WAVE-PARTICLE INTERACTIONS
AND THE DYNAMICS OF THE SOLAR WIND

by

CHARLES CARSON GOODRICH

S.B., Massachusetts Institute of Technology
(1972)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
SEPTEMBER, 1978

Signature redacted
Signature of Author

Signature redacted
Department of Physics
August 15, 1978

Signature redacted
Certified by
Thesis Supervisor

Signature redacted
Accepted by
Chairman, Departmental Committee

NOV 2 1978
LIBRARIES
Wave-particle interactions play an important role in determining the thermal state and dynamic properties of the solar wind. A number of these interactions can be satisfactorily described by standard plasma theory for an infinite homogeneous plasma. However, recent developments in fluid theory have shown that there are many wave processes for which the inhomogeneity of the solar wind is a dominant factor. There is no general kinetic formalism suitable to the solar wind which includes inhomogeneity in both space and time. Thus, wave-particle processes in the solar wind which are dependent on inhomogeneity have not been studied using a kinetic approach. To study such processes, we develop quasilinear kinetic equations describing the temporal and spatial evolution of the distribution functions for an inhomogeneous plasma in the presence of high frequency, short wavelength waves. Our theory includes new wave-particle interaction terms which arise solely from the inhomogeneity of the plasma. We demonstrate the usefulness of our formalism by considering the evolution of the proton distribution function in the presence of Alfvén waves in a radially symmetric solar wind model. We obtain numerical solutions for the radial evolution of the distribution function. These solutions exhibit the wave acceleration effects well known from fluid models. We find that the Alfvén wave pressure is strongly velocity dependent, leading to interesting distortions in the proton distribution function.

Thesis supervisor: John W. Belcher
Associate Professor of Physics
To Christine
# TABLE OF CONTENTS

| Abstract | 2 |
| List of Figures | 5 |
| Chapter I: Introduction | 6 |
| Chapter II: Quasilinear Kinetic Theory in an Inhomogeneous Plasma | 14 |
| 1. Introduction | 14 |
| 2. Formulation | 16 |
| 3. Expansion of \( W \) | 26 |
| 4. Evaluation of \( N \) | 37 |
| 5. Wave Equations and Conservation Laws | 41 |
| 6. Weak Magnetic Field Limit | 50 |
| 7. Summary | 53 |
| Chapter III: Kinetic Theory of Alfven Wave Pressure in the Solar Wind | 54 |
| 1. Introduction | 54 |
| 2. Basic Equations | 58 |
| a. Proton Equations | 59 |
| b. Electron Equations | 64 |
| c. Wave Equations | 65 |
| 3. Method of Solution | 68 |
| a. The Kinetic Equations | 68 |
| b. The Two Fluid Model | 71 |
| 4. The Radial Evolution of the Proton Distribution Function | 79 |
| 5. Summary | 89 |
| Bibliography | 90 |
| Acknowledgments | 92 |
LIST OF FIGURES

Figure 1.........................................................75
Figure 2.........................................................78
Figure 3a.......................................................81
Figure 3b.........................................................82
Figure 3c.....................................................83
Figure 3d.....................................................84
Figure 3e.....................................................85
The interactions of waves and instabilities with the solar wind have long been considered of importance in determining the dynamic state of the solar wind. Historically, such processes were first proposed in connection with the thermal anisotropy of the solar wind plasma. In his first paper on the solar wind, Parker (1958) noted that the collisionless expansion of the solar wind plasma in the interplanetary magnetic field could result in thermal anisotropies large enough to excite the well known "firehose" instability. In this case, the growth of magnetic field energy associated with the instability acts to reduce the thermal anisotropies, ultimately producing a state of marginal stability in the plasma. Numerous other instabilities, both resonant and non-resonant in nature, have since been proposed as influences on the thermal properties of the plasma. These include, for example, instabilities associated with the electron heat flux and instabilities associated with the relative streaming of the proton and alpha particle components of the solar wind. Details of these instabilities can be found in the excellent review by Hollweg (1975).

In addition to local instabilities, the damping of externally driven waves of solar origin play an important role in determining the thermal state and dynamic properties of the solar wind. Large amplitude Alfvén waves propagating outward
from the Sun are commonly observed at 1 A.U. These waves are found to dominate the microstructure of the solar wind at least 50% of the time (Belcher and Davis, 1971). The waves are almost certainly the undamped remnants of a broad spectrum of MHD waves generated at the Sun. The fast and slow MHD wave modes are subject to moderate and strong Landau damping, respectively (Stepanov, 1958; Barnes, 1966). This damping presumably accounts for the fact that the compressive MHD wave modes have not been observed in the solar wind at 1 A.U. (Belcher and Davis, 1971). In addition, Barnes et al. (1971) have shown that the fast waves damp over an extended region (10-20 solar radii). The dissipation of the waves efficiently heats the plasma in this region, leading to considerable acceleration of the plasma. Barnes (1978) has reviewed in detail these and numerous other examples of the importance of wave processes in the solar wind.

Wave processes in the solar wind are most appropriately described in terms of plasma kinetic theory, since the solar wind protons are essentially collisionless beyond a few solar radii. Previously kinetic studies of these interactions have been based on the standard expressions derived for an infinite homogeneous plasma. This approach is not always satisfactory, as it neglects the inherent inhomogeneity of the solar wind. It is valid only for interactions that can be described locally, that is, interactions which depend only on the local state of the plasma. However, there are a number of wave processes for
which the inhomogeneity of the plasma and fields is a major or even the dominate factor. We give several examples of such interactions below. Although these examples are taken from fluid theory, it is clear that they should also appear in any proper kinetic treatment.

1) The Alfvén waves commonly observed in the solar wind generally have frequencies that are extremely low compared to the proton gyrofrequency. In the context of the homogeneous theory, damping for such waves is negligible, and they do not interact with the plasma. However, Belcher (1971) and Alazraki and Couturier (1971) have shown that in an inhomogeneous plasma Alfvén waves exert a volume force, analogous to a radiation pressure. For a thermally isotropic plasma in the short wavelength approximation, the volume force is given by

\[ F = - \nabla \left\langle \frac{\delta B^2}{8\pi} \right\rangle \]  

(1)

Here \( \delta B \) is the magnitude of the magnetic field fluctuation associated with the wave, and the angle brackets denote the average over the wave oscillation time scale. Fluid calculations for solar wind models including the wave force by Belcher (1971), Hollweg (1973), and Jacques (1977) show that it can lead to significant acceleration of the plasma. We emphasize that the wave force occurs only in an inhomogeneous plasma, as explicitly shown by the form of equation (1).

2) A second example involves the development of the thermal anisotropy of the solar wind. This development in the
presence of Alfvén waves has been studied using a radially symmetric solar wind model by Patterson (1971). He found that the waves interact with the plasma in such a way as to oppose the increase in the anisotropy. To demonstrate this effect, we assume that the protons are collisionless and obey the double-adiabatic equations of state (Chew et al., 1956). Then, in the presence of Alfvén waves, the proton thermal anisotropy evolves according to

\[ \frac{P_n}{P_\perp} \left( \frac{B_\perp^3}{n^2} \right) \left( 1 + \frac{\delta B^2}{B_0^2} \right)^\frac{3}{2} = \text{constant.} \]  

(2)

Here \( P_n \) (\( P_\perp \)) is the proton pressure parallel (perpendicular) to the average (radial) magnetic field \( B_0 \), and \( n \) is the proton number density. If we assume that the electrons are isotropic, which is a good approximation in the solar wind, then the total thermal anisotropy of the plasma is \( P_n / P_\perp \). In the absence of waves, equation (2) predicts that \( P_n / P_\perp \) should increase with the distance \( r \) from the Sun as \( n^2 / B_0^3 \), which goes as \( r^2 \) for a radial field model. However, in the presence of the waves, the wave amplitude \( \delta B^2 \) evolves according to

\[ \frac{\delta B^2}{B_0^2} \frac{n}{r^2} \frac{\omega}{k} \left( \frac{\omega}{k} - V \right) = \text{constant} \]  

(3)

where \( V \) is the plasma bulk speed and \( \omega / k \) is the wave phase speed. For an outwardly propagating wave, the phase speed is
\[
\frac{\omega}{k} = V + \frac{B_o}{\sqrt{4\pi n m_p}} \sqrt{1 - \frac{4\pi (P_\parallel P_\perp)}{B_o^2}}, \tag{4}
\]

where \(m_p\) is the proton rest mass. Comparing equations (2), (3), and (4), the role of the waves in limiting the evolution of anisotropy in this model is clear. An increase in \(P_\parallel/P_\perp\) with increasing \(r\) leads through equation (4) to a decrease in \(\omega/k\). This leads to an increase in \(\delta B^2\) (cf. equation (3)), in turn to an increase in the factor in parentheses in equation (2), and thus to a decrease in \(P_\parallel/P_\perp\). This self-limiting aspect of the radial evolution of the thermal anisotropy is a further example of a wave process in the solar wind which is critically dependent on the inhomogeneity of the plasma and fields.

3) The final example is closely related to the examples above. The expression given in equation (1) for the volume force exerted by Alfvén waves is the net volume force felt by the plasma as a whole, rather than that felt by the individual plasma species in a multifluid plasma. The more general case of a multifluid plasma has been investigated by Hollweg (1974) for a radially symmetric plasma. In this work the individual plasma species are treated as cold beams moving along the radial average magnetic field. Hollweg's result for the volume force on the \(i^{th}\) species can be written in a form similar to equation (1), i.e.,

\[
F_i = n_i m_i \frac{d}{dr} \left[ \left( \frac{\omega^2}{k^2} - V_i^2 \right) \frac{\langle \delta B^2 \rangle}{B_o^2} \right]. \tag{5}
\]
Here \( n_i \), \( m_i \), and \( v_i \) are the number density, mass, and bulk speed, respectively, of the \( i \)th species.

From equation (5) we see that the acceleration felt by each species is dependent on its bulk speed. Thus fluid components of the plasma with different bulk speeds are accelerated differently by the waves. This property has interesting implications for a thermal distribution of a single species. If we consider such a thermal distribution to be composed of a large number of cold beams, then each beam (or, equivalently, each velocity space volume element of the distribution) would be accelerated differently by the waves. Strictly speaking, this argument applies only to particles with zero pitch angle. However it is clear that the wave acceleration effect has a kinetic (i.e., velocity dependent) nature in addition to its obvious dependence on the inhomogeneity of the plasma.

The above examples involve only effects associated with Alfvén waves, primarily because the properties of these waves have been studied extensively in the literature. Analogous effects are undoubtedly associated with other MHD waves, as well as with non-MHD plasma waves. In all of these examples, wave interactions that occur exclusively in an inhomogeneous plasma have been derived using fluid theory. Such interactions of course should also appear in a kinetic treatment of the solar wind which includes inhomogeneities. In fact, the third example above clearly implies that the interaction of Alfvén waves with
an inhomogeneous plasma is velocity dependent and can only be properly described in terms of a kinetic theory. The lack of a general kinetic theory which includes inhomogeneity in both space and time is a major failing of the theory of wave processes in the solar wind. The present work is an attempt to remedy this failing.

The only substantial past work in this area is due to Hollweg (1978). He has obtained a quasilinear kinetic theory which incorporates wave-particle interactions due to inhomogeneity of the plasma and fields in the WKB limit. In his work, however, only transverse waves propagating parallel to the average magnetic field are considered. The average electric and gravitational fields are constrained to be parallel to the average magnetic field. Furthermore, no illustrative numerical or analytic solutions to these equations are available.

Our purpose in this thesis is the development of a quasilinear kinetic formalism suitable for the description of wave-particle interactions in the solar wind, with numerical solutions in special cases to illustrate the range of possible physical effects. In Chapter II we obtain quasilinear equations describing the temporal and spatial evolution of the plasma distribution functions for a strongly magnetized inhomogeneous plasma in the presence of high frequency, short wavelength waves. We obtain new wave-particle interaction terms which arise from the inhomogeneity of the plasma and average magnetic field and
from the presence of average electric and gravitational fields. For completeness, we also consider the extension of our results for a weakly magnetized plasma.

In Chapter III, we demonstrate the usefulness of our formalism by considering the evolution of the proton distribution function in the presence of Alfvén waves in a radially symmetric solar wind model. We find that the effects due to the Alfvén wave on the proton distribution are strongly velocity dependent, resulting in novel and unexpected features in the evolving distribution function. Numerical solutions for a steady-state solar wind model are presented.
1. Introduction

The purpose of this chapter is the development of a kinetic formalism based on the Vlasov-Maxwell equations appropriate for the study of wave-particle interactions in the solar wind. In view of the discussion in Chapter I, we include the effects of the inhomogeneity of the plasma as well as those due to the background magnetic, electric, and gravitational fields. In our development, we use a quasilinear approach, in which terms to second order in wave amplitude are retained. The validity of this approach has been discussed extensively in the literature and will not be considered here. We further assume the wave time (length) scales are small compared to the evolution time (length) of the plasma. This is a good assumption for most of the waves observed in the solar wind.

We proceed in the following manner. In section 2 we consider the general formulation of the problem. We consider the requirements that our separation of wave and evolution time scales impose on the magnitude of the background magnetic, electric, and gravitational fields. We then specialize to the case of a strongly magnetized plasma. For this limit we obtain in sections 3 and 4 a description of the wave-particle interaction. In section 5 we consider the equations for the
wave amplitudes, frequencies, and wave numbers. We extend our results in section 6 to the case of a weakly magnetized plasma, and summarize our results in section 7.
2. Formulation

Consider a collisionless, quasineutral plasma, described by a set of distribution functions \( f_i(x,v,t) \), in the presence of a gravitational field \( g(x) \), an electric field \( E(x,t) \), and a magnetic field \( B(x,t) \). The system satisfies the Vlasov-Maxwell equations given by:

\[
\left[ \frac{\partial}{\partial t} + v \cdot \nabla + (q + \frac{q_i}{m_i} E + \frac{q_i}{m_i} v \times B) \cdot \nabla v \right] f_i = 0 ,
\]

\[
\nabla \cdot E = \sum_i q_i f_i \, d^3v ,
\]

\[
\nabla \cdot B = 0 ,
\]

\[
\nabla \times E = -\frac{1}{c} \frac{\partial}{\partial t} B ,
\]

\[
\nabla \times B = \frac{4\pi}{c} \sum_i q_i \int_v f_i \, d^3v \, + \frac{1}{c} \frac{\partial}{\partial t} E .
\]

Here \( q_i \) and \( m_i \) are the charge and mass, respectively, of the \( i \)th species.

We assume that there is a clear separation of time and length scales, so that the distribution functions and the electromagnetic fields can be divided into slowly varying parts, and rapidly varying parts. Slowly varying is taken here to indicate behavior with evolution time \( T \) and scale height \( L \). The rapidly varying wave-associated parts are
assumed to average to zero on the evolution time and distance scales, and to be small amplitude with typical period $T$ and wavelength $\lambda$ where

$$T \ll T \quad \text{and} \quad \lambda \ll L \quad . \quad (6)$$

It is convenient to define a measure of the departure of the system from temporal and spatial uniformity; the small parameter $\mu$ is taken to be the larger of $\frac{\lambda}{L}$ and $\frac{T}{T}$. We will assume for the development in this chapter that the two ratios are the same order of magnitude, i.e.,

$$\mu \sim \frac{\lambda}{L} \sim \frac{T}{T} \quad . \quad (7)$$

This ordering is desirable as it results in the retention of the effects of both temporal and spatial inhomogeneities in the theory. Hereafter, we will accordingly use the terms "homogeneity" and "inhomogeneity" in a broad sense to refer to a time stationary, spatially uniform state and departure from that state, respectively.

With these assumptions, we consider the non-linear equations (1)-(5) by expanding separately in the small parameter $\mu$ and the wave amplitude. In contrast to many quasilinear developments for homogeneous plasmas, we must consider expansions both in $\mu$ and in wave amplitude.

We proceed in the usual manner by writing $f_{0i}$, $E$, and $B$ as the sum of average and wave parts:

$$f_i = f_{0i} + \delta f_i \quad ,$$
The average distribution functions $f_{o1}$ are defined as the Eulerian averages of the $f_i$ centered at the point $(x, t)$ over a time interval $\Delta$ and a volume $V$ chosen such that

$$\omega \ll \Delta \ll T$$

and

$$\lambda \ll \sqrt{\omega} \ll L .$$

Thus,

$$f_{o1}(x, y, t) = \frac{1}{V \Delta} \int_{t-\Delta/2}^{t+\Delta/2} \int_{x-\epsilon/2}^{x+\epsilon/2} f_i(x', y, t') \, dx' \, dt' ,$$

where $V = \pi \sigma$. Similar definitions are made for $E_o$ and $B_o$.

The rapidly varying part of $f_i$ is then given by the difference between $f_i$ and $f_{o1}$, with similar definitions for $\delta E$ and $\delta B$.

In the high frequency, short wavelength limits assumed here, the solution of equations (2)-(5) for the wave fields $\delta E$ and $\delta B$ has been considered by Stix (1962), Bernstein (1975), Akhiezer et al. (1975), and Bernstein and Baldwin (1977). Following their results we assume $\delta E$ to be the sum of uncorrelated wave modes, each the product of a slowly varying complex amplitude and a rapidly oscillating amplitude

$$\delta E(x, t) = \sqrt{2} \sum_n \text{Re} \left( A_n(x, t) e^{i\psi_n(x, t)} \right) .$$

The phase $\psi_n(x, t)$ is real with slowly varying derivatives.
A similar ansatz can be made for $\delta B$.

To lowest order in wave amplitude, the rapidly varying part of the distribution function, $\delta f_i$, is obtained from the linearized Vlasov equation

$$\frac{D}{Dt} \delta f = -\frac{q}{m} \left( \delta E + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B} \right) \cdot \nabla \Psi f_o . \tag{12}$$

The operator $\frac{D}{Dt}$ in the above and following equations is given in terms of the gyrofrequency $\omega_c = \frac{qB_0}{mc}$ and magnetic field direction $\hat{e}_n = \frac{B_0}{|B_0|}$ by

$$\frac{D}{Dt} = \frac{3}{3t} + \mathbf{v} \cdot \nabla + \left( q + \frac{q}{m} \mathbf{E}_o + \omega_c (\mathbf{v} \times \hat{e}_n) \right) \cdot \nabla \nu . \tag{13}$$

The evolution of the averaged distribution function, $f_o$, is given by the local average of the Vlasov equation

$$\frac{D}{Dt} f_o = -\frac{q}{m} \left( \delta E + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B} \right) \cdot \nabla \nu \delta f \equiv -W , \tag{14}$$

where $W$ is the quasilinear wave-particle interaction term. Here and in the remainder of this chapter, the subscript $i$ denoting species is suppressed, except where explicitly necessary, to simplify the notation.

From equation (13) it is clear that the characteristic time (length) scales of equations (12) and (14) are strongly influenced by the average fields $E_o$, $q$, and $B_o$. These time (length) scales, and therefore the average fields, must be
consistent with the clear separation assumed here between wave and evolution time (length) scales, which is basic to the derivation of equations (12)-(14). That is, equation (12) must describe behavior on both the wave and evolution time scales, while equation (14) must describe behavior on the evolution time scale only. This fact forces certain requirements on the magnitudes of $E_0$, $g$, and $B_0$.

We consider first the requirements on the average magnetic field $B_0$. We note that the magnetic force term in equation (13) is oscillatory in nature, causing a rotation in velocity space around $B_0$ with characteristic time $\tilde{\omega}_c$. Two separate orderings of $\tilde{\omega}_c$ are consistent with our assumptions; $B_0$ can be taken either such that $(\tilde{\omega}_cT)^l \sim \mu$ or that $(\tilde{\omega}_cT)^l \gg 1$. In the former limit, the effect due to $B_0$ occurs on the wave time scale. In view of equation (9), $f_0$ is then nearly symmetric in velocity space around $\tilde{e}_n$, as shown below. In the latter limit, the effect due to $B_0$ occurs on the evolution time scale. We note that a value of $\omega_c$ between the above limits, i.e. $\mu \ll (\omega_cT)^l \ll 1$, is not consistent with our assumptions, as it would result in behavior with a time scale intermediate to the wave and evolution time scales.

Throughout the bulk of this work, we consider the strong $B_0((\omega_cT)^l \sim \mu)$ limit, which is valid for many interesting laboratory and astrophysical plasmas. The extension of our results to include the weak $B_0$ limit is straightforward, and is indicated in section 6.
We require the effects due to the remaining fields $E_o$ and $g$ to occur on the evolution time scale. For the strong $B_o$ limit considered here, this requires that $E_o$ and $g$ obey

$$\frac{|E_o c|}{B_o v_{av}} \sim \mu , \quad (15a)$$

$$\left| \frac{g}{\omega_c v_{av}} \right| \sim \mu , \quad (15b)$$

where $v_{av}$ is a typical velocity for which $f_o$ is non-zero. Equation (15b) is almost always a valid assumption. For equation (15a) to be valid, the average motion of the plasma across $B_o$ must be small ($O(\mu)$). Furthermore the time variation of both the magnitude and direction of $B_o$ must be negligible ($O(\mu^2)$). This latter condition can be seen by combining equations (4) and (15a)

$$\left| \frac{\partial}{\partial t} B_o \right| = \left| c \nabla \times E_o \right|$$

$$\sim \left| \frac{c E_o}{L} \right| \sim \frac{B_o}{T} \mu , \quad (16)$$

(assuming $L \sim v_{av} T$). Thus assumption (15a) places strong restrictions on the time dependence of $B_o$—sufficiently strong that time derivatives of $B_o$ will not appear in expansions to first order in $\mu$. Assumption (15a) could be relaxed somewhat to apply only to the component of $E_o$ parallel to $B_o$ (e.g., allowing zero order average plasma motion across $B_o$), but this introduces additional complexity which we would rather avoid.
at this stage.

Under the above assumptions regarding the averaged fields, \( f_0 \) is approximately symmetric in velocity space around the local direction of the magnetic field. To exploit this symmetry, it is convenient to work in a coordinate system locally aligned with \( \mathbf{B}_0 \). This system, hereafter referred to as the local system, has coordinate axes \( \hat{\mathbf{e}}_j \) with \( \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_n \) which in the original, or fixed, system are in general slowly varying functions of \( \mathbf{x} \) through the slow variation in \( \mathbf{B}_0 \). As discussed above, the time variation of \( \mathbf{B}_0 \), and therefore the \( \hat{\mathbf{e}}_j \), can be neglected. Of course in the local frame the \( \hat{\mathbf{e}}_j \) are constant.

We now consider the transformation of the differential equations for the wave fields and the plasma into the local system. To avoid confusion we use \( \frac{\partial}{\partial \mathbf{x}} \) to denote spatial derivatives taken in the local system, as contrasted to spatial derivatives taken in the fixed system, denoted by \( \nabla \). We note that \( \frac{\partial}{\partial \mathbf{x}} \hat{\mathbf{e}}_j \equiv 0 \).

The wave fields are described in the fixed system by equations (4) and (5). Substituting \( \nabla \mathbf{E} = \sum_j \delta \mathbf{E}_j \hat{\mathbf{e}}_j \) and \( \nabla \mathbf{B} = \sum_j \delta \mathbf{B}_j \hat{\mathbf{e}}_j \) in equation (4), there results

\[
\sum_j \left[ (\nabla \delta \mathbf{E}_j) \times \hat{\mathbf{e}}_j + \mathbf{E}_j \nabla \times \hat{\mathbf{e}}_j \right] = -\frac{1}{c} \sum_j \hat{\mathbf{e}}_j \frac{\partial}{\partial t} \delta \mathbf{B}_j.
\]

The summation of the first term on the right can be identified as the curl of \( \delta \mathbf{E} \) in the local system. Defining

\[
\Lambda (\mathbf{x}) = \sum_j (\nabla \times \hat{\mathbf{e}}_j) \hat{\mathbf{e}}_j \sim \frac{1}{L},
\] (17)
we obtain the equivalent of (4) in the local system

\[
\frac{3}{\delta x} \times \delta E + \nabla \cdot \delta E = -\frac{1}{c} \frac{\rho}{\delta E} \delta B.
\]

Similarly, equation (5) becomes

\[
\frac{3}{\delta x} \times \delta B + \nabla \cdot \delta B = \frac{4\pi}{c} \sum q_i \delta f_i \mathbf{v} d^3\mathbf{v} + \frac{1}{c} \frac{3}{\delta x} \delta E.
\]

To transform the plasma equations (12)-(14) into the local system, we express \( \mathbf{v} \) in local cylindrical coordinates, with

\[
\mathbf{v}_\parallel = \mathbf{v} \cdot \hat{e}_\parallel,
\]

\[
\mathbf{v}_\perp = \sqrt{(\mathbf{v} \cdot \hat{e}_1)^2 + (\mathbf{v} \cdot \hat{e}_2)^2},
\]

\[
\phi_v = \tan^{-1} \frac{\mathbf{v} \cdot \hat{e}_1}{\mathbf{v} \cdot \hat{e}_2}.
\]

Using

\[
\hat{e}_\perp = \cos \phi_v \hat{e}_1 + \sin \phi_v \hat{e}_2,
\]

\[
\hat{e}_\phi = -\sin \phi_v \hat{e}_1 + \cos \phi_v \hat{e}_2,
\]

we have

\[
\mathbf{v} = \mathbf{v}_\parallel \hat{e}_\parallel + \mathbf{v}_\perp \hat{e}_\perp,
\]

\[
\frac{3}{\delta v} = \hat{e}_\parallel \frac{3}{\delta v}_\parallel + \hat{e}_\perp \frac{3}{\delta v}_\perp + \hat{e}_\phi \frac{1}{v_\perp} \frac{3}{\delta \phi_v}.
\]

The local velocity variables \( \mathbf{v}_\parallel, \mathbf{v}_\perp, \phi_v \) are functions of \( \mathbf{x} \) in the fixed system through the \( \hat{e}_j \). This results in the
appearance of additional small \( O(\mu) \) acceleration terms in the operator \( \frac{D}{Dt} \) when equations (12)-(14) are transformed into the local system. We define the acceleration \( \dot{a} \) to include these terms and the \( O(\mu) \) terms due to \( E_o \) and \( g \). Then

\[
\begin{align*}
\dot{a}_n &= (\frac{2}{m} E_o + g) \cdot \hat{e}_n + v \cdot [v \cdot \nabla] \hat{e}_n,
\end{align*}
\]

\[
\begin{align*}
\dot{a}_\perp &= (\frac{2}{m} E_o + g) \cdot \hat{e}_\perp + v \cdot [v \cdot \nabla] \hat{e}_\perp,
\end{align*}
\]

\[
\begin{align*}
\dot{a}_\phi &= (\frac{2}{m} E_o + g) \cdot \hat{e}_\phi + [v \cdot \nabla] \phi_v.
\end{align*}
\]

Using equations (18) and (17), equation (13) becomes

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} + a_n \frac{\partial}{\partial v_n} + a_\perp \frac{\partial}{\partial v_\perp} + (a_\phi - \omega_c) \frac{\partial}{\partial \phi_v}.
\]

Substituting this definition and equation (20) into equation (12) we obtain the desired equation for \( \delta f(x,v_n,v_\perp,\phi_v,t) \). In order for the terms in equation (14) to be \( O(\mu) \) or smaller, we require that

\[
\frac{\partial}{\partial \phi_v} f_o \sim O(\mu).
\]

To allow \( f_o \) to depart slightly from axial symmetry in velocity space, we let

\[
f_o = F^{(s)}(x,v_n,v_\perp,\phi_v,t) + F^{(i)}(x,v_n,v_\perp,\phi_v,t) \cdots
\]

where the terms are ordered in ascending powers of \( \mu \).

Collecting the first order terms in equation (14), we obtain

\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} + a_n \frac{\partial}{\partial v_n} + a_\perp \frac{\partial}{\partial v_\perp} \right) F^{(o)} - \omega_c \frac{2}{\partial \phi_v} F^{(i)} = -\mathcal{W}.
\]
We can eliminate $F^{(n)}$ in the above equation by averaging over $\phi_v$.

$$\frac{\partial}{\partial t} F^{(0)} = \left( \frac{3}{3t} + \frac{2}{3\chi} + \frac{2}{3v} + \frac{3}{3v} \right) F^{(0)} = -\mathcal{W}$$ \hspace{1cm} (26)

where

$$\mathcal{W} = \frac{1}{2\pi} \int_0^{2\pi} W d\phi_v$$

and, using equation (21),

$$\bar{a}_n = \frac{a}{\alpha} \mathcal{E}_{\|} + g_{\|} + \mathbf{v} \cdot [\mathbf{v} \cdot \mathbf{\nabla}] \hat{e}_n$$

$$\bar{a}_\perp = \mathbf{v} \cdot [\mathbf{v} \cdot \mathbf{\nabla}] \hat{e}_\perp$$

An expression for the wave interaction term $W$ in the local system is obtained in the following sections in terms of $F^{(0)}$ and the wave amplitudes $A_n$, wave frequencies $\omega_n$, and wave numbers $k_n$. Thus equation (26) provides the basic equation for the evolution of the plasma. It is to be solved simultaneously with the equations for the wave parameters, which are considered, again in the local system, in section 5. If desired, the results can be used in equation (25) to calculate $F^{(n)}$. 
3. Expansion of $W$

In this section, the wave-particle interaction term

$$W = \left( \frac{1}{m} (\delta E + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\mathbf{B}}{\mathbf{B}} \delta f \right)$$

is evaluated to lowest order: quadratic in wave amplitude and first order in the inhomogeneity parameter $\mu$. We obtain $\delta f$ by integrating the linearized Vlasov equation (12) by the method of characteristics with the result

$$\delta f = -\frac{\gamma_m}{c} \int_{-\infty}^{t} dt^* (\delta E (x^*, t^*) + \frac{1}{c} \mathbf{v}^* \times \delta \mathbf{B} (x^*, t^*)) \cdot \frac{3}{2} f_0 (x^*, v^*, t^*) \cdot (27)$$

The integration is along the particle trajectories $x^*(t^*)$ and $v^*(t^*)$ defined for the total derivative (22).

Before proceeding, it is instructive to identify the factors that result in contributions to $W$. This is most easily done by considering the phase relationship between $\delta f$ and the wave acceleration $\gamma_m (\delta E + \frac{1}{c} \mathbf{v} \times \delta \mathbf{B})$. If the plasma is homogeneous (temporally and spatially), i.e. $f_0 = f_0 (v)$, and $\delta E$ and $\delta B$ are purely oscillatory in $x$ and $t$, then equation (27) indicates that $\delta f$ oscillates $90^\circ$ out of phase from the wave acceleration. Derivations from this phase relationship, and thus non-zero values of $W$, can be caused by two related sources: [1] inhomogeneity of $|\delta E|$, $|\delta B|$, or $f_0$ and [2] departure of the trajectories $x^*$ and $v^*$ in equation (27) from the trajectories for the homogeneous case due to wave-particle resonance and the average forces in plasma. Under our approximations, the effects of each source,
and accordingly \( W \), are first order in \( \mu \) or smaller.

Although both sources reflect the inhomogeneity of the system, it is convenient to use separate approaches in evaluating their effects to first order in \( \mu \). For clarity in the following development, we refer to the effects of [1] as inhomogeneity effects and the effects of [2] as trajectory effects. To consider the former, it is convenient to express \( \delta E \) and \( f_0 \) in terms of their transforms, \( \widetilde{\delta E}(\alpha, \xi) \) and \( \widetilde{f_0}(\alpha, s, \nu) \) defined in the local system with

\[
\delta E(x_\nu, t) = \int d\alpha d\xi \, \widetilde{\delta E}(\alpha, \xi) \, e^{-i\alpha t + i\nu \xi} ,
\]

\[
f_0(x_\nu, \nu, t) = \int d\alpha d\xi \, \widetilde{f_0}(\alpha, \xi, \nu) \, e^{-i\alpha t + i\nu \xi} .
\]

The separation of the wave and evolutionary time and distance scales assumed here for \( \delta E \) and \( f_0 \) requires that \( \widetilde{\delta E} \) and \( \widetilde{f_0} \) reflect equivalent separations in frequency and wave number space. That is, recalling the form for \( \delta E \) assumed in equations (10) and (11), \( \widetilde{\delta E} \) is strongly peaked for \( \alpha = \alpha_r + i\alpha_i \) and \( s = s_r + is_i \) such that

\[
|\alpha_r| \sim |\omega_n| \gg |\alpha_i| \sim \frac{1}{T}
\]

and

\[
|s_r| \sim |k_n| \gg |s_i| \sim \frac{1}{L} .
\]

(28a)
As \( f_0 \) is a slowly varying function, \( \tilde{f}_0(\alpha, s, v) \) is non-zero only for

\[
|\alpha| \sim \frac{1}{T},
\]

\[
|s| \sim \frac{1}{L}.
\]

(29b)

These properties of \( \tilde{\delta E} \) and \( \tilde{f}_0 \) can be exploited to expand the inhomogeneity effects in the small parameter \( \mu \) conveniently in frequency and wave number space. We note that \( \delta B \) can be expressed in terms of \( \delta E \) by substituting equation (28a) into equation (18) and integrating:

\[
\delta B(x,t) = \int d\alpha d\xi \left[ \frac{c}{\alpha} s \times \tilde{\delta E} - i \frac{c}{\alpha} A \cdot \tilde{\delta E} \right] e^{-i\omega t + i\xi \cdot x}. \tag{30}
\]

The quantity in brackets, though similar to the transform of \( \delta B \), retains a slow \( x \) dependence through \( A \). Using equation (30) the wave acceleration can be written as

\[
\frac{q}{m} (\delta E + \frac{1}{c} v \times \delta B) = \frac{q}{m} \int d\alpha d\xi \tilde{H} \cdot \tilde{\delta E} e^{-i\omega t + i\xi \cdot x}, \tag{31a}
\]

where \( \tilde{H}(\alpha, s, v, x) \) is defined as

\[
\tilde{H} = \tilde{H}^{(o)} + i\tilde{H}^{(i)}
\]

\[
= \left[ (1 - \frac{\xi \cdot v}{\alpha}) \tilde{H}^{(o)} + i\frac{\xi \cdot v}{\alpha} \right] + i \frac{1}{\alpha} (\xi \times A). \tag{31b}
\]

The slow \( x \) dependence of \( \tilde{H} \) arises again through \( A(x) \). From equation (17) we note that

\[
|\tilde{H}^{(o)}| / |\tilde{H}^{(i)}| \sim \mu. \tag{31c}
\]
The trajectory effects are obtained by explicitly evaluating the $t^*$ integration in equation (27) to first order in $\mu$. Substituting equations (28) and (31) into equation (27) we write

$$\frac{3}{\delta v} \delta f = i \frac{m}{q} \int d\alpha' ds' N^{(27)} (\alpha - \alpha', \xi - \xi', \alpha, \xi, \alpha', \xi', v, x)$$

$$\cdot \left[ \int d\alpha' ds' N^{(27)} (\alpha - \alpha', \xi - \xi') e^{-i\alpha t + i\xi \cdot x} \right]$$

(32)

The $t^*$ integration is incorporated in the matrix $N$, which is defined as

$$N^{(27)} (\alpha - \alpha', \xi - \xi', \alpha, \xi, \alpha', \xi', v, x)$$

$$= + \frac{q^2}{m} \frac{3}{\delta v} \int_{-\infty}^{t} dt^* i e^{-i\alpha (t^* - t)} \frac{H^{(27)} (\alpha - \alpha', \xi - \xi', \alpha')}{\delta v} (\alpha', \xi', v, x).$$

(33)

The integration is carried out to $O(\mu)$ in the next section, with the result

$$N = N^{(27)} + iN^{(27)}; \quad |N^{(27)}| / |N^{(27)}| \sim O(\mu).$$

We note that $N$ retains a slow $x$ and $t$ dependence arising through $H^{(27)}$ and the average force terms implicit in $x^*$ and $v^*$. The major term $N^{(27)}$ provides through equation (32) the zero order part of $\frac{3}{\delta v} \delta f$, i.e., the part which, in the homogeneous limit, oscillates out of phase with the wave force. The part of $\frac{3}{\delta v} \delta f$ in phase with the wave force comes from the first order, phase shifted part $\tilde{N}^{(27)}$, which, as anticipated above, results from wave-particle resonance and small corrections to
the integration trajectories caused by the average forces. The rather lengthy explicit expressions for $N^{(n)}$ and $N''^{(n)}$ are given in equations (43)-(45) in section 4. They are not necessary for the development in this section and are not repeated here.

Using equations (30) and (32), and the reality conditions

$$
\hat{\delta E}^*(\alpha, \xi) = \delta E (-\alpha, -\xi) ,
$$

$$
H^*(\alpha, \xi) = H (-\alpha, -\xi) ,
$$

$$
N^*(\alpha-\alpha', \xi-\xi', \alpha, \xi, \alpha', \xi', \ldots) = -N \left( (\alpha-\alpha'), (\xi-\xi'), -\alpha, -\xi, -\alpha', -\xi', \ldots \right) .
$$

$W$ can be written as

$$
W = \langle i \int d \Omega e^{-i(\alpha-\alpha')t + i(\xi-\xi') \cdot \xi} \hat{\delta E}^* (\alpha', \xi') \cdot \hat{H}^+ (\alpha', \xi') \cdot N (q, \beta, \varphi, \alpha'', \xi''), \hat{\delta E} (\alpha-\alpha'', \xi-\xi'') \rangle ,
$$

where

$$
d \Omega = d \alpha d \alpha' d \alpha'' d \xi d \xi' d \xi'' ,
$$

$$
\xi = \alpha - \alpha'' ,
$$

$$
q = \xi - \xi'' ,
$$

$$
\beta = \alpha ,
$$

$$
\varphi = \xi .
$$
Here \( H^\dagger = (H^*) \) is the hermitian adjoint of \( H \). The \( v, x, \) and \( t \) dependence of \( H \) and \( N \) have been suppressed to simplify the notation. The variables \( \alpha, \beta, \eta, \) and \( \varphi \) have been introduced to differentiate clearly for future use between the two separate dependences of \( N \) on both \( \alpha \) and \( \varphi \). From equation (33) we identify the \((\xi, \eta)\) dependence with the \( H^\dagger \) factor in \( N \), the \((\beta, \varphi)\) dependence with the exponential factor, and finally the \((\alpha'', \varphi'')\) with the \( \widetilde{\rho} \) factor in \( N \).

From the properties of \( \tilde{E} \) and \( \tilde{\rho} \) given in equation (29) we note that the integrand in equation (35) is non-zero only for large values of \( \alpha_r, \alpha'_r, \xi_r, \) and \( \xi'_r \) and for small values of \( \alpha_i, \alpha'_i, \xi_i, \xi'_i, \) and \( \xi'' \), i.e.,

\[
|\alpha_r|, |\alpha'_r| \gg |\alpha_i|, |\alpha'_i|, |\alpha''| \quad ;
\]

\[
|\xi_r|, |\xi'_r| \gg |\xi_i|, |\xi'_i|, |\xi''| \quad .
\]  

(36)

Thus the variation of the magnitude of the exponential factor, as well as the slow \( x \) dependence of \( H^\dagger \) and \( N \), can be neglected over the temporal and spatial averaging intervals. The averaging then only affects the exponential term with the resulting requirement that \( \alpha_r = \alpha'_r \) and \( \xi_r = \xi'_r \), as

\[
\frac{1}{\sqrt{\Delta}} \int_{t-t/2}^{t+t/2} \int_{\xi-\sigma/2}^{\xi+\sigma/2} e^{-i(\alpha-\alpha')t+i(\xi-\xi')(x-x')} dt' dx' 
\]

\[
= \delta_{\alpha_r, \alpha'_r} \delta_{\xi_r, \xi'_r} e^{(\alpha_i-\alpha'_i)t-(\xi_i-\xi'_i)(x-x')} ,
\]

(37)
where $\delta_{a,b}$ is the Kroneker delta function.

Using the above result and equation (36), we expand the
$(\alpha',\xi')$, $(\xi,\gamma)$, and $(\beta,\rho)$ dependences of $H^+ \cdot N$ in
equation (35) around $(\alpha_r,\xi_r)$. In light of equation (29),
this is equivalent to expanding in the inhomogeneity parameter
$\mu$. By definition the $(\gamma,\xi)$ and $(\beta,\rho)$ dependences of $N$
contribute separately to the expansion. We can express the
result in terms of
\[
\frac{\partial}{\partial \beta} N \quad \text{and} \quad \frac{\partial}{\partial \rho} N
\]
and the "complete" partial derivatives
\[
\frac{\partial}{\partial \alpha} N = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \beta} \right) N \quad \text{and} \quad \frac{\partial}{\partial \xi} N = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \rho} \right) N
\]

Relabelling the transform variables and eliminating
$\frac{\partial}{\partial \gamma} N$ and $\frac{\partial}{\partial \xi} N$ in favor of $\frac{\partial}{\partial \alpha} N$ and $\frac{\partial}{\partial \xi} N$, there results
to first order
\[
W = \int d\Omega \ \delta_{\alpha',\alpha} \ \delta_{\xi',\xi} \ \delta_{\xi,\xi'} \ \delta \mathcal{E}(\alpha',\xi') \cdot \left[ -2 \left( H_{\xi}^{t(0)} \cdot N^{(0)} \right) 
+ H_{\xi}^{t(0)} \cdot N^{(0)} \right] (\alpha' - \alpha) \left( H_{\xi}^{t(0)} \cdot \frac{1}{3\xi} \cdot N^{(0)} - N_{\xi}^{t(0)} \cdot \frac{1}{3\alpha} \cdot H^{(0)} \right) 
- (\xi' - \xi) \left( \frac{1}{3\xi} \cdot N_{\xi}^{t(0)} \cdot H^{(0)} - \frac{1}{3\xi} \cdot H_{\xi}^{t(0)} \cdot N^{(0)} \right) 
+ 2i \alpha' \cdot H^{t(0)} \cdot \frac{1}{3\rho} \cdot N^{(0)} 
+ i \cdot \frac{1}{2} \cdot N_{\xi}^{t(0)} \cdot H^{(0)} \cdot \delta \mathcal{E}(\alpha,\xi) \cdot e^{(\alpha' - \alpha - i\omega') t - (\xi' - \xi - i\omega') x}
\]

(38)
Here $H = H(\alpha_r, s_r, v, x)$ and

$$N = N(\psi, q, \rho, \omega, \alpha, s, v, x) \mid \psi = \rho = \alpha = \beta = q = \rho = s_r.$$  

We have set $\alpha_r' = \alpha_r$ and $s_r = s_r'$ within the brackets by virtue of the Kronecker delta functions.

In obtaining equation (38) we have used the property that $H^{(\omega)} N^{(\omega)}$ is hermitian, when evaluated at $\omega_r$ and $p_r$ as indicated above. This is equivalent to having the wave force oscillate out of phase with the part of $\frac{\partial \delta f}{\partial \psi}$ resulting from $N^{(\omega)}$ in the homogeneous limit.

In equation (38) the only dependence of $N$ on $\alpha''$ and $p''$ is through $\tilde{f}_0(\alpha'', p'')$, and the $\alpha''$ and $s''$ integrations can be easily performed. In the first three terms in equation (35) these integrations take the form

$$\int d\alpha'' d\delta'' \tilde{f}_0(\alpha'', s'', v) e^{-i\alpha'' t + i\delta'' \cdot x} = f_0(x, y, t).$$

In the fourth and fifth terms these integrations give the derivatives of $f_0$, i.e.,

$$i \int d\alpha'' d\delta'' \alpha'' f_0(\alpha'', \delta'', v) e^{-i\alpha'' t + i\delta'' \cdot x} = -\frac{\partial}{\partial t} f_0(x, y, t),$$

$$i \int d\alpha'' d\delta'' \delta'' \tilde{f}_0^*(\alpha'', \delta'', v) e^{-i\alpha'' t + i\delta'' \cdot x} = \frac{\partial}{\partial \delta} f_0(x, y, t).$$

The remaining integrations are dominated by $\tilde{\Delta E}(\alpha', s')$. 
and \( \tilde{\delta} E(\alpha, s) \). We first identify the second and third terms in equation (35) as derivatives of the slow \( x \) and \( t \) dependence determined through \( \tilde{\delta} E \) and \( \tilde{\delta} E^* \). We accordingly define the differential operators \( \frac{\partial}{\partial x} \lambda_w \) and \( \frac{\partial}{\partial x} \rho_w \) as temporal and spatial derivatives which act only on the time and space dependences of the wave parameters. After expressing the second and third terms in terms of these operators to eliminate the factors \((\alpha_i' - \alpha_i)\) and \((s_i' - s_i)\), the bracketed quantity in equation (38) depends only on the real parts of \( \alpha, \alpha', s, \) and \( s' \) through \( H \) and \( N \). The effect of the integration there, due to the narrowness of \( \tilde{\delta} E \), is to force \( \alpha_i', \alpha_i' \to \omega_n \) and \( s_i', s_i' \to k_n \) in \( H \) and \( N \). There results, valid to first order in \( \mu \),

\[
W = \sum_n \left\{ -A_n^* \cdot \left( H^{(\omega)} N^{(i)} + H^{(i)} N^{(\omega)} \right) \cdot A_n \right. \\
+ \frac{1}{2} \frac{\partial}{\partial x} \lambda_w \left[ A_n^* \cdot \left( N^{(\omega)} \frac{\partial}{\partial \omega} H^{(i)} - H^{(\omega)} \frac{\partial}{\partial \omega} N^{(i)} \right) \cdot A_n \right] \\
+ \frac{1}{2} \frac{\partial}{\partial x} \rho_w \cdot \left[ \left( \frac{\partial}{\partial x} N^{(\omega)} \cdot H^{(i)} - \frac{\partial}{\partial x} H^{(\omega)} \cdot N^{(i)} \right) \cdot (A_n A_n^*) \right] \\
- A_n^* \cdot \left[ \frac{\partial}{\partial x} N^{(\omega)} \frac{\partial}{\partial x} f_o - \sum_j \frac{\partial}{\partial x} N^{(\omega)} \frac{\partial}{\partial x} f_{o_j} \right] \cdot A_n \right\} .
\]

From the above discussion equation (36) is to be interpreted as follows. The matrices \( H \) and \( N \) and their derivatives are considered functions of \( \omega_n(x,t) \) and \( k_n(x,t) \), i.e.,

\[
H = H(\omega_n, k_n, \nu, x, t)
\]
and

\[ N = N(\gamma, \alpha, \beta, \rho, v, x, t) \mid \gamma = \beta = \omega_n, \quad \alpha = \rho = k_n. \]

The \( \beta \) and \( \rho \) derivatives of \( N^{(\omega)} \) are to be taken before this evaluation. The \( \omega_n \) and \( \rho_n \) derivatives act on both the \((\alpha, \gamma)\) and \((\beta, \rho)\) dependences of \( N \), i.e., the total \( \omega_n \) and \( k_n \) dependence of \( N \), and may be performed conveniently after the evaluation above. In the first three terms in equation (39) \( N \) is a function of \( f_0(x,v,t) \). In the final term this functional dependence is on \( \frac{2}{\beta x} f_0 \) and \( \frac{2}{\beta x} f_0 \) as indicated.

The derivatives \( \frac{2}{\beta x} f_0 \) and \( \frac{2}{\beta x} f_0 \), by definition, act on \( A_n(x,t) \) [see equation (10)] (and \( A_n^* \)), \( \omega_n(x,t) \), and \( k_n(x,t) \): the \( x \) and \( t \) dependences of \( N \) arising through \( f_0 \) and the averaged fields are to be held constant. As \( \omega_n \) and \( k_n \) occur in equation (36) as a result of the properties of \( \hat{\delta E}(\omega, \rho) \), they are affected by these differentiations.

Equation (39) gives the desired expression \( \hat{W} \), explicitly quadratic in wave amplitude \( A_n \) and first order in the inhomogeneity parameter \( \mu \). In the way of a summary, we analyze equation (39) with respect to the discussion of the sources of contributions to \( \hat{W} \) at the beginning of this section. The contributions due to the inhomogeneity of the waves and plasma are given by the second through fourth terms. The second and third are the contributions from the slow evolution of the wave parameters; the fourth is the contribution from the evolution of \( f_0 \). The first term gives
the contributions due to both wave particle resonance and the average forces. Included in this term are the effects due to the slow variation of both the magnitude and direction of $B_0$. 
4. Evaluation of $N$

We present here the calculation for the matrix $N$ which links $\frac{\partial f}{\partial \gamma}$ to $\ddot{\gamma} E$. The matrix $N$ is defined in section 3, equation (31)

$$N (\gamma, \xi, \beta, \rho, \alpha', \xi', \gamma, \xi, t) =$$

$$i \frac{q^2}{m} \frac{\partial}{\partial \gamma} \int_{-\infty}^{t} dt^* e^{-i \beta (t^* - t) + i \xi' \cdot (\xi' - \gamma)} H^T (\gamma, \xi, \gamma^*) \frac{\partial}{\partial \gamma^*} \tilde{f}_0 (\alpha', \xi', \gamma^*) .$$

In the integral, $\gamma^* (t^*)$ and $\gamma^* (t^*)$ are the single particle trajectories in the averaged fields $E_0$, $g$, and $B_0$. The exact evaluation of the integral is a very difficult calculation and is not attempted here. We rather make use of the ordering of the averaged fields discussed in section 2 to obtain approximate results, valid to first order in $\mu$.

The integration is performed by expanding around the zero order particle trajectories, i.e., for the homogeneous case. Thus the zero order contribution to $N$, $N_0$, is essentially the result for a homogeneous, time-independent system. The first order derivations from the zero order trajectories due to wave-particle resonance and the average forces give rise to a first order contribution, $iN^{(1)}$, phase shifted by a factor $i$ with respect to $N^{(0)}$.

The zero order orbits are those for a uniform magnetic field, i.e., gyromotion perpendicular to $B_0$ and free translation of the guiding center along $B_0$. Including the corrections to
The gyromotion is given in terms of $\tau = t^* - t$ by
\begin{align*}
\nu_1^* &= \nu_1 (1 + \bar{a}_\perp \tau) , \\
\phi_v^* &= \phi_v - \int_0^\tau \omega_c (x_v^*, \tau') d\tau' \approx \phi_v - \omega_c \tau - \frac{\bar{a}_\perp}{\nu_1} \omega_c \tau^2 ,
\end{align*}
where $\bar{a}_\perp$ is given in section 2.

The guiding center motion is
\begin{equation}
\nu_\parallel^* = \nu_\parallel + \bar{a}_\parallel \tau .
\end{equation}

Note that the solution for the guiding center velocity does not include the drift velocity $v_D$ resulting from $E_o$, $g$, and the gradient and curvature of $B_0$, since $v_D$ is independent of $\tau$ to first order. It is eliminated from the solution by the boundary condition $v^*(\tau = 0) = v$.

Integrating equations (41) we obtain
\begin{align*}
x_\parallel^* &= x_\parallel + \nu_\parallel \tau + \frac{1}{2} \bar{a}_\parallel \tau^2 , \\
x_1^* &= x_1 + \frac{\nu_1}{\omega_c} \sin (\phi_v^* + \Delta \phi_v) + \frac{\nu_1}{\omega_c} \sin (\phi_v + \Delta \phi_v) + v_D \cdot \hat{e}_i \tau , \\
x_2^* &= x_2 + \frac{\nu_2}{\omega_c} \cos (\phi_v^* + \Delta \phi_v) - \frac{\nu_2}{\omega_c} \cos (\phi_v + \Delta \phi_v) + v_D \cdot \hat{e}_2 \tau ,
\end{align*}
where
\begin{align*}
\nu_1 &= v_\perp (1 - \frac{\bar{a}_\perp}{\nu_1} \tau) , \\
\Delta \phi_v &= \frac{\bar{a}_\perp}{\omega_c} \nu_1 , \\

v_D &= \left( E_0 + \frac{m}{q} g \right) x \hat{e}_1 c \frac{g_B}{B_0} + \frac{1}{2} \frac{1}{\omega_c c^2} (v_\perp^2 + 2 v_\parallel^2) \left( \hat{e}_\parallel \times \frac{2}{3} \frac{\tau}{\tau^2} B_0 \right) .
\end{align*}

In equations (41) and (42), $\omega_c$, $\bar{a}_\parallel$, $\bar{a}_\perp$, and $v_D$ are
functions of \( x \) and \( t \).

Using equations (41) and (42), the integration in equation (40) can be accomplished. To lowest order, we note from section 2 that the \( \Phi_v \)-independent distribution function transform, \( \widehat{f}_\Phi(\alpha', \xi', v_n, v_t) \) can be substituted for \( f_v \). \( H \) is given by equation (31b). The result can be expressed as

\[
N = \frac{q^2}{\hbar^2} \frac{1}{\lambda_n} \sum_{m,l} \int_m \left( \frac{P_n v_n}{\omega_c} \right) e^{i(m-\phi_v - \phi_p)}
\]

\[
\cdot \left\{ \left( \Omega^{-1}(v_\xi, \xi, s) - i \frac{q_e}{\hbar} \Omega^{-1}(v_\xi, \xi, s) \right) \left( \tilde{a}_u \frac{3}{3v_\xi} + \tilde{a}_t \frac{3}{3v_n} - \tilde{a}_u \frac{3}{3v_n} \right) \right\}_{v_n}^{v_n}
\]

where

\[
G^4 = \sqrt{1 - \frac{q_e v_n^2}{\hbar^2}} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p}
\]

\[
G^b = \sqrt{1 - \frac{q_e v_n^2}{\hbar^2}} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p}
\]

\[
R = \sqrt{1 - \frac{q_e v_n^2}{\hbar^2}} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p} \left( \frac{P_n v_n}{\omega_c} \right) e^{i\phi_p}
\]

\[
\epsilon = \epsilon - \rho_n v_n - \omega_c\n\]

\[
\epsilon_t = \frac{1}{2} \rho_n \tilde{a}_u + l \omega_c \tilde{a}_u \n\]

\[
\hat{\epsilon}^4 = \hat{\epsilon} + i\hat{\epsilon}^2 \n\]

\[
\phi_p = \tan^{-1} \frac{\hat{\epsilon}^4}{\hat{\epsilon}^2} \n\]

\[
\phi_t = \tan^{-1} \frac{\hat{\epsilon}^4}{\hat{\epsilon}^2} \n\]
\( J_\ell(x) \) is the Bessel function of order \( \ell \).

The function \( \Omega^1 \) is defined in terms of the Fresnel integrals \( C(x) \) and \( S(x) \).

\[
\Omega^1(v, \epsilon) = \frac{\pi}{\sqrt{1-i^2}} \left[ i \left( \cos \left( \frac{v \epsilon}{4\epsilon_1^2} \right) \left( \frac{1}{2} - C \left( \frac{v \epsilon}{2\epsilon_1^2} \right) \right) + \sin \left( \frac{v \epsilon}{4\epsilon_1^2} \right) \left( \frac{1}{2} - S \left( \frac{v \epsilon}{2\epsilon_1^2} \right) \right) \right]
- \frac{\epsilon_2}{i\epsilon_1} \left( \cos \left( \frac{v \epsilon}{4\epsilon_1^2} \right) \left( \frac{1}{2} - S \left( \frac{v \epsilon}{2\epsilon_1^2} \right) \right) - \sin \left( \frac{v \epsilon}{4\epsilon_1^2} \right) \left( \frac{1}{2} - C \left( \frac{v \epsilon}{2\epsilon_1^2} \right) \right) \right].
\]

In the above equations, the first order term \( \Omega \) has been neglected compared to \( \beta \).

The leading contribution to \( \Omega^1 \) comes from the real part of \( \Omega^1 \) in the first term in the brackets in equation (43). We obtain \( \Omega^1 \) by taking the imaginary parts of \( \Omega^1 \) and \( i \frac{\beta}{\beta} \Omega^1 \) in the first term and the real part of \( \Omega^1 \) in the second and third terms.

Since \( \epsilon \) is proportional to the first order accelerations \( \bar{a}_\parallel \) and \( \bar{a}_\perp \), we see that \( v^2 \gg |\epsilon| \) except for small regions of velocity space in which \( p_\parallel v_\parallel \sim (\beta - l_\omega) \). This suggests that we make the usual separation of velocity space into resonant \( (\epsilon^2 << 1/\epsilon) \) and non-resonant \( (\epsilon^2 > 1/\epsilon) \) regions. Then \( \Omega^1 \) and \( i \frac{\beta}{\beta} \Omega^1 \) can be expanded to give

\[
\Omega^1(v, \epsilon) \approx - P \frac{i}{2\epsilon} + i \pi \left( \delta(v) - 2\epsilon_1^P \frac{i}{2\epsilon} \right),
\]

\[
i \frac{2}{\beta} \Omega^1(v, \epsilon) \approx i \frac{P}{2\epsilon^2}.
\]
5. Wave Equations and Conservation Laws

We now turn our attention to the complementary problem of the evolution of the wave parameters $A_n$, $\omega_n$, and $k_n$ described in the local frame by equations (18) and (19). These equations can be expanded in the inhomogeneity parameter $\mu$ using the expansion method used in section 3. This results in a hierarchy of equations for the wave parameters analogous to the formalism obtained above for the average distribution functions. To zero order in the expansion, well known dispersion relation results. To first order, we obtain transport equations for wave energy and momentum. These latter equations are used at the conclusion of this section to show that the result for $W$ obtained in section 2 conserves mass, energy, and momentum.

The behavior of the wave parameters is linked to the plasma behavior by the wave current density $\delta J$ term in equation (19), where

$$\delta J = \sum_i q_i \int \delta f_i d^3 \nu .$$

In view of equation (32) we can express $\delta J$ in terms of the transform of the wave electric field $\tilde{\Sigma E}$ by defining the conductivity tensor $\tilde{\Sigma}$ such that

$$\tilde{\Sigma}(\gamma, \beta, \rho, \alpha', \xi', \gamma') = -i \sum_i \frac{m_i}{d} \int d^3 \nu \nabla \cdot N(\gamma, \beta, \rho, \alpha', \xi', \gamma', \xi', \gamma')$$

$$= i \tilde{\Sigma}^{(c)} + \tilde{\Sigma}^{(sc)} , \quad (46)$$
where \( \gamma, \varphi, \beta, \) and \( \varphi \) are defined in equation (35b). Then

\[
\frac{\delta J}{\delta \gamma} = \int \frac{d\alpha d\xi}{d\alpha d\xi} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \gamma} \mathcal{F}(\gamma, \varphi, \beta, \alpha, \xi, \beta, \alpha) \exp(\frac{i\varphi t + i\beta}{\gamma}).
\]

(47)

The separation of \( \gamma \) and \( \beta \) dependences of \( \mathcal{F} \) on \( \alpha \), and the \( \varphi \) and \( \varphi \) dependences of \( \mathcal{F} \) on \( \gamma \), arise through \( N \) and, for the reasons discussed in section 3, are convenient for our expansion scheme. The zero and first order parts of \( \mathcal{F} \) come from \( N^{(0)} \) and \( N^{(1)} \) respectively. We have written the zero order part as \( i\mathcal{F}^{(0)} \) to emphasize that the zero order part of \( \delta J \) is out of phase with \( \mathcal{F} \).

Substituting equations (28a), (30), and (46) into equation (19), we obtain

\[
i \int d\Omega \mathcal{F} \left[ (1 - \frac{\alpha^2}{\gamma^2}) \int + \frac{\gamma^2}{\beta^2} \frac{\partial}{\partial \beta} + \frac{4\pi i}{\gamma} \mathcal{F}(\gamma, \varphi, \beta, \alpha, \xi, \beta, \alpha) \right. \\
- \frac{i}{\beta} (\varphi \times \omega + \omega \times \varphi) \right] \cdot \frac{\partial}{\partial \xi} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \gamma} \exp(\frac{i\varphi t + i\beta}{\gamma}) = 0.
\]

(48)

where \( d\Omega = d\alpha d\xi d\alpha d\xi \) and, again, \( \gamma, \varphi, \beta, \) and \( \varphi \) are defined in equation (35b).

We note from equations (46) and (33) that the \( \alpha^\prime \prime \) and \( \xi^\prime \prime \) are related to the slow \( t \) and \( \xi \) variation of \( f_0 \). The inequalities (36) are then valid here, i.e.,

\[
|\alpha| >> |\alpha^\prime|, |\alpha^\prime\prime|, \\
|\xi| >> |\xi^\prime|, |\xi^\prime\prime|.
\]
To evaluate equation (48) to zero order in $\mu$, we set $s = \beta = \alpha_r$ and $q = p = s_r$, and neglect the first order terms $\xi^{(1)}$ and $\Delta$. The integrations are then performed as discussed in section 2. Since $\delta E$ is the sum of uncorrelated wave modes (see equations (10) and (11)), equation (48) becomes a set of $n$ equations, one for each mode, of the form

$$D(\omega_n, k_n, x, t) A_n = 0 ,$$

(49a)

where

$$D(\omega_n, k_n, x, t) = \frac{i}{\rho} \left( 1 - \frac{k_n^2 c^2}{\omega_n^2} \right) + \frac{c^2}{\omega_n^2} k_n k_n - \frac{4\pi}{\omega_n} \sigma^{(o)}$$

and

$$\sigma^{(o)}(\omega_n, k_n, x, t) .$$

(49b)

The condition that equation (49a) have a non-trivial solution is

$$\det |D(\omega_n, k_n, x, t)| = 0 ,$$

(50)

which, as usual, we take to relate $\omega_n$ to $k_n$. We note that $D$ is a slowly varying function of $x$ and $t$ and, at each point $(x, t)$ equation (50) is the well known dispersion relation obtained for steady state, homogeneous plasma (e.g., Stix, 1962).

We now consider equation (50) to first order in $\mu$, in order to obtain transport equations for the wave energy and momentum densities. Beginning with the wave energy equation, we take the scalar product of equation (48) and the electric field of the $n^{th}$ wave mode $\delta E_n$ where

$$\delta E_n = \sqrt{\rho} \Re (\Lambda_n e^{i\psi_n}) .$$
Averaging over the wave time and distance scales, there results

$$
\langle i \int d\Omega \, \sum_{k_n} \langle \mathcal{E}_n \rangle \cdot \left[ \mathcal{X} \left( 1 - \frac{4 \xi^2}{z^2} \right) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi} \right] \mathcal{D} + \frac{4 \pi i}{\mathcal{E}} \, \mathcal{G} - \frac{i c}{\mathcal{E}} \left( \mathfrak{q} \times \mathfrak{q} + \Lambda \mathfrak{q} \right) \right] \cdot \mathcal{E} (x, \varphi) \, e^{-i(\alpha - \alpha')t + i(\xi - \xi')x} > 0 ,
$$

(51)

where \( d\Omega = d\omega d\xi d\alpha' d\alpha d\alpha'' d\alpha''. \) Equation (50) is similar in form to equation (35a) and can be evaluated in the same manner. Specifically, noting that the inequalities (36) are valid here, the expansion in \( \mu \) is performed by expanding \( \mathfrak{y} \) and \( \beta \) around \( \alpha_r \), \( \mathfrak{q} \) and \( \varphi \) around \( \mathfrak{s}_r \), \( \alpha' \) around \( \alpha_r' \), and \( \mathfrak{s}' \) around \( \mathfrak{s}_r' \). In light of equation (37) we see that the averaging sets \( \alpha_r' = \alpha_r \) and \( \mathfrak{s}_r' = \mathfrak{s}_r \).

Performing the integrations, and keeping terms to first order, we obtain

$$
\frac{1}{8 \pi} \frac{1}{2} \left( \mathcal{D} \cdot \frac{2}{\mathcal{E}} (\omega_n D) \cdot \mathcal{E} \right) - \frac{1}{8 \pi} \frac{1}{2} \left( \omega_n (\frac{2}{\mathcal{E}} D) \cdot (\mathcal{A}_n \mathcal{A}_n^*) \right) \\
+ \mathcal{A}_n \cdot \left[ \mathcal{G} - \frac{\mathcal{E}}{4 \pi} k_n \chi \left( \langle \mathcal{A} - \mathcal{A}^\dagger \rangle \cdot \mathcal{D} \rho_n \mathcal{F}_n \right) + \sum_{j,k} \frac{3}{\mathcal{E}} \mathcal{G} \langle \mathcal{F}_j \mathcal{F}_k \rangle \right] \mathcal{A}_n = 0 ,
$$

(52)

where \( \mathcal{D} = D(\omega_n, k_n, x) \) is given by equation (49b). Here \( \mathcal{G} \) and its derivatives are functions of \( \omega_n \) and \( k_n \), i.e.,

$$
\mathcal{G} (x, \varphi, \beta, \rho, \xi, t) \bigg|_{\beta = \omega_n, \xi = k_n} .
$$

We identify the first term in equation (52) as the time derivative of the total wave energy \( U_n \) where
\[ u_n = \frac{1}{2\pi} \mathbf{A}_n \cdot \frac{\partial}{\partial \mathbf{x}_n} (\omega_n \mathbf{D}) \cdot \mathbf{A}_n . \]  

(53)

We similarly identify the second term as the divergence of the total wave energy flux \( v_g \mathbf{U}_n \) where \( v_g \) is the wave group velocity

\[ v_g = -\frac{\omega_n}{\partial \mathbf{x}_n} (A_n A_n^*) / U_n . \]  

(54)

Equation (52) is then the desired transport equation for the wave energy density, with the remaining terms representing wave energy exchange with the plasma.

As noted in section 2, the problem of wave energy transport in the high frequency, short wavelength approximation has been considered by several authors (e.g., Akhiezer et al., 1975; Bernstein, 1975; Bernstein and Baldwin, 1977).

In order to compare equation (52) (which is expressed in the local system) with their results, it is convenient to transform it into the fixed system. With the help of a vector identity we note that

\[ \mathbf{k}_n \times (\mathbf{A}_n \cdot \mathbf{A}_n^T) = \sum_j (k_n \hat{e}_j \cdot \mathbf{k}_n) \mathbf{v} \cdot \mathbf{e}_j . \]  

(55)

Using equations (53) and (54), equation (54) becomes

\[ \frac{2}{\delta t} u_n + \nabla \cdot v_g u_n + A_n^* \cdot \mathbf{\sigma}' \cdot \mathbf{A}_n = 0 , \]  

(56a)

where

\[ \mathbf{\sigma}' = \mathbf{\sigma}'^{(1)} - \frac{1}{2} \mathbf{\sigma}'^{(2)} \mathbf{\gamma}_s^j \mathbf{\gamma}_s^j + \frac{1}{2} \left[ \nabla - \frac{\partial}{\partial \mathbf{x}_n} \right] \cdot \left( \frac{\partial}{\partial \mathbf{x}_n} \mathbf{\sigma}^{(3)} \right) \mathbf{\gamma}_s^j \mathbf{\gamma}_s^j - \frac{1}{2} \mathbf{\sigma}'^{(2)} \mathbf{\gamma}_s^j \mathbf{\gamma}_s^j + \sum_j \frac{\partial}{\partial \mathbf{x}_n} \mathbf{\sigma}'^{(2)} \mathbf{\gamma}_s^j \mathbf{\gamma}_s^j . \]  

(56b)
We note that the operator \( \left[ \nabla - \frac{2}{\mu} \right] \) acts on \( \varphi \) through its dependence on the average magnetic field \( B_0 \), as well as on the average plasma distribution functions. If we identify \( \varphi' \) as the hermitian part of the conductivity tensor \( \varphi_h \) assumed in the derivation of Bernstein (1975) and Bernstein and Baldwin (1977), equation (56a) is identical with their result. A general expression for \( \varphi_h \) is not given by these authors. However \( \varphi_h \) is meant to represent wave energy damping or growth due to slow \( x \) and \( t \) variations in the plasma and average fields. (Bernstein and Baldwin, 1977; Baldwin, private communication). Clearly \( \varphi' \), defined by equation (56b), is consistent with this role; we note that from equations (43)-(45) and (46) that \( \varphi^{(i)} \) is due to the effects of wave particle resonance and the average fields, while the remaining terms in equation (55b) explicitly depend on the slow variation of the plasma and average fields. We thus conclude that equation (55a) is equivalent to the results of these authors, with equation (56b) providing a new, explicit expression for \( \varphi_h \).

The transport equation for the wave momentum density is obtained by taking the vector product of equation (48) and the magnetic field of the \( n^{th} \) wave mode \( \frac{\delta B_n}{\delta t} \). Using equation (30) and averaging over the wave time and length scales, we obtain
The evaluation to first order in $\omega$ of equation (56), although complicated somewhat by its vector nature, is straightforward using our usual expansion method, and we omit the details. Using equations (49), (53) and (54) we obtain the simple result

$$\begin{align*}
&\frac{2}{\beta^2} \lambda_n (U_n \frac{k_n}{\omega_n}) + \frac{2}{\beta^2} \lambda_n \cdot (\varphi_n \frac{k_n}{\omega_n} U_n) + A_n^*(k_n \cdot \Delta \times k_n) \cdot A_n

&+ \frac{k_n}{\omega_n} A_n^* \left[ \sigma^{(-)} - \frac{c}{\sqrt{\pi}} k_n \times (\Delta - \Delta^T) - \frac{1}{\sqrt{\beta}} \sigma^{(-)} \frac{2}{\sqrt{\beta}} f_n \right]

&+ \sum_j \frac{2}{\beta_j^2} \sigma^{(-)} \frac{2}{\beta_j^2 f_n} \cdot A_n - A_n^* \frac{k_n}{\omega_n} \cdot \sigma^{(-)} A_n = 0 .
\end{align*}$$

This is the desired transport equation for the wave momentum density. We note that $\frac{k_n}{\omega_n} U_n$ is the total wave momentum density and $U_n \frac{\omega_n}{\omega_n}$ is the total wave pressure.

With the above transport equations (52) and (58), the expression found for the wave interaction term, given by equation (39), can be shown to conserve energy and momentum. That is, the first and second velocity moments of $W$, summed over species, give the expected wave contributions to momentum.
and energy conservation for the moments of equation (25). We note that mass conservation is a trivial result of equation (39). From the definitions of $H$ and $N$ in equations (31b) and (33) each term can be written as a divergence in velocity space, and the zero velocity moment of $W$ is identically zero.

Specifically, considering energy conservation first, we take the second moment of equation (39) and sum over species. With the definition of $\Sigma$ (equation (46)), we obtain

\[ \sum \frac{m_i}{2} \int d^3v v^2 W_i = \sum_n \left[ A_n^{(e)} \cdot \left( -\frac{\sigma}{\rho} + \frac{3}{3} \gamma^\xi \right) \left( \frac{\partial}{\partial x^j} f_i \right) - \frac{2}{3} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x^j} f_i \right) \right] A_n 
+ \frac{1}{2} \frac{3}{3} \left( A_n^{(e)} \cdot \frac{\partial}{\partial \omega_n} \gamma^\xi \right) \left( A_n \right) - \frac{1}{2} \frac{3}{3} \left( \frac{\partial}{\partial x^j} \gamma^\xi \right) \left( A_n A_n^{(e)} \right) \right]. \]

(58)

Using the wave energy density transport equation (52) and equations (10), (30), (49), and (55), this reduces to

\[ \sum \frac{m_i}{2} \int d^3v v^2 W_i = \frac{1}{4\pi} \nabla \cdot \left[ k_n A_n^2 - A_n \left( k \cdot A_n^{(e)} \right) \right] \]

\[ \frac{1}{4\pi} \nabla \cdot \left[ k_n A_n^2 - A_n \left( k \cdot A_n^{(e)} \right) \right] \]

\[ = \frac{1}{4\pi} \nabla \cdot \left( \delta B^2 + \delta E^2 \right) + \nabla \cdot S \]

where $S$ is the average wave Poynting flux

\[ S = \frac{c}{4\pi} \left( \delta E \times \delta B \right) \sim \frac{1}{4\pi} \sum \left( k_n A_n^2 - A_n \left( k \cdot A_n^{(e)} \right) \right). \]

The momentum calculation follows in a similar manner by
taking the first velocity moment of $W$ and summing over species. Substituting the wave momentum density transport equation (57) into the result, we obtain, after considerable algebra, the relation

$$\sum_i m_i \int d^3v \nu W_i = \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{S} - \frac{i}{4\pi} \nabla \cdot \left\langle \delta \mathbf{B} \delta \mathbf{B}^* + \delta \mathbf{E} \delta \mathbf{E}^* - \frac{i}{2} \mathbf{I} (\delta \mathbf{E}^2 + \delta \mathbf{B}^2) \right\rangle. \quad (60)$$

Here we used equations (30), (49), (55), and several vector identities. The terms on the right can be identified as the electromagnetic wave contributions to momentum balance. The first term is just the time derivative of the wave electromagnetic momentum density. The second is the divergence of the wave contribution to the Maxwell stress tensor.
6. Weak Magnetic Field Limit

As discussed in section 2, two orderings of the magnitude of \( B_0 \) are consistent with the separation of time and length scales assumed here: \( B_0 \) can be taken to be either strong \( ((\omega_c T)^{-n} \ll 1) \) or weak \( ((\omega_c T)^{-1} \gg 1) \). We are primarily interested in the former limit and, in the preceding sections have obtained the equations for the evolution of the distribution functions \( f_{oi} \) and the wave parameters \( A_n', \omega_n', \) and \( k_n \) appropriate for this limit. However, the extension of these results to the weak \( B_0 \) limit is straightforward and is outlined in this section.

In the weak \( B_0 \) limit we note that the effect due to the magnetic term in the operator \( \frac{D}{Dt} \) (equation (13)) is \( O(\mu) \), and there are no symmetry requirements on the \( f_{oi} \). Thus the \( f_{oi} \) and the wave parameters are to be considered in a fixed, cartesian system with the \( f_{oi} \) described by equations (12) and (14) and the wave parameters by equations (4) and (5).

The results in sections 3 and 5, in particular the expression for \( W \) and the wave equations, can be extended to the weak \( B_0 \) limit in the following manner. First as the coordinate axes are constant for this limit, we replace

\[
\frac{2}{\gamma x} \rightarrow \nabla_x , \\
\frac{2}{\gamma v} \rightarrow \nabla_v , \\
\Lambda \rightarrow 0 ,
\]

(61)
throughout sections 3 and 5. Thus from equation (31b)

$$\tilde{H} = \tilde{H}^{(v)} = \left[ (1 - \frac{\delta \cdot v}{\kappa}) I + \frac{\delta \cdot v}{\kappa} \right],$$

(62)

$$\tilde{H}^{(v)} \equiv 0.$$

The major change is in the matrix $\mathbf{N}$, defined by substituting equations (61) and (62) into equation (40). In the weak $B_0$ limit the zero order integration trajectories are those for a field-free system, i.e., free translation. The effects of $B_0$ as well as $E_0$ and $g$ can be treated as first order perturbations to these orbits. If we define

$$a = \frac{q}{m} E_0 + q + \frac{q}{mc} v \times B_0,$$

then the orbits can be written

$$v^* = v + a t,$$

$$x^* = x + v t + \frac{1}{2} a t^2.$$

(63)

Using equation (62) for $H$, equation (40) can be written in terms of $\nu = (\beta - p \cdot v)$ and $\xi = p \cdot a$ as

$$\tilde{N} = \frac{\delta^2}{\kappa m^2} \nabla \left[ \Omega^{-1}(\nu, \xi) - i \frac{1}{\delta^3} \Omega^{-1}(\nu, \xi) \cdot \nabla \nu \right]$$

$$\left( (1 - \frac{\delta \nu}{\kappa}) \nabla f_0 + \frac{\nu}{\delta} \xi \cdot \nabla f_0 \right),$$

(64)

where $\Omega^{-1}(\nu, \xi)$ is defined by equation (44).

As in section 4, $\tilde{N}_0$ is obtained from the real part of $\Omega^{-1}$. Taking the imaginary parts of $\Omega^{-1}$ and $i \frac{1}{\delta^3} \Omega^{-1}$ gives $N_a$. We
can again make the separation of velocity space into resonant \((\mathbf{v}^2 \ll |\mathbf{c}|)\) and non-resonant \((\mathbf{v}^2 \gg |\mathbf{c}|)\) regions. With this approximation, we obtain

\[
\mathcal{N}^{(n)} = -\frac{q^2}{m} \nabla \left[ \mathcal{P} \left( \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \right) \nabla \mathcal{F}_0 + \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \nabla \mathcal{F}_n \right) \right]
\]

(65a)

and

\[
\mathcal{N}^{(n)} = \frac{q^2}{m} \nabla \left[ \left( \pi f(\mathbf{v}) + \mathcal{P} \frac{1}{c} \mathbf{a} \cdot \nabla \frac{1}{c} \right) \left( \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \right) \nabla \mathcal{F}_0 + \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \nabla \mathcal{F}_n \right) \right]
\]

(65b)

With the replacements (61) and using equations (62) and (64) (or (65)) for \(\mathcal{H}\) and \(\mathcal{N}\), the expression for the wave interaction term \(W\) given by equation (39) and the wave equations given by equations (49), (50), (52), and (58) are valid in the weak \(B_0\) limit. If we consider electrostatic waves in a field-free plasma, i.e.,

\[
\mathcal{H} \rightarrow \frac{\mathcal{H}}{2},
\]

\[
E_0, \mathbf{q}, B_0 \rightarrow 0,
\]

then equations (14) with equation (39) and the wave equations reduce to the results obtained by Davidson (1972) if the mode-coupling terms \(O(\mathbf{E}^4)\) are neglected.
7. Summary

We have derived a quasilinear formalism describing the evolution of the plasma distribution functions for both strongly and weakly magnetized, inhomogeneous plasmas in the presence of high frequency, short wavelength waves. The principle new feature of this formalism is the appearance of new wave-particle interaction terms in the equations which arise from inhomogeneity in the plasma and magnetic field and from the presence of average electric and gravitational fields. Because these terms are critically dependent on the inhomogeneity and fields in the plasma, they do not appear in the quasilinear equations obtained for a homogeneous plasma.

As part of our formalism, we have also derived transport equations for the wave energy and momentum densities. With these equations, the formalism conserves mass, energy, and momentum. The wave energy density transport equation is equivalent to the equation obtained by Bernstein (1975) and Bernstein and Baldwin (1977). Our result provides an explicit expression for the previously unspecified hermitian part of the conductivity tensor for a collisionless plasma. This term is responsible for damping or growth of the wave energy density.
III - KINETIC THEORY OF ALFVÉN WAVE PRESSURE IN THE SOLAR WIND

1. Introduction

In this chapter we wish to demonstrate the application of the formalism developed in Chapter II to the solar wind. To this end we consider the kinetic effect of Alfvén waves on the evolution of the solar wind. We have chosen the Alfvén wave mode for this investigation for several reasons, as discussed in Chapter I. Briefly, Alfvén waves propagating outward from the sun account for much of the wave energy flux observed at 1 AU. The wave-plasma interaction is non-resonant in nature; resonant damping within 1 AU is negligible for the observed waves. The interaction is thus critically dependent on the inhomogeneity of the plasma and fields, and it is the inclusion of these effects that is the major strength of our formalism. Finally the transverse nature of the waves leads to a relatively simple mathematical description of the interaction, facilitating the physical interpretation of the interaction.

For the purpose of our analysis, we assume that the solar wind is a steady-state, radial, spherically symmetric flow of a proton-electron plasma. The plasma is strongly magnetized, i.e., the proton gyroradius is much smaller than the scale length of the system. The average magnetic field is taken to be radial; we neglect the rotation of the Sun. The Alfvén
wave energy flux is assumed to be spherically symmetric and directed radially outward from the Sun. For the low frequency Alfvén waves commonly observed in the solar wind (frequencies low compared to the proton gyro-frequency), the wave-particle interaction is insensitive to the frequency spectrum of the waves. We thus for simplicity assume the wave flux to be due to a single circularly polarized wave propagating parallel to the average magnetic field.

Within the context of this model we consider the radial evolution of the plasma above a radial distance \( r_0 \) at which the protons are collisionless and highly supersonic. We assume that electrons are both collision-dominated and highly subsonic throughout the spatial region of interest. We neglect collisional coupling between the electrons and protons. Under these conditions the Alfvén wave interaction with the electrons is macroscopic rather than kinetic in nature, and occurs through adjustments to the average electric potential of the plasma due to the presence of the wave. The kinetic properties of the wave-particle interaction appear solely in the evolution of the proton distribution function.

We are thus primarily interested here in calculating the evolution of the proton distribution function \( f_{op} \). The basic equations describing this evolution are obtained in section 2. The kinetic equation for \( f_{op} \) is obtained by specializing the results of Chapter II to our model and depends on the average electric potential, the wave amplitude, and the wave phase speed.
Since resonant damping is negligible in our model, these parameters are fluid in nature, i.e., they depend on the velocity moments of \( f_{\text{op}} \). The electric potential is found from consideration of the electron behavior. In light of the above discussion, the electrons are treated as a fluid with a polytropic equation of state. With the requirements that the plasma be electrically neutral and have zero average current, the electron momentum equation gives the electric potential in terms of the proton density. The equations for wave amplitudes are obtained from section 5 of Chapter II', and with the requirement of charge neutrality, can be expressed in terms of the proton density, velocity, and pressure tensor.

In section 3 we consider the solution to this set of equations. In order to avoid the complexity due to the appearance of both \( f_{\text{op}} \) and its velocity moments in the equations, we use a two fluid model to calculate the radial profiles of the electric potential, wave amplitude, and wave phase speed. For the protons in this model we obtain the momentum equation from the kinetic equation for \( f_{\text{op}} \) and use the double adiabatic equations of state (Chew et al., 1956). The use of this fluid description for the protons, and the neglect of the proton heat flux implicit in it, is justified since the electric potential, the wave amplitude, and the wave phase speed are sensitive only to the low order velocity moments of \( f_{\text{op}} \) (second order or smaller). Leer and Holzer (1972) have shown that essentially identical values for these moments, the proton density, velocity,
and pressure tensor, are obtained from both fluid and kinetic descriptions of the solar wind.

Using the values for the electric potential, wave amplitude, and wave phase speed obtained from the fluid model, we obtain numerical solutions for the radial evolution of $f_{op}$ in section 4. Our solutions show interesting distortions of $f_{op}$ which give rise to the familiar fluid effects. We summarize our results in section 5.
2. Basic Equations

We consider in this section the specialization of the results of Chapter II for a strongly magnetized plasma to the steady-state spherically symmetric solar wind model described above. The average magnetic field $B_\circ$ is radial, i.e.,

$$B_\circ = B_\circ(r) \hat{e}_r,$$

where

$$B_\circ(r) = B_\circ \left( \frac{R_\odot}{r} \right)^2.$$

The average electric and gravitational fields are also radially directed and can be written in terms of the electric and gravitational potential $\Phi_E$ and $\Phi_G$ as

$$E_\circ = -\hat{e}_r \frac{\partial}{\partial r} \Phi_E(r),$$

$$G = -\hat{e}_r \frac{\partial}{\partial r} \Phi_G(r),$$

where

$$\Phi_G = -\frac{GM_\odot}{r}.$$

Here $G$ is the gravitational constant and $M_\odot$ is the solar mass. To use the equations in Chapter II, it is necessary to work in a coordinate system locally aligned with $B_\circ$. In view of equation (1), a spherical polar coordinate system is the logical choice for our model. We thus take

$$\hat{e}_1 = \hat{e}_\theta,$$

$$\hat{e}_2 = \hat{e}_\phi,$$

$$\hat{e}_3 = \hat{e}_r = \hat{e}_r.$$
a) Proton Equations

We begin by considering the equations for the proton distribution function $f_{op}$, given by equations (II.23)-(II.26). With the assumption of spherical symmetry, we can work, without further loss of generality, in the plane $\theta = \frac{\pi}{2}$. We further note that for the circularly polarized Alfvén wave assumed here, the wave interaction term $W_p$ can be shown to be independent of the velocity phase angle $\phi_v$. In view of equations (2) and (3), equations (II.25) and (II.26) are then equivalent and the separation in equation (II.24) is unnecessary. Using equation (1b) to evaluate the acceleration terms in equation (II.26), we obtain

$$
\left( v_r \frac{2}{\delta r} + a_{\|} \frac{2}{\delta v_{\|}} + a_{\perp} \frac{2}{3v_{\perp}} \right) f_{op} = -W_p ,
$$

where

$$
a_{\|} = -\frac{q}{m} \frac{\delta}{\delta r} \frac{\Phi_\epsilon}{\Phi_\phi} - \frac{2}{3r} \frac{\delta}{\delta \phi} \frac{\Phi_\phi}{r} + \frac{v^2}{r} ,
$$

$$
a_{\perp} = -\frac{v_h v_{\|}}{r} .
$$

The wave interaction term $W_p$ is given by (II.39). For our model $W_p$ reduces to

$$
W_p = -A^* \left[ H^{(\omega)} + H^{(\epsilon)} - \frac{2}{\delta \phi} \frac{\partial}{\partial \phi} f_{op} \right] \cdot A 
- \frac{1}{2} \left[ \frac{2}{\delta \phi} \frac{\partial}{\partial \phi} \right] [ A^* (H^{(\epsilon)} - \frac{2}{3k} N_p) \cdot A ] ,
$$

As we consider here a single wave propagating along $B_0$, we have used
\[ k(r) = k(r) \hat{e}_n, \quad \text{(8)} \]

and have suppressed the wave mode index n. From section II.3, we note that the operator \( \frac{\partial}{\partial \chi_n} \) is the derivative with respect to \( r \) of \( A(r) \) (and \( A^*(r) \)) and \( k(r) \). The wave frequency \( \omega \) is constant (see equation (II.11)).

To evaluate equation (7), we first specify the wave electric field in the form of equation (II.10). In the plane \( \theta = \frac{\pi}{2} \), \( \delta E \) associated with the circularly polarized, radially propagating Alfvén wave can be written

\[ \delta E = \delta E(r) \mathbb{R} \left[ -i(\hat{e}_\phi - i\hat{e}_\theta) \exp i(\int k(r) dr - \omega t) \right], \quad \text{(9a)} \]

where

\[ |\omega| \ll |\omega_{\rho p}|. \quad \text{(9b)} \]

Here, as in Chapter II, \( v_{av} \) is a typical velocity for which \( f_{op} \) is finite. Comparing equation (9a) and (II.10) we identify \( A \) as

\[ A = i \delta E(r) \frac{(\hat{e}_\phi - i\hat{e}_\theta)}{\sqrt{2}}. \quad \text{(10)} \]

Using equations (4), (II.17), and (II.31b), we obtain

\[ H^{(e)} = (1 - \frac{k v_a}{\omega}) \mathbb{T} + \frac{\hat{e}_r v_a}{\omega} k \quad \text{(11a)} \]

\[ H^{(g)} = \frac{1}{\omega r} \left( v_n (\hat{e}_\theta \hat{e}_\phi + \hat{e}_\phi \hat{e}_\theta) + v_\theta \hat{e}_r \hat{e}_\phi + v_\phi \hat{e}_r \hat{e}_\theta \right). \quad \text{(11b)} \]

The matrix \( N \), defined by equations (II.43) - (II.45), is
simplified considerably for wave propagation along $B_0$. We note that only the elements of $N$ contributing to $N \cdot A$ are necessary here. In view of equation (8) we obtain

$$N^{(o)}_p = -\frac{q_p^2}{\alpha_p^2} \frac{1}{\nu} \left[ \frac{e^{-i\Phi_\nu v_\perp}}{v} \left( (1 - \frac{k v_\perp}{\omega}) \frac{\partial}{\partial v_\perp} + \frac{k}{2\omega} \frac{\partial}{\partial v_\parallel} \right) f_{op} \right] (\hat{e}_\phi + i\hat{e}_\phi), \quad (12a)$$

$$\frac{\partial}{\partial p_n} N^{(o)}_p (\frac{\partial}{\partial r} f_{op}) = -\frac{q_p^2}{\alpha_p^2} \frac{3}{\nu^2} \left[ \frac{e^{-i\Phi_\nu v_\perp}}{v^2} \left( (1 - \frac{k v_\perp}{\omega}) \frac{3}{\partial v_\perp} + \frac{1}{2} \frac{3}{\partial v_\parallel} \right) f_{op} \right] (\hat{e}_\phi + i\hat{e}_\phi), \quad (12b)$$

$$N^{(\phi)}_p = -\frac{q_p^2}{\alpha_p^2} \frac{\partial}{\partial \nu} \left[ (a_\parallel \frac{3}{\partial v_\parallel} + a_\perp \frac{3}{\partial v_\perp} + \frac{2\omega_c}{\nu}) \right]$$

$$\cdot \left( (1 - \frac{k v_\parallel}{\omega}) \frac{3}{\partial v_\parallel} + \frac{k}{2\omega} \frac{3}{\partial v_\parallel} \right) f_{op} - \frac{v_\perp}{\omega} \left( v_\perp \frac{3}{\partial v_\perp} - \frac{1}{2} \frac{3}{\partial v_\parallel} \right) f_{op} \right] (\hat{e}_\phi + i\hat{e}_\phi), \quad (12c)$$

where

$$\nu = \omega - k v_\parallel - \omega_c p.$$

The matrices $\frac{\partial}{\partial k} H^{(o)}_p$ and $\frac{\partial}{\partial k} N^{(o)}_p$ are obtained by straightforwardly differentiating equations (11a) and (12a) respectively.

Substituting equations (11) and (12) into equation (7), we obtain an expression for $W_p$. This expression is equivalent to the result for $W_p$ found by Hollweg (1978) using a WKB expansion valid for transverse circularly polarized waves propagating along a radial average magnetic field. This expression is, however, quite involved and will not be given here. Since $W_p$ is proportional to $\delta E^2$, we note that the left hand side of equation (5) is also $O(\delta E^2)$. Thus terms involving the expression

$$\left( v_\parallel \frac{3}{\partial r} + a_\parallel \frac{3}{\partial v_\parallel} + a_\perp \frac{3}{\partial v_\perp} \right) f_{op}$$
that occur in \( \mathcal{W}_p \) are \( O(\delta E^4) \) and can be neglected. After considerable algebraic manipulation, this property leads to a significantly simpler expression for \( \mathcal{W}_p \). Further simplification is obtained in view of the inequalities (9b) by expanding \( \nu \) around \( \nu = \nu_{cp} \) in equations (12), keeping terms to second order in \( \omega_c^{-1} \).

It is convenient to express the resulting expression in terms of the wave magnetic field amplitude \( \delta B(r) \) rather than \( \delta E(r) \). From equation (11.15), we see that

\[
\delta B^2 = \frac{k_c^2 c^2}{\omega^2} \delta E^2 + O(\mu) .
\]  

With this substitution, we obtain

\[
\mathcal{W}_p = \nu_{\|} \frac{D}{Dr} f_{ow} + \frac{q_p}{m_p} \frac{D}{Dr} \left( \frac{\omega}{2k_c} \frac{\delta B^2}{B_0^2} \right) \frac{1}{\nu_{\|}} \frac{D}{Dr} f_{op} + O(\delta B^4) ,
\]  

(14a)

where

\[
\frac{D}{Dr} = \frac{2}{3r} + \left( -\frac{q_p}{m_p} \frac{1}{3r} \Phi_E^1 - \frac{2}{3r} \frac{\nu_{\perp}}{r} \right) \frac{1}{\nu_{\|}} \frac{2}{\nu_{\|}} - \frac{\nu_{\perp}}{r} \frac{2}{\nu_{\perp}} ,
\]  

(14b)

\[
\Phi_E^1 = \Phi_E + \frac{1}{2} \frac{\omega}{k_c} \frac{\delta B^2}{B_0} ,
\]  

(14c)

\[
f_{ow} = \frac{\delta B^2}{B_0^2} \left\{ \nu_{\perp} \left[ \left( \frac{\nu_{\perp}}{k_c} - \frac{\nu_{\|}}{k_c} \right) \frac{3}{3 \nu_{\perp}^2} + \frac{1}{2} \frac{3}{3 \nu_{\perp}^2} \right] \left[ \left( \frac{\nu_{\perp}}{k_c} - \frac{\nu_{\|}}{k_c} \right) \frac{3}{3 \nu_{\perp}^2} + \frac{1}{2} \frac{3}{3 \nu_{\perp}^2} \right] f_{op} \right. \\
+ \left. \left( \frac{\nu_{\perp}}{k_c} - \frac{\nu_{\|}}{k_c} \right) \frac{3}{3 \nu_{\perp}^2} \right] \frac{2}{3 \nu_{\perp}^2} - \frac{1}{2} \nu_{\|} \frac{3}{3 \nu_{\perp}^2} \right] f_{op} \} .
\]  

(14d)

Inserting equation (14a) into equation (5), there results

\[
\frac{D}{Dr} f_{op} = 0 ,
\]  

(15)
where

\[ f'_{op} = f_{op} - f_{ow} \]

Equation (15) is the desired equation for \( f_{op} \). The wave effect occurs directly as a second order contribution, \( f_{ow} \), to \( f_{op} \). This contribution is analogous to the non-resonant adiabatic "broadening" associated with electrostatic waves in a homogeneous plasma (Kaufman, 1972; Krall and Trivelpiece, 1973). We note that \( f_{ow} \) is sensitive, through the first term in equation (14d), to the anisotropy of \( f_{op} \) in the rest frame of the wave, and, through the second term, to the anisotropy in the rest frame of the Sun. In addition, \( f_{o} \) depends on \( \omega \) and \( k \) only through the wave phase speed \( c/k \). Alfvén waves, such as assumed here, are non-dispersive (as shown below), and thus \( f_{ow} \) is independent of the wave frequency (and wave number).

There is also a wave contribution to the electric potential felt by the protons, changing it from \( \Phi_{E} \) to \( \Phi'_{E} \). This change is dependent on the polarization of the wave; the sign of the second term in equation (14c) is negative for lefthanded circular polarization, and the term is zero for linear polarization. It is also sensitive to the wave frequency, and thus to the wave frequency spectrum. However, as shown below, \( \Phi'_{E} \) is the electric potential felt by the electrons as well as the protons, and thus the wave contribution has no real effect. We note that the electric potential is to be determined from the requirement of charge neutrality in the plasma. In the presence of the wave, \( \Phi'_{E} \), rather than \( \Phi_{E} \), is seen to be the electric potential relevant to this requirement and is
determined through consideration of the electron equations. The individual values of $\Phi_E$ and the wave contribution are thus unimportant, and the wave effect is, as assumed here, independent of the wave frequency.

b) Electron Equations

Evaluating equation (11.39) for the electrons, we find that the wave interaction term is given by equation (14) where $f_{op}$ is replaced by the electron distribution function. We note that the major effect is the wave contribution to the electric potential. As the electrons are assumed here to be collisional and highly subsonic, the electron distribution function is nearly isotropic in both the rest frame of the Sun and the rest frame of the wave. Under these conditions the $f_{ow}$ term for the electrons can be seen from equation (14d) to be negligible.

The electrons may then be treated as a fluid. The electron continuity and momentum equations are

$$\frac{3}{2r} r^2 n_e V_e = 0 \; ,$$  \hspace{1cm} (16)

$$n_e m_e V_e \frac{3}{2r} V_e = -\frac{3}{2r} P_e + q_e n_e \frac{1}{2r} \Phi_E + n_e m_e \frac{3}{2r} \Phi_G \; ,$$ \hspace{1cm} (17)

where $n_e$ is the electron number density, and $P_e$ is the isotropic electron pressure. The inertial and gravitational terms can be neglected compared to the electron pressure and electric potential terms. Assuming a polytrop relationship between $P_e$ and $n_e$, i.e.,
\[
\frac{d}{dr} \left( P_e n_e^{-\alpha} \right) = 0 ,
\]  
\text{(18)}

Equation (17) becomes
\[
\frac{d}{dr} \left( \frac{\alpha}{\alpha+1} \frac{P_e}{n_e} - q_e \Phi E' \right) = 0 .
\]  
\text{(19)}

We require that the plasma be electrically neutral and have no average current, i.e.,
\[
\begin{align*}
n_e & = n_p = n , \\
v_e & = v_p = v ,
\end{align*}
\]  
\text{(20a)}

where
\[
\begin{align*}
n_p & = \int f_{op} d^3 v , \\
v_p & = \frac{1}{n_p} \int f_{op} v d^3 v .
\end{align*}
\]  
\text{(20b)}

Then equation (19) becomes the equation for \( \Phi E' \) in terms of the plasma density \( n \).

c) Wave Equations

To complete the mathematical description of our model, we obtain the equations for the wave phase speed \( \frac{\omega}{k} \) and amplitude \( \delta B \) by specializing the results of section 5 of Chapter II. We include the derivation of these equations primarily for completeness. As discussed in section 1, the behavior of \( \frac{\omega}{k} \) and \( \delta B \) is fluid in nature for our model. The evolution of these parameters for Alfvén waves in steady-
state, fluid models of the solar wind has received considerable attention in the literature, and equations similar to those derived here have been obtained by several authors. We refer the interested reader to the review by Hollweg (1975) for further details.

We begin by evaluating the conductivity tensor $\sigma$ using equations (12) for $N_p$ and its derivatives and the equivalent expressions for the electron matrix $N_e$ and its derivatives. We note that the inequalities (9b) are valid for the electrons as well as the protons, and expand $N_p$ and $N_e$ around $v = \omega_{cp}$ and $v = \omega_{ce}$ respectively. Substituting these results into equation (11.46) and using the charge neutrality and zero current conditions, there results

$$\sigma^{(0)} = \frac{1}{2}(\hat{e}_e - i\hat{e}_\phi)(\hat{e}_e + i\hat{e}_\phi)\frac{e^2}{B_0^2} \frac{\omega}{\omega} \left[\frac{\omega}{k}(-\nu)^2 + P_\parallel - P_\perp\right],$$

$$\frac{2}{\partial P_{ln}} \sigma^{(0)}(\frac{\partial}{\partial r} f_{ori}) = \frac{1}{2}(\hat{e}_e - i\hat{e}_\phi)(\hat{e}_e + i\hat{e}_\phi)\frac{e^2}{B_0^2} \left[\frac{\omega}{k} \frac{\partial}{\partial r} (nm_p V) - \frac{\partial}{\partial r} (nm_p V^2) - \frac{2}{r} (P_\parallel - P_\perp)\right],$$

$$\sigma^{(1)} = -\frac{1}{2}(\hat{e}_e - i\hat{e}_\phi)(\hat{e}_e + i\hat{e}_\phi)\frac{k}{\omega r} \left[\frac{\omega}{k} V(\frac{\omega}{k} - V) - (P_\parallel - P_\perp)\right],$$

where

$$P_\parallel = m_p \int v_\parallel^2 f_{\parallel p} d^3v,$$

$$P_\perp = \frac{1}{2} m_p \int v_\perp^2 f_{\perp p} d^3v.$$

In the above equations we have neglected terms proportional
Using equation (20a) to evaluate the dispersion relation equation (II.50), we obtain the well known result for the wave phase speed

\[
\frac{\omega}{k} = V + V_A \sqrt{1 - \frac{4\pi}{B_o} \left( P_\parallel - P_\perp \right)} \quad ,
\]

where \( V_A = B_0/\sqrt{4\pi nm_p} \) is the Alfvén speed. Substituting equations (10), (13), and (20) into the wave energy transport equation (II.50) and using equation (16), we obtain after considerable algebra the equation for the wave amplitude \( \delta B \)

\[
\frac{\partial}{\partial t} \left[ r^2 n \frac{\delta B^2}{B_o^2} \frac{\omega^2}{k^2} \left( \frac{\omega}{k} - V \right) \right] = 0 .
\]
3. Method of Solution

a) The Kinetic Equations

The results derived in the preceding section form a closed set of equations describing the radial evolution of the proton distribution function $f_{op}$ in terms of its velocity moments. The kinetic equation for $f_{op}$, equation (15), has the form of a conservation law for the wave-independent part of $f'_{op}$

$$\frac{D}{D\tau} f'_{op} = 0 \quad \text{(24a)}$$

with

$$f'_{op} = f_{op} - f_{ow} \quad \text{(24b)}$$

That is, $f'_{op}$ is conserved along the characteristic trajectories of the operator $\frac{D}{D\tau}$, which, from equation (14b), are defined by

$$\frac{dv_{\perp}}{dr} = -\frac{v_{\perp}}{r} \quad \text{(25a)}$$

$$\frac{dv_{\parallel}}{dr} = -\frac{1}{v_{\parallel}} \frac{2}{r} \left( \frac{q_r}{m_r} \Phi'_{E} + \Phi'_{G} \right) + \frac{v_{\parallel}}{v_{\parallel}r} \quad \text{(25b)}$$

The equations can be easily integrated to obtain the constants of the motion.

$$\frac{d}{dr} \left( \frac{v_{\perp}}{r} \right) = 0 \quad \text{(26a)}$$

$$\frac{d}{dr} \left[ \frac{1}{2} v_{\perp}^2 + \frac{1}{2} v_{\perp}^2 + \frac{q_r}{m_r} \Phi'_{E} + \Phi'_{G} \right] = 0 \quad \text{(26b)}$$
Noting that $B_0 \sim r^{-2}$, the constants can be identified as the magnetic moment and the total energy, as expected.

If $f_{op}'$ is known at the reference level $r_o$, then the solution to equation (24) at the point $r$ can be written as

$$f_{op}'(r, v_{||}, v_{\perp}) = f_{op}'(r_0, v_{||0}, v_{\perp0}),$$

where $v_{||}$ and $v_{\perp}$ are related to $v_{||0}$ and $v_{\perp0}$ by equation (26). That is,

$$v_{\perp} = v_{\perp0} \left( \frac{r}{r_0} \right),$$

$$v_{||} = \pm \left[ v_{||0}^2 + v_{\perp0}^2 \left( 1 - \frac{r}{r_0} \right)^2 + \frac{2}{m_p} \left( \Phi_{E}^{'}(r_0) - \Phi_{E}^{'}(r) \right) + 2 \frac{\Phi_{G}(r)}{r} \left( 1 - \frac{r}{r_0} \right) \right]^{1/2},$$

where we have used $\Phi_{G} = -\frac{GM}{r}$. The electric potential $\Phi_{E}^{'}$ is a decreasing function of $r$ (as shown below), so for $r > r_o$ the quantity in brackets is positive. However equation (27) is not well defined due to the positive and negative square roots appearing in equation (28b).

This ambiguity is resolved by the boundary conditions of our model. We consider here the evolution of $f_{op}$ above a reference level $r_o$, which we take to be 40 solar radii. The protons are highly supersonic at and above this point. Therefore $f_{op}$ (and $f_{op}'$) are negligibly small for $v_{||} < 0$, and we can ignore this part of velocity space. Thus the positive sign in equation (28b) is to be taken. With this choice, $f_{op}'(r)$ is unambiguously defined for $r > r_o$ in terms of $f_{op}'(r_o)$ by equations (27) and (28).
We obtain the desired result for $f_{op}(r)$ by substituting $f'_{op}(r)$ into equation (24b). From the definition of $f_{ow}(r)$ (equation (14d)), this equation is seen to be an inhomogeneous differential equation in velocity space for $f_{op}$ at each point $r$. We note, however, that $f_{op}(r)$ differs from $f'_{op}(r)$ by a term of $O(\delta B^2)$, i.e., $f_{ow}(r)$. Therefore, the error introduced by replacing $f_{op}(r)$ by $f'_{op}(r)$ in evaluating $f_{ow}$ is negligible, $O(\delta B^4)$. We can then obtain $f_{op}(r)$ directly from $f'_{op}(r)$, i.e.

$$f_{op}(r) = f'_{op}(r) + f_{ow}(f'_{op}(r), r).$$

(29)

There is an additional complication in that, in the above discussion, the parameters $\Phi_E(r)$, $\omega_K(r)$, and $\delta B(r)$ are implicitly assumed to be known functions of $r$. Actually these parameters are functions of the velocity moments of $f_{op}$ through equations (19), (22), and (23) respectively and are, ideally, to be obtained in a completely self-consistent manner by solving these equations simultaneously with equations (27) through (29) for $f_{op}$. This is a laborious procedure, requiring repeated integrations of $f_{op}$ over velocity space at each point $r$. This level of sophistication, however, is not necessary to obtain $\Phi'_E$, $\omega_K$, and $\delta B$ under the conditions assumed here. We note that $\Phi'_E$, $\omega_K$, and $\delta B$ are sensitive only to the low order velocity moments of $f_{op}$ (i.e., $n$, $V$, $P_\parallel$, and $P_\perp$) and can be adequately obtained from a two fluid model of the plasma in which the proton heat flux is neglected.
We thus take the following approach. We first obtain the radial profiles of $\Phi'_\xi$, $\omega_k$, and $\delta B$ from the fluid model of the solar wind described below. Then, using these results, $f_{op}$ can be obtained directly from equations (27) through (29). This method does not lead to a completely self-consistent solution for $f_{op}$, $\Phi'_\xi$, $\omega_k$, and $\delta B$ because of the neglect of the proton heat flux in the fluid model. The solution could be improved by using the moments of the $f_{op}$ so obtained to compute better estimates of $\Phi'_\xi$, $\omega_k$, and $\delta B$ and from them a better estimate of $f_{op}$. Presumably this iterative process would converge to the completely self-consistent solution. We obtain here only the first level of solution, since for the reasons stated in section 1 of this chapter we argue that the values of $\Phi'_\xi$, $\omega_k$, and $\delta B$ will not be significantly affected by the neglect of the proton heat flux. Thus the first estimate of $f_{op}$ will be very close to the completely self-consistent result.

b) The Two Fluid Model

The equations of the fluid model include the equations obtained in section 2 for the electrons and the wave parameters. For convenience we collect these equations here

$$\frac{1}{2r} (nVr^2) = 0 \ , \quad (30)$$

$$\frac{2}{2r} \left( \frac{\alpha}{\alpha - 1} \right) \frac{P_e}{n} - q_e \Phi'_\xi = 0 \ , \quad (31)$$
To complete the fluid model, the above equations must be complemented by the momentum equation and equation of state for the protons. We obtain the former by multiplying equation (24a) by $V$ and integrating over velocity space. There results after some algebra

$$\frac{2}{3r} \left[ \frac{1}{2} V^2 - \frac{1}{2} \left( \frac{\omega^2}{k^2} - V^2 \right) \frac{B^2}{B_0^2} + \frac{q}{m} \Phi'_{p} + \Phi'_{e} + \frac{3}{2} \frac{P_{||}}{n m_p} + \frac{P_{\perp}}{n m_p} \right] = 0 .$$

As the protons are collisionless, we assume the double adiabatic equations of state (Chew et al., 1956).

$$\frac{3}{3r} \left[ P_{||} n^{-3} (B_0^2 + \delta B^2) \right] = 0 ,$$

$$\frac{3}{3r} \left[ P_{\perp} n^{-1} (B_0^2 + \delta B^2)^{-\frac{1}{2}} \right] = 0 .$$

Equations (30) through (36) can be solved numerically to obtain $\Phi'_{e}(r)$, $\frac{\omega}{k(r)}$ and $\delta B(r)$ for $r > r_o$. To this end we take the adiabatic index $\alpha$ to be 1.223 and assume the following values for $n, V, P_{||}, P_{\perp}, P_{e}, B_0$, and $\delta B$ at the reference level $r_o = 40 r_0$, where $r_0$ is the radius of the Sun. At $r = r_0$: 

$$\frac{2}{3r} \left( P_{e} n^{-\alpha} \right) = 0 ,$$

$$\frac{\omega}{k} = V + V_A \sqrt{1 - \frac{4\pi (P_{e} - P_{\perp})}{B_0^2}} ,$$

$$\frac{2}{3r} \left[ n r^2 \frac{\delta B^2}{B_0^2} \frac{\omega^2}{k^2} \left( \frac{\omega}{k} - V \right) \right] = 0 .$$
$n_o = 320 \text{ cm}^{-3}$,

$V_o = 325 \text{ km/sec}$,

$P_{\|0} = P_{\perp0} = 7.25 \times 10^{-9} \text{ dynes/cm}^2$,

$P_{e0} = 3.52 \times 10^{-8} \text{ dynes/cm}^2$,

$B_{\|o} = 9.38 \times 10^{-4} \text{ gauss}$,

$\delta B_0 = 0.1 B_{\|o} = 9.38 \times 10^{-5} \text{ gauss}$.

The above values are chosen as typical of existing fluid models of the solar wind (e.g., Hollweg, 1973) and lead to reasonable values of $n$ and $V$ at 1 AU. For these values at $r = r_o$,

$V_A = 114 \text{ km/sec}$ and $\frac{\omega}{k} = 439 \text{ km/sec}$. For convenience, we define $\Phi'(r_o) = 0$.

Figure 1 shows the behavior of the wave phase speed $\frac{\omega}{k}$ for $40 < \frac{r}{r_o} < 120$. The plasma bulk speed $V$ is indicated for the purpose of comparison, as is the plasma density $n$. We note that $\frac{\omega}{k}$ changes from much greater than $V$ to comparable to $V$ over the range of $r$ shown. Specifically, $\frac{\omega}{k} - V$ exceeds twice the parallel thermal speed $W_{\|}$ ($\sim 52 \text{ km/sec}$) of the protons at $r = 40 r_o$, where

$$W_{\|} = \sqrt{\frac{P_{\|}}{2nm_p}}.$$  \hspace{1cm} (37)

With increasing $r$, $\frac{\omega}{k}$ gradually decreases, approaching the slowly increasing bulk speed $V$. We note that $W_{\|}$ decreases slightly with $r$, as can be inferred from equations (30) and (36a). At $r = 120 r_o \frac{\omega}{k} - V$ has become less than $W_{\|}$ ($\sim 45 \text{ km/sec}$). The decrease in $\frac{\omega}{k}$ is primarily due to the decrease in the Alfvén speed $V_A$. 
FIGURE 1: The radial evolution of the wave phase speed, $\frac{\omega}{k}$, the plasma bulk speed, $V$, and the plasma number density, $n$, from $40 \, r_\odot$ to $120 \, r_\odot$ in the fluid model described in the text.
FIGURE 1
Figure 2 shows the radial behavior of $\frac{\delta B^2}{B_0^2}$ and $\phi'_E$. We have plotted $\frac{\delta B^2}{B_0^2}$ here rather than $\delta B$, as the former is the relevant parameter in the calculation of $f_{ow}$ (see equation (14d)). Although both $\delta B$ and $B_0$ decrease with $r$, $\frac{\delta B^2}{B_0^2}$ increases slowly from 0.01 at $r = 40 r_\odot$ to 0.057 at $r = 120 r_\odot$. The electric potential $\phi'_E$ decreases gradually with $r$ with a drop of 150 volts over the range shown here. This change corresponds to an electric field directed radially outward of approximately $2.7 \times 10^{-9}$ volts/meter.
FIGURE 2: The radial evolution of the electric potential, \( \Phi'_E \), and the square of the ratio of the wave amplitude to the average magnetic field, \( \left( \frac{\delta B}{B_o} \right)^2 \), from 40 \( r_o \) to 120 \( r_o \) in the fluid model.
FIGURE 2
4. The Radial Evolution of the Proton Distribution Function

Using the radial profiles of $\Phi_E$, $\omega/k$, and $fB$ shown in Figures 1 and 2, we can solve equations (27) through (29) for $f_{op}$ using the procedure described in section 3a. We take $f_{op}' (r_o, v_{\|o}, v_{\perp o})$ to be a convected isotropic Maxwellian velocity distribution consistent with the boundary conditions at $r = r_o$ on the proton component of the fluid model, i.e.,

$$f_{op}' (r_o, v_{\|o}, v_{\perp o}) = \frac{m}{(2\pi)^{3/2}} \frac{e^{-\frac{(v_{\|o} - v_o)^2}{2w_o^2}}}{w_o} \exp \left( -\frac{v_{\perp o}^2}{w_o^2} \right),$$

where

$$w_o = \sqrt{\frac{P_{fo}}{2n m_p}}.$$  \hspace{1cm} (38b)

Numerical solutions to equations (27) through (29) for successively larger radial distances are shown in Figures 3a through 3e. In the lower panel of each figure the velocity contours of $f_{op}$ are given. The vertical axis is the velocity perpendicular to the radial direction. The horizontal axis is the velocity parallel to the radial direction. In the upper panel of each figure the contours of the wave-independent proton distribution function $f_{op}'$ are given for comparison. In both panels the innermost contour has a value of $e^{-1}$ of the maximum of $f_{op}$ or $f_{op}'$. The remaining contours are drawn at intervals of $1/e$ of the maximum.

The evolution of $f_{op}'$ corresponds to the evolution of the proton distribution function without the wave and is governed
FIGURE 3: Velocity space contours of the wave-independent proton distribution function $f'_{op}$ (upper panel) and of the proton distribution function $f_{op}$ in the presence of waves (lower panel). The proton distribution functions are shown at radial distances of 40 $r_\odot$, 60 $r_\odot$, 80 $r_\odot$, 100 $r_\odot$, and 120 $r_\odot$. 
FIGURE 3a
FIGURE 3b
FIGURE 3c
FIGURE 3d
FIGURE 3e
solely by the conservation of total energy and magnetic moment for the protons (equations (28)). As shown in the progression from Figures 3a to 3e, with increasing $r$, the bulk of distribution is accelerated to higher radial velocities due to the decrease in the electric potential. Simultaneously, the thermal spread of the distribution perpendicular to the radial direction (and $B_0$) decreases because of the decrease in $B_0$, causing $f'_\text{op}$ to become increasingly anisotropic in velocity space.

Comparing the evolution of $f_{\text{op}}$ with that of $f'_{\text{op}}$ in Figures 3a through 3e, one can easily see the effects due to the Alfvén wave. First of all, we see that at each radial distance the bulk of the distribution is shifted to higher radial velocities as compared to the case without waves, with the magnitude of this shift increasing with $r$. Thus the wave causes an increase in the proton bulk speed. This is a well known result of fluid theory (Belcher, 1971; Alazraki and Couturier, 1971). In addition, the wave decreases the thermal anisotropy of the protons, primarily by decreasing the thermal spread of the distribution in the radial direction. This effect of the wave is similar to that obtained by Patterson (1971) on the basis of fluid theory.

These macroscopic effects, well known from fluid theory, are actually manifestations of interesting detailed changes in the distribution function. To discuss these changes, it is convenient to consider the wave as transforming $f'_{\text{op}}$ into $f_{\text{op}}$ at
each point in space. This view is of course not strictly accurate. In fact, $f'_{op}$ and $f_{op}$ are the solutions for the proton distribution function under different physical conditions, i.e., in the absence and in the presence of the wave. Their differences are not due to the temporal growth of the wave at each point in space. However, this viewpoint simplifies the discussion of the qualitative aspects of wave effects.

The "transformation" of the wave independent $f'_{op}$ into $f_{op}$ is due to two distinct wave effects, which are represented mathematically by the two terms of $f_{ow}$ (see equation (14d)). The first term in $f_{ow}$ leads to a diffusion of protons away from the $v_\parallel$ axis ($v_\perp = 0$). This term is proportional to $v_\perp ^2$, and thus the diffusion is weak near the $v_\parallel$ axis and strong away from it. The second term leads to an acceleration of protons along the $v_\parallel$ axis. In contrast to the diffusion process, this effect is strongest along the $v_\parallel$ axis. Near the $v_\parallel$ axis, protons with $v_\parallel$ less than the wave phase speed $\omega / k$ are accelerated, while protons with $v_\parallel$ greater than $\omega / k$ are decelerated. In both cases, the magnitude of the acceleration increases as $|\omega / k - v_\parallel|$ increases. Thus the general tendency of this second term is to move the protons near the $v_\parallel$ axis into the rest frame of the wave.

The competition of these effects produces the most prominent feature of $f_{op}$. This feature is the low energy indentation of the contours near the $v_\parallel$ axis. The acceleration effect is
dominant near the $v_\parallel$ axis, whereas the diffusion effect is dominant for large $v_\perp$. Acceleration of the proton near the $v_\parallel$ axis causes the contours of $f_{\text{op}}$ to shift to the right. Strong diffusion at large $v_\perp$ tends to negate this effect, leading to the "horn" shapes at low energies.
5. Summary

We have specialized the general formalism developed in Chapter II to a problem of particular interest in the solar wind—the acceleration of the plasma by Alfvén waves of solar origin. Using the resulting quasilinear equations, we have found numerical solutions for the radial evolution of the proton distribution function in a simple solar wind model. For the first time, we have obtained the detailed features of the distribution function which gives rise to the familiar fluid results. We find that the Alfvén wave pressure is strongly velocity dependent, leading to the interesting distortions of the distribution function discussed above.

The obvious question is whether such exotic proton distributions could actually be observed in the solar wind. It is possible that these distributions are unstable to a variety of high frequency wave modes (e.g., the loss cone instability). In this case, the detailed shapes we have obtained would be unobservable. However, the Alfvén wave fluxes that tend to produce them would serve as an energy source for such instabilities. This question is beyond the scope of our investigation. Our purpose here was to isolate the changes in the proton distribution function due solely to the presence of low frequency Alfvén waves. The novel results we have obtained illustrate the desirability of our general kinetic approach to wave-particle interactions caused by inhomogeneity in the solar wind.
BIBLIOGRAPHY


ACKNOWLEDGMENTS

I am deeply grateful for the guidance and encouragement of Professor John Belcher. His personal support was especially invaluable during the final stages of this work. I have also benefited from many useful discussions with Professor Stan Olbert, and acknowledge his efforts, as well as those of Professor Miklos Porkolab, in the development of this thesis.

I am indebted to Professor Herb Bridge and the Interplanetary Plasmas Group for encouragement and financial support. I would also like to acknowledge the hospitality of Dr. Gerhard Haerendel and the Max Planck Institut für Extraterrestrische Physik during the preliminary stages of this work. I appreciate the financial support of the German Academic Exchange Service (DAAD) at that time.

During my graduate years at M.I.T. I have been fortunate to have had many stimulating discussions with Dr. Alan Lazarus, Dr. James Sullivan, and with my fellow graduate students Keith MacGregor, Ed Sittler, and Ralph McNutt. I am especially indebted to Ralph for his patience and willingness to listen.

Finally, I am deeply indebted to my wife, Christine. Her efforts in editing and typing, and more importantly, her patience and encouragement, have made the completion of this thesis possible.