EXTENSION OF GAUSS' METHOD FOR THE SOLUTION OF KEPLER'S EQUATION

by

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ABSTRACT

The solution of a transcendental equation known as "Kepler's equation," which relates position in orbit with time, requires an iterative procedure for solution. A method is developed based on one presented by Gauss in his Theoria Motus dealing with the problem of position determination for time since pericenter passage in cases where elliptic and hyperbolic orbits approached very near unity. The problem of interest here is the more general one of determining final position and velocity from given initial conditions and a specified time interval. Kepler's equation is transformed to an equation which is of the form of a cubic and which provides the nucleus of an efficient iteration algorithm. The final algorithm is a general form valid for any orbit of any eccentricity and requires no knowledge of the nature of the orbit for application. Universal formulae are developed relating final position and velocity to initial values in terms of variables defined in the transformation. Finally, the method is tested over a wide range of orbits to observe its performance and comparison is made with the proposed Kepler subroutine for the NASA Space Shuttle orbiter vehicle.

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SYMBOLS

a	Semi-major axis
α, β	Constants whose sum is unity
e	Eccentricity
E	Eccentric anomaly, angle used as position parameter in elliptic orbits
ΔE	$E - E_0$
f	Angle referred to as the true anomaly between the radius vector and eccentricity vector
F, F_t G, G_t	Lagrange Functions relating final position and velocity to initial position and velocity
γ	$\frac{1}{2} (1 - \alpha \frac{r_0}{q} (1-e))$
h	Massless angular momentum
H	Position parameter in hyperbolic orbits
ΔH	$H - H_0$
M	Mean anomaly
p	Semi-latus rectum or parameter
q	Magnitude of radius vector at pericenter or point of closest approach
r	Magnitude of radius vector
$\underline{r}$	Position vector
σ_0	$\frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu}}$

τ	Time of pericenter passage
t	$t_f - t_0$, Time interval from initial to final position
t_{max}	Time interval corresponding to Y_{max}
t_T	Sum of time steps
T	Time interval computed in the iteration procedure for convergence test
μ	Product of the universal gravitational constant and the sum of the masses of the two bodies
v	magnitude of velocity vector
$\underline{v}$	velocity vector

CHAPTER 1

INTRODUCTION

The determination of the position and velocity in two-body orbits leads to the solution of a transcendental equation commonly referred to as "Kepler's equation" which relates the dependence of position in orbit with time.

In classical analysis, the shape of these two-body orbits is described through the use of conics and corresponding to each conic Kepler's equation has a different form. A useful quantity in classifying conics is a constant e called the eccentricity. For a circle e = 0, for an ellipse e is between 0 and 1, e equals 1 for a parabola, and is greater than 1 for the hyperbola. Also obtainable from elementary considerations, is the general polar equation for the conic which can be stated as

$$r = \frac{h^2/\mu}{1 + e \cos f} = \frac{p}{1 + e \cos f} \qquad (1.1)$$

where h is the massless angular momentum, μ is the product of the universal gravitational constant and the sum of the masses of the two bodies, p is the semi-latus rectum or parameter, and f, called the true anomaly, is the angle between the radius vector and the direction of pericenter or point of closest approach of the two bodies.

For the parabola, Kepler's equation is simply

$$6\sqrt{\frac{\mu}{p^3}}\,(t - \tau) = \tan^3(f/2) + 3\tan(f/2) \tag{1.2}$$

where τ is the time of pericenter passage. Although Eq. (1.2) is a special form of Kepler's equation it is more commonly known as "Barker's formula." A graph of the parabola is shown in Fig. 1.1.

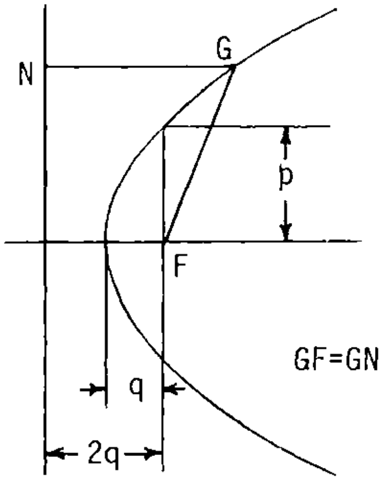

Figure 1.1 Parabola

In the case of the ellipse, use is made of an angle, denoted by E, called the eccentric anomaly, which is based on a reference circle referred to as the "auxiliary circle," and whose geometrical significance can be seen in Fig. 1.2.

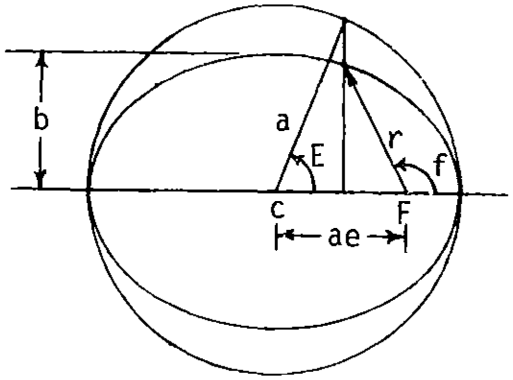

Figure 1.2 Ellipse

In terms of E, Kepler's equation may be expressed as

$$M = E - e \sin E \tag{1.3}$$

$$M = \sqrt{\frac{\mu}{a^3}}\,(t - \tau) \tag{1.4}$$

M is the mean anomaly and a is the semi-major axis.

For the hyperbola, instead of an angle an area is employed as the auxiliary variable and is also based on a reference geometric shape referred to as the "equilateral hyperbola" as shown in Fig. 1.3.

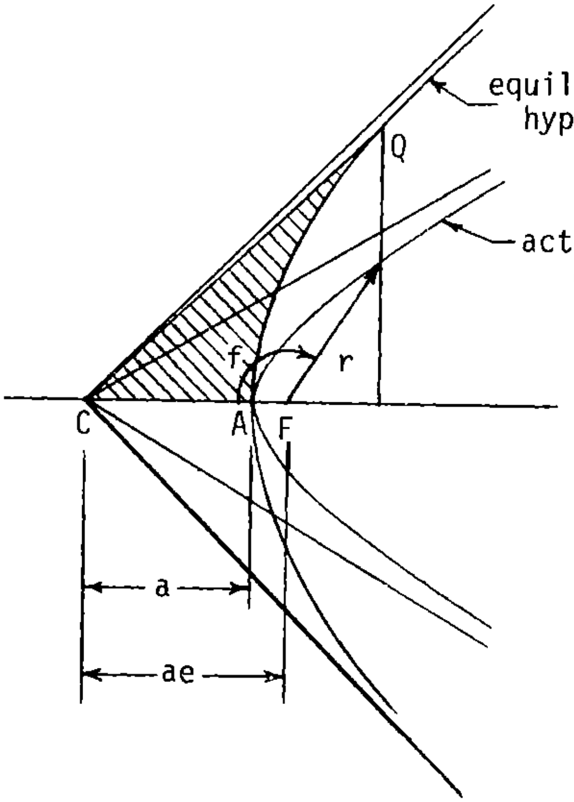

Figure 1.3 Hyperbola

Then the appropriate variable H is defined as

$$\text{Area CAQ} = \frac{a^2}{2} H$$

so that Kepler's equation may be written as

$$\sqrt{\frac{\mu}{a^3}}\ (t - \tau) = e \sinh H - H \tag{1.5}$$

Kepler's equation is transcendental, so that for a given time it cannot be solved algebraically for the position parameter. However, there is one and only one solution and for an analytic solution an iterative process must be employed.

In his Theoria Motus Gauss addressed the problem of determining the true anomaly from the time, for elliptic and hyperbolic orbits which are nearly parabolic. In such cases the conventional methods of solution could not give the precision required. As Gauss expressed it, "The methods above treated, both for the determination of the true anomaly from the time and for the determination of the time from the true anomaly, do not admit of all the precision that might be required in those conic sections of which the eccentricity differs but little from unity, that is, in ellipses and hyperbolas which approach very near to the parabola; indeed, unavoidable errors, increasing as the orbit tends to resemble the parabola, may at length exceed all limits. Larger tables, constructed to more than seven figures would undoubtedly diminish this uncertainty, but they would not remove it, nor would they prevent its surpassing all limits as soon as the orbit approached too near the parabola. Moreover, the methods given above become in this case very troublesome, since a part of them require the use of indirect trials frequently repeated, of which the tediousness is even greater if we work with the larger tables. It certainly, therefore, will not be superflous, to furnish a peculiar method by means of which the uncertainty in this case may be avoided, and sufficient precision may be obtained with the help of the common tables."

Gauss' method of solution is applicable to orbits of any eccentricity. The required iterative scheme is a "Picard" type iteration, i.e. successive substitution, there being no need for trials or tests which are so characteristic of many iterative schemes. Furthermore the method is applicable to all conic orbits, the advantage here being that the type of conic encountered need not be known in order to apply the formulae. Also, continuity is maintained during transition from one conic to another while at the same time being free from ambiguities or indeterminant forms. As will be seen, the speed of convergence is quite rapid. Gauss' method is briefly outlined here for the elliptic orbit.

Rewriting Eq. (1.4) as

$$M = \sqrt{\frac{\mu (1 - e)^3}{q^3}} (t - \tau) \tag{1.6}$$

q being the pericenter distance, Gauss then chose to replace E and sin E by the quantities

$$\begin{aligned} P &= E - \sin E \\ Q &= \frac{9}{10} E + \frac{1}{10} \sin E \end{aligned} \tag{1.7}$$

With these, Eq. (1.3) takes the form

$$(1 - e)\ P + \left(\frac{1}{10} + \frac{9}{10} e\right) Q = M \tag{1.8}$$

then as long as E is a quantity of first order,

$$P = \frac{1}{6} E^3 - \frac{1}{120} E^5 + \frac{1}{5040} E^7 - \ldots$$

is a quantity of the third order, while

$$Q = E - \frac{1}{60} E^3 + \frac{1}{1200} E^5 - \ldots$$

is a quantity of the first order. Then defining

$$Y = \frac{6\ P}{Q} \qquad B^2 = \frac{Q^3}{6\ P} = \frac{Q^2}{Y}$$

$$Y = E^2 - \frac{1}{30} E^4 - \frac{1}{5040} E^6 - \ldots$$

is a quantity of second order, while

$$B = 1 + \frac{3}{2800} E^4 + \frac{1}{16800} E^6 + \ldots$$

is a quantity which differs from unity by a quantity of the fourth order. Finally, Eq. (1.8) becomes

$$B \left\{(1 - e)\, Y^{1/2} + \frac{1}{60}\,(1 + 9\,e)\, Y^{3/2}\right\} = M \qquad (1.9)$$

Hence it is readily seen that the choice of $\frac{1}{10}$ and $\frac{9}{10}$ in the definition of Q was to obtain B, which multiplies the entire left side of Eq. (1.9), as nearly constant as possible. Eq. (1.9) is essentially an algebraic equation of third order.

It is easy to see that B can be considered as a function of Y. Furthermore, in the words of Gauss, "Now, although B may be finally known from Y by means of our auxiliary table, nevertheless it can be foreseen, owing to its differing so little from unity, that if the divisor B were wholly neglected from the beginning, Y would be affected with a slight error only. Therefore, we will first determine roughly Y, putting B = 1; with the approximate value of Y, we will find B in our auxiliary table, with which we will repeat more exactly the same calculations; most frequently, precisely the same value of B that had been found from the approximate value of Y will correspond to the value of Y thus corrected, so that a second repetition of the operation would be superfluous, those cases excepted in which the value of E may have been very considerable." The tables referred to by Gauss were constructed to further simplify the iteration process by reducing the amount of computation even more. Here corresponding values of B are listed for values of Y, which are in increments of .004 from 0 to 1.2. In this manner the tables provide a simple means of obtaining values of E up to 64° 7'.

To further illustrate the speed of convergence, consider an elliptic orbit in the x-y plane where pericenter is given by

$$\underline{r}_p = (2 \times 10^7 \text{ m})\, \underline{i}_x \qquad \underline{V}_p = (6.2 \times 10^3 \text{ m/sec})\, \underline{i}_y$$

then for the Earth as the central force and an arbitrary value of 5 hours for the time since pericenter passage

$$e = .928735 \qquad M = .0764383$$

For an initial guess of 1 for B, Y is obtained from Eq. (1.9) and is found to be

$$Y = .59995$$

Then from Gauss' table,

$$\log B = .0001734 \qquad \text{or} \qquad B = 1.000399$$

With this value of B, Eq. (1.9) gives

$$Y = .59998$$

and again from the table

$$B = 1.000399$$

Hence we have converged to the solution after just one correction at which point E may be calculated from

$$E = B\sqrt{Y}\left(1 + \frac{1}{60}Y\right)$$

with a value of 33.7039^{0}.

It is the purpose of this study to extend Gauss' method of solution of Kepler's equation in standard form to the general problem of determining final position and velocity vectors for a specified time interval from given initial conditions at any point in the orbit while at the same time preserving all or most of the qualities which were inherent in Gauss' method. Finally, to see just how practical this solution process is, comparison is made with the algorithm proposed for the on-board computer in the NASA Space Shuttle orbital vehicle.

CHAPTER 2

EXTENDED FORM OF GAUSS' METHOD

The extension of Gauss' method to the solution of Kepler's equation for some arbitrary interval of time to obtain the final position and velocity is presented here for the ellipse and hyperbola, respectively. The parabola is shown to be the limiting form of both the elliptic and hyperbolic solutions as the eccentricity tends to unity. Furthermore, universal formulae are derived which permit calculation of final position and velocity using the initial position and velocity without knowledge of the type of orbit encountered. Finally a generalized procedure of the iterative process is presented, the details being left as the topic for a later chapter.

2.1 The Ellipse

Kepler's equation for a time interval $t = t_f - t_0$ corresponding to an eccentric anomaly difference $\Delta E = E - E_0$, may be written as[2]

$$\sqrt{\frac{\mu}{a^3}}\, t = \Delta E + \frac{\sigma_0}{\sqrt{a}} (1 - \cos \Delta E) - (1 - \frac{r_0}{a}) \sin \Delta E \tag{2.1}$$

where the quantity σ_0 is defined as

$$\sigma_0 = \frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu}}$$

Since $q = a\ (1 - e)$ we have

$$\sqrt{\frac{\mu}{q^3}}\, t = (1 - e)^{-3/2} \left\{\Delta E + \frac{\sigma_0}{\sqrt{q}} (1 - e)^{1/2} (1 - \cos \Delta E) - \left(1 - \frac{r_0}{q} (1 - e)\right) \sin \Delta E \right\} \tag{2.2}$$

Defining the variables

$$P = \Delta E - \sin \Delta E \tag{2.3}$$

$$Q = \alpha\ \Delta E + \beta \sin \Delta E \tag{2.4}$$

$$R = 1 - \cos \Delta E \tag{2.5}$$

$$\gamma = \frac{1}{2} \left(1 - \alpha \frac{r_0}{q} (1 - e)\right) \tag{2.6}$$

where α and β are constants to be specified such that $\alpha + \beta = 1$. Then Eq. (2.2) becomes

$$\sqrt{\frac{\mu}{q^3}}\, t = (1 - e)^{-3/2} \left\{\frac{r_0}{q} (1 - e) (\alpha\ \Delta E + \beta \sin \Delta E) + \frac{\sigma_0}{\sqrt{q}} (1 - e)^{1/2} (1 - \cos \Delta E) + \left(1 - \alpha \frac{r_0}{q} (1 - e)\right) (\Delta E - \sin \Delta E) \right\}$$

or

$$\sqrt{\frac{\mu}{q^3}}\; t = (1 - e)^{-3/2} \left(\frac{r_0}{q} (1 - e)\, Q + \frac{\sigma_0}{\sqrt{q}} (1 - e)^{1/2} R + 2\, \gamma\, P\right) \tag{2.7}$$

Then, as did Gauss, defining the variables

$$Y = \frac{6\,P}{Q} \qquad\qquad B^2 = \frac{Q^3}{6P} \tag{2.8}$$

and also a new variable

$$C = \frac{\sigma_0}{2\,\sqrt{r_0}}\, C_0 \qquad\qquad C_0 = \sqrt{\frac{2}{3}}\, \frac{R}{\sqrt{P}\sqrt{Q}} = \frac{2\,R}{Y\,B} \tag{2.9}$$

it is easily verified that

$$Q = B\,\sqrt{Y} \qquad\qquad P = \frac{1}{6}\, B\, Y^{3/2}$$

Hence, we have in Eq. (2.7)

$$\sqrt{\frac{\mu}{q^3}}\; t = (1 - e)^{-3/2}\, B \left\{\frac{r_0}{q} (1 - e)\, Y^{1/2} + \sqrt{\frac{r_0}{q}}\, (1 - e)^{1/2}\, C\, Y + \frac{1}{3}\, \gamma\, Y^{3/2}\right\}$$

or, defining the variable D such that

$$Y = \frac{r_0}{q}(1 - e)\,D^2 \tag{2.10}$$

then

$$\sqrt{\frac{\mu}{r_0^{\,3}}}\; t = B\left(D + C\,D^2 + \frac{1}{3}\,\gamma\,D^3\right) \tag{2.11}$$

Hence, we have succeeded in reducing Kepler's equation to a form which resembles a cubic equation, the solution of which will be discussed in a subsequent chapter. Notice also that B, C_0 and Y are all functions of ΔE, so that we may regard B and C_0 as functions of Y. This fact, as will be seen, is of great importance in the solution of Eq. (2.11). Also, in the definition of Q we leave α and β unspecified for the moment instead of using Gauss' choice of $\frac{9}{10}$ and $\frac{1}{10}$. The functions B and C_0 are both dependent on Y and their sensitivity to particular values of α and β will be discussed in a later chapter.

2.2 The Hyperbola

Kepler's equation for the hyperbola may be treated in an analagous fashion. To the time interval $t = t_f - t_0$ corresponds the difference $\Delta H = H - H_0$ and the relevant form of Kepler's equation is

$$\sqrt{\frac{\mu}{a^3}}\; t = -\Delta H + \frac{\sigma_0}{\sqrt{a}}\left(\cosh \Delta H - 1\right) + \left(1 + \frac{r_0}{a}\right)\sinh \Delta H$$

since $q = a\,(e - 1)$

$$\sqrt{\frac{\mu}{q^3}}\; t \;=\; (e - 1)^{-3/2} \left\{-\Delta H + \frac{\sigma_0}{\sqrt{q}}\,(e - 1)^{1/2}\,\left(\cosh \Delta H - 1\right) + \left(1 + \frac{r_0}{q}\,(e - 1)\right) \sinh \Delta H \right\} \tag{2.12}$$

Again defining the variables

$$P \;=\; \sinh \Delta H - \Delta H \tag{2.13}$$

$$Q \;=\; \alpha\, \Delta H + \beta \sinh \Delta H \tag{2.14}$$

$$R \;=\; \cosh \Delta H - 1 \tag{2.15}$$

$$\gamma \;=\; \frac{1}{2}\left(1 + \alpha\, \frac{r_0}{q}\,(e - 1)\right) \tag{2.16}$$

gives

$$\sqrt{\frac{\mu}{q^3}}\; t \;=\; (e - 1)^{-3/2} \left\{\frac{r_0}{q}\,(e - 1)\, Q + \frac{\sigma_0}{\sqrt{q}}\,(e - 1)^{1/2}\, R + 2\,\gamma\, P\right\}$$

or, using Eqs. (2.8) and (2.9)

$$\sqrt{\frac{\mu}{q^3}}\; t \;=\; (e - 1)^{-3/2}\, B \left\{\frac{r_0}{q}\,(e - 1)\, Y^{1/2} + \sqrt{\frac{r_0}{q}}\,(e - 1)^{1/2}\, C\, Y + \frac{1}{3}\,\gamma\, Y^{3/2}\right\} \tag{2.17}$$

By letting

$$Y = \frac{r_0}{q} (e - 1) D^2 \tag{2.18}$$

Eq. (2.17) becomes

$$\sqrt{\frac{\mu}{r_0{}^3}}\ t = B \left(D + C D^2 + \frac{1}{3} Y D^3\right) \tag{2.19}$$

The universality of this method becomes apparent here since, for different type orbits, the same resulting equation is obtained. Furthermore, from Eqs. (2.18) and (2.10), it is readily seen that Y may be used to classify conics in a similar fashion to the eccentricity. If Eq. (2.10) is accepted as the definition of D then for the ellipse Y is greater than 0, Y is equal to 0 for the parabola, and is less than 0 for the hyperbola. Hence we now possess a general form for the solution of Kepler's equation for hyperbolic and elliptic orbits and will show subsequently that this general form is indeed also valid for parabolic orbits.

2.3 The Parabola

The parabola can be shown to be the limiting form of both the ellipse and hyperbola as e tends to unity. Here Kepler's equation is

$$2 \sqrt{\frac{\mu}{p^3}}\ t = \{\tan(f/2) - \tan(f_0/2)\} + \frac{1}{3} \{\tan^3(f/2) - \tan^3(f_0/2)\} \tag{2.20}$$

As e approaches unity from either the hyperbola or ellipse a approaches infinity and from the definition of D, Y tends to 0. It is easily verified that both B and C_0 approach unity. Hence we have from either Eq. (2.11) or Eq. (2.19)

$$\sqrt{\frac{\mu}{r_0{}^3}}\ t = D + \frac{\sigma_0}{2\sqrt{r_0}} D^2 + \frac{1}{6} D^3 \tag{2.21}$$

Using the fact that for parabolic motion

$$\sin f_0 = \frac{\sqrt{p}\ \sigma_0}{r_0} \qquad \cos f_0 = \frac{p}{r_0} - 1 \tag{2.22}$$

and that the root of Eq. (2.21) is

$$D = \frac{\sqrt{2}\sin\left(\frac{f - f_0}{2}\right)}{\cos(f/2)} \tag{2.23}$$

then substitution of Eq. (2.22) and Eq. (2.23) does indeed lead to Eq. (2.20). As a case in point; at pericenter $f_0 = 0$; hence

$$D = \sqrt{2}\tan(f/2) \qquad \sigma_0 = 0 \qquad \frac{p}{r_0} = 2$$

Therefore Eq. (2.21) becomes

$$\sqrt{\frac{\mu}{r_0{}^3}}\ (t - \tau) = \sqrt{2}\tan(f/2) + \frac{\sqrt{2}}{3}\tan^3(f/2)$$

or

$$2 \sqrt{\frac{\mu}{p^3}} \; (t - \tau) = \tan(f/2) + \frac{1}{3} \tan^3(f/2)$$

which is Barker's formula. Therefore as Y approaches 0 ($e \rightarrow 1$), the hyperbolic and elliptic forms reduce to the parabolic form.

2.4 Final Position and Velocity Vectors

Determination of the final position and velocity from given initial position and velocity vectors and a time interval may be done through the use of universal formulae expressed in terms of the variables Y, B, D, and C_0.

In general, the final position and velocity vectors may be written in terms of the Lagrange F and G functions as

$$\begin{aligned} \underline{r} &= F\, \underline{r}_0 + G\, \underline{v}_0 \\ \underline{v} &= F_t\, \underline{r}_0 + G_t\, \underline{v}_0 \end{aligned} \qquad (2.24)$$

For elliptic orbits F and G are found to be$^{(2)}$

$$F = 1 - \frac{a}{r_0} (1 - \cos \Delta E)$$

$$F_t = - \frac{\sqrt{\mu\, a}}{r\, r_0} \sin \Delta E$$

$$G = t - \sqrt{\frac{a^3}{\mu}} (\Delta E - \sin \Delta E)$$

$$G_t = 1 - \frac{a}{r} (1 - \cos \Delta E)$$

making use of Eq. (2.9) and Eq. (2.5)

$$1 - \cos \Delta E = \frac{1}{2} Y B C_0$$

Also from Eq. (2.3) and Eq. (2.4)

$$\sin E = Q - \alpha P = B \sqrt{Y} - \frac{\alpha}{6} B Y^{3/2}$$

$$= B \sqrt{Y} (1 - \frac{\alpha}{6} Y)$$

but since $Y = \frac{r_0}{a} D^2$

$$\sin \Delta E = B D \sqrt{\frac{r_0}{a}} (1 - \frac{\alpha}{6} Y)$$

and

$$P = \Delta E - \sin \Delta E = \frac{1}{6} B Y^{3/2} = \frac{D^3 B}{6} \sqrt{\frac{r_0^3}{a^3}}$$

Therefore, for an ellipse

$$\begin{aligned} F &= 1 - \frac{1}{2} D^2 B C_0 & G &= t - \frac{B D^3}{6} \sqrt{\frac{r_0^3}{\mu}} \\ F_t &= - \frac{B D}{r} \sqrt{\frac{\mu}{r_0}} (1 - \frac{\alpha}{6} Y) & G_t &= 1 - \frac{r_0}{r} (\frac{1}{2} D^2 B C_0) \end{aligned} \qquad (2.25)$$

For hyperbolic orbits F and G are

$$F = 1 - \frac{a}{r_0}(\cosh \Delta H - 1)$$

$$F_t = - \frac{\sqrt{\mu a}}{r\, r_0} \sinh \Delta H$$

$$G = t - \sqrt{\frac{a^3}{\mu}}(\sinh \Delta H - \Delta H)$$

$$G_t = 1 - \frac{a}{r}(\cosh \Delta H - 1)$$

making use of Eq. (2.9) and Eq. (2.15)

$$\cosh \Delta H - 1 = \frac{1}{2} Y B C_0 = \frac{r_0}{a} (\frac{1}{2} D^2 B C_0)$$

Also from Eq. (2.13) and Eq. (2.14)

$$\sinh \Delta H = Q + \alpha P = B \sqrt{Y} (1 + \frac{\alpha}{6} Y)$$

$$= B D \sqrt{\frac{r_0}{a}} (1 + \frac{\alpha}{6} Y)$$

and

$$P = \sinh \Delta H - \Delta H = \frac{1}{6} B Y^{3/2} = \frac{B D^3}{6} \sqrt{\frac{r_0^3}{a^3}}$$

Therefore, for the hyperbola

$$F = 1 - \frac{1}{2} D^2 B C_0 \qquad G = t - \frac{B D^3}{6}\sqrt{\frac{r_0^3}{\mu}}$$

$$F_t = -\frac{B D}{r}\sqrt{\frac{\mu}{r_0}}\left(1 + \frac{\alpha}{6} Y\right) \qquad G_t = 1 - \frac{r_0}{r}\left(\frac{1}{2} D^2 B C_0\right) \tag{2.26}$$

The only difference between Eq. (2.25) and Eq. (2.26) is in the expression for F_t. This sign difference may be avoided by the convention that $Y<0$ for the hyperbola, $Y=0$ for the parabola, and $Y>0$ for the ellipse. Thus, the final position and velocity vectors may be determined through the use of Eq. (2.24) and Eq. (2.25), where Y determines the conic. Notice that nowhere in Eq. (2.25) is there any need to know the nature of the conic.

One further point to be made here is that another means of calculating γ and also Y will be needed if the new form of Kepler's equation is to be valid for rectilinear orbits as well. This may be accomplished using the fact that

$$q = \frac{p}{(1 + e)} \qquad e^2 = \left(\frac{p}{r_0} - 1\right)^2 + \frac{p\,\sigma_0^2}{r_0} \tag{2.27}$$

Recall that

$$p = \frac{h^2}{\mu} = \frac{(\underline{r}_0 \times \underline{v}_0) \cdot (\underline{r}_0 \times \underline{v}_0)}{\mu}$$

Using the identity

$$(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) = (\underline{A} \cdot \underline{B})\,(\underline{B} \cdot \underline{D}) - (\underline{A} \cdot \underline{D})\,(\underline{B} \cdot \underline{C})$$

we have

$$p = \frac{(r_0 v_0)^2}{\mu} - \sigma_0^2$$

Now

$$(1 - e)^2 = \frac{2p}{r_0} - \frac{p(p + \sigma_0^2)}{r_0^2}$$

and

$$\frac{(1 - e)}{q} = \frac{(1 - e^2)}{p} = \frac{2}{r_0} - \frac{p + \sigma_0^2}{r_0^2} = \frac{2}{r_0} - \frac{v_0^2}{\mu}$$

Hence Eq. (2.6) and Eq. (2.16) become

$$\gamma = \frac{1}{2} \left(1 - \alpha \left(2 - \frac{r_0 v_0^2}{\mu}\right)\right) \qquad (2.28)$$

while

$$Y = \left(2 - \frac{r_0 v_0^2}{\mu}\right) D^2 \qquad (2.29)$$

2.5 General Summary of the Iterative Method

The purpose for the following general discussion of the method of solution is to summarize the work presented in this chapter and also to give purpose to the material presented in the following chapters.

Given an initial position, velocity and a time interval one may find the final position and velocity by performing the following sequence of steps:

1. Calculate r_0, v_0, σ_0, and γ.

2. For an initial guess set B and C_0 equal to unity.
3. Substitute these values into the cubic

$$\sqrt{\frac{\mu}{r_0^3}}\; t = B\left(D + C\,D^2 + \frac{1}{3}\gamma\,D^3\right)$$

   and solve for D.
4. With this value of D, calculate Y using Eq. (2.29).
5. From B and C_0 expressed as functions of Y, calculate new values for B and C_0.
6. Test for convergence by using the present values of B, C_0 and D in the cubic and recomputing t. Check the error between this value and the given time interval. If the error is larger than some specified value, return to step 3. If the error is smaller, then calculate the final position and velocity using Eq. (2.24) and Eq. (2.25).

As in Gauss' solution there must be a limit to the magnitude of ΔE or ΔH in the ellipse and hyperbola and hence on the time interval or else the number of iterations does not remain small. Therefore it is necessary to account for larger intervals of time. This aspect along with determining B and C_0 as functions of Y, and determining the root to the cubic, which in this case may not always possess one real root as was the case with Gauss' solution since C and γ are not constant, will be discussed in the subsequent chapters.

CHAPTER 3

SERIES EXPANSIONS

Probably the most convenient method of obtaining B and C_0 as functions of Y is by power series representation. The means of obtaining these series is discussed along with the selection of the constants α and β. A procedure to economize the series is then presented and a potential means by which further reduction in the amount of computation is obtained.

3.1 Selection of the Constants α and β.

Consideration in the selection of the constants α and β, in this instance, must be given not only to B but also to C_0. For elliptic motion, if B and C_0 are expanded as powers of ΔE in terms of α and β the resulting first terms are

$$B = 1 + \frac{1}{4}\left(\frac{1}{10} - \beta\right)(\Delta E)^2 + \ldots$$

$$C_0 = 1 + \frac{1}{12}\left(\beta - \frac{7}{10}\right)(\Delta E)^2 + \ldots$$

Clearly selection of $\alpha = \frac{9}{10}$, $\beta = \frac{1}{10}$ makes B a value which differs from unity by a quantity of fourth order in ΔE. But if $\alpha = \frac{3}{10}$, $\beta = \frac{7}{10}$ are selected then C_0 will differ from unity by a quantity of fourth order in ΔE. On the other hand if $\alpha = \frac{3}{4}$, $\beta = \frac{1}{4}$ are chosen, both B and C_0 differ from unity by the same quantity of second order in ΔE.

It is possible to observe both B and C_0 as functions of Y for these selections by evaluating B, C_0 and Y for different values of ΔE (ΔH) and then plotting B vs. Y and C_0 vs. Y. The resulting curves are shown in Figs. 3.1 through 3.4. It is easy to see, for both the hyperbola and ellipse, that to obtain either B or C_0 as nearly flat as possible the other varies significantly. Also notice that for $\alpha = \frac{3}{10}$, C_0 remains much flatter than B does for $\alpha = \frac{9}{10}$. These two choices of α will be compared in tests to see which gives better results.

The following procedure to obtain B and C_0 as series in powers of Y will be explained with $\alpha = \frac{9}{10}$ but results for both $\alpha = \frac{3}{10}$ and $\frac{1}{4}$ are also presented.

3.2 Methods of Derivation

Two methods of obtaining B and C_0 as a power series in Y were tried. The first uses the technique of series reversion. The procedure is to obtain Y as a series in ΔE (ΔH) and revert the series such that now ΔE (ΔH) is a series in powers of Y. A convenient expression can be obtained relating B to ΔE (ΔH) and Y in which case substitution of the reverted series gives the final result. The series for C_0 can then be obtained by making use of the reverted series and the B power series. This method may prove more applicable if the computation is to be done by

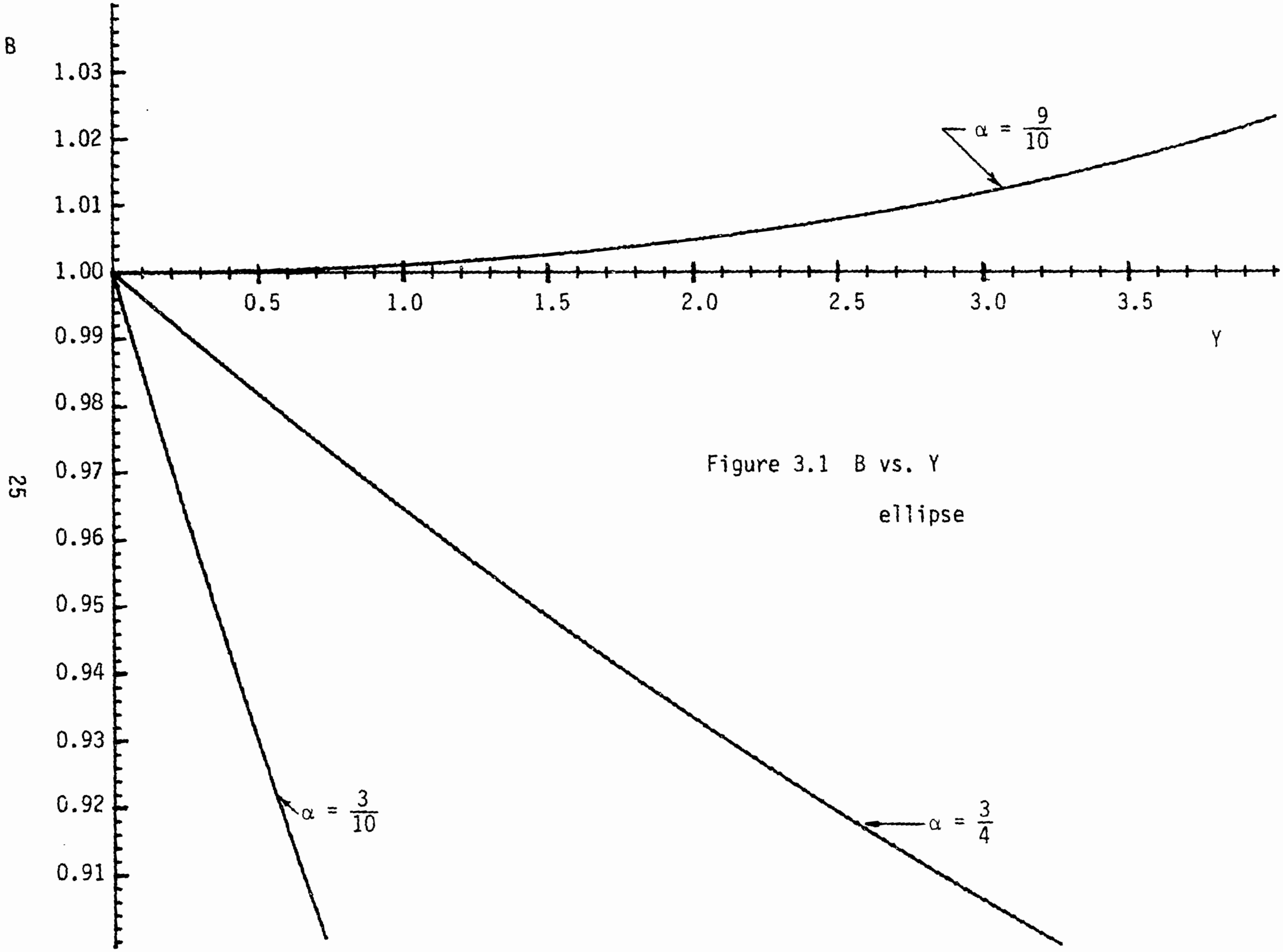

Figure 3.1 B vs. Y

ellipse

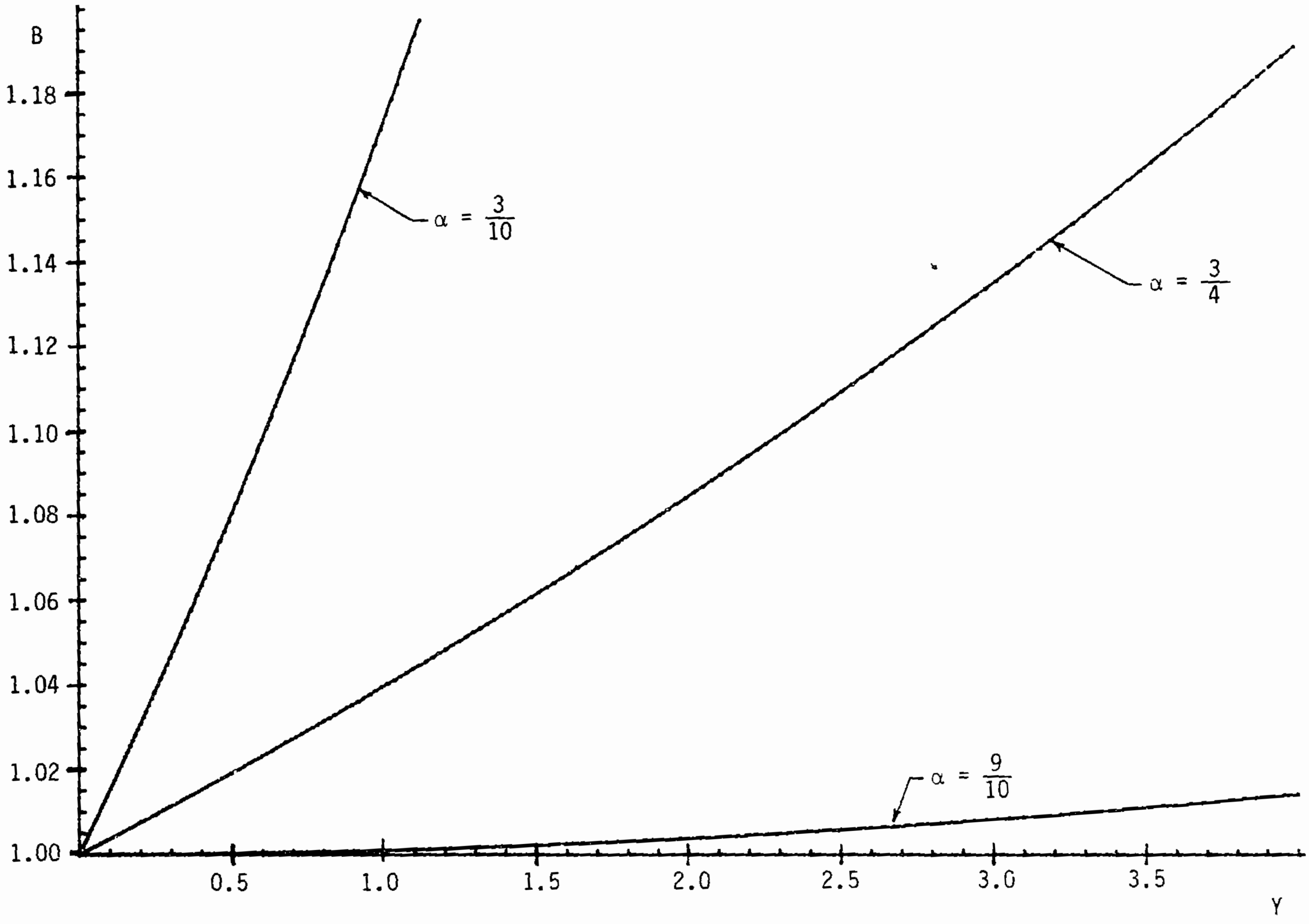

Figure 3.2 B vs. Y, hyperbola

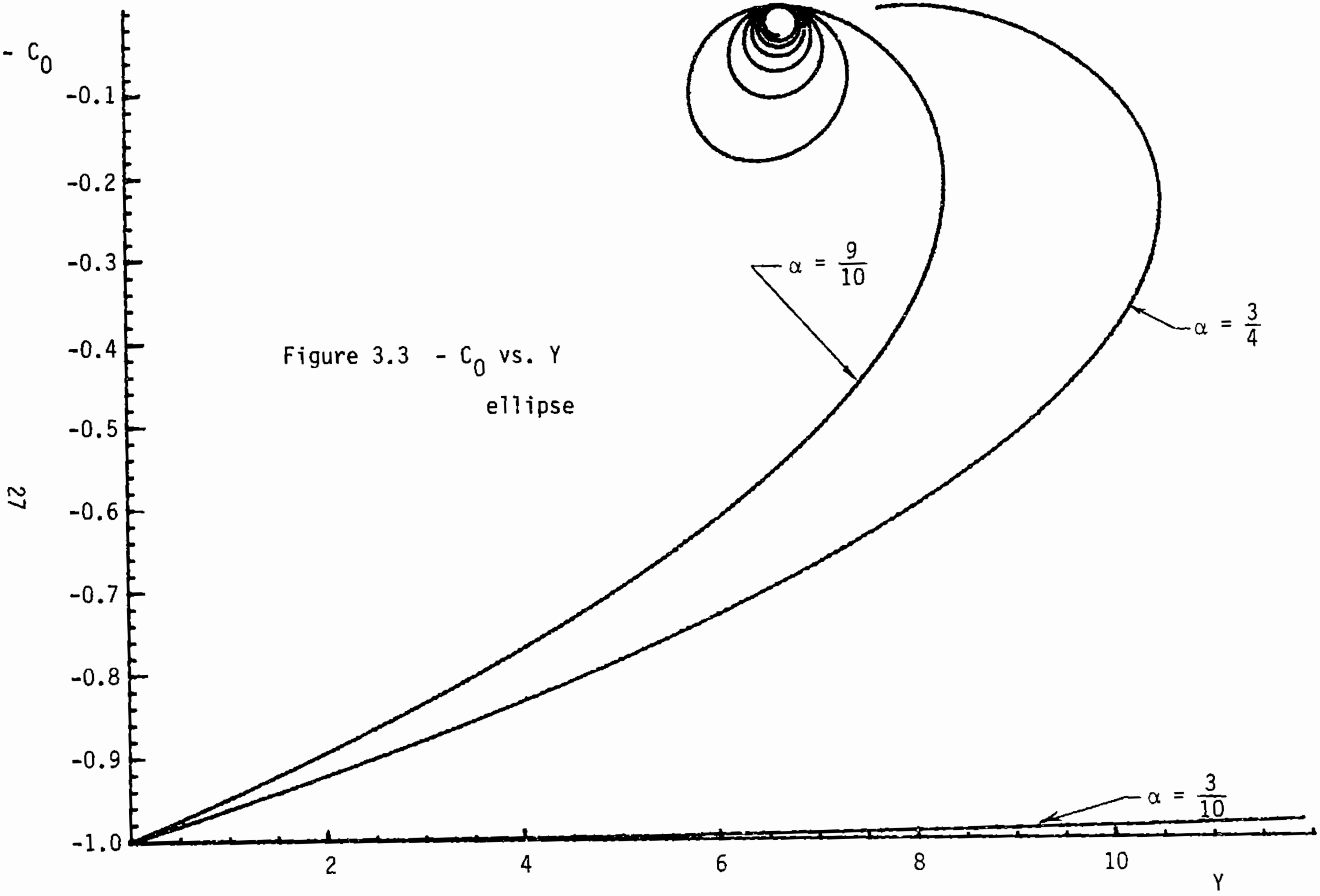

Figure 3.3 - C_0 vs. Y

ellipse

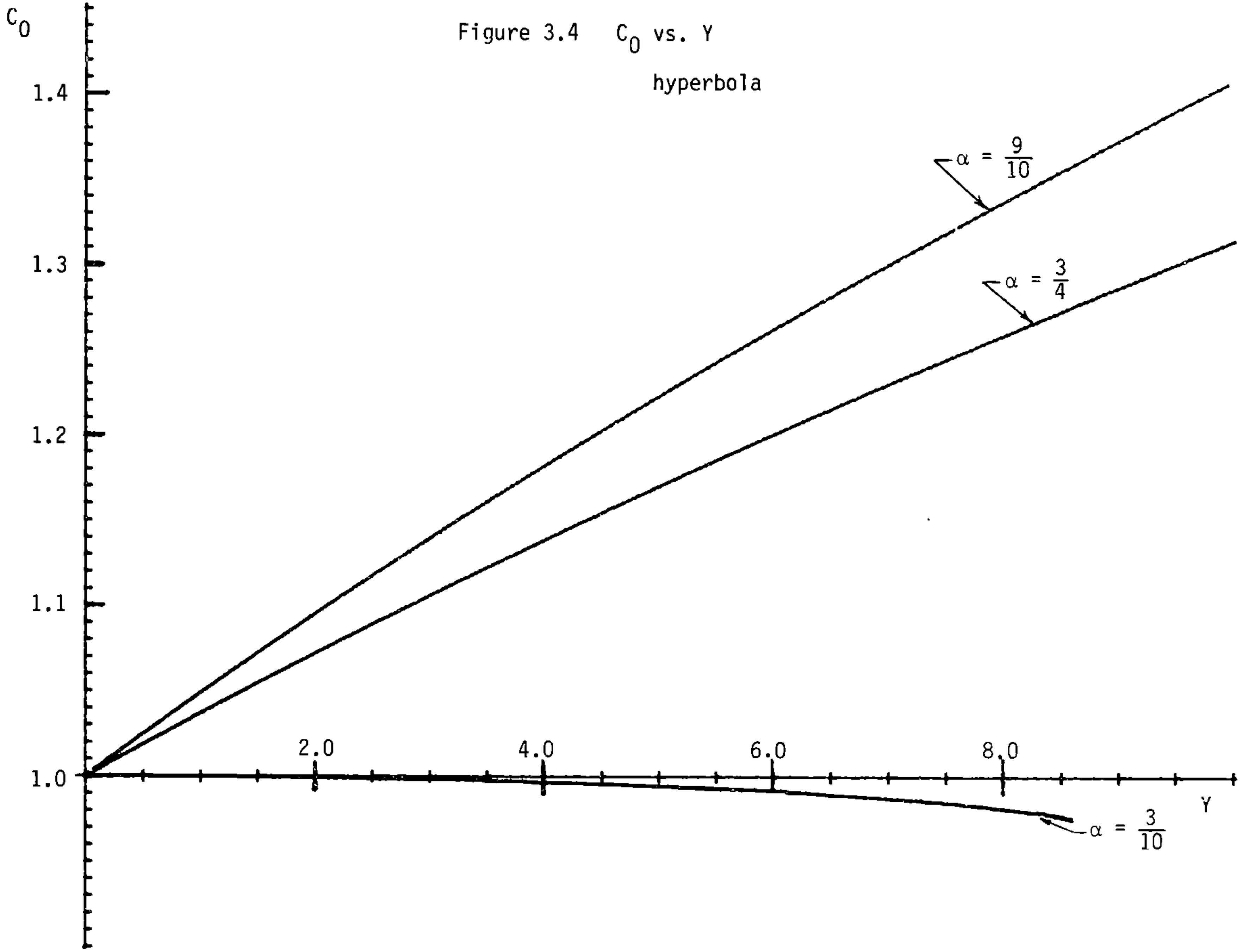

Figure 3.4 C_0 vs. Y

hyperbola

hand, but with the aid of a computer the second method proved to be much quicker and easier. For this reason the reversion method is explained in detail in Appendix A.

The second method was performed on MACSYMA (Project MAC's SYmbolic MAnipulation system), a large computer programming system used for performing symbolic as well as numerical mathematical manipulations and developed by the Mathlab Group Project MAC, of MIT. The procedure was to obtain the power series in ΔE, in the case of the ellipse, about $\Delta E = 0$, for Y, B and C_0 from their definitions using the algebra for power series. Hence,

$$Y = \Delta E^2 - \frac{\Delta E^2}{30} - \frac{\Delta E^4}{5040} + \frac{\Delta E^6}{36000} - \frac{79\ \Delta E^8}{498960000} - \frac{6469\ \Delta E^{10}}{340540200000} + \ldots$$

$$B^2 = 1 + \frac{3\ \Delta E^4}{2800} - \frac{\Delta E^6}{84000} + \frac{71\ \Delta E^8}{258720000} - \frac{527\ \Delta E^{10}}{100900800000} + \ldots$$

$$C_0 = 1 - \frac{\Delta E^2}{20} + \frac{\Delta E^4}{4200} + \frac{11\ \Delta E^6}{504000} - \frac{43\ \Delta E^8}{194040000} - \frac{8747\ \Delta E^{10}}{908107200000} + \ldots$$

Now, if a series for B is assumed in the form

$$B = \sum_{0}^{\infty} A_n Y^n \tag{3.1}$$

substitution of the power series for Y then gives B as a power series in ΔE which must be equal to the one above. Therefore, the coefficients of like powers must be equivalent. The end result is a system of simultaneous equations in the coefficients which may be solved for the coefficients of Eq. (3.1). The same procedure can be used to obtain C_0 as a power series in Y. Values for the coefficients of the B and C_0 power series in Y for the cases when $\alpha = \frac{9}{10}$, $\frac{3}{10}$ and $\frac{3}{4}$ are listed in tables 3.1 through 3.3.

Expansions for the hyperbola were found to differ from those for the ellipse only in the sign of the coefficients of the odd powers of Y. Therefore if the convention that $Y<0$ for the hyperbola is invoked, then the coefficients for the B and C_0 power series in Y for the hyperbola become identical to those listed in tables 3.1, 3.2, and 3.3.

In the remainder of this report this convention will be assumed unless otherwise stated.

3.3 Limits on Y

Before presenting a procedure to determine the number of terms needed in the series presented above, some discussion is needed on the range of Y. Clearly before determining the number of terms needed in the series, it is important to know the magnitude of Y.

Figs. 3.5 and 3.6 illustrate graphs of Y versus ΔE and ΔH, respectively. Clearly, for the ellipse, Y is not single-valued; therefore ΔE maximum must be limited such that no ambiguities arise in the series. Furthermore, from consideration of the figures presented earlier, Y must be limited in such a way that the maximum difference of B or C_0 from unity is fairly small.

Table 3.1

Coefficients of B and C_0 series for $\alpha = \frac{9}{10}$

n	$B = 1 + \sum_{1}^{\infty} A_n Y^n$	$C_0 = 1 + \sum_{1}^{\infty} A_n Y^n$
1	0	$-\frac{1}{20}$
2	$\frac{3}{2800}$	$-\frac{1}{700}$
3	$\frac{1}{16800}$	$-\frac{1}{12000}$
4	$\frac{471}{86240000}$	$-\frac{53}{8624000}$
5	$\frac{1363}{2802800000}$	$-\frac{223}{436800000}$
6	$\frac{434741}{9417408000000}$	$-\frac{4129}{90552000000}$
7	$\frac{2408477}{533653120000000}$	$-\frac{311177}{72770880000000}$
8	$\frac{403919063}{892268016640000000}$	$-\frac{37728329}{90841766400000000}$
9	$\frac{2145072731}{46329300864000000000}$	$-\frac{105033143}{2535919626240000000}$
10	$\frac{75471659629837}{15687142454151168000000000}$	$-\frac{5471610044533}{1297649651538240000000000}$
11	$\frac{132411358246191}{261452374235852800000000000}$	$-\frac{106817118047}{2447546669568000000000000}$
12	$\frac{5579979899245476 7}{10353514019739770880000000000000}$	$-\frac{11851702685582548}{2588378504934942720000000000000}$
13	$\frac{12414235743970997}{21446564755175239680000000000000}$	$-\frac{19027953195172973}{3916329216162435072000000000000}$

Table 3.2

Coefficients of B and C_0 series for $\alpha = \frac{3}{4}$

n	$B = 1 + \sum_{1}^{\infty} A_n Y^n$	$C_0 = 1 + \sum_{1}^{\infty} A_n Y^n$
1	$-\frac{3}{80}$	$-\frac{3}{80}$
2	$\frac{201}{89600}$	$-\frac{79}{89600}$
3	$-\frac{1471}{21504000}$	$-\frac{871}{21504000}$
4	$\frac{176613}{35323904000}$	$-\frac{252481}{105971712000}$
5	$-\frac{271451}{3748659200000}$	$-\frac{12351271}{78721843200000}$
6	$\frac{8488006389}{617179250688000000}$	$-\frac{6865898111}{617179250688000000}$
7	$\frac{13098502849}{55957585395712000000}$	$-\frac{2525021731}{30522319306752000000}$
8	$\frac{58803292553097}{1069269517504348160000000}$	$-\frac{330173488030733}{51818445848287641600000000}$
9	$\frac{9420554697114853}{3233471020933148835840000000}$	$-\frac{16288553229423227}{3233471020933148835840000000}$
10	$\frac{3883596108433264805167}{1.2632955714051083625037824E34}$	$-\frac{3020949675655576376649}{7.43115042003004919119872E32}$
11	$\frac{23098122874658499783029}{1.01063645712408669000302592E36}$	$\frac{-53217485590555768462627}{1.5957417744064526684258304E35}$
12	$\frac{6441756777920188837201637}{3.138917937420692778362339328E38}$	$\frac{-8704290755686248953605003}{3.138917937420692778362339328E39}$
13	$\frac{7006713442959367729144985491}{4.126630781729070772620355436544E42}$	$\frac{-1255133409038017992997674049}{5.38256188921183144254828969984E42}$

Table 3.3

Coefficients of B and C_0 series for $\alpha = \frac{3}{10}$

n	$B = 1 + \sum_1^\infty A_n Y^n$	$C_0 = 1 + \sum_1^\infty A_n Y^n$
1	$-\frac{3}{20}$	0
2	$\frac{111}{5600}$	$-\frac{1}{5600}$
3	$-\frac{841}{336000}$	$\frac{1}{168000}$
4	$\frac{42657}{137984000}$	$-\frac{157}{413952000}$
5	$-\frac{1352179}{35875840000}$	$\frac{71}{2745600000}$
6	$\frac{137316943}{30135705600000}$	$-\frac{59569}{30135705600000}$
7	$-\frac{9352130257}{17076899840000000}$	$\frac{4117}{25589760000000}$
8	$\frac{5346453665829}{81578790092800000000}$	$-\frac{5437831417}{395343367372800000000}$
9	$-\frac{482112405431887}{61673565310156800000000}$	$\frac{855068051}{700835969433600000000}$
10	$\frac{56022846609957208997}{60238627023940485120000000000}$	$-\frac{43869829127363}{393716516496343040000000000}$
11	$-\frac{1330256777885202910 01}{1204772540478809702400000000000}$	$\frac{435581052799}{41808185326043136000000000}$
12	$\frac{1809728048880088249343}{138286935081045983232000000000000}$	$\frac{-3159248718285685463}{3180599506864057614336000000000000}$
13	$-\frac{7326398441765058458 6453}{4.730122343541419016192E34}$	$\frac{175461881414420479}{1.822873235159242506 24E33}$

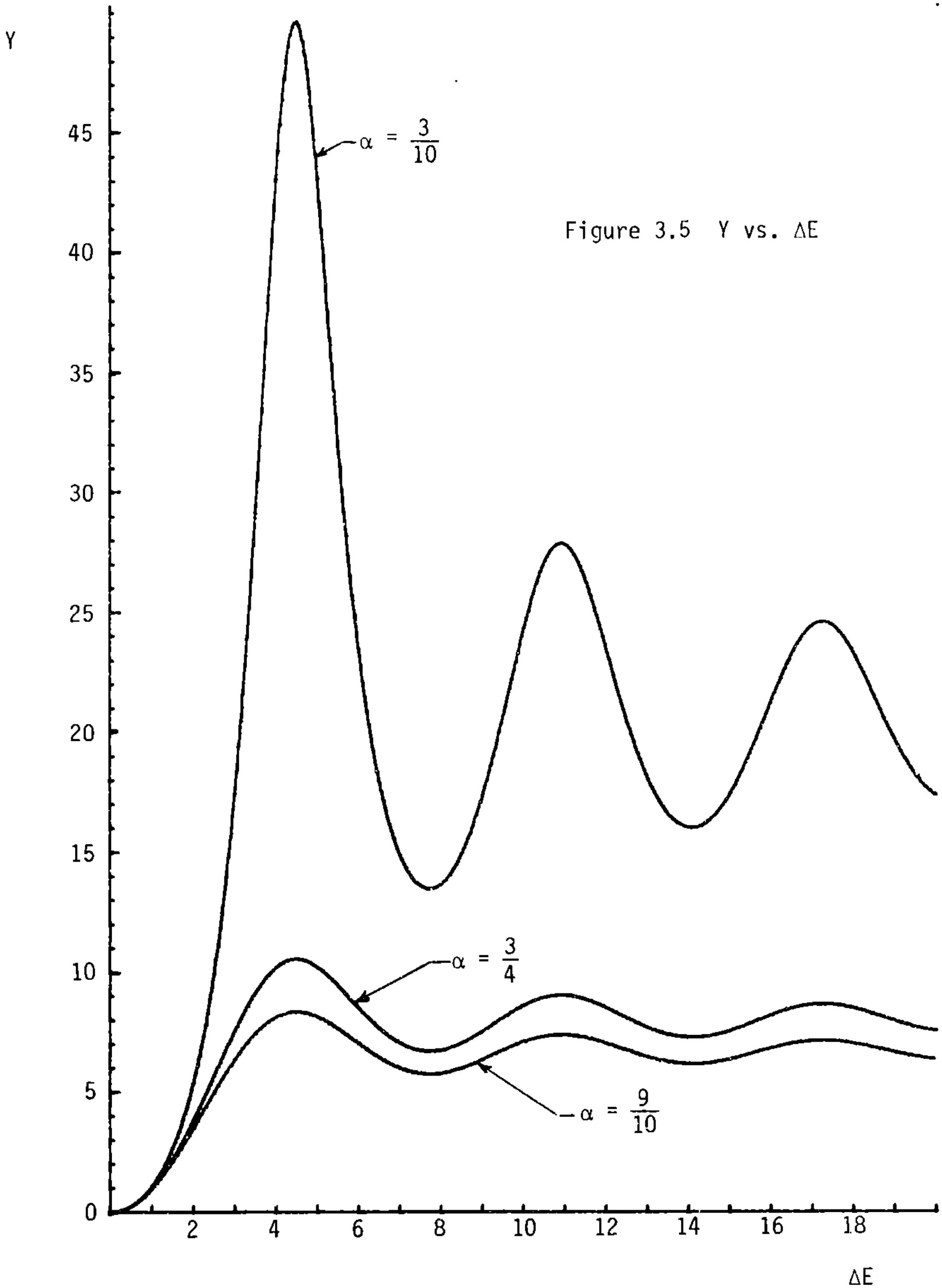

Figure 3.5 Y vs. ΔE

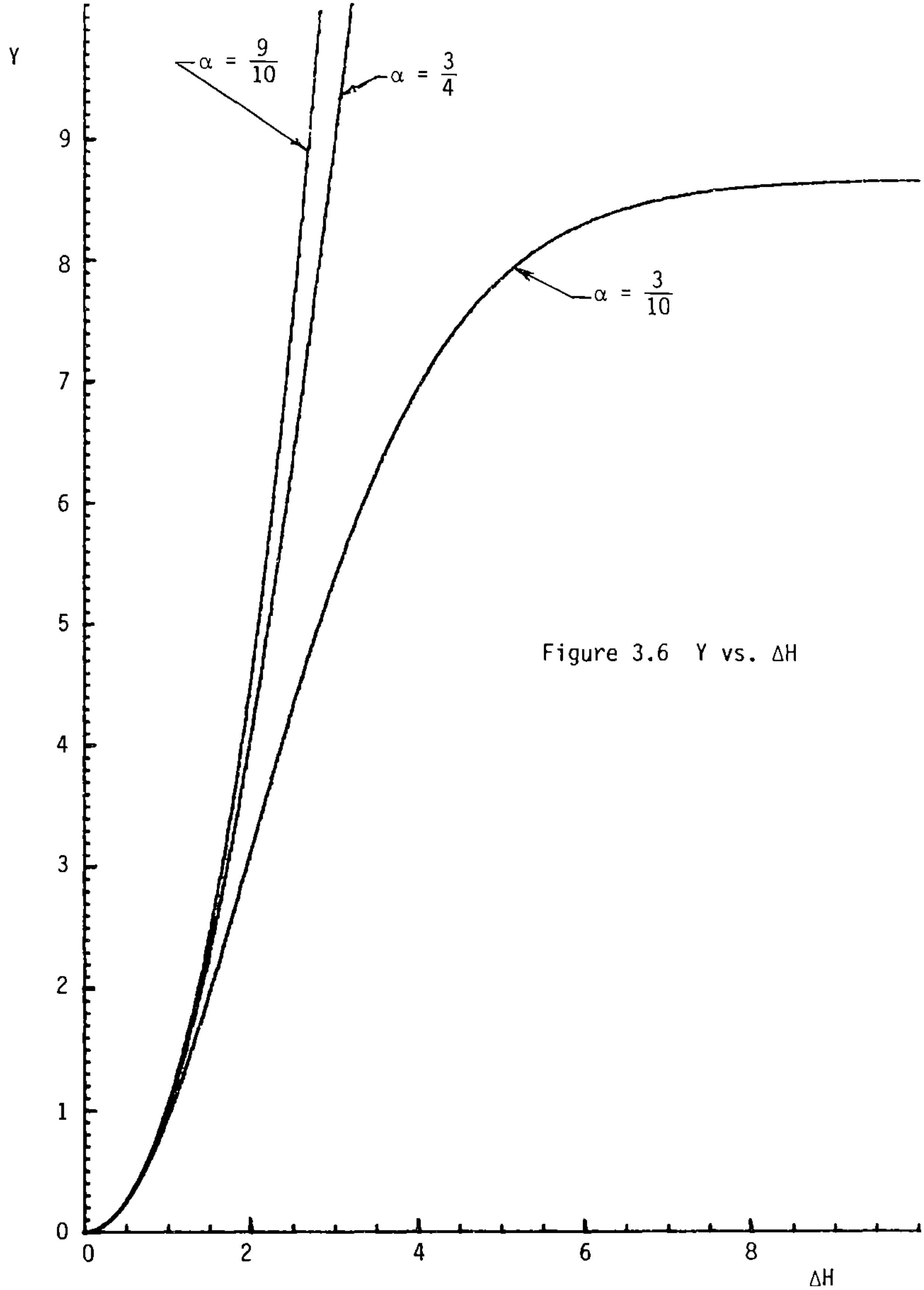

Figure 3.6 Y vs. ΔH

For comparison purposes, two ranges of Y were selected on the basis of the figures, to illustrate both the number of terms needed in the B and C_0 series and the number of iterations required in the basic algorithm for certain test cases to be presented subsequently. These two ranges are $-2 \le Y \le 2$ and $-1 \le Y \le 1$.

In the case of an ellipse; for $\alpha = \frac{9}{10}$, a Y of 2 corresponds to a ΔE of approximately 84° while for a Y of 1, ΔE is approximately 58°. These values are roughly the same for $\alpha = \frac{3}{10}$.

3.4 Series Economization

A simple procedure to predict the effect of a term in a power series is through the use of Chebyshev polynomials.(1) If a function is expanded in a series of Chebyshev polynomials, then the contribution of the nth Chebyshev (T_n) term will never be greater than the magnitude of its coefficient since the magnitude of any Chebyshev polynomial does not exceed unity. Specifically, for a given function

$$f(x) = \sum_{0}^{m} a_n x^n \quad \text{where } -1 \le x \le 1 \tag{3.2}$$

let it be required to find a function

$$g(x) = \sum_{0}^{K} b_n x^n \tag{3.3}$$

with K as small as possible, such that

$$|f(x) - g(x)| < \rho$$

where ρ is some maximum permissable error. This may be done by expressing f(x) in terms of Chebyshev polynomials

$$f(x) = \sum_{0}^{m} c_n T_n(x)$$

and since $|T_n(x)| \leq 1$, for $-1 \leq x \leq 1$, then

$$g(x) = \sum_{0}^{K} c_n T_n(x) \tag{3.4}$$

within the desired accuracy provided that

$$\sum_{n=K+1}^{m} |c_n| < \rho$$

After Eq. (3.4) is obtained, then by expressing the Chebyshev polynomials in terms of the powers of x, we obtain Eq. (3.3).

Economizations were performed to obtain 10 decimal place accuracy, for reasons which will become apparent later, for the two ranges of Y discussed earlier. For Y between ±2, when $\alpha = \frac{9}{10}$, the B series required 9 terms and the C_0 series 9 terms. When $\alpha = \frac{3}{10}$, the B series required 11 terms and the C_0 series 7 terms. For Y between ±1; when $\alpha = \frac{9}{10}$, the B series required 6 terms and the C_0 series 7 terms. When $\alpha = \frac{3}{10}$, the B series required 8 terms and the C_0 series 5 terms. The new coefficients for the four cases noted here are listed in Tables 3.4 through 3.7. Out of curiosity, the B series was economized for $|Y| < \pm \frac{1}{2}$ and $\alpha = \frac{9}{10}$. In this case only 4 terms were required.

3.5 Series About an Arbitrary Point Y_0

If a greater accuracy is required for the values of B and C_0 and the resulting number of terms required in the above series, expanded about Y = 0, is too large or still further reduction in the amount of computation in the B and C_0 series is desired, this may be had by expanding B and C_0 about some arbitrary point Y_0. In this way the range of Y may be divided into smaller ranges. For example, B and C_0 might be expanded about the points Y_0=0, ±1, ±2. Then, if Y varies from -2 to +2, this range may be subdivided, where the different expansions would cover the ranges

$$-\frac{5}{2} \le (Y + 2) \le -\frac{3}{2}$$

$$-\frac{3}{2} \le (Y + 1) \le -\frac{1}{2}$$

$$-\frac{1}{2} \le (Y - 0) \le \frac{1}{2}$$

$$\frac{1}{2} \le (Y - 1) \le \frac{3}{2}$$

$$\frac{3}{2} \le (Y - 2) \le \frac{5}{2}$$

Clearly the symmetry of the expansions is lost here in that different series are needed for negative values of Y, (i.e. for hyperbolic motion,

Table 3.4

Economized coefficients of B and C_0 series for $|Y| < 2$ and $\alpha = \frac{9}{10}$

n	$B = \sum_{0}^{\infty} b_n Y^n$	$C_0 = \sum_{0}^{\infty} b_n Y^n$
0	$\frac{7843571227151055659629837}{7843571227075584000000000}$	1
1	0	$-\frac{1}{10}$
2	$\frac{672306029706247570163}{6274856981660467200000000}$	$-\frac{1}{175}$
3	$\frac{1}{16800}$	$-\frac{1}{1500}$
4	$\frac{1713582915051373837}{3137428490830233600000000}$	$-\frac{53}{539000}$
5	$\frac{1363}{2802800000}$	$-\frac{223}{13650000}$
6	$\frac{68717400172199 21}{1494013567062016000000000}$	$-\frac{4129}{1414875000}$
7	$\frac{2408477}{533653120000000}$	$-\frac{311177}{568522500000}$
8	$\frac{785609842671397}{15687142454151168000000000}$	$-\frac{37728329}{354850650000000}$
9	$\frac{2145072731}{46239300864000000000}$	$-\frac{105033143}{4952968020000000}$

Table 3.5

Economized coefficients of B and C_0 series for $|Y| < 2$ and $\alpha = \frac{3}{10}$

n	$B = \sum_{0}^{\infty} b_n Y^n$	$C_0 = \sum_{0}^{\infty} b_n Y^n$
0	$\frac{6914346753871326356711991175065 7}{69143467540522991616000000000000}$	$\frac{197671683691837831417}{197671683686400000000}$
1	$-\frac{3}{20}$	$-\frac{855068051}{77870663270400000000}$
2	$\frac{76140129756137295600088249343}{3841303752251277312000000000000}$	$-\frac{4412319805831417}{24708960460800000000}$
3	$-\frac{841}{336000}$	$\frac{139055610908051}{23361198981120000000}$
4	$\frac{407147244803613763111750657}{1317018429343295078400000000000}$	$-\frac{7491676408583}{19767168368640000000}$
5	$-\frac{1352179}{35875840000}$	$\frac{2011135659847}{77870663270400000000}$
6	$\frac{5627889561901438168249343}{1234704777509339136000000000000}$	$-\frac{1340583750421}{642432971980800000000}$
7	$-\frac{9352130257}{17076899840000000}$	$\frac{13383263731}{77870663270400000000}$
8	$\frac{16602272369847774919065 7}{2560869168167518208000000000000}$	
9	$-\frac{482112405431887}{61673565310156800000000}$	
10	$\frac{2881242702080537493216 89}{265049958905338134528000000000000}$	
11	$\frac{-1330256777885202910 01}{1204772540478809702400000000000}$	

Table 3.6

Economized coefficients of B and C_0 series for $|Y| < 1$ and $\alpha = \frac{9}{10}$

n	$B = \sum_0^\infty b_n Y^n$	$C_0 = \sum_0^\infty b_n Y^n$
0	$\frac{2677272312165437747913889689}{2677272312175132672000000000}$	$\frac{11627746099237728329}{11627746099200000000}$
1	$\frac{1437326685903}{285525765324800000000}$	$-\frac{3606641246102966857}{72132824924160000000}$
2	$\frac{286850635854381151775093}{267727231217513267200000000}$	$-\frac{519095845728329}{363367065560000000}$
3	$\frac{23898422653221547}{401520607488000000000}$	$-\frac{450830260809143}{5409961869312000000}$
4	$\frac{36550962583721353843}{6693180780437831680000000}$	$-\frac{446586911671}{72673413120000000}$
5	$\frac{10178067028931}{20590800384000000000}$	$-\frac{2301313214171}{4508301557760000000}$
6	$\frac{2363485560315763573}{50198855853283737600000000}$	$-\frac{2108834729}{45420888320000000}$
7		$-\frac{126270321}{28899368960000000}$

Table 3.7

Economized coefficients of B and C_0 series for $|Y| < 1$ and $\alpha = \frac{3}{10}$

n	$B = \sum_{0}^{\infty} b_n Y^n$	$C_0 = \sum_{0}^{\infty} b_n Y^n$
0	$\frac{30842177036313551228049957208997}{30842177036257528381440000000000}$	$\frac{657851363267207815197053}{657851363308339200000000}$
1	$-\frac{263140544841223274568113}{1754270302155571200000000}$	$\frac{177959943673}{996744489861120000000}$
2	$\frac{1222672012620781642697079 1003}{61684354072515056762880000000}$	$-\frac{73420448973994290 53}{411157102067712000000000}$
3	$-\frac{3293177419590204718 87}{1315702726616678400000000}$	$\frac{2781028488704629}{46722397962240000000}$
4	$\frac{23836782488154859250008997}{771054425906438209536000000000}$	$-\frac{7858448465474947}{205578551033856000000000}$
5	$-\frac{137700448610720 8113}{3654729796157440000000}$	$\frac{8143355893811}{311482653081600000000}$
6	$\frac{12542260617507133815 1003}{275376580680870789120000000000}$	
7	$-\frac{1549340372354 7727}{274104734711808000000000}$	
8	$\frac{16351736425964604976 37}{240954508095761940480000000000}$	

B and C_0 must be expanded about $|Y_0|$ using the hyperbolic definitions and then the sign of the coefficients for the odd powers of Y changed.).

The methods presented earlier are no longer of practical use in this case. The method of obtaining these coefficients and their values are presented in Appendix B.

CHAPTER 4

ROOTS OF THE CUBIC

The cubic equation in Gauss' solution is monotonic in that the coefficients are such that only one real root exists. This is not the case here because of the existence of C and γ which are functions of initial position and velocity. It is now possible that three real roots may exist and clearly only one is valid. Two methods are presented here for obtaining the one correct solution of the cubic.

4.1 Method A

Equation (2.11) is

$$\frac{1}{3}\gamma D^3 + C D^2 + D = \frac{t}{B}\sqrt{\frac{\mu}{r_0^3}}$$

which may be written in the form

$$\frac{1}{3}(\gamma D)^3 + C(\gamma D)^2 + \gamma(\gamma D) = \frac{\gamma^2 t}{B}\sqrt{\frac{\mu}{r_0^3}}$$

Then by defining

$$\gamma D = x - C \tag{4.1}$$

$$F = \frac{t}{B}\sqrt{\frac{\mu}{r_0^3}} \tag{4.2}$$

we have

$$x^3 - 3\,\varepsilon\, x = 2\,b \tag{4.3}$$

where

$$\varepsilon = C^2 - \gamma \tag{4.4}$$

$$2\,b = 3\,\gamma^2 F + C^3 - 3\,\varepsilon\, C \tag{4.5}$$

From elementry algebra, the criterion for one real root in Eq. (4.3) is

$$b^2 - \varepsilon^3 > 0$$

In this case it is easily shown that

$$x = \left(b + \sqrt{b^2 - \varepsilon^3}\,\right)^{1/3} + \left(b - \sqrt{b^2 - \varepsilon^3}\,\right)^{1/3} \tag{4.6}$$

is the real root. This may be written in the form

$$x = \frac{2\,b\,m_0}{(m_0 - \varepsilon)^2 + m_0\varepsilon} \tag{4.7}$$

where

$$m_0 = \left(|b| + \sqrt{b^2 - \varepsilon^3}\,\right)^{1/3} \tag{4.8}$$

The absolute value of b results from taking the positive square root of b^2. Eqs. (4.7) and (4.8) are more desirable than Eq. (4.6) in computing x due to the fact that only one and not two cube roots is required and that m_0 is obtained by adding two positive numbers in contrast to the subtraction required in Eq. (4.6).

When $b^2 - \varepsilon^3 \leq 0$, three real roots exist, two of which are equal when the equality holds. Here the three roots are obtained by calculating

$$3\,\theta = \arccos\left(\frac{\sqrt{b^2}}{\sqrt{\varepsilon^3}}\right) \tag{4.9}$$

so that

$$\begin{aligned} x_1 &= \pm 2\sqrt{\varepsilon}\cos\theta \\ x_2 &= \pm 2\sqrt{\varepsilon}\cos\left(\theta + \frac{2\pi}{3}\right) \qquad \begin{matrix} + \text{ if } b > 0 \\ - \text{ if } b < 0 \end{matrix} \\ x_3 &= \pm 2\sqrt{\varepsilon}\cos\left(\theta + \frac{4\pi}{3}\right) \end{aligned}$$

are the three real roots.

Since ε must be greater than zero and b lies between $\pm\sqrt{\varepsilon^3}$ for for these three real roots to exist, selection of the proper root can be deduced, and easily verified, by plotting the three roots as functions of b and ε as is shown in Fig. 4.1. Using Eq. (4.1), if C is positive then this equation is equivalent to translation of the γ D axis to the right of the origin whereas if C is negative, translation is to the left. Rewriting Eq. (4.4) in the form

$$|C| = \sqrt{1 + \frac{\gamma}{\varepsilon}}\,\sqrt{\varepsilon}$$

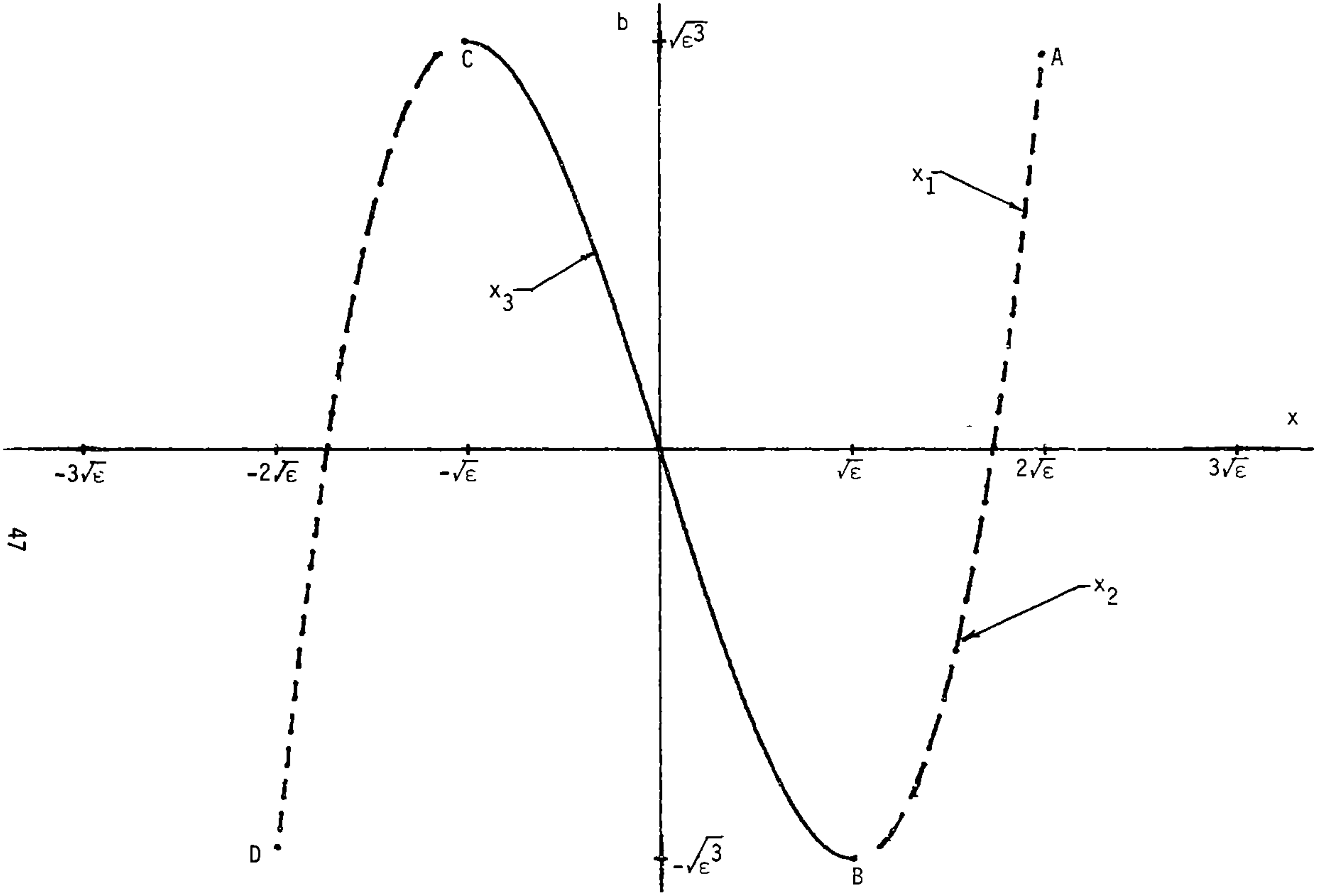

Figure 4.1 Graph of three real roots, b vs. x

it can be seen that if γ is negative then absolute C will always be less than $\sqrt{\varepsilon}$. Hence translation of the axis will be such that x_3 is the correct choice of the root. If γ is positive then selection of the roots x_1 or x_2 will be based on the signs of b and C.

A simple criterion for selection of the roots can be obtained by letting θ run from 0 to π, starting at pt. A in Fig. 4.1 and moving along the curve,instead of oscillating between 0^0 and 30^0 as is seen in Eq. (4.9). Therefore the roots may be computed using the following criteria: for $b^2 \leq \varepsilon^3$,

$$3\,\theta = \arccos\left(\frac{b}{\sqrt{\varepsilon^3}}\right)$$

then for $\gamma < 0$, $120^0 - \theta \rightarrow \theta$

for $\gamma > 0$ and $C < 0$, $\theta + 120^0 \rightarrow \theta$

otherwise, θ remains as calculated (always in the first quadrant) and

$$x = 2\sqrt{\varepsilon}\cos\theta$$

From a continuity standpoint, since small changes in t and hence in F and b , must correspond to small changes in the root D, then depending on the values of γ and C there are only certain ranges of t for which this solution method is valid. For example, consider C positive with b and ε being such that three real roots exist. If b is decreased (by changing t), some value will be reached such that $b = -\sqrt{\varepsilon^3}$ at which point the value of $x = \sqrt{\varepsilon}$ is the solution (pt. B in Fig. 4.1). Any further decrease in b results in the existence of one real root and Fig. 4.1 illustrates that a jump from pt. B to point D exists. Hence a discontinuity in x exists at the point where $b = -\sqrt{\varepsilon^3}$ when $c > 0$. Since this is not physically possible, t must be constrained, in this case, such that $b \geq -\sqrt{\varepsilon^3}$. (we know that requiring that b be less than $-\sqrt{\varepsilon^3}$ is incorrect from the simple fact that three real roots exist when $t = 0$ and hence continuity must be maintained from this point). In a similar fashion it is seen that when C is negative, t must be constrained such that $b \leq \sqrt{\varepsilon^3}$. Hence several tests must be incorporated in the algorithm to maintain these requirements. This is, of course, not desirable.

Depending on the selection of α , in particular if $\alpha > \frac{1}{2}$, it is possible for γ to be equal to or less than zero. If γ equals zero then Eq. (2.11) reduces to a quadratic with solution

$$D = \frac{-1 \pm \sqrt{1 + 4\,C\,F}}{2\,C} \tag{4.10}$$

Clearly the (+) sign is the correct choice from the simple fact that D must equal zero when F is zero.

When γ is in the vicinity of zero, computation of the root using Eq. (4.1) leads to an indeterminant form. In this case, an asymptotic expansion of the form

$$D = D_0 + D_1 \left(\frac{\gamma}{3}\right) + D_2 \left(\frac{\gamma}{3}\right)^2 + \ldots \tag{4.11}$$

is appropriate.(4) Substitution into Eq. (2.11) and equating powers of γ gives a set of expressions for the coefficients of Eq. (4.11) which may be written in the form

$$A = \sqrt{1 + 4\,c\,F}$$

$$A_1 = C\,D_1 + 3\,D_0^{\,2}$$

$$A_2 = C\,D_2 + 3\,D_0\,D_2$$

$$A_3 = C\,D_3 + 3\,D_0\,D_2 + D_1^{\,2}$$

$$\vdots$$

$$D_0 = \frac{2\,F}{1 + A}$$

$$D_1 = -\frac{D_0^{\,3}}{A}$$

$$D_2 = -\frac{D_1\,A_1}{A}$$

$$D_3 = -\frac{D_2\,A_1 + D_1\,A_2}{A}$$

$$D_4 = -\frac{D_3\,A_1 + D_2\,A_2 + D_1\,A_3}{A}$$

$$\vdots$$

A more convenient form which eliminates all of the indirect computation may be arrived at after some inspection of the above set of coefficients. Rewriting Eq. (4.11) in the form

$$D = D_0\,\{1 + \sum_{1}^{\infty} W_k \left(\frac{D_0^{\,2}\,\gamma}{A^2}\right)^k) = D_0\,H \qquad (4.12)$$

and expressing C and F in the forms

$$F = \frac{(1 + A)\,D_0}{2} \qquad\qquad C = \frac{A^2 - 1}{4F} = \frac{A - 1}{2\,D_0}$$

we substitute into Eq. (2.11) to obtain

$$\frac{1}{3} A^2 \left(\frac{D_0^2 \gamma}{A^2}\right) H^3 + \frac{(A - 1)}{2} H^2 + H = \frac{A + 1}{2}$$

The series definition of H then provides expressions for the coefficients W_k in terms of A. Table 4.1 provides a list of several of these coefficients.

The efficiency and practical use of the asymptotic series both in the amount of computation required and in the accuracy leaves much to be desired. The coefficients do alternate in sign so that the truncation error is smaller in absolute value than the first term omitted. On the other hand, no precise criterion is apparent to decide when the asymptotic series must be used or when obtaining the root using Eq. (4.1) is no longer valid. Also, when C is negative (or t is negative in which case F is also negative), then for γ equal zero, t must be such that the radical in Eq (4.10) is nonnegative. Furthermore, when γ is small, t must be such that A is not in the vicinity of zero, since A is present in the denominator of the asymptotic series.

4.2 Method B

The second method involves a change in variable from D to x according to

$$D = \frac{3 F}{1 + x} \tag{4.13}$$

Table 4.1

Coefficients of asympotic series

n	$1 + \sum_{1}^{\infty} W_n \left(\frac{D_0^2 \gamma}{A^2}\right)^n$
1	$-\frac{A}{3}$
2	$\frac{A + 5 A^2}{18}$
3	$-\frac{A + 7 A^2 + 16 A^3}{54}$
4	$\frac{5 A + 45 A^2 + 159 A^3 + 231 A^4}{648}$
5	$-\frac{7 A + 77 A^2 + 357 A^3 + 847 A^4 + 896 A^5}{1944}$
6	$\frac{7 A + 91 A^2 + 518 A^3 + 1638 A^4 + 2931 A^5 + 2431 A^6}{3888}$
7	$-\frac{11 A + 165 A^2 + 1110 A^3 + 4330 A^4 + 10455 A^5 + 15033 A^6 + 10240 A^7}{11664}$
8	$(429 A + 7293 A^2 + 56528 A^3 + 260865 A^4 + 780615 A^5 + 1529847 A^6$ $+ 1839739 A^7 + 1062347 A^8)/839808$

Substituting into Eq. (2.11) gives

$$x^3 - 3 \varepsilon x = 2 b \tag{4.14}$$

where now

$$\varepsilon = 1 + 3 C F \tag{4.15}$$

$$2 b = 2 + 9 C F + 9 \gamma F^2 \tag{4.16}$$

Note that Eq. (4.14) is identical to Eq. (4.3) with different definitions of the coefficients. Thus, if $b^2 - \varepsilon^3$ is positive, then the root is computed using Eq. (4.7).

When $b^2 - \varepsilon^3 \leq 0$, selection of the proper root can be made by plotting the three roots as was previously done in Fig. 4.1. When F = 0 (t = 0), the roots of Eq. (4.14) are $x_1 = 2$, $x_2 = x_3 = -1$ and $b^2 = \varepsilon^3$. Hence point A is the solution point for all values of C and γ. From a continuity standpoint, small changes in F, and hence in b, correspond to small changes in the root. Therefore if b decreases then the correct root will always be located on that portion of the curve from points A to B. Hence for three real roots, the correct root is simply calculated from

$$3 \theta = \arccos\left(\frac{b}{\sqrt{\varepsilon^3}}\right) \tag{4.17}$$

and

$$x = 2\sqrt{\varepsilon} \cos \theta \tag{4.18}$$

To avoid any discontinuities, the requirement that $b \geq -\sqrt{\varepsilon^3}$ must be met. This requirement avoids obtaining a value of x in the vicinity of $x = -1$ for which case Eq. (4.13) has a singularity.

The most important characteristic of this method is the elimination of division by γ and hence the need for an asymptotic expansion is avoided. Also computation of the real root has been simplified along with the number of criteria to maintain continuity. Another quality evident is the effect of errors resulting in the calculation of x. Small errors in x using method A are magnified by a factor of $\frac{1}{\gamma}$ which is not the case in this method.

CHAPTER 5

FINAL ALGORITHM

Incorporating any iterative procedure into an algorithm requires the use of tests along with certain logic operations to account for cases that may arise which could pose problems if precautions are not taken. In this solution of Kepler's equation only two such cases arise.

The first deals with time intervals which require a larger transfer angle than that which is permitted by limitations on the range of Y. This can be handled simply by computing the time interval corresponding to the maximum value of Y and comparing with the desired time interval. If the time interval corresponding to Y_{max} is greater than the desired time interval then the iteration process is initiated immediately to obtain the final position and velocity using the desired time interval. If, on the other hand, the opposite is true then a transfer angle step corresponding to Y_{max} is taken. This is done by computing a position and velocity by means of the universal formulae derived previously using the values of B, C_0, D, t and Y corresponding to Y_{max} . Then the time corresponding to Y_{max} is subtracted from the desired transfer time interval. This is continued until the time interval corresponding to Y_{max} is greater than the present desired time interval. This procedure will be referred to subsequently as "time stepping."

For the second case, recall that one requirement in the solution of

Eq. (4.14) was that $b \geq -\sqrt{\varepsilon^3}$ (when ε is positive), to maintain continuity and also to avoid the singularity at $x = -1$ in computing D. Note, b and ε are both functions of γ, C, and F while γ and C, for the most part, are functions of the initial position and velocity. Thus, b and ε vary with F and hence t. When ε is positive and a value of b which is less than $-\sqrt{\varepsilon^3}$ is encountered, a simple relation to determine that value of F such that $b = -\sqrt{\varepsilon^3}$ does not seem to exist.

The value of F where $b = -\sqrt{\varepsilon^3}$ is determined from the quadratic

$$b^2 - \varepsilon^3 = 9\,F^2\,\left(\frac{9}{4}\,\gamma^2\,F^2 + \frac{9}{2}\,C\,(\gamma - \frac{2}{3}\,C^2)\,F + (\gamma - \frac{3}{4}\,C^2)\right) = 0$$

and, unfortunately, division by γ is required. If γ is very nearly zero, problems will arise. Furthermore, when ε is negative, there being always only one real root in this case, there is continuity for all values of b and ε but now b must be restricted to only positive values to positively avoid the singularity at $x = -1$. Hence the first test is to see if b is positive or negative. If positive, there is no problem. If negative, then ε must be checked to see if it is positive or negative. If negative, then a value of F must be determined which will make b at least zero. Here b is a quadratic itself in F and also requires division by γ. If ε is positive then $b \geq -\sqrt{\varepsilon^3}$ must be tested. If it is true, there is no problem. If it is false then the above quadratic must be solved for that value of F where $b = -\sqrt{\varepsilon^3}$.

Clearly all this testing is very tedious and detracts from the appeal and potential of this method of solution. It was found that for all orbits tested that a root less than zero was never encountered and for the most part was in fact quite large. Thus, the need for the tests described above can be avoided by the simple expediency of assuring only positive roots. If a negative root is encountered then a prescribed fraction of the time interval is subtracted and the iteration is reinitiated. This fraction would be preferred to be near unity (say .95) since fairly small changes in t have a significant effect on the root. Also, a decrease, rather than an increase, in F (t) is taken due to the fact that as F approaches zero both b and ε approach unity while the root, x, approaches a value of 2. Maintaining x positive not only satisfies the continuity requirement when ε is positive but avoids the possibility of having to subtract a small quantity from unity when computing D.

5.1 Procedure

Before presenting the final algorithm the following quantities are defined:

t_{max} = time interval corresponding to Y_{max}

t = input time interval

t_T = sum of time steps (if any)

T = time interval computed in the iteration for convergence test

Figure 5.1 illustrates the flow chart diagram used in the tests. In the following discussion, an iteration will be referred to as executing blocks 12 to 20 in the flow diagram.

5.2 Tests and Data

All tests described below were made on a Hewlett Packard model 9820A calculator which employs 12 significant figures internally and displays 10.

The first set of 28 tests consist of a series of orbits which comprise the test package for the Kepler subroutine in the Apollo project. The characteristics of these orbits are listed in Table 5.1. These tests were first run to examine different ranges of Y. Two ranges were examined; $-1 \leq Y \leq 1$ and $-2 \leq Y \leq 2$, both for $\alpha = \frac{9}{10}$. The results of the number of time steps and iterations needed along with the results of the Kepler subroutine proposed for the NASA Space Shuttle orbiter vehicle are listed in Table 5.2.

The second set of tests was used to examine the performance of this method for orbits of very high eccentricities. These cases evolved from modelling the Earth as a series of point masses, resulting in orbits of very high eccentricities. The orbits and results are listed in Table 5.3.

Finally, we note that several cases were run for different values

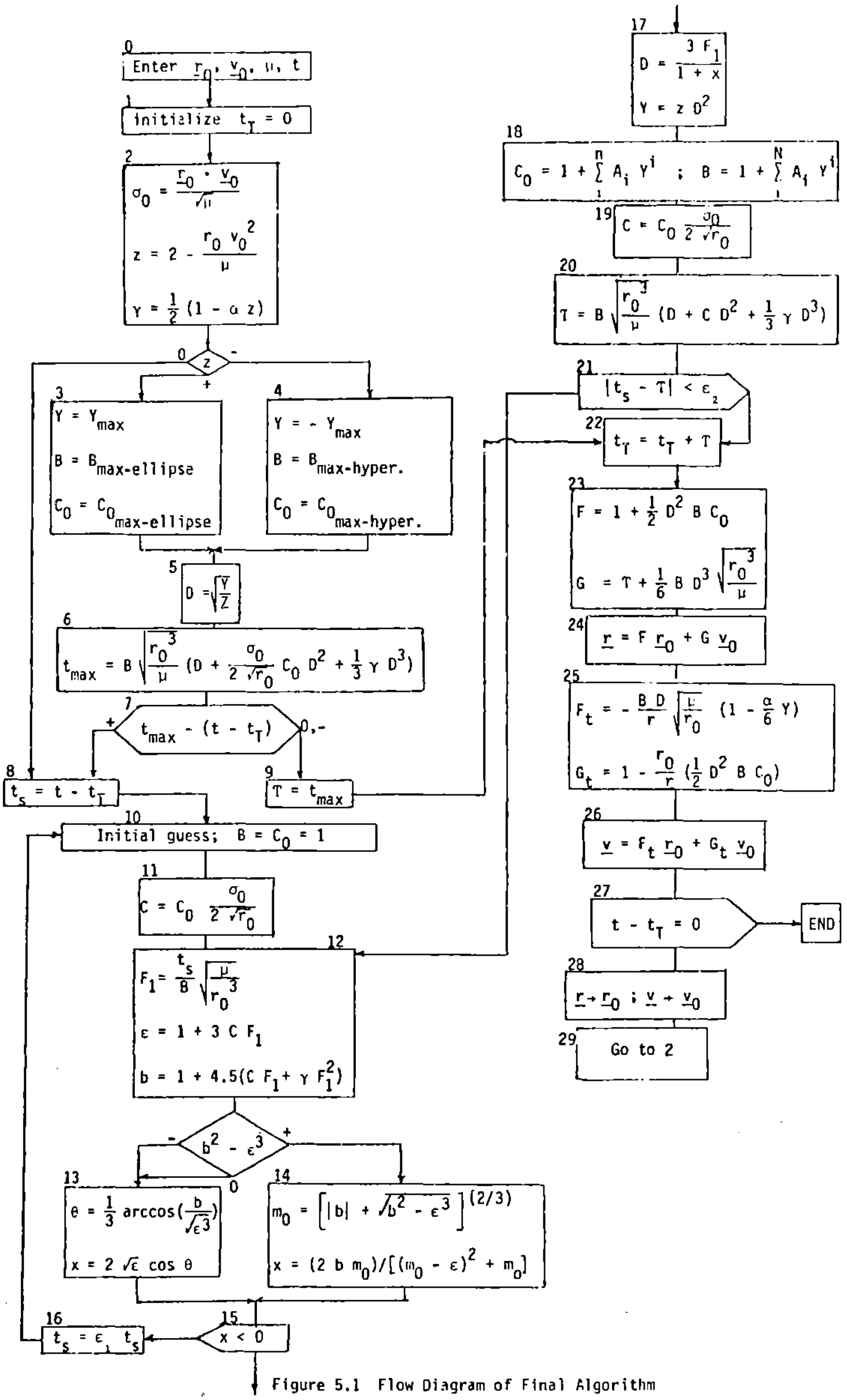

Figure 5.1 Flow Diagram of Final Algorithm

Table 5.1

Characteristics of test orbits in Apollo test package

Test Case	Transfer Angle, deg (True Anomaly)	Eccentricity	Transfer Time, sec
1	7.99998424968 E01	9.08472055853 E-02	1.14836999999 E03
2	1.00048154767 E-02	9.08473428889 E-02	1.49999999999 E-01
3	4.99987664764 E00	9.08473428889 E-02	7.53999999999 E01
4	1.20000175146 E02	9.08473428889 E-02	2.13718999999 E03
5	1.80000041363 E02	9.08473428889 E-02	3.33458999999 E03
6	2.40000321827 E02	9.08473428889 E-02	4.37899999999 E03
7	3.10000511619 E02	9.08473428889 E-02	5.40949999999 E03
8	3.59990041015 E02	9.08472106471 E-02	6.13027999999 E03
9	3.00001706630 E01	4.74644650401 E-02	5.93839999999 E02
10	3.00000052396 E02	4.74644650401 E-02	6.40665999999 E03
10A	1.51788202155 E02	9.99992103557 E-01	2.49214659999 E05
10B	1.51804097745 E02	9.99922247179 E-01	2.49386009999 E05
10C	1.79577137679 E02	9.99922247179 E-01	2.49733099999 E05
11	4.99999925706 E00	9.99999997122 E-01	5.58423099999 E04
12	1.60650689794 E02	9.99999997122 E-01	1.15658179999 E05
13	3.21302199029 E02	9.99999997122 E-01	2.31316369999 E05
14	2.00001935050 E01	1.00000008927 E00	2.81689999999 E02
15	1.60651090924 E02	1.00000008927 E00	1.15658209999 E05
16	1.99999947324 E01	2.12962970416 E00	3.15733599999 E04
17	2.38668401433 E02	1.87546300185 E00	7.75691699999 E04
18	1.99994918900 E01	2.12914297903 E00	2.26909999999 E02
19	1.08575407348 E02	2.82215957165 E00	2.99999999999 E04
20	1.51672933663 E02	9.99999876956 E-01	2.49192869999 E05
21	1.51672339021 E02	9.99999998429 E-01	2.49192869999 E05
22	1.79675677848 E02	9.99999998429 E-01	2.49542759999 E05
23	1.40807640275 E02	8.96059501396 E-01	1.80147099999 E04
24	1.15344978692 E02	2.12962970416 E00	3.47293699999 E04
25	1.09425122755 E02	2.70365025705 E00	2.99999999999 E04

Table 5.2

Number of iterations for Apollo test package

Test Case	Proposed Shuttle version	$-1 \leq Y \leq 1$		$-2 \leq Y \leq 2$	
		number of time steps	iterations	number of time steps	iterations
1	4	1	3	0	4
2	3	0	1	0	1
3	4	0	1	0	1
4	5	2	1	1	2
5	5	3	2	2	2
6	5	4	2	2	5
7	5	5	2	3	4
8	3	6	2	4	2
9	4	0	2	0	2
10	5	5	2	3	3
10A	14	0	2	0	2
10B	11	0	7	0	7
10C	10	0	7	0	7
11	6	0	1	0	1
12	9	0	2	0	2
13	10	0	2	0	2
14	5	0	1	0	1
15	13	0	1	0	1
16	8	2	2	1	5
17	12	7	3	5	3
18	5	0	2	0	2
19	7	3	5	2	6
20	14	0	2	0	2
21	10	0	2	0	2
22	10	0	2	0	2
23	6	1	4	1	2
24	10	3	4	2	6
25	7	3	5	2	6

Table 5.3

High Eccintricity Test Orbits resulting from use of Point Masses for a Circular Orbit around Earth

Characteristics

Time interval equals a quarter orbit
Point masses scaled to $(1/\mu_e)$

Test case	Point mass	Eccentricity
1	4.99999999999 E-6	2.50374472748 E5
2	2.49999999999 E-6	1.00220726562 E5
3	2.49999999999 E-6	6.96929475557 E5
4	2.49999999999 E-6	4.93695513720 E5

Results

	$-1 \leq Y \leq 1$		$-2 \leq Y \leq 2$	
Test Case	number of time steps	number of iterations	number of time steps	number of iterations
1	1	2	0	3
2	2	3	1	4
3	0	2	0	2
4	1	2	0	3

of α . The value of $\frac{9}{10}$ was superior to an α of $\frac{3}{10}$ for reasonable values of transfer angle although they were comparable for very small transfer angles. Hence, it was concluded that the value of $\frac{9}{10}$ was the proper choice, just as Gauss had determined for the more elementary problem.

5.3 Discussion of Results

The preceeding results clearly illustrate the application of this method to the solution of transfer problems for all types of orbits and for a wide range of eccentricity.

In the comparison of the two ranges of Y, the smaller range did tend to reduce the number of iterations in several instances. And, as would be expected, the smaller range increases the number of time steps but the amount of computation required in taking a time step is small. Also, the smaller range of Y requires fewer terms in the B and C_0 power series. Selection of the range of Y is largely up to the user although evidence points towards a smaller range being more benificial. Clearly there is a point of diminishing returns in reducing the allowable range of Y. Furthermore reduction of the range of Y below $\frac{1}{2}$ yields very little in the further reduction of the B and C_0 power series. It is felt that a range of Y between ±1 or possibly $\pm\frac{1}{2}$ would be the best selection.

5.4 Comparison with the Proposed NASA Shuttle Kepler Subroutine

The comparison with the proposed NASA Shuttle Shuttle subroutine shows a considerable decrease in the number of iterations in most cases.

It should be kept in mind that the amount of computation in an iteration for both methods is different. Interesting enough, this method shows a substantial decrease in both the amount of logic and computation in one iteration.

The convergence test in the Kepler subroutine was based on a relative error of 10^{-13} between the computed time in the iteration and the desired time. This error criteria was unobtainable due to the limitations of the HP 9820A model. Hence a convergence test was selected based on an absolute difference of 10^{-4} seconds. Cases were run where this error was tightened whenever possible and little or no increase was evidenced in the number of iterations. Hence the comparison of these results could be made with little or no hesitation that these results would differ significantly if the same criteria were used.

Of equal importance to the number of iterations is the accuracy of the final position and velocity vectors. Here again the calculator posed limitations. For instance, the desired time interval could not be posed to full accuracy since only 10 significant figures could be used without calculator round off. Nevertheless, with a convergence criteria of 10^{-4} seconds, accuracy of the final position and velocity vectors was at least 8 significant figures and sometimes even 10. Hence, this method is not only quicker but also quite accurate. The reason for this accuracy is due to an added feature in this algorithm. Referring to the flow chart of Fig. 5.1, the use of T, the computed time in the iteration for the convergence test, in the statement numbers 22 and 23 has the net effect, upon reaching statement 27 , of taking all the errors which remain in statement 21, on the final iteration of any number of steps, and iterating on that time interval error. Almost always this error is so small that only one iteration is required to obtain a good approximation of Y, B, and C_0 . Here again, in most cases, the limits of the calculator prevented the effectiveness of this feature from being realized since Y was quite small and the calculator set B and C_0 exactly to 1. In several cases, though, this was the reason for the increased accuracy of the final position and velocity.

CHAPTER 6

CONCLUSIONS

The method developed here for the solution of Kepler's equation for the general problem of determining final position and velocity vectors from given initial conditions for a specified time interval through the extension of Gauss' method in the standard form of position determination for time since pericenter passage also has resulted in a Picard type iteration, requiring only successive substitution. Furthermore, the form is general, thereby being applicable to all conics without knowledge of the conic encountered and at the same time is continuous during transition from one conic to another and is free from ambiguities or indeterminant forms. Also, it has proven itself to be applicable to both rectilinear motion and to all orbits of any eccentricity even in the cases where the eccentricity is very large in which case motion is nearly rectilinear. And, the resulting universal formulae, relating final position and velocity to initial values, are not only simple but are also expressed in terms of variables which have already been computed in the iteration process.

The final algorithm exhibits both simplicity and strong convergence and at the same time has very good accuracy in determining final position and velocity which results from internal correction of errors in the time interval accumulated in the iteration process.

Finally, comparison with the Apollo version of solving Kepler's equation showed not only a decrease in the number of iterations but also a reduction in the amount of logic and computation in any one iteration thereby further illustrating its simplicity and potential of finding a wide range of application in computer oriented problems and also presents itself as a simple method for hand or calculator computation.

Most important is the basic concept behind the method which was to transform Keplers equation to an equation which is nearly cubic and hence is solvable through algebraic methods, the result of which is a simple, straightforward and expedious means of obtaining the final position and velocity.

It is recommended that if increased accuracy is desired, say to 16 significant figures in which case better accuracy will be required for B and C_0 , then serious consideration should be given to using the B and C_0 power series about points other than Y = 0, since the number of terms needed in the series about Y = 0 could get quite large depending on the range of Y selected.

APPENDIX A

SERIES REVERSION

A procedure for obtaining the coefficients of the expansions of B and C_0 in powers of Y, which is useful if the coefficients are to be calculated by hand, is through the use of series reversion and the algebra of power series. The algebraic relations used here are:(1)

$$\text{If} \quad S_1 = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$S_2 = 1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

$$S_3 = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

then for

$$S_3 = S_1 S_2 \qquad c_n = b_n + a_n + \sum_{0}^{(n-1)} a_k b_{n-k}$$

$$S_3 = S_1 / S_2 \qquad c_n = a_n - b_n - \sum_{0}^{(n-1)} c_k b_{n-k}$$

$$S_1^2 = S_3 \qquad a_n = \frac{1}{2} c_n - \frac{1}{2} \sum_{0}^{(n-1)} a_k a_{n-k}$$

where $c_0 = b_0 = a_0$

The variable Y may be expanded in powers of ΔE using Eq. (2.8)

$$Y = \frac{6\ P}{Q} = \frac{60\ (\Delta E - \sin \Delta E)}{9\ \Delta E + \sin \Delta E}$$

or

$$\frac{Y}{\Delta E^2} = \frac{1 + a_1\ \Delta E^2 + a_2\ \Delta E^4 + a_3\ \Delta E^6 + \ .\ .\ .}{1 + b_1\ \Delta E^2 + b_2\ \Delta E^4 + b_3\ \Delta E^6 + \ .\ .\ .}$$

where

$$a_n = \frac{6\ (-1)^n}{(2\ n + 3)!} \qquad b_n = \frac{(-1)^n}{10\ (2\ n + 1)!} \tag{1}$$

Then dividing the two series yields

$$\frac{Y}{\Delta E^2} = 1 + c_1\ \Delta E^2 + c_2\ \Delta E^4 + c_3\ \Delta E^6 + \ .\ .\ . \tag{2}$$

where

$$c_n = a_n - b_n - \sum_{0}^{(n-1)} c_k\ b_{n-k} \tag{3}$$

The reversion theorem for a power series states that given an expansion of the form of Eq. (2) which is convergent in some interval, then if $c_1 \neq 0$ there exists one and only one function which can be expanded in the form

$$\frac{\Delta E^2}{Y} = 1 + d_1\ Y + d_2\ Y^2 + d_3\ Y^3 + d_4\ Y^4 + \ .\ .\ . \tag{4}$$

Clearly one method of obtaining the coefficients to Eq. (4) is by substituting Eq. (2) into Eq. (4) and equating powers. Unfortunately, no general expression for the n^th^ coefficient is obtainable through this process. An alternate procedure is based on the fact that

$$d_{n-1} = \frac{\left.\frac{d^n}{dY^n}(\Delta E^2)\right|_{Y=0}}{n!} \tag{5}$$

Differentiating Eq. (2) with respect to ΔE^2 and inverting gives

$$\frac{d}{dY}(\Delta E^2) = 1 + a_1\,\Delta E^2 + a_2\,\Delta E^4 + \ldots \tag{6}$$

where in general

$$a_n = -(n+1)\,c_n - \sum_{0}^{(n-1)} (k+1)\,c_k\,a_{n-k} \tag{7}$$

Now the $n^{\underline{th}}$ coefficient in Eq. (4) may be obtained as follows:

1. Compute k coefficients in Eq. (6).
2. Starting with n = 2, compute

$$\frac{d^n}{dY^n}(\Delta E^2) = \frac{d}{dY}(\Delta E^2)\ \frac{d}{d(\Delta E^2)}\left[\frac{d^{(n-1)}}{dY^{(n-1)}}(\Delta E^2)\right]$$

For example, with n = 2,

$$\frac{d^2}{dY^2}(\Delta E^2) = \frac{d}{dY}(\Delta E^2)\ \frac{d}{d(\Delta E^2)}\left[\frac{d}{dY}(\Delta E^2)\right]$$

$$= (1 + a_1\,\Delta E^2 + \ldots + a_{k-1}\,\Delta E^{2k-2})(a_1 + \ldots + k\,a_k\,\Delta E^{2k-2})$$

$$= a_1 \left(1 + b_1^2 \, \Delta E^2 + b_2^2 \, \Delta E^4 + \dots + b_i^2 \, \Delta E^{2i} + \dots\right)$$

where

$$b_i^2 = \frac{1}{a_1}\left[(i+1)\, a_{i+1} + i\, a_i\, a_1 + (i-1)\, a_{i-1}\, a_2 + \dots + a_1\, a_i\right]$$

$$1 \leq i \leq (k-1)$$

With n = 3,

$$\frac{d^3 \Delta E^2}{dY^3} = a_1 \left(1 + a_1 \, \Delta E^2 + \dots + a_{k-2} \, \Delta E^{2k-4}\right)\left(b_1^2 + 2\, b_2^2 \, \Delta E^2 + \dots + (k-1)\, b_{k-1}^2 \, \Delta E^{2k-4}\right)$$

$$= a_1\, b_1^2 \left(1 + b_1^3 \, \Delta E^2 + \dots + b_i^3 \, \Delta E^{2i} + \dots\right)$$

where

$$b_i^3 = \frac{1}{b_1^2}\left[(i+1)\, b_{i+1}^2 + \sum_{1}^{i} m\, b_m^2\, a_{i-m+1}\right] \qquad 1 \leq i \leq (k-2)$$

and in general

$$\frac{d^n \Delta E^2}{dY^n} = a_1\, b_1^2\, b_1^3 \dots b_1^{n-1} \left(1 + b_1 \, \Delta E^2 + \dots + b_i \, \Delta E^{2i} + \dots\right)$$

$$1 \leq i \leq (k + 1 - n)$$

where

$$b_i^n = \frac{1}{b_1^{n-1}}\left[(i+1)\, b_{i+1}^{n-1} + \sum_{1}^{i} m\, b_m^{n-1}\, a_{i-m+1}\right]$$

It can be seen that $1 \leq n \leq (k+1)$. At the same time

$$\left.\frac{d^n \Delta E^2}{dY^n}\right|_{Y=0} = a_1 \, b_1^2 \, b_1^3 \ldots b_1^{n-1}$$

$$= \left[\left.\frac{d^{n-1} \Delta E^2}{dY^{n-1}}\right|_{Y=0} \right] b_1^{n-1}$$

Hence, if k coefficients are calculated in Eq. (6), then k coefficients are obtainable in Eq. (4). With the series in Eq. (4) determined, B as a power series in powers of Y may be obtained since

$$B^2 = \frac{\Delta E^2}{Y} \left(1 + \frac{Y}{60}\right)^{-2}$$

Using the binomial expansion

$$\left(1 + \frac{Y}{60}\right)^{-2} = 1 + a_1 \, Y + a_2 \, Y^2 + \ldots$$

where

$$a_n = \frac{(-1)^n \, (n + 1)}{(60)^n}$$

then

$$B^2 = (1 + d_1 \, Y + d_2 \, Y^2 + \ldots) \, (1 + a_1 \, Y + a_2 \, Y^2 + \ldots)$$

$$= 1 + b_1 \, Y + b_2 \, Y^2 + b_3 \, Y^3 + \ldots$$

where

$$b_n = a_n + d_n + \sum_0^{n-1} a_i \, d_{n-i}$$

Hence

$$B = 1 + A_1 Y + A_2 Y^2 + A_3 Y^3 + \ldots$$

with

$$A_n = \frac{1}{2} b_n - \frac{1}{2} \sum_0^{n-1} b_i \, b_{n-i}$$

To obtain the C_0 power series in Y;

$$2R = 2(1 - \cos \Delta E) = \Delta E^2 \left(1 - \frac{2\,\Delta E^2}{4!} + \ldots + \frac{2\,(-1)^n \,\Delta E^{2n}}{(2\,n + 2)!} + \ldots \right)$$

and after substitution of Eq. (4) for ΔE^2 into the above series and expanding the result is

$$2R = \Delta E^2 \,(1 + b_1 Y + b_2 Y^2 + b_3 Y^3 + \ldots)$$

where the b's are the result of raising Eq. (4) to the corresponding power of ΔE^2 in the above series and adding the coefficients of similar powers in Y. Now using the definition of C_0

$$C_0 = \left(\frac{1}{B}\right) \left(\frac{\Delta E^2}{Y}\right) (1 + b_1 Y + b_2 Y^2 + b_3 Y^3 + \ldots)$$

or

$$C_0 = (1 + a_1 Y + a_2 Y^2 + \ldots)\,(1 + d_1 Y + \ldots)\,(1 + b_1 Y + \ldots)$$

where

$$a_n = - A_n - \sum_0^{n-1} a_k \, A_{n-k}$$

Hence, performing the required multiplication

$$C_0 = (1 + w_1 Y + w_2 Y^2 + \ldots) (1 + b_1 Y + b_2 Y^2 + \ldots)$$

where

$$w_n = a_n + d_n + \sum_0^{n-1} a_k d_{n-k}$$

and finally

$$C_0 = 1 + A_1 Y + A_2 Y^2 + A_2 Y^3 + \ldots$$

where

$$A_n = w_n + b_n + \sum_0^{n-1} w_k b_{n-k}$$

APPENDIX B

SERIES EXPANSION ABOUT AN ARBITRARY POINT Y_0

The methods presented earlier for obtaining B and C_0 as power series in Y about Y = 0 are no longer practical if the expansion is required about some arbitrary point Y_0. An alternate procedure can be used in this instance to obtain numerical values of the coefficients for these two series which makes use of the simplicity of the functions P, Q, and R and their derivatives and Leibnitz's formula for the differentiation of a product, which states

$$\frac{d^n}{dx^n}(u\ v) = \sum_0^n \binom{n}{k} u^{n-k}\ v^k \qquad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\ (n-k)!}$$

For convenience, let

$$P^{(i)} = \frac{d^i P}{d\ E^i} \quad E = E_0$$

be the convention for all functions, other than B and C_0, in which case

$$B^{(i,j)} = \frac{d^{i+j} B}{d\ E^i\ dY^j} \quad E = E_0$$

will be the convention

Leibnitz's formula for computing the $n^{\underline{th}}$ derivative of a function assumes that the values of the (n-1) derivatives are known. To compute

$Y^{(n)}$; from Eq. (2.8)

$$Q\ Y = 6\ P$$

Differentiating both sides n times gives

$$\sum_0^n \binom{n}{k} Q^{(n-k)}\ Y^{(k)} = 6\ P^{(n)} \qquad n = 0, 1, \ldots$$

or solving for $Y^{(n)}$

$$Q\ Y^{(n)} = 6\ P^{(n)} - \sum_0^{n-1} \binom{n}{k} Q^{(n-k)}\ Y^{(k)} \qquad n = 1, 2, \ldots \qquad (1)$$

For $\left.\frac{d^n B}{d\Delta E^n}\right|_{\Delta E_0} = B^{(n,0)}$; If we let

$$x = z^2 = B^2\ Y^2 = 6\ P\ Q \qquad (2)$$

Then differentiating $x = z^2$ gives

$$x^{(n)} = \sum_0^n \binom{n}{k} z^{(n-k)}\ z^{(k)} \qquad n = 0, 1, \ldots \qquad (3)$$

but $x = 6\ P\ Q$, hence

$$x^{(n)} = 6 \sum_0^n \binom{n}{k} P^{(n-k)}\ Q^{(k)} \qquad n = 0, 1, \ldots \qquad (4)$$

Equating Eqs. (3) and (4) gives, when n = 1

$$2\ z\ z^{(1)} = 6\ (P^{(1)}\ Q + P\ Q^{(1)}) \qquad (5)$$

and for $n \geq 2$,

$$2\,z\,z^{(n)} = 6\sum_{0}^{n}\binom{n}{k} P^{(n-k)}\,Q^{(k)} - \sum_{1}^{n-1}\binom{n}{k} z^{(n-k)}\,z^{(k)} \tag{6}$$

And since $z = B\,Y$, differentiating with respect to E gives

$$z^{(n)} = \sum_{0}^{n}\binom{n}{k} B^{(n-k,0)}\,Y^{(k)}$$

$$= \sum_{1}^{n}\binom{n}{k} B^{(n-k,0)}\,Y^{(k)} + B^{(n,0)}\,Y$$

or

$$Y\,B^{(n,0)} = z^{(n)} - \sum_{1}^{n}\binom{n}{k} B^{(n-k,0)}\,Y^{(k)} \qquad n = 1, 2, \ldots \tag{7}$$

Now $B^{(0,n)} = \left.\frac{d\,B}{dY^n}\right|_{\Delta E_0}$ may be determined using the chain rule

$$B^{(1,j)} = \frac{d}{d\Delta E}\left.\frac{d^jB}{dY^j}\right|_{\Delta E_0} = Y^{(1)}\,B^{(0,j+1)} \tag{8}$$

Differentiating i times with respect to ΔE gives

$$B^{(i,j)} = \sum_{0}^{i-1}\binom{i-1}{k} Y^{(i-k)}\,B^{(k,j+1)} \qquad i = 2, 3, \ldots$$

or

$$Y^{(1)}\,B^{(i-1,j+1)} = B^{(i,j)} - \sum_{0}^{i-2}\binom{i-1}{k} Y^{(i-k)}\,B^{(i,j+1)} \tag{9}$$

If j is replaced by n-1 in Eq. (8) then

$$B^{(1,n-1)} = Y^{(1)}\,B^{(0,n)} \tag{10}$$

hence, $B^{(1,n-1)}$ must be determined. The procedure for doing this can be seen by expanding Eq. (9) for several values of n:

For n = 1, $B^{(1,0)}$ is determined from Eq. (7)

from Eq. (10) $B^{(1,0)} = Y^{(1)} B^{(0,1)}$

For n = 2, $B^{(2,0)}$ is determined from Eq. (7)

In Eq. (9)

For i=2; j=0 $Y^{(1)} B^{(1,1)} = B^{(2,0)} - Y^{(2)} B^{(0,1)}$

From Eq. (10) $B^{(1,1)} = Y^{(1)} B^{(0,2)}$

For n = 3, $B^{(3,0)}$ is determined from Eq. (7)

In Eq. (9)

For i=3; j=0 $Y^{(1)} B^{(2,1)} = B^{(3,0)} - Y^{(3)} B^{(0,1)} - 2\, Y^{(2)} B^{(1,1)}$

For i=2; j=1 $Y^{(1)} B^{(1,2)} = B^{(2,1)} - Y^{(2)} B^{(0,2)}$

From Eq. (10) $B^{(1,2)} = Y^{(1)} B^{(0,2)}$

Hence in general for n $\geq$ 2, in Eq. (9)

$$Y^{(1)} B^{(n-j-1,\, j+1)} = B^{(n-j,j)} - \sum_{0}^{n-j-2} \binom{n-j-1}{k} Y^{(n-j-k)} B^{(k,\, j+1)}$$

$$j = 0, 1, \ldots, n-2 \qquad (11)$$

To determine $\left.\dfrac{d^n C_0}{d\Delta E^n}\right|_{\Delta E_0}$,

using Eq. (2.9) and Eq. (2)

$$z\,C_0 = 2\,R \tag{12}$$

Differentiating n times with respect to ΔE gives

$$\sum_0^n \binom{n}{k} C_0^{(k,0)}\, z^{(n-k)} = 2\,R^{(n)}$$

or

$$z\,C_0^{(n,0)} = 2\,R^{(n)} - \sum_0^{n-1} \binom{n}{k} C_0^{(k,0)}\, z^{(n-k)} \qquad n = 1,\ 2, \ldots \tag{13}$$

and now $\left.\dfrac{d^n C_0}{dY^n}\right|_{\Delta E_0}$ is obtained using Eqs. (8), (10) and (11) where B is simply replaced by C_0.

To summarize the procedure of determining the coefficients of the B and C_0 power series in Y about some arbitrary point Y_0; To calculate m coefficients to these series the sequence is as follows

1. Evaluate the variables P, Q and R and their m derivatives at the point ΔE_0 (ΔH_0).
2. Evaluate

$$Y^{(0)} = \frac{6\,P^{(0)}}{Q^{(0)}} \qquad B^{(0,0)} = \frac{Q^{(0)}}{Y^{(0)}} \qquad C_0^{(0,0)} = \frac{2\,R^{(0)}}{Y^{(0)}\,B^{(0,0)}}$$

3. For $n = 1$, evaluate

$$Y^{(1)} = (6\,P^{(1)} - Q^{(1)}\,Y^{(0)})/Q^{(0)}$$

$$z^{(1)} = 3\,(P^{(1)}\,Q^{(0)} + P^{(0)}\,Q^{(1)})/z^{(0)}$$

$$B^{(1,0)} = (z^{(1)} - B^{(0,0)}\,Y^{(1)})/Y^{(0)}$$

$$B^{(0,1)} = B^{(1,0)}/Y^{(1)}$$

$$C_0^{(1,0)} = (2\,R^{(1)} - C_0^{(0,0)}\,z^{(1)})/z^{(0)}$$

$$C_0^{(0,1)} = C_0^{(1,0)}/Y^{(1)}$$

4. For $n = 2, 3, 4, \ldots, m$

$$Q^{(0)}\,Y^{(n)} = 6\,P^{(n)} - \sum_{0}^{n-1} \binom{n}{k} Q^{(n-k)}\,Y^{(k)}$$

$$2\,z^{(0)}\,z^{(n)} = 6 \sum_{0}^{n} \binom{n}{k} P^{(n-k)}\,Q^{(k)} - \sum_{1}^{n-1} \binom{n}{k} z^{(n-k)}\,z^{(k)}$$

$$Y^{(0)}\,B^{(n,0)} = z^{(n)} - \sum_{1}^{n} \binom{n}{k} B^{(n-k,0)}\,Y^{(k)}$$

$$z^{(0)}\,C_0^{(n,0)} = 2\,R^{(n)} - \sum_{0}^{n-1} \binom{n}{k} C_0^{(k,0)}\,z^{(n-k)}$$

For $j = 0, 1, \ldots, (n-2)$

$$Y^{(1)}\,B^{(n-j-1,j+1)} = B^{(n-j,j)} - \sum_{0}^{n-j-2} \binom{n-j-1}{k} Y^{(n-j-k)}\,B^{(k,j+1)}$$

For j = 0, 1, ..., (n-2)

$$Y^{(1)} C_0^{(n-j-1)} = C_0^{(n-j,j)} - \sum_{0}^{n-j-2} \binom{n-j-1}{k} Y^{(n-j-k)} C_0^{(k,j+1)}$$

$$Y^{(1)} B^{(0,n)} = B^{(1,n-1)}$$

$$Y^{(1)} C_0^{(0,n)} = C_0^{(1,n-1)}$$

$$a_n = B^{(0,n)}/(n!) \qquad b_n = C_0^{(0,n)}/(n!)$$

where

$$B = 1 + \sum_{1}^{m} a_n (Y - Y_0)^n \qquad C_0 = 1 + \sum_{1}^{m} b_n (Y - Y_0)^n$$

Table B.1 lists the coefficients of the B and C_0 power series expanded about the points .5, 1.0, 1.5, 2.0. Table B.2 lists the coefficients for the points -.5, -1.0, -1.5, -2.0. In both tables the number of significant figures drops from 29 for the zeroth coefficient to 15 for the 15th. It will be hence safe to assume that the coefficients from 1 to 5 are correct to 25 figures; coefficients 6 through 10 are correct to 20 figures and those from 11 through 15 correct to 15 figures. Finally, it is important to take note of the fact that the sign of Y has not been accounted for in Table B.2. Hence for Y < 0, the sign of the odd powered coefficients must be changed.

Table B.1

B and C_0 series coefficients for $Y > 0$

Y_0 = 4.99920793653689037165097540983B-1

n	a_n
0	1.00027556629659726644430289588B0
1	1.11877805591153866712861977927B-3
2	1.16954380813962351712854294901B-3
3	7.1786525665411241568329493859B-5
4	6.87203163064854247174723199308B-6
5	6.52028433859334612908012003632B-7
6	6.56822195689472748648596925367B-8
7	6.82430353790162025933937716942B-9
8	7.27487389418031529408518296883B-10
9	7.90986039151248332596825027154B-11
10	8.73854370396782131647310642643B-12
11	9.78131643809698547504807697806B-13
12	1.10691164586115839767865058638B-13
13	1.26434527871678208599139086472B-14
14	1.45575149944960705844895462873B-15
15	1.68780150807908525717728459312B-16

n	b_n
0	9.74636118338569407231225165762B-1
1	- 5.14940651621667810950204247743B-2
2	- 1.56345074645680626916714147211B-3
3	- 9.70226638716167223960822394225B-5
4	- 7.61339746813898399539955168998B-6
5	- 6.73014761036327286623953750243B-7
6	- 6.39654905195183275317675747415B-8
7	- 6.38327063905106551650921467205B-9
8	- 6.59728111620994056153364482l6B-10
9	- 7.00096514595267222395201086224B-11
10	- 7.58408812770680620509108354011B-12
11	- 8.35274547649791652536794537l1B-13
12	- 9.32484402318727311063977333134B-14
13	- 1.05285071016817904803180005289B-14
14	- 1.20020131572573363999654808812B-15
15	- 1.37947026856651413829240253971B-16

Table B.1 Continued

Y_0 = 1.05631489201046008637152731699B0

n	a_n
0	1.00127316894124718653016167973B0
1	2.49197208826952211342132528528B-3
2	1.30335884274550136593427291279B-3
3	8.93508182056833183246520896056B-5
4	9.03757154796369830595426307523B-6
5	9.2376959466027251865330977384B-7
6	9.9915005631126676555239469581B-8
7	1.11609568524772285064902702864B-8
8	1.27938068378296321241441892018B-9
9	1.49619266525570159622529580109B-10
10	1.77817384248164148134169809988B-11
11	2.14143889357880898372004591775B-12
12	2.60758448726320558370280983698B-13
13	3.20511063407351690963263294498B-14
14	3.97140891052001117892837494838B-15
15	4.95543517221269086567566830471B-16

n	b_n
0	9.45483640826002146714229205519B-1
1	- 5.33295517655410408139391338996B-2
2	- 1.74079966745741804312355909275B-3
3	- 1.16294270187351996292626555513B-4
4	- 9.82616640226647326695592072312B-6
5	- 9.35376604257074116845604273967B-7
6	- 9.57344128597197692284716116376B-8
7	- 1.02877828012135613333092218354B-8
8	- 1.14496628535256265305616531547B-9
9	- 1.30836048419997795077262094159B-10
10	- 1.52618687778239016912829272615B-11
11	- 1.80993163746772264552166926718B-12
12	- 2.17569266206407314316875056524B-13
13	- 2.64509309774744925325001476404B-14
14	- 3.24670043914499550314204241958B-15
15	- 4.01800716304420496814271988482B-16

Table B.1 Continued

Y_0 = 1.4999931813019943176174666512 2B0

n	a_n
0	1.00264354023124009372503252719B0
1	3.70462959444615755062606734016B-3
2	1.43383139840157634613145688816B-3
3	1.07399360273910071898592366039B-4
4	1.14198215623961765058363062072B-5
5	1.24296137665114913642974485988B-6
6	1.42891252220445879853000744177 3B-7
7	1.69800869398863291689806009554B-8
8	2.07101534724894995366879055497B-9
9	2.57754536882356480622830715397B-10
10	3.26053735550372587996722039835B-11
11	4.17988044937423972803911734643B-12
12	5.41846352077061188379271480007B-13
13	7.09070175236391848909375975125B-14
14	9.35454057011311924796495381515B-15
15	1.24282598042311799289010476183B-15

n	b_n
0	9.21469244824267931208526675571B-1
1	− 5.49465640202236387482398574626B-2
2	− 1.90807366950267960015672685464B-3
3	− 1.35756561048117852843117785796B-4
4	− 1.22187421519529818204440317047B-5
5	− 1.23906844272228693842837008753 6B-6
6	− 1.35096653326951415234924180879B-7
7	− 1.54653897469038011596684788025B-8
8	− 1.83352737792592286710082244117B-9
9	− 2.23187977606480466013455407424B-10
10	− 2.77327806759923950996279559587B-11
11	− 3.50335846782929479050102959909B-12
12	− 4.48592236347018872785598706352B-13
13	− 5.80928537498712011794216819577B-14
14	− 7.59535042014752452035294742503B-15
15	− 1.00123709835241698326897581622B-15

Table B.1 Continued

Y_0 = 2.01043263543128048366832781608B0

n	a_n
0	1.00492321636159559555244765834B0
1	-1.54460970715929933875310911962B-1
2	5.47847550051100681389637110604B-2
3	- 2.21652988998282340332899850739B-2
4	9.53734566710033770280417122913B-3
5	- 4.23319514878923382326436478741B-3
6	1.91757473333574616864703698985B-3
7	- 8.81008246961441937749021702636B-4
8	4.09020980342156494717111730107B-4
9	- 1.91424139185167440048919403913B-4
10	9.01590609626623724183449085328B-5
11	- 4.26838636065894883260317498512B-5
12	2.02942917509888686752677451978B-5
13	- 9.68379711490458084034825235047B-6
14	4.63499844390808972227815900763B-6
15	- 2.22435043545974353905128509B-6

n	b_n
0	8.92906275284140892833654870823B-1
1	2.02763846453686228582203648149B-3
2	- 1.27696096745085217121869752328B-2
3	4.04307094638551618975216783925B-3
4	- 1.53105597286445110987185956798B-3
5	6.07021867967787597749757688578B-4
6	- 2.50420721102010438735608275647B-4
7	1.06116737624290910059617380613B-4
8	- 4.58881365125874689935191525423B-5
9	2.01568795385501870694289733598B-5
10	- 8.965589777798818557597122166105B-6
11	4.02873533884031373329369649411B-6
12	- 1.82577446603926687335858842478B-6
13	8.33377703174821292428294732209B-7
14	- 3.82740597107953460468838414015B-7
15	1.76716486189089742379333847142B-7

Table B.2

B and C_0 series coefficients for $Y < 0$

Y_0 = -4.99920793520485411987906408295B-1

n	a_n
0	1.00026066197468537939753602349B0
1	1.02921589292810133110808083643B-3
2	9.89779551716639081000171008884B-4
3	- 4.97116749287700354874317360708B-5
4	4.40108115063861797912742695581B-6
5	- 3.68679995324749193091788058081B-7
6	3.31082880926888281322773107638B-8
7	- 3.05641032596164985832988105746B-9
8	2.89495201851904472138726002951B-10
9	- 2.79532093137894114971756077949B-11
10	2.74178221283150258953865794217B-12
11	- 2.72411352911327179444871964417B-13
12	2.73590325735494340547940311447B-14
13	- 2.77302672639601154887752781931B-15
14	2.83288044919813681521146406491B-16
15	- 2.91390664164797642058818307112B-17

n	b_n
0	1.02464905308140179325054949036B0
1	4.86312150553980815754911322832B-2
2	- 1.31220899695435361234084905495B-3
3	7.22146551498396926447166221836B-5
4	- 5.02341972311691449936301607113B-6
5	3.93590810636812675530613112253B-7
6	- 3.31553024869202216494483439474B-8
7	2.93251404599881849827444778969B-9
8	- 2.68633453030940462735161086377B-10
9	2.52674535414594297830270071147B-11
10	- 2.42619376846459509420782030456B-12
11	2.36853212051101218925654119758B-13
12	- 2.34384006933099082090211122208B-14
13	2.34583295262263349472673335914B-15
14	- 2.37047259097064508957890292473B-16
15	2.41518074624473216800749914377B-17

Table B.2 Continued

Y_0 = -1.0563148923596519121895253947B0

n	a_n
0	1.0011315624811341735101861383B0
1	2.0873281041087005055044604211 2B-3
2	9.1438583902785514986648115982 8B-4
3	- 4.0953468670618284854659191661 2B-5
4	3.5125106124367730908927027294B-6
5	- 2.7553408639402452385882194570 1B-7
6	2.3358379532541385091799358957 8B-8
7	- 2.0302679082680603162822376614 1B-9
8	1.8110152986396160545655520461 3B-10
9	- 1.6463183645296716754148914011 9B-11
10	1.5200632675912574340256654450 4B-12
11	- 1.4215046368743074260045474642B-13
12	1.3436291942774247317366297126 5B-14
13	- 1.2816133107594229320375351090 5B-15
14	1.2320550082562734656315628137 5B-16
15	- 1.1924920366367075493113910822 6B-17

n	b_n
0	1.0513129251170215943497924253 9B0
1	4.7234789321154506373955566124B-2
2	- 1.2003669513188184271839014711 1B-3
3	6.2147829950845884147978046991B-5
4	- 4.0663895719942767025497473447 6B-6
5	2.9965508851566109227884618494 2B-7
6	- 2.3740395851738772768531018187 2B-8
7	1.9748432318147993691972441215 7B-9
8	- 1.7014267307998052106878051434 3B-10
9	1.5051490803244472019288870230 7B-11
10	- 1.3592924430554302949410652988 2B-12
11	1.2480741662387296928037160038 7B-13
12	- 1.1616271950282664557845948676 1B-14
13	1.0934959235692898116589397814 5B-15
14	- 1.0392967023663464899664352703 2B-16
15	9.9595944876738169151072483003 5B-18

Table B.2 Continued

Y_0 = - 1.49999318228542225920052038385B0

n	a_n
0	1.00223421662569913244859373925B0
1	2.87570529899299222402581591874B-3
2	8.63796213598232649483717298875B-4
3	- 3.52239474389793055571675613674B-5
4	2.96449141387002452502332183632B-6
5	- 2.20933798233012276124467950156B-7
6	1.79416240404420658111105113476B-8
7	- 1.48984870856662119245157862601B-9
8	1.27010891826803485178124975209B-10
9	- 1.10315355050486212841262514242B-11
10	9.73083823768386704187405174304B-13
11	- 8.69284463810276267765198183131B-14
12	7.84853377037132117707805709334B-15
13	- 7.15051465062561356769678522547B-16
14	6.56541204814375333534047212148B-17
15	- 6.06907915474831560590067580571B-18

n	b_n
0	1.07203895822449035520642292111B0
1	4.62049724840888113060279222222B-2
2	- 1.12220004914252999700323994177B-3
3	5.54822350252313889551413973233B-5
4	- 3.46613106670203016905134582051B-6
5	2.43856074974759899779653335158B-7
6	- 1.84445467084128299651873949898B-8
7	1.46480328755097733626810919573B-9
8	- 1.20483698213257108089811344052B-10
9	1.01757327422327407781271714902B-11
10	- 8.77350857095793998197745367377B-13
11	7.69091605368031532412258761486B-14
12	- 6.83415117303380434104704284526B-15
13	6.14211196815404983612093372584B-16
14	- 5.57346435641755083914395081736B-17
15	5.09934719495745558610508264009B-18

Table B.2 Continued

Y_0 = -2.01043263634649869748277864184B0

n	a_n
0	1.003922660135218550762684121 64B0
1	3.73150960527407278060975353576B-3
2	8.14214913581495502497743624384B-4
3	- 2.97023996454011018406673331318B-5
4	2.46436852638318960720009560 87B-6
5	- 1.73277438527933825395445647712B-7
6	1.34342305048743013203743512703B-8
7	- 1.06089281064664847609505220622B-9
8	8.60690610742836534683470804 63B-11
9	- 7.11122961239505800065045840235B-12
10	5.96663479349480515127955157556B-13
11	- 5.06947884230815765928848269895B-14
12	4.35290976372222710037189411179B-15
13	- 3.77129883136074089150755857356B-16
14	3.29273111868663222015579847152B-17
15	- 2.89427375976191729413909303847B-18

n	b_n
0	1.09533856333816603062186046174B0
1	4.51009447603430057094807815374B-2
2	- 1.04235111199156734458214680735B-3
3	4.89947984504316660100575618981B-5
4	- 2.90956051564959829782233241207B-6
5	1.94564462415510159332014308868B-7
6	- 1.39873652702463467702062480399B-8
7	1.05580251251383733313101734876B-9
8	- 8.25407470276356947223558870 05B-11
9	6.62590372620908740174331387146B-12
10	- 5.42994128460081994511883702642B-13
11	4.52423770095716487236443753 81B-14
12	- 3.82120563828076951732866060 36B-15
13	3.26426038064105013206919339 16B-16
14	- 2.91543845670845145715487072749B-17
15	2.44845096406449232428253442966B-18

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