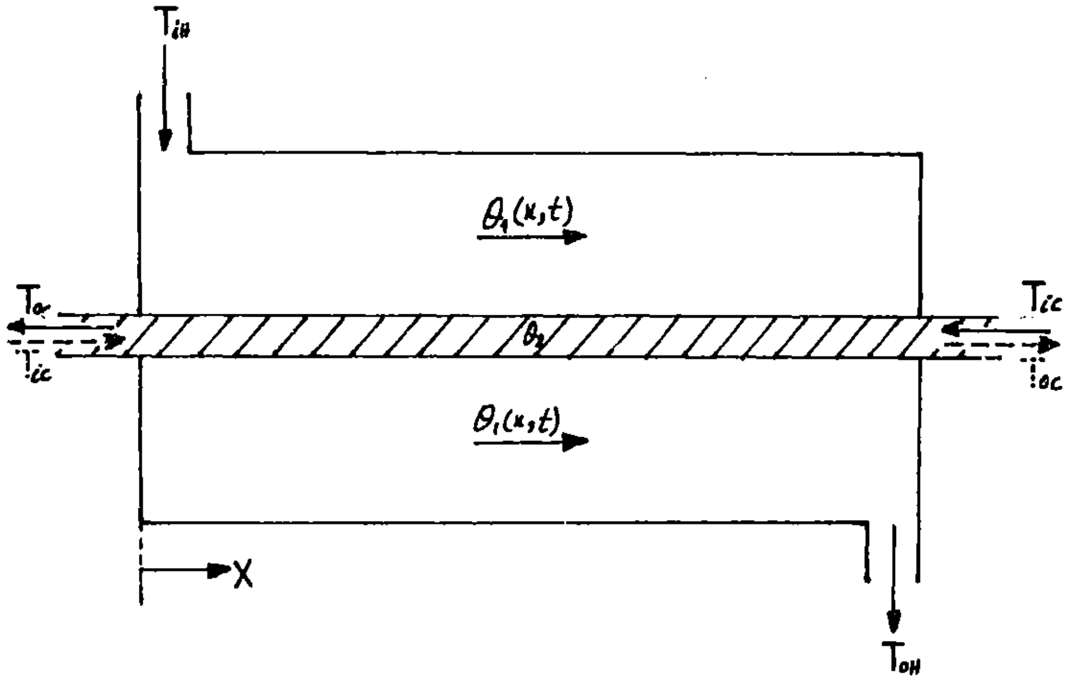
18. G.E. Myer,J.W. Mitchel, R.F. Norman, "The Transient Response of Cross-Flow Heat Exchangers and Condensers." ASME, Journal of Heat Transfer, Feb.1967, pp.75-80.

19. G.M. Dusinberre, "Calculation of Transients in Cross-Flow Heat Exchanger." ASME, Journal of Heat Transfer, Feb.1959, pp.61-69.

APPENDIX

a-THE EXACT SOLUTION:

Exact solution of dynamic responses of heat exchangers can be obtained by using the energy balance (First law of thermodynamics) for Fig. (A-1):



FIGURE(A-1): Counter Flow or Parallel Flow Heat Exchanger

Energy Balance:

Rate of heat accumulation = Heat Flow - Heat flow out ± Heat Transfered

This relation is obtained from first law of thermodynamics by following assumptions:

$$\dot{Q} - \dot{W}_{sh} = \dot{m}_{out}\left(h+\frac{V^2}{2}+gz\right)_{out} - \dot{m}_{in}\left(h+\frac{V^2}{2}+gz\right)_{in} + M \cdot \dot{E}$$

Mass conservation:

$$\rho A V = \text{Const.}$$

1- Assume constant geometry $A = \text{Const.}$

$V = \text{Const.}$

2- Assume constant fluid properties

$\rho = \text{Const.}$

Therefore, there is no change in kinematic energy. There is no shaft work transfer;therefore, we get:

$$\dot{Q} = \dot{M}(h_{out} - h_{in}) + M\dot{E}$$

$$M*\dot{E} = \dot{M}h + \dot{Q} - \dot{M}h_{out}$$

By using the Profos operator (7) and equation we get:

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t} + \dfrac{\partial \theta_1}{\partial x} = a_1(\theta_2 - \theta_1) \\ r\dfrac{\partial \theta_2}{\partial t} \pm \dfrac{\partial \theta_2}{\partial x} = a_2(\theta_1 - \theta_2) \end{cases}$$

Where r is ratio of cold flow rate and hot flow rate. Negative sign is for counter-flow and positive sign is for parallel-flow case.

Initial conditions:

$$\begin{cases} \theta_1 (x,0) = 0 \\ \theta_2 (x,0) = 0 \end{cases}$$

Boundary Conditions:

counter flow

$$\begin{cases} \theta_2 (1,t) = 0 \\ \theta_1 (0,t) = T_{lH} \end{cases}$$

Parallel Flow
$$\begin{cases} \theta_1(0,t) = T_{iH} \\ \theta_2(0,t) = 0 \end{cases}$$

The easiest way to solve partial differential equations is using Laplace transform, therefore by taking Laplace-transform of equation:

i- for Counter Flow:

$$\begin{cases} s\bar{\theta}_1 + \dfrac{d\bar{\theta}_1}{dx} = a_1\bar{\theta}_2 - a_1\bar{\theta}_1 \\ rs\bar{\theta}_2 - \dfrac{d\bar{\theta}_2}{dx} = a_2\bar{\theta}_1 - a_2\bar{\theta}_2 \end{cases}$$

or

$$\begin{cases} \bar{\theta}_2 = \bar{\theta}_1 + \dfrac{s}{a_1}\bar{\theta}_1 + \dfrac{1}{a_1}\dfrac{d\bar{\theta}_1}{dx} \\ (rs+a_2)\left(\bar{\theta}_1 + \dfrac{s}{a_1}\bar{\theta}_1 + \dfrac{1}{a_1}\dfrac{d\bar{\theta}_1}{dx}\right) - \dfrac{d}{dx}\left(\bar{\theta}_1 + \dfrac{s}{a_1}\bar{\theta}_1 + \dfrac{1}{a_1}\dfrac{d\bar{\theta}_1}{dx}\right) = a_2\bar{\theta}_1 \end{cases}$$

then:

$$\frac{d^2\bar{\theta}_1}{dx^2} - \left[s(r-1) + (a_2 - a_1)\right]\frac{d\bar{\theta}_1}{dx} - s(rs + ra_1 + a_2)\bar{\theta}_1 = 0$$

using $\bar{\theta}_1 = e^{px}$, it gives:

$$P^2 - \left[s(r-1)+(a_2-a_1)\right]P - s(rs+ra_1+a_2) = 0 \qquad \text{(A-I)}$$

Equation(A-I) gives two values P_1 , P_2 , then by boundary condition we have:

$$\bar{\theta}_1 = Ae^{P_1x} + Be^{P_2x}$$

$$\bar{\theta}_2 = \left(\frac{s+a_1}{a_1}\right)\left(Ae^{P_1x} + Be^{P_2x}\right) + \frac{1}{a_1}\left(AP_1e^{P_1x} + BP_2e^{P_2x}\right)$$

Inserting boundary condition:

$$A + B = \overline{T}_{iH}$$

$$A(s + a_1 + P_1)e^{P_1} + B(s + a_1 + P_2)e^{P_2} = 0$$

which gives:

$$\frac{\overline{\theta}_1}{\overline{T}_{iH}} = \frac{(s + a_1 + P_2)e^{P_1 x} - (s + a_1 + P_1)e^{P_2 x + P_1 - P_2}}{\left[s + a_1 + P_2 - (s + a_1 + P_1)e^{P_1 - P_2}\right]} \qquad (A-2)$$

$$\frac{\overline{\theta}_2}{\overline{T}_{iH}} = \frac{(s + a_1 + P_2)(s + a_1 + P_1)\left(e^{P_1 x} - e^{P_2 x + P_1 - P_2}\right)}{a_1\left[s + a_1 + P_2 - (s + a_1 + P_1)e^{P_1 - P_2}\right]} \qquad (A-3)$$

Putting x=1 and using proper parameters equations(A-2), (A-3) becomes the same as equation(2-3).

ii-Parallel flow: using Laplace transform:

$$\begin{cases} s\overline{\theta}_1 + \dfrac{d\overline{\theta}_1}{dx} = a_1\overline{\theta}_2 - a_1\overline{\theta}_1 \\ rs\overline{\theta}_2 + \dfrac{d\overline{\theta}_2}{dx} = a_2\overline{\theta}_1 - a_2\overline{\theta}_2 \end{cases}$$

By continue in the same manner of counter-flow, we get:

$$\frac{\overline{\theta}_1}{\overline{T}_{iH}} = \frac{(s + a_1 + P_2)e^{P_1 x} - (s + a_1 + P_1)e^{P_2 x}}{P_2 - P_1} \qquad (A-4)$$

$$\frac{\overline{\theta}_2}{\overline{T}_{iH}} = \frac{(s + a_1 + P_1)(s + a_1 + P_2)\left(e^{P_1 x} - e^{P_2 x}\right)}{a_1(P_2 - P_1)} \qquad (A-5)$$

Where: $P_1, P_2 = -\left(\frac{a_1 + a_2}{2}\right) - \frac{s(r+1)}{2} \pm \sqrt{\left(\frac{a_1 + a_2 + rs + s}{2}\right)^2 - s(rs + a_1 r + a_2)}$

FOR STATIC CASE:

To get the Static(steady state) case, S should be zero, then:

counter flow

$$\frac{\theta_1}{T_{iH}} = \frac{(P_2+a_1)e^{P_1x} - (P_1+a_1)e^{P_2x+P_1-P_2}}{\left[P_2+a_1-(P_1+a_1)\,e^{-P_2+P_1}\right]}$$

$$\frac{\theta_2}{T_{iH}} = \frac{(P_1+a_1)(P_2+a_1)\left(e^{P_1x} - e^{P_2x+P_1-P_2}\right)}{a_1\left[P_2+a_1-(P_1+a_1)\,e^{P_1-P_2}\right]}$$

parallel flow

$$\frac{\theta_1}{T_{iH}} = \frac{(P_2+a_1)e^{P_1x} - (P_1+a_1)\,e^{P_2x}}{P_2-P_1}$$

$$\frac{\theta_2}{T_{iH}} = \frac{(P_1+a_1)(P_2+a_1)\left(e^{P_1x} - e^{P_2x}\right)}{a_1(P_2-P_1)}$$

By using the values of P_1 , P_2 in each case the summary results are:

parallel flow

$$\frac{\theta_2(x)}{T_{iH}} = \frac{a_2\left[1-e^{-(a_1+a_2)x}\right]}{a_1+a_2}$$

$$\frac{\theta_1(x)}{T_{iH}} = \frac{a_2 + a_1\,e^{-(a_1+a_2)x}}{a_1+a_2}$$

Counter Flow

$$\left\{\begin{aligned} \frac{\theta_2(x)}{T_{iH}} &= \frac{\left[a_1 e^{-(a_1-a_2)x} - e^{-(a_1-a_2)}\right]}{a_1 - a_2 e^{-(a_1-a_2)}} \\ \frac{\theta_1(x)}{T_{iH}} &= \frac{a_2\left[e^{-(a_1-a_2)x} - e^{-(a_1-a_2)}\right]}{a_1 - a_2 e^{-(a_1-a_2)}} \end{aligned}\right.$$

b-TEMPERATURE OF EACH SECTION IN LUMPED MODELS:

For each section in Lumped Models, we assumed that they work as a stirred tank, as it's shown in Fig. (A-2); therefore, the temperature of tank is a linear combination of it s inlet and outlet temperatures say:

$$T = \alpha T_i + (1-\alpha) T_o \qquad \text{(A-7)}$$

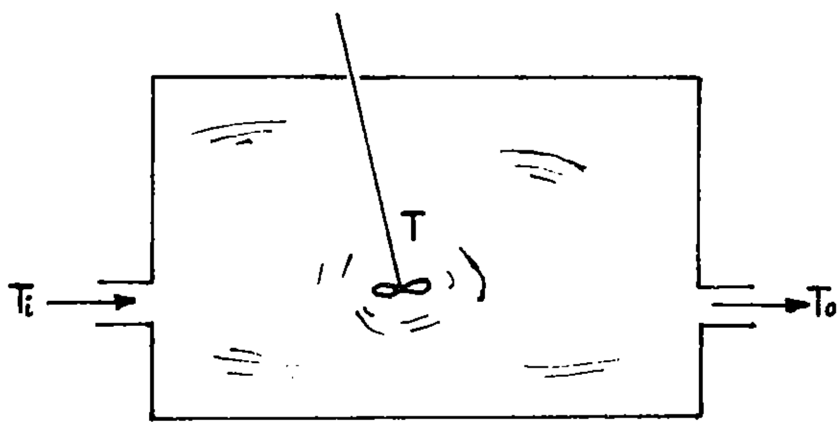


FIGURE (A-2): Stirred Tank

For tank as a control volume, the first law of thermo-dynamics is:

$$M\frac{du}{dt} = \dot{M}_{in}\left(h+\frac{v^2}{2}+gz\right)_{in} - \dot{M}_{out}\left(h+\frac{v^2}{2}+gz\right)_{out}$$

There is no mass accumulation and outlet and inlet pipes have the same geometry ;therefore, by continuiting we have:

$$V_{in} = V_{out}$$

In the steady state case it is shown that: $M = \dot{M} \times \tau$

where M is mass of tank and $\dot{M}$ is mass flow rate through control volume;therefore;

$$\tau c_p \frac{dT}{dt} = c_p (T_i - T_o)$$

or

$$\tau \frac{dT}{dt} = T_i - T_o$$

(A-8)

Combination of equations(A-7)&(A-8)gives:

$$T_o = \frac{1-\alpha\tau S}{1+(1-\alpha)\tau S} T_i \quad , \quad T = \frac{1}{-\alpha\tau S+1} \qquad \text{(A-9)}$$

Because α* is very small and τ is a small number;therefore, for perfect stirred tank α goes to zero and from equation(A-9) we get: $T \cong T_{out}$

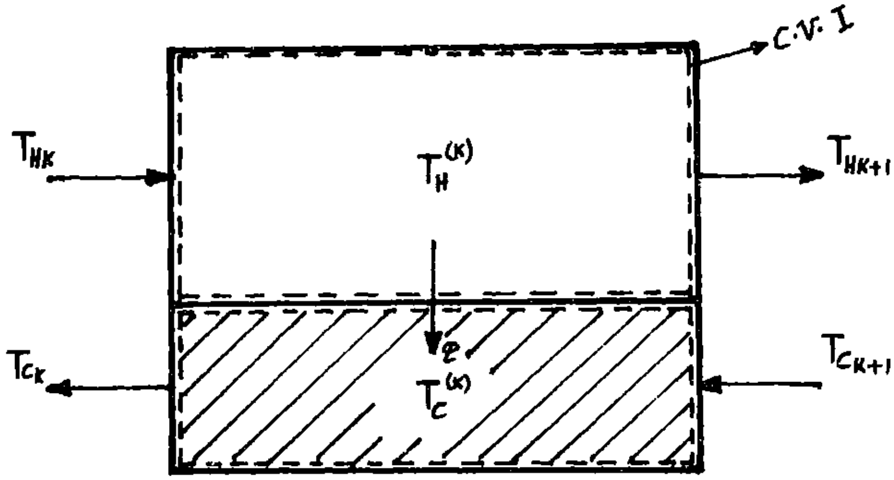
This is why, in all of cases(steady or dynamic) the temperature of each section is put the outlet temperature.

---

* α is small because fluid which leaves the tank has property of the fluid in the tank, but fluid which comes to tank has properties of outside the tank;therefore, the temperature of tankhas more tendency to outlet temperature than inlet temperature,which means α is a small number.

c-ENERGY EQUATION FOR EACH SECTION OF LUMPED MODEL

For each section of Lumped Model we can use the Profos (7) operator as shown for Kth section:



FIGURE(A-3): Kth Section of Lumped Model

Applying first law of thermodynamics for each of control volumes:

$$\dot{Q} - \dot{W}_{sh} = \frac{d}{dt}(MU)_{c.v.} - \dot{M}_{in}\left(h + \frac{V^2}{2} + gz\right)_{in} + \dot{M}_{out}\left(h + \frac{V^2}{2} + gz\right)_{out}$$

Where M is mass of fluid in side the control volume, U is internal energy of fluid inside the control volume and $\dot{M}$ is mass flow rate which comes in and out.

Geometry of control volume is constant(means rigid boundary) then from continuity: $\rho_{in} V_{in} = \rho_{out} V_{out} \implies V_{in} = V_{out}$

Because $\dot{W}_{sh}=0$ (no shaft work), and the internal energy of fluid and it s enthalpy can be written as temperature, then we get:

$$\dot{Q}=MC\frac{dT}{dt}+\dot{M}C(T_{out}-T_{in})$$

Therefore, for control volume of fig.(A-3) the results are:

$$C.V.\ I \Longrightarrow \dot{Q}=M_H C\frac{dT_H^{(K)}}{dt}+\dot{M}_H C(T_{H_{K+1}}-T_{H_K}) \qquad \text{(A-10)}$$

But $\dot{Q}$ can be written as:

$$\dot{Q}=-UA_K(T_H^{(K)}-T_c^{(K)})$$

and for relation between M_H and $\dot{M}_H$,they can be written as:

$$M_H=\rho_H A_s \ell_K$$

$$\dot{M}_H=\rho_H A_s V_H$$

Where A_s is cross-section area and ℓ_K is length of section then if τ_K defined as the time which fluid needs to go through section; therefore:

$$V_H=V_{in}=V_{out}=\frac{\ell_K}{\tau_K}$$

Then:

$$\dot{M}_H=\rho_H A_s\frac{\ell_K}{\tau_K}$$

or

$$\frac{M_H}{\dot{M}_H}=\tau_K$$

By definition $\varphi_K=\frac{\ell_K}{L}$, where L is the total length of tube, it concludes that the time which fluid takes to go through whole of the heat-exchanger is:

$$\tau = \frac{\tau_K}{\varphi_K}$$

Therefore, equation(A-10) becomes:

$$-UA_K\left(T_H^{(K)} - T_C^{(K)}\right) = M_H^{\cdot}\,\tau_K\, C\,\frac{dT_H^{(K)}}{dt} + M_H^{\cdot} C\left(T_{H_{K+1}} - T_{HK}\right)$$

or

$$\tau_K \frac{dT_H^{(K)}}{dt} = T_{HK} - T_{H_{K+1}} - \frac{UA_K}{M_H^{\cdot} C}\left(T_H^{(K)} - T_C^{(K)}\right)$$

(A-11)

Define: $$a_1 \equiv \frac{U \cdot A}{M_H^{\cdot} \cdot C} = \frac{U}{C} \cdot \frac{A_K}{\varphi_K} \times \frac{1}{M_H^{\cdot}}$$

$$\therefore\ a_1 = \frac{UA_K}{M_H^{\cdot} C} \times \frac{1}{\varphi_K}$$

$$\therefore\ \frac{UA_K}{M_H^{\cdot} C} = \varphi_K\, a_1$$

Define:

$$\theta_0 \equiv \frac{t}{\tau} \implies dt = \tau\, d\theta_0$$

$$\therefore\ dt = \frac{\tau_K}{\varphi_K}\, d\theta_0$$

Therefore, equation(A-11) becomes as:

$$\varphi_K \frac{dT_H^{(K)}}{d\theta_0} = T_{HK} - T_{H_{K+1}} - \varphi_K a_1\left(T_H^{(K)} - T_C^{(K)}\right) \qquad \text{(A-12)}$$

By taking Laplace transform of equation(A-12):

$$\varphi_K s \bar{T}_H^{(K)} = \bar{T}_{HK} - \bar{T}_{HK+1} - \varphi_K a_1 (\bar{T}_H^{(K)} - \bar{T}_C^{(K)})$$

Using assumption of perfect stirred tank:

$$\varphi_K s \bar{T}_{HK+1} = \bar{T}_{HK} - \varphi_K a_1 (\bar{T}_{HK+1} - \bar{T}_{CK}) - \bar{T}_{HK+1} \qquad \text{(A-13)}$$

By the same manner, for second control volume, the result become as:

$$r \varphi_K s \bar{T}_{CK} = \bar{T}_{CK+1} - \varphi_K a_2 (\bar{T}_{HK+1} - \bar{T}_{CK}) - \bar{T}_{CK} \qquad \text{(A-14)}$$

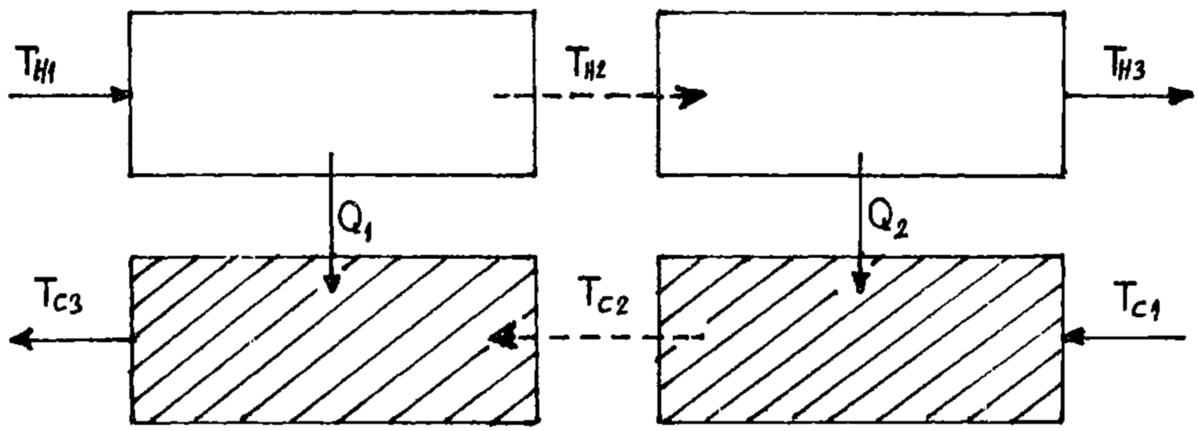
Where r is velocity ratio of two hot and cold fluid putting s=0, gives the results of steady-state case.For parallel-flow the policy is the same and results has been shown in both static and dynamic cases.

As an example for results of each section treatments, here is calculations for case N=2:

Counter Flow:

From preceding calculations, there are four equation for outlet temperatures as follows:

$$\begin{cases} T_{H1} - T_{H2} - \varphi_1 a_1 (T_{H2} - T_{C3}) = 0 \\ T_{C2} - T_{C3} + \varphi_1 a_2 (T_{H2} - T_{C3}) = 0 \\ T_{H2} - T_{H3} - \varphi_2 a_1 (T_{H3} - T_{C1}) = 0 \\ T_{C1} - T_{C2} + \varphi_2 a_2 (T_{H3} - T_{C1}) = 0 \end{cases}$$



FIGURE(A-4): Two Section Counter-Flow Heat Exchanger

Solving above equations gives following results:

$$\frac{T_{C3}}{T_{H1}} = \frac{\varphi_1 a_2 (1 + \varphi_2 a_1 + \varphi_2 a_2) + \varphi_2 a_2}{(1 + \varphi_1 a_1 + \varphi_1 a_2)(1 + \varphi_2 a_1 + \varphi_2 a_2) - \varphi_1 \varphi_2 a_1 a_2}$$

$$\frac{T_{H3}}{T_{H1}} = \frac{(1 + \varphi_1 a_2)(1 + \varphi_2 a_2)}{(1 + \varphi_1 a_1 + \varphi_1 a_2)(1 + \varphi_2 a_1 + \varphi_2 a_2) - \varphi_1 \varphi_2 a_1 a_2}$$

d-CALCULATIONS ABOUT FITTING STATIC CASE WITH EXACT VALUE:

i- Fitting Parallel-Flow:

case: $a_1=4$, $a_2=1$, $r=1$

For this case the exact results are:

$$\frac{T_{oC}}{T_{iH}} = .1986 \qquad \frac{T_{oH}}{T_{iH}} = .2054$$

Results of Lumped Models are:

"Parallel Flow"

$\varphi = 1/2$

$$\left\{ \begin{array}{l} \frac{T_{oC}}{T_{iH}} = .1837 \\ \frac{T_{oH}}{T_{iH}} = .2153 \end{array} \right. \Rightarrow \frac{T_{oC})_E}{T_{oC})_L} = \frac{.1986}{.1837} = 1.0811$$

Which gives: Fitting Factor = 1.0811

Therefore, all of results of parallel and counter-flow are multiplied by 1.0811, then:

"Counter-Flow"

$\varphi = 1/2$

$$\left\{ \begin{array}{l} \frac{T_{oC}}{T_{iH}} = .2 \\ \frac{T_{oC}}{T_{iH}} = .2 \end{array} \right.$$

$$\therefore \quad \frac{T_{oC})_{F-L}}{T_{iH}} = .2 \times 1.0811 = .2162 < .24 = \frac{T_{oC})_E}{T_{iH}}\Big|_{Counter}$$

As it shows, if we fit parallel-flow then we diminish the error in counter-flow by 6.75% . Then the new error is: 9.9% .

$$\varphi = 1/3 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel} = .18945 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.1986}{.18945} = 1.0483 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter\text{-}F} = .213 \times 1.0483 = .2233 \end{cases}$$

The error is decreased by 4.3% and new error is 7% .

$$\varphi = 1/4 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel} = .1922 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.1986}{.1922} = 1.033 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter\text{-}F} = .22 \times 1.033 = .227 \end{cases}$$

The error is decreased by 3% and new error is 5.4% .

Case: $a_1 = 1$, $a_2 = 1$, $r=1$

$$\varphi = 1/2 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel} = .375 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.432}{.375} = 1.152 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter\text{-}F} = .4 \times 1.152 = .4608 \end{cases}$$

The error is decrease by 12% and new error is 7.8% .

$$\varphi = 1/3 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel} = .392 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.432}{.392} = 1.102 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter\text{-}F} = .4286 \times 1.102 = .472 \end{cases}$$

The error is decreased by 7%, and new error is 5.6% .

$$\varphi = 1/4 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel} = .401 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.432}{.401} = 1.077 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter\text{-}F} = .444 \times 1.077 = .479 \end{cases}$$

The error is decreased by 7% andnew error is 4.2% .

ii-Fitting Counter-Flow:

Case: a =4 , a =1 , r=1

$$\varphi = 1/2 \quad \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .2 \Rightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.24}{.2} = 1.2$$

Fitting Factor=1.2

$$\left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel\text{-}F} = 1.2 \times .1837 = .22 > .1986 = \left.\frac{T_{oc}}{T_{iH}}\right|_{Exact}$$

Its value is more than exact value, but it is also farther to exact value than unfitted, and it increases the error by 3.2%, and new error is 10.7% .

$$\varphi = 1/3 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .213 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.24}{.213} = 1.126 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel\text{-}F} = .1894 \times 1.126 = .213 > .1986 \end{cases}$$

It increases the error by 2.6% , and new error is 7.25% .

$$\varphi = 1/4 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .22 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.24}{.22} = 1.09 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel-f} = .1922 \times 1.09 = .209 > .1986 \end{cases}$$

It increases the error by 2% , and new error is 5.2% .

For case: $a_1 = a_2 = 1$

$$\varphi \doteq 1/2 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .4 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.5}{.4} = 1.25 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel-f} = .375 \times 1.25 = .4687 > .432 \end{cases}$$

Because the new value is closer to exact value but higher then it decreasesthe error by 4.7% ,and new error is 8.5% .

$$\varphi = 1/3 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .4286 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.5}{.4286} = 1.16 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel-f} = .392 \times 1.16 = .457 > .432 \end{cases}$$

It decreases the error by 3.5% ,and new error is 5.8% .

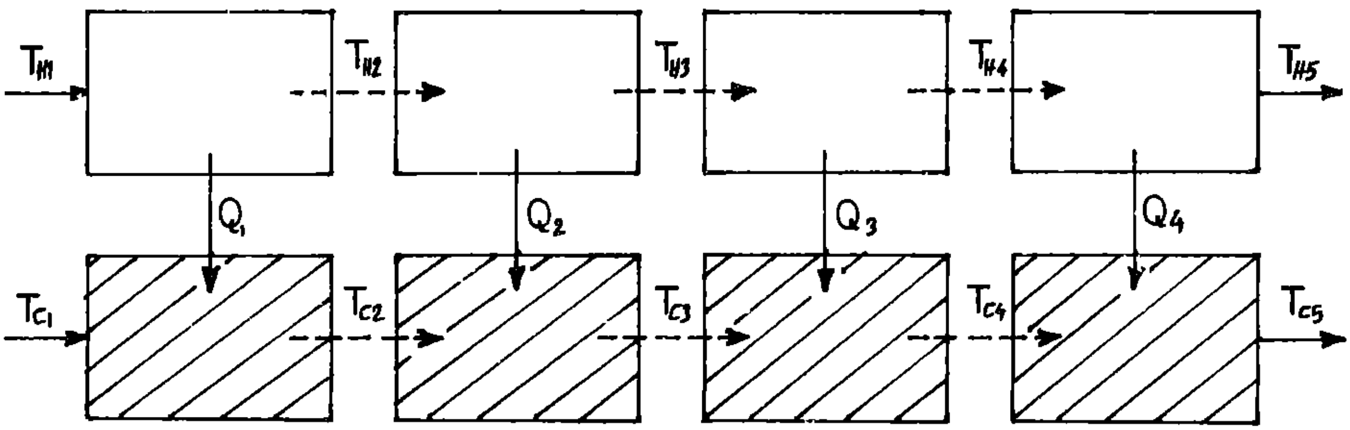
$$\varphi = 1/4 \begin{cases} \left.\frac{T_{oc}}{T_{iH}}\right|_{Counter} = .444 \Longrightarrow \frac{T_{oc})_E}{T_{oc})_L} = \frac{.5}{.444} = 1.125 \\ \left.\frac{T_{oc}}{T_{iH}}\right|_{Parallel-f} = .401 \times 1.125 = .451 > .432 \end{cases}$$

It decreases the error by 2.8% ,and new error is 4.4% .

e-CALCULATION OF DYNAMIC RESPONSES FOR FOUR LUMPED MODELS:

The general form of energy balance for each section of Lumped Models gives following equations for heat exchangers of Fig. (A-5)&(A-6).

I- Parallel-Flow:



FIGURE(A-5): Four Sections Heat-Exchanger

i-For case $a_1 = 4$, $a_2 = 1$, $r=1$

$$\begin{cases} (s+8)\,\bar{T}_{H2} = 4\,\bar{T}_{H1} + 4\,\bar{T}_{C2} \\ (s+5)\,\bar{T}_{C2} = 4\,\bar{T}_{C1} + \bar{T}_{H2} \\ (s+8)\,\bar{T}_{H3} = 4\,\bar{T}_{H2} + 4\,\bar{T}_{C3} \\ (s+5)\,\bar{T}_{C3} = 4\,\bar{T}_{C2} + \bar{T}_{H3} \\ (s+8)\,\bar{T}_{H4} = 4\,\bar{T}_{C4} + 4\,\bar{T}_{H3} \end{cases}$$

$$\begin{cases} (s+5)\bar{T}_{C4} = 4\bar{T}_{C3} + \bar{T}_{H4} \\ (s+8)\bar{T}_{H5} = 4\bar{T}_{H4} + 4\bar{T}_{C5} \\ (s+5)\bar{T}_{C5} = 4\bar{T}_{C4} + \bar{T}_{H5} \end{cases}$$

Solution of these equations gives following results:

$$G_1(s) = \frac{\bar{T}_{C5}}{\bar{T}_{H1}} = \frac{256(4s^3+78s^2+532s+1261)}{(s+9)^4(s+4)^4} \qquad \text{(A-15)}$$

$$G_2(s) = \frac{\bar{T}_{H5}}{\bar{T}_{H1}} = \frac{256\left[(s+5)^4+12(s+5)^2+8(s+8)(s+5)+4(s+8)^2+16\right]}{(s+9)^4(s+4)^4} \qquad \text{(A-16)}$$

For case: $a_1=1$, $a_2=1$, $r=1$

Equations for this case are:

$$\begin{cases} (s+5)\bar{T}_{H2} = 4\bar{T}_{H1} + \bar{T}_{C2} \\ (s+5)\bar{T}_{C2} = 4\bar{T}_{C1} + \bar{T}_{H2} \\ (s+5)\bar{T}_{H3} = 4\bar{T}_{H2} + \bar{T}_{C3} \\ (s+5)\bar{T}_{C3} = 4\bar{T}_{C2} + \bar{T}_{H3} \\ (s+5)\bar{T}_{H4} = 4\bar{T}_{H3} + \bar{T}_{C4} \\ (s+5)\bar{T}_{C4} = 4\bar{T}_{C3} + \bar{T}_{H4} \\ (s+5)\bar{T}_{H5} = 4\bar{T}_{H4} + \bar{T}_{C5} \\ (s+5)\bar{T}_{C5} = 4\bar{T}_{C4} + \bar{T}_{H5} \end{cases}$$

The solution of above equations gives following results:

$$G_1(s) = \frac{\bar{T}_{C5}}{\bar{T}_{H1}} = \frac{1024(s+5)(s^2+10s+26)}{(s+6)^4(s+4)^4} \quad \text{(A-17)}$$

$$G_2(s) = \frac{\bar{T}_{H5}}{\bar{T}_{H1}} = \frac{256(s+5)^4+1536(s+5)^2+256}{(s+6)^4(s+4)^4} \quad \text{(A-18)}$$

For case: $a_1=2$, $a_2=1$, $r=2$

$$\varphi_1=\varphi_2=\varphi_3=\varphi_4=1/4$$

The equations for each Lumped Model give following results:

$$\begin{cases} (s+6)\bar{T}_{H2} = 4\bar{T}_{H1} + 2\bar{T}_{C2} \\ (2s+5)\bar{T}_{C2} = 4\bar{T}_{C1} + \bar{T}_{H2} \\ (s+6)\bar{T}_{H3} = 4\bar{T}_{H2} + 2\bar{T}_{C3} \\ (2s+5)\bar{T}_{C3} = 4\bar{T}_{C2} + \bar{T}_{H3} \\ (s+6)\bar{T}_{H4} = 4\bar{T}_{H3} + 2\bar{T}_{C4} \\ (2s+5)\bar{T}_{C4} = 4\bar{T}_{C3} + \bar{T}_{H4} \\ (s+6)\bar{T}_{H5} = 4\bar{T}_{H4} + 2\bar{T}_{C5} \\ (2s+5)\bar{T}_{C5} = 4\bar{T}_{C4} + \bar{T}_{H5} \end{cases}$$

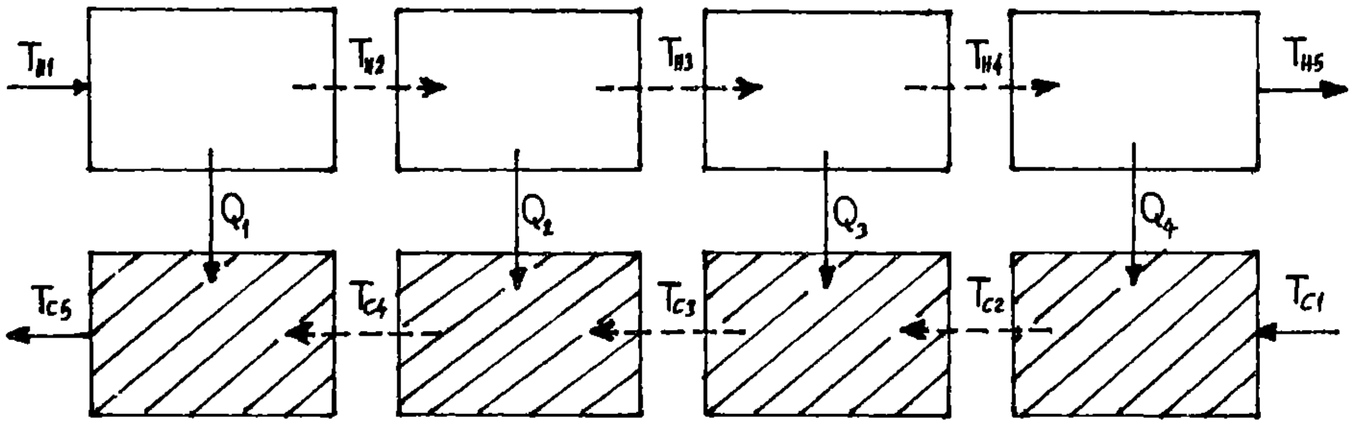
And by solving the above equations, following results are gained:

$$G_1(s) = \frac{\bar{T}_{C5}}{\bar{T}_{H1}} = \frac{16(2s+5)^3+16(s+6)^3+16(2s+5)(s+6)(3s+11)+64(3s+11)}{(s^2+8.5s+14)^4} \quad \text{(A-19)}$$

$$G_2(S) = \frac{\bar{T}_{H5}}{\bar{\bar{T}}_{H1}} = \frac{16(2S+5)^3 + 96(2S+5)^2 + 64(S+6)(2S+5) + 32(S+6)^2 + 64}{(S^2+8.5S+14)^4} \qquad \text{(A-20)}$$

II- Counter-Flow Heat-Exchanger:

Fig.(A-6) shows the counter flow case for four-Lumped Models.



FIGURE(A-6): Counter-Flow Four Lumped Models

For case: $a_1=4$, $a_2=1$, $r=1$

$$\varphi_1=\varphi_2=\varphi_3=\varphi_4=1/4$$

Using equation of each section for these eight sections , results are:

$$\begin{cases} (s+8)\,\bar{\bar{T}}_{H2} = 4\bar{\bar{T}}_{H1} + 4\bar{\bar{T}}_{C5} \\ (s+5)\,\bar{\bar{T}}_{C5} = 4\bar{\bar{T}}_{C4} + \bar{\bar{T}}_{H2} \\ (s+8)\,\bar{\bar{T}}_{H3} = 4\bar{\bar{T}}_{H2} + 4\bar{\bar{T}}_{C4} \\ (s+5)\,\bar{\bar{T}}_{C4} = 4\bar{\bar{T}}_{C3} + \bar{\bar{T}}_{H3} \\ (s+8)\,\bar{\bar{T}}_{H4} = 4\bar{\bar{T}}_{H3} + 4\bar{\bar{T}}_{C3} \\ (s+5)\,\bar{\bar{T}}_{C3} = 4\bar{\bar{T}}_{C2} + \bar{\bar{T}}_{H4} \end{cases}$$

$$\begin{cases} (s+8)\overline{T}_{H5} = 4\overline{T}_{H4} + 4\overline{T}_{C2} \\ (s+5)\overline{T}_{C2} = 4\overline{T}_{C1} + \overline{T}_{H5} \end{cases}$$

There are eight equations and eight unknows which solutions of them gives following results:

$$G(j\Omega) = \frac{4(s^6+39s^5+631s^4+5421s^3+26124s^2+6718s+74048)}{s^8+52s^7+1158s^6+14404s^5+109153s^4+513552s^3+1452448s^2+2219776s+1344768} \quad \text{(A-21)}$$

$$G(j\Omega) = \frac{256(s+5)(s^6+39s^5+627s^4+5317s^3+25080s^2+62400s+64000)}{(s+8)^3(s^8+52s^7+1158s^6+14404s^5+109153s^4+513552s^3+1452448s^2+2219776s+1344768)} \quad \text{(A-22)}$$

For case: $a_1=1$, $a_2=1$, $r=1$

$$\varphi_1=\varphi_2=\varphi_3=\varphi_4=1/4$$

There are following equations:

$$\begin{cases} (s+5)\overline{T}_{H2} = 4\overline{T}_{H1} + \overline{T}_{C5} \\ (s+5)\overline{T}_{C5} = 4\overline{T}_{C4} + \overline{T}_{H2} \\ (s+5)\overline{T}_{H3} = 4\overline{T}_{H2} + \overline{T}_{C4} \\ (s+5)\overline{T}_{C4} = 4\overline{T}_{C3} + \overline{T}_{H3} \\ (s+5)\overline{T}_{H4} = 4\overline{T}_{H3} + \overline{T}_{C3} \\ (s+5)\overline{T}_{C3} = 4\overline{T}_{C2} + \overline{T}_{H4} \\ (s+5)\overline{T}_{H5} = 4\overline{T}_{H4} + \overline{T}_{C2} \\ (s+5)\overline{T}_{C2} = 4\overline{T}_{C1} + \overline{T}_{H5} \end{cases}$$

And the results of solution are:

$$G(j\Omega)=\frac{4(s+4)(s+6)(s+5)^2(s^2+10s+39)+836s^2+8936s+34400}{s^8+40s^7+696s^6+6880s^5+42208s^4+164000s^3+391680s^2+520960s+288000} \qquad (A-23)$$

$$G(j\Omega)=\frac{64(16s^8+640s^7+11184s^6+57539s^5+694380s^4+2760716s^3+8338160s^2+9162200s+6\times10^6}{(s+4)(s+6)(s+5)^2(s^8+40s^7+696s^6+6880s^5+42208s^4+164000s^3+391680s^2+520960s+288000} \qquad (A-24)$$

For case: $a_1=2$, $a_2=1$, $r=2$

Equations for this case are:

$$\begin{cases}
(s+6)\,\overline{T}_{H2}=4\overline{T}_{H1}+2\overline{T}_{C5}\\
(2s+5)\,\overline{T}_{C5}=4\overline{T}_{C4}+\overline{T}_{H2}\\
(s+6)\,\overline{T}_{H3}=4\overline{T}_{H2}+2\overline{T}_{C4}\\
(2s+5)\,\overline{T}_{C4}=4\overline{T}_{C3}+\overline{T}_{H3}\\
(s+6)\,\overline{T}_{H4}=4\overline{T}_{H3}+2\overline{T}_{C3}\\
(2s+5)\,\overline{T}_{C3}=4\overline{T}_{C2}+\overline{T}_{H4}\\
(s+6)\,\overline{T}_{H5}=4\overline{T}_{H4}+2\overline{T}_{C2}\\
(2s+5)\,\overline{T}_{C2}=4\overline{T}_{C1}+\overline{T}_{H5}
\end{cases}$$

Solution of these equations gives following results:

$$G(j\Omega)=\frac{2(s^6+25.5s^5+266.75s^4+1464s^3+4472.5s^2+7310s+5368)}{s^8+34s^7+433.5s^6+38845s^5+18510s^4+53975s^3+93408s^2+86496s+31472} \quad \text{(A-25)}$$

$$G(j\Omega)=\frac{(2s+5)\left[(s^2+8.5s+14)^3(s^2+8.5s+238)+32(s^2+8.5s+14)(s^2+8.5s+22)(s^2+8.5s+36)-192(17s^2+144.5s+238)\right]}{2(s+6)^3\left[s^8+34s^7+433.5s^6+3884.5s^5+18510s^4+53975s^3+93408s^2+86496s+31472\right]} \quad \text{(A-26)}$$

TABLE(3-1): Results of Steady State for Parallel Flow

For case: $a_1 = 4$, $a_2 = 1$

"Two Lumped-Models"

Temp. Ratio	$\frac{T_{C2}}{T_{H1}}$	$\frac{T_{C3}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$
Exact	.184	.199	.266	.205
Lump	.143	.184	.428	.265

"Three Lumped-Models"

Temp. Ratio	$\frac{T_{C2}}{T_{H1}}$	$\frac{T_{C3}}{T_{H1}}$	$\frac{T_{C4}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$
Exact	.162	.193	.199	.351	.228	.205
Lump	.125	.172	.189	.500	.312	.242

"Four Lumped-Models"

Temp. Ratio	$\frac{T_{C2}}{T_{H1}}$	$\frac{T_{C3}}{T_{H1}}$	$\frac{T_{C4}}{T_{H1}}$	$\frac{T_{C5}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$	$\frac{T_{H5}}{T_{H1}}$
Exact	.143	.184	.195	.199	.43	.266	.219	.205
Lump	.111	.160	.182	.192	.555	.358	.270	.231

TABLE(3-2): Results of Steady State for Counter Flow

For case: $a_1 = 4$, $a_2 = 1$

"Two Lumped-Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$
Exact	.044	.240	.213	.038
Lump	.067	.200	.467	.200

"Three Lumped-Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$
Exact	.022	.080	.240	.360	.124	.038
Lump	.037	.100	.213	.550	.293	.147

"Four Lumped-Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{c5}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$	$\frac{T_{H5}}{T_{H1}}$
Exact	.014	.044	.107	.240	.466	.213	.094	.038
Lump	.023	.062	.122	.220	.612	.370	.214	.118

TABLE(3-3): Results of Steady State for Parallel Flow

For case: $a_1 = 1$, $a_2 = 1$

"Two Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$
Exact	.316	.432	.684	.568
Lump.	.250	.375	.750	.625

"Three Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$
Exact	.243	.368	.432	.757	.632	.568
Lump	.200	.320	.392	.800	.680	.608

"Four Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{c5}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$	$\frac{T_{H5}}{T_{H1}}$
Exact	.197	.316	.388	.432	.803	.684	.612	.568
Lump	.167	.278	.352	.400	.833	.722	.648	.600

TABLE(3-4): Results of Steady State for Counter-Flow

For case: $a_1=1$, $a_2=1$

"Two Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$
Exact	.25	.50	.75	.50
Lump	.20	.40	.80	.60

"Three Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$
Exact	.167	.333	.50	.833	.667	.50
Lump	.142	.289	.429	.857	.714	.571

"Four Lumped Models"

Temp. Ratio	$\frac{T_{c2}}{T_{H1}}$	$\frac{T_{c3}}{T_{H1}}$	$\frac{T_{c4}}{T_{H1}}$	$\frac{T_{c5}}{T_{H1}}$	$\frac{T_{H2}}{T_{H1}}$	$\frac{T_{H3}}{T_{H1}}$	$\frac{T_{H4}}{T_{H1}}$	$\frac{T_{H5}}{T_{H1}}$
Exact	.125	.25	.375	.50	.875	.75	.625	.50
Lump	.111	.222	.333	.444	.888	.777	.666	.555

TABLE(3-5): Results of Fitting Parallel-Flow

"Static Case"

For case: $a_1=4$, $a_2=1$

φ	$\frac{T_{oc}}{T_{HI}}\Big\|_{L-P}$	$\frac{T_{oc}}{T_{HI}}\Big\|_{L-C}$	$\frac{T_{oc}}{T_{HI}}\Big\|_{E-C}$	Unfitted Error in $\frac{T_{oc}}{T_{HI}}\Big\|_{L-C}$	Fitted Error in $\frac{T_{oc}}{T_{HI}}\Big\|_{L-C}$
1/2	.199	.216	.240	16.7%	10%
1/3	.199	.224	.240	11.25%	6.7%
1/4	.199	.228	.240	8.3%	5.0%

TABLE(3-6): Results of Fitting Parallel-Flow

"Static Case"

For case: $a_1=1$, $a_2=1$

φ	$\frac{T_{oc}}{T_{HI}}\Big\|_{L-C}$	$\frac{T_{oc}}{T_{HI}}\Big\|_{L-P}$	$\frac{T_{oc}}{T_{HI}}\Big\|_{E-P}$	Unfitted Error in $\frac{T_{oc}}{T_{HI}}\Big\|_{L-P}$	Fitted Error in $\frac{T_{oc}}{T_{HI}}\Big\|_{L-P}$
1/2	.432	.461	.50	20%	7.8%
1/3	.432	.473	.50	14.2%	5.4%
1/4	.432	.480	.50	11.1%	4.0%

TABLE(3-7): Results of Fitting Counter-Flow

"Static Case"

For case: $a_1=4$, $a_2=1$

φ	$\frac{T_{oc}}{T_{H1}}\Big\vert_{L-C}$	$\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$	$\frac{T_{oc}}{T_{H1}}\Big\vert_{E-P}$	Unfitted Error in $\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$	Fitted Error in $\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$
1/2	.240	.220	.199	7.5%	10.9%
1/3	.240	.213	.199	5.0%	7.0%
1/4	.240	.209	.199	3.5%	5.2%

TABLE(3-8): Results of Fitting Counter-Flow

"Static Case"

For case: $a_1=1$, $a_2=1$

φ	$\frac{T_{oc}}{T_{H1}}\Big\vert_{L-C}$	$\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$	$\frac{T_{oc}}{T_{H1}}\Big\vert_{E-P}$	Unfitted Error in $\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$	Fitted Error in $\frac{T_{oc}}{T_{H1}}\Big\vert_{L-P}$
1/2	.50	.469	.432	13.2%	8.6%
1/3	.50	.457	.432	9.2%	5.8%
1/4	.50	.450	.432	7.4%	4.2%

TABLE(4-1): Gain and Phase Angle

For Case: $a_1=4$, $a_2=1$, r=1

"Exact Results" "Friedly's Results"

COUNTER-FLOW

Ω	$\lvert G \rvert_N$	θ°	Ω	$\lvert G \rvert_N$	θ°
0.0	1.0	0.0	0.0	1.0	0.0
0.1	.990	-3.88	0.1	.990	-2.8
0.3	.970	-11.2	0.3	.984	-8.04
0.7	.880	-22.8	0.7	.946	-20.8
1.0	.820	-29.4	1.0	.900	-25.5
2.0	.630	-45.0	2.0	.707	-45.0
4.0	.416	-60.1	4.0	.458	-62.7
6.0	.300	-69.1	6.0	.325	-71.0
10.0	.200	-76.0	10.0	.202	-78.3
20.0	.100	-82.7	20.0	.102	-84.1
50.0	.041	-86.7	50.0	.041	-87.6
80.0	.026	-88.1	80.0	.026	-88.5

TABLE(4-2): Gain and Phase Angle

For Case: $a_1=4$, $a_2=1$, $r=1$

"Two Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N,Fitted}$
0.0	0.2	.832	0.0	.2405	1.000
0.1	0.199	.831	-5.1	.240	.997
0.3	0.195	.812	-15.2	.234	.974
0.7	0.175	.73	-33.5	.210	.880
1.0	0.157	.655	-45.0	.189	.780
2.0	0.106	.440	-71.5	.127	.536
4.0	0.055	.230	-99.5	.066	.268
6.0	0.034	.140	-115.5	.040	.169
10.0	0.016	.065	-135	.020	.078
20.0	.005	.020	-155	.006	.024
50.0	.001	.003	-169.7	.001	.004
80.0	.0003	.001	-173.6	.0003	.001

TABLE(4-3): Gain and Phase Angle

For Case: $a_1=4$, $a_2=1$, r=1

"Three Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.2133	.890	0.0	.2405	1.00
0.1	.213	.886	-4.6	.240	.997
0.3	.209	.870	-13.5	.236	.980
0.7	.192	.800	-30	.216	.900
1.0	.174	.725	-40.6	.196	.816
2.0	.122	.506	-65.5	.137	.570
4.0	.067	.280	-91.7	.076	.315
6.0	.043	.180	-107.5	.048	.202
10.0	.021	.090	-127.6	.024	.101
20.0	.007	.030	-147.5	.008	.034
50.0	.001	.005	-167.5	.001	.006
80.0	.0003	.001	-171.6	.0003	.001

TABLE(4-4): Gain and Phase Angle

For Case a_1=4 , a_2=1 , r=1

"Four Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N,Fitted}$
0.0	.2202	.918	0.0	.2405	1.0
0.1	.2200	.916	-4.2	.240	.998
0.3	.2160	.900	-12.6	.236	.981
0.7	.1990	.830	-28.0	.218	.905
1.0	.1830	.761	-38.0	.200	.830
2.0	.1310	.543	-61.8	.142	.592
4.0	.0750	.312	-86.0	.082	.340
6.0	.050	.210	-101.3	.055	.230
10.0	.026	.110	-121.4	.029	.120
20.0	.010	.040	-145.6	.010	.043
50.0	.001	.006	-165.3	.002	.007
80.0	-	-	-170	-	-

TABLE(4-5): Gain and Phase Angle

For Case: $a_1=1$, $a_2=1$, $r=1$

COUNTER-FLOW

"Exact Results"

Ω	$\lvert G\rvert_N$	θ°
0.0	1.00	0.0
0.1	.997	-4.9
0.3	.980	-14.2
0.7	.850	-37.8
1.0	.814	-46
2.0	.460	-73.3
4.0	.254	-69
6.0	.142	-84.3
10.0	.096	-76.5
20.0	.055	-81.6
50.0	.018	-93.4
80.0	.0141	-88.6

"Friedly's Results"

Ω	$\lvert G\rvert_N$	θ°
0.0	1.00	0.0
0.1	.999	-6.6
0.3	.980	-6.6
0.7	.880	-33.
1.0	.822	-44.
2.0	.523	-66.
4.0	.250	-75.5
6.0	.143	-80.
10.0	.087	-82.
20.0	.045	-82.
50.0	.020	-81.6
80.0	.014	-87.0

TABLE(4-6): Gain and Phase Angle

For case: $a_1=1$, $a_2=1$, $r=1$

"Two Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{fitted}$	$\|G\|_{N, fitted}$
0.0	.400	.800	0.0	.50	1.00
0.1	.398	.797	-8.0	.497	.995
0.3	.386	.772	-18.8	.483	.965
0.7	.338	.676	-40.5	.422	.845
1.0	.297	.593	-55.2	.371	.741
2.0	.177	.353	-87.7	.221	.442
4.0	.076	.151	-117.2	.094	.190
6.0	.041	.083	-131.9	.052	.104
10.0	.018	.035	-148.0	.022	.044
20.0	.005	.010	-162.7	.006	.012
50.0	.0008	.0016	-173.0	.001	.002
80.0	.0003	.0006	-175.7	.0004	.0008

TABLE(4-7): Gain and Phase Angle

For case: $a_1=1$, $a_2=1$, $r=1$

"Three Lumped Models"

COUNTER

Ω	$\lvert G \rvert$	$\lvert G \rvert_N$	θ°	$\lvert G \rvert_{Fitted}$	$\lvert G \rvert_{N, Fitted}$
0.0	.4286	.8672	0.0	.500	1.000
0.1	.4272	.8544	-5.8	.498	.996
0.3	.4165	.8330	-17.3	.486	.971
0.7	.3720	.7430	-38.4	.433	.867
1.0	.3280	.6550	-52.0	.382	.764
2.0	.1980	.3960	-83.16	.231	.462
4.0	.0886	.1770	-108.3	.103	.210
6.0	.0527	.1054	-122.0	.061	.123
10.0	.0243	.0486	-140.6	.0283	.057
20.0	.0070	.0141	-158.0	.0080	.016
50.0	.0012	.0024	-171.0	.001	.003
80.0	.0004	.0009	-174.6	.0005	.001

TABLE(4-8): Gain and Phase Angle

For case: $a_1=1$, $a_2=1$, $r=1$

"Four Lumped Models"

COUNTER

Ω	$\lvert G \rvert$	$\lvert G \rvert_N$	θ°	$\lvert G \rvert_{Fitted}$	$\lvert G \rvert_{N,Fitted}$
0.0	.444	.888	0.0	.500	1.00
0.1	.443	.886	-5.5	.499	.997
0.3	.434	.868	-16.3	.488	.976
0.7	.392	.785	-36.7	.441	.883
1.0	.350	.700	-58.0	.393	.787
2.0	.212	.414	-81.25	.239	.477
4.0	.094	.189	-102	.106	.212
6.0	.060	.120	-114.5	.067	.134
10.0	.030	.060	-132.75	.033	.066
20.0	.009	.018	-153	.010	.020
50.0	.0016	.0032	-166	.0018	.0036
80.0	.0006	.0012	-173	.0007	.0014

TABLE(4-9): Gain and Phase Angle

For Case: $a_1=2$, $a_2=1$, $r=2$

COUNTER-FLOW

"Exact Results"

Ω	$\lvert G \rvert_N$	θ°
0.0	1.00	0.0
0.1	.996	-6.7
0.3	.954	-49.3
0.7	.790	-41.0
1.0	.642	-53.4
2.0	.370	-72.2
4.0	.196	-77.1
6.0	.135	-82.6
10.0	.086	-87.1
20.0	.045	-87.5
50,0	.018	-87.1
80.0	.011	-89.87

"Friedly's Results"

Ω	$\lvert G \rvert_N$	θ°
0.0	1.00	0.0
0.1	.999	-7.0
0.3	.939	-20.0
0.7	.760	-40.5
1.0	.633	-50.5
2.0	.378	-67.5
4.0	.203	-74.2
6.0	.134	-76.0
10.0	.080	-84.3
20.0	.042	-85.0
50.0	.018	-87.6
80.0	.011	89.80

TABLE(4-10): Gain and Phase Angle

For Case: $a_1=2$, $a_2=1$, r=2

"Two Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.304	.786	0.0	.387	1.00
0.1	.301	.778	-9.16	.384	.990
0.3	.281	.726	-25.5	.358	.924
0.7	.217	.560	-53.7	.276	.712
1.0	.173	.448	-68.1	.220	.570
2.0	.092	.237	-95.0	.117	.300
4.0	.040	.101	-120.5	.050	.129
6.0	.021	.055	-134.4	027	.070
10.0	.009	.023	-150.5	.011	.030
20.0	.0024	.0063	-164.5	.0027	.007
50.0	.0004	.001	-173.75	.0005	.0013
80.0	.0002	.0004	-176.0	.0002	.0005

TABLE(4-11): Gain and Phase Angle

For case: $a_1=2$, $a_2=1$, $r=2$

"Three Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.328	.847	0.0	.387	1.00
0.1	.324	.837	-8.3	.383	.988
0.3	.307	.792	-24.25	.362	.934
0.7	.243	.627	-50.0	.286	.740
1.0	.197	.510	-63.75	.232	.600
2.0	.107	.276	-89.0	.126	.325
4.0	.050	.127	-112.3	.060	.150
6.0	.028	.074	-126.7	.034	.087
10.0	.013	.033	-143.7	.015	.038
20.0	.004	.009	-160.5	.004	.011
50.0	.0006	.0015	-172	.0007	.0018
80.0	.0002	.0006	-175	.0003	.0007

TABLE(4-12): Gain and Phase Angle

For case: $a_1=2$, $a_2=1$, $r=2$

"Four Lumped Models"

COUNTER

Ω	$\|G\|$	$\|G\|_N$	$\theta°$	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.341	.881	0.0	.387	1.000
0.1	.339	.874	-7.7	.384	.993
0.3	.321	.828	-22.0	.364	.940
0.7	.257	.665	-48.0	.292	.755
1.0	.210	.542	-61.5	.238	.616
2.0	.116	.300	-76.0	.132	.340
4.0	.050	.124	-81.7	.055	.160
6.0	.033	.085	-104.8	.037	.097
10.0	.021	.055	-135.0	.024	.062
20.0	.004	.012	-160.0	.005	.014
50.0	.0008	.002	-171.1	.0009	.0022
80.0	.0003	.0008	-174.2	.0003	.0009

TABLE(4-13): Gain and Phase Angle

For Case $a_1=4$, $a_2=1$, $r=1$

"Exact Results for Parallel Flow"

Ω	$\lvert G \rvert$	$\lvert G \rvert_N$	θ°
0.	.1986	1.0	-0.0
0.1	.1986	1.0	-5.7
0.3	.1986	1.0	-17.2
0.7	.1986	1.0	-40.3
1.0	.1986	1.0	-57.6
2.0	.1986	1.0	-114.6
4.0	.1986	1.0	-229.2
6.0	.1986	1.0	-343.7
10.0	.1986	1.0	-573

TABLE(4-14): Gain and phase Angle

For case: $a_1=4$, $a_2=1$, $r=1$

"Two Lumped Models"

PARALLEL

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N,Fitted}$
0	.1837	.925	0	.1986	1.0
.1	.183	.922	-6.1	.198	.997
.3	.180	.905	-18	.194	.978
.7	.164	.825	-41	.177	.893
1.0	.147	.743	-56.8	.159	.803
2.0	.093	.468	-98	.100	.506
4.0	.037	.186	-144.7	.040	.202
6.0	.018	.089	-171.2	.019	.096
10.0	.006	.03	-201.6	.006	.030

TABLE(4-15): Gain and phase Angle

For Case: $a_1=4$, $a_2=1$, $r=1$

"Three Lumped Models"

PARALLEL

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N,Fitted}$
0.0	.1894	.954	0	.1986	1.0
0.1	.189	.952	-5.9	.198	.998
0.3	.187	.940	-17.7	.196	.985
0.7	.175	.882	-40.8	.184	.925
1.0	.162	.817	-57.3	.170	.857
2.0	.110	.557	-105.3	.116	.583
4.0	.042	.214	-169.4	.046	.232
6.0	.018	.090	-207.7	.019	.094
10.0	.004	.022	-252	.005	.023

TABLE(4-16): Gain and phase Angle

For Case: $a_1=4$, $a_2=1$, $r=1$

"Four Lumped Models"

PARALLEL

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.1922	.968	0	.1986	1.0
0.1	.1920	.967	-5.9	.1985	.999
0.3	.1900	.957	-17.7	.196	.988
0.7	.181	.910	-40.9	.187	.941
1.0	.170	.857	-57.8	.176	.885
2.0	.123	.619	-109.5	.127	.640
4.0	.05	.245	-186.2	.051	.253
6.0	.02	.096	-235.7	.020	.099
10.0	.004	.020	-364.2	.004	.020

TABLE(4-17): Gain and Phase Angle (Parallel Flow)

For Case: $a_1=2$, $a_2=1$, $r=2$

"The Exact Results"

Ω	$\|G\|$	$\|G\|_N$	θ°
0	.3167	1.0	0
.1	.3166	.999	-8
.3	.3155	.996	-24
.7	.3110	.982	-63
1.0	.3055	.965	-89
2.0	.2730	.864	-177
4.0	.1720	.542	-324
6.0	.0623	.197	-572
10.0	.0525	.166	-688

TABLE(4-18): Gain and Phase Angle

For Case: $a_1=2$, $a_2=1$, $r=2$

"Two Lumped Models"

PARALLEL

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N,Fitted}$
0.0	.280	.884	0.0	.3167	1.00
0.1	.278	.878	-10.1	.314	.993
0.3	.263	.830	-30.0	.297	.940
0.7	.208	.656	-63.3	.235	.742
1.0	.165	.520	-84.8	.186	.588
2.0	.076	.240	-129	.085	.270
4.0	.023	.072	-173	.026	.082
6.0	.009	.030	-198	.011	.034
10.0	.0025	.008	-223	.003	.009

TABLE(4-19): Gain and Phase Angle

For Case: $a_1=2$, $a_2=1$, $r=2$

"Three Lumped Models"

PARALLEL

Ω	$\lvert G\rvert$	$\lvert G\rvert_N$	$\theta°$	$\lvert G\rvert_{Fitted}$	$\lvert G\rvert_{N, Fitted}$
0.0	.292	.920	0.0	.3167	1.0
0.1	.290	.916	-9.7	.315	.996
0.3	.280	.882	-29	.303	.960
0.7	.234	.740	-64.6	.255	.804
1.0	.193	.610	-88	.210	.663
2.0	.090	.283	-144.7	.100	.310
4.0	.023	.074	-205	.025	.080
6.0	.008	.026	-240	.009	.028
10.0	.0016	.0053	-279	.0018	.0057

TABLE(4-20): Gain and Phase Angle

For Case: $a_1=2$, $a_2=1$, $r=2$

"Four Lumped Models"

PARALLEL

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_{Fitted}$	$\|G\|_{N, Fitted}$
0.0	.298	.94	0.0	.3167	1.0
0.1	.297	.937	-9.5	.315	.996
0.3	.288	.910	-28.4	.306	.967
0.7	.250	.790	-64.5	.266	.840
1.0	.213	.673	-89.4	.227	.716
2.0	.164	.330	-154.6	.110	.350
4.0	.025	.080	-229.3	.026	.084
6.0	.008	.025	-274.0	.008	.027
10.0	.0013	.004	-328.0	.0014	.0043

TABLE(5-1): Gain and phase Angle (Using Different φ's)

For case: $a_1=4$, $a_2=1$, $r=1$ $\varphi_1=1/3$, $\varphi_2=2/3$

"Two Lumped Models"

COUNTER-FLOW

Ω	$\lvert G\rvert$	$\lvert G\rvert_N$	θ°	$\lvert G\rvert_{Fitted}$	$\lvert G\rvert_{N,Fitted}$
0.0	.198	.823	0.0	.2405	1.00
0.1	.197	.820	-4.7	.2400	.997
0.3	.193	.800	-13.9	.2340	.973
0.7	.174	.723	-30.0	.2110	.878
1.0	.157	.652	-40.0	.1900	.790
2.0	.110	.460	-62.3	.1350	.560
4.0	.065	.270	-87.0	.0800	.328
6.0	.043	.180	-104.6	.0520	.217
10.0	.022	.091	-125.0	.0260	.110
20.0	.007	.030	-149.5	.0090	.036
50.0	.001	.005	-167.5	.0010	.006
80.0	.0004	.002	-171.5	.0004	.002

TABLE(5-2): Gain and Phase Angle for Improved Case:

$a_1=4$, $a_2=1$, $r=1$

"Two Lumped-Models" "Exact"

COUNTER-FLOW

Ω	$\lvert G \rvert$	$\lvert G \rvert_N$	θ°	$\lvert G \rvert_N$	θ°
0.0	.2405	1.000	0.0	1.00	0.0
0.1	.2400	.997	-3.75	.99	-3.8
0.3	.2350	.977	-11.0	.97	-11.2
0.7	.2140	.890	-23.8	.88	-23.0
1.0	.195	.812	-31.0	.82	-29.5
2.0	.144	.600	-46.0	.63	-45.0
4.0	.094	.390	-58.0	.41	-60.0
6.0	.071	.295	-64.0	.30	-69.0
10.0	.048	.200	-71.0	.20	-76.0
20.0	.025	.106	-80.0	.10	-82.7
50.0	.010	.043	-86.0	.041	-86.7
80.0	.006	.027	-87.3	.026	-88.0

TABLE(5-3): Gain and Phase Angle for Improved Case:

$a_1=2$, $a_2=1$, $r=2$

"Two Lumped Models" "Exact"

COUNTER-FLOW

Ω	$\|G\|$	$\|G\|_N$	θ°	$\|G\|_N$	θ°
0.0	.400	1.037	0.0	1.00	0.0
0.1	.397	1.026	-8.0	.996	-7.0
0.3	.368	.950	-23.0	.954	-20.0
0.7	.280	.720	-46.0	.790	-41.0
1.0	.222	.573	-55.0	.642	-53.5
2.0	.125	.323	-67.0	.370	-72.0
4.0	.074	.190	-73.2	.196	-77.0
6.0	.051	.132	-78.5	.135	-82.0
10.0	.031	.081	-84.0	.086	-87.0
20.0	.016	.041	-87.4	.045	-87.5
50.0	.007	.018	-87.7	.018	-87.7
80.0	.004	.011	-89.7	.011	-89.7

TABLE(5-4): Step Response for Counter-Flow Heat-Exchanger

For Case: $a_1=4$, $a_2=1$, $r=1$

t	$\frac{T_{oc}}{T_{iH}}\Big\vert_{Lumped}$	$\frac{T_{oc}}{T_{iH}}\Big\vert_{Exact}$	$\frac{T_{oc}}{T_{iH}}\Big\vert_{Friedly}$
0.0	.000	.000	.000
0.25	.373	.375	.400
0.5	.580	.580	.644
1.0	.780	.785	.873
1.5	.884	.896	.955
2.0	.925	.94	.984
2.5	.968	.97	.989
3.0	.983	.983	.992
3.5	.991	.991	.993
4.0	.995	.995	.995

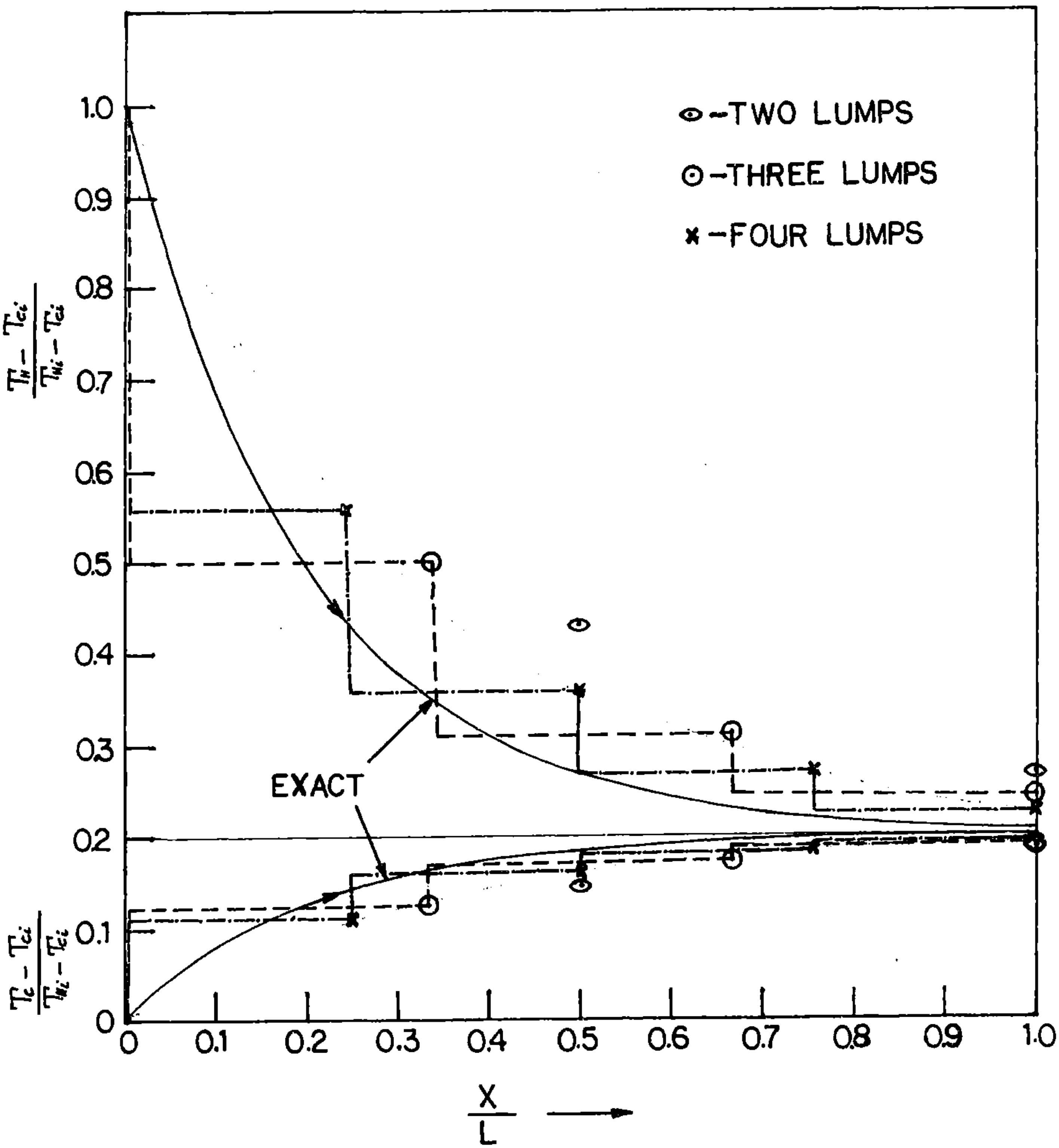
TABLE(5-5): Impulse Response for Counter Flow

For case: $a_1=4$, $a_2=1$, $r=1$

t	$\left.\frac{T_{oc}}{T_{iH}}\right\|_{Lumped}$	$\left.\frac{T_{oc}}{T_{iH}}\right\|_{Exact}$	$\left.\frac{T_{oc}}{T_{iH}}\right\|_{Exact}$
0.0	2.16	2.065	2.065
0.25	1.01	1.05	1.23
0.5	0.605	0.625	0.735
0.75	0.410	0.410	0.440
1.00	0.285	0.275	0.262
1.25	0.200	0.190	0.160
1.50	0.150	0.150	0.093
1.75	0.112	0.120	0.055
2.00	0.062	0.050	0.03
2.25	0.051	0.04	0.02
2.50	0.032	0.03	0.01
3.00	0.020	0.01	0.005

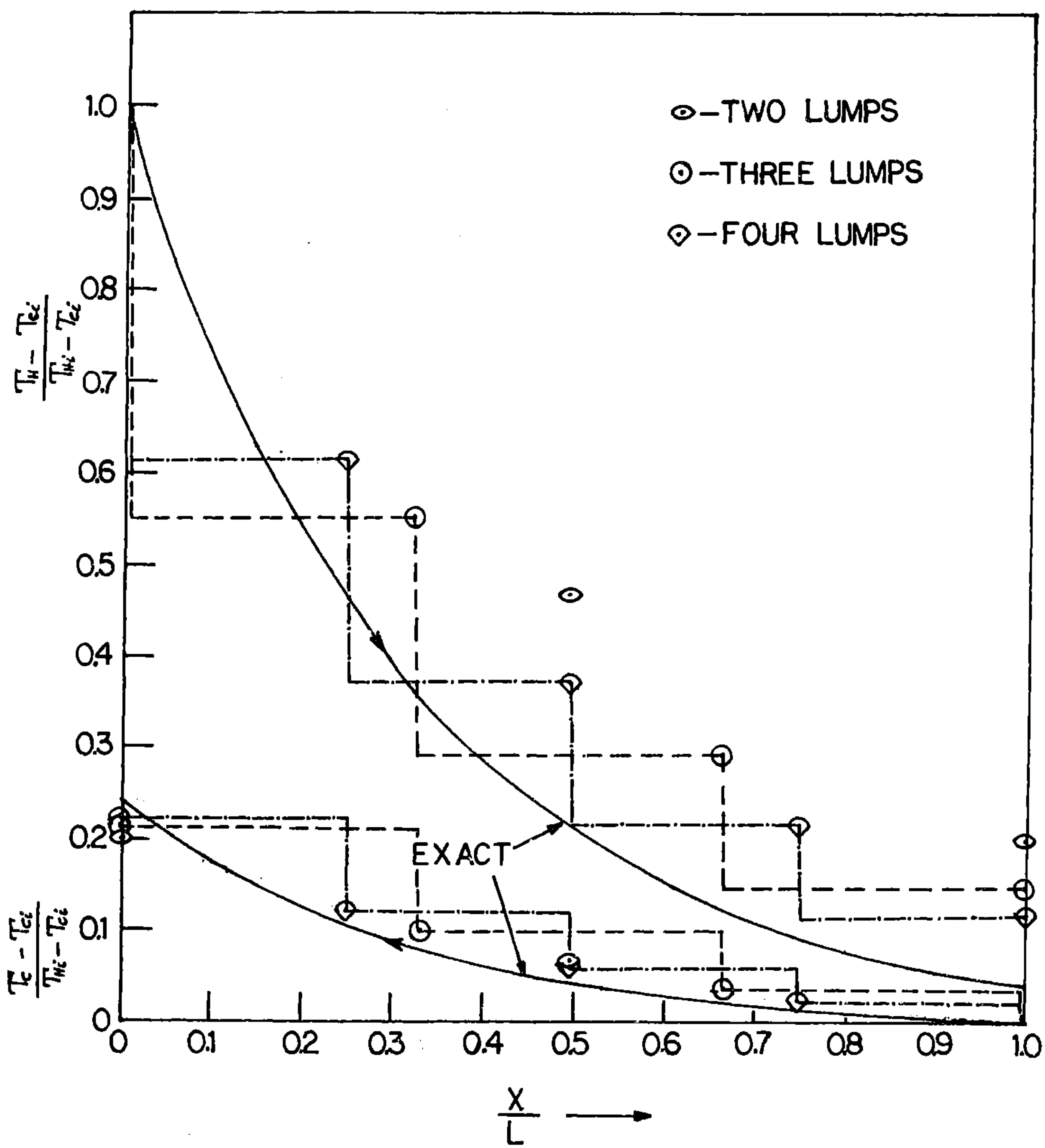
FIGURE(3-12a): Static Case of Parallel-Flow

For case: $a_1 = 4$, $a_2 = 1$



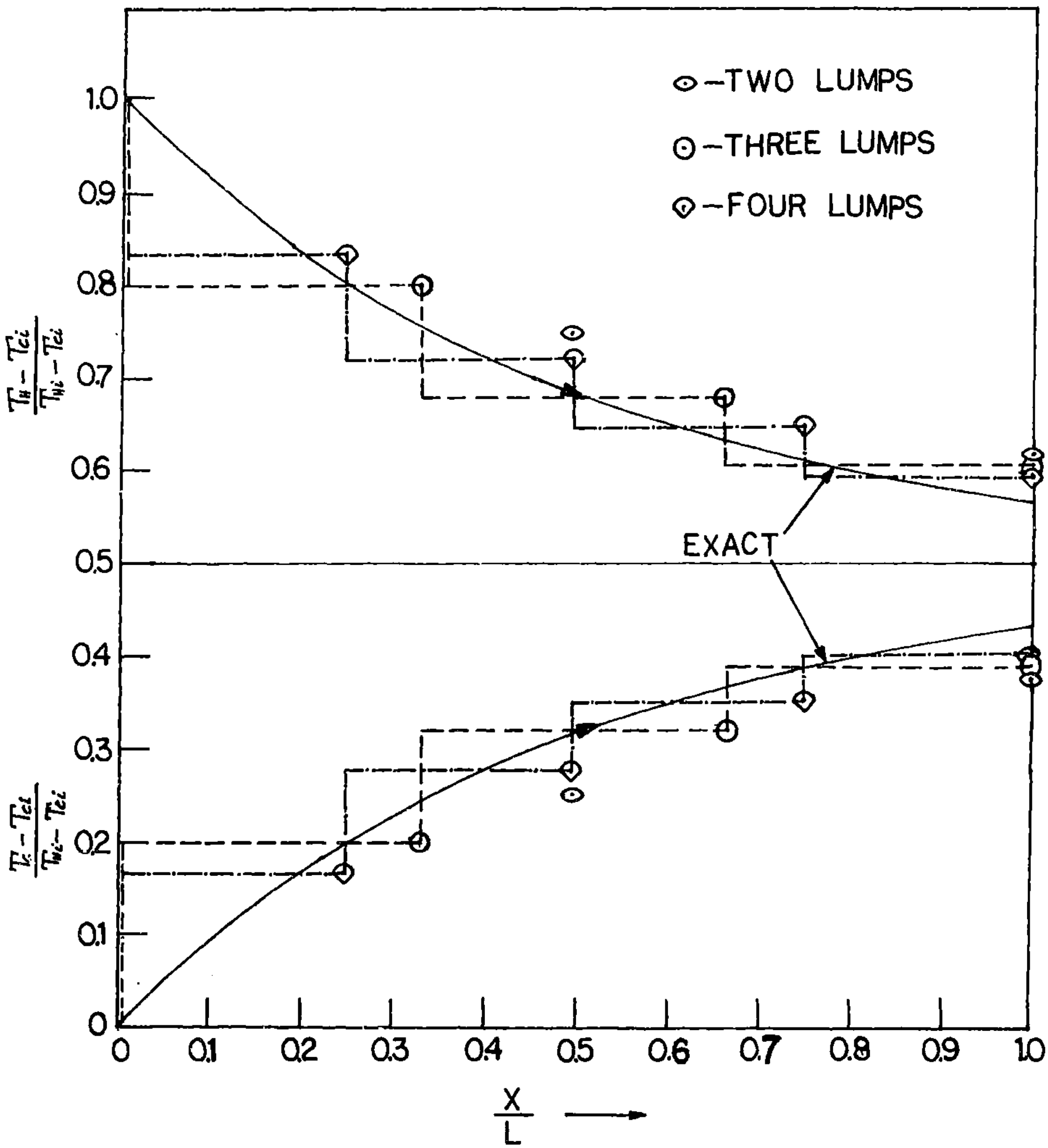
FIGURE(3-12b): Static Case of Counter-Flow

For case: $a_1 = 4$, $a_2 = 1$



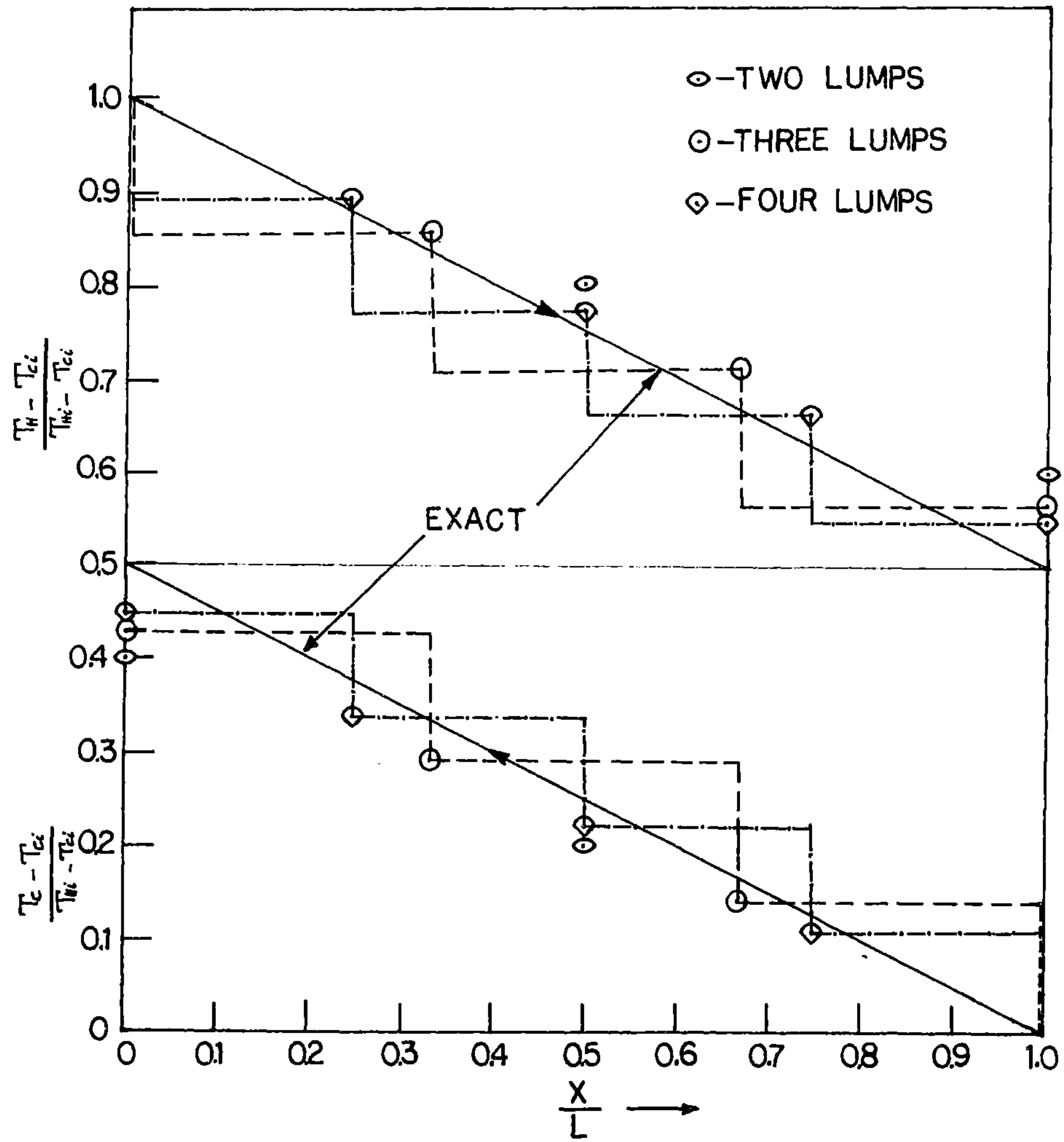
FIGURE(3-13a): Static Case of Parallel-Flow

For case: $a_1=1$, $a_2=1$

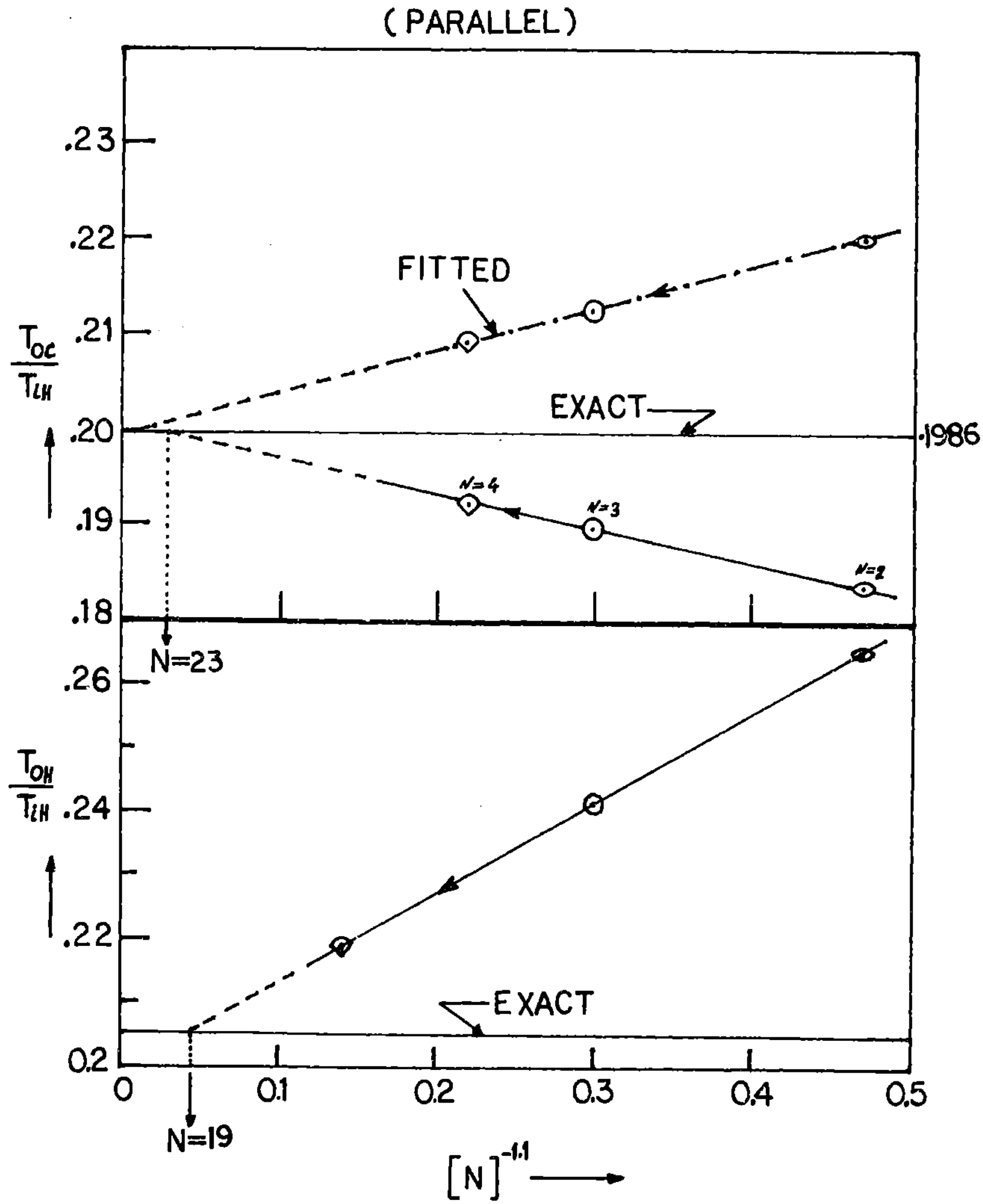


FIGURE(3-13b): Static Case of Counter-Flow

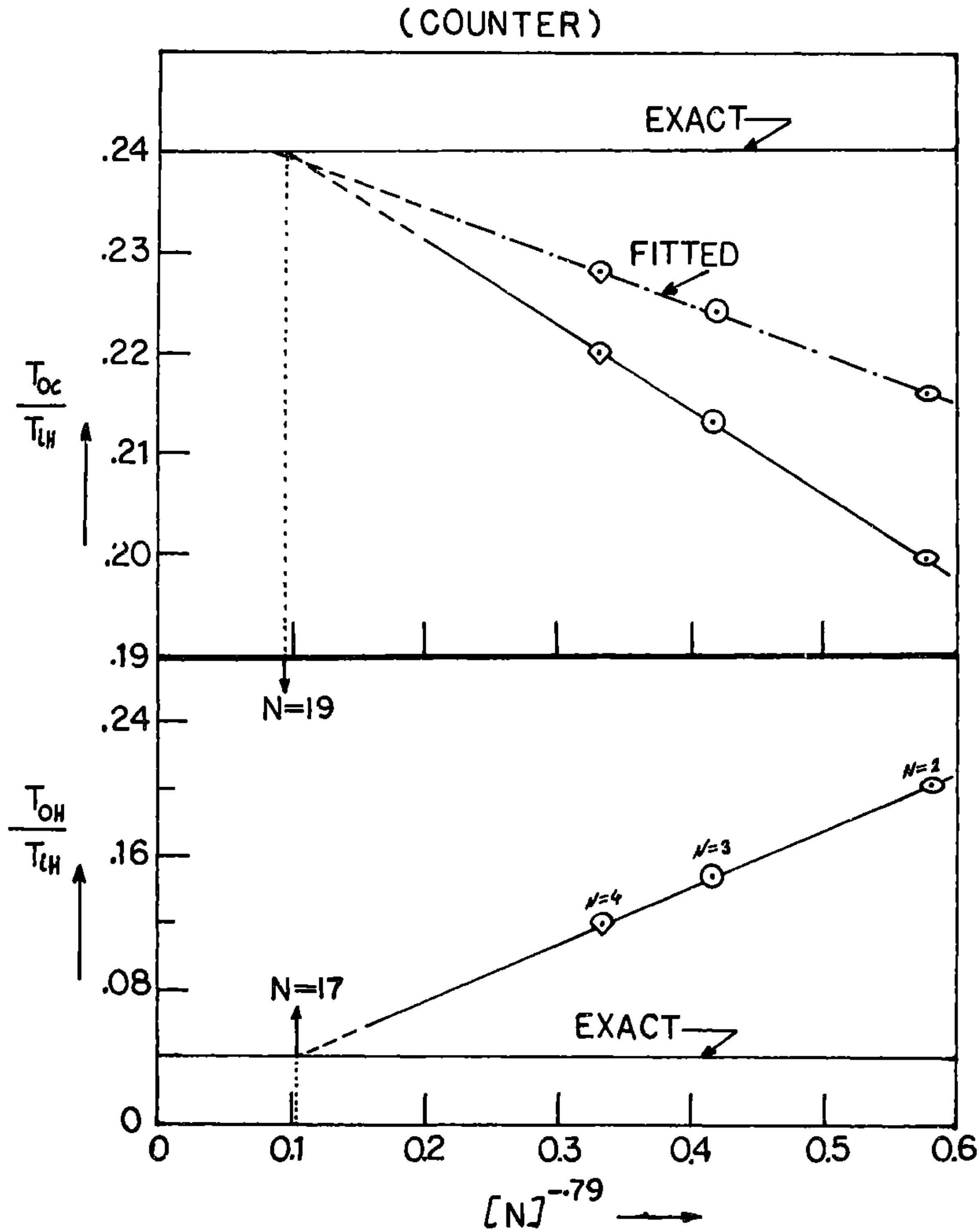
For case: $a_1=1$, $a_2=1$



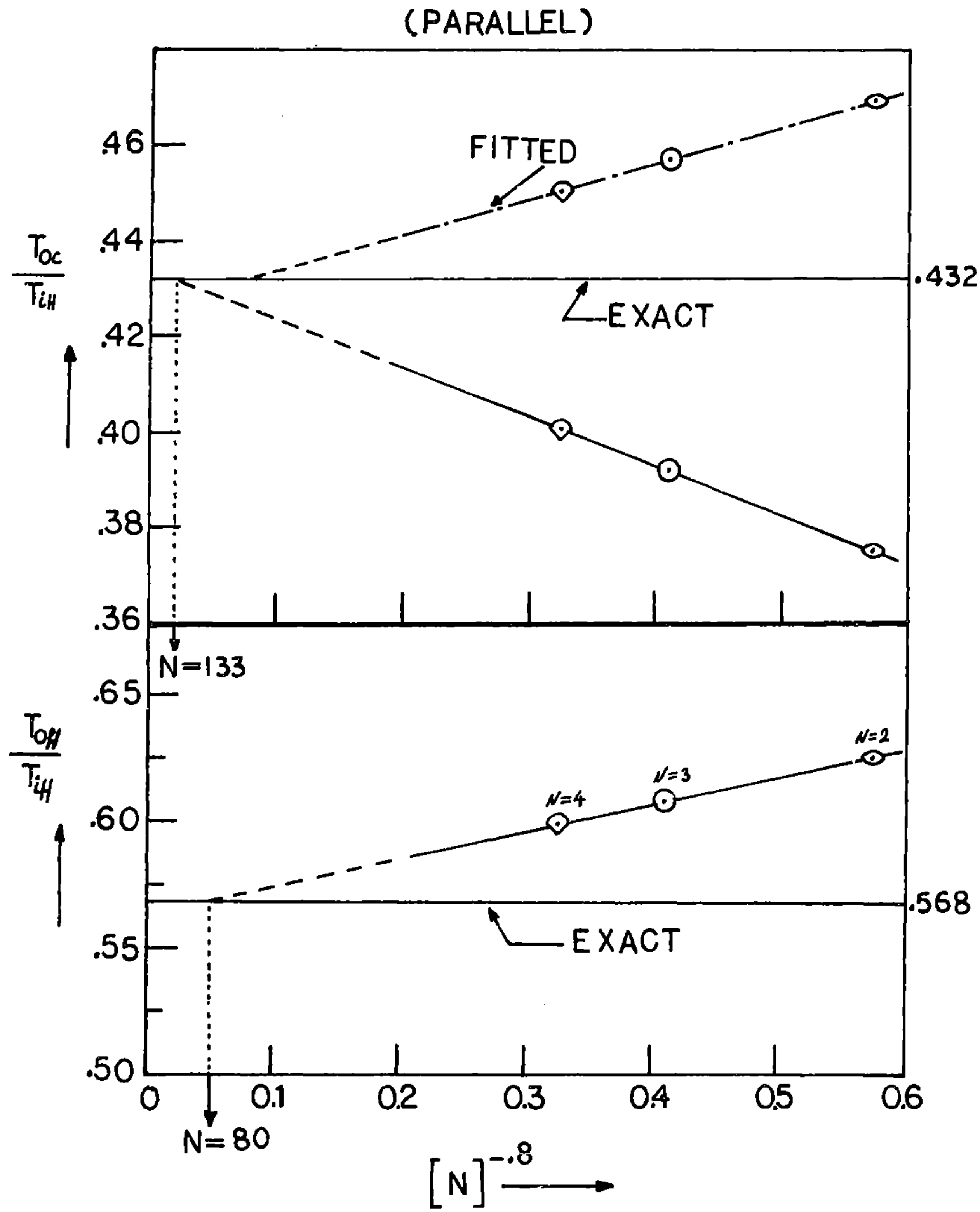
FIGURE(3-14): Effect of Fitting Counter-Flow by Exact Value. $a_1=4$, $a_2=1$



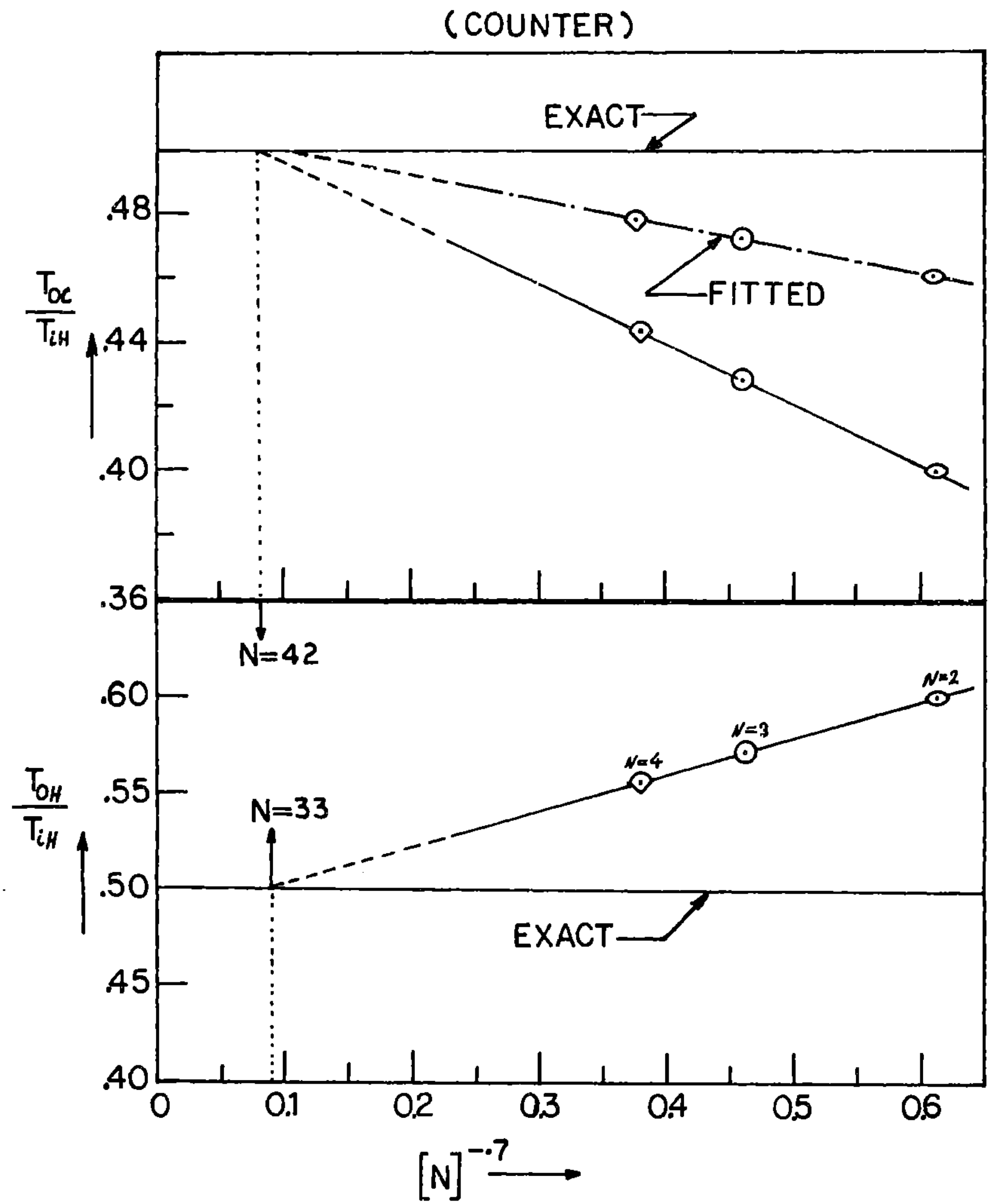
FIGURE(3-15): Effect of Fitting Parallel-Flow by Exact Values. $a_1=4$, $a_2=1$



FIGURE(3-16): Effect of Fitting Counter-Flow by Exact Value. $a_1=1$, $a_2=1$

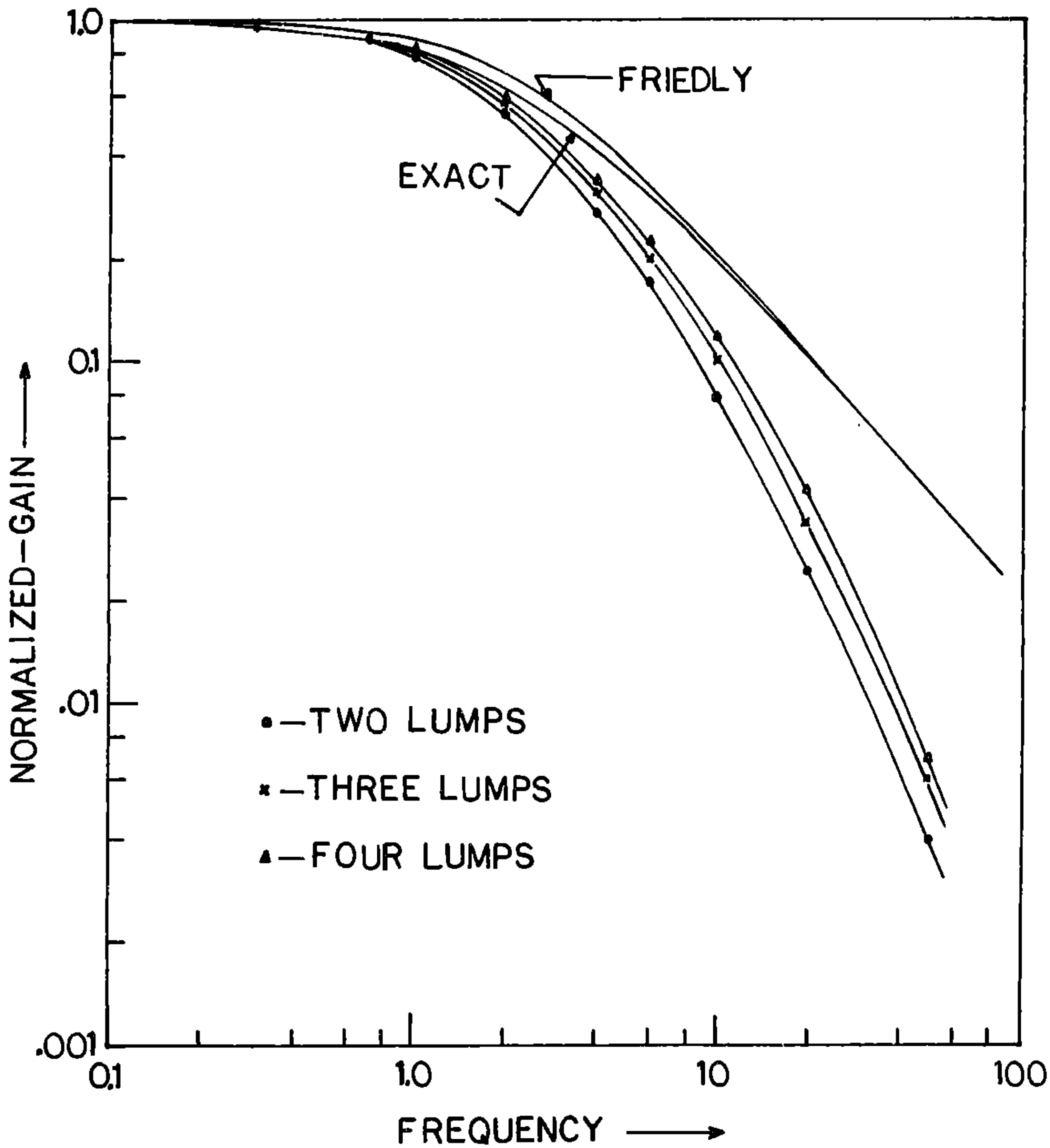


FIGURE(3-17): Effect of Fitting Parallel-Flow by Exact Value. $a_1=1$, $a_2=1$



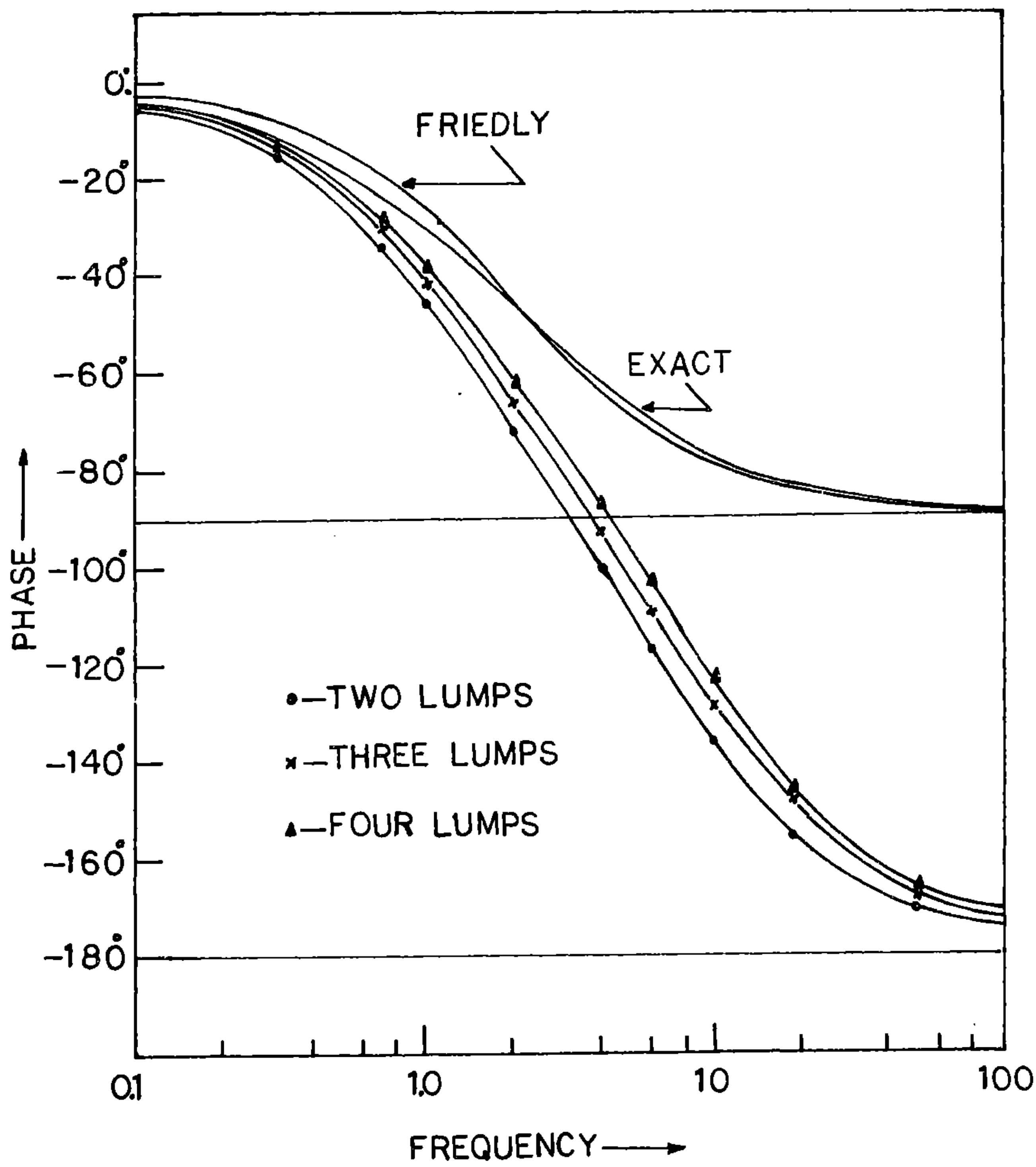
FIGURE(4-13): Gain of Frequency Response of Lumped-Models

"Counter-Flow" $a_1=4$, $a_2=1$, $r=1$



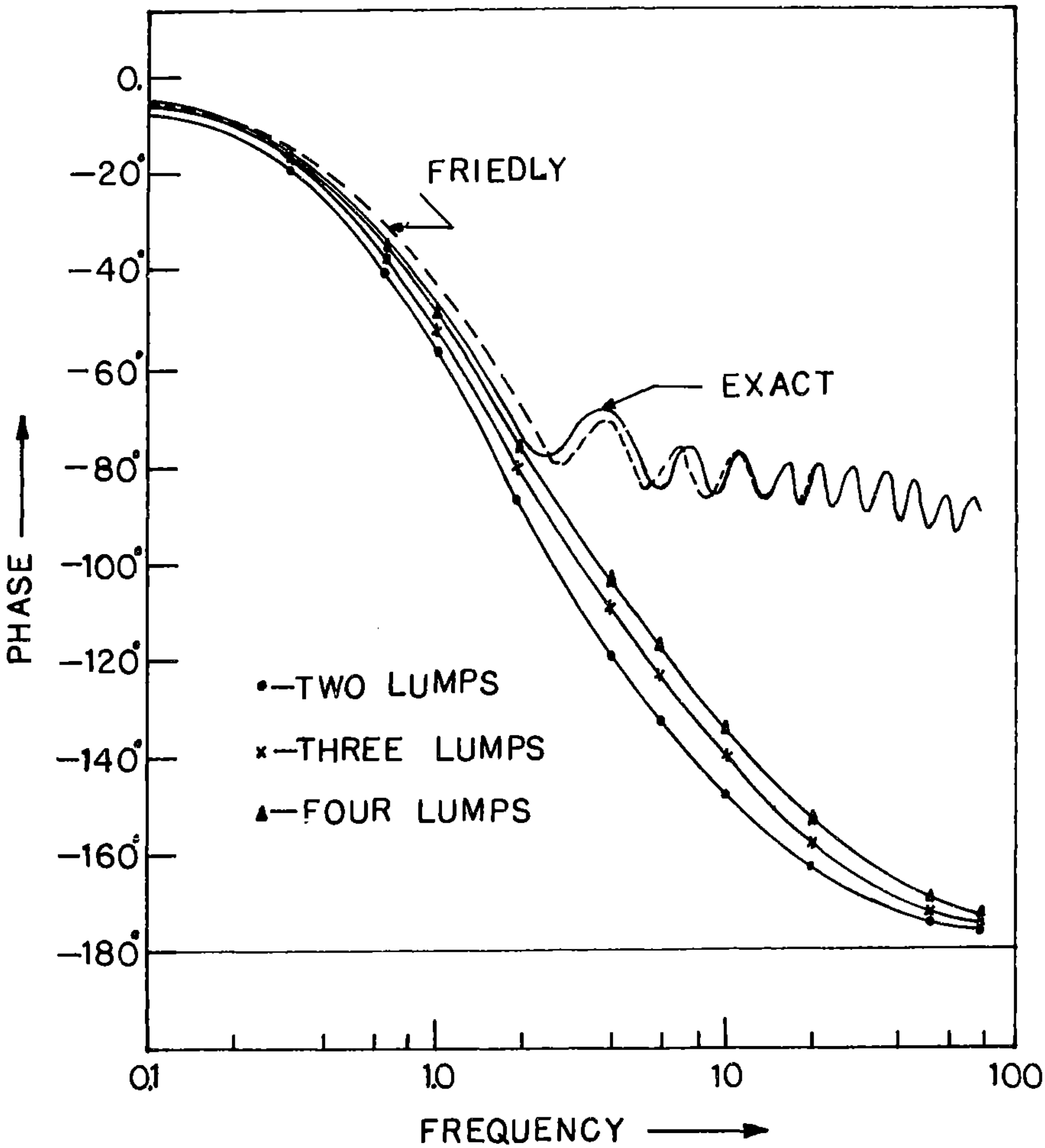
FIGURE(4-14): Phase Angle of Lumped Models "Counter-Flow"

For case: $a_1=4$, $a_2=1$, $r=1$

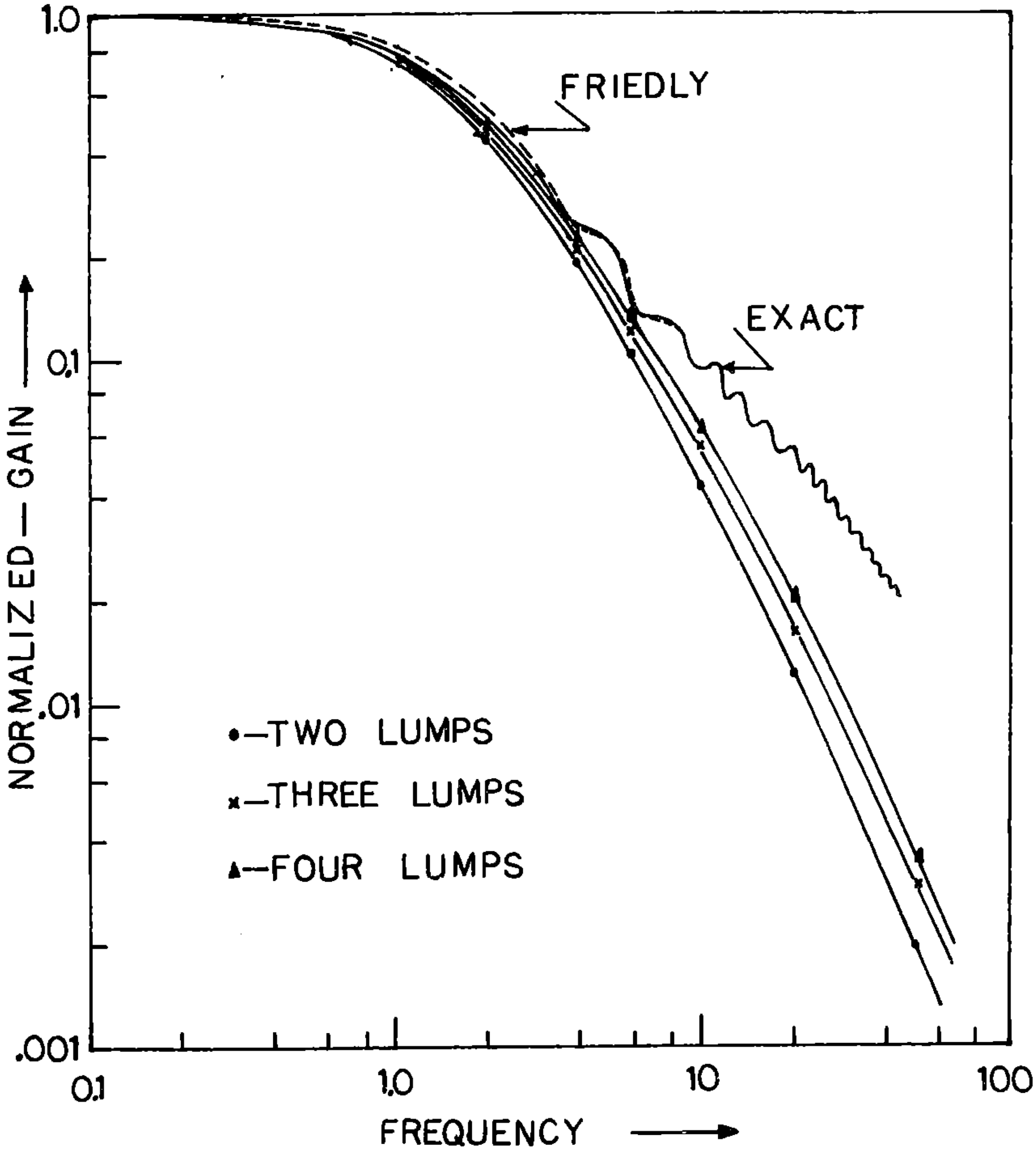


FIGURE(4-15): Phase Angle of Lumped-Models "Counter-Flow"

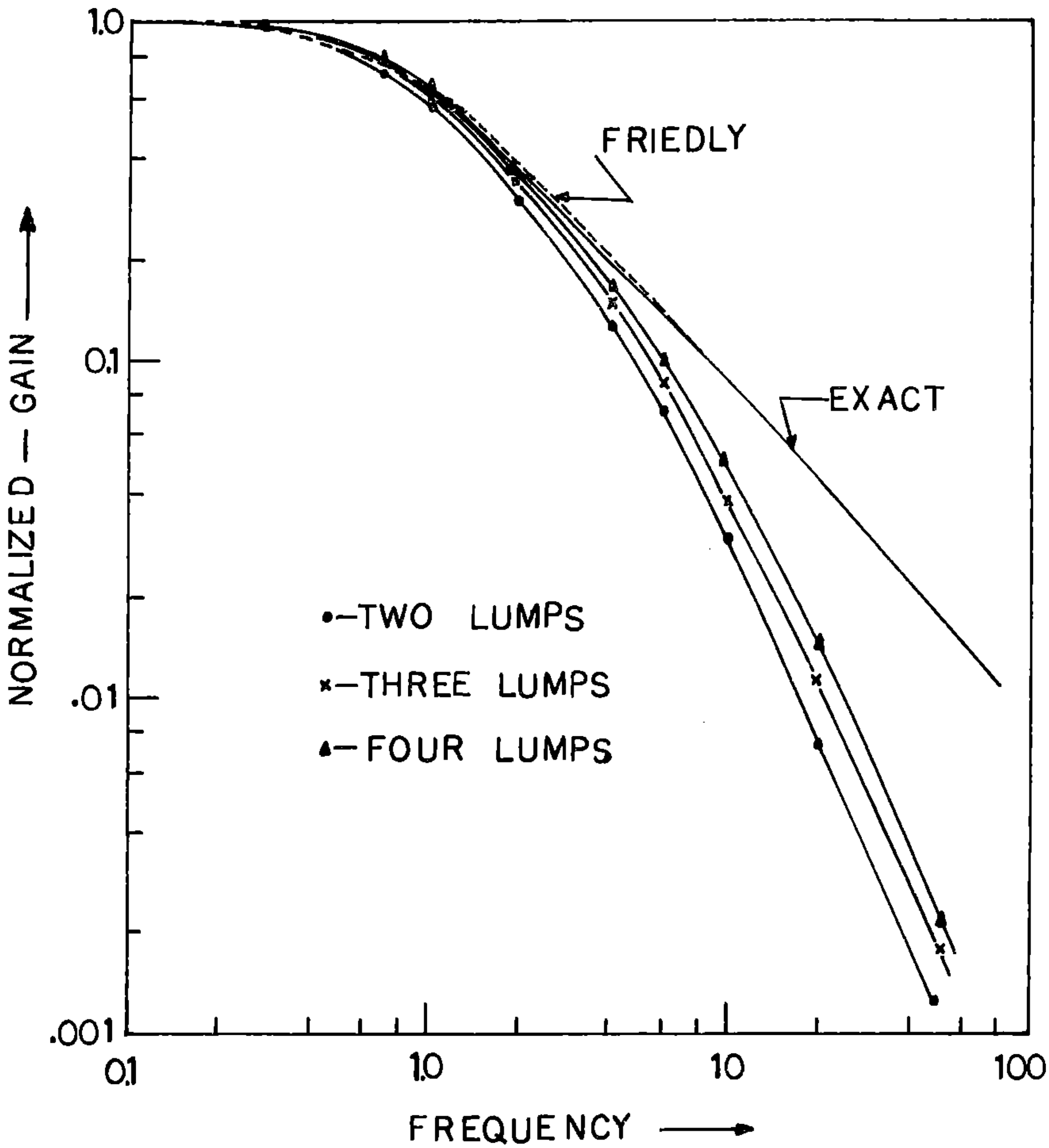
For case $a_1=1$, $a_2=1$, $r=1$



FIGURE(4-16): Gain of Frequency Response of Lumped-Models "Counter-Flow" $a_1=1$, $a_2=1$, $r=1$

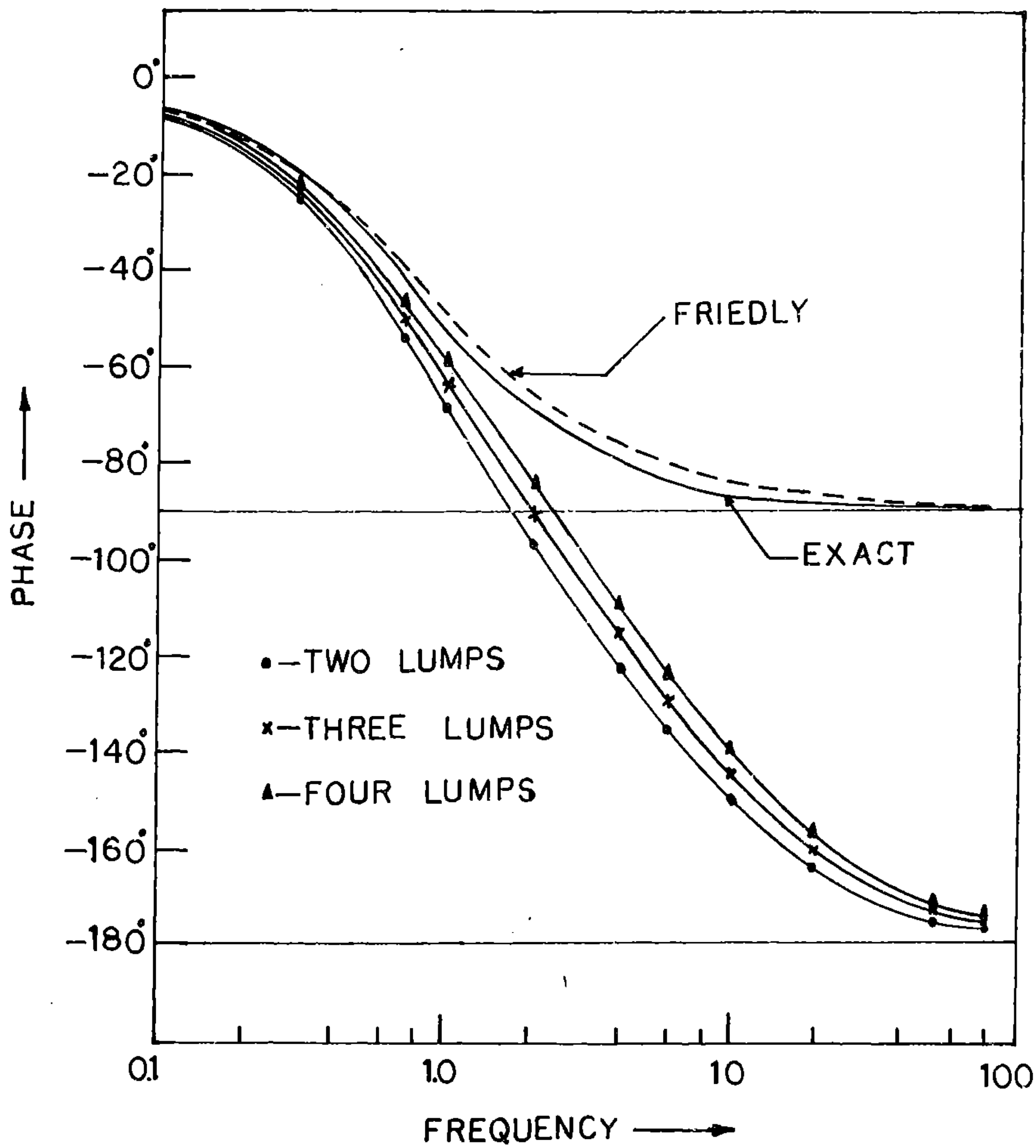


FIGURE(4-17): Gain of Frequency Response of Lumped-Models "Counter-Flow" $a_1=2$, $a_2=1$, $r=2$



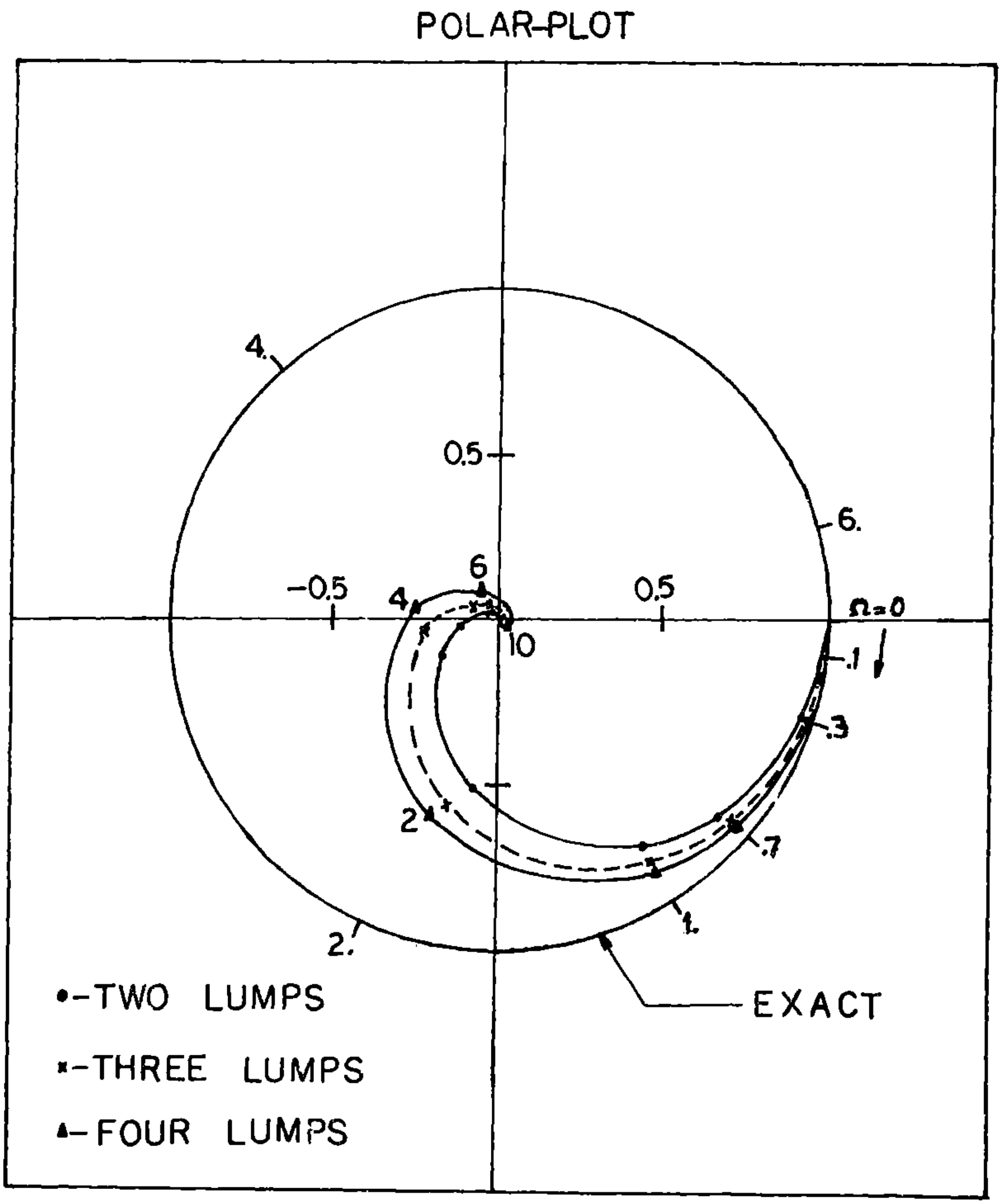
FIGURE(4-18): Phase Angle of Lumped Models"Counter Flow"

For case: $a_1=2$, $a_2=1$, $r=2$



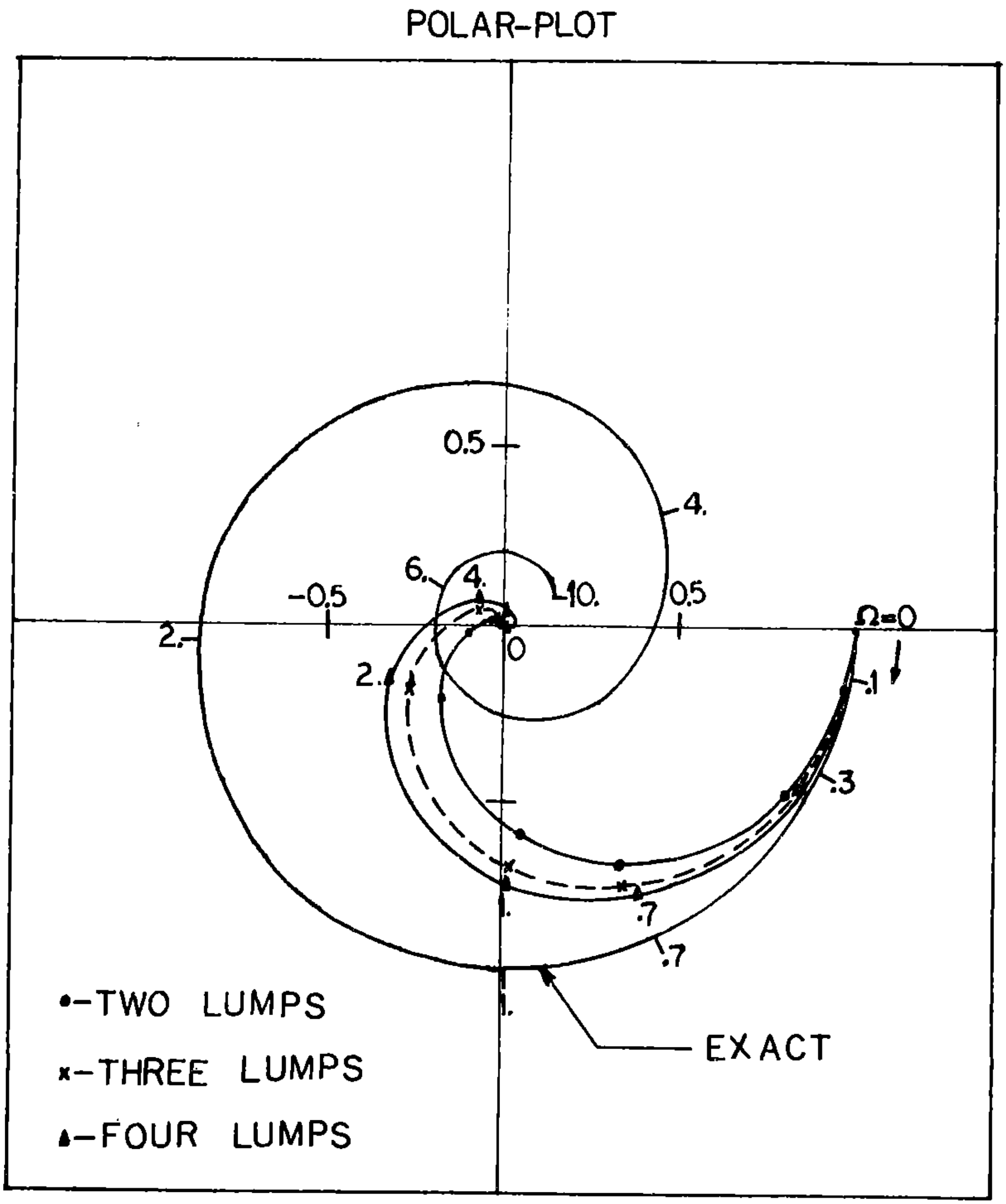
FIGURE(4-19): Frequency Response of Lumped-Models

"Parallel-Flow" $a_1=4$, $a_2=1$, $r=1$



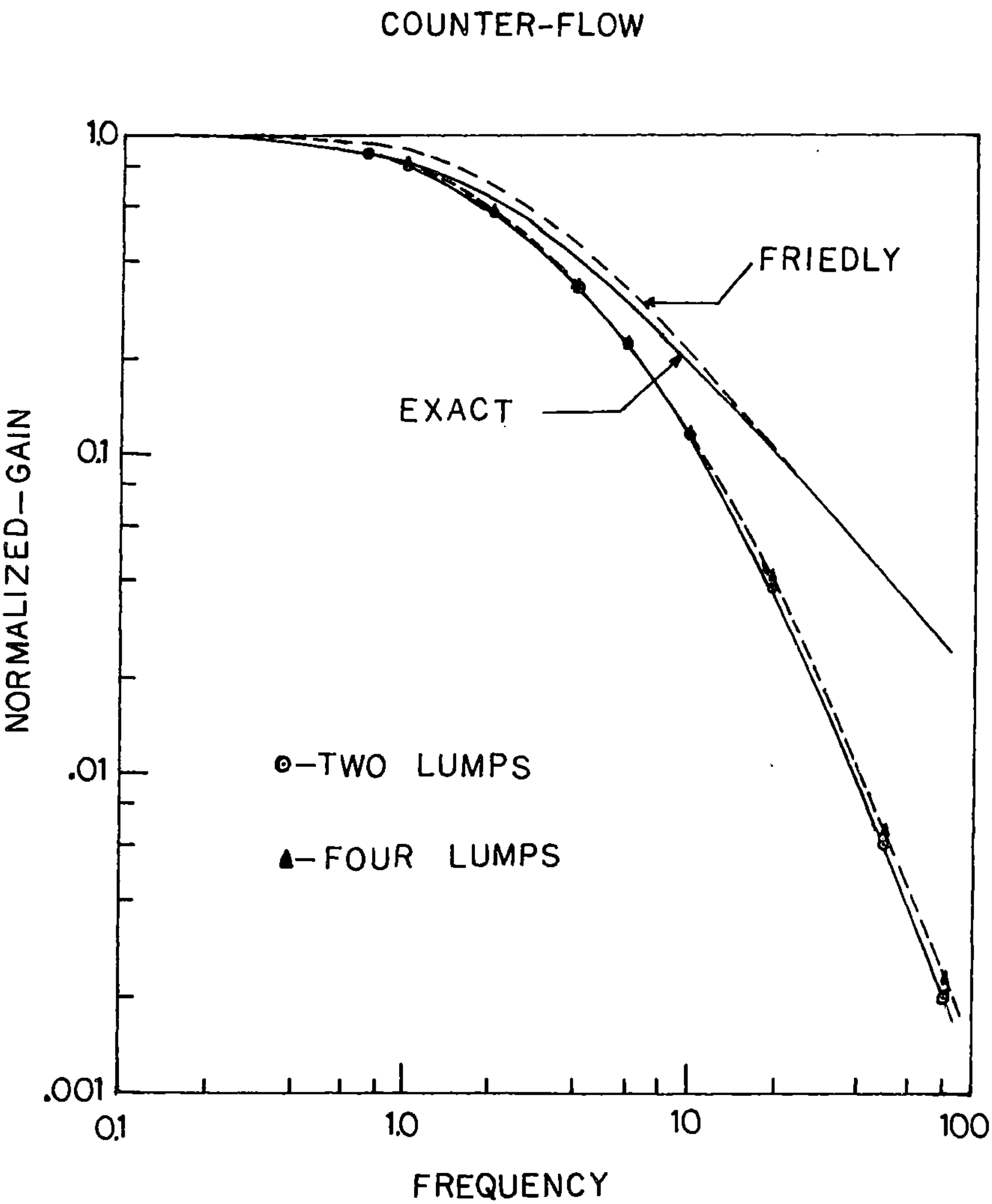
FIGURE(4-20): Frequency Response of Lumped-Models

"Parallel Flow" $a_1=2$, $a_2=1$, $r=2$



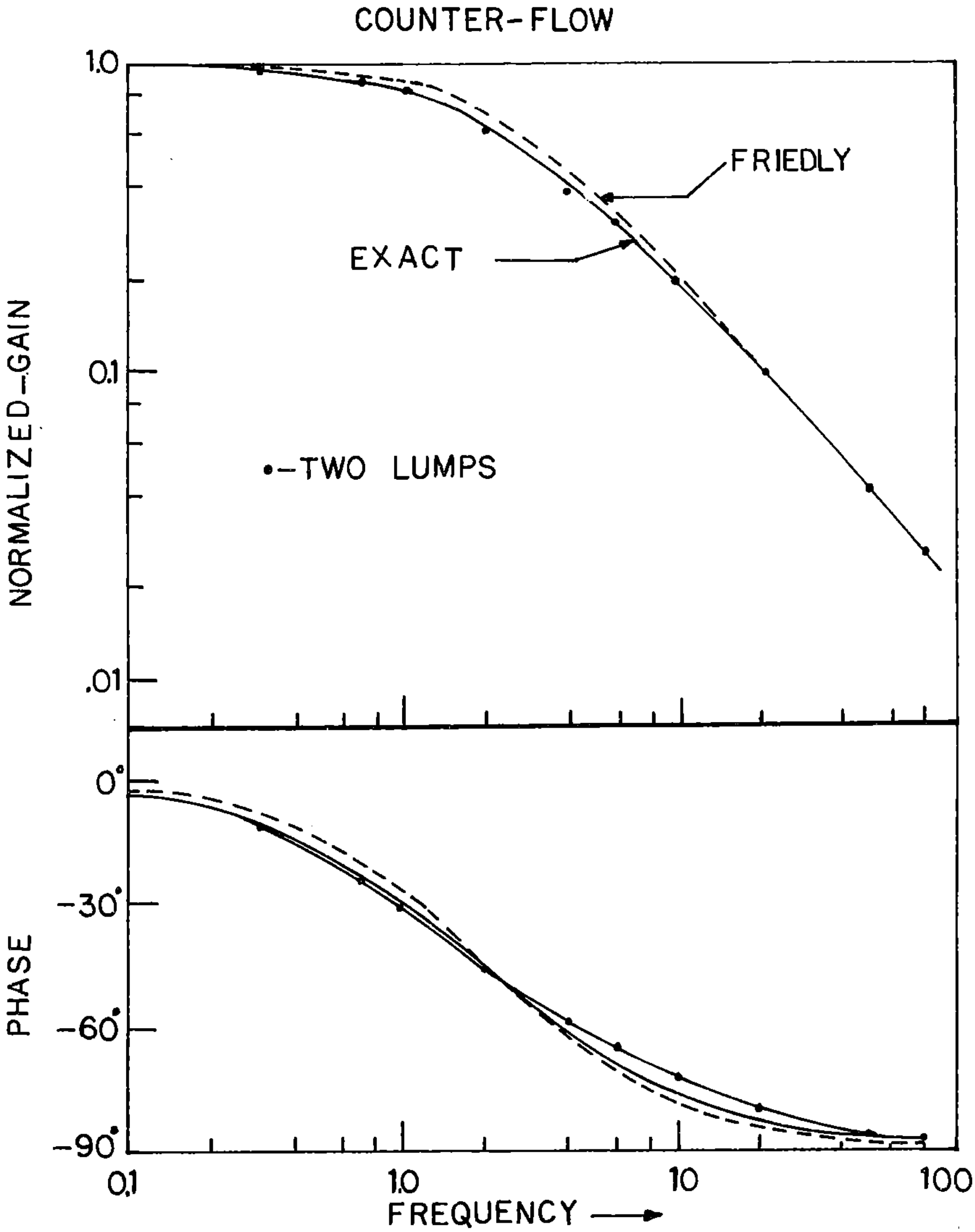
FIGURE(5-1): Frequency Responses of Fitted Lumped-Models

For case: a =4 , a =1 , r=1 , $\varphi_1 = 1/3$, $\varphi_2 = 2/3$



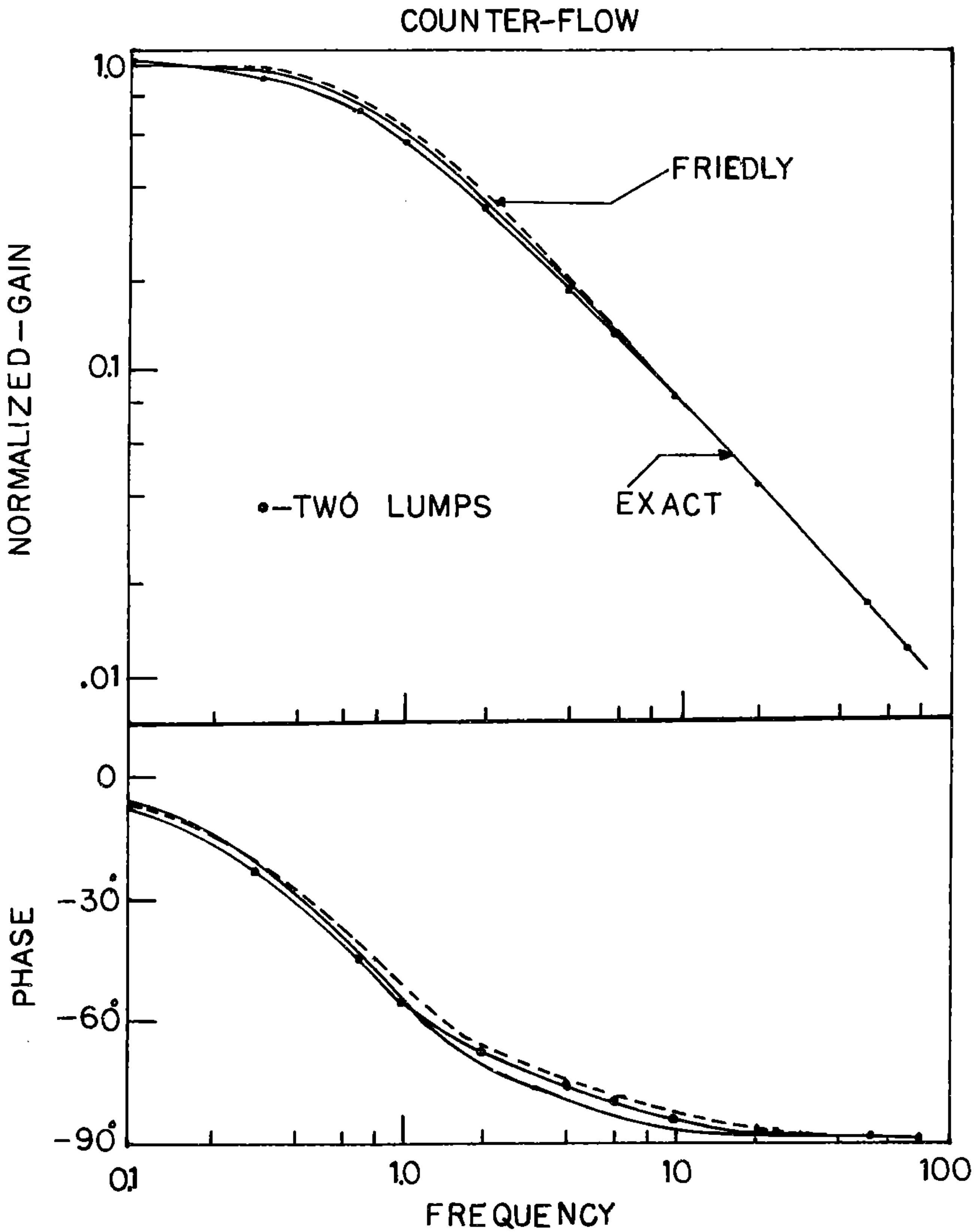
FIGURE(5-3): Gain and Phase Angle for Improved Case

For case: $a_1=4$, $a_2=1$, $r=1$



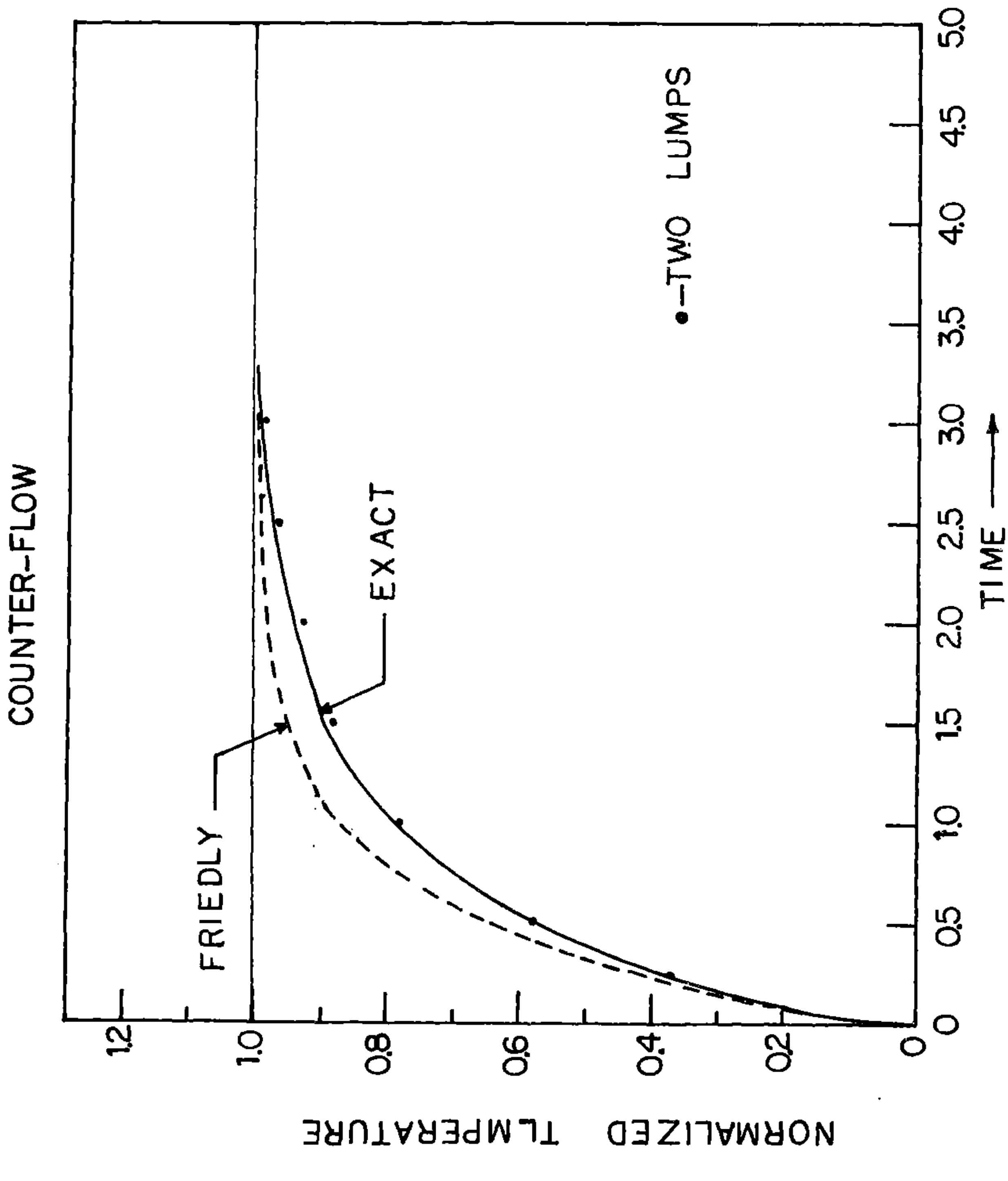
FIGURE(5-4): Gain and Phase Angle for Improved Case

For case: $a_1=2$, $a_2=1$, $r=1$



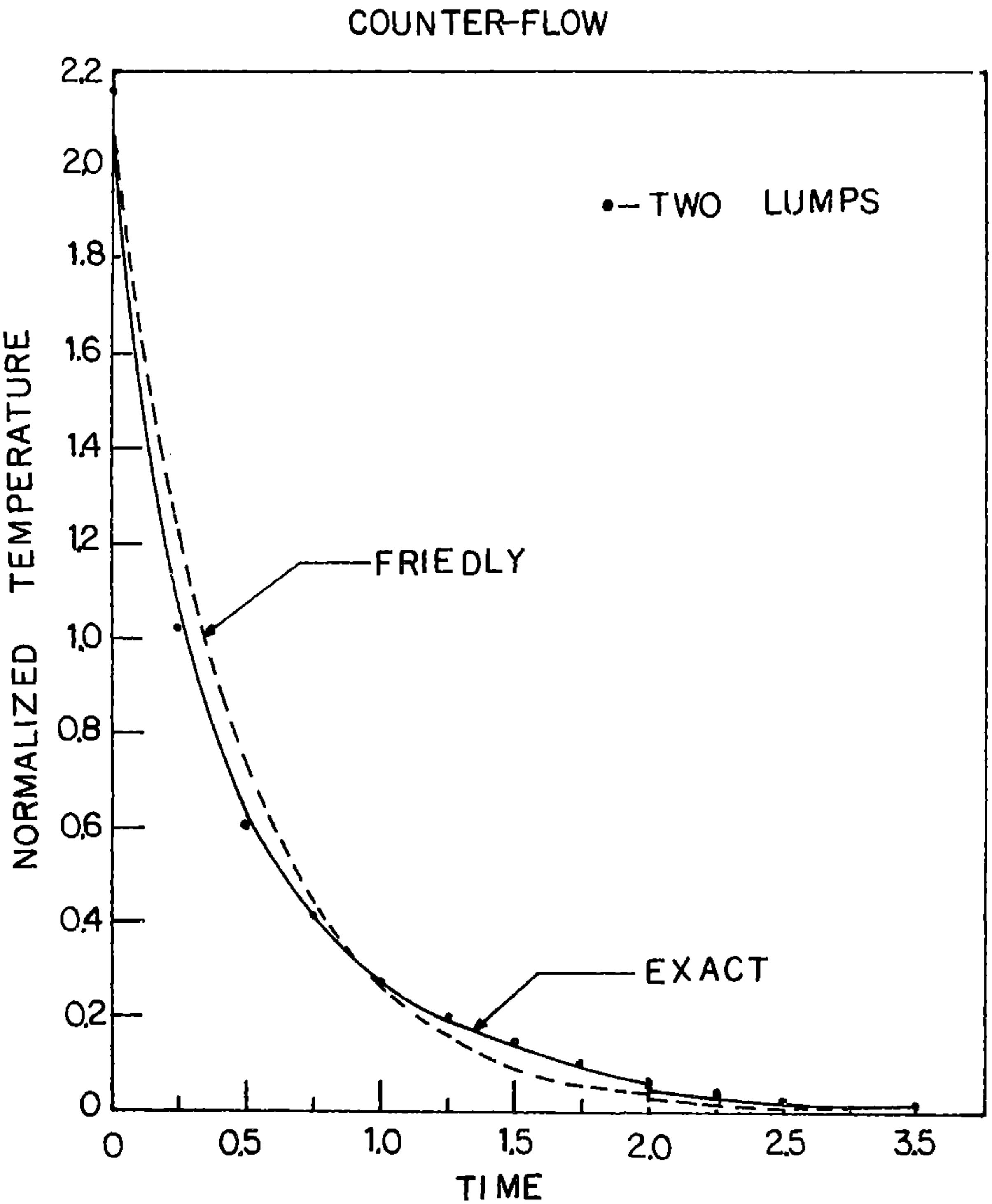
FIGURE(5-5): Step Response of Improved Case

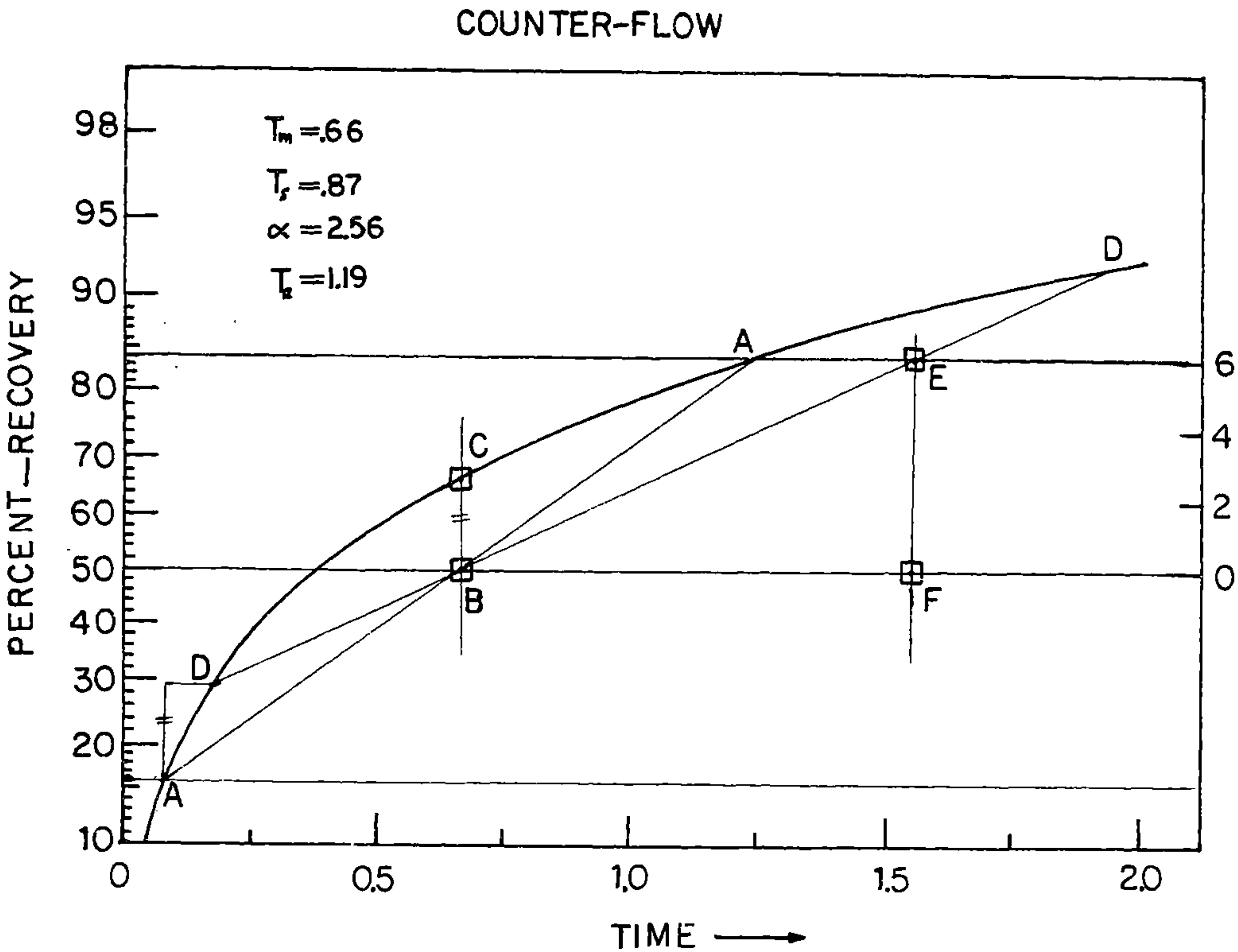
For case: $a_1=4$, $a_2=1$, $r=1$



FIGURE(5-6): Impulse Response of Improved Case

For case: $a_1 = 4$, $a_2 = 1$, $r = 1$



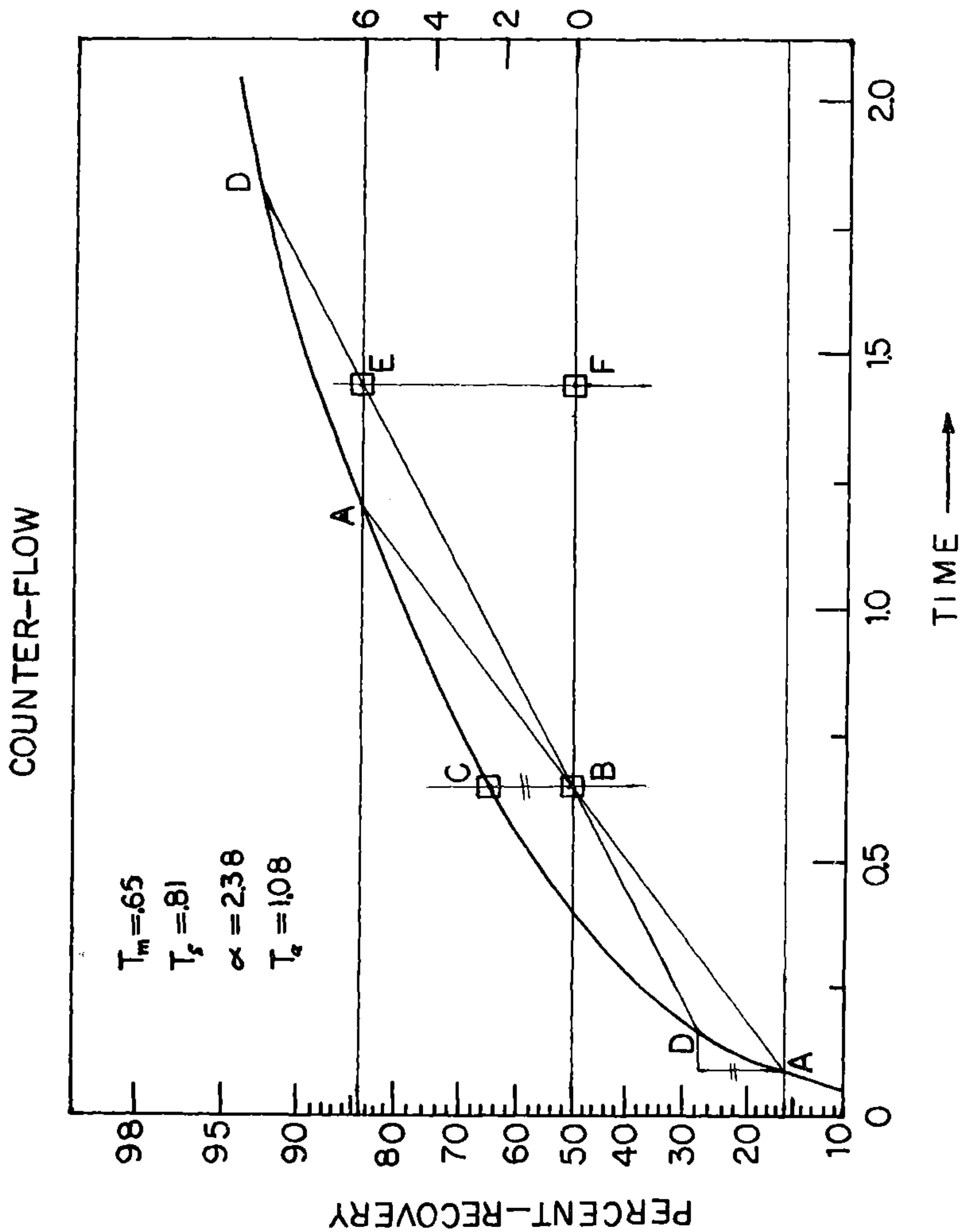


FIGURE(5-7): Monotonic Parameters of Lumped-Models

For case: $a_1=4$, $a_2=1$, $r=1$

FIGURE(5-8): Monotonic Parameters of Exact Solution

For case: $a_1=4$, $a_2=1$, $r=1$



FIGURE(5-9): Monotonic Parameters of Friedly's Method

For case: $a_1=4$, $a_2=1$, $r=1$