THE ALGEBRA OF SPINORS AND ITS APPLICATIONS
TO QUANTUM MECHANICS

by

WILLIAM DICKEY THACKER

Submitted in Partial Fulfillment
of the Requirements for the
Degree of Bachelor of Science
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ABSTRACT

It is well known that the discussion of spin phenomena and two state systems provides an approach to Quantum Mechanics that is simpler than the historical path which leads at once to the Schrödinger equation and the continuous spectrum. This thesis shows that these advantages are enhanced if the spinor theory is introduced in a geometric-algebraic context before the physical interpretation is undertaken. Moreover, among the existing versions of spinor theory we choose one in which non-relativistic and relativistic Quantum Mechanics are treated in close harmony with each other.

Thesis Supervisor: Laszlo Tisza
Title: Professor of Physics
DEDICATED TO THE MEMORY OF MY MOTHER
I would like to thank Professor Laszlo Tisza for instructing me in his approach to the topics discussed here and for countless valuable discussions.

I would also like to thank my father for making all this possible.
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INTRODUCTION

The purpose of this paper is to present a group theoretic introduction to Quantum Mechanics, using an approach recently advanced by Laszlo Tisza.\textsuperscript{1,2} This method is based on the concept of the object group, the formal expression of the identity concept implicit in atomic physics. Elementary particles and even composite structures in pure quantum states are recognized as identical representatives of distinct classes of objects. This "principle of generic identity" is fundamental to quantum theory and has no analog in classical mechanics where the identity of an object is based on the continuity of its trajectory through space.

The concept of object group arises because in quantum mechanics an object of a given identity can exist in a variety of states. We assume that the set of transformations connecting these states is closed, contains the identity element, and has a unique inverse for each of its elements: the set of transformations associated with a quantum mechanical object forms a group.

The object group can be interpreted in a passive or active sense. In the passive interpretation one considers how an object transforms as one varies the reference frame from which it is viewed. The active interpretation considers how an object transforms when viewed from a single frame of reference. These transformations are used to represent kinematic and dynamic
processes; we shall refer to them as the active kinematic and dynamic groups, respectively. This thesis is concerned with the Lorentz group and its subgroup of pure rotations in three dimensional space which shall be interpreted, throughout, in the active sense.

The standard applications of group theory to quantum mechanics call for considerable mathematical knowledge of representation theory and Lie groups, and the physical interpretation of the formalism is not always obvious. The object group approach offers an intuitive interpretation of the connection between group theory and quantum mechanics, which renders it more appropriate for our purposes.

The algebra of Pauli spin matrices and spinors has been introduced by Pauli, Dirac and many others in connection with physical situations, such as spin and antiparticles, which were, and to some extent still are, poorly understood. In this thesis we follow Tisza by developing this formalism in an elementary constructive fashion as a means of describing simple models as they transform under the active kinematic group. The formalism is first built up on a purely mathematical basis. This thesis then focuses on the applications of spinorial techniques to quantum mechanics.
1.1 Heuristic Introduction

Throughout this thesis, $x_0 = ct$ shall denote the time component of a position four vector and $p_0 = E/C$ the zeroth component of the energy-momentum four vector. Our task is to develop a convenient representation for transformations which leave the quantity $x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant, i.e. Lorentz transformations. It is widely recognized in the advanced literature that the algebra of two by two complex matrices is well suited to this purpose. Recent articles in the American Journal of Physics recognize the usefulness of this method for elementary purposes, also. Unfortunately, the method has been advanced in a number of variants, and a measure uniformization would be desirable. Accordingly, we start with elementary heuristic considerations in which the motivation is explained for the conventions chosen.

It is known that every Lorentz transformation can be decomposed into the product of a pure Euclidean rotation and a Lorentz boost. Lorentz boosts can be thought of as hyperbolic rotations. As a first step we consider two dimensional Euclidean rotations in the $x_1, x_2$ plane and hyperbolic rotations in the $x_0, x_3$ plane.
The rotation of the point \((x_1, x_2)\) into the point \((x_1', x_2')\) is defined by the invariance relation: \(x_1'^2 + x_2'^2 = x_1^2 + x_2^2\). Factoring and separating this into two equations we obtain

\[
\begin{align*}
x_1' - i x_2' &= \lambda (x_1 - i x_2) \\
x_1' + i x_2' &= \lambda^*(x_1 + i x_2)
\end{align*}
\]

The above invariance requires that \(\lambda^*\lambda = 1\) or \(\lambda = e^{-i\phi}, \lambda^* = e^{i\phi}\). We can rewrite the above relations as a matrix equation

\[
\begin{pmatrix}
0 & x_1' - i x_2' \\
x_1' + i x_2' & 0
\end{pmatrix}
= \begin{pmatrix}
e^{-i\phi} & 0 \\
0 & e^{i\phi}
\end{pmatrix}
\begin{pmatrix}
0 & x_1 - i x_2 \\
x_1 + i x_2 & 0
\end{pmatrix}
\]

The point \((x_1, x_2)\) is now represented by a two by two hermitian matrix and the rotation is represented by a unitary matrix.

The hyperbolic rotation of the point \((x_0, x_3)\) into \((x_0', x_3')\) is defined by the equivalence, \(x_0'^2 - x_3'^2 = x_0^2 - x_3^2\). Factoring and separating we obtain:

\[
\begin{align*}
x_0' + x_3' &= a (x_0 + x_3) \\
x_0' - x_3' &= a^{-1}(x_0 - x_3)
\end{align*}
\]

*Throughout this paper \(\lambda\) should be read as \(e\).*
By letting \(a = \zeta^\mu\) we exclude it from having negative values. Negative values would represent Lorentz transformations which reverse the order of past and future. We require Lorentz transformations to be orthochronic so that the time ordering is preserved. The real parameter, \(\mu\), is related with velocity as follows: \(\tanh \mu = \beta = v/c\). We map the point \((x_0', x_3')\) onto the hermitian matrix

\[
\begin{pmatrix}
  x_0 + x_3 & 0 \\
  0 & x_0 - x_3
\end{pmatrix}
\]

and represent the hyperbolic rotation as follows:

\[
\begin{pmatrix}
  x_0' + x_3' & 0 \\
  0 & x_0' - x_3'
\end{pmatrix} = \begin{pmatrix}
  \zeta^\mu & 0 \\
  0 & \zeta^{-\mu}
\end{pmatrix} \begin{pmatrix}
  x_0 + x_3 & 0 \\
  0 & x_0 - x_3
\end{pmatrix}
\]

The operator effecting this hyperbolic rotation is a hermitian rather than a unitary matrix.

We have found it convenient to represent the points \((x_1, x_2)\) and \((x_0, x_3)\) as two-by-two hermitian matrices. Putting the above two mappings together we represent the space-time point, \((x_0; x_1, x_2, x_3)\), with the hermitian matrix,

\[
\begin{pmatrix}
  x_0 + x_3 & x_1 - ix_2 \\
  x_1 + ix_2 & x_0 - x_3
\end{pmatrix}
\]

However if we multiply this matrix on the left by \(\begin{pmatrix}
  \zeta^{-i\phi} & 0 \\
  0 & \zeta^{i\phi}
\end{pmatrix}\) we do not produce a rotation in the \(x_1, x_2\) plane:
\[
\begin{pmatrix}
\ell^{-i\phi} & 0 \\
0 & \ell^{i\phi}
\end{pmatrix}
\begin{pmatrix}
x_0 - x_3 & x_1 - ix_2 \\
x_1 + ix_2 & x_0 + x_3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\ell^{-i\phi}(x_0 - x_3) & \ell^{-i\phi}(x_1 - ix_2) \\
\ell^{i\phi}(x_1 + ix_2) & \ell^{i\phi}(x_0 + x_3)
\end{pmatrix}
\]

Moreover this operation does not preserve the hermiticity of
\[
\begin{pmatrix}
x_0 - x_3 & x_1 - ix_2 \\
x_1 + ix_2 & x_0 + x_3
\end{pmatrix}
\]. These difficulties can be removed by using bilateral matrix multiplications:

\[
\begin{pmatrix}
\ell^{-i\phi/2} & 0 \\
0 & \ell^{i\phi/2}
\end{pmatrix}
\begin{pmatrix}
x_0 + x_3 & x_1 - ix_2 \\
x_1 + x_2 & x_0 - x_3
\end{pmatrix}
\begin{pmatrix}
\ell^{i\phi/2} & 0 \\
0 & \ell^{-i\phi/2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
x_0 + x_3 & x'_1 - ix'_2 \\
x'_1 + ix'_2 & x_0 - x_3
\end{pmatrix}
\]

The same device works for the Lorentz boost along the x_3-axis:

\[
\begin{pmatrix}
\ell^\mu/2 & 0 \\
0 & \ell^{-\mu/2}
\end{pmatrix}
\begin{pmatrix}
x_0 + x_3 & x_1 - ix_2 \\
x_1 + ix_2 & x_0 - x_3
\end{pmatrix}
\begin{pmatrix}
\ell^\mu/2 & 0 \\
0 & \ell^{-\mu/2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
x'_0 + x'_3 & x_1 - ix_2 \\
x'_1 + ix'_2 & x'_0 - x'_3
\end{pmatrix}
\]
In fact it can be generalized for arbitrary rotations and Lorentz boosts, however to show this we must develop the formalism further.

1.2 The Pauli Algebra

We introduce the three Pauli spin matrices along with the unit matrix as the basis for our algebra.

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(1.1)

The Pauli matrices obey the following relations:

\[ \sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k; \quad [\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k. \]

(1.2)

Letting \( \hat{\sigma} = \sigma_1 \hat{x}_1 + \sigma_2 \hat{x}_2 + \sigma_3 \hat{x}_3 \), we can represent the above matrix for the point, \((x_0, \hat{x})\), as \( x_0 1 + \hat{x} \cdot \hat{\sigma} \). In general any two by two complex matrix can be represented as \( A = a_0 1 + \hat{a} \cdot \hat{\sigma} \) where \( a_0 \) and \( \hat{a} \) are now complex. The components of \( A \) can be found by the relation: \( a_\mu = \frac{1}{2} \text{Tr} (A \sigma_\mu) \) where \( \text{Tr} \) stands for trace. The product of \( A = a_0 1 + \hat{a} \cdot \hat{\sigma} \) and \( B = b_0 1 + \hat{b} \cdot \hat{\sigma} \) is given by:

\[
AB = (a_0 b_0 + \hat{a} \cdot \hat{b}) 1 + (a_0 \hat{b} + b_0 \hat{a} + i \hat{a} x \hat{b}) \cdot \hat{\sigma}
\]

(1.3)

Their commutator is:

\[
[A, B] = 2i (\hat{a} x \hat{b}) \cdot \hat{\sigma}
\]

(1.4)
In analogy with the operation of complex conjugation, which acts on the complex numbers, we introduce three new involutions which act on the set of two-by-two complex matrices. The involutions are characterized by the fact that when they operate twice on a matrix they leave it unchanged. The involutions act on the component, \( a_\mu \), of the matrix \( A \) rather than on its elements, \( a_{mn} \). The Pauli matrices which make up our basis, are left unchanged by the involutions.* Our first involution, the hermitian adjoint, carries \( A = a_0 1 + \vec{a} \cdot \vec{s} \) into \( A^+ = a^*_0 1 + \vec{a}^* \cdot \vec{s} \). Hermitian matrices are defined by the relation, \( H^+ = H \). The operation of simple reversal carries \( A \) into \( \tilde{A} = a_0 1 - \vec{a} \cdot \vec{s} \). This operation is known from the theory of quaternions. Complex reversal, which is derived from the composition of the above two involutions, carries \( A \) into \( \bar{A} = a^*_0 1 - \vec{a}^* \cdot \vec{s} \). These three involutions, together with the identity, form a group where each involution is its own inverse.

1.3 Matrix Representations for Rotations and Lorentz Boosts

At this point we are ready to find matrix expressions for arbitrary Euclidean rotations and Lorentz boosts. To do this we represent the four vector to be transformed as a hermitian matrix and operate on it with a bilateral multiplication which preserves its hermiticity. First we try \( X' = VXW \), in this case

*This is a characteristic feature of Tisza's formalism which contrasts with the Van der Waerden formalism. See, for instance, Laporte and Uhlenbeck.4
\( x' = W^+ X V^+ \). For \( x' \) to be hermitian we must have: \( W = V^+ \).

If we add the condition that \( V \) be unimodular, i.e. that \( \det|V| = 1 \), then we find that a transformation of the form \( x' = V X V^+ \) leaves the quantity \( \det X = x_0^2 - \dot{x}_2^2 \) invariant. Therefore \( x' = V X V^+ \) is a Lorentz transformation, by definition.

Our next step is to find expressions for \( V \) which produce pure rotations and Lorentz boosts.

We have already found that a bilateral multiplication by
\[
\begin{pmatrix}
\ell^{-i\phi/2} & 0 \\
0 & \ell^{i\phi/2}
\end{pmatrix}
\]
and its hermitian adjoint produces a rotation through \( \phi \) about the \( x_3 \)-axis. This unitary matrix can be rewritten as:

\[
U(x_3, \phi/2) = \exp\left[-i \frac{\phi}{2} \dot{x}_3 \cdot \dot{\gamma}\right] = \cos \frac{\phi}{2} 1 - i \sin \frac{\phi}{2} \dot{x}_3 \cdot \dot{\gamma}
\]

Generalizing, we conjecture that a rotation through \( \phi \) about \( \hat{u} \) is produced by the following matrix:

\[
U(\hat{u}, \phi/2) = \exp\left[-i \frac{\phi}{2} \hat{u} \cdot \dot{\gamma}\right] = \cos \frac{\phi}{2} 1 - i \sin \frac{\phi}{2} \hat{u} \cdot \dot{\gamma}
\]

The second equivalence can be demonstrated by expanding the exponential in a power series and using the fact that \( (\hat{u} \cdot \dot{\gamma})^n \) is 1 if \( n \) is even and \( \hat{u} \cdot \dot{\gamma} \) if \( n \) is odd. To show that \( \dot{x}' \cdot \dot{\gamma} = U(\hat{u}, \phi/2) \dot{x} \cdot \dot{\gamma} U^+(\hat{u}, \phi/2) \) does represent a rotation, as conjectured, we separate \( \dot{x} \) into a component parallel to \( \hat{u} \) and one perpendicular to it: \( \dot{x} = \dot{x}_{11} + \dot{x}_\perp \). \( \dot{x}_{11} \cdot \dot{\gamma} \) commutes with \( \hat{u} \cdot \dot{\gamma} \) whereas \( (\dot{x}_\perp \cdot \dot{\gamma})(\hat{u} \cdot \dot{\gamma}) = - (\hat{u} \cdot \dot{\gamma})(\dot{x}_\perp \cdot \dot{\gamma}) \). Therefore \( \dot{x}_{11} \) and \( \dot{x}_\perp \)
transform as follows:

\[
\begin{align*}
\dot{x}_{11} \cdot \vec{\sigma} &= (\cos \frac{\phi}{2} \ l - i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}) \ \dot{x}_{11} \cdot \vec{\sigma} \ (\cos \frac{\phi}{2} \ l + i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}) \\
&= (\cos \frac{\phi}{2} \ l - i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma})(\cos \frac{\phi}{2} \ l + i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}) \ \dot{x}_{11} \cdot \vec{\sigma} \\
&= \dot{x}_{11} \cdot \vec{\sigma}
\end{align*}
\]

\[
\dot{x}_{11} = \dot{x}_{11}
\]

\[
\begin{align*}
\dot{x}_{\perp} \cdot \vec{\sigma} &= (\cos \frac{\phi}{2} \ l - i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}) \ \dot{x}_{\perp} \cdot \vec{\sigma} (\cos \frac{\phi}{2} \ l + i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}) \\
&= (\cos \frac{\phi}{2} \ l - i \sin \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma})^2 \ \dot{x}_{\perp} \cdot \vec{\sigma} \\
&= [(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2}) \ l - 2 i \sin \frac{\phi}{2} \ \cos \frac{\phi}{2} \ \hat{u} \cdot \vec{\sigma}] \ \dot{x}_{\perp} \cdot \vec{\sigma} \\
&= \cos \phi \ \dot{x}_{\perp} \cdot \vec{\sigma} + \sin \phi \ (\hat{u} \times \dot{x}_{\perp}) \cdot \vec{\sigma} \\
\dot{x}_{\perp} &= \cos \phi \ \dot{x}_{\perp} + \sin \phi \ \hat{u} \times \dot{x}_{\perp}
\end{align*}
\]

\[\dot{x}_{11}\] remains invariant under this operation while \[\dot{x}_{\perp}\] rotates through \(\phi\) about \(\hat{u}\). Q.E.D.

We recall that the hermitian matrix which effects a Lorentz boost along \(\hat{x}_3\) is:

\[
\begin{pmatrix}
\exp(\mu/2 \ \hat{x}_3 \cdot \vec{\sigma})
\end{pmatrix}
\]

We write it as \(H(\hat{x}_3, \mu/2)\) = \(\exp(\mu/2 \ \hat{x}_3 \cdot \vec{\sigma})\) = \(\cosh \mu/2 \ l + \sinh \mu/2 \ \hat{x}_3 \cdot \vec{\sigma}\). Generalizing, we obtain for a boost along an arbitrary axis \(\hat{h}\):

\[
H(\hat{h}, \mu/2) = \exp(\mu/2 \ \hat{h} \cdot \vec{\sigma}) = \cosh \mu/2 \ l + \sinh \mu/2 \ \hat{h} \cdot \vec{\sigma}
\]

As we did before, we break \(\dot{x}\) up into components parallel and
perpendicular to \( \mathbf{h} \). Now \( \mathbf{x}_1 \cdot \mathbf{\sigma} \) remains unchanged by the bilateral multiplication:

\[
x_1' \cdot \mathbf{\sigma} = (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma}) x_1 \cdot \mathbf{\sigma} (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma})
\]

\[
= (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma}) (\cosh \frac{\mu}{2} l - \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma}) x_{1} \cdot \mathbf{\sigma}
\]

\[
= x_1' \cdot \mathbf{\sigma}
\]

(1.7a)

\[
x_1' = x_1. \quad x_0 \text{ and } x_{11} \text{ transform together as follows:}
\]

\[
x_0' + x_{11}' \cdot \mathbf{\sigma} = (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma}) (x_0 + x_{11} \cdot \mathbf{\sigma}) (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma})
\]

\[
= (\cosh \frac{\mu}{2} l + \sinh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma})^2 (x_0 + x_{11} \cdot \mathbf{\sigma})
\]

\[
= [(\cosh^{2} \frac{\mu}{2} + \sinh^{2} \frac{\mu}{2}) l + 2 \sinh \frac{\mu}{2} \cosh \frac{\mu}{2} \hat{\mathbf{h}} \cdot \mathbf{\sigma}] 
\]

\[
(x_0 + x_{11} \cdot \mathbf{\sigma})
\]

\[
= (\cosh \mu x_0 + \sinh \mu x_{11}) l + (\cosh \mu x_{11} + \sinh \mu x_0 \hat{\mathbf{h}} \cdot \mathbf{\sigma})
\]

(1.7b)

Equating the scalar and vector components we obtain:

\[
x_0' = \cosh \mu x_0 + \sinh \mu x_{11}
\]

(1.8a)

\[
x_{11}' = \cosh \mu x_{11} + \sinh \mu x_0 \hat{\mathbf{h}} \cdot \mathbf{\sigma}
\]

(1.8b)
These are the standard relations for how $x_0$ and $\vec{x}_{\perp}$ transform under an active Lorentz boost: 

$$\cosh \mu = (1 - \beta^2)^{\frac{1}{2}}$$ \hspace{1cm} \sinh \mu = \beta (1 - \beta^2)^{-\frac{1}{2}}.$$

### 1.4 Introduction to the Spinor Concept

Thus far we have developed the Pauli algebra as a convenient means of representing four vectors and their transformation under Euclidean and hyperbolic rotations. A basic feature of this formalism is the appearance of half angles, both circular and hyperbolic, in the matrices which induce these transformations. We have found it necessary to represent the rotation or boost of a four vector with a bilateral multiplication in order that the hermiticity of the matrix representing the four vector be preserved. Thus, to produce a rotation through $\phi$ we use two multiplications, each with a matrix parameterized by $\phi/2$.

However, the half angles disappear from the equations describing the transformation of the components of a four vector. Our next step is to introduce mathematical entities which transform in such a way that the half angles remain important.

It seems natural to extend our formalism by introducing two component complex column vectors upon which the two by two matrices operate. These two component entities are called spinors. We use Dirac's bra-ket notation denoting the spinor, \( \phi \), with the ket \( |\phi> \). Spinors transform under Lorentz boosts and rotations as follows: \( |\phi'> = \mathbf{V}|\phi> \). Here the half angles are
important because only a single multiplication is used in transforming the spinor. In what follows we concern ourselves exclusively with Euclidean rotations and omit the discussion of relativistic spinors for a later chapter.

1.5 Unitary Spinors and the Orientable Object

Our present task is to use the algebraic methods developed so far, together with the spinor concept, to find a convenient parameterization for the orientation of objects in ordinary space. The mathematical model we consider is the triad, which consists of three orthogonal unit vectors attached to a common origin. We consider two frames: the space fixed frame, associated with the triad $\Sigma(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, and the body fixed frame, associated with the triad $\Sigma(\hat{l}_1, \hat{l}_2, \hat{l}_3)$.

In order to define the orientation of the body, $\Sigma(\hat{l}_1, \hat{l}_2, \hat{l}_3)$, with respect to fixed space, $\Sigma(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, we suppose that the two triads originally coincided and the body has been brought to its present position through a sequence of rotations through the three Euler angles. The Euler angles refer to a rotation first through the angle $\alpha$ about its $\hat{l}_3$-axis, then through $\beta$ about its $\hat{l}_2$-axis, and finally through $\gamma$ about its $\hat{l}_3$-axis. This sequence of rotations in the body frame corresponds to the same sequence, performed in the opposite order, in the space frame. The unitary matrix which describes the sequence of Euler rotations is therefore given by:
\[ V(\alpha, \beta, \gamma) = U(\hat{x}_3, \alpha/2) \, U(\hat{x}_2, \beta/2) \, U(\hat{x}_3, \gamma/2) \]

\[
= \begin{pmatrix}
\frac{\gamma}{2} - i \alpha/2 & 0 \\
0 & \frac{\gamma}{2} + i \alpha/2
\end{pmatrix}
\begin{pmatrix}
\cos \beta/2 & -\sin \beta/2 \\
\sin \beta/2 & \cos \beta/2
\end{pmatrix}
\begin{pmatrix}
\frac{\gamma}{2} - i \gamma/2 & 0 \\
0 & \frac{\gamma}{2} + i \gamma/2
\end{pmatrix}
= \begin{pmatrix}
\frac{\gamma}{2} - i \alpha/2 \cos \beta/2 & \frac{\gamma}{2} - i \gamma/2 \\
\frac{\gamma}{2} + i \alpha/2 \sin \beta/2 & \frac{\gamma}{2} + i \gamma/2
\end{pmatrix}
= \begin{pmatrix}
\frac{\gamma}{2} - i \alpha/2 \cos \beta/2 & \frac{\gamma}{2} - i \gamma/2 \\
\frac{\gamma}{2} + i \alpha/2 \sin \beta/2 & \frac{\gamma}{2} + i \gamma/2
\end{pmatrix}
\]

\[ V(\alpha, \beta, \gamma) \] describes the orientation of an object in space. However \( V(\alpha, \beta, \gamma) \) is the same kind of mathematical entity as the matrices which induce rotations. It is preferable to continue to use matrices to denote rotation operators and to introduce spinors which describe orientable objects. We therefore introduce the two basis spinors:

\[ |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\bar{\Omega}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Operating upon them with \( V(\alpha, \beta, \gamma) \) we obtain:

\[ |\xi\rangle = V(\alpha, \beta, \gamma) |1\rangle = \begin{pmatrix}
\frac{\gamma}{2} - i \alpha/2 \cos \beta/2 \\
\frac{\gamma}{2} + i \alpha/2 \sin \beta/2
\end{pmatrix} |1\rangle \]

\[ = \begin{pmatrix}
|\xi_1\rangle \\
|\xi_2\rangle
\end{pmatrix} \quad (1.9a) \]

\[ |\bar{\xi}\rangle = V(\alpha, \beta, \gamma) |\bar{\Omega}\rangle = \begin{pmatrix}
\frac{\gamma}{2} - i \alpha/2 \sin \beta/2 \\
\frac{\gamma}{2} + i \alpha/2 \cos \beta/2
\end{pmatrix} |\bar{\Omega}\rangle \]

\[ = \begin{pmatrix}
|\xi_2^*\rangle \\
|\xi_1^*\rangle
\end{pmatrix} \quad (1.9b) \]
We have defined the spinor components:
\[ \xi_1 = e^{-ia/2} \cos \beta/2 e^{-i\gamma/2} \quad \text{and} \quad \xi_2 = e^{ia/2} \sin \beta/2 e^{-i\gamma/2} \]
\[ |\xi\rangle \] and its conjugate, \[ |\bar{\xi}\rangle \], contain the same information as \[ V(\alpha, \beta, \gamma) \] and therefore describe orientation.

\[ V(\alpha, \beta, \gamma) \] can be written as the juxtaposition of \[ |\xi\rangle \] with its conjugate: \[ V(\alpha, \beta, \gamma) = (|\xi\rangle \mid \bar{\xi}\rangle) \]. Taking the hermitian adjoint we obtain:
\[ V^+(\alpha, \beta, \gamma) = \begin{pmatrix} \langle \xi | \\ \langle \xi | \end{pmatrix} = \begin{pmatrix} \xi_1^* & \xi_2^* \\ -\xi_2^* & \xi_1^* \end{pmatrix} \]

We have thus introduced the bra spinors \[ \langle \xi | = (\xi_1^*, \xi_2^*) \]
\[ \langle \bar{\xi} | = (-\xi_2^*, \xi_1^*) \]. Taking the inner products between \[ |\xi\rangle, |\bar{\xi}\rangle \] and their hermitian adjoints we obtain:

\[ \langle \xi | \xi \rangle = (\xi_1^*, \xi_2^*) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = |\xi_1|^2 + |\xi_2|^2 = 1 \quad (1.10a) \]

\[ \langle \bar{\xi} | \bar{\xi} \rangle = (-\xi_2^*, \xi_1^*) \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = |\xi_2|^2 + |\xi_1|^2 = 1 \quad (1.10b) \]

\[ \langle \bar{\xi} | \xi \rangle = (-\xi_2^*, \xi_1^*) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -\xi_1 \xi_2 + \xi_1 \xi_2 = 0 \quad (1.10c) \]

\[ \langle \xi | \bar{\xi} \rangle = (\xi_1^*, \xi_2^*) \begin{pmatrix} -\xi_2 \\ \xi_1^* \end{pmatrix} = -\xi_1^* \xi_2^* + \xi_1^* \xi_2^* = 0 \quad (1.10d) \]

These relations show that \[ |\xi\rangle \] and \[ |\bar{\xi}\rangle \] are orthogonal to each other and are each normalized to unity; \[ |\xi\rangle \] and its conjugate therefore form an orthonormal basis for our two dimensional spinor space. Because of their normalization we shall call \[ |\xi\rangle \]
and $|\vec{\xi}\rangle$ unitary spinors.

We now consider the operation of spin conjugation, which carries $|\xi\rangle$ into $|\vec{\xi}\rangle$. This operation can be represented as follows: 
\[-i\sigma_2 \kappa |\xi\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = \begin{pmatrix} \xi_2^* \\ -\xi_1^* \end{pmatrix} = |\vec{\xi}\rangle \]
where $\kappa$ denotes complex conjugation. The hermitian adjoint $<\xi|$ can be conjugated similarly: 
\[\kappa <\xi| \sigma_2 = (\xi_1^*, \xi_2^*) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-\xi_2^*, \xi_1^*) = <\vec{\xi}|.\]
Performing the operation of spin conjugation twice we find that:
\[|\xi\rangle = -i\sigma_2 \kappa \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = \begin{pmatrix} -\xi_1 \\ -\xi_2 \end{pmatrix} = -|\xi\rangle \quad (1.11a)\]
\[<\xi| = \kappa (-\xi_2^*, \xi_1^*) i\sigma_2 = (-\xi_1^*, -\xi_2^*) = -<\xi| \quad (1.11b)\]

These relations show that spinors are two valued under the operation of spin conjugation.

It will be shown in Chapter III that when a relativistic spinor transforms according to the matrix $V$, i.e. $|\eta^\prime\rangle = V|\eta\rangle$, the conjugate spinor transforms according to the complex reflected matrix $\bar{V}$, $|\bar{\eta}\rangle = \bar{V}|\bar{\eta}\rangle$. However, since a unitary matrix is unchanged by complex reflection, the transformation properties of unitary spinors are invariant under spin conjugation.

We have started with a triad, $\xi(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$, and associated it with a pair of conjugate spinors. Now we work the other way starting with $|\xi\rangle, |\vec{\xi}\rangle$ and finding the unit vectors of the triad.
To do this we consider the unit configuration, \( |l> = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( |\bar{l}> = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( <l| = (1,0) \), \( <\bar{l}| = (0,1) \), and find the outer products of these basis spinors:

\[
|l><l| = \begin{bmatrix} 10 \\ 00 \end{bmatrix} = \frac{1}{2} (1 + \hat{x}_3 \cdot \vec{\sigma})
\]

\[
|\bar{l}><\bar{l}| = \begin{bmatrix} 00 \\ 01 \end{bmatrix} = \frac{1}{2} (1 - \hat{x}_3 \cdot \vec{\sigma})
\]

\[
|l><\bar{l}| = \begin{bmatrix} 01 \\ 00 \end{bmatrix} = \frac{1}{2} (\hat{x}_1 + i \hat{x}_2) \cdot \vec{\sigma}
\]

\[
|\bar{l}><l| = \begin{bmatrix} 00 \\ 10 \end{bmatrix} = \frac{1}{2} (\hat{x}_1 - i \hat{x}_2) \cdot \vec{\sigma}
\]

The unit configuration is related with the body frame through the sequence of Euler rotations: \( |\xi> = V(\alpha, \beta, \delta) |l> \), \( |\bar{\xi}> = V(\alpha, \beta, \gamma) |\bar{l}> \), \( <\xi| = <l|V^+(\alpha, \beta, \gamma) \), \( <\bar{\xi}| = <\bar{l}|V^+(\alpha, \beta, \gamma) \), \( \hat{\lambda}_k \cdot \vec{\sigma} = V(\alpha, \beta, \gamma) \hat{x}_k \cdot \vec{\sigma} V^+(\alpha, \beta, \gamma) \). Transforming the outer products for the unit configuration, we obtain:

\[
|\xi><\xi| = V(\alpha, \beta, \gamma) |l><l| V^+(\alpha, \beta, \gamma) = \frac{1}{2} (1 + \hat{\lambda}_3 \cdot \vec{\sigma}) \quad (1.12a)
\]

\[
|\bar{\xi}><\bar{\xi}| = V(\alpha, \beta, \gamma) |\bar{l}><\bar{l}| V^+(\alpha, \beta, \gamma) = \frac{1}{2} (1 - \hat{\lambda}_3 \cdot \vec{\sigma}) \quad (1.12b)
\]

\[
|\xi><\bar{\xi}| = V(\alpha, \beta, \gamma) |l><\bar{l}| V^+(\alpha, \beta, \gamma) = \frac{1}{2} (\hat{\lambda}_1 + i \hat{\lambda}_2) \cdot \vec{\sigma} \quad (1.12c)
\]

\[
|\bar{\xi}><\xi| = V(\alpha, \beta, \gamma) |\bar{l}><l| V^+(\alpha, \beta, \gamma) = \frac{1}{2} (\hat{\lambda}_1 - i \hat{\lambda}_2) \cdot \vec{\sigma} \quad (1.12d)
\]

One can use these outer products to derive expressions for the unit vectors \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \). For example:
The important point here is that vectors are related to quantities quadratic in the spinor components.

The outer products listed above can be interpreted as operators on the space of spinors. We rewrite them as follows:

\[
\begin{align*}
E_3 &= |\xi><\xi| \\
\overline{E}_3 &= |\overline{\xi}><\overline{\xi}| \\
E^+ &= |\overline{\xi}<\xi| \\
E^- &= |\overline{\xi}<\xi|
\end{align*}
\]

\hspace{1cm} (1.13 a, b)

\[E_3 \text{ and } \overline{E}_3 \text{ are projection operators which act on an arbitrary spinor } |\eta> \text{ as follows:}
\]

\[
\begin{align*}
E_3 \ |\eta> &= a_1 \ |\xi> \quad \text{where } a_1 = <\xi|\eta> \\
\overline{E}_3 \ |\eta> &= a_2 \ |\overline{\xi}> \quad \text{where } a_2 = <\overline{\xi}|\eta>
\end{align*}
\]

\hspace{1cm} (1.14 a, b)

Since \( E_3 + \overline{E}_3 = \frac{1}{2} (1 + \hat{\lambda}_3 \cdot \hat{\sigma}) + \frac{1}{2} (1 - \hat{\lambda}_3 \cdot \hat{\sigma}) = 1 \), \(|\eta>\) can be expanded in terms of \(|\xi>\) and \(|\overline{\xi}>\): 
\[ |\eta> = [E_3 + \overline{E}_3] |\eta> , \]
which means that 
\[ |\eta> = a_0 |\xi> + a_1 |\overline{\xi}> . \]

\( E^+ \) and \( E^- \) operate on \(|\xi>\) and its conjugate as follows:

\[
\begin{align*}
E_+ \ |\overline{\xi}> &= |\xi> \\
E_- \ |\overline{\xi}> &= 0 \\
E_+ \ |\xi> &= 0 \\
E_- \ |\xi> &= |\xi>
\end{align*}
\]

\hspace{1cm} (1.15 a, b)
We now introduce a short hand notation for spinors as an alternative to the Euler angle parameterization. The first two Euler angles, $\alpha$ and $\beta$, define the direction in space of the preferred axis of the triad, $\hat{Z}_3$. A rotation of the triad through $\alpha$ about $\hat{Z}_3$ followed by one through $\beta$ about $\hat{Z}_2$ leave the $\hat{Z}_3$ axis pointing in the $(\alpha, \beta)$ direction where $\alpha$ is the azimuthal angle and $\beta$ the polar angle. The third angle, $\gamma$, defines the orientation of the triad about the preferred axis. We can therefore write the spinor, $|\xi\rangle$, in the axis angle form as $|\ell_3, \gamma\rangle$. It is easily seen from our expression for $|\xi\rangle$ that: $|\ell_3, \gamma\rangle = \ell^{-i\gamma/2} |\ell_3\rangle$. The conjugate spinor, $|\bar{\xi}\rangle$, is written in axis angle form as: $|\ell_3, \gamma\rangle = \ell^{i\gamma/2} |\ell_3\rangle$.

Let us determine the action of the operator $\hat{\ell}_3 \cdot \hat{\sigma}$ on $|\ell_3, \gamma\rangle$ and $|\ell_3, \gamma\rangle$. For the moment we omit the third angle, $\gamma$, from the discussion. We shall re-insert it into our results. We recall that $|\ell_3\rangle <\ell_3| = \frac{1}{2} (1 + \hat{\ell}_3 \cdot \hat{\sigma})$ and $|\ell_3\rangle <\ell_3| = \frac{1}{2} (1 - \hat{\ell}_3 \cdot \hat{\sigma})$, and we use the fact that $|\ell_3\rangle <\ell_3| \hat{\ell}_3\rangle = |\ell_3\rangle$ and $|\ell_3\rangle <\ell_3| \bar{\ell}_3\rangle = |\bar{\ell}_3\rangle$.

\[
\frac{1}{2} (1 + \hat{\ell}_3 \cdot \hat{\sigma}) |\ell_3\rangle = |\ell_3\rangle \quad \text{(1.16a)}
\]

\[
\hat{\ell}_3 \cdot \hat{\sigma} |\ell_3\rangle = |\ell_3\rangle
\]

\[
\frac{1}{2} (1 - \hat{\ell}_3 \cdot \hat{\sigma}) |\ell_3\rangle = |\bar{\ell}_3\rangle \quad \text{(1.16b)}
\]

\[
\hat{\ell}_3 \cdot \hat{\sigma} |\bar{\ell}_3\rangle = - |\bar{\ell}_3\rangle
\]
Multiplying $|k_3\rangle$ and $|\hat{k}_3\rangle$ by a phase factor does not change our results and therefore $\gamma$ can be reintroduced. Thus, $|\hat{k}_3,\gamma\rangle$ and $|\hat{k}_3,\gamma\rangle$ are eigen-spinors, for the direction operator $\hat{\gamma}$, with eigenvalues 1 and -1, respectively.

The spinor $|\xi\rangle = |\hat{k}_3,\gamma\rangle = |\alpha,\beta,\gamma\rangle$, and its conjugate, transform in a particularly simple way under rotations about the body axis, $\hat{k}_3$, and the space axis, $\hat{x}_3$:

\[
U(\hat{k}_3, \phi) |\hat{k}_3,\gamma\rangle = [\cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \hat{k}_3 \cdot \hat{g}] |\hat{k}_3,\gamma\rangle
\]

\[
= [\cos \frac{\phi}{2} - i \sin \frac{\phi}{2}] |\hat{k}_3,\gamma\rangle
\]

\[
= e^{-i\phi/2} e^{-i\gamma/2} |\hat{k}_3\rangle
\]

\[
= |\hat{k}_3,\gamma + \phi\rangle
\]

\[
U(\hat{k}_3, \phi) |\hat{k}_3,\gamma\rangle = [\cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \hat{k}_3 \cdot \hat{g}] |\hat{k}_3,\gamma\rangle
\]

\[
= [\cos \frac{\phi}{2} + i \sin \frac{\phi}{2}] |\hat{k}_3,\gamma\rangle
\]

\[
= e^{+i\phi/2} e^{+i\gamma/2} |\hat{k}_3\rangle
\]

\[
|\hat{k}_3,\gamma + \phi\rangle
\]
Unitary spinors therefore have a biaxial character where both \( \hat{\lambda}_3 \) and \( \hat{x}_3 \) play a privileged role. A rotation about \( \hat{\lambda}_3 \) changes the angle \( \gamma \) and one about \( \hat{x}_3 \) changes the angle \( \alpha \).

Finally, it should be mentioned that unitary spinors have the peculiar property that a rotation through \( 2\pi \) reverses their sign. This property is rooted in the appearance of the half angle in the matrices which induce spinor transformations. The matrix for a rotation through \( 2\pi \) about the arbitrary axis, \( \hat{u} \), is given by: \( U(\hat{u}, \frac{2\pi}{2}) = \cos \pi I - i \sin \pi \hat{u} \cdot \sigma = -I \). Therefore: \( |\xi'\rangle = U(\hat{u}, \frac{2\pi}{2}) |\xi\rangle = -|\xi\rangle \) and \( |\bar{\xi}'\rangle = -|\bar{\xi}\rangle \). However, since the
unit vectors of the triad, associated with $|\xi\rangle$ and $|\overline{\xi}\rangle$, are related with quantities quadratic in the spinor components, the factor of \(-1\) cancels out and these vectors are left unchanged by a rotation through \(2\pi\).
CHAPTER II

APPLICATIONS OF THE SPINOR FORMALISM TO
NON-RELATIVISTIC QUANTUM MECHANICS

2.1 The Quantum Mechanics of the Spin 1/2 State

Up to this point we have developed the formalism on a purely mathematical basis. Unitary spinors have been introduced within the context of a geometric problem, the parameterization of the orientation of a triad. We now make the transition to quantum mechanics by introducing a new prototype for the orientable object, the spin 1/2 state. This program shall be carried out in a series of steps. We first consider the description of stationary spin 1/2 states and measurements made on them. Then we discuss the kinematics of time dependent spin states and derive an analogue to the Schrödinger equation. A transition to dynamics follows in which spin is related to angular momentum. Finally, we extend the spinor formalism to account for states of higher angular momentum.

The transition from the triad, taken as a prototype for the orientable object, to the spin 1/2 state necessitates a new conceptual interpretation of the formalism. We recall that the conjugate spinors, $|\xi\rangle$ and $|\bar{\xi}\rangle$ were used for a complete description of the triad, but these ingredients always appeared jointly, without an independent meaning. By contrast the conjugate spinors take on a separate meaning in quantum mechanics: $|\xi\rangle$ refers to
a spin up state in the $(\alpha, \beta)$ direction and $|\bar{\xi}\rangle$ refers to a state with spin down. The third angle is irrelevant to the direction of the spin axis, and we omit it from the present discussion.

Spin in quantum mechanics differs from the triad in that one can only consider its orientation within the context of a measurement process. The component of spin in a particular direction is measured by means of the Stern-Gerlach experiment. A beam of spin $1/2$ magnetic dipoles is passed through an apparatus which contains an inhomogeneous magnetic field. This apparatus deflects the particles in one of two possible directions, thus splitting the beam in two. We do not go into the dynamical details of the Stern-Gerlach effect; this phenomenon is rooted in the fact that spin may be associated with a magnetic moment which is coupled to an external magnetic field. Rather, we treat the measurement of spin in a purely formal manner.\(^5\) The important point is that the inhomogeneous magnetic field sets up a preferred direction in space with respect to which a spin can take on two values, spin up and spin down, and the particles with these spin states are deflected in different ways.

A Stern-Gerlach apparatus can be used to produce a beam of particles in identical spin states. Now suppose that a beam in the state $|\xi\rangle = \xi_1|\uparrow\rangle + \xi_2|\downarrow\rangle$ is sent through a Stern-Gerlach apparatus which splits it into two beams, one in the
state $|l\rangle$ and the other in the state $|\bar{1}\rangle$. Each particle in the incident beam is in a state which can be thought of as a mixture, or more properly a superposition, of the two outgoing states. However, it is basic to quantum mechanics that an individual particle cannot be split into its constituent states; each particle must be considered as a unit when its spin is measured. For this reason the spinor components, which still describe orientation, must now be interpreted as probability amplitudes; $\xi_1$ and $\xi_2$ are related to the probabilities that a particle come out in the $|l\rangle$ or $|\bar{1}\rangle$ state.

The fact that the spin 1/2 state can take on two orientations, spin up and spin down, when it is sent through a measuring device renders it particularly suitable for treatment with the algebra of two-by-two matrices and two component spinors. Let us now translate the formalism developed in the first chapter into the language of quantum mechanics. We associate the Pauli matrices with the three components of the quantum mechanical operator for spin:

$$s_1 = \frac{1}{2}\sigma_1, \quad s_2 = \frac{1}{2}\sigma_2, \quad s_3 = \frac{1}{2}\sigma_3. \quad (2.1)$$

These operators obey the standard commutation relations for angular momentum: $[s_i, s_j] = i\xi_{ijk}s_k$. Spinors are now interpreted as state vectors associated with spin. The operator for spin can be defined in terms of the generator for an infinitesimal rotation of the spin 1/2 state. An infinitesimal rotation about the $\hat{k}$-axis is generated by the unitary operator:
\[ U(k, \frac{\delta \phi}{2}) = \exp[-i \frac{\delta \phi}{2} k \cdot \vec{\sigma}] \]

\[ = 1 - i \frac{\delta \phi}{2} k \cdot \vec{\sigma} \]

\[ = 1 - i \delta \phi \vec{\sigma} \cdot \hat{k}. \]

The Pauli matrices are chosen so that our representation is diagonal in the third component of \( \vec{s} \), \( s_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). We therefore choose as our basis spinors the two eigenstates of \( s_3 \):

\[ |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\bar{1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

These states are associated with the eigenvalues \( 1/2 \) and \( -1/2 \), respectively: \( s_3 |1\rangle = 1/2 |1\rangle \), \( s_3 |\bar{1}\rangle = -1/2 |\bar{1}\rangle \). We call them spin up and spin down states with respect to the \( \hat{x}_3 \) direction. The spin up and spin down states are connected by the raising and lowering operators,

\[ s_+ = |1\rangle\langle\bar{1}| = s_1 + is_2 \quad \text{and} \quad s_- = |\bar{1}\rangle\langle1| = s_1 - is_2: \]

\[ s_+ |\bar{1}\rangle = 0 \quad s_- |1\rangle = |1\rangle \quad (2.3 \ a,b) \]

\[ s_+ |1\rangle = |1\rangle \quad s_- |\bar{1}\rangle = 0 \quad (2.3 \ c,d) \]

It should also be mentioned that the operator \( \vec{s}^2 \) is diagonal:

\[ \vec{s}^2 = \frac{1}{4}(s_1^2 + s_2^2 + s_3^2) = \frac{3}{4}. \]

The results we have found for spin along the \( \hat{x}_3 \) direction can be generalized for the arbitrary direction \( \hat{k} \). The operator \( \vec{s} \cdot \hat{k} = \frac{1}{2} \hat{k} \cdot \vec{\sigma} \) has two eigenstates \( |\hat{k}\rangle \) and \( |\bar{k}\rangle \) with eigenvalues \( 1/2 \) and \( -1/2 \), respectively. In addition, raising and lowering operators can be defined: \( s_+ = |\hat{k}\rangle\langle\hat{k}| \) and \( s_- = |\bar{k}\rangle\langle\bar{k}|. \)
Let us now go back to the problem of how spin is measured. A Stern-Gerlach apparatus with its preferred axis in the \( \hat{k} \)-direction can be described by the projection operators:
\[
P_k^- = |\hat{k}\rangle \langle \hat{k}| \quad \text{and} \quad P_k^+ = |\tilde{k}\rangle \langle \tilde{k}|
\]
A spin in the state \( |\eta\rangle \) will emerge from this apparatus in the state \( |\hat{k}\rangle \) or \( |\tilde{k}\rangle \) with the respective probabilities given by:
\[
\langle \eta | P_k^- | \eta \rangle = |\langle \hat{k} | \eta \rangle|^2 \quad \text{and} \quad \langle \eta | P_k^+ | \eta \rangle = |\langle \tilde{k} | \eta \rangle|^2.
\]
When an ensemble of such spins is sent through this apparatus all the spins projected into the state \( |\hat{k}\rangle \) will contribute a value of \(+1/2\) to the expectation value \( \langle \hat{s} \cdot \hat{k} \rangle \) and all those projected into the \( |\tilde{k}\rangle \) state will contribute a value of \(-1/2\). The average value of spin in the \( \hat{k} \) direction for this ensemble is therefore:
\[
\langle \hat{s} \cdot \hat{k} \rangle = \frac{1}{2} \langle \eta | P_k^- | \eta \rangle - \frac{1}{2} \langle \eta | P_k^+ | \eta \rangle \quad (2.4)
\]
\[
= \langle \eta | \frac{1}{2} (P_k^- - P_k^+) | \eta \rangle
\]
\[
= \langle \eta | \frac{1}{2} (1 - \hat{k} \cdot \hat{\sigma}) - \frac{1}{2} (1 + \hat{k} \cdot \hat{\sigma}) | \eta \rangle
\]
\[
= \langle \eta | \frac{1}{2} \hat{k} \cdot \hat{\sigma} | \eta \rangle
\]
\[
= \langle \eta | \hat{s} \cdot \hat{k} | \eta \rangle
\]

It is instructive to consider as an example the measurement of the average value of \( s_3 \) for an ensemble of spins in the
state $|\lambda_3\rangle$. The measurement of this would be carried out with a Stern-Gerlach apparatus oriented along the $\hat{x}_3$-axis. The respective probabilities for a particle to be projected into the state, $|1\rangle$ or $|\bar{1}\rangle$, are given by:

$$\text{prob (spin up)} = \langle \lambda_3 | 1 \rangle \langle 1 | \lambda_3 \rangle$$

$$= (\lambda^{i\alpha/2}\cos\beta/2, \lambda^{-i\alpha/2}\sin\beta/2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^{i\alpha/2}\cos\beta/2 \\ \lambda^{i\alpha/2}\sin\beta/2 \end{pmatrix}$$

$$= \cos^2 \beta/2$$

$$\text{prob (spin down)} = \langle \lambda_3 | \bar{1} \rangle \langle \bar{1} | \lambda_3 \rangle$$

$$= \cos^2 \beta/2$$

Therefore: $\langle s_3 \rangle = (\frac{1}{2})\cos^2 \beta/2 + (-\frac{1}{2})\sin^2 \beta/2 = \frac{1}{2}\cos \beta$.

Notice that the half angle, which plays an integral role in the spinor formalism, appears in the expressions for the probabilities that $|\lambda_3\rangle$ be projected into the spin up or down state. However, only the full angle, $\beta$, appears in the expression for the expectation value of spin. $\frac{1}{2}\cos \beta$ is just equal to the $x_3$-component of a length $\frac{1}{2}$ vector in the direction of the spin axis, $\hat{x}_3$. This example clarifies the role that spinors and vectors play in describing spin $\frac{1}{2}$ states. A spinor is associated with the quantum mechanical state which gives the probabilities for the various possible outcomes of a measurement process. When the expectation value of a component of spin is
taken, a vector can be associated with the spin axis.

2.2 Time Dependent Spin States

The phenomena we have considered involve measurements taken on stationary spin states. Only two angles are required to specify the orientation of the spin axis. In order to consider the time evolution of a spin state we must reintroduce the third angle and express it as a frequency, $\omega$, multiplied by time: $\gamma = \omega t$. We thus describe a spin state with the time dependent spinor:

$$|\xi(t)\rangle = \begin{pmatrix} e^{-i\alpha/2 \cos \beta /2} & e^{-i\omega t/2} \\ e^{i\alpha/2 \sin \beta /2} & \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix}$$

This spinor can be interpreted as a two dimensional isotropic, harmonic oscillator. The two spinor components, $\xi_1(t)$ and $\xi_2(t)$, oscillate with the common frequency, $\omega$. These modes of oscillation have a phase difference of $\alpha$, and their respective amplitudes are $\cos \beta /2$ and $\sin \beta /2$. Therefore, the angles $\alpha$ and $\beta$, which describe the orientation in space of the spin axis, also determine the configuration space of an oscillator. The isotropic, harmonic oscillator is only a physical model which arises from our mathematical description of the time dependent spinor. However, it shall become apparent in what follows that
this mathematical description is useful in dealing with physical phenomena such as the spin procession. This writer suspects, therefore, that spin may be associated with a process of oscillation. Much more work needs to be done to understand the underlying dynamics of this oscillation, but that carries us beyond the scope of this thesis.

In dealing with time dependent spinors it is convenient to work in our short hand notation. We choose as our basis set the spinors: |k,0> and |k',0>. Then, as we derived in the first chapter:

\[ |\hat{k},\omega t\rangle = U(\hat{k}, \frac{\omega t}{2})|k,0\rangle \]

\[ = \lambda^{-i\omega t/2}|\hat{k},0\rangle \]

and

\[ |\hat{k},\omega t\rangle = U(\hat{k}, \frac{\omega t}{2})|\hat{k},0\rangle \]

\[ = \lambda^{+i\omega t}|\hat{k},0\rangle \]

An arbitrary time dependent spinor, |\eta(t)\rangle, can be written as the linear combination:

\[ |\eta(t)\rangle = a_1 \lambda^{-i\omega t/2}|\hat{k},0\rangle + a_2 \lambda^{i\omega t/2}|\hat{k},0\rangle, \]

where

\[ a_1 = \langle \hat{k},0|\eta(0) \rangle \quad \text{and} \quad a_2 = \langle \hat{k},0|\eta(0) \rangle \]
We now seek a differential equation for the time dependence of $|\eta(t)>$. Infinitesimal time displacements are generated by the unitary operator:

$$U(k, \frac{\omega dt}{2}) = \exp[-i \frac{\omega dt}{2} \hat{k} \cdot \vec{\sigma}]$$

$$= 1 - i \frac{\omega dt}{2} \hat{k} \cdot \vec{\sigma}$$

The choice of the axis, $\hat{k}$ depends on the physical situation. We proceed by operating on $|\eta(t)>$ with the infinitesimal time displacement operator:

$$|\eta(t + dt)> = U(k, \frac{\omega dt}{2}) |\eta(t)>$$

$$= [1 - i \frac{\omega dt}{2} \hat{k} \cdot \vec{\sigma}] |\eta(t)>$$

$$|\eta(t + dt)> - |\eta(t)> = - i \frac{\omega}{2} \hat{k} \cdot \vec{\sigma} |\eta(t)> dt$$

$$i |\eta(t)> \frac{\omega}{2} \hat{k} \cdot \vec{\sigma} |\eta(t)>$$

(2.7)

This is the analogue to Schrödinger's equation where the Hamiltonian is: $H = \frac{\omega}{2} \hat{k} \cdot \vec{\sigma}$. A similar equation can be derived for the bra spinor, $<\eta(t)|$
\[
\langle \eta(t + dt) \rangle = \langle \eta(t) | u^+(k, \frac{\omega dt}{2}) = \langle \eta(t) | \left[ 1 + i \frac{\omega dt}{2} \hat{k} \cdot \hat{\sigma} \right] \langle \eta(t) \rangle - r(t) | = \langle \eta(t) | i \frac{\omega}{2} \hat{k} \cdot \hat{\sigma} \rangle \] 

The bra and ket spinors, \(|\eta(t)\rangle\) and \(\langle \eta(t) |\), are related with the spin axis as follows:

\[
|\eta(t)\rangle \langle \eta(t) | = S(t) = \frac{1}{2} (1 + \hat{s}(t) \cdot \hat{\sigma}) \quad (2.8)
\]

The Schrödinger-like equations can be used to derive a differential equation for the time dependence of the spin axis:

\[
\dot{\hat{s}}(t) = \frac{1}{2} \hat{s}(t) \cdot \hat{\sigma} = |\eta(t)\rangle \langle \eta(t) | + |\eta(t)\rangle \langle \eta(t) | \dot{\hat{s}}(t) | = -i \frac{\omega}{2} \hat{k} \cdot \hat{\sigma} |\eta\rangle \langle \eta | + i |\eta\rangle \langle \eta | \frac{\omega}{2} \hat{k} \cdot \hat{\sigma}
\]

Since \(|\eta\rangle \langle \eta | = \frac{1}{2} (1 + \hat{s}(t) \cdot \hat{\sigma})\) and the commutator \([1, \frac{\omega}{2} \hat{k} \cdot \hat{\sigma}] = 0\), we have:

\[
\dot{\hat{s}}(t) \cdot \hat{\sigma} = i[\hat{s}(t) \cdot \hat{\sigma}, \frac{\omega}{2} \hat{k} \cdot \hat{\sigma}] = \hat{s}(t) \cdot \hat{\sigma} \quad (2.9)
\]
2.3 The Transition from Quantum Kinematics to Dynamics

We have obtained a kinematic analogue to the Schrödinger equation by considering infinitesimal time displacements of the state \( |\eta(t)\rangle \). In our Schrödinger-like equation the Hamiltonian has the dimensions of frequency. We can translate the kinematic Schrödinger-like equation into a dynamic one by multiplying it by Planck's constant \( \hbar \).

\[
\text{i}\hbar \frac{\partial}{\partial t} |\eta\rangle = i\hbar \frac{\omega}{2} \hat{k} \cdot \hat{\sigma} |\eta\rangle
\]  

(2.10)

In addition, we associate spin with angular momentum through Planck's constant: \( \hat{J} = \hbar \hat{\sigma} = \frac{\hbar}{2} \hat{\sigma} \). Then the Hamiltonian associated with spin is: \( H = \omega \hat{J} \cdot \hat{k} \).

In classical mechanics angular momentum is related to angular velocity about a spin axis through the moment of inertia. This physical interpretation does not apply in quantum mechanics because one cannot define an angular velocity associated with spin. However the angular momentum of a spin state does have observable consequences when it is linked with a magnetic moment. It is an experimental fact that a charge distribution with angular momentum has a magnetic moment given by: \( \hat{\mu} = \gamma \hat{J} \), where \( \gamma \) is a phenomenological constant. This relationship also holds in quantum mechanics where \( \hat{\mu} \) and \( \hat{J} \) are interpreted as operators. The Hamiltonian associated with a magnetic moment is: \( H = - \hat{\mu} \cdot \hat{B} \). If we let the magnetic field be directed in the \( \hat{k} \)-direction the Hamiltonian is
\[ H = - B \gamma \vec{j} \cdot \hat{k} \]
\[ B \hat{k} = \hat{B} \]
\[ = - \hbar \frac{B \gamma}{2} \hat{k} \cdot \vec{\sigma} . \]

We recognize that this Hamiltonian has the form \( H = \hbar \frac{\omega}{2} \hat{k} \cdot \vec{\sigma} \) where the frequency is defined as: \( \omega = -\gamma B \). This Hamiltonian has the eigenstates \( |\hat{k}, \frac{\omega t}{2} \rangle \) and \( |\hat{k}, \frac{\omega t}{2} \rangle \) with eigenvalues \( \frac{1}{2} \hbar \omega \) and \( -\frac{1}{2} \hbar \omega \), respectively. Therefore, while we cannot ascribe an intrinsic frequency to spin, we can define a "beat frequency" between spin up and spin down states in a magnetic field.

Let us now use these results to derive a relation for the time evolution of spin axis \( \hat{s}(t) \). Recalling our equation for the time rate of change of the spin axis.

\[ \hat{s}(t) = i \left[ \hat{s}(t), \frac{\omega}{2} \hat{k} \cdot \vec{\sigma} \right] \]
\[ = i \frac{\omega}{2} \left[ \hat{s} \cdot \vec{\sigma}, \hat{k} \cdot \vec{\sigma} \right] \]
\[ = i \frac{\omega}{2} \left( 2i (\hat{s} \times \hat{k}) \cdot \vec{\sigma} \right) \]
\[ = - \omega (\hat{s} \times \hat{k}) \cdot \vec{\sigma} \]
\[ \hat{s} = - \omega \hat{s} \times \hat{k} \quad (2.11) \]

Therefore the spin axis precesses about the magnetic field with a frequency \( |\omega| = \gamma B \).
2.4 Extension of the Formalism to States of Higher Angular Momentum

We have found the two component spinor formalism to be well-suited for a description of the spin 1/2 state. The effectiveness of this formalism is rooted in the fact that the spin 1/2 particle is a two state system when coupled with an external field.* An arbitrary spin 1/2 state can be expressed as the superposition of two conjugate basis states representing spin up and spin down with respect to a given direction in space.

The question now arises as to how one can use our formalism to deal with states of higher angular momentum. How can our two component formalism be extended to account for states which take on more than two configurations when coupled with an external field? In what follows, a procedure shall be outlined for using the algebra of spin 1/2 to build up states of higher angular momentum. This procedure is well known within the context of the theory of group representations. It is a standard method for building up unitary representations of the rotation group. We shall attach an intuitive interpretation to this procedure.

We recall that our spin 1/2 representation is spanned by two basis states representing spin up and spin down. We now interpret these basis states as the two components of a wave

*This has been extensively demonstrated by Richard Feynman.\textsuperscript{6,7}
function describing the motion of a spin:

\[ |\Psi(\vec{x},t)\rangle = \begin{pmatrix} u(\vec{x},t) \\ v(\vec{x},t) \end{pmatrix} = u(\vec{x},t) |1\rangle + v(\vec{x},t) |\overline{1}\rangle \quad . \tag{2.12} \]

The quantities

\[ |u(\vec{x},t)|^2 \text{ and } |v(\vec{x},t)|^2 \]

give the probability densities for a spin to be found in the up or down state, respectively, at the point \((\vec{x},t)\). The total probability density for a spin to be found at \((\vec{x},t)\) is therefore:

\[ \langle \Psi(\vec{x},t) | \Psi(\vec{x},t) \rangle = |u(\vec{x},t)|^2 + |v(\vec{x},t)|^2 . \]

This quantity, which we write as \(uu^* + vv^*\), must be invariant under spacial rotations.

To find the basis functions for states of higher angular momentum \(j\), we raise this invariant to the \(2j\) th power:

\[ (uu^* + vv^*)^{2j} = \sum_{n=1}^{2j} \binom{2j}{n} u^{2j-n} v^n u^{*2j-n} v^{*n} \]

Dividing through by \((2j)!\) and relabeling the terms of the sum:

\[ \frac{1}{(2j)!} (uu^* + vv^*)^{2j} = \sum_{m=-j}^{j} \frac{1}{(j + m)! (j - m)!} u^{(j+m)} v^{(j-m)} u^{*j+m} v^{*j-m} \quad . \tag{2.13} \]

This sum contains \(2j + 1\) terms each of which is associated with a probability density. Again the sum of these probability densities must be invariant under rotations. Thus, we write the \(2j + 1\) basis functions for angular momentum \(j\) in the form:
\[ \phi^j_m = \frac{1}{\sqrt{(j + m)! (j - m)!}} u^{(j+m)} v^{(j-m)}. \]

The state described by \( \phi^j_m \) is interpreted, in our approach, as a collection of \( 2j \) spin \( 1/2 \) excitations, \( j + m \) of which are in the state \( u \) and \( j - m \) of which are in the state \( v \). The importance of the suffix \( m \) shall become apparent shortly.

The angular momentum operator is now given by:

\[ J = \sum_{n=1}^{2j} \hat{S}^{(n)} = \frac{1}{2} \sum_{n=1}^{2j} \hat{\sigma}^{(n)}. \]

The commutation relations for the components of \( \hat{J} \) can be readily found from the commutation relations for the components of \( \hat{S} \):

\[ [J_i, J_k] = \sum_{n=1}^{2j} [s_i^{(n)}, s_k^{(n)}] = i \sum_{n=1}^{2j} \varepsilon_{ikl} s_l^{(n)} = i \varepsilon_{ikl} J_l. \]

(2.14)

Recalling that \( s_3 U = \frac{1}{2} U \) and \( s_3 V = -\frac{1}{2} V \), we now consider the action of the operator \( j_3 \) on the basis function \( \phi^j_m \):

\[ j_3 \phi^j_m = \sum_{n=1}^{2j} s_3^{(n)} \left\{ \frac{1}{\sqrt{(j + m)! (j - m)!}} u^{(j+m)} v^{(j-m)} \right\} \]

\[ = \left\{ \frac{1}{2} (j + m) - \frac{1}{2} (j - m) \right\} \phi^j_m \]

(2.15)

\[ = m \phi^j_m \]

*We are now working in units where \( \hbar = 1 \).*
The quantum number $m$ therefore denotes the eigenvalue of $J_3$ associated with the eigenfunction $\phi_m^j$.

The eigenfunctions $\phi_m^j$ can be connected with their adjacent states $\phi_{m+1}^j$ and $\phi_{m-1}^j$ through the step operators $J_+$ and $J_-$. These operators are defined in terms of the spin raising and lowering operators, $s_+ = s_1^1 + i s_2^1$ and $s_- = s_1^1 - i s_2^1$:

\[
J_+ = \sum_{n=1}^{2j} s_+^{(n)} = J_1^1 + i J_2^1 \\
J_- = \sum_{n=1}^{2j} s_-^{(n)} = J_1^1 - i J_2^1
\] (2.16a)

Each term of these two sums operates on a separate excitation. We recall the action of $s_+$ and $s_-$ on the states $u$ and $v$:

\[
s_+ v = u \quad s_- v = 0 \\
s_+ u = 0 \quad s_- u = v
\]

Now consider how $J_+$ acts on $\phi_m^j$:

\[
J_+ \phi_m^j = \sum_{n=1}^{2j} s_+^{(n)} \frac{1}{\sqrt{(j + m)! (j - m)!}} u^{(j+m)} v^{(j-m)}
\]

Since $\phi_m^j$ contains $j-m$ excitations in the state $v$, the above sum will have $j-m$ non-zero terms, each with a $v$-state raised to a $u$-state:
\[ J_+ \phi^j_m = (j-m) \frac{1}{\sqrt{(j+m)! (j-m)!}} u^{(j+m+1)} v^{(j-m-1)} \]  
\begin{align*}
&= (j-m) \sqrt{(j+m+1)! (j-m-1)!} \frac{u^{(j+m+1)} v^{(j-m-1)}}{\sqrt{(j+m+1)! (j-m-1)!}} \\
&= (j-m) \sqrt{\frac{j+m+1}{j-m}} \phi^j_{m+1} \\
&= \sqrt{(j+m+1)(j-m)} \phi^j_{m+1}
\end{align*}

Similarly:
\[ J_- \phi^j_m = \sum_{n=1}^{2j} s_n^{(n)} \frac{u^{(j+m)} v^{(j-m)}}{\sqrt{(j+m)! (j-m)!}} \]  
\begin{align*}
&= \frac{j+m}{\sqrt{(j+m)! (j-m)!}} u^{(j+m-1)} v^{(j-m+1)} \\
&= (j+m) \sqrt{(j+m-1)! (j-m+1)!} \phi^j_{m-1} \\
&= \sqrt{(j-m+1)(j+m)} \phi^j_{m-1}
\end{align*}

It should be noted that \( J_+ \phi^j_m = 0 \) and \( J_- \phi^j_{-j} = 0 \), \( m \) is restricted to take on integral or half integral values between
The $\phi^j_m$'s are eigenfunctions of the operator $\hat{J}^2 = \hat{J} \cdot \hat{J}$ with eigenvalues $j(j+1)$. To show this we find an expression for $\hat{J}^2$ in terms of operators whose action on $\phi^j_m$ has already been shown. The operator $J_- J_+$ operates on $\phi^j_m$ as follows:

$$J_- J_+ \phi^j_m = J_- \sqrt{(j+m+1)(j-m)} \phi^j_{m+1}$$

$$= (j+m+1)(j-m) \phi^j_m$$

Note that:

$$J_- J_+ = (J_1 - iJ_2)(J_1 + iJ_2)$$

$$= J_1^2 + J_2^2 + i[J_1, J_2]$$

$$= J_1^2 + J_2^2 - J_3$$

Therefore:

$$\hat{J}^2 = J_1^2 + J_2^2 + J_3^2 = J_- J_+ + J_3^2 + J_3^3.$$ (2.18)

This operator acts on $\phi^j_m$ as follows:

$$J^2 \phi^j_m = [J_- J_+ + J_3^2 + J_3] \phi^j_m$$ (2.19)

$$= j(j+1) \phi^j_m$$

Let us now consider the behavior of the basis functions $\phi^j_m$ under rotations. We recall that a rotation of the spin 1/2 state is generated by the operator:
\[ U(\hat{u}, \frac{\phi}{2}) = \exp[-i \frac{\phi}{2} \hat{u} \cdot \hat{\sigma}]. \] We rewrite this operator as:

\[ U(\hat{u}, \phi) = \exp[-i \hat{\sigma} \cdot \hat{u}]. \] The rotation of an angular momentum state should be generated by the operator:

\[ U(j) = \exp[-i \frac{\phi}{2} \hat{J} \cdot \hat{u}] \]

\[ = \exp[-i \frac{\phi}{2} \sum_{n=1}^{2j} \hat{u} \cdot \hat{\sigma}(n)] \quad (2.20a) \]

This operator can be written as the product:

\[ U(j) = \prod_{n=1}^{2j} \exp[-i \frac{\phi}{2} \hat{u} \cdot \hat{\sigma}(n)] \quad (2.20b) \]

Each of the above factors operates on an independent excitation. Therefore, to induce a rotation of the state

\[ \phi_j^m = \frac{1}{\sqrt{(j+m)! (j-m)!}} \ u(j+m) \ v(j-m), \quad (2.21) \]

one operates on each excitation sending \( u \) into \( u' = u a_{11} + v a_{21} \) and \( v \) into \( v' = u a_{12} + v a_{22} \). Then the rotated state is given by:

\[ \phi_j'^{m'} = \frac{1}{\sqrt{(j+m)! (j-m)!}} \ u'(j+m) \ v'(j-m) \]

\[ = \frac{1}{\sqrt{(j+m)! (j-m)!}} \ (u a_{11} + v a_{21})(j+m) (u a_{12} + v a_{22})(j-m) \]
The $a_{k\ell}$ are matrix elements for $\exp[-i \frac{\hat{\phi}}{2} \cdot \hat{\tau}]$.

To demonstrate this method for a concrete case we consider the problem of parameterizing the orientation of a spin 1 object with the 3 Euler angles. The eigenfunctions for angular momentum 1 are:

$$\phi_1^1 = \frac{1}{\sqrt{2}} u^2, \quad \phi_0^1 = uv, \quad \phi_{-1}^1 = \frac{1}{\sqrt{2}} v^2 \quad (2.23)$$

A rotation through the three Euler angles sends $u$ into $u' = \xi_1 u + \xi_2 v$ and $v$ into $v' = -\xi_2^* u + \xi_1^* v$, where

$$\xi_1 = \ell^{-i\alpha/2} \cos \beta/2 \; \ell^{-i\alpha/2} \quad \text{and} \quad \xi_2 = \ell^{+i\alpha/2} \sin \beta/2 \; -i\alpha/2.$$ The rotated angular momentum 1 eigenfunctions are therefore:

$$\phi_1^{1'} = \frac{1}{\sqrt{2}} (\xi_1 u + \xi_2 v)^2$$

$$= \frac{1}{\sqrt{2}} (\xi_1^2 u^2 + 2\xi_1 \xi_2 uv + \xi_2^2 v^2) \quad (2.24a)$$

$$= \xi_1^2 \phi_1^1 + \sqrt{2} \xi_1 \xi_2 \phi_0^1 + \xi_2^2 \phi_{-1}^1$$

$$\phi_0^{1'} = (\xi_1 u + \xi_2 v) (-\xi_2^* u + \xi_1^* v) \quad (2.24b)$$

$$= -\xi_1 \xi_2^* u^2 + (|\xi_1|^2 - |\xi_2|^2) uv + \xi_1^* \xi_2 v^2$$

$$= -\sqrt{2} \xi_1 \xi_2^* \phi_1^1 + (|\xi_1|^2 - |\xi_2|^2) \phi_0^1 + \sqrt{2} \xi_1^* \xi_2 \phi_{-1}^1.$$
\[
\phi_1' = \frac{1}{\sqrt{2}} (-\xi_2^* \ u + \xi_1^* \nu)^2 \\
= \frac{1}{\sqrt{2}} (\xi_2^* \ u^2 - 2 \xi_1^* \xi_2^* \ uv + \xi_1^* \nu^2) \\
= \xi_2^* \ (\phi_1' - \sqrt{2} \xi_1^* \phi_1 + \xi_1^* \phi_1') + \xi_1^* \phi_1 
\]

We can represent the angular momentum basis states as three component column vectors:

\[
\phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \phi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

A rotation of these basis states through the Euler angles is effected by the matrix:

\[
V^{(1)}(\alpha, \beta, \gamma) = \begin{pmatrix}
\xi_1^2 & -\sqrt{2} \xi_1 \xi_2^* & \xi_2^2 \\
\sqrt{2} \xi_1 \xi_2 & |\xi_1|^2 - |\xi_2|^2 & -\sqrt{2} \xi_1^* \xi_2^* \\
\xi_2^2 & \sqrt{2} \xi_1^* \xi_2^* & \xi_1^2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\lambda^{-i\alpha} & \frac{1+\cos\beta}{2} & \frac{-i\alpha \sin}{\sqrt{2}} & \frac{-i\alpha \cos}{2} & \lambda^{i\gamma} \\
\frac{-i\gamma \sin}{\sqrt{2}} & \cos\beta & \frac{-i\gamma \sin}{\sqrt{2}} & \lambda^{i\alpha} & \frac{1-\cos\beta}{2} \\
\frac{i\alpha \sin}{\sqrt{2}} & \frac{i\alpha \cos}{2} & \lambda^{-i\gamma} & \frac{i\alpha \sin}{\sqrt{2}} & \frac{i\alpha \cos}{2}
\end{pmatrix}
\]
We can write this matrix as the juxtaposition of three column vectors, each of which describe the orientation of an object with angular momentum \( l \). Notice that only the full angles \( \alpha, \beta, \) and \( \gamma \) appear in this representation. In general, the half angles will appear only in representations for half-integral angular momentum.
CHAPTER III
RELATIVISTIC SPINORS AND THEIR APPLICATIONS
TO RELATIVISTIC QUANTUM MECHANICS

3.1 Introduction

In the past several sections we have restricted ourselves to a consideration of unitary spinors and their applications to non-relativistic quantum mechanics. Unitary spinors are defined by the normalization condition: $\langle \xi | \xi \rangle = 1$. They can only transform meaningfully under pure rotations, which are represented by unitary matrices, in order that their normalization be preserved. Unitary spinors are therefore quite useful in dealing with non-relativistic phenomena in quantum mechanics but are not sufficient for use in relativistic quantum mechanics, which considers objects that transform under the Lorentz group.

In this chapter we introduce relativistic spinors and present some of their applications to relativistic quantum mechanics. The theory of relativistic spinors was first introduced by Van der Waerden in connection with the Dirac equation. The problem was that the four component Dirac wave functions did not fit into the tensor formalism based on Minkowski's space-time. Van der Waerden remedied the situation by introducing a complex spinor space along with a use of indices setup to resemble those in the traditional tensor calculus of relativity theory. Tisza has recently introduced an alternative approach in which the theory
of relativistic spinors is developed in a manner harmonious with our treatment of unitary spinors. In Tisza's formalism, relativistic spinors are written in the same bra-ket notation which we have been using, and Lorentz transformations are represented by the matrices derived in section 1.3.* In addition this method makes use of the involution group together with the operation of spin conjugation. This method seems to obtain equivalent results to the standard method within the domain of special relativity and relativistic quantum mechanics, even while maintaining the basic unity of the formalism of quantum mechanics. The Van der Waerden formalism has been applied also to general relativity, while no attempt has been made as yet to extend the present formalism in this direction. In this chapter we shall apply Tisza's method to the description of the motion of neutrinos and anti-neutrinos and give an intuitive motivation for the Dirac equation.

3.2 The Properties of Relativistic Spinors

Relativistic spinors can be defined formally by their relationship with null four vectors. We recall that the outer product between a unitary spinor and its hermitian adjoint defines a unit

*Remember that an arbitrary Lorentz transformation can be expressed as the composition of a pure rotation and a boost: \( V = H U \). This is a result of the polar decomposition theorem. See Tisza [2] pp. 18-20.
Similarly, we equate the outer product of a relativistic spinor and its adjoint with one half times the matrix for a real four vector:

\[ |\eta><\eta| = \frac{1}{2} A = \frac{1}{2} (a_o \ I + \hat{a} \cdot \hat{\sigma}) \]  

(3.1)

Notice that the four vector \( (a_o, \ a) \) is null because the determinant of \( |\eta><\eta| \) is zero:

\[
|\eta><\eta| = \begin{pmatrix} |\eta_1|^2 & \eta_1 \eta_2^* \\ \eta_1^* \eta_2 & |\eta_2|^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_o + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_o - a_3 \end{pmatrix}
\]

therefore \( 2 \det(|\eta><\eta|) = a_o^2 - a_1^2 - a_2^2 - a_3^2 = 0 \).

We call it a null vector. In analogy with our relation \( |\xi><\xi| = \frac{1}{2} (1 - \hat{\omega} \cdot \hat{\sigma}) \) for unitary spinors, we can define the conjugate spinor \( |\bar{\eta}| \) by the relation:

\[ |\bar{\eta}><\eta| = \frac{1}{2} \bar{A} \]

\[ = \frac{1}{2} \left( a_o \ I - \hat{a} \cdot \hat{\sigma} \right) \]

(3.2)

Expressions for the transformation of relativistic spinors and their conjugates can be derived from the transformation properties of four vectors:
\[ \frac{1}{2} \mathbf{A}' = \frac{1}{2} V \mathbf{A} V^\dagger \]

\[ |\eta'\rangle \langle \eta'| = V |\eta\rangle \langle \eta| V^\dagger \]

This is satisfied if

\[ |\eta'\rangle = V |\eta\rangle \text{ and } \langle \eta'| = \langle \eta| V^\dagger \] (3.3)

Therefore the Lorentz transformation of a null four vector can be dissociated into two spinor transformations and, again, the half angles, both circular and hyperbolic, come into play. If \( \mathbf{A} \) transforms according to the matrix \( V \) then \( \bar{\mathbf{A}} \) transforms according to the complex reflected matrix \( \bar{V} \):

\[ \frac{1}{2} \bar{\mathbf{A}}' = \frac{1}{2} \bar{V} \bar{\mathbf{A}} \bar{V}^\dagger \]

\[ |\bar{\eta}'\rangle \langle \bar{\eta}'| = \bar{V} |\bar{\eta}\rangle \langle \bar{\eta}| \bar{V}^{-1} \]

Therefore

\[ |\bar{\eta}'\rangle = \bar{V} |\bar{\eta}\rangle \text{ and } \langle \bar{\eta}'| = \langle \bar{\eta}| \bar{V}^{-1}. \] (3.4)

The transformation properties of the conjugate spinor \( |\bar{\eta}\rangle \) can also be derived by considering the operation of spin conjugation: \(-i\sigma_2 \kappa\). Suppose, again, that \( |\eta\rangle \) transforms according to the matrix \( V \): \( |\eta'\rangle = V |\eta\rangle \). Then, taking the spin conjugate of both sides of this equation:
\( -i\sigma_2 \langle \eta' | = -i\sigma_2 [\kappa| \eta >] \)

\[ |\eta' > = -i\sigma_2 (\kappa| \eta >) \]

\[ = [-i\sigma_2 (\kappa| \eta >) i\sigma_2 ] - i\sigma_2 |\eta > \]

\[ = [-i\sigma_2 (\kappa| \eta >) i\sigma_2 ] |\bar{\eta} > \]

To evaluate \(-i\sigma_2 (\kappa| \eta >) i\sigma_2 \) we note that if \( V = v_0 1 + \vec{v} \cdot \vec{\sigma} \),

then: \( \kappa|V > = v_0^* 1 + v_1^* \sigma_1 + v_2^* \sigma_2 + v_3^* \sigma_3 \)

\[ = v_0^* 1 + v_1^* \sigma_1 - v_2^* \sigma_2 + v_3^* \sigma_3 \]

\[ -i\sigma_2 (\kappa| \eta >) i\sigma_2 = \sigma_2 (\kappa| \eta >) \]

\[ = \sigma_2 [v_0^* 1 + v_1^* \sigma_1 - v_2^* \sigma_2 + v_3^* \sigma_3] \sigma_2 \]

\[ = \sigma_2^2 [v_0^* 1 - v_1^* \sigma_1 - v_2^* \sigma_2 - v_3^* \sigma_3] \]

\[ = v_0^* 1 - v_1^* \sigma_1 - v_2^* \sigma_2 - v_3^* \sigma_3 \]

\[ = \bar{V} . \]

Therefore: \( |\bar{\eta} ' > = \bar{V} |\bar{\eta} > \), the conjugate spinor transforms according to the complex reflected matrix.
It is of interest in relativity to consider quantities which are invariant under Lorentz transformations. One such invariant is the inner product of a conjugate spinor and a spinor
\[
\langle \chi | \eta \rangle = x_1 \eta_2 - x_2 \eta_1:
\]
\[
\langle \chi | \eta' \rangle = \langle \chi | y^{-1} v | \eta \rangle
\]
\[
= \langle \chi | \eta \rangle.
\]
(3.5)

In the Van der Waerden formalism this invariant is written as:
\[ x_1 \eta^2 + x_2 \eta^2. \]

3.3 The Motion of the Spin 1/2 Null Particles

Let us apply the relativistic spinor formalism to the task of describing the motion of spin 1/2 null particles. Null particles are characterized by the fact that they travel at the speed of light in all inertial reference frames, they cannot be transformed to rest. (See also Tisza\textsuperscript{2} III.2). The wave four vector \((k_0, \vec{k})\) is introduced to describe the motion of a null particle through space-time; \(k_0 = \frac{\omega}{c}\) and \(|\vec{k}| = \frac{2\pi}{\lambda}\), where \(\lambda\) and \(\omega\) are the wave length and angular frequency of a wave. For a null particle the wave four vector must satisfy the conditions: \(k_0^2 - k_1^2 - k_2^2 - k_3^2 = 0, k_0 \geq 0\). The wave function describing the motion of a null particle will be some function times the phase factor: \(\exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})].\)
Null particles with spin have an additional degree of freedom known as helicity. Helicity is defined by the relative orientation of the spin axis and wave vector:

\[
\text{helicity} = \frac{\hat{s} \cdot \hat{k}}{|\hat{s}| |\hat{k}|}
\]  

(3.6)

where \(\hat{s}\) is a vector rather than an operator in this equation. It can take on two values for spin 1/2 null particles:

+1 when \(\hat{s}\) is parallel to \(\hat{k}\) and -1 when \(\hat{s}\) is anti-parallel to \(\hat{k}\). Null particles with spin 1/2 are realized in nature as neutrinos and anti-neutrinos. (Actually there are two varieties of neutrino, anti-neutrino pairs but this has not been accounted for.) It is known that the neutrino has negative helicity and the anti-neutrino has positive helicity.

Since the neutrino and anti-neutrino have spin 1/2, we would expect each to be describable by a two component spinor. For the moment we denote the spinors associated with the neutrino and anti-neutrino by \(|\gamma\rangle\) and \(|\alpha\rangle\), respectively, and do not distinguish between conjugate spinors. Since \(|\hat{k}| = k_0\), we have from the definition of helicity:

\[
k_0 = \frac{\hat{s} \cdot \hat{k}}{|\hat{s}|}, \text{ for the anti-neutrino}
\]

and

\[
k_0 = -\frac{\hat{s} \cdot \hat{k}}{|\hat{s}|}, \text{ for the neutrino.}
\]
If we interpret \( \hat{s} \) as an operator, again, this means that:

\[
k_o |a> = \hat{k} \cdot \hat{\sigma} |a>, \text{which implies that } [k_o 1 - \hat{k} \cdot \hat{\sigma}] |a> = 0 \quad (3.7a)
\]
\[
k_o |\gamma> = \hat{k} \cdot \hat{\sigma} |\gamma>, \text{which implies that } [k_o 1 + \hat{k} \cdot \hat{\sigma}] |\gamma> = 0 \quad (3.7b)
\]

These equations are very suggestive of the formalism we have developed. Notice that \( k_o 1 - \hat{k} \cdot \hat{\sigma} \) and \( k_o 1 + \hat{k} \cdot \hat{\sigma} \) can be interpreted as projection operators. If we redefine \( |\gamma> \) as \( |\bar{\gamma}> \) and \( |a> \) as \( |\kappa> \) and let \( |\kappa><\kappa| = \frac{1}{2} (k_o 1 + \hat{k} \cdot \hat{\sigma}) \) and \( |\bar{\kappa}<\bar{\kappa}| = \frac{1}{2} (k_o 1 - \hat{k} \cdot \hat{\sigma}) \), then

\[
|\bar{\kappa}<\bar{\kappa}| \kappa> = 0, \text{for the anti-neutrino} \quad (3.7a)
\]
and
\[
|\kappa><\kappa| \bar{\kappa> = 0, \text{for the neutrino} \quad (3.7b)
\]

follow as identities.

Let us now find explicit expressions for \( |\kappa> \) and \( |\bar{\kappa}> \), the spin parts of the wave functions for the anti-neutrino and neutrino. We first solve for \( |\kappa> \) using the outer product relation:

\[
|\kappa><\kappa| = \frac{1}{2} (k_o 1 + \hat{k} \cdot \hat{\sigma})
\]
\[
\begin{pmatrix}
|\kappa_1|^2 & \kappa_1 \kappa_2^* \\
\kappa_1^* \kappa_2 & |\kappa_2|^2
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
k_o + k_3 & k_1 - ik_2 \\
k_1 + ik_2 & k_0 - k_3
\end{pmatrix}
\]

\[
|\kappa_1|^2 = \frac{k_o + k_3}{2} \text{ implies that } \kappa_1 = \sqrt{\frac{k_o + k_3}{2}} \phi_1
\]
\[
|\kappa_2|^2 = \frac{k_o - k_3}{2} \text{ implies that } \kappa_2 = \sqrt{\frac{k_o - k_3}{2}} \phi_2
\]
and $k_1 k_2 = \frac{k_1 + i k_2}{2}$ implies that $i(\phi_2 - \phi_1) = \frac{k_1 + k_2}{\sqrt{k_0^2 - k_3^2}}$

The phases $i \phi_1$ and $i \phi_2$ are arbitrary and only the phase difference is determined. If we set $\phi_1 = 0$ then we obtain:

$$|\kappa> = \begin{pmatrix} \sqrt{k_0 + k_3} \\ k_1 + i k_2 \\ \sqrt{2(k_0 + k_3)} \end{pmatrix}$$

Note that if another phase convention had been chosen then our expression for $|\kappa>$ in terms of the components of $(k_0, \vec{k})$ would have appeared quite different.

We can express the components of $\vec{k}$ in terms of the spherical angles $(\alpha, \beta)$:

$$k_1 = k_0 \sin \beta \cos \alpha$$
$$k_2 = k_0 \sin \beta \sin \alpha$$
$$k_3 = k_0 \cos \beta$$
Then:

\[ \kappa_1 = \sqrt{k_0} \sqrt{\frac{1 + \cos \beta}{2}} = \sqrt{k_0} \cos \beta/2 \]

\[ \kappa_2 = \sqrt{k_0} e^{i \alpha} \frac{\sin \beta}{\sqrt{2(1 + \cos \beta)}} = \sqrt{k_0} e^{i \alpha} \frac{2 \sin \beta/2 \cos \beta/2}{2 \cos \beta/2} = \sqrt{k_0} \sin \beta/2 e^{i \alpha} \]

\[ |\kappa\rangle = \sqrt{k_0} \begin{pmatrix} \cos \beta/2 \\ \sin \beta/2 e^{i \alpha} \end{pmatrix} \quad (3.8a) \]

Notice the similarity between this expression and the expression we found for \(|\lambda_3\rangle\), they differ only in their normalization and the phase convention used. Taking the spin conjugate of \(|\kappa\rangle\) we obtain:

\[ |\bar{\kappa}\rangle = \sqrt{k_0} \begin{pmatrix} -\sin \beta/2 e^{-i \alpha} \\ \cos \beta/2 \end{pmatrix} \quad (3.8b) \]

Our expressions for \(|\kappa\rangle\) and \(|\bar{\kappa}\rangle\) underscore the point that the anti-neutrino and neutrino are orientable objects with their orientation given by the direction of their wave vector and their helicity about that direction.

Reintroducing the space-time part of the wave functions for the anti-neutrino and neutrino, we obtain:
$$|\psi_a> = A \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] |k>$$  \hspace{1cm} (3.9a)

$$|\bar{\psi}> = A \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] |\bar{k}>,$$  \hspace{1cm} (3.9b)

respectively, where $A$ is a normalization constant. To obtain differential equations for the space-time dependence of $|\psi_a>$ and $|\bar{\psi}>$, similar to the Schrödinger-like equation derived in section 2.2, we introduce the differential operators:

$$D = \frac{\partial}{\partial x_0} \left[ 1 - \vec{v} \cdot \vec{\sigma} \right]$$  \hspace{1cm} (3.10a)

and

$$\bar{D} = \frac{\partial}{\partial x_0} \left[ 1 + \vec{v} \cdot \vec{\sigma} \right].$$  \hspace{1cm} (3.10b)

When $i D$ and $i \bar{D}$ operate on $|\psi_a>$ or $|\bar{\psi}>$ they bring out a factor of

$$K = k_0 1 + \vec{k} \cdot \vec{\sigma} \quad \text{and} \quad \bar{K} = k_0 1 - \vec{k} \cdot \vec{\sigma},$$

respectively:

$$i D \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] |\phi> = [k_0 1 + \vec{k} \cdot \vec{\sigma}] \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] |\phi>$$

$$i \bar{D} \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] = [k_0 1 - \vec{k} \cdot \vec{\sigma}] \exp[-i(k_0 x_0 - \vec{k} \cdot \vec{x})] |\phi>$$

where $|\phi>$ can be any spinor. This means that $D$ transforms like $K$ and $\bar{D}$ transforms like $\bar{K}$ under the Lorentz group. Referring to equations (3.7 a and b), we obtain the differential equations:
In order to translate these kinematic equations of motion into dynamic ones we note that the wave four vector is related to the four momentum through Planck's constant:

\[(p_0, \vec{p}) = \hbar(k_0, \vec{k})\]  \hspace{1cm} (3.12)

The operators associated with the four momentum are:

\[\mathbf{P} = i \hbar \frac{\partial}{\partial x_0} - \vec{\nabla} \cdot \vec{\sigma}\]  \hspace{1cm} (3.13a)

and

\[\bar{\mathbf{P}} = i \hbar \frac{\partial}{\partial x_0} + \vec{\nabla} \cdot \vec{\sigma}\]  \hspace{1cm} (3.13b)

Using these operators, we rewrite the equations of motion for the anti-neutrino and neutrino as follows:

\[\bar{\mathbf{P}}|\psi_a^\text{\textgreater} = (p_0 \mathbf{1} - \vec{p} \cdot \vec{\sigma})|\psi_a^\text{\textgreater} = 0\]  \hspace{1cm} (3.14a)

\[\mathbf{P}|\psi_\gamma^\text{\textgreater} = (p_0 \mathbf{1} + \vec{p} \cdot \vec{\sigma})|\psi_\gamma^\text{\textgreater} = 0\]  \hspace{1cm} (3.14b)

Notice that mass does not appear at all in these equations. Operating on (3.14a), we obtain:
\[ p \bar{\psi} | \psi_a > = (p_0 \gamma^0 + \mathbf{p} \cdot \mathbf{\hat{\sigma}}) (p_0 \gamma^0 - \mathbf{p} \cdot \mathbf{\hat{\sigma}}) | \psi_a > = 0 \]

\[ (p_0^2 - |\mathbf{p}|^2) | \psi_a > = 0 \]

Therefore:

\[ p_0^2 - |\mathbf{p}|^2 = 0 . \]

It is apparent that a single two component spinor can account only for massless particles. It will be shown in the next section that a pair of conjugate spinors is needed in order to introduce mass in a relativistically covariant fashion.

### 3.4 Massive Spin 1/2 Particles—The Dirac Equation*

The equations of motion for the neutrino and anti-neutrino describe waves of well defined helicity which travel at the speed of light. The neutrino and anti-neutrino have a Lorentz invariant helicity because their spin and wave vector have the same relative orientation in all inertial frames; these particles cannot be transformed to rest.

Mass appears in our approach because there exist objects which can be transformed to rest. Massive objects with spin do not have a Lorentz invariant helicity because their wave vector

*The argument presented here is similar to that which appears in Kaempfer. Section 19. However some aspects of the interpretation are different. (see Tisza₂).*
can be transformed away altogether while the absolute value of the spin is invariant. This is another reason why a single relativistic two-component spinor cannot describe a massive spin 1/2 object.

The question now arises as to how we can account for mass within our scheme of massless waves which travel at the speed of light. In answer to this, we hypothesize that a state with rest mass can be thought of as a standing wave pattern achieved through the superposition of two waves traveling in opposite directions. We therefore expect that a massive spin 1/2 particle will be describable by a pair of spinors obeying two wave equations of a form similar to (3.14 a and b). These equations must be relativistically covariant and must also be consistent with the fact that:

\[ p_0^2 - |p|^2 = m^2 c^2, \]  

(3.15)

for a particle with rest mass m.

Let us investigate the transformation properties of the two uncoupled equations (3.14 a and b):

\[ \tilde{F}' |\psi'_a> = \tilde{\nu} \tilde{F} \tilde{V}^{-1} \nu |\psi_a> = \tilde{\nu} \tilde{F} |\psi_a>. \]

Therefore \( \tilde{F} |\psi_a> \) transforms like a conjugate spinor. Similarly:

\[ \tilde{p}' |\overline{\psi}_\gamma> = \nu p \nu^* \overline{\psi}_\gamma = \nu p |\overline{\psi}_\gamma>. \]
Therefore $\Psi_1$ transforms like a spinor. The spinors $\Psi_a$ and $\Psi_y$ describe two different massless spin 1/2 particles. We expect a massive spin 1/2 particle to be described by the two conjugate spinors $|u>$ and $|\bar{v}>$ coupled by equations with the same transformation properties as (3.14 a and b):

$$\overline{\mathcal{P}}|u> = A|\bar{v}>$$

$$\mathcal{P}|\bar{v}> = A|u>$$

where $A$ is a coupling constant. Iterating each of these equations we obtain:

$$\overline{\mathcal{P}}\overline{\mathcal{P}}|u> = (p_0^2 - |\mathbf{p}|^2)|u> = A^2|u>$$

$$\mathcal{P}\overline{\mathcal{P}}|\bar{v}> = (p_0^2 - |\mathbf{p}|^2)|\bar{v}> = A^2|\bar{v}>$$

The condition (3.15) is satisfied if $A^2 = m^2c^2$. We omit negative energy solutions by selecting $A = mc$. We have therefore found that a spin 1/2 particle of mass $m$ should be describable by the two coupled equations:

$$\overline{\mathcal{P}}|u> = mc|\bar{v}> \quad (3.16a)$$

$$\mathcal{P}|\bar{v}> = mc|u> \quad (3.16b)$$

Equations (3.16 a and b) are normally written as a single four component equation known as the Dirac Equation.
SUMMARY AND CONCLUSIONS

In keeping with our stated purpose we have built up the mathematical formalism sufficiently to serve as the basis for subsequent physical applications. The Pauli algebra was developed as a convenient means of representing Lorentz transformations. Spinors were then introduced and used to represent objects which transform first under pure rotations (Chapter I) and then under Lorentz transformations (Chapter III). Only after we had developed the mathematics as a coherent structure did we undertake its physical interpretation. This allowed us to take full advantage of the inner unity of our formalism supplied by the involution group, in particular. An example of the unifying power of the involutions is provided by the operation of spin conjugation which was found to be associated with complex reflection. This operation was first introduced in reference to the triad and then found to convey the transformation between spin up and spin down states, in non-relativistic quantum mechanics, and finally the transformation between neutrino and anti-neutrino states in relativistic quantum mechanics.

A further unification was made possible by our use of the object group concept. Throughout this thesis we have focused on the problem of how to characterize objects which transform under the active kinematic group. The transition between the geometric-algebraic treatment of Chapter I and the applications
to quantum mechanics involved the introduction of a new kind of "object", the quantum mechanical state, and a new interpretation of the formalism in terms of the measurement process and probability amplitudes.

This thesis deals only with a fraction of the problems involved in the foundation of quantum mechanics. Many more problems can be treated within the existing framework and the framework itself is susceptible to extension, in particular, to deal with the continuum aspects of quantum theory.
REFERENCES

1. L. Tisza, Lecture Notes.

2. L. Tisza, "Toward a Reconstruction of the Mathematical Principles of Physics", (to be published).


