SYMMETRY PRINCIPLES IN SELECTED PROBLEMS OF FIELD THEORY

by

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B.S., Massachusetts Institute of Technology
(June 1973)

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ABSTRACT

The importance of understanding the symmetries of nature has been increasingly realized in the use of quantum field theory as a description of nature. These symmetry principles and the conservation laws which arise from them are often well understood before dynamics. Within this thesis several problems in which symmetry principles play a prominent role are investigated.

The problem of constructing the most general, scalar potential for an arbitrary compact, semisimple Lie group, G, is solved. The technique which is derived is then applied to the specific model of the weak and electromagnetic interactions. The scalar potential for this model is constructed and analyzed. This analysis points out the existence of a possible pseudosymmetry.

A new model is then constructed which incorporates the pseudosymmetry and extends the earlier model to include the hadrons. This new model is anomaly-free and offers some ideas as to the role which spin-0 exchange may play in the weak interaction.

The discussion of the weak interaction ends with a possible explanation for the $\Delta I=1/2$ rule among the hadrons and the anomalous strength of the weak non-leptonic decays.

The thesis then turns to a brief review of the new symmetry principle known as supersymmetry.

At the completion of the review, the spinor superfield is studied. The implications of this study are then used to derive a suitable Lagrangian for the superfield. Finally a discussion of supersymmetry and gauge invariance of the internal type are given. This discussion also points out the similarity of local invariance in both superspace and ordinary spacetime.

Finally, we study the geometry of global superspace. This study indicates that global superspace is a metric space, in the differential geometric sense, which posseses constant torsion and zero curvature. The study also indicates that a theory of curved superspace may be constructed as the superspace generalization of Einstein's unified field theory. This generalization is performed and it is shown that a non-Riemannian, superspace version of general relativity exists as a special case.

Thesis Supervisor: James E. Young
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Biography

The candidate was born on December 15, 1950 in Tampa, Florida to Mr. and Mrs. Sylvester J. Gates Sr. He attended several elementary schools in Texas and Florida. It was necessary for the family to move about since Mr. Gates, Sr. was in the army.

On May 24, 1969 the candidate graduated valedictorian from Jones High School in Orlando, Florida. He gave an address entitled "The Seeker" to the graduating class.

The following September, he was enrolled as a freshman at the Massachusetts Institute of Technology in Cambridge, Massachusetts. Four years later on June 1, 1973, he received a Bachelor of Science degree in applied mathematics. On September 12 of the same year, he received a second Bachelor's degree in physics. By this time he had begun his graduate education in the physics department of M.I.T.

As of this date, the candidate is expected to receive his doctorate degree in physics on June 1, 1977.

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How does one acknowledge the continuum that is one's life?

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I. On the Construction of Gauge Invariant, Scalar Potentials

Lagrangians which possess gauge invariance with respect to some compact, semisimple Lie group, G, have become objects of much study. In a theory where some or all of the symmetries are broken by the vacuum expectation values of elementary spin-0 fields, the scalar potential plays a crucial role. It is the purpose of this comment to describe a procedure for the construction of the most general, gauge-invariant, scalar potential for a given theory.

For simplicity, let all of the scalar fields in a theory be assembled into a real, n-component multiplet denoted by $\vec{\phi}$. Let t_α denote a $n\times n$ matrix representation of the α-th generator of the gauge group. Since the spin-0 multiplet is real, it follows that the matrices to t_α satisfy the following relations,

$$t_\alpha = t_\alpha^\dagger \qquad\qquad t_\alpha = -t_\alpha^*$$

$$[t_\alpha, t_\beta] = i f_{\alpha\beta}{}^{\gamma} t_\gamma \qquad (\alpha,\beta,\gamma = 1,\ldots,p) \tag{1.1}$$

The structure constants, $f^{\gamma}_{\alpha\beta}$, form a real, totally antisymmetric tensor. We may regard the transformation properties of $\vec{\phi}$ as arising, solely, from the transformation properties of the cononical basis elements denoted by $\hat{e}_\ell$. Indeed since the canonical basis is complete, we must have a relationship of the form

$$t_\alpha \hat{e}_\ell = i h_{\alpha\ell}{}^{\ell'} \hat{e}_{\ell'} \qquad (\ell,\ell' = 1,\ldots,n) \tag{1.2}$$

for some set of coefficients $h^{\ell'}_{\alpha\ell}$.

We may continue by considering the set of all second order tensors, $\{T_A\}$, which transform irreducibly under the action of the group. Once again, the completeness of this basis must imply that,

$$[t_\alpha, T_A] = i\, H_{\alpha A}{}^{A'} T_{A'} \tag{1.3}$$

Furthermore, the set $\{T_A\}$ may be written in the form

$$\{T_A\} = \{\mathcal{T}^1\} \cup \{\mathcal{T}^2\} \cup \cdots \cup \{\mathcal{T}^m\} \tag{1.4}$$

where each subset $\{\mathcal{T}^f\}$, f=1, ... , m, is invariant under gauge transformations. This implies that the coefficients $H^{A'}_{\alpha A}$ must only connect elements within the same subset. Also, the elements within a subset must transform as the members of a definite representation of the gauge group. The coefficients, $H^{A'}_{\alpha A}$, are obviously functions of the subsets $\{\mathcal{T}^f\}$. We could, therefore, display the dependence by writing these coefficients in the form

$$H_{\alpha A}{}^{A'}\{\mathcal{T}^f\} \tag{1.5}$$

These coefficients indicate to which representation of the group the subset $\{\ {}^f\}$ belongs.

Now our problem is to construct the most general polynomial mapping, U, such that $U:R^n \to R^1$ and $[U,G]=0$. In order to achieve our goal, we first introduce a set of group, bilinear covariants, Γ_A, defined by

$$\Gamma_A \equiv \vec{\Phi} \cdot T_A \cdot \vec{\Phi} \tag{1.6}$$

The requirement of renormalizability forces the mapping to be no more than quartic in the field $\vec{\phi}$. The only possible form that the quartic terms may have is

$$\tfrac{1}{8}\, a^{AB}\, \vec{\Phi} \cdot \Gamma_A \cdot \vec{\Phi}\;\; \vec{\Phi} \cdot \Gamma_B^{\dagger} \cdot \vec{\Phi} \tag{1.7}$$

where a is a dimensionless, hermitian matrix. If this is subjected to an infinitesimal gauge transformation, then the first order change is proportional to the expression

$$\frac{1}{8} a^{AB} (\mathcal{J}_\alpha)_{AB}^{A'B'} \vec{\Phi} \cdot T_{A'} \cdot \vec{\Phi} \, \vec{\Phi} \cdot T_{B'}^{\dagger} \cdot \vec{\Phi} \tag{1.8}$$

where

$$(\mathcal{J}_\alpha)_{AB}^{A'B'} \equiv [\, H_{\alpha A}^{\;\;A'} \delta_B^{B'} + (H_{\alpha B}^{\;\;B'})^* \delta_A^{A'} \,] \tag{1.9}$$

Clearly, in order for equation (1.7) to be invariant, we must require that

$$(\mathcal{J}_\alpha)_{AB}^{A'B'} a^{AB} = 0 \tag{1.10}$$

independent of the values of α, A', and B'. But in equation (1.7), we need not let the summation on A and B take on all of the values that are consistent with equation (1.10). In other words, there are essential subsets of $\{T_A\}$ which only need be considered in equation (1.7). To show this we consider the following. The elements of $\{T_A\}$ may be chosen so that the equation

$$\mathrm{Tr}\{ T_A T_B^{\dagger} \} = k_o \, \delta_{AB} \tag{1.11}$$

is satisfied for some constant k_o. This statement together with the completeness of the basis provided by $\{T_A\}$ implies

$$\delta_{im} \delta_{nj} = (k_o)^{-1} \sum_A (T_A)_{ij} (T_B^{\dagger})_{nm} \tag{1.12}$$

We may use this statement to derive others in the same fashion that the usual Fierz identities are derived. These group Fierz identities when fully contracted with $\vec{\phi}$ can then be used to express some couplings as linear combinations of others. In this way, we can see that the range of A and B may be restricted even further than implied by equation (1.10).

Next in the mapping, U, are the terms which are cubic in the scalar field. The presence or absence of such terms depends on whether the relation

$$H_{\alpha A}^{\ \ A'}\{\mathfrak{T}^f\} = \mathcal{h}_{\alpha A}^{\ \ A'} \tag{1.13}$$

can be satisfied for any of the subsets $\{\mathfrak{T}^f\}$. If this relation is satisfied then we may have cubic terms of the form

$$M\, \mathcal{B}^{A\ell}\, \vec{\Phi}\cdot\hat{e}_\ell\, \vec{\Phi}\cdot T_A\cdot\vec{\Phi} \; + \; h.c. \tag{1.14}$$

where M is a dimensional constant with the units of mass. We restrict the values of A so that the elements T_A are members of a subgroup $\{\mathfrak{T}\}$ which satisfies equation (1.13). Once again we may derive the condition for invariance which is given by

$$(\mathcal{K}_\alpha)_{A\ell}^{A'\ell'}\, \mathcal{B}^{A\ell} = 0$$

$$(\mathcal{K}_\alpha)_{A\ell}^{A'\ell'} \equiv [\; H_{\alpha A}^{\ \ A'}\,\delta_\ell^{\ell'} + \mathcal{h}_{\alpha \ell}^{\ \ \ell'}\,\delta_A^{A'}\;] \tag{1.15}$$

Since the number of subsets which satisfy equation (1.13) may not provide a complete basis, in general we need not find identities which play the role of the group Fierz identities.

Finally, we come to quadratic terms in the mapping. These terms will be present whenever any of the subsets $\{\mathfrak{T}^f\}$ satisfy the equation

$$H_{\alpha A}^{\ \ A'}\{\mathfrak{T}^f\} = 0 \tag{1.16}$$

The subset containing the identity always satisfies this equation. The quadratic terms are then given by the expression

$$\tfrac{1}{2} M^2 \mathcal{C}^A\, \vec{\Phi}\cdot T_A\cdot\vec{\Phi} \tag{1.17}$$

where, of course, T_A is an element of one of the subsets which satisfy equation (1.16). Without loss of generality we may assume that the coupling tensors, T_A, which appear in equation (1.17) are hermitian. Therefore, the coupling vector

C is real and dimensionless.

The full potential is simply the sum of equations (1.6), (1.14), and (1.17).

$$\mathcal{U}(\vec{\Phi}) = \tfrac{1}{8}\,\mathcal{A}^{AB}\,\vec{\Phi}\cdot T_A\cdot\vec{\Phi}\;\vec{\Phi}\cdot T_B^{\dagger}\cdot\vec{\Phi} + M\,\mathcal{B}^{A\ell}\,\vec{\Phi}\cdot\hat{e}_\ell\;\vec{\Phi}\cdot T_A\cdot\vec{\Phi} + h.c. + \tfrac{1}{2}M^2\,\mathcal{C}^A\,\vec{\Phi}\cdot T_A\cdot\vec{\Phi} \tag{1.18}$$

As examples of the method described in the preceding section, we will treat three scalar systems. The three systems are the Higgs model with a reducible scalar system, the Weinberg-Salam scalar system, and the Georgi-Glashow scalar system. Of course, these systems are so simple that the use of the construction procedure is purely pedagogical.

The Higgs Model has U(1) as its gauge group. The single group generator in the reducible representation that we are interested in is given by,

$$t = I \otimes \sigma^2 \tag{1.19}$$

The set of all coupling tensors may be defined once we define a basis given by;

$$\begin{aligned} \mathcal{S} &= \tfrac{1}{\sqrt{2}}\left[I \otimes (\sigma^3 + i\sigma^1)\right] \quad ; \qquad t \\ \vec{\mathcal{F}} &= \vec{\sigma} \otimes I \qquad\qquad\qquad ; \qquad I \otimes I \end{aligned} \tag{1.20}$$

Now we may define the full set of coupling tensors.

$$\{T_a\} \equiv \{I, t, \vec{\mathcal{F}}, t\vec{\mathcal{F}}\} \cup \{\mathcal{S}, \mathcal{S}\vec{\mathcal{F}}\} \cup \{\mathcal{S}^{\dagger}, \mathcal{S}^{\dagger}\vec{\mathcal{F}}\} \tag{1.21}$$

One may easily verify that the relation

$$\mathrm{Tr}\{T_a T_b^{\dagger}\} = 4\,\delta_{ab} \tag{1.22}$$

is satisfied. This will lead to the identity

$$(\mathbb{I})_{in}(\mathbb{I})_{mj} = \tfrac{1}{4}[\ \mathbb{I}_{ij}\,\mathbb{I}_{mn} + t_{ij}\,t_{mn} + (\vec{\mathcal{F}})_{ij}\cdot(\vec{\mathcal{F}})_{mn} + (t\vec{\mathcal{F}})_{ij}\cdot(t\vec{\mathcal{F}})_{mn} + \mathcal{S}_{ij}\,\mathcal{S}^{\dagger}{}_{mn} + \mathcal{S}^{\dagger}{}_{ij}\,\mathcal{S}_{mn} + (\mathcal{S}\vec{\mathcal{F}})_{ij}\cdot(\mathcal{S}^{\dagger}\vec{\mathcal{F}})_{mn} + (\mathcal{S}^{\dagger}\vec{\mathcal{F}})_{ij}\cdot(\mathcal{S}\vec{\mathcal{F}})_{mn}\ ] \tag{1.23}$$

which can be used to derive other group Fierz identities. These identities can then be used to show that one essential set of symmetric tensors is given by $\{T_A\}_\varepsilon$ where

$$\{T_A\}_\epsilon \equiv \{\, I\,,\ f^1,\ t f^2,\ f^3 \} \tag{1.24}$$

These may be contracted with the field to form the following group bilinear covariants.

$$\Gamma_0 \equiv |\vec{\Phi}|^2 \qquad \Gamma_1 \equiv \vec{\Phi}\cdot[f^1]\cdot\vec{\Phi}$$

$$\Gamma_2 \equiv \vec{\Phi}\cdot[t f^2]\cdot\vec{\Phi} \qquad \Gamma_3 \equiv \vec{\Phi}\cdot[f^3]\cdot\vec{\Phi} \tag{1.25}$$

Thus, the scalar potential is simply given by the expression,

$$U(\vec{\Phi}) = \tfrac{1}{8}\, a^{AB}\,\Gamma_A\,\Gamma_B + \tfrac{1}{2} M^2 C^A\,\Gamma_A \tag{1.26}$$

where a is a real, symmetric, 4×4 "matrix" and C a real "vector".

The gauge group for the Weinberg-Salam model is U(1)⊗SU(2). The generators of the gauge group may be taken to be

$$\tfrac{1}{2}\vec{t}_w = \tfrac{1}{2}(\sigma^2\otimes\sigma^1,\ I\otimes\sigma^2,\ \sigma^2\otimes\sigma^3)$$

$$t_Y = \sigma^2\otimes I \tag{1.27}$$

These generators satisfy the usual commutator algebra

$$[t^i_w, t^j_w] = i2\epsilon^{ijk} t^k_w \qquad [t_Y, \vec{t}_w] = 0 \tag{1.28}$$

The coupling tensors may be defined once we define one other matrix, S. The definition of S is

$$S \equiv \frac{-i}{\sqrt{2}}(\sigma^3 + i\sigma^1) \otimes \sigma^2 \tag{1.29}$$

Thus the coupling tensors are given by:

$$\{T_a\} = \{\mathbb{I}, t_Y\} \cup \{S\} \cup \{S^\dagger\} \cup \{\vec{t}_w\} \cup \{\vec{t}_w t_Y\} \cup \{\vec{t}_w S\} \cup \{\vec{t}_w S^\dagger\} \tag{1.30}$$

Once again there is a completeness relation which may be used to derive a number of gauge, Fierz identities. From these identities we may show that an essential set is given by $\{\mathbb{I}, t_Y, S, S^+\}$ but only the identity is symmetric. Thus,

$$U(\vec{\Phi}) = \frac{1}{8}\lambda_1 |\vec{\Phi}|^4 + \frac{1}{2}M^2\lambda_2 |\vec{\Phi}|^2 \tag{1.31}$$

is the most general, gauge-invariant potential.

For the Georgi-Glashow model, the group 0(3) is chosen. The generators may be represented by:

$$(t^i)^j{}_k = -i\epsilon^{ij}{}_k \tag{1.32}$$

The coupling matrices may be taken to be

$$\{T_a\} = \{\mathbb{I}\} \cup \{\vec{t}\} \cup \{\alpha(m)\} \tag{1.33}$$

where the matrices $\alpha(m)$ are defined by the relations

$$\alpha(0) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \alpha(\pm 1) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & 0 & i \\ \pm 1 & i & 0 \end{bmatrix} \quad \alpha(\pm 2) = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & \pm i & 0 \\ \pm i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.34}$$

The commutation relations between the α-matrices and the group generators are:

$$[\, t^3, \alpha(m) \,] = m \; \alpha(m)$$
$$[\, t^{\pm}, \alpha(m) \,] = \sqrt{(3 \pm m)(2 \mp m)} \;\; \alpha(m \pm 1) \qquad (1.35)$$

Thus, we see that the α-matrices are the components of a spin-2 tensor. The completeness relation is,

$$I_{in} I_{mj} = \frac{1}{3} I_{ij} I_{mn} + \frac{1}{2} (\vec{t})_{ij} \cdot (\vec{t})_{mn}$$
$$+ \frac{1}{2} \sum_p [\alpha(p)]_{ij} [\alpha^{\dagger}(p)]_{mn} \qquad (1.36)$$

We may use this relation to show that,

$$\sum_p \vec{\Phi} \cdot \alpha(p) \cdot \vec{\Phi} \;\; \vec{\Phi} \cdot \alpha^{\dagger}(p) \cdot \vec{\Phi} = \frac{4}{3} |\vec{\Phi}|^4 \qquad (1.37)$$

so we may neglect the α-couplings. Furthermore the basis vectors, $\hat{e}_i$, and group generators, t^i, belong to the same representation. Thus, in principle there could be trilinear couplings. But the antisymmetry of the generators implies such terms are identically zero. The potential is therefore given by:

$$U(\phi) = \frac{1}{4} \lambda_1 |\vec{\Phi}|^4 + \frac{1}{2} M^2 \lambda_2 |\vec{\Phi}|^2 \qquad (1.38)$$

II. The Scalar System of Dual Model I

In the Dual Model of Dicus, Teplitz and Young[1], the gauge group, G, was chosen to be $U_Y(1) \otimes SU_W(2) \otimes SU_F(2)$. In this spontaneously broken gauge theory model, four scalar multiplets were used. These multiplets, denoted by Ψ, ρ, ϕ, and χ, transformed as $(1, \frac{1}{2}, 0)$, $(0, 0, \frac{1}{2})$, $(1, \frac{1}{2}, \frac{1}{2})$, and $(1, \frac{1}{2}, \frac{1}{2})$ respectively under the gauge group. Since we are not concerned here with the vector or spinor sectors of the model, we have the following Lagrangian:

$$\mathcal{L} = -|\partial_\mu \Psi|^2 - |\partial_\mu \rho|^2 - \mathrm{Tr}\{|\partial_\mu \phi|^2 + |\partial_\mu \chi|^2\} - U(\Psi, \rho, \phi, \chi) \tag{2.1}$$

In order to construct the most general gauge invariant potential, $U(\Psi,\rho,\phi,\chi)$, it is convenient to write the potential in the form:

$$U(\Psi,\rho,\phi,\chi) = U_1(\Psi) + U_2(\rho) + U_3(\phi,\chi) + U_4(\Psi,\rho,\phi,\chi) \tag{2.2}$$

The first two "sub-potentials" are simply given by:

$$U_1(\Psi) = \tfrac{1}{2}\lambda_1 |\Psi|^4 + \lambda_2 M_G^2 |\Psi|^2 \tag{2.3a}$$

$$U_2(\rho) = \tfrac{1}{2}\lambda_3 |\rho|^4 + \lambda_4 M_G^2 |\rho|^2 \tag{2.3b}$$

where the λ_i's are dimensionless constants and M_G^2 a dimensional constant with units of squared mass.

As a first step in the construction of U_3, we may define group, bilinear covariants which we will denote by $\Gamma(a;b)$. These objects are classified according to; (a)how they transform under unitary redefinition of the multiplets ϕ and χ, (b)their transformation properties with respect to the gauge group. An essential set of these quantities are defined below.

$$w^0 = \phi\phi^\dagger + \chi\chi^\dagger \qquad w^2 = -i(\phi\chi^\dagger - \chi\phi^\dagger)$$

$$w^1 = \phi\chi^\dagger + \chi\phi^\dagger \qquad w^3 = \phi\phi^\dagger - \chi\chi^\dagger \tag{2.4a}$$

$$f^0 = \phi^\dagger\phi + \chi^\dagger\chi \qquad f^2 = -i(\phi^\dagger\chi - \chi^\dagger\phi)$$

$$f^1 = \phi^\dagger\chi + \chi^\dagger\phi \qquad f^3 = \phi^\dagger\phi - \chi^\dagger\chi \tag{2.4b}$$

$$W^0 = \phi\tilde{\phi}^\dagger + \chi\tilde{\chi}^\dagger \qquad W^2 = -i(\phi\tilde{\chi}^\dagger - \chi\tilde{\phi}^\dagger)$$

$$W^1 = \phi\tilde{\chi}^\dagger + \chi\tilde{\phi}^\dagger \qquad W^3 = \phi\tilde{\phi}^\dagger - \chi\tilde{\chi}^\dagger \tag{2.4c}$$

$$\mathcal{F}^0 = \tilde{\phi}^\dagger\phi + \tilde{\chi}^\dagger\chi \qquad \mathcal{F}^2 = -i(\tilde{\phi}^\dagger\chi - \tilde{\chi}^\dagger\phi)$$

$$\mathcal{F}^1 = \tilde{\phi}^\dagger\chi + \tilde{\chi}^\dagger\phi \qquad \mathcal{F}^3 = \tilde{\phi}^\dagger\phi - \tilde{\chi}^\dagger\chi \tag{2.4d}$$

where $\tilde{\phi}\equiv\sigma^2\phi^*\sigma^2$ and $\tilde{\chi}\equiv\sigma^2\chi^*\sigma^2$. The transformation laws of ϕ and χ are:

$$Y:\ \phi \longrightarrow \mathcal{U}_Y(\theta_Y)\,\phi \qquad W:\ \phi \longrightarrow \mathcal{U}_W(\vec{\theta}_W)\,\phi$$

$$F:\ \phi \longrightarrow \phi\,\mathcal{U}_F(-\vec{\theta}_F) \tag{2.5}$$

These can be used to derive the following transformation properties for the covariants defined above. For instance, the transformation laws for the w^α's are,

$$Y:\ w^\alpha \longrightarrow w^\alpha \qquad W:\ w^\alpha \longrightarrow \mathcal{U}_W\, w^\alpha\, \mathcal{U}_W^\dagger$$

$$F:\ w^\alpha \longrightarrow w^\alpha \tag{2.6}$$

and one can derive the laws for the other defined covariants. As can be seen, the matrices w^{α} and f^{α} are hermitian, while the matrices W^{α} and F^{α} are not. Since all of the covariants are 2×2 matrices, we would like to construct, using only w^{α}, f^{α}, W^{α}, F^{α} and their duals, the most general possible mapping denoted by U_3 such that $U_3 : C^2 \times C^2 \to R^1$ and $[U_3, G] = 0$. This mapping must have the form:

$$\begin{aligned} U_3(\phi, \chi) = {} & a^{w}_{\alpha\beta}\, \mathrm{Tr}\{w^{\alpha}\}\, \mathrm{Tr}\{w^{\beta\dagger}\} + w \to f, W, F \\ & + B^{w}_{\alpha\beta}\, \mathrm{Tr}\{w^{\alpha} w^{\beta\dagger}\} + w \to f, W, F \\ & + C^{w}_{\alpha\beta}\, \mathrm{Tr}\{w^{\alpha} \tilde{w}^{\beta\dagger}\} + w \to f, W, F \\ & + M_G^2\, \lambda^{w}_{\alpha}\, \mathrm{Tr}\{w^{\alpha}\} + w \to f \end{aligned} \qquad (2.7)$$

The coupling matrices a, B, and C are dimensionless as is the coupling vector λ. The determinant of the bilinear covariants has been neglected since we have the identity

$$\det\{M\} = \mathrm{Tr}\{M^2\} - [\mathrm{Tr}\{M\}]^2 \qquad (2.8)$$

for any 2×2 matrix, M. The terms where f is exchanged with w may be neglected, because the use of the cyclic property of traces implies that these only duplicate the w- couplings. The diagonal terms of the C's may also be neglected because of the identity

$$\mathrm{Tr}\{M \tilde{M}^{\dagger}\} = 2 \det\{M\} \qquad (2.9)$$

and the use of equation (2.8). The F-couplings may be ignored in favor of the W-couplings by the same reasoning which allowed the elimination of the f-couplings. Furthermore, since $w = w^{\dagger}$ we assume that a^{W}, B^{W}, and C^{W} are real and symmetric. Thus among these matrices there are 10+10+6=26 independent parameters. We may now turn to the terms proportional to

$$\mathrm{Tr}\{W^{\alpha}\}\,\mathrm{Tr}\{W^{\beta\dagger}\};\quad \mathrm{Tr}\{W^{\alpha}W^{\beta\dagger}\};\quad \mathrm{Tr}\{W^{\alpha}\tilde{W}^{\beta\dagger}\}$$

We can show that for any 2×2 matrix, M, we have,

$$M\tilde{M}^{\dagger} = \det\{M\}\,\mathbb{I} = \tilde{M}^{\dagger}M \tag{2.10}$$

which implies that $W^{0}=[\det\phi+\det\chi]\,\mathbb{I}$ and $W^{3}=[\det\phi-\det\chi]\,\mathbb{I}$. Because of these equalities and equation (8), we may let a^{W}, B^{W}, and C^{W} be hermitian, 2×2, coupling matrices with α and β taking only the values 1 and 2. Thus, the mapping U_3, which is the scalar potential, may be reduced to the form:

$$\begin{aligned} U_3 = {} & a_{\alpha\beta}\,\mathrm{Tr}\{w^{\alpha}\}\,\mathrm{Tr}\{w^{\beta}\} + a'_{\alpha\beta}\,\mathrm{Tr}\{W^{\alpha}\}\,\mathrm{Tr}\{W^{\beta\dagger}\} \\ & + B_{\alpha\beta}\,\mathrm{Tr}\{w^{\alpha}w^{\beta}\} + B'_{\alpha\beta}\,\mathrm{Tr}\{W^{\alpha}W^{\beta\dagger}\} \\ & + C_{\alpha\beta}\,\mathrm{Tr}\{w^{\alpha}\tilde{w}^{\beta}\} + C'_{\alpha\beta}\,\mathrm{Tr}\{W^{\alpha}\tilde{W}^{\beta\dagger}\} \\ & + M^{2}_{G}\,\lambda_{\alpha}\,\mathrm{Tr}\{w^{\alpha}\} \end{aligned} \tag{2.11}$$

The matrices a, B, and C are real and symmetric ($\alpha,\beta=0,1,2,3$). Matrices a', B', and C' are hermitian ($\alpha,\beta=1,2$). C and C' are further restricted to have no diagonal elements. Thus, we have a total of forty coupling constants in U_3!

We know that U_3 is the most general scalar potential which we may construct from w^{α}, f^{α}, W^{α}, F^{α}, and their duals. But, is it the most general scalar potential? The answer to this question is affirmative. To prove this statement, we first introduce another set of internal, bilinear covariants, $\Gamma'(\mu\nu;AB)$ which are linearly related to the first set. We assume that Γ' has the form:

$$\Gamma'(\mu\nu:AB) = \sigma^{\mu}A^{\dagger}\sigma^{\nu}B \tag{2.12}$$

where $\sigma^{\mu}=(I,\vec{\sigma})$ and the fields A and B may take on the identities $(\phi,\chi,\tilde{\phi},\tilde{\chi})$. Thus, in principle there are $4\cdot4\cdot4\cdot4=16^2$ covariants which could enter the potential. The covariants written in equations (2.4a) through (2.4d) are those possible with $\mu=\nu=0$. In order to have a gauge invariant potential, we will be forced to sum over μ and ν in such a way that we may utilize the identity:

$$I_{ij}\ I_{mn} = \frac{1}{2}\left[\ \sum_{\mu=0}^{3}\ (\sigma^{\mu})_{in}\ (\sigma^{\mu})_{mj}\ \right] \tag{2.13}$$

Thus, the potential constructed w^{α}, f^{α}, W^{α}, F^{α} and their duals is, indeed, the most general possible potential.

Finally, we come to U_4 which may be written by inspection.

$$\begin{aligned} U_4 = \ & \lambda_{1\alpha}|\psi|^2\,\mathrm{Tr}\{w^{\alpha}\} + \lambda_{2\alpha}|\rho|^2\,\mathrm{Tr}\{w^{\alpha}\} \\ & + \lambda_{3\alpha}\psi^{\dagger}w^{\alpha}\psi + \lambda_{4\alpha}\rho^{\dagger}f^{\alpha}\rho + \lambda_{5\alpha}\tilde{\rho}^{\dagger}f^{\alpha}\tilde{\rho} \\ & + \lambda_{6\alpha}[\tilde{\rho}^{\dagger}f^{\alpha}\rho+\rho^{\dagger}f^{\alpha}\tilde{\rho}] + i\,\lambda_{7\alpha}[\tilde{\rho}^{\dagger}f^{\alpha}\rho-\rho^{\dagger}f^{\alpha}\tilde{\rho}] \\ & + \lambda_8 M_G[\psi^{\dagger}\phi\rho+\rho^{\dagger}\phi^{\dagger}\psi] + i\,\lambda_9 M_G[\psi^{\dagger}\phi\rho-\rho^{\dagger}\phi^{\dagger}\psi] \\ & + \lambda_{10}M_G[\psi^{\dagger}\chi\rho+\rho^{\dagger}\chi^{\dagger}\psi] + i\,\lambda_{11}M_G[\psi^{\dagger}\chi\rho-\rho^{\dagger}\chi^{\dagger}\psi] \\ & + \lambda_{12}M_G[\psi^{\dagger}\phi\tilde{\rho}+\tilde{\rho}^{\dagger}\phi^{\dagger}\psi] + i\,\lambda_{13}M_G[\psi^{\dagger}\phi\tilde{\rho}-\tilde{\rho}^{\dagger}\phi^{\dagger}\psi] \\ & + \lambda_{14}M_G[\psi^{\dagger}\chi\tilde{\rho}+\tilde{\rho}^{\dagger}\chi^{\dagger}\psi] + i\,\lambda_{15}M_G[\psi^{\dagger}\chi\tilde{\rho}-\tilde{\rho}^{\dagger}\chi^{\dagger}\psi] \end{aligned} \tag{2.14}$$

Thus, the total potential $U(\Psi,\rho,\phi,\chi)$ has a total of seventy-eight terms.

The fact that both ρ and its dual, $\tilde{\rho}$, appear in the potential is necessary in order to break the pseudo-symmetry which was promoted to a symmetry in the Dual Model II.

III. Dual Model II

Summary

In Ref. [1] the authors proposed that duality might also apply to the electromagnetic and weak interactions as well as the strong interaction. In that spontaneously broken gauge theory model, the gauge group $U(1)\otimes SU(2)\otimes SU(2)$ was chosen. The first two subgroups refer to the usual Weinberg-Salam groups. The remaining group refers to the two different types of lepton number. The representation of the scalars used in the model had an interesting property: the existence of a possible pseudo-symmetry. This caused no problems, however, because the pseudo-symmetry could be broken by terms in the scalar potential. Thus, the existence of a pseudo-Goldstone boson was avoided.

In this paper, we shall consider a model where the pseudo-symmetry breaking terms in the gauge invariant, scalar potential are absent and the pseudo-Goldstone boson is used to "grow" another vector boson with mass. Thus the gauge group for the present model is $U(1)\otimes SU(2)\otimes SU(2)\otimes U(1)$. We will also retain the requirement of duality as was initially proposed. Furthermore, the requirements of duality and color invariance of the quark masses, which arise from symmetry breaking, conspire to give the new model a single primitive coupling constant, $\sqrt{2}e_o$. Scalar exchange is then used to explain the $\Delta I=1/2$ rule and the anomalous strength of the weak nonleptonic decays.

In addition the model also describes a "super weak" interaction where $|\Delta S|=2$ and neutral, strangeness-changing transitions are allowed. But this interaction is mediated by vector bosons which have masses that may be several orders of magnitude larger than that of the Z^o or W bosons.

Finally, the model predicts that the neutral, hadronic current which couples to the Z^o boson is composed of an isosinglet, vector current component and an isotriplet, axial vector current component.

Conventions

In this section we establish the conventions to be used throughout this chapter. We are using a metric $g_{\mu\nu}$ with non-zero elements -1,1,1,1 for $\mu=\nu=0,1,2,3$. We are using the Pauli spinors in the standard representation. We use $\vec{\sigma}$ to denote these matrices and I for the 2×2 identity.

The Dirac matrices that we are using are given by:

$$\gamma^0 = \sigma^3 \otimes I \qquad \vec{\gamma} = i\sigma^2 \otimes \vec{\sigma} \tag{3.1}$$

so that we have the following identities and definitions.

$$\{\gamma^\mu, \gamma^\nu\} = -2 g_{\mu\nu}$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^1 \otimes I \tag{3.2}$$

The following symbols are also used:

$\mathrm{Tr}\{\ \} \equiv$ Trace
$\dagger \equiv$ Hermitian conjugate
$* \equiv$ Complex conjugate
$t \equiv$ Transpositions

The Existence of a Pseudo-symmetry

In the model of Ref. [1], two complex quartets of scalars, denoted by ϕ and χ, which belong to the $(1, \frac{1}{2}, \frac{1}{2})$ representation of the $U(1)_Y \otimes SU(2)_W \otimes SU(2)_F$ gauge group are used. In addition, two complex doublets, denoted by Ψ and ρ^*, were also used. The first doublet belongs to the $(1, \frac{1}{2}, 0)$ representation, while the latter belongs to the $(0, 0, \frac{1}{2})$ representation. The transformation properties of these scalar fields under the gauge group are:

$$\phi \longrightarrow exp[\,i\tfrac{1}{2}\,g_Y\,\xi\,]\,\phi \qquad \text{Y-gauge}$$
$$\phi \longrightarrow exp[-i\tfrac{1}{2}\,g_W\,\vec{\sigma}\cdot\vec{\xi}\,]\,\phi \qquad \text{W-gauge}$$
$$\phi \longrightarrow \phi\,exp[\,i\tfrac{1}{2}\,g_F\,\vec{\sigma}\cdot\vec{\xi}\,] \qquad \text{F-gauge} \tag{3.3}$$

$$\psi \longrightarrow exp[\,i\tfrac{1}{2}\,g_Y\,\xi\,]\,\psi \qquad \text{Y-gauge}$$
$$\psi \longrightarrow exp[-i\tfrac{1}{2}\,g_W\,\vec{\sigma}\cdot\vec{\xi}\,]\,\psi \qquad \text{W-gauge}$$
$$\psi \longrightarrow \psi \qquad \text{F-gauge} \tag{3.4}$$

$$\rho^* \longrightarrow \rho^* \qquad \text{Y-gauge}$$
$$\rho^* \longrightarrow \rho^* \qquad \text{W-gauge}$$
$$\rho^* \longrightarrow exp[-i\tfrac{1}{2}\,g_F\,\vec{\sigma}\cdot\vec{\xi}\,]\,\rho^* \qquad \text{F-gauge} \tag{3.5}$$

where g_Y, g_W, and g_F are the coupling constants associated with each subgroup and ξ is the parameter of the transformation.

We presently propose to enlarge the gauge group to $U(1)_Y$■$SU(2)_W$■$SU(2)_F$■$U(1)_H$. We ask that these fields transform as:

$$\phi \longrightarrow \phi\,exp[-i\tfrac{1}{2}\,g_H\,\xi\,]$$
$$\psi \longrightarrow \psi$$
$$\rho^* \longrightarrow exp[\,i\tfrac{1}{2}\,g_H\,\xi\,]\,\rho^* \tag{3.6}$$

under the H-gauge transformation.

In the U(1)■SU(2)■SU(2) model, the most general gauge invariant scalar potential contains seventy-eight terms.

Among the terms allowed by gauge invariance are:

$$\psi^{\dagger}\phi\sigma^{2}\rho \qquad ; \qquad \sum_{i} \mathrm{Tr}\{\phi^{\dagger}\sigma^{i}\sigma^{2}\chi^{*}\sigma^{2}\}\, \psi^{\dagger}\sigma^{2}\sigma^{i}\psi$$

$$\psi^{\dagger}\chi\sigma^{2}\rho \tag{3.6}$$

We can readily verify that these terms are invariant under Y-, W-, and F-gauge transformations but not under H-gauge transformations. It is the presence of such terms which prevent the appearance of a pseudo-Goldstone boson in the U(1)⊗SU(2)⊗ SU(2) model

Having shown the existence of the pseudo-symmetry in the aforementioned model, we now present a model which incorporates the pseudo-symmetry as a gauge symmetry.

Fermions

(a) Quarks

We consider a model with six left-handed quartets of quarks, along with twenty-four right-handed singlets. These left-handed quartets we denoted by L_{qi} where i=1, ..., 6. The first three quartets provide a representation of color SU(3). While the last three quartets provide a representation of color SU($\bar{3}$). With each quartet, we associate four right-handed singlets. Thus we are considering a Han-Nambu model[2] which incorporates the suggestions of Glashow, Iliopoulos, and Maiani[3] and Pati and Salam[4]. The quartets L_{qi}, L_{q2} and L_{q6} we assign to the ($Y=1$, $W=\frac{1}{2}$, $F=\frac{1}{2}$, $H=1$) representation of our gauge group. The remaining quartets are assigned to the (-1, $\frac{1}{2}$, $\frac{1}{2}$, -1) representation. The right-handed singlets are either assigned to the (2,0,0,0), (0,0,0,2), (-2,0,0,0), or (0,0,0,-2) representations. We exhibit L_{q1} and its associated right-handed singlets below.

$$L_{q_1} \equiv \frac{1-\gamma^5}{2} \begin{bmatrix} p_1(\theta_c) & c_1(\theta_c) \\ n_1(\theta_c) & \lambda_1(\theta_c) \end{bmatrix}_{Y=1 \quad H=1}$$

$$R_{p_1} \equiv \tfrac{1}{2}(1+\gamma^5)p_{1\ Y=2\ H=0} \qquad R_{c_1} \equiv \tfrac{1}{2}(1+\gamma^5)c_{1\ Y=2\ H=0}$$

$$R_{n_1} \equiv \tfrac{1}{2}(1+\gamma^5)n_{1\ Y=0\ H=2} \qquad R_{\lambda_1} \equiv \tfrac{1}{2}(1+\gamma^5)\lambda_{1\ Y=0\ H=2} \tag{3.7}$$

where p, n, λ, and c refer to the proton, neutron, lambda, and charmed quarks, respectively. The Cabibbo rotation we use is defined by:

$$\begin{aligned} p_1(\theta_c) &\equiv p_1 \cos(\tfrac{1}{2}\theta_c) - c_1 \sin(\tfrac{1}{2}\theta_c) \\ n_1(\theta_c) &\equiv n_1 \cos(\tfrac{1}{2}\theta_c) + \lambda_1 \sin(\tfrac{1}{2}\theta_c) \\ c_1(\theta_c) &\equiv c_1 \cos(\tfrac{1}{2}\theta_c) + p_1 \sin(\tfrac{1}{2}\theta_c) \\ \lambda_1(\theta_c) &\equiv \lambda_1 \cos(\tfrac{1}{2}\theta_c) - n_1 \sin(\tfrac{1}{2}\theta_c) \end{aligned} \tag{3.8}$$

The second and sixth quartets along with their associated singlets have the same assignments as those above. The remaining quartets and singlets differ from those above only by the replacements $Y \to Y-2$ and $H \to H-2$.

As can be seen, the Adler-Bell-Jackiw anomalies [5], [6], [7], and [8] of the first three quartets are cancelled by those of the last three quartets. A similar statement also holds for the anomalies of the right-handed singlets.

With this sort of arrangement, the electric charge is related to the weak hypercharge and the third component of weak isospin by the usual relation

$$Q = W^3 + \tfrac{1}{2} Y \tag{3.9}$$

(b) Leptons

Keeping in mind that we want a model that is free of anomalies, we introduce two quartets of leptons. The first quartet is assigned to the $(-1, \frac{1}{2}, \frac{1}{2}, -1)$ representation. The second is in the $(1, \frac{1}{2}, \frac{1}{2}, 1)$ representation. The singlets belong to either the (-2,0,0,0) or (2,0,0,0) representations. Below we exhibit the leptons.

$$L_{\ell_1} \equiv \frac{1-\gamma^5}{2} \begin{bmatrix} \nu_{e_1} & \nu_{\mu_1} \\ e_1 & \mu_1 \end{bmatrix}_{Y=-1 \quad H=-1}$$

$$R_{e_1} \equiv \tfrac{1}{2}(1+\gamma^5)\, e_1 \;{}_{Y=-2 \quad H=0} \qquad R_{\mu_1} \equiv \tfrac{1}{2}(1+\gamma^5)\, \mu_1 \;{}_{\substack{Y=-2 \\ H=0}}$$

$$L_{\ell_2} \equiv \frac{1-\gamma^5}{2} \begin{bmatrix} e_2 & \mu_2 \\ \nu_{e2} & \nu_{\mu 2} \end{bmatrix}_{Y=1 \quad H=1} \tag{3.10}$$

$$R_{e_2} \equiv \tfrac{1}{2}(1+\gamma^5)\, e_2 \;{}_{Y=2 \quad H=0} \qquad R_{\mu_2} \equiv \tfrac{1}{2}(1+\gamma^5)\, \mu_2 \;{}_{\substack{Y=2 \\ H=0}}$$

Gauge Transformations and Spin-1 Bosons

Our gauge group is $U(1)_Y \otimes SU(2)_W \otimes SU(2)_F \otimes U(1)_H$. Accordingly, we must introduce an octet of vector bosons. We will denote these bosons by Y_α, $\vec{W}_\alpha$, $\vec{F}_\alpha$, and H_α.

In order to complete our "gauging" of the former pseudo-symmetry, we give the transformation properties of the fermions under gauge transformations of the first kind:

$$
\begin{array}{ll}
\text{Y-gauge} & L \to \exp[\tfrac{i}{2} g_Y \theta_Y \xi] L \\
\text{W-gauge} & L \to \exp[-\tfrac{i}{2} g_W \vec{\sigma}\cdot\vec{\xi}] L \\
\text{F-gauge} & L \to L \exp[\tfrac{i}{2} g_F \vec{\sigma}\cdot\vec{\xi}] \\
\text{H-gauge} & L \to L \exp[\tfrac{i}{2} g_H \theta_H \xi]
\end{array}
\tag{3.11}
$$

where $\theta_Y = \pm 1$ and $\theta_H = \pm 1$

$$
\begin{array}{ll}
\text{Y-gauge} & R \to \exp[i\tfrac{1}{2} g_Y \theta'_Y \xi] R \\
\text{W-gauge} & R \to R \\
\text{F-gauge} & R \to R \\
\text{H-gauge} & R \to \exp[i\tfrac{1}{2} g_H \theta'_H \xi] R
\end{array}
\tag{3.12}
$$

where $\theta'_Y = 2, 0, -2$ and $\theta'_H = 2, 0, -2$.

Scalar Bosons: Spurions and Higgs Scalars

The requirement that the two-body scattering amplitude be dual in the sense defined by equations (2.14a) and (2.14b) of Ref. [1] for fermions which belong to the same irreducible representation forces the use of more scalars than were used in Ref. [1]. These additional scalars are needed to avoid some very restrictive mass relations among the fermions.

Motivated by this and the desire to use these fields most efficiently, we propose that the additional scalars are physical fields as opposed to being non-physical scalars[8]. We will refer to these physical multiplets as "spurions"[9].

We ask that spurions be the explanation for (a)the $\Delta I=1/2$ rule among the hadrons and (b)the anomalous strength of the weak nonleptonic decays. The minimum number of spurions required for our purposes is three, which we will denote by ζ, ξ, and η.

In the present model, we use five spurion fields ζ, ξ, η, ϕ and χ, all of which we assign to the $(1,\frac{1}{2},\frac{1}{2},-1)$ representation of our gauge group. The fields ϕ and χ will be allowed to acquire non-zero vacuum expectation values, while ζ, ξ, and η will not.

The Higgs scalars in the present model are Ψ, σ, ρ_1 and ρ_2. These fields transform as $(-3,\frac{3}{2},0,-\frac{1}{3})$, $(0,0,0,1)$, $(0,0,1,0)$ and $(0,0,1,0)$ respectively.

The Lagrangian

The Lagrangian density for our model is given by:

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_F + \mathcal{L}_{SF} \tag{3.13}$$

where the decomposition denotes vector, scalar, fermion, and scalar-fermion interactions. Explicitly we have:

$$\mathcal{L}_V \equiv -\tfrac{1}{4} Y_{\mu\nu} Y^{\mu\nu} - \tfrac{1}{4} |\vec{W}_{\mu\nu}|^2 - \tfrac{1}{4} |\vec{F}_{\mu\nu}|^2 - \tfrac{1}{4} H_{\mu\nu} H^{\mu\nu}$$

$$Y_{\mu\nu} \equiv \partial_\mu Y_\nu - \partial_\nu Y_\mu \qquad\qquad H_{\mu\nu} \equiv \partial_\mu H_\nu - \partial_\nu H_\mu$$

$$\vec{W}_{\mu\nu} \equiv \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g_W \vec{W}_\mu \times \vec{W}_\nu$$

$$\vec{F}_{\mu\nu} \equiv \partial_\mu \vec{F}_\nu - \partial_\nu \vec{F}_\mu + g_F \vec{F}_\mu \times \vec{F}_\nu \tag{3.14}$$

$$\mathcal{L}_S \equiv -|D_\alpha\psi|^2 - |D_\alpha\sigma|^2 - \tfrac{1}{2}|D_\alpha\rho_1|^2 - \tfrac{1}{2}|D_\alpha\rho_2|^2$$

$$-\mathrm{Tr}\{|D_\alpha\zeta|^2 + |D_\alpha\xi|^2 + |D_\alpha\eta|^2 + |D_\alpha\chi|^2 + |D_\alpha\Phi|^2\}$$

$$-\,\mathcal{U}(\psi,\sigma,\rho_1,\rho_2,\zeta,\xi,\eta,\chi,\Phi)$$

$$D_\alpha\zeta \equiv (\partial_\alpha + i\tfrac{1}{2}g_Y Y_\alpha - i\tfrac{1}{2}g_W\vec{\sigma}\cdot\vec{W}_\alpha)\zeta - \zeta(i\tfrac{1}{2}g_H H_\alpha - i\tfrac{1}{2}g_F\vec{\sigma}\cdot\vec{F}_\alpha)$$

$$D_\alpha\psi \equiv (\partial_\alpha - i\tfrac{3}{2}g_Y Y_\alpha - ig_W\vec{T}\cdot\vec{W}_\alpha - i\tfrac{1}{6}g_H H_\alpha)\psi$$

$$D_\alpha\sigma \equiv (\partial_\alpha + i\tfrac{1}{2}g_H H_\alpha)\sigma \qquad D_\alpha\rho_i \equiv (\partial_\alpha + ig_F\vec{\mathcal{I}}\cdot\vec{F}_\alpha)\rho_i$$

$$\vec{\sigma}\cdot\vec{W}_\alpha \equiv \tfrac{1}{\sqrt{2}}(\sigma^+W_\alpha^+ + \sigma^-W_\alpha^-) + \sigma^3W_\alpha^3 \quad ; \quad W_\alpha^\pm = \tfrac{1}{\sqrt{2}}(W_\alpha^1 \mp iW_\alpha^2)$$

$$\vec{\sigma}\cdot\vec{F}_\alpha \equiv \tfrac{1}{\sqrt{2}}(\sigma^+F_\alpha^+ + \sigma^-F_\alpha^-) + \sigma^3F_\alpha^3 \quad ; \quad F_\alpha^\pm = \tfrac{1}{\sqrt{2}}(F_\alpha^1 \mp iF_\alpha^2)$$

$$\vec{T}\cdot\vec{W}_\alpha \equiv \tfrac{1}{\sqrt{2}}(T^+W_\alpha^+ + T^-W_\alpha^-) + T^3W_\alpha^3 \quad ; \quad |\vec{T}|^2 = \tfrac{15}{4}$$

$$\vec{\mathcal{I}}\cdot\vec{F}_\alpha \equiv \tfrac{1}{\sqrt{2}}(\mathcal{I}^+F_\alpha^+ + \mathcal{I}^-F_\alpha^-) + \mathcal{I}^3F_\alpha^3 \quad ; \quad |\vec{\mathcal{I}}|^2 = 2 \tag{3.15}$$

$$\mathcal{L}_F \equiv \tfrac{i}{2}\sum_i \mathrm{Tr}\{\bar{L}_{\ell i}\gamma^\alpha(\overleftrightarrow{\partial}_\alpha + ig_Y\theta_Y Y_\alpha - ig_W\vec{\sigma}\cdot\vec{W}_\alpha)L_{\ell i}$$

$$+\,\bar{L}_{\ell i}\gamma^\alpha L_{\ell i}(ig_F\vec{\sigma}\cdot\vec{F}_\alpha + ig_H\theta_H H_\alpha)\}$$

$$+\,i\tfrac{1}{2}\sum_{M,i}\bar{R}_{Mi}\gamma^\alpha(\overleftrightarrow{\partial}_\alpha + ig_Y\theta_Y' Y_\alpha)R_{Mi}$$

$$-\tfrac{1}{2}\sum_j \mathrm{Tr}\{\bar{L}_{qj}\gamma^\alpha(g_Y\theta_Y Y_\alpha - g_W\vec{\sigma}\cdot\vec{W}_\alpha)L_{qj}$$

$$+\,\bar{L}_{qj}\gamma^\alpha L_{qj}(g_F\vec{\sigma}\cdot\vec{F}_\alpha + g_H\theta_H H_\alpha)\}$$

$$-\tfrac{1}{2}\sum_{A,j}\bar{R}_{Aj}(g_Y\theta_Y' Y_\alpha + g_H\theta_H' H_\alpha)\gamma^\alpha R_{Aj}$$

where $A=p,n,\lambda,c$; $M=e,\mu$; $\quad \overleftrightarrow{\partial}_\alpha \equiv \partial_\alpha - \overleftarrow{\partial}_\alpha$ (3.16) and finally we have the scalar-fermion interaction terms:

$$\mathcal{L}_{SF} \equiv -\sum_{I,j} f^{q}_{Ij}\, \mathrm{Tr}\{\overline{L}_{qj}[\tilde{h}(\gamma_I,\delta_I,\varphi_I)]R_{I,j} + h.c.\}$$
$$-\sum_{J,j} f^{q}_{Jj}\, \mathrm{Tr}\{\overline{L}_{qj}[h(\gamma_J,\delta_J,\varphi_J)]R_{J,j} + h.c.\}$$
$$-\sum_{M} f^{\ell}_{M1}\, \mathrm{Tr}\{\overline{L}_{\ell_1}[\ell(\gamma_{M1},\delta_{M1})]R_{M1} + h.c.\}$$
$$-\sum_{M} f^{\ell}_{M2}\, \mathrm{Tr}\{\overline{L}_{\ell_2}[\tilde{\ell}(\gamma_{M2},\delta_{M2})]R_{M2} + h.c.\} \qquad (3.17)$$

where I=p,c $\qquad$ J=n,λ

$$h(\gamma,\delta,\varphi) \equiv (\xi\cos\varphi + \zeta\sin\varphi)\cos\gamma + (\phi\cos\delta + \chi\sin\delta)\sin\gamma$$

$$\ell(\gamma,\delta) \equiv \eta\cos\gamma + \sin\gamma\,(\phi\cos\delta + \chi\sin\delta)$$

$$\tilde{\ell} \equiv \sigma^2\ell^*\sigma^2 \qquad\qquad \tilde{h} \equiv \sigma^2 h^*\sigma^2$$

We require that the potential be such that a minimum occurs at:

$$\xi_0 = \zeta_0 = \eta_0 = 0 \qquad\qquad \phi_0 = \tfrac{1}{4} a M_G (I - \sigma^3)$$

$$\chi_0 = \tfrac{1}{4} a M_G (\sigma^1 - i\sigma^2) \qquad\qquad \sigma_0 = b M_G \tfrac{1}{\sqrt{2}}$$

$$(\rho)_0 = \tfrac{1}{2} c M_G \hat{\rho}_0 \qquad (\rho_2)_0 = \tfrac{1}{2} d M_G \hat{\rho}_1$$

$$\psi_0 = \tfrac{1}{\sqrt{2}} a M_G \hat{\psi}_{\frac{1}{2}} \qquad (3.18)$$

where M_G is a mass parameter used to characterize the spontaneous symmetry breaking and the constants a, b, c and d are dimensionless parameters.

Imposing Duality

We may now impose the requirement that the two-body scattering amplitudes be dual for members of the same irreducible representation of the group. In other words, we require that for every such process we want the lowest-order amplitude to have the form:

$$A = A_0\left(\frac{1}{t} + \frac{1}{u}\right) \tag{3.19}$$

for energies and angles such that all masses can be neglected. By considering the processes $p_L+n_L \to p_L+n_L$ and $n_L+\lambda_L \to n_L+\lambda_L$ we obtain the following equations:

$$g_Y^2 + g_W^2 - 3g_F^2 + g_H^2 = 0 \tag{3.20}$$

$$g_Y^2 - 3g_W^2 + g_F^2 + g_H^2 = 0 \tag{3.21}$$

These equations are sufficient to enforce duality for all scatterings of left-hand fermions. The scattering of right-hand particles into right-hand particles is slightly more difficult. The interested reader is referred, Ref. [1].

The scattering of right-handed fermions into left-handed fermions such as $p_R+n_L \to p_R+n_L$ will provide the relationship:

$$(f_p^1)^2 = g_Y^2 \tag{3.22}$$

and similarly the process $p_L+n_R \to p_L+n_R$ will require that:

$$(f_n^1)^2 = g_H^2 \tag{3.23}$$

These last two results may be summarized by the equation:

$$(f)^2 = \begin{cases} g_Y^2\text{: for charged right-handed singlets} \\ g_H^2\text{: for neutral right-handed singlets} \end{cases} \tag{3.24}$$

Massive Fermions

We now turn to the massive fermions which are generated by the spontaneous symmetry breaking of the electromagnetic and weak gauge group. Since the fields ϕ, χ, Ψ, σ, ρ_1 and ρ_2

all acquire non-zero V.E.V's, we may perform shifts of these fields. First we treat the quark masses which are generated by the following terms:

$$- \sum_{I,j} f_I^q \sin\gamma_I \cos\delta_I \, \mathrm{Tr}\{ \bar{L}_{q_j} \tilde{\Phi}_0 R_{I,j} + \bar{R}_{I,j} \tilde{\Phi}_0^{\dagger} L_{q_j} \}$$

$$- \sum_{I,j} f_I^q \sin\gamma_I \sin\delta_I \, \mathrm{Tr}\{ \bar{L}_{q_j} \tilde{\chi}_0 R_{I,j} + \bar{R}_{I,j} \tilde{\chi}_0^{\dagger} L_{q_j} \}$$

$$- \sum_{J,j} f_J^q \sin\gamma_J \cos\delta_J \, \mathrm{Tr}\{ \bar{L}_{q_j} \Phi_0 R_{J,j} + \bar{R}_{J,j} \Phi_0^{\dagger} L_{q_j} \}$$

$$- \sum_{J,j} f_J^q \sin\gamma_J \sin\delta_J \, \mathrm{Tr}\{ \bar{L}_{q_j} \chi_0 R_{J,j} + \bar{R}_{J,j} \chi_0^{\dagger} L_{q_j} \} \qquad (3.25)$$

Considering the proton quarks, initially, we find that the choice $\delta_p = -1/2\ \theta_c$ will eliminate the Cabibbo mixing of proton and charmed quarks. We then find the proton quarks masses to be given by the expression:

$$- \tfrac{1}{2}\, a\, M_G \sin\gamma_p [\ g_Y (\bar{p}_1 p_1 + \bar{p}_2 p_2) + g_H\, \bar{p}_3 p_3\ ] \qquad (3.26)$$

Before going on, we note that in a sense we may regard the δ-angles as the origins of the Cabibbo rotations. If we now consider the transformation properties of the above expression under color SU(3), we see that it may be written as:

$$- \tfrac{1}{2}\, a\, M_G \sin\gamma_p\, \bar{P} [\, (\tfrac{2}{3} g_Y + \tfrac{1}{3} g_H) I_c + (g_Y - g_H) \mathcal{Y}_c] P \qquad (3.27)$$

where I_C and $\mathcal{Y}_C$ are the color identity and hypercharge operators respectively and P denotes the color triplet of proton quarks. Now, if $g_Y = g_H = g$ then the contribution to the quark mass from the symmetry breaking will be a color SU(3) invariant. Applying the same considerations to the other quarks, we find by making the choices: $\delta_C = 1/2(\pi - \theta_C)$, $\delta_n = 1/2(\pi + \theta_C)$ and $\delta_\lambda = 1/2\theta_C$ that the mass terms for the quarks are:

$$- \tfrac{1}{2}\, a g M_G [\, \sin\gamma_p\, \bar{P} P + \sin\gamma_n\, \bar{N} N + \sin\gamma_c\, \bar{C} C + \sin\gamma_\lambda\, \bar{\Lambda} \Lambda\,] \qquad (3.28)$$

where P, N, C, and Λ denote color SU(3) triplets. Similar considerations apply to the quarks that provide the representation of color SU($\bar{3}$).

Now in a similar manner, we can find the massive leptons of the model. Needless to say, the lepton masses are also proportional to the sines of γ-angles. Furthermore, we may use the δ-angles to give Cabibbo-mixing of the leptons. For instance, with the choices $\delta_{\mu 1}=0$ and $\delta_{e1}=\pi/2$, we find that the electron and muon mass terms are:

$$-\tfrac{1}{2}\, a g M_G \left[\sin\gamma_{e_1} \bar{e}_1 e_1 + \sin\gamma_{\mu_1} \bar{\mu}_1 \mu_1 \right] \tag{3.29}$$

The massive leptons of $L_{\ell 2}$ are presumably more massive and can easily be accommodated in the model.

A Single Coupling Constant

In the previous section we saw how the color invariance of the quark mass matrix leads to the requirement that both charged and neutral right-handed singlets possess a single coupling constant. If we combine this with equations (3.20) and (3.21), we find that all of the vector coupling constants are the same. Furthermore, equation (3.23) then implies that the model possesses a single coupling constant, g.

Massive Vector Bosons

The shifted fields also cause seven vector bosons in the model to become massive. We find for the mass matrix of "neutral" bosons:

$$g^2 M_G^2 \begin{bmatrix} H_\alpha \\ Y_\alpha \\ W_\alpha^3 \end{bmatrix} \begin{bmatrix} b^2 + \frac{5}{17} a^2 & 0 & 0 \\ 0 & \frac{5}{2} a^2 & \frac{5}{2} a^2 \\ 0 & \frac{5}{2} a^2 & \frac{5}{2} a^2 \end{bmatrix} \begin{bmatrix} H_\alpha \\ Y_\alpha \\ W_\alpha^3 \end{bmatrix} \tag{3.30}$$

For the squared masses of vector bosons, the following values are obtained.

$$M_A^2 = 0 \qquad M_{Z^0}^2 = 5 g^2 M_G^2 a^2$$

$$M_{G^0}^2 = g^2 M_G^2 [b^2 + \tfrac{5}{18} a^2] \tag{3.31}$$

Thus, we have a massless, neutral, vector boson (the photon) given by:

$$A_\alpha \equiv \tfrac{1}{\sqrt{2}} [- Y_\alpha + W_\alpha^3] \tag{3.32a}$$

and two massive, neutral vector bosons.

$$Z_\alpha^0 \equiv \tfrac{1}{\sqrt{2}} [Y_\alpha + W_\alpha^3] \tag{3.32b}$$

$$G_\alpha^0 \equiv H_\alpha \tag{3.32c}$$

For the other bosons of the model, we find two electronically charged, spin-1 bosons at a mass squared value of:

$$M_{W^\pm}^2 = g^2 M_G^2 a^2 \tag{3.33}$$

and two electrically neutral, vector bosons at a mass squared value of:

$$M_{F^\pm}^2 = \tfrac{1}{4} g^2 M_G^2 [a^2 + c^2 + d^2] \tag{3.34}$$

Finally, the remaining F-boson, $F_\alpha^0 = F_\alpha^3$, acquires a mass squared value given by:

$$M_{F^0}^2 = \tfrac{1}{4} g^2 M_G^2 [a^2 + 2 d^2] \tag{3.35}$$

Model Parameters and Phenomenology

Now that we have found the photon, we may make the identification between the single coupling constant of the model and the fundamental unit of charge:

$$g = \sqrt{2}\ e_0 \tag{3.36}$$

Next we may consider μ-decay in the context of this model. By considering the lowest order contributions to this process, we may deduce that the mass parameter, M_G, which was used to characterize the spontaneous symmetry breaking, is related to the weak Fermi coupling constant.

$$M_G^2 = \frac{1}{4\sqrt{2}}\ G^{-1} \tag{3.37}$$

Furthermore the parameters a, c, and d are constrained by the equation:

$$1 = \left[\frac{1}{a^2} + \frac{4}{a^2 + c^2 + d^2} \right] \tag{3.38}$$

We may use this equation to define another parameter α via the equation:

$$\tan \alpha = [\ 4a^2 / a^2 + c^2 + d^2\ ]^{\frac{1}{2}} \tag{3.39}$$

Continuing by considering β-decay, we find that the effective coupling constant is:

$$G_\beta = G \cos\theta_c \cos^2\alpha \tag{3.40}$$

The effective coupling constant for μ-decay is just the weak Fermi coupling constant. Experimentally, these two constants are within two and one half percent of each other. This implies that the angle α must be very close to zero. In order to fulfill this condition we will assume that $a^2 \ll c^2+d^2$. We will also assume that the parameter b is much larger than a. With these conditions met we find the mass spectrum for the vector bosons that is indicated in Figure (#1).

With this mass spectrum, we see that the model describes the ordinary weak interaction with the exchange of either W or Z^O bosons and a "super weak" interaction where F or G^O bosons are exchanged. This is important to this model since both $|\Delta S|=2$ and neutral, strangeness-changing transitions are mediated by F-bosons. Such effects are severely suppressed owing to the large masses of the F-bosons. By using either Cabibbo or F-spin rotations of the hadrons we may introduce CP violation into the model.

Lepton-Lepton Scattering

In this section we consider the scattering of the "light" leptons. The part of the Lagrangian which is responsible for such processes is given by:

$$\begin{aligned}
\mathcal{L}_{int} = & -e_o[\bar{e}\gamma^\alpha e + \bar{\mu}\gamma^\alpha\mu]A_\alpha + \tfrac{1}{2}e_o[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)e + \\
& \bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\mu]W^+_\alpha - \tfrac{1}{2}e_o[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)\nu_\mu + \\
& e\gamma^\alpha(1-\gamma^5)\mu]F^-_\alpha - \tfrac{1}{2\sqrt{2}}e_o[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)\nu_e + \bar{e}\gamma^\alpha(1-\gamma^5)e \\
& -\bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\nu_\mu - \bar{\mu}\gamma^\alpha(1-\gamma^5)\mu]F^o_\alpha + \tfrac{1}{2}e_o[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)\nu_e \\
& +\bar{e}\gamma^\alpha(1+\gamma^5)e + \bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\nu_\mu + \bar{\mu}\gamma^\alpha(1+\gamma^5)\mu]Z^o_\alpha \\
& +\tfrac{e_o}{2\sqrt{2}}[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)\nu_e + \bar{e}\gamma^\alpha(1-\gamma^5)e + \bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\nu_\mu + \bar{\mu}\gamma^\alpha(1-\gamma^5)\mu]G^o_\alpha \\
& -\tfrac{1}{\sqrt{2}}e_o[\bar{\nu}_e(1+\gamma^5)e\,\ell_{\nu_e}(\gamma_e,\delta_e) + \bar{e}(1+\gamma^5)e\,\ell_e(\gamma_e,\delta_e)] \\
& -\tfrac{1}{\sqrt{2}}e_o[\bar{\nu}_\mu(1+\gamma^5)e\,\ell_{\nu_\mu}(\gamma_e,\delta_e) + \bar{\mu}(1+\gamma^5)e\,\ell_\mu(\gamma_e,\delta_e)] \\
& -\tfrac{1}{\sqrt{2}}e_o[\bar{\nu}_e(1+\gamma^5)\mu\,\ell_{\nu_e}(\gamma_\mu,\delta_\mu) + \bar{e}(1+\gamma^5)\mu\,\ell_e(\gamma_\mu,\delta_\mu)] \\
& -\tfrac{1}{\sqrt{2}}e_o[\bar{\nu}_\mu(1+\gamma^5)\mu\,\ell_{\nu_\mu}(\gamma_\mu,\delta_\mu) + \bar{\mu}(1+\gamma^5)\mu\,\ell_\mu(\gamma_\mu,\delta_\mu)] + h.c.
\end{aligned}$$

(3.41)

Figure (1)

$$M^2_{F^\circ} \cong g^2 M_G^2 \left[d^2 + \tfrac{1}{2}\left(1 + \tfrac{1}{2}\alpha^2\right)\right]$$

$$M^2_{F^\pm} \cong g^2 M_G^2 \left(\tfrac{1}{\alpha}\right)^2$$

$$M^2_{G^\circ} \cong g^2 M_G^2 \left[b^2 + \tfrac{5}{18}\left(1 + \tfrac{1}{2}\alpha^2\right)\right]$$

$$M^2_{Z^\circ} \cong 5 g^2 M_G^2 \left(1 + \tfrac{1}{2}\alpha^2\right)$$

$$M^2_{W} \cong g^2 M_G^2 \left(1 + \tfrac{1}{2}\alpha^2\right)$$

$$M^2_{A}$$

Mass spectrum of vector bosons in Dual Model II. The Parameter $g^2M_G^2$ is equal to $e_O^2/2\sqrt{2}G_F$ where G_F is the weak Fermi coupling constant.

We now turn to the process $\bar{\nu}_e+e\to\bar{\nu}_e+e$. This receives contributions from W, Z^O, F^O, G^O, and $\ell_{\nu e}$ exchanges. The effective Lagrangian for this process is:

$$\mathcal{L}_{eff} = \frac{G}{\sqrt{2}}\ \bar{\nu}_e \gamma^\alpha (1-\gamma^5)\nu_e\ \bar{e}\ \gamma_\alpha (C_V - \gamma^5 C_A)\, e \tag{3.42}$$

where

$$C_V = \frac{6}{5} - \sin^2\alpha + \frac{1}{2}\left(\frac{b^2+d^2}{b^2d^2}\right) + \frac{1}{5} M^2_{Z^0}\left(\frac{\cos^2\gamma_e}{M^2_{\eta\nu_e}} + \frac{\sin^2\gamma_e}{M^2_{\chi\nu_e}}\right) \tag{3.43a}$$

$$C_A = \frac{4}{5} - \sin^2\alpha + \frac{1}{2}\left(\frac{b^2+d^2}{b^2d^2}\right) - \frac{1}{5} M^2_{Z^0}\left(\frac{\cos^2\gamma_e}{M^2_{\eta\nu_e}} + \frac{\sin^2\gamma_e}{M^2_{\chi\nu_e}}\right) \tag{3.43b}$$

The angle α is constrained to be very small by our assumption that $c^2+d^2 >> a^2$. Furthermore, we may assume that $\frac{1}{2}(b^2+d^2)/b^2d^2$ is also negligible. Finally the angle γ_e is of the order of m_e/M_{Z^O}. Therefore, we may approximate the above expressions by;

$$C_V = \frac{6}{5} + \frac{1}{5}\left(M_{Z^0}/M_{\eta\nu_e}\right)^2 \tag{3.44a}$$

$$C_A = \frac{4}{5} - \frac{1}{5}\left(M_{Z^0}/M_{\eta\nu_e}\right)^2 \tag{3.44b}$$

This process has been measured by Gurr, Reines, and Sobel [10] and their results may be used to put bounds on the mass of the scalar $\eta_{\nu e}$. Our values for C_V and C_A are shown for $\alpha=0$, $b=d=\infty$ and for all values of the ratio $x=(M_{Z^O}/M\eta_{\nu e})$ in Figure(#2).

Next we may consider the process $\mu_\nu+e\to\nu_\mu+e$. This process receives contributions from Z^O, F^O, G^O, and $\ell\nu_\mu$ exchanges. The effective Lagrangian is:

$$\mathcal{L}_{eff} = \frac{-G}{\sqrt{2}}\ \bar{\nu}_\mu \gamma^\alpha (1-\gamma^5)\nu_\mu\ \bar{e}\ \gamma_\alpha (C'_V - \gamma^5 C'_A)\, e \tag{3.45}$$

where

$$C'_V = -\frac{1}{5} + \frac{1}{2}\left(\frac{b^2-d^2}{b^2d^2}\right) - \frac{1}{5} M^2_{Z^0}\left(\frac{\cos^2\gamma_e}{M^2_{\eta\nu_\mu}} + \frac{\sin^2\gamma_e}{M^2_{\chi\nu_\mu}}\right) \tag{3.46a}$$

$$C_A' = \frac{1}{5} + \frac{1}{2}\left(\frac{b^2-d^2}{b^2d^2}\right) + \frac{1}{5}M_{z^\circ}^2\left(\frac{\cos^2\gamma_e}{M^2_{\eta\nu_\mu}} + \frac{\sin^2\gamma_e}{M^2_{\chi\nu_\mu}}\right) \qquad (3.46b)$$

Once more we may make the same approximations and we find.

$$C_V' = -\frac{1}{5} - \frac{1}{5}\left(M_{z^\circ}/M_{\eta\nu_\mu}\right)^2 \qquad (3.47a)$$

$$C_A' = \frac{1}{5} + \frac{1}{5}\left(M_{z^\circ}/M_{\eta\nu_\mu}\right)^2 \qquad (3.47b)$$

In Figure (#3) we plot our values for C_V' and C_A' as a function of $x' \equiv (M_{ZO}/M\eta\nu_\mu)$ where a, b, and d are fixed at the same values as in the discussion of C_V and C_A given above

The $\Delta I=1/2$ Rule and the Enhancement of the Weak Nonleptonic Decays

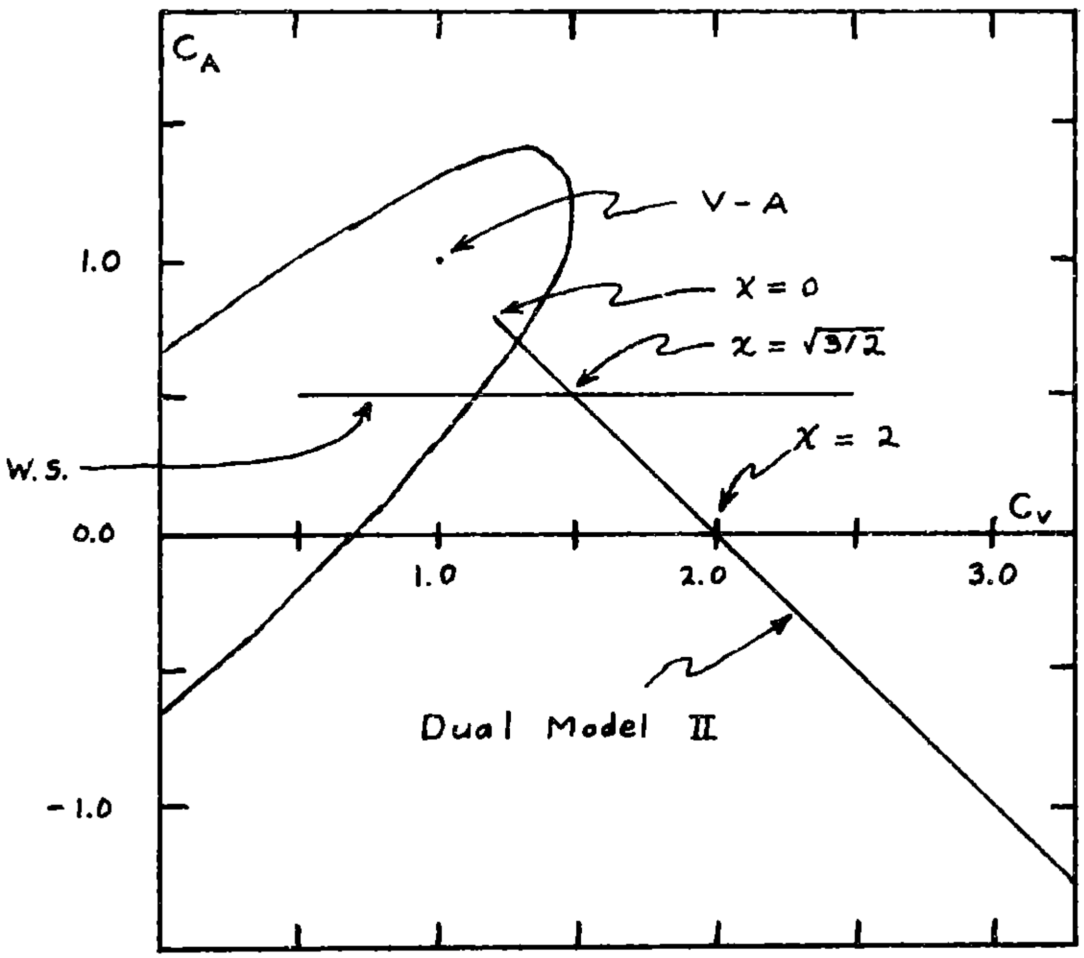
In this section we explicity demonstrate the role that scalar exchange plays, within the model, in explaining (a)the $\Delta I=1/2$ rule among the hadrons and (b)the anomalous strength of the weak nonleptonic decays. We will set the Cabibbo angle to zero and ignore color since the scalar interaction is a color singlet. Furthermore, we will assume that the masses of the Higgs spurion multiplets are large compared to the non-Higgs spurions ζ and ξ. The only other free parameters, aside from the masses of these multiplets, are the angles ϕ_I and ϕ_J. We make the choices $\phi_p=\phi_c=0$ and $\phi_n=\phi_\lambda=\pi/2$. Thus the scalar-quark interaction effectively becomes:

$$\begin{aligned}\mathcal{L}_{q\text{-}s} \cong \frac{-e}{\sqrt{2}}\{[\zeta_p\bar{p} + \zeta_n\bar{n} + \zeta_c\bar{c} + \zeta_\lambda\bar{\lambda}](1+\gamma^5)[n\cos\gamma_n + \lambda\cos\gamma_\lambda]\}\\ \frac{-e}{\sqrt{2}}\{[\xi_\lambda^\dagger\bar{p} - \xi_c^\dagger\bar{n} - \xi_n^\dagger\bar{c} + \xi_p^\dagger\bar{\lambda}](1+\gamma^5)[p\cos\gamma_p + c\cos\gamma_c]\}\\ + h.c.\end{aligned} \qquad (3.48)$$

Now by considering processes that are characterized by $|\Delta S|=1$ and $|\Delta C|=0$, we find the following effective interaction:

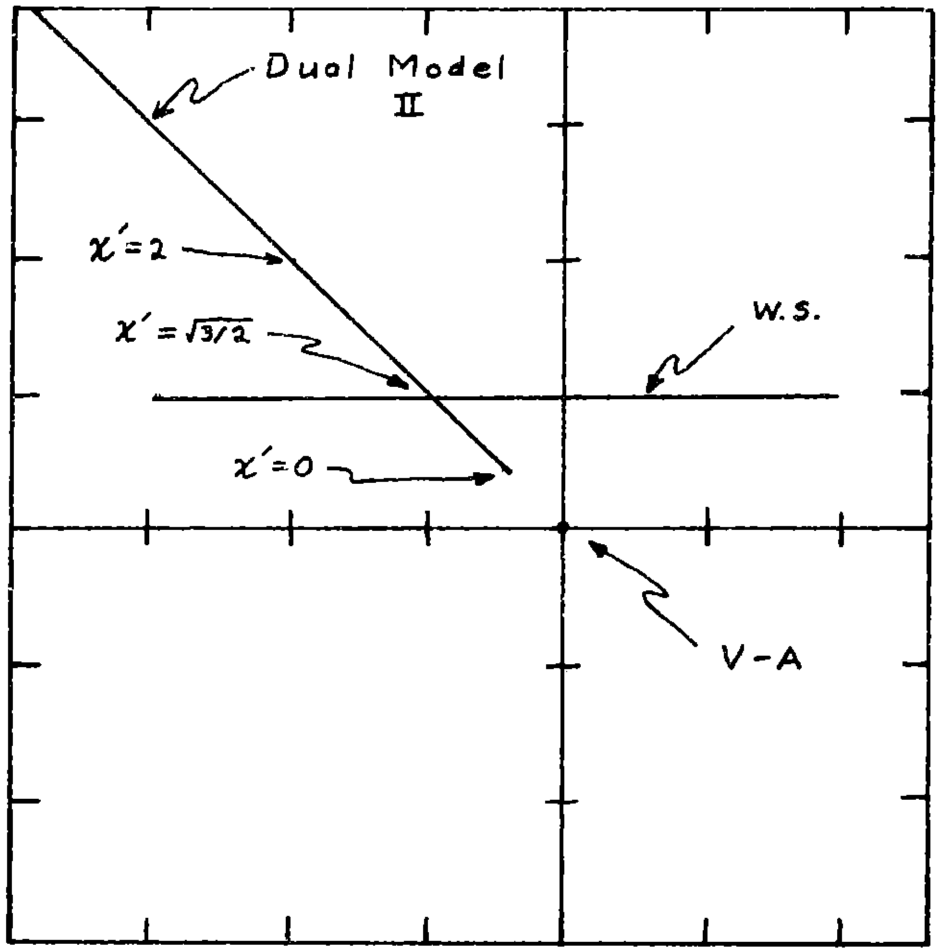
$$\mathcal{L}_{eff} \cong 2\Delta_\zeta\frac{G}{\sqrt{2}}\{\bar{n}(1-\gamma^5)[p\bar{p} + \bar{n}n + \bar{c}c + \bar{\lambda}\lambda](1+\gamma^5)\lambda + h.c. \qquad (3.49)$$

Figure (2)



A plot of C_V vs. C_A for the process $\bar{\nu}_e + e \to \bar{\nu}_e + e$. The shaded region is the experimentally allowed region. The diagonal line which ends at $(\frac{6}{5}, \frac{4}{5})$ is the prediction of the Dual Model for all values of the parameter x. In the V-A theory $C_V = C_A = 1$, while in the Weinberg-Salam theory $C_A = \frac{1}{2}$ and $\frac{1}{2} \leq C_V \leq \frac{5}{2}$. For $x < \frac{1}{2}$ the Dual Model is in the allowed region.

Figure (3)



A plot of C'_V vs. C'_A for the process $\nu_\mu+e\rightarrow\nu_\mu+e$. The diagonal line which ends at $(-\frac{1}{5}, \frac{1}{5})$ is the prediction of the Dual Model for all values of the parameter x'. In V-A theory $C'_V=C'_A=0$, while in the Weinberg-Salam theory $C'_A=\frac{1}{2}$ and $-\frac{3}{2}\leq C'_V\leq\frac{1}{2}$.

where $\Delta_\zeta \equiv 1/5(\frac{M_\xi}{M_\zeta})^2 \cos\gamma_n \cos\gamma_\lambda \cos^2\alpha$. We have assumed that no appreciable mass splitting occurs within the ζ multiplet. The above expression transforms purely as I=1/2. Note that the expression within the brackets is SU(4) invariant and therefore an isospin singlet also. In the above discussion we set the Cabibbo angle equal to zero. The appearance of the SU(4) singlet, however, insures that the above interaction is invariant under Cabibbo rotations.

Thus we see that our weak spurions are, indeed, successful in introducing terms in our weak effective Hamiltonian which explains the $\Delta I=1/2$ rule. Now we are able to understand how the $\Delta I=1/2$ transitions are able to avoid suppression by the Cabibbo angle. These processes may proceed through the above interaction. On the other hand, processes which have $\Delta I \geq 3/2$ and semileptonic processes proceed through vector exchange. The effective interaction Lagrangian for these processes is proportional to $\sin\theta_C \cos\theta_C$. Therefore, these processes appear anomalously weak when compared to the $\Delta I=1/2$ decays.

The appearance of the SU(4) singlet can be traced back to the assumption of the mass degeneracy of the ζ multiplet. This assumption is not only important in guaranteeing the pure I=1/2 nature of the interaction but it also insures the absence of $|\Delta S|=2$ processes. Such processes could arise if the masses of the scalars ζ_n and ζ_λ were considerably different.

One other point that we note is that the choice of ϕ-angles (and therefore coupling constants) is not unique. It is possible to produce a pure $|\Delta I|=1/2$, $|\Delta S|=1$ interaction for other values of these parameters. By making our choice, the masses of the ξ particles are determined by the decay rates of the charmed hadrons. Furthermore if the masses of the particles of the ξ multiplet are comparable to that of the ζ particles there exist a $\Delta I=1/2$ rule for the decay of the charmed hadrons. This scalar Lagrangian also allows the charmed hadrons to decay into strange and nonstrange channels with comparable ratios.

Returning now to the effective Lagrangian which may describe the usual $\Delta I=1/2$ rule, we may perform a Fierz transformation and obtain:

$$\mathcal{L}_{eff} = \Delta_\xi \frac{G}{\sqrt{2}} \; \bar{q}\, \gamma^\alpha (1+\gamma^5) F_6\, q \; \bar{q}\, \gamma_\alpha\, (1-\gamma^5)\, q \tag{3.50}$$

where $q\equiv(p,n,\lambda,c)$ and F_6 is the generator of SU(4) which transforms like the K_L^0 meson. In this form, it is easy to see that the Lee-Sugawara relation is satisfied[11]. Furthermore we also have the SU(4) generalization of octet dominance. This is therefore a satisfactory Lagrangian with which to explain the $\Delta I=1/2$ rule.

We may now address the question of how the effective Lagrangian in equation (3.49) transforms under chiral SU(4)$\otimes$SU(4). If we let q undergo the following infinitesimal transformations;

$$\text{I.} \quad q \longrightarrow q - i\,\tfrac{1}{2}\Lambda^a F_a\, q \qquad\qquad |\Lambda^a| << 1$$

$$\text{II.} \quad q \longrightarrow q - i\,\tfrac{1}{2}\Lambda^a F_a\, \gamma^5 q \tag{3.51}$$

then we find that the first order change in $\mathcal{L}_{eff}$ is given by;

$$i\,\tfrac{1}{2}\Lambda^a \Delta_\xi \frac{G}{\sqrt{2}} \; \bar{q}\, \gamma^\alpha (1+\gamma^5)[F_a, F_6]\, q \; \bar{q}\, \gamma_\alpha\, (1-\gamma^5)\, q \tag{3.52}$$

independent of which transformation we use. We can easily see that the generators F_a and $F_a\gamma^5$ satisfy the usual commutator algebra of chiral SU(2)$\otimes$SU(2). We may, therefore, conclude that

$$[\,Q_a, \mathcal{L}_{eff}\,] = [\,Q_a^5, \mathcal{L}_{eff}\,] \tag{3.53}$$

if Q_a and Q_a^5 are the generators of chiral SU(2)$\otimes$SU(2). In a paper by Golowich and Holstein[12], four classes of $|\Delta S|=1$ Lagrangians are defined. If we make allowances for our notation, then our total $|\Delta S|=1$ Lagrangian, derived from both scalar and vector exchange, is a member of the second class of Lagrangians. However, Golowich and Holstein concluded that, experimentally, class one Lagrangians are preferred. So our suggestion fails at this point. There is one final observation which we note. Our suggested, effective Lagrangian contains terms which are quartic in the "unflavored" quark operators. As far as we know, in this respect our suggestion is unique.

Conclusions

The present work extends the dual model of lepton-lepton scattering presented in Ref. [1]. The model has many attractive features but these are gained at an expensive price: the multitudinous spurions. The same ideas about scalar exchange used in this model may also be applied directly to the Weinberg-Salam model[13]. One could construct a hybrid model by taking the Weinberg-Salam model and coupling it to a spin-zero exchange model similar to that proposed by Dicus, Segré, and Teplitz[14]. A model of this sort would share many of the features of our dual model. But again the number of scalar multiplets (two) needed to explain the $\Delta I=1/2$ rule may tend to effect the belivability of such a model. This is not, however, the first time that spin-zero exchange has been proposed as the explanation of the $\Delta I=1/2$ rule among the hadrons[15]. Perhaps, the most interesting result of this paper is that spin-zero exchange may play a considerable role in the weak interaction.

IV. A $\Delta I=1/2$ Rule in the Weinberg-Salam Model

Here we show how the $\Delta I=1/2$ rule may be embedded in the Weinberg-Salam model without adding new quarks, invoking dynamical enhancement, or disturbing renormalizability. The strategy we pursue is the addition of more scalar multiplets, "spurions" to the usual model.

The $\Delta I=1/2$ rule for nonleptonic processes enjoys ample experimental support. If we take the viewpoint that this selection rule arises not as a dynamic effect but instead from the exchange of some elementary, weakly interacting particle, we are confronted with a problem. In constructing the simplest model of the weak interactions, the $\Delta I=1/2$ rule still appears anomalous. Various explanations, all using quarks of "exotic" flavors[16], have been suggested. Although these suggestions do, indeed, lead to effective Lagrangians which transform as isospin doublets, they all, necessarily, are only quadratic in the ordinary quark operators. As such, these Lagrangians require the spontaneous creation of ordinary quark-antiquark pairs to explain the $\Delta I=1/2$ rule among the observed hadrons. In this comment, we would like to offer a suggestion which leads to an effective interaction which is quartic in the ordinary quark operators.

Consider adding to the Weinberg-Salam model[13], the following terms.

$$\mathcal{L} = -2\sqrt{2} f \left[\bar{p}_R \zeta_p^\dagger + \bar{n}_R \zeta_n^\dagger + \bar{\lambda}_R \zeta_\lambda^\dagger + \bar{c}_R \zeta_c^\dagger \right] \eta_L + h.c. \tag{4.1}$$

Here η_L denotes the left-handed, weak, isotopic doublet composed of the proton quark and the Cabibbo-rotated neutron quark. The scalar multiplets ζ_p, ζ_n, ζ_λ, and ζ_c we will refer to as "spurions"[9]. The spurions ζ_p and ζ_c transform as $(-1, \frac{1}{2})$ under the $U_Y(1) \otimes SU_W(2)$ gauge group. The remaining spurions belong to the $(1,\frac{1}{2})$ representation. The first point we note is that the addition of these terms to the usual Weinberg-Salam model does not disturb its renormalizability.

Now, if we consider the effective $|\Delta S|=1$ Lagrangian arising from equation (1), we find:

$$\mathcal{L}_{eff} = (f/M^2) \cos\theta_c \sin\theta_c \; J^{\alpha}(\tfrac{1}{2}) J_{\alpha}(0) \tag{4.2a}$$

$$J^{\alpha}(\tfrac{1}{2}) = \bar{\lambda}\gamma^{\alpha}(1-\gamma^5)n + \bar{n}\gamma^{\alpha}(1-\gamma^5)\lambda \tag{4.2b}$$

$$\begin{aligned} J^{\alpha}(0) = {} & p\gamma^{\alpha}(1+\gamma^5)p + \bar{n}\gamma^{\alpha}(1+\gamma^5)n \\ & + \bar{\lambda}\gamma^{\alpha}(1+\gamma^5)\lambda + c\,\gamma^{\alpha}(1+\gamma^5)c \end{aligned} \tag{4.2c}$$

In arriving at equation (4.2a) we have performed a Fierz transformation and assumed that the masses of the spurion multiplets are degenerate. This interaction could also account for the anomalous strength of the nonleptonic decay modes of the strange particles. Of course, the two parameters f and M must be chosen so that the numerical factor $(f/M^2)\sin\theta_c$ is of the order of $G/\sqrt{2}$. We find a further restriction by requiring that equation (1) not lead to contributions to $\Delta S=2$ amplitudes which are intolerably large.

We now turn to the question of how this effective Lagrangian transforms under chiral $SU(4)\otimes SU(4)$. From the form of the above equations, we may verify that we have a left-handed current, transforming as an isotopic doublet, which is coupled to a right-handed current, transforming as an SU(4) singlet. Since this is so, the usual results on nonleptonic decays derived from current algebra and partial conservation of the axial-vector current (PCAC), remain intact. Furthermore, the total $|\Delta S|=1$, effective Lagrangian, derived from both spurions and vector exchange, belongs to the first class of Lagrangians as defined by Golowich and Holstein[12]. This is important since Class I Lagrangians seem experimentally preferred. We may note from equation (2a) that we have quindecuplet dominance, the SU(4) generalization of octet dominance.

We make a final observation. The form of equation (1) is exactly what we would expect if the right-handed quarks are the members of some multiplet which carry additional weak quantum numbers. The conservation of these numbers would then explain the mass degeneracy of the spurion multiplets.

V. Basic Supersymmetry

Graded Lie algebras arise naturally with an extension of the complex number system. In fact, this is the crucial step in the representation of such algebras. The idea of extending known number systems to new number systems is as primitive as passing from the counting numbers to rational numbers. In order to represent the elements of graded Lie algebras it is convenient to introduce the concept of a fermionic number. We will customarily denote such a number by the symbol θ. This is very similar to the procedure by which we introduce the symbol i in order to be able to represent the complex numbers. But for our purposes we introduce four of these fermionic numbers $\theta^a (a=-4, \ldots, -1)$. We must specify the algebraic properties of these quantities. Once again we have the analogous procedure for the complex numbers where we specify

$$i^2 = -1 \tag{5.1}$$

But for our purposes, since we have four of these "new" numbers, we must follow a procedure which more closely resembles the situation with quaternions. Thus, we specify that the fermionic numbers satisfy the equation below.

$$\{\theta^a, \theta^b\} = 0 \tag{5.2}$$

Thus the product of two independent fermi numbers is independent of both 1 and θ^a. Similarly the product of three independent fermi numbers is independent of 1, θ^a, and $\theta^a\theta^b (a<b)$. Thus we see that the elements

$$1 \qquad \theta^a \qquad \theta^a\theta^b \qquad \theta^a\theta^b\theta^c \qquad \theta^a\theta^b\theta^c\theta^d \tag{5.3}$$

for $a<b<c<d$ form a Grassman algebra. Also we see that the product of more than four θ's must vanish.

Having introduced these four independent fermi numbers we now consider another structure, a fermi-bose superspace. A superspace of particular interest is an eight-dimensional space $\{X^M : X^M(\theta^m, \chi^\mu)\}$ where the θ^m's are the fermionic co-ordinates of superspace and the χ^μ's are the bosonic co-ordinates. Furthermore, we identify the bosonic co-ordinates with the co-ordinates of space time.

There is additional structure to this space. In identifying the bosonic co-ordinates of superspace with the co-ordinates of spacetime, we have specified how that sector of superspace transforms under Lorentz transformations. We do not want the fermionic of superspace to be an internal space but to be a nontrivial representation of the Lorentz group. This is achieved by assigning the fermionic co-ordinates to the spinor representation of O(3,1). Thus, the generators of the Lorentz transformations on the superspace are given by

$$\begin{aligned} M_{\mu\nu} &= -i\,(\chi_\mu\,\partial_\nu - \chi_\nu\,\partial_\mu) + \tfrac{1}{2}\,(\bar\theta\,\sigma_{\mu\nu}\,\bar\partial) \\ &= (L_{\mu\nu})_\alpha{}^\beta\,\chi^\alpha\,\partial_\beta + \tfrac{1}{2}\,(\gamma^\circ\sigma_{\mu\nu}\,\gamma^\circ)_a{}^b\,\theta^a\,\partial_b \\ &= (L_{\mu\nu})_A{}^B\,X^A\,\partial_B \end{aligned} \tag{5.4}$$

where we have introduced the constant super matrix $(L_{\mu\nu})_A^B$. In order to interpret this expression we have had to introduce a Majorana representation of the Dirac matrices. Furthermore, we have chosen a representation where complex conjugation of a Dirac spinor is equivalent charge conjugation. This implies that the charge conjugation matrix is simply $-(\gamma^o)$. The negative of this matrix has been used to raise and lower the indices of the fermionic co-ordinates

$$\bar\theta_m = (\gamma^\circ)_{\ell m}\,\theta^\ell \tag{5.5}$$

and the gradient with respect to these co-ordinates

$$\bar\partial^\ell = (\gamma^\circ)^{\ell m}(\partial/\partial\theta^m) = (\gamma^\circ)^{\ell m}\,\partial_m \tag{5.6}$$

according to these rules. Thus we have a generalization of the Minkowskian metric given by

$$\eta_{MN} \equiv \begin{bmatrix} (\gamma^0)_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{bmatrix} \tag{5.7}$$

With this eight dimensional superspace as a carrier space, it is possible to construct a differential representation of a graded Lie algebra. The generators of this algebra are

$$S^m = -i[\, (\gamma^0)^{mn}\partial_n + i\tfrac{1}{2}(\gamma^\nu\theta)^m \partial_\nu \,]$$

$$P_\mu = -i\,\partial_\mu$$

$$M_{\mu\nu} = -i\,(x_\mu\partial_\nu - x_\nu\partial_\mu) + \tfrac{1}{2}(\bar\theta\,\sigma_{\mu\nu}\,\bar\partial) \tag{5.8}$$

But the bracketing operation for these generators is not the usual Lie commutator. Instead, it is necessary to introduce a graded Lie commutator which we denote by $[\ ,\ \}$. Before defining this operator, we define a mapping σ from the set of generators to the set $(0,1)$. Let A be a generator of a graded Lie algebra. We define σ such that

$$\sigma(A) \equiv \begin{cases} 1; & \text{if } A \text{ contains an odd number of fermionic factors} \\ 0; & \text{if } A \text{ contains an even number of fermionic factors and/or contains bosonic factors} \end{cases}$$

The graded Lie bracket is defined by the relation

$$[A, B\} = AB - (-)^{\sigma(A)\sigma(B)} BA \tag{5.9}$$

so that it is simply a commutator unless both operators are fermionic. Under this bracketing operation we find the following algebra.

$$[P_\mu, P_\nu \} = 0$$

$$[P_\lambda, M_{\mu\nu} \} = -i\, \eta_{\mu\lambda} P_\nu + i\, \eta_{\nu\lambda} P_\mu$$

$$[M_{\kappa\lambda}, M_{\mu\nu} \} = -i\, \eta_{\mu\kappa} M_{\nu\lambda} + i\, \eta_{\nu\kappa} M_{\mu\lambda} - i\, \eta_{\mu\lambda} M_{\kappa\nu} + i\, \eta_{\nu\lambda} M_{\kappa\mu}$$

$$[P_\mu, S^m \} = 0$$

$$[M_{\mu\nu}, S^m \} = -\tfrac{1}{2} (\sigma_{\mu\nu})^m{}_n S^n$$

$$[S^m, S^n \} = -(\gamma^\nu \gamma^0)^{mn} P_\nu \tag{5.10}$$

The first three relationships are the usual ones obtained for the Poincaré group. The fifth one shows that S^m transforms like the fermionic sector of superspace as the spinor representation of the Poincaré group. It is the last relationship which is most remarkable. We see that the product of two successive fermionic translations is an ordinary bosonic translation. We may apply the fermionic translation to the eight-dimensional superspace. Let ϵ^m be a constant, real fermi number. Under the transformation, $\mathfrak{T}(\epsilon)$

$$\mathfrak{T}(\epsilon) \equiv \exp[\, i \bar{\epsilon} S\,] = \exp[\, i \epsilon^m (\gamma^0)_{mn} S^n\,] \tag{5.11}$$

the superspace transforms as

$$X'^M = \exp[\, i\bar{\epsilon} S\,]\, X^M \exp[-i\bar{\epsilon} S\,]$$

$$(\theta'^m, x'^\mu) = (\theta^m + \epsilon^m,\ x^\mu + i\tfrac{1}{2}(\bar{\epsilon}\gamma^\mu\theta)\,) \tag{5.12}$$

where the second line is for infinitesimal values of ϵ. In order for the transformed superspace to be real, we must interpret the action of complex conjugation in a new manner. That is, complex conjugation on the superspace not only has its usual action ($i \leftrightarrow -i$) but also re-orders products of fermionic numbers. With this redefinition, the factor $i\frac{1}{2}(\bar{\epsilon}\gamma^{\mu}\theta)$ may be considered real and the reality of the superspace is preserved.

Thus, we see that the eight-dimensional superspace provides an adequate carrier space over which representations of the Poincaré group may be constructed. The entire conformal group may also be represented in superspace. Consider the set of generators given by

$$D \equiv -i\left[x^{\lambda}\partial_{\lambda} + \tfrac{1}{2}\theta^{\ell}\partial_{\ell} \right]$$

$$P_{\mu} \equiv -i\,\partial_{\mu}$$

$$K_{\mu} \equiv -i\left[x^{2}\partial_{\mu} - 2x_{\mu}x^{\nu}\partial_{\nu} + x^{\nu}\bar{\theta}\gamma_{\mu}\gamma_{\nu}\bar{\partial} - i\tfrac{1}{2}\bar{\theta}\theta\,\bar{\theta}\gamma_{\mu}\bar{\partial} + \tfrac{1}{8}(\bar{\theta}\theta)^{2}\partial_{\mu} \right]$$

$$M_{\mu\nu} \equiv -i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) + \tfrac{1}{2}(\bar{\theta}\sigma_{\mu\nu}\bar{\partial}) \tag{5.13}$$

It may be verified that these generators satisfy the commutator algebra

$$[D,D\} = [P_{\mu},P_{\nu}\} = [K_{\mu},K_{\nu}\} = [D,M_{\mu\nu}\} = 0$$

$$[D,P_{\mu}\} = iP_{\mu} \qquad\qquad [D,K_{\mu}\} = -iK_{\mu}$$

$$[P_{\lambda},M_{\mu\nu}\} = -i\eta_{\mu\lambda}P_{\nu} + i\eta_{\nu\lambda}P_{\mu}$$

$$[K_{\lambda},M_{\mu\nu}\} = -i\eta_{\mu\lambda}K_{\nu} + i\eta_{\nu\lambda}K_{\mu}$$

$$[P_{\mu},K_{\nu}\} = i2\eta_{\mu\nu}D - i2M_{\mu\nu}$$

$$[M_{\kappa\lambda},M_{\mu\nu}\} = -i\eta_{\mu\kappa}M_{\nu\lambda} + i\eta_{\nu\kappa}M_{\mu\lambda} - i\eta_{\mu\lambda}M_{\kappa\nu} + i\eta_{\nu\lambda}M_{\kappa\mu} \tag{5.14}$$

which is, of course, the algebra of the conformal group. But, we have not considered the results of bracketing the fermionic translations with all of the generators of the conformal subgroup. If we bracket D with S^m we find.

$$[\, D, S^m \} = i \tfrac{1}{2} S^m \tag{5.15}$$

On the other hand, the computation of the special conformal generator bracketed with the fermionic translation generator yields

$$[\, K_\mu, S^m \} = -(\gamma_\mu)^m{}_n \, R^n \tag{5.16}$$

where

$$R^n \equiv -i[\; x^\nu (\gamma_\nu S)^n - \tfrac{1}{2}(\bar\theta\theta) D^n - \tfrac{1}{2}(\bar\theta\gamma^5\theta)(\gamma^5\bar\partial)^n + \tfrac{1}{4}\,\bar\theta\gamma^5\gamma^\rho\theta\;(\gamma^5\gamma_\rho\bar\partial)^n \;]$$

$$D^n \equiv \bar\partial^n - i\tfrac{1}{2}(\not\partial\theta)^n \tag{5.17}$$

Thus, we must add another fermionic generator, R^n, which we may refer to as the fermionic special conformal generator. This new spinorial generator satisfies the following bracket algebra with the conformal subgroup generators.

$$[\, D, R^m \} = -i\tfrac{1}{2} R^m$$

$$[\, P_\mu, R^m \} = -(\gamma_\mu)^m{}_n \, S^n$$

$$[\, K_\mu, R^m \} = 0$$

$$[\, M_{\mu\nu}, R^m \} = -\tfrac{1}{2}(\sigma_{\mu\nu})^m{}_n \, R^n \tag{5.18}$$

If we bracket R^n with S^m, we find

$$[S^m, R^n\} = -i(\gamma^0)^{mn} D - \tfrac{1}{2}(\sigma^{\mu\nu}\gamma^0)^{mn} M_{\mu\nu} + \tfrac{3}{2}(\gamma^5\gamma^0)^{mn} \Pi$$

$$\Pi \equiv \bar{\theta}\gamma^5\bar{\partial} \qquad (5.19)$$

Now we find a new bosonic generator, Π, which we may call the generator of chiral transformations. This generator has the algebra

$$[D,\Pi\} = [P_\mu,\Pi\} = [K_\mu,\Pi\} = [M_{\mu\nu},\Pi\} = 0 \qquad (5.20)$$

with the conformal subgroup generators. For the spinorial generators we find

$$[\Pi, S^m\} = -(\gamma^5)^m{}_n S^n$$

$$[\Pi, R^m\} = (\gamma^5)^m{}_n R^n \qquad (5.21)$$

Finally the bracket of R^m with R^n gives the result

$$[R^m, R^n\} = -(\gamma^\mu\gamma^0)^{mn} K_\nu \qquad (5.22)$$

and we see that the graded algebra has finally closed. Thus, we have obtained the full, twenty-four element, graded Lie algebra of Wess and Zumino[17].

It can be seen that the eight-dimensional superspace admits this graded algebra in the same way as Minkowski space admits the conformal algebra. It should be noted that the conformal subgroup does not completely fix the superconformal group. This can be seen, for example, by noting that the transformation

$$K_\mu \longrightarrow K_\mu + \alpha\ \bar{\theta}\theta\,\bar{\theta}\gamma_\mu\bar{\partial} + \beta\ (\bar{\theta}\theta)^2\partial_\mu \qquad (5.23)$$

will leave the conformal subalgebra invariant for arbitrary bosonic values of α and β. But under this transformation the complete algebra would require the existence of spinor-vector generators in order to close.

It is possible to illustrate the superconformal group in a very concise manner. We may classify all of the generators according to their Lorentz transformation properties and dilation properties. The result of this classification is illustrated in the diagram (4). This diagram shows a marked similarity to a weight diagram in SU(3). It can be seen how the fermionic generators are the "square root" of the bosonic translation and special conformal generators. If we borrow on the SU(3) analogy a bit more, we may say that the fermionic co-ordinates of superspace describe the "internal degrees" of freedom for a point in spacetime just as quarks describe the internal variables of hadrons. Thus, superspace may be regarded as an attempt to describe a spacetime composed of nonclassical points.

Within supersymmetry, an important role is played by an operator which may be referred to as the fermionic gradient. Explicitly, this operator is given by

$$D^m = \bar{\partial}^m - i\tfrac{1}{2}(\not{\partial}\theta)^m \tag{5.24}$$

and it may be shown that this operator satisfies the relations below.

$$[P_\mu, D^m\} = [S^m, D^m\} = 0$$

$$[M_{\mu\nu}, D^m\} = -\tfrac{1}{2}(\sigma_{\mu\nu})^m{}_n D^n \tag{5.25}$$

This gradient operator, unlike the bosonic gradient, is a representation of a nontrivial algebra. We may demonstrate that

$$[D^a, D^b\} = i(\gamma^\mu\gamma^0)^{ab}\partial_\mu \tag{5.26}$$

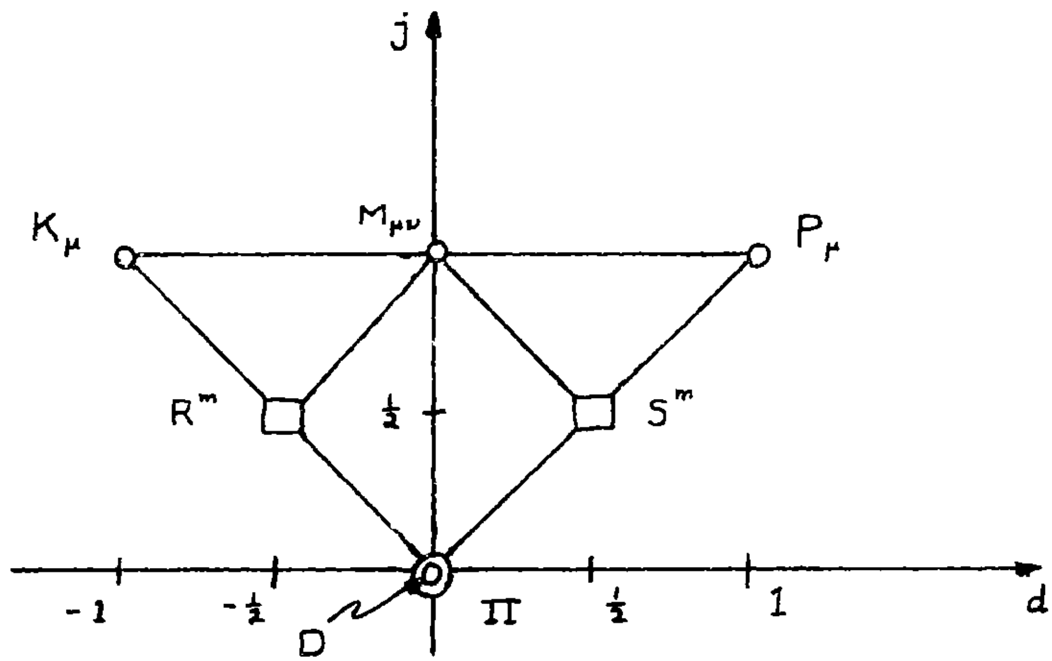
from which it follows that

$$[D^a_\pm, D^b_\pm\} = 0; [D^a_\pm, D^b_\mp\} = [\tfrac{1}{2}(1 \pm \gamma^5)\gamma^\mu\gamma^0]^{ab}\partial_\mu \tag{5.27}$$

where $D^a_\pm \equiv [\frac{1}{2}(1\pm\gamma^5)D]^a$.

Additionally, it follows from definition that

Figure (4)



○ ≡ Bosonic generator

□ ≡ Fermionic generator

Diagram illustrating the generators of the superconformal group. The classification of these operators has been made according to intrinsic spin and dimensionality.

$$D^a D^b = i\tfrac{1}{2}(\gamma^\mu\gamma^0)^{ab}\partial_\mu + \tfrac{1}{4}(\gamma^0)^{ab}(\bar{D}D)$$
$$+ \tfrac{1}{4}(\gamma^5\gamma^0)^{ab}(\bar{D}\gamma^5 D) + \tfrac{1}{4}(\gamma^5\gamma^\mu\gamma^0)^{ab}(\bar{D}\gamma^5\gamma_\mu D) \quad (5.28)$$

With this relation as a starting point, there are a number of identities which one may prove for the fermionic gradient. Some of the most important ones are

$$\bar{D}\gamma_\mu D = i2\,\partial_\mu$$
$$\bar{D}\sigma_{\mu\nu}D = 0$$
$$D^a(\bar{D}D) = (\bar{D}D)D^a - i2(\not{\partial}D)^a$$
$$D^a(\bar{D}\gamma^5 D) = -(\bar{D}D)(\gamma^5 D)^a + i2(\gamma^5\not{\partial}D)^a$$
$$(\bar{D}\gamma^5\gamma_\mu D)D^a = -(\bar{D}D)(\gamma^5\gamma_\mu D)^a - 2\partial^\lambda(\gamma^5\sigma_{\lambda\mu}D)^a$$
$$(\bar{D}D)(\gamma^5 D)^a = -(\bar{D}\gamma^5 D)D^a$$
$$D^a(\bar{D}\gamma^5\gamma_\mu D) = -(\bar{D}D)(\gamma^5\gamma_\mu D)^a - i2\partial_\mu(\gamma^5 D)^a$$

$$(\bar{D}D)(\bar{D}\gamma^5\gamma_\mu D) = -(\bar{D}\gamma^5\gamma_\mu D)(\bar{D}D) = -i2\partial_\mu(\bar{D}\gamma^5 D)$$

$$(\bar{D}D)(\bar{D}\gamma^5 D) = -(\bar{D}\gamma^5 D)(\bar{D}D) = i2\partial_\mu(\bar{D}\gamma^5\gamma^\mu D)$$

$$(\bar{D}\gamma^5\gamma_\mu D)(\bar{D}\gamma^5\gamma_\nu D) = -\eta_{\mu\nu}(\bar{D}D)^2 + 4(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)$$
$$+ 2\epsilon_{\mu\nu\kappa\lambda}\partial^\kappa(\bar{D}\gamma^5\gamma^\lambda D)$$

$$(\bar{D}D)^2 = -(\bar{D}\gamma^5 D)^2$$

$$(D^a)(\bar{D}D)^2 = -i2(\not{\partial}D)^a(\bar{D}D) \quad (5.29)$$
$$(\bar{D}D)^3 = 4\partial^2(\bar{D}D)$$

The algebra of this derivative will play an important role in the construction of Lagrangian models.

Since this derivative is fermionic it obeys a generalized Liebnitz rule,

$$D^a(\Phi_1 \Phi_2) = (D^a \Phi_1)\Phi_2 \pm \Phi_1 (D^a \Phi_2) \tag{5.30}$$

where the plus or minus sign is chosen depending on whether Φ_1 is bosonic or fermionic.

The representation of the Dirac matrices which shall be used throughout is given by

$$\begin{aligned}
\gamma^\mu &\equiv (\sigma^3 \otimes \sigma^2, iI \otimes \sigma^1, i\sigma^2 \otimes \sigma^2, iI \otimes \sigma^3) \\
\gamma^5 &\equiv i\tfrac{1}{4}\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \sigma^1 \otimes \sigma^2 \\
\sigma^{\mu\nu} &\equiv i\tfrac{1}{2}[\gamma^\mu, \gamma^\nu] = i\tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)
\end{aligned} \tag{5.31}$$

then the full set of Dirac matrices is given by

$$\mathbf{1}, \quad \gamma^\mu, \quad \sigma^{\mu\nu}, \quad \gamma^5\gamma^\mu, \quad \gamma^5 \tag{5.32}$$

In this representation we find

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \qquad diag(\eta^{\mu\nu}) = (-1, 1, 1, 1) \tag{5.33}$$

Under hermitian conjugation, these matrices transform as

$$\begin{aligned}
\mathbf{1}^\dagger &= \mathbf{1} & \gamma^{5\dagger} &= \gamma^5 \\
\gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0 & (\gamma^5\gamma^\mu)^\dagger &= \gamma^0\gamma^5\gamma^\mu\gamma^0
\end{aligned}$$

$$(\sigma_{\mu\nu})^\dagger = \gamma^0\sigma_{\mu\nu}\gamma^0 \tag{5.34}$$

Furthermore, under transposition these Dirac matrices have the properties

$$(\gamma^0)^t = -(\gamma^0) \qquad (\gamma^0\gamma^5)^t = -(\gamma^0\gamma^5)$$

$$(\gamma^0\gamma^\mu)^t = (\gamma^0\gamma^\mu) \qquad (\gamma^0\gamma^5\gamma^\mu)^t = -(\gamma^0\gamma^5\gamma^\mu)$$

$$(\gamma^0\sigma_{\mu\nu})^t = (\gamma^0\sigma^{\mu\nu}) \tag{5.35}$$

where t denotes transposition. For this representation, there is an orthogonality relation,

$$\frac{1}{4}\,\mathrm{Tr}\{\Gamma_A\Gamma_B^{\dagger}\} = \delta_{AB} \tag{5.36}$$

if we restrict $\sigma^{\mu\nu}$ so that $\mu\leq\nu$. This in turn implies a completeness relation given by,

$$\delta_j^a\,\delta_b^i = \frac{1}{4}\sum_A (\Gamma_A)^a{}_b\,(\Gamma_A^{\dagger})^i{}_j \tag{5.37}$$

This equation can next be used to derive the following Fierz rearrangement matrix.

$$-\frac{1}{4}\begin{array}{cc} \begin{array}{ccccc} S & V & T & A & P \end{array} & \\ \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ -4 & -2 & 0 & -2 & 4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & -2 & -4 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix} & \begin{array}{c} S \\ V \\ T \\ A \\ P \end{array} \end{array} \tag{5.38}$$

The bilinear covariants appropriate here are

$$\begin{aligned}
S(1\,2\,3\,4) &\equiv (\bar\psi_1\psi_2)(\bar\psi_3\psi_4)\\
V(1\,2\,3\,4) &\equiv (\bar\psi_1\gamma^\mu\psi_2)(\bar\psi_3\gamma_\mu\psi_4)\\
T(1\,2\,3\,4) &\equiv \tfrac{1}{2}(\bar\psi_1\sigma^{\mu\nu}\psi_2)(\bar\psi_3\sigma_{\mu\nu}\psi_4)\\
A(1\,2\,3\,4) &\equiv (\bar\psi_1\gamma^5\gamma^\mu\psi_2)(\bar\psi_3\gamma^5\gamma_\mu\psi_4)\\
P(1\,2\,3\,4) &\equiv (\bar\psi_1\gamma^5\psi_2)(\bar\psi_3\gamma^5\psi_4)
\end{aligned} \tag{5.39}$$

Note the minus sign preceding the matrix is the consequence of the anticommutivity of Ψ_1, Ψ_2, Ψ_3, and Ψ_4. Furthermore, the sum on μ and ν in the tensor T is unrestricted.

Under charge conjugation, time reversal, and parity transformations, the following transformations may be defined for a Dirac field $\Psi(t,\vec{\chi})$

$$C: \quad \psi(t,\vec{x}) \longrightarrow -(\gamma^\circ)[\bar{\psi}(t,\vec{x})]^t = \psi^*(t,\vec{x})$$

$$T: \quad \psi(t,\vec{x}) \longrightarrow \gamma^5\gamma^\circ\,\psi^*(-t,\vec{x})$$

$$P: \quad \psi(t,\vec{x}) \longrightarrow i\gamma^\circ\,\psi(t,-\vec{x}) \tag{5.40}$$

Therefore it may be seen that

$$CTP: \quad \psi(t,\vec{x}) \longrightarrow i\gamma^5\,\psi(-t,-\vec{x}) \tag{5.41}$$

The fermionic co-ordinates of superspace transform as the components of a relativistic spinor. This implies that the basis of a grassman algebra which is given by

$$1 \;;\; \theta^a \;;\; \theta^a\theta^b \;;\; \theta^a\theta^b\theta^c \;;\; \theta^a\theta^b\theta^c\theta^d \tag{5.42}$$

$(a\leq b\leq c\leq d)$ does not transform irreducibly under a Lorentz transformation. But the basis given by

$$1 \;;\; \theta^a \;;\; \bar{\theta}\theta \;,\; \bar{\theta}\gamma^5\theta \;,\; \bar{\theta}\gamma^5\gamma_\mu\theta \;;\; \bar{\theta}\theta\,\theta^a \;;\; (\bar{\theta}\theta)^2 \tag{5.43}$$

does transform irreducibly. The elements of this basis are simply linear combinations of the elements of the previous basis. It may be noted that the symmetry properties of $(\gamma^0\gamma_\mu)$ and $(\gamma^0\sigma_{\mu\nu})$ preclude the possibility of the vector or tensor from entering this basis.

With the use of the Fierz transformation, it may be shown that

$$\theta^a\theta^b = -\tfrac{1}{4}\left[\,(\gamma^\circ)^{ab}\,\bar{\theta}\theta + (\gamma^5\gamma^\circ)^{ab}\,\bar{\theta}\gamma^5\theta + (\gamma^5\gamma^\rho\gamma^\circ)^{ab}\,\bar{\theta}\gamma^5\gamma_\rho\theta\,\right]$$

$$\bar{\theta}\gamma^5\theta\,\theta^a = -\bar{\theta}\theta\,(\gamma^5\theta)^a \quad , \quad \bar{\theta}\gamma^5\gamma^\rho\theta\,\theta^a = -\bar{\theta}\theta\,(\gamma^5\gamma^\rho\theta)^a$$

$$\theta^a\theta^b\theta^c\theta^d = \frac{1}{16}\left[\,(\gamma^0)^{ab}(\gamma^0)^{cd} - (\gamma^5\gamma^0)^{ab}(\gamma^5\gamma^0)^{cd} - (\gamma^5\gamma^\rho\gamma^0)^{ab}(\gamma^5\gamma_\rho\gamma^0)^{cd}\right](\bar{\theta}\theta)^2$$

$$= \frac{1}{8}\,\epsilon^{abcd}\,(\bar{\theta}\theta)^2 \qquad (5.44)$$

where the ϵ-tensor here has its usual properties with $\epsilon^{-4,-3,-2,-1} \equiv +1$.

The following multiplication table for bilinear products of the fermionic co-ordinates may be derived.

	$\bar{\theta}\theta$	$\bar{\theta}\gamma^5\theta$	$\bar{\theta}\gamma^5\gamma_\nu\theta$
$\bar{\theta}\theta$	$(\bar{\theta}\theta)^2$	0	0
$\bar{\theta}\gamma^5\theta$	0	$-(\bar{\theta}\theta)^2$	0
$\bar{\theta}\gamma^5\gamma_\mu\theta$	0	0	$-\eta_{\mu\nu}(\bar{\theta}\theta)^2$

(5.45)

In manipulations of superfields the properties described above will often be used without reference.

VI. Superfields and Known Models

Just as it is possible to define fields in ordinary Minkowski space, it is possible to define fields in superspace. If such a field may be expanded in a power series, then because of the anticommutivity of the fermionic co-ordinates the expansion must terminate in the θ variables. Thus, we are led to define a superfield by the expansion

$$\Phi_J(X) = A_J(x) + \bar{\theta}\psi_J(x) + \tfrac{1}{4}\bar{\theta}\theta F_J(x) + i\tfrac{1}{4}\bar{\theta}\gamma^5\theta\, G_J(x)$$
$$+ \tfrac{1}{4}\bar{\theta}\gamma^5\gamma_\mu\theta A^\mu_J(x) + \tfrac{1}{4}\bar{\theta}\theta\,\bar{\theta}\chi_J(x) + \tfrac{1}{32}(\bar{\theta}\theta)^2 D_J(x) \tag{6.1}$$

where the label J may be an internal index or a Lorentz index, either vector or spinor. A single superfield contains boson and fermion fields as components. It should be noted that this form is unique with the assumption that the superfield has a power series expansion. Superfields may be transformed in a manner that is analogous to the procedure for transforming ordinary fields. For instance, a superfield may be subjected to an infinitesimal dilation. The first order variation in the superfield will be given by

$$\delta\Phi_J = (i\lambda D + \lambda d)\,\Phi_J$$
$$= \lambda(x^\nu\partial_\nu + \tfrac{1}{2}\theta^n\partial_n + d)\,\Phi_J \tag{6.2}$$

where $\lambda \ll 1$ and d is the intrinsic dimensionality of the superfield.

Let a scalar superfield be subjected to an infinitesimal fermionic translation. When this is performed on the expansion given above, we find

$$\delta A(x) = \bar{\epsilon}\,\psi(x) \qquad ; \qquad \delta D(x) = -i\,\bar{\epsilon}\,\not{\partial}\,\chi(x)$$

$$\delta F(x) = \tfrac{1}{2}\,\bar{\epsilon}\,[\,\chi(x) - i\,\not{\partial}\,\psi(x)]\ ;\ \delta G(x) = i\tfrac{1}{2}\,\bar{\epsilon}\,\gamma^5[\,\chi(x) - i\not{\partial}\,\psi(x)]$$

$$\delta A_\lambda(x) = -\tfrac{1}{2}\,\bar{\epsilon}\,\gamma^5\,[\,\gamma_\lambda\,\chi(x) - i\,\not{\partial}\,\gamma_\lambda\,\psi(x)]$$

$$\delta\psi(x) = \tfrac{1}{2}\,[\,F(x) - i\,\gamma^\lambda\,\partial_\lambda\,A(x) + \gamma^5\gamma^\lambda A_\lambda(x) + i\,\gamma^5 G(x)\,]\,\epsilon$$

$$\begin{aligned}\delta\chi(x) = \tfrac{1}{2}[\,D(x) &- i\,\gamma^\lambda\partial_\lambda F(x) + i\,\tfrac{1}{2}\,\sigma^{\lambda\mu}\,\epsilon_{\lambda\mu\alpha\beta}\,\partial^\alpha A^\beta(x) \\ &- \gamma^5\gamma^\lambda\partial_\lambda G(x) - i\,\gamma^5\,\partial^2 A_\lambda(x)\,]\,\epsilon\end{aligned} \tag{6.3}$$

It can be seen that the fermionic translation induces a rearrangement of the components of the superfield. Now it may be noted that the variation in the last component of the superfield may be written as

$$\delta D(x) = \partial_\mu[\,-i\,\bar{\epsilon}\,\gamma^\mu\,\chi(x)] \tag{6.4}$$

for a constant spinorial parameter ϵ. This property will be of particular importance to the construction of supersymmetric Lagrangians.

It can be seen that the multiplication of superfields is closed. That is, the product of a number of superfields is again a superfield. More interestingly, the fermionic derivative acting on a superfield produces also a superfield. This property is trivially satisfied by the bosonic gradient. From these properties it follows that a Lagrangian constructed from products of fermionic derivatives, bosonic derivatives and superfields is itself a superfield. The last component of the Lagrangian superfield will therefore change by a bosonic gradient under a fermionic translation. Thus, an expression of the form

$$\int d^4x\ \tfrac{1}{4}(\bar{D}D)^2\,\mathcal{L}(\,\Phi,\,\partial_\mu\Phi,\,D^n\Phi) \tag{6.5}$$

will be invariant under a fermionic translation. The four powers of the fermionic derivative just extract the last component of the Lagrangian superfield. Thus, the quantity

$$\int d^4x \ \tfrac{1}{4}(\bar{D}D)^2 \tag{6.6}$$

should be thought of as a generalized measure for superspace.

In eight-dimensional superspace, however, the expansion of a superfield given previously does not transform irreducibly under a fermionic translation. It can be shown that superfields which satisfy the equations

$$\tfrac{1}{2}(1 \pm \gamma^5)^{k}{}_{l}\, D^{l}\, \Phi_{J\mp}(X) = 0 \tag{6.7}$$

transform irreducibly. Such superfields are known as chiral superfields. All chiral superfields may be written in the form

$$\Phi_{J\pm}(X) = exp[\mp \tfrac{i}{4}\bar{\theta}\gamma^5 \not{\partial}\theta]\{ A_{J\pm}(x) + \bar{\theta}\tfrac{1}{2}(1\pm\gamma^5)\psi_{J\pm}(x) + \tfrac{1}{4}\bar{\theta}(1\pm\gamma^5)\theta\ F_{J\pm}(x)\} \tag{6.8}$$

where the exponential is defined by its power series expansion. For chiral superfields one may deduce that the term of order $(\bar{\theta}\theta)$ is transformed by a pure bosonic divergence under a fermionic translation. Furthermore, the product of any number of superfields, all of the same chirality class, is again a superfield of the same chirality class. The fermionic derivative of a chiral superfield is not a chiral superfield, however. On the other hand, the bosonic derivative acting on a chiral superfield does produce a chiral superfield. The product of two superfields of different chirality produces a general superfield and not a chiral superfield.

With some care, it is possible to produce chiral Lagrangian superfields. For these Lagrangian superfields the quantity

$$\int d^4x \ \tfrac{1}{2}(\bar{D}D) \tag{6.9}$$

is the generalized superspace measure.

For each of these superspace measures it is possible to define a super-delta function. For the non-chiral measure we define

$$\delta(X) \equiv \frac{1}{16}\,(\bar{\theta}\theta)^2\,\delta^{(4)}(x) \tag{6.10}$$

as the super-delta function. It may be verified that

$$\int d^4x\,\frac{1}{4}(\bar{D}D)^2\,\delta(X) = 1 \tag{6.11}$$

For the chiral measure the super-delta functions are defined by

$$\delta_{\pm}(X) \equiv \frac{1}{4}\,\bar{\theta}(1 \pm \gamma^5)\theta\;\;\delta^{(4)}(x) \tag{6.12}$$

and can be shown to satisfy the equations below.

$$\int d^4x\,\frac{1}{2}(\bar{D}D)\;\delta_{\pm}(X) = 1 \tag{6.13}$$

Now we may begin to consider Lagrangian models in superspace. The simplest superfield is the scalar superfield.

$$\Phi(X) = A(x) + \bar{\theta}\psi(x) + \frac{1}{4}\bar{\theta}\theta\,F(x) + i\,\frac{1}{4}\bar{\theta}\gamma^5\theta\,G(x)$$
$$+\frac{1}{4}\bar{\theta}\gamma^5\gamma^\mu\theta\,A_\mu(x) + \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\chi(x) + \frac{1}{32}(\bar{\theta}\theta)^2 D(x) \tag{6.14}$$

But as we stated previously this superfield is not irreducible. It has been shown by Sokatchev[18] that for superfields of arbitrary spin that it is possible to construct projection operators which project irreducible components from arbitrary superfields. For the scalar superfield these operators are

$$\Pi_{\pm} = \frac{1}{8\partial^2}\,(\bar{D}D)\,\bar{D}(1 \pm \gamma^5)D \tag{6.15}$$

$$\Pi_0 = \left[\,1 - \frac{1}{4\partial^2}(\bar{D}D)^2\,\right] \tag{6.16}$$

and it can be seen that these are integro-differential operators. It can be shown with the properties of the fermionic gradient that these operators satisfy the relations

$$\Pi_+ \Pi_- = \Pi_+ \Pi_\circ = \Pi_- \Pi_\circ = 0$$

$$(\Pi_+)^2 = \Pi_+ \qquad (\Pi_-)^2 = \Pi_-$$

$$(\Pi_\circ)^2 = \Pi_\circ \tag{6.17}$$

This justifies the identification of these operators as projection operators. Using the operators Π_+ or Π_- we may define chirally positive or negative, Lorentz scalar superfields.

$$\Phi_\pm(X) \equiv \Pi_\pm \Phi(X) \tag{6.18}$$

These superfields are given by the simpler expansions

$$\begin{aligned}\Phi_\pm &= A_\pm(x) + \tfrac{1}{2}\bar\theta(1\pm\gamma^5)\psi_\pm(x) + \tfrac{1}{4}\bar\theta(1\pm\gamma^5)\theta\, F_\pm(x) \\ &\quad \mp \tfrac{i}{4}\bar\theta\gamma^5\gamma^\lambda\theta\,\partial_\lambda A_\pm(x) - i\tfrac{1}{8}\bar\theta\theta\,\bar\theta\,\partial\!\!\!/(1\pm\gamma^5)\psi_\pm(x) \\ &\quad + \tfrac{1}{32}(\bar\theta\theta)^2\,\partial^2 A_\pm(x) \\ &= \exp\left[\mp\tfrac{i}{4}\bar\theta\gamma^5\partial\!\!\!/\theta\right]\left\{A_\pm(x) + \tfrac{1}{2}\bar\theta(1\pm\gamma^5)\psi_\pm(x) + \tfrac{1}{4}\bar\theta(1\pm\gamma^5)\theta F_\pm\right\}\end{aligned} \tag{6.19}$$

It should be noted that chiral superfields are intrinsically complex.

The simplest supersymmetric model known is the Wess-Zumino Φ^3 model. The Lagrangian for this model is

$$\begin{aligned}\mathcal{L}_{WZ} = \int d^4x\,\Big[&\tfrac{1}{8}(\bar D D)^2\{\Phi_+\Phi_-\} + \tfrac{1}{4}M_\circ(\bar D D)\{\Phi_+^2 + \Phi_-^2\} \\ &+ \tfrac{1}{6}g_\circ(\bar D D)\{\Phi_+^3 + \Phi_-^3\}\Big]\end{aligned} \tag{6.20}$$

where $\Phi_+ = (\Phi_-)^*$. It can be seen that the first term is of the nonchiral type; whereas the final two terms are chiral types. The restriction that relates the positive chiral superfield to the negative chiral superfield insures the hermiticity of the Lagrangian. In terms of components this restriction implies that $A_+=(A_-)^*$, $F_+=(F_-)^*$, and Ψ must be a Majorana spinor. The Lagrangian may be expressed in terms of the component fields. Some calculation reveals

$$\frac{1}{8}(\bar{D}D)^2\{\Phi_+\Phi_-\} = -|\partial_\mu A|^2 + i\frac{1}{2}\bar{\Psi}\not{\partial}\Psi + |F|^2$$

$$\frac{1}{2}M_0(\bar{D}D)\{\Phi_+^2+\Phi_-^2\} = M_0(AF + A^*F^* - \frac{1}{2}\bar{\Psi}\Psi)$$

$$\frac{1}{6}g_0(\bar{D}D)\{\Phi_+^3+\Phi_-^3\} = g_0[A^2F + (A^*)^2F^* - \frac{1}{2}A\bar{\Psi}(1+\gamma^5)\Psi - \frac{1}{2}A^*\bar{\Psi}(1-\gamma^5)\Psi] \tag{6.21}$$

Now it can be seen that the field F is an auxiliary field which may be eliminated by the use of its equation of motion.

$$F + M_0A^* + g_0(A^*)^2 = 0 \tag{6.22}$$

When this is done one obtains the Lagrangian in the form

$$\begin{aligned}\mathcal{L}_{WZ} = &-[|\partial_\mu A|^2 + M_0^2|A|^2] + \bar{\Psi}(i\not{\partial} - M_0)\Psi \\ &- g_0^2|A|^4 - g_0M_0(A + A^*)|A|^2 \\ &- g_0[A\bar{\Psi}(1+\gamma^5)\Psi + A^*\bar{\Psi}(1-\gamma^5)\Psi]\end{aligned} \tag{6.23}$$

where the spinor field has been rescaled by a factor. This model has been studied in detail[19] and it can be shown that it possesses some rather remarkable properties when quantized.

Aside from the work of Adjei and Akyeampong[20], the only theories studied thus far are those in which matter superfields are assumed to be chiral, Lorentz-scalar superfields. Since we are interested in the spinor superfield, in particular, it is of some importance that we review the spinor superfield model of Adjei and Akyeampong. This model describes the interaction of a chiral, Lorentz-spinor superfield with the chiral, Lorentz-scalar superfield of the model described above. The chiral, Lorentz-spinor superfield is given by

$$\Psi_{\pm}(\theta,x) = \exp[\mp\tfrac{i}{4}\bar{\theta}\gamma^5\not\partial\theta]\{\ \phi_{\pm}(x) + \tfrac{1}{2}\ \alpha(x)(1\pm\gamma^5)\theta$$

$$+ \tfrac{1}{4}\bar{\theta}(1\pm\gamma^5)\theta\ \zeta_{\pm}(x)\ \} \tag{6.24}$$

where $\phi_{\pm}$ and $\zeta_{\pm}$ are Dirac fields which are not themselves of a chiral nature. The sixteen-component Duffin-Kemmer-Petiau field is given by

$$\alpha(x) = s(x) - i\gamma^{\mu}v_{\mu}(x) + i\tfrac{1}{2}\sigma^{\mu\nu}t_{\mu\nu}(x)$$

$$+ \gamma^5\gamma^{\mu}a_{\mu}(x) + i\gamma^5 p(x) \tag{6.25}$$

The Lagrangian for the free spinor superfield of this model can be expressed in the form

$$\int d^4x\,[\tfrac{1}{2}(\bar{D}D)\{\ \bar{\Psi}_{+}(i\not\partial - 2M_0)\Psi_{-} + \bar{\Psi}_{-}(i\not\partial - 2M_0)\Psi_{+}\}$$

$$\tfrac{1}{4}(\bar{D}D)^2\{\ \bar{\Psi}_{+}\Psi_{+} + \bar{\Psi}_{-}\Psi_{-}\}\ \ ] \tag{6.26}$$

and the interaction of the spinor superfield with the scalar superfield is given by the equation

$$\int d^4x\ \tfrac{1}{2}(\bar{D}D)\,f_0[\ \Phi_{+}\bar{\Psi}_{-}\Psi_{+} + \Phi_{-}\bar{\Psi}_{+}\Psi_{-}\ ] \tag{6.27}$$

with f_0 as the scalar-fermion coupling constant.

Unfortunately this model turns out to be unrenormalizable. This can be partially understood by considering the propagator for the free spinor superfield. One may add to the Lagrangian the following chiral source terms

$$\int d^4x\ \tfrac{1}{2}(\bar{D}D)\,[\ \bar{\Psi}_{-}\eta_{+} + \bar{\Psi}_{+}\eta_{-} + h.c.\ ] \tag{6.28}$$

To obtain the free spinor superfield equation of motion, we simply vary the part of the Lagrangian that is quadratic in the spinor superfield. This leads to the equations

$$- \tfrac{1}{2} (\bar{D}D) \Psi_{+} + (-i \not{\partial} + 2M_0) \Psi_{-} = \eta_{-}$$

$$- \tfrac{1}{2} (\bar{D}D) \Psi_{-} + (-i \not{\partial} + 2M_0) \Psi_{+} = \eta_{+} \qquad (6.29)$$

These coupled equations may be replaced by two "second order" uncoupled equations

$$[\tfrac{1}{4} (\bar{D}D)^2 - (-i\not{\partial} + 2M_0)^2] \Psi_{\pm} =$$

$$- (-i\not{\partial} + 2M_0) \eta_{\pm} - \tfrac{1}{2} (\bar{D}D) \eta_{\mp} \qquad (6.30)$$

We may evaluate the first line of this equation by noting that the projection operators $\Pi_{\pm}$ may be inserted between the "second order" operator and the spinor superfield. Then by referring to the properties of the fermionic derivative we note

$$\tfrac{1}{4} (\bar{D}D)^2 \, \Pi_{\pm} = \partial^2 \, \Pi_{\pm} \qquad (6.31)$$

which implies that solutions to the coupled equations are,

$$\Psi_{\pm} = \frac{1}{4M_0} \left[\frac{1}{-i\not{\partial} + M_0} \right] \left[(-i \not{\partial} + 2M_0) \eta_{\pm} + \tfrac{1}{2} (\bar{D}D) \eta_{\mp} \right] \qquad (6.32)$$

If we let the source functions become chiral, super-delta functions, these become the propagators of the respective spinor superfields. By recalling that the fermionic gradient is the "square root" of the bosonic gradient, we see that for large momenta this super-propagator approaches a constant value and is undamped. Thus, naively, we expect to encounter quartic divergences in this model. A detailed analysis of this model has shown that, in fact, only quadratic and logarithmic divergences survive and render the model nonrenormalizable.

VII. The Spinor Superfield and the Hemitrion

The spinor superfield has not been extensively studied thus far. In the work of Adjei and Akyeampong[20], a Lagrangian for the interaction of the chiral, spinor superfield with the chiral, scalar superfield was examined in detail. As far as we are aware, this is the only work that has been done in this direction.

We find the spinor superfield of particular interest for two reasons. First of all, the spinor superfield is the simplest superfield which contains a component which has not been extensively used in models of the elementary particles. This component is the Rarita-Schwinger or hemitrion field. We have wondered whether simple supersymmetric considerations can generate a nontrivial model which contained the hemitrion in interaction with other fields. For instance, is there a supersymmetric model, for the electrodynamics of the hemitrion, which avoids the inconsistencies which are present in a nonsupersymmetric model[21]?

Secondly, since there exist a fermionic derivative in supersymmetry, naively, we expect to find a great variety in the possible Lagrangians one may construct for the spinor superfield.

Thus motivated by these reasons we shall begin a study of the spinor superfield.

To begin, we first give the expansion of the spinor superfield in terms of component fields

$$\Psi(\theta,x) = \phi(x) + \alpha(x)\theta + \tfrac{1}{4}\bar{\theta}\theta\,\zeta(x) + \tfrac{1}{4}\bar{\theta}\gamma^5\theta\;\gamma^5\eta(x) + i\tfrac{1}{4}\bar{\theta}\gamma^5\gamma^\lambda\theta\;\gamma^5\psi_\lambda(x) + \tfrac{1}{4}\bar{\theta}\theta\,\beta(x)\theta + \tfrac{1}{32}(\bar{\theta}\theta)^2\pi(x) \tag{7.1}$$

where $\alpha(x)$ and $\beta(x)$ are Duffin-Kemmer-Petiau fields. These matrix fields may be expanded in terms of the Dirac-Γ matrices.

$$\alpha(x) = \mathcal{s}(x) - i\gamma^{\mu}\mathcal{v}_{\mu}(x) + i\tfrac{1}{2}\sigma^{\mu\nu}\mathcal{t}_{\mu\nu}(x) + \gamma^{5}\gamma^{\mu}\mathcal{a}_{\mu}(x) + i\gamma^{5}\mathcal{p}(x)$$

$$\beta(x) = s(x) - i\gamma^{\mu}v_{\mu}(x) + i\tfrac{1}{2}\sigma^{\mu\nu}t_{\mu\nu}(x) + \gamma^{5}\gamma^{\mu}a_{\mu}(x) + i\gamma^{5}p(x) \quad (7.2)$$

Thus, we see that the spinor superfield contains a hemitrion (Ψ_λ), four hemidions ($\phi,\zeta,\eta,$ and π), two scalars ($\mathcal{s}$, and s), two vectors ($\mathcal{v}_\mu$ and v_μ), two antisymmetric tensors ($\mathcal{t}_{\mu\nu}$ and $t_{\mu\nu}$), two axial vectors ($\mathcal{a}_\mu$ and a_μ), and finally two pseudoscalars ($\mathcal{p}$ and p).

Keeping in mind that it is the hemitrion which motivated the study of this superfield, it is appropriate that we recall some features about the Rarita-Schwinger field.

The Rarita-Schwinger field describes a particle of spin 3/2. There are two ways in which we may describe such a field. We may use a multi-spinor Ψ_{abc}, where a, b, and c are spinor indices or we may use a spinor-vector $\Psi_{b\mu}$ where b is a spinor index and μ is a vector index. We shall use the latter description. In either case, however, there are subsidiary conditions which must be satisfied to insure that the field is an irreducible representation of the Poincaré group. Stated another way, these conditions insure that the field $\Psi_{b\mu}$ is purely a spin 3/2 field and not a mixture of spins 3/2 and 1/2. These conditions are

$$\gamma^{\mu}\psi_{\mu} = 0 \qquad\qquad \partial^{\mu}\psi_{\mu} = 0 \quad (7.3)$$

and give a total of eight conditions. The spinor-vector has a total of sixteen components and we see that these conditions imply that only eight components are independent. This is just the correct number to describe a particle and antiparticle of spin 3/2. The Lagrangian for a massive hemitrion may be expressed in the form

$$\bar{\psi}_{\alpha}[-\epsilon^{\alpha\beta\gamma\delta}\gamma^{5}\gamma_{\beta}\,\partial_{\gamma} + i\,m_{0}\,\sigma^{\alpha\delta}]\,\psi_{\delta} \quad (7.4)$$

and the subsidiary conditions are implied by this expression.

The simplest supersymmetric Lagrangian for the spinor superfield that we may write is

$$\int d^4x \, \tfrac{1}{4}(\bar{D}D)^2 \{ \bar{\Psi} [i \not{\partial} - M_0] \Psi \} \tag{7.5}$$

but this expression completely neglects the conditions which must be satisfied by the hemitrion. For this reason this is an unacceptable Lagrangian.

Thus, we must search for an alternate Lagrangian. After some work we can convince ourselves that

$$\int d^4x \, \tfrac{1}{4}(\bar{D}D)^2 \{ \bar{\Psi} [\tfrac{1}{2} \gamma^5 \gamma^\mu (\bar{D}\gamma^5 \gamma_\mu D) \; - \; \sqrt{2}\, M_0] \Psi \} \tag{7.6}$$

is a much more promising expression. To see this we may expand this expression in terms of the components. After some algebra we find the equation below. The details of the calculation are found in Appendix D.

$$\begin{aligned}
\int d^4x \, [& -\bar{\psi}^\lambda \epsilon_{\lambda\mu\nu\rho} \gamma^5 \gamma^\mu \partial^\nu \psi^\rho - \; i \, (\bar{\pi} \gamma_\lambda \psi^\lambda - \bar{\psi}^\lambda \gamma_\lambda \pi) \\
& + i \tfrac{1}{2} \bar{\phi} \gamma^\mu (\eta_{\mu\nu} \partial^2 - 2 \partial_\mu \partial_\nu) \psi^\nu - i \tfrac{1}{2} \psi^\nu \gamma^\mu (\eta_{\mu\nu} \partial^2 - 2 \partial_\mu \partial_\nu) \phi \\
& + i (\bar{\zeta} \not{\partial} \eta + \bar{\eta} \not{\partial} \zeta) \\
& + \tfrac{1}{2} \mathrm{Tr} \{ \tilde{\alpha}^* \gamma^5 \gamma^\mu (\eta_{\mu\nu} \partial^2 - 2 \, \partial_\mu \partial_\nu) \alpha \, \gamma^5 \gamma^\nu \\
& \quad + i \tilde{\alpha}^* \gamma^5 \gamma^\mu \partial^\nu \beta \, \gamma^5 \gamma_\nu \gamma_\mu \\
& \quad - \tilde{\beta}^* \gamma^5 \gamma^\mu (i \, \partial^\nu \alpha \, \gamma^5 \gamma_\mu \gamma_\nu - \beta \, \gamma^5 \gamma_\mu) \} \\
& - \sqrt{2}\, M_0 (\bar{\zeta} \zeta - \bar{\eta} \eta) \; + \; \sqrt{2}\, M_0 \, \bar{\psi}_\lambda \bar{\psi}^\lambda \\
& + \sqrt{2}\, M_0 \, \mathrm{Tr} \{ \tilde{\alpha}^* \beta + \tilde{\beta}^* \alpha \}
\end{aligned} \tag{7.7}$$

Thus, we see that the usual expression for the kinetic energy of the hemitrion has appeared. Furthermore, we see that the unphysical components of the hemitrion are coupled to other unphysical fields. ϕ and π. We could vary this Lagrangian with respect to the component fields to obtain equations of motion but rather than doing this for all of the fields, we consider only the equations of the two hemidions ζ and η.

$$-i\,\partial\!\!\!/\,\eta + \sqrt{2}\,M_0\,\zeta = 0$$

$$-i\,\partial\!\!\!/\,\zeta - \sqrt{2}\,M_0\,\eta = 0 \qquad (7.8)$$

which then implies that,

$$(-\partial^2 - 2M_0^2)\zeta = (-\partial^2 - 2M_0^2)\eta = 0 \qquad (7.9)$$

Therefore, these massive spinors are tachyonic ($m^2_{\text{fermion}}<0$). So, it appears that the massive system is unstable. If we ignore this presently, we may solve for the propagator for the entire superfield. As in our discussion of the chiral, spinor superfield model, we first add source terms to the Lagrangian and then obtain the equation of motion for the superfield in the presence of the source. We find

$$\left[-\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) + \sqrt{2}\,M_0\right]\Psi = \mathcal{J}(\theta, x) \qquad (7.10)$$

where $\mathcal{J}$ is the spinor superfield source term. The solution of this equation is

$$\begin{aligned}\Psi(\theta, x) = \frac{-1}{2\sqrt{2}\,M_0}\left\{\frac{1}{-\partial^2+M_0^2}\right\}\Big\{&\left[1 - i\tfrac{3}{4}\,\frac{\sqrt{2}\,M_0}{\partial^2+2M_0^2}\,\partial\!\!\!/\right](\bar{D}D)^2 \\ &-\tfrac{1}{\sqrt{2}}M_0\left[\gamma^5\gamma^\mu + i\,3\,\frac{\sqrt{2}\,M_0}{\partial^2+2M_0^2}\,\gamma^5\partial^\mu\right](\bar{D}\gamma^5\gamma_\mu D) \\ &-\left[\partial^2 + 2M_0^2\right]\Big\}\,\mathcal{J}(\theta, x)\end{aligned} \qquad (7.11)$$

Once again we relegate the derivation to an appendix. We may note the presence of $(\partial^2+2M_o^2)^{-1}$ in this expression. This is an indication that there are tachyonic particles in the multiplet. From the work of Sokatchev, we know that it is possible to construct four projection operators for the spinor superfield which decompose a general spinor superfield into its irreducible components. Two of these operators are the chiral operators which we have used previously.

$$\Pi_{\frac{1}{2}}^{\pm} = \frac{1}{8\partial^2}(\bar{D}D)\bar{D}(1 \pm \gamma^5)D \tag{7.12}$$

However, for the spinor superfield there are two additional operators

$$\Pi_1 = \left[1 - \frac{1}{4\partial^2}(\bar{D}D)^2\right]\left[\frac{3}{4} + \frac{1}{8\partial^2}\gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\right]$$

$$\Pi_0 = \left[1 - \frac{1}{4\partial^2}(\bar{D}D)^2\right]\left[\frac{1}{4} - \frac{1}{8\partial^2}\gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\right] \tag{7.13}$$

and by using some of the identities of the fermionic derivative given previously, we can convince ourselves that these are orthogonal projection operators.

These operators may be used to project the spinor superfield into its irreducible representations

$$\Psi_\gamma^P = \Pi_\gamma^P \Psi(\theta, x) \tag{7.14}$$

or more importantly we may decompose the Lagrangian by defining

$$\mathcal{L}_{\gamma\gamma'}^{PP'} \equiv \Pi_\gamma^P\left[-\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) + \sqrt{2}\,M_o\right]\Pi_{\gamma'}^{P'} . \tag{7.15}$$

Obviously the mass term is diagonal, leaving the kinetic term as the nontrivial part of the calculation. After a substantial bit of algebra we can show that

$$\Pi_1 \mathcal{L}_K \Pi_1 = \left[-\frac{3}{16}\mathcal{L}_K - i\frac{3}{4}\not{\partial}\right]\left[\Pi_1 + \Pi_0\right]$$

$$\Pi_0 \mathcal{L}_K \Pi_0 = \left[\frac{5}{16}\mathcal{L}_K + i\frac{3}{4}\not{\partial}\right]\left[\Pi_1 + \Pi_0\right]$$

$$\Pi_0 \mathcal{L}_K \Pi_1 = \Pi_1 \mathcal{L}_K \Pi_0 = \tfrac{7}{16} \mathcal{L}_K [\Pi_1 + \Pi_0] \tag{7.16}$$

$$\Pi^{\rho}_{\frac{1}{2}} \mathcal{L}_K \Pi^{\rho'}_{\frac{1}{2}} = -i\gamma^5 \not{\partial}\, \Pi^{\rho}_{\frac{1}{2}} \delta^{\rho\rho'} \tag{7.17}$$

with all other terms vanishing. Using the first four of these relation, we can see that,

$$[\Pi_1 + \Pi_0]\, \mathcal{L}_K [\Pi_1 + \Pi_0] = \mathcal{L}_K [\Pi_1 + \Pi_0] \tag{7.18}$$

Thus, this Lagrangian only makes a distinction between the chiral and non-chiral parts of the spinor superfield. Therefore, the propagator should also possess this feature. On making this observation, we may define for the propagator

$$[\mathcal{S}(\partial,\theta)]_{1+0} = [\Pi_1 + \Pi_0]\, \mathcal{S}(\partial,\theta) [\Pi_1 + \Pi_0] \tag{7.19}$$

$$[\mathcal{S}(\partial,\theta)]^{\rho\rho'}_{\frac{1}{2}\frac{1}{2}} = \Pi^{\rho}_{\frac{1}{2}}\, \mathcal{S}(\partial,\theta)\, \Pi^{\rho'}_{\frac{1}{2}}$$

as the relevant projected propagators. When these expressions are evaluated we find

$$[\mathcal{S}]_{1+0} = \tfrac{1}{4}\left[\frac{1}{-\partial^2 + M_0^2}\right]\Big[\gamma^5\gamma^{\mu}\Big(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\Big)(\bar{D}\gamma^5\gamma^{\nu}D) + \frac{2}{\sqrt{2}M_0}(\partial^2 + 2M_0^2)\Big] \tag{7.20}$$

for the non-chiral sector. For the chiral sector we may write the propagator as a 2×2 matrix

$$\frac{1}{J}\begin{bmatrix} \frac{-1}{2\sqrt{2}M_0}[K(\bar{D}D)^2 - (\partial^2 + 2M_0^2)] & \frac{1}{2}\gamma^5[\not{\partial} + \frac{i6\sqrt{2}M_0}{\partial^2 + 2M_0^2}\partial^2] \\ -\frac{1}{2}\gamma^5[\not{\partial} + \frac{i6\sqrt{2}M_0}{\partial^2 + 2M_0^2}\partial^2] & \frac{-1}{2\sqrt{2}M_0}[K(\bar{D}D)^2 - (\partial^2 + 2M_0^2)] \end{bmatrix} \tag{7.21}$$

where $\quad J \equiv \partial^2 + M_0^2 \qquad K \equiv \Big[1 - i\frac{3}{2}\frac{\sqrt{2}M_0^2}{\partial^2 + 2M_0^2}\Big]$

Now it is important to notice that the non-chiral propagator is independent of the operator $[\partial^2+2M_o^2]^{-1}$ which means that that sector of the propagator is free from tachyonic particles. Thus we see that even our second guess at the Lagrangian must be supplemented by the condition.

$$[\ 1 - \frac{1}{4\partial^2}(\bar{D}D)^2]\ \Psi = \Psi$$

$$(\bar{D}D)\Psi = 0 \tag{7.22}$$

At this point, we shall not make another guess. It is more important to consider the information we have learned about the spinor superfield. As we have just seen, it is possible to write an expression which, naively, looks to be a perfectly good Lagrangian. However, when we solve for the propagator we find that it has "bad" ultra-violet behavior and in order to be free from particles of imaginary mass we must put a constraint on the superfield. The constraint is a gauge-like condition and the defects of the propagator showing up simultaneously are giving strong hints about the nature of the spinor superfield.

In ordinary spacetime, if we had written a naive Lagrangian for a massive vector field, we would find ourselves in the same situation. Thus, we are led to the idea that the spinor superfield must be a gauge superfield.

VIII. Spinor Yang-Mills Superfields

As we have just seen there is some justification for believing that the spinor superfield is a gauge superfield. If this is the case then we have a way of generating a reasonable Lagrangian for the superfield.

We may recall that the fermionic derivative transforms as a relativistic spinor under the Lorentz group. This suggests that perhaps the spinor superfield may be able to play a role that is analogous to that played by gauge vector fields in ordinary theories. We shall see, shortly, that in exact analogy with the covariant derivative of usual Yang-Mills theories, one may define a "supercovariant derivative" in the fermionic sector of superspace. More, remarkably, the existence of this fermionic, Yang-Mills covariantized derivative implies the existence of a bosonic Yang-Mills covariantized derivative. The truly remarkable feature about this relation is that it does not require the introduction of independent gauge, vector superfields for the bosonic components of the supercovariant derivative.

Before we embark on a derivation of this Lagrangian, however, let us recall some features of supersymmetric gauge theories. Within the context of supersymmetry there are two possible viewpoints as to the origin of Yang-Mills invariance. One of these viewpoints might be called the "geometric" view. This scheme is implemented by proposing that the fermionic co-ordinates of superspace provide a nontrivial representation of some internal symmetry group G. This has been proposed by Salam and Strathdee[22]. Unfortunately there is presently no renormalizable model along these lines[23]. The second viewpoint, which might be called "nongeometrical", is essentially the same as that encountered in ordinary Yang-Mills theories within four dimensional spacetime. Here it is the fields which provide nontrivial representations of the internal group. This problem has been solved for chiral superfields by Salam and Strathdee and Ferrara and Zumino [24]. It is this mode in which we are suggesting that the spinor superfield acts as a gauge superfield.

Results From Ordinary Gauge Theories

We would like to recall some results from other theories which possess gauge invariance. We begin from the simplest of gauge theories, quantum electrodynamics. In this theory, by the minimal coupling prescription, we introduce a covariant derivative via the definition,

$$D_\mu \equiv \partial_\mu + ieA_\mu \qquad (8.1)$$

It is natural to introduce a vector field since the operator ∂_μ transforms as a vector under the Lorentz group. With this definition of the covariant dorivative, we can insure the existence of a local invariance under redefinition of the phase of the electron field. The well known transformation is given by;

$$\psi_e' = exp[iq\Lambda(x)]\,\psi_e$$

$$A_\mu' = A_\mu - \partial_\mu\Lambda \qquad (8.2)$$

where $\Lambda(x)$ is an arbitrary local function. Next, we need an expression for the kinetic energy of the gauge field , A_μ, which is invariant under the gauge transformation. This is done by defining the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (8.3)$$

and contracting it with itself. With these definitions, we observe that the identities

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \qquad (8.4)$$

$$F_{\mu\nu} = -i\,(L_{\mu\nu})^\alpha{}_\beta\,\partial_\alpha A^\beta \qquad (8.5)$$

are valid.

If we now consider some non-Abelian group[25], we introduce a multiplet of vector gauge fields which transform as the adjoint representation of the group. The covariant derivative in equation (8.1) is redefined so that

$$D_\mu \equiv \partial_\mu + i g A^a{}_\mu T_a \tag{8.6}$$

where T_a is some representation of the group. In analogy with equation (8.4), we find

$$[D_\mu, D_\nu] = i g F^a{}_{\mu\nu} T_a \tag{8.7}$$

$$\begin{aligned} F^a{}_{\mu\nu} &= \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - g f^a{}_{bc} A^b{}_\mu A^c{}_\nu \\ &= -i (L_{\mu\nu})^{\alpha\beta} [\partial_\alpha A_\beta - \tfrac{1}{2} g f^a{}_{bc} A^b{}_\alpha A^c{}_\beta] \end{aligned} \tag{8.8}$$

where f^a_{bc} are the totally antisymmetric structure constants of some Lie algebra.

Gauge Spinor Superfields

We can easily see that the fermionic derivative of supersymmetry transforms like a Dirac spinor under the Lorentz group. For, we find the relation

$$[M^{\alpha\beta}, D^\ell] = -\tfrac{1}{2} (\sigma^{\alpha\beta} D)^\ell \tag{8.9}$$

is satisfied. Thus, if we think about the fermionic derivative, D^ℓ, as being a projection of a superspace gradient, $\partial/\partial X^L$, onto the fermion sector of superspace and the ordinary derivative, ∂_λ, the projection onto the bose sector; then, in the case where superfields[22] possess some internal symmetry, it does not seem unreasonable to add to the supergradient the following quantity

$$\mathbb{V}^M = [\tfrac{1}{2} \Lambda^{am}, \mathbb{G}^{a\mu}] T_a \tag{8.10}$$

to form a supercovariant gradient.

$$\mathcal{D}^M \equiv \partial / \partial \mathbf{X}_M + i g \mathbb{V}^M \tag{8.11}$$

In equation (8.10) the multiplet of spinor superfields, $\Lambda^a(X)$, must transform as the adjoint representation of the internal symmetry group. The vector superfields G^a_μ are defined by the equation below.

$$\mathbb{G}^a{}_\mu \equiv -\frac{i}{4}[\bar{D}\gamma_\mu \Lambda^a + \frac{1}{4} g f^a{}_{bc} \bar{\Lambda}^b \gamma_\mu \Lambda^c] \quad (8.12)$$

The quantities T_a have their usual meanings. We may require that the spinor superfields are constrained to be real. With our conventions, this implies that the vector superfields are also real. A priori, in equation (8.11) we could assume that the vector superfields are independent of the spinor superfields. We will have to justify equation (8.12) below. The analogy between equations (8.11) and (8.6) is more striking if we recall that an arbitrary spinor superfield contains a spinor superfield which is the fermionic derivative of a scalar superfield. By thinking of this as the analog of the transformation of the photon field in equation (8.2), we are led to require that the Lagrangian for the gauge spinor superfields be invariant with respect to the transformation:

$$\Lambda^a \longrightarrow \Lambda^a - 2 D \delta\Phi^a - g f^a{}_{bc} \delta\Phi^b \Lambda^c \quad (8.13)$$

where $\delta\Phi^a$ is an infinitesimal multiplet of scalar superfields. Under this transformation, the vector superfields $\mathbb{G}^a_\mu$ change as

$$\mathbb{G}^a{}_\mu \longrightarrow \mathbb{G}^a{}_\mu - \partial_\mu \delta\Phi^a - g f^a{}_{bc} \delta\Phi^b \mathbb{G}^c{}_\mu \quad (8.14)$$

which justifies the identification made in equation (8.12). Next, we need to construct the Lagrangian for the gauge spinor superfield. To this end, we need to employ the generalized Lie bracket. This Lie bracket is defined by the relation,

$$[A, B\} \equiv A B - (-)^{\alpha_A \alpha_B} B A \quad (8.15)$$

where $\alpha_A = (1,0)$ depending on whether A is a fermi or bose operator. Using this operator on the supercovariant gradient then leads to,

$$([\mathcal{D}_M, \mathcal{D}_N\}) ^i{}_j = -(T_{MN}{}^L)(\mathcal{D}_L)^i{}_j + ig(\mathbb{R}_{MN})^i{}_j \tag{8.16}$$

where we have used the following definitions

$$(\mathbb{R}_{MN})^i{}_j \equiv \begin{bmatrix} -\frac{i}{2}(\gamma^0\sigma^{\mu\nu})_{mn}\mathbb{E}^a{}_{\mu\nu} & \mathbb{F}^a{}_{m\nu} \\ \mathbb{F}^a{}_{\mu n} & \mathbb{G}^a{}_{\mu\nu} \end{bmatrix} (T_a)^i{}_j$$

$$\mathbb{E}^a{}_{\mu\nu} \equiv -\frac{i}{4}[\bar{D}\sigma_{\mu\nu}\Lambda^a + \frac{1}{4}gf^a{}_{bc}\bar{\Lambda}^b\sigma_{\mu\nu}\Lambda^c]$$

$$\mathbb{F}^{amv} \equiv D^m\mathbb{G}^{a\nu} - \frac{1}{2}\partial^\nu\Lambda^{am} - \frac{1}{2}gf^a{}_{bc}\Lambda^{bm}\mathbb{G}^{c\nu}$$

$$\mathbb{F}^{am\nu} \equiv -\mathbb{F}^{a\nu m}$$

$$\mathbb{G}^a{}_{\mu\nu} \equiv \partial_\mu\mathbb{G}^a{}_\nu - \partial_\nu\mathbb{G}^a{}_\mu - gf^a{}_{bc}\mathbb{G}^b{}_\mu\mathbb{G}^c{}_\nu$$

$$T_{MN}{}^L \equiv \begin{cases} i(\gamma^0\gamma^\lambda)_{mn} : & \text{if } L=\lambda,\ M=m,\ N=n \\ 0 : & \text{otherwise} \end{cases} \tag{8.17}$$

The term proportional to D_L on the right hand side of equation (8.16) might be called the "anomalous term". It is anomalous in the sense that it does not have an analog in equations (8.4) and (8.7). But, the presence of such a term has an interesting interpretation within the context of differential geometry. Such a term can arise from the fact that we are describing superspace in terms of a noncommuting coordinate basis and therefore the components of the invariant superspace gradient are the directional derivatives of such a basis.

There are no nonzero scalars which may be formed from $(\mathbb{R}_{MN})^i{}_j$. Therefore, we may form a quadratic and take the trace over the internal elements to obtain

$$\mathbb{R}^a{}_{KL}\mathbb{R}^b{}_{MN}\ \delta_{ab} \tag{8.18}$$

Therefore, we may take as the gauge Lagrangian the expression,

$$\mathcal{L}'_{gauge} = \frac{1}{4}(\bar{D}D)^2\{\mathbb{R}^a{}_{KL}\mathbb{R}^b{}_{MN}\}\delta_{ab}A^{KLMN} \qquad (8.19)$$

where A^{KLMN} is the most general constant tensor such that $\mathcal{L}'_{gauge}$ is invariant. Thus, we have constructed a manifestly supersymmetric Lagrangian for the gauge spinor superfields. As can be seen, there remains quite a bit of ambiguity in this equation. We expect, however, that the requirement of renormalizability will place further restrictions on the arbitrary supertensor. We may note that the various sectors of the superfield strength tensor $\mathbb{E}^a_{\mu\nu}$, $\mathbb{F}^a_{m\nu}$, and $\mathbb{G}^a_{\mu\nu}$ have dimensionalities of d+1/2, d+1, and d+3/2 respectively, where d=1/2 is the dimensionality of the spinor superfield in units of mass. Therefore, various sectors of A must differ by powers of inverse mass. Thus, we may argue that A must be chosen so that $\mathcal{L}'_{gauge}$ is proportional only to the square of fermion-fermion sector of the superfield strength tensor. So we may assume that

$$\mathcal{L}'_{gauge} = \frac{1}{4}(\bar{D}D)^2\{\mathbb{E}^a{}_{\mu\nu}\mathbb{E}_a{}^{\mu\nu}\}c_0 \qquad (8.20)$$

is the form of the gauge Lagrangian.

However, when this expression is expanded in terms of component fields, it is found not to contain a term which may be interpreted as the kinetic energy of a vector gauge field. Thus, by following the procedure which leads to a gauge theory in ordinary Minkowski space, we have not, as yet, a complete gauge Lagrangian. On the other hand, the expansion of the quantity

$$\mathcal{L}''_{gauge} = \frac{1}{4}(\bar{D}D)^2\{\mathbb{G}^a{}_\mu\mathbb{G}_a{}^\mu\} \qquad (8.21)$$

is found to contain the kinetic energy term of a gauge vector field but is not invariant under a gauge transformation. Under an infinitesimal gauge transformation this quantity is changed by an amount

$$- 2 \, \mathbb{G}^{a}{}_{\mu} \, \partial^{\mu} (\delta \Phi_{a}) \tag{8.22}$$

Therefore in order to have a Lagrangian which is gauge invariant we must add an additional term to equation (8.21). This additional term should have the same dimensionality as equation (8.20). We note that in equations (8.20) and (8.21) two powers of the fermionic gradient act on the gauge superfield. We also know from equation (5.28) that two powers of the fermionic derivative may be combined to yield the bosonic derivative. This suggests that we may try to add to equation (8.21) a term which is linear in ∂_{μ}. The simplest such term is of the form below.

$$\bar{\Lambda}^{a} \not{\partial} \Lambda^{b} \, \delta_{ab} \tag{8.23}$$

We may subject this to the gauge transformation and find it is changed by the amount below plus two pure divergence terms.

$$i \, 16 \, \mathbb{G}^{a}{}_{\mu} \, \partial^{\mu} (\delta \Phi_{a}) \tag{8.24}$$

Thus, it is clear that the expression

$$\mathcal{L}''_{gauge} = \tfrac{1}{4} (\bar{D} D)^{2} \{ \mathbb{G}^{a}{}_{\mu} \mathbb{G}_{a}{}^{\mu} - i \tfrac{1}{8} \bar{\Lambda}^{a} \not{\partial} \Lambda_{a} \} \tag{8.25}$$

will change by pure divergences under a gauge transformation. But this is exactly the manner a supersymmetric Lagrangian transforms under a fermionic translation. Therefore, the gauge Lagrangian for the spinor superfield is

$$\mathcal{L}_{gauge} = \tfrac{1}{4} (\bar{D} D)^{2} \{ c_{1} [\mathbb{G}^{a}{}_{\mu} \mathbb{G}_{a}{}^{\mu} - i \tfrac{1}{8} \bar{\Lambda}^{a} \not{\partial} \Lambda_{a}] + c_{0} \, \mathbb{E}^{a}{}_{\mu\nu} \mathbb{E}_{a}{}^{\mu\nu} \} \tag{8.26}$$

Using either the bosonic or fermionic sectors of the "super-covariant derivative" we may couple the gauge fields to matter superfields, provided that the pure kinetic terms for the matter superfields are only expressable with the use of fermionic and/or bosonic components of the invariant supergradient. An example of an interacting model is provided by;

$$\mathcal{L}_{gauge} + \frac{1}{4}(\bar{D}D)^2\{\Phi^\dagger[(\mathcal{D}^a(\gamma^c)_{ab}\mathcal{D}^b) + M_0]\Phi\} \tag{8.27}$$

where Φ is a complex scalar superfield belonging to some representation of the group.

Thus, formally at least, it appears that we have a solution to the problem of implementing Yang-Mills invariance for non-chiral superfields. The gauge, spinor superfields allow the Yang-Mills transformation of the gauge superfields to be realized linearly in a manner that is consistent with global supersymmetry. In previous works done on supersymmetry and Yang-Mills invariance by Salam and Strathdee and Ferrara and Zumino[24], the Yang-Mills transformation of the gauge superfield is implemented nonlinearly with respect to the supermultiplet, by introducing the gauge fields as components of a multiplet of real pseudo-scalar superfields, $\mathbb{V}^a(X)$. This allows the definition of two "phase factors" via the equations

$$exp[\pm g\mathbb{V}(X)] \tag{8.28}$$

where $\mathbb{V} \equiv \mathbb{V}^a(X)T_a$. Using chiral matter superfields permits the gauge and matter superfields to be coupled.

$$\frac{1}{4}(\bar{D}D)^2\{\Phi^\dagger_+ exp[g\mathbb{V}]\Phi_+ + \Phi^\dagger_- exp[-g\mathbb{V}]\Phi_-\} \tag{8.29}$$

We will return to this point at the end of the next section. It remains to be seen whether our linear approach will prove as useful in model building as the nonlinear one. We are presently studying this question.

Yang-Mills n-beins and Supersymmetry

In ordinary Yang-Mills theories, we have a set of gauge fields, $A^a_\mu(x)$, and a set of generators, T_a, which belong to some representation of a compact, semisimple Lie algebra. At each point in spacetime, we associate a set of "internal n-bein" fields, $e^{i}{}_{I}(x)$, which are given by the expression

$$e^{i}{}_{I}(x) \equiv \left\{ exp[-ig\int_0^x dy^{\lambda} A^{a}{}_{\lambda}(y)\, T_a\,] \right\}^{i}{}_{I} \tag{8.30}$$

where we have explicitly exhibited the matrix indices i and I. That we should recognize this as an n-bein for the internal space is made plausible by observing that we may define a connection $\Gamma_{\mu}{}^{J}{}_{I}$ in the usual manner,

$$d e^{i}{}_{I} = dx^{\mu}\, \Gamma_{\mu I}{}^{J}\, e^{i}{}_{J} \tag{8.31}$$

so that the equation below is valid.

$$0 = dx^{\mu} D_{\mu} e^{i}{}_{I} = dx^{\mu}\left(\partial_{\mu}\, \delta^{J}_{I} - \Gamma_{\mu I}{}^{J}\right) e^{i}{}_{J} \tag{8.32}$$

Now we may perform the differentiation that is indicated in equation (8.31) and substitute the result into equation (8.32) to find

$$D_{\mu} = \partial_{\mu} + ig\, A^{a}{}_{\mu}\, T_a \tag{8.33}$$

which is the usual expression for the covariant derivative in a Yang-Mills theory.

We now observe that the "internal n-bein" concept easily generalizes in a flat, bose-fermi superspace. Indeed, we may replace equation (8.30) by the expression,

$$e^{i}{}_{I}(x) \equiv \left\{ exp[-ig\int_0^X (\tfrac{1}{2}\, d\bar{\theta}\, \Lambda^{a} + dx^{\mu}\, \mathbb{G}^{a}{}_{\mu})\, T_a\,] \right\}^{i}{}_{I} \tag{8.34}$$

where X is some point in the superspace. Here we can see that it is crucial that both spinor and vector superfields are present in order to define the supersymmetric generalization of the line integral.

Equation (8.34) is reminiscent of the phase factor in equation (8.28). It appears that we may make some identification between $\mathbb{V}^a$ and the supersymmetric line integral. The fact that chiral superfields couple to the Yang-Mills group beins is analogous to the coupling of ordinary spinors to the vierbeins of gravitational theories. Thus, chiral scalar superfields may be viewed as Yang-Mills spinors.

It is obvious that the "internal n-beins" are nothing but Yang's gauge phase factors[26]. This in turn implies that the usual supersymmetric "phase factors" discussed in the previous section may also be identified as a supersymmetric version of the Yang gauge phase factor for chiral theories.

This viewpoint suggests a whole class of supersymmetric, chiral gauge models which have not, as yet, been explored. One could consider a chiral model where the matter superfields are chiral spinor superfields. The gauge superfields may couple to these matter superfields through the chiral, gauge vector superfields. An example of such a model is given by

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr} \{ (\bar{D}D)^2 (\mathbb{V}^{\mu} \mathbb{V}_{\mu} + \mathbb{V}^{\dagger \mu} \mathbb{V}^{\dagger}{}_{\mu}) \}$$

$$+ \frac{1}{4} (\bar{D}D) \{ \bar{\Psi}_{-} [i \gamma^{\mu} (\partial_{\mu} + \tfrac{1}{2} \mathbb{V}_{\mu}) - M_{0}] \Psi_{+} + h.c. \} \tag{8.35}$$

where for simplicity we have used the notation of Salam and Strathdee[24]. In this expression Ψ_+ and Ψ_- are independent chiral spinor superfields which belong to a representation of the group. An interesting point about such a model is that it easily admits the existence of a conserved fermion number. It would also be of some interest to see if this model is renormalizable in view of the model of Adjei and Akyeampong[20]. It is clear that the free propagator for the chiral spinor superfield here is just the Dirac propagator. This is to be compared with the propagator for the aforementioned model. Therefore, naively, we might suspect that the model of equation (8.35) may be renormalizable.

As we have just seen on a formal level the spinor superfield in eight-dimensional superspace may be a gauge superfield for internal symmetries.

At this point, we may continue our investigation of the spinor superfield as a gauge superfield, but in 1+1 dimensions. The superspace appropriate here is four-dimensional. We will shortly see that in this superspace that the analog of a chiral superfield does not exist. This is a consequence of the absence of a nontrivial γ^5 matrix. However, the Clifford algebra represented by the fermionic gradient in four-dimensional superspace is similar enough to that of the eight-dimensional superspace so that there exists a gauge transformation on spinor superfields which involves the fermionic derivative. It is this transformation which is used to implement the Yang-Mills transformation of the gauge superfield. We will repeat the argument of the previous sections and construct the Lagrangian for a spinor superfield which is the gauge superfield for a U(1) symmetry. When this Lagrangian is expressed in terms of ordinary fields we shall find that it contains a gauge vector field, a Majorana spinor field, and a Hertzian tensor field.

There are some differences between the gauge, spinor superfield in the two superspaces, however, For instance, we will find that it is possible to use more components of the superfield strength tensor in four-dimensional superspace than in the eight-dimensional space. This is possible because the dimensionality of the gauge spinor superfield is smaller in four dimensions. Also we will find that the Majorana spinor which is associated with the gauge field may be massive without the breaking of the symmetry. This feature appears unique to the four-dimensional superspace.

IX. The Spinor Superfield in 1+1 Dimensions

A Four Dimensional Superspace

Let us consider a world which possesses a single temporal dimension and a single spatial dimension. Therefore, the analogs of Minkowskian four-vectors are two vectors of the form;

$$x^\mu = (x^0, x^1) \tag{9.1}$$

We may introduce a metric which has diagonal elements -1 and 1 for $\mu=\nu=0$ and 1, respectively. The Lorentz group will consist of a single boost, M^{01}. But the Lie algebra of the Poincaré group will retain its form since it is, explicitly, independent of the number of spatial dimensions.

$$[P_\mu, P_\nu] = 0 \qquad [M_{\lambda\mu}, P_\nu] = i\,\eta_{\lambda\nu} P_\mu - i\,\eta_{\nu\mu} P_\lambda$$

$$[M_{\kappa\lambda}, M_{\mu\nu}] = i\,\eta_{\kappa\mu} M_{\lambda\nu} - i\,\eta_{\kappa\nu} M_{\lambda\mu} + i\,\eta_{\lambda\mu} M_{\nu\kappa} - i\,\eta_{\lambda\nu} M_{\mu\kappa} = 0 \tag{9.2}$$

Next, we observe that the fermion components of the generalized translation operator is given by the usual expression,

$$S^m = -i\,[\, (\gamma^0)^{mn}\, \partial_n + i\,(\gamma^\nu \theta)^m\, \partial_\nu \,] \tag{9.3}$$

if we now understand that the fermion space is also two-dimensional. In order for this expression to have meaning, we must introduce a set of Dirac matrices in a Majorana representation. One such representation is given by

$$\mathbb{1}\ , \quad \gamma^\mu = (\sigma^2, i\,\sigma^1), \quad \sigma^{\mu\nu} = \tfrac{i}{2}\, (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{9.4}$$

where $\mathbb{1}$ is the two by two identity matrix and $\vec{\sigma}$ are the Pauli spinor matrices in the standard representation. In this representation γ^0 is antisymmetric, while $\gamma^0\gamma^\mu$ and $\gamma^0\sigma_{\mu\nu}$ are symmetric. Thus, the elements of our superspace are four componented super-vectors

$$X^M = (\theta^m ; x^\mu) \tag{9.5}$$

with two fermion components and two boson components. The concept of a superfield may also be generalized. The scalar superfield, for instance, is given by the expansion;

$$\Phi(\theta,x) = A(x) + \bar{\theta}\,\psi(x) + \tfrac{1}{4}\,\bar{\theta}\theta\, F(x) \tag{9.6}$$

The fermion components of the invariant superspace gradient will then be given by the usual expression

$$D^{m} = [\,(\gamma^{0})^{mn}\,\partial_{n} - i\tfrac{1}{2}\,(\gamma^{\nu}\theta)^{m}\,\partial_{\nu}\,] \tag{9.7}$$

and will satisfy the identity

$$D^{a}D^{b} = i\tfrac{1}{2}\,(\gamma^{\mu}\gamma^{0})^{ab}\,\partial_{\mu} + \tfrac{1}{4}\,(\gamma^{0})^{ab}\,(\bar{D}D) \tag{9.8}$$

where $\bar{D}_a \equiv (\gamma^0)_{ab}D^b$. This equation is very similar to its four dimension analog which possesses two additional terms, on the right hand side, that are proportional to γ^5. We can understand the absence of these terms here by making the observation that for two-dimensional spinors there does not exist a γ^5 matrix. From the above identity it follows that $\bar{D}\gamma_\mu D = i\partial_\mu$ and $\bar{D}\sigma_{\mu\nu}D = 0$.

In supersymmetric theories in 1+3 dimensions, chiral superfields have been used extensively in constructing supersymmetric models. Here we see that in 1+1 dimension, such superfields can not even be defined. From the work of Sokatchev [18], it is known that in 1+3 dimension it is possible to construct projection operators which decompose an arbitrary superfield into its irreducible parts. We may pursue the same strategy here and we find that for the scalar superfield in 1+1 dimensions these projection operators are given by:

$$\begin{aligned} \Pi_0 &\equiv [\,1 - \tfrac{1}{\partial^2}(\bar{D}D)^2\,] = 0 \\ \Pi_1 &\equiv \tfrac{1}{\partial^2}(\bar{D}D)^2 = 1 \end{aligned} \tag{9.9}$$

Thus, the analog of the nonchiral part of the scalar superfield vanishes and the scalar superfield is composed of the sum of the analogs of the chirally positive and chirally negative parts.

Now having demonstrated the existence of a well-defined supersymmetry in 1+1 dimensions, we may begin to consider the construction of supersymmetric models. The simplest such model is the analog of the four-dimensional Wess-Zumino Φ^3 theory [17]. Here, we find that the following supersymmetric action is such an analog.

$$S = \int d^2x \left[-(\bar{D}D) \left\{ \tfrac{1}{2} \Phi \left[(\bar{D}D) - M_0 \right] \Phi + \tfrac{1}{6} g_0 \Phi^3 \right\} \right] \tag{9.10}$$

But since there are no terms proportional to γ^5, we may not consider this as the dimensional continuation of the original theory.

Gauge Spinor Superfields

We now pose for ourselves the task of constructing Yang-Mills invariant theories in our four-dimensional superspace. Immediately we see that we do not have the option of following the approach discovered by Salam and Strathdee and independently by Ferrara and Zumino[24]. This approach depends crucially on the existence of a nontrivial chirality operator. But, on the other hand, the approach which we have described for non-chiral superfields in eight-dimensional superspace can be applied to four-dimensional superspace.

We observed that the invariant superspace gradient which is given by

$$\frac{\partial}{\partial X_M} = (D^m, \partial^\mu) \tag{9.11}$$

may be made covariant with respect to Yang-Mills transformations if we add to it a supervector $\mathbb{V}^M$ of the form

$$\mathbb{V}^M(\theta, x) = (\tfrac{1}{2} \Lambda^{am}, G^{a\mu}) T_a \tag{9.12}$$

where T_a forms a matrix representation of the generators of some compact Lie algebra. Furthermore, in this expression $\mathbb{\Lambda}^{am}$ and $\mathbb{G}^a_\mu$ are spinor and vector superfield multiplets, respectively, which satisfy a certain differential equation. The analog of that equation in our four-dimensional superspace is given by,

$$\mathbb{G}^a{}_\mu = -\tfrac{i}{2}\left[\, \bar{D}\gamma_\mu \mathbb{\Lambda}^a + \tfrac{1}{4} g f^a{}_{bc}\, \bar{\mathbb{\Lambda}}^b \gamma_\mu \mathbb{\Lambda}^c \,\right] \tag{9.13}$$

and we see that there is only one independent gauge field multiplet, the spinor superfield multiplet. The quantities, f^a_{bc}, are the structure constants for the Lie algebra. Thus, we are allowed to define a super-covariant derivative, $\mathcal{D}/\mathcal{D}X_M$ via the equation,

$$\frac{\mathcal{D}}{\mathcal{D}X_M} \equiv \frac{\partial}{\partial X_M} + i g \mathbb{V}^M(\theta, x) \tag{9.14}$$

and propose that under a Yang-Mills transformation it transforms as

$$\frac{\mathcal{D}'}{\mathcal{D}X_M} = exp[\, i g \Phi\,]\, \frac{\mathcal{D}}{\mathcal{D}X_M}\, exp[-i g \Phi\,] \tag{9.15}$$

where $\Phi(\theta,\chi) \equiv T_a\, \Phi^a(\theta,\chi)$ is an arbitrary scalar superfield. Now the remarkable thing about equation (9.13) is that if we look at the transformation induced on the fermion components in equation (9.15), then we find that the vector superfield defined by equation (9.13) transforms properly as a vector Yang-Mills, gauge superfield. The proof of this fact depends crucially on the relations $\bar{D}\gamma_\mu D = i\partial_\mu$ and $(\gamma^0\gamma_\mu)_{ab} = (\gamma^0\gamma_\mu)_{ba}$. Next we may define a supertensor which is the analog of the usual field strength tensor. This supertensor is explicitly given by the expression,

$$\mathbb{R}^{aMN} \equiv \begin{bmatrix} -\frac{i}{2}(\gamma^0\sigma^{\alpha\beta})^{mn}\mathbb{E}^a{}_{\alpha\beta} & \mathbb{F}^{am\nu} \\ \mathbb{F}^{a\mu n} & \mathbb{G}^a{}_{\mu\nu} \end{bmatrix} \tag{9.16}$$

$$\mathbb{E}^{a}{}_{\mu\nu} \equiv -\tfrac{i}{2}[\,\bar{D}\,\sigma_{\mu\nu}\mathbb{\Lambda}^{a} + \tfrac{1}{4}\,g\,f^{a}{}_{bc}\,\bar{\mathbb{\Lambda}}^{b}\sigma_{\mu\nu}\mathbb{\Lambda}^{c}$$

$$\mathbb{F}^{a}{}_{\mu n} \equiv -\,\mathbb{F}^{a}{}_{n\mu}$$

$$\mathbb{F}^{a\,m\nu} \equiv D^{m}\mathbb{G}^{a\nu} - \tfrac{1}{2}\,\partial^{\nu}\mathbb{\Lambda}^{am} - \tfrac{1}{2}\,g\,f^{a}{}_{bc}\,\mathbb{\Lambda}^{bm}\mathbb{G}^{c\nu}$$

$$\mathbb{G}^{a}{}_{\mu\nu} \equiv \partial_{\mu}\mathbb{G}^{a}{}_{\nu} - \partial_{\nu}\mathbb{G}^{a}{}_{\mu} - g\,f^{a}{}_{bc}\,\mathbb{G}^{b}{}_{\mu}\mathbb{G}^{c}{}_{\nu} \tag{9.17}$$

and the action for the gauge fields is given by,

$$S_{gauge} = \int d^{2}x\,(\bar{D}D)\{\,\mathbb{R}^{a}{}_{KL}\,\mathbb{R}^{b}{}_{MN}\,\}\,\delta_{ab}\,A^{KLMN} \tag{9.18}$$

where A^{KLMN} is the most general constant supertensor which is consistent with super Poincaré invariance.

An Abelian Example

In order to examine the content of the previous results in terms of component fields, we need the expansion of a spinor superfield. If $\mathbb{\Lambda}^{a}(\theta,\chi)$ denotes such a superfield then we find

$$\mathbb{\Lambda}^{a}(\theta,x) = 2\,\phi^{a}(x) + \alpha^{a}(x)\,\theta + \tfrac{1}{4}\,\bar{\theta}\theta[\,4\zeta^{a}(x) + i\,2\,\not{\partial}\phi^{a}(x)] \tag{9.19}$$

where a is an internal index. The fields ϕ^{a} and ζ^{a} are Majorana spin-1/2 fields and $\alpha^{a}(x)$ is a four-componented matrix field. This latter field may also be expanded.

$$\alpha^{a}(x) = s^{a}(x) - i\,\gamma^{\mu}\,v^{a}{}_{\mu}(x) + i\,\tfrac{1}{2}\,\sigma^{\mu\nu}\,t^{a}{}_{\mu\nu}(x) \tag{9.20}$$

Thus, we see that the spinor superfield also contains multiplets of scalar, vector, and antisymmetric tensor fields. Let us, for simplicity, consider a theory with a U(1) internal symmetry. If we now use the expansion of the scalar superfield and examine the transformation of the equation (3.5) we find that the component fields transform as

$$\phi' = \phi - \psi \qquad\qquad s' = s - F$$
$$\zeta' = \zeta \qquad\qquad v'_\mu = v_\mu - \partial_\mu A$$
$$t'_{\mu\nu} = t_{\mu\nu} \tag{9.21}$$

under infinitesimal transformations. From this we see that the vector field v_μ is subjected to an ordinary gauge transformation as expected. In equation (9.18) we may choose the tensor A^{KLMN} so that the action assumes the form

$$S_{gauge} = \int d^2x\, \tfrac{1}{4}(\bar{D}D)\{\, 2\mathbb{F}_\mu \gamma^\nu \gamma^\mu \mathbb{F}_\nu + \mathbb{E}^{\mu\nu}\mathring{\mathbb{G}}_{\mu\nu} + \tfrac{1}{2} M_0\, \mathbb{E}^{\mu\nu}\mathbb{E}_{\mu\nu} \} \tag{9.22}$$

When this is expressed in component form we find,

$$S_{gauge} = \int d^2x \{\, \bar{\zeta}(i\partial\!\!\!/ - M_0)\zeta - \tfrac{1}{4}|\partial_\mu v_\nu - \partial_\nu v_\mu|^2 - \tfrac{1}{4}|\partial_\mu t_{\alpha\beta}|^2 - \tfrac{1}{4} M_0 t^{\mu\nu}(\partial_\mu v_\nu - \partial_\nu v_\mu)\} \tag{9.23}$$

Thus, we see that the gauge system consists of a massive Majorana spin-1/2 field, a massless spin-1 gauge field, and a Hertzian tensor field. In passing from equations (9.18) to (9.22) a judicious choice of the tensor A^{KLMN} has been made. Several factors have governed this choice. First of all we have neglected terms proportional to $|\mathring{\mathbb{G}}_{\mu\nu}|^2$. This term leads to a quadratic derivative, self-interaction for the spinor field. Secondly, the relative coefficients of the terms proportional to $\mathbb{F}^2$ and $\mathbb{E}^{\mu\nu}\mathring{\mathbb{G}}_{\mu\nu}$ have been chosen so as to eliminate a term in equation (9.23) which would be proportional to the divergence of the tensor field.

Appendix A

The Ratio m_e/m_μ

In Ref. [1], the question of whether anything could be learned, within the context of the dual, gauge model, about the ratio m_e/m_μ was addressed. In particular the authors investigated the possibility that an electron which is massless in zeroth order might acquire a mass due to radiative corrections caused by the exchange of weak vector bosons. Several variations of the basic model were considered but none were successful in this respect.

If we take spin-zero exchange seriously, as in the rest of this work, we find that it is relatively easy to achieve the above goal. For simplicity we consider a model with only the ordinary leptons. Next we introduce two spurion quartets, ϕ and η, which both couple to the singlets μ_R and e_R. We then allow ϕ to acquire a non-zero vacuum expectation value in such a way that the e-μ mass matrix has the form

Since this matrix has a zero eigenvalue only one fermion, which we may take to be the muon, gains mass in zeroth order. Now, however, there exist second order processes due to the exchange of η which gives the electron mass. The Feynman diagram for this process is given in Figure 5. Obviously the mass of the electron will be proportional to e_o^2 times the mass of the muon. We decided against using such a process in the bulk of the model because when we add the heavy leptons, e' and μ', we find that $m_{e'}/m_{\mu'}=m_e/m_\mu$. This implies that the heavy muon should have a mass on the order of a few hundred billion electron volts!

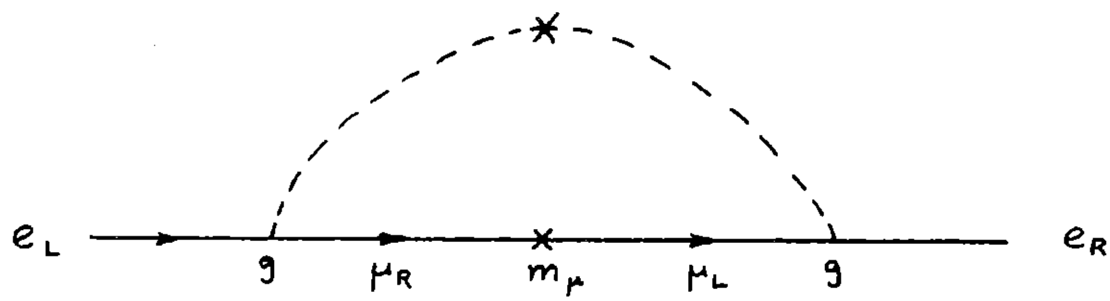


Figure 5. The higher order process which may be utilized to produce a mass for an electron which is massless in zeroth order.

Appendix B

Higher Isospin Representations of the Weinberg-Salam Gauge Group

In this extended footnote we want to discuss two points. If a scalar multiplet which transforms as (2j,j) under the Weinberg-Salam U(1)$\otimes$SU(2) gauges, suffers a spontaneous symmetry breakdown in a gauge model; then the ratio of the squared masses of the W and Z^o bosons is given by:

$$\left(\frac{M_W}{M_{Z^o}}\right)^2 = \frac{g_W^2}{2j\,[\,g_Y^2 + g_W^2\,]} = \left(\frac{1}{2j}\right)\cos^2\phi_W$$

provided that only this scalar multiplet spontaneously breaks the symmetry group. This change is significant, since in a pure Weinberg-Salam model C_V and C_A would be given by:

$$C_V = 1 + \frac{1}{2j}\left(2\sin^2\phi_W - \frac{1}{2}\right)$$

$$C_A = 1 - \frac{1}{2j}\left(\frac{1}{2}\right)$$

The other point we note is, that if we use one of these higher representations in a model it can not couple to the fermions in the lower isospin representations. So these new particles interact only with the gauge bosons and as such may not be produced except in associated production with the vector bosons. Weinberg has shown (Phys. Rev. Lett. 36, #6, 294) that the minimum mass of the Higgs boson is of order $\alpha G^{-1/2}$. Thus, if these particles are lighter than the vector boson, they will be absolutely stable!

Appendix C

Alternate F-spin Scalar System for Dual Model II

In the body of this work we arranged the scalar system so that the F bosons gained the major part of their masses from the two nonvanishing V.E.V.'s of the scalar multiplets ρ_1 and ρ_2. These multiplets transform according to the F=1 representation of $SU_F(2)$. Here we propose a slightly more economical scheme. We may replace ρ_1 and ρ_2 by a single multiplet ρ which transforms according to the F=2 representation. Explicitly, the multiplet ρ is given by:

$$\rho \equiv \begin{bmatrix} \frac{1}{\sqrt{6}}\rho_0 - \frac{1}{\sqrt{2}}\rho_4 & \frac{1}{\sqrt{2}}\rho_3 & \frac{1}{\sqrt{2}}\rho_2 \\ \frac{1}{\sqrt{2}}\rho_3 & \frac{1}{\sqrt{6}}\rho_0 + \frac{1}{\sqrt{2}}\rho_4 & \frac{1}{\sqrt{2}}\rho_1 \\ \frac{1}{\sqrt{2}}\rho_2 & \frac{1}{\sqrt{2}}\rho_1 & -\sqrt{\frac{2}{3}}\rho_0 \end{bmatrix}$$

which is constructed from the five real components $\rho_0,\ldots,\rho_4$. Since ρ is to transform as the F=2 representation of $SU_F(2)$, we may take as the generators:

$$(\mathcal{T}^a)^{bc} \equiv i\epsilon^{abc}$$

The gauge covariant derivative is then given by:

$$D_\alpha\rho \equiv \partial_\alpha\rho - ig[(\vec{F}_\alpha\cdot\vec{\mathcal{T}}),\rho]$$

Thus, we may construct the following gauge invariant, scalar Lagrangian;

$$\mathcal{L} = \mathrm{Tr}\{-\tfrac{1}{2}|D_\alpha\rho|^2 - U(\rho)\}$$

where the potential $U(\rho)$ is given by:

$$U(\rho) = \tfrac{1}{4}\lambda_1\rho^4 + \tfrac{1}{3}\lambda_2 M_G\rho^3 + \tfrac{1}{2}\lambda_3 M_G^2\rho^2$$

If we now allow ρ_4 to acquire a nonvanishing V.E.V. given by $\frac{1}{2}$ cM_G, we find the following contribution to the F boson mass matrix:

$$-\frac{1}{2} g^2 M_G^2 \left[\frac{1}{4} c^2 \left(|F_\alpha^+|^2 + |F_\alpha^-|^2 \right) + c^2 (F_\alpha^0)^2 \right]$$

This change causes the F-bosons masses to be given by:

$$M_{F^\pm}^2 = \frac{1}{4} g^2 M_G^2 \left[a^2 + c^2 \right]$$

$$M_{F^0}^2 = \frac{1}{4} g^2 M_G^2 \left[a^2 + 4c^2 \right]$$

Thus, the parameter d is eliminated from the model and we may express the parameters a and c in terms of the angle α.

$$a = 1 / \cos \alpha$$

$$c = 2\sqrt{5} \left[\cos^2 \alpha - \frac{1}{5} \right]^{\frac{1}{2}} / \sin(2\alpha)$$

Since we require c to be real, we have the following restriction on the angle α.

$$0 < \alpha < 0.35 \pi \text{ rad.}$$

But of course we know from β-decay that α is much smaller than the maximum given above.

Appendix D

Derivation of Component Form of a Lagrangian for the Spinor Superfield

We may begin this derivation by observing that

$$\bar{D}\gamma^5\gamma^\lambda D \equiv (\gamma^5\gamma^\lambda)^l{}_m \bar{D}_l D^m$$

$$= (\gamma^5\gamma^\lambda)^l{}_m [\partial_l - i\tfrac{1}{2}(\gamma^0\not\partial\theta)_l][\bar{\partial}^m - i\tfrac{1}{2}(\not\partial\theta)^m]$$

$$= \partial\gamma^5\gamma^\lambda\bar{\partial} + i(\bar{\theta}\gamma^5\not\partial\gamma^\lambda\bar{\partial}) + \tfrac{1}{4}\bar{\theta}\gamma^5\gamma_\kappa\theta[\eta^{\lambda\kappa}\partial^2 - 2\partial^\lambda\partial^\kappa]$$

Now we proceed to calculate the action of these operators on the spinor superfield.

$$\partial\gamma^5\gamma_\lambda\bar{\partial}\,\Psi = i2\gamma^5\psi_\lambda - \beta\gamma^5\gamma_\lambda\theta - \tfrac{1}{4}\bar{\theta}\gamma^5\gamma_\lambda\theta\,\pi$$

$$i(\bar{\theta}\gamma^5\not\partial\gamma_\lambda\bar{\partial})\Psi = i\partial^\rho\alpha\gamma^5\gamma_\lambda\gamma_\rho\theta - i\tfrac{1}{2}\bar{\theta}\theta\,\gamma^5\partial_\lambda\eta$$

$$- i\tfrac{1}{2}\bar{\theta}\gamma^5\theta\,\partial_\lambda\zeta + i\tfrac{1}{2}\bar{\theta}\gamma^5\gamma^\alpha\theta\,\epsilon_{\alpha\lambda\gamma\delta}\,\gamma^5\partial^\gamma\psi^\delta$$

$$- i\tfrac{1}{4}\bar{\theta}\theta\,\partial^\rho\beta\gamma^5\gamma_\rho\gamma_\lambda\theta$$

$$\tfrac{1}{4}\bar{\theta}\gamma^5\gamma^\kappa\theta[\eta_{\kappa\lambda}\partial^2 - 2\partial_\kappa\partial_\lambda]\Psi = \tfrac{1}{4}\bar{\theta}\gamma^5\gamma^\kappa\theta(\eta_{\kappa\lambda}\partial^2 - 2\partial_\kappa\partial_\lambda)\phi$$

$$- \tfrac{1}{4}\bar{\theta}\theta(\eta_{\kappa\lambda}\partial^2 - 2\partial_\kappa\partial_\lambda)\alpha\gamma^5\gamma^\kappa\theta$$

$$- i\tfrac{1}{16}(\bar{\theta}\theta)^2(\eta_{\kappa\lambda}\partial^2 - 2\partial_\kappa\partial_\lambda)\gamma^5\psi^\kappa$$

When these three equations are combined and multipled by $\frac{1}{2}\gamma^5\gamma^\lambda$ we find

$$\tfrac{1}{2}\gamma^5\gamma^\lambda\bar{D}\gamma^5\gamma_\lambda D\,\Psi = -i\gamma^\alpha\psi_\alpha + \tfrac{1}{2}\gamma^5\gamma^\rho(i\partial^\nu\alpha\gamma^5\gamma_\rho\gamma_\nu - \beta\gamma^5\gamma_\rho)\theta$$

$$+ i\tfrac{1}{4}\bar{\theta}\theta\not\partial\eta - i\tfrac{1}{4}\bar{\theta}\gamma^5\theta\,\gamma^5\not\partial\zeta$$

$$+ \tfrac{1}{4}\bar{\theta}\gamma^5\gamma^\alpha\theta[-i\epsilon_{\alpha\beta\gamma\delta}\gamma^\beta\partial^\gamma\psi^\delta - \tfrac{1}{2}\eta_{\alpha\beta}\,\gamma^5\gamma^\beta\,\pi$$

$$+ \tfrac{1}{2}\,\gamma^5\gamma^\beta(\eta_{\alpha\beta}\,\partial^2 - 2\,\partial_\alpha\partial_\beta)\,\Phi\ ]$$

$$+ \tfrac{1}{8}\,\bar\theta\theta\,\gamma^5\gamma^\rho[-i\,\partial^\lambda\beta\,\gamma^5\gamma_\lambda\gamma_\rho - (\eta_{\lambda\rho}\,\partial^2 - 2\,\partial_\lambda\partial_\rho)\,\alpha\,\gamma^5\gamma^\lambda]\theta$$

$$+ i\,\tfrac{1}{32}\,(\bar\theta\theta)^2\,(\eta_{\alpha\beta}\,\partial^2 - 2\,\partial_\alpha\partial_\beta)\,\gamma^\alpha\psi^\beta$$

Now we multiply by $\bar\Psi$ and consider only the term proportional to $(\bar\theta\theta)^2$ in the resulting expression

$$\bar\Psi\,\tfrac{1}{2}\,\gamma^5\gamma^\lambda(\bar D\,\gamma^5\gamma_\lambda\,D)\,\Psi =$$

$$\tfrac{1}{16}(\bar\theta\theta)^2[-\epsilon_{\alpha\beta\gamma\delta}\,\bar\psi^\alpha\gamma^5\gamma^\beta\partial^\gamma\psi^\delta - i\,\tfrac{1}{2}(\bar\pi\,\gamma_\alpha\,\psi^\alpha - \bar\psi^\alpha\gamma_\alpha\,\pi)$$

$$-i\,\tfrac{1}{2}(\bar\psi^\alpha\gamma^\beta(\eta_{\alpha\beta}\,\partial^2 - 2\,\partial_\alpha\partial_\beta)\,\Phi - \bar\Phi\,(\eta_{\alpha\beta}\,\partial^2 - 2\,\partial_\alpha\partial_\beta)\,\gamma^\beta\psi^\alpha)$$

$$+ i\,(\bar\zeta\,\partial\!\!\!/\,\eta + \bar\eta\,\partial\!\!\!/\,\zeta)$$

$$-\tfrac{1}{2}\,\mathrm{Tr}\{\tilde\beta^*\gamma^5\gamma^\rho(i\,\partial^\lambda\alpha\,\gamma^5\gamma_\rho\gamma_\lambda - \beta\,\gamma^5\gamma_\rho)\}$$

$$+\tfrac{1}{2}\,\mathrm{Tr}\{\tilde\alpha^*\gamma^5\gamma^\rho[i\,\partial^\lambda\beta\,\gamma^5\gamma_\lambda\gamma_\rho + (\eta_{\lambda\rho}\,\partial^2 - 2\,\partial_\lambda\partial_\rho)\,\alpha\,\gamma^5\gamma^\lambda]\}$$

where $\tilde\alpha^*$ is defined by the relation

$$(\gamma^0\,\tilde\alpha)_{\ell m} = (\gamma^0\,\alpha)_{m\ell}$$

The above equation will lead to the desired result quoted.

Appendix E

Derivation of Superpropagator from an Equation of Motion

The Lagrangian

$$\bar{\Psi}\left[\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) - \sqrt{2}\,M_o\right]\Psi$$

leads, in the presence of a source $\mathcal{J}(\theta,\chi)$, to the equation of motion

$$\left[-\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) + \sqrt{2}\,M_o\right]\Psi = \mathcal{J}$$

Formally, the solution of this equation is simply

$$\Psi(\theta,x) = \left[-\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) + \sqrt{2}\,M_o\right]^{-1}\mathcal{J}(\theta,x)$$

but the content of this equation is not clear. However, we can imagine expanding this expression about $\sqrt{2}M_o$ and in this way we are led to an infinite power series in the fermionic derivatives. Now we may recall the multiplication properties of the operator $\bar{D}\gamma^5\gamma_\lambda D$ and this will then imply that the inverse operator appearing above must be equivalent to

$$\Psi(\theta,x) = \frac{1}{\sqrt{2}M_o}\left[\, C(\partial) - \frac{2}{\sqrt{2}M_o}\gamma^5 B^\nu(\partial)(\bar{D}\gamma^5\gamma_\nu D) + \frac{1}{2M_o^2}A(\partial)\,(\bar{D}D)^2\,\right]\mathcal{J}(\theta,x)$$

for some choice of coefficients $A(\partial)$, $B^\nu(\partial)$, and $C(\partial)$. These coefficients are matrices in Dirac space and are only functions of the bosonic gradient. In order to find these coefficients we need only substitute this equation into the equation of motion. Thus, the coefficients must satisfy the equation.

$$\left[-\tfrac{1}{2}\gamma^5\gamma^\lambda(\bar{D}\gamma^5\gamma_\lambda D) + \sqrt{2}\,M_o\right]\times$$

$$\frac{1}{\sqrt{2}M_o}\left[\, C - \frac{2}{\sqrt{2}M_o}\gamma^5 B^\kappa(\bar{D}\gamma^5\gamma_\kappa D) + \left(\frac{1}{\sqrt{2}M_o}\right)^2 A(\bar{D}D)^2\right]\mathcal{J} = \mathcal{J}$$

Now we may use some of the properties of the fermionic gradient and this implies that we must have

$$C - 2\left(\frac{1}{M_0}\right)^2 \gamma^\lambda B^\mu (\eta_{\lambda\mu}\partial^2 - \partial_\lambda \partial_\mu) = 1$$

$$A + \gamma^\lambda B_\lambda = 0$$

$$-\left(\frac{1}{M_0}\right)^2 \gamma^5 \not{\partial} A\, \partial_\beta - \frac{\sqrt{2}}{M_0} \epsilon_{\lambda\mu\alpha\beta} \gamma^\lambda B^\mu \partial^\alpha - 2\gamma^5 B_\beta - \frac{1}{2}\gamma^5 \gamma_\beta C = 0$$

The first two of these equations imply that A and C may be expressed in terms of B^μ.

$$C = 1 + 2\left(\frac{1}{M_0}\right)^2 \gamma^\alpha B^\beta (\eta_{\alpha\beta}\partial^2 - \partial_\alpha \partial_\beta)$$

$$A = -\gamma^\lambda B_\lambda$$

The remaining unknown coefficient is required to satisfy the unhomogeneous equation

$$\left(\frac{1}{M_0}\right)^2 \gamma^5 \not{\partial} \gamma^\lambda B_\lambda \partial_\beta - 2\gamma^5 B_\beta - \frac{\sqrt{2}}{M_0} \epsilon_{\lambda\mu\alpha\beta} \gamma^\lambda B^\mu \partial^\alpha$$
$$- \left(\frac{1}{M_0}\right)^2 \gamma^5 \gamma_\beta \gamma^\lambda B^\mu (\eta_{\lambda\mu}\partial^2 - \partial_\lambda \partial_\mu) = \frac{1}{2}\gamma^5 \gamma_\beta$$

Now we may multiply by $\gamma^5 \gamma^\beta$ and use various identities for the Dirac matrices to obtain

$$[(-3\partial^2 + 2M_0^2)\gamma_\rho + 4\not{\partial}\partial_\rho + \sqrt{2} M_0 \sigma_{\mu\rho} \partial^\mu] B^\rho$$
$$= 2M_0^2$$

At this point we could expand B_ρ in terms of the Dirac matrices and solve for the respective parameters which would enter in such an expansion. Rather than doing this, however, let us assume that the solution has the form

$$B^\rho = c_0 \gamma^\rho + c_1 \frac{1}{\sqrt{2}M_0} i \partial^\rho$$

and substitute this into the equation for B_ρ. Again we may use some of the properties of the Dirac matrices to show that

$$c_0 = \frac{1}{4} M_0^2 \left(\frac{1}{\partial^2 - M_0^2} \right)$$

$$c_1 = \frac{3}{2} M_0^4 \left(\frac{1}{\partial^2 + 2M_0^2} \right) \left(\frac{1}{\partial^2 - M_0^2} \right)$$

Thus for B^ρ we find,

$$B^\rho = \frac{M_0^2}{4} \left[\frac{1}{\partial^2 - M_0^2} \right] \left[\gamma^\rho + \frac{i 3\sqrt{2} M_0}{\partial^2 + 2M_0^2} \partial^\rho \right]$$

At this point, we may go back and check to see whether this solution is unique. This we can do by attempting to solve the homogeneous version of equation E9. When this is done, one finds an overspecified system of equations for parameters that are analogous to c_0 and c_1. Therefore, the only solution to the homogeneous system is the trivial solution and the form of B^ρ is unique.

Now that we have obtained the nontrivial part of the superpropagator, we see that A and C are given by

$$A = M_0^2 \left[\frac{1}{\partial^2 - M_0^2} \right] \left[1 - \frac{i 3\sqrt{2} M_0}{4(\partial^2 + 2M_0^2)} \not\partial \right]$$

$$C = -\frac{1}{2} \left[\frac{\partial^2 + 2M_0^2}{\partial^2 - M_0^2} \right]$$

At last we may write equation E4 in the form:

$$\begin{aligned} \Psi(\theta, x) = \frac{1}{2\sqrt{2} M_0} \left[\frac{1}{\partial^2 - M_0^2} \right] \Big\{ & \left[1 - \frac{i 3\sqrt{2} M_0}{4(\partial^2 + 2M_0^2)} \not\partial \right] (\bar{D}D)^2 \\ & - \frac{\sqrt{2} M_0}{2} \left[\gamma^5 \gamma^\nu + \frac{i 3\sqrt{2} M_0}{\partial^2 + 2M_0^2} \gamma^5 \partial^\nu \right] (\bar{D}\gamma^5 \gamma_\nu D) \\ & - \left[\partial^2 + 2M_0^2 \right] \Big\} \mathbb{J}(\theta, x) \end{aligned}$$

This is the form of the propagator given in the text.

Appendix F

Projection Operators for the Spinor Superfield

As stated in the text, the spinor superfield may be projected onto four irreducible representations of the supersymmetric Poincaré group. The operators which effect the projections are

$$\Pi_{\pm} \equiv \frac{1}{8\partial^2} \bar{D}D \; \bar{D}(1 \pm \gamma^5)D$$

$$\Pi_0 \equiv \left[1 - \frac{1}{4\partial^2}(\bar{D}D)^2 \right]\left[\frac{3}{4} + \frac{1}{8\partial^2} \gamma^5\sigma^{\mu\nu} \partial_\mu (\bar{D}\gamma^5\gamma_\nu D) \right]$$

$$\Pi_1 \equiv \left[1 - \frac{1}{4\partial^2}(\bar{D}D)^2 \right]\left[\frac{1}{4} - \frac{1}{8\partial^2} \gamma^5\sigma^{\mu\nu} \partial_\mu (\bar{D}\gamma^5\gamma_\nu D) \right]$$

For a complete discussion of the projection algebra for the superfield of arbitrary spin, the interested reader is referred to the pioneering work of Sokatchev[18].

We may first satisfy ourselves that these operators are orthogonal projection operators

$$\begin{aligned}
\Pi_{\pm}\Pi_{\pm} &= \frac{1}{8\partial^2}\bar{D}D\,\bar{D}(1\pm\gamma^5)D \,\frac{1}{8\partial^2}\bar{D}D\,\bar{D}(1\pm\gamma^5)D \\
&= \frac{1}{64\partial^4}(\bar{D}D)^2\,\bar{D}(1\mp\gamma^5)D\,\bar{D}(1\pm\gamma^5)D \\
&= \frac{1}{64\partial^4}\left[(\bar{D}D)^3 \mp (\bar{D}D)^2\bar{D}\gamma^5 D \right] \bar{D}(1\pm\gamma^5)D \\
&= \frac{1}{64\partial^4}(4\partial^2)\,\bar{D}(1\mp\gamma^5)D\,\bar{D}(1\pm\gamma^5)D \\
&= \frac{1}{16\partial^2}\left[\bar{D}D\,\bar{D}(1\pm\gamma^5)D \mp \bar{D}\gamma^5 D\,\bar{D}(1\pm\gamma^5)D \right] \\
&= \frac{1}{16\partial^2}\left[\bar{D}D\,\bar{D}(1\pm\gamma^5)D \mp \bar{D}\gamma^5 D\bar{D}D - (\bar{D}\gamma^5 D)^2 \right] \\
&= \frac{1}{16\partial^2}\left[\bar{D}D\,\bar{D}(1\pm\gamma^5)D \pm \bar{D}D\,\bar{D}\gamma^5 D + (\bar{D}D)^2 \right] \\
&= \frac{1}{8\partial^2}\bar{D}D\,\bar{D}(1\pm\gamma^5)D = \Pi_{\pm}
\end{aligned}$$

By making the appropriate changes in this proof, we see that

$$\Pi_{\pm}\,\Pi_{\mp} = 0$$

Furthermore, we see that the chiral projectors are orthogonal to the non-chiral projectors by noting the identities

$$\bar{D}D\,[\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2] = [\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2]\,\bar{D}D = 0$$

$$\bar{D}D\ \ \gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D) = \gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\,\bar{D}D = 0$$

Now we need only to consider the non-chiral projection operators in order to complete the proof of the projection algebra. Thus, we need to evaluate

$$\begin{aligned}
&[\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2]\,[\,\tfrac{a}{4} + \tfrac{b}{8\partial^2}\gamma^5\sigma^{\mu\nu}\,\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\,]\times\\
&[\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2]\,[\,\tfrac{c}{4} + \tfrac{d}{8\partial^2}\gamma^5\sigma^{\alpha\beta}\,\partial_\alpha(\bar{D}\gamma^5\gamma_\beta D)\,] \quad =\\
&\{\,\tfrac{ac}{16}[\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2] \;+\; \tfrac{ad+bc}{4}\,\tfrac{1}{8\partial^2}\,\gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\\
&\qquad \tfrac{bd}{64\partial^2}\,(\sigma^{\mu\nu}\sigma^{\alpha\beta})\,\partial_\mu\partial_\alpha\ \bar{D}\gamma^5\gamma_\nu D\,\bar{D}\gamma^5\gamma_\beta D\,\}
\end{aligned}$$

$$= [\,1 - \tfrac{1}{4\partial^2}(\bar{D}D)^2][\,\tfrac{ac+3bd}{16} + \tfrac{ad+bc-2bd}{4}\,\tfrac{1}{8\partial^2}\,\gamma^5\sigma^{\mu\nu}\partial_\mu(\bar{D}\gamma^5\gamma_\nu D)\,]$$

In arriving at this final form we have used the multiplication table for $\bar{D}\gamma^5\gamma_\nu D\bar{D}\gamma^5\gamma_\beta D$ and used properties of the Dirac algebra. By choosing the parameters a, b, c, and d appropriately, we may convince ourselves that

$$(\Pi_1)^2 = \Pi_1 \qquad\qquad (\Pi_0)^2 = \Pi_0$$

$$\Pi_1\,\Pi_0 = 0$$

This completes the proof of the algebraic properties of the projection operators.

Appendix G
On the Geometry of Superspace

Introduction and Summary

There are two well known approaches to the construction of a model which weds local supersymmetry to a gravitational theory.

One approach is the gauge supersymmetry approach of Arnowitt and Nath[27] where one essentially postulates that the geometry of curved superspace is an extension of the Riemannian geometry of ordinary spacetime. From this viewpoint, one is led to introduce a supermetric, $g_{MN}(\theta,\chi)$, which governs the geometry of curved superspace.

The other approach is the supergravity approach of Freedman, van Nieuwenhuizen, and Ferrara[28] and Deser and Zumino[29]. Here starting from the principles of local supersymmetry and general covariance, one constructs a gravitation theory from a Rarita-Schwinger field, the vierbein fields, and the connection coefficients.

We will now present an alternative to these approaches. This alternative is very much in the spirit of gauge supersymmetry. Indeed, we find the basic ideas which underlie gauge supersymmetry very plausible. Thus, we will attempt to formulate a theory of a curved, eight-dimensional, fermi-bose superspace. A few months ago[30], we made the observation that a factorization of the most general form of the supermetric, that is consistent with global supersymmetry, strongly suggests that a theory of curved superspace should be constructed as the generalization of Einstein's unified field theory[31]. It is this generalization to superspace to which we shall address ourselves in this paper.

In the second section, we will study the geometry of global supersymmetry. This we will do first by investigating the supermetric of global supersymmetry. Next we will recall some well known results about differentiation within global supersymmetry. These results are presented so as to facilitate their interpretation from a differential geometric viewpoint. Finally, we study some Lagrangian models of global supersymmetry. This we do keeping in mind that it is the geometry of superspace which is our ultimate goal. The conclusions we reach at the end of this section are that global superspace is a metric space, in the differential geometric sense, which possesses zero curvature and constant torsion. It is these final two features which we interpret as excluding a Riemannian geometry for curved superspace. We further conclude that simplicity dictates a theory which is the supersymmetric generalization of Einstein's unified field theory.

In the third section, we review general relativity from a very simplistic viewpoint. We undertake this review for two reasons. First of all, we are interested in demonstrating the interplay between gauge invariance and differential geometry. We will show, in an intuitive way, that it is possible to use only gauge invariance to construct a theory of curved spacetime. The second reason for this survey is to formulate a stratagem which we may apply to superspace.

In the fourth section, we address the problem of constructing a theory which generalizes Einstein's unified field theory. This construction proceeds in exact analogy with general relativity. It is shown that the requirement that the local isometries of the affine super connection coincide with those of the supermetric places a restriction on the isometries which may be gauged. This in turns reduces the number of local gauge fields needed and further allows for the construction of a theory with vanishing nonmetricity. We also find that global supersymmetry can be recovered as a continuous limit of this curved superspace theory. It is noted that complex conjugation seems to play an important role in the theory of a curved superspace. We suspect that this has some relation to the fact that there is a connection between supersymmetry and twistors[32]. We end by proposing an action for the gravitational interaction in a fermi-bose superspace.

In conclusion, we give a brief discussion of this approach and point out some relevant features.

On the Geometry of Global Supersymmetry

As is well known, the fermionic translations of global supersymmetry induce the following transformation on the superspace $\{X^M : X^M = (\theta^m, x^\mu)\}$,

$$\begin{aligned} \theta'^m &= \theta^m + \epsilon^m \\ x'^\mu &= x^\mu + i\tfrac{1}{2}(\bar{\epsilon}\gamma^\mu\theta) \end{aligned} \tag{1}$$

By choosing two points in superspace which are infinitesimally separated, we may deduce that for differential elements the transformation law is given by

$$\begin{aligned} d\theta'^m &= d\theta^m \\ dx'^\mu &= dx^\mu + i\tfrac{1}{2}(\epsilon\gamma^\mu d\theta) \end{aligned} \tag{2}$$

With the use of both sets of equations, we conclude that the square of the generalized line element

$$ds^2 = d\theta^m(\gamma^0 N)_{mn}\, d\theta^n + [\, dx^\mu - i\tfrac{1}{2}(\bar{\theta}\gamma^\mu d\theta)]^2 \tag{3}$$

is invariant under the super Poincaré group.

The super Poincaré group is the graded group which possesses as its generators the Lorentz boosts and rotations, the ordinary bosonic translations and the fermionic translations. It is illustrated, concisely, in Figure (#6).

In equation (3), the matrix $(\gamma^0 N)_{mn}$ must be chosen so that it is antisymmetric in its indices. This and a generalized reality are, a priori, the only properties required of this matrix. As such, the most general form of this matrix is

$$(\gamma^0 N)_{mn} = (\gamma^0)_{mn}\, \Phi^S(\theta, x) + i\,(\gamma^0\gamma^5)_{mn}\, \Phi^P(\theta, x) + (\gamma^0\gamma^5\gamma^\lambda)_{mn}\, \Phi^A_\lambda(\theta, x) \tag{4}$$

where Φ^S, Φ^P, and Φ^A_λ are unspecified scalar, pseudo-scalar, and axial vector superfields. We may rewrite equation (3) in the form

$$ds^2 = dX^{\dot M}\, \mathring{h}_{\dot M \dot N}\, dX^{\dot N} \tag{5}$$

where $\mathring{h}_{\dot M \dot N}$ is a supermatrix appropriate for the given line element. It can be seen that the following supermetric satisfies this equation.

$$\mathring{h}_{\dot M \dot N} \equiv \begin{bmatrix} (\gamma^0 N)_{\dot m \dot n} + \frac{1}{4}(\gamma^0\gamma_\lambda\theta)_{\dot m}(\bar\theta\gamma^\lambda)_{\dot n} & i\,\frac{1}{2}(\gamma^0\gamma_{\dot\nu}\theta)_{\dot m} \\ -i\,\frac{1}{2}(\gamma^0\gamma_{\dot\mu}\theta)_{\dot n} & \eta_{\dot\mu\dot\nu} \end{bmatrix} \tag{6}$$

However, we may also proceed one step further and factor the supermetric. This factorization could proceed in the same manner as does the factorization of the metric in general relativity where we write

$$g_{\dot\mu\dot\nu} = \eta_{\alpha\beta}\, e^\alpha{}_{\dot\mu}(x)\, e^\beta{}_{\dot\nu}(x) \tag{7}$$

But, here we shall find it simpler to factor the supermetric in the form below.

$$\mathring{h}_{\dot M \dot N} = \mathring{E}^K{}_{\dot M}\, \eta_{KL}\, (\mathring{E}^*)^L{}_{\dot N} \tag{8}$$

We will justify this choice shortly.

At this point we must make a choice to satisfy this equation. Fortunately, however, within global supersymmetry there are many hints as to the form of the octad $\mathring{E}^K{}_{\dot M}$. The name octad or achtbein is appropriate for this supermatrix since it plays a role which is analogous to that played by the usual tetrads. The solution which seems appropriate for global supersymmetry is

$$\mathring{E}^{K}{}_{\dot{\mu}}(\theta,\chi) = \begin{bmatrix} \delta^{k}_{\dot{m}} & 0 \\ i\frac{1}{2}(\gamma^{0}\gamma^{\kappa}\theta)_{\dot{m}} & \delta^{\kappa}_{\dot{\mu}} \end{bmatrix} \tag{9}$$

and this implies that the supermatrix denoted by η_{KL} is block diagonal and given by

$$\eta_{KL} = \begin{bmatrix} (\gamma^{0}N)_{k\ell} & 0 \\ 0 & \eta_{\kappa\lambda} \end{bmatrix} \tag{10}$$

Thus, making our choice of factorization and global octads leads to this very simple result.

Let us momentarily consider the results which would be obtained if we replace equation (8) by an equation which contained two factors of the octad. If this had been done and if we assume the same form for the global octad, then we would find that η_{KL} would contain an additional term of $-i(\gamma^{0}\gamma_{\mu}\theta)_{\ell}$ in the lower left hand corner. Having made this observation, let us return to the justification of the form of the global octad.

The form of the global octad given above appears, explicitly, in global supersymmetry[33]. To see this, let us consider a function, $f(\theta,\chi)$, which is defined over superspace. A supergradient operator may be defined via the equation for infinitesimal $d\theta$ and dx. The components of this naive supergradient may be identified as

$$\partial_{\dot{M}} = \left(\frac{\partial}{\partial\theta^m}, \frac{\partial}{\partial x^\mu} \right) \equiv (\partial_{\dot{m}}, \partial_{\dot{\mu}}) \tag{12}$$

Within global supersymmetry, however, it is not this operator which is used in the construction of Lagrangian field theories. Instead, one introduces an invariant gradient which we may denote by ∇_M. Explicitly, this operator is of the form

$$\nabla_M = (\partial_m - i\tfrac{1}{2}(\gamma^0\gamma^\nu\theta)_m \partial_\nu, \partial_\mu) = (\bar{D}_m, \partial_\mu) \tag{13}$$

and we may verify that this gradient operator is related to the naive gradient via the equation

$$\partial_{\dot{M}} = \mathring{\mathbb{E}}^K{}_{\dot{M}} \nabla_K \tag{14}$$

The invariant gradient is invariant with respect to both fermion and boson components of the supertranslation operator, $\mathcal{P}_M \equiv (\bar{S}_m, P_\mu)$, and therefore satisfies the equation

$$[\mathcal{P}_M , \nabla_N \} = 0 \tag{15}$$

where $[\; , \; \}$denotes the graded commutator. The invariant derivative also possesses the property that its action on a superfield produces another superfield. The naive gradient does not have this property.

It may be verified that

$$\partial_{\dot{K}} \mathring{\mathbb{E}}^A{}_{\dot{L}} = \mathbb{L}_{\dot{K}\dot{L}}{}^{\dot{R}} \mathbb{E}^A{}_{\dot{R}} \tag{16}$$

where the only nonvanishing components of the supertensor $L_{\dot{K}\dot{L}}{}^{\dot{R}}$ are given by $i\frac{1}{2}(\gamma^0\gamma^\rho)_{\dot{k}\dot{\ell}}$. We may also verify that the relations

$$[\partial_{\dot{K}}, \partial_{\dot{L}} \} = 0 \tag{17}$$

$$[\nabla_K, \nabla_L \} = -2 T_{KL}{}^M \nabla_M \tag{18}$$

are satisfied. The only nonvanishing components of $T_{KL}{}^{M}$ are given by $i\frac{1}{2}(\gamma^0\gamma^\mu)_{k\ell}$. We may note also that the relation

$$T_{KL}{}^{M} = \tfrac{1}{2}\left[L_{KL}{}^{M} + L_{KL}{}^{M} \right] \tag{19}$$

is valid. Finally, we make the trivial observation that equation (8) may be differentiated, assuming that $\partial_{\dot{L}}\eta_{KL}=0$, to obtain

$$\partial_{\dot{L}} \mathring{h}_{\dot{M}\dot{N}} = L_{\dot{L}\dot{M}}{}^{\dot{R}}\, \mathring{h}_{\dot{R}\dot{N}} + (-)^{\pi}\, \mathring{h}_{\dot{M}\dot{R}}\, L^{*}_{\dot{L}\dot{N}}{}^{\dot{R}} \tag{20}$$

where $\pi\equiv\sigma(\dot{L})[\sigma(\dot{M})+\sigma(\dot{N})+1]$. The assumption that $\partial_{\dot{L}}\eta_{KL}=0$ implies that the previously undetermined fermion-fermion sector of the supermetric may assume the form below.

$$(\gamma^0 N)_{k\ell} = K(\gamma^0)_{k\ell} + i\,L(\gamma^0\gamma^5)_{k\ell} + M_\mu(\gamma^0\gamma^5\gamma^\mu)_{k\ell} \tag{21}$$

Now if we make the additional assumptions that $L=M_\mu=0$ and $K=1$, we obtain for η_{KL}

$$\eta_{KL} \equiv \begin{bmatrix} (\gamma^0)_{k\ell} & 0 \\ 0 & \eta_{\kappa\lambda} \end{bmatrix} \tag{22}$$

With these assumptions, we may define an inverse supermetric which is given by

$$\mathring{h}^{\dot{M}\dot{N}}(\theta,x) \equiv \begin{bmatrix} (\gamma^0)^{\dot{m}\dot{n}} & -i\frac{1}{2}(\gamma^{\dot{\nu}}\theta)^{\dot{m}} \\ -i\frac{1}{2}(\gamma^{\dot{\mu}}\theta)^{\dot{n}} & \eta^{\dot{\mu}\dot{\nu}}\left[1-\frac{1}{4}(\bar{\theta}\theta)\right] \end{bmatrix} \tag{23}$$

and satisfies the condition $\mathring{h}_{\dot{L}\dot{M}}\mathring{h}^{\dot{M}\dot{N}}=\delta_{\dot{L}}^{\dot{N}}$.

At this point, we relinquish, temporarily, the discussion of the geometry of global supersymmetry. We shall survey, now, some Lagrangian models of global supersymmetry. However, we shall keep in mind that it is the geometry of global supersymmetry which we are trying to understand.

The simplest of models involves the scalar superfield. The kinetic energy term of a Lagrangian may be expressed in the form below.

$$\mathcal{L} = [\, \mathring{h}^{\dot{A}\dot{B}} - T^{*K\dot{L}\dot{A}} T_{\dot{L}K}{}^{\dot{B}}] (\partial_{\dot{B}} \Phi)(\partial_{\dot{A}} \Phi) \tag{24}$$

A mass term may be included without difficulty. This Lagrangian may be compared with the Lagrangian of a scalar field in ordinary curved spacetime

$$\mathcal{L} = g^{\dot{\alpha}\dot{\beta}} (\partial_{\dot{\alpha}} \phi)(\partial_{\dot{\beta}} \phi) \tag{25}$$

It may be noted that in the supersymmetric case the derivative of superfield couples not only to the inverse supermetric but also to a bilinear form involving the supertensor T.

Let us turn now to the gauge spinor superfield[34]. It has been shown that it is possible to make the invariant supergradient covariant with respect to internal symmetries by introducing a supervector field. It is possible then to define a supercovariant derivative

$$\mathcal{D}_M \equiv \nabla_M + i g \mathbb{V}^i{}_M t_i \tag{26}$$

where t_i denotes a representation of the generators of the internal group. The bracket of this operator with itself yields.

$$[\mathcal{D}_L, \mathcal{D}_M\} = -2 T_{LM}{}^N \mathcal{D}_N + i g \mathbb{R}_{LM} \tag{27}$$

where

$$\begin{aligned} \mathbb{R}_{LM} \equiv\ & \nabla_L \mathbb{V}_M - (-)^{\sigma(L)\sigma(M)} \nabla_M \mathbb{V}_L \\ & + [\mathbb{V}_L, \mathbb{V}_M\} + 2 T_{LM}{}^N \mathbb{V}_N \end{aligned} \tag{28}$$

Furthermore, a Lagrangian for the gauge supervector field has been proposed. This Lagrangian may be written as

$$\mathcal{L} = \mathrm{Tr}\{ c_1 \mathbb{T}^{LMA}[\mathbb{T}_{LM}{}^{B} \mathbb{V}_A \mathbb{V}_B + \mathbb{V}_L \mathring{\mathcal{D}}_A \mathbb{V}_M] + c_0 P_-^{ABLM} P_{-AB}{}^{IJ} [\mathbb{R}_{LM} \mathbb{R}_{IJ}] \} \tag{30}$$

where c_0 and c_1 are constants and $P_\pm^{ABKM}$, $\mathring{\mathcal{D}}_A \mathbb{V}_M$ are defined by the equations.

$$P_\pm^{ABKM} \equiv [\, \mathbb{T}^{KLA} \mathbb{T}_L{}^{MB} \pm \mathbb{T}^{KLB} \mathbb{T}_L{}^{MA}]$$

$$\mathring{\mathcal{D}}_A \mathbb{V}_M \equiv (\nabla_A \delta_M^B - \mathbb{L}_{AM}{}^{B}) \mathbb{V}_B \tag{31}$$

In this model, it has also been noted that the supervector fields $\mathbb{V}^i{}_M \equiv (\frac{1}{2} \bar{\Lambda}^i{}_m, \mathring{\mathbb{G}}^i{}_\mu)$ may be constrained to satisfy the condition below.

$$\mathbb{T}^{LMN} \mathbb{R}_{LM} = 0 \tag{32}$$

This condition permits the vector components, $\mathring{\mathbb{G}}^i{}_\mu$, of the supervector fields to be expressed totally in terms of the spinor components, $\bar{\Lambda}^i{}_m$. Once again we see that the supertensor T plays a prominent role. With this we conclude this survey of globally supersymmetric models.

At this point, it is necessary to assess the information which is presently at our disposal. We have seen that the global supermetric may be factored as an octad, conjugate octad, and a tangent space metric. If this is done, the tangent space metric can assume the form

$$\begin{bmatrix} (\gamma^0)_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{bmatrix} \tag{33}$$

Thus, the tangent space has zero curvature and zero torsion. This is very similar to the situation in general relativity. In general relativity, the tangent space metric is the Minkowski metric. This is why local frames are characterized by the group SO(3,1). If we assume that the tangent space metric for local supersymmetry has the form above, then the local superframe is characterized by OSp(4I3,1)[35]. This is not, however, the full isometric group of the supermetric. This follows from the fact that both the octad and its conjugate appear in equation (8). In the boson-boson sector, the isometric group is U(3,1) which contain SO(3,1) as a subgroup.

On the other hand, if we factor the supermetric into octad, octad, and tangent space metric, then the tangent space metric can assume the form

$$\begin{bmatrix} (\gamma^{0})_{mn} & 0 \\ -i(\gamma^{0}\gamma_{\mu}\theta)_{n} & \eta_{\mu\nu} \end{bmatrix} \tag{34}$$

This tangent space must, at least, have nonzero torsion.

Thus, we are faced with a choice and if we use general relativity as a guide, we make the choice in favor of the foremost tangent space. This choice implies that if we regard global supersymmetry as a continuous limit of a curved superspace theory, then the curved superspace possesses a non-Riemannian geometry. This is implied by the presence of both the achtbein and its conjugate in the supermetric. Additionally if we look at equation (20) we are led to conclude that global supersymmetry arises as a limit of a non-Riemannian curved superspace which possesses a complex, graded group as its local group. Global supersymmetry arises when this space has zero curvature, zero nonmetricity, and constant torsion. The fact that the full, local group is complex and therefore may not be realized on the co-ordinate basis is in some sense expected. It is well-known that internal symmetries may be combined non-trivially with global supersymmetry[36]. On the other hand, in ordinary spacetime unified field theories have been formulated. A well-known example of such a theory is the work of Einstein[31]. In such a theory the metric may be described in terms of complex tetrads. Also, there have been indications that curved superspace is non-Riemannian every since the work of Woo[37] and Srivastava[38].

Gauge Theory Concepts and Differential Geometry

At this point, we shall review general relativity and demonstrate, in a very simplistic manner, the interplay between gauge invariance and differential geometry. For a rigorous treatment of this topic, the interested reader is referred to an excellent paper by Cho[39]. Our goal here is to show that by applying a few simple ideas which are derived from gauge theories, we are led directly to general relativity.

All gauge theories are characterized by the presence of bein fields and gauge fields. The gauge fields appear as coefficients in the definition of a Lie valued operator which may be referred to as the fully covariant derivative. The bein fields may be denoted by e(x) and we must specify the group or groups for which these fields provide a representation. In all gauge theories presently known, we require that

$$\mathfrak{D}\, e(x) = 0 \qquad (35)$$

where $\mathfrak{D}$ is a symbolic notation for the fully covariant derivative. The operator $\mathfrak{D}$ may also provide a representation for groups other than those represented by the bein fields.

As an illustration of these points, let us consider electromagnetism. For electromagnetism, the group is U(1). If we choose a complex representation, the bein field may be represented by a complex field, e(x). The group U(1) possesses a single generator. Thus, we introduce a single gauge field and define the covariant derivative by the relation

$$D_\mu = \partial_\mu + i g A_\mu(x)\, t \tag{36}$$

where t is simply a real number, In general, we introduce one gauge field for each independent group generator. The requirement that the covariant derivative annihilate the bein field is simply given by,

$$[\, \partial_\mu + i g A_\mu(x)\, t \,]\, e(x) = 0 \tag{37}$$

This equation is easily solved to find

$$e(x) = \exp[-ig \int_{x_0}^{x} dy^\lambda A_\lambda(y)\, t\,] \tag{38}$$

which simply is the Yang gauge phase factor[26], for the group U(1). Thus, we see that the bein field is completely determined by the gauge field. This very simple argument generalizes to all internal symmetry groups in a straightforward fashion. We simply replace t by representation matrices of the group generators. The condition that the covariant derivative annihilate the bein is now

$$\partial_\mu e^i{}_\ell + i g A^a{}_\mu [(t_a)^i{}_j\, e^j{}_\ell - (t_a)^j{}_\ell\, e^i{}_j] = 0 \tag{39}$$

and once again we note that the bein is completely determined by the gauge field. The feature which allows this is the fact that the bein field is only a representation of a single group.

Let us now turn to general relativity. Here the bein fields $e^\alpha{}_{\dot\mu}(x)$ are simultaneously a representation of SO(3,1) and GL(4,R). Stated another way, we may say that the bein transforms as SO(3,1) on the undotted index and as GL(4,R) on the dotted index. Thus, in the definition of the fully covariant derivative we need a set of gauge fields for both groups. The condition that the covariant derivative annihilate the bein field now takes the form

$$\partial_{\dot{\kappa}} e^{\alpha}{}_{\dot{\lambda}} + (\omega_{\dot{\kappa}})^{\alpha}{}_{\beta}\, e^{\beta}{}_{\dot{\lambda}} - L_{\dot{\kappa}\dot{\lambda}}{}^{\dot{\mu}}\, e^{\alpha}{}_{\dot{\mu}} = 0 \tag{40}$$

where we have introduced all the necessary gauge fields. The new feature which has entered here is that we still have two fields which may be considered as independent variables. For instance, L may be expressed in terms of e and ω.

$$L_{\dot{\kappa}\dot{\lambda}\dot{\mu}} \equiv L_{\dot{\kappa}\dot{\lambda}}{}^{\dot{\rho}}\, g_{\dot{\rho}\dot{\mu}} = \eta_{\alpha\beta}[\,\partial_{\dot{\kappa}} e^{\alpha}{}_{\dot{\lambda}} + (\omega_{\dot{\kappa}})^{\alpha}{}_{\gamma} e^{\gamma}{}_{\dot{\lambda}}]e^{\beta}{}_{\dot{\mu}} \tag{41}$$

In this equation we have introduced the metric as the following bilinear form.

$$g_{\dot{\rho}\dot{\mu}} = \eta_{\alpha\beta}\, e^{\alpha}{}_{\dot{\rho}}(x)\, e^{\beta}{}_{\dot{\mu}}(x) \tag{42}$$

It can readily be seen that the metric possesses a group of isometries. That is, we may perform the transformation

$$e^{\alpha}{}_{\dot{\mu}} \longrightarrow U^{\alpha}{}_{\mu}\, e^{\mu}{}_{\dot{\mu}} \tag{43}$$

and if U is an element of SO(3,1), the metric will remain invariant. The requirement that L should also be invariant under this set of transformations leads to the transformation law

$$(\omega_{\dot{\kappa}})^{\alpha}{}_{\beta} \longrightarrow [\,U\omega_{\dot{\kappa}}U^{-1} - (\partial_{\dot{\kappa}}U)U^{-1}]^{\alpha}{}_{\beta} \tag{44}$$

which is the usual one for a gauge vector field.

Since the fields $\omega_{\dot{k}}$ transform like the generators of SO(3,1), it then follows from definition that

$$\partial_{\dot{\kappa}} g_{\dot{\lambda}\dot{\mu}} - L_{\dot{\kappa}\dot{\lambda}}{}^{\dot{\rho}}\, g_{\dot{\rho}\dot{\mu}} - L_{\dot{\kappa}\dot{\mu}}{}^{\dot{\rho}}\, g_{\dot{\lambda}\dot{\rho}} = 0 \tag{45}$$

Thus, we see that the gauge field L acts as the connection coefficient for the metric in the differential geometric sense.

It can further be seen that under the transformation

$$e^{\lambda}{}_{\dot\lambda} \longrightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\lambda}}\, e^{\lambda}{}_{\dot\alpha}$$

$$\omega_{\dot\kappa} \longrightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}}\, \omega_{\dot\alpha} \tag{46a}$$

the response of the connection is given by

$$L_{\dot\kappa\dot\lambda\dot\mu} \longrightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}} \frac{\partial f^{\dot\beta}}{\partial x^{\dot\lambda}} \frac{\partial f^{\dot\gamma}}{\partial x^{\dot\mu}} L_{\dot\alpha\dot\beta\dot\gamma} + \frac{\partial^2 f^{\dot\alpha}}{\partial x^{\dot\kappa}\partial x^{\dot\lambda}} g_{\dot\alpha\dot\beta} \frac{\partial f^{\dot\beta}}{\partial x^{\dot\mu}} \tag{46b}$$

This may readily be identified as the transformation law of an affine connection coefficient. We may decompose L into two other quantities, Γ and T where

$$\Gamma_{\dot\kappa\dot\lambda\dot\mu} \equiv \tfrac{1}{2}[\, L_{\dot\kappa\dot\lambda\dot\mu} + L_{\dot\lambda\dot\kappa\dot\mu}\,]$$

$$T_{\dot\kappa\dot\lambda\dot\mu} \equiv \tfrac{1}{2}[\, L_{\dot\kappa\dot\lambda\dot\mu} - L_{\dot\lambda\dot\kappa\dot\mu}\,] \tag{47a}$$

The Riemannian part of the connection, Γ, is not a true tensor since it transforms just as L. On the other hand, the tensor T transforms as

$$T_{\dot\kappa\dot\lambda\dot\mu} \longrightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}} \frac{\partial f^{\dot\beta}}{\partial x^{\dot\lambda}} \frac{\partial f^{\dot\gamma}}{\partial x^{\dot\mu}}\, T_{\dot\alpha\dot\beta\dot\gamma} \tag{47b}$$

and is seen to be a true tensor.

At this point, it is convenient to introduce a partially covariant derivative. We define this operator through the equation

$$\hat{D}_{\dot\kappa} \equiv \partial_{\dot\kappa}\, \delta^{\alpha}_{\beta} + (\omega_{\dot\kappa})^{\alpha}{}_{\beta} \tag{48}$$

The connection, Γ, and the tensor, T, may be expressed in terms of this operator.

$$\Gamma_{\dot\kappa\dot\lambda\dot\mu} = \tfrac{1}{2}\, \eta_{\alpha\beta}\, [\, \hat{D}_{\dot\kappa}\, e^{\alpha}{}_{\dot\lambda} + \hat{D}_{\dot\lambda}\, e^{\alpha}{}_{\dot\kappa}\,]\, e^{\beta}{}_{\dot\mu} \tag{49}$$

$$T_{\dot{\kappa}\dot{\lambda}\dot{\mu}} = \tfrac{1}{2}\,\eta_{\alpha\beta}\,[\,\hat{D}_{\dot{\kappa}}\, e^{\alpha}{}_{\dot{\lambda}} - \hat{D}_{\dot{\lambda}}\, e^{\alpha}{}_{\dot{\kappa}}\,]\, e^{\beta}{}_{\dot{\mu}} \tag{50}$$

The partially covariant derivative may be treated as any covariant derivative. Thus, we find the usual expressions.

$$(\,[\,\hat{D}_{\dot{\kappa}}, \hat{D}_{\dot{\lambda}}\,]\,)^{\alpha}{}_{\beta} = (\,R_{\dot{\kappa}\dot{\lambda}}\,)^{\alpha}{}_{\beta} \tag{51}$$

$$(R_{\dot{\kappa}\dot{\lambda}})^{\alpha}{}_{\beta} \equiv \partial_{\dot{\kappa}}(\omega_{\dot{\lambda}})^{\alpha}{}_{\beta} - \partial_{\dot{\lambda}}(\omega_{\dot{\kappa}})^{\alpha}{}_{\beta} + (\,[\,\omega_{\dot{\lambda}}, \omega_{\dot{\kappa}}\,])^{\alpha}{}_{\beta}$$

It may be verified that under the transformations

$$\omega_{\dot{\lambda}} \longrightarrow \frac{\partial f^{\dot{\alpha}}}{\partial x^{\dot{\lambda}}}\,\omega_{\dot{\alpha}} \tag{52}$$

$$\omega_{\dot{\lambda}} \longrightarrow \mathcal{U}\,\omega_{\dot{\lambda}}\,\mathcal{U}^{-1} - (\,\partial_{\dot{\lambda}}\mathcal{U})\,\mathcal{U}^{-1} \tag{53}$$

the quantity $R_{\dot{\kappa}\dot{\lambda}}$ undergoes the transformations

$$R_{\dot{\kappa}\dot{\lambda}} \longrightarrow \frac{\partial f^{\dot{\alpha}}}{\partial x^{\dot{\kappa}}}\,\frac{\partial f^{\dot{\beta}}}{\partial x^{\dot{\lambda}}}\,R_{\dot{\alpha}\dot{\beta}} \tag{54}$$

$$R_{\dot{\kappa}\dot{\lambda}} \longrightarrow \mathcal{U}\,R_{\dot{\kappa}\dot{\lambda}}\,\mathcal{U}^{-1} \tag{55}$$

These transformation laws imply that the simplest invariant quantity which we may form is given by

$$\mathcal{L} = M^{2}\,\eta_{\alpha\gamma}\,(R_{\dot{\kappa}\dot{\lambda}})^{\alpha}{}_{\beta}\,e^{\beta\dot{\lambda}}\,e^{\gamma\dot{\kappa}} \tag{56}$$

where M^2 is a constant with the dimensions of squared mass. It should be noted that the inverse metric has been introduced to raise the dotted indices of the vierbeins.

We could also form higher order invariants such as the Yang action[26].

$$\mathcal{L}' = (\,R^{\dot{\mu}\dot{\nu}}R_{\dot{\mu}\dot{\nu}}\,)^{\alpha}{}_{\alpha} \tag{57}$$

Here, we have once again used the metric to raise a pair of dotted indices.

Thus, by using arguments based on the gauge invariance of a theory with a bein which provides a nontrivial representation of SO(3,1) and GL(4,R) we are led to general relativity. Therefore, we conclude that the representations provided by the bein field are very important. For instance, let us consider a bein field, $e^{i\alpha}{}_{\dot{\mu}}(x)$, which transforms as SO(2) on the index i, SO(3,1) on the index α, and as GL(4,R) on the $\dot{\mu}$ index. If this bein is factorizable into the form

$$e^{i\alpha}{}_{\dot{\mu}}(x) = e^{i}(x)\, e^{\alpha}{}_{\dot{\mu}}(x) \tag{58}$$

then we may construct a system that is like the coupled Maxwell-Einstein system. If the bein is not factorizable, then we have a theory of the unified field type. So we observe that bein fields play a particularly important role in gauge theories.

A Unified Field Theory in Fermi-Bose Superspace

We being by assuming that the constant supermatrix η_{AB} which is given by

$$\eta_{AB} \equiv \begin{bmatrix} (\gamma^{\circ})_{ab} & 0 \\ 0 & \eta_{\alpha\beta} \end{bmatrix} \tag{59}$$

is the supersymmetric generalization of the usual Minkowski metric. As we have seen, a supermetric which is compatible with global supersymmetry may be expressed as

$$\mathring{h}_{\dot{M}\dot{N}}(\theta,x) = \mathring{\mathbb{E}}^{K}{}_{\dot{M}}\, \eta_{KL}\, (\mathring{\mathbb{E}}^{*})^{L}{}_{\dot{N}} \tag{60}$$

where the octad, $\mathring{\mathbb{E}}$, takes the form given in equation (9).

It can be noted that the flat supermetric possesses a graded isometry group. To see this, we may subject the global achtbein to the transformation

$$\mathring{\mathbb{E}}^{K}{}_{\dot{M}} \longrightarrow \mathring{\mathbb{E}}^{A}{}_{\dot{M}}\, (\mathcal{U})^{K}{}_{A} \tag{61}$$

where $\mathcal{U}$ is an arbitrary supermatrix. Under this transformation the supermetric transforms as

$$\overset{\circ}{h}_{\dot{M}\dot{N}} \longrightarrow \overset{\circ}{\mathbb{E}}{}^{A}{}_{\dot{M}} (\mathcal{U})^{K}{}_{A}\, \eta_{KL} (\mathcal{U}^{*})^{L}{}_{B} (\overset{\circ}{\mathbb{E}}{}^{*})^{B}{}_{\dot{N}} \qquad (62)$$

If the supermatrix $\mathcal{U}$ satisfies the equation

$$(\mathcal{U})^{K}{}_{A}\, \eta_{KL} (\mathcal{U}^{*})^{L}{}_{B} = \eta_{AB} \qquad (63)$$

then the supermetric is unchanged. This equation defines the isometries of the supermetric. This graded group plays the same role as does SO(3,1) for general relativity. However, it should be noted that the full graded group of isometries is not realizable on the co-ordinates of superspace. There is a subgroup which may be represented on this basis. This subgroup contains SO(3,1).

In order to have a curved superspace, we assume that the octad fields may deviate from the configuration given by $\overset{\circ}{\mathbb{E}}$. That is, we assume that in curved superspace

$$\mathbb{E}^{A}{}_{\dot{M}}(\theta, x) = \overset{\circ}{\mathbb{E}}{}^{A}{}_{\dot{M}}(\theta, x) + \tilde{\mathbb{E}}^{A}{}_{\dot{M}}(\theta, x) \qquad (64)$$

where $\tilde{\mathbb{E}}$ is the nontrivial part of the octad. Thus, in curved superspace we have

$$h_{\dot{M}\dot{N}}(\theta, x) = \mathbb{E}^{A}{}_{\dot{M}}\, \eta_{AB} (\mathbb{E}^{*})^{B}{}_{\dot{N}} \qquad (65)$$

for the fully interacting supermetric. It should be noted that the graded group of isometries is a local group. That is, the supermatrices $\mathcal{U}$ may be functions of X.

We are now in position to define a co-ordinate transformation in superspace. We may define the co-ordinates $X^{\dot{M}}$ so that

$$X^{\dot{M}} = F^{\dot{M}}(Z) \qquad (66)$$

where $Z^{\dot{N}}$ is an alternate set of co-ordinates which label superspace. This equation implies that

$$d\mathbb{X}^{\dot M} = d\mathbb{Z}^{\dot K}\,\frac{\partial F^{\dot M}}{\partial \mathbb{Z}^{\dot K}}$$

$$d\mathbb{X}^{\dot M}\,\mathbb{E}^{A}{}_{\dot M}(\mathbb{X}) = d\mathbb{Z}^{\dot K}\,\frac{\partial F^{\dot M}}{\partial \mathbb{Z}^{\dot K}}\,\mathbb{E}^{A}{}_{\dot M}(\mathbb{X}) \tag{67}$$

If we now define $\mathbb{E}^{A}{}_{\dot K}(\mathbb{Z})$ so that

$$\mathbb{E}^{A}{}_{\dot K}(\mathbb{Z}) = \frac{\partial F^{\dot M}}{\partial \mathbb{Z}^{\dot K}}\,\mathbb{E}^{A}{}_{\dot M}(\mathbb{X})\Big|_{\mathbb{X}=F(\mathbb{Z})} \tag{68}$$

then we have the following equation.

$$d\mathbb{X}^{\dot M}\,\mathbb{E}^{A}{}_{\dot M}(\mathbb{X}) = d\mathbb{Z}^{\dot K}\,\mathbb{E}^{A}{}_{\dot K}(\mathbb{Z}) \tag{69}$$

Furthermore, by applying the operation of complex conjugation we conclude that

$$[\,\mathbb{E}^{B}{}_{\dot L}(\mathbb{Z})]^{*} = [\,\mathbb{E}^{B}{}_{\dot N}(\mathbb{X})]^{*}\left(\frac{\partial F^{\dot N}}{\partial \mathbb{Z}^{\dot L}}\right)^{*}\Big|_{\mathbb{X}=F(\mathbb{Z})} \tag{70}$$

Therefore, the supermetric expressed in the X co-ordinates is related to that in the Z co-ordinates through the equation

$$h_{\dot K\dot L}(\mathbb{Z}) = \frac{\partial F^{\dot M}}{\partial \mathbb{Z}^{\dot K}}\,h_{\dot M\dot N}(\mathbb{X})\left(\frac{\partial F^{\dot N}}{\partial \mathbb{Z}^{\dot L}}\right)^{*}\Big|_{\mathbb{X}=F(\mathbb{Z})} \tag{71}$$

It may be noted how complex conjugation simplifies the discussion of transformations properties. This is a consequence of the fact that complex conjugation in superspace is not the same as ordinary complex conjugation. To illustrate this let us consider the following example. Let θ^{a} and $\in^{b}$ be independent Majorana spinors. The quantity J^{ab} is defined by

$$J^{ab} = i\,\theta^{a}\in^{b} \tag{72}$$

and is real. To see this we perform complex conjugation

$$(J^{ab})^{*} = -i\,(\theta^{a}\in^{b})^{*} = -i\in^{b}\theta^{a} = i\,\theta^{a}\in^{b} \tag{73}$$

But the fermionic derivative of this quantity is imaginary.

On differentiating J^{ab} we find the equation

$$\frac{\partial}{\partial\theta^c} J^{ab} = i\,\delta^a_c\,\epsilon^b \tag{74}$$

Now we take the complex conjugate of this expression to find,

$$\left(\frac{\partial}{\partial\theta^c} J^{ab}\right)^* = -\left(\frac{\partial}{\partial\theta^c} J^{ab}\right) \tag{75}$$

Thus, even though the two sets of co-ordinates X and Z, are real and functionally related by the equation X=F(Z); the quantity $(\partial F/\partial Z)^*$ is in general not equal to $(\partial F/\partial Z)$. It can be seen that our notation of derivative and conjugate derivative is equivalent to the left and right derivatives of Arnowitt and Nath[27].

Having obtained the transformation properties of the supermetric under its graded group of isometries and under general co-ordinate transformations of superspace, we take the next step and define a fully covariant derivative such that $\mathfrak{D}_{\dot{K}}\mathbb{E}^A{}_{\dot{L}}=0$. Thus we require

$$\partial_{\dot{K}}\mathbb{E}^A{}_{\dot{L}} + (\mathbb{W}_{\dot{K}})^A{}_C\,\mathbb{E}^C{}_{\dot{L}} - \mathbb{L}_{\mathfrak{a}\dot{K}\dot{L}}{}^{\dot{R}}\,\mathbb{E}^A{}_{\dot{R}} = 0 \tag{76}$$

Just as before, we may solve this equation to express $\mathbb{L}_{\mathfrak{a}}$ in terms of $\mathbb{E}$ and $\mathbb{W}$. The result is simply given by

$$\mathbb{L}_{\mathfrak{a}\dot{K}\dot{L}}{}^{\dot{R}}\,\hbar_{\dot{R}\dot{M}} \equiv \mathbb{L}_{\mathfrak{a}\dot{K}\dot{L}\dot{M}} = \eta_{AB}\,(\hat{\mathfrak{D}}_{\dot{K}}\mathbb{E}^A{}_{\dot{L}})\,\mathbb{E}^{*B}{}_{\dot{M}} \tag{77}$$

$$\hat{\mathfrak{D}}_{\dot{K}}\mathbb{E}^A{}_{\dot{L}} = \eta_{AB}\left[\partial_{\dot{K}}\mathbb{E}^A{}_{\dot{L}} + (\mathbb{W}_{\dot{K}})^A{}_C\,\mathbb{E}^C{}_{\dot{L}}\right] \tag{78}$$

where once again we introduce a partially covariant derivative.

As we have seen, the supermetric possesses a graded group of isometries. We therefore expect the connections to also possess this property. However, we find that under the transformation

$$\mathbb{E}^A{}_{\dot{M}} = \mathbb{E}'^B{}_{\dot{M}}\,(\mathcal{U})^A{}_B \tag{79}$$

$\mathbb{L}_0$ cannot remain invariant. This stems from the fact that if $\mathcal{U}$ is assumed to have both fermionic and bosonic elements, then it is impossible to define a gauge transformation on $\mathbb{W}_{\dot{K}}$ alone which restores the form of $\mathbb{L}_0$. If, however, the supermatrix $\mathcal{U}$ is block diagonal like η then we may define

$$(\mathbb{W}'_{\dot{K}})^{A}{}_{B} = [\,(\partial_{\dot{K}}\mathcal{U}^{C}{}_{B}) + (\mathbb{W}_{\dot{K}})^{C}{}_{D}\,\mathcal{U}^{D}{}_{B}]\,\eta_{CE}\,\mathcal{U}^{*E}{}_{F}\,\eta^{FA} \tag{80}$$

and the form of $\mathbb{L}_0$ will remain unchanged. This can be seen to be a supersymmetric version of the usual transformation law of a gauge field.

Thus, we find the unexpected result that the entire graded group of isometries may not be made into local symmetries of $\mathbb{L}_0$. Actually, we may make a stronger statement. It turns out that even if $\mathcal{U}$ is independent of X, the supermatrix $\mathcal{U}$ must still be block diagonal in order for $\mathbb{L}_0$ to remain invariant. So it is only the block diagonal subgroup of isometries which may be local symmetries. Therefore, we need only introduce block diagonal gauge fields $(\mathbb{W}_{\dot{K}})^{B}{}_{A}$; one such gauge field for each block diagonal generator. If we use the index $\mathcal{I}$ to denote these generators then we have

$$(\mathbb{W}_{\dot{K}})^{A}{}_{B} = \mathbb{W}^{\mathcal{I}}_{\dot{K}}\,(t_{\mathcal{I}})^{A}{}_{B} \tag{81}$$

where the supermatrices $(t_{\mathcal{I}})$ satisfy the relations

$$(t_{\mathcal{I}})^{C}{}_{B}\,\eta_{CA} + \eta_{BC}\,(t^{*}{}_{\mathcal{I}})^{C}{}_{A} = 0 \tag{82}$$

$$(t_{\mathcal{I}})^{A}{}_{B} = (-)^{[\sigma(A)-\sigma(B)]}\,(t_{\mathcal{I}})^{A}{}_{B} \tag{83}$$

We now differentiate the supermetric to find the result below.

$$\partial_{\dot{K}}\mathcal{A}_{\dot{L}\dot{M}} = \mathbb{L}_{0\dot{K}\dot{L}}{}^{\dot{R}}\,\mathcal{A}_{\dot{R}\dot{M}} + (-)^{\pi}\,\mathcal{A}_{\dot{L}\dot{R}}\,\mathbb{L}^{*}_{0\dot{K}\dot{M}}{}^{\dot{R}} - \mathbb{Q}_{\dot{K}\dot{L}\dot{M}}$$

$$\mathbb{Q}_{\dot{K}\dot{L}\dot{M}} \equiv \mathbb{E}^{A}{}_{\dot{L}}[(-)^{\pi'}(\mathcal{W}_{\dot{K}})^{C}{}_{A}\,\eta_{CB} + (-)^{\pi''}\eta_{AC}(\mathcal{W}^{*}_{\dot{K}})^{C}{}_{B}]\,\mathbb{E}^{*B}{}_{\dot{M}}$$

$$\pi \equiv \sigma(\dot{K})[\,\sigma(\dot{L})+\sigma(\dot{M})+1\,]$$

$$\pi' \equiv \sigma(\dot{K})[\,\sigma(C)+\sigma(\dot{L})\,]$$

$$\pi'' \equiv \sigma(\dot{K})[\,\sigma(B)+\sigma(\dot{L})+1\,] \tag{85}$$

In arriving at the above result, we have made use of the following identity

$$\partial_{\dot{K}}\,\mathbb{E}^{*B}{}_{\dot{M}} = (-)^{\sigma(\dot{K})[\sigma(B)-\sigma(\dot{M})-1]}\,(\,\partial_{\dot{K}}\,\mathbb{E}^{B}{}_{\dot{M}})^{*} \tag{86}$$

Equation (84) may be expressed in the form below.

$$\mathcal{D}_{\dot{K}}\,h_{\dot{L}\dot{M}} = -\,\mathbb{Q}_{\dot{K}\dot{L}\dot{M}} \tag{87}$$

Thus, it appears as though our curved superspace must also be nonmetric in the differential geometric sense.

Let us now show that, in fact, the curved superspace need not be nonmetric. We recall that the fields $\mathcal{W}_{\dot{K}}$ are block diagonal. The product of any number of such matrices is also block diagonal. This implies that the nonmetricity may be rewritten in the simple form below.

$$\mathbb{Q}_{\dot{K}\dot{L}\dot{M}} = (-)^{\pi'}\,\mathbb{E}^{A}{}_{\dot{L}}[\,(\mathcal{W}_{\dot{K}})^{C}{}_{A}\,\eta_{CB} + (-)^{\sigma(\dot{K})}\eta_{AC}(\mathcal{W}^{*}_{\dot{K}})^{C}{}_{B}]\,\mathbb{E}^{*B}{}_{\dot{M}} \tag{88}$$

In this form it is clear that the nonmetricity will vanish if

$$0 = (\mathcal{W}_{\dot{K}})^{C}{}_{A}\,\eta_{CB} + (-)^{\sigma(\dot{K})}\,\eta_{AC}\,(\mathcal{W}^{*}_{\dot{K}})^{C}{}_{B} \tag{89}$$

We recall now equations (81) and (82) and conclude that the equation above will be satisfied if

$$\mathcal{W}^{I}_{\dot{K}} = +(-)^{\sigma(\dot{K})}\,(\mathcal{W}^{I}_{\dot{K}})^{*} \tag{90}$$

This equation simply implies that the fermi components are purely imaginary and the boson components are purely real. This requirement may seem somewhat artificial. But, we may recall that it is by an analogous requirement that Einstein's unified field theory is able to avoid nonmetricity. Thus, we may formulate a metric theory of curved superspace. In this metric theory, we are encouraged to identify the quantity $\mathbb{L}_{\square}$ as the super connection coefficient. It may be noted that this connection is complex.

Now we turn to the question of the transformation properties of $\mathbb{L}_{\square}$ under a general co-ordinate transformation of superspace. To this end, we subject the octad to the transformation of equation (68) and substitute this into equation (77). The response of the connection is

$$\mathbb{L}_{\square\,\dot K \dot L \dot M} \longrightarrow (-)^{\sigma(\dot A)[\sigma(\dot B)-\sigma(\dot L)]} \frac{\partial F^{\dot A}}{\partial Z^{\dot K}} \frac{\partial F^{\dot B}}{\partial Z^{\dot L}} \mathbb{L}_{\square\,\dot A \dot B \dot C} \left(\frac{\partial F^{\dot C}}{\partial Z^{\dot K}}\right)^{*} + \frac{\partial^2 F^{\dot A}}{\partial Z^{\dot K}\partial Z^{\dot L}} h_{\dot A \dot B} \left(\frac{\partial F^{\dot B}}{\partial Z^{\dot M}}\right)^{*} \tag{91}$$

We may write $\mathbb{L}_{\square}$ as the sum of two other quantities Γ and T. These quantities are defined by the equations

$$\Gamma^{\square}_{\dot K \dot L \dot M} = \tfrac{1}{2}\left[\, \mathbb{L}_{\square\,\dot K \dot L \dot M} + (-)^{\sigma(\dot K)\sigma(\dot L)} \mathbb{L}_{\square\,\dot L \dot K \dot M} \right] \tag{92}$$

$$\mathbb{T}^{\square}_{\dot K \dot L \dot M} = \tfrac{1}{2}\left[\, \mathbb{L}_{\square\,\dot K \dot L \dot M} - (-)^{\sigma(\dot K)\sigma(\dot L)} \mathbb{L}_{\square\,\dot L \dot K \dot M} \right] \tag{93}$$

The first of these is the generalization of the torsionless connection of a Riemannian manifold. The second is the generalized torsion tensor. The torsionless part of the connection transforms just as does $\mathbb{L}_{\square}$ under a general co-ordinate transformation. But $\mathbb{T}^{\square}$ transforms as a true tensor

$$\mathbb{T}^{\square}_{\dot K \dot L \dot M} \longrightarrow (-)^{\sigma(\dot A)[\sigma(\dot B)-\sigma(\dot L)]} \frac{\partial F^{\dot A}}{\partial Z^{\dot K}} \frac{\partial F^{\dot B}}{\partial Z^{\dot L}} \mathbb{T}^{\square}_{\dot A \dot B \dot C} \left(\frac{\partial F^{\dot C}}{\partial Z^{\dot M}}\right)^{*} \tag{94}$$

under a general co-ordinate transformation.

At this point, let us consider a limit where

$$\mathbb{E}^{A}{}_{\dot{M}} = \mathring{\mathbb{E}}^{A}{}_{\dot{M}} \tag{95}$$

$$(\mathbb{W}_{\dot{K}})^{A}{}_{B} = 0 \tag{96}$$

In this limit, equations (76), (84), and (93) go over to equation (16), (20), and (19) respectively. In this limit, we recover global supersymmetry. Furthermore, we now realize that equation (30) imples that gauge superfields are allowed to couple to the torsion tensor. From equation (24) we see that the scalar superfield is also coupled to the torsion tensor.

Now we return to the arguments which led to general relativity in the second section. We utilize the partially covariant derivative to define a curvature tensor.

$$([\hat{\mathfrak{D}}_{\dot{K}}, \hat{\mathfrak{D}}_{\dot{L}}\})^{A}{}_{B} \equiv (\mathbb{R}_{\dot{K}\dot{L}})^{A}{}_{B} \tag{97}$$

$$(\mathbb{R}_{\dot{K}\dot{L}})^{A}{}_{B} = \partial_{\dot{K}}(\mathbb{W}_{\dot{L}})^{A}{}_{B} - (-)^{\sigma(\dot{K})\sigma(\dot{L})}\partial_{\dot{L}}(\mathbb{W}_{\dot{K}})^{A}{}_{B} + ([\mathbb{W}_{\dot{K}}, \mathbb{W}_{\dot{L}}\})^{A}{}_{B} \tag{98}$$

This tensor may be subjected to the transformation

$$(\mathbb{W}_{\dot{K}})^{A}{}_{B} = \mathcal{U}^{A}{}_{M}(\mathbb{W}'_{\dot{K}})^{M}{}_{N}(\mathcal{U}^{-1})^{N}{}_{B} - (\partial_{\dot{K}}\mathcal{U}^{A}{}_{M})(\mathcal{U}^{-1})^{M}{}_{B}$$

$$(\mathcal{U}^{-1})^{M}{}_{B} \equiv \eta_{BD}(\mathcal{U}^{*})^{D}{}_{E}\,\eta^{EM}$$

$$\eta_{AE}\,\eta^{EB} = \delta_{A}^{B} \tag{99}$$

The response is the expected result.

$$(\mathbb{R}_{\dot{K}\dot{L}})^{A}{}_{B} = \mathcal{U}^{A}{}_{M}(\mathbb{R}'_{\dot{K}\dot{L}})^{M}{}_{N}(\mathcal{U}^{-1})^{N}{}_{B} \tag{100}$$

Thus, the quantity $\mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}}$ which is defined by the equation

$$\mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}} = \eta_{AC}(\mathbb{R}_{\dot{K}\dot{L}})^{A}{}_{B}\,\mathbb{E}^{B}{}_{\dot{M}}(\mathbb{E}^{*})^{C}{}_{\dot{N}} \tag{101}$$

is invariant under the local isometries. This is, of course, the supercurvature tensor. In the limit of global supersymmetry it must vanish owing to the vanishing of $\mathbb{W}_{\dot{L}}$. Thus, global superspace is flat.

Finally we perform the transformation

$$\mathbb{E}^{B}{}_{\dot{M}} \longrightarrow \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} \mathbb{E}^{B}{}_{\dot{C}}$$

$$\partial_{\dot{K}} \longrightarrow \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \partial_{\dot{A}}$$

$$\mathbb{W}_{\dot{L}} \longrightarrow \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \mathbb{W}_{\dot{B}} \tag{102}$$

and deduce that the transformation law for the supercurvature tensor is given by

$$\mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}} \longrightarrow \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} \mathbb{R}_{\dot{A}\dot{B}\dot{C}\dot{D}} \left(\frac{\partial F^{\dot{D}}}{\partial Z^{\dot{N}}}\right)^{*} (-)^{\phi} \tag{103}$$

where $\phi \equiv \sigma(\dot{A})[\sigma(\dot{B})+\sigma(\dot{L})]+[\sigma(\dot{A})+\sigma(\dot{B})][\sigma(\dot{C})+\sigma(\dot{M})]$. This may be reexpressed in the more symmetric form

$$\mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}} \longrightarrow (-)^{\phi'} \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \mathbb{R}_{\dot{A}\dot{B}\dot{C}\dot{D}} \left(\frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}}\right)^{*} \left(\frac{\partial F^{\dot{D}}}{\partial Z^{\dot{N}}}\right)^{*}$$

$$\phi' \equiv \sigma(\dot{A})[\sigma(\dot{B})+\sigma(\dot{L})] + [\sigma(\dot{D})+1][\sigma(\dot{C})+\sigma(\dot{M})] \tag{104}$$

Under complex conjugation this tensor has the following transformation law.

$$(\mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}})^{*} = (-)^{\epsilon} \mathbb{R}_{\dot{L}\dot{K}\dot{N}\dot{M}}$$

$$\epsilon \equiv [\sigma(\dot{K})+\sigma(\dot{L})][\sigma(\dot{M})+\sigma(\dot{N})+1] \tag{105}$$

An inverse supermetric may be introduced via the definition below.

$$\mathcal{h}_{\dot{K}\dot{L}} \mathcal{h}^{\dot{L}\dot{M}} = \delta_{\dot{K}}^{\dot{M}} \tag{106}$$

In order to preserve this property under co-ordinate transformation, we require the following statement be valid.

$$\frac{\partial F^{\dot{R}}}{\partial Z^{\dot{K}}} \frac{\partial Z^{\dot{M}}}{\partial F^{\dot{R}}} = \delta^{\dot{M}}_{\dot{K}} = \left(\frac{\partial Z^{\dot{M}}}{\partial F^{\dot{R}}}\right)^* \left(\frac{\partial F^{\dot{R}}}{\partial Z^{\dot{K}}}\right)^* \tag{107}$$

Furthermore, as the supermetric undergoes the transformation

$$h_{\dot{L}\dot{M}} \longrightarrow \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{L}}} h_{\dot{A}\dot{B}} \left(\frac{\partial F^{\dot{B}}}{\partial Z^{\dot{M}}}\right)^* \tag{108}$$

the inverse supermetric must transform also.

$$h^{\dot{L}\dot{M}} \longrightarrow \left(\frac{\partial Z^{\dot{L}}}{\partial F^{\dot{A}}}\right)^* h^{\dot{A}\dot{B}} \left(\frac{\partial Z^{\dot{M}}}{\partial F^{\dot{B}}}\right) \tag{109}$$

A contraction between the inverse supermetric and the supercurvature tensor may be formed.

$$\mathbb{R}_{\dot{L}\dot{N}} \equiv (-)^{\sigma(\dot{L})\sigma(\dot{M})} h^{\dot{M}\dot{K}} \mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}} \tag{110}$$

It may be verified with the use of equations (109) and (103) that this quantity transforms as does the supermetric.

$$\mathbb{R}_{\dot{L}\dot{N}} \longrightarrow \left(\frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}}\right) \mathbb{R}_{\dot{B}\dot{D}} \left(\frac{\partial F^{\dot{D}}}{\partial Z^{\dot{N}}}\right)^* \tag{111}$$

Finally this tensor may be contracted to form a scalar $\mathbb{R}$, where

$$\begin{aligned} \mathbb{R} &\equiv (-)^{\sigma(\dot{L})} \mathbb{R}_{\dot{L}\dot{N}} h^{\dot{N}\dot{L}} \\ &= (-)^{\sigma(\dot{L})[\sigma(\dot{M})+1]} h^{\dot{M}\dot{K}} \mathbb{R}_{\dot{K}\dot{L}\dot{M}\dot{N}} h^{\dot{N}\dot{L}} \end{aligned} \tag{112}$$

It is apparent that $\mathbb{R}$ is the supercurvature and a true scalar under local and global transformations.

Thus, we may take as an action

$$\int d^8X \, [\, det(\mathbb{E})\, det(\mathbb{E}^*)]^{\frac{1}{2}} \; \mathbb{R}(\mathbb{E},\mathbb{E}^*,\mathbb{W}) \tag{113}$$

where the definition of the superdeterminant has been developed by Arnowitt, Nath, and Zumino[40]. Once again we have the option of forming other invariants such as the Yang action or modifications such as those indicated by Boal and Moffat[41].

Conclusion

We are now in position to make some comparisons between this approach and gauge supersymmetry. It was first shown by Woo[11] that if the geometry of curved superspace is assumed to be Riemannian then global superspace must be identified as a singular limit. This may be interpreted in a relatively straightforward fashion. Global superspace possesses torsion. This is implicit in the fact that the anticommutator of two fermionic derivatives is a bosonic derivative. But a Riemannian space cannot possess torsion. In order to produce the torsion of global superspace the geometry of the Riemannian superspace must be severely distorted. We believe this is the meaning of the singular limit proposed by Woo.

On the other hand, the theory outlined here does not lead to general relativity in the bose sector of superspace. Instead, we are led to Einstein's unified field theory. Whether this is an asset or liability we are presently unable to discern. This theory does possess global superspace as a continuous limit, however, Also we have had to introduce a whole new set of fields, the super Fock-Ivanenko coefficients which we denoted by $\mathcal{W}_{K}$. Thus, the number of fields has been increased beyond the prodigious number already present in gauge supersymmetry.

At this point there are many questions which must be answered. We are encouraged, however, that our earlier speculation concerning Einstein's unified field theory and local supersymmetry has now been proven.

Figure (6)

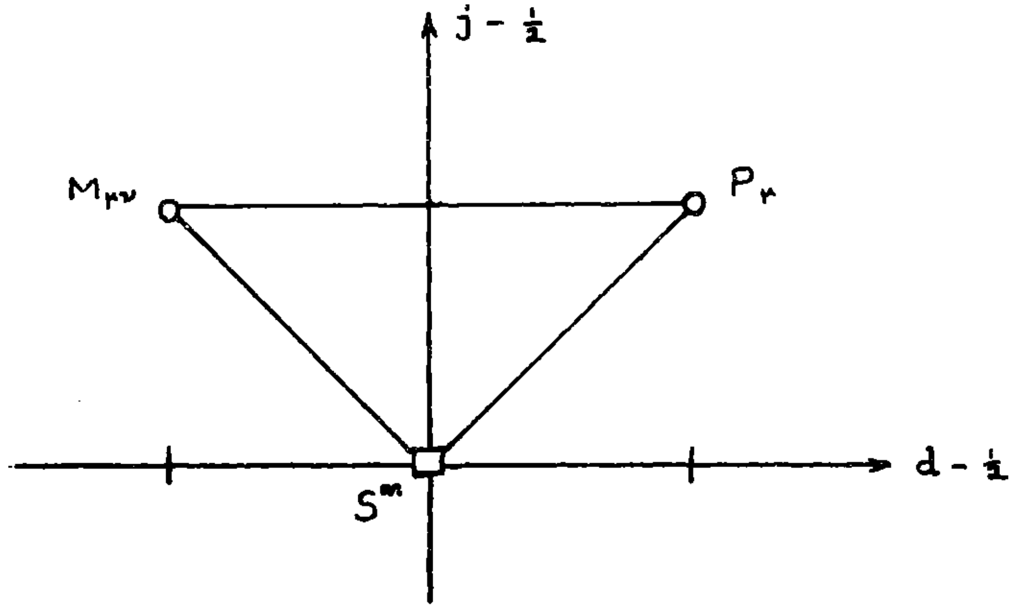


Diagram illustrating the super Poincaré group.

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