LARGE DEFLECTION ANALYSIS OF THIN ELASTIC STRUCTURES BY THE ASSUMED STRESS HYBRID FINITE ELEMENT METHOD

by

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ABSTRACT

Two models for analyzing the large deflection, linear elastic, static behavior of structures have been developed. They are the consistent and inconsistent assumed stress hybrid finite element models. The consistent model satisfies the entire stress equilibrium equation while the inconsistent model satisfies only the linear portion of this equation. These models are consistently derived from the Principle of Virtual Work for two separate coordinate frames: a stationary system and a convected (moving), updated system. Throughout the development "correction" terms are maintained in all the functionals to minimize the approximate solution drifting away from the true solution. These correction terms correspond to checks on the stress equilibrium and compatibility in the reference state. The former check is recast into a more convenient form, for these models, than is usually used. Utilizing a tangent stiffness approach (updated at every solution step) various incremental and incremental-iterative solution procedures are used. When used the check conditions are represented as equivalent loads.

Careful attention is given to properly define various coordinate systems and reference frames. In conjunction with this the various definitions of stress and strain are discussed. Energy and work terms consistent with each system is considered in detail. The proper statements of the Principle of Virtual Work are defined.

The actual applications utilize flat and shallow elements to analyze the large deflection (moderate rotation), small strain behavior of thin, linearly elastic beams, plates, and shells. For the beam problems two elements are derived: flat and shallow curved, two node, six degree of freedom beam elements. For the plate and shell analysis two elements are derived: flat and shallow shell, three node, fifteen degree of freedom, triangular elements.

Several example problems are given and compared to independent solutions. Some of the studies in this work include the comparison of the consistent and inconsistent models, the flat and shallow elements, the Kirchhoff-Love and Marguerre shallow shell theories, the two coordinate systems, the effectiveness of the correction terms and solution procedures, and the adequacy of the models and methods. The results demonstrate that the consistent and inconsistent models yield essentially the same results, however, much less computational effort is required by the latter. Shallow elements perform better than flat elements, however, for fine meshes the results are comparable. The two coordinate systems yield slightly different results caused by the approximations made in the elements. For large deflection analysis the most effective solution technique is an incremental-iterative procedure with all the correction terms utilized. Overall, the models yield satisfactory results with simple elements.

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SECTION 1

INTRODUCTION

1.1 Background

The advent of computers enlarged the scope of structural mechanics, as well as a multitude of other diverse fields, enormously. The mathematics of numerical methods grew rapidly. Solutions to problems so complex that only a short time ago were intractable are now solvable. Many numerical techniques flourish, but one of the most useful of these, for solid mechanics, is the newest. It is the finite element method.

The finite element method essentially recognizes that structural mechanics problems may be reduced, mathematically, to boundary value problems [Strang and Fix, 1973; Crandall, 1956; Courant and Hilbert, 1953]. Hence a solution of the governing field equations in the interior of a continuum is sought subject to prescribed quantities on the boundaries. This method divides a solid continuum into a finite number of regions called elements. Each of these elements in themselves is a continuum and elements must be connected to each other in very special ways. This piecewise scheme results in an idealization yielding a finite number of simultaneous equations which can easily be formulated.

The finite element method is most generally based on the method of weighted residuals [Crandall, 1956]. For solid mechanics problems this is equivalent to the variational principles. The Principle of Virtual Work is most commonly used. Courant [1943] first used such principles to solve the St. Venant torsion problem. It was, however, the works done by Turner et al. [1956] and Argyris [1960], to name a few, which gave the finite element method impetus. The term "finite element method" was first introduced by Clough [1960]. During the 1960's its popularity grew exponentially. Since then, the basic principle has undergone various modifications, all of which yield alternative variational principles and associated finite element models. Washizu [1975], Pian and Tong [1972], and Pian [1972] describe many of these principles and their interrelationships.

The most popular methods derived from these principles are the compatible displacement model [Melosh, 1963], the equilibrium model [Fraeijs de Veubeke, 1964], and the mixed models [Herrmann, 1965]. The displacement model is based on the Principle of Minimum Total Potential Energy. A displacement field is chosen to be continuous in the entire continuum. This is a minimum principle. Fraeijs de Veubeke [1964] and Tong and Pian [1967], as well as others, have shown that for linear analysis the direct flexibility influence coefficients converge monotonically from below (i.e. too stiff) to the exact solution. The equilibrium model is based on the Principle of Stationary Total Complementary Energy. This is a force method where a stress field is chosen such that it satisfies stress equilibrium over the entire domain. Being a maximum principle, Fraeijs de Veubeke [1964] has shown that for linear analysis the direct flexibility influence coefficients monotonically converge from above (i.e. too flexible) to the exact solution. The mixed model is based on Reissner's variational principle which is a stationary condition. Here, one chooses displacement fields and stress fields in such a way as to allow the integrals in the principle to be defined. Thus, the analyst has more lattitude in describing these fields. Details may be found in Pian and Tong [1972] and Washizu [1975].

These basic models require various continuity conditions throughout the entire continuum Since, in an effort to simplify the analysis, the finite element method makes use of an assemblage of discrete continua (elements) to form the whole, the variables must be continuous from element to element. This, of course, may be difficult to establish. It is possible to generate a noncompatible model [Zienkiewicz, 1971] as long as the requirements of the patch test [Strang and Fix, 1973] are met. This model will no longer yield a lower bound to the solution, however, it will, in general, give better results than the compatible model. One of the major advances in finite element analysis was the concept of relaxing such continuity conditions across element boundaries. Naturally, the variables must still be fully continuous on the interior of each element. Pian [1972] describes many such possible arrangements which are named hybrid methods.

For instance, the assumed stress hybrid model, developed by Pian [1964, 1965], is based on a Modified Complementary Energy Principle. Compatible displacement fields must be chosen only along the interelement boundaries and stress fields on the interior of the elements. This too is a stationary principle and Tong and Pian [1969, 1970] have shown that while the solution can not be said to converge monotonically it does converge to the exact solution. In fact, as these authors have demonstrated, the convergence of this solution, for linear problems, is generally faster than that of the displacement or equilibrium models.

For linear analysis much work has been done utilizing many of the possible finite element models. Systematic classification of the various schemes and the variational principles upon which they are based are given by Pian and Tong [1969a], Pian [1971], Atluri and Pian [1972], Washizu [1975] and Knothe [1974]. This last paper shows an enormous variety of potential schemes, however, some may not be practical. In these works it can be seen how each consistently derived functional is interrelated with each of the others.* Many diverse fields such as heat transfer, fluid mechanics, and biomechanics [Atluri et al., 1975] have made use of these techniques. Since it is not possible here to discuss these applications in detail, reference is given to excellent reviews by Desai and Abel [1972], Zienkiewicz [1971], and Pian and Tong [1972].

1.2 Review of the Pertinent Finite Element Literature

Nonlinear theory has not received as much attention as the linear theory until very recently. The bulk of this work deals with the displacement (compatible and noncompatible) model. Reviews of this work may be found in Stricklin et al. [1971a, 1971b, 1972], Martin [1965, 1971], Felippa [1966], Oden [1969], and Haisler et al. [1972].

In solid mechanics nonlinearities have two basic forms: those due to geometric nonlinearities which result from nonlinear strain-displacement relations, and those due to material nonlinearities which result from nonlinear

*At the completion of this work it was brought to the writer's attention that Horrigmoe and Bergan [1975] and Horrigmoe [1975] just completed research extending this classification to the nonlinear range.

constitutive relations. Various solution techniques for the nonlinear energy methods include direct search schemes, iterative methods, incremental methods, incremental-iterative schemes, and the so-called "self correcting" schemes. Extensive reference lists on these methods are given by Stricklin et al. [1971a, 1971b, 1972, 1973], Tillerson et al. [1973], Desai and Abel [1972], and Pirotin [1971]. Other techniques include the Rungge-Kutta and predictor-corrector methods, [Ralston, 1965; Pian and Tong, 1971], as well as energy minimization methods [Schmit et al., 1968]. Hofmeister et al. [1971] suggests keeping terms in the functionals to act as checks on equilibrium at each stage of the analysis. These "corrective" terms keep the approximation from drifting away from the true solution.

Iterative schemes attempt to solve the nonlinear governing equations by techniques such as the Newton-Raphson procedure and perturbation methods. Incremental schemes attempt to solve the nonlinear problems as a series of piecewise linear problems. This allows one to make use of the vast amount of existing work and experience. Combination schemes of the incremental-iterative type try to step up the rate of convergence by making iterative corrections after a certain number of increments have been performed.

In conjunction with these techniques, one may associate an initial stiffness, tangent stiffness, or combination approach. With the initial stiffness procedure, the initial stiffness is preserved and all nonlinearities take the form of equivalent loads. This approach is usually associated with an iterative solution scheme. The tangent stiffness procedure forms a new tangent stiffness at every step of the solution (incremental and/or iterative) associated with the current state of deformation. This may be used with any solution scheme. Combination approaches retain the same tangent stiffness for a certain number of solution steps and then form a new tangent stiffness before continuing. Connor et al. [1968] shows that purely initial stiffness approaches may be mathematically unstable. Murray and Wilson [1968, 1969] state that, for some cases, if the tangent stiffness approach contains correction terms and the solution is allowed to iterate, an exact tangent stiffness is not required. Although this is apparently true, Martin [1971] points out that this sort of approximation requires a high degree of insight for a specific problem (or class of problems).

Geometric nonlinearities are the concern of this work. Such nonlinearities include the effects of instability (linear and nonlinear bifurcation buckling and limit load buckling) and large deflections (translations and/or rotations). Large deflections may be caused by large deformations (strains) or by large rigid body, small strain behavior. The important concept for geometric nonlinearities is that the equilibrium equations in the undeformed and deformed states are referred to different geometries. The term "large", however, can be misleading. For instance, a clamped or simply supported plate under a uniform load exhibits significant deviations from the linear theory when the maximum transverse deflection is of the order of half the plate thickness. When this deflection is one to two times the plate thickness, the problem is well into the nonlinear range.

Since the deformed geometries are no longer coincident, different coordinate systems may be used to describe the states of deformation. These systems are based on the fact that nonlinear analysis may be decomposed into a series of linear problems. Either a single stationary system may be used, or as deformation procedes in a step by step fashion, one may associate a new coordinate frame with each new, known state. There are several variations available for the latter system. One must be careful, however, to ensure that the proper stresses and strains are used with each appropriate coordinate system.

Although, as mentioned, most of the work in this area has utilized the displacement model (the first of which was by Turner et al. [1960]), other functionals have been used successfully. Some of these include the Reissner model, the modified Reissner model, the various mixed models, and the assumed displacement hybrid model which originally was developed for linear theory by Tong [1970], and has many of the same features as the assumed stress hybrid model. In the summary, Subsection 9.1, a partial survey list is tabulated. This list is mainly concerned with the literature discussing the large deflection, small strain, linear elastic behavior of thin structures. Although a variety of functionals and numerical schemes are presented this list is by no means complete.

Since the main discussion here centers about the assumed stress hybrid method, a short history of it is in order. For the linear theory, the method was introduced for plane stress and plate bending problems by Pian [1964, 1965] and Pian and Tong [1969a, 1969b]. Severn and Taylor [1966] applied it to the flexure of slabs. Allwood and Cornes [1969] utilized the technique for polygonal elements in plate bending. The method was extended to the bending of laminated (composite) plates by Boland [1971], Spilker [1972], Pian and Mau [1972] and Mau et al. [1972]. Mau and Witmer [1972] next extended the method to the static vibration problems of plates and shells using flat elements. The free vibration problem has been studied by Tabarrok [1971] who employed the Toupin principle. Then Mau and Pian [1973] solved the linear dynamics problem. Atluri [1973a] studied the same problem utilizing convolution integrals in time. Meanwhile Tanaka [1969] solved the linear static shell problem using deep, doubly curved, triangular elements. Crack analysis was also studied using this method by Pian et al. [1972]. Modifications of this principle were discussed by Wolf [1971, 1972], and Atluri [1971]. Further, Wolf [1974] discussed alternate, extended, and generalized assumed stress hybrid models.

The first attempt at using the assumed stress hybrid method for buckling problems was by Lundgren [1967]. This work was not consistently derived. To solve the linear prebuckling problem for flat, sandwiched plates, he utilized a linear elastic stiffness matrix by standard assumed stress hybrid methods and simply chose the geometric stiffness matrix associated with the displacement model. His results, however, were reasonable. Tong et al. [1973] used alternate hybrid models to generate geometric and mass matrices by a unified approach.

Among other models, Pirotin [1971] derived what he called a modified stress hybrid method to solve the geometric nonlinear problem. Although he alluded to a consistent assumed stress hybrid model, the method he actually uses is an "inconsistent" assumed stress hybrid model derived from a modified Reissner principle. This functional is the basis of one of the functionals in this work. Pirotin derived only the basic functional, for a convected coordinate frame, with no check conditions. In addition to equilibrium checks, and the use of two coordinate frames, the functional to be derived here allows for noncompatible displacement fields and other displacement mismatches. Additionally, Pirotin used only an incremental approach. In this work, several solution schemes are used and compared. While Pirotin used a rectangular, doubly curved, shell element, the present work utilizes simpler, but perhaps more general, flat and shallow triangular elements.

The first to report on a fully consistent assumed stress hybrid model for large deflection analysis was Atluri [1973b]. Here the basic functional along with a Hofmeister et al. [1971] type stress equilibrium check was derived for a convected coordinate system. There appears to be an error in this paper which is pointed out in Section 5.* Additionally, a more convenient stress equilibrium check is developed here.

Recently, Spilker [1974] reported on the assumed stress hybrid method for plasticity under small deflection theory. Presently, work on creep behavior under large deflection theory is proceding in the Aeroelastic and Structures Research Laboratory at Massachusetts Institute of Technology.

The present paper will investigate the large deflection behavior of structures. From the Principle of Virtual Work, various variational principles will be developed for two coordinate systems until, ultimately, the assumed stress hybrid functionals (consistent and inconsistent) are derived. The consistent model satisfies the entire, nonlinear stress equilibrium equation, while the inconsistent model satisfies only the linear portion of this equation. These functionals shall be cast in the tangent stiffness, incremental form of an initial stress solution. Extensive use of stationary and moving coordinate systems will be used. The general derivations will include all correction terms.

The general equations will be reduced to analyze the large deflection (moderate rotation), small strain behavior of thin, linearly elastic structures such as beams, plates, and shells. The elements utilized will be of the flat and shallow type. The concept of using flat and shallow elements for the analysis of curved structures is not new. For the analysis of shells Clough and Johnson [1968] and Aldstedt [1969] have used planar triangular elements; Megard [1969] makes some comparisons of planar and curved elements; Johnson

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*The consistent assumed stress hybrid model from the recent work by Horrigmoe and Bergan [1975] follows from Atluri.

[1967] utilizes a quadrilateral element formulated from four flat triangular subelements. These solutions are all based on the displacement model. Mau and Witmer [1972] formulated four node planar elements for linear shell analysis based on the assumed stress hybrid method. Various incremental and incremental-iterative procedures shall be used. Investigations shall be performed comparing accuracy, efficiency, convergence, physical modelling limitations, and possible problem dependencies among these elements and with independent sources.

1.3 Synopsis of the Research

Two models for analyzing the large deflection, static behavior of structures have been developed. They are the inconsistent and consistent assumed stress hybrid finite element models. Each case has been considered in two separate coordinate frames: a stationary system and an updated (moving) system. The functionals are cast in the initial stress, incremental form. Various incremental and incremental-iterative solution techniques are used in conjunction with a tangent stiffness procedure. The actual applications utilize flat and shallow elements to analyze the large deflection (moderate rotation), small strain behavior of thin, linearly elastic structures.

Unlike linear analysis, it is important to realize that in large deflection analysis, the geometries of the undeformed and deformed states do not coincide. Thus, it becomes obvious that various coordinate systems may be used to describe the deformation process. Section 2 is an introductory section in that it presents various coordinate system descriptions and all its implications. Specifically, Subsection 2.2 describes four common coordinate frames. To define equivalent forms of work and energy terms, various definitions of strain and stress are required for each coordinate system. These definitions are the subjects of Subsections 2.3 and 2.4 respectively. In order that these work and energy terms are indeed equivalent, consistent strain and stress definitions must be used concurrently as discussed in Subsection 2.5. Subsection 2.6 makes a brief attempt at comparing the definitions given here with some selected authors in the literature. Finally, the effects of some of the simplifying assumptions to be used later are discussed in Subsection 2.7.

Section 3 contains various considerations for thin, flat and shallow structures which will be required in the analysis. Subsection 3.2 discusses the equations of elasticity under flat and shallow theory. For the latter both the Kirchhoff-Love and Marguerre theories are presented. Further approximations are made here to further facilitate the analysis. These equations are found to apply on the element level. Subsection 3.3 discusses the assembly procedure for various coordinate frames and the associated transformations which are necessary.

With this information at hand attention is now turned to the derivation of the functionals which govern the element matrix generation. First, in Section 4, the functionals are developed for a Stationary Lagrangian system. Starting from the Principle of Virtual Work, Subsection 4.2 derives the Principle of Stationary Total Potential Energy and generalizes it to the Hu-Washizu principle. From this, Reissner's principle and the modified Reissner principle for an assemblage of elements is developed in Subsection 4.3. Making appropriate modifications of these the consistent and inconsistent assumed stress hybrid functionals are derived in Subsection 4.4. The consistent model satisfies the entire, nonlinear stress equilibrium equation while the inconsistent model satisfies only the linear part of this equation. This latter model may be thought of as a special case of the modified Reissner principle. In fact, in some instances they coincide. All correction terms (stress equilibrium and compatibility checks) are maintained throughout the entire derivation. In Subsection 4.4.3 these equilibrium checks are all identified and the functionals are written so the checks may easily be removed. It is also emphasized that the stress equilibrium check can be written in a much more convenient form than the typical Hofmeister et al. [1971] corrections.

In Section 5 a parallel development is given for an updated, moving coordinate frame referred to as the Convected, Updated Lagrangian system. The derivation was written so that simple comparisons could be made between the functionals in the two coordinate systems. However, the same simplified notation is used in both Sections 4 and 5. It is to be understood that the variables (strain, stress, displacement, etc.) have different definitions in each coordinate system as indicated in Section 2. The assumed stress hybrid functionals can also be derived directly from the Principle of Virtual Complementary Work and the associated Principle of Stationary Total Complementary Energy derived in Appendix B.

Section 6 discusses the general finite element matrix equations associated with the functionals of Sections 4 and 5. The element level matrices are discussed for the consistent and inconsistent models for the stationary and the updated systems in Subsections 6.1 and 6.2 respectively. Some problems associated with the consistent model and the stationary system are discussed. Subsection 6.3 describes the assembly procedure in more detail. Finally, Subsection 6.4 discusses the various incremental and incremental-iterative solution procedures.

These general matrix equations are now reduced to the actual ones to be used in Section 7. All the approximations and simplifications made in previous sections are employed here. Subsection 7.1 generates the appropriate matrices for two node (6 dof) shallow beam elements and notes that these can easily be reduced to flat beam elements. Both coordinate systems and assumed stress hybrid models are discussed. Although the Marguerre theory is chosen for the bulk of the work, comments are made concerning the Kirchhoff-Love theory. Because of the complexity and apparent lack of efficiency of the consistent model only the inconsistent model is discussed for plates and shells. Here, general three node (15 dof) triangular elements are used. This model, in both coordinate systems, is the subject of Subsection 7.2. Additionally, the satisfaction of the linear stress equilibrium equations is discussed. Subsection 7.3 simplifies the general large deflection analysis to the linear prebuckling of a flat plate. Here the consistent model is also considered. Subsection 7.4 discusses, in detail, the computational and updating procedures for each coordinate system and solution technique.

Applications, evaluation and discussion of these methods are presented in Section 8. The problems considered consist of the linear prebuckling of a simply supported, square, flat plate; and the large deflection analysis of a shallow, sinusoidal arch under a sinusoidal load; a shallow, circular arch under a central concentrated load; a simply supported and a clamped, square, flat plate under uniform load; a shallow, clamped, cylindrical panel under uniform load; a spherical cap under a central concentrated load; and a shallow, cylindrical panel under edge compression. The shallow, sinusoidal arch problem was used for the bulk of the investigations made. Such effects included the comparison of the consistent and inconsistent models, the flat and shallow elements, the Kirchhoff-Love and Marguerre theories, the two coordinate systems, the effectiveness of the correction terms and solution procedures, and the adequacy of the models and methods.

Section 9 is a statement of the summary and conclusions of this work. The summary includes a partial listing of the literature in this area. This section concludes with some suggestions for further research.

Finally, Appendices A-E provide some supportive comments.

SECTION 2

DESCRIPTIONS AND DEFINITIONS FOR LARGE DEFLECTION ANALYSIS

2.1 Introduction

For general three dimensional elasticity the assumptions of linear analysis have led to a unified theory. Small displacements imply small strains and negligable rotations. Thus, the deformed configuration and initial, undeformed configuration can be thought of as coinciding. Only one definition of stress is, therefore, necessary. Virtual work and the many other alternate energy formulations are straight forwardly derived. So are the finite element schemes based on these energy principles. With the single extension of allowing for large deflections (extensions, rotations or both) the theory of elasticity becomes considerably more complicated.

As a structure deforms with large deflections, its deformed and undeformed states are no longer coincident. Thus, it is obvious that different coordinate frames may be used to describe the deformation process. In addition, various definitions of strain and stress must be considered for each of these coordinate reference frames. One must be assured that the strain and stress definitions are consistent with each other. Furthermore, since energy considerations are the ultimate goal of the analysis, virtual work must be dealt with in a consistent manner.

There has been a considerable amount of work done in the area of large deflection, small strain analysis. In many of the earlier works not enough attention was given to carefully defining terms leading to consistent energy principles. A survey of some early formulations is given by Martin [1971]. As more and more work was done, based on earlier work, more and more confusion seemed to develop. More recently more care has been taken in this respect. However, since a firm foundation was never laid, one finds conflict in the literature. Often authors referring to the same definitions of strain, stress, etc. call them by different names.

This chapter will attempt to carefully define some of the coordinate systems, strains and stresses, commonly found in the literature. In addition, consistent virtual work terms will be given.

2.2 Coordinate System Descriptions

The two most popular coordinate systems would be the Lagrangian and Eulerian systems. In the Lagrangian system, one stationary reference frame is established for all time. All variables are measured in this one system. It may, of course, consist of simple rectangular Cartesian coordinates or, more generally, curvilinear coordinates. The Eulerian system is one which follows the ongoing process in time. It is always associated with the moving body. Generally, curvilinear coordinates are most appropriate here. The former system is used extensively for solid mechanics while the latter one is popular in fluid mechanics.

Restricting attention to solid mechanics, the Eulerian approach is not useful in that the state of deformation is generally not known. However, the Lagrangian approach is not the only alternative. In fact, for large displacement analysis there are four commonly used coordinate systems. These systems are based on the fact that nonlinear analysis may be decomposed into a series of linear problems for which initial conditions exist. Thus, the concept of initial values for the problem variables and incremental values is established. For instance, a parameter, u, may be decomposed into an initial value, u^o, and an incremental value, Δu, so that

$$u = u^o + \Delta u \tag{2.1}$$

Although the governing equations may be nonlinear in u^o, they can be considered linear in Δu. Since the initial values are known (from previous solution steps) the equations in the unknown incremental quantities are linear.

If the initial values are known at some state 'N', one may consider a state 'N+1' which is incrementally close to 'N' (Fig. 2.1). Defining "incrementally close" to mean that the governing equations may be written as linear functions of the unknown incremental quantities one may easily solve the system of equations by standard techniques. Once the solution determines the incremental values, Eq. 2.1 may be used to define the total quantities at state 'N+1' and incremental values define the difference between states 'N+1' and 'N+2'.

Throughout the solution procedure intermediate values of the problem variables are known. This implies that as deformation procedes in a step by step fashion one may consider each set of coordinate systems associated with each known state. Thus, the concept of using an updated or moving coordinate system is established. It is imperative to realize that this is not the same as an Eulerian approach, for an updated system is always in the last known configuration. This updated system has three common alternate approaches which will be described herein.

By far the most popular coordinate system in solid mechanics is the Lagrangian system. (For convenience, this system will be referred to herein as the Stationary Lagrangian (S.L.) system or the Total Lagrangian (T.L.) system [Bathe et al., 1973, 1974, 1975]). This was the first system used stemming from linear analysis. This system is used to describe the initial, undeformed configuration of the structure. Once it is established, it remains as a stationary reference frame regardless of how the structure deforms. All variables are measured from this system for all time. All derivatives and integrations are performed with respect to this one system. Essentially, as the solution proceeds step by step the structure is always thought of as remaining in the same initial configuration. To account for deformation the structure is considered with new sets of initial conditions applied to it. This is quite convenient in that although the distorted structure might take on quite complicated shapes, for the purposes of analysis, only the simpler initial configuration need be viewed.

In practice, however, there is a distinct disadvantage to such a system. If one is concerned with only moderately large deflections a great simplification arises from the general large deflection (possibly large strain) theory of elasticity. Thus, it is convenient and economical to take advantage of such simplifications. If now a problem is posed where very large deflections occur, this theory is now invalid and the approach compromised. Two alternatives are available. Either a more general theory is used, which may be very costly, or an updated coordinate system is used.

The first of these systems is the Updated Lagrangian (U.L.) system [Bathe et al. 1973, 1974, 1975]. In actuality it is not a true moving system. Its coordinate directions remain the same as the S.L. system. However, as deformations occur the coordinates of the structure are updated. The coordinate values in the U.L. system may be given as

$$(x_i)_{U.L.} = (x_i + u_i)_{S.L.} \tag{2.2}$$

Thus, all derivatives and integrations are performed with respect to the U.L. coordinates. The effect of this procedure is to eliminate initial displacement and strain terms from the formulation. Another significant effect involves the constitutive relations. These will be described in more detail in the following subsections. However, because it is not a true moving coordinate system the previously mentioned problem for very large deflections is inherent.

Another variation of coordinate systems is the Convected Coordinate (C.C.) system [Fung, 1965; Pirotin, 1971; Atluri, 1973b]. This is a true moving system in that each step the coordinate system follows the deforming system. However, unlike the U.L. system, the coordinate values are not changed as Eq. 2.2 would suggest. The undeformed and deformed configurations are referenced to the same set of coordinates although the metrics of each system are, of course, different. This is typical of differential geometry used in shell analysis. The effect here, as in all updated or moving systems, is to remove initial displacement and strain terms from the formulations. Note that with such a moving system the effect of large deflections is accounted for by the movement of the system. For example, large rigid body motions can be achieved here easily even though the theory may only account for small strains which imply small incremental deformations. So here a simplified theory is useful even for very large displacements.

The present work utilizes simple elements which are easily described in rectangular Cartesian coordinates. The U.L. formulation seems to be a good choice for such a situation. However, since the simplifications of small strain theory were to be assumed the drawback of the U.L. system needed to be avoided.

For the purposes of this work a compromise system was instituted. The ideas of the U.L. formulation and a convected system were combined. That is, an updated rectangular Cartesian system such as that of the U.L. system was used, but, instead of having the same directions as the original reference frame, it was convected to follow the deforming system. Such a system will be referred to as a Convected, Updated Lagrangian (C.U.L.) system. The advantages of this system will be shown in the following subsections.

The concept of large rigid body displacements but small deformation modes (strains) has led to a specially tailored C.U.L. system [Belytschko et al., 1973a, 1973b; Murray et al., 1969]. (Note that these authors designate the system C.C.) In this system the displacements are actually separated into two parts: rigid body and deformation displacements, i.e.,

$$u = u_{r.b.} + u_{def} \tag{2.3}$$

With this type of decomposition one can determine directly what part of the motion is due to rigid body displacements and what part is actually strain producing. If the strains are assumed small then they are only linearly related to the deformations, u_{def}, although they may be nonlinearly related to the total displacement, u. This can lead to further simplifications and still allow for very large displacements in the form of rigid body modes. For the displacement model in finite element analysis it is convenient to separate out rigid body displacements from total displacements. For other approaches, such as assumed stress hybrid models, this is not the case.

The analysis performed in this work was performed by using two of the above systems, the S.L. and the C.U.L., so that some comparisons could be made. Therefore, in the following subsections greater attention will be paid to these.

2.3 Definitions of Strain

Consider a body which in its original configuration, state 'O', is undeformed. At some later state 'N' the body has undergone deformation. It is assumed that the position and state of strain and stress are completely known at state 'N'. One now desires to determine these values for the yet unknown state 'N+1'. Let it also be assumed that states 'N' and 'N+1' are incrementally close to one another (Fig. 2.1).

Following the notation developed by Yaghmai and Bathe [Yaghmai, 1969; Bathe et al., 1973] a rectangular Cartesian coordinate system shall exist in state 'O' with base vectors ${}^{O}\bar{i}_i$. The coordinates and displacements measured along these directions will be ${}^{O}x_i$ and ${}^{O}u_i$ (${}^{O}\Delta u_i$) respectively. Similarly, a rectangular Cartesian coordinate system shall exist in state 'N', however, its base vectors will be ${}^{N}\bar{i}_i$. The coordinates and displacements measured along these directions will be ${}^{N}x_i$ and ${}^{N}u_i$ (${}^{N}\Delta u_i$) respectively. Thus, one may define radius vectors to each state as

$$ {}^{o}\bar{r} = {}^{o}x_k \, {}^{o}\bar{i}_k \qquad = ({}^{N}x_k - {}^{N}u_k)\,{}^{N}\bar{i}_k \tag{2.4a} $$

$$ {}^{N}\bar{r} = ({}^{o}x_k + {}^{o}u_k)\,{}^{o}\bar{i}_k \qquad = {}^{N}x_k \, {}^{N}\bar{i}_k \tag{2.4b} $$

$$ {}^{N+1}\bar{r} = ({}^{o}x_k + {}^{o}u_k + {}^{o}\Delta u_k)\,{}^{o}\bar{i}_k \qquad = ({}^{N}x_k + {}^{N}\Delta u_k)\,{}^{N}\bar{i}_k \tag{2.4c} $$

where

${}^{A}(\ )_b \equiv$ refers to configuration 'A' measured along base vector 'b' in state 'A'

Note that, in general, ${}^{O}\bar{i}_i$ and ${}^{N}\bar{i}_i$ do not coincide.

Green (Lagrange) strain is defined as follows when referred to state 'O'.

$$ 2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = {}^{N+1}_{o}\bar{r}_{,i} \cdot {}^{N+1}_{o}\bar{r}_{,j} - {}^{o}_{o}\bar{r}_{,i} \cdot {}^{o}_{o}\bar{r}_{,j} \tag{2.5} $$

where

$$ {}^{N+1}_{o}\bar{r}_{,i} = \frac{\partial({}^{N+1}\bar{r})}{\partial({}^{o}x_i)} \tag{2.6} $$

Placing the leftmost Eqs. 2.4a and 2.4c into Eq. 2.5 gives

$$ 2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = (\delta_{ki} + {}^{o}_{o}u_{k,i} + {}^{o}_{o}\Delta u_{k,i}) \cdot (\delta_{\ell j} + {}^{o}_{o}u_{\ell,j} + {}^{o}_{o}\Delta u_{\ell,j})\,\delta_{k\ell} - \delta_{ki}\delta_{\ell j}\delta_{k\ell} \tag{2.7} $$

or

$$ 2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = (\delta_{ki} + {}^{o}_{o}u_{k,i} + {}^{o}_{o}\Delta u_{k,i}) \cdot (\delta_{kj} + {}^{o}_{o}u_{k,j} + {}^{o}_{o}\Delta u_{k,j}) - \delta_{ki}\delta_{kj} \tag{2.8} $$

Finally,

$$ 2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = {}^{o}_{o}u_{i,j} + {}^{o}_{o}u_{j,i} + {}^{o}_{o}u_{k,i}\,{}^{o}_{o}u_{k,j} + {}^{o}_{o}\Delta u_{i,j} + {}^{o}_{o}\Delta u_{j,i} + {}^{o}_{o}u_{k,i}\,\Delta u_{k,j} + {}^{o}_{o}\Delta u_{k,i}\,{}^{o}_{o}u_{k,j} + {}^{o}_{o}\Delta u_{k,i}\,{}^{o}_{o}\Delta u_{k,j} \tag{2.9} $$

One may identify from Eq. 2.9 the total Green strain in state 'N' and the incremental Green strain from state 'N' to 'N+1'.

$$2\,{}^{0}e_{ij} = {}^{0}_{0}u_{i,j} + {}^{0}_{0}u_{j,i} + {}^{0}_{0}u_{k,i}\,{}^{0}_{0}u_{k,j} \tag{2.10}$$

and

$$\begin{aligned} 2\,{}^{0}\Delta e_{ij} = {}& {}^{0}_{0}\Delta u_{i,j} + {}^{0}_{0}\Delta u_{j,i} + {}^{0}_{0}u_{k,i}\,{}^{0}_{0}\Delta u_{k,j} \\ & + {}^{0}_{0}\Delta u_{k,i}\,{}^{0}_{0}u_{k,j} + {}^{0}_{0}\Delta u_{k,i}\,{}^{0}_{0}\Delta u_{k,j} \end{aligned} \tag{2.11}$$

Alternately one may define an Almansi strain referred to state 'N' as

$$2({}^{N}e_{ij} + {}^{N}\Delta e_{ij}) = {}^{N+1}_{N}\bar{r}_{,i}\cdot{}^{N+1}_{N}\bar{r}_{,j} - {}^{0}_{N}\bar{r}_{,i}\cdot{}^{0}_{N}\bar{r}_{,j} \tag{2.12}$$

Placing the outer relationships of Eq. 2.4a and 2.4c into Eq. 2.12 gives

$$\begin{aligned} 2({}^{N}e_{ij} + {}^{N}\Delta e_{ij}) = {}& (\delta_{ki} + {}^{N}_{N}\Delta u_{k,i})(\delta_{\ell j} + {}^{N}_{N}\Delta u_{\ell,j})\,\delta_{k\ell} \\ & - (\delta_{ki} - {}^{N}_{N}u_{k,i})(\delta_{\ell j} - {}^{N}_{N}u_{\ell,j})\,\delta_{k\ell} \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} 2({}^{N}e_{ij} + {}^{N}\Delta e_{ij}) = {}& (\delta_{ki} + {}^{N}_{N}\Delta u_{k,i})(\delta_{kj} + {}^{N}_{N}\Delta u_{k,j}) \\ & - (\delta_{ki} - {}^{N}_{N}u_{k,i})(\delta_{kj} - {}^{N}_{N}u_{k,j}) \end{aligned} \tag{2.14}$$

Finally

$$\begin{aligned} 2({}^{N}e_{ij} + {}^{N}\Delta e_{ij}) = {}& {}^{N}_{N}u_{i,j} + {}^{N}_{N}u_{j,i} - {}^{N}_{N}u_{k,i}\,{}^{N}_{N}u_{k,j} \\ & + {}^{N}_{N}\Delta u_{i,j} + {}^{N}_{N}\Delta u_{j,i} + {}^{N}_{N}\Delta u_{k,i}\,{}^{N}_{N}\Delta u_{k,j} \end{aligned} \tag{2.15}$$

where one may identify the total Almansi strain in state 'N' and the incremental updated Green strain from state 'N' to 'N+1' as

$$2\,{}^{N}e_{ij} = {}^{N}_{N}u_{i,j} + {}^{N}_{N}u_{j,i} - {}^{N}_{N}u_{k,i}\,{}^{N}_{N}u_{k,j} \tag{2.16}$$

and

$$2\,{}^{N}\Delta e_{ij} = {}^{N}_{N}\Delta u_{i,j} + {}^{N}_{N}\Delta u_{j,i} + {}^{N}_{N}\Delta u_{k,i}\,{}^{N}_{N}\Delta u_{k,j} \tag{2.17}$$

Note that ${}^{0}e_{ij}$ and ${}^{0}\Delta e_{ij}$ measure displacements along the ${}^{0}\bar{i}_{i}$ base vectors and derivatives are taken with respect to ${}^{0}x_{i}$. The updated strains ${}^{N}e_{ij}$ and

${}^{N}\Delta e_{ij}$ measure displacements along the ${}^{N}\bar{i}_i$ base vectors with derivatives taken with respect to the ${}^{N}x_i$ coordinates.

The strains referred to the initial configuration (${}^{O}e_{ij}$) are an obvious choice for use with the S.L. system. The updated strains referred to the convected, updated system (${}^{N}e_{ij}$) are most appropriate for the C.U.L. system. These sets of strains must be related to one another since they represent the same state of strain. To determine this relationship one must consider the relationship between the two coordinate systems (S.L. and C.U.L.).

In general two rectangular Cartesian coordinate systems can be related to each other via the direction cosines between their axes. For example

$$ {}^{O}\bar{i}_k = {}^{O,N}T_{km}\, {}^{N}\bar{i}_m \tag{2.18} $$

where

${}^{O,N}T_{km}$ ≡ array of direction cosines relating the base vectors in state 'O' to those of state 'N'

A unique inverse to this relation exists and is expressed as

$$ {}^{N}\bar{i}_k = {}^{N,O}T_{km}\, {}^{O}\bar{i}_m \tag{2.19} $$

where

$$ {}^{N,O}T_{km} = \left[{}^{O,N}T_{km}\right]^{-1} = \left[{}^{O,N}T_{km}\right]^{T} = {}^{O,N}T_{mk} \tag{2.20} $$

Note also that the displacements measured in one system may be related to components in another system in the same manner

$$ {}^{O}u_k = {}^{O,N}T_{km}\, {}^{N}u_m \tag{2.21} $$

and

$$ {}^{N}u_k = {}^{N,O}T_{km}\, {}^{O}u_m \tag{2.22} $$

A property of this array of direction cosines is

$$ {}^{O,N}T_{km}\, {}^{O,N}T_{kn} = \delta_{mn} \tag{2.23} $$

In light of Eq. 2.23 one may observe that

$$ \begin{aligned} {}^{O}u_k\, {}^{O}u_k &= {}^{O,N}T_{km}\, {}^{N}u_m\, {}^{O,N}T_{kn}\, {}^{N}u_n \\ &= {}^{O,N}T_{km}\, {}^{O,N}T_{kn}\, {}^{N}u_m\, {}^{N}u_n \end{aligned} \tag{2.24} $$

or

$${}^{o}u_k {}^{o}u_k = {}^{N}u_k {}^{N}u_k \tag{2.25}$$

Consider the position vector in state 'N'

$${}^{N}\bar{r} = ({}^{o}x_k + {}^{o}u_k)\, {}^{o}\bar{i}_k = {}^{N}x_m {}^{N}\bar{i}_m$$

Placing Eq. 2.18 into this relation yields

$${}^{N}x_m {}^{N}\bar{i}_m = ({}^{o}x_k + {}^{o}u_k)\, {}^{o,N}T_{km} {}^{N}\bar{i}_m \tag{2.26}$$

Taking the derivative of this with respect ${}^{O}x_i$ gives

$$\frac{\partial({}^{N}x_m)}{\partial({}^{o}x_i)} = (\delta_{ki} + {}^{o}_{o}u_{k,i})\, {}^{o,N}T_{km} \tag{2.27}$$

where

$$\delta_{ki} = \left\{ \begin{matrix} 0 & \text{if} & k \neq i \\ 1 & \text{if} & k = i \end{matrix} \right\} \equiv \quad \text{Kronecker delta} \tag{2.28}$$

It will now be demonstrated that the strains in the two systems may be related as follows

$$2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = 2\, \frac{\partial({}^{N}x_m)}{\partial({}^{o}x_i)} \frac{\partial({}^{N}x_n)}{\partial({}^{o}x_j)} ({}^{N}e_{mn} + {}^{N}\Delta e_{mn}) \tag{2.29}$$

Placing Eq. 2.15 into the above yields

$$2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = \frac{\partial({}^{N}x_m)}{\partial({}^{o}x_i)} {}^{N}_{o}u_{m,j} + \frac{\partial({}^{N}x_n)}{\partial({}^{o}x_j)} {}^{N}_{o}u_{n,i} - {}^{N}_{o}u_{k,i} {}^{N}_{o}u_{k,j}$$
$$+ \frac{\partial({}^{N}x_m)}{\partial({}^{o}x_i)} {}^{N}_{o}\Delta u_{m,j} + \frac{\partial({}^{N}x_n)}{\partial({}^{o}x_j)} {}^{N}_{o}\Delta u_{n,i} + {}^{N}_{o}\Delta u_{k,i} {}^{N}_{o}\Delta u_{k,j}$$

Placing Eqs. 2.25 and 2.27 into this gives

$$2({}^{o}e_{ij} + {}^{o}\Delta e_{ij}) = (\delta_{ki} + {}^{o}_{o}u_{k,i})\, {}^{o,N}T_{km} {}^{N}_{o}u_{m,j}$$
$$+ (\delta_{kj} + {}^{o}_{o}u_{k,j})\, {}^{o,N}T_{kn} {}^{N}_{o}u_{n,i} - {}^{o}_{o}u_{k,i} {}^{o}_{o}u_{k,j}$$
$$+ (\delta_{ki} + {}^{o}_{o}u_{k,i})\, {}^{o,N}T_{km} {}^{N}_{o}\Delta u_{m,j}$$
$$+ (\delta_{kj} + {}^{o}_{o}u_{k,j})\, {}^{o,N}T_{kn} {}^{N}_{o}\Delta u_{n,i} + {}^{o}_{o}\Delta u_{k,i} {}^{o}_{o}\Delta u_{k,j} \tag{2.30}$$

Finally, recalling Eq. 2.21 and simplifying gives

$$2({}^{\circ}e_{ij} + {}^{\circ}\Delta e_{ij}) = {}^{0}_{0}u_{i,j} + {}^{0}_{0}u_{j,i} + 2\,{}^{0}_{0}u_{k,i}\,{}^{0}_{0}u_{k,j} - {}^{0}_{0}u_{k,i}\,{}^{0}_{0}u_{k,j}$$

$$+ {}^{0}_{0}\Delta u_{i,j} + {}^{0}_{0}\Delta u_{j,i} + {}^{0}_{0}u_{k,i}\,{}^{0}_{0}\Delta u_{k,j} + {}^{0}_{0}\Delta u_{k,i}\,{}^{0}_{0}u_{k,j}$$

$$+ {}^{0}_{0}\Delta u_{k,i}\,{}^{0}_{0}\Delta u_{k,j} = 2({}^{\circ}e_{ij} + {}^{\circ}\Delta e_{ij}) \qquad (2.31)$$

Thus, Eq. 2.29 is shown to be correct and furthermore

$${}^{\circ}e_{ij} = \frac{\partial({}^{N}x_m)}{\partial({}^{0}x_i)}\,\frac{\partial({}^{N}x_n)}{\partial({}^{0}x_j)}\,{}^{N}e_{mn} \qquad (2.32)$$

and

$${}^{\circ}\Delta e_{ij} = \frac{\partial({}^{N}x_m)}{\partial({}^{0}x_i)}\,\frac{\partial({}^{N}x_n)}{\partial({}^{0}x_j)}\,{}^{N}\Delta e_{mn} \qquad (2.33)$$

also can be shown to hold.

At this point it is straight forward to introduce the strain tensor for the simpler U.L. system. Since in this system the base vectors are always taken to be in the same directions, i.e.

$${}^{\circ}\bar{i}_i \equiv {}^{N}\bar{i}_i \qquad (2.34)$$

then the displacements

$${}^{\circ}u_i \equiv {}^{N}u_i \qquad (2.35)$$

Introducing Eq. 2.35 into Eq. 2.15 one observes

$$2({}^{N}e_{ij} + {}^{N}\Delta e_{ij}) = {}^{0}_{N}u_{i,j} + {}^{0}_{N}u_{j,i} - {}^{0}_{N}u_{k,i}\,{}^{0}_{N}u_{k,j}$$

$$+ {}^{0}_{N}\Delta u_{i,j} + {}^{0}_{N}\Delta u_{j,i} + {}^{0}_{N}\Delta u_{k,i}\,{}^{0}_{N}\Delta u_{k,j} \qquad (2.36)$$

where

$${}^{0}_{N}u_{i,j} = \frac{\partial({}^{0}u_i)}{\partial({}^{N}x_j)} \qquad (2.37)$$

From Eq. 2.18 the direction cosine matrix becomes an identity matrix

$${}^{0,N}T_{km} = \delta_{km} \qquad (2.38)$$

and so Eq. 2.27 becomes

$$\frac{\partial({}^N x_m)}{\partial({}^0 x_i)} = \left(\delta_{mi} + {}^0_0 u_{m,i}\right) \tag{2.39}$$

Placing Eqs. 2.36–2.39 into Eq. 2.29 would yield an identity and, therefore, the same strain transformation law would hold for the U.L. system. It is to be understood that these transformations reduce from Eq. 2.27 to Eq. 2.39.

All of the strain displacement relations given so far involve displacement measures which include both rigid body modes and actual strain producing deformation modes. If these two modes could be completely separated, as Eq. 2.3 would suggest, then an alternative strain displacement (deformation) relation could be written. The advantage here would be under small strain assumptions. Under these conditions the strain would be only linearly related to the deformation modes [Belytschko et al., 1973a, 1973b; Murray and Wilson, 1969], i.e.

$$2\,{}^N\Delta e_{ij} = {}^N_N u^d_{i,j} + {}^N_N u^d_{j,i}$$

where

$$u^d = u - u_{r.b.} \qquad = \text{acutal deformation}$$

This definition of strain is appropriately associated with the C.U.L. system. The advantage of this system would, of course, be lost for large strain analysis where even this relation would become nonlinear.

2.4 Definitions of Stress

In order that energy, or more specifically, virtual work be consistently defined for various coordinate systems, one must careful in defining appropriate stresses. Before doing so some preliminaries must be discussed.

Corresponding to the deformed state 'N' are curvilinear coordinates associated with the rectangular Cartesian coordinates. Their base vectors are defined by [Green and Zerna, 1968]

$$\bar{G}_i = \frac{\partial({}^N\bar{r})}{\partial({}^0 x_i)} = \left(\delta_{ki} + {}^0_0 u_{k,i}\right) {}^0\bar{i}_k \tag{2.40}$$

Or, considering Eqs. 2.18 and 2.27

$$\bar{G}_i = \frac{\partial({}^N\bar{r})}{\partial({}^0 x_i)} = \left(\delta_{ki} + {}^0_0 u_{k,i}\right) {}^{0,N}T_{km} {}^N\bar{i}_m = \frac{\partial({}^N x_m)}{\partial({}^0 x_i)} {}^N\bar{i}_m \tag{2.41}$$

Also, the metric associated with the deformed state 'N' is

$$G_{ij} = \bar{G}_i \cdot \bar{G}_j = \frac{\partial(^N\bar{r})}{\partial(^0X_i)} \cdot \frac{\partial(^N\bar{r})}{\partial(^0X_j)} = \frac{\partial(^NX_m)}{\partial(^0X_i)} \frac{\partial(^0X_n)}{\partial(^0X_j)} \delta_{mn} \tag{2.42}$$

The base vectors of the rectangular Cartesian system may be related to the position vectors by (see Eq. 2.4a)

$$^0\bar{i}_i = \frac{\partial(^0\bar{r})}{\partial(^0X_i)} \tag{2.43}$$

The metric of this system is simply

$$\delta_{ij} = {}^0\bar{i}_i \cdot {}^0\bar{i}_j = \frac{\partial(^0\bar{r})}{\partial(^0X_i)} \cdot \frac{\partial(^0\bar{r})}{\partial(^0X_j)} \tag{2.44}$$

Considering the definition of Eqs. 2.42 and 2.44 with Eq. 2.5

$$2\,{}^0e_{ij} = \frac{\partial(^N\bar{r})}{\partial(^0X_i)} \cdot \frac{\partial(^N\bar{r})}{\partial(^0X_j)} - \frac{\partial(^0\bar{r})}{\partial(^0X_i)} \cdot \frac{\partial(^0\bar{r})}{\partial(^0X_j)} = G_{ij} - \delta_{ij} \tag{2.45}$$

Elements of area and volume should now be considered for the undeformed and deformed configurations. The differential of area may be defined as

$$d\,{}^0A_k = d\,{}^0X_i\, d\,{}^0X_j \qquad (i \neq j \neq k) \tag{2.46}$$

and

$$d\,{}^NA_k = \sqrt{G\,G^{kk}}\; d\,{}^0X_i\, d\,{}^0X_j = \sqrt{G\,G^{kk}}\; d\,{}^0A_k \quad \text{(no sum)} \tag{2.47}$$

where

$$G = \det|G_{ij}| \tag{2.48}$$

$$G_{mp}\,G^{mn} = \delta^n_p \tag{2.49}$$

A differential in volume may be expressed as

$$d\,{}^0V = d\,{}^0X_1\, d\,{}^0X_2\, d\,{}^0X_3$$

and

$$d\,{}^NV = d\,{}^NX_1\, d\,{}^NX_2\, d\,{}^NX_3 = \sqrt{G}\; d\,{}^0X_1\, d\,{}^0X_2\, d\,{}^0X_3 = \sqrt{G}\; d\,{}^0V \tag{2.50}$$

With these preliminaries completed the descriptions of stress become more straight forward [Green et al. 1970]. Since the notion of curvilinear coordinates (and differential geometry) will be used here a stricter adherence to tensor notation will follow in this subsection.

Consider a stress resultant $\bar{T}^i$ exerted on the deformed surfaces in state 'N' by a neighboring element in the continuum. This may be expressed in terms of two stress vectors by (Fig. 2.2)

$$\bar{T}^i = {}^N_0\bar{\sigma}^i\sqrt{g\,g^{ii}} = {}^N_N\bar{\sigma}^i\sqrt{G\,G^{ii}} \qquad \text{(no sum)} \qquad (2.51)$$

Note that here

$$\bar{g}_i = {}^0\bar{i}_i \longrightarrow g = 1 \quad \text{and} \quad g^{ii} = 1 \qquad (2.52)$$

where

${}^N_0\bar{\sigma}^i$ ≡ stress vector acting on deformed surface but measured per unit undeformed area

${}^N_N\bar{\sigma}^i$ ≡ stress vector acting on deformed surface and measured per unit deformed area

The true Eulerian stress is defined as

$${}^N_N\bar{\sigma}^i\sqrt{G^{ii}} = {}^N_N\tau^{ij}\,\bar{G}_j \qquad (2.53)$$

while the first Kirchhoff stress or simply Kirchhoff stress is defined by

$${}^N_0\bar{\sigma}^i\sqrt{g^{ii}} = {}^N_0\bar{\sigma}^i = {}^N_0 S^{ij}\,\bar{G}_j \qquad (2.54)$$

Note that both the Eulerian and Kirchhoff stresses are defined to act along the deformed base vectors. However, while the Eulerian stress is based on deformed areas the Kirchhoff stress is based on undeformed areas. From Eqs. 2.51, 2.53 and 2.54 one observes a relationship between these stresses.

$$\bar{T}^i = \sqrt{g}\;{}^N_0 S^{ij}\,\bar{G}_j = {}^N_0 S^{ij}\,\bar{G}_j = \sqrt{G}\;{}^N_N\tau^{ij}\,\bar{G}_j$$

or

$${}^N_0 S^{ij} = \sqrt{\frac{G}{g}}\;{}^N_N\tau^{ij} = \sqrt{G}\;{}^N_N\tau^{ij} \qquad (2.55)$$

Since the Eulerian stress is known to be symmetric [Green et al. 1970], it follows that the Kirchhoff stress is symmetric. Through Eq. 2.40 these stresses can be written with new components in the ${}^{O}\bar{i}_i$ directions, namely

$$ {}^{N}_{O}S^{ij}\,\bar{G}_j = {}^{N}_{O}S^{ij}\left(\delta_{kj} + {}^{O}_{O}u_{k,j}\right){}^{O}\bar{i}_k = {}^{O}_{O}T^{ik}\,{}^{O}\bar{i}_k \qquad (2.56) $$

${}^{O}_{O}T^{ik}$ is known as the Lagrange or first Piola stress and should not be confused with the direction cosines ${}^{O,N}T_{km}$. Note that this stress is unsymmetric. Similarly

$$ {}^{N}_{N}\tau^{ij}\,\bar{G}_j = {}^{N}_{N}\tau^{ij}\left(\delta_{kj} + {}^{O}_{O}u_{k,j}\right){}^{O}\bar{i}_k = {}^{O}_{N}T^{ik}\,{}^{O}\bar{i}_k \qquad (2.57) $$

${}^{O}_{N}T^{ik}$ may be referred to as a second Piola stress. Like ${}^{O}_{O}T^{ik}$ it is unsymmetric. The two Piola stresses may be related by

$$ {}^{O}_{O}T^{ik} = \sqrt{\frac{G}{g}}\;{}^{O}_{N}T^{ik} \qquad (2.58) $$

Since both of these stresses are unsymmetric they are of little use in the finite element procedure. However, they have been used to describe Complementary Energy [Langhaar, 1953; Levinson, 1965; Lubov, 1970; Koiter, 1973; Fraeijs de Veubeke, 1972].

Since this work is primarily interested in using rectangular Cartesian coordinates it would be very useful to define stresses associated with ${}^{N}_{O}S^{ij}$ and ${}^{N}_{N}\tau^{ij}$ which are in the ${}^{O}\bar{i}_i$ or ${}^{N}\bar{i}_i$ systems but are symmetric. This may be done by use of the tensor transformation Eq. 2.39 (${}^{O,N}T_{mn}=\delta_{mn}$) or Eq. 2.27. Considering the more general case of ${}^{N}\bar{i}_i$ the stress tensor transformation is

$$ {}^{N}_{N}\sigma^{ij} = \frac{\partial({}^{N}x_i)}{\partial({}^{O}x_m)}\,\frac{\partial({}^{N}x_j)}{\partial({}^{O}x_n)}\;{}^{N}_{N}\tau^{mn} \qquad (2.59) $$

or

$$ {}^{N}_{N}\sigma^{ij} = \sqrt{\frac{g}{G}}\,\frac{\partial({}^{N}x_i)}{\partial({}^{O}x_m)}\,\frac{\partial({}^{N}x_j)}{\partial({}^{O}x_n)}\;{}^{N}_{O}S^{mn} \qquad (2.60) $$

where

$$ \frac{\partial({}^{N}x_i)}{\partial({}^{O}x_m)} = \left(\delta_{km} + {}^{O}_{O}u_{k,m}\right){}^{O,N}T_{ki} $$

It is understood that while the left superscript of ${}^N_N\tau^{mn}$ and ${}^N_O S^{mn}$ refer to directions $\bar{G}_n$, for ${}^N_N\sigma^{ij}$ it refers to the ${}^N\bar{i}_j$, directions. These stresses are symmetric and will be useful for updated coordinate systems. They are referred to as Cauchy stresses.

2.5 Virtual Work and the Constitutive Relations

Considering a body in the deformed state under the internal stress resultants $\bar{T}^i$ and prescribed body forces per unit undeformed volume $\tilde{\bar{F}}$, the equation of equilibrium may be written in vector form as

$$\bar{T}^i_{,i} + \tilde{\bar{F}} = 0 \tag{2.61}$$

where the undeformed configuration is measured in a rectangular Cartesian coordinate system and Eq. 2.52 applies. The boundary tractions corresponding to these stress resultants are

$$\bar{P} = \bar{T}^i \nu_i \tag{2.62}$$

where

ν_i = outward normal from the boundary

And, if S_1 is that part of the boundary S upon which external tractions occur then the mechanical boundary condition may be expressed as

$$\bar{P} = \tilde{\bar{P}} \qquad \text{on} \quad S_1 \tag{2.63}$$

where

$\tilde{\bar{P}}$ = prescribed surface traction on S_1

S_2 is that portion of the boundary S where the geometric boundary conditions are given. If one allows virtual displacements from the equilibrium configuration to take place without violating the conditions on S_2, then one may write the Principle of Virtual Work referred to the initial reference configuration as

$$-\int_V (\bar{T}^i_{,i} + \tilde{\bar{F}}) \cdot \delta^o\bar{u}\, d^oV + \int_{S_1} (\bar{P} - \tilde{\bar{P}}) \cdot \delta^o\bar{u}\, d^oA = 0 \tag{2.64}$$

where

$\delta^o\bar{u}$ = represents a virtual displacement

From Eqs. 2.4a and 2.4b one may observe that

$$ {}^{N}\bar{r} = {}^{o}\bar{r} + {}^{o}\bar{u} \tag{2.65} $$

Taking the variation of this

$$ \delta\, {}^{N}\bar{r} = \delta\, {}^{o}\bar{u} \tag{2.66} $$

Introducing Eq. 2.66 into Eq. 2.64 and separating the terms of the first integral gives

$$ -\int_V \bar{T}^{i}_{,i} \cdot \delta\, {}^{N}\bar{r}\, d^{o}V - \int_V \tilde{\bar{F}} \cdot \delta\, {}^{o}\bar{u}\, d^{o}V + \int_{S_1} (\bar{P} - \tilde{\bar{P}}) \cdot \delta\, {}^{o}\bar{u}\, d^{o}A = 0 \tag{2.67} $$

Integrating the first integral by parts with Eq. 2.62 yields

$$ -\int_V \bar{T}^{i}_{,i} \cdot \delta\, {}^{N}\bar{r}\, d^{o}V = \int_V \bar{T}^{i} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V - \int_{S_1+S_2} \bar{P} \cdot \delta\, {}^{N}\bar{r}\, d^{o}A \tag{2.68} $$

Placing this into Eq. 2.67 bearing in mind that virtual displacements are not permitted on S_2 then

$$ \int_V \bar{T}^{i} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V - \int_V \tilde{\bar{F}} \cdot \delta\, {}^{o}\bar{u}\, d^{o}V - \int_{S_1} \tilde{\bar{P}} \cdot \delta\, {}^{o}\bar{u}\, d^{o}A = 0 \tag{2.69} $$

Resolving the stress resultant as in the leftmost relation of Eq. 2.55 with $\sqrt{g}=1$ the first integral of Eq. 2.69 becomes

$$ \int_V \bar{T}^{i} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V = \int_V {}^{N}_{o}S^{ij}\, \bar{G}_j \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V \tag{2.70} $$

Placing Eq. 2.40 into this gives

$$ \begin{aligned} \int_V \bar{T}^{i} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V &= \int_V {}^{N}_{o}S^{ij}\, {}^{N}_{o}\bar{r}_{,j} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V \\ &= \int_V {}^{N}_{o}S^{ij} \left[\tfrac{1}{2}\, \delta ({}^{N}_{o}\bar{r}_{,i} \cdot {}^{N}_{o}\bar{r}_{,j})\right] d^{o}V \end{aligned} \tag{2.71} $$

In light of Eq. 2.42 this becomes

$$ \int_V \bar{T}^{i} \cdot (\delta\, {}^{N}_{o}\bar{r}_{,i})\, d^{o}V = \int_V {}^{N}_{o}S^{ij}\, \tfrac{1}{2}\, \delta G_{ij}\, d^{o}V \tag{2.72} $$

Taking the variation of Eq. 2.45 one observes that

$$ \delta\, {}^{o}e_{ij} = \tfrac{1}{2}\, \delta G_{ij} \tag{2.73} $$

Placing this result into Eq. 2.72 gives

$$\int_V \bar{T}^i \cdot (\delta {}^N_o \bar{F}_{,i}) \, d^oV = \int_V {}^N_o S^{ij} \delta^o e_{ij} \, d^oV \tag{2.74}$$

Now, placing this into Eq. 2.69 results in

$$\int_V {}^N_o S^{ij} \delta^o e_{ij} \, d^oV - \int_V \tilde{\bar{F}} \cdot \delta^o \bar{u} \, d^oV - \int_{S_1} \tilde{\bar{P}} \cdot \delta^o \bar{u} \, d^oA = 0 \tag{2.75}$$

Resolving the body forces, prescribed tractions and displacements in the ${}^o\bar{i}_i$ directions

$$\begin{aligned} \tilde{\bar{F}} &= {}^o\tilde{F}^i \, {}^o\bar{i}_i \\ {}^o\bar{u} &= {}^ou^i \, {}^o\bar{i}_i \\ \tilde{\bar{P}} &= {}^N_o\tilde{P}^i \, {}^o\bar{i}_i = {}^N_o\tilde{S}^{ik} (\delta^j_k + {}^o_o u^j_{,k}) \nu_j \, {}^o\bar{i}_i \end{aligned} \tag{2.76}$$

Finally, placing Eqs. 2.76 into Eq. 2.75 gives the scalar form of the Principle of Virtual Work.

$$\int_V [{}^N_o S^{ij} \delta^o e_{ij} - {}^o\tilde{F}^i \delta^o u_i] \, d^oV - \int_{S_1} {}^N_o \tilde{P}^i \delta^o u_i \, d^oA = 0 \tag{2.77}$$

From this derivation of virtual work it has been shown that for an S.L. system the Kirchhoff stress corresponds to the Green strain. Since an incremental analysis will ultimately be performed (see Sections 4 and 5) it is appropriate to consider that possibility now. Eq. 2.77 represents the virtual work at state 'N'. If we study a state "N+1' which is an incremental distance away from state 'N', then one may assume that the state variables change incrementally, i.e.

$$\begin{aligned} {}^N_o S^{ij} &\longrightarrow {}^N_o S^{ij} + {}_o\Delta S^{ij} \\ {}^o e_{ij} &\longrightarrow {}^o e_{ij} + {}^o\Delta e_{ij} \\ {}^o\tilde{F}^i &\longrightarrow {}^o\tilde{F}^i + {}^o\Delta\tilde{F}^i \\ {}^N_o\tilde{P}^i &\longrightarrow {}^N_o\tilde{P}^i + {}_o\Delta\tilde{P}^i \end{aligned} \tag{2.78}$$

Expanding Eq. 2.77 to account for Eqs. 2.78 results in

$$\int_V \left\{ \left[{}^N_0 S^{ij} + {}_0\Delta S^{ij} \right] \delta \left[{}^0 e_{ij} + {}^0\Delta e_{ij} \right] - \left[{}^0\tilde{F}^i + {}^0\Delta\tilde{F}^i \right] \delta \left[{}^0 u_i + {}^0\Delta u_i \right] \right\} d^0V$$

$$- \int_{S_1} \left\{ \left[{}^N_0 \tilde{P}^i + {}_0\Delta\tilde{P}^i \right] \delta \left[{}^0 u_i + {}^0\Delta u_i \right] d^0A = 0 \qquad (2.79)$$

Eq. 2.79 represents an incremental expansion of the Principle of Virtual Work for an S.L. system which will be used as a basis for the derivations in Section 4. However, one may wish to express a similar result in a C.U.L. system. To do this all expressions must be consistently transformed to the updated system so that the Principle of Virtual Work is consistent in both systems.

First, consider the strain. In the updated system one wants to express displacements and their derivatives with respect to the ${}^N\bar{i}_i$ directions and their corresponding ${}^N x_i$ coordinates. Thus, from Eq. 2.15, the total Almansi and the incremental updated Green strains are required. These strains are related to the Green strains of Eq. 2.79 by Eq. 2.29. Considering the first term of Eq. 2.79 with the strain transformation of Eq. 2.29, one observes

$$\int_V \left[{}^N_0 S^{ij} + {}_0\Delta S^{ij} \right] \frac{\partial ({}^N x^m)}{\partial ({}^0 x^i)} \frac{\partial ({}^N x^n)}{\partial ({}^0 x^j)} \delta \left[{}^N e_{mn} + {}^N e_{mn} \right] d^0V \qquad (2.80)$$

Recalling Eq. 2.60 the Kirchhoff stress will transform to the Cauchy stress by the transformation appearing in Eq. 2.80. Thus,

$$\int_V \left[{}^N_N \sigma^{mn} + {}_N\Delta\sigma^{mn} \right] \delta \left[{}^N e_{mn} + {}^N\Delta e_{mn} \right] \sqrt{G}\; d^0V \qquad (2.81)$$

But, noting Eq. 2.50

$$\int_V \left[{}^N_0 S^{ij} + {}_0\Delta S^{ij} \right] \delta \left[{}^0 e_{ij} + {}^0\Delta e_{ij} \right] d^0V$$

$$= \int_V \left[{}^N_N \sigma^{ij} + {}_N\Delta\sigma^{ij} \right] \delta \left[{}^N e_{ij} + {}^N\Delta e_{ij} \right] d^NV \qquad (2.82)$$

where

${}_N\Delta\sigma^{ij}$ = increment of second Kirchhoff stress

Note that ${}_{N}\Delta\sigma^{ij}$ is similar to the Kirchhoff stress, however, it is measured in the new updated state 'N', as opposed to the original state 'O'. Thus, for the C.U.L. system (and the special case of this, the U.L. system) Cauchy stress is consistent with Almansi strain.

The other terms in Eq. 2.79 must also be updated to the new system. Since they are simply vectors this is straightforward. One must remember that the body forces and prescribed tractions must be based on the updated volume and area respectively. Thus, for the C.U.L. system Eq. 2.79 would transform to

$$\int_V \left\{ \left[{}^N_N\sigma^{ij} + {}_N\Delta\sigma^{ij} \right] \delta\left[{}^N e_{ij} + {}^N\Delta e_{ij} \right] - \left[{}^N\tilde{F}^i + {}^N\Delta\tilde{F}^i \right] \delta\left[{}^N u_i + {}^N\Delta u_i \right] \right\} d^N V$$

$$- \int_{S_1} \left\{ \left[{}^N_N\tilde{P}^i + {}_N\Delta\tilde{P}^i \right] \delta\left[{}^N u_i + {}^N\Delta u_i \right] \right\} d^N A \qquad (2.83)$$

This Principle of Virtual Work will form the basis of the derivations of Section 5.

The last consideration is the relations between stress and strain in the two systems. Suppose that the Kirchhoff stresses may be related to the Green strains by

$$ {}^N_O S^{ij} = {}^O C^{ijk\ell}\ {}^O e_{k\ell} \qquad (2.84)$$

and the Cauchy stresses to the Almansi strains by

$$ {}^N_N \sigma^{mn} = {}^N C^{mnpq}\ {}^N e_{pq} \qquad (2.85)$$

then a relationship between ${}^O C^{ijk\ell}$ and ${}^N C^{mnpq}$ must exist. Placing Eqs. 2.29 and 2.60 into Eq. 2.84 gives

$$\sqrt{G}\ \frac{\partial({}^O x^i)}{\partial({}^N x^m)}\ \frac{\partial({}^O x^j)}{\partial({}^N x^n)}\ {}^N_N\sigma^{mn} = {}^O C^{ijk\ell}\ \frac{\partial({}^N x^p)}{\partial({}^O x^k)}\ \frac{\partial({}^N x^q)}{\partial({}^O x^\ell)}\ {}^N e_{pq} \qquad (2.86)$$

or

$$ {}^N_N\sigma^{mn} = \frac{1}{\sqrt{G}}\ {}^O C^{ijk\ell}\ \frac{\partial({}^N x^m)}{\partial({}^O x^i)}\ \frac{\partial({}^N x^n)}{\partial({}^O x^j)}\ \frac{\partial({}^N x^p)}{\partial({}^O x^k)}\ \frac{\partial({}^N x^q)}{\partial({}^O x^\ell)}\ {}^N e_{pq} \qquad (2.87)$$

Comparing Eqs. 2.85 and 2.87 one observes that

$$ {}^{N}C^{mnpq} = \frac{1}{\sqrt{G}} \, {}^{0}C^{ijkl} \frac{\partial({}^{N}x^{m})}{\partial({}^{0}x^{i})} \frac{\partial({}^{N}x^{n})}{\partial({}^{0}x^{j})} \frac{\partial({}^{N}x^{p})}{\partial({}^{0}x^{k})} \frac{\partial({}^{N}x^{q})}{\partial({}^{0}x^{l})} \qquad (2.88) $$

Thus, if the material properties are known for one set of stresses and strains, then they may be determined for their corresponding set in the other coordinate system by Eq. 2.88. For this analysis, which is assumed linear elastic, one would expect the material properties to be constant. If this is true for one system, then, in general, in the other system they would be variable. However, it can be shown that for small strain the material properties may be taken as equal constants in each system (see Subsection 2.7).

It should be pointed out that when material nonlinearities are present the stress strain laws may only be valid incrementally. More care is required for these situations as well as those for general large strain problems [Bathe et al. 1973].

2.6 Comparisons of Definitions with the Literature

To further demonstrate the conflict which exists in the literature, the definitions given in the previous subsections for coordinate systems, strains, and stresses, are compared to those of some selected authors. These works are consistently defined mathematically, however, an entire range of names are used. Some of these are very misleading.

The information is presented in tabular form for quick reference. The first table shows comparisons for coordinate systems and strains. The second one gives comparisons for stresses. This latter variable seems the most abused. In the literature one often finds variables identified by name only. In light of the comparisons shown here it seems that this can be dangerous and mathematical definitions should always be given (or trusted). Names should only be used for convenience.

REFERENCE	COORDINATE SYSTEMS				STRAIN DEFINITIONS		
PRESENT WORK	S.L.	U.L.	C.C.	C.U.L.	GREEN (LAGRANGE)	ALMANSI	UPDATED GREEN
WASHIZU [1975]					GREEN		
BATHE ET AL. [1973]	T.L.	U.L.			GREEN-LAGRANGE	ALMANSI	GREEN*
YAGHMAI [1969]					LAGRANGE		LAGRANGE*
ATLURI [1973b]			C.C.				GREEN*
ATLURI ET AL. [1975]		U.L.			LAGRANGE		
PIROTIN [1971]			C.C.				GREEN*
WUNDERLICH [1972]	LAGRANGE		C.C.		LAGRANGE		LAGRANGE
BELYTSCHKO AND HSIEH [1973b]	LAGRANGE	EULER		C.C.	GREEN		CONVECTED

*AUTHOR NOTES THAT THESE ARE BASED ON UPDATED VALUES.

REFERENCE	STRESS DEFINITIONS					
PRESENT WORK	EULER	FIRST KIRCHHOFF (KIRCHHOFF)	SECOND KIRCHHOFF	CAUCHY	FIRST PIOLA (LAGRANGE)	SECOND PIOLA
WASHIZU [1975]		KIRCHHOFF	TRUESDELL	EULER	PIOLA (LAGRANGE; FIRST KIRCHHOFF)	
BATHE ET AL. [1973]		SECOND PIOLA KIRCHHOFF	SECOND* PIOLA KIRCHHOFF	CAUCHY		
YAGHMAI [1969]		SYMMETRIC PIOLA	SYMMETRIC* PIOLA	CAUCHY		
ATLURI [1973b]			PIOLA KIRCHHOFF			
ATLURI ET AL. [1975]			PIOLA KIRCHHOFF	PIOLA		
PIROTIN [1971]			KIRCHHOFF			
WUNDERLICH [1972]			TRUE	KIRCHHOFF	ϱ_t^{AB}	
BELYTSCHKO AND HSIEH [1973b]						

*AUTHOR NOTES THAT THESE ARE BASED ON UPDATED VALUES.

2.7 Discussion of Some Approximations

The actual class of problems to be solved in this work are subject to the following assumptions.

a. Thin, linear elastic (no material nonlinearities) structures such as beams, plates, and shells with no transverse shear. The Kirchhoff hypotheses hold.

b. The structures should not deform beyond the limits of shallow shell theory (within an element).

c. Although the deflections may be large the strain remains small.

d. Only moderate rotations are permitted (inplane displacements are small).

A real structure when it is thin and undergoes large deflection, small strain behavior does not, in general, exhibit plasticity. Since this analysis is static, no creep behavior exists, and the assumption of linear elastic behavior is justified. Many thin structures may deflect well into the nonlinear range (5-10 times its characteristic thickness) and still exhibit only moderate rotations. Such structures have some restraint on boundary displacements. Here the assumption of moderate rotation (and small inplane displacement) is justified. These same structures usually exhibit only small strains. For a structure such as a cantilevered plate undergoing cylindrical bending this assumption rapidly becomes invalid as the free edge rotates through large angles. This case is generally less significant. Finally, with the above geometric assumptions made, it is very unlikely that such a structure would become non-shallow on the element level to the degree of invalidating the analysis. Thus, the assumptions made are reasonably consistent with one another for a wide variety of practical problems.

While the previous analysis has been general, and is not restricted to any of these assumptions, the actual analysis becomes much more tractable when the assumptions apply. Much of the simplification can be seen in the equations of elasticity given in Subsection 3.2. Some remarks are more appropriately stated here.

a. Linear elastic material:

(1) If a material exhibits a nonlinear constitutive relation, by

means of stress or strain dependent material properties, or through higher order terms, Eqs. 2.84 (or 2.85) may not be stated in the total sense, in general. At best, it can only be stated incrementally, and procedures and material laws must be developed in order to update the relations. Under the assumption of a linear constitutive law this problem is avoided and the constitutive law holds for incremental as well as total values.

b. Small strain (moderate rotations):

(1) From Eq. 2.45, if ${}^{0}e_{ij}$ is truly small then

$$G_{ij} \approx \delta_{ij} \qquad G \approx 1 \tag{2.89}$$

$$d^{N}A_{k} \approx d^{0}A_{k} \qquad d^{N}V \approx d^{0}V \tag{2.90}$$

(2) Even for a linear elastic material, while the material properties are constant for one system (S.L. or U.L.) they will be variable in the other. This can be demonstrated by Eq. 2.88. From Eqs. 2.27 and 2.39 the Jacobian transforms may be written as

$$\left(\frac{\partial^{N}x_m}{\partial^{0}x_i}\right)_{C.U.L.} = \left(\delta_{ki} + {}^{0}_{0}u_{k,i}\right) {}^{0,N}T_{km} \tag{2.27}$$

$$\left(\frac{\partial^{N}x_m}{\partial^{0}x_i}\right)_{U.L.} = \left(\delta_{mi} + {}^{0}_{0}u_{m,i}\right) \tag{2.39}$$

Also from Eqs. 2.40

$$\bar{G}_i = \left(\delta_{mi} + {}^{0}_{0}u_{m,i}\right) {}^{0}\bar{i}_i \tag{2.40}$$

Normalizing this vector and considering Eq. 2.45

$$\left(\frac{G_i}{\sqrt{G_{ii}}}\right) = \left(\delta_{mi} + {}^{0}_{0}u_{m,i}\right)/\sqrt{G_{ii}} \quad {}^{0}\bar{i}_m$$

$$= \frac{\delta_{mi} + {}^{0}_{0}u_{m,i}}{\sqrt{1+2\,{}^{0}e_{ii}}} \; {}^{0}\bar{i}_m \qquad \text{(no sum on i)} \tag{2.91}$$

Taking $\bar{G}_3$ as the unit normal $(\bar{n})$ to the shell

$$\bar{G}_3 = \bar{n} = \left(-{}^{0}_{0}u_{3,1}\,{}^{0}\bar{i}_1 - {}^{0}_{0}u_{3,2}\,{}^{0}\bar{i}_2 + {}^{0}\bar{i}_3\right)/C \tag{2.92}$$

where

$$C = \sqrt{1 + ({}^{\circ}_{\circ}u_{3,1})^2 + ({}^{\circ}_{\circ}u_{3,2})^2}$$

One may write

$$\frac{\bar{G}_1}{\sqrt{G_{11}}} = \left[(1+{}^{\circ}_{\circ}u_{1,1})\,{}^{\circ}\bar{i}_1 + {}^{\circ}_{\circ}u_{2,1}\,{}^{\circ}\bar{i}_2 + {}^{\circ}_{\circ}u_{3,1}\,{}^{\circ}\bar{i}_3\right] / \sqrt{1+2\overset{\circ}{e}_{11}}$$

$$\frac{\bar{G}_2}{\sqrt{G_{22}}} = \left[{}^{\circ}_{\circ}u_{1,2}\,{}^{\circ}\bar{i}_1 + (1+{}^{\circ}_{\circ}u_{2,2})\,{}^{\circ}\bar{i}_2 + {}^{\circ}_{\circ}u_{3,2}\,{}^{\circ}\bar{i}_3\right] / \sqrt{1+2\overset{\circ}{e}_{22}} \qquad (2.93)$$

$$\bar{G}_3 = \left[-{}^{\circ}_{\circ}u_{3,1}\,{}^{\circ}\bar{i}_1 - {}^{\circ}_{\circ}u_{3,2}\,{}^{\circ}\bar{i}_2 + {}^{\circ}\bar{i}_3\right] / C$$

Or, more conveniently as

$$\begin{Bmatrix} \frac{\bar{G}_1}{\sqrt{G_{11}}} \\ \frac{\bar{G}_2}{\sqrt{G_{22}}} \\ \bar{G}_3 \end{Bmatrix} = \left[{}^{\bar{G},\,{}^{\circ}\bar{i}}T\right] \begin{Bmatrix} {}^{\circ}\bar{i}_1 \\ {}^{\circ}\bar{i}_2 \\ {}^{\circ}\bar{i}_3 \end{Bmatrix} \qquad (2.94)$$

Since the ${}^{N}\bar{i}_i$ system follows the deforming structure, it approximates the unit $\bar{G}_i/\sqrt{G_{ii}}$ vectors. Thus, regardless of the magnitude of the rotations, the ${}^{N}\bar{i}_i$ vectors can be thought of as "average" $\bar{G}_i/\sqrt{G_{ii}}$ vectors. The finer the finite element mesh is the better this approximation is. Under the assumption of small strain (small deformation) further improvement is realized. Thus, from Eq. 2.19

$$\begin{Bmatrix} {}^{N}\bar{i}_1 \\ {}^{N}\bar{i}_2 \\ {}^{N}\bar{i}_3 \end{Bmatrix} = \left[{}^{N,O}T\right] \begin{Bmatrix} {}^{\circ}\bar{i}_1 \\ {}^{\circ}\bar{i}_2 \\ {}^{\circ}\bar{i}_3 \end{Bmatrix} \approx \left[{}^{\bar{G},\,{}^{\circ}\bar{i}}T\right] \begin{Bmatrix} {}^{\circ}\bar{i}_1 \\ {}^{\circ}\bar{i}_2 \\ {}^{\circ}\bar{i}_3 \end{Bmatrix} \qquad (2.95)$$

Therefore

$${}^{N,O}T_{km} \approx {}^{\bar{G},\,{}^{\circ}\bar{i}}T_{km} \qquad {}^{O,N}T_{km} \approx {}^{{}^{\circ}\bar{i},\,\bar{G}}T_{km} \qquad (2.96)$$

Placing Eq. 2.96 into Eq. 2.27 yields

$$\left(\frac{\partial^N x_m}{\partial^o x_i}\right)_{C.U.L.} \approx \left(\delta_{ki} + {}^o_o u_{k,i}\right)^{o\bar{i},\bar{G}} T_{km} \tag{2.97}$$

With Eqs. 2.93 and 2.94 one obtains

$$\left(\frac{\partial^N x_m}{\partial^o x_i}\right)_{C.U.L.} \approx \left(\delta_{ki} + {}^o_o u_{k,i}\right)\left(\delta_{km} + {}^o_o u_{k,m}\right) \Big/ \sqrt{1+2\,{}^o e_{mm}} \tag{2.98}$$

Where there is no sum on m. This result may be rewritten as

$$\left(\frac{\partial^N x_m}{\partial^o x_i}\right)_{C.U.L.} \approx \left(\delta_{mi} + {}^o_o u_{m,i} + {}^o_o u_{i,m} + {}^o_o u_{k,m}\,{}^o_o u_{k,i}\right) \Big/ \sqrt{1+2\,{}^o e_{mm}}$$

$$\approx \left(\delta_{mi} + 2\,{}^o e_{mi}\right) \Big/ \sqrt{1+2\,{}^o e_{mm}} \qquad (\text{no sum on } m) \tag{2.99}$$

Under the assumption of small strain this can be approximated as

$$\left(\frac{\partial^N x_m}{\partial^o x_i}\right)_{C.U.L.} \approx \delta_{mi} \tag{2.100}$$

Note that

$$\left(\frac{\partial^N x_m}{\partial^o x_i}\right)_{C.C.} \equiv 1 \tag{2.101}$$

and thus, the C.U.L. system is a compromise between the C.C. and the U.L. systems. Comparing this with Eq. 2.39 one can see that for large or even moderate displacement gradients the U.L. system has a non-diagonal Jacobian while for the C.U.L. system the Jacobian is essentially the identity matrix.

One further comment should be made here. For the assumption of thin plates and shells under moderate transverse rotation and small inplane rotation, the Jacobian for the U.L. system can also be approximated by the identity matrix.

Under the assumptions stated here, with Eq. 2.84, Eq. 2.88 may be reduced to

$${}^N C^{mnpq} \approx {}^o C^{mnpq} \tag{2.102}$$

(3) Furthermore, considering Eq. 2.55, one observes that there is essentially no difference between the Kirchhoff stresses (first or second) and the Eulerian stress. Eqs. 2.59 and 2.60 demonstrate that the Cauchy stress in the C.U.L. system is approximately equal to the Kirchhoff and Eulerian stresses.

c. Use of C.U.L. as an updated system for thin structures:

(1) For thin plate and shell structures it is more convenient to use the natural coordinate frame associated with the midsurface rather than a general coordinate system (see Fig. 2.1). Thus, since the ${}^{N}\bar{i}_{i}$ system approximates the natural coordinates (especially for flat and shallow elements) the C.U.L. system seems more appropriate than the U.L. system.

(2) Even if rotations are moderately large, the C.U.L. system allows the structure to remain shallow on the element level. Thus, if shallow shell theory is to be adhered to then the C.U.L. would allow larger rigid body rotations than the S.L. system does.

(3) See b. (2).

Recapping the above simplifications.

Linear elastic material:

$$ {}_{o}\Delta S^{ij} = {}^{o}C^{ijk\ell}\,\Delta^{o}e_{k\ell} \qquad {}^{N}_{o}S^{ij} = {}^{o}C^{ijk\ell}\,{}^{o}e_{k\ell} $$

$$ {}_{N}\Delta\sigma^{ij} = {}^{N}C^{ijk\ell}\,\Delta^{N}e_{k\ell} \qquad {}^{N}_{N}\sigma^{ij} = {}^{N}C^{ijk\ell}\,{}^{N}e_{k\ell} $$

Small strain:

$$ G_{ij} \approx \delta_{ij} \qquad G \approx 1 $$

$$ d^{N}A_{k} \approx d^{o}A_{k} \qquad d^{N}V \approx d^{o}V $$

$$ {}^{N}C^{ijk\ell} \approx {}^{o}C^{ijk\ell} \qquad \text{(for C.U.L.)} $$

$$ {}^{N}_{o}S^{ij} \approx {}^{N}_{N}\tau^{ij} $$

$$ {}^{N}_{N}\sigma^{ij} \approx {}^{N}_{N}\tau^{ij} \approx {}^{N}_{o}S^{ij} \qquad \text{(for C.U.L.)} $$

Small strain and moderate rotation (small inplane gradients):

$$ \left.\begin{aligned} {}^{N}C^{ijk\ell} &\approx {}^{o}C^{ijk\ell} \\ {}^{N}_{N}\sigma^{ij} &\approx {}^{N}_{N}\tau^{ij} \approx {}^{N}_{o}S^{ij} \end{aligned}\right\} \; {}^{N}\bar{i}_{i} \rightarrow {}^{o}\bar{i}_{i} \qquad \text{(for U.L.)} $$

Further discussion for other approximations is more appropriately discussed in Subsection 3.2.

SECTION 3

GENERAL FINITE ELEMENT CONSIDERATIONS FOR THIN FLAT AND SHALLOW CURVED ELEMENTS

3.1 Introduction

In this section the general equations of elasticity will be simplified for thin, flat and shallow structures. While the general equations shall be used to develop the general matrix finite element equations for Section 6, the simplified equations shall be used for a corresponding analysis in Section 7. The reductions introduced for flat elements are quite straight forward. Those for shallow structures may be done in several ways. Two of these, the Kirchhoff-Love theory and Marguerre theory, will be discussed.

Since both two dimensional (plates and shells) and one dimensional (beam) problems are analyzed, the two dimensional equations will be discussed and the one dimensional cases can be taken as special cases of these.

Furthermore, since a finite element scheme is to be utilized, a discussion of the various coordinate systems for element generation and assembly purposes will be presented.

3.2 Equations of Elasticity

Throughout this subsection analysis will be performed without regard to a specific coordinate system. (Although when necessary for convenience the S.L. system is assumed.) Thus, all superscripts and subscripts referring to such systems will be dropped. In addition, all notation here does not necessarily refer to that of other sections and, therefore, will be defined herein. The assumptions of large deflection (moderate rotation), small strain elasticity will be made. The concepts of initial and incremental quantities will also be abandoned here in favor of total quantities only.

As shown in Section 2 the Principle of Virtual Work can be written as

$$\int_V [\sigma_{ij}\,\delta e_{ij} - \bar{F}_i\,\delta u_i]\,dV - \int_{S_\sigma} \bar{T}_i\,\delta u_i\,ds = 0 \qquad (3.1)$$

where

σ_{ij} = stress tensor

e_{ij} = strain tensor

$\bar{F}_i$ = prescribed body force per unit volume

$\bar{T}_i$ = prescribed boundary traction

u_i = displacement

dV = element volume

dS = element of bounding surface

The strain displacement relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \right) \tag{3.2}$$

and the kinematic (geometric) boundary conditions

$$u_i = \bar{u}_i \quad \text{on} \quad S_u \tag{3.3}$$

are assumed to be satisfied exactly

where $\quad u_{i,j} = \dfrac{\partial u_i}{\partial x_j} \qquad S = S_\sigma + S_u$

Placing Eq. 3.2 into Eq. 3.1 gives

$$\int_V \left\{ \frac{1}{2}\sigma_{ij}\left[\delta u_{i,j} + \delta u_{j,i} + \delta(u_{k,i} u_{k,j}) \right] - \bar{F}_i \delta u_i \right\} dV - \int_{S_\sigma} \bar{T}_i \delta u_i \, ds = 0 \tag{3.4}$$

Noting the following forms for integration by parts

$$\int_V \sigma_{ij} \delta u_{i,j} \, dV = -\int_V \sigma_{ij,j} \delta u_i \, dV + \int_S \sigma_{ij} \delta u_i \nu_j \, ds \tag{3.5}$$

and

$$\begin{aligned} \frac{1}{2}\int_V \sigma_{ij} \delta(u_{k,i} u_{k,j}) \, dV &= -\int_V (\sigma_{ij} u_{k,i})_{,j} \delta u_k \, dV + \int_S \sigma_{ij} u_{k,i} \nu_j \delta u_k \, ds \\ &= -\int_V (\sigma_{kj} u_{i,k})_{,j} \delta u_i \, dV + \int_S \sigma_{kj} u_{i,k} \nu_j \delta u_i \, ds \end{aligned} \tag{3.6}$$

Eq. 3.4 becomes (recall that σ_{ij} is symmetric)

$$-\int_V \{[\sigma_{kj}(\delta_{ik}+u_{i,k})]_{,j} + \bar{F}_i\}\,\delta u_i\, dV + \int_S [\sigma_{kj}(\delta_{ik}+u_{i,k})]\,\nu_j\,\delta u_i\, ds - \int_{S_\sigma} \bar{T}_i\,\delta u_i\, ds = 0 \tag{3.7}$$

or

$$-\int_V \{[\sigma_{kj}(\delta_{ik}+u_{i,k})]_{,j} + \bar{F}_i\}\,\delta u_i\, dV - \int_{S_\sigma} [\bar{T}_i - \sigma_{kj}(\delta_{ik}+u_{i,k})\,\nu_j]\,\delta u_i\, ds = 0 \tag{3.8}$$

since $\delta\bar{u}_i=0$ on S_u. Since the variations of displacement in V and on S_σ are arbitrary, then the individual integrands of Eq. 3.2 must be zero. Thus, the stress equilibrium equation

$$[\sigma_{kj}(\delta_{ik}+u_{i,k})]_{,j} + \bar{F}_i = 0 \tag{3.9}$$

and the mechanical boundary conditions

$$\bar{T}_i = [\sigma_{kj}(\delta_{ik}+u_{i,k})]\,\nu_j \tag{3.10}$$

result. These form the basic equations of elasticity in general terms.

3.2.1 Reductions for Flat Plate Elements

Following the analysis of Washizu [1975] the large deflection theory for plates proposed by Von Karmen will be employed. This theory assumes that the deflections may be large compared to the plate thickness h but small compared to its characteristic lengths. Thus, only moderate rotations are permitted and inplane displacements are assumed small. This allows one to analyze the large deflection of a simply supported plate, but not a cantilevered plate whose free end rotates through large angles. Therefore, the strain displacement relations of Eq. 3.2 in rectangular Cartesian coordinates may be expressed as (see Fig. 3.1)

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2} \\ e_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\,\frac{\partial^2 w}{\partial y^2} \\ 2e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\,\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{3.11}$$

where

$$u_1 = u_x = u - z\,\partial w/\partial x$$
$$u_2 = u_y = v - z\,\partial w/\partial y \qquad (3.12)$$
$$u_3 = w$$
$$u_{,x} \sim u_{,y} \sim v_{,x} \sim v_{,y} \sim (w_{,x})^2 \sim (w_{,y})^2 \sim w_{,x}w_{,y} << 1$$

and the transverse shear strains are assumed zero. Also, higher order terms are neglected in Eq. 3.11. Because of the thinness of the plate, the body forces $\bar{F}_i$ as well as any distributed loads applied to the top and/or bottom surfaces, may be expressed as

$$\int_v \bar{F}_i\,\delta u_i\,dv = \int_{S_m} (\bar{p}_x\,\delta u + \bar{p}_y\,\delta v + \bar{p}_z\,\delta w)\,dx\,dy \qquad (3.13)$$

where

S_m = the midsurface of the plate

Considering the strain energy

$$\int_v \sigma_{ij}\,\delta e_{ij}\,dv = \int_v [\sigma_{xx}\,\delta e_{xx} + \sigma_{yy}\,\delta e_{yy} + 2\sigma_{xy}\,\delta e_{xy}]\,dv \qquad (3.14)$$

and placing Eqs. 3.11 into this yields

$$\int_v \sigma_{ij}\,\delta e_{ij}\,dv = \int_v \Big\{\sigma_{xx}\Big[\delta\Big(\frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big) - z\,\delta\frac{\partial^2 w}{\partial x^2}\Big] + \cdots\Big\}\,dx\,dy\,dz \qquad (3.15)$$

Defining stress resultants as

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}\,dz \qquad M_x = \int_{-h/2}^{h/2} \sigma_{xx}\,z\,dz$$
$$N_y = \int_{-h/2}^{h/2} \sigma_{yy}\,dz \qquad M_y = \int_{-h/2}^{h/2} \sigma_{yy}\,z\,dz \qquad (3.16)$$
$$N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\,dz \qquad M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\,z\,dz$$

Eq. 3.15 becomes

$$\int_v \sigma_{ij}\,\delta e_{ij}\,dv = \int_{S_m} \Big\{N_x\,\delta\Big[\frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big] + N_y\,\delta\Big[\frac{\partial v}{\partial y} + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]$$
$$+ N_{xy}\,\delta\Big[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big] - M_x\,\delta\Big(\frac{\partial^2 w}{\partial x^2}\Big)$$
$$- M_y\,\delta\Big(\frac{\partial^2 w}{\partial y^2}\Big) - 2M_{xy}\,\delta\Big(\frac{\partial^2 w}{\partial x\,\partial y}\Big)\Big\}\,dx\,dy \qquad (3.17)$$

Integrating this by parts gives rise to the following stress resultant equilibrium equations in S_m

$$N_{x,x} + N_{xy,y} + \bar{p}_x = 0$$
$$N_{y,y} + N_{xy,x} + \bar{p}_y = 0 \qquad (3.18)$$
$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + (N_x w_{,x} + N_{xy} w_{,y})_{,x} + (N_y w_{,y} + N_{xy} w_{,x})_{,y} + \bar{p}_z = 0$$

and the mechanical boundary conditions on C_σ $(S_\sigma \to C_\sigma)$

$$\bar{N}_{x\nu} = N_{x\nu} \qquad \bar{N}_{y\nu} = N_{y\nu}$$
$$\bar{V} = Q_{x\nu} + Q_{y\nu} + N_{x\nu} w_{,x} + N_{y\nu} w_{,y} \qquad (3.19)$$
$$\bar{M}_{x\nu} = M_{x\nu} \qquad \bar{M}_{y\nu} = M_{y\nu}$$

where

$$N_{x\nu} = N_x \nu_x + N_{xy} \nu_y \qquad N_{y\nu} = N_y \nu_y + N_{xy} \nu_x$$
$$Q_{x\nu} = (M_{x,x} + M_{xy,y}) \nu_x \qquad Q_{y\nu} = (M_{y,y} + M_{xy,x}) \nu_y \qquad (3.20)$$
$$M_{x\nu} = M_x \nu_x + M_{xy} \nu_y \qquad M_{y\nu} = M_y \nu_y + M_{xy} \nu_x$$

ν_x = direction cosine between outward normal of C and x axis

ν_y = direction cosine between outward normal of C and y axis

Note that the strains of Eq. 3.11 may be written in terms of midsurface strains $[\underset{\sim}{e}_o]$, and curvatures, $[\underset{\sim}{k}]$.

$$[\underset{\sim}{e}] = [\underset{\sim}{e}_o] - z[\underset{\sim}{K}] \qquad (3.21)$$

where

$$e_{xx_o} = u_{,x} + \frac{1}{2} w_{,x}^2 \qquad K_{xx} = w_{,xx}$$
$$e_{yy_o} = v_{,y} + \frac{1}{2} w_{,y}^2 \qquad K_{yy} = w_{,yy} \qquad (3.22)$$
$$2e_{xy_o} = u_{,y} + v_{,x} + w_{,x} w_{,y} \qquad K_{xy} = w_{,xy}$$

For an isotropic material the constitutive relations are

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left[e_{xx} + \nu e_{yy}\right] \qquad \sigma_{yy} = \frac{E}{(1-\nu^2)}\left[e_{yy} + \nu e_{xx}\right] \tag{3.23}$$

$$\sigma_{xy} = 2G\, e_{xy} \qquad G = \frac{E}{2(1+\nu)}$$

Combining this with Eqs. 3.16 and 3.22 gives the familiar relations

$$N_x = \frac{Eh}{(1-\nu^2)}\left[e_{xx_0} + \nu e_{yy_0}\right] \qquad M_x = -D\left[K_{xx} + \nu K_{yy}\right]$$

$$N_y = \frac{Eh}{(1-\nu^2)}\left[e_{yy_0} + \nu e_{xx_0}\right] \qquad M_y = -D\left[K_{yy} + \nu K_{xx}\right] \tag{3.24}$$

$$N_{xy} = 2Gh\, e_{xy_0} \qquad M_{xy} = -D(1-\nu)K_{xy}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

E = Young's modulus

ν = Poisson's ratio

Eqs. 3.11, 3.18–3.22, and 3.24 form the basic equations that will be necessary to form the energy principles utilizing flat plate elements.

3.2.2 Reductions for Shallow Shell Elements

Two popular theories are used for shallow shell theory. Kirchhoff-Love theory is developed through the use of shell theory with appropriate reductions for shallowness. While Marguerre theory can be developed from formal shell theory, it can also be derived from extensions of plate theory. The latter was used for the bulk of the shallow element analysis although some beam cases were formulated by the former for comparisons.

3.2.2.1 Kirchhof-Love Theory

Let $z=z(x,y)$ represent the midsurface of a thin, shallow shell, and ζ be a coordinate normal to this midsurface (Fig. 3.2). The shell shall be considered shallow and thin if

$$\left|\frac{\partial z}{\partial x}\right|^2 \sim \left|\frac{\partial z}{\partial y}\right|^2 \sim \left|\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right| \ll 1$$

and

$$\left|\zeta\frac{\partial^2 z}{\partial x^2}\right| \sim \left|\zeta\frac{\partial^2 z}{\partial y^2}\right| \sim \left|\zeta\frac{\partial^2 z}{\partial x\partial y}\right| \ll 1 \tag{3.25}$$

From shell theory with these assumptions and those of small inplane displacements the strain displacement relations are

$$
\begin{aligned}
e_{11} &= \frac{\partial u_1}{\partial x} - w\frac{\partial^2 z}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \zeta\frac{\partial^2 w}{\partial x^2} \\
e_{22} &= \frac{\partial u_2}{\partial y} - w\frac{\partial^2 z}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \zeta\frac{\partial^2 w}{\partial y^2} \\
2e_{12} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} - 2w\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2\zeta\frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\tag{3.26}
$$

where

$(\ )_{\alpha\beta}$ = refer to ξ^1 and ξ^2 coordinates in the shell which correspond to the x and y axis

$\bar{u}$ = $u_1\bar{a}_1 + u_2\bar{a}_2 + w\bar{n}$

$\bar{a}_\alpha$ = base vectors along ξ^α (in midsurface)

$\bar{n}$ = unit normal to midsurface of shell

Note that the displacements and strains are measured in the shell coordinates (ξ^1, ξ^2, ζ).

Defining stress resultants as

$$
\begin{aligned}
N_1 &= \int_{-h/2}^{h/2} \sigma_{11}\, d\zeta & M_1 &= \int_{-h/2}^{h/2} \sigma_{11}\,\zeta\, d\zeta \\
N_2 &= \int_{-h/2}^{h/2} \sigma_{22}\, d\zeta & M_2 &= \int_{-h/2}^{h/2} \sigma_{22}\,\zeta\, d\zeta \\
N_{12} &= \int_{-h/2}^{h/2} \sigma_{12}\, d\zeta & M_{12} &= \int_{-h/2}^{h/2} \sigma_{12}\,\zeta\, d\zeta
\end{aligned}
\tag{3.27}
$$

Then the equations of stress resultant equilibrium in the base plane (x,y plane) are

$$
\begin{aligned}
&\frac{\partial N_1}{\partial x} + \frac{\partial N_{12}}{\partial y} + \bar{p}_1 = 0 \\
&\frac{\partial N_2}{\partial y} + \frac{\partial N_{12}}{\partial x} + \bar{p}_2 = 0 \\
&\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_2}{\partial y^2} + 2\frac{\partial^2 M_{12}}{\partial x \partial y} + N_1\frac{\partial^2 z}{\partial x^2} + N_2\frac{\partial^2 z}{\partial y^2} + 2N_{12}\frac{\partial^2 z}{\partial x \partial y} \\
&\qquad + \frac{\partial}{\partial x}\left[N_1\frac{\partial w}{\partial x} + N_{12}\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[N_2\frac{\partial w}{\partial y} + N_{12}\frac{\partial w}{\partial x}\right] + \bar{p}_3 = 0
\end{aligned}
\tag{3.28}
$$

where stresses are also measured in the shell. The mechanical boundary conditions are

$$\bar{N}_{1\nu} = N_{1\nu} \qquad \bar{N}_{2\nu} = N_{2\nu}$$

$$\bar{V} = Q_{1\nu} + Q_{2\nu} + N_{1\nu}\frac{\partial w}{\partial x} + N_{2\nu}\frac{\partial w}{\partial y} \tag{3.29}$$

$$\bar{M}_{1\nu} = M_{1\nu} \qquad \bar{M}_{2\nu} = M_{2\nu}$$

where

$$N_{1\nu} = N_1 \nu_x + N_{12}\nu_y \qquad N_{2\nu} = N_2\nu_y + N_{12}\nu_x$$

$$Q_{1\nu} = \left(\frac{\partial M_1}{\partial x} + \frac{\partial M_{12}}{\partial y}\right)\nu_x \qquad Q_{2\nu} = \left(\frac{\partial M_2}{\partial y} + \frac{\partial M_{12}}{\partial x}\right)\nu_y \tag{3.30}$$

$$M_{1\nu} = M_1\nu_x + M_{12}\nu_y \qquad M_{2\nu} = M_2\nu_y + M_{12}\nu_x$$

The strains of Eq. 3.26 may be separated into inplane (shell) strains and curvatures, where

$$e_{11_0} = \frac{\partial u_1}{\partial x} - w\frac{\partial^2 z}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad K_{11} = \frac{\partial^2 w}{\partial x^2}$$

$$e_{22_0} = \frac{\partial u_2}{\partial y} - w\frac{\partial^2 z}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad K_{22} = \frac{\partial^2 w}{\partial y^2} \tag{3.31}$$

$$2\,e_{12_0} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} - 2w\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad K_{12} = \frac{\partial^2 w}{\partial x\partial y}$$

Finally, the constitutive relations are

$$N_1 = \frac{Eh}{(1-\nu^2)}\left[e_{11_0} + \nu e_{22_0}\right] \qquad M_1 = -D\left[K_{11} + \nu K_{22}\right]$$

$$N_2 = \frac{Eh}{(1-\nu^2)}\left[e_{22_0} + \nu e_{11_0}\right] \qquad M_2 = -D\left[K_{22} + \nu K_{11}\right] \tag{3.32}$$

$$N_{12} = 2Gh\,e_{12_0} \qquad M_{12} = -D(1-\nu)K_{12}$$

Eqs. 3.26-3.32 define the basic relations necessary for finite element formulations based on Kirchhoff-Love shallow shell theory.

3.2.2.2 Marguerre Theory

This theory is a much more natural extension of flat plate theory and, in fact, was originally developed for plates with small initial deflections

[Marguerre, 1938]. Although it can be formulated through shell considerations [Washizu, 1975] it is worthwhile to derive it from the plate equations.

Consider the strain displacement relations Eq. 3.11 with an initial displacement, z, such that (see Fig. 3.2)

$$w \longrightarrow w + z \tag{3.33}$$

Thus,

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\frac{\partial(w+z)}{\partial x}\right]^2 - \zeta \frac{\partial^2 (w+z)}{\partial x^2} \tag{3.34}$$

where ζ in this relation replaces the role of z in Eq. 3.11 and $z=z(x,y)$ here represents the midsurface of the shell. Eq. 3.34 becomes

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial z}{\partial x}\right]^2 - \zeta\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 z}{\partial x^2}\right] \\ &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial w}{\partial x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2\right] - \zeta\frac{\partial^2 w}{\partial x^2} - \zeta\frac{\partial^2 z}{\partial x^2} \end{aligned} \tag{3.35}$$

Recalling the restrictions defining shallowness, Eqs. 3.25 reduces Eq.3.35 to

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \zeta\frac{\partial^2 w}{\partial x^2} \tag{3.36a}$$

Similarly

$$\begin{aligned} e_{yy} &= \frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \zeta\frac{\partial^2 w}{\partial y^2} \\ 2e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2\zeta\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{3.36b}$$

Note that here the previous curvilinear system ξ^1, ξ^2, ζ is approximated by the rectangular Cartesian system x, y, ζ (where the ζ's play different roles in each case).

Defining stress resultants as

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_{11}\, d\zeta & M_x &= \int_{-h/2}^{h/2} \sigma_{11}\, \zeta\, d\zeta \\ N_y &= \int_{-h/2}^{h/2} \sigma_{22}\, d\zeta & M_y &= \int_{-h/2}^{h/2} \sigma_{22}\, \zeta\, d\zeta \\ N_{xy} &= \int_{-h/2}^{h/2} \sigma_{12}\, d\zeta & M_{xy} &= \int_{-h/2}^{h/2} \sigma_{12}\, \zeta\, d\zeta \end{aligned} \tag{3.37}$$

The resulting equations of stress resultant equilibrium are

$$N_{x,x} + N_{xy,y} + \bar{p}_x = 0$$
$$N_{y,y} + N_{xy,x} + \bar{p}_y = 0 \tag{3.38}$$
$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + [N_x(w_{,x} + z_{,x}) + N_{xy}(w_{,y} + z_{,y})]_{,x}$$
$$+ [N_y(w_{,y} + z_{,y}) + N_{xy}(w_{,x} + z_{,x})]_{,y} + \bar{p}_z = 0$$

and the mechanical boundary conditions are

$$\bar{N}_{x\nu} = N_{x\nu} \qquad \bar{N}_{y\nu} = N_{y\nu}$$
$$\bar{V} = Q_{x\nu} + Q_{y\nu} + N_{x\nu}(w_{,x} + z_{,x}) + N_{y\nu}(w_{,y} + z_{,y}) \tag{3.39}$$
$$\bar{M}_{x\nu} = M_{x\nu} \qquad \bar{M}_{y\nu} = M_{y\nu}$$

where Eqs. 3.20 for plates still hold. Separating out the inplane strains and curvatures

$$e_{xx_o} = u_{,x} + z_{,x}w_{,x} + \tfrac{1}{2}(w_{,x})^2 \qquad K_{xx} = w_{,xx}$$
$$e_{yy_o} = v_{,y} + z_{,y}w_{,y} + \tfrac{1}{2}(w_{,y})^2 \qquad K_{yy} = w_{,yy} \tag{3.40}$$
$$2e_{xy_o} = u_{,y} + v_{,x} + z_{,x}w_{,y} + z_{,y}w_{,x} + w_{,x}w_{,y} \qquad K_{xy} = w_{,xy}$$

The constitutive relations are the same as those for plates, Eqs. 3.24. Eqs. 3.36–3.40, 3.20 and 3.24 define the basic relations necessary for finite element formulations based on Marguerre theory for shallow shells.

Note that in all the theories given here the geometric boundary conditions on S_u (or C_u) are the same, namely

$$\bar{u} = u \qquad \bar{v} = v \qquad w = \bar{w} \tag{3.41}$$
$$\frac{\partial \bar{w}}{\partial x} = \frac{\partial w}{\partial x} \qquad \frac{\partial \bar{w}}{\partial y} = \frac{\partial w}{\partial y}$$

However, for Kirchhoff-Love theory $u=u_1$, $v=v_1$, and $w=\bar{w}$ are measured in the ξ^1, ξ^2, and $\zeta(\bar{n})$ directions.

3.2.3 Summary of Some Approximations

The approximations discussed in the previous subsections here will be listed in a convenient form for quick reference. Additional remarks were made in Subsection 2.7.

a. Thinness (Kirchhoff assumptions hold):

(1) No transverse shear stress or strain, i.e.,

$$\sigma_{xz} \sim \sigma_{yz} \sim \sigma_{zz} \approx 0 \qquad e_{xz} \sim e_{yz} \sim e_{zz} \approx 0$$

(2) Higher order terms in the thickness variable, ζ, are omitted, i.e., $O(\zeta^2) \approx 0$

(3) Volumetric body forces and distributed loads acting on the top and bottom surfaces are replaced with distributed loads acting over the middle surface of the structure (see Eq. 3.13).

b. Small strain moderate transverse rotations (small inplane displacements):

(1) The following terms are considered small in the strain displacement relations.

$$u_{,x} \sim u_{,y} \sim v_{,x} \sim v_{,y} \sim (w_{,x})^2 \sim (w_{,y})^2 \sim w_{,x}w_{,y} \ll 1$$

(2) Only the stress equilibrium equation in the normal direction must be altered while the others remain linear (i.e. see Eqs. 3.18).

c. Shallowness:

(1) Let z=z(x,y) represent the middle surface of a structure, and ζ be the normal coordinate to this midsurface, then the following terms are omitted in the strain displacement relations

$$\left|\frac{\partial z}{\partial x}\right|^2 \sim \left|\frac{\partial z}{\partial y}\right|^2 \sim \left|\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right| \ll 1$$
$$\left|\zeta\frac{\partial^2 z}{\partial x^2}\right| \sim \left|\zeta\frac{\partial^2 z}{\partial y^2}\right| \sim \left|\zeta\frac{\partial^2 z}{\partial x \partial y}\right| \ll 1$$

(See Eq. 3.25).

(2) Again, only the stress equilibrium equation in the normal direction must be altered (i.e. see Eqs. 3.28).

3.3 Coordinate Systems and Transformations

The coordinate systems to be discussed here are those required for the generation of element level matrices and element assembly procedures. Since this work predominently makes use of flat plate theory and Marguerre theory the majority of the following will deal with rectangular Cartesian systems. However, since some work involved the use of Kirchhoff-Love theory, and for the sake of completeness, Subsection 3.3.3 will make some comments regarding curvilinear coordinate systems. More detail on this may be found in Appendix A.

3.3.1 Basic Considerations

In the general case of shell analysis there are three basic coordinate systems (four for the Kirchhoff-Love case) which, in general, do not coincide. They are

1. a global system which remains fixed. All quantities and processes (differentiations and/or integrations) can ultimately be defined with respect to it. This system is usually chosen so that the initially undeformed structure can be easily described.
2. a local system where individual element properties and processes can easily be defined. For thin structures this is usually taken as the natural reference frame.
3. a common system where all the elements will be assembled for solution purposes.

This common system may have several alternatives. One could allow the common and global systems to coincide [Aldstedt, 1969]. Or, allow it to represent a tangent plane system to the real shell surface [Megard, 1969]. Alternately, some system could be arrived at by an averaging process of local systems at a point on the shell surface [Mau and Witmer, 1972]. (See Fig.3.3).

The systems one uses is generally a matter of preference and convenience. Considerations to be made, however, are what kind of loading will be used and what type of boundary conditions will be applied. It appears from this work that choosing a common system which essentially coincides with the real shell system (utilizing rectangular Cartesian coordinates) is, in general, the most useful. This allows for the simplest handling of the majority of boundary conditions and loadings.

It should be pointed out that the local system, or base plane, is not taken, in general, coincident with the common or global systems. While this would be legitimate for an entire shell which was shallow, for deep shells this is not the case. In fact, for deep shells, shallow shell theory is, of course, invalid. Thus, the local base planes follow the shell (at the nodal points) so that even for deep shells the elements will be locally shallow.

This approximation of shallowness gets better and better as the finite element mesh becomes finer. Theoretically, any deep shell can be represented by a series of locally shallow shells. For that matter, at the risk of greater approximation, any shell can be approximated by a series of locally flat planes (a faceted shell).

3.3.2 Assembly in the Common System

The set of coordinate systems presented here are for flat plates and shallow shells derived from Maguerre theory. Consider three sets of rectangular Cartesian coordinates: a global set $({}^{G}x, {}^{G}y, {}^{G}z)$; a local set $({}^{L}x, {}^{L}y, {}^{L}z)$; and a common set $({}^{C}x, {}^{C}y, {}^{C}z)$. Any two sets of such systems may be related to each other via the direction cosines between their axes. Thus

$$^{G}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad \begin{bmatrix} \cos({}^{G}x, {}^{L}x) & \cos({}^{G}x, {}^{L}y) & \cos({}^{G}x, {}^{L}z) \\ \cos({}^{G}y, {}^{L}x) & \cos({}^{G}y, {}^{L}y) & \cos({}^{G}y, {}^{L}z) \\ \cos({}^{G}z, {}^{L}x) & \cos({}^{G}z, {}^{L}y) & \cos({}^{G}z, {}^{L}z) \end{bmatrix}^{L} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{3.42}$$

where

$\cos({}^{G}y, {}^{L}z)$ = direction cosine between the global y axis and the local z axis

or, in contracted form

$$^{G}\{x\} = {}^{GL}[T]^{L}\{x\} \tag{3.43}$$

where

$$\{x\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{3.44}$$

Similarly

$$^{G}\{x\} = {}^{GC}[T]^{C}\{x\} \tag{3.45}$$

and

$$^{L}\{x\} = {}^{LC}[T]\,{}^{C}\{x\} \tag{3.46}$$

If the nodal points corresponding to an element are each located with respect to the global system, then the normal to the flat plane defined by the three nodes can easily be determined [Zienkiewicz, 1971; Mau and Witmer, 1972]. This normal then becomes ^{L}z. The two inplane axes, ^{L}x and ^{L}y, can be arbitrarily defined in a plane perpendicular to ^{L}z. In this work it was decided to locate the ^{L}x axis such that it would always remain perpendicular to ^{G}y. Thus, the direction cosine cos $(^{G}y, {}^{L}x)$ vanishes. Once the position of the ^{L}x and ^{L}z axes are determined with respect to the global axes then ^{L}y may simply be determined as the cross product between the ^{L}z and ^{L}x directions. One must remember that the vectors defining such directions must be unit vectors. It was also chosen, arbitrarily, that the origin of this local axis system be at the centroid of its respective element. With this information at hand the transformation of Eq. 3.43 $^{GL}[T]$ is completely determined for each element.

In a similar fashion the common system is defined with respect to the global system. First the ^{C}z axis must be determined by arranging it in the same direction as the normal to the shell. It can be shown (see Appendix A) that the normal to the shell with respect to the global axes is

$$^{C}z \longrightarrow \bar{n} = \frac{1}{C_1}\left[-{}^{G}_{G}z_{,x}\,{}^{G}\bar{i}_x - {}^{G}_{G}z_{,y}\,{}^{G}\bar{i}_y + {}^{G}\bar{i}_z\right] \tag{3.47}$$

where

$$C_1 = \sqrt{1 + {}^{G}_{G}z^2_{,x} + {}^{G}_{G}z^2_{,y}}$$

$${}^{G}_{G}z_{,x} = \frac{\partial(^{G}z)}{\partial(^{G}x)}$$

$^{G}\bar{i}$ = base vectors for the global system.

Again, the ^{C}x direction was chosen to be perpendicular to the ^{G}y direction. Thus, cos $(^{G}y, {}^{C}x)$ vanishes. ^{C}y is determined by taking the cross product of ^{C}z and ^{C}z. This information completely determines the transformation, $^{GC}[T]$, of Eq. 3.45.

Finally the transformation ${}^{LC}[T]$ may be determined from the previous two. Noting that a property of the direction cosine transformations states that the inverse of the transformation matrix is equal to its transpose, i.e.

$$ {}^{GL}[T]^{-1} = {}^{GL}[T]^{T} \tag{3.48} $$

one may write

$$ {}^{L}\{x\} = {}^{GL}[T]^{T}\,{}^{G}\{x\} = {}^{GL}[T]^{T}\,{}^{GC}[T]\,{}^{C}\{x\} \tag{3.49} $$

Comparing Eqs. 3.46 and 3.49 yields

$$ {}^{LC}[T] = {}^{GL}[T]^{T}\,{}^{GC}[T] \tag{3.50} $$

Thus, all the necessary transformations are given by the above.

A comment should be made at this point explaining why assembly is performed in the common system (the details of which will be presented in Section 7). Many authors prefer to use the global system. One should recall that thin plate and shell theory have only five degrees of freedom (dofs) per node. There is no stiffness contribution for a rotational dof about the normal. To assemble in the global system, which may be located at a considerable rotation from the local system, requires the use of six dofs. This is because the stiffness in the five local dofs will have contributions to all six dofs in the global system. The displacement solutions are also in the global directions. If the common system is used, which is some average position between neighboring elements, then only five dofs are required because the loss of stiffness in this transformation is negligible. Thus, for every node point one dof (out of five) is saved. This can lead to substantial savings in computational effort. Furthermore, the displacement solutions are in the approximate shell directions which seems more natural for shell analysis. Additionally, for global assembly, if a flat plate is to be analyzed such that neighboring elements are coplanar then no stiffness contribution is added to the fictitious sixth dof and it must be restrained. For the common system this problem does not exist.

3.3.3 Assembly in the Shell System

When Kirchhoff-Love theory is used to formulate the problem, the displacements, stresses, and strains are in the shell coordinates. For shallow shell theory the curvilinear coordinates associated with each element, which correspond to their Cartesian counterparts in the base plane, are nearly orthogonal and can be taken as such. For the entire shell there is also a set of curvilinear coordinates which correspond to the global rectangular Cartesian system. The details of these curvilinear systems are given in Appendix A.

The element matrices are associated with the local curvilinear coordinates. This set of coordinates has its normal direction coincident with the normal to the real shell. The inplane coordinates are in the shell surface. The global curvilinear set of coordinates also has its normal direction coincident with the normal to the shell. An orthgonal set of coordinates can be formed where the other two coordinates are in the surface of the shell. Since both sets of curvilinear coordinates have the same normal directions (to the shell) and both of their other coordinates lie in the surface of the shell then only a simple inplane rotation is necessary for assembly. Although the local and global inplane coordinates are not coincident, they may both be taken as orthogonal and thus, only a simple inplane direction cosine transformation is required.

Since both of these curvilinear coordinate systems can be referred to the same rectangular Cartesian global system, it is a simple matter to determine the angle or rotation required for transformation.

SECTION 4

A GENERAL INCREMENTAL ASSUMED STRESS HYBRID FORMULATION FOR LARGE DEFLECTION ANALYSIS BASED ON A STATIONARY LAGRANGIAN COORDINATE SYSTEM

4.1 Introduction

The original variational formulation of the assumed stress hybrid functional [Pian, 1964] was limited to linear elastic, static equilibrium problems. The formulation was completely consistent and it was shown that, although the functional only satisfied a stationary condition, it converged [Tong and Pian, 1969]. The beauty of this approach lies in the fact that no compatible displacement field need be assumed on the interior of an element -- a problem which has plagued the use of the displacement model for many years, particularly for plate and shell problems. However, the functional does require that stress equilibrium on the interior be satisfied exactly. For the finite element analysis of linear elastic problems this is easily accomplished. It has been pointed out [Langhaar, 1953; Washizu, 1975] that for complementary energy principles this condition no longer exists for nonlinear problems. From Eq. 3.9 one observes that a nonlinear coupling exists in the stress equilibrium equation and satisfaction of this constraint is extremely difficult. This coupling can be eliminated by the use of Piola stresses, however, these are unsymmetric. Fraeijs de Veubeke [1972], has shown the resulting functionals.

The first attempt at extending the assumed stress hybrid formulation to geometrically nonlinear problems was done by Lundgren [1967]. In an attempt to solve a bifurcation buckling problem, he used a hybrid elastic stiffness matrix with a geometric stiffness constructed from a standard displacement method. Although the method was successful, it was inconsistently derived. Later Pirotin [1971] derived an inconsistent approach from a modified Hellinger-Reissner Principle and applied it to the large deflection analysis of beams and shells. This was a first step and only the basic functional was considered. Finally Atluri [1973b] presented a finite element approach based on a consistent assumed stress hybrid functional. This paper is in error and in the formulation to be presented, the differences will be pointed out.

In this section an assumed stress hybrid functional will be derived from the Principle of Virtual Work. Although a more direct approach can be realized by utilizing the Principle of Virtual Complementary Work, derived in Appendix B, starting with the Principle of Virtual Work allows one to consider a variety of functionals. A stationary Lagrangian coordinate system will be utilized here. As in the works by Pirotin and Atluri, an incremental initial stress procedure will be incorporated. It should be noted that both Pirotin and Atluri used a convected coordinate system. In the next section a similar system will be used to derive an alternate updated approach. As discussed in Section 2, for a stationary Langrangian coordinate system the Kirchhoff stress is consistent with the Green (Lagrangian) strain. Thus, these shall be used in the derivation.

4.2 The Hu-Washizu Principle

The Principle of Virtual Work has been a cornerstone in the derivation of energy principles [Zienkiewicz, 1971; Washizu, 1975]. It has been well established and will be used as the starting point for the present formulation. Consider the volume, V, of a continuum which has the boundary surface ∂V composed of segments. Let S_σ be that portion of ∂V upon which prescribed tractions are applied while S_u is that portion of ∂V upon which displacements are prescribed. The Principle of Virtual Work states that the variation of the functional π with respect to e_{ij} and u_i may be written as

$$\delta\pi = \int_V [\sigma_{ij}\,\delta e_{ij} - \bar{F}_i\,\delta u_i]\,dV - \int_{S_\sigma} \bar{T}_i\,\delta u_i\,dS = 0 \tag{4.1}$$

where

σ_{ij} = elastic stress tensor

e_{ij} = elastic strain tensor

$\bar{F}_i$ = prescribed body force

u_i = displacements

$\bar{T}_i$ = prescribed surface tractions

$\bar{(\ )}$ = prescribed quantity

Eq. 4.1 states that the sum of the virtual work, done by external loads acting on the body moving through admissible virtual displacements, is equal to the change in external strain energy of the body, effected by the virtual strains. Now, consider a state incrementally close to the present state. The variables in this state may be written as

$$\sigma_{ij} \longrightarrow \sigma_{ij} + \Delta\sigma_{ij}$$
$$e_{ij} \longrightarrow e_{ij} + \Delta e_{ij}$$
$$\bar{F}_i \longrightarrow \bar{F}_i + \Delta\bar{F}_i$$
$$u_i \longrightarrow u_i + \Delta u_i$$
$$\bar{T}_i \longrightarrow \bar{T}_i + \Delta\bar{T}_i$$

where

( ) = initial quantity in reference state

Δ( ) = incremental quantity.

The initial quantities are assumed to be known from the previous state. The incremental quantities are the changes in the corresponding variables from the known state to the incrementally close next state. The Principle of Virtual Work may now be written as

$$\delta\pi = \int_V \left[(\sigma_{ij} + \Delta\sigma_{ij})\,\delta(e_{ij} + \Delta e_{ij}) - (\bar{F}_i + \Delta\bar{F}_i)\,\delta(u_i + \Delta u_i)\right] dV - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta(u_i + \Delta u_i)\,dS = 0 \tag{4.2}$$

It should be noted that only incremental quantities are subject to variation and, therefore, Eq. 4.2 may simply be written as

$$\delta\pi = \int_V \left[(\sigma_{ij} + \Delta\sigma_{ij})\,\delta\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\delta\Delta u_i\right] dV - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta u_i\,dS = 0 \tag{4.3}$$

One may now define a state function $A(\Delta e_{ij})$ such that

$$\delta A(\Delta e_{ij}) = \Delta\sigma_{ij}\,\delta\Delta e_{ij} \longrightarrow \Delta\sigma_{ij} = \frac{\partial A(\Delta e_{ij})}{\partial \Delta e_{ij}} \tag{4.4}$$

With this strain energy potential defined, the Principle of Stationary Total Potential Energy (PSTPE) may be formed. It states that the functional π_p, defined as

$$\Pi_p = \int_V [A(\Delta e_{ij}) + \sigma_{ij}\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i]\,dV - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i\,dS \tag{4.5}$$

is stationary with respect to a variation of displacement increments, Δu_i when the strain displacement relations in V and the kinematic boundary conditions on S_u are satisfied exactly. The stress equilibrium equations in V and the mechanical boundary conditions on S_σ are the Euler equations of π_p after appropriate integration by parts.

The first step in the procedure is to relax the equations of strain displacement and kinematic boundary conditions by allowing them to be satisfied only approximately. For large displacement analysis, the total* Green strain displacement relations may be written as

$$e_{ij} + \Delta e_{ij} = \tfrac{1}{2}[(u_i+\Delta u_i)_{,j} + (u_j+\Delta u_j)_{,i} + (u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j}] \tag{4.6}$$

or

$$(e_{ij} + \Delta e_{ij}) - \tfrac{1}{2}[u_{i,j} + u_{j,i} \cdot u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}] = 0 \tag{4.7}$$

The kinematic boundary condition on S_u can be written as

$$u_i + \Delta u_i = \bar{u}_i + \Delta\bar{u}_i \tag{4.8}$$

---

*The total strain displacement relations are introduced, as opposed to the incremental relations, so that a compatibility check may be obtained consistently. (See Subsection 4.4.3).

Through the use of Lagrange multipliers these conditions may be incorporated into the previous functional, π_p. The resulting generalized variational principle states that the functional, π_I^*, which is stationary with respect to variations in the incremental quantities $\Delta\sigma_{ij}$, and Δu_i, where

$$\begin{aligned}
\pi_I^* = \int_V \Big\{ & A(\Delta e_{ij}) + \sigma_{ij}\,\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\Delta u_i \\
& + \lambda_{ij}\Big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \\
& + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})\Big]\Big\}\,dV \\
& - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta u_i\,ds \\
& + \int_{S_u} \mu_i\big[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)\big]\,ds
\end{aligned} \tag{4.9}$$

Taking the variation of π_I^* recalling that only incremental quantities are subject to variation.

$$\begin{aligned}
\delta\pi_I^* = \int_V \Big\{ & \Delta\sigma_{ij}\,\delta\Delta e_{ij} + \sigma_{ij}\,\delta\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\delta\Delta u_i \\
& + \delta\lambda_{ij}\Big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \\
& + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})\Big] \\
& + \lambda_{ij}\Big[\delta\Delta e_{ij} - \tfrac{1}{2}(\delta\Delta u_{i,j} + \delta\Delta u_{j,i} + u_{k,i}\,\delta\Delta u_{k,j} \\
& \qquad + \delta\Delta u_{k,i}\,u_{k,j} + \delta(\Delta u_{k,i}\Delta u_{k,j}))\Big]\Big\}\,dV \\
& - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta u_i\,ds \\
& + \int_{S_u} \Big\{ \delta\mu_i\big[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)\big] + \mu_i\,\delta\Delta u_i \Big\}\,ds = 0
\end{aligned} \tag{4.10}$$

Upon rearranging

$$\delta\pi_I^* = \int_V \{ \delta\Delta e_{ij}(\sigma_{ij} + \Delta\sigma_{ij} + \lambda_{ij}) + \delta\lambda_{ij}[e_{ij} + \Delta e_{ij}$$
$$- \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j}$$
$$+ \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})] - \tfrac{1}{2}\lambda_{ij}[\delta\Delta u_{i,j} + \delta\Delta u_{j,i}$$
$$+ u_{k,i}\delta\Delta u_{k,j} + \delta\Delta u_{k,i}u_{k,j} + \delta(\Delta u_{k,i}\Delta u_{k,j})] - (\bar{F}_i + \Delta\bar{F}_i)\delta\Delta u_i \} dV$$
$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta u_i \, ds$$
$$+ \int_{S_u} \{ \delta\mu_i[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)] + \mu_i\delta\Delta u_i \} ds = 0 \qquad (4.11)$$

Terms containing the variation of incremental displacement gradients must be integrated by parts such that

$$\int_V \lambda_{ij}\delta\Delta u_{i,j} dV = \int_{\partial V} \lambda_{ij}\nu_j \delta\Delta u_i ds - \int_V \lambda_{ij,j}\delta\Delta u_i dV \qquad (4.12)$$

and

$$\tfrac{1}{2}\int_V \lambda_{ij}\delta(\Delta u_{k,i}\Delta u_{k,j}) dV = \int_{\partial V} (\lambda_{kj}\Delta u_{i,k})\nu_j \delta\Delta u_i ds$$
$$- \int_V (\lambda_{kj}\Delta u_{i,k})_{,j} \delta\Delta u_i dV \qquad (4.13)$$

Placing Eqs. 4.12 and 4.13 into Eq. 4.11 and rearranging yields

$$\delta\pi_I^* = \int_V \{ \delta\Delta e_{ij}(\sigma_{ij} + \Delta\sigma_{ij} + \lambda_{ij}) + \delta\lambda_{ij}[e_{ij} + \Delta e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i}$$
$$+ u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})]$$
$$+ \delta\Delta u_i\{\lambda_{ij,j} + [\lambda_{kj}(u_{i,k} + \Delta u_{i,k})]_{,j} - (\bar{F}_i + \Delta\bar{F}_i)\}\} dV$$
$$- \int_{\partial V} \delta\Delta u_i[\lambda_{ij} + \lambda_{kj}(u_{i,k} + \Delta u_{i,k})]\nu_j ds$$
$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta u_i ds$$
$$+ \int_{S_u} \{ \delta\mu_i[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)] + \mu_i\delta\Delta u_i \} ds = 0 \qquad (4.14)$$

From the above equation and recalling that $\partial V = S_\sigma + S_u$ the Lagrange multipliers can be found by

$$\sigma_{ij} + \Delta\sigma_{ij} + \lambda_{ij} = 0 \longrightarrow \lambda_{ij} = -(\sigma_{ij} + \Delta\sigma_{ij}) \qquad (4.15)$$

and

$$\mu_i = [\lambda_{ij} + \lambda_{kj}(u_{i,k} + \Delta u_{i,k})]\,\nu_j = -(T_i + \Delta T_i) \tag{4.16}$$

Placing these equations into Eq. 4.14 the Euler equations of the functional π_I^* are as follows.

In the volume, V: $\lambda_{ij} = -(\sigma_{ij} + \Delta\sigma_{ij})$

Strain displacement

$$e_{ij} + \Delta e_{ij} = \tfrac{1}{2}\{u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}\} \tag{4.17}$$

Stress equilibrium

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{4.18}$$

On the boundary: $\mu_i = [\lambda_{ij} + \lambda_{kj}(u_{i,k} + \Delta u_{i,k})]\,\nu_j = -(T_i + \Delta T_i)$

Tractions on ∂V

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]\,\nu_j \tag{4.19}$$

Mechanical boundary condition on S_σ

$$T_i + \Delta T_i = \bar{T}_i + \Delta\bar{T}_i \tag{4.20}$$

Kinematic boundary condition on S_u

$$u_i + \Delta u_i = \bar{u}_i + \Delta\bar{u}_i \tag{4.21}$$

Placing Eqs. 4.15 and 4.16 into Eq. 4.9 and relaxing the stress strain relations (Eq. 4.4) yields the Hu-Washizu functional, π_I which is stationary with respect to variations in the incremental quantities Δe_{ij}, $\Delta\sigma_{ij}$, and Δu_i. After some reduction (and eliminating constant terms)

$$\pi_I = \int_V \{ A(\Delta e_{ij}) - \Delta\sigma_{ij}\Delta e_{ij} - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i$$

$$+ \frac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i}$$

$$+ u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\} dV$$

$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i \, ds$$

$$- \int_{S_u} (T_i + \Delta T_i)[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)] \, ds \qquad (4.22)$$

Note that the stress strain relations (Eq. 4.4) and Eqs. 4.17-4.21 are the Euler equations for the Hu-Washizu functional.

4.3 The Modified Reissner Principle

The next step in the derivation is to obtain Reissner's principle from the Hu-Washizu principle. First Eq. 4.4 must be satisfied exactly and a new state function, $B(\Delta\sigma_{ij})$, is defined so that

$$B(\Delta\sigma_{ij}) = \Delta\sigma_{ij}\Delta e_{ij} - A(\Delta e_{ij}) \qquad (4.23)$$

Placing this into Eq. 4.22 gives the functional π_R associated with the Reissner principle. This variational principle is stationary with respect to variations in the incremental quantities $\Delta\sigma_{ij}$, ΔT_i and Δu_i.

$$\pi_R = \int_V \{ -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i$$

$$+ \frac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i}$$

$$+ u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\} dV$$

$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i \, ds - \int_{S_u} (T_i + \Delta T_i)[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)] \, ds \qquad (4.24)$$

Before continuing it is now convenient to recognize that these functionals shall ultimately be used to generate matrix equations for the finite element method. Thus, the possibility of dividing the continuum into n segments is considered. If each segment is treated as a continuum and if it is connected to a neighboring segment properly, then the entire domain may be treated by summing the individual segments [Zienkiewicz, 1971]. Thus, π_R can be taken to represent one segment and the total principle would involve summing over all segments. Upon doing this one can envision the creation of internal boundaries. If one refers to these segments as elements, then the internal boundaries may be referred to as interelement boundaries. Letting ∂V_n represent the total bounding surface of an element, one can decompose it as follows

$$\partial V_n = S_{\sigma_n} + S_{u_n} + S_{n_n} \tag{4.25}$$

where S_{σ_n} and S_{u_n} are the same as S_σ and S_u respectively with the understanding that they now refer to the n^{th} element rather than the entire domain. S_{n_n} is that part of ∂V_n corresponding to internal (interelement) boundaries.

Bearing this in mind the modified Reissner principle is derived by relaxing the boundary displacement conditions. For all the previous functionals it was necessary that the displacements be continuous over the entire domain. With the domain divided into separate elements, each one being continuous, it is convenient to choose displacements which are continuous in each element but discontinuous across interelement boundaries. To extend this concept further, one can relax the continuity conditions between the displacements on the interior of an element and those on the entire boundary ∂V_n. Of course, it must be understood that upon relaxing these conditions, the boundary displacements from one element must be continuous with the boundary displacements of all neighboring elements. Consider the equation on ∂V_n

$$u_i + \Delta u_i = \tilde{u}_i + \Delta \tilde{u}_i \tag{4.26}$$

or

$$(u_i + \Delta u_i) - (\tilde{u}_i + \Delta \tilde{u}_i) = 0 \tag{4.27}$$

where

$u_i + \Delta u_i$ = displacement on interior of element

$\tilde{u}_i + \Delta \tilde{u}_i$ = displacement on boundary

Relaxing Eq. 4.27 by the Lagrange multiplier technique, adding the corresponding term to Eq. 4.24 and recalling that one must now sum over all the elements, the modified Reissner principle results.

$$
\begin{aligned}
\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ & -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i \\
& + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} \\
& + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\} dV \\
& - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i \, ds \\
& - \int_{S_{u_n}} (T_i + \Delta T_i)[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)] ds \\
& + \int_{\partial V_n} \mu_i [(u_i + \Delta u_i) - (\tilde{u}_i + \Delta\tilde{u}_i)] ds \Big\}
\end{aligned}
\tag{4.28}
$$

Taking the variation of π_{mR} with respect to $\Delta\sigma_{ij}$, ΔT_i, and Δu_i gives

$$
\begin{aligned}
\delta\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ & -\Delta e_{ij}\delta\Delta\sigma_{ij} - \delta\Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\delta\Delta u_i \\
& + \tfrac{1}{2}\delta\Delta\sigma_{ij}[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} \\
& + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}] + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[\delta\Delta u_{i,j} + \delta\Delta u_{j,i} \\
& + u_{k,i}\delta\Delta u_{k,j} + \delta\Delta u_{k,i}u_{k,j} + \delta(\Delta u_{k,i}\Delta u_{k,j})]\} dV \\
& - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta\tilde{u}_i \, ds - \int_{S_{u_n}} \delta\Delta T_i[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)] ds \\
& - \int_{S_{u_n}} (T_i + \Delta T_i)\delta\Delta\tilde{u}_i \, ds + \int_{\partial V_n} \delta\mu_i[(u_i + \Delta u_i) - (\tilde{u}_i + \Delta\tilde{u}_i)] ds \\
& + \int_{\partial V_n} \mu_i[\delta\Delta u_i - \delta\Delta\tilde{u}_i] ds \Big\} = 0
\end{aligned}
\tag{4.29}
$$

Integrating the variation of the incremental displacement gradient terms by parts (as before) and rearranging

$$
\begin{aligned}
\delta\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ & -\delta\Delta\sigma_{ij}[e_{ij} + \Delta e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} \\
& + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})] \\
& - \delta\Delta u_i\{(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]_{,j} + (\bar{F}_i + \Delta\bar{F}_i)\}\} dV \\
& + \int_{\partial V_n} \delta\Delta u_i[\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]\nu_j \, ds \\
& + \int_{\partial V_n} \delta\mu_i[(u_i + \Delta u_i) - (\tilde{u}_i + \Delta\tilde{u}_i)] ds + \int_{\partial V_n} \mu_i[\delta\Delta u_i - \delta\Delta\tilde{u}_i] ds \\
& - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta\tilde{u}_i \, ds - \int_{S_{u_n}} \delta\Delta T_i[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)] ds \\
& - \int_{S_{u_n}} (T_i + \Delta T_i)\delta\Delta\tilde{u}_i \, ds \Big\} = 0
\end{aligned}
\tag{4.30}
$$

Recalling that $\partial V_n = S_{\sigma_n} + S_{u_n} + S_{n_n}$ the Lagrange multiplier is determined as

$$\mu_i + [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})] \nu_j = 0 \tag{4.31}$$

or

$$\mu_i = -(T_i + \Delta T_i) \tag{4.32}$$

Placing Eqs. 4.25 and 4.32 into Eq. 4.30 and rearranging

$$\begin{aligned}
\delta \Pi_{mR} = \sum_n \Big\{ & \int_{V_n} \{ -\delta\Delta\sigma_{ij} [e_{ij} + \Delta e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \\
& + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i} \Delta u_{k,j} + \Delta u_{k,i} u_{k,j} + \Delta u_{k,i} \Delta u_{k,j})] \\
& - \delta\Delta u_i \{ (\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]_{,j} \\
& + (\bar{F}_i + \Delta\bar{F}_i) \} \} dV \\
& + \int_{\partial V_n} \delta\Delta u_i \{ -(T_i + \Delta T_i) + [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})] \nu_j \} ds \\
& + \int_{\partial V_n} \delta\mu_i [(u_i + \Delta u_i) - (\tilde{u}_i + \Delta\tilde{u}_i)] ds \\
& + \int_{S_{\sigma_n}} \delta\Delta\tilde{u}_i [(T_i + \Delta T_i) - (\bar{T}_i + \Delta\bar{T}_i)] ds \\
& - \int_{S_{u_n}} \delta\Delta T_i [(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)] ds \\
& + \int_{S_{n_n}} \delta\Delta\tilde{u}_i (T_i + \Delta T_i) ds \Big\} = 0
\end{aligned} \tag{4.33}$$

Thus, the Euler equations of the functional are as follows.

In the volume, V_n:

Strain displacement

$$\begin{aligned}
e_{ij} + \Delta e_{ij} = \tfrac{1}{2} (& u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} \\
& + u_{k,i} \Delta u_{k,j} + \Delta u_{k,i} u_{k,j} + \Delta u_{k,i} \Delta u_{k,j}
\end{aligned} \tag{4.34}$$

Stress equilibrium

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{4.35}$$

On the boundary:

Tractions on ∂V_n

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})]\,\nu_j \tag{4.36}$$

Mechanical boundary conditions on S_{σ_n}

$$T_i + \Delta T_i = \bar{T}_i + \Delta\bar{T}_i \tag{4.37}$$

Kinematic boundary condition on S_{u_n}

$$\tilde{u}_i + \Delta\tilde{u}_i = \bar{u}_i + \Delta\bar{u}_i \tag{4.38}$$

Relaxation of displacements on ∂V_n

$$u_i + \Delta u_i = \tilde{u}_i + \Delta\tilde{u}_i \tag{4.39}$$

The last term in Eq. 4.33 represents the relaxation of boundary tractions across the interelement boundaries. If one considers two neighbor elements (Fig. 4.1), one designated 'a' and the other designated 'b', then the interelement tractions are T_i^a and T_i^b. This term, therefore, states that on S_{n_n}

$$(T_i + \Delta T_i)^a + (T_i + \Delta T_i)^b = 0 \tag{4.40}$$

Upon summing over all the elements and their interelement boundaries these corresponding work terms would cancel each other if Eq. 4.40 were exactly satisfied.

Placing Eq. 4.32 into Eq. 4.28 the functional π_{mR} corresponding to the modified Reissner principle results in the stationary condition

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i \\ + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \\ + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\}\,dV \\ - \int_{\partial V_n} (T_i + \Delta T_i)[(u_i + \Delta u_i) - (\tilde{u}_i + \Delta\tilde{u}_i)]\,ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i\,ds \\ - \int_{S_{u_n}} (T_i + \Delta T_i)[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)]\,ds \Big\}\end{aligned} \tag{4.41}$$

It should be noted here that while the Euler equations for $\delta\pi_I=0$ (Eqs. 4.17-4.21) and those of $\delta\pi_{mR}=0$ (Eqs. 4.34-4.38) look identical they refer to different domains. The equations for π_I refer to the entire domain as one continuum. The equations for π_{mR} refer to the domain of one element or subdomain. Since, of course, the elasticity equations govern the entire domain these sets of equations must be the same. However, in the case of an element, simpler assumptions may be made on the space variations of the variables and, in fact, they must only be continuous in the element domain. This advantage is paid for by the necessary addition of the last term in Eq. 4.28. This term results in the relaxation of displacement compatibility (Eq. 4.39) and boundary tractions (Eq. 4.40).

4.4 The Assumed Stress Hybrid Functional

The final phase of this derivation is the formulation of the assumed stress hybrid functionals. For a fully consistent functional, it is required that the total stress equilibrium equation (Eq. 4.35) and the total boundary tractions (Eq. 4.36) be satisfied exactly. These equations are nonlinear. Although in the incremental analysis they may be linearized they are still difficult equations to always satisfy. This leads to several compromises in satisfying the complete stress equilibrium equations. Such functionals will be deemed inconsistent assumed stress hybrid functionals.

In each of the inconsistent approaches, only the linear portion of the stress equilibrium equation is satisfied. As will be shown when discussing particular cases of finite element models (in Section 7) for plates and shells, the linear portion of the stress equilibrium equation may be accounted for in different ways. For the purposes of the general derivations given here it will suffice to say the linear stress equilibrium equation is satisfied exactly. This condition results in an artificial constraint on the total stress for the S.L. system. Computational expediency, however, warrants the study of an inconsistent model.

4.4.1 The Consistent Model

Although Pirotin [1971] alluded to a consistent assumed stress hybrid model the details were not carried out due to the complexity of his element (a deep, doubly curved four sided shell element). The first attempt at describing a consistent approach in detail was by Atluri [1973b]. However, the writer feels this paper is in error and will point out the differences between Atluri's functional and that of the present work in the next section. Since both Pirotin and Atluri used a convected coordinate system, comparisons to the present work will be deferred to Section 5 for a convected, updated coordinate system.

Before constructing the consistent functional it is convenient to first write π_{mR} differently. That is, the terms in the volume integral which contain incremental displacement gradients will be integrated by parts. In addition, it will be necessary that the term

$$\frac{1}{2}\int_{V_n}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})\,dV$$

be added and subtracted to Eq. 4.41 to preserve the functional. Recalling Eqs. 4.12 and 4.13, π_{mR} may be written as

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ \int_{V_n} \Big\{ & -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}e_{ij} - (\bar{F}_i+\Delta\bar{F}_i)\,\Delta u_i \\ & - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}) \\ & - \big\{(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]_{,j}\big\}\Delta u_i \Big\}\, dV \\ & + \int_{\partial V_n}[\sigma_{ij}+\Delta\sigma_{ij} + (\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]\,\nu_j\,\Delta u_i\,dS \\ & - \int_{\partial V_n}(T_i+\Delta T_i)[(u_i+\Delta u_i)-(\tilde{u}_i+\Delta\tilde{u}_i)]\,dS - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\Delta\tilde{u}_i\,dS \\ & - \int_{S_{u_n}}(T_i+\Delta T_i)[(\tilde{u}_i+\Delta\tilde{u}_i)-(\bar{u}_i+\Delta\bar{u}_i)]\,dS \Big\}\end{aligned} \tag{4.42}$$

or

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ & \int_{V_n}[-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})]\,dV \\ & - \int_{V_n}\big\{(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]_{,j} + (\bar{F}_i+\Delta\bar{F}_i)\big\}\Delta u_i\,dV \\ & - \int_{\partial V_n}\big\{(T_i+\Delta T_i) - [\sigma_{ij}+\Delta\sigma_{ij} + (\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]\nu_j\big\}\Delta u_i\,dS \\ & + \int_{\partial V_n}(T_i+\Delta T_i)\Delta\tilde{u}_i\,dS - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\Delta\tilde{u}_i\,dS - \int_{S_{u_n}}(T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,dS \\ & + \int_{\partial V_n}(T_i+\Delta T_i)(\tilde{u}_i-u_i)\,dS - \int_{S_{u_n}}(T_i+\Delta T_i)(\tilde{u}_i-\bar{u}_i)\,dS \\ & - \int_{V_n}\Delta\sigma_{ij}[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})]\,dV + \text{constants}\Big\}\end{aligned} \tag{4.43}$$

In the spirit of the assumed stress hybrid functional, the entire stress equilibrium equation (Eq. 4.35) and the boundary traction requirement (Eq. 4.36) must be satisfied exactly. Subject to these conditions Eq. 4.43 becomes the consistent modified complementary energy principle π^c_{mc}, which is stationary with respect to the incremental quantities $\Delta\sigma_{ij}$, $\Delta\tilde{u}_i$, and Δu_i, where,

$$\begin{aligned}\pi^c_{mc} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})]\,dV \\ &+\int_{\partial V_n}(T_i+\Delta T_i)\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\Delta\tilde{u}_i\,ds - \int_{S_{u_n}}(T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds \\ &+\int_{\partial V_n}(T_i+\Delta T_i)(\tilde{u}_i-u_i)\,ds - \int_{S_{u_n}}(T_i+\Delta T_i)(\tilde{u}_i-\bar{u}_i)\,ds \\ &-\int_{V_n}\Delta\sigma_{ij}\,[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})]\,dV \Big\}\end{aligned} \qquad (4.44)$$

The constant terms are dropped because they are not subject to variation. Eq. 4.44 is subject to Eqs. 4.35 and 4.36. Eq. 4.44 contains very complicated expressions in light of the fact that Eq. 4.35 and 4.36 must be satisfied exactly. Note that all these equations, unlike linear analysis, exhibit coupling between stresses and interior displacements. If the increments are kept small, it is possible to linearize these equations and, although the result is still complicated, it is worthwhile considering them. If initial quantities are of O(1) assume the increments small enough so that incremental quantities are of $O(\varepsilon)$ where $\varepsilon \ll 1$.

Consider Eq. 4.44 again. Retaining terms up to $O(\varepsilon^2)$ only.

$$\begin{aligned}\pi^c_{mc} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}]\,dV \\ &+\int_{\partial V_n}(T_i+\Delta T_i)\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\Delta\tilde{u}_i\,ds - \int_{S_{u_n}}(T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds \\ &+\int_{\partial V_n}(T_i+\Delta T_i)(\tilde{u}_i-u_i)\,ds - \int_{S_{u_n}}(T_i+\Delta T_i)(\tilde{u}_i-\bar{u}_i)\,ds \\ &-\int_{V_n}\Delta\sigma_{ij}\,[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})]\,dV \Big\}\end{aligned} \qquad (4.45)$$

One must now also linearize Eqs. 4.35 and 4.36

$$(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [\sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]_{,j} + (\bar{F}_i+\Delta\bar{F}_i) = 0 \qquad (4.46)$$

and

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}(u_{i,k} + \Delta u_{i,k}) + \Delta\sigma_{kj} u_{i,k}] \nu_j \tag{4.47}$$

If one takes the variation of Eq. 4.45 with respect to stress, after some manipulation it will be seen that Eq. 4.34 becomes

$$\begin{aligned} e_{ij} + \Delta e_{ij} = \tfrac{1}{2} (& u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \\ & + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i} \Delta u_{k,j} + \Delta u_{k,i} u_{k,j}) \end{aligned} \tag{4.48}$$

which linearizes the incremental strain as expected.

It should be noted that all the above functionals are complete in that they make no assumptions on stress equilibrium or compatibility in the initial configuration. Alternate functionals can be constructed in a similar fashion to the ones above by assuming different conditions are satisfied in the reference state. Further comments are made in this regard in Subsection 4.4.3.

4.4.2 The Inconsistent Model

In an attempt to strike a compromise between the fully consistent assumed stress hybrid model, with all its complexities, and the modified Reissner principle, with no constraints on stress equilibrium, an inconsistent model was pursued. Pirotin [1971], who used a basic form of this functional, showed that for beam analysis this inconsistent procedure yielded better results than a conventional Reissner approach.

To construct this model it is again convenient to write π_{mR} differently. Considering Eq. 4.41, integrate only the first two incremental displacement gradient terms in V by parts.

$$\begin{aligned} \pi_{mR} = \sum_n \Big\{ & \int_{V_n} \{ -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i) \Delta u_i \\ & + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij}) [u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} + u_{k,i} \Delta u_{k,j} + \Delta u_{k,i} u_{k,j} \\ & + \Delta u_{k,i} \Delta u_{k,j}] - (\sigma_{ij} + \Delta\sigma_{ij})_{,j} \Delta u_i \} dV + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij}) \nu_j \Delta u_i \, dS \\ & + \int_{\partial V_n} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta u_i) \, dS - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i) \Delta\tilde{u}_i \, dS - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) dS \\ & + \int_{\partial V_n} (T_i + \Delta T_i)(\tilde{u}_i - u_i) \, dS - \int_{S_{u_n}} (T_i + \Delta T_i)(\tilde{u}_i - \bar{u}_i) dS \Big\} \end{aligned} \tag{4.49}$$

Or, after rearranging

$$\begin{aligned}\Pi_{mR} = \sum_n \Big\{ & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}\Delta u_{k,j} + \Delta u_{k,i} u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})] dV \\ & - \int_{V_n} [(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i)] \Delta u_i \, dV \\ & - \int_{\partial V_n} [(T_i + \Delta T_i) - (\sigma_{ij}+\Delta\sigma_{ij})\nu_j] \Delta u_i \, ds \\ & + \int_{\partial V_n} (T_i + \Delta T_i)\Delta\tilde{u}_i \, ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i \, ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) ds \\ & + \int_{\partial V_n} (T_i + \Delta T_i)(\tilde{u}_i - u_i) ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\tilde{u}_i - \bar{u}_i) ds \\ & - \int_{V_n} \Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})] dV + \text{constants} \Big\} \end{aligned}$$

(4.50)

From Eq. 4.50 it is obvious that one would satisfy only the linear part of the stress equilibrium equation, namely

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{4.51}$$

Assuming Eq. 4.36 to be satisfied exactly, one may write

$$(T_i + \Delta T_i) - (\sigma_{ij} + \Delta\sigma_{ij})\nu_j = (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})\nu_j \tag{4.52}$$

Placing Eqs. 4.51 and 4.52 into Eq. 4.50 yields the inconsistent modified complementary energy principle, π^I_{mc}, which is also stationary with respect to the incremental quantities, $\Delta\sigma_{ij}$, $\Delta\tilde{u}_i$ and Δu_i, where

$$\begin{aligned}\Pi^I_{mc} = \sum_n \Big\{ & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})] dV \\ & + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j \Delta\tilde{u}_i \, ds + \int_{\partial V_n} (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k})\nu_j (\Delta\tilde{u}_i - \Delta u_i) ds \\ & - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i \, ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) ds \\ & + \int_{\partial V_n} (T_i + \Delta T_i)(\tilde{u}_i - u_i) ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\tilde{u}_i - \bar{u}_i) ds \\ & - \int_{V_n} \Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})] dV \Big\} \end{aligned}$$

(4.53)

where the constant terms are dropped since they are not subject to variation. In light of Eq. 4.51 the functional π^I_{mc} would be simpler to implement than that of π^C_{mc}.

Assuming that the increments are small the coupling between stress and displacement increments can be removed. As in the previous subsection, if initial quantities are of O(1) the increments can be chosen small enough so that incremental valu . are of $O(\varepsilon)$ where $\varepsilon \ll 1$. Eq. 4.53 can thus be reduced to

$$\begin{aligned}\pi^I_{mc} = \sum_n \Big\{ & \int_{V_n} \big[-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})(u_{k,i}\,\Delta u_{k,j} + \Delta u_{k,i}\,u_{k,j}) \\ & + \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\big]dV + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\,\nu_j\,\Delta\tilde{u}_i\,ds \\ & + \int_{\partial V_n} \big[\sigma_{kj}(u_{i,k} + \Delta u_{i,k}) + \Delta\sigma_{kj}\,u_{i,k}\big]\nu_j\,(\Delta\tilde{u}_i - \Delta u_i)\,ds \\ & - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds \\ & + \int_{\partial V_n} (T_i + \Delta T_i)(\tilde{u}_i - u_i)\,ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\tilde{u}_i - \bar{u}_i)\,ds \\ & - \int_{V_n} \Delta\sigma_{ij}\big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j})\big]dV \Big\}\end{aligned} \tag{4.54}$$

where π^I_{mc} is subject to Eqs. 4.51 and 4.47 which are already linear.

Comparing the consistent and inconsistent functionals (Eqs. 4.45 and 4.54 respectively) one may observe that the initial displacements appear explicitly in π^I_{mc} while they appear implicitly through stress equilibrium in π^C_{mc}. Furthermore, for π^C_{mc}, the initial displacements appear explicitly in the stress equilibrium equations (Eq. 4.46). They do not appear at all in the corresponding equation (Eqs. 4.51) for π^I_{mc}. At this point it is obvious that π^I_{mc} would be simpler to implement than π^C_{mc}. This will become even more clear when the matrix equations are considered for each functional in Section 6.

4.4.3 Equilibrium Checks

The consistent model, π^C_{mc}, and the inconsistent model, π^I_{mc}, are given by Eqs. 4.44 and 4.53 respectively. As previously stated these equations make no assumptions about what conditions may be satisfied in the initial, reference configuration. Although both these functionals require constraints on the stress equilibrium equations there is no guarantee that stress equilibrium will satisfied in the n^{th} state. While in linear analysis where a one step procedure is used and the stresses satisfy equilibrium exactly there is no problem. It is certainly conceivable that the stresses may drift away from the true solution during the incremental process. In fact, this is quite commonly observed in nonlinear analysis [Hofmeister et al., 1971]. Thus, an equilibrium check might be appropriate. In addition, the strain displacement relations are Euler equations in all of these assumed stress hybrid functionals. Since they are only satisfied in an average sense within an increment they cannot be expected to satisfy these conditions in the total sense. It is, therefore, necessary to consider a compatibility check in the functionals. Finally, a third equilibrium check is generated if a displacement mismatch occurs between either the interior and interelement boundary displacement fields or the interelement boundary and prescribed displacement fields. This check should be considered only in conjunction with the compatibility check. The equilibrium checks developed here differ somewhat from those of Hofmeister et al. This is intentional to allow a more natural form of checks for the assumed stress hybrid models.

Eqs. 4.44 and 4.53 already have all of these checks in them. For both functionals, the compatibility check and its associated displacement mismatch checks are most easily identified. They are the last three integrals in each equation. If the strain displacement relations were satisfied exactly in the reference configuration the compatibility term would be identically zero.

The stress equilibrium check is a bit harder to see, but it is taken into account by maintaining an initial stress term, the initial tractions in the surface integrals of the functionals and the corresponding terms in the constraint equations. The artificial constraint on the total stress for the inconsistent model in the S.L. system is reflected in the stress equilibrium check. To identify the correction terms one must first consider the Principle of Virtual Work again. Repeating Eq. 4.3

$$\delta\pi = \int_V [(\sigma_{ij} + \Delta\sigma_{ij})\,\delta\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\delta\Delta u_i]\,dv - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta u_i\,ds \qquad (4.3)$$

In Eq. 4.3 the strain displacement relations are assumed to be satisfied exactly. From Eq. 4.7 the incremental strain displacement relations may be identified as

$$2\Delta e_{ij} = \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j} \tag{4.55}$$

Placing Eq. 4.55 into Eq. 4.3 and integrating by parts yields

$$\begin{aligned} \delta\Pi = &- \int_V \Big\{ (\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]_{,j} \\ &\qquad + (\bar{F}_i + \Delta\bar{F}_i)\Big\} \delta\Delta u_i \, dV \\ &+ \int_{\partial V_n} \Big\{ (\sigma_{ij}+\Delta\sigma_{ij}) + [(\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]\Big\} \nu_j \, \delta\Delta u_i \, ds \\ &- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\, \delta\Delta u_i \, ds = 0 \end{aligned} \tag{4.56}$$

Recalling that $\partial V = S_\sigma + S_u$ and that $\delta\Delta u_i$ on S_u is zero gives

$$\begin{aligned} \delta\Pi = &- \int_V \Big\{ (\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]_{,j} + (\bar{F}_i+\Delta\bar{F}_i)\Big\} \delta\Delta u_i \, dV \\ &- \int_{S_\sigma} \Big\{ (\bar{T}_i + \Delta\bar{T}_i) - [(\sigma_{ij}+\Delta\sigma_{ij}) \\ &\qquad + (\sigma_{kj}+\Delta\sigma_{kj})(u_{i,k}+\Delta u_{i,k})]\nu_j \Big\} \delta\Delta u_i \, ds = 0 \end{aligned} \tag{4.57}$$

Identifying the conditions in the initial state as

$$\sigma_{ij,j} + (\sigma_{kj} u_{i,k})_{,j} + \bar{F}_i = 0 \tag{4.58}$$

and

$$T_i = [\sigma_{ij} + \sigma_{kj} u_{i,k}]\nu_j \tag{4.59}$$

the terms in Eq. 4.57 may be separated

$$\begin{aligned} \delta\Pi = &- \int_V \Big\{ \Delta\sigma_{ij,j} + [\sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k}+\Delta u_{i,k})]_{,j} + \Delta\bar{F}_i \Big\} \delta\Delta u_i \, dV \\ &- \int_{S_\sigma} \Big\{ \Delta\bar{T}_i - [\Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k}+\Delta u_{i,k})]\nu_j \Big\} \delta\Delta u_i \, ds \\ &- \int_V \Big\{ \sigma_{ij,j} + (\sigma_{kj} u_{i,k})_{,j} + \bar{F}_i \Big\} \delta\Delta u_i \, dV \\ &- \int_{S_\sigma} \Big\{ \bar{T}_i - [\sigma_{ij} + \sigma_{kj} u_{i,k}]\nu_j \Big\} \delta\Delta u_i \, ds = 0 \end{aligned} \tag{4.60}$$

If it is required that Eqs. 4.58 and 4.59 are to be satisfied exactly, then the last two integrals in Eq. 4.60 vanish. Integrating by parts again gives

$$\delta\pi = \int_V [\Delta\sigma_{ij}\,\delta\Delta e_{ij} + \tfrac{1}{2}\sigma_{ij}\,\delta(\Delta u_{k,i}\Delta u_{k,j}) - \Delta\bar{F}_i\,\delta\Delta u_i]\,dV$$
$$- \int_{S_\sigma} \Delta\bar{T}_i\,\delta\Delta u_i\,ds = 0 \qquad (4.61)$$

Following the development of Subsection 4.2 one finds

$$\pi_p = \int_V [A(\Delta e_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i\Delta u_i]\,dV$$
$$- \int_{S_\sigma}\Delta\bar{T}_i\,\Delta u_i\,ds \qquad (4.62)$$

and

$$\pi_I = \int_V \{A(\Delta e_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i\Delta u_i$$
$$+ \lambda_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j}$$
$$+ \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})]\}\,dV$$
$$- \int_{S_\sigma}\Delta\bar{T}_i\Delta u_i\,ds + \int_{S_u}\mu_i[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)]\,ds \qquad (4.63)$$

It is obvious here that the Lagrange multipliers will be identified by

$$\lambda_{ij} = -\Delta\sigma_{ij} \qquad (4.64)$$

and

$$\mu_i = -\Delta T_i = [\lambda_{ij} + \lambda_{kj}(u_{i,k} + \Delta u_{i,k})]\,\nu_j + (\sigma_{kj}\Delta u_{i,k})\nu_j \qquad (4.65)$$

Placing Eqs. 4.64 and 4.65 into Eq. 4.63 gives

$$\pi_I = \int_V \{A(\Delta e_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i\Delta u_i$$
$$- \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j}$$
$$+ \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j})]\}\,dV$$
$$- \int_{S_\sigma}\Delta\bar{T}_i\Delta u_i\,ds - \int_{S_u}\Delta T_i[(u_i + \Delta u_i) - (\bar{u}_i + \Delta\bar{u}_i)]\,ds \qquad (4.66)$$

For completeness the following functionals are written but no further explanation is necessary

$$\Pi_R = \int_V \{-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} + \Delta\bar{F}_i \Delta u_i$$
$$+ \tfrac{1}{2}\Delta\sigma_{ij}[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i}$$
$$+ u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\}dV$$
$$- \int_{S_\sigma} \Delta\bar{T}_i \Delta u_i \, ds - \int_{S_u} \Delta T_i[(u_i+\Delta u_i) - (\bar{u}_i+\Delta\bar{u}_i)]ds \tag{4.67}$$

$$\Pi_{mR} = \sum_n \{\int_{V_n}[-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i \Delta u_i$$
$$+ \tfrac{1}{2}\Delta\sigma_{ij}[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j}$$
$$+ \Delta u_{k,i}\Delta u_{k,j}]\}dV - \int_{\partial V_n}\Delta T_i[(u_i+\Delta u_i) - (\tilde{u}_i+\Delta\tilde{u}_i)]ds$$
$$- \int_{S_{\sigma_n}}\Delta\bar{T}_i \Delta\tilde{u}_i ds - \int_{S_{u_n}}\Delta T_i[(\tilde{u}_i+\Delta\tilde{u}_i) - (\bar{u}_i+\Delta\bar{u}_i)]ds \tag{4.68}$$

or,

$$\Pi_{mR} = \sum_n \{\int_{V_n}[-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})\Delta u_{k,i}\Delta u_{k,j}]dV$$
$$- \int_{V_n}\{\Delta\sigma_{ij,j} + [\sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k}+\Delta u_{i,k})]_{,j} + \Delta\bar{F}_i\}\Delta u_i dV$$
$$- \int_{\partial V_n}\{\Delta T_i - [\Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k}+\Delta u_{i,k})]\nu_j\}\Delta u_i ds$$
$$+ \int_{\partial V_n}\Delta T_i \Delta\tilde{u}_i ds - \int_{S_{\sigma_n}}\Delta\bar{T}_i\Delta\tilde{u}_i ds - \int_{S_{u_n}}\Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds$$
$$+ \int_{\partial V_n}\Delta T_i(\tilde{u}_i - u_i)ds - \int_{S_{u_n}}\Delta T_i(\tilde{u}_i - \bar{u}_i)ds$$
$$- \int_{V_n}\Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})]dV\} \tag{4.69}$$

Finally

$$\Pi^c_{mc} = \sum_n \{\int_{V_n}[-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})\Delta u_{k,i}\Delta u_{k,j}]dV$$
$$- \int_{\partial V_n}\Delta T_i\Delta\tilde{u}_i ds - \int_{S_{\sigma_n}}\Delta\bar{T}_i\Delta\tilde{u}_i ds - \int_{S_{u_n}}\Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds$$
$$+ \int_{\partial V_n}\Delta T_i(\tilde{u}_i - u_i)ds - \int_{S_{u_n}}\Delta T_i(\tilde{u}_i - \bar{u}_i)ds$$
$$- \int_{V_n}\Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})]dV \tag{4.70}$$

where Eq. 4.70 is subject to

$$\Delta\sigma_{ij,j} + [\sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k})]_{,j} + \Delta\bar{F}_i = 0 \tag{4.71}$$

and

$$\Delta T_i = [\Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k})]\nu_j \tag{4.72}$$

Note that these last two equations are identical to Eqs. 4.35 and 4.36 if in the initial state Eqs. 4.58 and 4.59 are satisfied exactly. Comparing Eqs. 4.70 and 4.44 one can see the effect of the equilibrium check on the functional.

By similar argument it can be shown that the inconsistent model without the equilibrium check can be written as

$$\begin{aligned}\pi^{I}_{mc} = \sum_n \Big\{ & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\Delta\sigma_{ij}(u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j}) \\ & + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})\Delta u_{k,i}\Delta u_{k,j}]\,dV + \int_{\partial V_n}\Delta\sigma_{ij}\nu_j\Delta\tilde{u}_i\,ds \\ & + \int_{\partial V_n}[\sigma_{kj}\Delta u_{i,k} + \Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k})]\nu_j(\Delta\tilde{u}_i - \Delta u_i)\,ds \\ & - \int_{S_{\sigma_n}}\Delta\bar{T}_i\Delta\tilde{u}_i\,ds - \int_{S_{u_n}}\Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds + \int_{\partial V_n}\Delta T_i(\tilde{u}_i - u_i)\,ds \\ & - \int_{S_{u_n}}\Delta T_i(\tilde{u}_i - \bar{u}_i)\,ds - \int_{V_n}\Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})]\,dV \Big\}\end{aligned} \tag{4.73}$$

which is subject to the conditions

$$\Delta\sigma_{ij,j} + \Delta\bar{F}_i = 0 \tag{4.74}$$

and Eq. 4.72.

Eqs. 4.70-4.73 may be linearized in the same fashion as in the previous subsections if the increments are kept small. The compatibility check can now be seen as playing an interesting role in these linearized functionals. While the incremental strain is linear in incremental displacements, the total strain in the reference configuration is nonlinear in the known total displacements. Thus, if incremental steps are taken to be large, the compatibility check, which form imbalance load terms, tends to correct the solution to the proper value. In fact, when all equilibrium checks are utilized, the entire load may be taken in one increment and in very few steps the equilibrium checks will iterate the solution to the correct one. This will be demonstrated in Section 8 where results are given.

SECTION 5

A GENERAL INCREMENTAL ASSUMED STRESS HYBRID FORMULATION FOR LARGE DEFLECTION ANALYSIS BASED ON A CONVECTED, UPDATED LAGRANGIAN COORDINATE SYSTEM

5.1 Introduction

An assumed stress hybrid functional based on an updated coordinate system will be derived from the Principle of Virtual Work in this section. Essentially a parallel discussion to that of Section 4 will be given. An initial stress, incremental procedure shall be incorporated. As discussed in Section 2, a Convected, Updated Lagrangian coordinate system requires the use of Cauchy stresses, referred to a rotated coordinate system, as initial stresses, and second Kirchhoff stresses as incremental stresses. Consistent with these are the Almansi strains as initial strains and updated Green strains as incremental strains. These Green strains are different from those of Section 4 in that the displacements and their derivatives are referred to the updated coordinates as opposed to the initial coordinates. It should also be noted that the reference configuration is always the last known configuration and must not be confused with the initial configuration. In the last section they were synonomous. This updated system should not be confused with either the convected system [Fung, 1965; Pirotin, 1971; Atluri, 1973b] or the standard update Lagrangian system [Bathe et al. 1973]. (See Section 2.) Since the same functionals will be derived here as in Section 4, it will simply be stated at this point that all the variational principles here are stationary with respect to variations in the same incremental quantities as before (referred to the C.U.L. system).

It is to be understood that although the same symbols will be used in this section as those used in Section 4 their interpretations are, in general, different. This is done for convenience so that comparisons between the two systems become more obvious.

5.2 The Hu-Washizu Principle

In a similar manner to the development in Subsection 4.2 this principle may be derived from the Principle of Virtual Work which is stated as

$$\delta\pi = \int_V [\sigma_{ij}\,\delta e_{ij} - \bar{F}_i \delta u_i]\,dv \;\; - \int_{S_\sigma} \bar{T}_i\,\delta u_i\,ds = 0 \tag{5.1}$$

Considering a state incrementally close to the present (reference) state the variables may be expanded as

$$\sigma_{ij} \longrightarrow \sigma_{ij} + \Delta\sigma_{ij}$$

$$e_{ij} \longrightarrow e_{ij} + \Delta e_{ij} \longrightarrow \Delta e_{ij}$$

$$\bar{F}_i \longrightarrow \bar{F}_i + \Delta \bar{F}_i$$

$$u_i \longrightarrow u_i + \Delta u_i \longrightarrow \Delta u_i$$

$$\bar{T}_i \longrightarrow \bar{T}_i + \Delta \bar{T}_i$$

where

( ) = initial quantity in reference state

Δ( ) = incremental quantity from reference state

and

σ_{ij} = Cauchy stress tensor

$\Delta\sigma_{ij}$ = incremental second Kirchhoff stress tensor

e_{ij} = Almansi strain tensor

Δe_{ij} = updated Green strain tensor

$\bar{F}_i$ and $\Delta\bar{F}_i$ = prescribed body forces

u_i and Δu_i = displacements

$\bar{T}_i$ and $\Delta\bar{T}_i$ = prescribed surface tractions

$\bar{(\;)}$ = prescribed quantities

Note that for the updated system the initial displacements are inherent to the system by virtue of updating the coordinates. Also, the updated reference state may be thought of as one in which there are initial stresses but no initial strains. Thus, the initial strain and displacement quantities are dropped. However, for the reference state at time n there are total displacements from the initial reference frame and these may be related to the Almansi strain. This shall form the basis for the compatibility check in the updated coordinate system.

The Principle of Virtual Work may now be written as

$$\delta\pi = \int_V [(\sigma_{ij} + \Delta\sigma_{ij})\delta(e_{ij} + \Delta e_{ij}) - (\bar{F}_i + \Delta\bar{F}_i)\delta(u_i + \Delta u_i)]dV$$
$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta(u_i + \Delta u_i)ds = 0 \tag{5.2}$$

or, with initial strains and displacements assumed to be zero

$$\delta\pi = \int_V [(\sigma_{ij} + \Delta\sigma_{ij})\delta\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\delta\Delta u_i]dV$$
$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta u_i ds = 0 \tag{5.3}$$

The volume, V, and the bounding surfaces, ∂V, refer to the reference configuration and, therefore, are the updated volume and surfaces respectively.

Defining a state function such that

$$\delta A(\Delta e_{ij}) = \Delta\sigma_{ij}\,\delta\Delta e_{ij} \longrightarrow \Delta\sigma_{ij} = \frac{\partial A(\Delta e_{ij})}{\partial\Delta e_{ij}} \tag{5.4}$$

The Principle of Total Potential Energy, π_p, is thus

$$\pi_p = \int_V [A(\Delta e_{ij}) + \sigma_{ij}\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\Delta u_i]dV$$
$$- \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta u_i ds \tag{5.5}$$

For this principle the appropriate strain displacement relations in V and kinematic boundary conditions on S_u are satisfied exactly. The stress equilibrium equations in V and mechanical boundary conditions on S_σ, for the reference state, are Euler equations. The next step is to relax the strain displacement relations and kinematic boundary conditions via Lagrange multipliers. The initial Almansi strain in the present reference configuration is

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \tag{5.6}$$

The increment in updated Green strain from the present reference configuration may be written as

$$\Delta e_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}) \tag{5.7}$$

Also, the kinematic boundary conditions on S_u are

$$\Delta u_i = \Delta\bar{u}_i \tag{5.8}$$

Although the initial strain in the reference state is assumed zero through the process of updating, Eqs. 5.6 and 5.7 can be combined and added to π_p when multiplied by an appropriate Lagrange multiplier. (See footnote on page 58.) In addition, Eq. 5.8 is relaxed resulting in the generalized functional π_I^*.

$$\begin{aligned}\pi_I^* = \int_V \Big\{ & A(\Delta e_{ij}) + \sigma_{ij}\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i + \lambda_{ij}\big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} \\ & - u_{k,i}u_{k,j}) + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})\big]\Big\} dV \\ & - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i \, ds + \int_{S_u} \mu_i (\Delta u_i - \Delta\bar{u}_i)\, ds \end{aligned} \tag{5.9}$$

Taking the variation of π_I^* with respect to incremental quantities only yields

$$\begin{aligned}\delta\pi_I^* = \int_V \Big\{ & \Delta\sigma_{ij}\delta\Delta e_{ij} + \sigma_{ij}\delta\Delta e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\delta\Delta u_i \\ & + \delta\lambda_{ij}\big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \\ & + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})\big] \\ & + \lambda_{ij}\big[\delta\Delta e_{ij} - \tfrac{1}{2}(\delta\Delta u_{i,j} + \delta\Delta u_{j,i} + \delta(\Delta u_{k,i}\Delta u_{k,j}))\big]\Big\} dV \\ & - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta u_i \, ds \\ & + \int_{S_u} \{ \delta\mu_i (\Delta u_i - \Delta\bar{u}_i) + \mu_i \delta\Delta u_i \} ds = 0 \end{aligned} \tag{5.10}$$

or, rearranging

$$\begin{aligned}\delta\pi_I^* = \int_V \Big\{ & \delta\Delta e_{ij}(\sigma_{ij} + \Delta\sigma_{ij} + \lambda_{ij}) + \delta\lambda_{ij}\big[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} \\ & - u_{k,i}u_{k,j}) + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}) \\ & - \tfrac{1}{2}\lambda_{ij}\big[\delta\Delta u_{i,j} + \delta\Delta u_{j,i} + \delta(\Delta u_{k,i}\Delta u_{k,j})\big] - (\bar{F}_i + \Delta\bar{F}_i)\delta\Delta u_i \Big\} dV \\ & - \int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\delta\Delta u_i \, ds \\ & + \int_{S_u} \{ \delta\mu_i (\Delta u_i - \Delta\bar{u}_i) + \mu_i \delta\Delta u_i \} ds = 0 \end{aligned} \tag{5.11}$$

Integrating by parts terms that contain incremental displacement gradients such that

$$\int_V \lambda_{ij}\,\delta\Delta u_{i,j}\,dV = \int_{\partial V} \lambda_{ij}\nu_j\,\delta\Delta u_i\,ds - \int_V \lambda_{ij,j}\,\delta\Delta u_i\,dV \tag{5.12}$$

and

$$\frac{1}{2}\int_V \lambda_{ij}\,\delta(\Delta u_{k,i}\Delta u_{k,j})\,dV = \int_{\partial V}(\lambda_{kj}\Delta u_{i,k})\nu_j\,\delta\Delta u_i\,ds - \int_V(\lambda_{kj}\Delta u_{i,k})_{,j}\,\delta\Delta u_i\,dV \tag{5.13}$$

Consider Eqs. 5.12 and 5.13 with Eq. 5.11 and rearranging gives

$$\begin{aligned}\delta\pi_I^* = \int_V \Big\{ &\delta\Delta e_{ij}\,(\sigma_{ij}+\Delta\sigma_{ij}+\lambda_{ij}) + \delta\lambda_{ij}\big[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})\\ &+\Delta e_{ij}-\tfrac{1}{2}(\Delta u_{i,j}+\Delta u_{j,i}+\Delta u_{k,i}\Delta u_{k,j})\big]\\ &+\delta\Delta u_i\big[\lambda_{ij,j}+(\lambda_{kj}\Delta u_{i,k})_{,j}-(\bar{F}_i+\Delta\bar{F}_i)\big]\Big\}\,dV\\ &-\int_{\partial V}\delta\Delta u_i\,[\lambda_{ij}+\lambda_{kj}\Delta u_{i,k}]\,\nu_j\,ds - \int_{S_\sigma}(\bar{T}_i+\Delta\bar{T}_i)\,\delta\Delta u_i\,ds\\ &+\int_{S_u}\{\delta\mu_i(\Delta u_i-\Delta\bar{u}_i)+\mu_i\,\delta\Delta u_i\}\,ds = 0\end{aligned} \tag{5.14}$$

Recalling that $\partial V = S_\sigma + S_u$ the Lagrange multipliers are determined from Eq. 5.14 as

$$\sigma_{ij}+\Delta\sigma_{ij}+\lambda_{ij}=0 \longrightarrow \lambda_{ij} = -(\sigma_{ij}+\Delta\sigma_{ij}) \tag{5.15}$$

and

$$\mu_i = [\lambda_{ij}+\lambda_{kj}\Delta u_{i,k}]\,\nu_j = -(T_i+\Delta T_i) \tag{5.16}$$

Placing these equations into Eq. 5.14, the functional π_I^* has the following Euler equations.

In the volume, V: $\lambda_{ij} = -(\sigma_{ij}+\Delta\sigma_{ij})$

Strain displacement

$$e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})$$

$$\Delta e_{ij} = \frac{1}{2}(\Delta u_{i,j}+\Delta u_{j,i}+\Delta u_{k,i}\Delta u_{k,j}) \tag{5.17}$$

Stress equilibrium

$$(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]_{,j} + (\bar{F}_i+\Delta\bar{F}_i) = 0 \tag{5.18}$$

On the boundary: $\mu_i = [\lambda_{ij} + \lambda_{kj}\Delta u_{i,k}]\nu_j = -(T_i+\Delta T_i)$

Tractions on ∂V

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]\nu_j \tag{5.19}$$

Mechanical boundary conditions on S_σ

$$T_i + \Delta T_i = \bar{T}_i + \Delta\bar{T}_i \tag{5.20}$$

Kinematic boundary conditions on S_u

$$\Delta u_i = \Delta\bar{u}_i \tag{5.21}$$

Placing the Lagrange multipliers (Eqs. 5.15 and 5.16) into Eq. 5.9 and relaxing the stress strain relations (Eq. 5.4) yields the Hu-Washizu principle, π_I. After reducing and eliminating constant terms (not subject to variation)

$$\begin{aligned}\pi_I = \int_V \{ & A(\Delta e_{ij}) - \Delta\sigma_{ij}\Delta e_{ij} - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i+\Delta\bar{F}_i)\Delta u_i \\ & + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})[u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}+\Delta u_{i,j}+\Delta u_{j,i}+\Delta u_{k,i}\Delta u_{k,j}]\}\,dv \\ & - \int_{S_\sigma}(\bar{T}_i+\Delta\bar{T}_i)\Delta u_i\,ds - \int_{S_u}(T_i+\Delta T_i)(\Delta u_i-\Delta\bar{u}_i)\,ds\end{aligned} \tag{5.22}$$

Eq. 5.22 should be compared to Eq. 4.22 for the corresponding functional, π_I, in the Stationary Lagrangian system. Recall that the variables must be interpreted in a consistent manner corresponding to the proper coordinate system.

5.3 The Modified Reissner Principle

Introducing another state function $B(\Delta\sigma_{ij})$, and satisfying Eq. 5.4 exactly, Reissner's principle may be obtained from the Hu-Washizu principle. Defining the state function as

$$B(\Delta\sigma_{ij}) = \Delta\sigma_{ij}\Delta e_{ij} - A(\Delta e_{ij}) \tag{5.23}$$

This in conjunction with Eq. 5.22 gives the Reissner functional,

$$\Pi_R = \int_V \{-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}\, e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\Delta u_i$$
$$+\tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]\}dV$$
$$-\int_{S_\sigma} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta u_i\, ds$$
$$-\int_{S_u} (T_i + \Delta T_i)(\Delta u_i - \Delta\bar{u}_i)\, ds \tag{5.24}$$

At this point the concept of subdomains or elements is introduced as it was in Subsection 4.3. With the subsequent addition of interelement boundaries the bounding surface, ∂V_n, may be decomposed as

$$\partial V_n = S_{\sigma_n} + S_{u_n} + S_{n_n} \tag{5.25}$$

where again S_{n_n} represents the interelement boundaries. Allowing for discontinuities between displacements on the interior of an element and those on the boundary ∂V_n one must relax the following condition

$$\Delta u_i = \Delta\tilde{u}_i \tag{5.26}$$

or

$$\Delta u_i - \Delta\tilde{u}_i = 0 \tag{5.27}$$

where

Δu_i = incremental displacement on interior of element

$\Delta\tilde{u}_i$ = incremental displacement on boundary ∂V_n

Relaxing Eq. 5.27 via Lagrange multiplier and summing over all elements π_R becomes the modified Reissner principle π_{mR}.

$$\Pi_{mR} = \sum_n \Big\{ \int_{V_n} \{-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}\, e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\Delta u_i$$
$$+\tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]\}dV$$
$$-\int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\, ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\, ds$$
$$+ \int_{\partial V_n} \mu_i (\Delta u_i - \Delta\tilde{u}_i)\, ds \Big\} \tag{5.28}$$

The variation of π_{mR} with respect to incremental quantities gives

$$\delta\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ -\Delta e_{ij}\,\delta\Delta\sigma_{ij} - \delta\Delta\sigma_{ij}\, e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\,\delta\Delta u_i$$
$$+ \tfrac{1}{2}\,\delta\Delta\sigma_{ij}\,[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]$$
$$+ \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})\,[\delta\Delta u_{i,j} + \delta\Delta u_{j,i} + \delta(\Delta u_{k,i}\Delta u_{k,j})]\}\,dV$$
$$- \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta\tilde{u}_i\,ds - \int_{S_{u_n}} \delta\Delta T_i\,(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds$$
$$- \int_{S_{u_n}} (T_i + \Delta T_i)\,\delta\Delta\tilde{u}_i\,ds + \int_{\partial V_n} \delta\mu_i\,(\Delta u_i - \Delta\tilde{u}_i)\,ds$$
$$+ \int_{\partial V_n} \mu_i\,(\delta\Delta u_i - \delta\Delta\tilde{u}_i)\,ds \Big\} = 0 \qquad (5.29)$$

Integrating the variation of the incremental displacement gradient terms by parts and rearranging gives

$$\delta\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ -\delta\Delta\sigma_{ij}\,[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})$$
$$+ \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})]$$
$$- \delta\Delta u_i\,\{(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})\,\Delta u_{i,k}]_{,j} + (\bar{F}_i + \Delta\bar{F}_i)\}\Big\}\,dV$$
$$+ \int_{\partial V_n} \delta\Delta u_i\,[\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})\,\Delta u_{i,k}]\,\nu_j\,ds$$
$$+ \int_{\partial V_n} \delta\mu_i\,(\Delta u_i - \Delta\tilde{u}_i)\,ds + \int_{\partial V_n} \mu_i\,(\delta\Delta u_i - \delta\Delta\tilde{u}_i)\,ds$$
$$- \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\delta\Delta\tilde{u}_i\,ds - \int_{S_{u_n}} \delta\Delta T_i\,(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds$$
$$- \int_{S_{u_n}} (T_i + \Delta T_i)\,\delta\Delta\tilde{u}_i\,ds \Big\} = 0 \qquad (5.30)$$

The Lagrange multiplier may be determined from Eq. 5.30 by recalling that $\partial V_n = S_{\sigma_n} + S_{u_n} + S_{n_n}$.

$$\mu_i + [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})\,\Delta u_{i,k}]\,\nu_j = 0 \qquad (5.31)$$

or

$$\mu_i = -(T_i + \Delta T_i) \qquad (5.32)$$

Placing this into Eq. 5.30 and rearranging

$$\begin{aligned}\delta\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ \delta\Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \\ + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})] \\ - \delta\Delta u_i \{(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})\Delta u_{i,k}]_{,j} + (\bar{F}_i + \Delta\bar{F}_i)\}\} dV \\ + \int_{\partial V_n} \delta\Delta u_i \{-(T_i + \Delta T_i) + [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})\Delta u_{i,k}] \nu_j\} ds \\ - \int_{\partial V_n} \delta\Delta T_i (\Delta u_i - \Delta\tilde{u}_i) ds + \int_{S_{\sigma_n}} \delta\Delta\tilde{u}_i [(T_i + \Delta T_i) - (\bar{T}_i + \Delta\bar{T}_i)] ds \\ - \int_{S_{u_n}} \delta\Delta T_i (\Delta\tilde{u}_i - \Delta\bar{u}_i) ds + \int_{S_{n_n}} \delta\Delta\tilde{u}_i (T_i + \Delta T_i) ds \Big\} = 0 \end{aligned} \qquad (5.33)$$

The Euler equations of π_{mR} are thus

In the volume, V_n:

Strain displacement

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \qquad (5.34)$$

$$\Delta e_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})$$

Stress equilibrium

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})\Delta u_{i,k}]_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \qquad (5.35)$$

On the boundary:

Tractions on ∂V_n

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})\Delta u_{i,k}]\nu_j \qquad (5.36)$$

Mechanical boundary conditions on S_{σ_n}

$$T_i + \Delta T_i = \bar{T}_i + \Delta\bar{T}_i \qquad (5.37)$$

Kinematic boundary conditions on S_{u_n}

$$\Delta\tilde{u}_i = \Delta\bar{u}_i \qquad (5.38)$$

Relaxation of displacement on ∂V_n

$$\Delta u_i = \Delta\tilde{u}_i \qquad (5.39)$$

Relaxation of interelement tractions on S_{n_n}

$$(T_i + \Delta T_i)^a + (T_i + \Delta T_i)^b = 0 \tag{5.40}$$

One should compare these equations (Eqs. 5.34-5.40) with the corresponding ones for a stationary system (Eqs. 4.34-4.40).

Placing the Lagrange multiplier (Eq. 5.32) into Eq. 5.28 results in the modified Reissner principle, π_{mR}.

$$\begin{aligned} \pi_{mR} = \sum_n \Big\{ & \int_{V_n} \{ -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij} e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i \\ & + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]\} dV \\ & - \int_{\partial V_n} (T_i + \Delta T_i)(\Delta u_i - \Delta\tilde{u}_i) ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i ds \\ & - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) ds \Big\} \end{aligned} \tag{5.41}$$

This may be compared with its corresponding principle for a stationary system, Eq. 4.41. Again it is pointed out that the Euler equations (Eqs. 5.17-5.21) for π_I refer to the entire domain while Eqs. 5.34-5.40 for π_{mR} refer to the domain of an element including its boundaries.

5.4 The Assumed Stress Hybrid Functional

Derivation of the assumed stress hybrid functionals requires some form of constraint on the stress equilibrium equations and boundary traction requirements. For a fully consistent model Eqs. 5.35 and 5.36 must be satisfied exactly. An inconsistent approach only requires partial restraints on the first of these equations. Namely, only the linear portion is satisfied. For the latter model several choices exist and are further discussed in Section 7. In the C.U.L. system this condition results in an artificial constraint on the incremental stresses. This approach is computationally more attractive than the consistent model.

5.4.1 The Consistent Model

Although the updated coordinate system utilized here is different than the convected system of Atluri [1973b] it will be shown that the basic functional derived is similar. Upon completing this derivation an important discrepency will be pointed out.

To construct the consistent model one first rewrites π_{mR} by integrating by parts the volume terms containing incremental displacement gradients. For convenience the following term will be added and subtracted from the functional.

$$\frac{1}{2}\int_{V_n}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})\,dV$$

Performing these operations, π_{mR} may be written as

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ \int_{V_n} \{ & -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}\, e_{ij} - (\bar{F}_i+\Delta\bar{F}_i)\,\Delta u_i \\ & -\tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}) \\ & - \{(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]_{,j}\}\,\Delta u_i \Big\}\,dV \\ & + \int_{\partial V_n}[\sigma_{ij}+\Delta\sigma_{ij}+(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]\,\nu_j\,\Delta u_i\,ds \\ & - \int_{\partial V_n}(T_i+\Delta T_i)(\Delta u_i-\Delta\tilde{u}_i)\,ds - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds \\ & - \int_{S_{u_n}}(T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds \Big\}\end{aligned} \tag{5.42}$$

or

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ & \int_{V_n}[-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})]\,dV \\ & - \int_{V_n}\{(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]_{,j} + (\bar{F}_i+\Delta\bar{F}_i)\}\,\Delta u_i\,dV \\ & - \int_{\partial V_n}\{(T_i+\Delta T_i) - [\sigma_{ij}+\Delta\sigma_{ij}+(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]\,\nu_j\}\,\Delta u_i\,ds \\ & + \int_{\partial V_n}(T_i+\Delta T_i)\,\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}}(\bar{T}_i+\Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds \\ & - \int_{S_{u_n}}(T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds \\ & - \int_{V_n}\Delta\sigma_{ij}\,[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})]\,dV + \text{constants} \Big\}\end{aligned} \tag{5.43}$$

The consistent model assumes that stress equilibrium and boundary traction conditions (Eqs. 5.35 and 5.36 respectively) are exactly satisfied. These requirements reduce π_{mR} to the consistent assumed stress hybrid functional π^{c}_{mc}.

$$\pi_{mc}^{c} = \sum_{n}\Big\{ \int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\,\Delta u_{k,j})]\,dV$$
$$+\int_{\partial V_n} (T_i+\Delta T_i)\,\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}} (\bar{T}_i+\Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds$$
$$-\int_{S_{u_n}} (T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds$$
$$-\int_{V_n} \Delta\sigma_{ij}\,[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})]\,dV\Big\} \qquad (5.44)$$

where the constant terms not subject to variation are dropped. The functional π_{mc}^{c} (Eq. 5.44) is subject to the conditions of Eqs. 5.35 and 5.36.

Comparing this functional with its corresponding principle in the stationary coordinate system (Eq. 4.44) one observes the striking similarity. Of course, the stresses and strains are defined differently which accounts for the minus sign in the compatibility check term of Eq. 5.44. The differences in these two functionals show up in the conditions of constraint. Comparing the stress equilibrium equations (Eqs. 4.35 and 5.35) one sees that the former is a function of the initial displacements while the latter is not. This should be expected because in updating the coordinates and basing the functional in the current reference state (as opposed to the initial one) the initial displacements (and strains) are accounted for. A similar statement can be made for the comparison of the boundary traction requirements (Eqs. 4.36 and 5.36).

Even though the initial displacement quantities are removed from the equations in this section nonlinear coupling still occurs between the unknown stress and displacement increments. Thus, a linearization is required through the assumptions of small increments. If incremental quantities have magnitudes much smaller than the initial quantities, Eq. 5.44 may be written as

$$\pi_{mc}^{c} = \sum_{n}\Big\{ \int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}]\,dV$$
$$+\int_{\partial V_n} (T_i+\Delta T_i)\,\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}} (\bar{T}_i+\Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds$$
$$-\int_{S_{u_n}} (T_i+\Delta T_i)(\Delta\tilde{u}_i-\Delta\bar{u}_i)\,ds$$
$$-\int_{V_n} \Delta\sigma_{ij}\,[e_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})]\,dV\Big\} \qquad (5.45)$$

subject to

$$(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\sigma_{kj}\Delta u_{i,k})_{,j} + (\bar{F}_i+\Delta\bar{F}_i) = 0 \tag{5.46}$$

and

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k}]\nu_j \tag{5.47}$$

Furthermore, taking the variation of π_{mc}^{c} with respect to stress would yield the linearized incremental strain displacement equation

$$\Delta e_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \tag{5.48}$$

as expected.

The basic functional in Eq. 5.45 is similar to that derived by Atluri with the major discrepency of the term

$$-\tfrac{1}{2}\int_{V_n} \sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\,dV$$

It appears that during an integration by parts Atluri inadvertantly left out this term.

5.4.2 The Inconsistent Model

The constraint conditions which π_{mc}^{c} is subject to are still difficult to satisfy exactly even in the linearized form for the updated system (no initial displacement terms present). Thus, in a compromising fashion an inconsistent assumed stress hybrid model is derived by satisfaction of only the linear part of Eq. 5.35.

If one were to integrate by parts only the linear incremental displacement gradient terms in Eq. 5.41 then π_{mR} may be rewritten as

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ \int_{V_n} \big\{ &-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}\, e_{ij} - (\bar{F}_i + \Delta\bar{F}_i)\Delta u_i \\ &+ \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}] \\ &- (\sigma_{ij} + \Delta\sigma_{ij})_{,j}\Delta u_i \big\} dV + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j\,\Delta u_i\, ds \\ &- \int_{\partial V_n} (T_i + \Delta T_i)(\Delta u_i - \Delta\tilde{u}_i)ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\, ds \\ &- \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds \Big\}\end{aligned} \tag{5.49}$$

Upon rearranging

$$\begin{aligned}\pi_{mR} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\Delta u_{k,j})] dV \\ &- \int_{V_n} [(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i)] \Delta u_i \, dV \\ &- \int_{\partial V_n} [(T_i + \Delta T_i) - (\sigma_{ij}+\Delta\sigma_{ij})\nu_j] \Delta u_i \, ds \\ &+ \int_{\partial V_n} (T_i + \Delta T_i) \Delta\tilde{u}_i \, ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i) \Delta\tilde{u}_i \, ds \\ &- \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) ds \\ &- \int_{V_n} \Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})] dV + \text{constants} \Big\}\end{aligned} \tag{5.50}$$

Thus, if only the linear stress equilibrium equation is satisfied exactly, namely

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{5.51}$$

and assuming Eq. 5.36 to be satisfied exactly, one may write

$$(T_i + \Delta T_i) - (\sigma_{ij} + \Delta\sigma_{ij})\nu_j = (\sigma_{kj} + \Delta\sigma_{kj})\Delta u_{i,k}\,\nu_j \tag{5.52}$$

Placing these two equations into Eq. 5.50 yields the inconsistent assumed stress hybrid model π^I_{mc}

$$\begin{aligned}\pi^I_{mc} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\Delta u_{k,j})] dV \\ &+ \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j \Delta\tilde{u}_i \, ds + \int_{\partial V_n} (\sigma_{kj} + \Delta\sigma_{kj}) \Delta u_{i,k}\, \nu_j (\Delta\tilde{u}_i - \Delta u_i) ds \\ &- \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i) \Delta\tilde{u}_i \, ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i) ds \\ &- \int_{V_n} \Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})] dV \Big\}\end{aligned} \tag{5.53}$$

where the constant terms are dropped.

Assuming that the increments are taken small enough the coupling occurring in Eq. 5.53 can be removed. The linearized version of π^I_{mc} may be written as

$$
\begin{aligned}
\pi^I_{mc} = \sum_n \Big\{ & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\Delta u_{k,j}]\,dV \\
& + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j\,\Delta\tilde{u}_i\,ds + \int_{\partial V_n} \sigma_{kj}\,\Delta u_{i,k}\,\nu_j(\Delta\tilde{u}_i - \Delta u_i)\,ds \\
& - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds \\
& - \int_{V_n} \Delta\sigma_{ij}\,[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})]\,dV \Big\} \qquad (5.54)
\end{aligned}
$$

The constraint conditions are linear. π^I_{mc} here is subject to the constraint conditions Eqs. 5.51 and 5.47.

Comparing π^c_{mc} and π^I_{mc} for the updated system one observes that these two functionals are subject to different constraint conditions. Comparing these equations, it is obvious that π^I_{mc} would be considerably easier to implement than π^c_{mc}.

The equations presented in this subsection should be compared to their counterparts for the stationary system in Subsection 4.4.2.

5.4.3 Equilibrium Checks

The consistent model, π^c_{mc}, and the inconsistent model, π^I_{mc}, are given by Eqs. 5.44 and 5.53 respectively. The corresponding linearized functionals are given by Eqs. 5.45 and 5.54 respectively. These equations contain both the stress equilibrium and the compatibility checks in the updated reference configuration. The necessity of these checks is discussed in Subsection 4.4.3.

The compatibility check is again easily identified as the last integral in each equation. Note that since the reference state is updated the strain displacement relations are of the Almansi type. Note that the associated displacement mismatch terms discussed in Subsection 4.4.3 are omitted here because of the updated system. They could, however, be included in much the same manner as the compatibility check.

Furthermore, the stress equilibrium check can be identified by the initial boundary tractions in π^c_{mc} and π^I_{mc} as well as the initial stress terms in the constraint equations. In the C.U.L. system the stress equilibrium check is exact for both functionals. To identify the correction terms one would have to start with the Principle of Virtual Work assuming that stress equilibrium was exactly satisfied in the reference state. The same procedure can be followed as that of Subsection 4.4.3. The details will be omitted here (the reader is referred to Subsection 4.4.3) but for completeness only the functionals will be written without the equilibrium check.

$$\pi_p = \int_V [A(\Delta e_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i \Delta u_i]dV - \int_{S_\sigma} \Delta\bar{T}_i \Delta u_i ds \tag{5.55}$$

$$\begin{aligned}\pi_I = \int_V \{ & A(\Delta e_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i \Delta u_i \\ & - \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) \\ & + \Delta e_{ij} - \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j})]\}dV \\ & - \int_{S_\sigma} \Delta\bar{T}_i \Delta u_i ds - \int_{S_u} \Delta T_i(\Delta u_i - \Delta\bar{u}_i)ds \end{aligned} \tag{5.56}$$

$$\begin{aligned}\pi_R = \int_{V_n} \{ & -B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}e_{ij} + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i \Delta u_i \\ & + \tfrac{1}{2}\Delta\sigma_{ij}[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]\}dV \\ & - \int_{S_\sigma} \Delta\bar{T}_i \Delta u_i ds - \int_{S_u} \Delta T_i(\Delta u_i - \Delta\bar{u}_i)ds \end{aligned} \tag{5.57}$$

$$\begin{aligned}\pi_{mR} = \sum_n \{ & \int_{V_n} [-B(\Delta\sigma_{ij}) - \Delta\sigma_{ij}e_{ij} + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j} - \Delta\bar{F}_i \Delta u_i \\ & + \tfrac{1}{2}\Delta\sigma_{ij}[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} + \Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}]]dV \\ & - \int_{\partial V_n} \Delta T_i(\Delta u_i - \Delta\tilde{u}_i)ds - \int_{S_{\sigma_n}} \Delta\bar{T}_i \Delta\tilde{u}_i ds - \int_{S_{u_n}} \Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds\} \end{aligned} \tag{5.58}$$

Finally

$$\begin{aligned}\pi_{mc}^c = \sum_n \{ & \int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})(\Delta u_{k,i}\Delta u_{k,j})]dV \\ & + \int_{\partial V_n} \Delta T_i \Delta\tilde{u}_i ds - \int_{S_{\sigma_n}} \Delta\bar{T}_i \Delta\tilde{u}_i ds \\ & - \int_{S_{u_n}} \Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds \\ & - \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})]dV\} \end{aligned} \tag{5.59}$$

which is subject to

$$\Delta\sigma_{ij,j} + [(\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]_{,j} + \Delta\bar{F}_i = 0 \tag{5.60}$$

and

$$\Delta T_i = [\Delta\sigma_{ij} + (\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}]\nu_j \tag{5.61}$$

where it has been assumed that

$$\sigma_{ij,j} + \bar{F}_i = 0 \tag{5.62}$$

and

$$T_i = \sigma_{ij}\nu_j \tag{5.63}$$

are satisfied exactly in the reference state.

Similarly

$$\begin{aligned}\pi^{I}_{mc} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(\Delta u_{k,i}\Delta u_{k,j})]\,dV \\ &+ \int_{\partial V_n} \Delta\sigma_{ij}\nu_j \Delta\tilde{u}_i\,ds + \int_{\partial V_n} (\sigma_{kj}+\Delta\sigma_{kj})\Delta u_{i,k}\nu_j(\Delta\tilde{u}_i-\Delta u_i)\,ds \\ &- \int_{S_{\sigma_n}} \Delta\bar{T}_i \Delta\tilde{u}_i\,ds - \int_{S_{u_n}} \Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds \\ &- \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})]\,dV \Big\}\end{aligned} \tag{5.64}$$

which is subject to

$$\Delta\sigma_{ij,j} + \Delta\bar{F}_i = 0 \tag{5.65}$$

and Eq. 5.61

again assuming Eqs. 5.62 and 5.63 are satisfied exactly in the reference state.

Linearizing these last two functionals and their corresponding constraint conditions.

$$\begin{aligned}\pi^{c}_{mc} = \sum_n \Big\{ &\int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}]\,dV \\ &+ \int_{\partial V_n} \Delta T_i \Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}} \Delta\bar{T}_i \Delta\tilde{u}_i\,ds - \int_{S_{u_n}} \Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds \\ &- \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j})]\,dV \Big\}\end{aligned} \tag{5.66}$$

which is subject to

$$\Delta\sigma_{ij,j} + (\sigma_{kj}\Delta u_{i,k})_{,j} + \Delta\bar{F}_i = 0 \tag{5.67}$$

and

$$\Delta T_i = [\Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k}]\nu_j \tag{5.68}$$

Also

$$\begin{aligned}\pi^{f}_{mc} = \sum_{n}\Big\{ & \int_{V_n}[-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}]dV \\ & + \int_{\partial V_n}\Delta\sigma_{ij}\nu_j\Delta\tilde{u}_i ds + \int_{\partial V_n}\sigma_{kj}\Delta u_{i,k}\nu_j(\Delta\tilde{u}_i - \Delta u_i)ds \\ & - \int_{S_{\sigma_n}}\Delta\bar{T}_i\Delta\tilde{u}_i ds - \int_{S_{u_n}}\Delta T_i(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds \\ & - \int_{V_n}\Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})]dV\Big\}\end{aligned} \tag{5.69}$$

which is subject to Eqs. 5.65 and 5.68.

The same comments apply here as those at the end of Subsection 4.4.3.

SECTION 6

GENERAL FINITE ELEMENT MATRIX EQUATIONS FOR AN ELEMENT

6.1 The Stationary Lagrangian System

The equations derived in Section 4 for the assumed stress hybrid functionals based on a Stationary Lagrangian coordinate system shall be used as a basis for this subsection. Although the linearized equations for the consistent functional decouple the incremental stresses and incremental displacements (both these quantities are unknown), the unknown stresses still couple with total displacements. This leads to a difficulty, which will be stated herein, which compromises the practicality of such a functional. Thus, a general set of matrix equations will be developed for the inconsistent model only. The consistent model will be discussed in detail for the updated system in Subsection 6.2.1.

The complete equations including the equilibrium checks will be utilized and the terms corresponding to these checks shall be pointed out. In Section 7 these equations will be specialized to the specific structures considered.

6.1.1 The Consistent Assumed Stress Hybrid Model

Specifically, the linearized equations developed in Subsection 4.4.1 will be discussed here. The functional of Eq. 4.45 may be written for an element as

$$\begin{aligned}\pi^{c}_{mc_n} = \int_{V_n} \left[-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij}\, \Delta u_{k,i}\, \Delta u_{k,j} \right] dv \\ + \int_{\partial V_n} (T_i + \Delta T_i)\, \Delta\tilde{u}_i\, ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\, \Delta\tilde{u}_i\, ds \\ - \int_{S_{u_n}} (T_i + \Delta T_i)\left[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i) \right] ds + \int_{\partial V_n} (T_i + \Delta T_i)(\tilde{u}_i - u_i)\, ds \\ - \int_{V_n} \Delta\sigma_{ij} \left[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}) \right] dv \end{aligned} \qquad (6.1)$$

where it is understood that

$$\pi^{c}_{mc} = \sum_{n} \pi^{c}_{mc_n} \qquad (6.2)$$

Eq. 6.1 is subject to the constraint conditions

$$(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + [\sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]_{,j} + (\bar{F}_i+\Delta\bar{F}_i) = 0 \qquad (4.46)$$

and

$$T_i+\Delta T_i = [\sigma_{ij}+\Delta\sigma_{ij}+\sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]\nu_j \qquad (4.47)$$

Eq. 4.46 may be written as

$$\Delta\sigma_{ij,j} = -\sigma_{ij,j} - (\bar{F}_i+\Delta\bar{F}_i) - [\sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]_{,j} \qquad (6.3)$$

or

$$[(\sigma_{kj}+\Delta\sigma_{kj})(\delta_{ik}+u_{i,k})]_{,j} = [(\sigma_{kj}+\Delta\sigma_{kj})\frac{\partial^N x_i}{\partial^\circ x_k}]_{,j}$$
$$= -(\bar{F}_i+\Delta\bar{F}_i) - (\sigma_{kj}\Delta u_{i,k})_{,j} \qquad (6.4)$$

Recalling the definition of the Piola stress, Eq. 2.56, Eq. 6.4 may be written as

$$(p_{ji}+\Delta p_{ji})_{,j} = -(\bar{F}_i+\Delta\bar{F}_i) - (\sigma_{kj}\Delta u_{i,k})_{,j} \qquad (6.5)$$

Before continuing, it is convenient to introduce matrix notation to replace the indicial notation above. Also, the variables shall be interpolated in terms of unknown parameters in the usual way [Zienkiewicz, 1971; Pian and Tong, 1972]. Thus, allow the homogeneous stress be represented by stress parameters

$$\underset{\sim}{\Delta\sigma}_H = \underset{\sim}{P}\,\underset{\sim}{\beta} \qquad (6.6)$$

where stress $\underset{\sim}{\Delta\sigma}$ is represented in vector form. Also, note that vectors and matricies shall be designated simply by underscoring with a tilde, for example

$$\{\Delta\sigma\} = \underset{\sim}{\Delta\sigma} \qquad [P] = \underset{\sim}{P} \qquad (6.7)$$

Similarly, the boundary homogeneous stresses shall be represented by

$$(\Delta\sigma_{ij})_H\nu_j = (\Delta\sigma_\nu)_H = \underset{\sim}{R}\,\underset{\sim}{\beta} \qquad (6.8)$$

The displacements on the interior of an element, u_i, Δu_i, shall be interpolated in terms of the nodal displacements, q_i, Δq_i respectively. These nodes are common with neighboring elements.

$$\underset{\sim}{u} = \underset{\sim}{L}\,\underset{\sim}{q} \qquad\qquad \Delta\underset{\sim}{u} = \underset{\sim}{L}\,\Delta\underset{\sim}{q} \tag{6.9}$$

The boundary displacements, $\Delta\tilde{u}_i$, which may be independent from the interior displacements, Δu_i, may be interpolated with respect to the same set of nodal displacements [Pian, 1972].

$$\Delta\tilde{\underset{\sim}{u}} = \tilde{\underset{\sim}{L}}\,\Delta\underset{\sim}{q} \tag{6.10}$$

Derivatives of displacements may be represented by taking derivatives of the corresponding interpolation functions, i.e.,

$$\underset{\sim}{u}' = \underset{\sim}{L}'\,\underset{\sim}{q} \tag{6.11}$$

Also note that the transpose of a matrix is expressed as

$$\text{Transpose of } [u] = [u]^T = \underset{\sim}{u}^T \tag{6.12}$$

and the inverse as

$$\text{inverse of } [M] = [M]^{-1} = \underset{\sim}{M}^{-1} \tag{6.13}$$

Expressing a solution of Eq. 6.5 in matrix form

$$\underset{\sim}{p} + \Delta\underset{\sim}{p} = \underset{\sim}{P}\,\underset{\sim}{\beta} + \underset{\sim}{\sigma}_p + \underset{\sim}{A}\,\Delta\underset{\sim}{q} + \underset{\sim}{C}_I\,\Delta\underset{\sim}{q} \tag{6.14}$$

It can be shown that

$$\underset{\sim}{p} + \Delta\underset{\sim}{p} = \left[\frac{\partial^N x}{\partial^o x}\right](\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma}) \qquad (\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma}) = \left[\frac{\partial^o x}{\partial^N x}\right](\underset{\sim}{p} + \Delta\underset{\sim}{p}) \tag{6.15}$$

where it should be remembered that $\underset{\sim}{p}$ is unsymmetric. Placing Eq. 6.15 into Eq. 6.14 yields

$$\left[\frac{\partial^N x}{\partial^o x}\right](\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma}) = \underset{\sim}{P}\,\underset{\sim}{\beta} + \underset{\sim}{\sigma}_p + \underset{\sim}{A}\,\Delta\underset{\sim}{q} + \underset{\sim}{C}_I\,\Delta\underset{\sim}{q} \tag{6.16}$$

Or,

$$\Delta\underset{\sim}{\sigma} = -\underset{\sim}{\sigma} + \left[\frac{\partial^o x}{\partial^N x}\right](\underset{\sim}{P}\,\underset{\sim}{\beta} + \underset{\sim}{\sigma}_p + \underset{\sim}{A}\,\Delta\underset{\sim}{q} + \underset{\sim}{C}_I\,\Delta\underset{\sim}{q}) \tag{6.17}$$

where

$\underset{\sim}{\sigma}$ = corresponds to the initial stress term

$\underset{\sim}{P}\underset{\sim}{\beta}$ = corresponds to the homogeneous solution

$\underset{\sim}{\sigma}_p$ = corresponds to the forces $(\bar{F}_i+\Delta\bar{F}_i)$

$\underset{\sim}{A}\Delta\underset{\sim}{q}$ = corresponds to the $(\sigma_{kj}\Delta u_{i,k})_{,j}$ term

$\underset{\sim}{C}_I\Delta\underset{\sim}{q}$ = corresponds to the constant of integration which is only a function of the boundary variable held constant. (This will yield a needed boundary term -- see Eq. 6.58).

Note that, for instance, (from Eq. 6.5)

$$\begin{aligned} p_{ji} + \Delta p_{ji} &= -\int_{x_j} \{(\bar{F}_i + \Delta\bar{F}_i) + (\sigma_{kj}\Delta u_{i,k})_{,j}\} dx_j \\ &= \underset{\sim}{P}(x_i, x_j)\underset{\sim}{\beta} + \underset{\sim}{\sigma}_p(x_i, x_j) \\ &\quad + \underset{\sim}{A}(x_i, x_j)\Delta\underset{\sim}{q} + \underset{\sim}{C}_I(x_i)\Delta\underset{\sim}{q} \end{aligned}$$

and

$$\frac{\partial[\underset{\sim}{C}_I(x_i)\Delta\underset{\sim}{q}]}{\partial x_j} \equiv 0$$

Recognizing the fact that the first integral of Eq. 6.1 is

$$\int_{V_n} B(\Delta\sigma_{ij})\, dV = \frac{1}{2}\int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \Delta\underset{\sim}{\sigma}\, dV \tag{6.18}$$

where the constitutive law is stated as (for linear elastic materials)

$$\Delta\underset{\sim}{e} = \underset{\sim}{S}\Delta\underset{\sim}{\sigma} \qquad\qquad \underset{\sim}{e} = \underset{\sim}{S}\,\underset{\sim}{\sigma} \tag{6.19}$$

and the unsymmetric stresses must be assumed leads to a questionable procedure. The former condition leads to an unattractive situation computationally while the latter may complicate the technique in general.

The situation can be greatly improved by removing at least the $(\Delta\sigma_{kj}u_{i,k})_{,j}$ term. Since this term is removed automatically in the updated system, further discussion of this functional will be deferred until Subsection 6.2.1.

6.1.2 The Inconsistent Assumed Stress Hybrid Model

Unlike the difficulties encountered with the consistent functional the inconsistent model is relatively straight forward. Considering the linearized equations of Subsection 4.4.2, the functional Eq. 4.54 may be written for an element as

$$\pi^I_{mc_n} = \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}]dV$$
$$+ \int_{\partial V_n} (\sigma_{ij}+\Delta\sigma_{ij})\nu_j \Delta\tilde{u}_i\, ds + \int_{\partial V_n} [\sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]\nu_j(\Delta\tilde{u}_i - \Delta u_i)dS$$
$$- \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i\, ds - \int_{S_{u_n}} (T_i+\Delta T_i)[(\tilde{u}_i+\Delta\tilde{u}_i) - (\bar{u}_i+\Delta\bar{u}_i)]ds$$
$$+ \int_{\partial V_n} (T_i+\Delta T_i)(\tilde{u}_i - u_i)ds$$
$$- \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})]dV \qquad (6.20)$$

where it is again understood that

$$\pi^I_{mc} = \sum_n \pi^I_{mc_n} \qquad (6.21)$$

Eq. 6.20 is subject to

$$(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \qquad (4.51)$$

and

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}(u_{i,k}+\Delta u_{i,k}) + \Delta\sigma_{kj}u_{i,k}]\nu_j \qquad (4.47)$$

Eq. 4.51 may be written as

$$\Delta\sigma_{ij,j} = -\sigma_{ij,j} - (\bar{F}_i + \Delta\bar{F}_i) \qquad (6.22)$$

A solution to this in matrix form is

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\underset{\sim}{\beta} - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_p \qquad (6.23)$$

where the definitions are the same as those of Eq. 6.17. Note that no constant of integration is required here. Utilizing Eqs. 6.8–6.13 and placing Eq. 6.23 into Eq. 6.20 and considering the terms of Eq. 6.20 on an individual basis

$$-\int_{V_n} B(\Delta\sigma_{ij})dV = -\tfrac{1}{2}\int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \Delta\underset{\sim}{\sigma}\, dV$$
$$= -\tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T\underset{\sim}{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_p^T)\underset{\sim}{S}(\underset{\sim}{P}\underset{\sim}{\beta} - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_p)dV$$
$$= -\tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T\underset{\sim}{P}^T\underset{\sim}{S}\underset{\sim}{P}\underset{\sim}{\beta} - 2\underset{\sim}{\beta}^T\underset{\sim}{P}^T\underset{\sim}{S}\underset{\sim}{\sigma} + 2\underset{\sim}{\beta}^T\underset{\sim}{P}^T\underset{\sim}{S}\underset{\sim}{\sigma}_p + \text{constants})dV$$
$$= -\tfrac{1}{2}\underset{\sim}{\beta}^T\underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{\beta}^T\underset{\sim}{H}_\sigma - \underset{\sim}{\beta}^T\underset{\sim}{H}_{\sigma_p} \qquad (6.24)$$

where

$$\underset{\sim}{H} = \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{P}\, dv$$

$$\underset{\sim}{H}_\sigma = \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma}\, dv \qquad (6.25)$$

$$\underset{\sim}{H}_{\sigma_p} = \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma}_p\, dv$$

and the constant terms are dropped since they are not subject to variation.

$$\frac{1}{2}\int_{V_n} (\sigma_{ij} + \Delta\sigma_{ij})(u_{k,i}\Delta u_{k,j} + \Delta u_{k,i} u_{k,j})\, dv$$
$$= \int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T + \underset{\sim}{\sigma}_p^T)(\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}' \Delta\underset{\sim}{q})\, dv = \underset{\sim}{\beta}^T \underset{\sim}{C} \Delta\underset{\sim}{q} + \underset{\sim}{Q}_p^T \Delta\underset{\sim}{q} \qquad (6.26)$$

where

$$\underset{\sim}{C} = \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}'\, dv$$

$$\underset{\sim}{Q}_p^T = \int_{V_n} \underset{\sim}{\sigma}_p^T \underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}'\, dv \qquad (6.27)$$

Note that in actually evaluating these quantities care must be taken. (See Subsection 7.2.2.)

$$\frac{1}{2}\int_{V_n} \sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}\, dv = \frac{1}{2}\int_{V_n} \Delta\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{\sigma} \underset{\sim}{L}' \Delta\underset{\sim}{q}\, dv$$
$$= \frac{1}{2}\Delta\underset{\sim}{q}^T \underset{\sim}{K}_q \Delta\underset{\sim}{q} \qquad (6.28)$$

where

$$\underset{\sim}{K}_q = \int_{V_n} \underset{\sim}{L}'^T \underset{\sim}{\sigma} \underset{\sim}{L}'\, dv \qquad (6.29)$$

$$\int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j \Delta\tilde{u}_i\, ds = \int_{\partial V_n} (\underset{\sim}{\beta}^T \underset{\sim}{R}^T + \underset{\sim}{\sigma}_{p\nu}^T)\, \underset{\sim}{\tilde{L}}\, \Delta\underset{\sim}{q}\, ds$$
$$= \underset{\sim}{\beta}^T \underset{\sim}{G} \Delta\underset{\sim}{q} + \underset{\sim}{Q}_{p\nu}^T \Delta\underset{\sim}{q} \qquad (6.30)$$

where

$$\underset{\sim}{G} = \int_{\partial V_n} \underset{\sim}{R}^T \underset{\sim}{\tilde{L}}\, ds \qquad \underset{\sim}{R} = \underset{\sim}{\nu}^T \underset{\sim}{P}$$

$$\underset{\sim}{Q}_{p\nu}^T = \int_{\partial V_n} \sigma_{p\nu}^T \underset{\sim}{\tilde{L}}\, ds \qquad \underset{\sim}{\sigma}_{p\nu} = \underset{\sim}{\nu}^T \underset{\sim}{\sigma}_p \tag{6.31}$$

From Eqs. 4.47, 6.23, and 6.31 the boundary tractions may be written in matrix form as

$$\underset{\sim}{T} + \Delta \underset{\sim}{T} = \underset{\sim}{R}\underset{\sim}{\beta} + \underset{\sim}{\sigma}_{p\nu} + \underset{\sim}{A}_b(\underset{\sim}{q} + \Delta\underset{\sim}{q}) + \underset{\sim}{B}_b\underset{\sim}{\beta} - \underset{\sim}{B}_\sigma - \underset{\sim}{B}_{\sigma_p} \tag{6.32}$$

where

$\underset{\sim}{R}\underset{\sim}{\beta}+\underset{\sim}{\sigma}_{p\nu}$ corresponds to the linear $(\sigma_{ij}+\Delta\sigma_{ij})\nu_j$ terms

$\underset{\sim}{A}_b(\underset{\sim}{q}+\Delta\underset{\sim}{q})$ corresponds to the $\sigma_{kj}(u_{i,k}+\Delta u_{i,k})\nu_j$ terms

$\underset{\sim}{B}_b\underset{\sim}{\beta}$ corresponds to the linear part of $\Delta\sigma_{kj}u_{i,k}\nu_j$

$\underset{\sim}{B}_\sigma$ corresponds to the initial stress part of $\Delta\sigma_{kj}u_{i,k}\nu_j$

$\underset{\sim}{B}\sigma_p$ corresponds to the particular solution of $\Delta\sigma_{kj}u_{i,k}\nu_j$

Thus,

$$\begin{aligned}\int_{\partial V_n} [\sigma_{kj}(u_{i,k} + \Delta u_{i,k}) + \Delta\sigma_{kj} u_{i,k}]\, \nu_j\, (\Delta\tilde{u}_i - \Delta u_i)\, ds \\ = \int_{\partial V_n} [(\underset{\sim}{q}^T + \Delta\underset{\sim}{q}^T)\underset{\sim}{A}_b^T + \underset{\sim}{\beta}^T\underset{\sim}{B}_b^T - \underset{\sim}{B}_\sigma^T - \underset{\sim}{B}_{\sigma_p}^T](\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\, \Delta\underset{\sim}{q}\, ds \\ = \tfrac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{M}_b + \underset{\sim}{M}_b^T)\Delta\underset{\sim}{q} + \underset{\sim}{q}^T\underset{\sim}{M}_b\Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T\underset{\sim}{S}_B\Delta\underset{\sim}{q} - \underset{\sim}{Q}_{B_\sigma}^T\Delta\underset{\sim}{q} - \underset{\sim}{Q}_{B_{\sigma_p}}^T\Delta\underset{\sim}{q}\end{aligned} \tag{6.33}$$

where

$$\begin{aligned}\underset{\sim}{M}_b &= \int_{\partial V_n} \underset{\sim}{A}_b^T(\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\, ds \\ \underset{\sim}{S}_B &= \int_{\partial V_n} \underset{\sim}{B}_b^T(\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\, ds \\ \underset{\sim}{Q}_{B_\sigma}^T &= \int_{\partial V_n} \underset{\sim}{B}_\sigma^T(\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\, ds \\ \underset{\sim}{Q}_{B_{\sigma_p}}^T &= \int_{\partial V_n} \underset{\sim}{B}_{\sigma_p}^T(\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\, ds\end{aligned} \tag{6.34}$$

$$-\int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\, \Delta\tilde{u}_i\, ds = -\underset{\sim}{Q}_{\bar{T}}^T\, \Delta\underset{\sim}{q} \tag{6.35}$$

where

$$Q_{\bar{T}}^{T} = \int_{S_{\sigma_n}} (\bar{T} + \Delta\bar{T}) \tilde{L} \, ds \tag{6.36}$$

$$\begin{aligned}
-\int_{S_{u_n}} (T_i + \Delta T_i)&[(\tilde{u}_i + \Delta\tilde{u}_i) - (\bar{u}_i + \Delta\bar{u}_i)] \, ds \\
&= -\int_{S_{u_n}} [\beta^T R^T + \sigma_{PV}^T + (q^T + \Delta q^T) A_b^T + \beta^T B_b^T - B_\sigma^T - B_{\sigma_P}^T][\tilde{L}(q + \Delta q) - (\bar{u} + \Delta\bar{u})] \, ds \\
&= -\beta^T G_u (q + \Delta q) + \beta^T V - Q_{PV_u}^T \Delta q - \tfrac{1}{2} \Delta q^T (\tilde{M}_b + \tilde{M}_b^T) \Delta q - \Delta q^T (\tilde{M}_b + \tilde{M}_b^T) q \\
&\quad + \Delta q^T \bar{M}_b - \beta^T \tilde{S}_B (q + \Delta q) + \beta^T \bar{S}_B + \tilde{Q}_{B_\sigma}^T \Delta q + \tilde{Q}_{B_{\sigma_P}}^T \Delta q
\end{aligned} \tag{6.37}$$

where

$$\begin{aligned}
G_u &= \int_{S_{u_n}} R^T \tilde{L} \, ds & \tilde{S}_B &= \int_{S_{u_n}} B_b^T \tilde{L} \, ds \\
V &= \int_{S_{u_n}} R^T (\bar{u} + \Delta\bar{u}) \, ds & \bar{S}_B &= \int_{S_{u_n}} B_b^T (\bar{u} + \Delta\bar{u}) \, ds \\
Q_{PV_u}^T &= \int_{S_{u_n}} \sigma_{PV}^T \tilde{L} \, ds & \tilde{Q}_{B_\sigma}^T &= \int_{S_{u_n}} B_\sigma^T \tilde{L} \, ds \\
\tilde{M}_b &= \int_{S_{u_n}} A_b^T \tilde{L} \, ds & \tilde{Q}_{B_{\sigma_P}}^T &= \int_{S_{u_n}} B_{\sigma_P}^T \tilde{L} \, ds \\
\bar{M}_b &= \int_{S_{u_n}} A_b^T (\bar{u} + \Delta\bar{u}) \, ds & &
\end{aligned} \tag{6.38}$$

Note that Eqs. 6.36 and 6.38 require special integration routines in that they are integrated over only portions of the element boundary. These are completely consistent terms. One may evaluate such terms in an inconsistent fashion by simply lumping equivalent loads at the nodes directly. This will be discussed in Section 7. Also, prescribed displacement conditions are usually quite simple. Or, elements are generally chosen so that applied displacements can be handled easily. In any case, it is extremely rare that $\Delta\tilde{u}_i = \Delta\bar{u}_i$ is not satisfied exactly. However, for the sake of generality they will be maintained here. Next

$$\begin{aligned}
\int_{\partial V_n} (T_i + \Delta T_i)&(\tilde{u}_i - u_i) \, ds \\
&= \int_{\partial V_n} [\beta^T R^T + \sigma_{PV}^T + (q^T + \Delta q^T) A_b^T + \beta^T B_b^T - B_\sigma^T - B_{\sigma_P}^T](\tilde{L} - L) q \, ds \\
&= \beta^T G q - \beta^T G^* q + \Delta q^T M_b q + \beta^T S_B q
\end{aligned} \tag{6.39}$$

where

$$\underset{\sim}{G}^{*} = \int_{\partial V_n} \underset{\sim}{R}^T \underset{\sim}{L}\, ds \tag{6.40}$$

$$\begin{aligned} -\int_{V_n} \Delta\sigma_{ij} e_{ij}\, dv &= -\int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \underset{\sim}{\sigma}\, dv \\ &= -\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_p^T) \underset{\sim}{S}\, \underset{\sim}{\sigma}\, dv \\ &= -\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{S}\, \underset{\sim}{\sigma} + \text{constants})\, dv = -\underset{\sim}{\beta}^T \underset{\sim}{H}_\sigma \end{aligned} \tag{6.41}$$

where $\underset{\sim}{H}_\sigma$ is defined by Eq. 6.25

$$\begin{aligned} \tfrac{1}{2}\int_{V_n} \Delta\sigma_{ij} (u_{i,j} + u_{j,i})\, dv &= \int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{L}' \underset{\sim}{q}\, dv \\ &= \int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_p^T) \underset{\sim}{L}' \underset{\sim}{q}\, dv \\ &= \int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{L}' \underset{\sim}{q} + \text{constants})\, dv = \underset{\sim}{\beta}^T \underset{\sim}{C}_1 \underset{\sim}{q} \end{aligned} \tag{6.42}$$

where

$$\underset{\sim}{C}_1 = \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{L}'\, dv \tag{6.43}$$

$$\begin{aligned} \tfrac{1}{2}\int_{V_n} \Delta\sigma_{ij} u_{k,i} u_{k,j}\, dv &= \tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_p^T)(\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}' \underset{\sim}{q})\, dv \\ &= \tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}' \underset{\sim}{q} + \text{constants})\, dv = \tfrac{1}{2} \underset{\sim}{\beta}^T \underset{\sim}{C}\, \underset{\sim}{q} \end{aligned} \tag{6.44}$$

where $\underset{\sim}{C}$ is defined by Eq. 6.27 and the same caution must be heeded here as before.

Placing all these integral evaluations into Eq. 6.20 yields

$$\begin{aligned} \pi^I_{mc_n}(\underset{\sim}{\beta}, \Delta\underset{\sim}{q}) = & -\tfrac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\, \underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{H}_\sigma^S - \underset{\sim}{\beta}^T \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{\beta}^T \underset{\sim}{C}\Delta\underset{\sim}{q} + \underset{\sim}{Q}_P^T \Delta\underset{\sim}{q} + \tfrac{1}{2}\Delta\underset{\sim}{q}^T \underset{\sim}{K}_g \Delta\underset{\sim}{q} \\ & + \underset{\sim}{\beta}^T \underset{\sim}{G}\, \Delta\underset{\sim}{q} + \underset{\sim}{Q}_{PV}^T \Delta\underset{\sim}{q} + \tfrac{1}{2}\Delta\underset{\sim}{q}^T (\underset{\sim}{M}_b + \underset{\sim}{M}_b^T)\Delta\underset{\sim}{q} + \underset{\sim}{q}^T \underset{\sim}{M}_b \Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{S}_B \Delta\underset{\sim}{q} \\ & - \underset{\sim}{Q}_{B\sigma}^T \Delta\underset{\sim}{q} - \underset{\sim}{Q}_{B\sigma_p}^T \Delta\underset{\sim}{q} - \underset{\sim}{Q}_{\bar{T}}^T \Delta\underset{\sim}{q} - \underset{\sim}{\beta}^T \underset{\sim}{G}_u (\underset{\sim}{q} + \Delta\underset{\sim}{q}) + \underset{\sim}{\beta}^T \underset{\sim}{U} - \underset{\sim}{Q}_{P\nu_u}^T \Delta\underset{\sim}{q} \\ & - \tfrac{1}{2}\Delta\underset{\sim}{q}^T (\tilde{\underset{\sim}{M}}_b + \tilde{\underset{\sim}{M}}_b^T)\Delta\underset{\sim}{q} - \Delta\underset{\sim}{q}^T (\tilde{\underset{\sim}{M}}_b + \tilde{\underset{\sim}{M}}_b^T)\underset{\sim}{q} + \Delta\underset{\sim}{q}^T \bar{\underset{\sim}{M}}_b - \underset{\sim}{\beta}^T \bar{\underset{\sim}{S}}_B (\underset{\sim}{q} + \Delta\underset{\sim}{q}) + \underset{\sim}{\beta}^T \bar{\underset{\sim}{S}}_B \\ & + \tilde{\underset{\sim}{Q}}_{B\sigma}^T \Delta\underset{\sim}{q} + \underset{\sim}{Q}_{B\sigma_p}^T \Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{\beta}^T \underset{\sim}{G}^{*} \underset{\sim}{q} + \Delta\underset{\sim}{q}^T \underset{\sim}{M}_b \underset{\sim}{q} \\ & + \underset{\sim}{\beta}^T \underset{\sim}{S}_B \underset{\sim}{q} - \underset{\sim}{\beta}^T \underset{\sim}{H}_\sigma^C + \underset{\sim}{\beta}^T \underset{\sim}{C}_1 \underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{C}\, \underset{\sim}{q} \end{aligned} \tag{6.45}$$

where

$$\underset{\sim}{H}_\sigma^S \equiv \underset{\sim}{H}_\sigma \quad \text{from stress equilibrium check}$$

$$\underset{\sim}{H}_\sigma^C \equiv \underset{\sim}{H}_\sigma \quad \text{from compatibility equilibrium check}$$

Although these two terms are identical they will be treated separately for purposes of identification only.

Eq. 6.45 contains two unknown vectors. These are the stress parameters $\underset{\sim}{\beta}$ and the incremental nodal displacements $\Delta\underset{\sim}{q}$. While the $\underset{\sim}{\beta}$'s are independent on the element level the $\Delta\underset{\sim}{q}$'s are not. Thus, one may take the variation of $\pi^I_{mc_n}$ with respect to $\underset{\sim}{\beta}$.

$$\begin{aligned}\frac{\partial \pi^I_{mc_n}}{\partial \underset{\sim}{\beta}} = & -\underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{H}_\sigma^S - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{C}\Delta\underset{\sim}{q} + \underset{\sim}{G}\Delta\underset{\sim}{q} + \underset{\sim}{S}_B\Delta\underset{\sim}{q} - \underset{\sim}{G}_u(\underset{\sim}{q}+\Delta\underset{\sim}{q}) \\ & + \underset{\sim}{V} - \underset{\sim}{\tilde{S}}_B(\underset{\sim}{q}+\Delta\underset{\sim}{q}) + \underset{\sim}{\bar{S}}_B + \underset{\sim}{G}\underset{\sim}{q} - \underset{\sim}{G}^*\underset{\sim}{q} + \underset{\sim}{S}_B\underset{\sim}{q} \\ & - \underset{\sim}{H}_\sigma^c + \underset{\sim}{C}_1\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q} = 0\end{aligned} \tag{6.46}$$

Solving for $\underset{\sim}{\beta}$ gives

$$\begin{aligned}\underset{\sim}{\beta} = & \underset{\sim}{H}^{-1}\left[\underset{\sim}{H}_\sigma^S - \underset{\sim}{H}_\sigma^c - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{V} + \underset{\sim}{\bar{S}}_B + (\underset{\sim}{G} - \underset{\sim}{G}^* - \underset{\sim}{G}_u + \underset{\sim}{C}_1 + \tfrac{1}{2}\underset{\sim}{C} + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B)\underset{\sim}{q}\right] \\ & + \underset{\sim}{H}^{-1}\left[\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B\right]\Delta\underset{\sim}{q}\end{aligned} \tag{6.47}$$

Placing this back into Eq. 6.45, and realizing that $\pi^I_{mc_n}$ will only be a function of the $\Delta\underset{\sim}{q}$'s one obtains

$$\begin{aligned}\pi^I_{mc_n}(\Delta\underset{\sim}{q}) = & -\tfrac{1}{2}\Delta\underset{\sim}{q}^T\left[\underset{\sim}{G}^T + \underset{\sim}{C}^T - \underset{\sim}{G}_u^T + \underset{\sim}{S}_B^T - \underset{\sim}{\tilde{S}}_B^T\right]\underset{\sim}{H}^{-1}\left[\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B\right]\Delta\underset{\sim}{q} \\ & + \Delta\underset{\sim}{q}^T\left[\underset{\sim}{G}^T + \underset{\sim}{C}^T - \underset{\sim}{G}_u^T + \underset{\sim}{S}_B^T - \underset{\sim}{\tilde{S}}_B^T\right]\underset{\sim}{H}^{-1}\left[\underset{\sim}{H}_\sigma^S - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{C}\Delta\underset{\sim}{q} + \underset{\sim}{G}\Delta\underset{\sim}{q} + \underset{\sim}{S}_B\Delta\underset{\sim}{q}\right. \\ & - \underset{\sim}{G}_u(\underset{\sim}{q}+\Delta\underset{\sim}{q}) + \underset{\sim}{V} - \underset{\sim}{\tilde{S}}_B(\underset{\sim}{q}+\Delta\underset{\sim}{q}) + \underset{\sim}{\bar{S}}_B + \underset{\sim}{G}\underset{\sim}{q} - \underset{\sim}{G}^*\underset{\sim}{q} + \underset{\sim}{S}_B\underset{\sim}{q} \\ & \left. - \underset{\sim}{H}_\sigma^c + \underset{\sim}{C}_1\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q}\right] + \tfrac{1}{2}\Delta\underset{\sim}{q}^T\left[\underset{\sim}{K}_g + \underset{\sim}{M}_b + \underset{\sim}{M}_b^T - \underset{\sim}{\tilde{M}}_b - \underset{\sim}{\tilde{M}}_b^T\right]\Delta\underset{\sim}{q} \\ & + \Delta\underset{\sim}{q}^T\left[\underset{\sim}{Q}_p + \underset{\sim}{Q}_{pv} + (\underset{\sim}{M}_b + \underset{\sim}{M}_b^T)\underset{\sim}{q} - \underset{\sim}{Q}_{pv_u} - (\underset{\sim}{\tilde{M}}_b + \underset{\sim}{\tilde{M}}_b^T)\underset{\sim}{q} - \underset{\sim}{Q}_{B\sigma}\right. \\ & \left. - \underset{\sim}{Q}_{B\sigma_p} - \underset{\sim}{Q}_{\bar{T}} + \underset{\sim}{\bar{M}}_b + \underset{\sim}{\tilde{Q}}_{B\sigma} + \underset{\sim}{\tilde{Q}}_{B\sigma_p}\right] + \text{constants}\end{aligned} \tag{6.48}$$

Or, upon rearranging and dropping the constants not subject to variation with respect to $\Delta \underset{\sim}{q}$

$$\pi^{I}_{mc_n} = \tfrac{1}{2}\Delta \underset{\sim}{q}^T \Big[(\underset{\sim}{G}^T + \underset{\sim}{C}^T - \underset{\sim}{G}_u^T + \underset{\sim}{S}_B^T - \underset{\sim}{\tilde{S}}_B^T)\underset{\sim}{H}^{-1}(\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B) + \underset{\sim}{K}_q$$
$$+ \underset{\sim}{M}_b + \underset{\sim}{M}_b^T - \underset{\sim}{\tilde{M}}_b - \underset{\sim}{\tilde{M}}_b^T \Big] \Delta \underset{\sim}{q}$$
$$+ \Delta \underset{\sim}{q}^T \Big\{ (\underset{\sim}{G}^T + \underset{\sim}{C}^T - \underset{\sim}{G}_u^T + \underset{\sim}{S}_B^T - \underset{\sim}{\tilde{S}}_B^T)\underset{\sim}{H}^{-1}\big[\underset{\sim}{H}_\sigma^s - \underset{\sim}{H}_\sigma^c - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{V} + \underset{\sim}{\bar{S}}_B$$
$$+ (\underset{\sim}{G} - \underset{\sim}{G}^* - \underset{\sim}{G}_u + \underset{\sim}{C}_1 + \tfrac{1}{2}\underset{\sim}{C} + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B)\underset{\sim}{q}\big] + \underset{\sim}{Q}_P + \underset{\sim}{Q}_{PV} - \underset{\sim}{Q}_{PV_u}$$
$$- \underset{\sim}{Q}_{\bar{T}} - \underset{\sim}{Q}_{B\sigma} - \underset{\sim}{Q}_{B\sigma_p} + \underset{\sim}{\tilde{Q}}_{B\sigma} + \underset{\sim}{\tilde{Q}}_{B\sigma_p} + \underset{\sim}{\bar{M}}_b$$
$$+ (\underset{\sim}{M}_b + \underset{\sim}{M}_b^T - \underset{\sim}{\tilde{M}}_b - \underset{\sim}{\tilde{M}}_b^T)\underset{\sim}{q} \Big\} \qquad (6.49)$$

Writing Eq. 6.49 as

$$\pi^{I}_{mc_n}(\Delta \underset{\sim}{q}) = \tfrac{1}{2}\Delta \underset{\sim}{q}^T \underset{\sim}{K}_T \Delta \underset{\sim}{q} - \Delta \underset{\sim}{q}^T \underset{\sim}{Q} \qquad (6.50)$$

$\underset{\sim}{K}_T$ becomes the element stiffness matrix and $\underset{\sim}{Q}$ becomes the element load vector, where

$$\underset{\sim}{K}_T = (\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B)^T \underset{\sim}{H}^{-1}(\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B) + \underset{\sim}{K}_q$$
$$+ \underset{\sim}{M}_b + \underset{\sim}{M}_b^T - \underset{\sim}{\tilde{M}}_b - \underset{\sim}{\tilde{M}}_b^T \qquad (6.51)$$
$$\underset{\sim}{Q} = -(\underset{\sim}{G} + \underset{\sim}{C} - \underset{\sim}{G}_u + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B)^T \underset{\sim}{H}^{-1}\big[\underset{\sim}{H}_\sigma^s - \underset{\sim}{H}_\sigma^c - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{V} + \underset{\sim}{\bar{S}}_B$$
$$+ (\underset{\sim}{G} - \underset{\sim}{G}^* - \underset{\sim}{G}_u + \underset{\sim}{C}_1 + \tfrac{1}{2}\underset{\sim}{C} + \underset{\sim}{S}_B - \underset{\sim}{\tilde{S}}_B)\underset{\sim}{q}\big] - \big[\underset{\sim}{Q}_P + \underset{\sim}{Q}_{PV} - \underset{\sim}{Q}_{PV_u} - \underset{\sim}{Q}_{\bar{T}}$$
$$- \underset{\sim}{Q}_{B\sigma} + \underset{\sim}{\tilde{Q}}_{B\sigma} - \underset{\sim}{Q}_{B\sigma_p} + \underset{\sim}{\tilde{Q}}_{B\sigma_p} + \underset{\sim}{\bar{M}}_b + (\underset{\sim}{M}_b + \underset{\sim}{M}_b^T - \underset{\sim}{\tilde{M}}_b - \underset{\sim}{\tilde{M}}_b^T)\underset{\sim}{q}\big]$$

These matrices must then be assembled as shown in Subsection 6.3. Note that if certain conditions exist, various terms drop out of Eqs. 6.51. They are as follows

Condition	Terms Removed
1. $\tilde{u}_i + \Delta\tilde{u}_i \equiv \bar{u}_i + \Delta\bar{u}_i$ on S_{u_n}	$\underset{\sim}{G}_u$, $\underset{\sim}{V}$, $\underset{\sim}{Q}_{PV_u}$, $\underset{\sim}{\tilde{M}}_b$, $\underset{\sim}{\bar{M}}_b$, $\underset{\sim}{\tilde{S}}_B$, $\underset{\sim}{\bar{S}}_B$, $\underset{\sim}{\tilde{Q}}_{B\sigma}$, $\underset{\sim}{\tilde{Q}}_{B\sigma_p}$
2. $\underset{\sim}{\sigma}_p$ is lumped at nodes	$\underset{\sim}{H}_{\sigma_p}$, $\underset{\sim}{Q}_P$, $\underset{\sim}{Q}_{PV}$, $\underset{\sim}{Q}_{PV_u}$, $\underset{\sim}{Q}_{B\sigma_p}$, $\underset{\sim}{\tilde{Q}}_{B\sigma_p}$

3.	$\bar{T}_i+\Delta\bar{T}_i$ is lumped at nodes	$Q_{\bar{T}}$
4.	No compatibility check (or associated mismatch terms)	H_σ^c, $(G-G^*-G_u+C_1+\frac{1}{2}C)q$, $(M_b+M_b^T-\tilde{M}_b-\tilde{M}_b^T+S_B-\tilde{S}_B)q$, $\bar{M}_b=\int_{S_{u_n}} A_b^T\Delta\bar{u}\,ds$, $\bar{S}_B=\int_{S_{u_n}} B_b^T\Delta\bar{u}\,ds$
5.	No stress equilibrium check	H_σ^s, $Q_{B\sigma}$, $\tilde{Q}_{B\sigma}$, and σ_p corresponds to $\Delta\bar{F}_i$ only, $Q_{\bar{T}}^T=\int_{S_{\sigma_n}}\Delta\bar{T}\,\tilde{L}\,ds$
6.	$\tilde{u}_i+\Delta\tilde{u}_i\equiv u_i+\Delta u_i$ on ∂V_n	M_b, S_B, $Q_{B\sigma}$, $Q_{B\sigma_p}$, $(G-G^*)q$

The last of these conditions assumes a compatible displacement field. For certain kinds of analysis, such as three dimensional or plane stress problems, compatible displacement fields are easily chosen. Even for the bending problem some of the displacements will be compatible. This, in conjunction with other considerations, allows one to remove some of the matrices listed in 6 above.

A much simplified form is achieved by assuming the first three conditions above are met and by recalling that $H_\sigma^S\equiv H_\sigma^C$. For this situation Eqs. 6.51 become

$$
\begin{aligned}
K_T &= (G+C+S_B)^T H^{-1}(G+C+S_B)+K_q+M_b+M_b^T \\
Q &= -(G+C+S_B)^T H^{-1}\left[G-G^*+C_1+\tfrac{1}{2}C+S_B\right]q+Q_{B\sigma}-(M_b+M_b^T)q
\end{aligned}
\tag{6.52}
$$

If the last condition is also met, Eqs. 6.52 reduce further to

$$
\begin{aligned}
K_T &= (G+C)^T H^{-1}(G+C)+K_q \\
Q &= -(G+C)^T H^{-1}(C_1+\tfrac{1}{2}C)\,q
\end{aligned}
\tag{6.53}
$$

Under the approximations chosen for actual computation in Section 7 it will be shown that because some displacement compatibility is present, Eqs. 6.53 form the basis for the S.L. system.

6.2 The Convected, Updated Lagrangian System

This subsection is concerned with the equations derived in Section 5 for the assumed stress hybrid functionals based on a Convected, Updated Lagrangian system. Here a general set of matrix equations will be developed for the linearized, consistent and inconsistent models. The number of terms will be somewhat reduced due to the lack of initial displacement terms (except in the compatibility check). This fact allows for a more easily derived consistent model.

It is important to note that although the same symbols will be used here to describe these models as were used for a stationary system their definitions are different. The appropriate definitions are given in Section 2.

6.2.1 The Consistent Assumed Stress Hybrid Model

The linearized equations developed in Subsection 5.4.1 will be utilized here. The functional of Eq. 5.45 may be written for an element as

$$\begin{aligned}\pi^c_{mc_n} = & \int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}]\,dV \\ & + \int_{\partial V_n} (T_i + \Delta T_i)\Delta\tilde{u}_i\,ds - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\Delta\tilde{u}_i\,ds \\ & - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds \\ & - \int_{V_n} \Delta\sigma_{ij}\{e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})\}\,dV\end{aligned} \tag{6.54}$$

where Eq. 6.2 is understood. Eq. 6.54 is subject to

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + (\sigma_{kj}\Delta u_{i,k})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{5.46}$$

and

$$T_i + \Delta T_i = \{\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\Delta u_{i,k}\}\nu_j \tag{5.47}$$

Eq. 5.46 may be written as

$$\Delta\sigma_{ij,j} = -\sigma_{ij,j} - (\bar{F}_i + \Delta\bar{F}_i) - (\sigma_{kj}\Delta u_{i,k})_{,j} \tag{6.55}$$

Comparing this to its counterpart in a stationary system, Eq. 6.4, one can observe the simplification realized by removing the initial displacement terms. A solution to Eq. 6.55 in matrix form is

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\underset{\sim}{\beta} - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_p + \underset{\sim}{A}\,\Delta\underset{\sim}{q} + \underset{\sim}{C}_I\,\Delta\underset{\sim}{q} \tag{6.56}$$

where the definitions are the same as those previously mentioned (for Eq. 6.17). Placing Eq. 6.56 into Eq. 5.47 yields in matrix form

$$\begin{aligned} \underset{\sim}{T} + \Delta \underset{\sim}{T} &= \underset{\sim}{\nu}^T \left[\underset{\sim}{P} \underset{\sim}{\beta} + \underset{\sim}{\sigma}_P + \underset{\sim}{A} \Delta \underset{\sim}{q} + \underset{\sim}{C}_I \Delta \underset{\sim}{q} - \underset{\sim}{A} \Delta \underset{\sim}{q} \right] \\ &= \underset{\sim}{\nu}^T \left[\underset{\sim}{P} \underset{\sim}{\beta} + \underset{\sim}{\sigma}_P + \underset{\sim}{C}_I \Delta \underset{\sim}{q} \right] \\ &= \underset{\sim}{R} \underset{\sim}{\beta} + \underset{\sim}{\sigma}_{P\nu} + \underset{\sim}{A}_\nu \Delta \underset{\sim}{q} \end{aligned} \tag{6.57}$$

where

$$\underset{\sim}{A}_\nu = \underset{\sim}{\nu}^T \underset{\sim}{C}_I \tag{6.58}$$

The nature of the term in Eq. 6.58 is not clearly stated or defined by Atluri [1973b]. Using similar types of interpolation functions for the variables of the functional, consider the terms of Eq. 6.54 individually.

$$\begin{aligned} -\int_{V_n} B(\Delta\sigma_{ij})\, dV &= -\frac{1}{2} \int_{V_n} \Delta \underset{\sim}{\sigma}^T \underset{\sim}{S} \Delta \underset{\sim}{\sigma}\, dV \\ &= -\frac{1}{2} \int_{V_n} \left[\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_P^T + \Delta \underset{\sim}{q} (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T) \right] \underset{\sim}{S} \left\{ \underset{\sim}{P} \underset{\sim}{\beta} - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_P + (\underset{\sim}{A} + \underset{\sim}{C}_I) \Delta \underset{\sim}{q} \right\} dV \\ &= -\frac{1}{2} \int_{V_n} \Big[\underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{P} \underset{\sim}{\beta} - 2 \underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma} + 2 \underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma}_P + 2 \underset{\sim}{\beta}^T \underset{\sim}{P}^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I) \Delta \underset{\sim}{q} \\ &\quad - 2 \underset{\sim}{\sigma}^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I) \Delta \underset{\sim}{q} + 2 \underset{\sim}{\sigma}_P^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I) \Delta \underset{\sim}{q} \\ &\quad + \Delta \underset{\sim}{q}^T (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T) \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I) \Delta \underset{\sim}{q} + \text{constants} \Big\} dV \\ &= -\frac{1}{2} \underset{\sim}{\beta}^T \underset{\sim}{H} \underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{H}_\sigma - \underset{\sim}{\beta}^T \underset{\sim}{H}_{\sigma_P} - \underset{\sim}{\beta}^T \underset{\sim}{H}_A \Delta \underset{\sim}{q} + \underset{\sim}{D}_\sigma \Delta \underset{\sim}{q} \\ &\quad - \underset{\sim}{D}_{\sigma_P} \Delta \underset{\sim}{q} - \frac{1}{2} \Delta \underset{\sim}{q}^T \underset{\sim}{D} \Delta \underset{\sim}{q} \end{aligned} \tag{6.59}$$

where

$$\begin{aligned} \underset{\sim}{H} &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{P}\, dV & \underset{\sim}{D} &= \int_{V_n} (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T) \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I)\, dV \\ \underset{\sim}{H}_\sigma &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma}\, dV & \underset{\sim}{D}_\sigma &= \int_{V_n} \underset{\sim}{\sigma}^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I)\, dV \\ \underset{\sim}{H}_{\sigma_P} &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{\sigma}_P\, dV & \underset{\sim}{D}_{\sigma_P} &= \int_{V_n} \underset{\sim}{\sigma}_P^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I)\, dV \\ \underset{\sim}{H}_A &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S} (\underset{\sim}{A} + \underset{\sim}{C}_I)\, dV \end{aligned} \tag{6.60}$$

and the constants, not subject to variation, are dropped.

$$-\frac{1}{2}\int_{V_n}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\,dV = -\frac{1}{2}\int_{V_n}\Delta \underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{\sigma}\,\underset{\sim}{L}'\,\Delta \underset{\sim}{q}\,dV$$

$$= -\frac{1}{2}\,\Delta \underset{\sim}{q}^T \underset{\sim}{K}_g\,\Delta \underset{\sim}{q} \qquad (6.61)$$

where

$$K_g = \int_{V_n} \underset{\sim}{L}'^T \underset{\sim}{\sigma}\,\underset{\sim}{L}'\,dV \qquad (6.62)$$

Recall that although Eq. 6.62 appears identical to Eq. 6.29 the definition of stress is different as shown in Section 2.

$$\int_{\partial V_n}(T_i + \Delta T_i)\,\Delta\tilde{u}_i\,d$$

$$= \int_{\partial V_n}(\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\,\Delta u_{i,k})\,\nu_j\,\Delta\tilde{u}_i\,ds$$

$$= \int_{\partial V_n}(\underset{\sim}{\beta}^T \underset{\sim}{R}^T + \underset{\sim}{\sigma}_{P\nu}^T + \Delta \underset{\sim}{q}^T \underset{\sim}{A}_\nu^T)\,\tilde{\underset{\sim}{L}}\,\Delta \underset{\sim}{q}\,ds$$

$$= \underset{\sim}{\beta}^T \underset{\sim}{G}\,\Delta \underset{\sim}{q} + \underset{\sim}{Q}_{P\nu}^T\,\Delta \underset{\sim}{q} + \frac{1}{2}\Delta \underset{\sim}{q}^T(\underset{\sim}{M} + \underset{\sim}{M}^T)\,\Delta \underset{\sim}{q} \qquad (6.63)$$

where

$$\underset{\sim}{G} = \int_{\partial V_n} \underset{\sim}{R}^T \tilde{\underset{\sim}{L}}\,ds$$

$$\underset{\sim}{Q}_{P\nu}^T = \int_{\partial V_n} \underset{\sim}{\sigma}_{P\nu}^T\,\tilde{\underset{\sim}{L}}\,ds \qquad (6.64)$$

$$\underset{\sim}{M} = \int_{\partial V_n} \underset{\sim}{A}_\nu^T\,\tilde{\underset{\sim}{L}}\,ds$$

$$-\int_{S_{\sigma_n}}(\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds = -\underset{\sim}{Q}_{\bar{T}}^T\,\Delta \underset{\sim}{q} \qquad (6.65)$$

where

$$\underset{\sim}{Q}_{\bar{T}}^T = \int_{\partial V_n}(\bar{T}_i + \Delta\bar{T}_i)\,\tilde{\underset{\sim}{L}}\,ds \qquad (6.66)$$

$$-\int_{S_{u_n}}(T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds = -\int_{S_{u_n}}(\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\,\Delta u_{i,k})\,\nu_j\,(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds$$

$$= -\int_{S_{u_n}}(\underset{\sim}{\beta}^T \underset{\sim}{R}^T + \underset{\sim}{\sigma}_{P\nu}^T + \Delta \underset{\sim}{q}^T \underset{\sim}{A}_\nu^T)(\tilde{\underset{\sim}{L}}\,\Delta \underset{\sim}{q} - \Delta\bar{\underset{\sim}{u}})\,ds$$

$$= \underset{\sim}{\beta}^T \underset{\sim}{G}_u\,\Delta \underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{V} - \underset{\sim}{Q}_{P\nu_u}^T\,\Delta \underset{\sim}{q} - \frac{1}{2}\Delta \underset{\sim}{q}^T(\underset{\sim}{M}_u + \underset{\sim}{M}_u^T)\,\Delta \underset{\sim}{q} + \Delta \underset{\sim}{q}^T \underset{\sim}{V}_A \qquad (6.67)$$

where

$$\underset{\sim}{G}_u = \int_{S_{u_n}} \underset{\sim}{R}^T \underset{\sim}{\tilde{L}}\, ds \qquad \underset{\sim}{M}_u = \int_{S_{u_n}} \underset{\sim}{A}_v^T \underset{\sim}{\tilde{L}}\, ds$$

$$\underset{\sim}{\mathcal{V}} = \int_{S_{u_n}} \underset{\sim}{R}^T \Delta \underset{\sim}{\bar{u}}\, ds \qquad \underset{\sim}{\mathcal{V}}_A = \int_{S_{u_n}} \underset{\sim}{A}_v^T \Delta \underset{\sim}{\bar{u}}\, ds \tag{6.68}$$

$$\underset{\sim}{Q}^T_{Pv_u} = \int_{S_{u_n}} \underset{\sim}{\tilde{\sigma}}^T_{Pv} \underset{\sim}{\tilde{L}}\, ds$$

and the same comments apply to Eqs. 6.66 and 6.68 as did to Eqs. 6.36 and 6.38.

$$\begin{aligned}
-\int_{V_n} \Delta\sigma_{ij}\, e_{ij}\, dV &= -\int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \underset{\sim}{\sigma}\, dV \\
&= -\int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_P^T + \Delta\underset{\sim}{q}^T (\hat{A}^T + \underset{\sim}{C}_I^T)\right] \underset{\sim}{S}\, \underset{\sim}{\sigma}\, dV \\
&= -\int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T \underset{\sim}{S}\, \underset{\sim}{\sigma} + \Delta\underset{\sim}{q}^T (\hat{A}^T + \underset{\sim}{C}_I^T) \underset{\sim}{S}\, \underset{\sim}{\sigma} + \text{constants}\right] dV \\
&= -\underset{\sim}{\beta}^T \underset{\sim}{H}_\sigma - \underset{\sim}{D}_\sigma \Delta\underset{\sim}{q}
\end{aligned} \tag{6.69}$$

where $\underset{\sim}{H}_\sigma$ and $\underset{\sim}{D}_\sigma$ are defined by Eqs. 6.60.

$$\begin{aligned}
\frac{1}{2}\int_{V_n} \Delta\sigma_{ij} (u_{i,j} + u_{j,i})\, dV &= \int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{L}' \underset{\sim}{q}\, dV \\
&= \int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_P^T + \Delta\underset{\sim}{q}^T (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T)\right] \underset{\sim}{L}' \underset{\sim}{q}\, dV \\
&= \int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T \hat{L}' \underset{\sim}{q} + \Delta\underset{\sim}{q}^T (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T) \underset{\sim}{L}' \underset{\sim}{q} + \text{constants}\right] dV \\
&= \underset{\sim}{\beta}^T \hat{C}_1 \underset{\sim}{q} + \Delta\underset{\sim}{q}^T \underset{\sim}{C}_2 \underset{\sim}{q}
\end{aligned} \tag{6.70}$$

where

$$\begin{aligned}
\underset{\sim}{C}_1 &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{L}'\, dV \\
\underset{\sim}{C}_2 &= \int_{V_n} (\underset{\sim}{A}^T + \underset{\sim}{C}_I^T)\, \underset{\sim}{L}'\, dV
\end{aligned} \tag{6.71}$$

$$\begin{aligned}
-\frac{1}{2}\int_{V_n} \Delta\sigma_{ij}\, u_{k,i}\, u_{k,j}\, dV &= -\frac{1}{2}\int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T - \underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}_P^T + \Delta\underset{\sim}{q}^T (\hat{A}^T + \hat{C}_I^T)\right] (\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}' \underset{\sim}{q})\, dV \\
&= -\frac{1}{2}\int_{V_n} \left[\underset{\sim}{\beta}^T \hat{P}^T \hat{q}^T \hat{L}'^T \hat{L}' \underset{\sim}{q} + \Delta\underset{\sim}{q}^T (\hat{A}^T + \hat{C}_I^T)\, \underset{\sim}{q}^T \underset{\sim}{L}'^T \hat{L}' \underset{\sim}{q} + \text{constants}\right] dV \\
&= -\frac{1}{2} \underset{\sim}{\beta}^T \hat{C} \underset{\sim}{q} - \frac{1}{2} \Delta\underset{\sim}{q}^T \underset{\sim}{F} \underset{\sim}{q}
\end{aligned} \tag{6.72}$$

where

$$C = \int_{V_n} P^T q^T L'^T L' \, dV$$
$$F = \int_{V_n} (A^T + C_I^T) \, q^T L'^T L' \, dV \qquad (6.73)$$

Note the cautions for Eq. 6.27.

Placing these integral evaluations into Eq. 6.54 gives

$$\begin{aligned} \pi^c_{mc_n}(\beta, \Delta q) = & -\tfrac{1}{2}\beta^T H \beta + \beta^T H^s_\sigma - \beta^T H_{\sigma_p} - \beta^T H_A \Delta q + D^s_\sigma \Delta q - D_{\sigma_p} \Delta q - \tfrac{1}{2}\Delta q^T D \Delta q \\ & - \tfrac{1}{2}\Delta q^T K_g \Delta q + \beta^T G \Delta q + Q^T_{PV} \Delta q + \tfrac{1}{2}\Delta q^T (M + M^T) \Delta q - Q^T_T \Delta q \\ & - \beta^T G_u \Delta q + \beta^T V - Q^T_{PV_u} \Delta q - \tfrac{1}{2}\Delta q^T (M_u + M_u^T) \Delta q + \Delta q^T V_A - \beta^T H^c_\sigma \\ & + D^c_\sigma \Delta q + \beta^T C_1 \Delta q + \Delta q^T C_2 q - \tfrac{1}{2}\beta^T C q - \tfrac{1}{2}\Delta q^T F q \end{aligned} \qquad (6.74)$$

where

$H^S_\sigma = H_\sigma$ from stress equilibrium check

$D^S_\sigma = D_\sigma$ from stress equilibrium check

$H^C_\sigma = H_\sigma$ from compatibility equilibrium check

$D^C_\sigma = D_\sigma$ from compatibility equilibrium check

These terms are separated for identification purposes only.

Taking the variation of $\pi^C_{mc_n}$ with respect to the independent β's yields

$$\begin{aligned} \frac{\partial \pi^c_{mc_n}}{\partial \beta} = & \; H\beta + H^s_\sigma - H_{\sigma_p} - H_A \Delta q + G \Delta q - G_u \Delta q + V \\ & - H^c_\sigma + C_1 q - \tfrac{1}{2} C q = 0 \end{aligned} \qquad (6.75)$$

Solving for the β's

$$\begin{aligned} \beta = & \; H^{-1}\left(H^s_\sigma - H^c_\sigma - H_{\sigma_p} + V + C_1 q - \tfrac{1}{2} C q\right) \\ & + H^{-1}\left(G - G_u - H_A\right) \Delta q \end{aligned} \qquad (6.76)$$

Placing Eq. 6.76 into Eq. 6.74 gives

$$\pi^{c}_{mc_n}(\Delta \underset{\sim}{q}) = -\tfrac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{G}^T-\underset{\sim}{G}_u^T-\underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}(\underset{\sim}{G}-\underset{\sim}{G}_u-\underset{\sim}{H}_A)\Delta\underset{\sim}{q} + \Delta\underset{\sim}{q}^T(\underset{\sim}{G}^T-\underset{\sim}{G}_u^T-\underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}$$
$$(\underset{\sim}{H}^S_\sigma - \underset{\sim}{H}_{\sigma_p} - \underset{\sim}{H}_A\Delta\underset{\sim}{q} + \underset{\sim}{G}\Delta\underset{\sim}{q} - \underset{\sim}{G}_u\Delta\underset{\sim}{q} + \underset{\sim}{V} - \underset{\sim}{H}^c_\sigma + \underset{\sim}{C}_1\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q})$$
$$-\tfrac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{D} + \underset{\sim}{K}_q - \underset{\sim}{M} - \underset{\sim}{M}^T + \underset{\sim}{M}_u + \underset{\sim}{M}_u^T)\Delta\underset{\sim}{q}$$
$$+\Delta\underset{\sim}{q}^T(\underset{\sim}{D}^{S^T}_\sigma - \underset{\sim}{D}^T_{\sigma_p} + \underset{\sim}{Q}_{PV} - \underset{\sim}{Q}_{\bar{T}} - \underset{\sim}{Q}_{PV_u} + \underset{\sim}{V}_A$$
$$-\underset{\sim}{D}^{c^T}_\sigma + \underset{\sim}{C}_2\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{F}\underset{\sim}{q}) + \text{constants} \tag{6.77}$$

Upon rearranging and dropping the constants not subject to variation with respect to $\Delta\underset{\sim}{q}$

$$\pi^{c}_{mc_n}(\Delta\underset{\sim}{q}) = \tfrac{1}{2}\Delta\underset{\sim}{q}^T[(\underset{\sim}{G}^T-\underset{\sim}{G}_u^T-\underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}(\underset{\sim}{G}-\underset{\sim}{G}_u-\underset{\sim}{H}_A) + \underset{\sim}{M} + \underset{\sim}{M}^T - \underset{\sim}{M}_u - \underset{\sim}{M}_u^T - \underset{\sim}{D} - \underset{\sim}{K}_q]\Delta\underset{\sim}{q}$$
$$+\Delta\underset{\sim}{q}^T[(\underset{\sim}{G}^T-\underset{\sim}{G}_u^T-\underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}(\underset{\sim}{H}^S_\sigma - \underset{\sim}{H}^c_\sigma - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{V} + \underset{\sim}{C}_1\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q})$$
$$+\underset{\sim}{Q}_{PV} - \underset{\sim}{Q}_{PV_u} - \underset{\sim}{Q}_{\bar{T}} + \underset{\sim}{D}^{S^T}_\sigma - \underset{\sim}{D}^{c^T}_\sigma - \underset{\sim}{D}^T_{\sigma_p}$$
$$+\underset{\sim}{V}_A + \underset{\sim}{C}_2\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{F}\underset{\sim}{q}] \tag{6.78}$$

Or in the form

$$\pi^{c}_{mc_n}(\Delta\underset{\sim}{q}) = \tfrac{1}{2}\Delta\underset{\sim}{q}^T\underset{\sim}{K}_T\Delta\underset{\sim}{q} - \Delta\underset{\sim}{q}^T\underset{\sim}{Q} \tag{6.79}$$

where

$$\underset{\sim}{K}_T = (\underset{\sim}{G}-\underset{\sim}{G}_u-\underset{\sim}{H}_A)^T\underset{\sim}{H}^{-1}(\underset{\sim}{G}-\underset{\sim}{G}_u-\underset{\sim}{H}_A) + \underset{\sim}{M} + \underset{\sim}{M}^T - \underset{\sim}{M}_u - \underset{\sim}{M}_u^T - \underset{\sim}{D} - \underset{\sim}{K}_q$$
$$\underset{\sim}{Q} = -(\underset{\sim}{G}-\underset{\sim}{G}_u-\underset{\sim}{H}_A)^T\underset{\sim}{H}^{-1}(\underset{\sim}{H}^S_\sigma - \underset{\sim}{H}^c_\sigma - \underset{\sim}{H}_{\sigma_p} + \underset{\sim}{V} + \underset{\sim}{C}_1\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q})$$
$$-(\underset{\sim}{Q}_{PV} - \underset{\sim}{Q}_{PV_u} - \underset{\sim}{Q}_{\bar{T}} + \underset{\sim}{D}^{S^T}_\sigma - \underset{\sim}{D}^{c^T}_\sigma - \underset{\sim}{D}^T_{\sigma_p} + \underset{\sim}{V}_A + \underset{\sim}{C}_2\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{F}\underset{\sim}{q}) \tag{6.80}$$

These element level matrices must be assembled as shown in Subsection 6.3. If certain conditions exist, various terms may be dropped or redefined in Eqs. 6.80. They are as follows

Condition	Terms Removed
1. $\Delta\tilde{u}_i \equiv \Delta\bar{u}_i$ on S_{u_n}	$\underset{\sim}{G}_u, \underset{\sim}{V}, \underset{\sim}{Q}_{PV_u}, \underset{\sim}{M}_u, V_A$
2. $\underset{\sim}{\sigma}_p$ is lumped at nodes	$\underset{\sim}{H}_{\sigma_p}, \underset{\sim}{D}_{\sigma_p}, \underset{\sim}{Q}_{PV}, \underset{\sim}{Q}_{PV_u}$
3. $\bar{T}_i + \Delta\bar{T}_i$ is lumped at nodes	$\underset{\sim}{Q}_{\bar{T}}$
4. No compatibility check	$\underset{\sim}{H}^c_\sigma, \underset{\sim}{D}^c_\sigma, \underset{\sim}{C}_1, \underset{\sim}{C}_2, \underset{\sim}{C}, \underset{\sim}{F}$
5. No stress equilibrium check	$\underset{\sim}{H}^s_\sigma, \underset{\sim}{D}^s_\sigma$ and σ_p corresponds to $\Delta\bar{F}_i$ only, $\underset{\sim}{Q}^T_{\bar{T}} = \int_{S_{\sigma_n}} \Delta\underset{\sim}{\bar{T}} \, \underset{\sim}{\tilde{L}} \, ds$

Assuming the first three conditions are met and both equilibrium checks are utilized a much simplified form of Eqs. 6.80 are realized, i.e.

$$\begin{aligned} \underset{\sim}{K}_T &= (\underset{\sim}{G} - \underset{\sim}{H}_A)^T \underset{\sim}{H}^{-1} (\underset{\sim}{G} - \underset{\sim}{H}_A) + \underset{\sim}{M} + \underset{\sim}{M}^T - \underset{\sim}{D} - \underset{\sim}{K}_g \\ \underset{\sim}{Q} &= -(\underset{\sim}{G} - \underset{\sim}{H}_A)^T \underset{\sim}{H}^{-1} (\underset{\sim}{C}_1 - \tfrac{1}{2}\underset{\sim}{C}) \underset{\sim}{q} + (\underset{\sim}{C}_2 - \tfrac{1}{2}\underset{\sim}{F}) \underset{\sim}{q} \end{aligned} \tag{6.81}$$

6.2.2 The Inconsistent Assumed Stress Hybrid Model

The linearized equations of Subsection 5.4.2 will be discussed here. The functional of Eq. 5.54 may be written for an element as

$$\begin{aligned} \pi^I_{mc_n} = & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}]\,dv \\ & + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij}) \nu_j \,\Delta\tilde{u}_i \, ds + \int_{\partial V_n} \sigma_{kj}\,\Delta u_{i,k}\, \nu_j (\Delta\tilde{u}_i - \Delta u_i)\, ds \\ & - \int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\, ds - \int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\, ds \\ & - \int_{V_n} \Delta\sigma_{ij} [e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i} u_{k,j})]\, dv \} \end{aligned} \tag{6.82}$$

where Eq. 6.21 is understood. Eq. 6.82 is subject to

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \tag{5.51}$$

and

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\,\Delta u_{i,k}]\,\nu_j \tag{5.47}$$

Writing Eq. 5.51 as

$$\Delta\sigma_{ij,j} = -\sigma_{ij,j} - (\bar{F}_i + \Delta\bar{F}_i) \tag{6.83}$$

a solution in matrix form is

$$\underset{\sim}{\Delta\sigma} = \underset{\sim}{P}\,\underset{\sim}{\beta} - \underset{\sim}{\sigma} + \underset{\sim}{\sigma}_P \tag{6.84}$$

Following the concepts of Subsection 6.1.2 one may write

$$-\frac{1}{2}\int_{V_n} B(\Delta\sigma_{ij})\,dV = -\frac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\,\underset{\sim}{\beta} + \underset{\sim}{\beta}\,\underset{\sim}{H}\sigma - \underset{\sim}{\beta}\,\underset{\sim}{H}\sigma_P \tag{6.85}$$

$$\frac{1}{2}\int_{V_n} \sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\,dV = \frac{1}{2}\Delta\underset{\sim}{q}^T\,\underset{\sim}{K}g\,\Delta\underset{\sim}{q} \tag{6.86}$$

$$\int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\,\nu_j\,\Delta\tilde{u}_i\,ds = \underset{\sim}{\beta}^T \underset{\sim}{G}\,\Delta\underset{\sim}{q} + \underset{\sim}{Q}_{P\nu}^T\,\Delta\underset{\sim}{q} \tag{6.87}$$

From Eqs. 5.47 the boundary tractions may simply be written as

$$\underset{\sim}{T} + \Delta\underset{\sim}{T} = \underset{\sim}{R}\,\underset{\sim}{\beta} + \underset{\sim}{\sigma}_{P\nu} + \underset{\sim}{A}_b\,\Delta\underset{\sim}{q} \tag{6.88}$$

Thus,

$$\int_{\partial V_n} \sigma_{kj}\,\Delta u_{i,k}\,\nu_j\,(\Delta\tilde{u}_i - \Delta u_i)\,ds = \frac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{M}_b + \underset{\sim}{M}_b^T)\,\Delta\underset{\sim}{q} \tag{6.89}$$

$$-\int_{S\sigma_n} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\,ds = -\underset{\sim}{Q}_{\bar{T}}^T\,\Delta q \tag{6.90}$$

$$\begin{aligned} -\int_{S_{u_n}} (T_i + \Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,ds = &-\underset{\sim}{\beta}^T \underset{\sim}{G}_u\,\Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{V} - \underset{\sim}{Q}_{P\nu}^T\,\Delta\underset{\sim}{q} \\ &- \frac{1}{2}\Delta\underset{\sim}{q}^T(\tilde{\underset{\sim}{M}}_b + \tilde{\underset{\sim}{M}}_b^T)\,\Delta\underset{\sim}{q} + \Delta\underset{\sim}{q}^T\,\bar{\underset{\sim}{M}}_b \end{aligned} \tag{6.91}$$

where here,

$$\underset{\sim}{V} = \int_{S_{u_n}} \underset{\sim}{R}^T\,\Delta\bar{\underset{\sim}{u}}\,ds \qquad\qquad \bar{\underset{\sim}{M}}_b = \int_{S_{u_n}} \underset{\sim}{A}_b^T\,\Delta\bar{\underset{\sim}{u}}\,ds \tag{6.92}$$

$$-\int_{V_n} \Delta\sigma_{ij}\left[e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})\right]dV = -\underset{\sim}{\beta}^T\underset{\sim}{H}\sigma + \underset{\sim}{\beta}^T\underset{\sim}{C}_1\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{\beta}^T\underset{\sim}{C}\underset{\sim}{q} \tag{6.93}$$

Placing these evaluations into Eq. 6.82 yields

$$\pi^{I}_{mc_n}(\beta, \Delta q) = -\tfrac{1}{2}\beta^T H \beta + \beta^T H^s_\sigma - \beta^T H_{\sigma_p} + \tfrac{1}{2}\Delta q^T K_g \Delta q + \beta^T \hat{G} \Delta q$$
$$+ Q^T_{pv}\Delta q + \tfrac{1}{2}\Delta q^T (M_b + M_b^T)\Delta q - \bar{Q}^T_{\mp}\Delta q - \beta^T G_u \Delta q + \beta^T V$$
$$- \bar{Q}^T_{pv_u}\Delta q - \tfrac{1}{2}\Delta q^T(\tilde{M}_b + \tilde{M}_b^T)\Delta q + \Delta q^T \bar{M}_b - \beta^T H^c_\sigma + \beta^T \hat{C}_1 q - \tfrac{1}{2}\beta \hat{C} q \qquad (6.94)$$

Taking the variation of this with respect to the independent β's yields

$$\frac{\partial \pi^{I}_{mc_n}}{\partial \beta} = -\hat{H}\beta + H^s_\sigma - H_{\sigma_p} + \hat{G}\Delta q - G_u \Delta q + V - H^c_\sigma + \hat{C}_1 q - \tfrac{1}{2}\hat{C} q = 0 \qquad (6.95)$$

Solving for the β's gives

$$\beta = H^{-1}(H^s_\sigma - H^c_\sigma - H_{\sigma_p} + V + C_1 q - \tfrac{1}{2}\hat{C} q) + H^{-1}(G - G_u)\Delta q \qquad (6.96)$$

Placing this into Eq. 6.94 yields

$$\pi^{I}_{mc_n}(\Delta q) = -\tfrac{1}{2}\Delta q^T (G^T - G_u^T) H^{-1} (G - G_u)\Delta q + \Delta q^T (G^T - G_u^T) H^{-1}$$
$$(H^s_\sigma - H_{\sigma_p} + \hat{G}\Delta q - G_u \Delta q + V - H^c_\sigma + \hat{C}_1 q - \tfrac{1}{2}\hat{C} q)$$
$$+ \tfrac{1}{2}\Delta q^T (K_g + M_b + M_b^T - \tilde{M}_b - \tilde{M}_b^T)\Delta q$$
$$+ \Delta q^T (Q_{pv} - Q_{pv_u} - Q_{\mp} + \bar{M}_b) \qquad (6.97)$$

Or, in the form

$$\pi^{I}_{mc_n}(\Delta q) = \tfrac{1}{2}\Delta q^T K_T \Delta q - \Delta q^T Q \qquad (6.98)$$

where

$$K_T = (\hat{G} - G_u)^T H^{-1} (\hat{G} - G_u) + K_g + M_b + M_b^T - \tilde{M}_b - \tilde{M}_b^T$$
$$Q = -(G - G_u)^T H^{-1}(H^s_\sigma - H^c_\sigma - H_{\sigma_p} + V + C_1 q - \tfrac{1}{2}\hat{C} q) - (Q_{pv} - Q_{pv_u} - Q_{\mp} + \bar{M}_b) \qquad (6.99)$$

The definitions of these terms are the same as previous subsections (except where noted) but it is to be understood that the variables are associated with the C.U.L. system. Comparing Eq. 6.99 to Eq. 6.80, one may observe the reduction in complication and computation that the inconsistent model allows.

Eqs. 6.99 may be simplified. Assuming the first three conditions at the end of Subsection 6.1.2 are met, Eqs. 6.99 become

$$\begin{aligned} \underset{\sim}{K}_T &= \hat{G}^T \hat{H}^{-1} \underset{\sim}{G} + \underset{\sim}{K}_q + \underset{\sim}{M}_b + \underset{\sim}{M}_b^T \\ \hat{Q} &= -\hat{G}^T \hat{H}^{-1} (\underset{\sim}{c}_1 - \tfrac{1}{2}\underset{\sim}{c}) \hat{q} \end{aligned} \tag{6.100}$$

If additionally the last condition is met, Eqs. 6.100 reduce to

$$\begin{aligned} \underset{\sim}{K}_T &= \hat{G}^T \hat{H}^{-1} \underset{\sim}{G} + \underset{\sim}{K}_q \\ \hat{Q} &= -\underset{\sim}{G}^T \underset{\sim}{H}^{-1} (\underset{\sim}{c}_1 - \tfrac{1}{2}\hat{c}) \underset{\sim}{q} \end{aligned} \tag{6.101}$$

Comparison of these with the consistent model (Eq. 6.81) show a significant simplification. Additionally, compared to the inconsistent model for the S.L. system (Eqs. 6.52 or 6.53) these equations show more simplification. The comparisons of this model with the very similar modified Reissner model are discussed in Appendix C.

6.3 Assembly Procedures

For all the models discussed here an equation of the form

$$\pi_n(\Delta \underset{\sim}{q}) = \tfrac{1}{2} \Delta \underset{\sim}{q}^T \underset{\sim}{K}_T \Delta \underset{\sim}{q} - \Delta \underset{\sim}{q}^T \underset{\sim}{Q} \tag{6.102}$$

results. Since these equations were developed by considering functionals for one element then Eqs. 6.102 represents the tangent stiffness, incremental displacements, and loads for one element. Since local coordinate systems are used to perform all the differentiations and integrations Eqs. 6.102 are referred to these local axes. (See Subsection 2.2.)

To analyze a problem the contributions from all the elements must be summed (i.e. see Eq. 6.2). In order to do this the $\Delta \underset{\sim}{q}$'s for neighboring elements must be the same. Since this is not the case the contributions from each element must be suitably transformed before being added to the total system. However, the $\Delta \underset{\sim}{q}$'s for each element given in the local systems can be transformed to $\Delta \underset{\sim}{q}$'s in a common system where contributions may be directly added.

From Subsection 3.3 one may write

$$ {}^{L}\Delta \underset{\sim}{q} = {}^{LC}\underset{\sim}{T} \; {}^{C}\Delta \underset{\sim}{q} \tag{6.103} $$

Although the ${}^{L}\Delta \underset{\sim}{q}$'s do not correspond with neighboring elements the ${}^{C}\Delta \underset{\sim}{q}$'s do. Considering Eqs. 6.102 for an element

$$ \Pi_n(\Delta \underset{\sim}{q}) = \frac{1}{2}\, {}^{L}\Delta \underset{\sim}{q}^T \; {}^{L}\underset{\sim}{K}_T \; {}^{L}\Delta \underset{\sim}{q} - {}^{L}\Delta \underset{\sim}{q}^T \; {}^{L}\underset{\sim}{Q} \tag{6.104} $$

Placing Eq. 6.103 into Eq. 6.104 yields

$$ \Pi_n(\Delta \underset{\sim}{q}) = \frac{1}{2}\, {}^{C}\Delta \underset{\sim}{q}^T \; {}^{LC}\underset{\sim}{T}^T \; {}^{L}\underset{\sim}{K}_T \; {}^{LC}\underset{\sim}{T} \; {}^{C}\Delta \underset{\sim}{q} - {}^{C}\Delta \underset{\sim}{q}^T \; {}^{LC}\underset{\sim}{T}^T \; {}^{L}\underset{\sim}{Q} \tag{6.105} $$

or

$$ \Pi_n(\Delta \underset{\sim}{q}) = \frac{1}{2}\, {}^{C}\Delta \underset{\sim}{q}^T \; {}^{C}\underset{\sim}{K}_T \; {}^{C}\Delta \underset{\sim}{q} - {}^{C}\Delta \underset{\sim}{q}^T \; {}^{C}\underset{\sim}{Q} \tag{6.106} $$

where

$$ \begin{aligned} {}^{C}\underset{\sim}{K}_T &= {}^{LC}\underset{\sim}{T}^T \; {}^{L}\underset{\sim}{K} \; {}^{LC}\underset{\sim}{T} \\ {}^{C}\underset{\sim}{Q} &= {}^{LC}\underset{\sim}{T}^T \; {}^{L}\underset{\sim}{Q} \end{aligned} \tag{6.107} $$

For an element Eqs. 6.107 represents the contribution of an element. Since these evaluations are referenced to $\Delta \underset{\sim}{q}$'s common to all neighboring elements then the contributions of each element may be directly added to the total system representing the entire structure. This would result in one set of total equations for the structure. Recalling that the total structure may be represented as the sum of the individual elements, from Eq. 6.106

$$ \begin{aligned} \Pi({}^{T}\Delta \underset{\sim}{q}) &= \sum_n \Pi_n(\Delta \underset{\sim}{q}) = \sum_n \left[\frac{1}{2}\, {}^{C}\Delta \underset{\sim}{q}^T \; {}^{C}\underset{\sim}{K}_T \; {}^{C}\Delta \underset{\sim}{q} - {}^{C}\Delta \underset{\sim}{q}^T \; {}^{C}\underset{\sim}{Q} \right] \\ &= \frac{1}{2}\, {}^{T}\Delta \underset{\sim}{q}^T \; {}^{T}\underset{\sim}{K}_T \; {}^{T}\Delta \underset{\sim}{q} - {}^{T}\Delta \underset{\sim}{q}^T \; {}^{T}\underset{\sim}{Q} \end{aligned} \tag{6.108} $$

where the left superscript here refers to the total system and ${}^{T}\Delta \underset{\sim}{q}$ represents an assembled vector of independent, unknown nodal displacements. These may be solved for by taking the variation of Eq. 6.108 with respect to ${}^{T}\Delta \underset{\sim}{q}$.

$$ \frac{\partial \Pi({}^{T}\Delta \underset{\sim}{q})}{\partial\, {}^{T}\Delta \underset{\sim}{q}} = {}^{T}\underset{\sim}{K}_T \; {}^{T}\Delta \underset{\sim}{q} - {}^{T}\underset{\sim}{Q} = 0 \tag{6.109} $$

Eqs. 6.109 represent the total (system), assembled equilibrium equations.

6.4 Solution Techniques

The total system of equations may be written as

$$ {}^{T}\underset{\sim}{K}_{T}\,{}^{T}\Delta\underset{\sim}{q} = {}^{T}\underset{\sim}{Q} \tag{6.110}$$

Therefore, both ${}^{T}\Delta\underset{\sim}{q}$ and ${}^{T}\underset{\sim}{Q}$ are vectors which are the total number of degrees of freedom long and ${}^{T}\underset{\sim}{K}_{T}$ is the corresponding total tangent stiffness matrix.

There are several numerical procedures which can be used to solve the geometrically nonlinear problem. One must remember that although the problem is nonlinear, Eq. 6.110 has been constructed to be a linear equation in the unknown ${}^{T}\Delta\underset{\sim}{q}$'s. Thus, incremental and/or iterative techniques are employed to solve the total problem [Haisler, et al., 1972; Zienkiewicz, 1971; Desai and Abel, 1972]. For the purposes of this work three basic forms of solution have been used.

The first of these is to utilize the basic equations with no equilibrium checks at all. As will be seen in Section 8 on results this is not an economical scheme. Even with small increments the solutions tend to drift. The next procedure makes use of the equilibrium checks either separately or combined. This solution technique only uses incremental steps. This is more efficient than the first procedure, but there is still room for improvement. The last technique is to combine incremental and iterative steps while using both equilibrium checks. This is by far the most efficient scheme for both coordinate systems.

Section 8, through the use of some sample problems, demonstrates the relative problems and merits of each of these solution procedures for beam, plate, and shell problems. Figure 6.1 schematically demonstrates the procedures.

6.4.1 Incremental Solutions with No Equilibrium Checks

This method of solution assumes that both stress equilibrium and compatibility are exactly satisfied in the reference state for all time. Thus, choosing the correct equilibrium equation, (Eqs. 6.51, 6.80, or 6.99) depending on the functional and coordinate system to be used, one must remove the terms corresponding to the equilibrium checks. This can be done as indicated in the previous subsections. Since the basic functionals are linearized in the unknowns then the assumption of small increments must be adhered to.

Obviously, since the state of stress and strain is assumed to be correct in the reference state the smaller the increment the better the assumption.

The procedure then is as follows. First, the appropriate tangent stiffness is generated. Next a small increment in external load is applied. This load may either be consistently derived or systematically lumped at the element nodes. Note that this load vector is only a function of the external load since no equilibrium checks are utilized to form an equilibrium imbalance load. The ${}^{T}\Delta \underset{\sim}{q}$'s are solved by the use of Eq. 6.110. From these an increment of stress can be obtained (see Section 7) and finally, through the constitutive relations, an increment of strain is calculated. These are then added to the total quantities and any appropriate updating of geometry, etc. (Section 7) are carried out. This state is now the reference configuration and since it is deemed correct no equilibrium balance terms are calculated. Therefore, the new tangent stiffness matrix and incremental load vector are calculated and Eq. 6.110 applied to obtain new incremental quantities and so on.

Since the increments must be kept small the total number of incremental solutions required may be large. For each increment a new tangent stiffness matrix and load vector is found. This can be an extremely costly procedure even though there is no need to calculate the equilibrium check matrices. However, for small increments, reasonable results can be obtained. (See Fig. 6.1.)

6.4.2 Incremental Solutions with Equilibrium Checks

Realizing that numerical solutions of approximate methods are being carried out one should expect solutions obtained with no check conditions to drift from the true solution. The assumed stress hybrid methods maintain some form of restraint on the stress equilibrium equation and, therefore, drifting due to lack of this check should be small although definitely significant. The inclusion of a stress equilibrium check is more simply accomplished for these functionals as can be seen by comparing the stress equilibrium check derived here to that of Hofmeister et al. [1971].

The strain displacement relations, however, are only an Euler equation of the functional and only can be satisfied in an average sense. It is expected that a compatibility check may have a greater significance than a stress equilibrium check. An additional benefit can be derived from such a check.

While the incremental strain displacement relations must be linearized to establish Eq. 6.110, the compatibility check may maintain a nonlinear relation since the total quantities are known. This means that if larger increments are used a self correcting effect will take place.

The procedure, being incremental only, is identical to that of the previous subsection with the following exceptions. After a solution of incremental quantities is obtained and added to the total state, this state is then checked to see if in fact stress equilibrium and/or compatibility is maintained. If these conditions are not met then there is an imbalance in Eq. 6.110. This imbalance is treated as a corrective load and is added to the increment in external load for the next step. Thus, as the solution proceeds a constant check is maintained to assure that the solution does not tend to drift. Note that the stress equilibrium equation is not used directly to form an imbalance term. It is simply assumed that in the next step the initial stresses do not satisfy equilibrium and, therefore, are carried over with the incremental stresses of the next step. In other words, at every step the total stress is required to satisfy the constraint condition rather than just the incremental stresses.

These equilibrium checks allow for a more accurate solution regardless of the increment size. Although a new tangent stiffness and load vector are still calculated at every step, as well as the matrices necessary for the checks, this system is more efficient because fewer solution steps are required. In fact, if the compatibility check is used the increment size can become quite large and still yield reasonable results. This allows this scheme to be much more efficient than its predecessor. (See Fig. 6.1.)

6.4.3 Incremental-Iterative Solution Procedure

Although the last procedure is quite an improvement its corrections are actually somewhat in error. Any imbalance that exists should be corrected within an incremental external load step. This assures that one is on the proper load path before continuing. Also, just as the procedure with no equilibrium checks tends to drift, there is no assurance that a one step equilibrium imbalance check will correct the drift completely. These solutions, therefore, also tend to drift although not nearly as much as solutions without checks.

This concept lead to combining an incremental and iterative procedure.

The solution proceeds as follows. An external load increment is applied and a solution of incremental quantities obtained. These are added to the totals to obtain a new reference state. As before, stress equilibrium and compatibility are checked. If an imbalance exists this equivalent imbalance load vector is applied to Eq. 6.110 with no new increase in external loads. An option becomes apparent to the analyst for the generation of a tangent stiffness matrix. One may either use the same stiffness that was used while the external load was added (at the beginning of a new load increment) or generate a new one. From Fig. 6.1 one can observe that while creating a new stiffness matrix may be costly, generally much fewer iterations are required to correct the solution. Thus, between each new external load increment a series of iterative steps are performed utilizing imbalance loads. The iteration process is stopped when convergence of the solution is reached. A solution is considered converged when the imbalance loads, or the incremental displacements due to such loads, are within a predetermined percentage of the external load increment, or displacement from such a load increment.

It has been demonstrated that even for highly geometrically nonlinear problems only a relatively small number of solutions are required for this approach. In this work it was decided to create a new tangent stiffness matrix for every incremental-iterative solution step. It was possible to obtain nonlinear results using only one external load step (increment) and only a couple of iterations. This scheme is by far the most efficient, where efficiency may be defined as obtaining the most accurate solutions at the least expense.

SECTION 7

FINITE ELEMENT MATRIX EQUATIONS

FOR THIN ELASTIC STRUCTURES

7.1 Shallow Curved (Marguerre) and Flat Beam Elements

As seen in the last section for a stationary system there are problems associated with the consistent model. When discussing this system only the inconsistent model will be utilized. For the updated system, however, both consistent and inconsistent models will be discussed.

For each subsection dealing with beams the general equations of Section 6 will be reduced to the one dimensional case. The reduction from the general case to that for shallow structures subject to moderate rotations requires care for the consistent models. For the curved beam elements, governed by the inconsistent functional, the complete linear stress equilibrium equation is satisfied. In the interest of simplicity, it was assumed that:

(a) The prescribed displacements on S_{u_n} were identical to the boundary displacements, $\tilde{u}_i+\Delta\tilde{u}_i \equiv \bar{u}_i+\Delta\bar{u}_i$;

(b) the body forces and external pressure forces were lumped at the nodes;

(c) the surface tractions, $\bar{T}_i+\Delta\bar{T}_i$ were lumped at the nodes; and

(d) the displacements are continuous, $u_i+\Delta u_i \equiv \tilde{u}_i+\Delta\tilde{u}_i$. Thus, Eq. 6.110 may be written as

$$^{T}\underset{\sim}{K}_{T}\, {}^{T}\Delta\underset{\sim}{q} = {}^{T}\underset{\sim}{Q}_{E} + {}^{T}\underset{\sim}{Q} \tag{7.1}$$

where

$^{T}\underset{\sim}{Q}_{E}$ = global lumped external loads

$^{T}\underset{\sim}{Q}$ = consistently generated equivalent check loads

7.1.1 The Stationary Lagrangian System

With the basic assumptions stated above, the inconsistent functional, Eq. 6.20, becomes for a one dimensional shallow beam

$$\pi^{I}_{mc_n} = \int_0^{\ell} \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} + (N+\Delta N)(w_{,x}\,\Delta w_{,x}) + \frac{1}{2} N\,\Delta w_{,x}^2 \right] dx$$
$$+ \left[(N+\Delta N)\,\Delta\tilde{u} + (S+\Delta S)\,\Delta\tilde{w} - (M+\Delta M)\,\Delta\tilde{w}_{,x} \right]_0^{\ell}$$
$$- \int_0^{\ell} \left[\frac{\Delta N(N)}{EA} + \frac{\Delta M(M)}{EI} \right] dx$$
$$+ \int_0^{\ell} \left[\Delta N (u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} + \frac{1}{2}\Delta N\, w_{,x}^2 \right] dx \tag{7.2}$$

See Fig. 7.1 for sign conventions and definitions. Under Marguerre theory (and this formulation)

$$^{*}(S+\Delta S) = (M+\Delta M)_{,x} + (N+\Delta N)\, z_{,x} \tag{7.3}$$

If one integrates the linear terms of the last integral by parts,

$$\int_0^{\ell} \left[\Delta N (u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} \right] dx$$
$$= - \int_0^{\ell} \left[\Delta N_{,x} u + (\Delta N z_{,x})_{,x}\, w - \Delta M_{,x}\, w_{,x} \right] dx$$
$$+ \left[\Delta N u + \Delta N z_{,x} w - \Delta M w_{,x} \right]_0^{\ell}$$
$$= - \int_0^{\ell} \left\{ \Delta N_{,x} u + \left[\Delta M_{,xx} + (\Delta N z_{,x})_{,x} \right] w \right\} dx$$
$$+ \left[\Delta N u + (\Delta M_{,x} + \Delta N z_{,x})\, w - \Delta M w_{,x} \right]_0^{\ell} \tag{7.4}$$

Eq. 7.2 may be rewritten as

$$\pi^{I}_{mc_n} = \int_0^{\ell} \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} + (N+\Delta N)(w_{,x}\,\Delta w_{,x}) + \frac{1}{2} N\,\Delta w_{,x}^2 \right] dx$$
$$+ \left[(N+\Delta N)\,\Delta\tilde{u} + (S+\Delta S)\,\Delta\tilde{w} - (M+\Delta M)\,\Delta\tilde{w}_{,x} \right]_0^{\ell}$$
$$- \int_0^{\ell} \left[\frac{\Delta N(N)}{EA} + \frac{\Delta M(M)}{EI} - \frac{1}{2}\Delta N\, w_{,x}^2 \right] dx$$
$$+ \left[\Delta N u + \Delta S w - \Delta M w_{,x} \right]_0^{\ell} \tag{7.5}$$

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*Since the lateral displacement w is compatible at the interelement boundaries, not all nonlinear boundary traction terms have contributions to boundary work. Throughout this section S and S_ν shall refer only to those shear terms whose corresponding boundary work terms do not vanish in the appropriate functionals.

or, in matrix form

$$\pi^{I}_{mc_n} = \int_0^{\ell} \Big[-\tfrac{1}{2} \begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}^T \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} + (N+\Delta N)(w_{,x}^T \Delta w_{,x})$$
$$+ \tfrac{1}{2} \Delta w_{,x}^T N \Delta w_{,x} \Big] dx$$
$$+ \left[\begin{Bmatrix} N+\Delta N \\ S+\Delta S \\ M+\Delta M \end{Bmatrix}^T \begin{Bmatrix} \Delta\tilde{u} \\ \Delta\tilde{w} \\ -\Delta\tilde{w}_{,x} \end{Bmatrix} \right]_0^{\ell} - \int_0^{\ell} \Big[\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}^T \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$
$$- \tfrac{1}{2} \Delta N\, w_{,x}^T w_{,x} \Big] dx + \left[\begin{Bmatrix} \Delta N \\ \Delta S \\ \Delta M \end{Bmatrix}^T \begin{Bmatrix} u \\ w \\ -w_{,x} \end{Bmatrix} \right]_0^{\ell} \tag{7.6}$$

Note that Eqs. 7.2 or 7.5 are subject to

$$(N+\Delta N)_{,x} = 0 \tag{7.7}$$

and

$$(M+\Delta M)_{,xx} + \left[(N+\Delta N) z_{,x} \right]_{,x} = 0 \tag{7.8}$$

These equations show that the integral on the right hand side of Eq. 7.4 is zero. (Recall that constants such as $N_{,x}u$ may be added without effecting the final result which is obtained by variation of the functional.)

The stress resultants must now be interpolated in terms of the unknown stress parameters, $\underset{\sim}{\beta}$. From Eqs. 7.7 and 7.8 an obvious choice is

$$\Delta N = \beta_1 - N \tag{7.9}$$

$$\Delta M = -z\beta_1 + \beta_2 + x\beta_3 - M \tag{7.10}$$

Let

$\alpha = x/\ell$

ℓ = length of beam in base plane

Then, the displacements Δu and Δw may be interpolated in terms of the unknown nodal values in a linear and cubic fashion respectively.

$$\Delta u = (1-\alpha)\Delta q_1 + \alpha \Delta q_4 \tag{7.11}$$

$$\Delta w = (1-3\alpha^2+2\alpha^3)\Delta q_2 - \ell(\alpha-2\alpha^2+\alpha^3)\Delta q_3 + (3\alpha^2-2\alpha^3)\Delta q_5 - \ell(\alpha^3-\alpha^2)\Delta q_6 \tag{7.12}$$

$$\Delta w_{,x} = \frac{6}{\ell}(\alpha^2-\alpha)\Delta q_2 - (1-4\alpha+3\alpha^2)\Delta q_3 + \frac{6}{\ell}(\alpha-\alpha^2)\Delta q_5 - (3\alpha^2-2\alpha)\Delta q_6 \tag{7.13}$$

The initial displacement quantities would be interpolated by the same functions but with respect to the initial $\underset{\sim}{q}$'s. Note that the interpolation of Δw in Eq. 7.12 is the Hermitian interpolation function. Thus, one may write

$$\Delta\underset{\sim}{\sigma} = \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = \underset{\sim}{P}\underset{\sim}{\beta} - \begin{Bmatrix}N\\ M\end{Bmatrix} = \begin{Bmatrix}\underset{\sim}{P}_N\\ \underset{\sim}{P}_M\end{Bmatrix}\underset{\sim}{\beta} - \begin{Bmatrix}N\\ M\end{Bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -z & 1 & x\end{bmatrix}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} - \begin{Bmatrix}N\\ M\end{Bmatrix} \tag{7.14}$$

$$\underset{\sim}{S} = \begin{bmatrix}1/EA & 0\\ 0 & 1/EI\end{bmatrix} \tag{7.15}$$

$$\begin{Bmatrix}N+\Delta N\\ S+\Delta S\\ M+\Delta M\end{Bmatrix}_V = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -z & 1 & x\end{bmatrix}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} = \underset{\sim}{R}\,\underset{\sim}{\beta} \tag{7.16}$$

$$\begin{Bmatrix}\Delta\tilde{u}\\ \Delta\tilde{w}\\ -\Delta\tilde{w}_{,x}\end{Bmatrix} = \begin{bmatrix}1-\alpha & 0 & 0 & \alpha & 0 & 0\\ 0 & 1-3\alpha^2+2\alpha^3 & -\ell(\alpha-2\alpha^2+\alpha^3) & 0 & 3\alpha^2-2\alpha^3 & -\ell(\alpha^3-\alpha^2)\\ 0 & -6/\ell(\alpha^2-\alpha) & (1-4\alpha+3\alpha^2) & 0 & -6/\ell(\alpha-\alpha^2) & 3\alpha^2-2\alpha\end{bmatrix}\begin{Bmatrix}\Delta q_1\\ \Delta q_2\\ \vdots\\ \Delta q_6\end{Bmatrix} = \underset{\sim}{\tilde{L}}\,\Delta\underset{\sim}{q} \tag{7.17}$$

$$\Delta w_{,x} = \lfloor 0 \quad 6/\ell(\alpha^2-\alpha) \quad -(1-4\alpha+3\alpha^2) \quad 0 \quad 6/\ell(\alpha-\alpha^2) \quad -(3\alpha^2-2\alpha) \rfloor \{\Delta\underset{\sim}{q}\} = \underset{\sim}{L}'\,\Delta\underset{\sim}{q} \tag{7.18}$$

Placing Eqs. 7.14-7.18 into Eq. 6.50 where, from Eq. 6.53

$$\underset{\sim}{K}_T = (\underset{\sim}{G}+\underset{\sim}{C})^T \underset{\sim}{H}^{-1} (\underset{\sim}{G}+\underset{\sim}{C}) + \underset{\sim}{K}_q \tag{7.19}$$

$$\underset{\sim}{Q} = -(\underset{\sim}{G}+\underset{\sim}{C})^T \underset{\sim}{H}^{-1} (\underset{\sim}{G}+\tfrac{1}{2}\underset{\sim}{C}) \underset{\sim}{q} \tag{7.20}$$

Note that the integration by parts allows one to write $\underset{\sim}{C}_1$ of Eq. 6.53 as $\underset{\sim}{G}$ in Eq. 7.20 because for a beam these interpolation functions for displacements are continuous (assumption d). The terms of Eqs. 7.19 and 7.20 may be more simply defined as

$$\begin{aligned} \underset{\sim}{H} &= \int_0^\ell \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{P}\, dx \\ \underset{\sim}{G} &= [\underset{\sim}{P}^T \underset{\sim}{\tilde{L}}]_0^\ell \\ \underset{\sim}{C} &= \int_0^\ell \underset{\sim}{P}_N^T \underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}'\, dx = \underset{\sim}{P}_N^T \underset{\sim}{q}^T \int_0^\ell \underset{\sim}{L}'^T \underset{\sim}{L}'\, dx \\ \underset{\sim}{K}_q &= \int_0^\ell N \underset{\sim}{L}'^T \underset{\sim}{L}'\, dx = N \int_0^\ell \underset{\sim}{L}'^T \underset{\sim}{L}'\, dx \end{aligned} \tag{7.21}$$

Note that an assumption must be made for describing z, the shallow arch height. For convenience it was interpolated in the same fashion as w (for an element). Namely

$$z = \underset{\sim}{L}_w \, {}^L\underset{\sim}{q} \tag{7.22}$$

where

$\underset{\sim}{L}_w$ is given by the part of $\underset{\sim}{\tilde{L}}_w$ in Eq. 7.17 representing $\Delta\underset{\sim}{\tilde{w}}$

${}^L\underset{\sim}{q}$ = the local nodal $\underset{\sim}{q}$'s (from the base plane)

Once the local tangent stiffness and loads are calculated from Eqs. 7.19 and 7.20 respectively, they must be transformed and assembled as stated in Subsection 6.3.

Assuming the displacements are known (see Subsection 7.4) then the stresses may be determined by reducing Eq. 6.47 appropriately to

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1} (\underset{\sim}{G}+\tfrac{1}{2}\underset{\sim}{C}) \underset{\sim}{q} + \underset{\sim}{H}^{-1} (\underset{\sim}{G}+\underset{\sim}{C}) \Delta\underset{\sim}{q} \tag{7.23}$$

And finally Eq. 7.14 is utilized. Note that rather than calculating the incremental stress resultants one may calculate the total initial stress for the next increment by rewriting Eq. 7.14 as

$$\begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \underset{\sim}{P} \underset{\sim}{\beta} \tag{7.24}$$

From this and Eq. 7.15 the strain may be calculated from the constitutive law, Eq. 2.84, or simply

$$\begin{Bmatrix} e+\Delta e \\ -(K+\Delta K) \end{Bmatrix} = \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \underset{\sim}{S} \underset{\sim}{P} \underset{\sim}{\beta} \tag{7.25}$$

Note that while the displacements are not coupled, the stress resultants are in Eq. 7.8 by the curvature. If one wishes to use flat elements only then the value of z is zero.

$$z \equiv 0 \tag{7.26}$$

Placing Eq. 7.26 into all the appropriate equations above would yield a finite element model for flat elements. This further simplifies the matrices and no coupling at all occurs.

7.1.2 The Convected, Updated Lagrangian System

For the stationary system the differentiations and integrations are all performed with respect to the initial configuration. Kirchhoff stresses and Green strains are used. In this subsection the coordinates must be updated at each step. The initial stresses (stress resultants) are Cauchy quantities while incremental stresses are of the second Kirchhoff type. Additionally, initial strains are Almansi values while incrementally updated Green strains which are referred to the updated configuration, are used. For small strain, moderate rotations, the differentiation of stresses is not critical but should be recognized. The same symbols will be used here as those of the previous subsections.

7.1.2.1 The Consistent Assumed Stress Hybrid Model

With the same assumptions of Subsection 7.1, the consistent functional for the updated system, Eq. 6.54, becomes for the one dimensional shallow beam case

$$\Pi^c_{mc_n} = \int_0^\ell \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} - \frac{1}{2} N \Delta w_{,x}^2 \right] dx$$
$$+ \left[(N+\Delta N)\,\Delta\tilde{u} + (S+\Delta S)\,\Delta\tilde{w} - (M+\Delta M)\,\Delta\tilde{w}_{,x} \right]_0^\ell$$
$$- \int_0^\ell \left[\frac{\Delta N (N)}{EA} + \frac{\Delta M (M)}{EI} \right] dx$$
$$+ \int_0^\ell \left[\Delta N (u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} - \frac{1}{2} \Delta N\, w_{,x}^2 \right] dx \qquad (7.27)$$

where here

$$(S+\Delta S) = (M+\Delta M)_{,x} + (N+\Delta N)\, z_{,x} + N\, \Delta w_{,x} \qquad (7.28)$$

For the consistent case, Eq. 7.27 is subject to

$$(N+\Delta N)_{,x} = 0 \qquad (7.7)$$

and

$$(M+\Delta M)_{,xx} + \left[(N+\Delta N) z_{,x} \right]_{,x} + (N\Delta w_{,x})_{,x} = 0 \qquad (7.29)$$

If the linear portion of the last integral in Eq. 7.27 is integrated by parts and Eq. 7.29 is made use of, then

$$\int_0^\ell \left[\Delta N (u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} \right] dx$$
$$= \int_0^\ell (N \Delta w_{,x})_{,x}\, w\, dx + \left[\Delta N u + (\Delta M_{,x} + \Delta N z_{,x}) w - \Delta M w_{,x} \right]_0^\ell \qquad (7.30)$$

Integrating the integral on the right hand side of Eq. 7.30 by parts yields, finally

$$\int_0^\ell \left[\Delta N (u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} \right] dx$$
$$= - \int_0^\ell N\, w_{,x}\, \Delta w_{,x}\, dx + \left[\Delta N u + \Delta S w - \Delta M w_{,x} \right]_0^\ell \qquad (7.31)$$

Placing this into Eq. 7.27 gives

$$\Pi^{c}_{mc_n} = \int_0^{\ell} \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} - N\,w_{,x}\Delta w_{,x} - \frac{1}{2} N \Delta w_{,x}^2 \right] dx$$
$$+ \left[(N+\Delta N)\Delta\tilde{u} + (S+\Delta S)\Delta\tilde{w} - (M+\Delta M)\Delta\tilde{w}_{,x} \right]_0^{\ell}$$
$$- \int_0^{\ell} \left[\frac{\Delta N (N)}{EA} + \frac{\Delta M (M)}{EI} + \frac{1}{2}\Delta N w_{,x}^2 \right] dx$$
$$+ \left[\Delta N u + \Delta S w - \Delta M w_{,x} \right]_0^{\ell} \tag{7.32}$$

Note that similarity between this expression and that of Eq. 7.5. It is to be understood that all initial displacement terms in Eq. 7.32 come from the compatibility check only. This is not the case with Eq. 7.5. This latter equation is written in this form for computational convenience only.

The stress resultants may now be expressed as

$$\Delta N = \beta_1 - N \tag{7.9}$$

and

$$\Delta M = -z_1\beta_1 + \beta_2 + x\beta_3 - N\Delta w - M \tag{7.33}$$

Note here that the constant of integration $\underset{\sim}{C}_I$, of Eq. 6.56 is taken to be zero. The stress equilibrium equations, Eqs. 7.7 and 7.29 are still satisfied. The displacements may be interpolated as before by Eqs. 7.11-7.13. In matrix form one may write

$$\Delta \underset{\sim}{\sigma} = \begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = \underset{\sim}{P}\,\underset{\sim}{\beta} + \underset{\sim}{A}\,\Delta \underset{\sim}{q} - \begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{P}_N \\ \underset{\sim}{P}_M \end{Bmatrix} \underset{\sim}{\beta} + \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{A}_M \end{Bmatrix} \Delta \underset{\sim}{q} - \begin{Bmatrix} N \\ M \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -z & 1 & x \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{0} \\ -N\,\underset{\sim}{L}_w \end{bmatrix} \begin{Bmatrix} \Delta q_1 \\ \Delta q_2 \\ \vdots \\ \Delta q_6 \end{Bmatrix} - \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{7.34}$$

where

$$\Delta w = \underset{\sim}{L}_w\,\Delta \underset{\sim}{q} \tag{7.35}$$

and $\underset{\sim}{L}_w$ may be determined by comparing Eq. 7.35 to Eq. 7.12. Also

$$\begin{Bmatrix} N+\Delta N \\ S+\Delta S \\ M+\Delta M \end{Bmatrix}_{\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -z & 1 & x \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ -N\underset{\sim}{L}_w \end{bmatrix} \begin{Bmatrix} \Delta q_1 \\ \Delta q_2 \\ \vdots \\ \Delta q_6 \end{Bmatrix} = \underset{\sim}{R}\underset{\sim}{\beta} + \underset{\sim}{A}_{\nu}^{*} \Delta \underset{\sim}{q} \tag{7.36}$$

It should be pointed out that the general boundary traction term is given by Eq. 5.47, namely

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + \sigma_{kj}\, \Delta u_{i,k}]\, \nu_j \tag{5.47}$$

For a shallow beam undergoing moderate rotations the boundary tractions may be written in terms of the axial load, shear, and moment at the nodes. Large deflection analysis under the above restrictions allows one to write these terms as (see Subsection 3.2)

$$\underset{\sim}{T} + \Delta\underset{\sim}{T} = \begin{Bmatrix} N+\Delta N \\ S+\Delta S \\ M+\Delta M \end{Bmatrix}_{\nu} = \begin{Bmatrix} N+\Delta N \\ (M+\Delta M)_{,x} + (N+\Delta N)z_{,x} + N\Delta W_{,x} \\ M+\Delta M \end{Bmatrix}_{\nu}$$

Thus, the nonlinear portion of Eq. 5.47 only applies to the shear term. As shown in Eq. 6.58 $\underset{\sim}{A}_{\nu}$ is normally derived from $\underset{\sim}{C}_I$ in the general case. Here, $\underset{\sim}{C}_I$ was chosen to be zero. However, the moment term of the boundary traction has the form on the interior (from Eq. 7.34)

$$M + \Delta M = \underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\,\Delta\underset{\sim}{q}$$

To evaluate this on the boundary so that the tractions may be obtained, equivalent matrices must exist, i.e.

$$(M+\Delta M)_{\nu} = \underset{\sim}{R}\underset{\sim}{\beta} + \underset{\sim}{A}_{\nu}^{*}\,\Delta \underset{\sim}{q}$$

where

$$\underset{\sim}{A}_{\nu}^{*} = \underset{\sim}{\nu}^{T}\underset{\sim}{A} \neq \underset{\sim}{A}_{\nu} \qquad \text{in general.}$$

Thus, to properly define $\underset{\sim}{M}$ here one should replace $\underset{\sim}{A}_{\nu}$ by $\underset{\sim}{A}_{\nu}^{*}$ in Eq. 6.64. Considering Eqs. 6.81 in light of the rewritten functional Eq. 7.32

$$\underset{\sim}{K}_T = (\underset{\sim}{G}-\underset{\sim}{H}_A)^T \underset{\sim}{H}^{-1} (\underset{\sim}{G}-\underset{\sim}{H}_A) + \underset{\sim}{M} + \underset{\sim}{M}^T - \underset{\sim}{D} - \underset{\sim}{K}_q \tag{7.37}$$

$$\underset{\sim}{Q} = -(\underset{\sim}{G} - \underset{\sim}{H}_A)^T \underset{\sim}{H}^{-1}(\underset{\sim}{G} - \tfrac{1}{2}\underset{\sim}{C})\underset{\sim}{q} - (\underset{\sim}{M} - \underset{\sim}{K}_g)\underset{\sim}{q} \tag{7.38}$$

where matrices in Eq. 6.81 are replaced by matrices in Eqs. 7.38 as follows

$$\begin{aligned} &\underset{\sim}{C}_1 \longrightarrow G \\ &\underset{\sim}{C}_2 - \tfrac{1}{2}\underset{\sim}{E} \longrightarrow \underset{\sim}{M} - \underset{\sim}{K}_g \end{aligned} \tag{7.39}$$

and $\underset{\sim}{K}_g$ in $\underset{\sim}{Q}$ is an additional term formed during the transformation process. One may observe that this is computationally more efficient than (while still exactly equivalent to) Eq. 6.81 since it requires the formation of fewer different matrices. These matrices may simply be evaluated as

$$\begin{aligned} \underset{\sim}{H}_A &= \int_0^\ell \underset{\sim}{P}^T \underset{\sim}{S} \underset{\sim}{A} \, dx &&= \int_0^\ell \begin{Bmatrix} \underset{\sim}{P}_N \\ \underset{\sim}{P}_M \end{Bmatrix}^T \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} 0 \\ -N\underset{\sim}{L}_w \end{Bmatrix} dx \\ \underset{\sim}{M} &= \left[\underset{\sim}{A}_v^{*T} \tilde{\underset{\sim}{L}}_w \right]_0^\ell &&= -N \left[\underset{\sim}{L}_w^T \underset{\sim}{L}' \right]_0^\ell \\ \underset{\sim}{D} &= \int_0^\ell \underset{\sim}{A}^T \underset{\sim}{S} \underset{\sim}{A} \, dx &&= N^2 \int_0^\ell \frac{1}{EI} \underset{\sim}{L}_w^T \underset{\sim}{L}_w \, dx \end{aligned} \tag{7.40}$$

and $\underset{\sim}{H}$, $\underset{\sim}{G}$, $\underset{\sim}{C}$, $\underset{\sim}{K}_g$ are defined by Eq. 7.21. Note that since for the beam problem $\underset{\sim}{u}+\Delta\underset{\sim}{u} = \tilde{\underset{\sim}{u}}+\Delta\tilde{\underset{\sim}{u}}$ this implies that $\tilde{\underset{\sim}{L}}_w = \underset{\sim}{L}_w$. Eqs. 7.37 and 7.38 must now be transformed and assembled into a global (common) set of equations.

With the displacements obtained from Eqs. 7.1 the stress parameters may be obtained by reducing Eq. 6.76 appropriately.

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}(\underset{\sim}{G} - \tfrac{1}{2}\underset{\sim}{C})\underset{\sim}{q} + \underset{\sim}{H}^{-1}(\underset{\sim}{G} - \underset{\sim}{H}_A)\Delta\underset{\sim}{q} \tag{7.41}$$

The total stress resultants may be determined by placing the displacement solution and Eq. 7.41 into Eq. 7.34.

$$\begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\,\Delta\underset{\sim}{q} \tag{7.42}$$

Total strains are calculated via the constitutive law

$$\begin{Bmatrix} e+\Delta e \\ -(K+\Delta K) \end{Bmatrix} = [\underset{\sim}{S}] \begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} [\underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\,\Delta\underset{\sim}{q}] \tag{7.43}$$

Applying Eq. 7.26 to all the above generates a consistent model for flat beam elements.

7.1.2.2 The Inconsistent Assumed Stress Hybrid Model

The inconsistent functional for an updated system (Eq. 6.82) becomes, under the previously stated assumptions, for a one dimensional shallow beam element

$$\begin{aligned}\Pi^{I}_{mc_n} = &\int_0^{\ell}\left[-\frac{1}{2}\frac{\Delta N^2}{EA}-\frac{1}{2}\frac{\Delta M^2}{EI}+\frac{1}{2}N\,\Delta w_{,x}^2\right]dx \\ &+\left[(N+\Delta N)\,\Delta\tilde{u}+(S+\Delta S)\,\Delta\tilde{w}-(M+\Delta M)\,\Delta\tilde{w}_{,x}\right]_0^{\ell} \\ &-\int_0^{\ell}\left[\frac{\Delta N(N)}{EA}+\frac{\Delta M(M)}{EI}\right]dx \\ &+\int_0^{\ell}\left[\Delta N(u_{,x}+z_{,x}w_{,x})-\Delta M\,w_{,xx}-\frac{1}{2}\Delta N\,w_{,x}^2\right]dx \end{aligned} \tag{7.44}$$

The same form of the stress equilibrium equations and shear term apply here as for the stationary system. (They are not functions of initial displacements.) These are Eqs. 7.7, 7.8 and Eq. 7.3 respectively. Integrating by parts the linear terms of the last integral in Eq. 7.44 yields the same result as Eq. 7.4. The same interpolation functions, Eqs. 7.14-7.18, are also used.

Placing these results into Eq. 6.101 they become

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T\underset{\sim}{H}^{-1}\underset{\sim}{G} + \underset{\sim}{K}_g \tag{7.45}$$

$$\underset{\sim}{Q} = -\underset{\sim}{G}^T\underset{\sim}{H}^{-1}(\underset{\sim}{G}-\tfrac{1}{2}\underset{\sim}{C})\underset{\sim}{q} \tag{7.46}$$

where again $\underset{\sim}{C}_1$ from Eq. 6.101 transforms to $\underset{\sim}{G}$ in Eq. 7.46. The definitions of Eqs. 7.45 and 7.46 are similar to those of Eqs. 7.21. The difference being that the latter equations are always referred to an updated coordinate system. Comparing Eqs. 7.45 and 7.46 to Eqs. 7.19 and 7.20 one observes that the forms are similar except for the initial displacement terms in the tangent stiffness of the latter. Although the equivalent load vector in the updated system is different than that of the stationary system, it is still a function of the initial displacements due to the compatibility check. Note the change in sign of the $\frac{1}{2}\underset{\sim}{C}\underset{\sim}{q}$ term in Eqs. 7.20 and 7.46. This is due to the fact that Eq. 7.20 is based on Green strain while Eq. 7.46 is based on Almansi strain.

$$\underset{\sim}{Q} = -(\underset{\sim}{G}-\underset{\sim}{H}_A)^T\underset{\sim}{H}^{-1}(\underset{\sim}{G}-\tfrac{1}{2}\underset{\sim}{C})\underset{\sim}{q} - (\underset{\sim}{M}-\underset{\sim}{K}_g)\underset{\sim}{q} \tag{7.38}$$

where matrices in Eq. 6.81 are replaced by matrices in Eqs. 7.38 as follows

$$\begin{aligned} &\underset{\sim}{C}_1 \longrightarrow G \\ &\underset{\sim}{C}_2 - \tfrac{1}{2}\underset{\sim}{E} \longrightarrow \underset{\sim}{M} - \underset{\sim}{K}_g \end{aligned} \tag{7.39}$$

and $\underset{\sim}{K}_g$ in $\underset{\sim}{Q}$ is an additional term formed during the transformation process. One may observe that this is computationally more efficient than (while still exactly equivalent to) Eq. 6.81 since it requires the formation of fewer different matrices. These matrices may simply be evaluated as

$$\begin{aligned} \underset{\sim}{H}_A &= \int_0^\ell \underset{\sim}{P}^T\underset{\sim}{S}\underset{\sim}{A}\,dx &&= \int_0^\ell \begin{Bmatrix} \underset{\sim}{P}_N \\ \underset{\sim}{P}_M \end{Bmatrix}^T \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} 0 \\ -N\underset{\sim}{L}_w \end{Bmatrix} dx \\ \underset{\sim}{M} &= \left[\underset{\sim}{A}_v^{*T}\tilde{\underset{\sim}{L}}_w\right]_0^\ell &&= -N\left[\underset{\sim}{L}_w^T \underset{\sim}{L}'\right]_0^\ell \\ \underset{\sim}{D} &= \int_0^\ell \underset{\sim}{A}^T\underset{\sim}{S}\underset{\sim}{A}\,dx &&= N^2\int_0^\ell \frac{1}{EI}\underset{\sim}{L}_w^T \underset{\sim}{L}_w\,dx \end{aligned} \tag{7.40}$$

and $\underset{\sim}{H}$, $\underset{\sim}{G}$, $\underset{\sim}{C}$, $\underset{\sim}{K}_g$ are defined by Eq. 7.21. Note that since for the beam problem $\underset{\sim}{u}+\Delta\underset{\sim}{u} = \tilde{\underset{\sim}{u}}+\Delta\tilde{\underset{\sim}{u}}$ this implies that $\tilde{\underset{\sim}{L}}_w = \underset{\sim}{L}_w$. Eqs. 7.37 and 7.38 must now be transformed and assembled into a global (common) set of equations.

With the displacements obtained from Eqs. 7.1 the stress parameters may be obtained by reducing Eq. 6.76 appropriately.

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}(\underset{\sim}{G}-\tfrac{1}{2}\underset{\sim}{C})\underset{\sim}{q} + \underset{\sim}{H}^{-1}(\underset{\sim}{G}-\underset{\sim}{H}_A)\Delta\underset{\sim}{q} \tag{7.41}$$

The total stress resultants may be determined by placing the displacement solution and Eq. 7.41 into Eq. 7.34.

$$\begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\Delta\underset{\sim}{q} \tag{7.42}$$

Total strains are calculated via the constitutive law

$$\begin{Bmatrix} e+\Delta e \\ -(K+\Delta K) \end{Bmatrix} = [\underset{\sim}{S}]\begin{Bmatrix} N+\Delta N \\ M+\Delta M \end{Bmatrix} = \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix}\left\{\underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\Delta\underset{\sim}{q}\right\} \tag{7.43}$$

Applying Eq. 7.26 to all the above generates a consistent model for flat beam elements.

7.1.2.2 The Inconsistent Assumed Stress Hybrid Model

The inconsistent functional for an updated system (Eq. 6.82) becomes, under the previously stated assumptions, for a one dimensional shallow beam element

$$\begin{aligned}\Pi^{I}_{mc_n} = & \int_0^{\ell}\left[-\frac{1}{2}\frac{\Delta N^2}{EA}-\frac{1}{2}\frac{\Delta M^2}{EI}+\frac{1}{2}N\,\Delta w_{,x}^2\right]dx \\ & +\left[(N+\Delta N)\,\Delta\tilde{u}+(S+\Delta S)\,\Delta\tilde{w}-(M+\Delta M)\,\Delta\tilde{w}_{,x}\right]_0^{\ell} \\ & -\int_0^{\ell}\left[\frac{\Delta N(N)}{EA}+\frac{\Delta M(M)}{EI}\right]dx \\ & +\int_0^{\ell}\left[\Delta N(u_{,x}+z_{,x}w_{,x})-\Delta M\,w_{,xx}-\frac{1}{2}\Delta N\,w_{,x}^2\right]dx\end{aligned} \tag{7.44}$$

The same form of the stress equilibrium equations and shear term apply here as for the stationary system. (They are not functions of initial displacements.) These are Eqs. 7.7, 7.8 and Eq. 7.3 respectively. Integrating by parts the linear terms of the last integral in Eq. 7.44 yields the same result as Eq. 7.4. The same interpolation functions, Eqs. 7.14-7.18, are also used.

Placing these results into Eq. 6.101 they become

$$\underset{\sim}{K}_T = \hat{\underset{\sim}{G}}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} + \underset{\sim}{K}_g \tag{7.45}$$

$$\underset{\sim}{Q} = -\hat{\underset{\sim}{G}}^T \underset{\sim}{H}^{-1} \left(\underset{\sim}{G}-\frac{1}{2}\underset{\sim}{C}\right)\underset{\sim}{q} \tag{7.46}$$

where again $\underset{\sim}{C}_1$ from Eq. 6.101 transforms to $\underset{\sim}{G}$ in Eq. 7.46. The definitions of Eqs. 7.45 and 7.46 are similar to those of Eqs. 7.21. The difference being that the latter equations are always referred to an updated coordinate system. Comparing Eqs. 7.45 and 7.46 to Eqs. 7.19 and 7.20 one observes that the forms are similar except for the initial displacement terms in the tangent stiffness of the latter. Although the equivalent load vector in the updated system is different than that of the stationary system, it is still a function of the initial displacements due to the compatibility check. Note the change in sign of the $\frac{1}{2}\underset{\sim}{C}\underset{\sim}{q}$ term in Eqs. 7.20 and 7.46. This is due to the fact that Eq. 7.20 is based on Green strain while Eq. 7.46 is based on Almansi strain.

$$Q = -(G - H_A)^T H^{-1} (G - \tfrac{1}{2} C) q - (M - K_g) q \tag{7.38}$$

where matrices in Eq. 6.81 are replaced by matrices in Eqs. 7.38 as follows

$$\begin{aligned} C_1 &\longrightarrow G \\ C_2 - \tfrac{1}{2} E &\longrightarrow M - K_g \end{aligned} \tag{7.39}$$

and K_g in Q is an additional term formed during the transformation process. One may observe that this is computationally more efficient than (while still exactly equivalent to) Eq. 6.81 since it requires the formation of fewer different matrices. These matrices may simply be evaluated as

$$\begin{aligned} H_A &= \int_0^l P^T S A \, dx &&= \int_0^l \begin{Bmatrix} P_N \\ P_M \end{Bmatrix}^T \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \begin{Bmatrix} 0 \\ -N L_w \end{Bmatrix} dx \\ M &= \left[A_v^{*T} \tilde{L}_w \right]_0^l &&= -N \left[L_w^T L' \right]_0^l \\ D &= \int_0^l A^T S A \, dx &&= N^2 \int_0^l \frac{1}{EI} L_w^T L_w \, dx \end{aligned} \tag{7.40}$$

and E, G, C, K_g are defined by Eq. 7.21. Note that since for the beam problem $u+\Delta u = \tilde{u}+\Delta\tilde{u}$ this implies that $\tilde{L}_w = L_w$. Eqs. 7.37 and 7.38 must now be transformed and assembled into a global (common) set of equations.

With the displacements obtained from Eqs. 7.1 the stress parameters may be obtained by reducing Eq. 6.76 appropriately.

$$\beta = H^{-1} (G - \tfrac{1}{2} C) q + H^{-1} (G - H_A) \Delta q \tag{7.41}$$

The total stress resultants may be determined by placing the displacement solution and Eq. 7.41 into Eq. 7.34.

$$\begin{Bmatrix} N + \Delta N \\ M + \Delta M \end{Bmatrix} = P \beta + A \Delta q \tag{7.42}$$

Total strains are calculated via the constitutive law

$$\begin{Bmatrix} e + \Delta e \\ -(K + \Delta K) \end{Bmatrix} = [S] \begin{Bmatrix} N + \Delta N \\ M + \Delta M \end{Bmatrix} = \begin{bmatrix} 1/EA & 0 \\ 0 & 1/EI \end{bmatrix} \{ P \beta + A \Delta q \} \tag{7.43}$$

Applying Eq. 7.26 to all the above generates a consistent model for flat beam elements.

7.1.2.2 The Inconsistent Assumed Stress Hybrid Model

The inconsistent functional for an updated system (Eq. 6.82) becomes, under the previously stated assumptions, for a one dimensional shallow beam element

$$\begin{aligned}\Pi^{I}_{mc_n} &= \int_0^{\ell} \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} + \frac{1}{2} N\, \Delta w_{,x}^2 \right] dx \\ &+ \left[(N+\Delta N)\,\Delta\tilde{u} + (S+\Delta S)\,\Delta\tilde{w} - (M+\Delta M)\,\Delta\tilde{w}_{,x} \right]_0^{\ell} \\ &- \int_0^{\ell} \left[\frac{\Delta N(N)}{EA} + \frac{\Delta M(M)}{EI} \right] dx \\ &+ \int_0^{\ell} \left[\Delta N(u_{,x} + z_{,x} w_{,x}) - \Delta M\, w_{,xx} - \frac{1}{2}\Delta N\, w_{,x}^2 \right] dx \end{aligned} \tag{7.44}$$

The same form of the stress equilibrium equations and shear term apply here as for the stationary system. (They are not functions of initial displacements.) These are Eqs. 7.7, 7.8 and Eq. 7.3 respectively. Integrating by parts the linear terms of the last integral in Eq. 7.44 yields the same result as Eq. 7.4. The same interpolation functions, Eqs. 7.14-7.18, are also used.

Placing these results into Eq. 6.101 they become

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} + \underset{\sim}{K}_g \tag{7.45}$$

$$\underset{\sim}{Q} = -\underset{\sim}{G}^T \underset{\sim}{H}^{-1} (\underset{\sim}{G} - \tfrac{1}{2}\underset{\sim}{C})\, \underset{\sim}{q} \tag{7.46}$$

where again $\underset{\sim}{C}_1$ from Eq. 6.101 transforms to $\underset{\sim}{G}$ in Eq. 7.46. The definitions of Eqs. 7.45 and 7.46 are similar to those of Eqs. 7.21. The difference being that the latter equations are always referred to an updated coordinate system. Comparing Eqs. 7.45 and 7.46 to Eqs. 7.19 and 7.20 one observes that the forms are similar except for the initial displacement terms in the tangent stiffness of the latter. Although the equivalent load vector in the updated system is different than that of the stationary system, it is still a function of the initial displacements due to the compatibility check. Note the change in sign of the $\frac{1}{2}\underset{\sim}{C}\underset{\sim}{q}$ term in Eqs. 7.20 and 7.46. This is due to the fact that Eq. 7.20 is based on Green strain while Eq. 7.46 is based on Almansi strain.

Although it may appear that the updated system would have small computational advantages over the stationary system because it has fewer terms this may not be the case. This is further explored in Subsection 7.4.

7.1.3 Comments on Kirchhoff-Love Theory

The equations of elasticity are slightly different under Kirchhoff-Love theory as opposed to those of Marguerre theory. (See Subsection 3.2.2.) The basic reason for this change is that for the former theory displacements, more like true shell theory, are measured in the shell coordinates, while in the latter case the displacements are measured in the base plane. Thus, the strain displacement relations and expressions for shear resultants change.

As an example, consider $\pi^{I}_{mc_n}$ for an updated system. Eq. 7.44 was written for Marguerre theory. Under Kirchhoff-Love theory (with the same assumptions) it would appear as

$$\begin{aligned}\pi^{I}_{mc_n} = & \int_0^l \left[-\frac{1}{2}\frac{\Delta N^2}{EA} - \frac{1}{2}\frac{\Delta M^2}{EI} + \frac{1}{2} N \Delta w_{,x}^2 \right] dx \\ & + \left[(N+\Delta N)\,\Delta\tilde{u} + (S+\Delta S)\,\Delta\tilde{w} - (M+\Delta M)\,\Delta\tilde{w}_{,x} \right]_0^l \\ & - \int_0^l \left[\frac{\Delta N(N)}{EA} + \frac{\Delta M(M)}{EI} \right] dx \\ & + \int_0^l \left[\Delta N(u_{,x} - z_{,xx}w) - \Delta M w_{,xx} - \frac{1}{2}\Delta N w_{,x}^2 \right] dx\end{aligned} \tag{7.47}$$

where

$$(S+\Delta S) = (M+\Delta M)_{,x} \tag{7.48}$$

Note that although quantities are measured in the shell system, differentiation and integration are still performed in the base plane as shallow shell theory permits.

Thus, for this case, the major differences are in the evaluation of the boundary shear term, the directions of the local displacements, and, correspondingly the assembly procedure as indicated in Subsection 3.3.3. To be precise the stress equilibrium equations are also slightly different as one observes by comparing Eqs. 3.28 to Eqs. 3.38. However, if the membrane stress resultant is constant within an element, as it is here, then the form of the equations are identical.

Integrating the linear terms of the last integral of Eq. 7.47 by parts yields

$$\int_0^\ell [\Delta N(u_{,x} - z_{,xx} w) - \Delta M w_{,xx}] dx$$
$$= -\int_0^\ell [\Delta N_{,x} u + \Delta N z_{,xx} w - \Delta M_{,x} w_{,x}] dx$$
$$+ [\Delta N u - \Delta M w_{,x}]_0^\ell$$
$$= -\int_0^\ell [\Delta N_{,x} u + (\Delta M_{,xx} + \Delta N z_{,xx}) w] dx$$
$$+ [\Delta N u + \Delta M_{,x} w - \Delta M w_{,x}]_0^\ell \tag{7.49}$$

Eq. 7.47 is subject to

$$(N+\Delta N)_{,x} = 0 \tag{7.7}$$

$$(M+\Delta M)_{,xx} + (N+\Delta N) z_{,xx} = 0 \tag{7.50}$$

Therefore, the inconsistent functional becomes

$$\pi^I_{mc_n} = \int_0^\ell [-\tfrac{1}{2}\frac{\Delta N^2}{EA} - \tfrac{1}{2}\frac{\Delta M^2}{EI} + \tfrac{1}{2} N \Delta w_{,x}^2] dx$$
$$+ [(N+\Delta N)\Delta\tilde{u} + (S+\Delta S)\Delta\tilde{w} - (M+\Delta M)\Delta\tilde{w}_{,x}]_0^\ell$$
$$- \int_0^\ell [\frac{\Delta N(N)}{EA} + \frac{\Delta M(M)}{EI} - \tfrac{1}{2}\Delta N w_{,x}^2] dx$$
$$+ [\Delta N u + \Delta S w - \Delta M w_{,x}]_0^\ell \tag{7.51}$$

Thus, the tangent stiffness and equivalent loads would be determined in the same manner as Eqs. 7.45 and 7.46 except that $\underset{\sim}{R}$ would be defined differently.

$$\begin{Bmatrix} N+\Delta N \\ S+\Delta S \\ M+\Delta M \end{Bmatrix}, = \begin{bmatrix} 1 & 0 & 0 \\ -z_{,x} & 0 & 1 \\ -z & 1 & x \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \underset{\sim}{R}\,\underset{\sim}{\beta} \tag{7.52}$$

and

$$\underset{\sim}{G} = [\underset{\sim}{R}^T \underset{\sim}{\tilde{L}}]_0^\ell$$

Of course, the directions they correspond to (the dofs) would also be different so that an appropriate assembly procedure would be necessary.

7.2 Shallow Shell (Marguerre) and Flat Plate Elements

In this subsection the first three assumptions stated in Subsection 7.1 will be observed. Also, the translational dofs will be compatible while the rotations are not. In addition, only the inconsistent models will be discussed for the two dimensional cases. As shown in the previous subsection, the consistent models require more matrix generation and, therefore, more computational effort. As will be seen in Section 8 the additional accuracy obtained by these models is negligible. Thus, the consistent models were abandoned here.

The equations shall be generated for shallow elements so by simple reduction flat elements may be obtained. Being a more straight forward extension of flat plate theory only Marguerre theory will be used. This allows the same form of transformation and assembly procedure for both flat and shallow elements. A three node flat triangular element and a doubly curved three node shallow shell element will be utilized.

7.2.1 The Linear Stress Equilibrium Equations

The inconsistent models require that only the linear portion of the stress equilibrium equations be satisfied exactly. While for flat elements this is straight forward, for shallow shells some choices become obvious. Consider the stress equilibrium equation for a shallow shell under Marguerre theory.

$$
\begin{aligned}
&N_{x,x} + N_{xy,y} + \bar{p}_x = 0 \\
&N_{y,y} + N_{xy,x} + \bar{p}_y = 0 \\
&M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + [N_x(w_{,x} + z_{,x}) + N_{xy}(w_{,y} + z_{,y})]_{,x} \\
&\qquad + [N_y(w_{,y} + z_{,y}) + N_{xy}(w_{,x} + z_{,x})]_{,y} + \bar{p}_z = 0
\end{aligned}
\tag{3.38}
$$

For the case at hand all body loads are lumped at nodes so that they would not appear in these equations. The inplane stress resultants simply must satisfy

$$N_{x,x} + N_{xy,y} = 0$$
$$N_{y,y} + N_{xy,x} = 0 \tag{7.53}$$

For the bending stress resultants the nonlinear portions must be removed (and placed back into the functional) yielding

$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + [N_x z_{,x} + N_{xy} z_{,y}]_{,x} + [N_y z_{,y} + N_{xy} z_{,x}]_{,y} = 0 \tag{7.54}$$

For a flat plate, z≡0 and there is no choice involved

$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} = 0 \tag{7.55}$$

However, since the total stress equilibrium equation Eq. 3.38 is not being satisfied, then there is no reason that the entire linear portion, Eq. 7.54, must be satisfied. Thus, two choices become obvious. Either one satisfies Eq. 7.54 or Eq. 7.55. In the latter case the terms removed from the stress equilibrium equation must be replaced in the functional.

For these reasons it is actually more convenient to consider the modified Reissner principle with certain stress equilibrium constraints. This way the proper terms remain in the functionals.

7.2.2 The Stationary Lagrangian System

The functional to be considered here is Eq. 4.50. If the linear portion of the compatibility check is integrated by parts, if Eq. 4.47 is satisfied exactly, and if the above assumptions are made, then Eq. 4.50 may be written for an element as

$$\begin{aligned}\Pi_{mR_n} = \int_{V_n} \Big[&-B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij})(u_{k,i}\,\Delta u_{k,j} + \Delta u_{k,i}\,u_{k,j}) \\ &+ \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\Big] dV - \int_{V_n} [(\sigma_{ij} + \Delta\sigma_{ij})_{,j}\,\Delta u_i]\, dV \\ &+ \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\nu_j\,\Delta\bar{u}_i\, ds - \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}u_{k,i}u_{k,j}]\, dV \\ &- \int_{V_n} \Delta\sigma_{ij,j}\,u_i\, dV + \int_{\partial V_n} \Delta\sigma_{ij}\,\nu_j\,\tilde{u}_i\, ds\end{aligned} \tag{7.56}$$

Note that the displacement discontinuity terms on ∂V_n vanish. As shown for the beam problem, in Subsection 7.1.2.1, the only nonlinear boundary traction terms come from the shear. Since the shear is multiplied by the transverse displacement, w, and these displacements are compatible then all nonlinear boundary terms vanish. (This implies that the matrices involving $\underset{\sim}{A}_b$, $\underset{\sim}{B}_b$, $\underset{\sim}{B}_\sigma$, and $\underset{\sim}{B}_{\sigma_p}$ vanish.) Computationally this implies a substantial savings. If only Eq. 7.55 is satisfied then (for an isotropic material) Eq. 7.56 may be written as

$$
\begin{aligned}
\pi_{mR_n} = & -\frac{1}{2}\int_{A_n}\Big\{\frac{1}{Eh}\big[(\Delta N_x+\Delta N_y)^2+2(1+\nu)(\Delta N_{xy}^2-\Delta N_x\Delta N_y)\big] \\
& +\frac{12}{Eh^3}\big[(\Delta M_x+\Delta M_y)^2+2(1+\nu)(\Delta M_{xy}^2-\Delta M_x\Delta M_y)\big]\Big\}dx\,dy \\
& +\int_{A_n}\Big\{(N_x+\Delta N_x)(w_{,x}\Delta w_{,x})+(N_y+\Delta N_y)(w_{,y}\Delta w_{,y}) \\
& \quad +(N_{xy}+\Delta N_{xy})(w_{,x}\Delta w_{,y}+w_{,y}\Delta w_{,x})\Big\}dx\,dy \\
& +\frac{1}{2}\int_{A_n}\Big\{N_x\Delta w_{,x}^2+N_y\Delta w_{,y}^2+2N_{xy}\Delta w_{,x}\Delta w_{,y}\Big\}dx\,dy \\
& -\int_{A_n}\Big\{\big[(N_x+\Delta N_x)z_{,x}+(N_{xy}+\Delta N_{xy})z_{,y}\big]_{,x} \\
& \quad +\big[(N_y+\Delta N_y)z_{,y}+(N_{xy}+\Delta N_{xy})z_{,x}\big]_{,y}\Big\}\Delta w\,dx\,dy \\
& +\int_{C_n}\Big\{(N_x+\Delta N_x)_\nu\Delta\tilde{u}+(N_y+\Delta N_y)_\nu\Delta\tilde{v}+(S+\Delta S)_\nu\Delta\tilde{w} \\
& \quad -(M_x+\Delta M_x)_\nu\Delta\tilde{w}_{,x}-(M_y+\Delta M_y)_\nu\Delta\tilde{w}_{,y}\Big\}ds \\
& -\int_{A_n}\Big\{\Delta N_x e_{xx_0}+\Delta N_y e_{yy_0}+\Delta N_{xy}e_{xy_0} \\
& \quad -\Delta M_x K_{xx}-\Delta M_y K_{yy}-\Delta M_{xy}K_{xy}\Big\}dx\,dy \\
& +\frac{1}{2}\int_{A_n}\Big\{\Delta N_x w_{,x}^2+\Delta N_y w_{,y}^2+2\Delta N_{xy}w_{,x}w_{,y}\Big\}dx\,dy \\
& -\int_{A_n}\Big\{\big[\Delta N_x z_{,x}+\Delta N_{xy}z_{,y}\big]_{,x}+\big[\Delta N_y z_{,y}+\Delta N_{xy}z_{,x}\big]_{,y}\Big\}w\,dx\,dy \\
& +\int_{C_n}\Big\{\Delta N_{x_\nu}\tilde{u}+\Delta N_{y_\nu}\tilde{v}+\Delta S_\nu\tilde{w}-\Delta M_{x_\nu}\tilde{w}_{,x}-\Delta M_{y_\nu}\tilde{w}_{,y}\Big\}ds
\end{aligned}
\tag{7.57}
$$

where

$$S_\nu = Q_{x_\nu}+Q_{y_\nu}+N_{x_\nu}z_{,x}+N_{y_\nu}z_{,y} \tag{7.58}$$

See Fig. 7.2 for sign conventions.

Interpolation functions must now be established in terms of the unknowns $\tilde{\beta}$ and $\Delta\tilde{q}$. It has been demonstrated that flat triangular elements by assumed stress hybrid models tend to be too stiff [Mau and Witmer, 1972]. It has also been shown that for a given number of displacements parameters, $\Delta\tilde{q}$, such an element can be made more flexible by reducing the number of stress parameters $\tilde{\beta}$ [Pian and Tong, 1969b]. However, there is a condition which governs the minimum number of $\tilde{\beta}$'s which can be used [Pian and Tong, 1969a]. If this condition is violated then kinematic modes may exist leading to erroneous results [Spilker, 1972]. The relation may be expressed as

$$N_\beta \geq N_{\Delta q} - N_c \tag{7.59}$$

where

N_β = number of stress parameters

$N_{\Delta q}$ = number of displacement parameters

N_c = number of rigid body constraints

Since a flat triangular element is to be considered then the stresses and displacements in and out of plane completely decouple. Thus, Eq. 7.59 must be applied to each set, i.e.

$$\begin{aligned} &\text{Inplane:} \quad N_\beta \geq 6-3=3 \\ &\text{Bending:} \quad N_\beta \geq 9-3=6 \end{aligned} \tag{7.60}$$

Therefore, the minimum condition for membrane stress resultants is constant, while for bending stress resultants at least three β's would have to be added to constant values. To maintain symmetry it was decided that a constant membrane resultant and fully linear bending resultant would be used. Thus, using Eqs. 7.53 and 7.55 one may write

$$\begin{Bmatrix} N_x + \Delta N_x \\ N_y + \Delta N_y \\ N_{xy} + \Delta N_{xy} \\ M_x + \Delta M_x \\ M_y + \Delta M_y \\ M_{xy} + \Delta M_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & 1 & & & X & & & Y & & \\ & & & & 1 & & & X & & & Y & \\ & & & & & 1 & & & X & & & Y \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_{12} \end{Bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{P}_N & \\ & \underset{\sim}{P}_M \end{bmatrix} \left\{ \underset{\sim}{\beta} \right\} = \underset{\sim}{P}\,\underset{\sim}{\beta} \tag{7.61}$$

where blank entries are zero. Note that this automatically satisfies Eqs. 7.53 and 7.55.

Interpolations for displacement quantities are more easily generated using the natural triangular (or area) coordinates, ξ_i [Zienkiewicz, 1971]. They are related to the local x,y coordinates as follows. (See Fig. 7.3.)

$$\begin{aligned} x &= \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \\ y &= \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 \\ 1 &= \xi_1 + \xi_2 + \xi_3 \end{aligned} \tag{7.62}$$

or, the unique inverse relations are

$$\begin{aligned} \xi_1 &= (a_1 + b_1 x + c_1 y)/2A \\ \xi_2 &= (a_2 + b_2 x + c_2 y)/2A \\ \xi_3 &= (a_3 + b_3 x + c_3 y)/2A \end{aligned} \tag{7.63}$$

where

x_i, y_i = nodal values of x and y respectively

$a_i = x_j y_k - x_k y_j$

$b_i = y_j - y_k$

$c_i = x_k - x_j$

$i,j,k = (1,2,3) \quad (i \neq j \neq k)$

A = area of triangle

and the values of a_i, b_i, c_i are determined by permutation of indices.

The inplane displacements are interpolated in a linear fashion on the interior. (See Fig. 7.2.)

$$\begin{aligned} \Delta u &= \xi_1 \Delta q_1 + \xi_2 \Delta q_6 + \xi_3 \Delta q_{11} = \underset{\sim}{L}_u \Delta \underset{\sim}{q} \\ \Delta v &= \xi_1 \Delta q_2 + \xi_2 \Delta q_7 + \xi_3 \Delta q_{12} = \underset{\sim}{L}_v \Delta \underset{\sim}{q} \end{aligned} \tag{7.64}$$

The interior transverse displacements are interpolated as cubic functions [Zienkiewicz, 1971].

$$\Delta w = \left\{ \begin{array}{l} \xi_1 + \xi_1^2 \xi_2 + \xi_1^2 \xi_3 - \xi_1 \xi_2^2 - \xi_1 \xi_3^2 \\ b_2 (\xi_3 \xi_1^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - b_3 (\xi_1^2 \xi_2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \\ c_2 (\xi_3 \xi_1^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - c_3 (\xi_1^2 \xi_2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \\ \xi_2 + \xi_2^2 \xi_3 + \xi_2^2 \xi_1 - \xi_2 \xi_3^2 - \xi_2 \xi_1^2 \\ b_3 (\xi_1 \xi_2^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - b_1 (\xi_2^2 \xi_3 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \\ c_3 (\xi_1 \xi_2^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - c_1 (\xi_2^2 \xi_3 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \\ \xi_3 + \xi_3^2 \xi_1 + \xi_3^2 \xi_2 - \xi_3 \xi_1^2 - \xi_3 \xi_2^2 \\ b_1 (\xi_2 \xi_3^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - b_2 (\xi_3^2 \xi_1 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \\ c_1 (\xi_2 \xi_3^2 + \frac{1}{2} \xi_1 \xi_2 \xi_3) - c_2 (\xi_3^2 \xi_1 + \frac{1}{2} \xi_1 \xi_2 \xi_3) \end{array} \right\}^T \left\{ \begin{array}{l} \Delta q_3 \\ \Delta q_4 \\ \Delta q_5 \\ \Delta q_8 \\ \Delta q_9 \\ \Delta q_{10} \\ \Delta q_{13} \\ \Delta q_{14} \\ \Delta q_{15} \end{array} \right\}$$

$$= \underset{\sim}{L}_w \Delta \underset{\sim}{q} \tag{7.65}$$

The derivatives of Δw with respect to x and y may be obtained by chain rule.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi_1} \frac{\partial \xi_1}{\partial x} + \frac{\partial}{\partial \xi_2} \frac{\partial \xi_2}{\partial x} + \frac{\partial}{\partial \xi_3} \frac{\partial \xi_3}{\partial x} \tag{7.66}$$

Placing Eqs. 7.63 into this yields

$$\frac{\partial}{\partial x} = \left(b_1 \frac{\partial}{\partial \xi_1} + b_2 \frac{\partial}{\partial \xi_2} + b_3 \frac{\partial}{\partial \xi_3} \right) / 2A$$

Similarly (7.67)

$$\frac{\partial}{\partial y} = \left(c_1 \frac{\partial}{\partial \xi_1} + c_2 \frac{\partial}{\partial \xi_2} + c_3 \frac{\partial}{\partial \xi_3} \right) / 2A$$

Thus, applying the operators of Eqs. 7.66 and 7.67 to Eq. 7.65 yields the appropriate functions, which can be written as

$$\Delta w_{,x} = \underset{\sim}{L}_{w,x}\, \Delta \underset{\sim}{q}$$
$$\Delta w_{,y} = \underset{\sim}{L}_{w,y}\, \Delta \underset{\sim}{q} \qquad (7.68)$$

The initial displacements, u, v, and w, are similarly interpolated in terms of the initial nodal values $\underset{\sim}{q}$.

Although the boundary displacements are interpolated in terms of the same nodal values as the interior displacements, they may be chosen differently. It is required that the boundary displacements be continuous among neighboring elements. This may be done by choosing the inplane displacements as linear. In fact, if one uses the same interpolation as for the interior displacements, properly evaluating the functions along the boundary, a linear boundary interpolation results. An additional benefit is that the inplane displacements are continuous from the interior of the element out to the boundaries, i.e.

$$u + \Delta u \equiv \tilde{u} + \Delta\tilde{u} \qquad\qquad v + \Delta v \equiv \tilde{v} + \Delta\tilde{v} \qquad (7.69)$$

For continuity in the transverse directions it is necessary to choose $\Delta\tilde{w}$ as cubic and the tangential slope as its derivative. The normal slope must be linear. Thus, [Pian and Tong, 1972]

$$\Delta\tilde{w}_p = H^1_{01}\,\Delta w_{p1} + H^1_{11}\,\Delta w_{p1,s} + H^1_{02}\,\Delta w_{p2} + H^1_{12}\,\Delta w_{p2,s} \qquad (7.70)$$

$$\Delta\tilde{w}_{p,s} = H^1_{01,s}\,\Delta w_{p1} + H^1_{11,s}\,\Delta w_{p1,s} + H^1_{02,s}\,\Delta w_{p2} + H^1_{12,s}\,\Delta w_{p2,s} \qquad (7.71)$$

$$\Delta\tilde{w}_{p,n} = (1-s)\,\Delta w_{p1,n} + s\,\Delta w_{p2,n} \qquad (7.72)$$

where

$\Delta\tilde{w}_p$ = transverse boundary displacement for side p

$\Delta\tilde{w}_{pi}$ = nodal value of $\Delta\tilde{w}_p$ for side p and node i

$\Delta\tilde{w}_{p,s}$ = tangential slope

$\Delta\tilde{w}_{p,n}$ = normal slope

H^1_{ij} = Hermitian interpolation functions (see Eq. 7.12).

It can be shown that if the interior interpolation function for Δw is properly evaluated at the boundary p then

$$\Delta w_p \equiv \Delta \tilde{w}_p \tag{7.73}$$

Thus, the transverse displacement is also continuous from the interior of the element out to the boundary. This is not the case for the rotations. The rotations are those with respect to the x and y axis, not the s and n system. Thus, the transformation

$$\begin{Bmatrix} \Delta \tilde{w}_{,x} \\ \Delta \tilde{w}_{,y} \end{Bmatrix} = \begin{bmatrix} -n & m \\ m & n \end{bmatrix} \begin{Bmatrix} \Delta \tilde{w}_{,s} \\ \Delta \tilde{w}_{,n} \end{Bmatrix} \tag{7.74}$$

must be used. Where

$m = \cos\theta$

$n = \sin\theta$

θ = angle between local x axis and the outward normal to the side

Thus, one may finally write

$$\begin{aligned} \Delta u &= \Delta \tilde{u} = \underset{\sim}{L}_u \, \Delta \underset{\sim}{q} \\ \Delta v &= \Delta \tilde{v} = \underset{\sim}{L}_v \, \Delta \underset{\sim}{q} \\ \Delta w &= \Delta \tilde{w} = \underset{\sim}{L}_w \, \Delta \underset{\sim}{q} \\ \Delta w_{,x} &= \underset{\sim}{L}_{w,x} \, \Delta \underset{\sim}{q} \qquad \Delta \tilde{w}_{,x} = \underset{\sim}{\tilde{L}}_{w,x} \, \Delta \underset{\sim}{q} \\ \Delta w_{,y} &= \underset{\sim}{L}_{w,y} \, \Delta \underset{\sim}{q} \qquad \Delta \tilde{w}_{,y} = \underset{\sim}{\tilde{L}}_{w,y} \, \Delta \underset{\sim}{q} \end{aligned} \tag{7.75}$$

where the initial quantities are interpolated in exactly the same manner except with respect to the initial nodal values $\underset{\sim}{q}$.

Next an interpolation for the height above the base plane, z, must be developed. It seems natural that the original displacement above the base plane should be of the same order as the displacement. This, however, is not necessary. In fact, for the stationary system, where it is convenient to obtain midside information (since the element is always referred to its initial configuration), a quadratic distribution was used. Additionally, a cubic distribution was developed for comparison and, since it depends only on nodal information, would be more useful in an updated system. For a quadratic distribution the z values at the three nodes plus three midside values must be determined. (See Fig. 7.4.) Thus,

$$z = \begin{Bmatrix} (2\xi_1 - 1)\xi_1 \\ (2\xi_2 - 1)\xi_2 \\ (2\xi_3 - 1)\xi_3 \\ 4\xi_1\xi_2 \\ 4\xi_2\xi_3 \\ 4\xi_3\xi_1 \end{Bmatrix}^T \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix} = \underset{\sim}{L}_{zQ}\, \underset{\sim}{z}_Q \tag{7.76}$$

For the cubic case

$$z = \underset{\sim}{L}_w\, \underset{\sim}{z}_c \tag{7.77}$$

where: L_w is given by Eq. 7.65

$$\underset{\sim}{z}_c = \lfloor z_1, z_{1,y}, -z_{1,x}, z_2, z_{2,y}, -z_{2,x}, z_3, z_{3,y}, -z_{3,x} \rfloor^T$$

In Section 6 Eq. 6.45 was constructed assuming the entire linear equilibrium equation was satisfied. Since this is not the case here it is worthwhile to reconsider Eq. 6.45 properly reduced for the assumptions of this subsection and with the two additional terms (resulting from $z \neq 0$) in Eq. 7.57.

$$\begin{aligned} \pi^I_{mc_n}(\underset{\sim}{\beta}, \Delta\underset{\sim}{q}) = & -\tfrac{1}{2}\underset{\sim}{\beta}\underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{\beta}^T\underset{\sim}{C}\Delta\underset{\sim}{q} + \tfrac{1}{2}\Delta\underset{\sim}{q}^T\underset{\sim}{K}_g\Delta\underset{\sim}{q} - \underset{\sim}{\beta}^T\underset{\sim}{C}_z\Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T\underset{\sim}{G}\Delta\underset{\sim}{q} \\ & + \underset{\sim}{\beta}^T\underset{\sim}{G}_z\Delta\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{\beta}^T\underset{\sim}{C}\underset{\sim}{q} - \underset{\sim}{\beta}^T\underset{\sim}{C}_z\underset{\sim}{q} + \underset{\sim}{\beta}^T\underset{\sim}{G}\underset{\sim}{q} + \underset{\sim}{\beta}^T\underset{\sim}{G}_z\underset{\sim}{q} \end{aligned} \tag{7.78}$$

where the additional terms are $\underset{\sim}{C}_z$ and $\underset{\sim}{G}_z$ which will be subsequently defined. Proceding as before

$$\begin{aligned} \frac{\partial \pi^I_{mc_n}}{\partial \underset{\sim}{\beta}} = & -\underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{C}\Delta\underset{\sim}{q} - \underset{\sim}{C}_z\Delta\underset{\sim}{q} + \underset{\sim}{G}\Delta\underset{\sim}{q} + \underset{\sim}{G}_z\Delta\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{C}\underset{\sim}{q} \\ & - \underset{\sim}{C}_z\underset{\sim}{q} + \underset{\sim}{G}\underset{\sim}{q} + \underset{\sim}{C}_z\underset{\sim}{q} = 0 \end{aligned} \tag{7.79}$$

The solution for the $\underset{\sim}{\beta}$'s is

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\tfrac{1}{2}\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]\underset{\sim}{q} + \underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]\Delta\underset{\sim}{q} \tag{7.80}$$

Placing this into Eq. 7.78 gives

$$\begin{aligned}\pi^{I}_{mc_n}(\Delta\underset{\sim}{q}) = &-\tfrac{1}{2}\left[(\underset{\sim}{G}^T+\underset{\sim}{C}^T)+(\underset{\sim}{G}_z^T-\underset{\sim}{C}_z^T)\right]\underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]\Delta\underset{\sim}{q}\\ &+\Delta\underset{\sim}{q}^T\left[(\underset{\sim}{G}^T+\underset{\sim}{C}^T)+(\underset{\sim}{G}_z^T-\underset{\sim}{C}_z^T)\right]\underset{\sim}{H}^{-1}\left[\underset{\sim}{C}-\underset{\sim}{C}_z+\underset{\sim}{G}+\underset{\sim}{G}_z\right]\Delta\underset{\sim}{q}\\ &+\tfrac{1}{2}\Delta\underset{\sim}{q}^T\underset{\sim}{K}_q\Delta\underset{\sim}{q}+\Delta\underset{\sim}{q}^T\left[(\underset{\sim}{G}^T+\underset{\sim}{C}^T)+(\underset{\sim}{G}_z^T-\underset{\sim}{C}_z^T)\right]\underset{\sim}{H}^{-1}\\ &\left[\tfrac{1}{2}\underset{\sim}{C}-\underset{\sim}{C}_z+\underset{\sim}{G}+\underset{\sim}{G}_z\right]\underset{\sim}{q} + \text{constants}\end{aligned} \tag{7.81}$$

Rearranging and dropping the constants not subject to variation

$$\begin{aligned}\pi^{I}_{mc_n}(\Delta\underset{\sim}{q}) = &\tfrac{1}{2}\Delta\underset{\sim}{q}^T\left\{\left[(\underset{\sim}{G}^T+\underset{\sim}{C}^T)+(\underset{\sim}{G}_z^T-\underset{\sim}{C}_z^T)\right]\underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]+\underset{\sim}{K}_q\right\}\Delta\underset{\sim}{q}\\ &+\Delta\underset{\sim}{q}^T\left[(\underset{\sim}{G}^T+\underset{\sim}{C}^T)+(\underset{\sim}{G}_z^T-\underset{\sim}{C}_z^T)\right]\underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\tfrac{1}{2}\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]\underset{\sim}{q}\end{aligned} \tag{7.82}$$

Or, in the form of Eq. 6.50

$$\pi^{I}_{mc_n}(\Delta\underset{\sim}{q}) = \tfrac{1}{2}\Delta\underset{\sim}{q}^T\underset{\sim}{K}_T\Delta\underset{\sim}{q} - \Delta\underset{\sim}{q}^T\underset{\sim}{Q} \tag{6.50}$$

where

$$\begin{aligned}\underset{\sim}{K}_T &= \left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]^T\underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]+\underset{\sim}{K}_q\\ \underset{\sim}{Q} &= -\left[(\underset{\sim}{G}+\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]^T\underset{\sim}{H}^{-1}\left[(\underset{\sim}{G}+\tfrac{1}{2}\underset{\sim}{C})+(\underset{\sim}{G}_z-\underset{\sim}{C}_z)\right]\underset{\sim}{q}\end{aligned} \tag{7.83}$$

Placing Eqs. 7.61-7.77 into these

$$\begin{aligned}\underset{\sim}{H} &= \int_{A_n}\underset{\sim}{P}^T\underset{\sim}{S}\underset{\sim}{P}\,dx\,dy\\ \underset{\sim}{G} &= \int_{C_n}\underset{\sim}{R}^T\tilde{\underset{\sim}{L}}\,ds\\ \underset{\sim}{G}_z &= \int_{C_n}\underset{\sim}{R}_{Nz}^T\tilde{\underset{\sim}{L}}_w\,ds\end{aligned}$$

$$\tilde{C} = \int_{A_n} [\tilde{P}_{N_x}^T \tilde{q}^T \tilde{L}_{w,x}^T \tilde{L}_{w,x} + \tilde{P}_{N_y}^T \tilde{q}^T \tilde{L}_{w,y}^T \tilde{L}_{w,y} + \tilde{P}_{N_{xy}}^T \tilde{q}^T (\tilde{L}_{w,x}^T \tilde{L}_{w,y} + \tilde{L}_{w,y}^T \tilde{L}_{w,x})] dx\,dy$$

$$\tilde{C}_z = \int_{A_n} [z_{,xx} \tilde{P}_{N_x}^T + z_{,yy} \tilde{P}_{N_y}^T + 2 z_{,xy} \tilde{P}_{N_{xy}}^T] \tilde{L}_w \, dx\,dy$$

$$\tilde{K}_q = \int_{A_n} \begin{bmatrix} \tilde{L}_{w,x} \\ \tilde{L}_{w,y} \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \tilde{L}_{w,x} \\ \tilde{L}_{w,y} \end{bmatrix} dx\,dy$$

$$\tilde{S} = \begin{bmatrix} \tilde{S}_N & \tilde{0} \\ \tilde{0} & \tilde{S}_M \end{bmatrix}$$

$$\tilde{S}_N = \frac{1}{Eh} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \qquad \tilde{S}_M = \frac{12}{h^2} \tilde{S}_N \tag{7.84}$$

$$\begin{Bmatrix} (N_x + \Delta N_x) \\ (N_y + \Delta N_y) \\ (S + \Delta S) \\ (M_x + \Delta M_x) \\ (M_y + \Delta M_y) \end{Bmatrix}_\nu = \tilde{R} \tilde{\beta} = \begin{Bmatrix} \tilde{R}_N \\ \tilde{R}_S \\ \tilde{R}_M \end{Bmatrix} \tilde{\beta}$$

$$\tilde{L} = \lfloor \tilde{L}_u , \tilde{L}_v , \tilde{L}_w , \tilde{L}_{w,x} , \tilde{L}_{w,y} \rfloor^T$$

$$\tilde{R}_{N_z} = z_{,x} \tilde{R}_{N_x} + z_{,y} \tilde{R}_{N,y}$$

Thus, for a shallow shell element satisfying Eq. 7.55, Eqs. 7.83 are used. If instead the full linear equations (Eqs. 7.54) are satisfied, or if the element is flat ($z \equiv 0$) then note that $\tilde{R}_{N_z} = 0$ and

$$\tilde{G}_z = 0 \qquad\qquad \tilde{C}_z = 0 \tag{7.85}$$

Placing this into Eqs. 7.83 gives the reduced form

$$\begin{aligned} \tilde{K}_T &= (\tilde{G}+\tilde{C})^T \tilde{H}^{-1} (\tilde{G}+\tilde{C}) + \tilde{K}_q \\ \tilde{Q} &= -(\tilde{G}+\tilde{C})^T \tilde{H}^{-1} (\tilde{G} + \tfrac{1}{2}\tilde{C}) \tilde{q} \end{aligned} \tag{7.86}$$

which correspond to Eqs. 6.53 of Subsection 6.1.2 with $\underset{\sim}{C}_1 \Rightarrow \underset{\sim}{G}$ as a result of the integration by parts. ($\underset{\sim}{S}_B$, $\underset{\sim}{M}_b$, $\underset{\sim}{Q}_{B_\sigma}$ = 0). Note that the additional matrices $\underset{\sim}{G}_z$ and $\underset{\sim}{C}_z$ are simple in form and should not add much to the computational effort.

It must be noted that for shallow elements where Eq. 7.54 is satisfied and then reduced Eqs. 7.86 are used that the $\underset{\sim}{P}$ matrix must reflect the change in stress assumptions. One simple choice would be

$$\underset{\sim}{P} = \begin{bmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ -z & & & 1 & & & x & & & y & & \\ & -z & & & 1 & & & x & & & y & \\ & & -z & & & 1 & & & x & & & y \end{bmatrix} \tag{7.87}$$

There are alternatives to this. One would be to use the static-geometric analogy commonly used in shell theory. (See Appendix D.)

Once the displacements are determined, $\underset{\sim}{\beta}$ may be determined from Eq. 7.80 (possibly with $\underset{\sim}{G}_z = \underset{\sim}{C}_z = 0$). The stresses are obtained by Eq. 7.61 where the appropriate $\underset{\sim}{P}$ matrix must be used. Finally, the strains may be obtained through the constitutive relations.

7.2.3 The Convected, Updated Lagrangian System

The corresponding functional for π_{mR} is Eq. 5.50. Integrating the linear portion of the compatibility check by parts, satisfying Eq. 5.47 exactly, and recognizing the assumptions of this section, Eq. 5.50 may be rewritten for an element as

$$\begin{aligned} \pi_{mR_n} = & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}]\, dV \\ & - \int_{V_n} [(\sigma_{ij} + \Delta\sigma_{ij})_{,j}\,\Delta u_i]\, dV \\ & + \int_{\partial V_n} (\sigma_{ij} + \Delta\sigma_{ij})\,\nu_j\,\Delta\tilde{u}_i\, ds - \int_{V_n} \Delta\sigma_{ij}(e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j})\, dV \\ & - \int_{V_n} \Delta\sigma_{ij,j}\,u_i\, dV + \int_{\partial V_n} \Delta\sigma_{ij}\,\nu_j\,u_i\, ds \end{aligned} \tag{7.88}$$

Comparing this to Eq. 7.56 for a stationary system, three differences can be observed. Firstly, the initial displacement terms in the first volume integral are removed. Secondly, the sign of the nonlinear term in the compatibility check has changed. Thirdly, the displacement in the last integral is that of the interior, Δu_i.

The equivalent set of equations to Eqs. 7.83 may, therefore, be written directly as

$$\underset{\sim}{K}_T = [\underset{\sim}{G} + (\underset{\sim}{G}_z - \underset{\sim}{C}_z)]^T \underset{\sim}{H}^{-1} [\underset{\sim}{G} + (\underset{\sim}{G}_z - \underset{\sim}{C}_z)] + \underset{\sim}{K}_q$$
$$\underset{\sim}{Q} = -[\underset{\sim}{G} + (\underset{\sim}{G}_z - \underset{\sim}{C}_z)]^T \underset{\sim}{H}^{-1} [(\underset{\sim}{G}^* - \tfrac{1}{2}\underset{\sim}{C}) + (\underset{\sim}{G}_z - \underset{\sim}{C}_z)] \underset{\sim}{q} \tag{7.89}$$

where $\underset{\sim}{G}^*$ is given by Eq. 6.40 and all the interpolation functions and comments of Subsection 7.2.2 apply.

7.3 The Linear Prebuckling of Flat Plates

The determination of linear prebuckling requires the solution of an eigenvalue problem. Unlike a limit load buckling situation which requires a full nonlinear analysis, linear bifurcation buckling is linear and is treated as such. Thus, there is no need to differentiate between initial and incremental displacements. In fact, for a flat plate only transverse and rotational degrees of freedom need be considered. The eigenvalue of the problem is a load (or stress) factor and so an initial stress problem must be formulated.

Since the nonlinear analysis developed up to this point is of the initial stress class appropriate simplifications of these functionals will yield the proper equations. Because initial displacements are of no interest here the functionals for the updated system will be considered. Removal of the equilibrium checks is also necessary.

7.3.1 The Consistent Assumed Stress Hybrid Model

Considering the functional presented in Eq. 6.54, a basic version would become

$$\pi^c_{mc_n} = \int_{V_n} [-B(\Delta\sigma_{ij}) - \tfrac{1}{2}\sigma_{ij} u_{k,i} u_{k,j}]\, dV + \int_{\partial V_n} \Delta T_i \tilde{u}_i \, dS \tag{7.90}$$

where Eq. 7.90 is subject to

$$\Delta\sigma_{ij,j} + (\sigma_{kj} u_{i,k})_{,j} = 0 \tag{7.91}$$

and

$$\Delta T_i = (\Delta\sigma_{ij} + \sigma_{kj} u_{i,k}) \nu_j \tag{7.92}$$

Using similar interpolations as in previous subsections

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\underset{\sim}{\beta} + \underset{\sim}{A}\underset{\sim}{q} \tag{7.93}$$

$$\Delta\underset{\sim}{T} = \underset{\sim}{R}\underset{\sim}{\beta} + \underset{\sim}{A_\nu}\underset{\sim}{q} \tag{7.94}$$

$$\tilde{\underset{\sim}{u}} = \tilde{\underset{\sim}{L}}\underset{\sim}{q} \tag{7.95}$$

$$\underset{\sim}{u}' = \underset{\sim}{L}'\underset{\sim}{q} \tag{7.96}$$

Placing Eqs. 7.93-7.96 into Eq. 7.90 yields

$$\pi^c_{mc_n}(\underset{\sim}{\beta},\underset{\sim}{q}) = -\tfrac{1}{2}\underset{\sim}{\beta}\underset{\sim}{H}\underset{\sim}{\beta} - \underset{\sim}{\beta}^T\underset{\sim}{H_A}\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{q}^T\underset{\sim}{D}\underset{\sim}{q} - \tfrac{1}{2}\underset{\sim}{q}^T\underset{\sim}{K_g}\underset{\sim}{q}$$
$$+ \underset{\sim}{\beta}^T\underset{\sim}{G}\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{q}^T(\underset{\sim}{M}+\underset{\sim}{M}^T)\underset{\sim}{q} \tag{7.97}$$

where these matrices are defined in Subsection 6.2.1. (Note also Eq. 7.36.) Taking the variation of $\pi^c_{mc_n}$ with respect to the independent $\underset{\sim}{\beta}$'s yields

$$\frac{\partial\pi^c_{mc_n}}{\partial\underset{\sim}{\beta}} = -\underset{\sim}{H}\underset{\sim}{\beta} - \underset{\sim}{H_A}\underset{\sim}{q} + \underset{\sim}{G}\underset{\sim}{q} = 0 \tag{7.98}$$

Solving for the $\underset{\sim}{\beta}$'s

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}(\underset{\sim}{G} - \underset{\sim}{H_A})\underset{\sim}{q} \tag{7.99}$$

Placing Eq. 7.99 into Eq. 7.97 yields

$$\pi^c_{mc_n}(\underset{\sim}{q}) = -\tfrac{1}{2}\underset{\sim}{q}^T(\underset{\sim}{G}^T - \underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}(\underset{\sim}{G} - \underset{\sim}{H_A})\underset{\sim}{q} + \underset{\sim}{q}^T(\underset{\sim}{G}^T - \underset{\sim}{H}_A^T)\underset{\sim}{H}^{-1}(\underset{\sim}{G} - \underset{\sim}{H_A})\underset{\sim}{q}$$
$$- \tfrac{1}{2}\underset{\sim}{q}^T(\underset{\sim}{D} + \underset{\sim}{K_g} - \underset{\sim}{M} - \underset{\sim}{M}^T)\underset{\sim}{q} \tag{7.100}$$

Or, writing Eq. 7.100 as

$$\pi^c_{mc_n}(\underset{\sim}{q}) = \tfrac{1}{2}\underset{\sim}{q}^T\underset{\sim}{K_T}\underset{\sim}{q} \tag{7.101}$$

the element tangent stiffness may be expressed as

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} - (\underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{H}_A + \underset{\sim}{H}_A^T \underset{\sim}{H}^{-1} \underset{\sim}{G}) + \underset{\sim}{H}_A^T \underset{\sim}{H}^{-1} \underset{\sim}{H}_A + \underset{\sim}{M} + \underset{\sim}{M}^T - \underset{\sim}{D} - \underset{\sim}{K}_g \tag{7.102}$$

Note from the definitions of these matrices that

$\underset{\sim}{H}, \underset{\sim}{G}$ are not functions of $\underset{\sim}{\sigma}$

$\underset{\sim}{H}_A, \underset{\sim}{M}, \underset{\sim}{K}_g$ are functions of $\underset{\sim}{\sigma}$

$\underset{\sim}{D}$ is a function of $(\underset{\sim}{\sigma})^2$

Thus, one may rewrite Eq. 7.102 as

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} + \lambda[\bar{\underset{\sim}{M}} + \bar{\underset{\sim}{M}}^T - \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \bar{\underset{\sim}{H}}_A - \bar{\underset{\sim}{H}}_A^T \underset{\sim}{H}^{-1} \underset{\sim}{G} - \bar{\underset{\sim}{K}}_g] + \lambda^2 [\bar{\underset{\sim}{H}}_A^T \underset{\sim}{H}^{-1} \bar{\underset{\sim}{H}}_A - \bar{\underset{\sim}{D}}] \tag{7.103}$$

where

λ = critical load parameter

$\lambda \bar{\underset{\sim}{M}} = \underset{\sim}{M}$

etc.

Note that this leads to a quadratic eigenvalue problem which can be solved [Przemieniecki, 1966]. The inplane displacements may be removed and the matrices of Eq. 7.103 may be defined by

$$\begin{aligned}
\underset{\sim}{H} &= \int_{A_n} \underset{\sim}{P}_M^T \underset{\sim}{S}_M \underset{\sim}{P}_M \, dx\,dy \\
\underset{\sim}{G} &= \int_{C_n} [\underset{\sim}{R}_S^T \tilde{\underset{\sim}{L}}_w + \underset{\sim}{R}_{Mx}^T \tilde{\underset{\sim}{L}}_{w,x} + \underset{\sim}{R}_{My}^T \tilde{\underset{\sim}{L}}_{w,y}]\, ds \\
\underset{\sim}{M} &= \int_{C_n} [\underset{\sim}{A}_{Mx}^T \tilde{\underset{\sim}{L}}_{w,x} + \underset{\sim}{A}_{My}^T \tilde{\underset{\sim}{L}}_{w,y}]\, ds \\
\underset{\sim}{H}_A &= \int_{A_n} \underset{\sim}{P}_M^T \underset{\sim}{S}_M \underset{\sim}{A}_M \, dx\,dy \\
\underset{\sim}{K}_g &= \int_{A_n} \begin{bmatrix} \underset{\sim}{L}_{w,x} \\ \underset{\sim}{L}_{w,y} \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \underset{\sim}{L}_{w,x} \\ \underset{\sim}{L}_{w,y} \end{bmatrix} dx\,dy \\
\underset{\sim}{D} &= \int_{A_n} \underset{\sim}{A}_M^T \underset{\sim}{S}_M \underset{\sim}{A}_M \, dx\,dy
\end{aligned} \tag{7.104}$$

where the definitions of the previous subsection hold and

$$\begin{aligned}
\underset{\sim}{A}_{Mx} &= (N_x \underset{\sim}{L}_w)\nu_x + (N_{xy} \underset{\sim}{L}_w)\nu_y \\
\underset{\sim}{A}_{My} &= (N_y \underset{\sim}{L}_w)\nu_y + (N_{xy} \underset{\sim}{L}_w)\nu_x \\
\underset{\sim}{A}_M &= \begin{bmatrix} \underset{\sim}{A}_{Mx} \\ \underset{\sim}{A}_{My} \end{bmatrix}
\end{aligned} \tag{7.105}$$

7.3.2 The Inconsistent Assumed Stress Hybrid Model

The basic version of the inconsistent functional in Eq. 6.82 may be written

$$\pi^{I}_{mc_n} = \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij} u_{k,i} u_{k,j}]\, dV + \int_{\partial V_n} \Delta\sigma_{ij} \nu_j \tilde{u}_i \, ds \tag{7.106}$$

where the constraint conditions are simply

$$\Delta\sigma_{ij,j} = 0 \tag{7.107}$$

and

$$\Delta T_i = (\Delta\sigma_{ij} + \sigma_{kj} u_{i,k}) \nu_j \tag{7.108}$$

Using the simple interpolations

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\underset{\sim}{\beta} \tag{7.109}$$

$$\Delta\underset{\sim}{T} = \underset{\sim}{R}\underset{\sim}{\beta} + \underset{\sim}{A}_{\nu}\underset{\sim}{q} \tag{7.110}$$

$$\underset{\sim}{\tilde{u}} = \underset{\sim}{\tilde{L}}\,\underset{\sim}{q} \tag{7.111}$$

$$\underset{\sim}{u}' = \underset{\sim}{L}'\underset{\sim}{q} \tag{7.112}$$

Eq. 7.106 becomes

$$\pi^{I}_{mc_n}(\underset{\sim}{\beta}, \underset{\sim}{q}) = -\tfrac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\underset{\sim}{\beta} + \tfrac{1}{2}\underset{\sim}{q}^T \underset{\sim}{K}_q \underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{G}\underset{\sim}{q} \tag{7.113}$$

Taking the variation with respect to $\underset{\sim}{\beta}$ yields

$$\frac{\partial \pi^{I}_{mc_n}}{\partial \underset{\sim}{\beta}} = -\underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{G}\underset{\sim}{q} = 0 \tag{7.114}$$

Thus

$$\underset{\sim}{\beta} = \underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q} \tag{7.115}$$

Placing Eq. 7.115 into Eq. 7.113 gives

$$\pi^{I}_{mc_n}(\underset{\sim}{q}) = -\tfrac{1}{2}\underset{\sim}{q}^T \underset{\sim}{G}^T \underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q} + \tfrac{1}{2}\underset{\sim}{q}^T \underset{\sim}{K}_q \underset{\sim}{q} + \underset{\sim}{q}^T \underset{\sim}{G}^T \underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q} \tag{7.116}$$

The element tangent stiffness may be written as

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T \underset{\sim}{H}^{-1}\underset{\sim}{G} + \underset{\sim}{K}_q \tag{7.117}$$

or

$$\underset{\sim}{K}_T = \underset{\sim}{G}^T \underset{\sim}{H}^{-1}\underset{\sim}{G} + \lambda \underset{\sim}{\bar{K}}_q \tag{7.118}$$

which leads to a standard linear eigenvalue problem. These matrices are defined similarly to those of the previous subsection.

The eigenvalue problem of Eq. 7.118 was first used by Lundgren [1967] although he did not consistently derive the functional upon which it is based. This looks very much like the eigenvalue problem derived from the displacement model. While the geometric stiffnesses are the same (for the same interpolation functions of the slopes), the elastic stiffness is that of the linear assumed stress hybrid model.

Comparing Eq. 7.118 with Eq. 7.103 one can appreciate the tremendous increase in computational effort required by the consistent model.

7.4 Computational Procedures

The computational procedures and subsequent updating of information varies with solution technique and functional formulation. A brief, general description of the significant points will be given in the following subsections.

7.4.1 Incremental and/or Iterative Procedures

For the specific example elements given in this section, the global (common) equilibrium equation was given as

$$ {}^{T}\underset{\sim}{K}_{T}\,{}^{T}\Delta\underset{\sim}{q} = {}^{T}\underset{\sim}{Q}_{E} + {}^{T}\underset{\sim}{Q} \tag{7.1}$$

which represents the total, properly assembled equations. As stated in Subsection 6.4 there are three general categories of solution procedures that were used here regardless of the basic functional.

The first was a purely incremental scheme with no equilibrium checks of any kind. For this case the proper element stiffness matrix must be obtained and assembled to form ${}^{T}\underset{\sim}{K}_{T}$. An increment of external load is chosen and lumped at the proper nodes in the proper directions (dofs), forming ${}^{T}\underset{\sim}{Q}_{E}$. Since no equilibrium checks are to be used, ${}^{T}\underset{\sim}{Q}\equiv 0$ for all time. With this information Eq. 7.1 is solved for ${}^{T}\Delta\underset{\sim}{q}$. This is then in turn used to obtain incremental stresses and strains as indicated in the previous subsections. At this time the present state is deemed in equilibrium (which is not entirely true, hence the need for equilibrium checks). With this new information, element stiffnesses are generated to form a new ${}^{T}\underset{\sim}{K}_{T}$ and a new (next) increment of external load is assembled to form ${}^{T}\underset{\sim}{Q}_{E}$. ${}^{T}\underset{\sim}{Q}$ is, of course, assumed zero so that Eq. 7.1

is again solved to obtain new ${}^{T}\Delta \underset{\sim}{q}$'s. This process continues until the total external load has been reached.

The second scheme is also purely incremental. Here, however, it is recognized that the new state will, in general, not be in equilibrium and, therefore, equilibrium checks are necessary. This procedure does assume that the exact equilibrium imbalance can be determined at each incremental step. Under this assumption the imbalance is determined and transformed into an equivalent load term ${}^{T}Q$. Thus, the element stiffnesses are generated as before and assembled to form ${}^{\tilde{T}}\underset{\sim}{K}_{T}$. At the first increment of external load, ${}^{T}\underset{\sim}{Q}$ is zero since no imbalance yet exists. This first step is solved in the same manner as in the previous procedure. A ${}^{T}\Delta \underset{\sim}{q}$ is determined and stresses and strains obtained from it. Realizing an imbalance to exist it is calculated and assembled to form ${}^{T}\underset{\sim}{Q}$. New element stiffnesses are formed and assembled as before. Now the next increment of external load is added to the first. The two load vectors ${}^{T}\underset{\sim}{Q}_{E}$ and ${}^{T}\underset{\sim}{Q}$ are added and Eq. 7.1 is solved for a new set of ${}^{T}\Delta \underset{\sim}{q}$'s. The reason that the subtotal external load (at the present time) must be used is the equilibrium checks involve the total initial quantities. These quantities will develop loads ${}^{T}\underset{\sim}{Q}$ which ideally should have been equal and opposite to ${}^{T}\underset{\sim}{Q}_{E}$ from the previous step. This additional imbalance load is calculated as shown in the previous subsections. Again it is assumed that the imbalance can be exactly determined, so that at the end of each step it is calculated and simply added to the subtotal external load of the next step.

The final scheme recognizes that not only are equilibrium checks necessary, but that the imbalance can not, at any one step, be calculated exactly. Thus, the combination of increments and iterations is formulated. The very first step is the same as the previous procedures. At the end of the first step an imbalance load ${}^{T}\underset{\sim}{Q}$ is calculated as in the last procedure. However, rather than adding this to a new subtotal external load, it is added to the same, old external load term. Solving Eq. 7.1 with this load gives a correction term in ${}^{T}\Delta \underset{\sim}{q}$ due to the imbalance of the first external load increment. At this point a new imbalance load is calculated and again is only added to the old external load term. Solving Eq. 7.1 again gives a further correction to ${}^{T}\Delta \underset{\sim}{q}$ still for the first load step (increment). This iterative process within load increments is continued until some predetermined convergence for that load step is reached.

Once convergence is reached the next increment of external load is added to the first to obtain a new subtotal. Now the iterative process takes over again until convergence is reached. This incremental-iterative procedure continues until the final total external load is reached.

For the purposes of this work the following scheme was used to determine if convergence was reached. At load increment "m" the increase in displacement due to the increase in external load was stored as $\Delta \underset{\sim}{q}_o^m$. The magnitude of this vector was obtained as

$$|\Delta \underset{\sim}{q}_o^m|^2 = (\Delta \underset{\sim}{q}_o^m) \cdot (\Delta \underset{\sim}{q}_o^m) \tag{7.119}$$

The first correction to this displacement on the first iteration within this incremental load step, $\Delta \underset{\sim}{q}_1^m$, is stored. Its magnitude was similarly obtained as

$$|\Delta \underset{\sim}{q}_1^m|^2 = (\Delta \underset{\sim}{q}_1^m) \cdot (\Delta \underset{\sim}{q}_1^m) \tag{7.120}$$

Finally for the i^{th} iteration

$$|\Delta \underset{\sim}{q}_i^m|^2 = (\Delta \underset{\sim}{q}_i^m) \cdot (\Delta \underset{\sim}{q}_i^m) \tag{7.121}$$

A ratio was formed such that

$$\underset{\sim}{R} = |\Delta \underset{\sim}{q}_i^m|^2 \Big/ |\Delta \underset{\sim}{q}_o^m|^2 \tag{7.122}$$

When this ratio (which should approach zero) gets below a predetermined number (i.e. 0.001) then convergence was assumed.

7.4.2 Updating the Displacements and Geometry

For all of the elements, procedures, and coordinate systems used it is convenient to refer the nodal displacements and coordinate locations to a stationary reference frame. Although for some shell problems it might be better to give the displacements in the shell system, for the purposes of plotting deformations it is simpler to use the fixed rectangular Cartesian coordinate system. Since the solution vector of displacements is in the common (or shell) system, to reduce the number of dofs, a transformation is necessary. These transformation matrices already exist, as they are necessary in the computational process (see Subsection 3.3), and stored for access. Once the five degrees of freedom at a node are known in the common system, use may be made of Eq. 3.45 to obtain the corresponding six degrees of freedom in the global reference frame.

$$
{}^{G}\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta \theta_x \\ \Delta \theta_y \\ \Delta \theta_z \end{Bmatrix}
\begin{bmatrix}
({}^{G}x,{}^{C}x) & ({}^{G}x,{}^{C}y) & ({}^{G}x,{}^{C}z) & 0 & 0 \\
({}^{G}y,{}^{C}x) & ({}^{G}y,{}^{C}y) & ({}^{G}y,{}^{C}z) & 0 & 0 \\
({}^{G}z,{}^{C}x) & ({}^{G}z,{}^{C}y) & ({}^{G}z,{}^{C}z) & 0 & 0 \\
0 & 0 & 0 & ({}^{G}x,{}^{C}x) & ({}^{G}x,{}^{C}y) \\
0 & 0 & 0 & ({}^{G}y,{}^{C}x) & ({}^{G}y,{}^{C}y) \\
0 & 0 & 0 & ({}^{G}z,{}^{C}x) & ({}^{G}z,{}^{C}y)
\end{bmatrix}
{}^{C}\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta \theta_x \\ \Delta \theta_y \end{Bmatrix}
\qquad (7.123)
$$

where

$({}^{G}x, {}^{C}y)$ = direction cosine of the angle between the global x and common y axes

$$\Delta \theta_x = \frac{\partial \Delta w}{\partial y}$$

$$\Delta \theta_y = -\frac{\partial \Delta w}{\partial x}$$

and

$${}^{C}\Delta \theta_z = 0$$

The transformation matrix varies from node to node so that at every node such a matrix must be stored. Only a 3x3 actually is stored since the other terms are the same.

For a stationary system the common system remains stationary for all time and the transformations need be calculated once. In the updated system the reference (common) system follows the deformation pattern and, therefore, must be recalculated at every solution step. In either case the matrices are previously needed and stored so that they will be available.

Since all increments for all time are finally referred to the same set of coordinates (the global system) they may be directly added to the initial quantities already in that system. Thus, the total displacements are established with respect to the fixed frame.

Although the stationary system utilizes the original configuration for differentiation and integrations, many of the matrices to be evaluated are functions of the initial displacements. The element level matrices are calculated using an individual, local rectangular Cartesian frame. Thus, the initial displacements which are the present total displacements in the global frame, must be transformed to these local frames. In this instance the transformations are for individual elements rather than nodes as before. Since these local axes are the natural systems only five dofs are non-zero. Utilizing Eq. 3.49

$$^{L}\begin{Bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} ({}^{G}x,{}^{L}x) & ({}^{G}y,{}^{L}x) & ({}^{G}z,{}^{L}x) & 0 & 0 & 0 \\ ({}^{G}x,{}^{L}y) & ({}^{G}y,{}^{L}y) & ({}^{G}z,{}^{L}y) & 0 & 0 & 0 \\ ({}^{G}x,{}^{L}z) & ({}^{G}y,{}^{L}z) & ({}^{G}z,{}^{L}z) & 0 & 0 & 0 \\ 0 & 0 & 0 & ({}^{G}x,{}^{L}x) & ({}^{G}y,{}^{L}x) & ({}^{G}z,{}^{L}x) \\ 0 & 0 & 0 & ({}^{G}x,{}^{L}y) & ({}^{G}y,{}^{L}y) & ({}^{G}z,{}^{L}y) \end{bmatrix} {}^{G}\begin{Bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \quad (7.124)$$

Note that the inverse of the direction cosine transformation matrix is equal to its transpose. These matrices are formed once and for all and stored as 3x3's for future use.

The updated system also requires initial displacement terms for the compatibility check. Here, however, since these local systems move with the deforming body the transformations are calculated for every solution step. It is not necessary to store the matrices in this instance because the transformations must be generated and used during element generation.

Once the total displacements are known then the new location on the structure is known. For the stationary system only the original coordinates are necessary so that no changes occur here. For the updated system, however, the new geometry is used as a reference frame. The coordinate locations of the nodal points are always given with respect to a fixed global frame. Since the total displacements are in the same system, the x, y, z coordinate locations may be continually updated by adding the global increment values of

Δu, Δv, Δw to them respectively. For flat elements this is all that is necessary. Shallow elements in addition must have updated slopes. The global slopes at the nodes may be obtained from the global rotations as

$$
\begin{aligned}
{}^{G}_{G}z_{,x} &= \mathrm{TAN}\,(-{}^{G}\theta_{y}) \\
{}^{G}_{G}z_{,y} &= \mathrm{TAN}\,({}^{G}\theta_{x})
\end{aligned}
\tag{7.125}
$$

With this information the local slopes with respect to the base planes (${}^{L}_{L}z_{,x}$ and ${}^{L}_{L}z_{,y}$) can be obtained by (see Appendix A)

$$
\begin{aligned}
{}^{L}_{L}z_{,x} &= \frac{1}{C}\left[({}^{G}z,{}^{L}x) - ({}^{G}x,{}^{L}x)\,{}^{G}_{G}z_{,x} - ({}^{G}y,{}^{L}x)\,{}^{G}_{G}z_{,y}\right] \\
{}^{L}_{L}z_{,y} &= \frac{1}{C}\left[({}^{G}z,{}^{L}y) - ({}^{G}x,{}^{L}y)\,{}^{G}_{G}z_{,x} - ({}^{G}y,{}^{L}y)\,{}^{G}_{G}z_{,y}\right]
\end{aligned}
\tag{7.126}
$$

where

$$
C = ({}^{G}x,{}^{L}z)\,{}^{G}_{G}z_{,x} + ({}^{G}y,{}^{L}z)\,{}^{G}_{G}z_{,y} - ({}^{G}z,{}^{L}z)
$$

7.4.3 Updating the Stresses and Strains

Once the incremental displacements have been obtained and the previous total displacements are known, then the stress parameters $\tilde{\beta}$ may be determined by the appropriate equations previously given. A caution is given here in that these $\tilde{\beta}$'s were obtained on the element level and, therefore, the displacements must be in the local system. Either the displacements can be brought into the local system, or the premultiplying product to determine the $\tilde{\beta}$'s can be transformed when the element stiffnesses are so that the displacements will be referred to the common system (or global if easier). The way in which this is best handled depends largely on the functional (and its corresponding check terms) used.

Assuming the $\tilde{\beta}$'s are known they must now be premultiplied by the element $\tilde{P}$ matrix. For flat elements and small strains these matrices do not change much and may be considered constant (for a given integration point in the element). This is, of course, true for a stationary system where no change is encountered. For shallow shells in the updated system where the $\tilde{P}$ matrix is a function of the local z height above the base plane, careful consideration must be given to evaluating $\underset{\sim}{P}$.

In addition, for the consistent models the stresses are also a function of the incremental displacements (see Eq. 6.56 and Eq. 7.33 for example) and, thus, to account for the complete stress this additional term must be accounted for. When using a stress equilibrium check the new total stress can be calculated directly rather than having to add increments to it. (See for example Eq. 7.61).

Once the total stresses are known, the strains are simply obtained through the constitutive relations. It must be recognized that the strain displacement relations are an Euler equation of the assumed stress hybrid functionals and cannot be used. (This is the basis of the compatibility check.)

7.4.4 Comments

Upon cursory examination of some of the computations required it may appear that the stationary system and the updated system could be competitive computationally, especially when compatibility checks are involved. While the updated system has fewer calculations for generating element stiffnesses and loads, it requires more transformations. In general this may be the case. However, for the simpler elements chosen in Section 7 the stationary system has significant computational savings.

The most obvious is for uniform meshes. If the mesh is uniform then only one block of elements (two triangular elements) need be generated and the rest are the same. In a stationary system this can always be taken advantage of for those matrices which are not functions of initial stresses or displacements. In the updated system the elements are constantly distorting, so after the first step the mesh is no longer uniform. Even further, however, the integrations and interpolations are so simple that once they are performed in the stationary system the initial displacement and stress terms can be moved outside the integrals. This means that the integrations need be performed only on the first step. In every successive step the matrices may simply be ratioed up rather than reintegrated. Numerical integration is still required for every step in the updated system and this can be very time consuming. A notable difference in time between solutions in the stationary and updated systems for two dimensional problems can be observed in the results of Section 8.

The integrations which are performed are done analytically for the beam elements and numerically for the plate and shell elements. Numerical integration over the triangular regions is performed by the Hammer rule [Hammer, et al., 1956; Lyness and Jaspersen, 1975]. In the latter reference integration coefficients are given for polynomials of up to eleventh order. This is quite an advance. The Hammer rule is more efficient over triangular regions than the standard Gauss schemes. The line integrals are performed by Gauss quadrature.

SECTION 8

APPLICATIONS, EVALUATION AND DISCUSSION

8.1 Introduction

This section makes use of the models and procedures of Section 7. The functionals were programmed in Fortran and run on the IBM 370/165 computer at the Massachusetts Institute of Technology's Information Processing Center. Extensive use is made of FEABL [Orringer and French, 1972]. It is a modular package of basic routines required by finite element analysis.

Beam, plate and shell problems were run to demonstrate various aspects of the analysis. Since the beam problems are the least expensive to run, they were used to make several investigations. These studies include comparisons of the consistent and inconsistent assumed stress hybrid models, flat and shallow elements, the Kirchhoff-Love and Marguerre theories, the stationary and updated coordinate systems, the value of the equilibrium checks, and the solution procedures. From these results only certain models were deemed useful for plate and shell analysis. In particular the following problems were run:

a. the linear prebuckling of a flat plate. This is basically a linear problem in that it involves only one solution step (an eigenvalue problem). The purpose here was to check out the accuracy of the geometric stiffness matrix. (Fig. 8.1.)

b. a shallow, sinusoidal arch under sinusoidal load. This problem was used for the bulk of the investigations because reliable independent solutions exist. (Fig. 8.2a.)

c. a shallow, circular arch under a central concentrated load. The best procedures from b. were used here and comparisons were made with independent solutions. The problem can demonstrate the great efficiency of the models. (Fig. 8.28.)

d. simply supported and clamped, initially flat, square plates under uniform load. Reliable results for displacements and fairly reliable results for stresses exist. (Figs. 8.30 and 8.37.)

e. a shallow cylindrical panel under uniform load. Fairly reliable results for displacements exist but stress results are only sketchy. (Fig. 8.38.)

f. a moderately shallow, spherical cap under a central concentrated load. This is the only doubly curved shell tested. (Fig. 8.39.)

g. a shallow, cylindrical panel under edge compression. This problem was chosen to test the inplane load capability of the analysis. (Fig. 8.41.)

In the following subsections each problem with all the appropriate comparisons are discussed in detail. All the nomenclature and definitions in the following tables and figures may be found in the figures, given above, associated with each problem.

As stated in Section 7 it is assumed that distributed loads are lumped at the nodes. For truly consistent lumping of loads, the following should be considered. In plate theory distributed loads occurring on the top and/or bottom surfaces of the plate are considered as body forces. Thus, the external load distribution can be discretized as follows.

$$\int_A \bar{F}_i u_i \, dA = \int_A F_z w \, dA = \underset{\sim}{Q}^T \underset{\sim}{q} \tag{8.1}$$

where

$F_z = p_o f(x,y)$

p_o = load parameter

$f(x,y)$ = load distribution

$$w = \underset{\sim}{L} \underset{\sim}{q} \tag{8.2}$$

$\underset{\sim}{L}$ = interpolation functions on the element interior

$\underset{\sim}{q}$ = element nodal displacements

$\underset{\sim}{Q}$ = element lumped load vector

Placing Eq. 8.2 into Eq. 8.1 gives

$$\underset{\sim}{Q}^T \underset{\sim}{q} = \int_A p_o f(x,y) \underset{\sim}{L} \underset{\sim}{q} \, dA = p_o \int_A f(x,y) \underset{\sim}{L} \, dA \, \underset{\sim}{q} \tag{8.3}$$

or

$$\underset{\sim}{Q}^T = p_o \int_A f(x,y) \underset{\approx}{L} \, dA \tag{8.4}$$

Eq. 8.4 yields a lumped load vector consistent with the assumed displacements on the interior of the element. Note that both shear and moment terms result, in general, from such an analysis. These element load vectors can then be assembled to form a global, lumped load vector.

For a uniformly distributed load acting on elements of uniform thickness, the moment terms, corresponding to the rotations, would cancel during the assembly process. Also, as the mesh is refined a nonuniformly distributed load can be approximated locally as a uniform load. Therefore, the assembled moment terms would tend to vanish (except at the boundaries). With this in mind, a simpler lumped loading was used in this work. Integrating the load distribution over the element the equivalent shear force is appropriately distributed to each node. The moment terms are neglected. All of the distributed loadings in this section are uniform with the exception of the shallow arch problem. In this case, sufficiently fine meshes are used so that the simpler inconsistent lumped load is reasonable.

Furthermore, to properly consider pressure loadings (nonconservative), the load should follow the deformed geometry regardless of the coordinate system used (S.L. or C.U.L.). In this work, cases considered in the C.U.L. system allow the load to follow the deformed path. In the S.L. system, however, the loads are always considered to act on the initial configuration for convenience. Since the deflections and rotations are restricted this is not a serious drawback, however, this in part accounts for different solutions obtained in each system. For the problems actually considered in this section, the maximum error in the normal load is less than one half percent. The change of inplane load would be approximately three orders of magnitude less than the inplane loads generated from the nonlinear effects.

8.2 Linear Prebuckling of a Flat Plate

The first attempt at using the assumed stress hybrid model for such an analysis was done by Lundgren [1967]. Although he applied his method to complicated sandwiched panels, his formulation was not consistently derived.

The linear stiffness matrix was that of the assumed stress hybrid method but the geometric stiffness matrix was derived from the displacement model. The consistent model derived in this work demonstrates that the associated geometric stiffness is considerably more complicated than this and, in fact, leads to a quadratic eigenvalue problem. However, since Lundgren only satisfies the linear stress equilibrium equation then he is actually making use of the inconsistent model which is also derived in this work. In this case the geometric stiffness matrix is that associated with the displacement model (for similar displacement interpolations). The interior displacement field for the inconsistent hybrid model may be chosen independently of the interelement (boundary) displacements. If the same displacement field is chosen for both models, then the geometric stiffness would be the same. This does not mean the critical loads would be the same because the elastic stiffnesses are, in general, different.

As a test case a flat, square plate was loaded uniaxially as shown in Fig. 8.1. Observing the double symmetry a uniform mesh was used in one quarter of the plate only. Table 8.1 shows comparisons of the consistent and inconsistent models with other independent solutions and the exact solution. The rate of convergence for each model can be observed as the mesh is refined. Since Allman [1971] uses a linear moment distribution, his modified Reissner principle is identical to the inconsistent assumed stress hybrid method. Using the same displacement distributions, the results should be the same.

Among the triangular elements the assumed stress models are better than the noncompatible displacement models and about the same as the modified Reissner model and the assumed displacement hybrid model. There is essentially no difference between the consistent and inconsistent assumed stress models although computationally the inconsistent model is more efficient. This can be seen by comparing the number of computational operations required to solve the eigenvalue problems expressed by Eqs. 7.103 and 7.118 for the consistent and inconsistent models respectively. The interior displacements were chosen to be cubic with dependent rotations. (Note that for hybrid models this is not necessary.) Thus, the interior displacement fields were similar to the other models in the table. Upon detail investigation, the geometric stiffness from the complicated consistent model was practically identical to that of the simpler inconsistent model which in turn was similar to the other models. The noncompatible displacement models are much too flexible and, therefore, their elastic stiffnesses are not accurately represented. The modified Reissner model assumes stresses such that the linear stress equilibrium equation is automatically satisfied. Thus, in actuality it is the same as the inconsistent model. The assumed displacement hybrid model can provide a more accurate representation of elastic stiffness than by the standard displacement model and, therefore, yields very good results. Note that for triangular elements there are two basic variations in meshes. (See. Fig. 8.1.) The results in Table 8.1 are for the preferred orientations for each model when applicable.

In addition, some results are given for rectangular elements. While rectangular elements generally give better results than triangular elements it appears that only the fully compatible displacement model is superior. The inconsistent assumed stress hybrid model for the rectangular case is given by an independent source. This result should be better than that given in the table, however, all the details are not available.

8.3 Shallow, Sinusoidal Arch Under Sinusoidal Load

This problem was first considered by Fung and Kaplan [1952]. The authors applied Marguerre theory to the entire arch depicted in Fig. 8.2a. It is a symmetric structure with both ends pinned-fixed. The initial midsurface is sinusoidal in shape. Similarly, the load distribution is sinusoidal. To generate an exact Marguerre theory solution it is necessary to assume that the load distribution acts in the vertical direction as shown in the first of Figs. 8.2b.

It has been shown by Fung and Kaplan, for the problem they considered, that depending upon a geometric parameter of the beam, different buckling modes may occur. This parameter may be given by Λ expressed as

$$\Lambda = \frac{w_o L}{2} \sqrt{\frac{A}{I}} \qquad (8.5)$$

where

w_o = the initial central rise of the arch

L = length of the global base plane

A = arch cross-sectional area

I = arch cross-sectional area moment of inertia

Thus:

a. if $1 < \Lambda < \sqrt{5.5}$ the arch exhibits a limit load buckling behavior as shown in Fig. 8.3.

b. if $\Lambda > \sqrt{5.5}$ the arch exhibits a bifurcation buckling behavior also demonstrated in Fig. 8.3.

For the purposes of comparisons, only case a. was considered with $\Lambda=1.5$.

A finite element analysis could be carried out by discretizing the arch into shallow elements and using a single base plane for all the elements. Applying a vertical load distribution a solution corresponding to Fung and Kaplan's was obtained. The results for a load just below the buckling load are presented in Table 8.2. These solutions were obtained using an incremental-iterative procedure with all the equilibrium checks to ensure convergence. (See Subsection 8.3.6.) The six element case yields good displacement and axial load solutions, however, the moments have a considerable error. The eighteen element solution provides a very good representation of Fung and Kaplan's solution.

A more general finite element scheme would allow each element to have its own individual base plane as shown in the second of Figs. 8.2b. Here, even if the arch were nonshallow, Marguerre theory would still be valid on the element level. Since the element displacements would be measured in the local base planes they would more accurately represent the displacements in the arch midsurface rather than those in the global x-z directions. This would correspond to the Kirchhoff-Love theory. Additionally, a normal sinusoidal pressure load was to be studied for comparative purposes with another source. Since no exact Kirchhoff-Love solution exists for this arch problem under a normal load distribution, a finite element reference solution was needed. The best eighteen element solution procedure which generated the results in Table 8.2 was chosen as such a reference. Also, since slightly different results are obtained in each coordinate system, an eighteen shallow element (individual base plane) solution utilizing ten load increments and a convergence ratio of R = 0.001 was run for each system. These cases are taken as the reference solutions for each system and will be used as such for comparative purposes.

Another independent solution for this problem (under actual pressure loading) was carried out by Pirotin [1971]. His study utilized curved beam elements by the standard displacement model, a Reissner model, and what he called a modified stress hybrid model. This last model is, in fact, the basic inconsistent assumed stress hybrid model. All of his functionals are purely incremental and only include the basic forms (in that no equilibrium checks were used). According to his results the latter model gave the best results and, with this in mind, only it will be presented with the results given here.

One must be careful about interpreting some of the results to be given in the following subsections. It might appear, for instance, that a nonconverged solution for a coarse mesh is better than that for a finer mesh when compared to the reference solution. This situation is not paradoxical at all. When a curved beam is approximated by a series of straight beams some error always exists. For these models this approximation makes the model more flexible (as can be seen from the linear results) than the actual structure. As the mesh gets finer the approximation gets smaller and the model more realistically represents the actual structure. Thus, for a coarse mesh one would expect the converged solution for this model to be more flexible than the reference solution. Thus, one must be sure to compare only fully converged solutions. More of these cautions will be given in the following subsections.

8.3.1 Comparison of the Consistent and Inconsistent Assumed Stress Hybrid Models

Since a consistent model was not developed for the S.L. system, the following comparisons will be given for the C.U.L. system only. Also, the consistent model was only run for purely incremental procedures with no equilibrium checks or with only stress equilibrium checks. For the purposes of comparing these two models the following parameters were considered:

a. two load steps. A Δp = 5 lb./in. and Δp = 1.25 lb./in. (corresponding to 10 and 40 solutions steps respectively for a total load, p_o, of 50 lb./in.). The solution was allowed to continue until buckling occurred.

b. cases with and without a stress equilibrium check (no compatibility check was used).

c. Two uniform mesh sizes of 6 and 18 elements for the entire beam. (Symmetry condition was not used.)

d. flat and shallow elements.

Firstly, flat beam elements were considered. Figures 8.4-8.10 show plots of a load parameter vs. deflection, axial load, and moment parameters at the center of the arch for various increment sizes, equilibrium checks, and (in succeeding figures) mesh sizes. The specific arch considered is depicted in Fig. 8.2. In each plot the reference solution is also given.* Pirotin's best solution results are only available for displacements. Secondly, Figs. 8.11-8.17 show similar plots for shallow elements.

The results obtained from the consistent and inconsistent models are essentially the same and no plotable difference is observed in general. The only exception to this is the case of the central moment for the six element solutions (Figs. 8.8 and 8.9; 8.15 and 8.16; 8.24, 8.25 and 8.26). Here the differences are still small and for the finer mesh the difference diminishes greatly. Since for most practical situations a finer mesh would be required (especially since low order elements are being used) then one may conclude that there is no significant difference between the two models. This conclusion might have been expected in light of the similar findings of Subsection 8.2.

8.3.2 Comparison of Flat and Shallow Elements

While Figs. 8.4-8.17 may be used indirectly to obtain comparisons for various conditions, Figs. 8.18-8.27 show direct comparisons between flat and shallow elements when equilibrium checks (stress only) are used. Figs. 8.19, 8.22, and 8.26 are expanded scale plots to provide a clear distinction of the

---

*The reference solution is obtained by using eighteen shallow elements and an incremental-iterative procedure. A load increment of $\Delta p = 5$ lb./in. and a convergence ratio of $R = 0.001$ is used.

various solutions. These plots show the effects of various load steps and mesh sizes for a purely incremental procedure.

Observing Fig. 8.19 for a coarse mesh, one may see that while the flat elements converge to a solution which is too flexible and exhibits a relatively poor buckling load, the shallow elements essentially converge to the reference solution. Note also that even the flat elements with a stress equilibrium check are not in much greater error than Pirotin's curved element without any checks (with the exception of the buckling load). The shallow elements with checks are superior for converged solutions.

Similar comments can be made concerning the axial load and the bending moments. Shallow elements exhibit a significant improvement over flat elements for axial load (Fig. 8.22), but only a slight improvement for moments (Fig. 8.26). Furthermore, for the coarse mesh, the consistent model is essentially the same as the inconsistent model with the biggest differences occurring in the moments (Fig. 8.26).

For a finer mesh, Figs. 8.20, 8.23, and 8.27 demonstrate that there is very little difference between flat and shallow elements for both displacements and stresses. Additionally, no perceptable difference exists between the inconsistent and consistent models for either flat or shallow elements.

If fine meshes need to be used there appears to be no advantage of choosing the shallow element over the simpler, flat element. However, with the exception of moments, the shallow elements can produce very good results with even a coarse mesh. Thus, for shallow structures, a coarse mesh of shallow elements would be an inexpensive way of determining good approximations to the buckling load. Note that no attempt is made in this work to determine the exact buckling loads (or postbuckling behavior) for limit load buckling problems. Thus, only approximations, at best, of the buckling load can be expected.

8.3.3 Comparison of Kirchhoff-Love and Marguerre Theories

The Kirchhoff-Love and Marguerre theories for shallow structures are discussed in Subsection 3.2.2. Although both theories are valid under shallow shell theory a study was made to determine the numerical differences. This analysis is performed on the element level and should not be confused with comparing these theories on a global level. See Subsection 8.3 for further comments. Since the Kirchhoff-Love theory uses the shell coordinates, fewer transformations are necessary. Under Marguerre theory (for finite element analysis) the element stiffnesses require small transformations which may involve some loss of stiffness. Although this should not be significant, it is expected that this latter theory would be more flexible.

To determine the difference the shallow arch problem of Fig. 8.2a, with a fine mesh and a stress equilibrium check only, was run. Using a purely incremental procedure and an inconsistent model, eighty load steps were used to obtain a total load of 50 lb./in. (just under the buckling load). The results are shown in Table 8.3. There is no significant difference, as expected, between these two theories. Furthermore, the Kirchhoff-Love theory does produce a slightly stiffer solution. Admittedly this comparison could become problem dependent, however, for shallow structures it may be taken as valid.

The Marguerre theory was used for the bulk of this analysis since it is a more logical extension of flat plate theory.

8.3.4 Comparison of the S.L. and C.U.L. Systems

Tables 8.4-8.7 show results obtained for flat and shallow elements, for various conditions and solution procedures, at a load of 50 lb./in. For the nomenclature on these tables refer to Fig. 8.2.

Since the S.L. system requires fewer calculations, in that stiffnesses may simply be ratioed (see Subsection 7.1), it can be observed that solutions times were faster than the C.U.L. system for corresponding models, elements, and procedures. The total number of solution steps (increments plus iterations) are, in general, about the same.

The most important observation here is that similar solutions in both coordinate systems do not converge to the same result. As Bathe et al. [1975] have pointed out, if the two systems are consistently derived they must give the same result. This statement is, of course, true within the approximations made in the models. When modeling curved structures with flat, or even shallow elements, the model tends to be more flexible than the actual structure. This was verified by considering the linear solutions. The greater the curvature the greater the flexibility for a given mesh. If a curved structure is modelled using the S.L. system, the inherent "flexibility" error remains constant throughout the deformation process since the initial geometry is always referred to. When the C.U.L. system is incorporated, the model, geometry, and base planes are constantly changing. Thus, if the deformation is such that the curvature becomes greater, then the flexibility error will increase. In this case an S.L. system should give better results. If the reverse occurs and the deformed geometry reduces its curvature, then the flexibility error decreases rendering a better solution than the S.L. system would yield. If this error could be eliminated and if the assembly procedure was exact (for Marguerre theory) both systems would yield the same results.

Looking at the tables one observes this phenomenon taking place. The arch has an initial curvature, however, during the deformation process this curvature reduces. As expected the C.U.L. system produces better results, especially for the coarse mesh. The differences are much less pronounced with the shallow elements. This is also expected since the flexibility error is less drastic with shallow elements than with flat ones.

This phenomenon will be observed in plate and shell problems to be discussed later. Thus, if one must choose a coordinate system to use this may be an important criterion. If a structure deforms such that curvatures increase, then the S.L. should be used. If curvatures decrease the C.U.L. is superior. However, cost is always an important factor in decision making. Looking at the tables where C.P.U. times are available, one may observe that the C.U.L. is more expensive to run. For this beam problem or, in general, for any small problem the difference in cost is minimal and, perhaps, is not a factor. Larger problems, such as plates and shells, have considerable differences in expense. Thus, cost may tend to favor the S.L. system, especially where relatively small changes in curvature are expected.

8.3.5 Discussion of Equilibrium Checks

Since the C.U.L. system yields better results it was used to carry out the bulk of the comparisons of equilibrium checks. Tables 8.8 and 8.9 show results for two mesh sizes and for flat and shallow elements respectively. In addition, two increment sizes were basically used. While most of the cases were purely incremental, two cases of the incremental-iterative scheme are also shown. Note that for comparison purposes, solutions should be compared with the reference solution for the particular model. On these tables the reference solutions are the I = 10, R = 0.001 cases.

For the purely incremental cases the stress equilibrium check seems to be the most significant. The total equilibrium check (i.e. the addition of the compatibility check and the stress equilibrium check) seems to bring the solution even closer to the reference solution. However, even with a total check more than forty increments would be necessary for full convergence. On the other hand, when an incremental-iterative scheme with a total check is used, excellent results are obtained with only one load step or a total of four or five solution steps. Thus, one can see that this procedure is by far the most efficient since it combines high accuracy with great economy. The disadvantage of this approach is the lack of total information. Results are obtainable for the one final load only.

The value of the compatibility equilibrium check is most evident in the incremental-iterative scheme where large load steps are taken. Table 8.10 shows some solutions obtained for six flat elements and the S.L. system. Note that here the compatibility check makes a great difference in the solutions. As the step size increases, the compatibility check becomes even more important. In fact, without this check an unacceptable result may be obtained.

8.3.6 Comparison of Solution Procedures

Considering Tables 8.4-8.7 again it is apparent that for either coordinate system the incremental-iterative schemes are, by far, the most efficient. It is obvious that for a model utilizing either element the incremental procedure without equilibrium checks is totally unacceptable. If one chooses to use many, many increments the solutions may converge. Of course the risk of numerical round-off and solution drift will always be present. Although the stress equilibrium check alone is quite helpful, when both equilibrium checks are used an incremental procedure can yield reasonable results with a tolerable number of incremental steps. However, if these checks are to be incorporated, the incremental-iterative scheme gives the fastest, most accurate solutions. However, both checks must be incorporated to provide convergence.

The convergence ratio, R, plays an important role. Since, by definition, it compares the latest correction in incremental displacement to the initial incremental displacement for a load step (see Subsection 7.4.1), the ratio is dependent on the load step size. As can be observed in Tables 8.4-8.7, if the load steps are small the convergence ratio is not critical. However, for large load steps, the value of this parameter may become critical.

The advantages of this latter system are many. Besides the obvious one of efficiency, this scheme virtually allows the analyst to choose as many or as few information points (load steps) as is necessary and still be assured that at each one the solution is converged to within the ability of the model. Since the cost (number of solution steps) is proportional to the number of information points, the analyst can make a better decision relating to the information/cost trade-off. Thus, if only maximum load information is needed an inexpensive technique is available without giving non-essential data.

It should be noted here that when small increments must be used, such as in the case of material nonlinearities defined by the incremental theory of plasticity, this approach fails from an efficiency standpoint (in that small increments must be used).

8.3.7 Adequacy of Models and Methods

In dealing with curved structures one ideally should use curved elements. However, such elements are often complicated and time consuming (computationally). Thus, flat or shallow elements are well worth considering. To model a curved structure with these simplified elements generally requires using a fine mesh. Although the system of equations increases with mesh size, such a substantial savings can be realized from the simplicity of element generation that the overall cost may be comparable to higher order elements. For complicated structures or situations where fine detail in distributions is required a fine mesh would be necessary regardless of the element sophistication.

Considering Tables 8.4 and 8.5 one may observe that for fine meshes the flat elements are reasonably adequate. Since these elements are so simple to generate this is a very appealing approach to curved structures. One finds, however, that with a small increase in sophistication, shallow elements give quite good results (Tables 8.6 and 8.7) even for coarse meshes (of a shallow structure). There does seem to be a degradation in the moments. To obtain good displacements and stresses one must utilize a fine mesh. For a given mesh size the shallow elements always produce better results. It might be stated that for crude results using a crude (coarse) mesh one would be better off using shallow elements, while for fine, detail meshes either flat or shallow elements may be used.

It is worthwhile mentioning here that the consistent models yield essentially the same results as the inconsistent models. This is true for both flat and shallow elements. However, with a much greater computational effort required, the efficiency of the consistent models is low. Because of this observation, as well as the one in Subsection 8.2, it was decided to eliminate this model from further numerical study.

The comparisons given in Subsections 8.3.3, 8.3.5, and 8.3.6 led to the decision that for plate and shell analysis the Marguerre theory would be used and, with the exception of a few cases, a total equilibrium check with an incremental-iterative procedure would be used for further analysis. The comments of Subsection 8.3.4 indicated that although for fine meshes the S.L. and C.U.L. systems give approximately the same results, the better choice is somewhat problem dependent. It must be remembered that this effect is due solely to the approximations in the elements and not because of any inconsistencies in the general theory.

Overall, however, the efficiency of the inconsistent flat and shallow models seems quite adequate. Note that Pirotin's best result, where no checks are used, is still in error. Considering Fig. 8.11 it can be seen that the simple shallow elements with no checks give comparable results to Pirotin's curved elements. When a stress equilibrium check is added better solutions are obtained with the shallow elements.

8.4 Shallow, Circular Arch Under a Central Concentrated Load

The shallow circular arch shown in Fig. 8.28 was studied. Two mesh sizes of three and nine elements/half span utilizing shallow elements and the S.L. system were used. A total equilibrium check was incorporated in an incremental-iterative scheme. Two load step sizes were used. In the first, ten load increments were used to demonstrate the overall shape of the solution. The second used just one step corresponding to the maximum load used in the previous case. For both load step sizes a convergence ratio of R = 0.001 was used. Both meshes yield excellent results as shown in Fig. 8.29 differing from each other only near the buckling load. Note that when only one load step is used, its solution corresponds exactly to the total load of the ten step solution. These solutions are extremely efficient. The symmetry condition was not utilized and the C.P.U. times, therefore, refer to a solution for the entire arch. By modelling only half the arch these execution times would all be below one second. The independent solutions are by Dupuis [1971], Bathe [1973], and Mallet [1966].

8.5 Flat Plates Under Uniform Loads

The large deflection solutions of a simply supported plate by triangular elements, were studied in some detail. The effects of mesh size, solution technique, coordinate system, and element type were studied. Based on these results only a limited study was conducted for a clamped plate.

Stresses are generally given at the node corresponding to the point of interest. Since the moments vary linearly, in general, within an element, the nodal value can be determined. If two or more elements meet at the same node, then nodal averaging is used. The membrane stresses are only constant within an element. Thus, the average of all element values connected to a node are used. If only one element contains the node of interest, then the values from the two elements forming a quadrilateral containing that node are averaged.

8.5.1 Simply Supported Plates

Levy [1942a] was one of the first investigators of this problem. His series solutions are used as a reference. Later investigators include Bäcklund [1973], Bergan [1972], and Kikuchi and Ando [1973]. The example considered is a thin, square, initially flat plate shown in Fig. 8.30. Bäcklund used a modification of Reissner's principle, Bergan used a displacement model, and Kikuchi and Ando used the assumed displacement hybrid approach. While the two former writers included the effects of equilibrium checks the latter used only a basic functional. Bergan utilized quadrilateral elements where the other finite element schemes utilized triangular elements. Bäcklund and Bergan used 4x4 meshes of flat and shallow elements respectively in a symmetric quarter of the plate while Kikuchi and Ando used a 5x5 mesh of shallow elements.

Figures 8.31 and 8.32 show some solutions at the center of the plate by the present method. Here a simple 2x2 mesh of flat elements (π^{I}_{mc}) and a C.U.L. system was run varying the increment size in a purely incremental scheme. Additionally, the effect of a stress equilibrium check was measured. Although the stress equilibrium check aided the displacement solution, using a smaller increment size had a more pronounced effect. It was possible to achieve a reasonable displacement solution with this coarse mesh. For the stress solutions the same general comments apply. It appears, however, that the shape of the moment curve is substantially different from Levy's.

Figures 8.33 and 8.34 show the same solutions for a 4x4 mesh. Essentially little improvement is apparent in the displacement and membrane stresses. While the moment solution has shifted considerably, its shape is still substantially different than Levy's. It turns out, however, that the present moment solution agrees very well with Bäcklund and Bergan as shown. Others [Murray and Wilson, 1969] have also shown Levy's moment solution to be in error.

From these plots two facts became evident. Firstly, that while displacements and membrane stresses benefitted very little from a finer mesh, the moment solution was highly dependent upon mesh size. Secondly, that the stress equilibrium check was necessary for an improved solution and small load steps are necessary. Since equilibrium checks are necessary, a more efficient scheme would be to utilize the incremental-iterative methods with a total equilibrium check (to ensure convergence).

Figures 8.35 and 8.36 show such a case. Using an S.L. system, and a 4x4 mesh, three load steps, and a convergence ratio of 0.001, the analysis was run out to a considerable degree of nonlinearity. Excellent agreement was achieved for both displacements and stresses (noting the error in Levy's moment solutions).

Using the same basic procedure a variety of other cases were run for a load parameter of 150 for easy comparison. Various load steps, both coordinate systems for flat elements, and a shallow element in the C.U.L. system were used. For the S.L. system only flat elements (or a shallow element degenerated to a flat element) can be used because the initial geometry is flat. Table 8.11 presents these solutions with others previously mentioned as references. The nomenclature on this table corresponds to that of Fig. 8.30.

The flexibility error discussed in Subsection 8.3.4 can be observed here. Since the deformation causes an increase in curvature it is expected that the S.L. system will produce better results than the C.U.L. system. This is observed for flat elements although the effect is small. Also, the shallow element should be little affected by this. One notes that the shallow element result is, in fact, stiffer than the flat one. Another significant fact is the cost differential between the two coordinate systems. The reasons for this are given in Subsection 8.3.4.

While all the displacement solutions agree well, there are discrepancies in the stresses. The present solutions agree reasonably well with Bäcklund and Bergan. Although Kikuchi and Ando use a 5x5 mesh and many more incremental steps (this is a purely incremental procedure) the membrane stress (no moment available) seems a bit higher than the other finite element solutions. This solution may not be fully converged since no checks or iteration are used. Again, it is difficult to say what the exact moment solution is, but reasonable agreement with Bäcklund and Bergan is achieved. A curious phenomenon occurs with the moment solution for the shallow element. It turns out that the moment distribution within the element is poor and, therefore, this result is only marginal. The element centroidal moment is in good agreement with the other results.

8.5.2 Clamped Plates

Way [1938] and Levy [1942b] solved this problem by a series solution. A finite element solution was carried out by Kikuchi and Ando using the same system as for simply supported plates. Prato [1968] utilizes a mixed model based on Reissner's principle. In a quarter of the plate an 8x8 mesh of flat, triangular elements and an S.L. system are utilized. Equilibrium checks are incorporated in an iterative scheme.

Table 8.12 shows some results obtained with a 4x4 mesh (π_{mc}^{I}) by the present methods compared to some independent solutions. The plate considered is shown in Fig. 8.37. Basically, a total equilibrium check and an incremental-iterative procedure were used. For the S.L. system a flat element case was run and for the C.U.L. system a flat and shallow element case were run. Note that two different values of Poisson's ratio were run because Way and Levy ran two different values. Such a small change (.300 to .316) should not significantly alter the results, however, the two series solutions show a discrepancy in the membrane stresses. In the present work no such difference occurred.

Only limited data is available for Kikuchi and Ando, and Prato. The displacements all agree reasonably well. One observes the flexibility error coming into play for the flat elements. The membrane stress solutions also agree well with the reference finite element schemes. They also fall somewhere between the two series solutions. For the moments, the solutions by the present method seem to agree reasonably well with each other and both series solutions. Prato's result seems somewhat high.

It is worthwhile mentioning that although the total number of solution steps required for each of the cases run under the present method is the same, the total execution time is quite different. This is accounted for by the greater computational effort (in terms of integrations) required by the C.U.L. system.

While the flat elements in the S.L. system give adequate results, the shallow elements seem to yield superior stresses. From an efficiency standpoint, however, the flat elements in the S.L. system is the superior choice.

8.6 Shallow, Cylindrical Panel Under Uniform Load

Although no series solution for this problem is given, there are three finite element schemes used for comparison. In addition to the works of Kikuchi and Ando, and Prato (from the previous subsection), a displacement solution is presented by Brebbia and Connor [1969]. The particular panel under consideration is shown in Fig. 8.38. Brebbia and Connor use the standard displacement method in an S.L. system. For a quarter of the panel an 8x8 mesh of rectangular, shallow, cylindrical panel elements are used. Prato used a 6x6 mesh.

Table 8.13 shows various solutions by the present method using a 4x4 mesh compared to the above references. The location of these values is noted in Fig. 8.38. Note that in the S.L. system two types of shallow shell elements are used. In the first case, only the flat plate, linear moments are used with the z terms (height above base plane) kept in the functional. The second case satisfies the total shallow, linear moment equations. This was discussed in Subsection 7.2.1. For the C.U.L. system a flat element and a shallow element of the latter type were used.

The two types of shallow elements exhibit basically the same behavior. In fact, with the exception of one of the moment results, they agree quite well. These shallow elements have a stiffer behavior than the flat elements. The flat elements again exhibit the flexibility error effect betweeen the two coordinate systems. Since, in this case, the deformation tends to reduce the curvature, the C.U.L. system yields better results. However, considering the cost factor involved with the C.U.L. system the S.L. results yield more efficient solutions.

The displacement solutions for the shallow elements seem to agree relatively well with Brebbia and Connor, and Prato. Kikuchi and Ando seem to exhibit a flexible solution, however, since this information was extracted from a plot of low resolution this value may not actually be so high. The membrane stress solutions all seem to agree well regardless of the model. Note again the discrepancy in moments. Comparison is only available from Prato and again his result seems to be considerably high. Since his resulting moments were high in the previous subsection when compared to other independent solutions, it is suggested that the moments obtained by the present methods are more accurate.

8.7 Spherical Cap Under Central Concentrated Load

The problem depicted in Fig. 8.39 was solved by Leicester [1966] using a series solution, by Dhatt [1970] and by Thomas and Gallagher [1975] using finite element schemes and an S.L. system. These latter models are modifications of the displacement method, utilizing triangular shell elements.

It is the only example of a doubly curved shell given in this work. The structure actually exhibits a snap through buckling behavior as discussed by the above authors. The present analysis does not include post buckling behavior, but the example is a good one in that buckling loads are often difficult to obtain. Utilizing a 4x4 mesh in a quarter of the shell (as opposed to a 3x3 mesh used by the above writers) flat and shallow elements were run for the S.L. system while just a flat element was run for the C.U.L. system. A total equilibrium check incorporated into an incremental-iterative scheme was used.

For the S.L. system, four load steps with a convergence ratio of 0.001 were used to ensure that the shape of the solutions was correct. In the C.U.L. system only one load step with a convergence ratio of 0.00001 was used to ensure convergence. The flat elements in the S.L. system yield good results (Fig. 8.40) until near the buckling load. Even at this point, the error is of the order of three percent. However, since the deformation of the actual structure tends to reduce the curvature it is expected that the C.U.L. system would yield better results. To show this, a C.U.L. system using flat elements and only one load step very near the buckling load was run. As can be seen, a stiffening affect does occur. A shallow element, where the moment satisfies the complete linear equation, was run next using an S.L. system. This solution is in excellent agreement with the other independent sources.

8.8 Shallow, Cylindrical Panel Under Axial Compression

Shown in Fig. 8.41 this problem has been considered by Schmit et al. [1968] (see also Bogner, 1968) utilizing an energy minimization scheme in the S.L. system and by Pirotin [1971] using what actually amounts to an inconsistent assumed stress hybrid finite element scheme. Both writers use rectangular shell elements. Schmit et al. has a high order assumed displacement shell element with 48 dofs. Pirotin's element has 20 dofs and uses a C.C. system for solution. While the former uses an iterative solution procedure, the latter uses a purely incremental scheme with no equilibrium check terms, requiring a large number of load steps.

The boundary conditions of this problem attempt to simulate the edgewise loading of a cylindrical panel in a testing machine. The compression-deformation behavior is demonstrated in Fig. 8.42. Note that the central node first begins to rise and then reverses its direction until snap through buckling occurs. In this figure Pirotin's solution corresponds reasonably well with Schmit's. Both these authors used 2x2 meshes in a quarter of the shell and it is not clear what would happen if more elements were used.

Using an S.L. system and flat elements, by the present method the solution using a 2x2 mesh is very poor. Although a 4x4 mesh yields a reasonable result, a 6x6 mesh gives a good comparison. Furthermore, when one considers the average edge load, $\overline{N}_x$, versus the edge compression (Fig. 8.43) it becomes evident that the load carrying capability predicted by the present method converges to a considerably lower value than the independent solutions.

SECTION 9

SUMMARY AND CONCLUSIONS

9.1 Summary

Numerical methods have been developed to study the large deflection, small strain behavior of thin, linearly elastic structures. These procedures are based upon two variations (the inconsistent and the consistent models) of the assumed stress hybrid finite element method.

Unlike small deflection theory, in large deflection analysis the deformed and undeformed structure may no longer be thought of as existing in coincident coordinate frames. Even under small strain restrictions the rotations, corresponding to rigid body mdoes, may be large. Thus, different coordinate systems may be used to describe the deformed and undeformed geometries. Basic descriptions of these coordinate systems are presented. The relative advantages and disadvantages of using a single fixed frame (stationary systems) or using combinations of coordinate frames (updated systems) are discussed. Furthermore, the concept of moving coordinate frames (convected systems) to further facilitate the analysis is discussed.

Of course, the use of such coordinate frames involves different definitions of stresses and strains. These tensors, as well as the constitutive relations, are carefully defined with respect to their reference frames. Since energy principles are sought, it becomes necessary to insure that consistent sets of stresses and strains are used. Additionally, the energy principles are shown to be equivalent regardless of the reference frame used. Although these concepts were introduced in a completely general sense, the convenient reductions in complexity due to certain approximations are stated. Namely the approximations of linear elasticity, small strains, and moderate rotations are discussed. Because of some confusion which seems to exist, comparisons in terminology of the present work and a few selected authors in the literature are tabulated.

Since even the simple initial geometry of a structure may become complex after deformation, elements which could conform to such distortion are ideally required. However, general, doubly curved elements are difficult and costly to generate. This, coupled with the normally costly process of nonlinear analysis, suggested the use of simpler elements which could approximate the general geometry. The simplest such element is a flat one. Adding only a slight degree of sophistication results in a shallow element. Thus, two node, six degree of freedom flat and shallow beam elements were developed. For plate and shell analysis, three node, fifteen degree of freedom flat and shallow triangular elements were developed. The reasons for this were twofold. Firstly, they permitted an inexpensive tool for comparative purposes. Secondly, the effectiveness of these simple elements could be tested.

The linear theory was, therefore, extended to large deflection, small strain, moderate rotation theory for thin structures obeying the Kirchhoff hypotheses. The equations of elasticity are discussed under two shallow structure theories, namely the Kirchhoff-Love and Marguerre theories. The further reductions in the general theory applicable to thin plate and shell structures are discussed. The relationships of the various coordinate systems are discussed in prospective with the appropriate theories and approximations. A result of these discussions is to choose the coordinate systems most appropriate for the ensuing analysis.

Two such systems were chosen. The first of these is the Stationary Lagrangian (S.L.) system. Based on this frame of reference a general incremental formulation of the variational principles is given. Starting with the Principle of Virtual Work the derivation develops the appropriate variational statements for the Principle of Stationary Total Potential Energy and the more general Hu-Washizu principle. From this, Reissner's principle is derived and, by appropriate expansion, the modified Reissner principle for an assemblage of elements is established. The analysis is then extended to the consistent assumed stress hybrid functional and its counterpart, the inconsistent assumed stress hybrid functional. Through the entire development no approximations are made until the very end. At this point the variational statements are appropriately linearized under the assumptions of small increment size. Additionally, the complete equilibrium checks are identified. It is further shown how the stress equilibrium check, in particular, can be more easily incorporated in these hybrid formulations than the conventional Hofmeister et al. type check.

The second coordinate system chosen for study is a combination of two more popular updated systems. It is the Convected, Updated Lagrangian (C.U.L.) system. A parallel derivation to that of the S.L. system is developed for easy comparison. Thus, the corresponding variational statements, and comments, are given. It is noted that while these assumed stress hybrid functionals could have been developed more directly from the Principle of Virtual Complementary Work, given in Appendix B, the more conventional approach is used so that a variety of variational statements could be shown.

The general matrix equations associated with the consistent and inconsistent assumed stress hybrid models is developed in detail for each of the two coordinate systems. The matrices associated with the basic functionals, and each type of equilibrium check (stress and compatibility) are separately identified. It is then suggested that under certain assumptions some matrix terms may be removed or altered to simplify the analysis. They include assuming that stress equilibrium and/or compatibility is exactly satisfied in the reference state; that external loads may be lumped at the nodes; and that certain displacement mismatch conditions are removed.

From these matrix equations, the element level tangent stiffness matrix and load vector are identified. Utilizing the appropriate coordinate transformations, it is shown how these element level matrices may be assembled into global matrices and finally into global equations necessary for solution. Once the global equations are established the general methods and procedures for solution are discussed. Various incremental and incremental-iterative schemes with various equilibrium checks are explored.

These general matrix equations are then considered for the actual thin linear elastic elements to be used. Flat and shallow (Marguerre) beam elements for the consistent and inconsistent models in the S.L. and C.U.L. systems are discussed. Further comments about shallow beam elements based on the Kirchhoff-Love theory are made. The details of flat and shallow (Marguerre) triangular elements appropriate for plate and shell analysis are given. For nonlinear plate and shell analysis only the inconsistent model was deemed useful. It was thus developed for both coordinate systems. For the linear prebuckling of plates, however, both the consistent and inconsistent models were utilized.

With the detail matrices now available the fine points of the computational procedures are discussed. The merits and drawbacks of three solution schemes are examined carefully. The details of the step by step methods are shown. The updating of the displacements, stresses, strains, and geometry for both coordinate systems is discussed. General comments are given concerning what the most efficient processes are. Finally, this information is put to use to make actual numerical investigations.

The first example considered was the linear prebuckling of a flat, square plate under uniform, uniaxial compression. Both models were run with varying mesh sizes, and comparisons were made with various independent solutions. For the first nonlinear case a simple shallow, sinusoidal arch under sinusoidal pressure was run. This problem was used to make several investigations relatively inexpensively. Such studies included comparisons of the inconsistent and consistent models, of the flat and shallow elements, of the Kirchhoff-Love and Marguerre theories, of the S.L. and C.U.L. systems, of the equilibrium checks, and of the solution procedures. From this information the adequacy of the models, elements, and procedures was discussed. Decisions as what to use for further analysis was also made. Utilizing the most efficient schemes from this problem, a shallow, circular arch under a central concentrated load was considered. This problem, which is compared to other independent solutions, demonstrates the great efficiency which can be achieved by the present work.

The large deflection of simply supported and clamped, square, flat plates was considered next. Some of the investigations made for the sinusoidal arch problem were run here to further verify some decisions previously made. Only brief investigations were made for three shell problems. These studies generally included performance comparisons of flat and shallow elements and of the S.L. and C.U.L. systems. The first of these was a shallow, cylindrical panel under uniform pressure. This problem was useful for element and coordinate system comparisons. The next was a spherical cap under a central concentrated load. This was the only doubly curved shell considered. Additionally, this problem exhibts a limit load buckling phenomena. Since buckling loads are often difficult to obtain, this case gave useful information as to the accuracy of the elements. Lastly, a shallow, cylindrical panel under axial end shortening was run.

The conclusions based upon these numerical calculations are the subject of the next subsection. Following that are some suggestions for future research. Finally, the table at the end of this subsection is a partial listing of the work which can be found in the literature which pertains to the finite element analysis of the large deflection, small strain behavior of thin, linearly elastic structures.

The nomenclature used in the table may be interpreted as follows:

$\underline{\pi}$: This represents the variational statements upon which the finite element method is based. The functionals in the table correspond to the models listed below.

π_p = displacement model (compatible and noncompatible)

π_{hd} = assumed displacement hybrid model (based on a modification of π_p)

π_R = mixed model (based on Reissner's principle)

π_{mR} = mixed model (based on modified Reissner principle)

π_{mc}^I = inconsistent assumed stress hybrid model

π_{mc}^c = consistent assumed stress hybrid model

<u>COORD</u>: This corresponds to the coordinate system used.

SL = Stationary Lagrangian system

CC = Convected Coordinate system

UL = Updated Lagrangian system

CUL = Convected, Updated Lagrangian system

<u>SOLN</u>: This designates the general solution procedure used. Note that there are many ways to incorporate such solution procedures. Thus, the details of each solution technique can be found in each reference. They may not be quite the same as in this work.

ES = energy search (energy minimization)

SS = successive substitution

SC = self correcting

PC = predictor corrector

I = incremental

I-I = incremental-iterative

Note that this last case may be specialized to either a purely incremental or a purely iterative procedure.

<u>CHK:</u> This refers to the type of equilibrium check used. Since there are many ways to apply such checks one must refer to each reference for the exact type of check, designated here by 'E'.

- = no check terms

E = equilibrium check used (see reference for specifics)

T = total equilibrium check used (stress and compatibility)

In this work, this last case may be specialized to include each check separately.

<u>ELEMENT:</u> This refers to the type of elements to be found in each reference. The code may generally be split into the following:

X Y Z..

- Z.. — specific element type
- Y — curvature of element
- X — type of analysis used for

X = B: beam (or arch) analysis

P: flat plate analysis

S: shell analysis

ISO: isoparametric elements

Y = F: flat element

S: shallow element

C: curved element

A: axisymmetric element

Z = CP: cylindrical panel

SR: shell of revolution

T: triangular

DCT: doubly curved triangle

DCR: doubly curved rectangle

π	COORD:	SOLN.	CHK.	ELEMENT	REFERENCE
π_p	SL	ES	-	P, SCCP	SCHMIT ET AL. [1968]
π_p	SL	SS	-	SASR	STRICKLIN ET AL. [1968]
π_p	SL	SC	E	SASR	STRICKLIN ET AL. [1972]
π_p	SL	PC	-	BS	PIAN AND TONG [1971]
π_p	SL	I-I	E	ISO	BATHE ET AL. [1974]
π_p	SL	I-I	E	SST	YAO [1968]
π_p	SL	I-I	E	SSCP	BREBBIA AND CONNOR [1969]
π_p	SL	I-I	E	SCDCT	RODRIQUEZ [1969]
π_p	SL	I-I	E	SCDCT	THOMAS AND GALLAGHER [1975]
π_p	CC	I	-	BC	PIROTIN [1971]

π	COORD:	SOLN.	CHK.	ELEMENT	REFERENCE
π_p	UL	I	-	SASR	YAGHMAI [1969]
π_p	UL	I-I	E	ISO	BATHE ET AL. [1974]
π_p	CUL	I-I	E	SFT	MURRAY AND WILSON [1969]
π_p	CUL	I	-	SASR	BELYTSCHKO AND HSIEH [1973a]
π_p	CUL	I	-	BF, SFT	BELYTSCHKO AND HSIEH [1973b]
π_{hd}	SL	I	-	SST	KIKUCHI AND ANDO [1973]
π_{hd}	UL	I-I	E	ISO	ATLURI ET AL. [1975]
π_R	SL	PC	-	BS, SASR	PIAN AND TONG [1971]
π_R	CC	I	-	BC	PITOTIN [1971]
π_{mR}	SL	I-I	E	SST	PRATO [1968]

π	COORD:	SOLN.	CHK.	ELEMENT	REFERENCE
π_{mR}	CUL	I-I	E	BF, SF	BÄCKLUND [1973]
π^{I}_{mc}	SL	I-I	T	BF, BS	PRESENT WORK
π^{I}_{mc}	SL	I-I	T	SFT, SST	PRESENT WORK
π^{I}_{mc}	CC	I	-	BC, SCDCR	PIROTIN [1971]
π^{I}_{mc}	CUL	I-I	T	BF, BS	PRESENT WORK
π^{I}_{mc}	CUL	I-I	T	SFT, SST	PRESENT WORK
π^{c}_{mc}	CC	I	-	BC	ATLURI [1973b]
π^{c}_{mc}	CUL	I-I	T	BF, BS	PRESENT WORK

9.2 Conclusions

Based upon the general derivations given in Sections 4 and 5 as well as those given in Appendix B, it becomes obvious that unlike linear analysis, stresses and displacements couple through the stress equilibrium equations. This becomes a drawback of the assumed stress hybrid methods, for now displacement fields must be chosen on the interior of the elements. Additionally, for the consistent model, the stress interpolations must contain both unknown stress and displacement quantities. As discussed in Section 6 this leads to two difficulties. Firstly, for an S.L. system unsymmetric stresses must be assumed which may, in general, be complicated. This is highly unattractive and alternative stresses (first Piola) may have to be used throughout. Secondly, in both coordinate systems the computational effort to generate element level matrices is enormous compared to more standard approaches, or the inconsistent model. The only justification for such an increase in expense would be a comparable increase in accuracy.

The linear prebuckling of a flat plate and the sinusoidal arch problems clearly demonstrate that the results obtained by the consistent model (in the C.U.L. system) are only, at best, slightly better than those obtained by the inconsistent model. The conclusion must be drawn that, for these simple elements, the former model is definitely less efficient. Although the inconsistent model appears to be very similar to a modified Reissner model, it should be computationally slightly more efficient as may be seen from Appendix C. Pirotin [1971] has also demonstrated the superiority of the basic inconsistent model to a basic model derived from Reissner's principle.

Comparisons demonstrate that, based on these assumed stress hybrid functionals, shallow elements perform better than flat elements for a given mesh size. It is also shown that the solution of a problem requires approximately the same number of solution steps for both elements. Yet the shallow elements require only an insignificant amount of additional computation time. Thus, from an efficiency standpoint, it appears that shallow elements are superior. However, if fine meshes are used flat elements will yield comparable results. Either of these elements will yield, in general, satisfactory results for curved structures. If extreme accuracy is needed then perhaps a more sophisticated element is required. For most practical engineering applications, these simple elements will suffice.

Although not much difference is observed between the Kirchhoff-Love and Marguerre theories utilized on the element level it is more convenient to use the latter for simple elements. While the former might be slightly more accurate and more elegant, the latter is more straightforward and a more logical extension of plate theory.

For the elements and assumptions considered in this work the S.L. system has considerable computational advantage over the C.U.L. system. Thus, from an efficiency point of view it is generally better. However, because of the approximations in the elements (especially the flat ones) different solutions are obtained by each system. This is basically attributed to the "flexibility" errors discussed in Section 8 and not to the basic derivations. If accuracy is the only criterion for the choice of system to be used, then a simple rule generally applies. If the structure deforms such that its curvature increases, then the S.L. system will yield more accurate results. If deformation causes a decrease in curvature, then the C.U.L. is more appropriate.

From an efficiency point of view, all the equilibrium checks should be included regardless of the element type or coordinate system considered. In conjunction with this, the incremental-iterative procedure is by far the most efficient. It must be stressed that these statements pertain only to large deflection, small strain, linear elasticity, where large incremental steps may be used.

The inconsistent assumed stress hybrid model seems to yield comparable results with other models and methods previously considered. It is difficult to make definitive appraisals as to its efficiency. However, it is felt that this model should be at least as efficient as the more widely used schemes for most problems.

9.3 Suggestions for Further Research

Since this work was limited to rather simple elements the inconsistent model should be used to formulate higher order elements. As a first type, the elements of Appendix E should be considered in more detail. Ultimately, deep, doubly curved elements should be developed (see Piortin [1971] and

Tanaka [1969]). Additionally, it would be interesting to see how fully three dimensional elements would behave.

Models using the modified Reissner principle, outlined in matrix form in Appendix C, should also be investigated. Since the only difference between this model and the inconsistent assumed stress hybrid model lies in the satisfaction of the linear stress equilibrium equations, it would be interesting to compare them to each other. Although the former model requires a small addition in computation time, the latter may suffer from imposing an inconsistent constraint.*

Continuing on this line of thought, it might be appropriate in the inconsistent model for the boundary tractions to be equated to the linear boundary stress only (in a manner corresponding to the stress equilibrium equations). The effect of this would be to eliminate nonlinear boundary traction terms in the inconsistent functionals. Since such terms are always multiplied by displacement mismatch terms, their effect may always be small. If it can be shown that these terms are negligible (perhaps for just a class of problems) great simplifications can be made.

The behavior of such models should also be studied under combined nonlinearities. The effects of plasticity alone have been studied by Spilker [1974]. Combining material nonlinearities, such as plasticity and creep, with geometric nonlinearities is an essential engineering problem.

Although the consistent assumed stress hybrid model has proven to be difficult to deal with, perhaps an alternative stress approach could be investigated. As shown in the derivation of the Principle of Virtual Complementary Work (Appendix B), the first Piola stress is used. It may be that this, or some other definition of stress, would yield a rather simplified functional. In fact, written in the form of the Piola stresses the stress-displacement coupling is eliminated and, perhaps, a functional with the advantages the linear systems exhibit may emerge. Of course, the drawback here is that the Piola stresses are unsymmetric. However, this may not be a problem.

*Recently, Horrigmoe [1975] has applied the modified Reissner principle to large deflection analysis.

The most logical first extension of this work would be to include the postbuckling behavior of structures under the same assumptions and approximations made here. This would involve changing the solution scheme at (or actually just before) the point of buckling. Before this change the analysis could be carried out as in this work. When large softening occurs (a large number of solution steps required in a load increment) the independent variable for solution must become the displacement rather than the load.

Choosing one displacement, which has the largest rate of change with respect to the load, the corresponding load increment and the remaining nodal displacements can be determined using a scheme presented by Pian and Tong [1971]. The procedure must be used for an incremental step with no equilibrium imbalance load included. This is so because the load distribution must be known within a single multiplicative constant which is to be determined and not known a priori. Once this step is carried out, the analysis may return to an iterative procedure until convergence for the determined load step is achieved. From this point on one should always proceed at the beginning of a load step by changing the independent variable since the load path may change directions again (a change in sign of the stiffness matrix determinant). It is conceivable that upon iteration the solution will tend to diverge. If this is the case, a smaller incremental displacement must be chosen to determine the size of the load step.

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TABLE 8.1

COMPARISON OF MODELS FOR THE LINEAR PREBUCKLING OF A FLAT, SQUARE, SIMPLY SUPPORTED PLATE UNDER A UNIFORM, UNIAXIAL EDGE COMPRESSION (EXACT SOLUTION = 4.000)

NO. OF DIV. FOR ONE QUARTER OF PLATE	TRIANGULAR ELEMENTS					
	DISPLACEMENT		MOD. REISSNER	HYB. DISPL.	ASSUMED STRESS HYBRID	
	NONCOMPATIBLE				INCONSISTENT	CONSISTENT
	ANDERSON ET AL. [1968]	ALLMAN [1971]	ALLMAN [1971]	KIKUCHI AND ANDO [1972]	PRESENT WORK	PRESENT WORK
1x1				4.479		
2x2	3.72	3.72	4.031	4.021	4.030	
4x4		3.94	4.006	4.003	4.005	4.005
5x5	3.90					
6x6						

	RECTANGULAR ELEMENTS				
	DISPLACEMENT				HYB. STRESS
	NONCOMPATIBLE		COMPOUND	COMPATIBLE	INCONSISTENT
	KAPUR AND HARTZ [1966]	DAWE [1969]	CLOUGH AND FELIPPA [1969]	CARSON AND NEWTON [1969]	LUNDGREN [1969]
1x1					
2x2	3.770	3.978	4.126	4.001	
4x4	3.933	3.993	4.031	4.000	3.945
5x5					
6x6	3.977				

TABLE 8.2

COMPARISON OF A FINITE ELEMENT SOLUTION (UTILIZING SHALLOW ELEMENTS IN THE STATIONARY LAGRANGIAN SYSTEM) WITH EXACT MARGUERRE SHALLOW ARCH THEORY FOR A SHALLOW, SINUSOIDAL ARCH UNDER VERTICAL SINUSOIDAL PRESSURE

($\bar{p}$ = 187.5, SEE FIG. 8.2b)

	FUNG AND KAPLAN [1952]	6 ELEMENT SOLUTION		18 ELEMENT SOLUTION	
		SOLUTION	% ERROR	SOLUTION	% ERROR
$\bar{W}$	0.62600	0.62719	0.190	0.62583	-0.027
$\bar{N}$	2.60755	2.60943	0.072	2.60717	-0.015
$\bar{M}$	1.64757	1.82102	10.528	1.66613	1.127

$$\bar{W} = \frac{W_c}{2} \quad \frac{A}{I}$$

$$\bar{N} = \frac{N}{EA} (10^3)$$

$$\bar{M} = \frac{ML}{EI} (10)$$

$$\bar{p} = \frac{p_o L^4}{2EI} \quad \frac{A}{I}$$

TABLE 8.3

COMPARISON OF THE KIRCHHOFF-LOVE AND MARGUERRE THEORIES FOR THE SHALLOW, SINUSOIDAL ARCH PROBLEM ($\bar{p}$ = 187.5)

(SEE FIGURE 8.2)

PARAMETER	KIRCHHOFF-LOVE	MARGUERRE	% DIFFERENCE
$\bar{W}$	0.635	0.637	0.315
$\bar{N}$	2.634	2.636	0.076
$\bar{M}$	1.673	1.675	0.121

$$\bar{W} = \frac{W_c}{2} \quad \frac{A}{I}$$

$$\bar{N} = \frac{N}{EA}\,(10^3)$$

$$\bar{M} = \frac{ML}{EI}\,(10)$$

TABLE 8.4

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF VARIOUS SOLUTION PROCEDURES AND MESH SIZES FOR FLAT ELEMENTS IN THE STATIONARY LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 6 FLAT ELEMENTS (π^{I}_{mc})								
		S=10 t=0.38	S=40 t=1.01	S=10 t=0.38	S=40 t=1.14	S=25 t=0.77	S=33 t=0.93	S=15 t=0.48	S=18 t=0.53	S=6 t=0.21
$\bar{W}$	0.644	0.601	0.699	0.679	0.763	0.827	0.828	0.826	0.828	0.825
$\bar{N}$	2.648	2.517	2.754	2.711	2.902	3.043	3.044	3.040	3.044	3.039
$\bar{M}$	1.692	1.597	1.856	1.803	2.025	2.196	2.197	2.192	2.197	2.191

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 18 FLAT ELEMENTS (π^{I}_{mc})								
		S=10 t=1.03	S=40 t=3.11	S=10 t=1.08	S=40 t=3.27	S=22 t=1.94	S=29 t=2.46	S=13 t=1.18	S=16 t=1.35	S=5 t=0.49
$\bar{W}$	0.644	0.556	0.619	0.613	0.649	0.655	0.655	0.654	0.655	0.655
$\bar{N}$	2.648	2.422	2.583	2.571	2.658	2.672	2.672	2.671	2.672	2.672
$\bar{M}$	1.692	1.448	1.612	1.597	1.690	1.705	1.705	1.704	1.705	1.705

SEE FIG. 8.2a FOR NOMENCLATURE

TABLE 8.5

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF VARIOUS SOLUTION PROCEDURES AND MESH SIZES FOR FLAT ELEMENTS IN THE CONVECTED, UPDATED LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 6 FLAT ELEMENTS $(\pi\frac{I}{mc})$								
		S=10 t=0.41	S=40 t=1.25	S=10 t=0.45	S=40 t=1.35	S=22 t=0.86	S=29 t=1.02	S=13 t=0.57	S=16 t=0.64	S=5 t=0.21
$\bar{W}$	0.638	0.592	0.677	0.649	0.692	0.696	0.696	0.697	0.696	0.696
$\bar{N}$	2.634	2.497	2.708	2.647	2.749	2.760	2.760	2.762	2.760	2.760
$\bar{M}$	1.684	1.578	1.806	1.737	1.854	1.867	1.867	1.868	1.866	1.866

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 18 FLAT ELEMENTS $(\pi\frac{I}{mc})$								
		S=10 t=1.12	S=40 t=3.75	S=10 t=1.17	S=40 t=4.01	S=22 t=2.37	S=29 t=3.08	S=13 t=1.55	S=16 t=1.74	S=4 t=0.52
$\bar{W}$	0.638	0.554	0.617	0.608	0.640	0.644	0.644	0.644	0.644	0.644
$\bar{N}$	2.634	2.419	2.578	2.558	2.637	2.647	2.647	2.647	2.647	2.647
$\bar{M}$	1.684	1.450	1.613	1.593	1.679	1.691	1.691	1.691	1.691	1.691

SEE FIG. 8.2a FOR NOMENCLATURE

TABLE 8.6

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF VARIOUS SOLUTION PROCEDURES AND MESH SIZES FOR SHALLOW ELEMENTS IN THE STATIONARY LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 6 SHALLOW ELEMENTS (π^{I}_{mc})								
		S=10 t=0.42	S=40 t=1.08	S=10 t=0.39	S=40 t=1.13	S=22 t=0.67	S=29 t=0.84	S=13 t=0.42	S=16 t=0.49	S=5 t=0.18
$\bar{W}$	0.644	0.552	0.612	0.607	0.640	0.645	0.645	0.645	0.645	0.645
$\bar{N}$	2.648	2.412	2.567	2.557	2.638	2.650	2.650	2.650	2.650	2.650
$\bar{M}$	1.692	1.584	1.756	1.743	1.835	1.849	1.849	1.849	1.849	1.849

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 18 SHALLOW ELEMENTS (π^{I}_{mc})								
		S=10 t=1.05	S=40 t=3.16	S=10 t=1.09	S=40 t=3.32	S=22 t=2.02	S=29 t=2.60	S=13 t=1.20	S=16 t=1.38	S=5 t=0.53
$\bar{W}$	0.644	0.551	0.611	0.606	0.639	0.644	0.644	0.644	0.644	0.644
$\bar{N}$	2.648	2.411	2.566	2.556	2.636	2.648	2.648	2.647	2.648	2.648
$\bar{M}$	1.692	1.448	1.606	1.593	1.679	1.692	1.692	1.692	1.692	1.692

SEE FIG. 8.2a FOR NOMENCLATURE

TABLE 8.7

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF VARIOUS SOLUTION PROCEDURES AND MESH SIZES FOR SHALLOW ELEMENTS IN THE CONVECTED, UPDATED LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 6 SHALLOW ELEMENTS (π^I_{mc})								
		S=10 t=0.46	S=40 t=1.41	S=10 t=0.46	S=40 t=1.49	S=22 t=0.95	S=29 t=1.19	S=13 t=0.60	S=16 t=0.68	S=4 t=0.22
$\bar{W}$	0.638	0.551	0.611	0.604	0.635	0.639	0.639	0.639	0.639	0.639
$\bar{N}$	2.634	2.410	2.565	2.550	2.626	2.636	2.636	2.636	2.636	2.636
$\bar{M}$	1.684	1.574	1.738	1.703	1.782	1.790	1.790	1.790	1.790	1.790

	REF. SOLN.	I=10	I=40	I=10 TEQCK	I=40 TEQCK	I=10 R=0.01	I=10 R=0.001	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001
		PRESENT WORK 18 SHALLOW ELEMENTS (π^I_{mc})								
		S=10 t=1.24	S=40 t=4.12	S=10 t=1.32	S=40 t=4.49	S=22 t=2.65	S=29 t=3.30	S=13 t=1.66	S=16 t=1.89	S=4 t=0.59
$\bar{W}$	0.638	0.550	0.610	0.603	0.634	0.638	0.638	0.638	0.638	0.638
$\bar{N}$	2.634	2.409	2.564	2.548	2.624	2.634	2.634	2.634	2.634	2.634
$\bar{M}$	1.684	1.450	1.608	1.590	1.673	1.684	1.684	1.684	1.684	1.684

SEE FIG. 8.2a FOR NOMENCLATURE

TABLE 8.8

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF EQUILIBRIUM CHECKS, SOLUTION PROCEDURES, AND MESH SIZES FOR FLAT ELEMENTS IN THE CONVECTED, UPDATED LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN.	I=40	I=10	I=40	I=10 SEQCK	I=40 SEQCK	I=10 TEQCK	I=40 TEQCK	I=10 R=0.001	I=1 R=0.001
		PIROTIN [1971]	PRESENT WORK 6 FLAT ELEMENTS (π^{I}_{mc})							
		S=40	S=10 t=0.41	S=40 t=1.25	S=10	S=40	S=10 t=0.45	S=40 t=1.35	S=29 t=1.02	S=5 t=0.21
$\bar{W}$	0.638	0.590	0.592	0.677	0.634	0.695	0.649	0.692	0.696	0.696
$\bar{N}$	2.634	-	2.497	2.708	2.640	2.757	2.647	2.749	2.760	2.760
$\bar{M}$	1.684	-	1.578	1.806	1.693	1.854	1.737	1.854	1.867	1.866

	REF. SOLN.	I=40	I=10	I=40	I=10 SEQCK	I=40 SEQCK	I=10 TEQCK	I=40 TEQCK	I=10 R=0.001	I=1 R=0.001
		PIROTIN [1971]	PRESENT WORK 18 FLAT ELEMENTS (π^{I}_{mc})							
		S=40	S=10 t=1.12	S=40 t=3.75	S=10	S=40	S=10 t=1.17	S=40 t=4.01	S=29 t=3.08	S=4 t=0.52
$\bar{W}$	0.638	0.590	0.554	0.617	0.594	0.635	0.608	0.640	0.644	0.644
$\bar{N}$	2.634	-	2.419	2.578	2.552	2.637	2.558	2.637	2.647	2.647
$\bar{M}$	1.684	-	1.450	1.613	1.549	1.664	1.593	1.679	1.691	1.691

SEE FIG 8.2a FOR NOMENCLATURE

TABLE 8.9

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISON OF EQUILIBRIUM CHECKS, SOLUTION PROCEDURES AND MESH SIZES FOR SHALLOW ELEMENTS IN THE CONVECTED, UPDATED LAGRANGIAN SYSTEM ($\bar{p}$ = 187.5)

	REF. SOLN.	I=40	I=10	I=40	I=10 SEQCK	I=40 SEQCK	I=10 TEQCK	I=40 TEQCK	I=10 R=0.001	I=1 R=0.001
		PIROTIN [1971]	PRESENT WORK 6 SHALLOW ELEMENTS (π^{I}_{mc})							
		S=40	S=10 t=0.46	S=40 t=1.41	S=10	S=40	S=10 t=0.46	S=40 t=1.49	S=29 t=1.19	S=4 t=0.22
$\bar{W}$	0.638	0.590	0.551	0.611	0.589	0.638	0.604	0.635	0.639	0.639
$\bar{N}$	2.634	-	2.410	2.565	2.543	2.636	2.550	2.626	2.636	2.636
$\bar{M}$	1.684	-	1.574	1.738	1.660	1.780	1.703	1.782	1.790	1.790

	REF. SOLN.	I=40	I=10	I=40	I=10 SEQCK	I=40 SEQCK	I=10 TEQCK	I=40 TEQCK	I=10 R=0.001	I=1 R=0.001
		PIROTIN [1971]	PRESENT WORK 18 SHALLOW ELEMENTS (π^{I}_{mc})							
		S=40	S=10 t=1.24	S=40 t=4.12	S=10	S=40	S=10 t=1.32	S=40 t=4.49	S=29 t=3.30	S=4 t=0.59
$\bar{W}$	0.638	0.590	0.550	0.610	0.587	0.629	0.603	0.634	0.638	0.638
$\bar{N}$	2.634	-	2.409	2.564	2.540	2.622	2.548	2.624	2.634	2.634
$\bar{M}$	1.684	-	1.450	1.608	1.545	1.654	1.590	1.673	1.684	1.684

SEE FIG. 8.2a FOR NOMENCLATURE

TABLE 8.10

SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE-COMPARISONS OF THE EQUILIBRIUM CHECKS FOR THE INCREMENTAL-ITERATIVE PROCEDURE (SIX FLAT ELEMENTS IN THE STATIONARY LAGRANGIAN SYSTEM) ($\bar{p}$ = 187.5)

(SEE FIG. 8.2 FOR NOMENCLATURE)

	PRESENT WORK (π^{I}_{mc})					CONVERGED SOLUTION FOR THIS MODEL
	I=10 INCREM.	I=10 SEQCK R=0.001	I=10 TEQCK R=0.001	I=5 SEQCK R=0.001	I=5 TEQCK R=0.001	
	S=10 t=0.38	S=27 t=0.84	S=33 t=0.93	S=16 t=0.49	S=18 t=0.53	
$\bar{W}$	0.601	0.712	0.828	0.677	0.828	0.828
$\bar{N}$	2.517	2.817	3.044	2.770	3.044	3.044
$\bar{M}$	1.597	1.889	2.197	1.800	2.197	2.197

TABLE 8.11

SIMPLY SUPPORTED FLAT PLATE UNDER UNIFROM PRESSURE-COMPARISONS OF SOLUTIONS FOR VARIOUS LOAD STEP SIZES, SOLUTION PROCEDURES, EQUILIBRIUM CHECKS, COORDINATE SYSTEMS, AND ELEMENTS (4x4 MESH) WITH INDEPENDENT SOLUTIONS ($\bar{p}$ = 150)

(SEE FIG. 8.30 FOR NOMENCLATURE)

	LEVY [1942a]	BÄCKLUND [1973] BERGAN [1972]	KIKUCHI AND ANDO [1973]	PRESENT WORK (π^{I}_{mc})						
	SERIES SOLN.	MIXED EQCK	HYB. DISPL.	I=10	I=10 TEQCK	I=5 R=0.01	I=5 R=0.001	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001
		U.L	S.L.	S.L.	S.L.	S.L.	S.L.	S.L.	C.U.L.	C.U.L.
		FLAT	SHALLOW	FLAT	FLAT	FLAT	FLAT	FLAT	FLAT	M^L-NZ
				S=10 t=16.2	S=10 t=17.4	S=14 t=24.3	S=15 t=25.7	S=5 t=8.8	S=5 t=36.10	S=5 t=35.98
$\bar{W}_c$	1.46	1.47	1.47	1.61	1.46	1.46	1.46	1.46	1.48	1.46
$\bar{N}_c$	6.92	6.58	6.70	6.20	6.50	6.49	6.49	6.50	6.28	6.43
$\bar{M}_c$	8.63	7.86	-	8.47	7.54	7.54	7.54	7.55	8.10	8.88*

*MOMENT AT CENTROID OF ELEMENT = 7.58

TABLE 8.12

CLAMPED, FLAT PLATE UNDER UNIFORM PRESSURE-COMPARISONS OF SOLUTIONS FOR VARIOUS COORDINATE SYSTEMS AND ELEMENTS (4x4 MESH) WITH INDEPENDENT SOLUTIONS ($\overline{p}$ = 150)

(SEE FIG. 8.37 FOR NOMENCLATURE)

	WAY [1938]	LEVY [1942b]	KIKUCHI AND ANDO [1973]	PRATO [1968]	PRESENT WORK (π^{I}_{mc})			
	SERIES SOLN.	SERIES SOLN.	HYB. DISPL.	MIXED EQCK	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001
	ν=.300	ν=.316	ν=.300	ν=.300	ν=.316	ν=.300	ν=.300	ν=.300
			S.L.	S.L.	S.L.	S.L.	C.U.L.	C.U.L.
			SHALLOW	SHALLOW	FLAT	FLAT	FLAT	M^{L}-NZ
					S=3 t=5.39	S=3 t=5.55	S=3 t=21.21	S=3 t=21.51
$\overline{W}_c$	1.160	1.170	1.16	1.18	1.171	1.178	1.230	1.182
$\overline{N}_c$	1.121	0.875	0.963	-	0.938	0.938	0.915	0.948
$\overline{N}_{s_x}$	1.269	0.758	-	0.838	0.892	0.890	0.615	0.812
$\overline{M}_c$	2.294	2.390	-	-	2.180	2.140	2.300	2.395
$\overline{M}_{s_x}$	7.648	7.580	-	10.85	6.500	6.490	6.700	6.695

TABLE 8.13

CLAMPED, SHALLOW, CYLINDRICAL PANEL UNDER UNIFORM PRESSURE-COMPARISON OF SOLUTIONS FOR VARIOUS COORDINATE SYSTEMS AND ELEMENTS (4x4 MESH) WITH INDEPENDENT SOLUTIONS (p_o = 0.15 psi)

(SEE FIG. 8.38 FOR NOMENCLATURE)

	BREBBIA AND CONNOR [1969]	KIKUCHI AND ANDO [1973]	PRATO [1968]	PRESENT WORK (π^I_{mc})				
	DISPL. EQCK	HYB. DISPL.	MIXED EQCK	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001	I=1 R=0.001
	S.L.	S.L.	S.L.	S.L.	S.L.	S.L.	C.U.L.	C.U.L.
	SHALLOW	SHALLOW	SHALLOW	FLAT	M^L ONLY	M^L-NZ	FLAT	M^L-NZ
				S=3 t=5.56	S=3 t=5.47	S=3 t=6.00	S=3 t=22.33	S=3 t=21.92
W_c	0.0528	0.0595	0.0528	0.0560	0.0526	0.0528	0.0548	0.0534
N_{c_x}	-	16.67	15.70	16.69	16.50	16.51	16.54	16.56
N_{c_y}	-	-	3.44	3.12	3.27	3.26	3.25	3.27
M_{s_x}	-	-	0.395	0.226	0.211	0.169	0.241	0.150
M_{s_y}	-	-	1.000	0.638	0.613	0.616	0.630	0.621

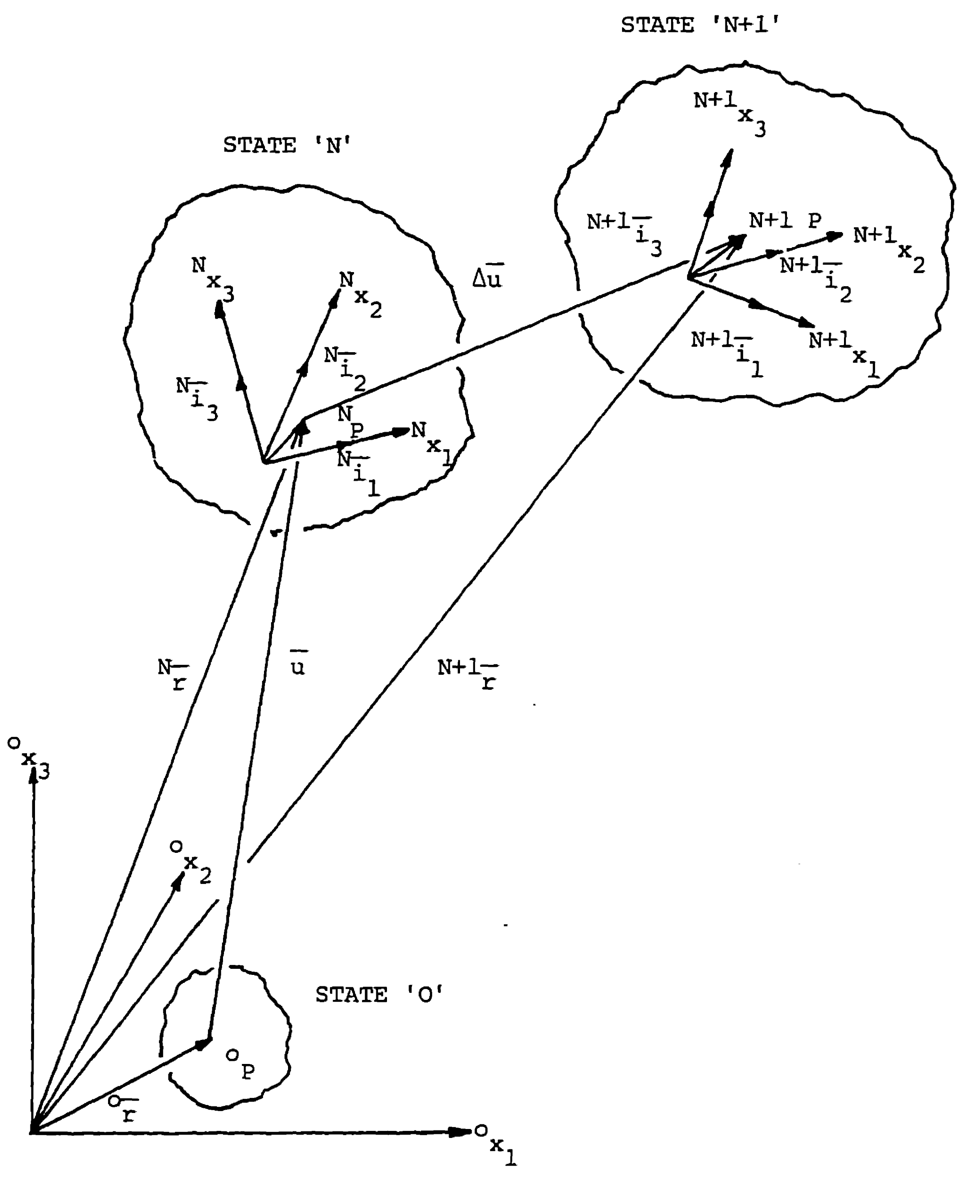


FIG. 2.1 DESCRIPTION OF DEFORMATION STATES AND ASSOCIATED COORDINATE SYSTEMS

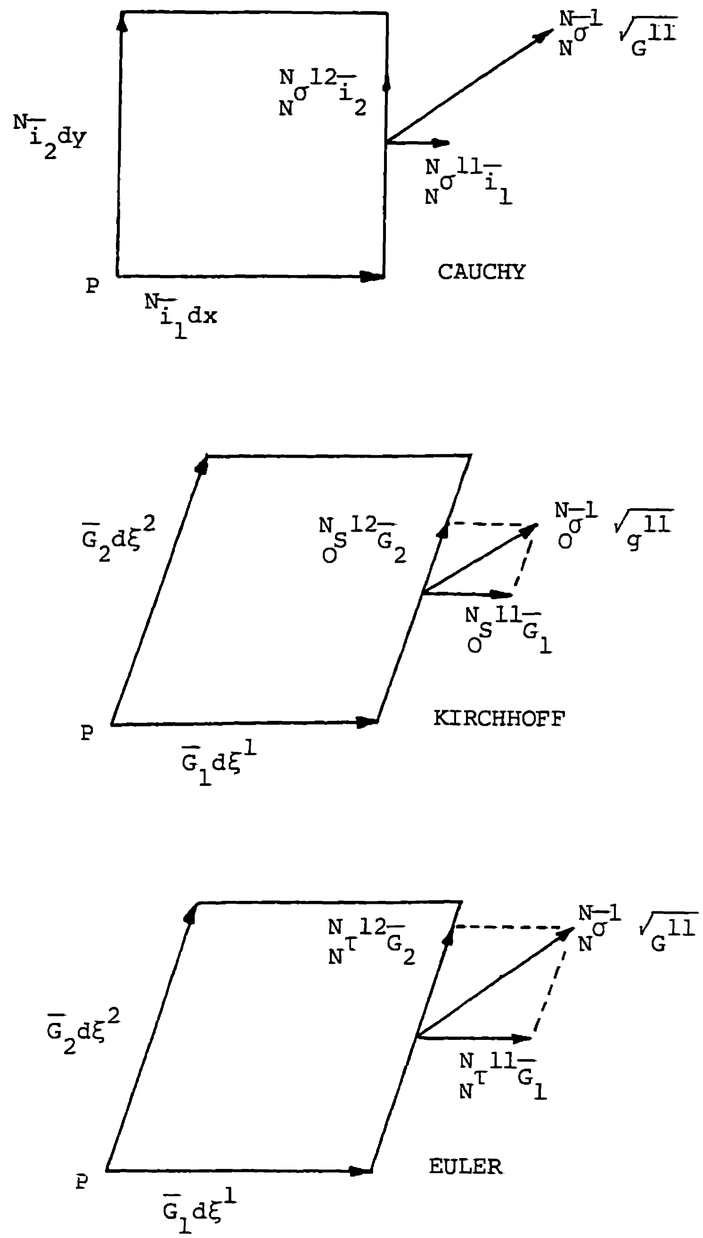


FIG. 2.2 DEFINITIONS OF SYMMETRIC STRESSES

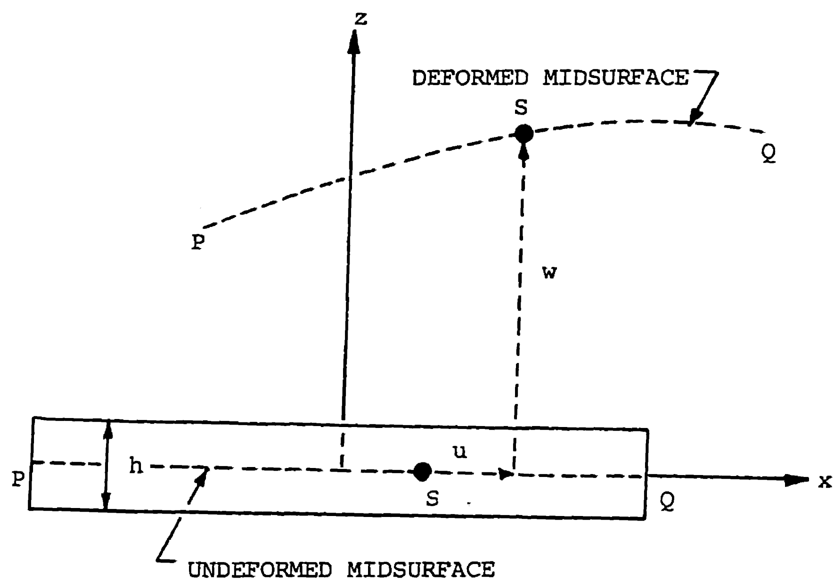


FIG. 3.1 GEOMETRY AND COORDINATES FOR FLAT PLATES

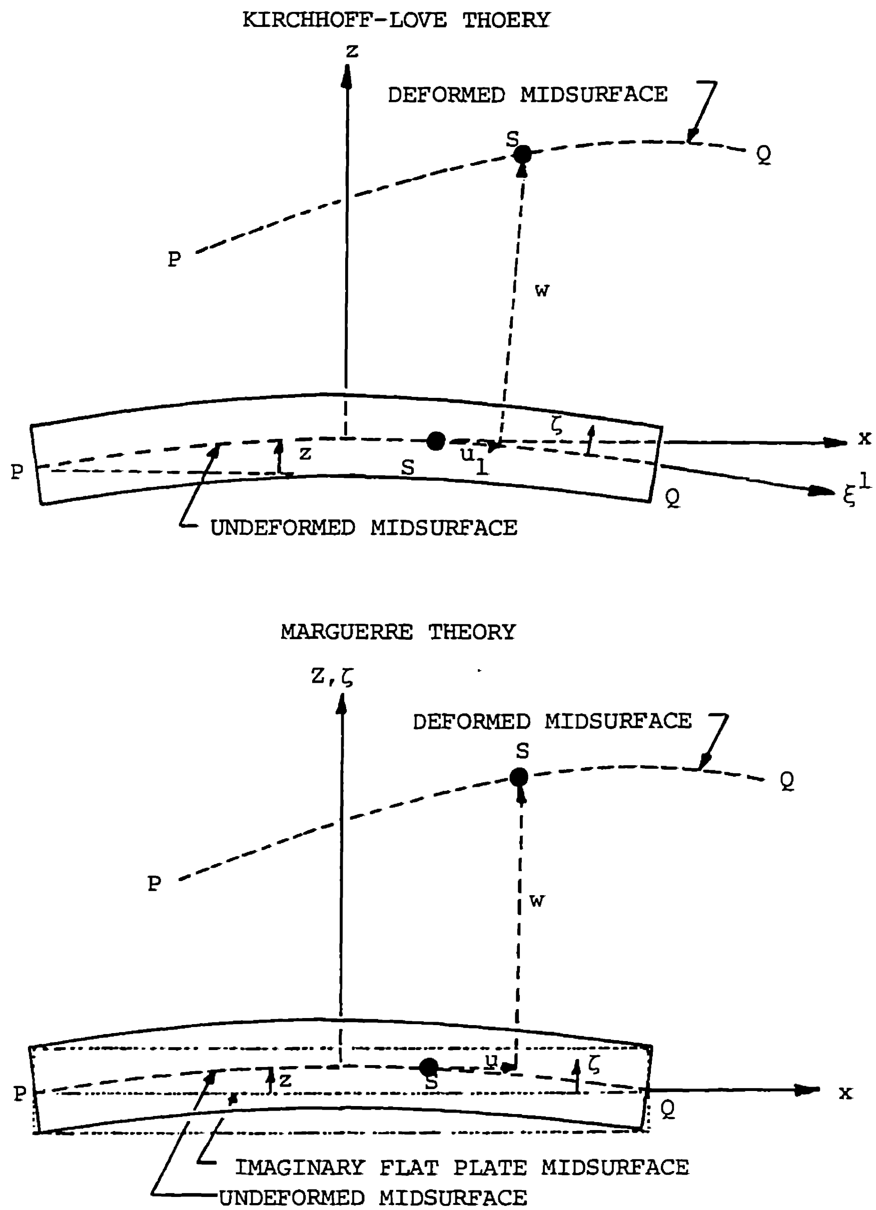


FIG. 3.2 GEOMETRY AND COORDINATES FOR SHALLOW SHELLS

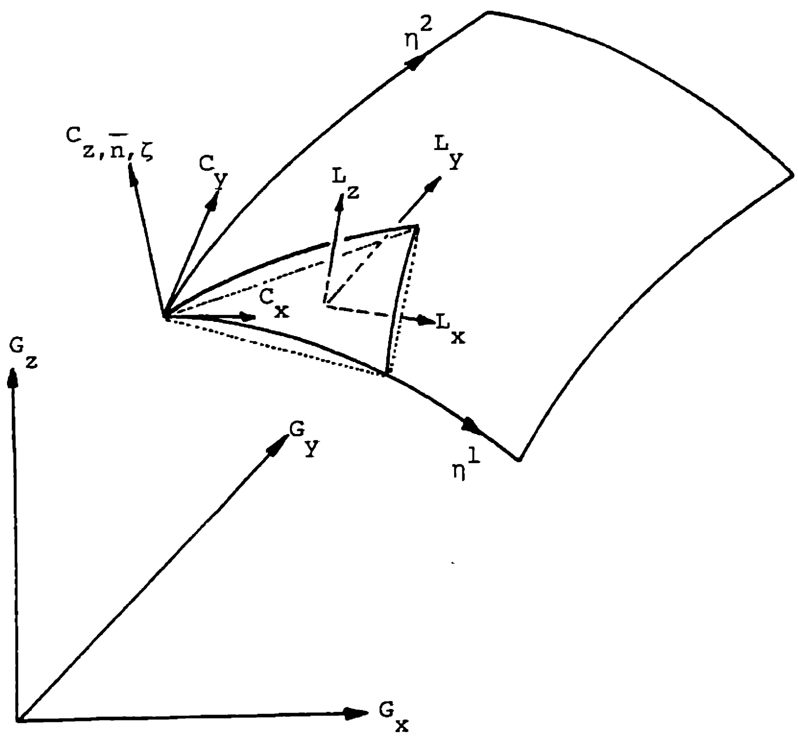


FIG. 3.3 THE LOCAL, GLOBAL, AND COMMON COORDINATE SYSTEMS

$$T_i^a(S) + T_i^b(S) = 0 \qquad (i = 1,2,3)$$

$$\longrightarrow \int_{PQ} \mu_i(S)\ [T_i^a(S) + T_i^b(S)]dS$$

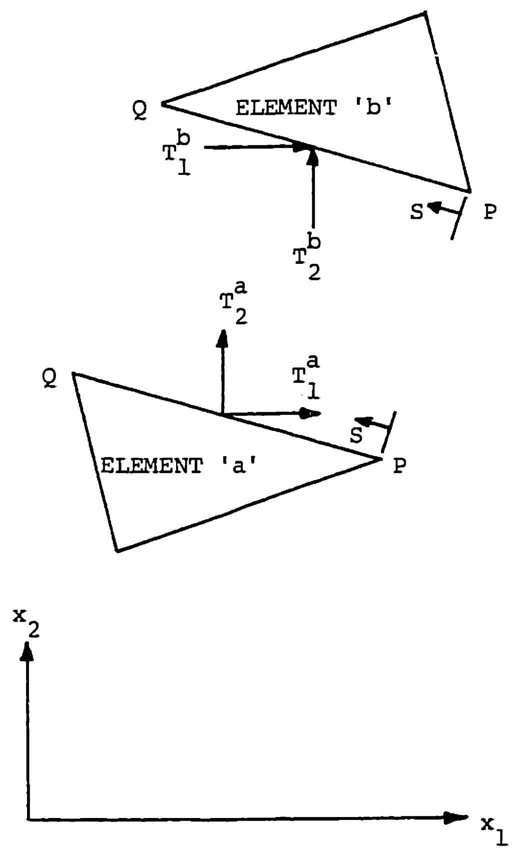
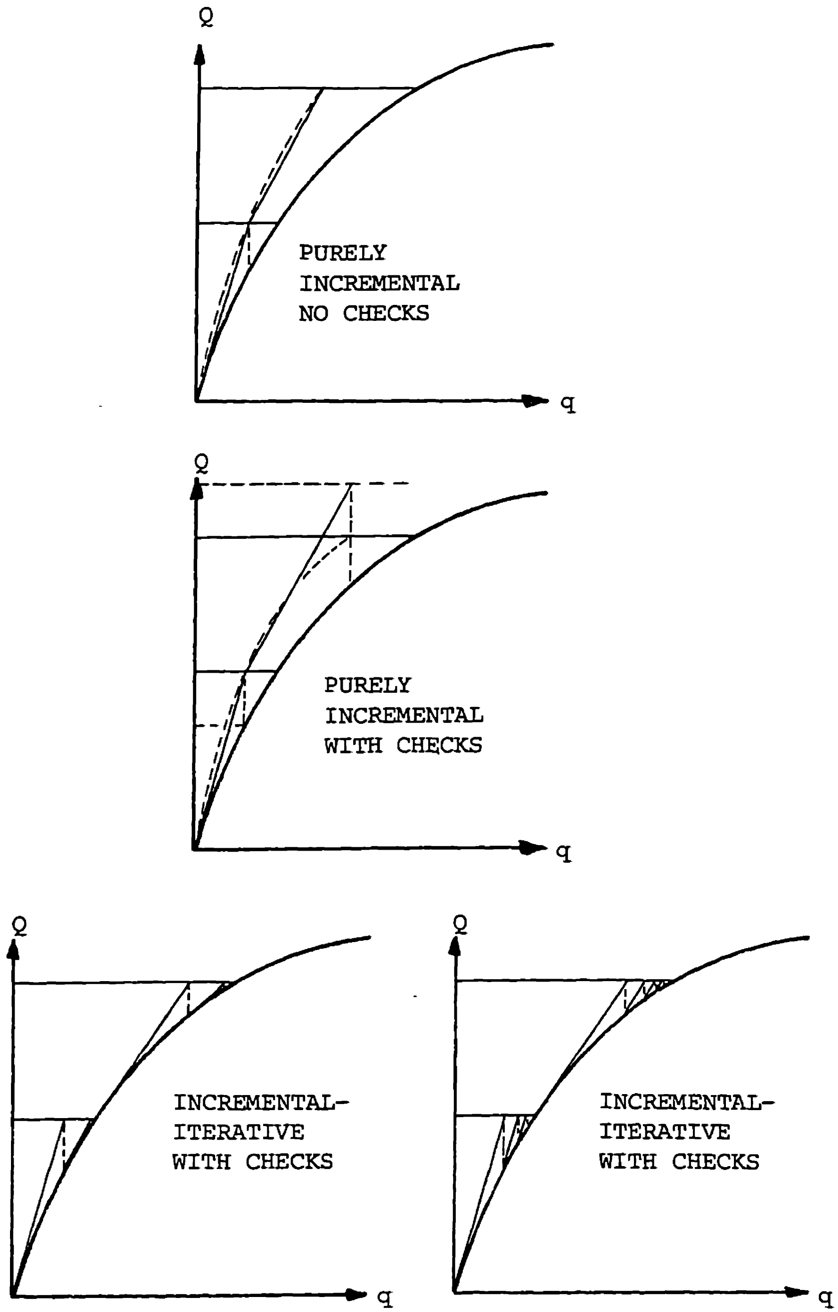


FIG. 4.1 INTERELEMENT BOUNDARY TRACTIONS



NEW TANGENT STIFFNESS AT EVERY SOLUTION STEP (USED IN THIS WORK)

NEW TANGENT STIFFNESS AT BEGINNING OF LOAD STEP

FIG. 6.1 GENERAL SOLUTION PROCEDURES

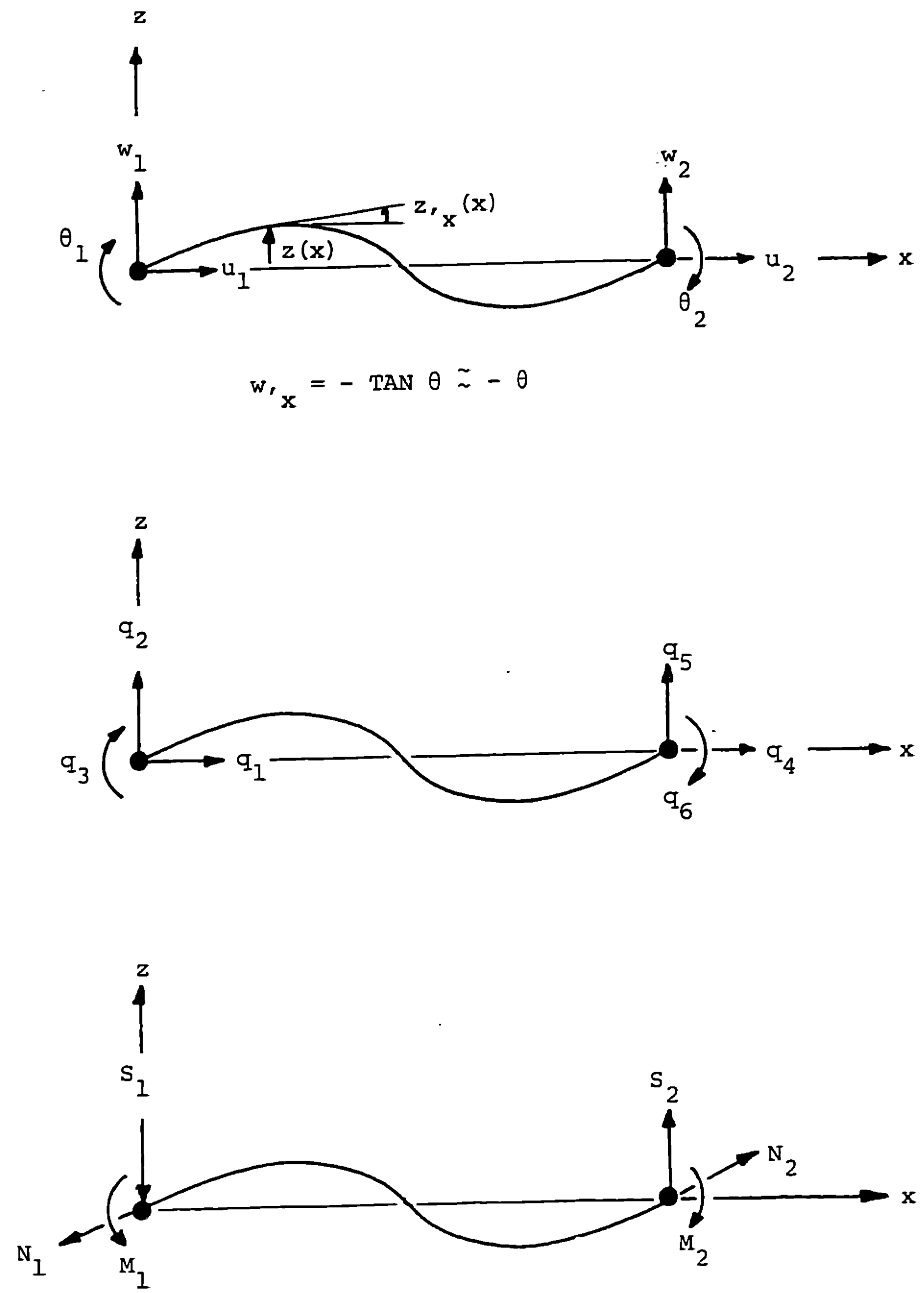
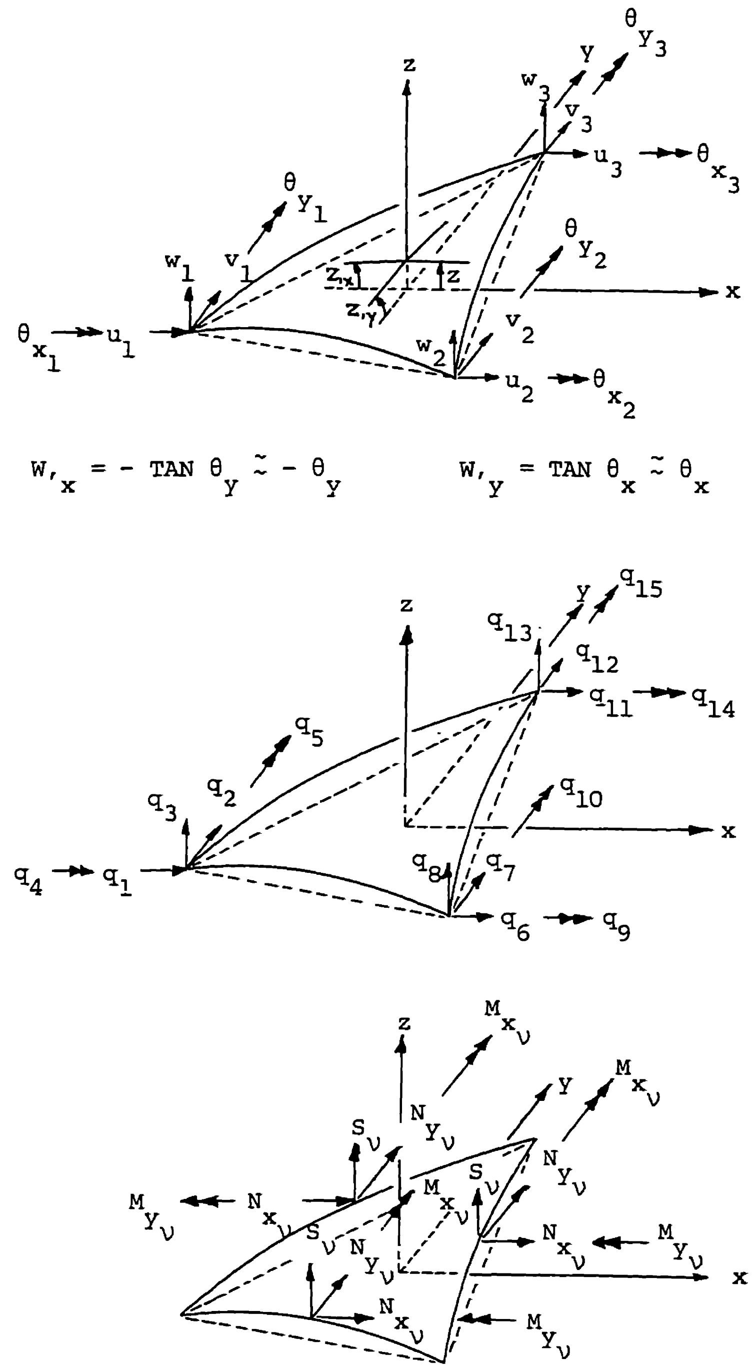
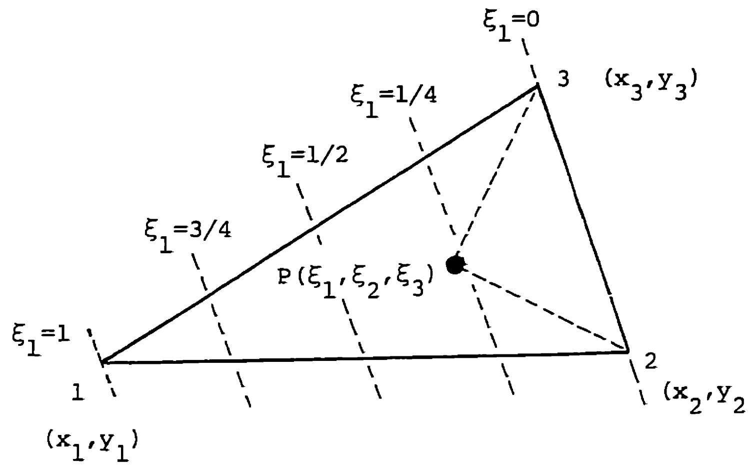


FIG. 7.1 SIGN CONVENTIONS FOR BEAM ELEMENTS



NOTE: STRESS RESULTANTS ARE ACTUALLY IN SHELL SURFACE
(SEE PAGE 124 FOR REMARKS ON S_ν)

FIG. 7.2 SIGN CONVENTION FOR SHELL ELEMENTS



$$\xi_1 = \frac{\text{AREA (P23)}}{\text{AREA (123)}}$$

$$x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$$

$$y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3$$

$$1 = \xi_1 + \xi_2 + \xi_3$$

$$\xi_1 = (a_1+b_1x+c_1y)/2\Delta$$

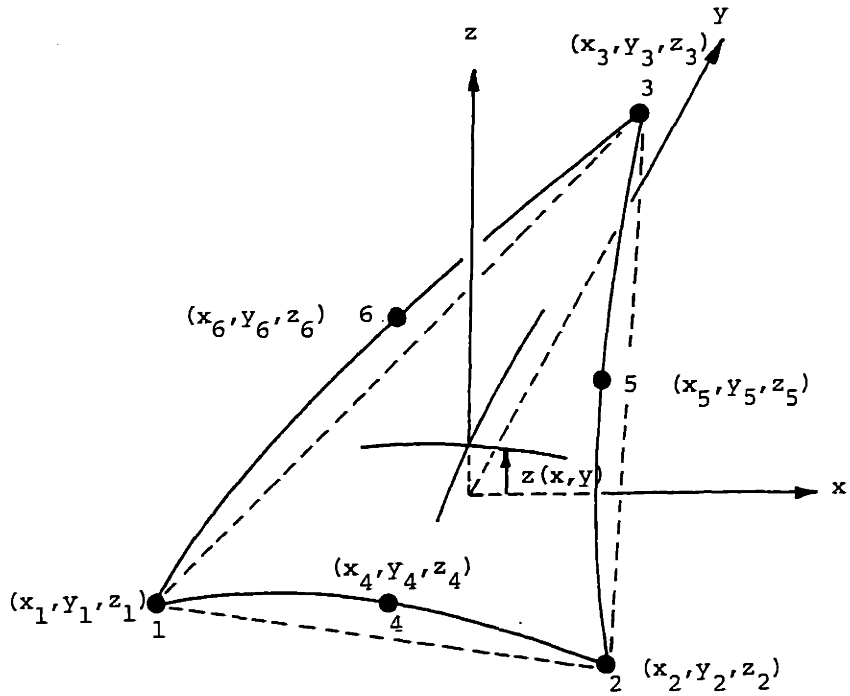
$$\xi_2 = (a_2+b_2x+c_2y)/2\Delta$$

$$\xi_3 = (a_3+b_3x+c_3y)/2\Delta$$

$$\left.\begin{aligned} a_1 &= x_2y_3-x_3y_2 \\ b_1 &= y_2-y_3 \\ c_1 &= x_3-x_2 \end{aligned}\right\} \quad \text{USE CYCLIC PERMUTATION FOR OTHERS}$$

$$\Delta = \frac{1}{2}\,\text{DET}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{AREA (123)}$$

FIG. 7.3 DEFINITION OF AREA COORDINATES (OR TRIANGULAR COORDINATES)

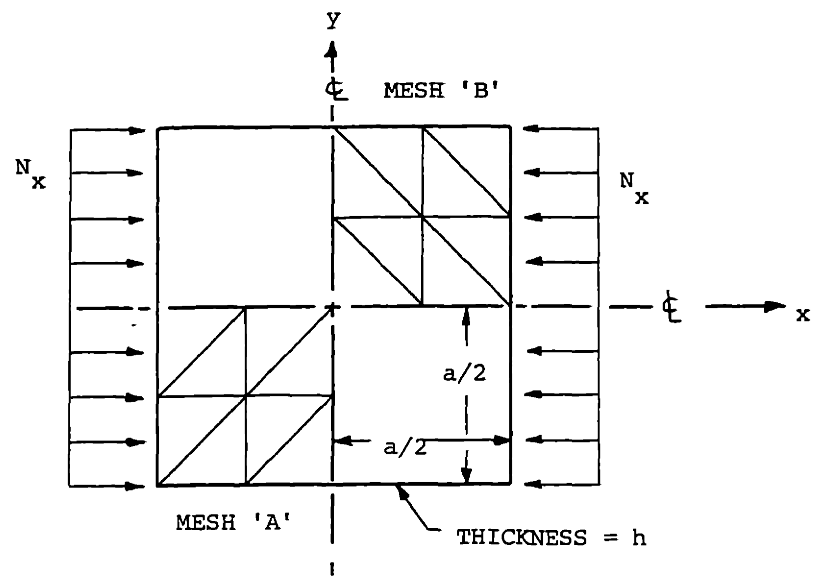


CORNER NODES: 1,2,3

MIDPOINT NODES: 4,5,6

$$z_Q(x,y) = \begin{Bmatrix} (2\xi_1-1)\xi_1 \\ (2\xi_2-1)\xi_2 \\ (2\xi_3-1)\xi_3 \\ 4\xi_1\xi_2 \\ 4\xi_2\xi_3 \\ 4\xi_3\xi_1 \end{Bmatrix}^T \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix}$$

FIG. 7.4 COORDINATE POINTS USED FOR A QUADRATIC DISTRIBUTION



TYPICAL 2x2 MESH

$$\lambda_{cr} = \frac{(N_x)_{cr}\, a^2}{\Pi^2 D^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

FOR PRESENT WORK PERFERRED MESH IS TYPE 'A'

BOUNDARY CONDITIONS: SIMPLY SUPPORTED - ALL EDGES

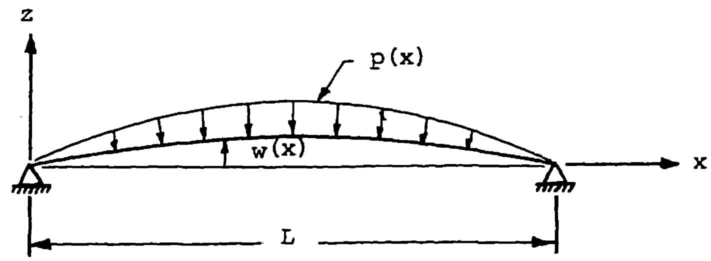
PROBLEM DATA: $E = 10^7$ PSI

$\nu = 0.300$

$h = 0.100$ IN.

$a = 10.$ IN.

FIG. 8.1 DESCRIPTION OF FLAT PLATE FOR LINEAR PREBUCKLING PROBLEM



$w(x) = w_o \text{ SIN } \frac{\pi x}{L}$ $\qquad$ $p(x) = p_o \text{ SIN } \frac{\pi x}{L}$

w_o = INITIAL CENTRAL RISE FROM BASE PLANE = 4 IN.

p_o = MAXIMUM PRESSURE LB./IN.

L = LENGTH OF BASE PLANE = 100 IN.

BOUNDARY CONDITIONS: PINNED-FIXED

NONDIMENSIONAL PARAMETERS:

$$\overline{p} = \frac{p_o L^4}{2EI} \sqrt{\frac{A}{I}} \qquad \overline{W} = \frac{w}{2} \sqrt{\frac{A}{I}} \qquad \overline{N} = \frac{N}{EA} (10^3) \qquad \overline{M} = \frac{ML}{EI} (10)$$

A = ARCH CROSS SECTIONAL AREA = 9/16 IN.^2

I = ARCH CROSS SECTIONAL AREA MOMENT OF INERTIA = 1 IN.^4

E = YOUNG'S MODULUS = 10^7 PSI

EQCK = GENERAL, ITERATIVE EQUILIBRIUM CHECK IN LITERATURE

SEQCK = PRESENT WORK: STRESS EQUILIBRIUM CHECK ONLY

TEQCK = PRESENT WORK: TOTAL EQUILIBRIUM CHECK

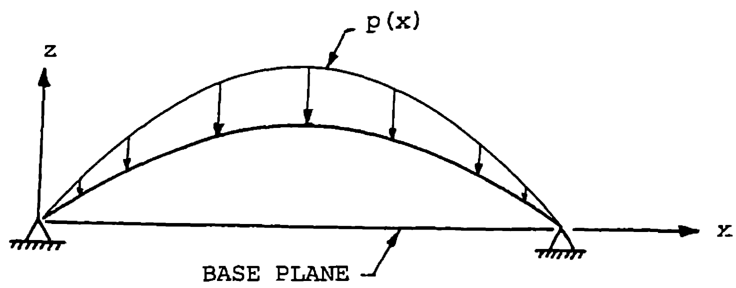
I = NUMBER OF INCREMENTS (LOAD STEPS) TO TOTAL LOAD

R = CONVERGENCE RATIO ON ITERATIONS WITHIN AN INCREMENT

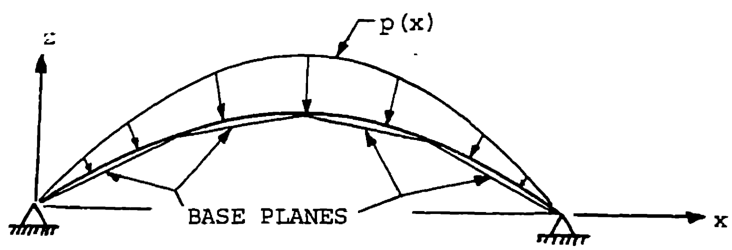
S = TOTAL NUMBER OF SOLUTION STEPS (INCREMENTS AND ITERATIONS)

t = TOTAL EXECUTION TIME (SECONDS)

FIG. 8.2a DESCRIPTION OF SHALLOW, SINUSOIDAL ARCH PROBLEM

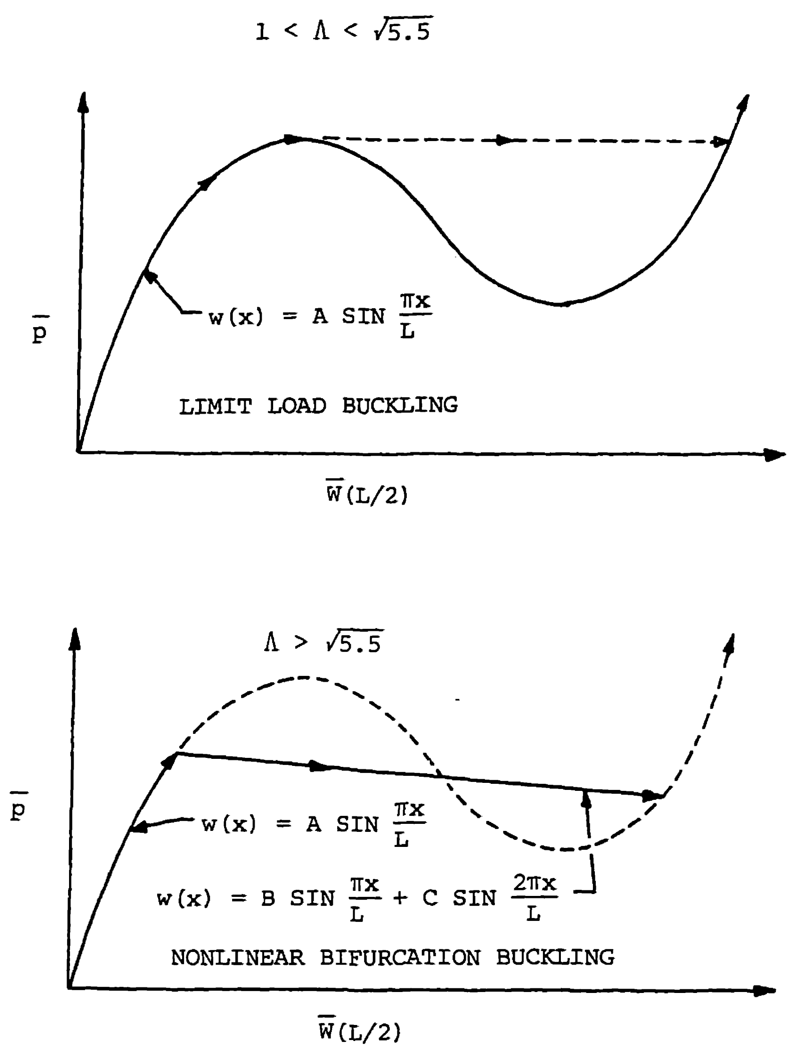


FUNG AND KAPLAN [1952] SOLUTION



SHALLOW SINUSOIDAL ARCH UNDER
SINUSOIDAL PRESSURE

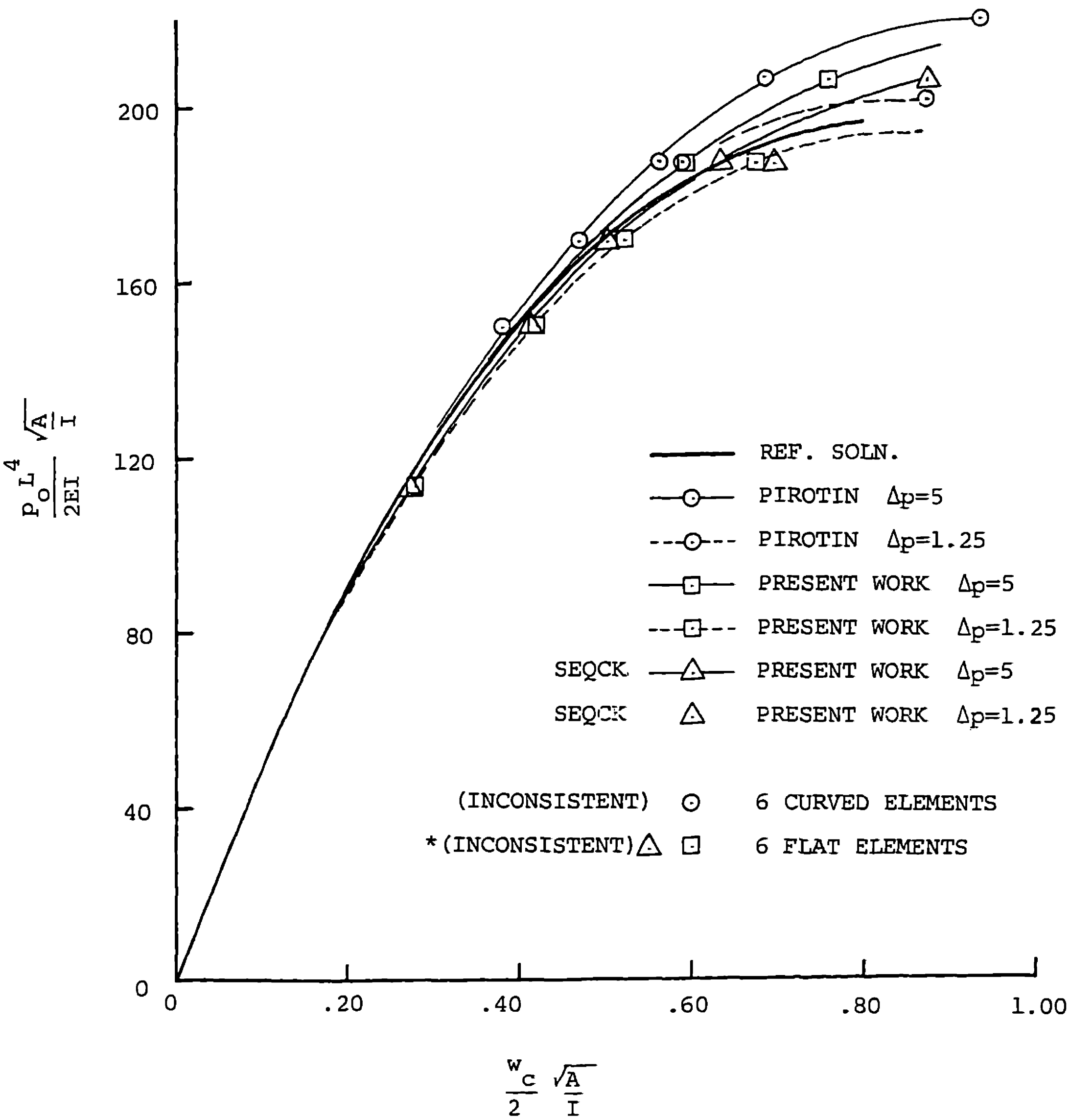
FIG. 8.2b FINITE ELEMENT DESCRIPTION OF SHALLOW, SINUSOIDAL ARCH PROBLEMS



$$\Lambda = \frac{w_c}{2} \sqrt{\frac{A}{I}}$$

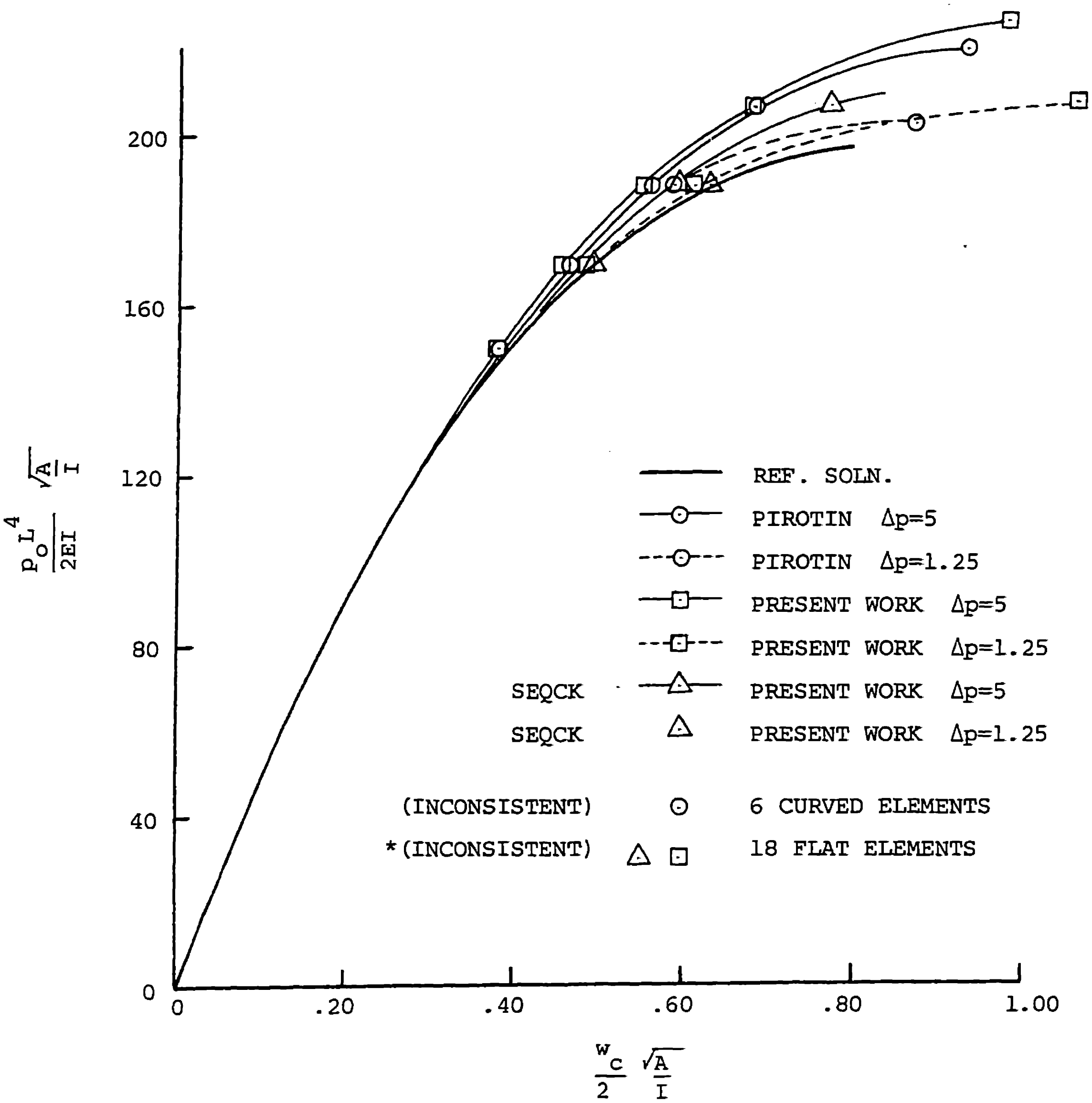
FOR PRESENT WORK: $\Lambda = 1.5$ (SEE FIG. 8.2)

FIG. 8.3 BUCKLING BEHAVIOR OF SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE



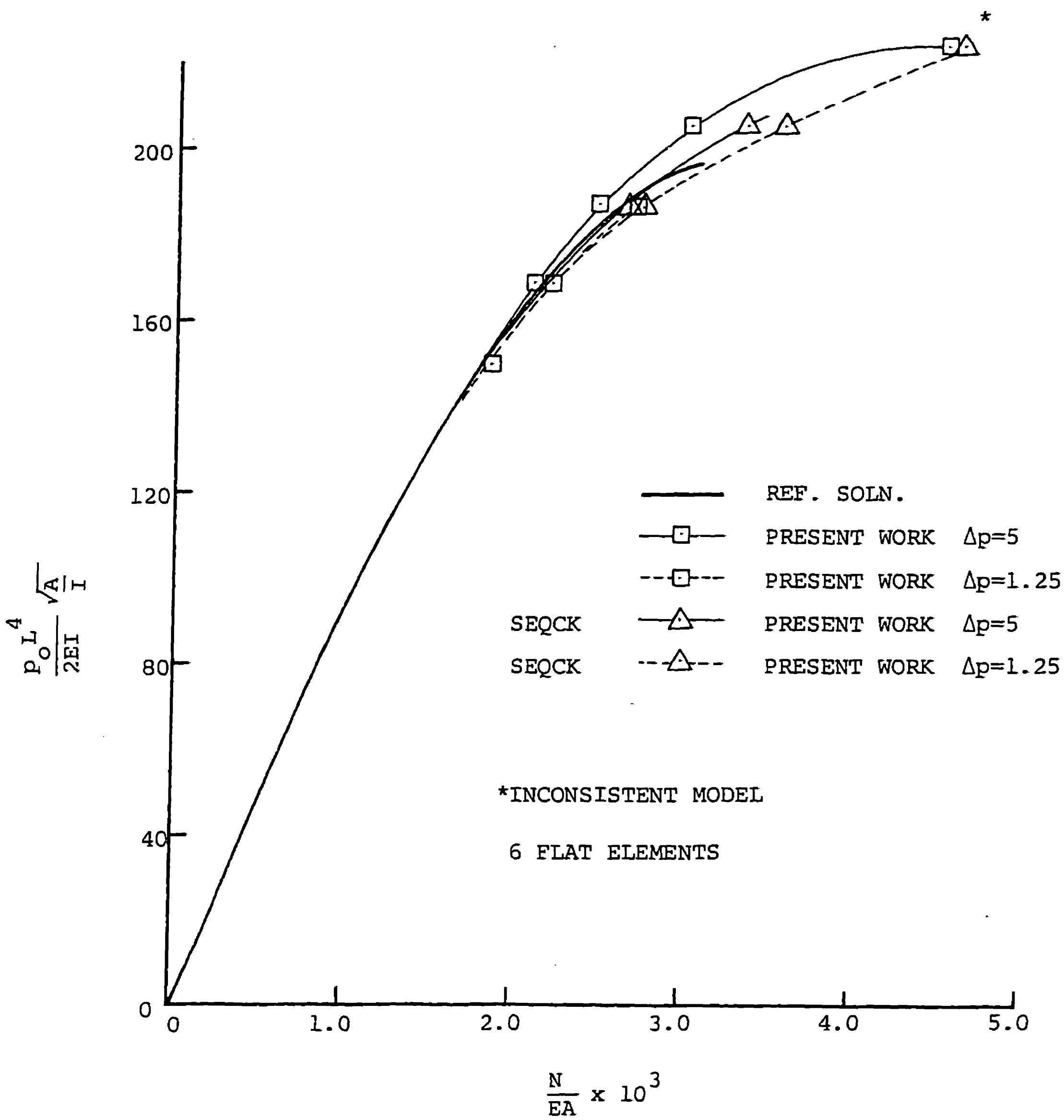
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.4 SHALLOW, SINUSOIDAL ARCH UNDER SINSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (SIX FLAT ELEMENTS)



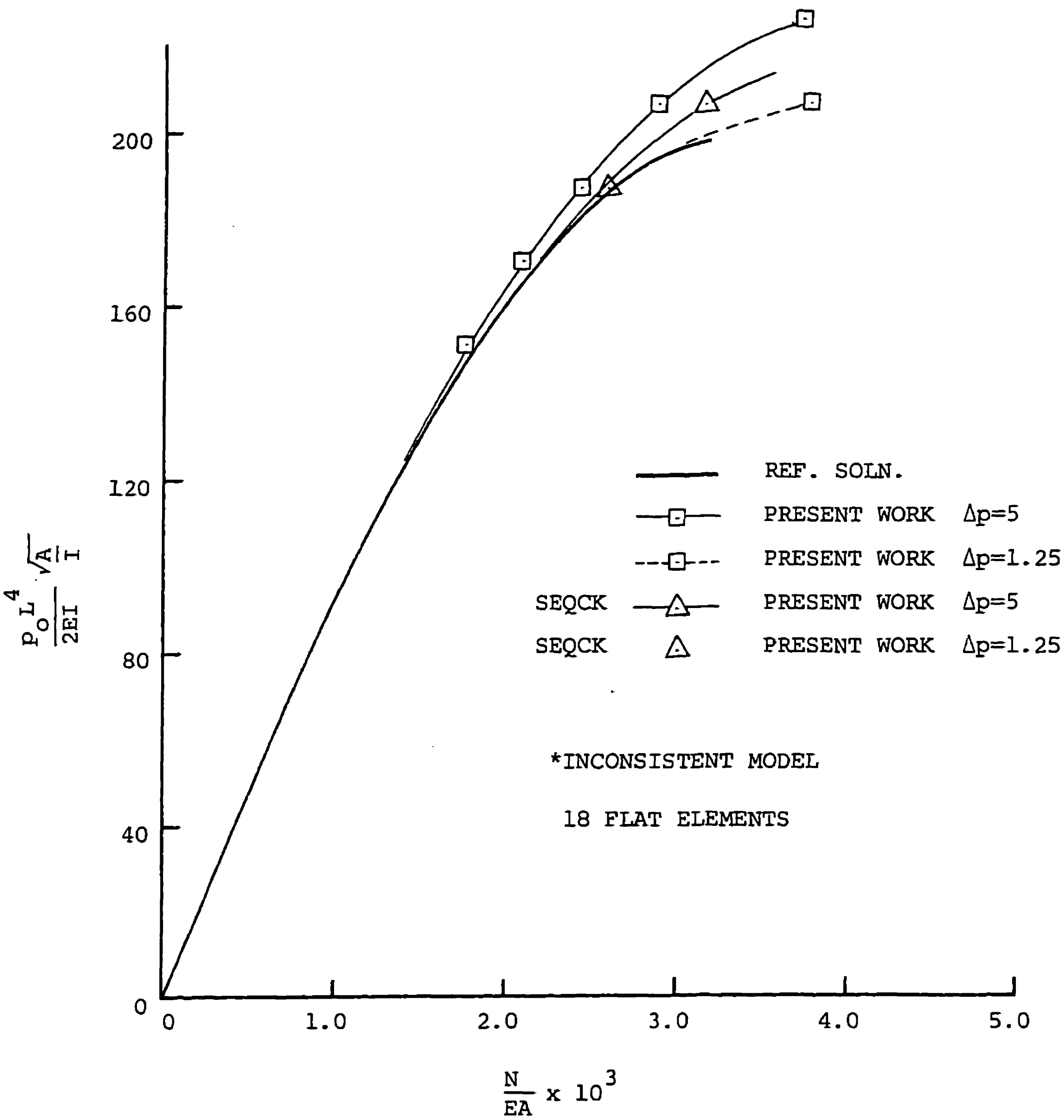
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.5 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (EIGHTEEN FLAT ELEMENTS)



*<u>NOTE</u>: WITH THE EXCEPTION OF THIS POINT, CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS

FIG. 8.6 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL AXIAL LOAD (SIX FLAT ELEMENTS)



*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.7 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL AXIAL LOAD (EIGHTEEN FLAT ELEMENTS)

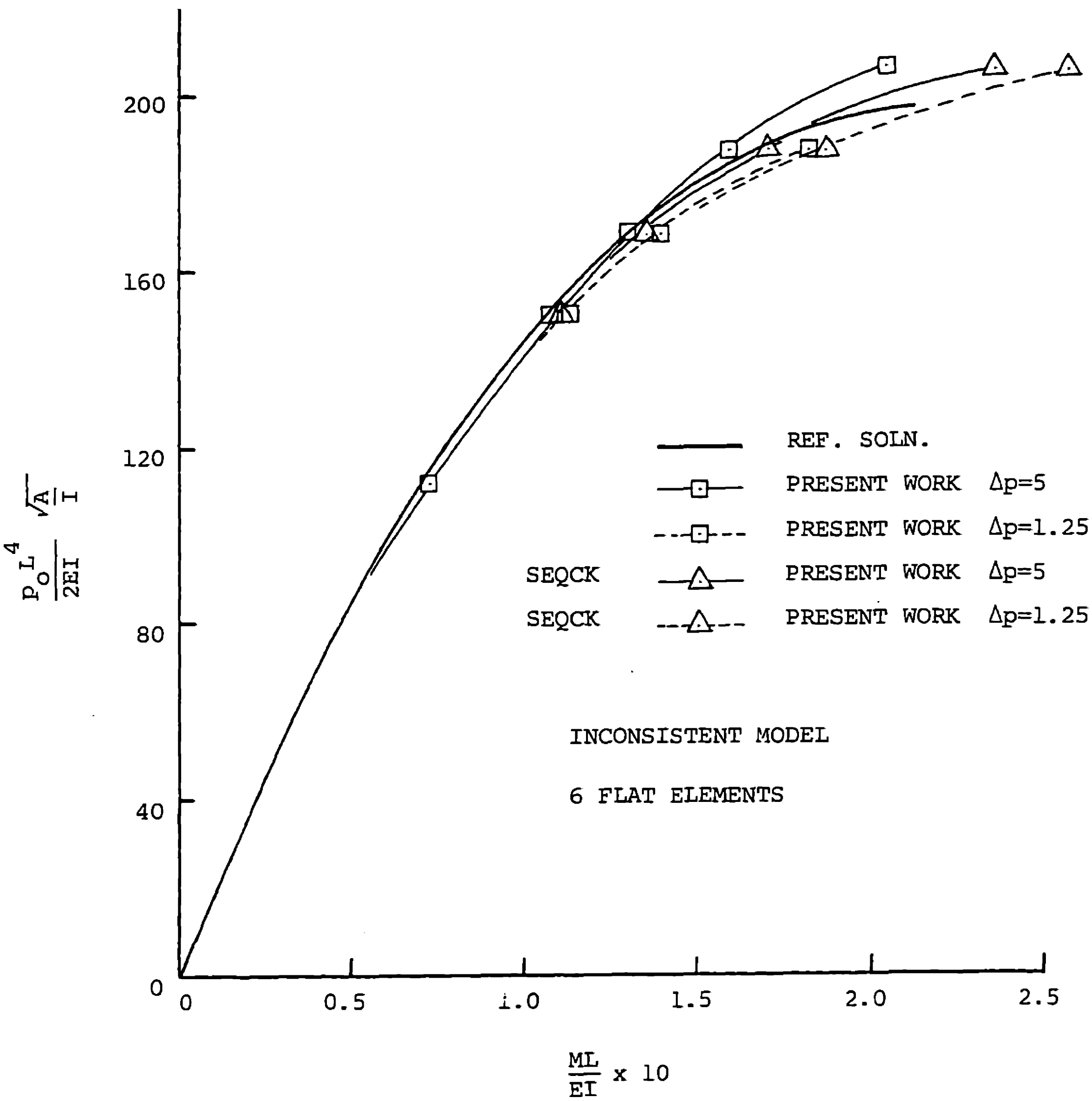


FIG. 8.8 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENT (SIX FLAT ELEMENTS; INCONSISTENT MODEL)

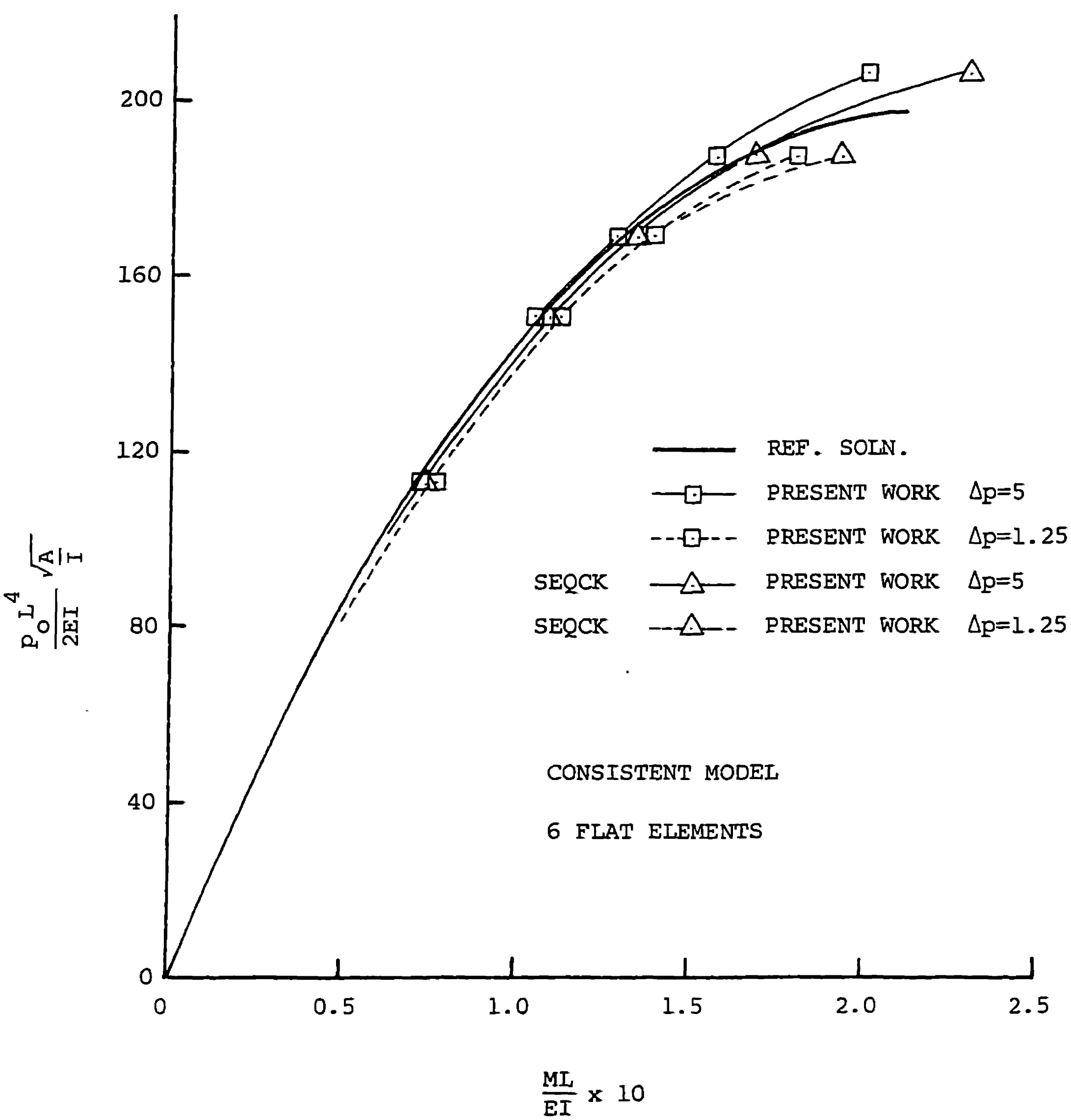
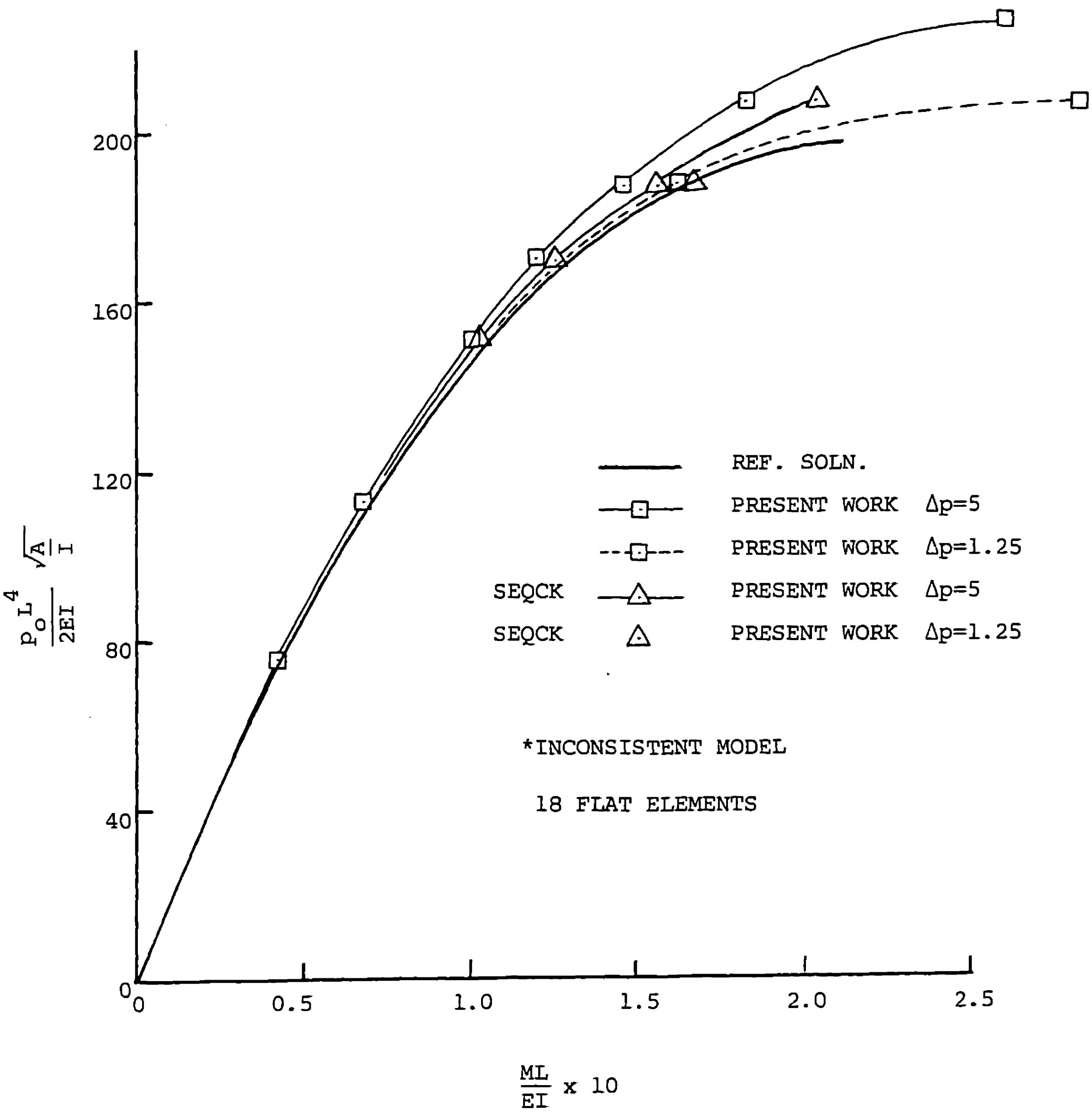
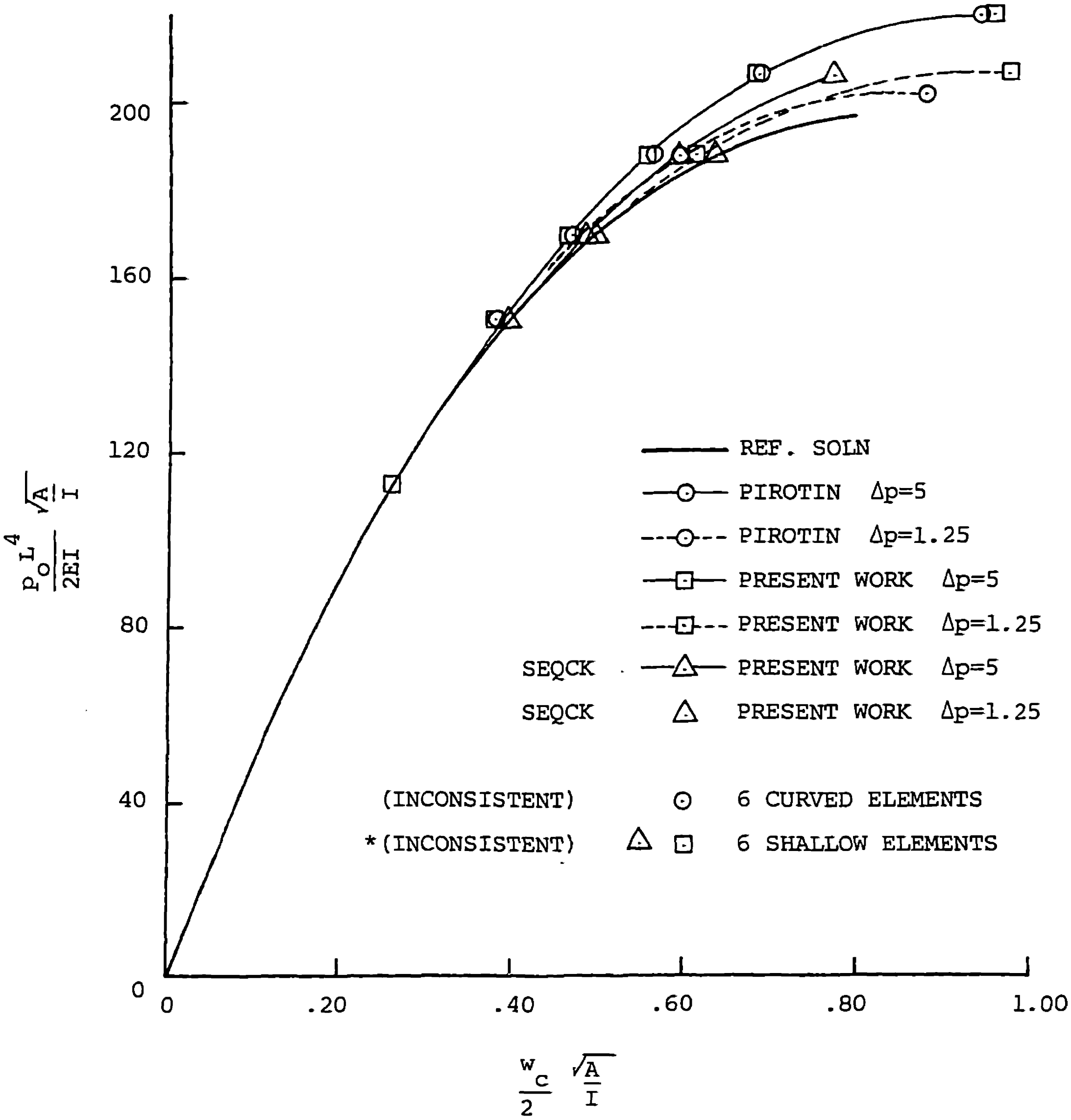


FIG. 8.9 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENT (SIX FLAT ELEMENTS; CONSISTENT MODEL)



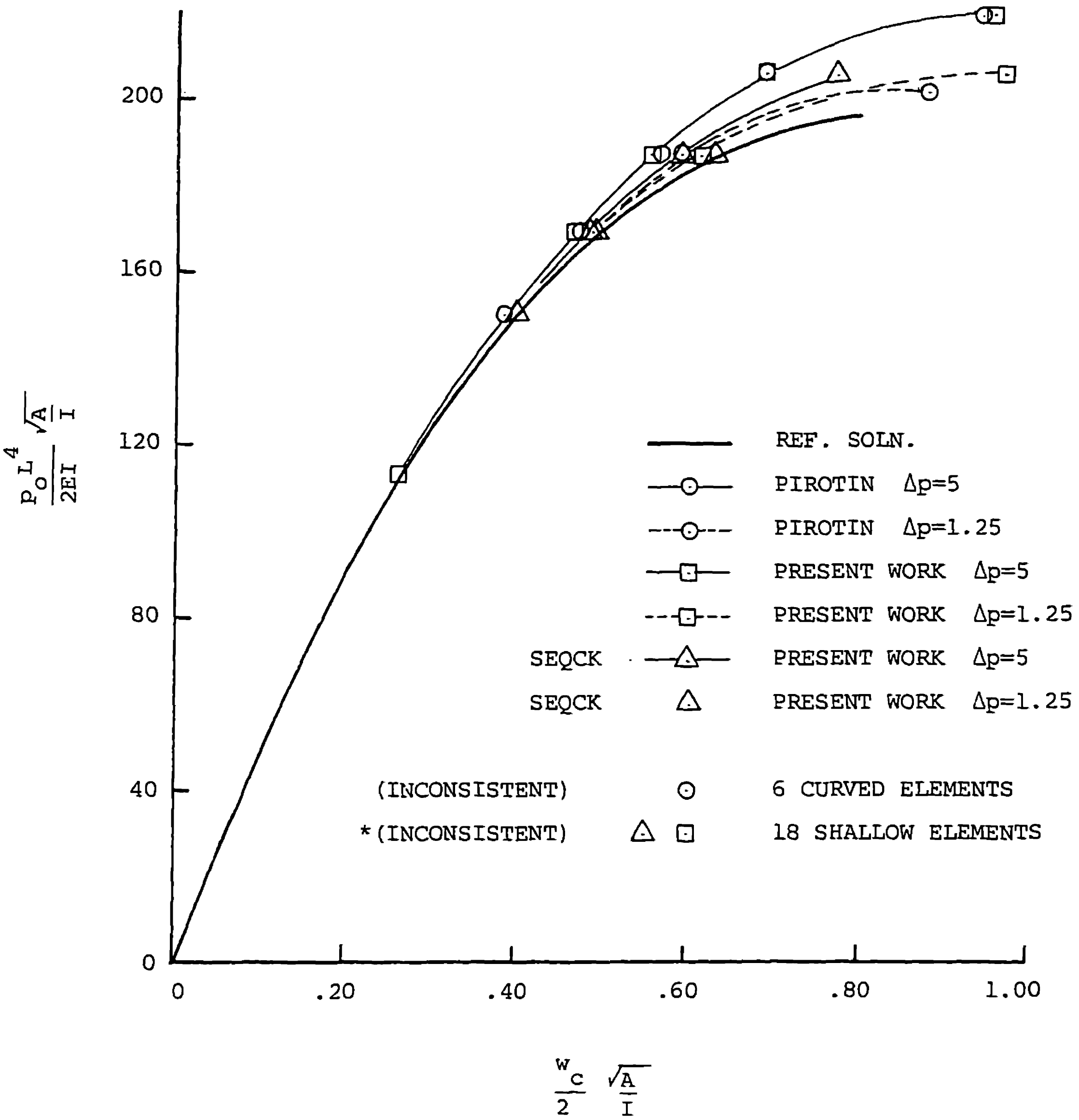
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.10 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENTS (EIGHTEEN FLAT ELEMENTS)



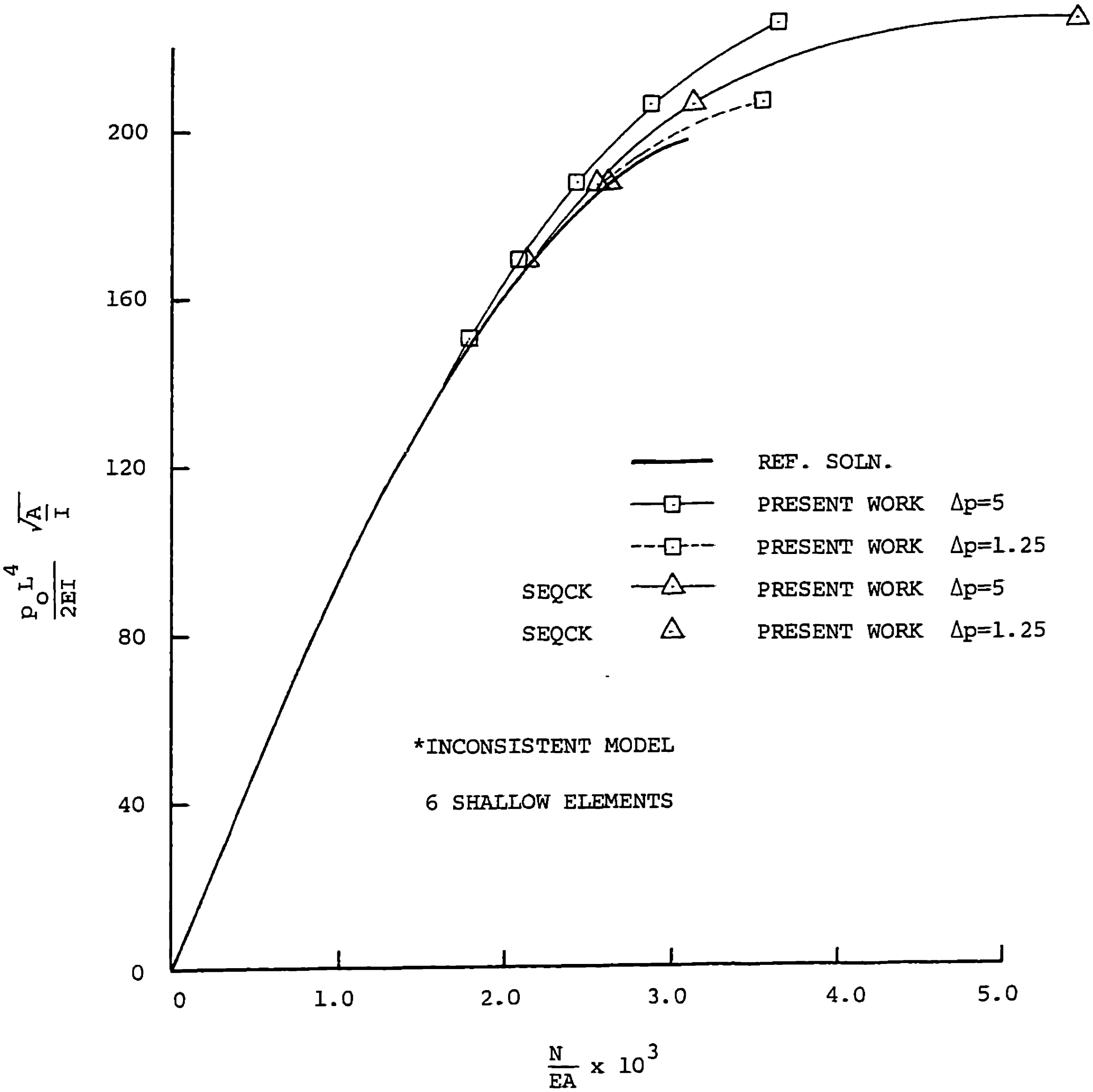
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.11 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (SIX SHALLOW ELEMENTS)



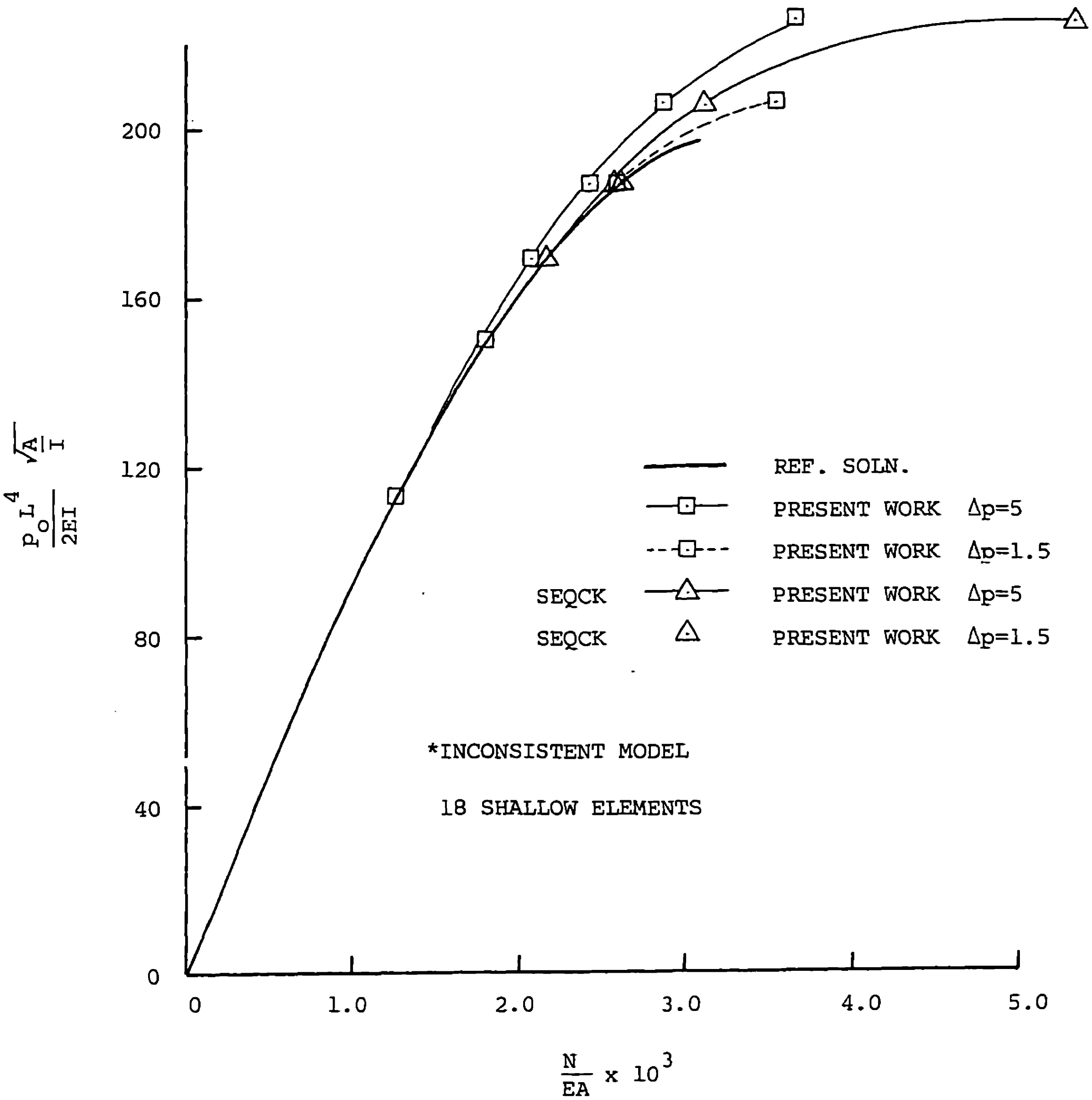
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.12 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (EIGHTEEN SHALLOW ELEMENTS)



*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.13 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL AXIAL LOAD (SIX SHALLOW ELEMENTS)



*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.14 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL AXIAL LOAD (EIGHTEEN SHALLOW ELEMENTS)

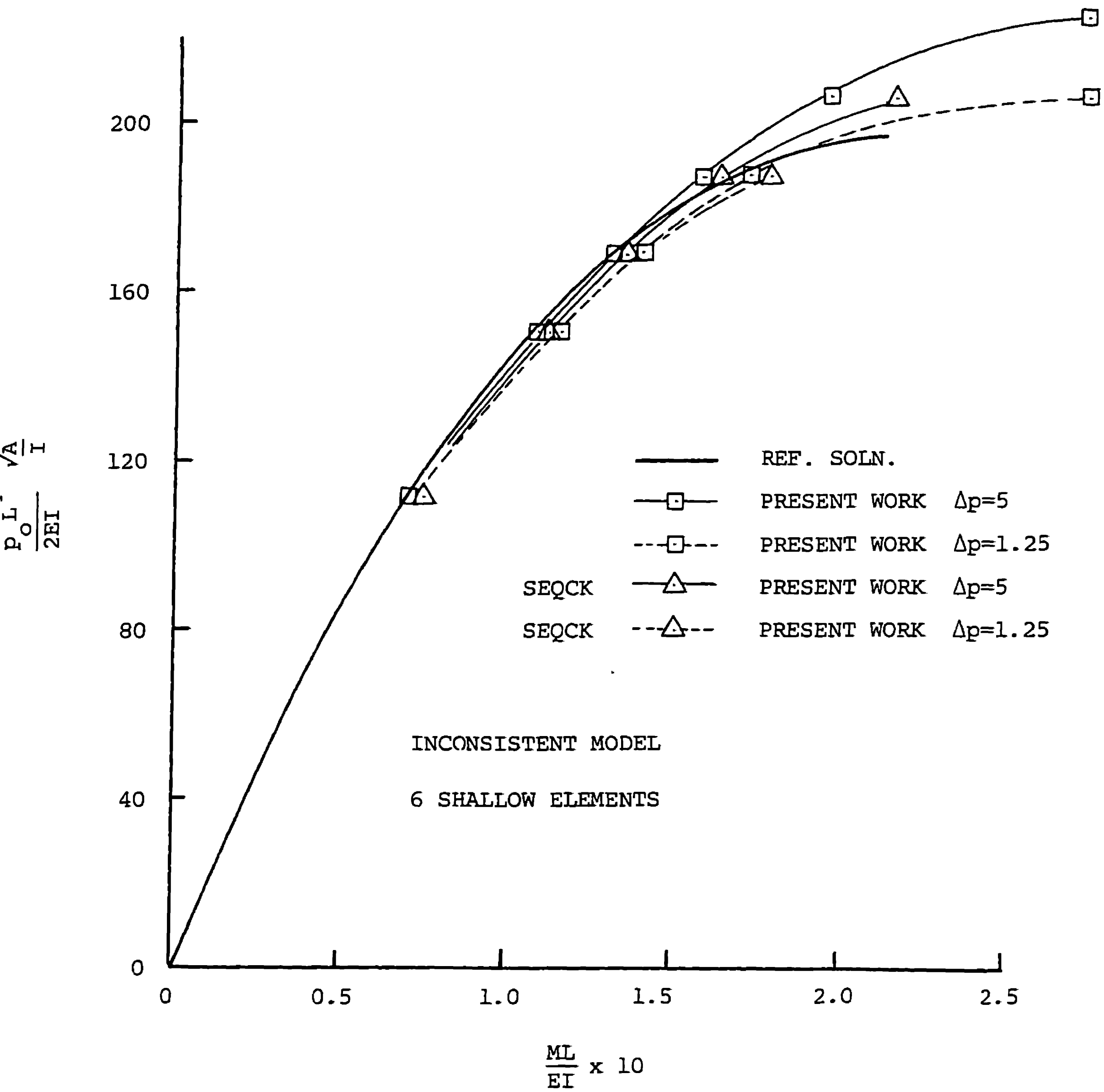


FIG. 8.15 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENT (SIX SHALLOW ELEMENTS; INCONSISTENT MODEL)

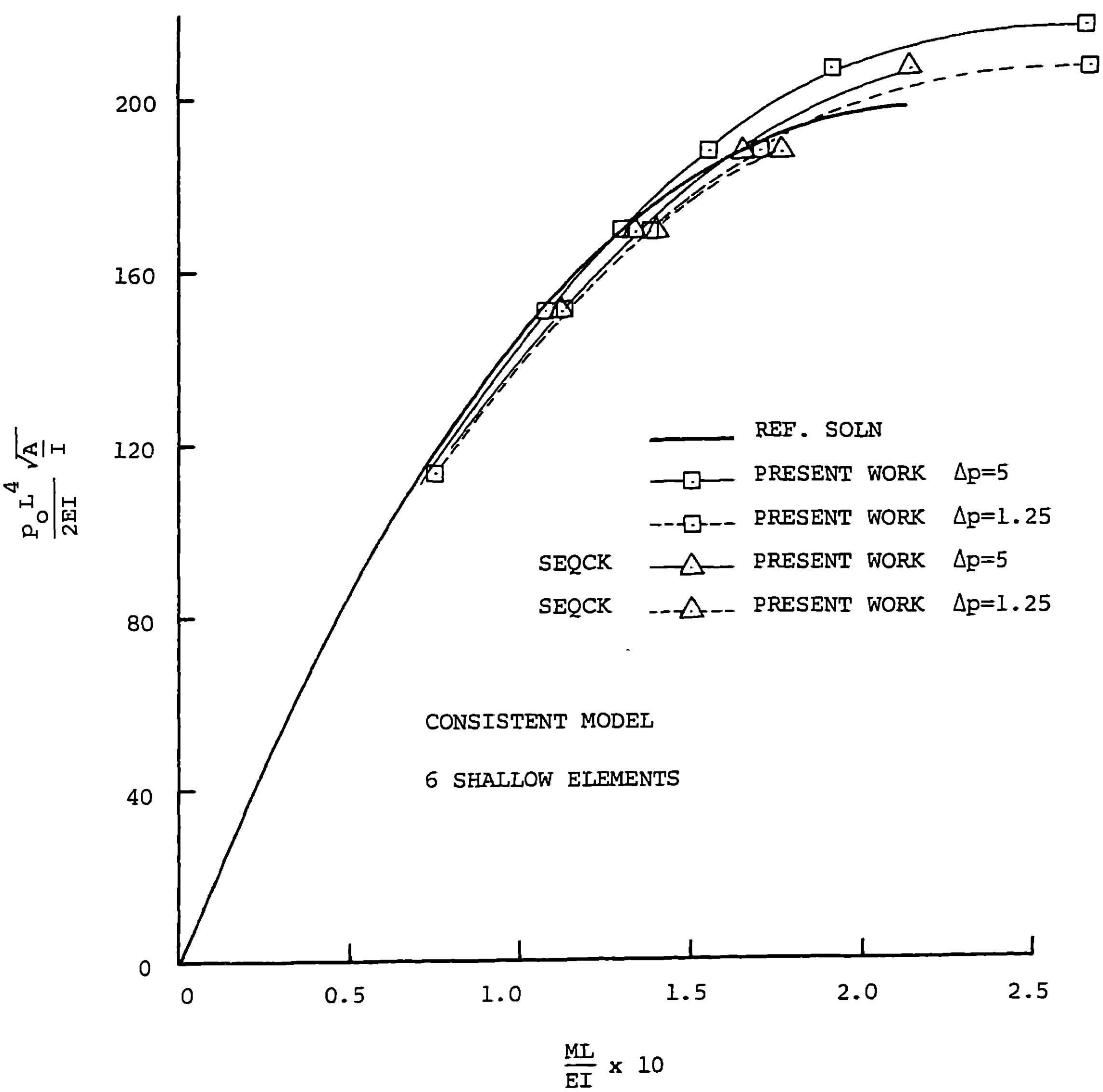
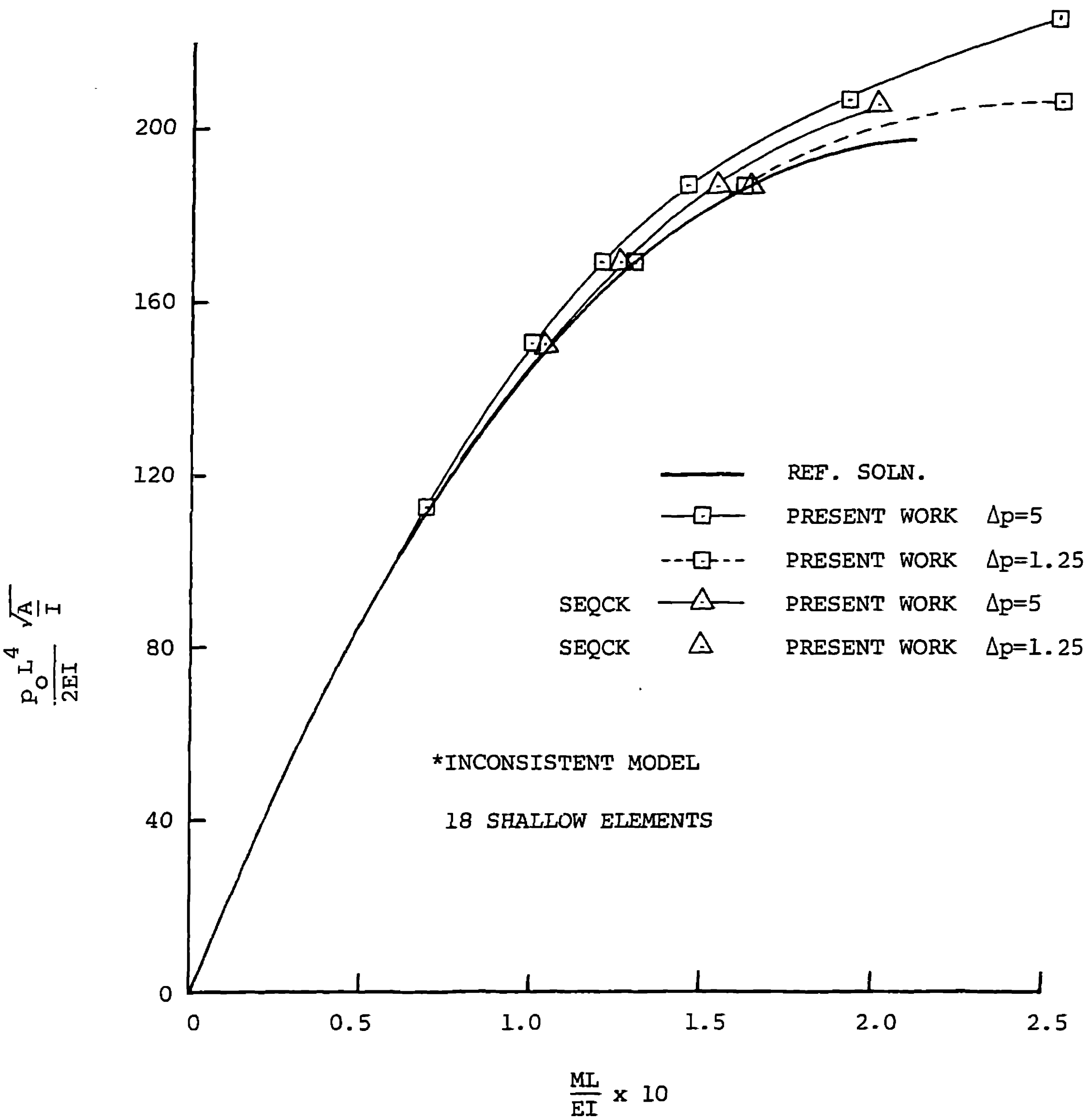
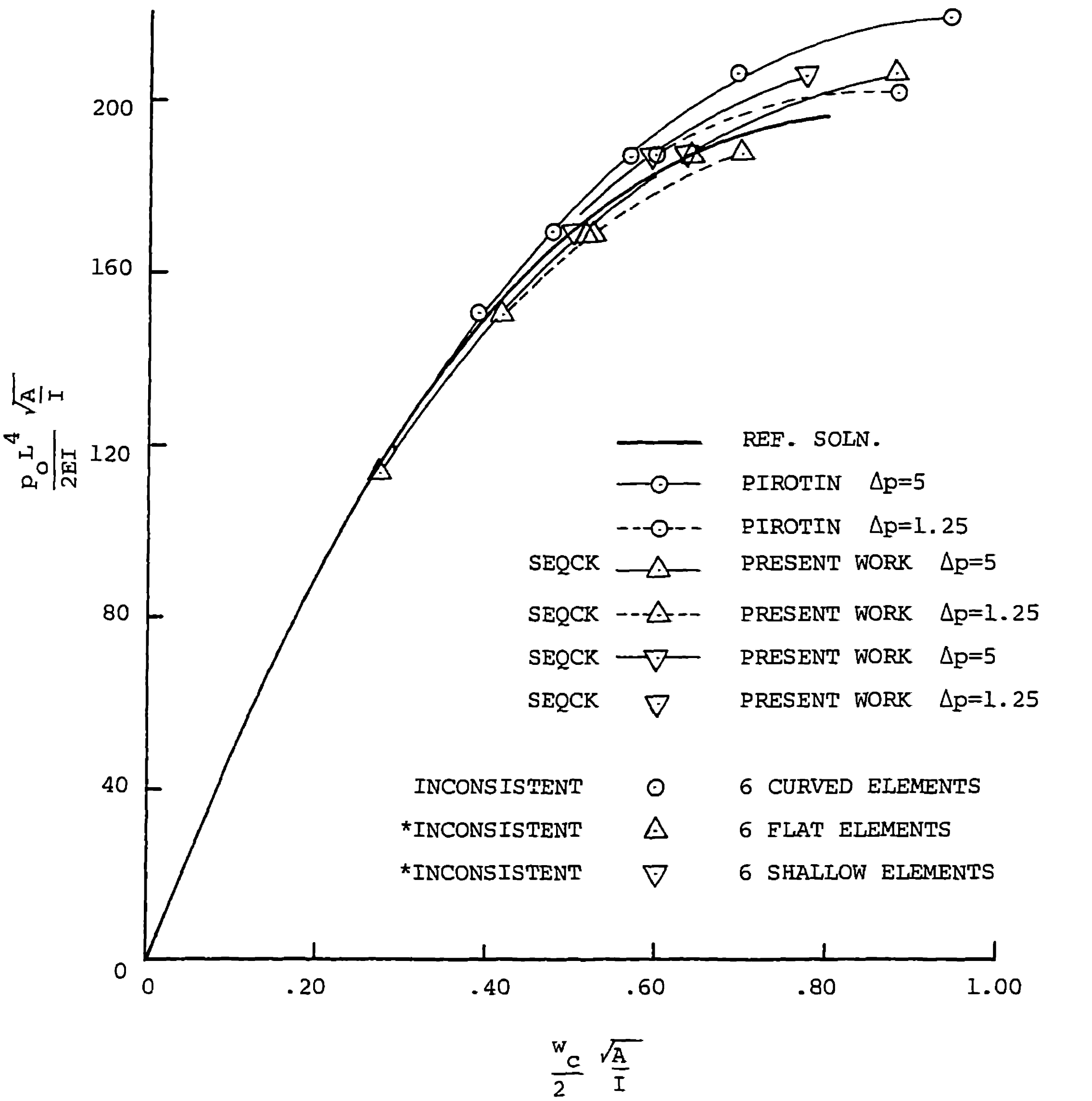


FIG. 8.16 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENT (SIX SHALLOW ELEMENTS; CONSISTENT MODEL)



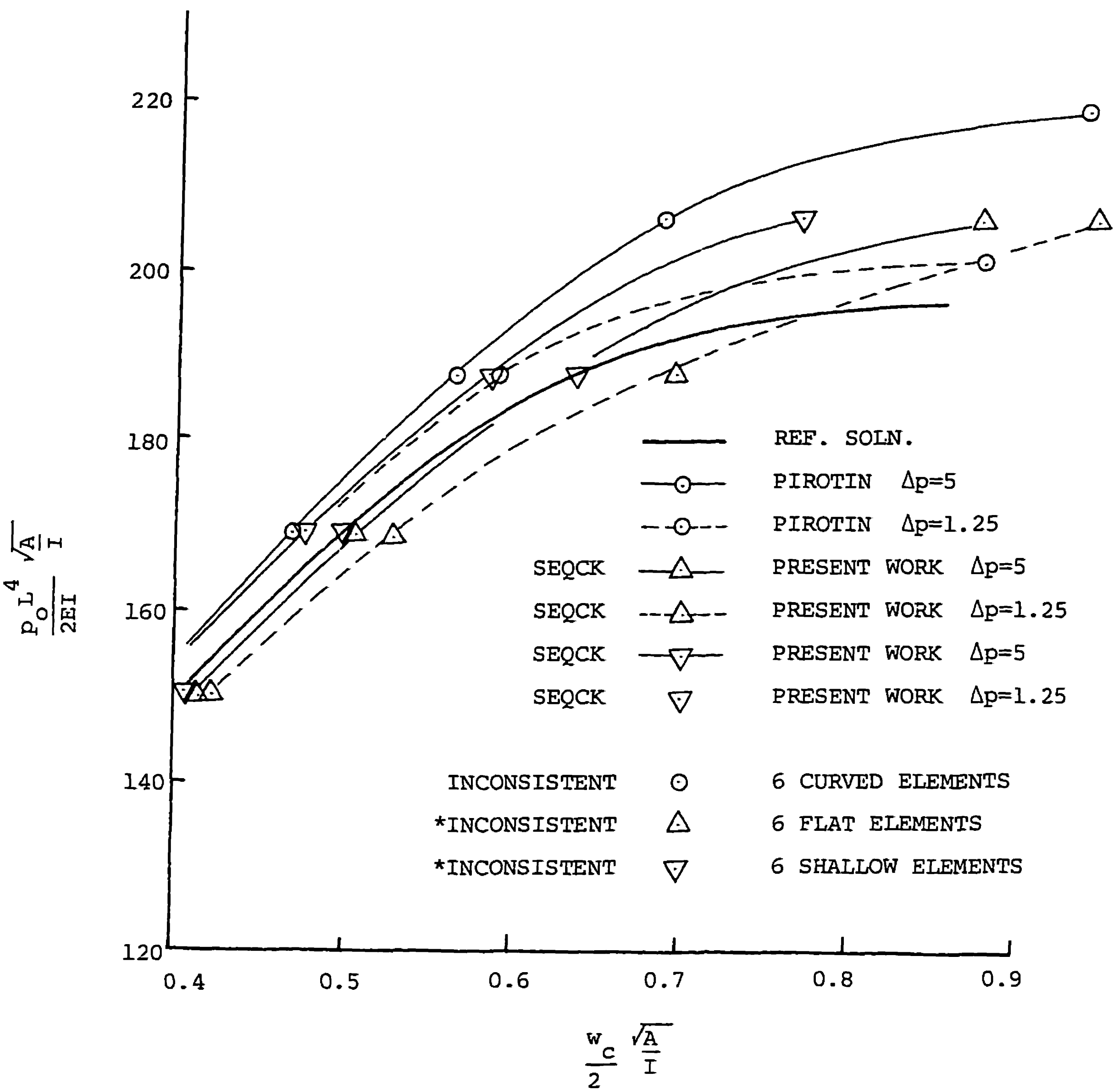
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.17 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL BENDING MOMENT (EIGHTEEN SHALLOW ELEMENTS)



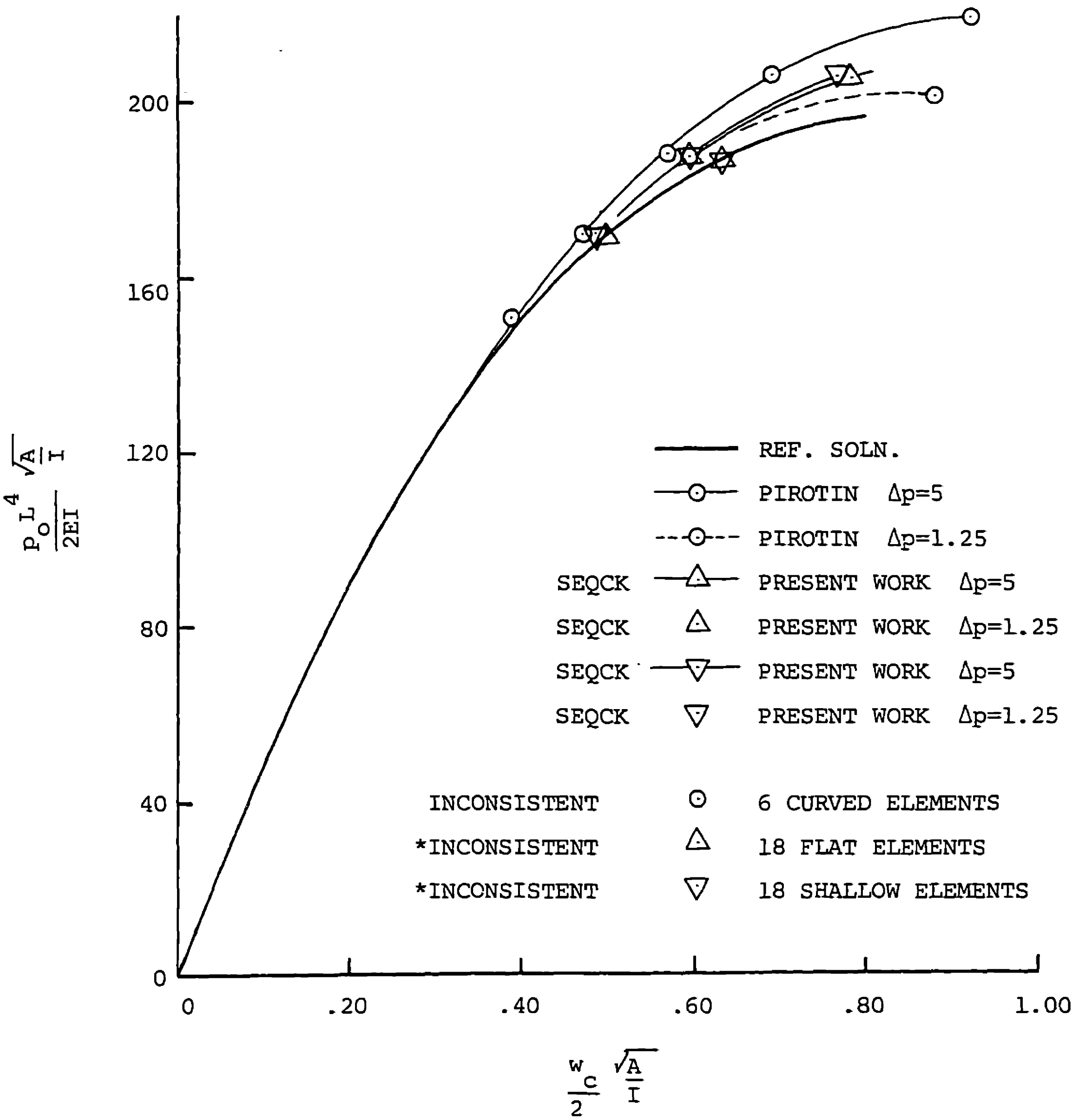
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS

FIG. 8.18 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL DISPLACEMENT (SIX ELEMENTS)



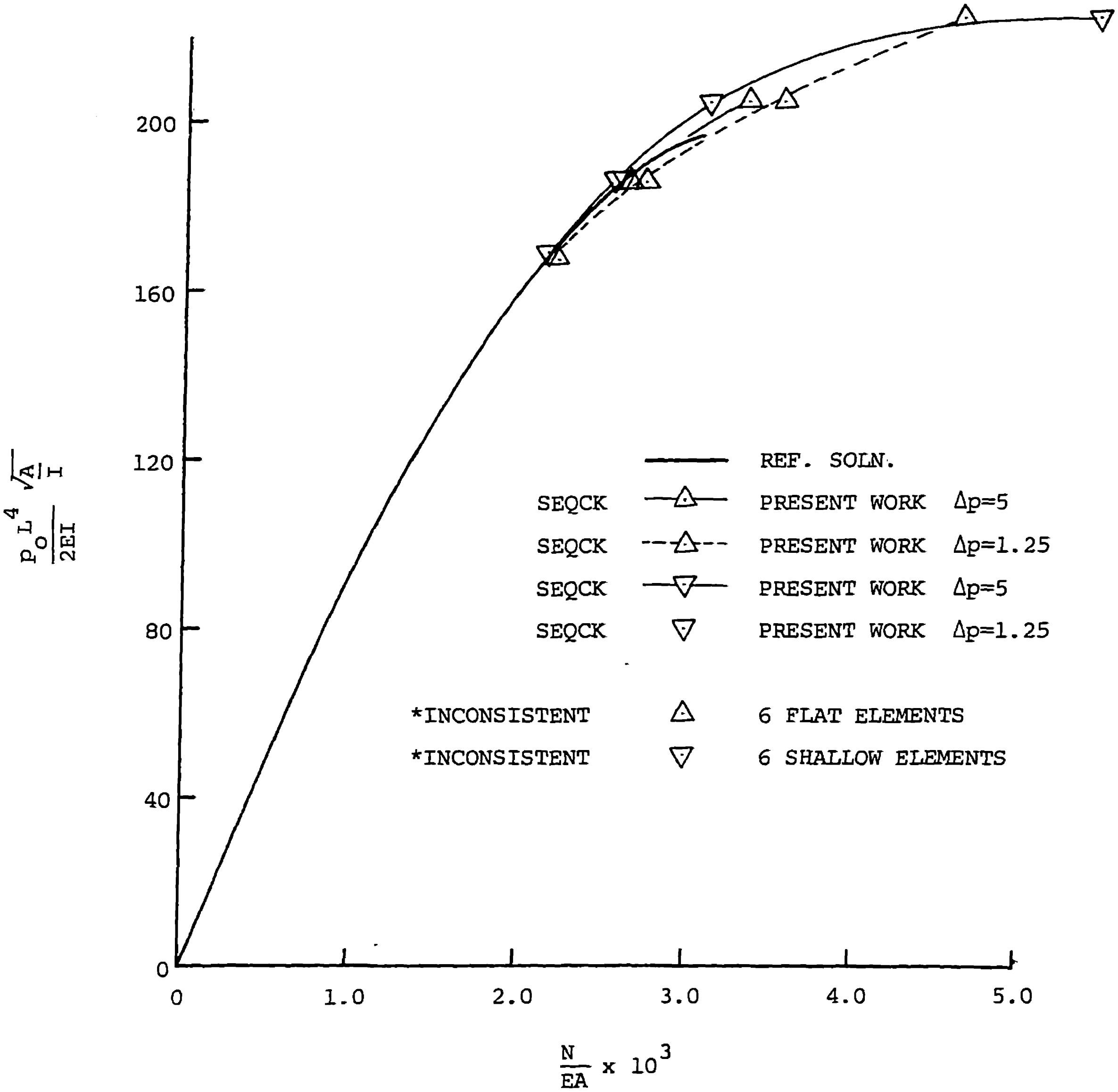
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS

FIG. 8.19 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL DISPLACEMENT (SIX ELEMENTS; EXPANDED PLOT)



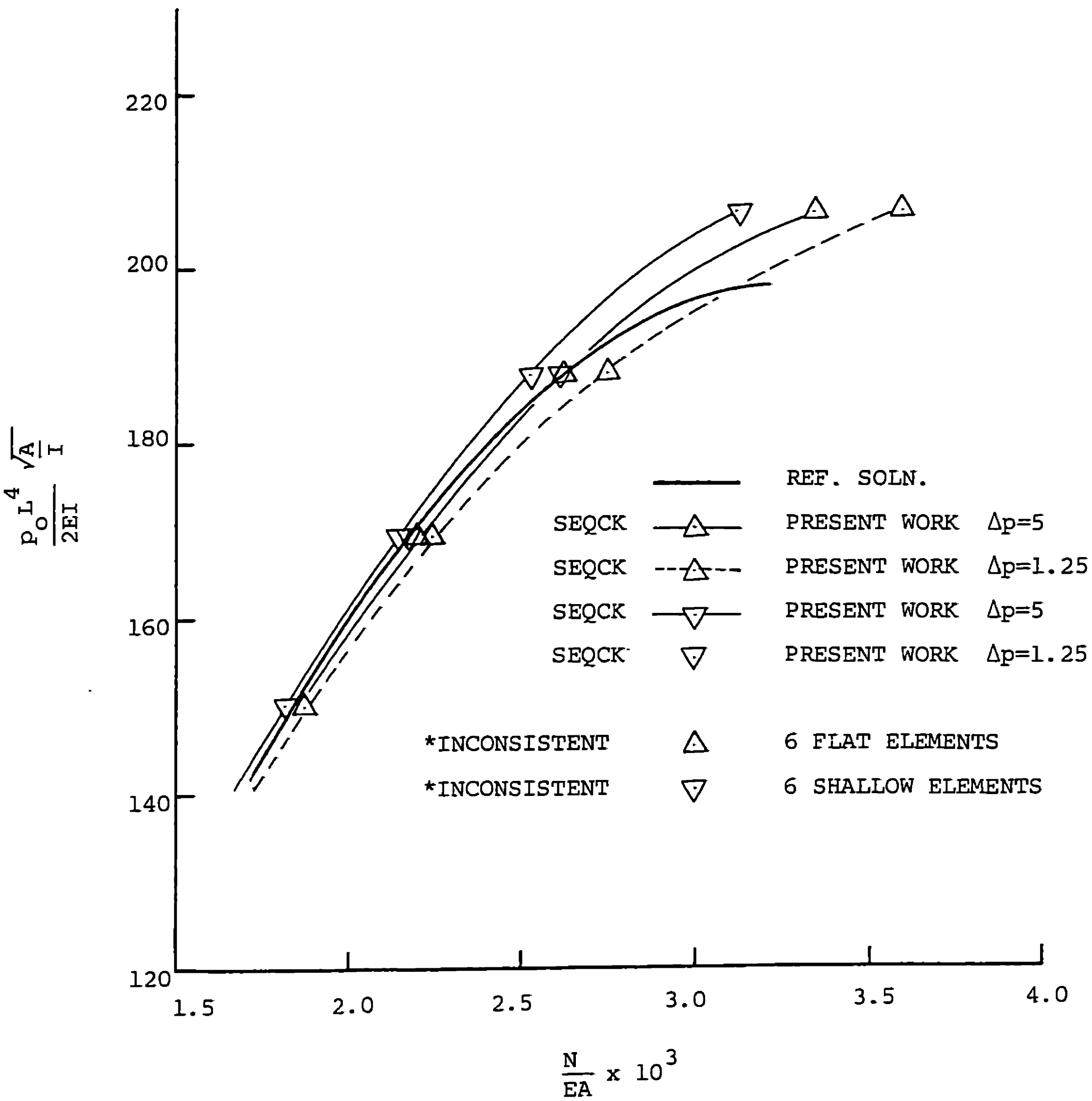
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.20 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL DISPLACEMENT (EIGHTEEN ELEMENTS)



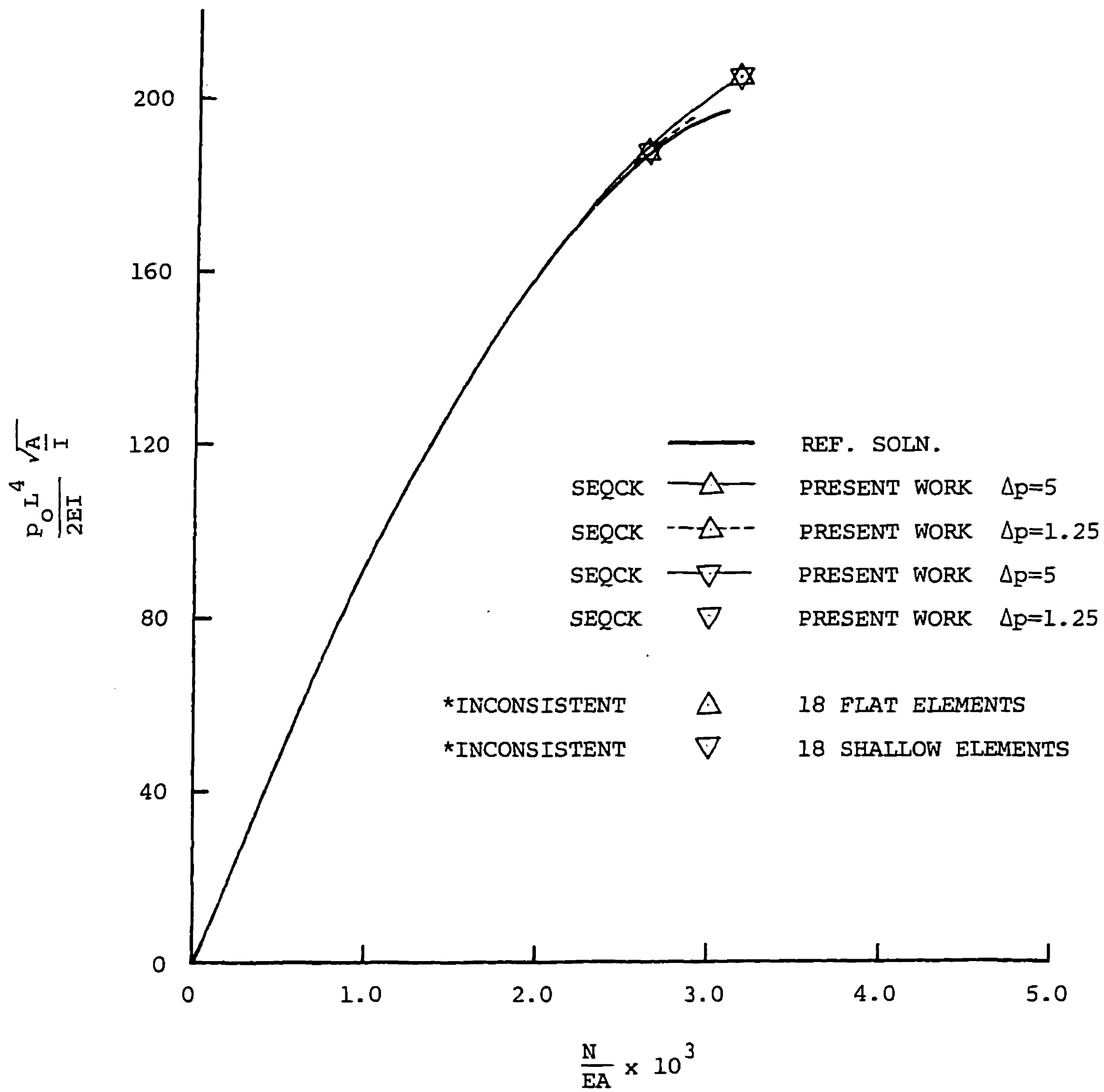
*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.21 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL AXIAL LOAD (SIX ELEMENTS)



*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.22 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISON OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL AXIAL LOAD (SIX ELEMENTS; EXPANDED PLOT)



*NOTE: CONSISTENT MODEL YIELD ESSENTIALLY THE SAME RESULTS.

FIG. 8.23 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL AXIAL LOAD (EIGHTEEN ELEMENTS)

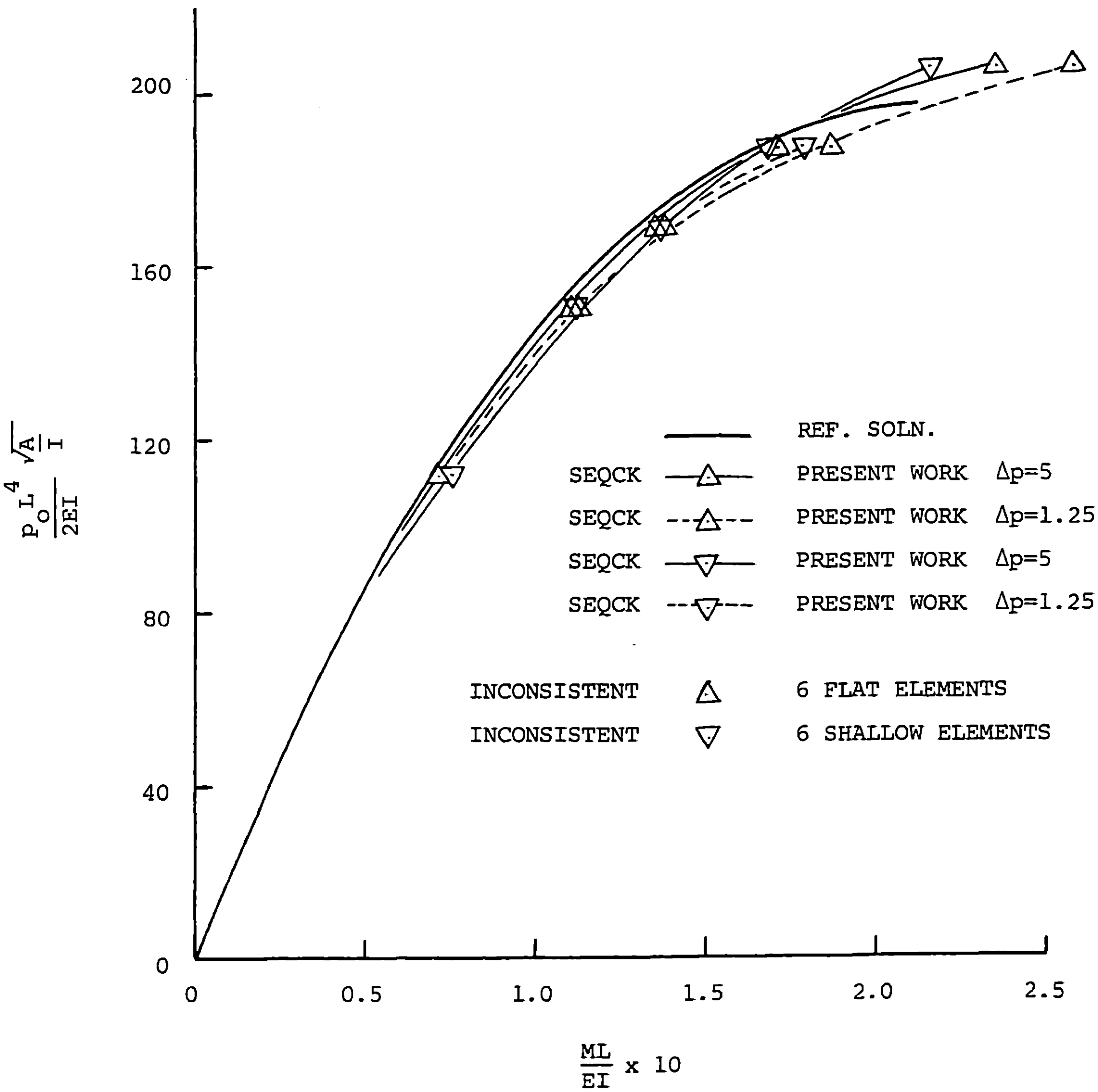


FIG. 8.24 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL BENDING MOMENT (SIX ELEMENTS; INCONSISTENT MODEL)

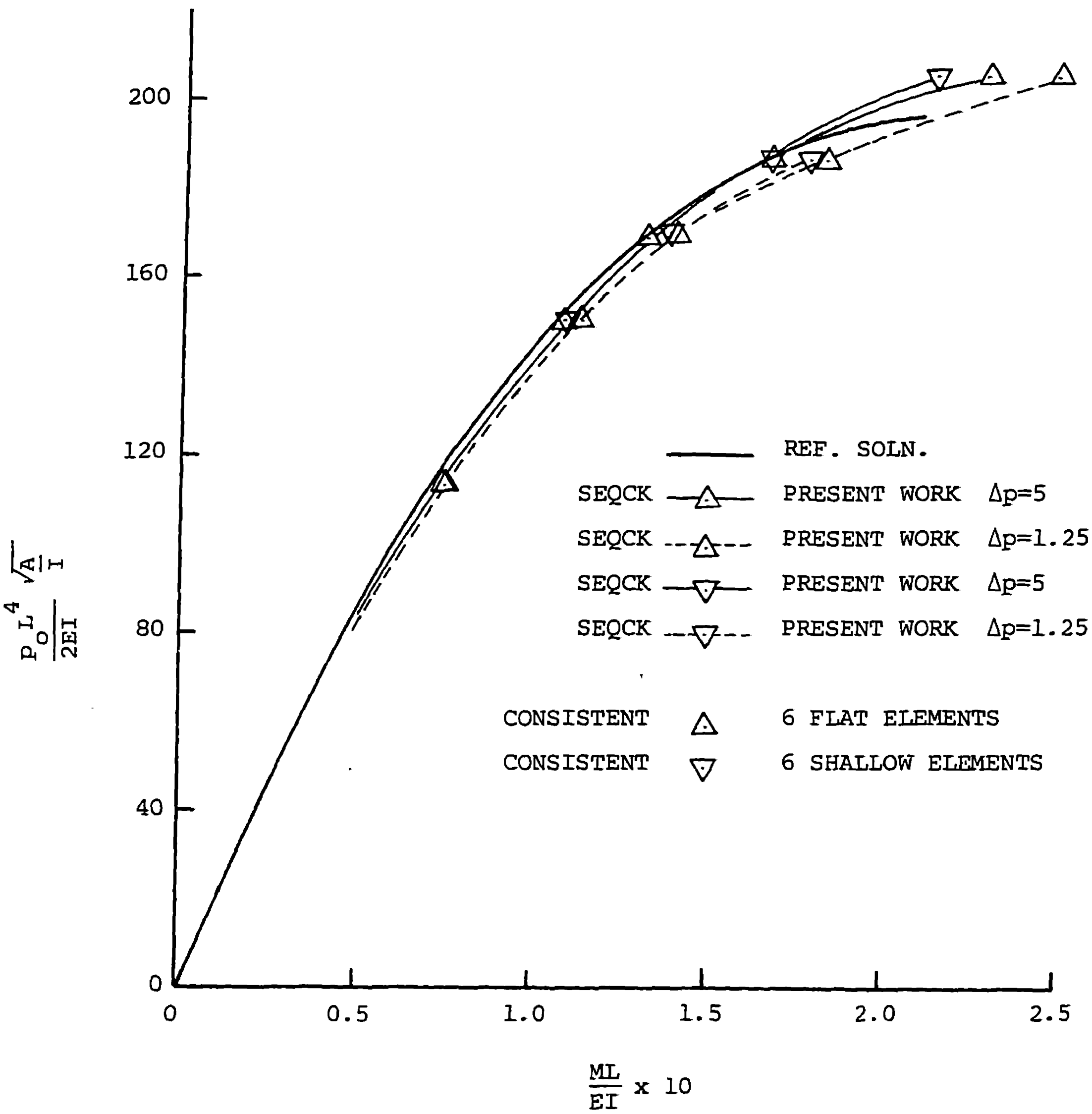


FIG. 8.25 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL BENDING MOMENT (SIX ELEMENTS; CONSISTENT MODEL)

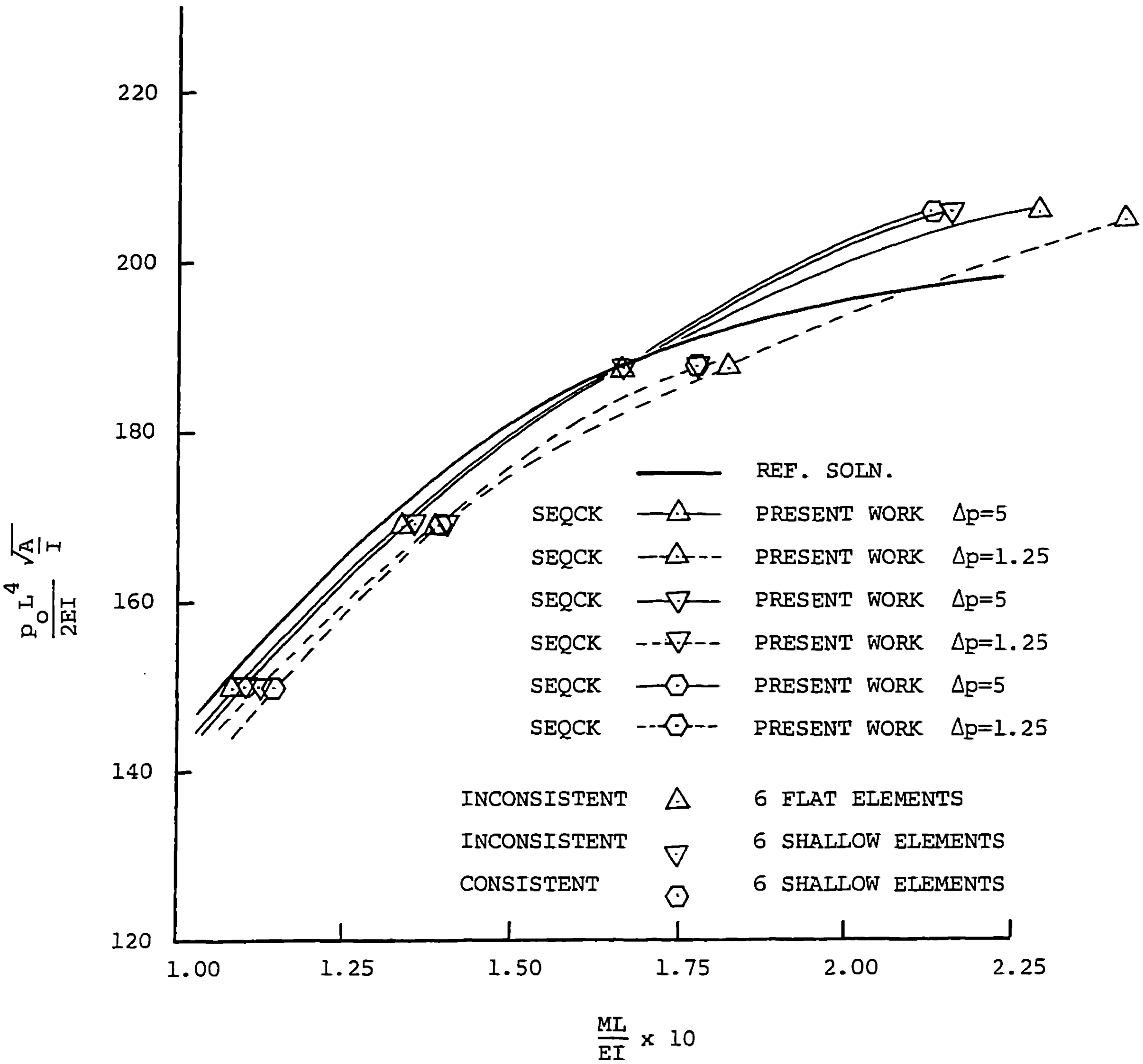
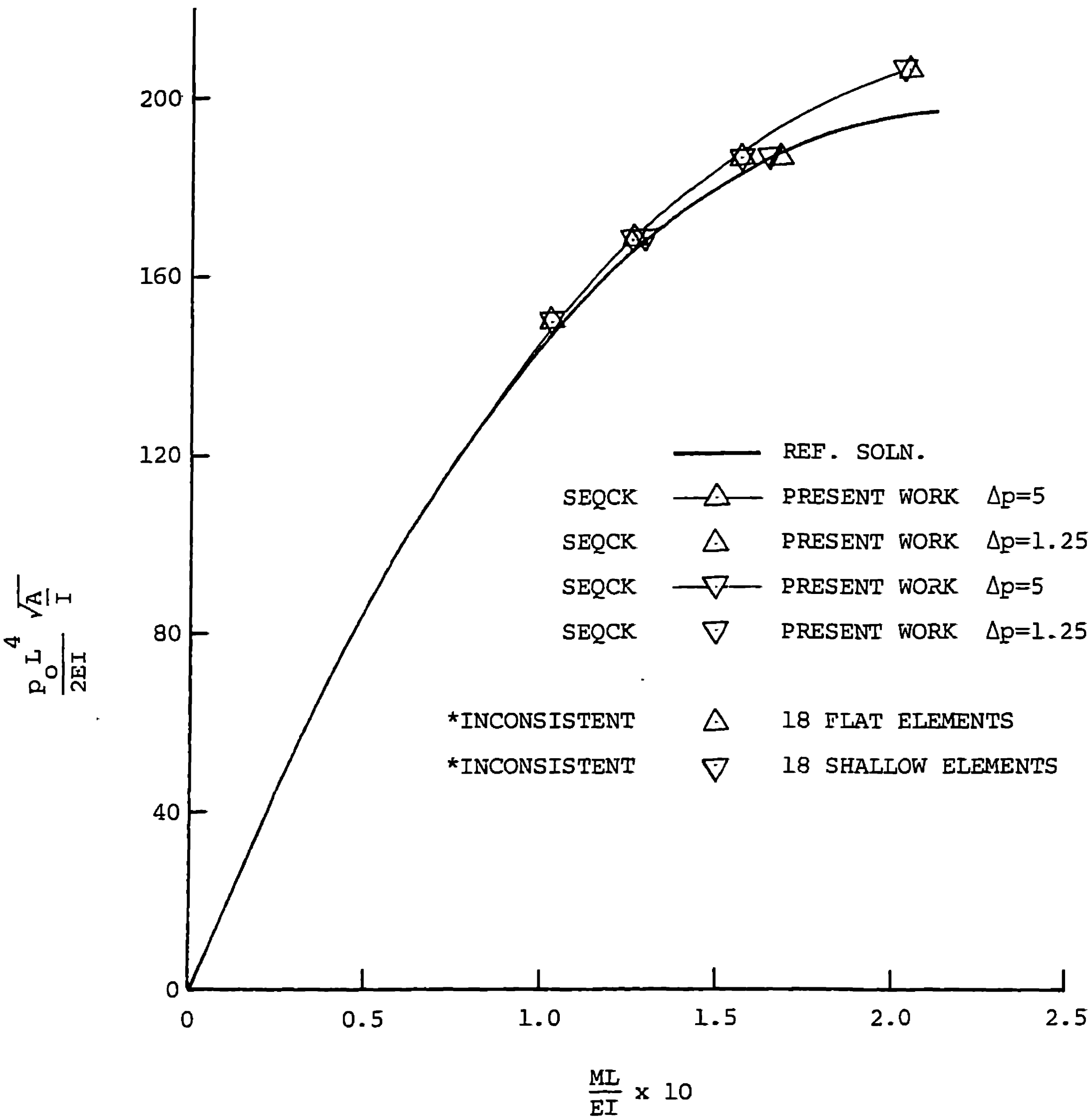
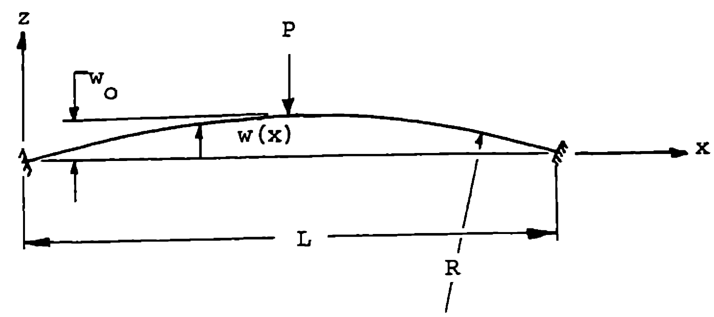


FIG. 8.26 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL BENDING MOMENT (SIX ELEMENTS; EXPANDED PLOT)



*NOTE: CONSISTENT MODEL YIELDS ESSENTIALLY THE SAME RESULTS.

FIG. 8.27 SHALLOW, SINUSOIDAL ARCH UNDER SINUSOIDAL PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF FLAT AND SHALLOW ELEMENTS FOR VARIOUS INCREMENT SIZES (STRESS EQUILIBRIUM CHECK INCLUDED) FOR CENTRAL BENDING MOMENT (EIGHTEEN ELEMENTS)



w_o = INITIAL CENTRAL RISE FROM BASE PLANE = 1.09 IN.

P = CENTRAL CONCENTRATED LOAD

L = LENGTH OF BASE PLANE = 34.0 IN.

R = RADIUS OF ARCH = 133.114 IN.

E = YOUNG'S MODULUS = 10^7 PSI

A = ARCH CROSS SECTIONAL AREA = 0.188 IN^2.

I = ARCH CROSS SECTIONAL AREA MOMENT OF INERTIA = 0.00055 IN^4.

BOUNDARY CONDITIONS: CLAMPED

TOTAL EQUILIBRIUM CHECK AND AN INCREMENTAL-ITERATIVE PROCEDURE IS USED FOR SHALLOW BEAM ELEMENTS

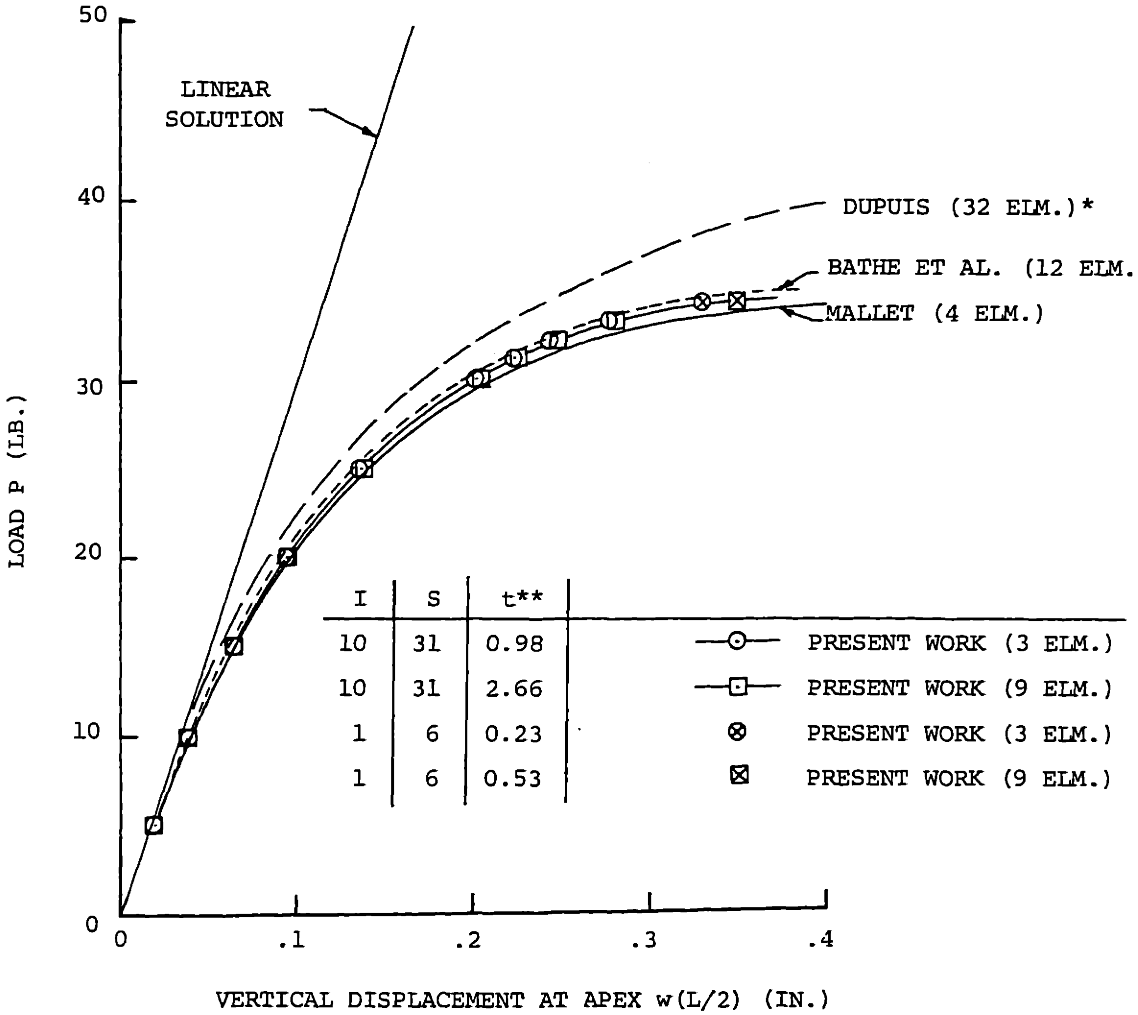
I = NUMBER OF INCREMENTS (LOAD STEPS) TO TOTAL LOAD

R = CONVERGENCE RATIO = 0.001

S = TOTAL NUMBER OF SOLUTION STEPS (INCREMENTS AND ITERATIONS)

t = TOTAL EXECUTION TIME (SECONDS)

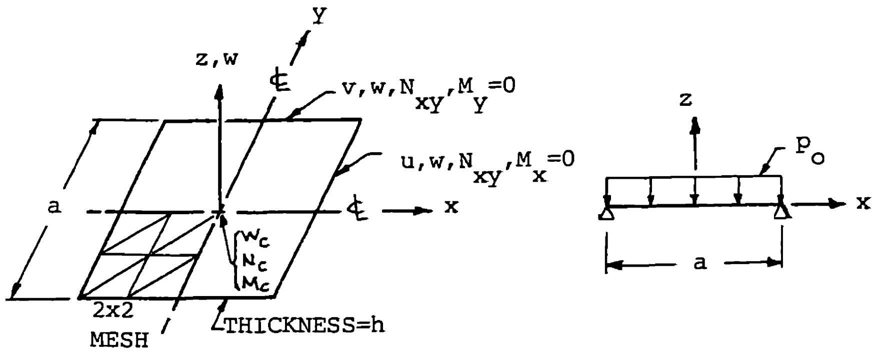
FIG. 8.28 DESCRIPTION OF SHALLOW, CIRCULAR ARCH PROBLEM



*ELM. = NUMBER OF ELEMENTS FOR <u>HALF</u> OF THE ARCH

**TIME (SECS.) ARE GIVEN FOR MODEL OF COMPLETE ARCH

FIG. 8.29 SHALLOW, CIRCULAR ARCH UNDER CENTRAL CONCENTRATED LOAD UTILIZING THE STATIONARY LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND MESH SIZE WITH INDEPENDENT SOLUTIONS FOR CENTRAL DISPLACEMENT



p_o = UNIFORM PRESSURE OVER ENTIRE PLATE

a = SIDE LENGTH OF PLATE = 10 IN.

h = THICKNESS OF PLATE = 0.100 IN.

E = YOUNG'S MODULUS = 10^7 PSI

ν = POISSON'S RATIO = 0.316 (UNLESS OTHERWISE SPECIFIED)

BOUNDARY CONDITIONS: SIMPLY SUPPORTED (ALL EDGES)

NONDIMENSIONAL PARAMETERS:

$$\bar{p} = \frac{p_o a^4}{Eh^4} \qquad \bar{W} = \frac{w_c}{t} \qquad \bar{N} = \frac{Na^2}{Eh^3} \qquad \bar{M} = \frac{6Ma^2}{Eh^4}$$

EQCK = GENERAL, ITERATIVE EQUILIBRIUM CHECK IN LITERATURE

SEQCK = PRESENT WORK: STRESS EQUILIBRIUM CHECK ONLY

TEQCK = PRESENT WORK: TOTAL EQUILIBRIUM CHECK

I = NUMBER OF INCREMENTS (LOAD STEPS) TO TOTAL LOAD

R = CONVERGENCE RATIO = 0.001

S = TOTAL NUMBER OF SOLUTION STEPS (INCREMENTS AND ITERATIONS)

t = TOTAL EXECUTION TIME (SECONDS)

FIG. 8.30 DESCRIPTION OF SIMPLY SUPPORTED FLAT PLATE PROBLEM

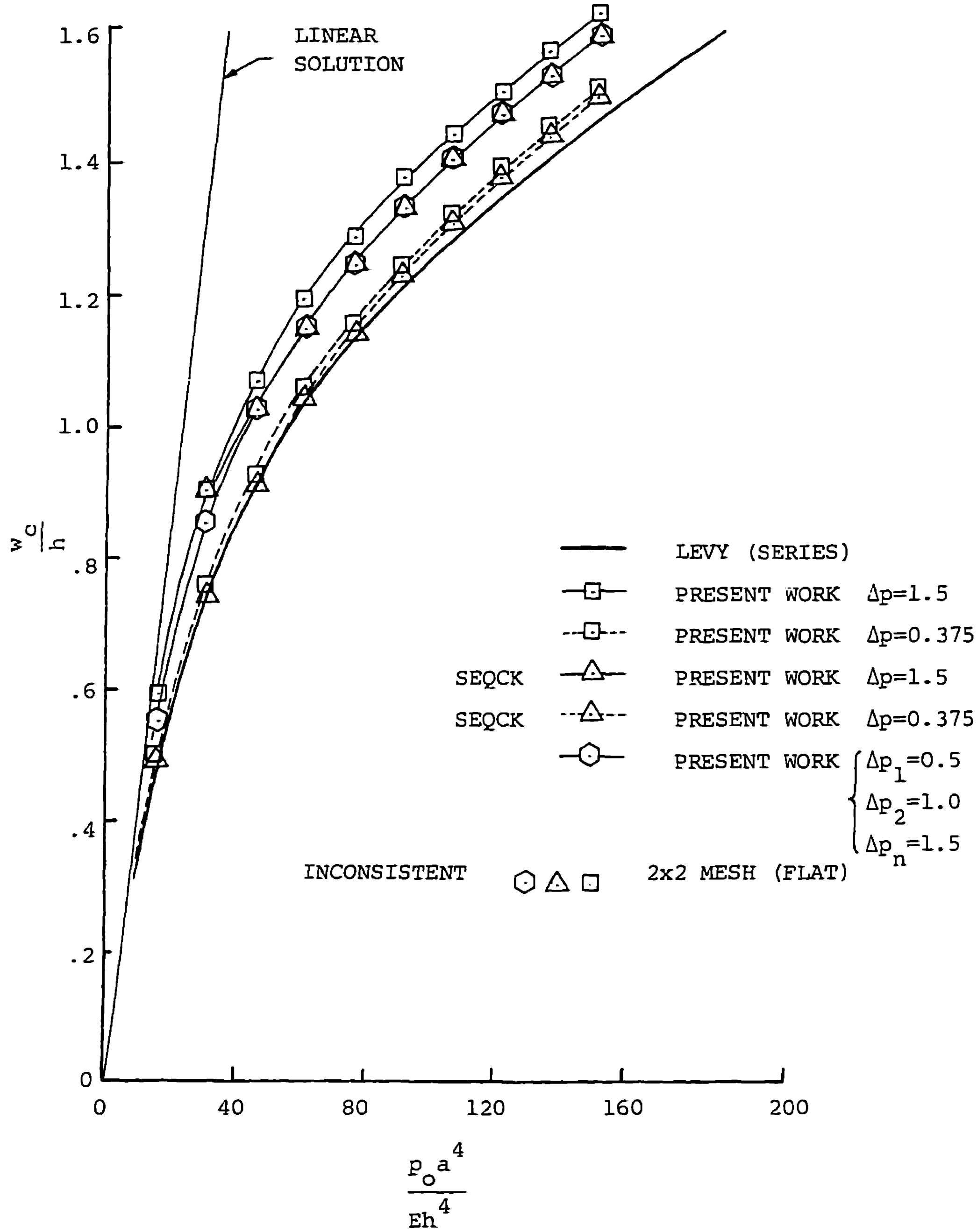


FIG. 8.31 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (2X2 MESH; FLAT ELEMENTS)

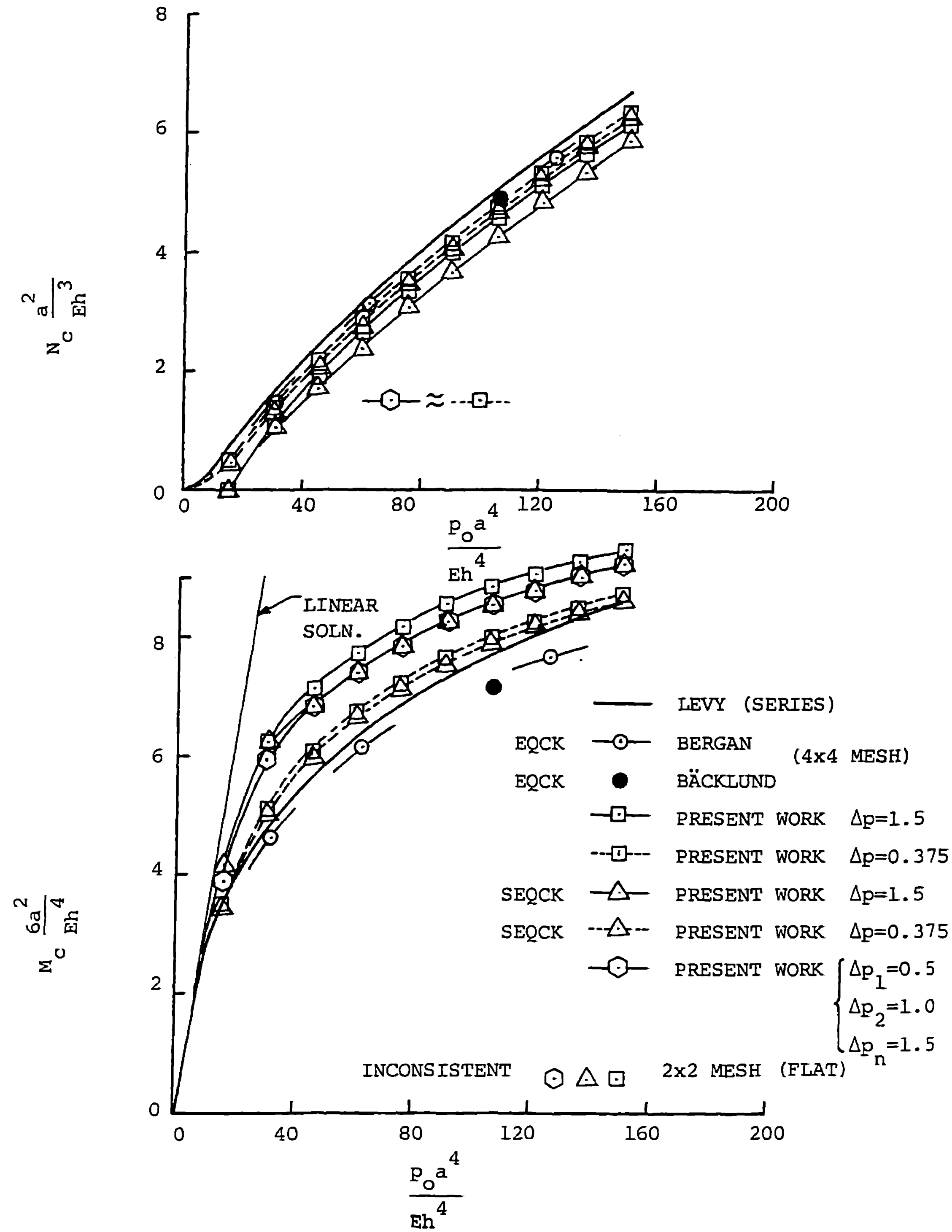


FIG. 8.32 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL STRESS RESULTANTS (2X2 MESH; FLAT ELEMENTS)

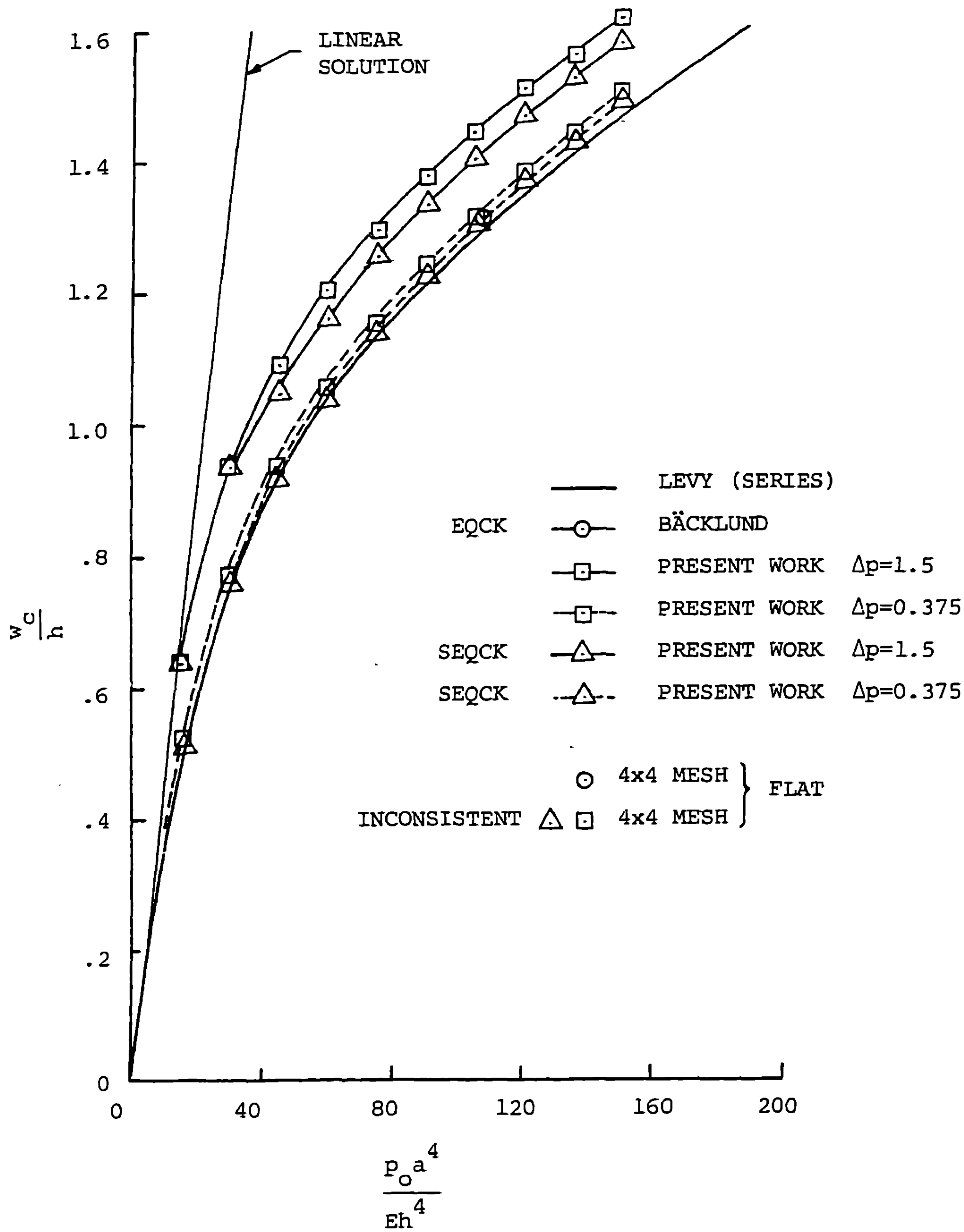


FIG. 8.33 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL DISPLACEMENT (4X4 MESH; FLAT ELEMENTS)

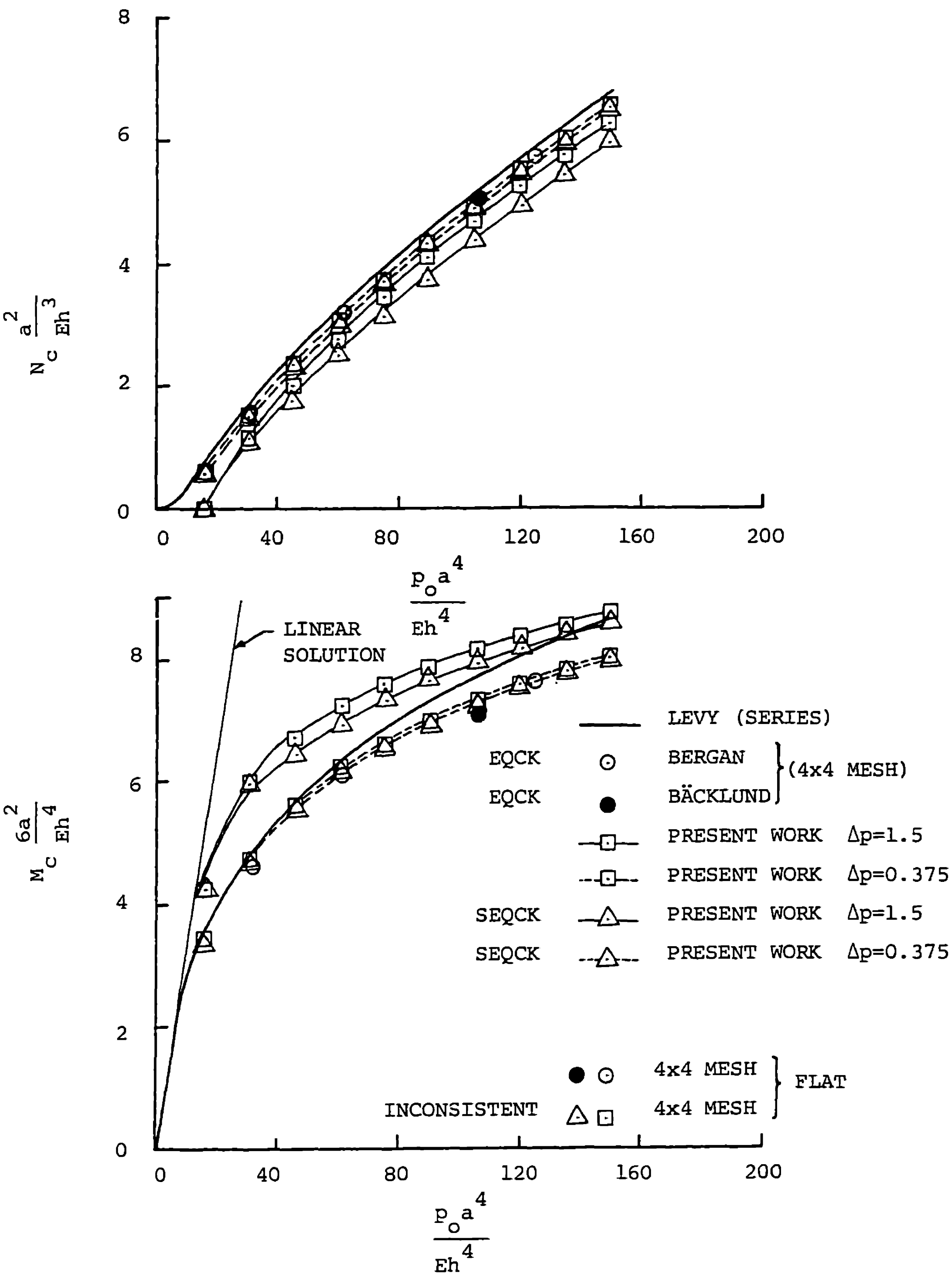


FIG. 8.34 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE CONVECTED, UPDATED LAGRANGIAN SYSTEM - COMPARISONS OF INCREMENT SIZE AND STRESS EQUILIBRIUM CHECK FOR CENTRAL STRESS RESULTANTS (4X4 MESH; FLAT ELEMENTS)

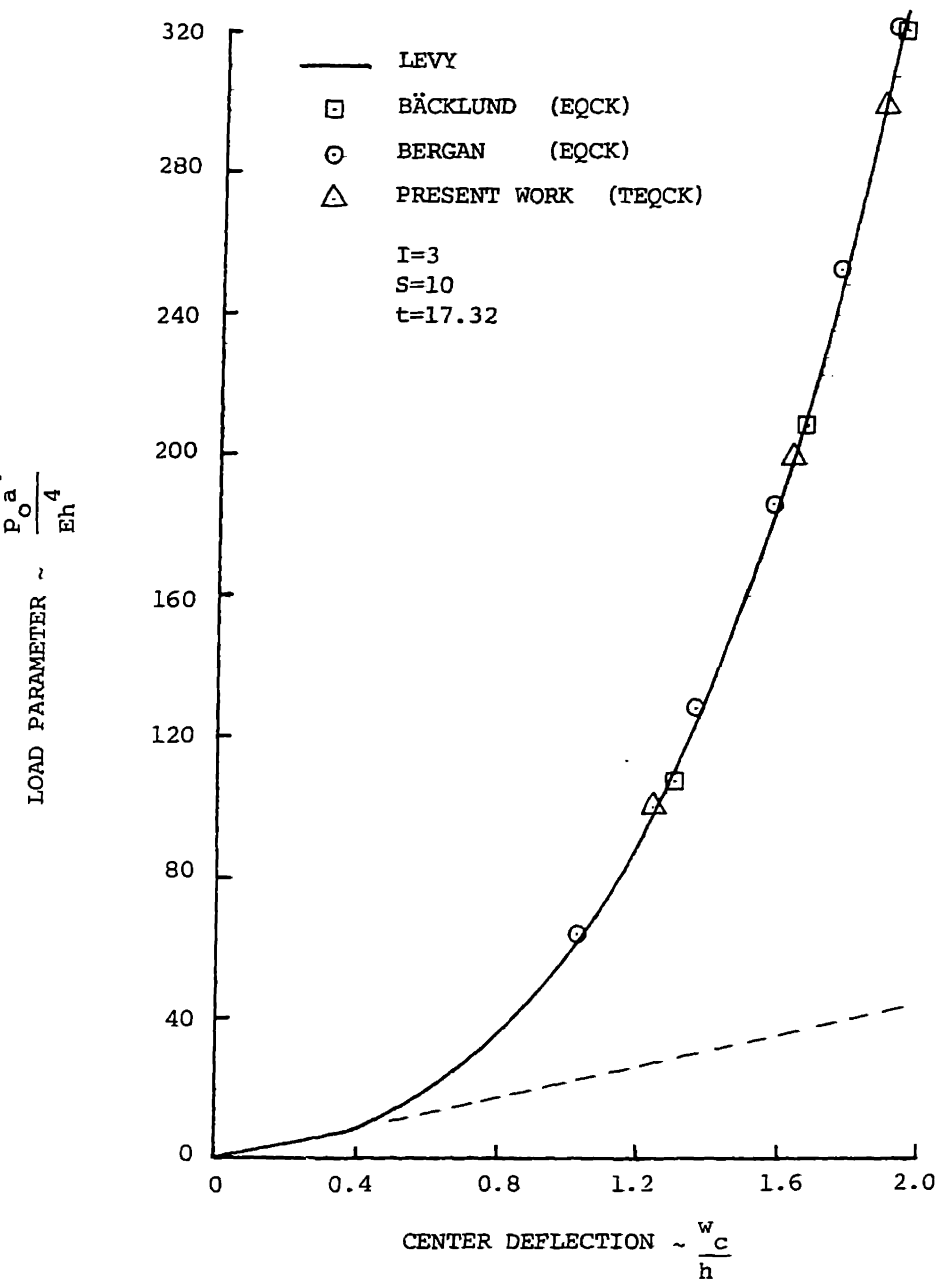


FIG. 8.35 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE STATIONARY LAGRANGIAN SYSTEM - COMPARISONS OF CENTRAL DEFLECTION WITH INDEPENDENT SOLUTIONS

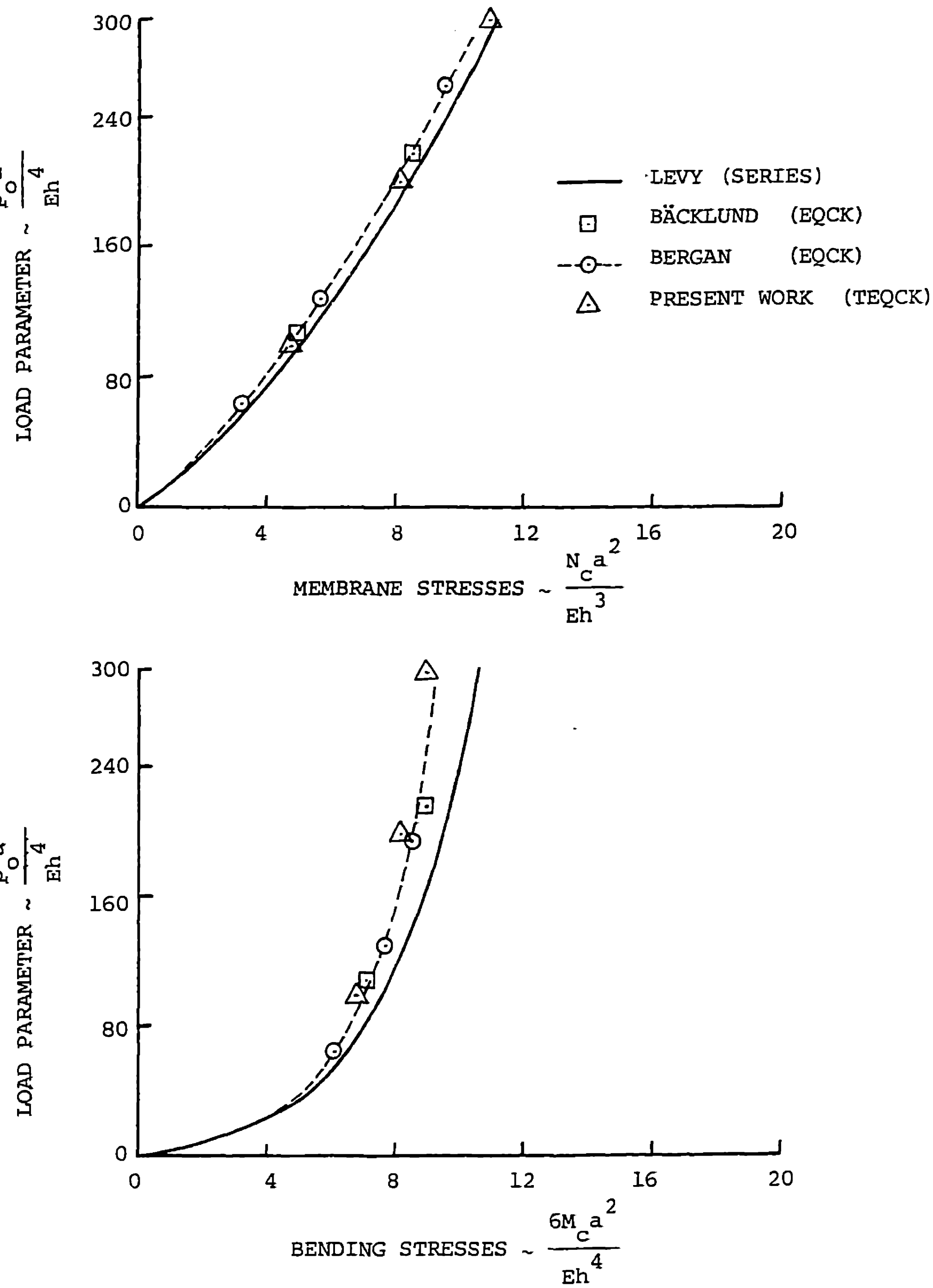
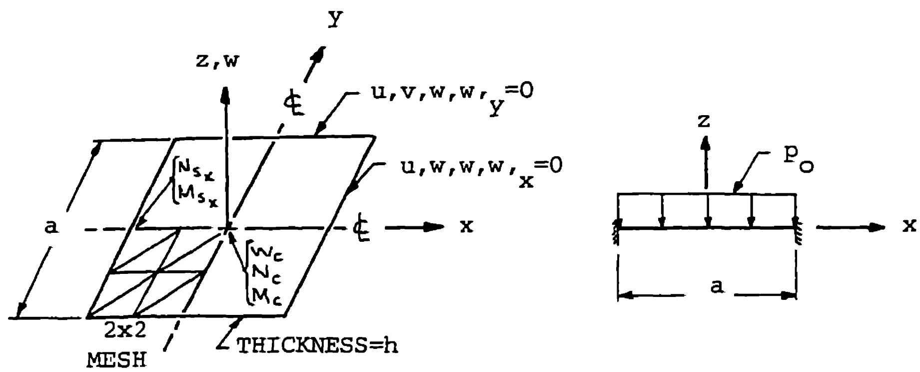


FIG. 8.36 SIMPLY SUPPORTED PLATE UNDER UNIFORM PRESSURE UTILIZING THE STATIONARY LAGRANGIAN SYSTEM - COMPARISONS OF CENTRAL STRESS RESULTANTS WITH INDEPENDENT SOLUTIONS



p_o = UNIFORM PRESSURE OVER ENTIRE PLATE

a = SIDE LENGTH OF PLATE = 10 IN.

h = THICKNESS OF PLATE = 0.100 IN.

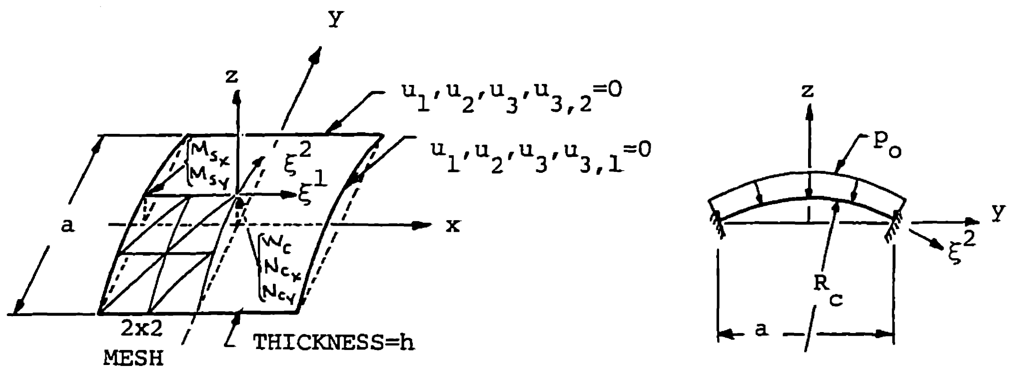
E = YOUNG'S MODULUS = 10^7 PSI

ν = POISSON'S RATIO = AS STATED ON TABLE 8.12

BOUNDARY CONDITIONS: CLAMPED (ALL EDGES)

NONDIMENSIONAL PARAMETERS AND CODES ARE THE SAME AS IN FIG. 8.30

FIG. 8.37 DESCRIPTION OF CLAMPED FLAT PLATE PROBLEM



p_o = UNIFORM PRESSURE OVER ENTIRE PANEL

a = SIDE LENGTH IN BASE PLANE = 20 IN.

h = THICKNESS OF PANEL = 0.125 IN.

E = YOUNG'S MODULUS = 4.50×10^5 PSI

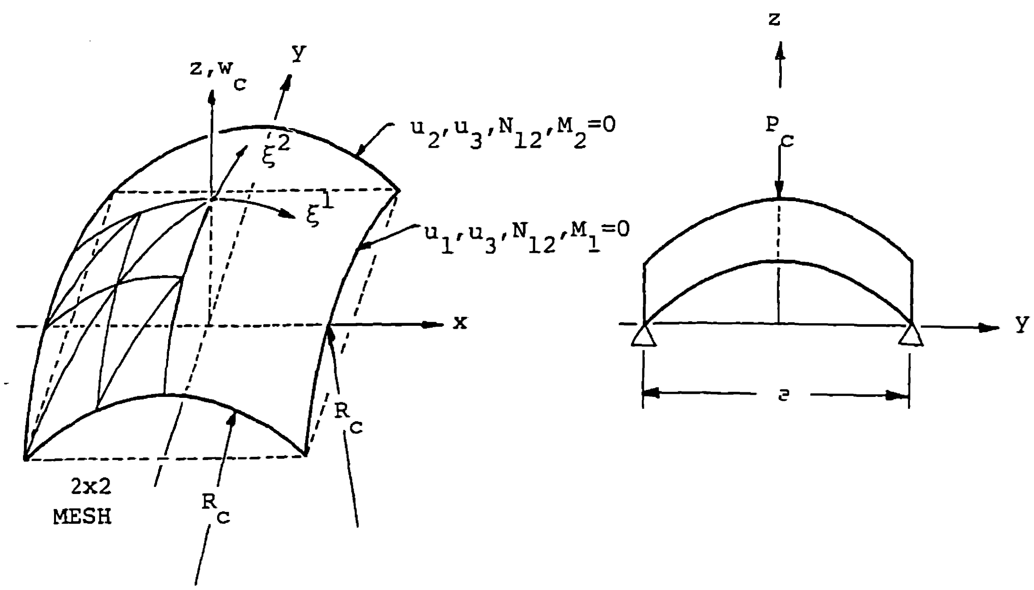
ν = POISSON'S RATIO = 0.300

R_c = RADIUS OF PANEL = 100 IN.

BOUNDARY CONDITIONS: CLAMPED (ALL EDGES)
(NOTE B.C.'S ACT IN SHELL SURFACE)

CODES ARE THE SAME AS IN FIG. 8.30

FIG. 8.38 DESCRIPTION OF CLAMPED, SHALLOW, CYLINDRICAL PANEL PROBLEM



P_c = CENTRAL CONCENTRATED LOAD (LBS.)

a = SIDE LENGTH OF BASE PLANE = 61.8034 IN.

h = THICKNESS OF SHELL = 3.9154 IN.

R_c = RADIUS OF CIRCLE AT INTERSECTION OF SPHERE AND PLANES PARALLEL TO GLOBAL x,y,z AXES = 100 IN.

E = YOUNG'S MODULUS = 10^5 PSI

ν = POISSON'S RATIO = 0.300

BOUNDARY CONDITIONS: SIMPLY SUPPORTED (ALL EDGES)
(NOTE B.C.'S ACT IN SHELL SURFACE)

FIG. 8.39 DESCRIPTION OF SPHERICAL CAP PROBLEM

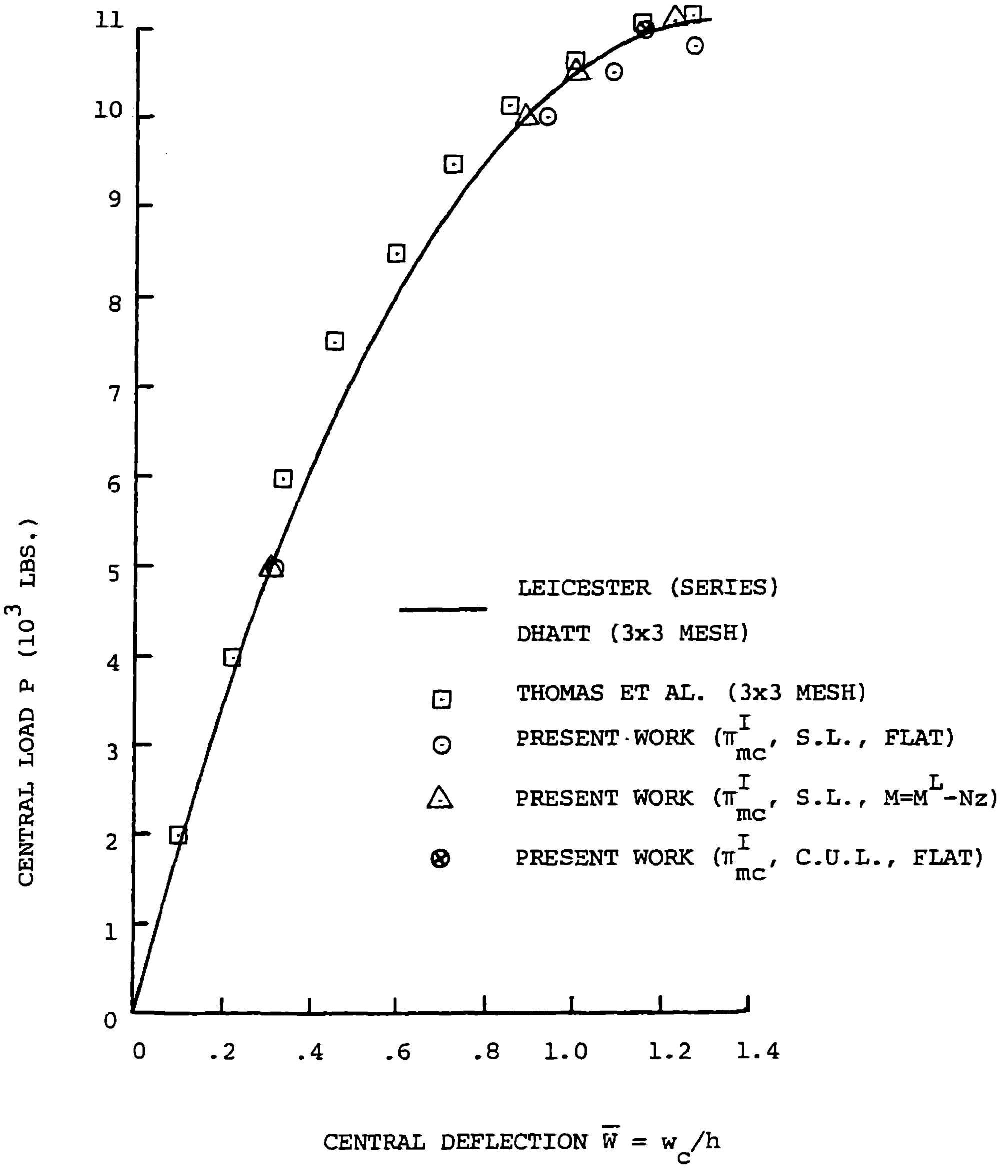
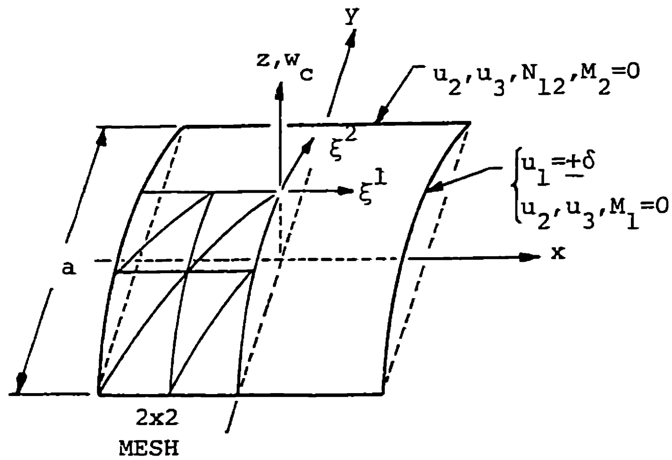


FIG. 8.40 SPHERICAL CAP UNDER CENTRAL CONCENTRATED LOAD - COMPARISON OF CENTRAL DEFLECTION BY VARIOUS MODELS (4X4 MESH) WITH INDEPENDENT SOLUTIONS



δ = UNIFORM EDGE COMPRESSION

a = SIDE LENGTH IN BASE PLANE = 23.9424 IN.

h = THICKNESS OF PANEL = 0.100 IN.

E = YOUNG'S MODULUS = 3×10^{7} PSI

ν = POISSON'S RATIO = 0.300

R_c = RADIUS OF PANEL = 500 IN.

BOUNDARY CONDITIONS: SIMPLY SUPPORTED

$u_1(\xi^1=a/2)=-\delta$ $\quad$ $u_1(\xi^1=-a/2)=\delta$

(NOTE B.C.'S ACT IN SHELL SURFACE)

FIG. 8.41 DESCRIPTION OF SHALLOW, CYLINDRICAL PANEL UNDER AXIAL COMPRESSION PROBLEM

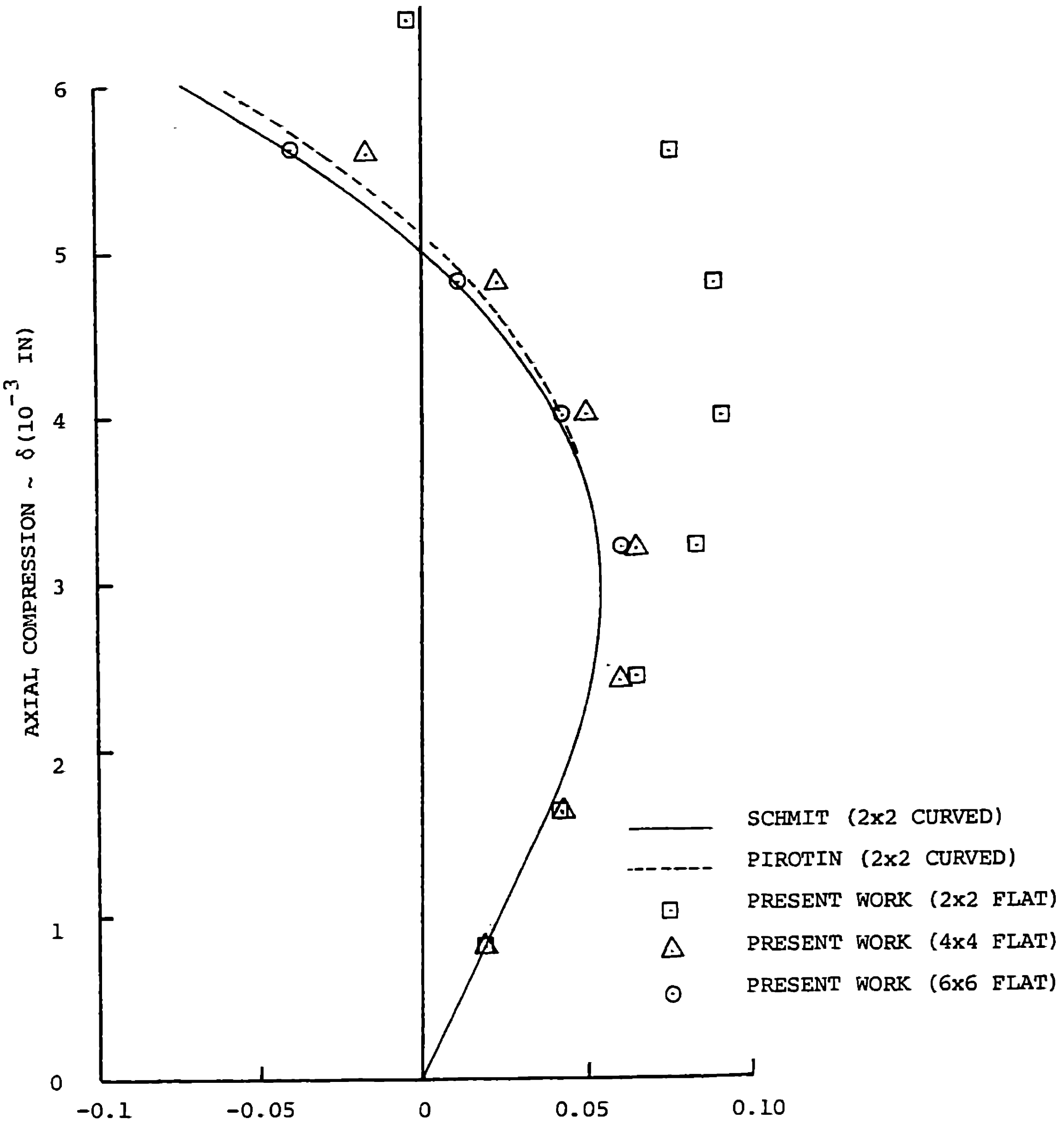


FIG. 8.42 SHALLOW, CYLINDRICAL PANEL UNDER AXIAL COMPRESSION UTILIZING THE STATIONARY LAGRANGIAN SYSTEM - COMPARISON OF MESH SIZE ON CENTRAL DEFLECTION

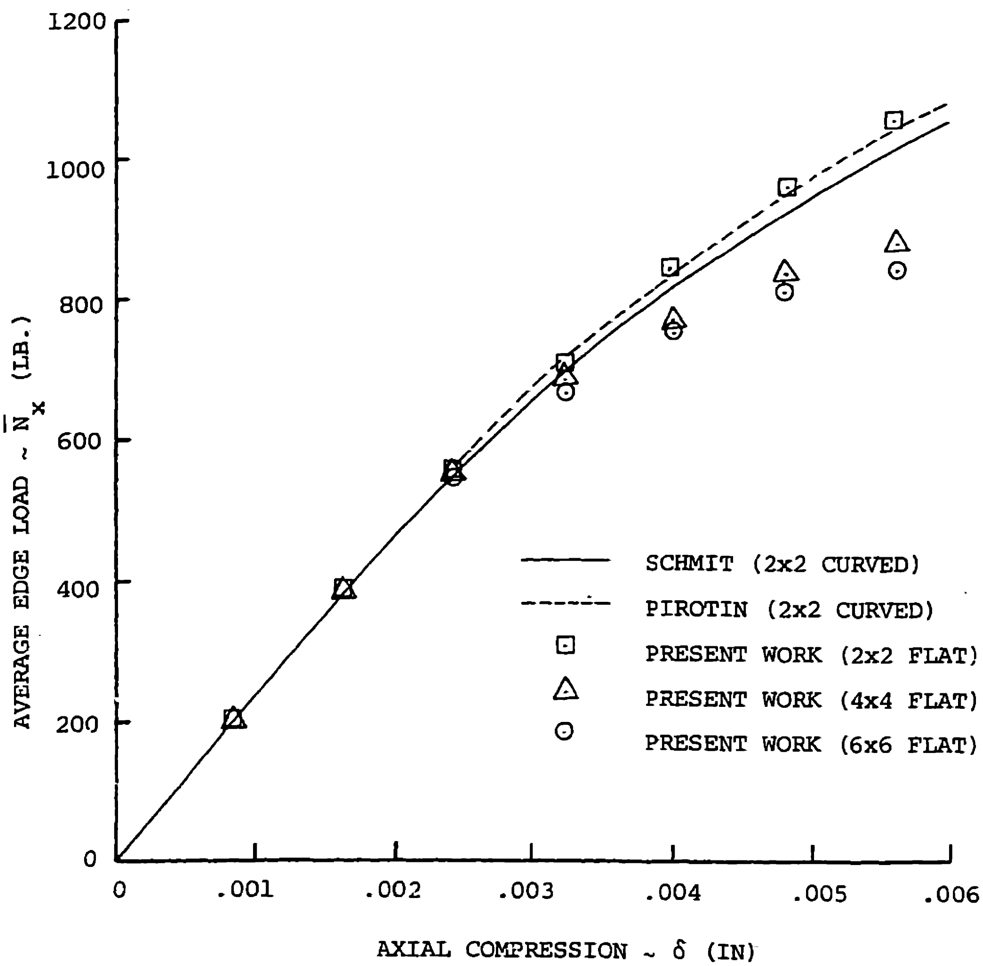


FIG. 8.43 SHALLOW, CYLINDRICAL PANEL UNDER AXIAL COMPRESSION UTILIZING THE STATIONARY LAGRANGIAN SYSTEM - COMPARISON OF MESH SIZE ON AVERAGE END LOAD (LOAD CARRYING CAPABILITY)

APPENDIX A

CURVILINEAR COORDINATES FOR THE KIRCHHOFF-LOVE SHALLOW SHELL THEORY

Four coordinate systems shall be described here. The first is a fixed, rectangular, Cartesian global system. This system remains stationary in space for all time and will be referred to as the G.R. system. Next is a curvilinear global system. This system, referred to as G.C., always remains in the natural shell coordinates. It represents the shell as a continuum and corresponds uniquely to the G.R. system. Two local systems also exist. The first of these, referred to as the L.R. system, is a rectangular Cartesian set of coordinates corresponding to the local base plane of an element. The L.R. system is uniquely related to the fixed G.R. system at all times. Finally, corresponding to L.R. is the L.C. system which is a local curvilinear set. See Fig. A.1.

Let the coordinates of the G.R. system be ${}^{G}x$, ${}^{G}y$, ${}^{G}z$. Also, let the coordinates of the G.C. system be ${}^{G}\xi^{1}$, ${}^{G}\xi^{2}$, ζ. The latter coordinates will be defined to form an orthogonal set. Let the ζ coordinate be along the outward normal to the shell ${}^{G}\bar{n}$. The ${}^{G}\xi^{1}$ and ${}^{G}\xi^{2}$ coordinates must lie in the midsurface of the shell. Choose ${}^{G}\xi^{1}$ such that its orthogonal projection on to the G.R. system corresponds to the ${}^{G}x$ axis. Along ${}^{G}\xi^{1}$ will be the base vector ${}^{G}\bar{t}_1$. With this information an orthogonal coordinate ${}^{G}\xi^{2}$ can be obtained if it is along the base vector ${}^{G}\bar{t}_2$ where

$$ {}^{G}\bar{t}_2 = {}^{G}\bar{n} \times {}^{G}\bar{t}_1 \tag{A.1} $$

To define these more precisely, consider a position vector ${}^{G}\bar{r}$ from the G.R. system to the shell midsurface. (See Fig. A.1.) In terms of the base vectors of the G.R. system, ${}^{G}\bar{i}_x$, ${}^{G}\bar{i}_y$, ${}^{G}\bar{i}_z$

$$ {}^{G}\bar{r} = {}^{G}x\,{}^{G}\bar{i}_x + {}^{G}y\,{}^{G}\bar{i}_y + {}^{G}z\,{}^{G}\bar{i}_z \tag{A.2} $$

Define base vectors $\bar{a}_1$ and $\bar{a}_2$ in the shell midsurface as

$$ \bar{a}_1 = {}^{G}_{G}\bar{r}_{,x} = {}^{G}\bar{i}_x + {}^{G}_{G}z_{,x}\,{}^{G}\bar{i}_z \tag{A.3} $$

$$ \bar{a}_2 = {}^{G}_{G}\bar{r}_{,y} = {}^{G}\bar{i}_y + {}^{G}_{G}z_{,y}\,{}^{G}\bar{i}_z \tag{A.4} $$

Thus, a unit normal vector to the shell may be defined by

$$ {}^{G}\bar{n} = \bar{a}_1 \times \bar{a}_2 / |\bar{a}_1 \times \bar{a}_2| \tag{A.5} $$

Placing Eqs. A.3 and A.4 into Eq. A.5 yields

$$ {}^{G}\bar{n} = \frac{1}{c_1}\left[-{}^{G}_{G}z_{,x}\,{}^{G}\bar{i}_x - {}^{G}_{G}z_{,y}\,{}^{G}\bar{i}_y + {}^{G}\bar{i}_z\right] \tag{A.6} $$

where

$$ c_1 = \sqrt{1 + {}^{G}_{G}z^2_{,x} + {}^{G}_{G}z^2_{,y}} \tag{A.7} $$

Since $\bar{a}_1$ is projected along ${}^{G}x$ then a unit tangent vector may be obtained as

$$ {}^{G}\bar{t}_1 = \bar{a}_1 / |\bar{a}_1| = \frac{1}{c_2}\left[{}^{G}\bar{i}_x + {}^{G}_{G}z_{,x}\,{}^{G}\bar{i}_z\right] \tag{A.8} $$

where

$$ c_2 = \sqrt{1 + {}^{G}_{G}z^2_{,x}} \tag{A.9} $$

Because ${}^{G}\bar{n}$ and ${}^{G}\bar{t}_1$ are unit vectors ${}^{G}\bar{t}_2$ may be defined as

$$ \begin{aligned} {}^{G}\bar{t}_2 &= {}^{G}\bar{n} \times {}^{G}\bar{t}_1 \\ &= \frac{1}{c_1 c_2}\left[-{}^{G}_{G}z_{,x}\,{}^{G}_{G}z_{,y}\,{}^{G}\bar{i}_x + (1 + {}^{G}_{G}z^2_{,x})\,{}^{G}\bar{i}_y + {}^{G}_{G}z_{,y}\,{}^{G}\bar{i}_z\right] \end{aligned} \tag{A.10} $$

Therefore, ${}^{G}\bar{t}_1$, ${}^{G}\bar{t}_2$, and ${}^{G}\bar{n}$ are a triad of unit orthogonal curvilinear base vectors corresponding to the entire shell. And via Eqs. A.8, A.10, and A.6 respectively are uniquely related to the G.R. system as long as the slopes are defined.

In a completely analogous procedure a position vector may be established from the L.R. system to the portion of the shell (element) corresponding to it. (See Fig. A.1.)

$$ {}^{L}\bar{r} = {}^{L}x\,{}^{L}\bar{i}_x + {}^{L}y\,{}^{L}\bar{i}_y + {}^{L}z\,{}^{L}\bar{i}_z \tag{A.11} $$

One may define base vectors ${}^{L}\bar{t}_1$ and ${}^{L}\bar{t}_2$ as

$$ {}^{L}\bar{t}_1 = {}^{L}_{L}\bar{r}_{,x} = {}^{L}\bar{i}_x + {}^{L}_{L}z_{,x}\,{}^{L}\bar{i}_z \tag{A.12} $$

$$ {}^{L}\bar{t}_2 = {}^{L}_{L}\bar{r}_{,y} = {}^{L}\bar{i}_y + {}^{L}_{L}z_{,y}\,{}^{L}\bar{i}_z \tag{A.13} $$

Since these vectors are in the shell midsurface the local normal is

$$ {}^{L}\bar{n} = {}^{L}\bar{t}_1 \times {}^{L}\bar{t}_2 = -{}^{L}_{L}z_{,x}\,{}^{L}\bar{i}_x - {}^{L}_{L}z_{,y}\,{}^{L}\bar{i}_y + {}^{L}\bar{i}_z \tag{A.14} $$

Note that these vectors are related to the local slopes which for locally shallow elements are small. Recalling that a condition of shallowness is (from Eq. 3.25)

$$|{}^L_L z_{,x}|^2 \sim |{}^L_L z_{,y}|^2 \sim |{}^L_L z_{,x}\, {}^L_L z_{,y}| << 1 \tag{A.15}$$

One can see that Eqs. A.12-A.14 are approximately unit vectors and will be considered as such. Additionally, one may observe that these base vectors are approximately orthogonal. ${}^L\bar{t}_1$, ${}^L\bar{t}_2$, ${}^L\bar{n}$ will be considered a triad of unit, orthogonal curvilinear base vectors uniquely related to the L.R. system. The L.R. system is also uniquely related to the G.R. system by

$${}^G\{x\} = {}^{GL}[T]\,{}^L\{x\} \tag{3.43}$$

The global slopes, ${}^G_G z_{,x}$ and ${}^G_G z_{,y}$ can be easily obtained by Eqs. 7.125 so it only remains to determine the local slopes.

It should be recognized that the global and local normals must be normal to the same shell surface and, therefore, must coincide.

$${}^G\bar{n} = {}^L\bar{n} \tag{A.16}$$

Placing Eqs. A.6 and A.14 into this gives

$$\frac{1}{C_1}\left[-{}^G_G z_{,x}\,{}^G\bar{i}_x - {}^G_G z_{,y}\,{}^G\bar{i}_y + {}^G\bar{i}_z\right] = \left[-{}^L_L z_{,x}\,{}^L\bar{i}_x - {}^L_L z_{,y}\,{}^L\bar{i}_y + {}^L\bar{i}_z\right] \tag{A.17}$$

Since the base vectors of the G.R. and L.R. systems are related the same way as the corresponding coordinates, from Eq. 3.43

$${}^G\{\bar{i}\} = {}^{GL}[T]\,{}^L\{\bar{i}\} \tag{A.18}$$

Placing this into Eq. A.17 yields

$$\begin{aligned} \frac{1}{C_1}\Big\{ &{}^G_G z_{,x}\left[({}^Gx,{}^Lx)\,{}^L\bar{i}_x + ({}^Gx,{}^Ly)\,{}^L\bar{i}_y + ({}^Gx,{}^Lz)\,{}^L\bar{i}_z\right] \\ &+ {}^G_G z_{,y}\left[({}^Gy,{}^Lx)\,{}^L\bar{i}_x + ({}^Gy,{}^Ly)\,{}^L\bar{i}_y + ({}^Gy,{}^Lz)\,{}^L\bar{i}_z\right] \\ &- \left[({}^Gz,{}^Lx)\,{}^L\bar{i}_x + ({}^Gz,{}^Ly)\,{}^L\bar{i}_y + ({}^Gz,{}^Lz)\,{}^L\bar{i}_z\right]\Big\} \\ &= {}^L_L z_{,x}\,{}^L\bar{i}_x + {}^L_L z_{,y}\,{}^L\bar{i}_y - {}^L\bar{i}_z \end{aligned} \tag{A.19}$$

Comparing coefficients of the local base vectors one obtains

$$
\begin{aligned}
{}^{L}_{L}z_{,x} &= \frac{1}{c}\left[({}^{G}z,{}^{L}x) - ({}^{G}x,{}^{L}x)\,{}^{G}_{G}z_{,x} - ({}^{G}y,{}^{L}x)\,{}^{G}_{G}z_{,y}\right] \\
{}^{L}_{L}z_{,y} &= \frac{1}{c}\left[({}^{G}z,{}^{L}y) - ({}^{G}x,{}^{L}y)\,{}^{G}_{G}z_{,x} - ({}^{G}y,{}^{L}y)\,{}^{G}_{G}z_{,y}\right] \\
c &= ({}^{G}x,{}^{L}z)\,{}^{G}_{G}z_{,x} + ({}^{G}y,{}^{L}z)\,{}^{G}_{G}z_{,y} - ({}^{G}z,{}^{L}z)
\end{aligned}
\qquad \text{(A.20)}
$$

Thus, each system can be related to the others and all the terms necessary for the analysis can be obtained.

According to the Kirchhoff-Love theory the local element displacements ${}^{L}\Delta u$, ${}^{L}\Delta v$, ${}^{L}\Delta w$ would be measured along ${}^{L}\bar{t}_1$, ${}^{L}\bar{t}_2$, and ${}^{L}\bar{n}$ respectively. The assembled system of equations is referred to displacements measured along ${}^{G}\bar{t}_1$, ${}^{G}\bar{t}_2$, and ${}^{G}\bar{n}$. Thus, a transformation is required to bring the element level matrices into the assembly system (G.C.). Since the normals coincide and since both systems (G.C. and L.C.) are taken to be unit orthogonal systems then only a simple planar rotation is required. This plane would be the tangent plane to the shell at the point of the outward normal. The angle of rotation required is simply obtained by

$$
\begin{aligned}
\cos\theta &= {}^{G}\bar{t}_1 \cdot {}^{L}\bar{t}_1 \\
&= \frac{1}{c_2}\left[{}^{G}\bar{i}_x + {}^{G}_{G}z_{,x}\,{}^{G}\bar{i}_z\right]\cdot\left[{}^{L}\bar{i}_x + {}^{L}_{L}z_{,x}\,{}^{L}\bar{i}_z\right] \\
&= \frac{1}{c_2}\Big[({}^{G}x,{}^{L}x) + ({}^{G}x,{}^{L}z)\,{}^{L}_{L}z_{,x} + ({}^{G}z,{}^{L}x)\,{}^{G}_{G}z_{,x} \\
&\qquad + ({}^{G}z,{}^{L}z)\,{}^{G}_{G}z_{,x}\,{}^{L}_{L}z_{,x}\Big]
\end{aligned}
\qquad \text{(A.21)}
$$

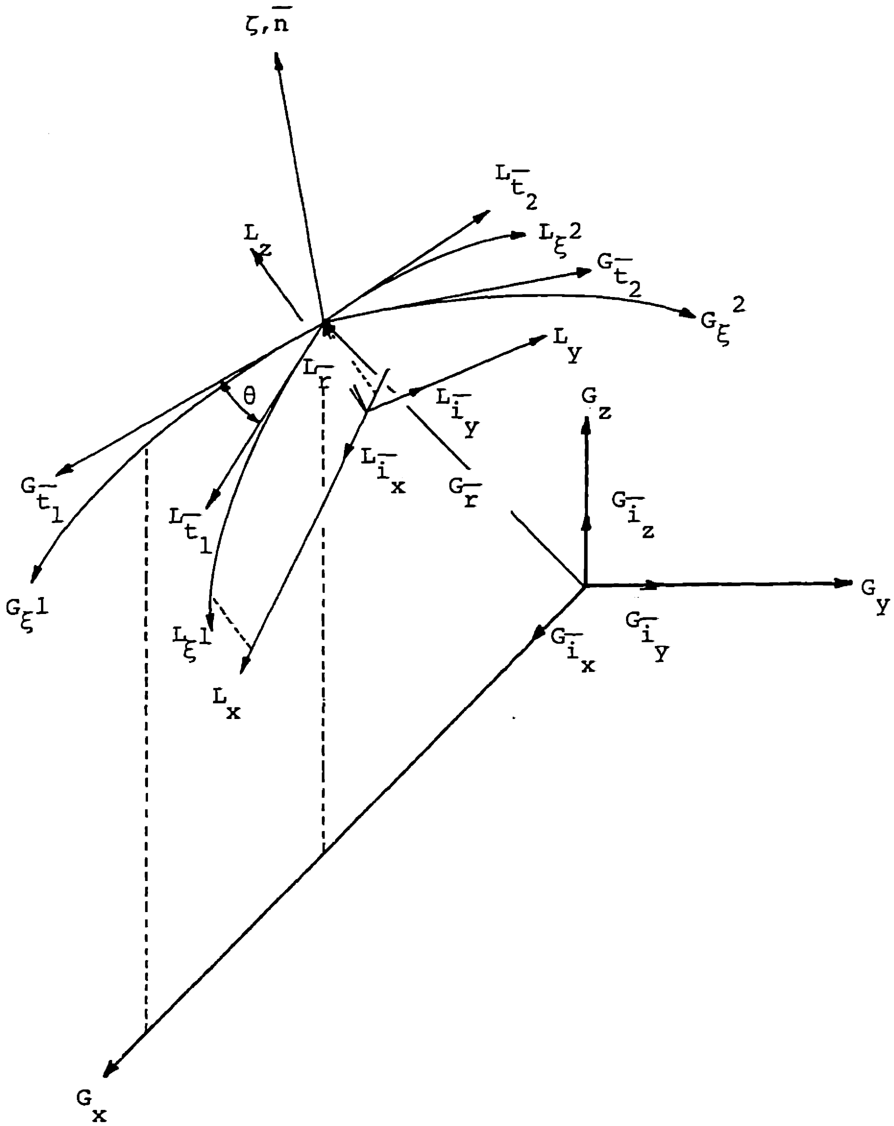


FIG. A.1 DESCRIPTION OF GLOBAL AND LOCAL FRAMES FOR RECTANGULAR CARTESIAN AND ORTHOGONAL CURVILINEAR COORDINATES AND THEIR ASSOCIATED UNIT BASE VECTORS

APPENDIX B

THE PRINCIPLE OF VIRTUAL COMPLEMENTARY WORK AND THE PRINCIPLE OF STATIONARY TOTAL COMPLEMENTARY ENERGY

The Principle of Virtual Complementary Work, in a total sense, for large deflection analysis has been discussed by Langhaar [1953], Levinson [1965], and more recently by Koiter [1973]. A derivation will be given here which will correspond to the incremental functionals, π^{c}_{mc}, of Section 4, for the S.L. system. In a total sense this principle corresponds to π_c derived by Washizu [1971].

The external work on a continuum may be expressed as

$$W_E = \int_V \bar{F}_i u_i \, dv + \int_{\partial V} T_i u_i \, ds \tag{B.1}$$

Expanding this in initial and incremental quantities

$$W_E + \Delta W_E = \int_V (\bar{F}_i + \Delta\bar{F}_i)(u_i + \Delta u_i) \, dv + \int_{\partial V} (T_i + \Delta T_i)(u_i + \Delta u_i) \, ds \tag{B.2}$$

The Principle of Virtual Complementary Work states that the sum of the complementary work done by virtual surface forces is equal to the virtual complementary work done under the exact state of strain by virtual stresses satisfying the stress equilibrium conditions. Considering only incremental forces and stresses subject to variation, Eq. B.2 becomes

$$\delta \Delta W_E = \int_V \delta\Delta\bar{F}_i (u_i + \Delta u_i) \, dv + \int_{\partial V} \delta\Delta T_i (u_i + \Delta u_i) \, ds \tag{B.3}$$

The equations of stress equilibrium may be written as

$$\sigma_{ij,j} + (\sigma_{kj} u_{i,k})_{,j} + \bar{F}_i = 0 \tag{B.4}$$

and the surface tractions as

$$T_i = (\sigma_{ij} + \sigma_{kj} u_{i,k}) \nu_j \tag{B.5}$$

Since ultimately one seeks a principle which will be stationary, the variations taken must be on independent variables. As can be seen from Eq. B.4 the first Kirchhoff stress is dependent on the displacements. Therefore, one may not take the variation with respect to this stress. However, making use of the unsymmetric first Piola stress of Eq. 2.56 one may write the stress equilibrium equations as

$$p_{ji,j} + \bar{F}_i = 0 \quad \text{(B.6)}$$

where

$$p_{ji} \equiv \overset{\circ}{\circ} T^{ji} \quad \text{(of Eq. 2.56)}$$

Additionally, the surface tractions may be written as

$$T_i = p_{ji} \nu_j \quad \text{(B.7)}$$

Eqs. B.6 and B.7 may be expanded to incremental form as

$$(p_{ji} + \Delta p_{ji})_{,j} + (\bar{F}_i + \Delta \bar{F}_i) = 0 \quad \text{(B.8)}$$

and

$$(T_i + \Delta T_i) = (p_{ji} + \Delta p_{ji}) \nu_j \quad \text{(B.9)}$$

Taking the variation of Eqs. B.8 and B.9 with respect to the independent, incremental stresses only

$$(\delta \Delta p_{ji})_{,j} + \delta \Delta \bar{F}_i = 0 \quad \text{(B.10)}$$

and

$$\delta \Delta T_i = (\delta \Delta p_{ji}) \nu_j \quad \text{(B.11)}$$

Placing these back into Eq. B.3 yields

$$\delta \Delta W_E = -\int_V (\delta \Delta p_{ji})_{,j} (u_i + \Delta u_i)\, dv + \int_{\partial V} (\delta \Delta p_{ji}) \nu_j (u_i + \Delta u_i)\, ds \quad \text{(B.12)}$$

Integrating the first term by parts gives

$$\delta \Delta W_E = \int_V (\delta \Delta p_{ji})(u_i + \Delta u_i)_{,j}\, dv = \delta \Delta U_c \quad \text{(B.13)}$$

Note that since Δp_{ij} is unsymmetric Eq. B.13 must be left in terms of the displacement gradients. Equating Eqs. B.3 and B.13 and realizing that $\delta\Delta\bar{F}_i=0$ one obtains

$$\int_V (\delta \Delta p_{ji})(u_i + \Delta u_i)_{,j}\, dv - \int_{\partial V} \delta \Delta T_i (u_i + \Delta u_i)\, ds = 0 \quad \text{(B.14)}$$

The Principle of Virtual Complementary Work becomes the Principle of Stationary Complementary Energy which states that of all the states of stress satisfying the stress equilibrium equations, the actual state of stress is that which makes the complementary energy stationary or

$$\delta\pi_c = \int_V (\delta\Delta p_{ji})(u_i+\Delta u_i)_{,j}\, dV - \int_{\partial V} \delta\Delta T_i (u_i+\Delta u_i)\, ds = 0 \qquad \text{(B.15)}$$

Expanding Eq. 2.56 for incremental analysis gives

$$p_{ji} + \Delta p_{ji} = \sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj})(u_{i,k} + \Delta u_{i,k}) \qquad \text{(B.16)}$$

Separating this into initial and incremental parts

$$p_{ji} = \sigma_{ij} + \sigma_{kj} u_{i,k} \qquad \text{(B.17)}$$

$$\Delta p_{ji} = \Delta\sigma_{ij} + \Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k}) + \sigma_{kj}\Delta u_{i,k} \qquad \text{(B.18)}$$

The variation of Eq. B.18 is not a simple variation of the first Kirchhoff stress but

$$\delta\Delta p_{ji} = \delta\Delta\sigma_{ij} + \delta[\Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k})] + \delta[\sigma_{kj}\Delta u_{i,k}] \qquad \text{(B.19)}$$

Placing this into Eq. B.15 yields

$$\delta\pi_c = \int_V \{\delta\Delta\sigma_{ij} + \delta[\Delta\sigma_{kj}(u_{i,k} + \Delta u_{i,k})] + \delta[\sigma_{kj}\Delta u_{i,k}]\}(u_i+\Delta u_i)_{,j}\, dV$$
$$- \int_{\partial V} \delta\Delta T_i (u_i+\Delta u_i)\, ds = 0$$

or

$$\delta\pi_c = \int_V \{\delta\Delta\sigma_{ij}\tfrac{1}{2}[(u_i+\Delta u_i)_{,j} + (u_j+\Delta u_j)_{,i}] + \delta[\Delta\sigma_{ij}(u_k+\Delta u_k)_{,i}](u_k+\Delta u_k)_{,j}$$
$$+ \delta[\sigma_{ij}\Delta u_{k,i}](u_k+\Delta u_k)_{,j}\}\, dV - \int_{\partial V} \delta\Delta T_i (u_i+\Delta u_i)\, ds$$
$$= \int_V \{\delta\Delta\sigma_{ij}\tfrac{1}{2}[(u_i+\Delta u_i)_{,j} + (u_j+\Delta u_j)_{,i} + (u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j}]$$
$$+ \tfrac{1}{2}\delta\Delta\sigma_{ij}(u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j} + \tfrac{1}{2}\Delta\sigma_{ij}\,\delta[(u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j}]$$
$$+ \tfrac{1}{2}\sigma_{ij}\,\delta[u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j} + \Delta u_{k,i}\Delta u_{k,j}]\}\, dV - \int_{\partial V} \delta\Delta T_i (u_i+\Delta u_i)\, ds = 0$$

or

$$\delta\pi_c = \int_V \{(e_{ij} + \Delta e_{ij})\delta\Delta\sigma_{ij} + \tfrac{1}{2}\delta[(\sigma_{ij}+\Delta\sigma_{ij})(u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j}]\}\, dV$$
$$- \int_{\partial V} \delta\Delta T_i (u_i+\Delta u_i)\, ds = 0 \qquad \text{(B.20)}$$

Note that $\partial V = S_\sigma + S_u$, $\delta\overline{T} = 0$ on S_σ and $\delta(\overline{u}_i + \Delta\overline{u}_i) = 0$ on S_u. Eq. B.20 becomes

$$\delta\pi_c = \int_V \left\{(e_{ij}+\Delta e_{ij})\delta\Delta\sigma_{ij} + \tfrac{1}{2}\delta\left[(\sigma_{ij}+\Delta\sigma_{ij})(u_k+\Delta u_k)_{,i}(u_k+\Delta u_k)_{,j}\right]\right\} dV - \int_{S_u} \delta\left[\Delta T_i(\overline{u}_i+\Delta\overline{u}_i)\right] dS = 0 \tag{B.21}$$

Assuming a potential function exists such that

$$\delta B(\Delta\sigma_{ij}) = \Delta e_{ij}\,\delta\Delta\sigma_{ij} \tag{B.22}$$

the Eq. B.20 becomes

$$\begin{aligned}\delta\pi_c = {} & \int_V \left\{\delta B(\Delta\sigma_{ij}) + \tfrac{1}{2}\delta\left[(\sigma_{ij}+\Delta\sigma_{ij})\Delta u_{k,i}\Delta u_{k,j}\right]\right\} dV - \int_{\partial V}\delta\Delta T_i\,\Delta u_i\,dS \\ & + \int_V \left\{e_{ij}\,\delta\Delta\sigma_{ij} + \tfrac{1}{2}\delta\left[(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}u_{k,j} + u_{k,i}\Delta u_{k,j} + \Delta u_{k,i}u_{k,j})\right]\right\} dV \\ & - \int_{\partial V}\delta\Delta T_i\,u_i\,dS = 0\end{aligned} \tag{B.23}$$

The last two integrals are zero, as can be shown by using the divergence theorem on the surface integral. From Eq. B.11

$$\begin{aligned}-\int_{\partial V}\delta\Delta T_i\,u_i\,dS &= -\int_V(\delta\Delta p_{ji}u_i)_{,j}\,dV = -\int_V(\delta\Delta p_{ji})_{,j}\,u_i\,dV \\ &\quad - \int_V(\delta\Delta p_{ji})u_{i,j}\,dV = -\int_V \delta\Delta p_{ji}\,u_{i,j}\,dV\end{aligned} \tag{B.24}$$

Since Eq. B.10 holds with $\delta\Delta\overline{F}_i = 0$. Placing Eq. B.19 into Eq. B.24 gives

$$\begin{aligned}-\int_{\partial V}\delta\Delta T_i\,u_i\,dS &= -\int_V\left\{\delta\Delta\sigma_{ij} + \delta\left[\Delta\sigma_{kj}(u_{i,k}+\Delta u_{i,k})\right] + \delta\left[\sigma_{kj}\Delta u_{i,k}\right]\right\}u_{i,j}\,dV \\ &= -\int_V\left\{\delta\Delta\sigma_{ij}\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}) + \tfrac{1}{2}\delta\Delta\sigma_{ij}u_{k,i}u_{k,j}\right. \\ &\qquad \left. + \delta\left[(\sigma_{ij}+\Delta\sigma_{ij})\Delta u_{k,i}\right]u_{k,j}\right\}dV \\ &= -\int_V\left\{e_{ij}\,\delta\Delta\sigma_{ij} + \tfrac{1}{2}\delta\left[(\sigma_{ij}+\Delta\sigma_{ij})u_{k,i}u_{k,j}\right]\right. \\ &\qquad \left. + \tfrac{1}{2}\delta\left[(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}\Delta u_{k,j}+\Delta u_{k,i}u_{k,j})\right]\right\}dV\end{aligned} \tag{B.25}$$

where variations on constants can be moved in or out of the variation brackets as well as added with no change in the virtual principle. Finally, Eq. B.25 may be written as

$$-\int_{\partial V} \delta\Delta T_i\, u_i\, ds = -\int_V \Big\{ e_{ij}\, \delta\Delta\sigma_{ij} + \tfrac{1}{2}\, \delta\big[(\sigma_{ij}+\Delta\sigma_{ij})(u_{k,i}\, u_{k,j} + u_{k,i}\,\Delta u_{k,j} + \Delta u_{k,i}\, u_{k,j})\big]\Big\}\, dV \qquad (B.26)$$

Placing Eq. B.26 into Eq. B.23 yields

$$\delta\pi_c = \int_V \Big\{ \delta B(\Delta\sigma_{ij}) + \tfrac{1}{2}\,\delta\big[(\sigma_{ij}+\Delta\sigma_{ij})\,\Delta u_{k,i}\,\Delta u_{k,j}\big]\Big\}\, dV - \int_{\partial V} \delta\Delta T_i\, \Delta u_i\, ds = 0 \qquad (B.27)$$

Recalling the statements following Eq. B.20

$$\delta\pi_c = \delta\Big\{ \int_V \big[B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})\,\Delta u_{k,i}\,\Delta u_{k,j}\big]\, dV - \int_{S_u} \Delta T_i\, \Delta\bar{u}_i\, ds \Big\} = 0 \qquad (B.28)$$

Finally the Principle of Total Minimum Complementary Energy may be written as

$$\pi_c(\Delta\sigma_{ij}, \Delta u_i) = \int_V \big[B(\Delta\sigma_{ij}) + \tfrac{1}{2}(\sigma_{ij}+\Delta\sigma_{ij})\,\Delta u_{k,i}\,\Delta u_{k,j}\big]\, dV - \int_{S_u} \Delta T_i\, \Delta\bar{u}_i\, ds \qquad (B.29)$$

Therefore, π_c is not just a function of $\Delta\sigma_{ij}$ as in linear analysis. If the increments are taken to be small with respect to the initial quantities then Eq. B.29 may be linearized as

$$\pi_c(\Delta\sigma_{ij}, \Delta u_i) = \int_V \big[B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\big]\, dV - \int_{S_u} \Delta T_i\, \Delta\bar{u}_i\, ds \qquad (B.30)$$

The functionals of Eqs. B.29 and B.30 correspond to the modified functionals, π^c_{mc}, of Eqs. 4.44 and 4.45 respectively. A similar (and simpler) derivation holds for the updated system. For convenience, π_c in the updated system is the same as π_c of Eq. B.29 (or Eq. B.30) except that the proper defintions of the variables must be used. (See Section 2.) Additionally, in the updated system, stress equilibrium and surface traction terms are given as

$$(p_{ji} + \Delta p_{ji})_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0$$

or

$$(\sigma_{ij} + \Delta\sigma_{ij})_{,j} + [(\sigma_{kj} + \Delta\sigma_{kj})\,\Delta u_{i,k}]_{,j} + (\bar{F}_i + \Delta\bar{F}_i) = 0 \qquad (B.31)$$

and

$$T_i + \Delta T_i = (p_{ji} + \Delta p_{ji}) \nu_j$$

or

$$T_i + \Delta T_i = [\sigma_{ij} + \Delta\sigma_{ij} + (\sigma_{kj} + \Delta\sigma_{kj}) \Delta u_{i,k}] \nu_j \qquad \text{(B.32)}$$

Note that π_c has none of the correction terms in it because it is assumed that the initial state is in equilibrium.

APPENDIX C

DISCUSSION OF THE MODIFIED REISSNER PRINCIPLE AND ITS COMPARISON TO THE ASSUMED STRESS HYBRID FUNCTIONALS

For purposes of discussion and comparison a finite element model based on π_{mR} for an updated system will be developed. The functional used as a basis is Eq. 5.50. The results will be compared to both the consistent and inconsistent assumed stress hybrid models. Eq. 5.50 written for an element is

$$
\begin{aligned}
\pi_{mR_n} = & \int_{V_n} [-B(\Delta\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}\Delta u_{k,i}\Delta u_{k,j}]dV \\
& - \int_{V_n} [(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\bar{F}_i + \Delta\bar{F}_i)]\Delta u_i dV \\
& - \int_{\partial V_n} [(T_i+\Delta T_i) - (\sigma_{ij}+\Delta\sigma_{ij})\nu_j]\Delta u_i ds \\
& + \int_{\partial V_n} (T_i+\Delta T_i)\Delta\tilde{u}_i ds - \int_{S_{\sigma_n}} (\bar{T}_i+\Delta\bar{T}_i)\Delta\tilde{u}_i ds \\
& - \int_{S_{u_n}} (T_i+\Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)ds \\
& - \int_{V_n} \Delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j}+u_{j,i} - u_{k,i}u_{k,j})]dV
\end{aligned} \tag{C.1}
$$

In a similar fashion to Section 6 the variables shall be interpolated in terms of the unknowns $\underset{\sim}{\beta}$ and $\Delta\underset{\sim}{q}$ and Eq. 5.47 will be satisfied exactly.

$$
\begin{aligned}
&\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\,\underset{\sim}{\beta} && (\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma})' = \underset{\sim}{P}'\underset{\sim}{\beta} \\
&\underset{\sim}{T} + \Delta\underset{\sim}{T} = \underset{\sim}{R}\,\underset{\sim}{\beta} + \underset{\sim}{A}_b\,\Delta\underset{\sim}{q} && \\
&\Delta\underset{\sim}{u} = \underset{\sim}{L}\,\Delta\underset{\sim}{q} && \underset{\sim}{u} = \underset{\sim}{L}\,\underset{\sim}{q} \\
&\Delta\tilde{\underset{\sim}{u}} = \tilde{\underset{\sim}{L}}\,\Delta\underset{\sim}{q} && \\
&\Delta\underset{\sim}{u}' = \underset{\sim}{L}'\,\Delta\underset{\sim}{q} && \underset{\sim}{u}' = \underset{\sim}{L}'\,\underset{\sim}{q}
\end{aligned} \tag{C.2}
$$

Placing Eq. C.2 into Eq. C.1 gives term by term

$$-\int_{V_n} B(\Delta\sigma_{ij})\,dV = -\int_{V_n} \tfrac{1}{2}\Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \Delta\underset{\sim}{\sigma}\,dV$$
$$= -\tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \sigma^T)\, \underset{\sim}{S}\, (\underset{\sim}{P}\,\underset{\sim}{\beta} - \underset{\sim}{\sigma})\,dV$$
$$= -\tfrac{1}{2}\,\underset{\sim}{\beta}^T \underset{\sim}{H}\,\underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{H}_{\sigma}^{S} \qquad \text{(C.3)}$$

$$\tfrac{1}{2}\int_{V_n} \sigma_{ij}\,\Delta u_{k,i}\,\Delta u_{k,j}\,dV = \tfrac{1}{2}\int_{V_n} \Delta\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{\sigma}\, \underset{\sim}{L}'\,\Delta\underset{\sim}{q}\,dV$$
$$= \tfrac{1}{2}\,\Delta\underset{\sim}{q}^T \underset{\sim}{K}_q\,\Delta\underset{\sim}{q} \qquad \text{(C.4)}$$

$$-\int_{V_n} [(\sigma_{ij}+\Delta\sigma_{ij})_{,j} + (\bar{F}_i+\Delta\bar{F}_i)]\,\Delta u_i\,dV$$
$$= -\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}'^T + \underset{\sim}{\sigma}_P^T)\,\underset{\sim}{L}\,\Delta\underset{\sim}{q}\,dV = -\underset{\sim}{\beta}^T \underset{\sim}{J}\,\Delta\underset{\sim}{q} - \underset{\sim}{J}_{\sigma}\,\Delta\underset{\sim}{q} \qquad \text{(C.5)}$$

$$\int_{\partial V_n} (\sigma_{ij}+\Delta\sigma_{ij})\,\nu_j\,\Delta\tilde{u}_i\,dS = \int_{\partial V_n} \underset{\sim}{\beta}^T \underset{\sim}{R}^T \underset{\sim}{\tilde{L}}\,\Delta\underset{\sim}{q}\,dS = \underset{\sim}{\beta}^T \underset{\sim}{G}\,\Delta\underset{\sim}{q} \qquad \text{(C.6)}$$

$$\int_{\partial V_n} \sigma_{kj}\,\Delta u_{i,k}\,\nu_j\,(\Delta\tilde{u}_i - \Delta u_i)\,dS = \int_{\partial V_n} \Delta\underset{\sim}{q}^T \underset{\sim}{A}_b^T (\underset{\sim}{\tilde{L}} - \underset{\sim}{L})\,\Delta\underset{\sim}{q}\,dS = \tfrac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{M}_b + \underset{\sim}{M}_b^T)\,\Delta\underset{\sim}{q} \qquad \text{(C.7)}$$

$$-\int_{S_{\sigma_n}} (\bar{T}_i + \Delta\bar{T}_i)\,\Delta\tilde{u}_i\,dS = -\underset{\sim}{Q}_{\bar{T}}^T\,\Delta\underset{\sim}{q} \qquad \text{(C.8)}$$

$$-\int_{S_{u_n}} (T_i+\Delta T_i)(\Delta\tilde{u}_i - \Delta\bar{u}_i)\,dS = -\int_{S_{u_n}} (\underset{\sim}{\beta}^T \underset{\sim}{R}^T + \Delta\underset{\sim}{q}^T \underset{\sim}{A}_b^T)(\underset{\sim}{\tilde{L}}\,\Delta\underset{\sim}{q} - \Delta\underset{\sim}{\bar{u}})\,dS$$
$$= -\underset{\sim}{\beta}^T \underset{\sim}{G}_u\,\Delta\underset{\sim}{q} + \underset{\sim}{\beta}^T \underset{\sim}{U} - \tfrac{1}{2}\Delta\underset{\sim}{q}^T(\underset{\sim}{\tilde{M}}_b + \underset{\sim}{\tilde{M}}_b^T)\,\Delta\underset{\sim}{q} + \Delta\underset{\sim}{q}^T \underset{\sim}{\bar{M}}_b \qquad \text{(C.9)}$$

$$-\int_{V_n} \Delta\sigma_{ij}\,e_{ij}\,dV = -\int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\,\underset{\sim}{\sigma}\,dV = -\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T)\,\underset{\sim}{S}\,\underset{\sim}{\sigma}\,dV = -\underset{\sim}{\beta}^T \underset{\sim}{H}_{\sigma}^{C} \qquad \text{(C.10)}$$

$$\tfrac{1}{2}\int_{V_n} \Delta\sigma_{ij}(u_{i,j} + u_{j,i})\,dV = \int_{V_n} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{L}'\,\underset{\sim}{q}\,dV$$
$$= \int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T)\,\underset{\sim}{L}'\,\underset{\sim}{q}\,dV = \underset{\sim}{\beta}^T \underset{\sim}{C}_1\,\underset{\sim}{q} \qquad \text{(C.11)}$$

$$-\tfrac{1}{2}\int_{V_n} \Delta\sigma_{ij}\,u_{k,i}\,u_{k,j}\,dV = -\tfrac{1}{2}\int_{V_n} (\underset{\sim}{\beta}^T \underset{\sim}{P}^T - \underset{\sim}{\sigma}^T)(\underset{\sim}{q}^T \underset{\sim}{L}'^T \underset{\sim}{L}'\,\underset{\sim}{q})\,dV = -\tfrac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{C}\,\underset{\sim}{q} \qquad \text{(C.12)}$$

Placing Eqs. C.3–C.12 into Eq. C.1 yields

$$\pi_{mR_n} = -\frac{1}{2}\beta^T H \beta + \beta^T H_\sigma^s + \frac{1}{2}\Delta q^T K_g \Delta q - \beta^T J \Delta q - J_\sigma \Delta q$$
$$+ \beta^T G \Delta q + \frac{1}{2}\Delta q^T (M_b + M_b^T)\Delta q - Q_\tau^T \Delta q - \beta^T G_u \Delta q + \beta^T V$$
$$- \frac{1}{2}\Delta q^T(\tilde{M}_b + \tilde{M}_b^T)\Delta q + \Delta q^T \tilde{M}_b - \beta^T H_\sigma^c + \beta^T C_1 q - \frac{1}{2}\beta^T C q \quad \text{(C.13)}$$

where all the matrix definitions are given in Subsection 6.2.1 and

$$J = \int_{V_n} P'^T L \, dv$$
$$J_\sigma = \int_{V_n} \sigma_P^T L \, dv$$

Taking the variation of Eq. C.13 with respect to β

$$\frac{\partial \pi_{mR_n}}{\partial \beta} = -H\beta + H_\sigma^s - J\Delta q + G\Delta q - G_u \Delta q + V$$
$$- H_\sigma^c + C_1 q - \frac{1}{2} C q = 0 \quad \text{(C.14)}$$

Solving for the β's

$$\beta = H^{-1}(H_\sigma^s - H_\sigma^c + V + C_1 q - \frac{1}{2}Cq) + H^{-1}(G - G_u - J)\Delta q \quad \text{(C.15)}$$

Placing Eq. C.15 into Eq. C.13 yields

$$\pi_{mR_n}(\Delta q) = -\frac{1}{2}\Delta q^T(G^T - G_u^T - J^T)H^{-1}(G - G_u - J)\Delta q$$
$$+ \Delta q^T(G^T - G_u^T - J^T)H^{-1}(H_\sigma^s - J\Delta q + G\Delta q - G_u\Delta q + V - H_\sigma^c + C_1 q - \frac{1}{2}Cq)$$
$$+ \frac{1}{2}\Delta q^T(K_g + M_b + M_b^T - \tilde{M}_b - \tilde{M}_b^T)\Delta q$$
$$+ \Delta q^T(-J_\sigma^T - Q_\tau + \tilde{M}_b) + \text{constants} \quad \text{(C.16)}$$

Upon rearranging and dropping the constants not subject to variation with respect to Δq

$$\pi_{mR_n}(\Delta q) = \frac{1}{2}\Delta q^T[(G^T - G_u^T - J^T)H^{-1}(G - G_u - J) + K_g + M_b + M_b^T - \tilde{M}_b - \tilde{M}_b^T]\Delta q$$
$$+ \Delta q^T[(G^T - G_u^T - J^T)H^{-1}(H_\sigma^s - H_\sigma^c + V + C_1 q - \frac{1}{2}Cq)$$
$$- J_\sigma^T - Q_\tau + \tilde{M}_b] \quad \text{(C.17)}$$

APPENDIX D

ALTERNATE P MATRIX BY THE STATIC-GEOMETRIC ANALOGY

In Section 7 it may be desired to satisfy the entire homogeneous stress equilibrium equation, Eq. 7.54. If this is the case, then the $\underset{\sim}{P}$ matrix must reflect the additional terms which are funtions of z. A simple choice of $\underset{\sim}{P}$ was given as Eq. 7.87. An alternative $\underset{\sim}{P}$ matrix may be established through the use of the static-geometric analogy for shells [Southwell, 1950]. Briefly, this concept utilizes the analogy which exists between the stress-stress function relations and the strain displacement relations.

Although the ultimate goal is to establish a $\underset{\sim}{P}$ matrix for a shallow shell element satisfying Eq. 7.54, the flat element properties must not be disregarded. For a flat plate the relation between the stress resultants, N_x, N_y, N_{xy}, M_x, M_y, M_{xy} and the stress functions, U, V, W is

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -\partial^2/\partial y^2 \\ 0 & 0 & -\partial^2/\partial x^2 \\ 0 & 0 & \partial^2/\partial x\partial y \\ 0 & -\partial/\partial y & 0 \\ -\partial/\partial x & 0 & 0 \\ \frac{1}{2}\partial/\partial y & \frac{1}{2}\partial/\partial x & 0 \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} \qquad \text{(D.1)}$$

Choose the stress functions to be complete quadratics in x-y space.

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} 1 & & & x & & & y & & \\ & 1 & & & x & & & y & \\ & & 1 & & & x & & & y \end{array}\right.$$

$$\left.\begin{array}{c|c|c|c|c|c|c|c|c} x^2 & & & xy & & & y^2 & & \\ & x^2 & & & xy & & & y^2 & \\ & & x^2 & & & xy & & & y^2 \end{array}\right] \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{18} \end{Bmatrix} \qquad \text{(D.2)}$$

Placing this into Eq. D.1 yields

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} =
\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
 & & & & & & & & & & & & & & & & & -2 \\
 & & & & & & & & & & & -2 & & & & & & \\
 & & & & & & & & & & & & & & 1 & & & \\
 & & & & & & & -1 & & & & & & -X & & & -2Y & \\
 & & & -1 & & & & & & -2X & & & -Y & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & X & & \frac{X}{2} & \frac{Y}{2} & & Y & &
\end{array}\right]
\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{18} \end{Bmatrix} \qquad \text{(D.3)}
$$

Note that five β's can be eliminated immediately and since two of the columns of P are dependent, one of the corresponding β's may be eliminated. Thus, the stress functions can be expressed in terms of only twelve independent β's. After some rearranging

$$
\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} =
\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
 & & & & X & Y & & X^2 & & & XY & Y^2 \\
 & & & Y & & X & XY & & X^2 & Y^2 & & \\
Y^2 & X^2 & XY & & & & & & & & &
\end{array}\right]
\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{12} \end{Bmatrix} \qquad \text{(D.4)}
$$

This equation guarantees that even if the shallow shell reduces to a flat plate, stress equilibrium will still be satisfied.

For a shallow shell the relation between the stress resultants and stress functions is

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -\partial^2/\partial y^2 \\ 0 & 0 & -\partial^2/\partial x^2 \\ 0 & 0 & \partial^2/\partial x \partial y \\ 0 & -\partial/\partial y & \partial^2 z/\partial y^2 \\ -\partial/\partial x & 0 & \partial^2 z/\partial x^2 \\ \frac{1}{2}\partial/\partial y & \frac{1}{2}\partial/\partial x & -\partial^2 z/\partial x \partial y \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{D.5}$$

Placing Eq. D.4 into Eq. D.5 one obtains

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} -z & & & & & & & & & & & \\ & -z & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ y^2 z_{,yy} & x^2 z_{,yy} & xyz_{,yy} & -1 & & & -x & & & -2y & & \\ y^2 z_{,xx} & x^2 z_{,xx} & xyz_{,xx} & & -1 & & & -2x & & & -y & \\ -y^2 z_{,xy} & -x^2 z_{,xy} & -xyz_{,xy} & & & 1 & y/2 & & x & & x/2 & y \end{array}\right] \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{12} \end{Bmatrix}$$

$$= [\underset{\sim}{P}]\{\underset{\sim}{\beta}\} \tag{D.6}$$

In Equation 7.87 the coupling in the stresses was through z only. Here, the coupling is through the curvature of the shell surface. For a flat element $z \equiv 0$ and although this does not quite look the same as the standard $\underset{\sim}{P}$ matrix it is equivalent. There are other $\underset{\sim}{P}$ matrices that could be formed and as long as Eq. 7.54 is satisfied any one of them is legitimate.

APPENDIX E

A HIGHER ORDER ELEMENT BY THE INCONSISTENT ASSUMED STRESS HYBRID FUNCTIONAL

Finite element theory allows one to choose from a whole range of element sophistication as long as the governing equations are valid and all assumptions are realized. Elements range from the very simple flat plane stress type to highly sophisticated doubly curved deep shell elements. The elements used here are simple in shape (flat or shallow) and interpolation functions. It is useful to study the effects of varying the number of stress parameters, degrees of freedom per node and the number of nodes per element. One may also combine various elements, condense out certain variables, etc., thereby creating new elements. In addition to this, displacement interpolations may be varied, assumptions on the severity of curvature may be made, etc.

For the assumed stress hybrid model under linear theory, Mau and Witmer [1972] have experimented with a variety of flat elements. Unfortunately, triangular elements which are most useful in all types of analyses, are not as accurate as the rectangular (or quadrilateral) elements. Mau and Witmer demonstrate this for flat elements. Tanaka [1969] shows that for linear shell analysis using doubly curved triangular shell elements, often an averaging process is necessary to yield good results.

Pirotin [1971] has used the basic inconsistent assumed stress hybrid formulation (purely incremental with no checks) to analyze the large deflection of shells with a doubly curved four noded shell element. This was somewhat limited in that the curvilinear coordinates were always required to be orthogonal. These deep shell elements are fairly complicated for large deflection analysis. Thus, more sophistication on a simpler plane is worthwhile considering.

One possibility would be a six node, shallow, triangular shell element, the interior displacements of which are defined by three translations at each node. Such a displacement field will, of course, maintain the continuity of the shell surface that the elements are representing. Being an assumed stress hybrid model the rotations along the boundary can be independently assumed.

In fact, the distribution of the normal slope θ_n at each edge may be assumed as constant, linear, or quadratic and correspondingly one, two, or three degrees of freedom will be needed for each edge. The boundaries may also be curved. For example, the following types of interpolations might be used.

$$(X, Y, Z, U, V, W) = \begin{Bmatrix} (2\xi_1 - 1)\xi_1 \\ (2\xi_2 - 1)\xi_2 \\ (2\xi_3 - 1)\xi_3 \\ 4\xi_1\xi_2 \\ 4\xi_2\xi_3 \\ 4\xi_3\xi_1 \end{Bmatrix}^T \begin{bmatrix} X_1, & Y_1, & Z_1, & U_1, & V_1, & W_1 \\ X_2, & Y_2, & Z_2, & U_2, & V_2, & W_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_6, & Y_6, & Z_6, & U_6, & V_6, & W_6 \end{bmatrix} \tag{E.1}$$

where the ξ_i's are the triangular (area) coordinates. Since the midside information would be required, perhaps a stationary system would be simpler to use.

Recalling that the minimum number of $\tilde{\beta}$'s depends on the number of $\tilde{q}$'s (see Eq. 7.59) and since the number of $\tilde{q}$'s have increased then the stress assumptions must change. Considering that the shell may become flat and the equations would completely decouple, then the inplane stress resultants would require nine independent $\tilde{\beta}$'s while the bending stress resultant would require six, nine, or twelve independent $\tilde{\beta}$'s if the distribution of the normal rotation at each edge is constant, linear, or quadratic respectively.

As can be seen from this appendix one must be extremely careful in choosing elements which will yield an increase in accuracy comparable to the increase in complexity and computational cost. The simple elements presented in this work, while useful and efficient for many problems, have drawbacks. Certain types of problems require extreme accuracy. Some of the higher order shell elements used in linear theory are just too prohibitively expensive for nonlinear analysis. Some good intermediate elements are needed.

BIOGRAPHY

Mr. Peter Lewis Boland was born in Brooklyn, New York on April 12, 1947. He received the degree of Bachelor of Mechanical Engineering (Magna Cum Laude) in February 1969 from the City College of New York, and the degree of Master of Science in Aeronautics and Astronautics in February 1971 from the Massachusetts Institute of Technology.

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