Localization of nonlinear water waves over a random bottom

by

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Abstract

The localization of harmonics generated in nonlinear shallow water waves over a random seabed is studied. A bathymetry which fluctuates randomly from a constant mean adds multiple scattering to resonant interactions and harmonic generation. By the method of multiple scales, nonlinear evolution equations for the harmonic amplitudes are derived. Effects of multiple scattering are shown to be represented by certain linear damping terms with complex coefficients related to the correlation function of the seabed disorder. For any finite number of harmonics, an equation governing the total wave energy variation is derived and used to verify the accuracy of numerical solutions. The effects of spatial attenuation (localization) on harmonic generation are studied by numerical solution of the harmonic amplitude equations.

The localization of nonlinear water waves over a random seabed of slowly varying mean depth is studied. From the Boussinesq equations and by the method of multiple scales, governing equations and solutions are obtained for different orders of approximation. The solvability conditions derived from the governing equations yield a modified Schrödinger equation for the free surface displacement amplitude. An analytical solution for the case of a steady train of attenuated Stokes waves is derived.

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Chapter 1

Introduction

1.1 Background and motivation

Several characteristics of long waves in shallow water are of general interest to wave physics in many different contexts. The interplay between nonlinearity and dispersion has, on one hand, led to impressive advances in soliton dynamics and the inverse scattering theory [1]. On the other hand independent discoveries in wave-wave interactions ushered the new age of oceanography [29], [30] and nonlinear optics [3]. In particular the mechanism of harmonic generation, first found in optics, is known to have a close cousin in shallow-water waves [24], [6].

Anderson localization which was originated in the study of transport in disordered quantum systems [2], is still an expanding topic in wave propagation in random media [33], [34]. Many mathematical studies on nonlinear waves in random media have also appeared. In particular Devillard & Souillard [7] have studied the one-dimensional nonlinear Schrödinger equation with a random potential. Extensions of this work for incident solitons and other types of random potentials have been advanced by many others (e.g. [8], [17], [5], [10], [12]). For extensive reviews, see [14] and [4]. Relevant to long waves in shallow water, a theory for the KdV equation [18] with a weak random potential has also been studied by Garnier [11]. In these mathematical models, a common feature is that the final differential equation has one or more stochastic coefficients.

In coastal oceanography, it is important to understand how the surface wave-spectra are affected by variations of bathymetry. Existing studies have largely been limited to classical aspects such as refraction and/or scattering by deterministic depth variations and by friction-induced attenuation. Since some complex bathymetries can be best described as a random function of space, it is of practical value to see how multiple scattering by random bathymetry can cause attenuation by radiation damping. Only a few papers on the linearized aspects have appeared in the literature ([15], [9], [27], [25], [26], [28]). For nonlinear long waves in shallow water, the only known theories are Howe [16] and Rosales & Papanicolaou [32]. For intermediate depth, the perturbation method of multiple scales, known as the theory of homogenization in some contexts, has been shown to be an effective tool for analyzing weakly nonlinear waves in a weakly disordered medium of large spatial extent. The basic ideas were first explained for the simple case of a taut string imbedded in a nonlinearly elastic surrounding whose elastic properties contain a random component [23]. It has been shown that the wave envelope is governed by a cubic Schrödinger equation modified by a linear term with a complex damping coefficient, which is related to the statistical correlation function of the random perturbations.

Effects of localization on the evolution of soliton envelopes and side-band instability have been examined. Similar analysis have been reported for small-amplitude water waves of intermediate wavelength over a seabed with weak disorder in depth. One- and two-dimensional nonlinear Schrödinger equations with a linear damping term have been derived for the envelope of a narrow-banded wavetrain. In one dimension, complex diffraction is found after a bi-soliton passes over a finite strip of random seabed [22]. When the random bathymetry is two-dimensional and confined in an elongated area of large width and length, the envelope of a uniform wave train is found to turn to a number of dark solitons in the shadow [31].

1.2 Physics and important assumptions

Water waves over a random medium are studied for constant and slowly varying bathymetries.

The starting point for the study of the effect of localization on harmonic generation over a bottom of constant average depth are the Boussinesq equations. One-dimensional long waves in shallow water are considered. $h(x,t)$ is taken to be the local depth beneath the still water level, $\eta(x,t)$ the free surface displacement above and $u(x,t)$ the depth-averaged horizontal velocity of the water. To the leading order of accuracy, for dimensional variables and for a varying depth, the modified Boussinesq equations are found to be.

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(h+\eta)\,u\right] = 0 \tag{1.2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{h}{2}\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u}{\partial t}\right) - \frac{h^2}{6}\frac{\partial^3 u}{\partial x^2 \partial t} \tag{1.2.2}$$

Two small independent parameters ϵ and μ^2 characterizing nonlinearity and dispersion respectively are defined to obtain dimensionless governing equations. They are considered to be of the same order of magnitude and simulations are run for an Ursell number μ^2/ϵ of order 1. The bathymetry is assumed to be slowly varying, the depth h is defined as the sum of a constant average depth and a fluctuating depth taken to be a stationary random function of zero mean.

The study of localization of nonlinear water waves over a slowly varying bottom is based on the boundary value problem defined for the velocity potential ϕ by the Laplace equation and the no flux conditions at the free surface defined by $z = \zeta$ and through the bottom defined by $z = -H + \epsilon b$. ζ is the free surface displacement, H is the slowly varying average bottom depth, b is a random function of zero mean and ϵ is the small parameter characterizing dispersion.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for} \quad -H + \epsilon b < z < \zeta \tag{1.2.3}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t} \quad \text{at} \quad z = \zeta \tag{1.2.4}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x}\left[\epsilon\frac{\partial b}{\partial x} - \frac{\partial H}{\partial x}\right] \quad \text{at} \quad z = -H + \epsilon b \tag{1.2.5}$$

Once the boundary value problem is solved for ϕ, a solution for the free surface displacement ζ can be obtained from Bernouilli's equation at the free surface.

$$\zeta = -\frac{1}{g}\frac{\partial \phi}{\partial t} - \frac{1}{g}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] \quad \text{at} \quad z = \zeta \tag{1.2.6}$$

Mathematical tools are used on the governing equations in both problems to simplify them to specified orders of accuracy.

1.3 Mathematics used for simplification

Tools such as multiple scale expansion or the method of induction are used in this thesis to simplify equations and to obtain analytical and numerical solutions to the problems described above.

1.3.1 The method of multiple scales

It is often the case for physical problems to have more than one length or time scale and it can be important to decompose the problem into multiple scales. As an example, let us consider the case of water waves: a short time scale of the order of a few seconds is the wave period, a longer time scale of the order of an hour could be the tide period and an even longer time scale could be defined in years for the evolution of the environment itself. All three time scales are relevant to understanding certain aspects of wave dynamics. Assuming ϵ to be a small parameter characteristic of the problem solved, a multiple scale expansion consists in creating several ficticious new independent variables

$$t, \quad t_1 = \epsilon t, \quad t_2 = \epsilon^2 t ... \tag{1.3.1}$$

as factors of the original variable (t) and powers of ϵ; and in expanding in powers of ϵ the unknown(s) (for example velocity u, displacement ζ) for which the problem is being solved:

$$u = u_1 + \epsilon u_2 + \cdots \tag{1.3.2}$$

The governing equations at every order of ϵ can be obtained and solved either analytically or numerically.

1.3.2 Taylor expansion around $z = z_0$

Surface or bottom boundary conditions are often defined at $z = z_0$ (or $z = -H + z_0$) where z_0 is small compared to the total depth H. It can therefore be useful to rewrite the equations at $z = 0$ (or $z = -H$). Taylor expansions at $z = z_0$ are used for this purpose, where for a function f c^n at $z = z_0$,

$$f(z) = f(z_0) + (z - z_0)\left[\frac{df}{dz}\right]_{z_0} + \cdots + \frac{(z - z_0)^n}{n!}\left[\frac{d^n f}{dz^n}\right]_{z_0} \tag{1.3.3}$$

A function c^n at $z = z_0$ is a function that is continuous and has n derivatives that are defined and continuous around $z = z_0$.

1.3.3 Green's identity

Green's functions are used several times to solve non obvious boundary value problems, a detailed chapter about them can be found in [21]. Only Green's identity is given here: for two twice differentiable functions F and G over a surface Σ surrounded by the closed line ℓ:

$$\iint_{\Sigma} \left(G \bigtriangledown^2 F - F \bigtriangledown^2 G\right) d\Sigma = \oint_{\ell} \left(G\frac{\partial F}{\partial n} - F\frac{\partial G}{\partial n}\right) d\ell \tag{1.3.4}$$

where n is the outgoing normal to ℓ.

1.3.4 The method of induction

In order to prove an n^{th} equation is true for any integer n, it is useful to understand the method of induction. Let us consider an assertion A_n dependent on the integer n. It is usually straightforward to prove A_n is satisfied for small values of n such as 0, 1 or 2 but it becomes more complicated for larger numbers. The idea behind this method is to avoid proving A_n directly. If A_1 is true, proving that A_{n+1} is satisfied as long as A_n is satisfied, shows that for any given n the assertion A_n is true.

1.3.5 Complex analysis

The complex conjugate Z^* of the complex number Z is such that $Z + Z^*$ is real. By definition, any complex number can be separated into a real and an imaginary part and taking the imaginary part of a real number is zero. The following therefore holds true and is used many times.

$$Z + Z^* \in \Re \Rightarrow \Im\left(Z + Z^*\right) = 0 \tag{1.3.5}$$

Another very useful tool is the evaluation of integrals by Cauchy's residue theorem. The theorem and details of Jordan's Lemma are not given here as they can be found in [21] but the following result is given. If f is analytic in the closed curve $\mathcal{C}$ of the complex plane, then.

$$\oint_{\mathcal{C}} f(z)dz = 0 \tag{1.3.6}$$

If there are singularities for f within the curve $\mathcal{C}$, the residue theorem may be applicable and the residues at the singularities should be calculated. A detailed example of this application is given in appendix B.

1.3.6 Leibniz' rule

It is well known that integration is the inverse operator of differentiation, so that the following

$$\frac{d}{dx}\left(\int_a^x f(u)du\right) = f(x) \tag{1.3.7}$$

holds true and seems very natural. The integral between 2 constants of an integrand that depends on x and is differentiable in x is also well known.

$$\frac{\partial}{\partial x}\left(\int_a^b f(u,x)du\right) = \int_a^b \frac{\partial f}{\partial x}(u,x)du \tag{1.3.8}$$

But what happens if instead of x, the upper bound is a function $g(x)$ that is differentiable in x and if the integrand is also a differentiable function of x? This is where Leibniz' rule can be used:

$$\frac{\partial}{\partial x}\left(\int_a^{g(x)} f(u,x)du\right) = \frac{dg}{dx} f(u,g(x)) + \int_a^{g(x)} \frac{\partial f}{\partial x}(u,x)du \tag{1.3.9}$$

1.4 Thesis outline

The first part of this thesis focuses on how harmonic generation by nonlinearity is counteracted by localization. The starting point is the Boussinesq equations [20], they account for weak nonlinearity and dispersion to the leading order. After normalization, they are combined into a single equation governing the free surface displacement. A multiple scale expansion yields governing equations for each order of accuracy. The solution for the wave amplitude at the first order is a Fourier expansion of time harmonics. To avoid resonance, the coefficients of the secular terms in the governing equations are set equal to zero, this results in coupled evolution equations for each of an infinite number of harmonics.

The system for an infinite number of interacting harmonics is truncated to a limited number n of harmonics so that it can be solved numerically. An analytical expression for the evolution of the wave energy is derived for n harmonics, this is then used to verify the numerical simulations run for 6 and 10 harmonics. The Fourier expansion is however not truncated at the first and second harmonic as was done in [24] for a smooth and horizontal bottom.

In the governing equations, linear damping terms are found analytically by accounting for multiple scattering by disorder. The complex damping coefficients are ensemble averages of certain correlations of the random bathymetry, hence the evolution equations are still deterministic, unlike those with random potentials. Physical implications are examined through numerical solutions of the systems for 6 and 10 harmonics.

The second part of this thesis is purely analytical and focuses on deriving equations for the localization of water waves over a random seabed of slowly varying average depth. It follows very closely the methods used by Lo [19] and Mei & Hancock [22]. The free surface boundary condition and Bernouilli's equation are first combined into a single equation for the velocity potential ϕ so that the boundary value problem is defined for one variable only. After a Taylor expansion of the resulting boundary conditions and a multiple scale expansion in the horizontal direction and in time, governing equations are obtained for every order of accuracy.

Analytical solutions are found for the first two orders of accuracy. These are then used in the governing equations for the next order to obtain solvability conditions which can in turn be intepreted as a governing equation for the zeroth harmonic and a modified Schrödinger equation with nonlinear terms, a linear damping term with a complex coefficient due to the seabed randomness and a linear term due to the slowly varying bottom. No simulations have been done so far for this section.

Chapter 2

Localization of harmonics generated in nonlinear shallow water waves

2.1 Boussinesq equations for long water waves

Consider one-dimensional long waves in shallow water. Using primes to distinguish quantities with physical dimensions, $h'(x',t')$ denotes the local depth beneath the still water level, $\eta'(x',t')$ the free surface displacement above, and $u'(x',t')$ the depth-averaged horizontal velocity of the water. It is well known that, to the leading order of nonlinearity and dispersion, the laws of mass and momentum conservation are approximated by

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial}{\partial x'}\left[(h' + \eta')\, u'\right] = 0 \tag{2.1.1}$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + g\frac{\partial \eta'}{\partial x'} = \frac{h'}{2}\frac{\partial^2}{\partial x'^2}\left(h'\frac{\partial u'}{\partial t'}\right) - \frac{h'^2}{6}\frac{\partial^3 u'}{\partial x'^2 \partial t'} \tag{2.1.2}$$

The accuracy of these equations can be made explicit by employing the following dimensionless variables without primes:

$$\begin{aligned} x &= Kx', \quad t = t'K\sqrt{gH}, \quad \eta = \frac{\eta'}{a}, \\ h &= \frac{h'}{H}, \quad u = \frac{u'}{A\sqrt{\frac{g}{H}}} \end{aligned} \tag{2.1.3}$$

where K, a and H are respectively the typical wavenumber, wave amplitude, and mean depth. Equations (2.1.1) and (2.1.2) can then be normalized in the following form :

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(h + \epsilon\eta)\, u\right] = 0 \tag{2.1.4}$$

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\mu^2 h}{2}\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u}{\partial t}\right) - \frac{\mu^2 h^2}{6}\frac{\partial^3 u}{\partial x^2 \partial t} \tag{2.1.5}$$

where

$$\epsilon = \frac{a}{H} \ll 1, \quad \mu = KH \ll 1 \tag{2.1.6}$$

It is clear that these equations are accurate to the leading order in ϵ and μ^2, which are small but independent parameters characterizing respectively nonlinearity and dispersion.

The sea depth is assumed to deviate only slightly from a constant mean value. In dimensionless terms, h fluctuates from the constant 1 by $\sqrt{\epsilon}b(x)$, i.e.,

$$h(x) = 1 - \sqrt{\epsilon}b(x) \tag{2.1.7}$$

where b is a stationary random function of x with zero mean, $\langle b(x)\rangle = 0$. The Boussinesq equations can be rewritten as:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[\left(1 - \sqrt{\epsilon}b + \epsilon\eta\right) u\right] = 0 \tag{2.1.8}$$

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \mu^2\left[\frac{1 - 2\sqrt{\epsilon}b + \epsilon b^2}{3}\frac{\partial^3 u}{\partial x^2 \partial t} - \sqrt{\epsilon}\left(1 - \sqrt{\epsilon}\right)\left(\frac{db}{dx}\frac{\partial^2 u}{\partial x \partial t} + \frac{1}{2}\frac{d^2 b}{dx^2}\frac{\partial u}{\partial t}\right)\right] \tag{2.1.9}$$

These are, in principle, stochastic differential equations. Assuming that $\epsilon = O(\mu^2)$, (2.1.9) can be truncated to the leading orders in ϵ and μ^2.

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\mu^2}{3}\frac{\partial^3 u}{\partial x^2 \partial t} + \text{h.o.t.} \tag{2.1.10}$$

Using approximations of the governing equations to the leading order:

$$\frac{\partial \eta}{\partial x} = -\frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t} \tag{2.1.11}$$

a combination of equations (2.1.8) and (2.1.10) yields a single equation for the free surface displacement η.

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = -\sqrt{\epsilon}\left(b\frac{\partial \eta}{\partial x}\right) + \frac{\epsilon}{2}\left(\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial t^2} + \frac{\partial^2 \eta^2}{\partial t^2}\right) + \frac{\mu^2}{3}\frac{\partial^4 \eta}{\partial x^4} \tag{2.1.12}$$

2.2 Asymptotic expansions

Approximations for the propagation of a wavetrain are sought. The wavetrain is a simple harmonic at some station to the left of the region of disorder. At the leading order the incident wave is:

$$(\eta, u) \propto e^{ikx - i\omega t} \tag{2.2.1}$$

η and u are expanded in ascending powers of $\sqrt{\epsilon}$ using the method of multiple scales:

$$\begin{aligned} \eta &= \eta_0 + \epsilon^{1/2}\eta_1 + \epsilon\eta_2 + \cdots \\ u &= u_0 + \epsilon^{1/2}u_1 + \epsilon u_2 + \cdots \end{aligned} \tag{2.2.2}$$

where each unknown function depends on t and the fast and slow variables in space x and $X = \epsilon x$. For successive orders, perturbation equations are found from (2.1.12):

$$\frac{\partial^2 \eta_0}{\partial t^2} - \frac{\partial^2 \eta_0}{\partial x^2} = 0 \tag{2.2.3}$$

$$\frac{\partial^2 \eta_1}{\partial t^2} - \frac{\partial^2 \eta_1}{\partial x^2} = -\frac{\partial}{\partial x}\left(b\frac{\partial \eta_0}{\partial x}\right) \tag{2.2.4}$$

$$\frac{\partial^2 \eta_2}{\partial t^2} - \frac{\partial^2 \eta_2}{\partial x^2} = -\frac{\partial}{\partial x}\left(b\frac{\partial \eta_1}{\partial x}\right) + 2\frac{\partial^2 \eta_0}{\partial X \partial x} + \frac{1}{2}\left(\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial t^2} + \frac{\partial^2 \eta_0^2}{\partial t^2}\right) + \frac{\nu}{3}\frac{\partial^4 \eta_0}{\partial x^4} \tag{2.2.5}$$

where $\nu = \frac{\mu^2}{\epsilon} = 0(1)$ is the Ursell number.
The perturbation problems are now solved sequentially.

2.2.1 Equation and solution at $O(\epsilon^0)$

Consider the evolution of a train of progressive waves whose harmonics have the amplitudes A_j, the frequencies $\omega_m = m\omega$ and the wavenumbers k_m, with m = 1,2,...:

$$\eta_0 = \frac{1}{2}\sum_{m=-\infty}^{\infty} A_m(X)e^{i\theta_m} \quad \text{and} \quad u_0 = \frac{1}{2}\sum_{m=-\infty}^{\infty} B_m(X)e^{i\theta_m} \tag{2.2.6}$$

such that

$$\theta_m = k_m x - \omega_m t, \quad k_{-m} = -k_m \quad \text{and} \quad \omega_{-m} = -\omega_m \tag{2.2.7}$$

and the amplitudes obey

$$A_{-m} = A_m^* \quad \text{and} \quad B_{-m} = B_m^* \tag{2.2.8}$$

where A^* refers to the complex conjugate of A. In order to have the normalized mean depth equal to unity, A_0 is set equal to zero.

The dispersion relation: $\omega_m = k_m$ is obtained from the first order equation (2.2.3). The following normalized wavenumbers and frequencies are implied $k_m = mk = k'_m/K$ and $\omega_m = \omega'_m/K\sqrt{gH} = m\omega$, k and ω are both of order unity.

At the leading order in ϵ, mass conservation

$$\frac{\partial \eta_0}{\partial t} = -\frac{\partial u_0}{\partial x} \tag{2.2.9}$$

leads to $B_m = \frac{\omega_m}{k_m} A_m = A_m$ for all m non zero. So that

$$\frac{\partial^2 u_0}{\partial x^2} = \frac{\partial^2 u_0}{\partial t^2} = \frac{\partial^2 \eta_0}{\partial t^2} \tag{2.2.10}$$

and the forcing terms of equation 2.2.5 can be simplified.

$$\frac{\partial^2 \eta_2}{\partial t^2} - \frac{\partial^2 \eta_2}{\partial x^2} = -\frac{\partial}{\partial x}\left(b\frac{\partial \eta_1}{\partial x}\right) + 2\frac{\partial^2 \eta_0}{\partial X \partial x} + \frac{3}{2}\frac{\partial^2 \eta_0^2}{\partial t^2} + \frac{\nu}{3}\frac{\partial^4 \eta_0}{\partial x^4} \tag{2.2.11}$$

2.2.2 Equation and solution at $O(\epsilon^{1/2})$

The forcing terms in the governing equation (2.2.4) for η_1 can be expanded and separated into time harmonics

$$\frac{\partial^2 \eta_1}{\partial t^2} - \frac{\partial^2 \eta_1}{\partial x^2} = -\frac{\partial}{\partial x}\left(b\frac{\partial \eta_0}{\partial x}\right) = -\sum_{m=-\infty}^{\infty} F_m e^{-i\omega_m t} \tag{2.2.12}$$

where the coefficients F_m are random functions of x.

$$F_m = \frac{1}{2} i k_m A_m(X) \frac{d}{dx}\left[b(x)e^{ik_m x}\right] \tag{2.2.13}$$

Let us note that $F_0 = 0$ because $A_0 = 0$.

Assuming the solution of (2.2.12) to be of the form:

$$\eta_1 = \sum_{m=-\infty}^{\infty} \eta_1^{(m)} e^{-i\omega_m t} \tag{2.2.14}$$

where the series does not include $m = 0$, for every $\eta^{(m)}$ where $m \neq 0$, a governing equation can be found to be.

$$\frac{d^2 \eta_1^{(m)}}{dx^2} + k_m^2 \eta_1^{(m)} = F_m(x) \tag{2.2.15}$$

These equations can be solved by using the Green functions

$$G_m(|x-x'|) = \frac{e^{ik_m|x-x'|}}{2ik_m} \tag{2.2.16}$$

which behave as outgoing waves at infinity with the result :

$$\eta_1 = \sum_{m=-\infty}^{\infty} e^{-i\omega_m t} \int_{-\infty}^{\infty} G_m(|x-x'|) ik_m \frac{A_m(X)}{2} \frac{d}{dx'} \left[b(x') e^{ik_m x'} \right] dx' \tag{2.2.17}$$

The sum does not include $m = 0$ as $A_0 = 0$.

Clearly the ensemble average of the $O(\epsilon^{1/2})$ solution vanishes.

2.2.3 Equation at $O(\epsilon)$

The ensemble average of equation (2.2.11) is taken,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \langle \eta_2 \rangle = -\frac{\partial}{\partial x} \langle b \frac{\partial \eta_1}{\partial x} \rangle + 2 \frac{\partial^2 \eta_0}{\partial X \partial x} + \frac{3}{2} \frac{\partial^2 \eta_0^2}{\partial t^2} + \frac{\nu}{3} \frac{\partial^4 \eta_0}{\partial x^4} \tag{2.2.18}$$

For more clarity, terms on the right-hand-side of (2.2.18) are calculated separately. Using the known solution for η_1, for each time harmonic, it can be written.

$$\begin{aligned} &\langle b \frac{\partial \eta_1^{(m)}}{\partial x} \rangle \\ &= \int_{-\infty}^{\infty} ik_m \mathrm{sgn}(x-x') G_m(|x-x'|) ik_m \frac{A_m}{2} \frac{d}{dx'} \left[\langle b(x) b(x') \rangle e^{ik_m x'} \, dx' \right] \\ &= A_m e^{ik_m x} \int_{-\infty}^{\infty} \mathrm{sgn}(x-x') \frac{1}{4} e^{ik_m(|x-x'|)} \frac{d}{dx'} \left[\gamma(x-x') e^{ik_m(x'-x)} \right] dx' \\ &= -A_m e^{ik_m x} \int_{-\infty}^{\infty} \frac{1}{4} \mathrm{sgn}(\xi) e^{ik_m|\xi|} \frac{d}{d\xi} \left[\sigma^2 \gamma(\xi) e^{-ik_m \xi} \right] d\xi \end{aligned} \tag{2.2.19}$$

The assumption was made that b is a stationary random function of x on the fast scale so that $\langle b(x) b(x') \rangle$ depends on $\xi = x - x'$, where $\langle b(x) b(\xi) \rangle = \sigma^2 \gamma(\xi)$ can be defined. γ is an even function that vanishes at infinity. Note that (2.2.19) is zero for $m = 0$, and the result for $m < 0$ is equal to the complex conjugate of the result for $m > 0$ so that.

$$-\frac{\partial}{\partial x} \left[\sum_{m=-\infty}^{\infty} e^{-i\omega_m t} \langle b \frac{\partial \eta_1^{(m)}}{\partial x} \rangle \right] = \sum_{m=1}^{\infty} ik_m A_m(X) \beta_m e^{i\theta_m} + c.c. \tag{2.2.20}$$

where the constant β_m is defined by.

$$\beta_m = \frac{\sigma^2}{4} ik_m \int_{-\infty}^{\infty} \mathrm{sgn}(\xi) \left(\frac{d\gamma}{d\xi} - ik_m \gamma \right) e^{ik_m(|\xi|-\xi)} \, d\xi \tag{2.2.21}$$

The second right-hand-side term in equation (2.2.18) is straightforward.

$$2\frac{\partial^2\eta_0}{\partial X\partial x} = \sum_{m=1}^{\infty} ik_m \frac{dA_m}{dX} e^{i\theta_m} + c.c. \tag{2.2.22}$$

Two series are now multiplied together to obtain the third term in equation (2.2.18).

$$\begin{aligned}\eta_0^2 &= \frac{1}{4}\left(\sum_{j=-\infty}^{\infty} A_j e^{i\theta_j}\right)\left(\sum_{l=-\infty}^{\infty} A_l e^{i\theta_l}\right) = \frac{1}{4}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} A_j A_l e^{i(\theta_j+\theta_l)} \\ &= \frac{1}{4}\sum_{m=1}^{\infty} e^{i\theta_m}\left[2\sum_{l=1}^{\infty} A_l^* A_{m+l} + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l}\right] + \frac{1}{4}\sum_{l=1}^{\infty} A_l A_l^* + c.c. \end{aligned} \tag{2.2.23}$$

$[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$ and α_l is a coefficient such that it is equal to 1 only for $l = [\frac{m}{2}]$, it is equal to 2 for all other values of l. The details are given in the following subsection. The third forcing term is then obtained.

$$\frac{3}{2}\frac{\partial^2\eta_0^2}{\partial t^2} = \sum_{m=1}^{\infty} -\frac{3}{8}\omega_m^2 e^{i\theta_m}\left[\sum_{l=1}^{\infty} 2A_l^* A_{m+l} + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l}\right] + c.c. \tag{2.2.24}$$

and

$$\frac{\nu}{3}\frac{\partial^4\eta_0}{\partial x^4} = \sum_{m=1}^{\infty} \frac{\nu}{6} k_m^4 A_m e^{i\theta_m} + c.c. \tag{2.2.25}$$

so that equation (2.2.18) can be rewritten.

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\langle\eta_2\rangle &= \sum_{m=1}^{\infty} ik_m A_m \beta_m e^{i\theta_m} + \sum_{m=1}^{\infty} ik_m \frac{dA_m}{dX} e^{i\theta_m} \\ &\quad - \sum_{m=1}^{\infty} \frac{3}{8}\omega_m^2 e^{i\theta_m}\left[\sum_{l=1}^{\infty} 2A_l^* A_{m+l} + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l}\right] \\ &\quad + \sum_{m=1}^{\infty} \frac{\nu}{6} k_m^4 A_m e^{i\theta_m} + c.c. \end{aligned} \tag{2.2.26}$$

2.2.4 Calculating η_0^2

Calculating η_0^2 involves a multiplication of two infinite series and some manipulations on the indices of the series that are detailed here.

$$\eta_0^2 = \left(\sum_{j=-\infty}^{\infty} A_j e^{i\theta_j}\right)\left(\sum_{l=-\infty}^{\infty} A_l e^{i\theta_l}\right) = \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} A_j A_l e^{i(\theta_j+\theta_l)} \tag{2.2.27}$$

where $\theta_j + \theta_l = \theta_{j+l}$ for all j and l. Let us use the new variable $m = j + l$ in the series.

$$\eta_0^2 = \sum_{m=-\infty}^{\infty} e^{i\theta_m} \sum_{l=-\infty}^{\infty} A_l A_{m-l} \tag{2.2.28}$$

The first sum should be separated into three terms: a sum for $m > 0$, another for $m < 0$ and a term for $m = 0$.

m positive integer

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = \sum_{l=-\infty}^{0} A_l A_{m-l} + \sum_{l=1}^{m-1} A_l A_{m-l} + \sum_{l=m}^{\infty} A_l A_{m-l} \tag{2.2.29}$$

Let us replace $l' = -l$ in the first series on the right-hand-side of (2.2.29) and $l' = l-m$ in the third series.

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = \sum_{l'=0}^{\infty} A_{-l'} A_{m+l'} + \sum_{l=1}^{m-1} A_l A_{m-l} + \sum_{l'=0}^{\infty} A_{m+l'} A_{-l'} \tag{2.2.30}$$

The first and last series on the right-hand-side of (2.2.30) can be grouped together. Remembering that $A_{-l} = A_l^*$, (2.2.30) can be simplified.

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = 2\sum_{l=0}^{\infty} A_l^* A_{m+l} + \sum_{l=1}^{m-1} A_l A_{m-l} \tag{2.2.31}$$

The second series in (2.2.31) depends on the parity of m.

- m even

  If m is even, then it can be written as $m = 2p$ where p is an integer.

$$\begin{aligned}\sum_{l=1}^{2p-1} A_l A_{2p-l} &= A_1 A_{2p-1} + A_2 A_{2p-2} + \cdots + A_p A_p + \cdots \\ &\quad + A_{2p-2} A_2 + A_{2p-1} A_1 \\ &= \sum_{l=1}^{p} \alpha_l A_l A_{2p-l} = \sum_{l=1}^{m/2} \alpha_l A_l A_{n-l} \end{aligned} \tag{2.2.32}$$

  where $\alpha_l = 1$ if $l = p = \frac{m}{2}$ and $\alpha_l = 2$ otherwise.

- m odd

  If m is odd, then it can be written as $m = 2p + 1$ where p is an integer.

$$\begin{aligned}\sum_{l=1}^{2p} A_l A_{2p+1-l} &= A_1 A_{2p} + A_2 A_{2p-1} + \cdots + A_{p-1} A_{p+1} + \cdots \\ &\quad + A_{2p-1} A_2 + A_{2p} A_1 \\ &= \sum_{l=1}^{p} 2 A_l A_{2p+1-l} = \sum_{l=1}^{(m-1)/2} \alpha_l A_l A_{m-l} \end{aligned} \tag{2.2.33}$$

  where $\alpha_l = 2$ for every l here.

Both formulas can be grouped into one if $[m/2]$ is defined as the integer part of $m/2$: it is equal to $\frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd.

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = 2\sum_{l=1}^{\infty} A_l^* A_{m+l} + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l} \tag{2.2.34}$$

m negative integer

For m negative, the sum over m can be manipulated as.

$$\sum_{m=-\infty}^{-1} e^{i\theta_m} \sum_{l=-\infty}^{\infty} A_l A_{m-l} = \sum_{m=1}^{\infty} e^{-i\theta_m} \sum_{l=-\infty}^{\infty} A_l A_{-m-l} \tag{2.2.35}$$

where the l series should be expanded.

$$\sum_{l=-\infty}^{\infty} A_l A_{-m-l} = \sum_{l=-\infty}^{-m} A_l A_{-m-l} + \sum_{l=-m+1}^{-1} A_l A_{-m-l} + \sum_{l=0}^{\infty} A_l A_{-m-l} \tag{2.2.36}$$

Let us replace $l' = l - m$ in the first series on the right-hand-side of (2.2.36), it becomes identical to the third series, they can be grouped together. Replacing $l' = -l$ in the second series yields.

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = 2\sum_{l=0}^{\infty} A_{-m-l} A_l + \sum_{l=1}^{m-1} A_{-l} A_{-m+l} \tag{2.2.37}$$

Recalling $A_{-l} = A_l^*$ can simplify the equation.

$$\sum_{l=-\infty}^{\infty} A_l A_{m-l} = 2\sum_{l=0}^{\infty} A_{m+l}^* A_l + \sum_{l=1}^{m-1} A_l^* A_{m-l}^* \tag{2.2.38}$$

The result is the complex conjugate of the expression obtained for m positive, there is no need for further separation into odd and even values of m.

m is zero

For $m = 0$ the exponential term is one, and the l-series in (2.2.28) can be easily evaluated.

$$\begin{aligned} \sum_{l=-\infty}^{\infty} A_l A_{-l} &= \sum_{l=-\infty}^{-1} A_l A_{-l} + A_0 A_0^* + \sum_{l=1}^{\infty} A_l A_{-l} \\ &= \sum_{l=1}^{\infty} 2 A_l A_l^* \end{aligned} \tag{2.2.39}$$

because the series over l negative and over l positive are identical and $A_0 = 0$. The total l-series is real, it can easily be written in complex form as

$$\sum_{l=-\infty}^{\infty} A_l A_{-l} = \sum_{l=1}^{\infty} A_l A_l^* + c.c. \tag{2.2.40}$$

Result

Summing over m negative, zero and positive yields.

$$\eta_0^2 = \sum_{m=1}^{\infty} e^{i\theta_m} \left[2\sum_{l=1}^{\infty} A_l A_{m+l}^* + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l} \right] + \sum_{l=1}^{\infty} A_l A_l^* + c.c. \tag{2.2.41}$$

where $[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$ and $\alpha_l = 1$ if $l = [\frac{m}{2}]$ or $\alpha_l = 2$ otherwise.

2.2.5 Evolution equations

On the right-hand side of (2.2.26), terms proportional to $\exp(i\theta_m)$ for each m^{th} harmonic are resonance-forcing and must be removed to ensure solvability. The coefficients of secular terms are equated to zero and use is made of the dispersion relation $\omega_m = k_m$.

$$\frac{dA_m}{dX} + \beta_m A_m - i\frac{\nu}{6}k_m^3 A_m + \frac{3}{8}i\omega_m \left[\sum_{l=1}^{\infty} 2A_l^* A_{m+l} + \sum_{l=1}^{[\frac{m}{2}]} \alpha_l A_l A_{m-l} \right] = 0 \quad m = 1, 2, \cdots \infty \tag{2.2.42}$$

Equations (2.2.42) constitute an infinite number of nonlinear differential equations describing the slow evolutions of the amplitudes in X. The linear terms with complex coefficients β_m are the result of multiple scattering by disorder and are related to the correlation coefficients of $b(x)$. It will be confirmed shortly that $\Re\beta_m > 0$ so that disorder leads to exponential attenuation (localization). These equations are similar to those in nonlinear optics [3] where damping was briefly mentioned and included in a generic sense. For $\beta_m = 0$, the limiting equations have been derived by [24].

For a limited number of harmonics n, where n can be any positive nonzero integer, (2.2.42) must be truncated at the n^{th} harmonic: the amplitudes for harmonics above n are neglected. The nonlinear differential equation system is therefore composed of

n equations, and the equation governing each harmonic m is.

$$\frac{dA_m}{dX}+\beta_m A_m-i\frac{\nu}{6}k_m^3 A_m+\frac{3}{8}i\omega_m\left[\sum_{l=1}^{n-m}2A_l^*A_{m+l}+\sum_{l=1}^{[\frac{m}{2}]}\alpha_l A_l A_{m-l}\right]=0\quad m=1,2,\cdots,n \tag{2.2.43}$$

The summation of the infinite series in (2.2.42) is truncated at $n-m$ because all the harmonics above $n=m$ are neglected, which can be modelled as an infinite number of harmonics such that the amplitude of any harmonic above n is zero: $A_m=0$, $\forall m>n$. The $A_l^*A_{m+l}$ terms can therefore only be summed up to $m+l\leq n$ or $l\leq n-m$.

In the case of smooth and horizontal seabed, $b(x)\equiv 0$ for all x, hence $\beta_m\equiv 0$ for all m. If $n=2$, ie if only 2 harmonics are kept, the exact analytical solution by [3] agrees with experiments by [13] for small to moderate values of ϵ and μ^2. Starting from the Korteweg-de-Vries equation [18], C.C. Mei [20] derived a very similar governing equation for the amplitude of the m^{th} harmonic of an infinite number of harmonics.

$$\frac{dA_m}{dx}-i\frac{\mu^2}{6}m^3A_m=-i\epsilon\frac{3}{8}\left[\sum_{l=1}^{[\frac{m}{2}]}(m\alpha_l)\,A_lA_{m-l}+\sum_{l=1}^{\infty}2mA_l^*A_{m+l}\right]\qquad m=1,2,\cdots\infty \tag{2.2.44}$$

where x is the fast varying coordinate, it corresponds to X/ϵ.

2.3 The β_m coefficients

For analytical convenience the correlation function of the random fluctuations is considered to be Gaussian:

$$\langle b(x)b(x')\rangle=\sigma^2\exp\left(-\frac{(x-x')^2}{2l^2}\right)=\sigma^2\exp\left(-\frac{\xi^2}{2l^2}\right) \tag{2.3.1}$$

where $\sigma(X)$ is the root-mean-square and l is the correlation distance.

2.3.1 Calculating β_m analytically

To calculate β_m let us note that

$$\langle b(x)\frac{db}{dx'}\rangle=\frac{-\sigma^2}{l^2}\xi e^{-\frac{\xi^2}{2l^2}}=-\langle b(x')\frac{db}{dx}\rangle \tag{2.3.2}$$

then evaluate the integrals,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{sgn}(\xi)\gamma(\xi)e^{ik_m(|\xi|-\xi)}d\xi &= -\int_{-\infty}^{0} e^{-\frac{\xi^2}{2l^2}}e^{-2ik_m\xi}\,d\xi + \int_0^{\infty} e^{-\frac{\xi^2}{2l^2}}\,d\xi \\ &= l\sqrt{2}\left(-\int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,du + \int_0^{\infty} e^{-u^2}\,du\right) \\ &= l\sqrt{2}\left(\frac{\sqrt{\pi}}{2} - \int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,du\right) \qquad (2.3.3)\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{sgn}(\xi)\frac{d\gamma}{d\xi}e^{ik_m(|\xi|-\xi)}d\xi &= \frac{1}{l^2}\left(\int_{-\infty}^{0} \xi e^{-\frac{\xi^2}{2l^2}}e^{-2ik_m\xi}\,d\xi - \int_0^{\infty} \xi e^{-\frac{\xi^2}{2l^2}}\,d\xi\right) \\ &= -\int_{-\infty}^{0} -2ue^{-u^2}e^{-2i\sqrt{2}k_m lu}\,du + \int_0^{\infty} -2ue^{-u^2}\,du \\ &= \int_{-\infty}^{0} (2u + 2i\sqrt{2}k_m l)e^{-u^2-2i\sqrt{2}k_m lu}\,du \\ &\quad - 2i\sqrt{2}k_m l\int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,du - 1 \\ &= -2 - 2ik_m l\sqrt{2}\int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,du \qquad (2.3.4)\end{aligned}$$

It follows that

$$\beta_m = \frac{\sigma^2}{4}ik_m\left(-2 - \frac{1}{2}i\sqrt{2\pi}k_m l - i\sqrt{2}k_m l\int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,du\right) \qquad (2.3.5)$$

where the integral can be further simplified as

$$\begin{aligned}\int_{-\infty}^{0} e^{-u^2-2i\sqrt{2}k_m lu}\,d\xi &= e^{-2k_m^2l^2}\int_{-\infty}^{0} e^{-(u+i\sqrt{2}k_m l)^2}\,du = e^{-2k_m^2l^2}\int_{-\infty}^{i\sqrt{2}k_m l} e^{-u^2}\,du \\ &= e^{-2k_m^2l^2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^{\sqrt{2}k_m l} e^{u^2}\,du\right) \qquad (2.3.6)\end{aligned}$$

The final expression for β_m is therefore.

$$\beta_m = \frac{\sigma^2}{8}k_m^2 l\sqrt{2\pi}\left(1 + e^{-2k_m^2l^2}\right) - i\frac{\sigma^2}{2}k_m\left(1 - \frac{k_m l}{\sqrt{2}}e^{-2k_m^2l^2}\int_0^{\sqrt{2}k_m l} e^{u^2}\,du\right) \qquad (2.3.7)$$

It has now become obvious that for each harmonic m, the real part of the damping coefficient β_m is positive. For sufficiently high harmonics, the exponential term can be neglected and the real part becomes proportional to m^2. It is therefore expected that the higher the harmonic, the larger the localization effect. Higher harmonics will attenuate faster in space. Harmonics are also more localized for either large σ

or small l, i.e., when the depth irregularity is either very large in amplitude or very random.

Let us note here that $\beta_m \ell / \sigma^2$ is a function of ℓ and $k_m \ell$ only. For given values of σ and ℓ, it is therefore possible to plot the real and imaginary parts of β_m as functions of the product $k_m \ell = m\ell$ for $k_1 = 1$.

2.3.2 Numerical results for real and imaginary parts of β_m

Real and imaginary parts of β_m are plotted here for $k_1 = 1$ and $\sigma = 0.2$ because these values will be the most frequently used in the rest of the simulations. A parabolic behavior for the real part and a linear behavior for the imaginary part can be observed in Figure 2-1.

The integral

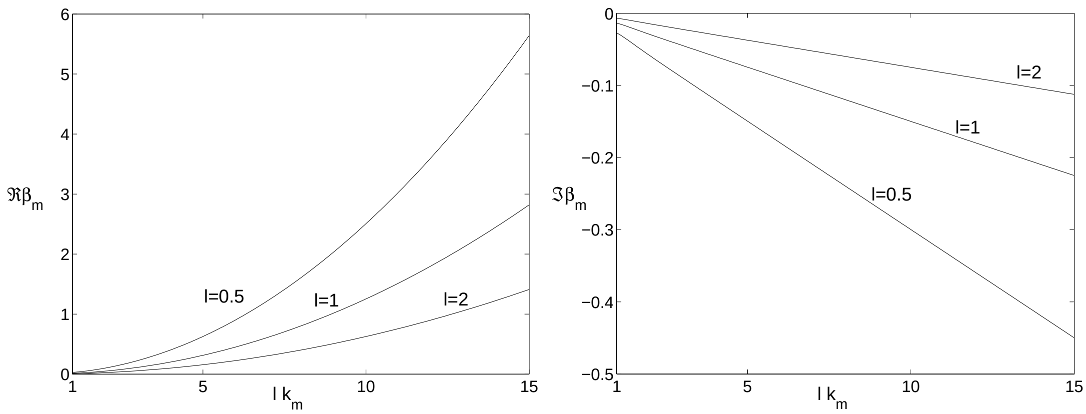

Figure 2-1: Real (left) and Imaginary (right) parts of β_m for $\ell = 0.5, 1, 2$, $\sigma = 0.2$ and $k_1 = 1$.

$$i \int_0^{\sqrt{2} k_m l} e^{u^2} \, du = \int_0^{i\sqrt{2} k_m l} e^{-u^2} \, du \tag{2.3.8}$$

is the complex error function. In order to calculate it easily for each value of m, use has been made of the algorithm discussed by Weideman in [35]. The matlab code used for this purpose can be found in appendix A

2.4 Evolution of total wave energy

Energy conservation is an important issue in the localization problem. If wave amplitudes are damped, where does the energy go? PJ Bryant [6] had attempted to answer this question for a smooth bottom, but could only prove energy conservation for the first two out of three and three out of four harmonics, possibly because of some missing terms in his governing equations. The equation he gave for the m^{th} harmonic seems correct.

$$\frac{dA_m}{dt} - i\frac{\mu^2}{6}m^3 A_m = -i\epsilon \left(\sum_{l=1}^{[\frac{m}{2}]} \frac{3}{8}\alpha_l A_l A_{m-l} + \sum_{l=1}^{\infty} \frac{3}{4} m A_l^* A_{m+l} \right) + O(\epsilon^2) \tag{2.4.1}$$

but the equations he gave for 2, 3 and 4 harmonics had terms missing that did not make it possible to observe the energy conservation relation. Let us give the example of three harmonics. Using primes to represent time derivatives, the terms underlined in the following equations were missing in PJ Bryant's equations:

$$A_1' - i\frac{\mu^2}{6}A_1 = -i\epsilon\frac{3}{4}A_1^*A_2 \underline{-i\epsilon\frac{3}{4}A_2^*A_3} \tag{2.4.2}$$

$$A_2' - i\frac{\mu^2}{6}2^3 A_2 = -i\epsilon\frac{3}{8}2A_1^2 \underline{-i\epsilon\frac{3}{4}2A_1^*A_3} \tag{2.4.3}$$

$$A_3' - i\frac{\mu^2}{6}3^3 A_3 = -i\epsilon\frac{3}{8}3A_1A_2 \tag{2.4.4}$$

The system of n differential equations (2.2.43) can be used to answer the question. A total of two harmonics is examined to understand the issues and the potential pattern before tackling n harmonics.

2.4.1 Total wave energy for 2 harmonics

The governing diferential equations for two harmonics are

$$\frac{dA_1}{dX} + \left(\beta_1 - i\frac{\nu}{6}k_1^3\right) A_1 + \frac{3}{4}ik_1A_1^*A_2 = 0 \tag{2.4.5}$$

$$\frac{dA_2}{dX} + \left(\beta_2 - i\frac{\nu}{6}k_2^3\right) A_2 + \frac{3}{8}ik_2A_1^2 = 0 \tag{2.4.6}$$

Let us first use the dispersion relation to calculate the group velocity C_g.

$$C_{g_m} = \frac{d\omega_m}{dk_m} = 1 \tag{2.4.7}$$

Without disorder it is known that:

$$\sum_{m=1}^{2} C_{g_m} A_m A_m^* = \sum_{m=1}^{2} A_m A_m^* = |A_1|^2 + |A_2|^2 = \text{constant} \tag{2.4.8}$$

which is called Manley-Rowe relation in nonlinear optics. Multiplying the equation for the first harmonic (2.4.5) by A_1^* and adding the resulting equation to its complex conjugate yields:

$$A_1^* \frac{dA_1}{dX} + A_1 \frac{dA_1^*}{dX} + 2\Re\left(\beta_1 - i\frac{\nu}{6}k_1^3\right) A_1 A_1^* + \frac{3}{4}\Re\left(ik_1 A_1^{*2} a_2\right) = 0 \tag{2.4.9}$$

This can easily be simplified into.

$$\frac{dA_1 A_1^*}{dX} + 2\Re(\beta_1) A_1 A_1^* - \frac{3}{4} k_1 \Im\left(A_1^{*2} A_2\right) = 0 \tag{2.4.10}$$

Similarly for the second harmonic,

$$\frac{dA_2 A_2^*}{dX} + 2\Re(\beta_2) A_2 A_2^* - \frac{3}{8} k_2 \Im\left(A_1^2 A_2^*\right) = 0 \tag{2.4.11}$$

Adding equations (2.4.10) and (2.4.11), the following energy relation is obtained.

$$\sum_{m=1}^{2}\left(\frac{dA_m A_m^*}{dX} + 2\Re\beta_m A_m A_m^*\right) = \frac{3}{4} k_1 \Im\left(A_1^{*2} A_2\right) + \frac{3}{8} 2k_1 \Im\left(A_1^2 A_2^*\right) \tag{2.4.12}$$

The right hand side of (2.4.12) is zero because the imaginary part of the sum of two complex conjugates is zero. For two harmonics, the energy equation shows a damping term.

$$\frac{d}{dX}\sum_{m=1}^{2} A_m A_m^* = -2\sum_{m=1}^{2} \Re(\beta_m) A_m A_m^* \tag{2.4.13}$$

For a smooth bottom, the β_m are zero and the energy conservation law is proved. As $\Re(\beta_m) > 0$, it can be observed that for a random bottom, energy is no longer conserved; it is lost by scattering.

2.4.2 Total wave energy for n harmonics

As a result of the energy equation obtained for two harmonics, a general equation for n harmonics is expected to be of the form:

$$\frac{d}{dX}\sum_{m=1}^{n} A_m A_m^* = -2\sum_{m=1}^{n} \Re(\beta_m) A_m A_m^* \tag{2.4.14}$$

Let us first show that it is true for one harmonic. The governing equation for one harmonic is simply.

$$\frac{dA}{dX} + \left(\beta - i\frac{\nu}{6}k^3\right) A = 0 \tag{2.4.15}$$

It is trivial to prove that it satisfies (2.4.14) for $n = 1$. Let us use the method of induction as follows. Assuming (2.4.14) is valid for n harmonics, proving that it is also valid for $n+1$ harmonics will prove it is valid for any number of harmonics. For n harmonics (2.2.43) is multiplied by A_m^*, added to its complex conjugate, and all the equations are summed for m from 1 to n.

$$\begin{aligned}&\sum_{m=1}^{n}\left(\frac{d}{dX}|A_m|^2 + 2\Re(\beta_m)|A_m|^2\right)\\ &\quad = \sum_{m=1}^{n}\frac{3}{8}k_m \Im\left[\sum_{l=1}^{n-m} A_l^* A_{m+l} A_m^* + \sum_{l=1}^{[m/2]} \alpha_l A_l A_{m-l} A_m^*\right]\end{aligned} \tag{2.4.16}$$

In order to prove (2.4.14), the task is to prove that for any n.

$$H_n = \sum_{m=1}^{n} k_m \Im\left[\sum_{l=1}^{n-m} A_l^* A_{m+l} A_m^* + \sum_{l=1}^{[m/2]} \alpha_l A_l A_{m-l} A_m^*\right] = 0 \tag{2.4.17}$$

Let us first write.

$$\begin{aligned}H_{n+1} &= \sum_{m=1}^{n+1} k_m \Im\left[\sum_{l=1}^{n+1-m} A_l^* A_{m+l} A_m^* + \sum_{l=1}^{[m/2]} \alpha_l A_l A_{m-l} A_m^*\right]\\ &= H_n + (n+1)k_1 \Im\left[\sum_{l=1}^{[(n+1)/2]} \alpha_l A_l A_{n+1-l} A_{n+1}^*\right]\\ &\qquad + \sum_{m=1}^{n} m k_1 \Im\left[2A_{n+1-m}^* A_{n+1} A_m^*\right]\end{aligned} \tag{2.4.18}$$

where $\alpha_l = 1$ for $l = (n+1)/2$. It is therefore necessary to make a distinction between odd and even values of n.

Odd number of harmonics

If n is odd, it can be written as $n = 2p - 1$ where p is a positive integer. Recall that $\alpha_l = 1$ only for $l = p$, it is equal to 2 otherwise. Assuming $H_n = H_{2p-1} = 0$ in equation (2.4.18), it shall now be proved that the rest of $H_{n+1} = H_{2p}$ is zero, where,

$$H_{2p} = k_1 \Im \left[2pA_{2p}^* \sum_{l=1}^{p-1} 2A_l A_{2p-l} + 2pA_{2p}^* A_p A_p + 2A_{2p} \sum_{m=1}^{2p-1} mA_{2p-m}^* A_m^* \right] \tag{2.4.19}$$

Expanding the sum over m will make it easier to simplify.

$$\begin{aligned} \sum_{m=1}^{2p-1} mA_{2p-m}^* A_m^* &= A_1^* A_{2p-1}^* + 2A_2^* A_{2p-2}^* + \cdots + pA_p^* A_p^* + \cdots \\ &\qquad + (2p-2)A_{2p-2}^* A_2^* + (2p-1)A_{2p-1}^* A_1^* \\ &= 2pA_1^* A_{2p-1}^* + 2pA_2^* A_{2p-2}^* + \cdots + 2pA_{p-1}^* A_{p+1}^* + pA_p^* A_p^* \\ &= 2p \sum_{m=1}^{p-1} A_m^* A_{2p-m}^* + pA_p^* A_p^* \end{aligned} \tag{2.4.20}$$

so that

$$\begin{aligned} H_{2p} &= k_1 \Im \left[2pA_{2p}^* \sum_{l=1}^{p-1} 2A_l A_{2p-l} + 2pA_{2p}^* A_p A_p \right. \\ &\qquad \left. + 2pA_{2p} \left(2 \sum_{m=1}^{p-1} A_m^* A_{2p-m}^* + A_p^* A_p^* \right) \right] \\ &= k_1 \Im \left[4pA_{2p}^* \sum_{l=1}^{p-1} A_l A_{2p-l} + 2pA_{2p}^* A_p A_p + c.c. \right] = 0 \end{aligned} \tag{2.4.21}$$

The last simplification was possible because the term in the brackets is real.

Let us consider an even number of harmonics.

Even number of harmonics

If n is even, it can be written as $n = 2p$ where p is an integer.

$\nu = 2$ for every value of l as $(2p+1)/2$ is not an integer:

$$H_{2p+1} = k_1 \Im \left[(2p+1)A_{2p+1}^* \sum_{l=1}^{p} 2A_l A_{2p+1-l} + 2A_{2p+1} \sum_{m=1}^{2p} mA_{2p+1-m}^* A_m^* \right] \tag{2.4.22}$$

Expanding the sum over m will make it easier to simplify.

$$\sum_{m=1}^{2p} mA_{2p+1-m}^* A_m^* = A_1^* A_{2p}^* + 2A_2^* A_{2p-1}^* + \cdots$$

$$+ (2p-1)A^*_{2p-1}A^*_2 + (2p)A^*_{2p}A^*_1$$
$$= (2p+1)\left[A^*_1A^*_{2p-1} + A^*_2A^*_{2p-2} + \cdots + A^*_{p+1}A^*_p\right]$$
$$= (2p+1)\sum_{m=1}^{p} A^*_m A^*_{2p+1-m} \tag{2.4.23}$$

so that

$$H_{2p+1} = k_1\Im\left[(2p+1)A^*_{2p+1}\sum_{l=1}^{p} 2A_l A_{2p+1-l} + c.c.\right] = 0 \tag{2.4.24}$$

for the same reasons as $H_{2p} = 0$ in (2.4.21).

Any number of harmonics

Proving $H_{n+1} = 0$ for n odd or even has proved that for any number of harmonics n, the amplitudes of the n harmonics obey the energy equation (2.4.14), which becomes the energy conservation equation for a smooth bottom.

$$\sum_{m=1}^{n} |A_m|^2 = \text{constant} \tag{2.4.25}$$

2.5 Using the energy relation to verify the numerical results

The energy conservation relation (2.4.25) for a perfectly smooth bottom should be satisfied by the numerical simulations for $\sigma = 0$. Setting σ equal to zero simulates the case of a perfectly smooth bottom: all the β_m coefficients are equal to zero. The numerical simulation is run and it evaluates the following quantity expected to be close to zero.

$$\sum_{m=1}^{6} \frac{d}{dX}|A_m|^2 \tag{2.5.1}$$

The results in Figure 2-2 show an error of the order of 10^{-12}. Even if this error is for the square of an amplitude, it can be neglected if compared to amplitudes of many orders of magnitude larger.

The equations for a perfectly smooth bottom are the same as those for a random bottom where the β_m coefficients have been set equal to zero. The error for the

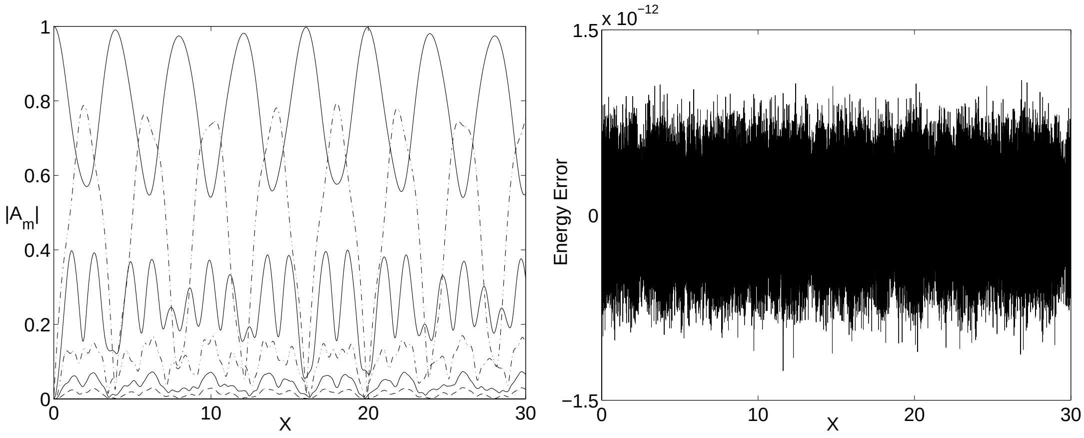

Figure 2-2: Left: Harmonics 1 to 6 for a perfectly smooth bottom. $\nu = 1$, $\sigma = 0$, and $l = k_1 = 1$. Right: X-derivative of the energy from equation (2.5.1)

smooth bottom being of the order of 10^{-12}, additional errors from accounting for a random bottom should be expected. The quantity caculated here is:

$$\sum_{m=1}^{6} \left(\frac{d}{dX} |A_m|^2 + 2\Re(\beta_m)|A_m|^2 \right) \tag{2.5.2}$$

Figure 2-3 shows a plot of the 6 first harmonics and the corresponding error for

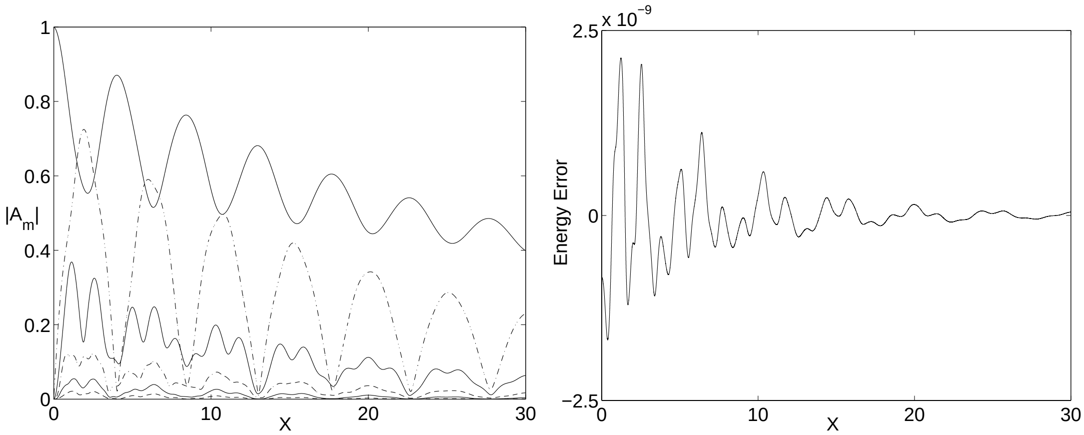

Figure 2-3: Left: Harmonics 1 to 6 for a random bottom. $\sigma = 0.2$, $\nu = 1$ and $l = k_1 = 1$. Right: energy relation from equation (2.5.2)

$\sigma = 0.2$. The error is at most of the order of $2.5\ 10^{-9}$ and dies out as the harmonics are damped. It can therefore be considered that the numerical simulations are sufficiently accurate for the purposes sought here.

2.6 Evolution of the first six harmonics over finite and semi-infinite regions of disorder

Analytical solutions for 2 harmonics over a perfectly smooth bottom ($\beta_m = 0$) can be derived, but for a random bottom, following [3], the analytical solution could be derived only if $\beta_1 = \beta_2$ which is impossible from the definition of β_m in (2.3.7). For numerical solutions, more harmonics can be added. An infinite number cannot be simulated and a truncation at n harmonics is necessary. $n = 6$ is chosen for the following simulations. As a verification, solutions are compared to simulations with larger values of n. The equations for six harmonics are given here.

$$\frac{dA_1}{dX} + \beta_1 A_1 - i\frac{\nu}{6}k_1^3 A_1 + \frac{3}{4}ik_1\left(A_1^* A_2 + A_2^* A_3 + A_3^* A_4 + A_4^* A_5 + A_5^* A_6\right) = 0 \tag{2.6.1}$$

$$\frac{dA_2}{dX} + \beta_2 A_2 - i\frac{\nu}{6}k_2^3 A_2 + \frac{3}{4}ik_2\left(A_1^* A_3 + A_2^* A_4 + A_3^* A_5 + A_4^* A_6 + \frac{1}{2}A_1^2\right) = 0 \tag{2.6.2}$$

$$\frac{dA_3}{dX} + \beta_3 A_3 - i\frac{\nu}{6}k_3^3 A_3 + \frac{3}{4}ik_3\left(A_1^* A_4 + A_2^* A_5 + A_3^* A_6 + A_1 A_2\right) = 0 \tag{2.6.3}$$

$$\frac{dA_4}{dX} + \beta_4 A_4 - i\frac{\nu}{6}k_4^3 A_4 + \frac{3}{4}ik_4\left(A_1^* A_5 + A_2^* A_6 + A_1 A_3 + \frac{1}{2}A_2^2\right) = 0 \tag{2.6.4}$$

$$\frac{dA_5}{dX} + \beta_5 A_5 - i\frac{\nu}{6}k_5^3 A_5 + \frac{3}{4}ik_5\left(A_1^* A_6 + A_1 A_4 + A_2 A_3\right) = 0 \tag{2.6.5}$$

$$\frac{dA_6}{dX} + \beta_6 A_6 - i\frac{\nu}{6}k_6^3 A_6 + \frac{3}{4}ik_6\left(A_1 A_5 + A_2 A_4 + \frac{1}{2}A_3^2\right) = 0 \tag{2.6.6}$$

Figure 2-4 shows a typical result for a seabed with disorder of uniform root-mean-square height $\sigma = 0.2$ on the right $X > 0$ and no disorder on the left $X < 0$. For more clarity, only harmonics 1 to 4 have been included but the result is obtained with a 6-harmonic simulation. Other parameters are chosen to be $k_1 = 1,\ \nu = \mu^2/\epsilon = 1,\ l = 1$. The initial conditions are that $A_1(0) = 1$ and $A_m(0) = 0$ for $m = 2 \cdots 6$ so that the higher harmonics start from zero at the left edge of disorder $X = 0$ and quickly grow at the expense of the first. Slow attenuation of all harmonics is evident.

For sufficiently large distance from the point of entry $X = 0$, amplitudes of all harmonics are reduced enough so that oscillatory exchange of energy no longer occurs, only exponential attenuation remains. For still larger X the highest harmonics disappear totally; only the first harmonic remains. In all the simulations, the first harmonic is the largest, followed by the lower harmonics: the higher the harmonic, the smaller its amplitude.

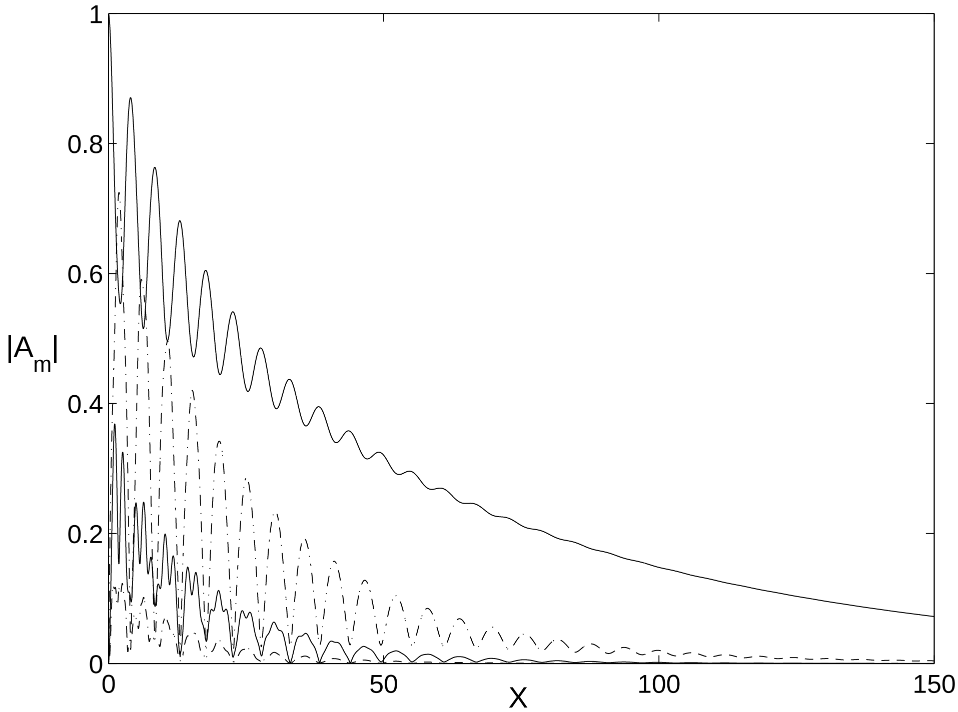

Figure 2-4: Typical result of harmonic localization over a semi-infinite region of disorder. $\sigma = 0.2$, $\nu = 1$ and $l = k_1 = 1$.

The problem depends on several parameters. Their effects on the propagation of the waves on the random seabed are studied. The parameters are the position and size of the region of disorder (space coordinates), the root mean square σ of the Gaussian correlation, the correlation length l and the ratio of dispersion to nonlinearity $\mu^2/\epsilon = \nu$.

2.6.1 Varying the size and location of the region of disorder

In the governing equations for the amplitudes, the β_m coefficients are constants over the region of disorder, and have been calculated previously, but over a perfectly smooth region, these coefficients are set equal to zero. In this section, the following results are obtained keeping k_1, σ and l constant but varying the length and position of the region of randomness.

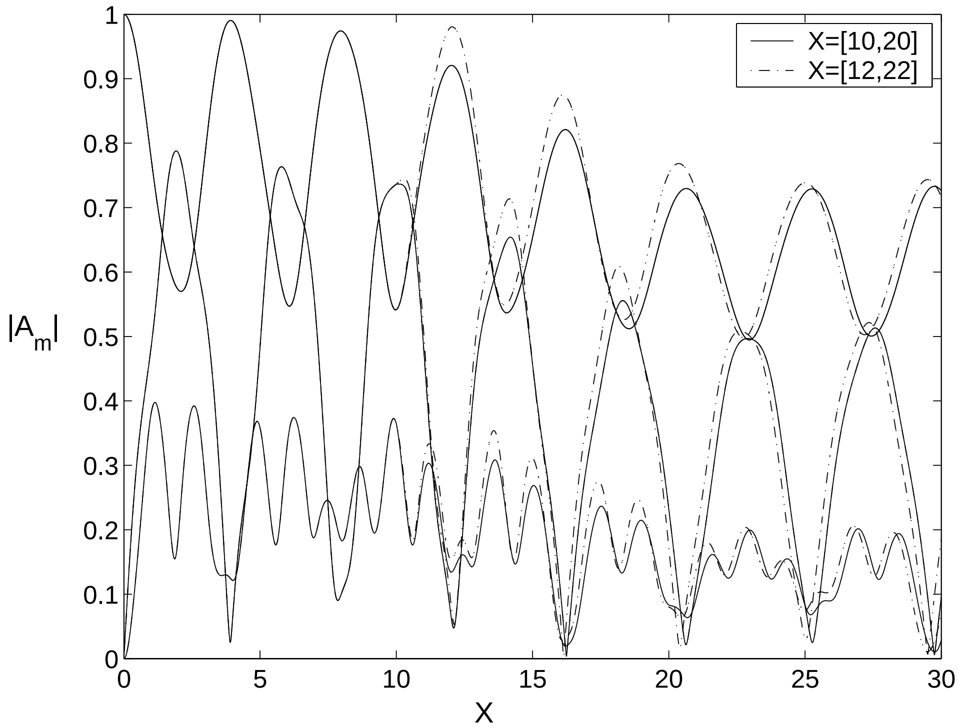

Figure 2-5: Incident wave phase effect for the first three harmonics. Full line: the disorder is in $X = [10, 20]$, dash-dotted line, the disorder is in $X = [12, 22]$. $\sigma = 0.2$, $\nu = 1$ and $l = k_1 = 1$.

The generation of the higher harmonics has taken place over some distance before entry into the zone of disorder in figures 2-5 and 2-6. They both compare cases with the same incident waves for which only the first three harmonics are plotted.

Figure 2-5 compares two regions of randomness of total length equal to 10 where for the first case the random bottom extends over $X = [10, 20]$ so that the second harmonic is the largest at the entry point $X = 10$. In the second case the region of disorder is $X = [12, 22]$; the first harmoinc is the greatest at the point of entry $X = 12$. After the region of randomness, the two results differ in phase but not in amplitude.

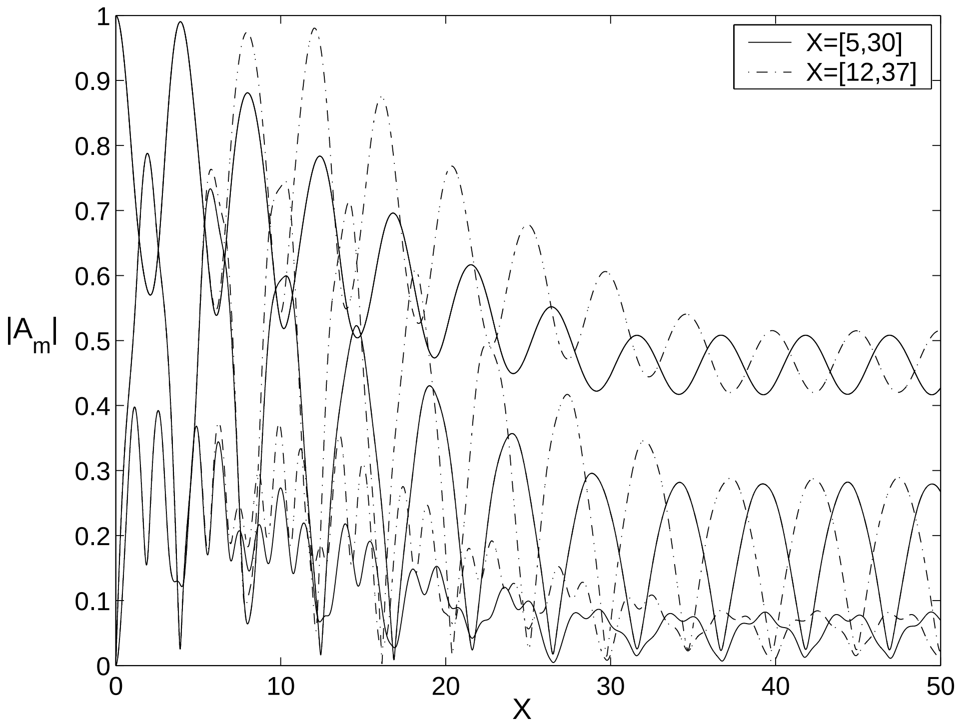

Figure 2-6: Incident wave phase effect for the first three harmonics. Full line: the disorder is in $X = [5, 30]$, dash-dotted line, the disorder is in $X = [12, 37]$. $\sigma = 0.2$, $\nu = 1$ and $l = k_1 = 1$.

Figure 2-6 compares two regions of randomness of total length equal to 25 where for the first case the random bottom extends over $X = [5, 30]$ and the second case the region of disorder is $X = [12, 37]$. The first harmonic is neither greatest nor smallest at each point of entry in the region of randomness, but there is a phase difference in the first harmonic between $X = 5$ and $X = 12$. After the region of randomness, the two results also differ in phase but not in amplitude. It has been shown in both figures that phase lag has only little effect on localization.

2.6.2 Varying the root mean square σ of the Gaussian correlation

One of the most important factors affecting localization is the amplitude of the disorder represented by σ. As expected from the expressions of β_m, larger σ leads to faster localization. Figure 2-7 compares solutions obtained for different values of σ. Randomness starts at $X = 3.75$ for which the first harmonic is maximum.

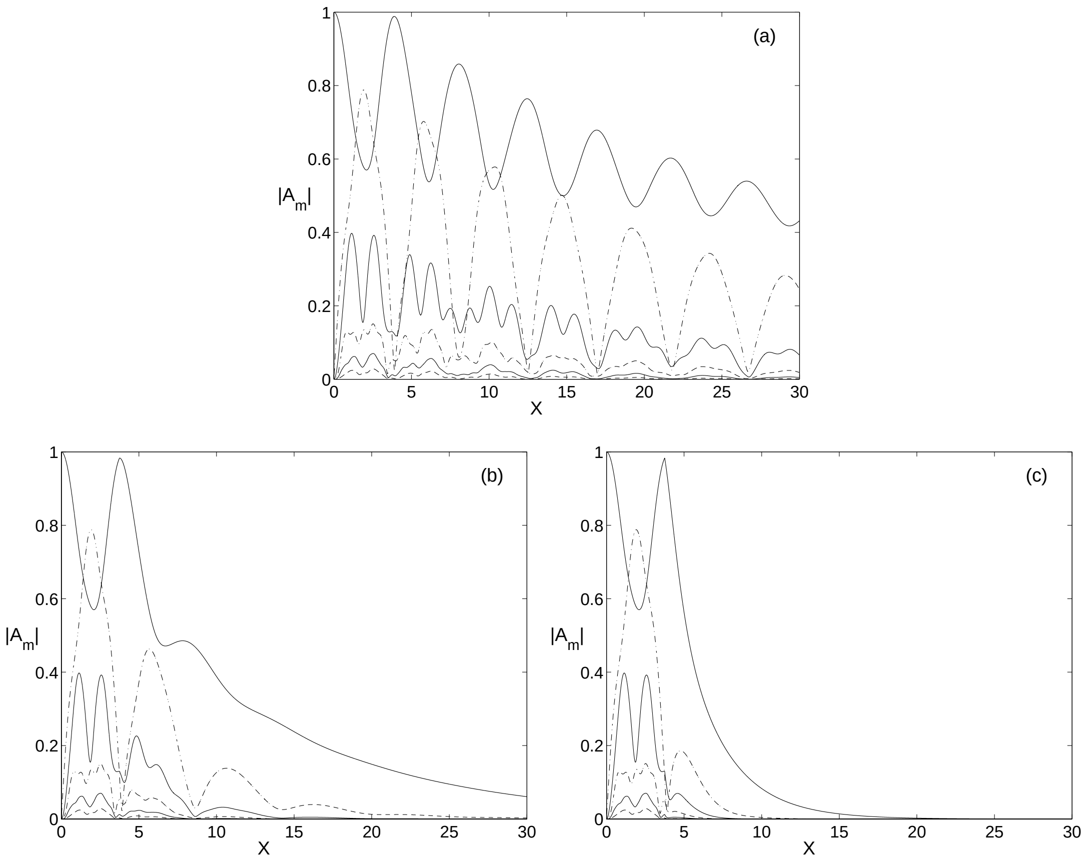

Figure 2-7: Harmonics 1 to 6 for different values of σ. Case (a) $\sigma = 0.2$; case (b) $\sigma = 0.5$ and case (c) $\sigma = 1$. For all cases, $\nu = 1$ and $l = k_1 = 1$. Solid line: odd harmonics (1,3,5), dash-dotted line: even harmonics (2,4,6).

2.6.3 Varying l: ratio of correlation length to wavelength

The effects of the ratio l of the correlation length to the wavelength are examined. Greater l means greater randomness, hence localization occurs in a shorter distance as shown in the comparison of $l = 0.5, 1, 2$ for $\sigma = 0.2$ in Figure 2-8 and $\sigma = 0.5$ in Figure 2-9. In both figures, $\nu = k_1 = 1$ and the region of randomness starts at $X = 3.75$, the peak of the first harmonic.

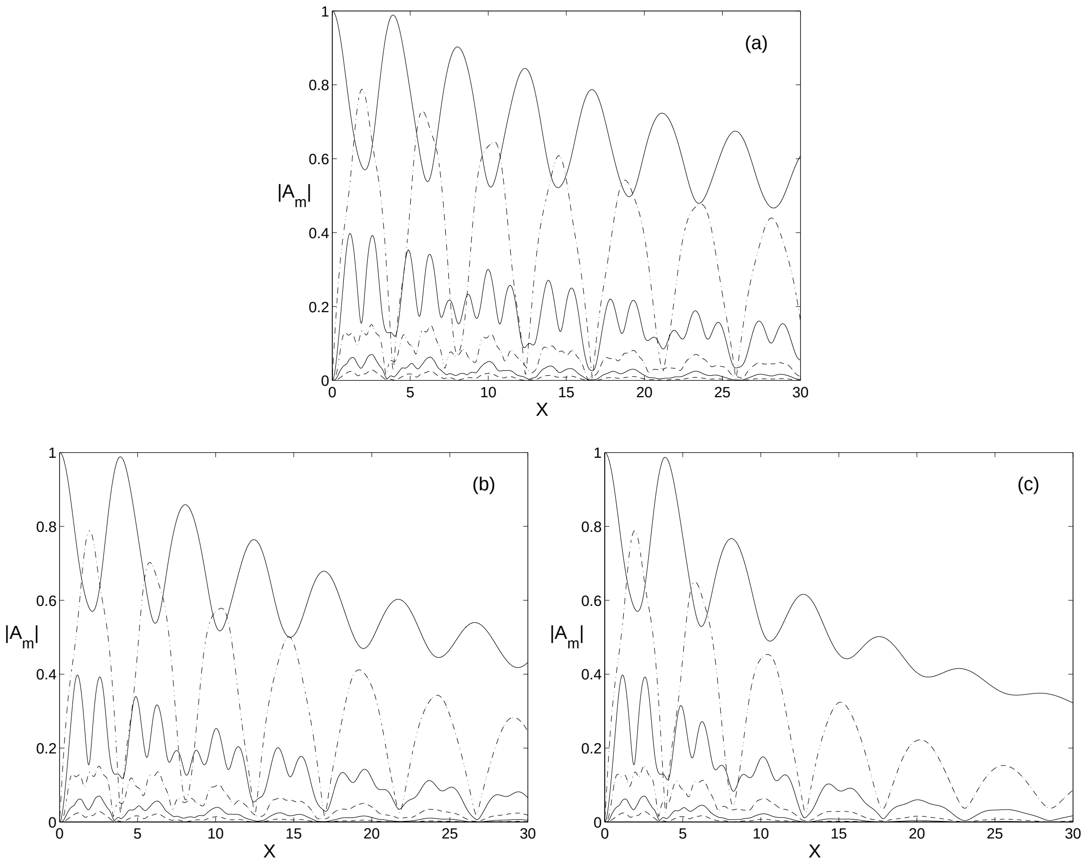

Figure 2-8: Harmonics 1 to 6 for $\sigma = 0.2$ and different values of l. Case (a): $l = 0.5$; case (b): $l = 1$ and case (c): $l = 2$. For all cases $\nu = k_1 = 1$ and randomness starts at $X = 3.75$, the peak of the first harmonic. Solid line: odd harmonics (1,3,5), dash-dotted line: even harmonics (2,4,6).

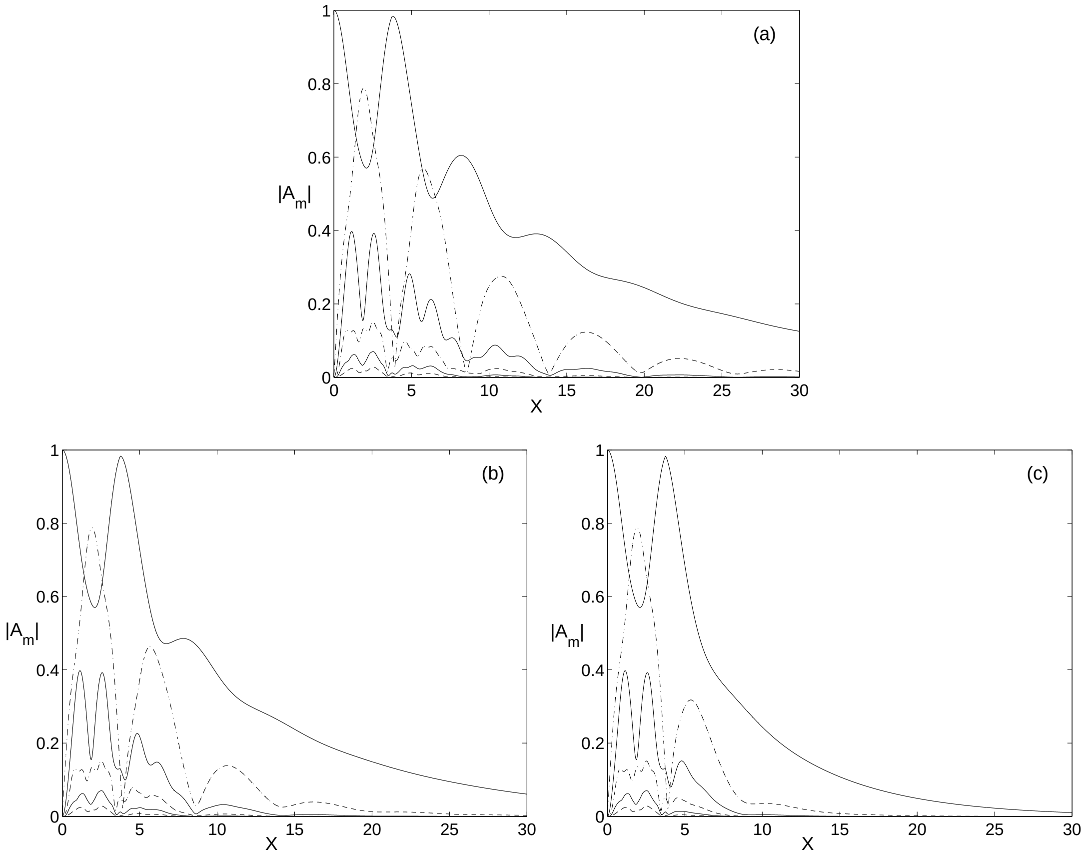

Figure 2-9: Harmonics 1 to 6 for $\sigma = 0.5$ and different values of l. Case (a): $l = 0.5$; case (b): $l = 1$ and case (c): $l = 2$. For all cases $\nu = k_1 = 1$ and randomness starts at $X = 3.75$, the peak of the first harmonic. Solid line: odd harmonics (1,3,5), dash-dotted line: even harmonics (2,4,6).

2.6.4 Varying the ratio $\mu^2/\epsilon = \nu$

σ is now set equal to 0.2 and 0.5 and the effect of different ratios of $\mu^2/\epsilon = \nu$ (dispersion vs. nonlinearity) on the evolution is studied. Figure 2-10 for which $\sigma = 0.2$ and Figure 2-11 for which $\sigma = 0.5$ display the results for $\nu = 0.5, 1, 2$, with $k_1 = l = 1$.

It is clear that as ν increases, the second harmonic amplitude decreases and the frequency of amplitude oscillations of all the harmonics increases. Physically, greater ν corresponds to weaker nonlinearity and/or shorter waves (stronger dispersion). Either cause leads to less generation of the higher harmonics even before entry into the region

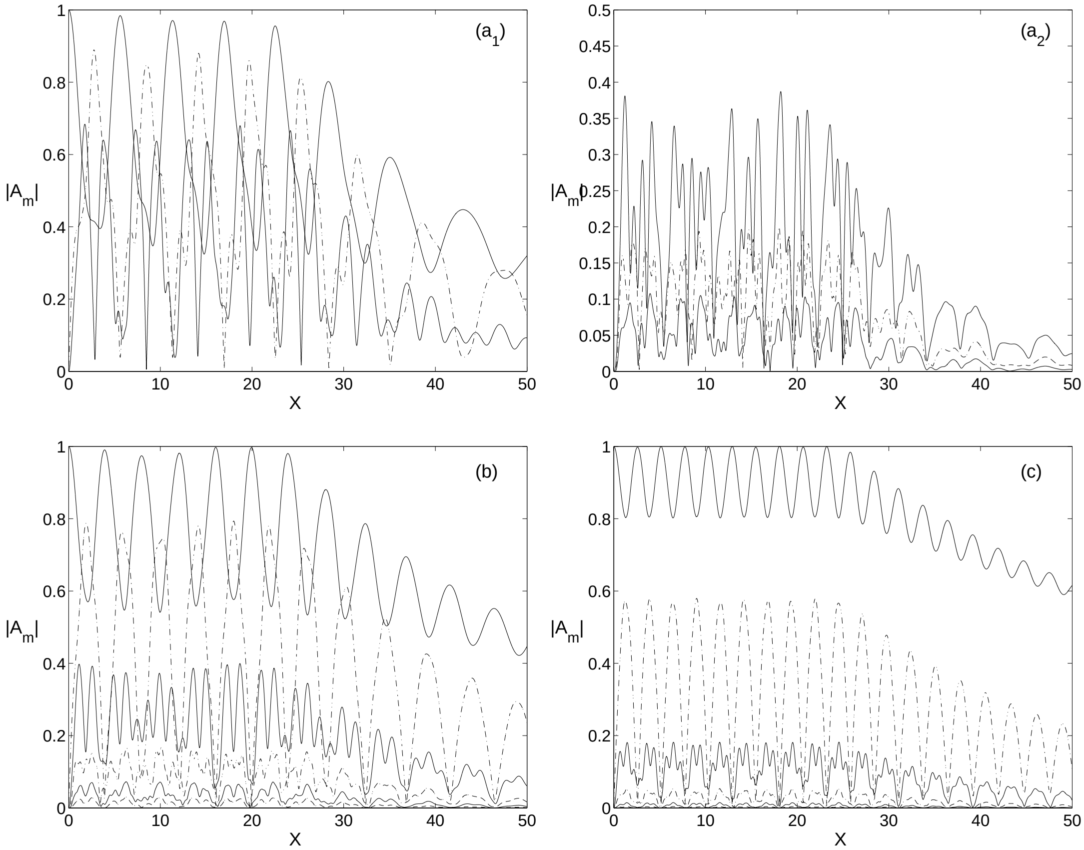

Figure 2-10: Harmonics 1 to 6 for $\sigma = 0.2$ and different values of ν. Cases (a_1) harmonics 1 to 3 and (a_2) harmonics 4 to 6: $\nu = 0.5$; case (b): $\nu = 1$ and case (c): $\nu = 2$. For all cases $l = k_1 = 1$ and the random region starts at $X = 25$. Solid line: odd harmonics (1,3,5), dash-dotted line: even harmonics (2,4,6) except for (a_2).

of disorder $X \geq 25$. In the region of disorder the higher harmonics are attenuated at comparable rates because of the same values of σ and l.

In both figures, the case (a_2) has a full line for harmonics 4 and 6 and a dash-dotted line for harmonic 5 for more clarity. The results for $\nu = 0.5$ are rather striking, the second harmonic is almost as large as the first. This is a case where nonlinearity effects are greater than dispersion effects.

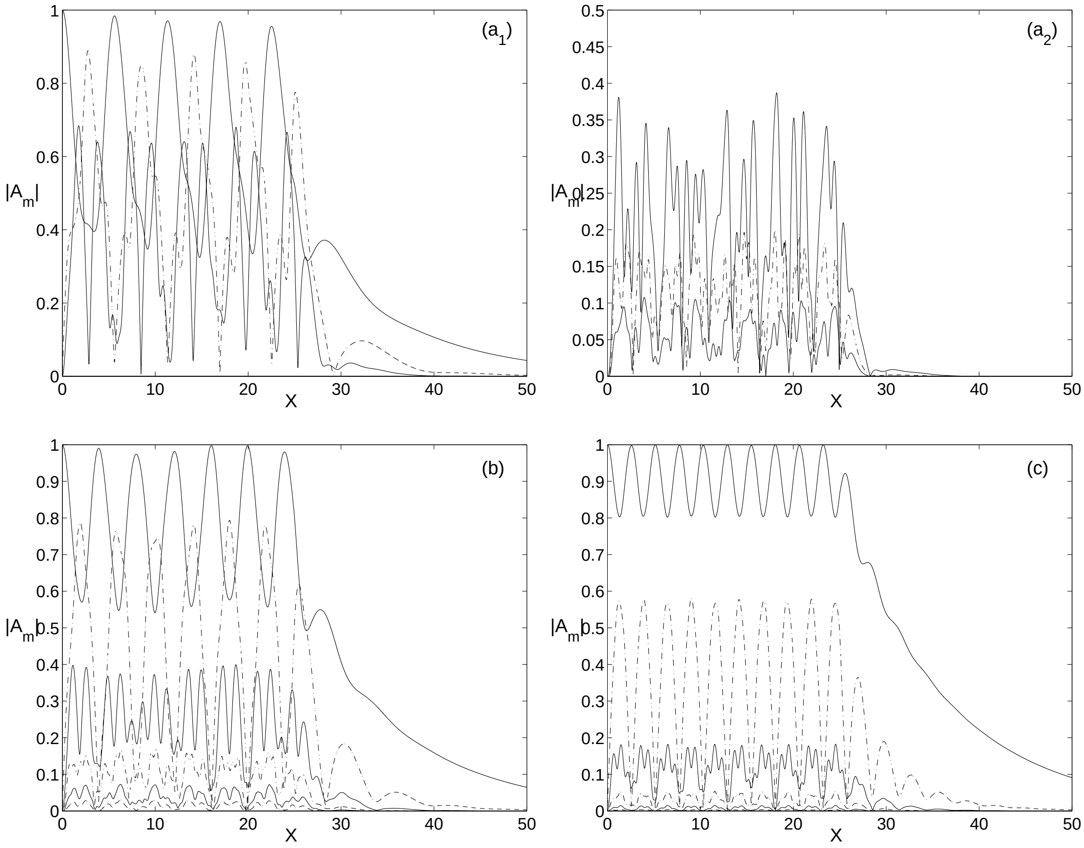

Figure 2-11: Harmonics 1 to 6 for $\sigma = 0.5$ and different values of ν. Cases (a_1) harmonics 1 to 3 and (a_2) harmonics 4 to 6: $\nu = 0.5$; case (b): $\nu = 1$ and case (c): $\nu = 2$. For all cases $l = k_1 = 1$ and the random region starts at $X = 25$. Solid line: odd harmonics (1,3,5), dash-dotted line: even harmonics (2,4,6) except for (a_2).

2.7 Comparing the results with a larger number of harmonics

In order to check the numerical accuracy of the six-harmonic approximation, the numerical analysis is extended by keeping the first ten harmonics. The ten governing equations are not stated as they can be obtained from equation (2.2.43). The final results are recorded below.

It has been observed that the higher harmonics are larger for higher nonlinearity,

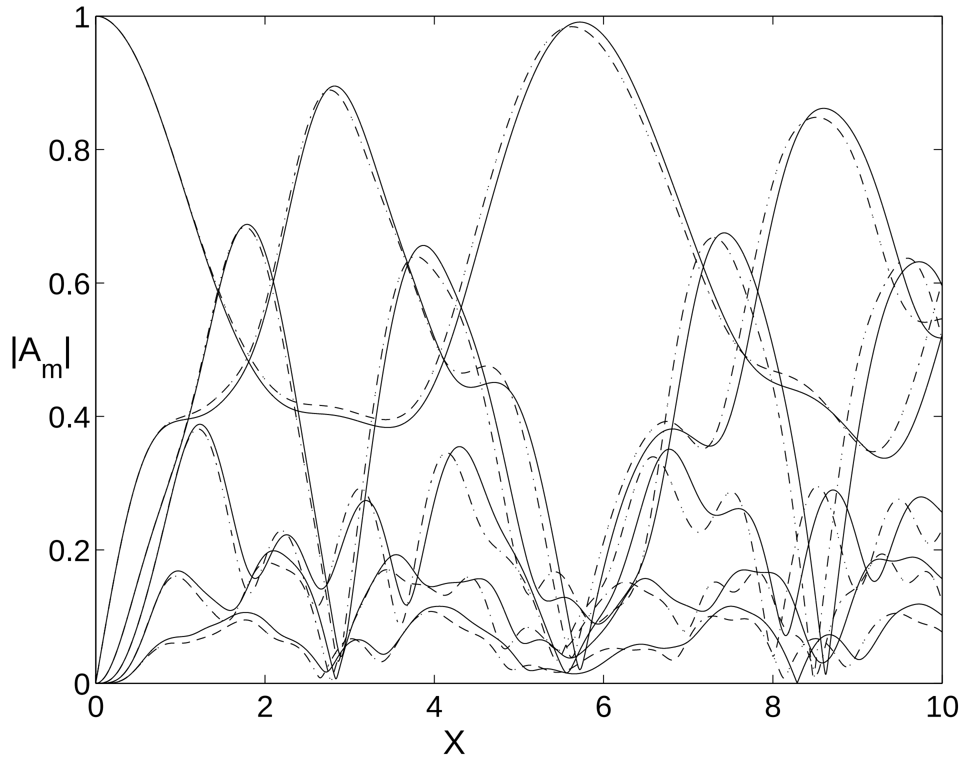

Figure 2-12: Harmonics 1 to 6 for a random bottom simulated for a total of 6 (dash-dotted line) and 10 harmonics (full line). $\sigma = 0$, $\nu = 0.5$, and $l = k_1 = 1$.

so that accounting for more harmonics should have the most effect for a small value of ν. In addition, accounting for randomness of the seabed damps all the harmonics, especially the higher harmonics. Figure 2-12 compares the simulations of the 6 (dash-dotted line) and 10-harmonic problem (full line) for $\nu = 0.5$, $l = k_1 = 1$ and a perfectly smooth bottom: $\sigma = 0$. Only the first six harmonics are plotted for more clarity. There does not seem to be a very important difference in the amplitude of the harmonics, the main difference is observed in the phase difference. The energy errors calculated for both simulations are not shown here as they are very similar to the error found in Figure 2-2: the errors are of the order 10^{-12}.

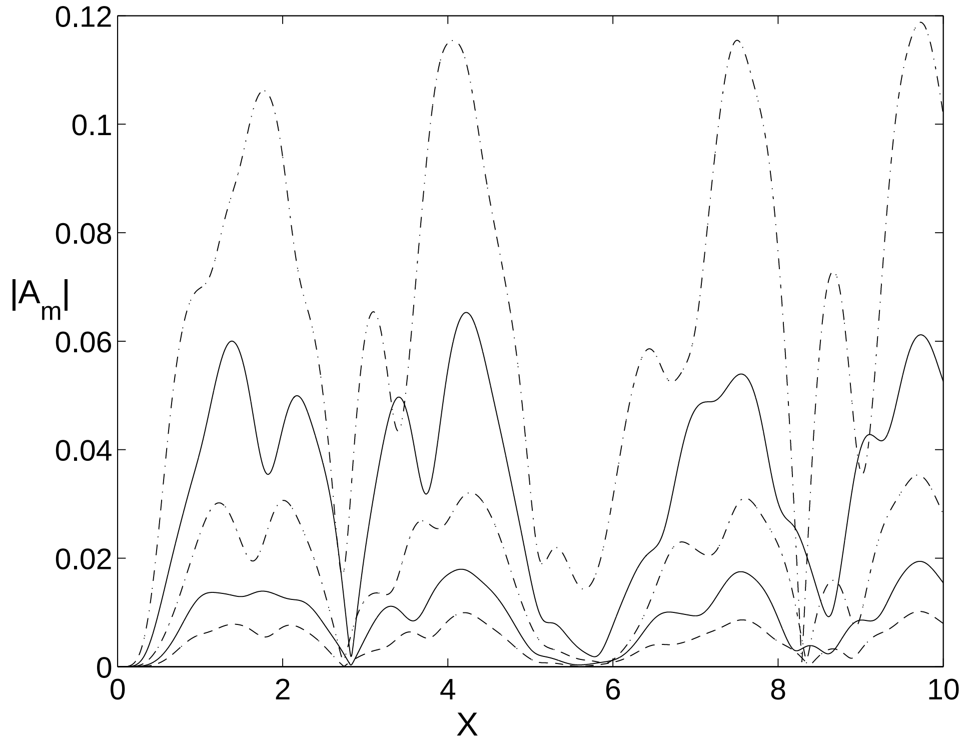

Figure 2-13: Harmonics 6 to 10 simulated for a total of 10 harmonics over a perfectly smooth bottom. $\sigma = 0$, $\nu = 0.5$ and $l = k_1 = 1$

Figure 2-13 shows harmonics 6 to 10 for the 10-harmonic simulation in the case of no randomness for $\nu = 0.5$. It is easy to observe the relatively important non-linearity in this case. The amplitude of harmonic 7 is at most 0.06, and harmonic 10 hardly reaches 0.02 which is very small compared to the amplitude of the first harmonic, almost two orders of magnitude larger. The simulations for strong nonlinearity and a perfectly smooth bottom are a boarder case of the theory studied here as one of the main assumptions is weak nonlinearity and the interest here is to look at localization. Let us therefore compare results for larger values of ν.

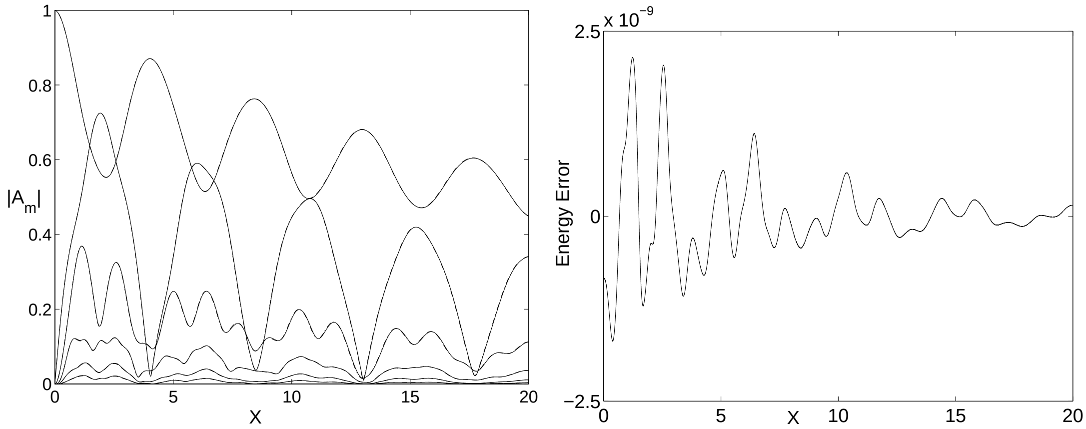

Figure 2-14: Left: Harmonics 1 to 6 for a random bottom ($\sigma = 0.2$) simulated for a total of 6 (dash-dotted line) and 10 harmonics (full line) Right: Energy error for the 10-harmonic simulation. $\nu = l = k_1 = 1$.

Figure 2-14 shows simulations run for 6 (dash-dotted line) and 10 harmonics (full line) for a case of weaker nonlinearity and stronger dispersion: $\nu = 1$. The energy error in the figure is for the 10-harmonic simulation, but it is very similar to the 6-harmonic simulation which is not reported to avoid redundance. The results for the harmonic amplitudes are very similar and it is almost impossible to distinguish the full line from the dash-dotted line. The 6th harmonic seems to be very small: at least one order of magnitude smaller than the first harmonic, let us look at the higher harmonics in more detail.

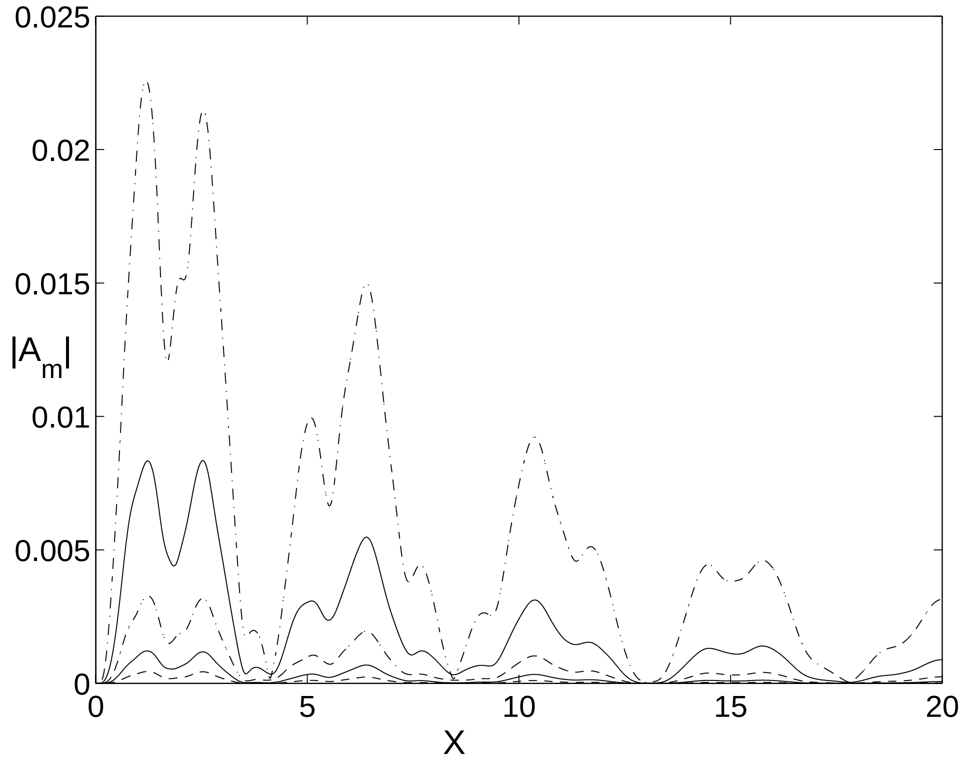

Figure 2-15: Harmonics 6 to 10 simulated for a total of 10 harmonics. $\sigma = 0.2$, $\nu = 1$ and $l = k_1 = 1$.

Figure 2-15 shows the amplitude of harmonics 6 to 10 for the 10-harmonic simulation. Harmonics 7 to 10 are between 2 and 3 orders of magnitude smaller than the first harmonic, they are also all smaller than 0.01 or 1% of the first harmonic. This explains why these higher harmonics have a very small influence on the simulation and the simulations for 6 or 10 harmonics are so similar. Results obtained for 6 harmonics seem accurate enough for weak nonlinearity and a random bathymetry, which is the purpose of the study.

2.8 Concluding remarks

In showing how harmonic generation and localization affect each other, two major advances of modern physics are seen to intertwine in a problem of classical origin. The method of multiple scales, developed primarily for classical problems involving periodic media, is also seen to lend itself easily to a medium with a slight disorder. The same technique can likely be used in other problems on waves through disordered media.

In deriving the equation for the evolution of the total wave energy, it was possible to obtain an analytical expression of energy conservation among a finite and infinite number of harmonics for a perfectly smooth bottom. This represents an extension of the Manley-Rowe relation obtained in nonlinear optics.

Chapter 3

Localization of nonlinear waves over a slowly varying bottom. Derivation of evolution equations

Nonlinear equations in the general shallow water case where the bottom is random and slowly varying are derived. The derivation is strongly inspired by previous work by [19] and [22]

3.1 Governing equations for the flow

The bottom depth is taken as $z = -H + \epsilon b$ where H is the average bottom depth, it is slowly varying in the horizontal direction, b is a random function of zero mean, and ϵ is the small parameter. A function $F_b = z + H - \epsilon b$ describing the bottom is defined. The free surface height will be called ζ so that the free surface is described by $F_s = z - \zeta$.

3.1.1 General nonlinear governing equations

Considering an irrotational flow for an incompressible fluid and using the Boussinesq approximation, the governing equations are the following.

The Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{3.1.1}$$

No flux condition at the bottom $z=-H+\epsilon b$

$$\bigtriangledown\phi.\bigtriangledown F_b=0 \qquad \text{or} \qquad \frac{\partial\phi}{\partial z}=\frac{\partial\phi}{\partial x}\left[\epsilon\frac{\partial b}{\partial x}-\frac{\partial H}{\partial x}\right] \tag{3.1.2}$$

Bernoulli equation at the surface $z=\zeta$

$$\frac{\partial\phi}{\partial t}+g\zeta+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0 \tag{3.1.3}$$

No flux condition through the free surface $z=\zeta$

$$\frac{\partial F_s}{\partial t}+\bigtriangledown\phi.\bigtriangledown F_s=0 \qquad \text{or} \qquad \frac{\partial\phi}{\partial z}=\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\zeta}{\partial t} \tag{3.1.4}$$

The combination of the free surface and the Bernouilli conditions gives a unique surface condition involving only the potential ϕ.

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}+\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0 \tag{3.1.5}$$

3.1.2 Taylor expansions at the surface and bottom

The no-flux condition at the bottom (3.1.2), the Bernoulli equation (3.1.3) and the combined condition (3.1.5) at the surface can be expanded in Taylor series at $z=-H$, $z=0$ and $z=0$ respectively.

Taylor expansion of (3.1.2) at $z=-H$.

$$\begin{aligned}&\frac{\partial\phi}{\partial z}+\epsilon b\frac{\partial^2\phi}{\partial z^2}+\frac{\epsilon^2}{2}b^2\frac{\partial^3\phi}{\partial z^3}\\ &\quad=\left(\frac{\partial\phi}{\partial x}+\epsilon b\frac{\partial^2\phi}{\partial z\partial x}+\frac{\epsilon^2b^2}{2}\frac{\partial^3\phi}{\partial z^2\partial x}\right)\left(\epsilon\frac{\partial b}{\partial x}-\frac{\partial H}{\partial x}\right)\end{aligned} \tag{3.1.6}$$

Taylor expansion of (3.1.3) at $z=0$.

$$\begin{aligned}\frac{\partial\phi}{\partial t}\;&+\;\zeta\frac{\partial^2\phi}{\partial t\partial z}+\frac{1}{2}\zeta^2\frac{\partial^3\phi}{\partial t\partial z^2}+g\zeta+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]\\ &+\;\frac{\zeta}{2}\frac{\partial}{\partial z}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0\end{aligned} \tag{3.1.7}$$

Taylor expansion of (3.1.5) at $z = 0$.

$$\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right) + \zeta\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right) + \frac{\zeta^2}{2}\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right)$$
$$+\left(\frac{\partial}{\partial t} + \zeta\frac{\partial^2}{\partial z\partial t} + \frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \quad (3.1.8)$$

3.2 Multiple scale expansion in the horizontal direction and in time

Let us now define the following new variables $x, x_1 = \epsilon x, x_2 = \epsilon^2 x, \cdots$ for the horizontal direction and $t, t_1 = \epsilon t, t_2 = \epsilon^2 t, \cdots$ for time, and use the perturbation method on the governing equations. ϕ is expanded as $\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots$, and the equations are separated in orders of ϵ. ϕ will then be separated into two components, an average component and a fluctuating component corresponding to the movements induced by the randomness of the bottom: $\phi = \langle\phi\rangle + \phi'$.

3.2.1 Governing equations and solution at order ϵ

The governing equations for ϕ_1 are:

$$\begin{cases} \text{Laplace} & \phi_{1xx} + \phi_{1zz} = 0 = F_1 & \text{for } -H \leq z \leq 0 \\ \text{Surface condition} & \phi_{1tt} + g\phi_{1z} = 0 = G_1 & \text{at } z = 0 \\ \text{Bottom condition} & \phi_{1z} = 0 = I_1 & \text{at } z = -H \\ \text{Bernoulli equation} & -g\zeta_1 = \phi_{1t} & \text{at } z = 0 \end{cases} \quad (3.2.1)$$

It can be observed that the equations at this order contain no random or nonlinear component, so that ϕ_1 will have a zero random component : $\phi_1' = 0$. By taking the leading order solution to be a monochromatic wave train propagating from left to right, the following solution is obtained.

$$\phi_1 = \langle\phi_1\rangle = \phi_{10} + \left(\phi_{11}e^{iS-i\omega_t} + *\right) \quad (3.2.2)$$

The above equations give a solution for the first harmonic

$$\phi_{11} = -i\frac{g}{2\omega}\frac{\cosh k\,(z+H)}{\cosh kH}A \quad (3.2.3)$$

and

$$\zeta_1 = \langle \zeta_1 \rangle = \frac{A}{2} e^{iS - i\omega t} + * \tag{3.2.4}$$

where $*$ denotes the complex conjugate of the preceding expression and S is defined by:

$$S = S(x) = \int_0^x k(\epsilon^2 l)\, dl = \frac{1}{\epsilon^2} \int_0^{x_2} k(u)\, du \tag{3.2.5}$$

Unless otherwise specified, S will be considered a function of x.

The dispersion relation is obtained from the surface condition and the solution for ϕ_1.

$$\omega^2 = gk \tanh kH \tag{3.2.6}$$

3.2.2 Governing equations and solution at order ϵ^2

The governing equations for ϕ_2 are:

$$\begin{cases} \phi_{2xx} + \phi_{2zz} = F_2 & -H \leq z \leq 0 \\ \phi_{2tt} + g\phi_{2z} = G_2 & z = 0 \\ \phi_{2z} = I_2 & z = -H \\ -g\zeta_2 = \phi_{2t} + \phi_{1t_1} + \zeta_1 \phi_{1tz} + \frac{1}{2}\left(\phi_{1x}^2 + \phi_{1z}^2\right) & z = 0 \end{cases} \tag{3.2.7}$$

where

$$F_2 = -2k\phi_{1x_1x} = \langle F_2 \rangle \tag{3.2.8}$$

$$G_2 = -2\phi_{1tt_1} - \zeta_1 \left(\phi_{1tt} + g\phi_{1z}\right)_z - \left(\phi_{1x}^2 + \phi_{1z}^2\right)_t = \langle G_2 \rangle \tag{3.2.9}$$

and

$$I_2 = \left(b\phi_{1x}\right)_x = I_2^{'} \tag{3.2.10}$$

F_2, G_2 and I_2 can be calculated from the results obtained at $O(\epsilon)$.

$$F_2 = -\frac{gk}{\omega} \frac{\cosh k(z+H)}{\cosh kH} \left(\frac{\partial A}{\partial x_1} e^{iS - i\omega t} + * \right) \tag{3.2.11}$$

$$G_2 = g\left(\frac{\partial A}{\partial t_1} e^{iS - i\omega t} + * \right) + \frac{3\omega^3}{4\sinh^2 kH} \left(iA^2 e^{2iS - 2i\omega t} + * \right) \tag{3.2.12}$$

and

$$I_2 = \frac{gk}{2\omega \cosh kH} \left[A \left(b e^{iS} \right)_x e^{-i\omega t} + * \right] \tag{3.2.13}$$

First a solution is derived for the ensemble average component $\langle\phi_2\rangle$ before considering the fluctuating solution $\phi_2^{'}$. In order to solve for $\langle\phi_2\rangle$, the ensemble average of the equations above are taken. They result in the same system of equations in which ϕ_2 can be replaced by $\langle\phi_2\rangle$, except for the bottom boundary condition for which $\langle I_2\rangle = 0$ because b has a zero mean.

The solution for $\langle\phi_2\rangle$ can be decomposed as

$$\langle\phi_2\rangle = \phi_{20} + \left(\phi_{21}e^{iS-i\omega t} + *\right) + \left(\phi_{22}e^{2iS-2i\omega t} + *\right) \tag{3.2.14}$$

Solving the system is done by solving separately for each harmonic.

The first harmonic ϕ_{21} and the solvability condition

ϕ_{21} obeys the following governing equations.

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi_{21} = -\frac{gk}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\frac{\partial A}{\partial x_1} \tag{3.2.15}$$

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g}\right)\phi_{21} = \frac{\partial A}{\partial t_1} \quad \text{at } z = 0 \tag{3.2.16}$$

$$\frac{\partial \phi_{21}}{\partial z} = 0 \quad \text{at } z = -H \tag{3.2.17}$$

The solution can be found to be

$$\phi_{21} = -\frac{\omega}{2k}(z+H)\frac{\sinh k(z+H)}{\sinh kH}\frac{\partial A}{\partial x_1} \tag{3.2.18}$$

Invoking the solvability condition, or simply using the surface boundary condition (3.2.16) and defining

$$C_g = \frac{d\omega}{dk} = \frac{\omega}{2k}\left(1 + \frac{2kH}{\sinh 2kH}\right) \tag{3.2.19}$$

as the group velocity of the propagating wave, the solvability condition is obtained:

$$\frac{\partial A}{\partial t_1} + C_g\frac{\partial A}{\partial x_1} = 0 \tag{3.2.20}$$

The second harmonic ϕ_{22}

ϕ_{22} obeys the following governing equations.

$$\left(\frac{\partial^2}{\partial z^2} - 4k^2\right)\phi_{22} = 0 \tag{3.2.21}$$

$$\left(\frac{\partial}{\partial z} - \frac{4\omega^2}{g}\right)\phi_{22} = \frac{3\omega^2}{4\sinh^2 kH} iA^2 \quad \text{at } z = 0 \tag{3.2.22}$$

$$\frac{\partial \phi_{22}}{\partial z} = 0 \quad \text{at } z = -H \tag{3.2.23}$$

The solution can be found to be

$$\phi_{22} = -i\frac{3\omega}{16}\frac{\cosh 2k(z+H)}{\sinh^4 kH}A^2 \tag{3.2.24}$$

A solution for $\langle\phi_2\rangle$ is obtained by grouping terms together:

$$\begin{aligned}\langle\phi_2\rangle &= \phi_{20} - \frac{\omega}{2k}(z+H)\frac{\sinh k(z+H)}{\sinh kH}\left(\frac{\partial A}{\partial x_1}e^{iS-i\omega t} + *\right) \\ &\quad - \frac{3\omega}{16}\frac{\cosh 2k(z+H)}{\sinh^4 kH}\left(iA^2 e^{2iS-2i\omega t} + *\right)\end{aligned} \tag{3.2.25}$$

The fluctuating part $\phi_2^{'}$

In averaging 3.2.7, the fluctuating terms are ommitted. They must be taken into account in this section when solving for ϕ_2'. $\phi_2^{'}$ obeys the following governing equations.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_2^{'} = O \quad -H \le z \le 0 \tag{3.2.26}$$

$$\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi_2^{'} = 0 \quad z = 0 \tag{3.2.27}$$

$$\frac{\partial \phi_2^{'}}{\partial z} = \frac{gk}{2\omega\cosh kH}\left[A\left(be^{iS}\right)_x e^{-i\omega t} + *\right] \quad z = -H \tag{3.2.28}$$

From the right hand side forcing terms of the governing equations, it can be predicted that the fluctuating part only has first harmonic components, so that:

$$\phi_2^{'} = \psi e^{-i\omega t} + * \tag{3.2.29}$$

The system of equations to be solved reduces to.

$$\begin{aligned} \psi_{xx} + \psi_{zz} &= 0 && -H \le z \le 0 \\ \psi_z &= \frac{gkA}{2\omega \cosh kH}\left[b(x)e^{iS}\right]_x && z = -H \\ \psi_z - \frac{\omega^2}{g}\psi &= 0 && z = 0 \end{aligned} \tag{3.2.30}$$

Using Green's theorem to solve the system for the fluctuating part, a solution for Green's function $G(x, x', z)$ is sought.

$$\begin{cases} G_{xx} + G_{zz} = 0 & -H \le z \le 0 \\ G_z = \delta(x - x') & z = -H \\ G_z - \frac{\omega^2}{g} G = 0 & z = 0 \end{cases} \tag{3.2.31}$$

Introducing the exponential Fourier transform on the equations for Green's function, $\widetilde{G}$ is defined as the Fourier transform of G:

$$\widetilde{G}(\alpha, x', z) = \int_{-\infty}^{\infty} G(x, x', z)\, e^{-i\alpha x}\, dx \tag{3.2.32}$$

The inverse Fourier transform of $\widetilde{G}$ is G itself by definition.

$$G(x, x', z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{G}(\alpha, x', z)\, e^{i\alpha x}\, d\alpha \tag{3.2.33}$$

$\widetilde{G}$ must then obey the system of equations:

$$\begin{cases} \widetilde{G}_{zz} - \alpha^2 \widetilde{G} = 0 & -H \le z \le 0 \\ \widetilde{G}_z = e^{-i\alpha x'} & z = -H \\ \widetilde{G}_z - \frac{\omega^2}{g}\widetilde{G} = 0 & z = 0 \end{cases} \tag{3.2.34}$$

because the integral of a delta function is calculated as.

$$\int_{-\infty}^{\infty} \delta(x - x')\, e^{-i\alpha x}\, dx = e^{-i\alpha x'} \tag{3.2.35}$$

$\widetilde{G}$ is determined as

$$\widetilde{G}(\alpha, x', z) = e^{i\alpha x'} \frac{\cosh \alpha z + \frac{\omega^2}{g\alpha}\sinh \alpha z}{\frac{\omega^2}{g}\cosh \alpha H - \alpha \sinh \alpha H} \tag{3.2.36}$$

The inverse Fourier transform of $\widetilde{G}$ gives G.

$$G(x, x', z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x - x')} \frac{\cosh \alpha z + \frac{\omega^2}{g\alpha}\sinh \alpha z}{\frac{\omega^2}{g}\cosh \alpha H - \alpha \sinh \alpha H}\, d\alpha \tag{3.2.37}$$

But G should be symmetric and should represent an outgoing wave at infinity, so

$$G(x, x', z) = G(|x - x'|, z) \tag{3.2.38}$$

Using Green's identity,

$$\int_{-\infty}^{\infty}\int_{-H}^{0}\left(\psi \nabla^2 G - G \nabla^2 \psi\right) dz dx = \oint_C \left(\psi \frac{\partial G}{\partial n} - G\frac{\partial \psi}{\partial n}\right) d\ell \tag{3.2.39}$$

and systems (3.2.30) and (3.2.31), it is possible to obtain a solution for ψ and then ϕ_2'.

$$\phi_2'(x, z) = \frac{gkAe^{iS-i\omega t}}{2\omega \cosh kH}\int_{-\infty}^{\infty} G(|x - x'|, z)\frac{d}{dx'}\left[b(x')e^{i[S(x')-S(x)]}\right] dx' + * \tag{3.2.40}$$

Applying the residue theorem yields the value of

$$G(|x - x'|, -H) = -\frac{i\omega^2}{gk}\frac{e^{ik|x-x'|}}{\frac{\omega^2 H}{g} + \sinh^2 kH} - \sum_{n=1}^{\infty}\frac{\omega^2}{gk_n}\frac{e^{-k_n|x-x'|}}{\frac{\omega^2 H}{g} + \sinh^2 k_n H} \tag{3.2.41}$$

where ik_n are the complex roots of the dispersion relation such that

$$\omega^2 = -gk_n \tan k_n H \tag{3.2.42}$$

The analytical details of the residue theorem are given in Appendix B.

ϕ_2' can then be calculated at $z = -H$.

Free surface displacement ζ_2

Equations (3.2.25) and (3.2.40) can now be used in the Bernouilli equation to determine the average free surface displacement.

$$\begin{aligned}\langle\zeta_2\rangle \quad = \quad & -\frac{\partial \phi_{10}}{\partial t_1} - \frac{k|A^2|}{2\sinh 2kH} - \frac{H\tanh kH}{2}\left(i\frac{\partial A}{\partial x_1}e^{iS-i\omega t} + *\right) \\ & + \frac{1}{2\omega}\left(i\frac{\partial A}{\partial t_1}e^{iS-i\omega t} + *\right) \\ & + \frac{k}{8}\frac{\cosh kH}{\sinh^3 kH}\left(1 + 2\cosh^2 kH\right)\left(A^2 e^{2iS-2i\omega t} + *\right)\end{aligned} \tag{3.2.43}$$

and the fluctuating free surface displacement.

$$\zeta_2' = -i\frac{gkAe^{iS-i\omega t}}{2\cosh kH}\int_{-\infty}^{\infty} G(|x - x'|, -H)\frac{d}{dx'}\left[b(x')e^{i[S(x')-S(x)]}\right] dx' + * \tag{3.2.44}$$

3.2.3 Equations at order ϵ^3

The governing equations for ϕ_3 are the following.

$$\begin{cases} \phi_{3xx} + \phi_{3zz} = F_3 & -H \leq z \leq 0 \\ \phi_{3tt} + g\phi_{3z} = G_3 & z = 0 \\ \phi_{3z} = I_3 & z = -H \\ -g\zeta_3 = L_3 & z = 0 \end{cases} \tag{3.2.45}$$

where

$$F_3 = -2\phi_{2xx_1} - \phi_{1xx_2} - \phi_{1x_2x} - \phi_{1x_1x_1} \tag{3.2.46}$$

$$\begin{aligned} G_3 \;=\; & -2\phi_{2tt_1} + 2\phi_{1tt_2} + \phi_{1t_1t_1} - \zeta_1\left(\phi_{2tt} + g\phi_{2z}\right)_z - \zeta_2\left(\phi_{1tt} + g\phi_{1z}\right)_z \\ & -\frac{\zeta_1^2}{2}\left(\phi_{1tt} + g\phi_{1z}\right)_{zz} - 2\left(\phi_{1x}\phi_{1x_1} + \phi_{1x}\phi_{2x} + \phi_{1z}\phi_{2z}\right)_t - \left(\phi_{1x}^2 + \phi_{1z}^2\right)_{t_1} \\ & -\zeta_1\left(\phi_{1x}^2 + \phi_{1z}^2\right)_{tz} - \frac{1}{2}\left(\phi_{1x}\frac{\partial}{\partial x} + \phi_{1z}\frac{\partial}{\partial z}\right)\left(\phi_{1x}^2 + \phi_{1z}^2\right) \end{aligned} \tag{3.2.47}$$

$$I_3 = \left(\frac{b^2}{2}\phi_{1xz}\right)_x + \left(b\phi_{2x}\right)_x + \left(b\phi_{1x_1}\right)_x + b\phi_{1xx_1} - \phi_{1x}H_{x_2} \tag{3.2.48}$$

and

$$L_3 = \phi_{3t} + \phi_{2t_1} + \phi_{1t_2} + \phi_{1x}\phi_{1x_1} + \zeta_1\left[\phi_{2t} + \phi_{1t_1} + \frac{1}{2}\left(\phi_{1x}^2 + \phi_{1z}^2\right)\right]_z + \zeta_2\phi_{1tz} + \frac{1}{2}\zeta_1^2\phi_{1tz^2} \tag{3.2.49}$$

Calculating $\langle F_3 \rangle$

The ensemble average of (3.2.46) is taken

$$\langle F_3 \rangle = -2\langle \phi_{2xx_1} \rangle - \phi_{1xx_2} - \phi_{1x_2x} - \phi_{1x_1x_1} \tag{3.2.50}$$

and each of the four terms is evaluated separately:

$$\begin{aligned} -2\langle \phi_2 \rangle_{xx_1} \;=\; & w(z+H)\frac{\sinh k(z+H)}{\sinh kH}\left(i\frac{\partial^2 A}{\partial x_1^2}e^{iS - i\omega t} + *\right) \\ & + \frac{3\omega k}{4}\frac{\cosh 2k(z+H)}{\sinh^4 kH}\left(A\frac{\partial A}{\partial x_1}e^{2iS - 2i\omega t} + *\right) \end{aligned} \tag{3.2.51}$$

Let us note here that ϕ_{1xx_2} is different from ϕ_{1x_2x} because the wavenumber k is a function of x_2 through the dispersion relation. They must be calculated separately.

$$\begin{aligned} -\phi_{1xx_2} &= -\frac{g}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{k\cosh k(z+H)}{\cosh kH}\right)\left(Ae^{iS-i\omega t}+*\right) \\ &\quad -\frac{gk}{2\omega}\frac{\cosh k(z+H)}{\cosh kH}\left(\frac{\partial A}{\partial x_2}e^{iS-i\omega t}+*\right) \end{aligned} \tag{3.2.52}$$

$$\begin{aligned} -\phi_{1x_2x} &= -\frac{gk}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)\left(Ae^{iS-i\omega t}+*\right) \\ &\quad -\frac{gk}{2\omega}\frac{\cosh k(z+H)}{\cosh kH}\left(\frac{\partial A}{\partial x_2}e^{iS-i\omega t}+*\right) \end{aligned} \tag{3.2.53}$$

$$-\phi_{1x_1x_1} = \frac{g}{2w}\frac{\cosh k(z+H)}{\cosh kH}\left(i\frac{\partial^2 A}{\partial x_1^2}e^{iS-i\omega t}+*\right) \tag{3.2.54}$$

The result is:

$$\begin{aligned} \langle F_3\rangle &= w(z+H)\frac{\sinh k(z+H)}{\sinh kH}\left(i\frac{\partial^2 A}{\partial x_1^2}e^{iS-i\omega t}+*\right) \\ &\quad -\frac{g}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{k\cosh k(z+H)}{\cosh kH}\right)\left(Ae^{iS-i\omega t}+*\right) \\ &\quad -\frac{gk}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)\left(Ae^{iS-i\omega t}+*\right) \\ &\quad -\frac{gk}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\left(\frac{\partial A}{\partial x_2}e^{iS-i\omega t}+*\right) \\ &\quad +\frac{3\omega k}{4}\frac{\cosh 2k(z+H)}{\sinh^4 kH}\left(A\frac{\partial A}{\partial x_1}e^{2iS-2i\omega t}+*\right) \end{aligned} \tag{3.2.55}$$

Calculating $\langle G_3\rangle$

Similarly, the ensemble average of (3.2.47) is taken

$$\begin{aligned} \langle G_3\rangle &= -2\langle\phi_2\rangle_{tt_1} - 2\phi_{1tt_2} - \phi_{1t_1t_1} - \zeta_1\left(\langle\phi_2\rangle_{tt} + g\langle\phi_2\rangle_z\right)_z \\ &\quad -\langle\zeta_2\rangle\left(\phi_{1tt}+g\phi_{1z}\right)_z - \frac{\zeta_1^2}{2}\left(\phi_{1tt}+g\phi_{1z}\right)_{zz} \\ &\quad -2\left(\phi_{1x}\phi_{1x_1} - \phi_{1x}\phi_{2x} - \phi_{1z}\phi_{2z}\right)_t - \left(\phi_{1x}^2+\phi_{1z}^2\right)_{t_1} \\ &\quad -\zeta_1\left(\phi_{1x}^2+\phi_{1z}^2\right)_{tz} - \frac{1}{2}\left(\phi_{1x}\frac{\partial}{\partial x}+\phi_{1z}\frac{\partial}{\partial z}\right)\left(\phi_{1x}^2+\phi_{1z}^2\right) \end{aligned} \tag{3.2.56}$$

and each group of terms is evaluated separately in Appendix C at $z = 0$, so that.

$$\begin{aligned}
\langle G_3 \rangle \quad = \quad & -\frac{\partial^2 \phi_{10}}{\partial t_1^2} + \frac{\omega^3}{2k} \frac{\cosh^2 kH}{\sinh^2 kH} \frac{\partial AA^*}{\partial x_1} - \frac{\omega^2}{4 \sinh^2 kH} \frac{\partial AA^*}{\partial t_1} \\
& + \left(\frac{g}{\omega} C_g^2 + \frac{\omega^2 H}{k} C_g \right) \left(i \frac{\partial^2 A}{\partial x_1^2} e^{iS - i\omega t} + * \right) + g \left(\frac{\partial A}{\partial t_2} e^{iS - i\omega t} + * \right) \\
& + gk \left(\frac{\partial \phi_{10}}{\partial x_1} - \frac{k}{2\omega \cosh^2 kH} \frac{\partial \phi_{10}}{\partial t_1} \right) \left(iAe^{iS - i\omega t} + * \right) \\
& + \frac{\omega^3 k}{16} \frac{\cosh kH}{\sinh^5 kH} \left(\cosh 4kH + 8 - 2 \tanh^2 kH \right) |A|^2 \left(iAe^{iS - i\omega t} + * \right) \\
& + \left(\langle G_{32} \rangle e^{2iS - 2i\omega t} + * \right) + \left(\langle G_{33} \rangle e^{3iS - 3i\omega t} + * \right) \qquad (3.2.57)
\end{aligned}$$

Use has been made of the dispersion relation, the definition of the group velocity C_g and the solvability condition (3.2.20) derived for the previous order.

Let us note that $\langle G_{30} \rangle$ is real.

$\langle G_{32} \rangle$ and $\langle G_{33} \rangle$ are not expanded here because they are not needed in the following calculations. They can be obtained from the terms expanded in appendix C.

Calculating $\langle I_3 \rangle$

The ensemble of average of I_3 is:

$$\langle I_3 \rangle = \langle b\phi_{2x} \rangle_x - \phi_{1x} H_{x_2} = \langle b\phi_2 \rangle_{xx} - \langle b_x \phi_2 \rangle_x - \phi_{1x} H_{x_2} \qquad (3.2.58)$$

A correlation for the randomness on the bottom must be specified to calculate $\langle I_3 \rangle$. The function γ and the new variable ξ are defined as:

$$\langle b(x) b(x') \rangle = \sigma^2 \gamma(\xi) \qquad \text{and} \qquad \xi = x - x' \qquad (3.2.59)$$

where σ is the root mean square of the correlation.

$$\begin{aligned}
\langle b\phi_2 \rangle_{xx} |_{z=-H} \quad = \quad & \frac{gkAe^{-i\omega t}}{2\omega \cosh kH} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{d}{dx'} \left[\langle b(x') b(x) \rangle e^{iS(x')} \right] \\
& \qquad G \left(|x - x'| , -H \right) \, dx' + * \\
= \quad & -\frac{gkA\sigma^2 e^{-i\omega t}}{2\omega \cosh kH} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{d}{d\xi} \left[\gamma(\xi) e^{i(S' - S)} \right] \\
& \qquad G \left(|\xi| , -H \right) \, d\xi e^{iS} + * \qquad (3.2.60)
\end{aligned}$$

$S' - S = S(x') - S(x)$ is approximated by $k(x' - x) = -k\xi$:

$$S' - S = \int_0^{x'} k(\epsilon^2 \ell) d\ell - \int_0^{x} k(\epsilon^2 \ell) d\ell = \int_x^{x'} k(\epsilon^2 \ell) d\ell \qquad (3.2.61)$$

so that

$$\langle b\phi_2\rangle_{xx}|_{z=-H} = \frac{gk^3 A\sigma^2 e^{iS-i\omega t}}{2\omega\cosh kH}\int_{-\infty}^{\infty}\frac{d}{d\xi}\left[\gamma(\xi)e^{-ik\xi}\right]G\left(|\xi|,-H\right)d\xi + * \tag{3.2.62}$$

Similarly for $\langle b_x\phi_2\rangle_x$,

$$\langle b_x\phi_2\rangle_x|_{z=-H} = -i\frac{gk^2 A\sigma^2 e^{iS-i\omega t}}{2\omega\cosh kH}\int_{-\infty}^{\infty}\frac{d}{d\xi}\left[\frac{d\gamma}{d\xi}e^{-ik\xi}\right]G\left(|\xi|,-H\right)d\xi + * \tag{3.2.63}$$

A constant β is defined as

$$\beta = \frac{gk^2\sigma^2}{\omega\cosh^2 kH}\int_{-\infty}^{\infty}\left[\left(\frac{d}{d\xi}-ik\right)^2\gamma\right]G\left(|\xi|,-H\right)e^{-ik\xi}\,d\xi \tag{3.2.64}$$

so that $\langle I_3\rangle$ can be simplified

$$\langle I_3\rangle = \frac{i}{2}\beta\cosh kH A e^{iS-i\omega t} + * - \frac{gk}{2\omega\cosh kH}\frac{\partial H}{\partial x_2}Ae^{iS-i\omega t} + * \tag{3.2.65}$$

3.3 Solvability conditions for the O(ϵ^3)

The restrictions of the system of governing equations to the zeroth and first harmonic lead to solvability conditions.

3.3.1 The zeroth harmonic

The system of governing equations for the zeroth harmonic $\langle\phi_{30}\rangle$ reduces to the following.

$$\frac{\partial^2\langle\phi_{30}\rangle}{\partial z^2} = F_{30} = -\frac{\partial^2\phi_{10}}{\partial x_1^2} \tag{3.3.1}$$

$$\left.\frac{\partial\langle\phi_{30}\rangle}{\partial z}\right|_{z=0} = \frac{1}{g}G_{30} = -\frac{1}{g}\frac{\partial^2\phi_{10}}{\partial t_1^2} + \frac{\omega^3}{2gk}\frac{\cosh^2 kH}{\sinh^2 kH}\frac{\partial AA^*}{\partial x_1} - \frac{\omega^2}{4g\sinh^2 kH}\frac{\partial AA^*}{\partial t_1} \tag{3.3.2}$$

$$\left.\frac{\partial\langle\phi_{30}\rangle}{\partial z}\right|_{z=-H} = 0 \tag{3.3.3}$$

The solvability condition for this system is straightforward. By definition, (3.3.1) can be integrated between $-H$ and zero:

$$\int_{-H}^{0}\frac{\partial^2\langle\phi_{30}\rangle}{\partial z^2}dz = \left[\frac{\partial\langle\phi_{30}\rangle}{\partial z}\right]_{z=-H}^{z=0} \tag{3.3.4}$$

Replacing with the governing equations above,

$$\int_{-H}^{0} F_{30} dz = \frac{1}{g} G_{30} \tag{3.3.5}$$

and integrating a constant on the left hand side of (3.3.5) yields the solvability condition obtained by Mei [20].

$$\frac{\partial^2 \phi_{10}}{\partial t_1^2} - gH \frac{\partial^2 \phi_{10}}{\partial x_1^2} = \frac{\omega^3}{2k} \frac{\cosh^2 kH}{\sinh^2 kH} \frac{\partial AA^*}{\partial x_1} - \frac{\omega^2}{4 \sinh^2 kH} \frac{\partial AA^*}{\partial t_1} \tag{3.3.6}$$

3.3.2 The first harmonic

In this section, let us define the operator $\mathcal{L}$ such that.

$$\mathcal{L} = \frac{\partial^2}{\partial z^2} - k^2 \tag{3.3.7}$$

The governing equations for the first harmonic are simplified into

$$\begin{cases} \mathcal{L}\langle \phi_{31} \rangle = \langle F_{31} \rangle & -H \leq z \leq 0 \\ \langle \phi_{31} \rangle_z - \frac{\omega^2}{g} \langle \phi_{31} \rangle = \langle G_{31} \rangle & z = 0 \\ \langle \phi_{31} \rangle_z = \langle I_{31} \rangle & z = -H \end{cases} \tag{3.3.8}$$

A new function

$$f = \frac{\cosh k(z+H)}{\cosh kH} \tag{3.3.9}$$

is defined such that f obeys the following system.

$$\begin{cases} \mathcal{L} f = 0 & -H \leq z \leq 0 \\ f_z - \frac{\omega^2}{g} f = 0 & z = 0 \\ f_z = 0 & z = -H \end{cases} \tag{3.3.10}$$

Using Green's theorem

$$\int_{-H}^{0} \left(f \mathcal{L} \langle \phi_{31} \rangle - \langle \phi_{31} \rangle \mathcal{L} f \right) dz = \left[f \frac{\partial \langle \phi_{31} \rangle}{\partial z} - \langle \phi_{31} \rangle \frac{\partial f}{\partial z} \right]_{-H}^{0} \tag{3.3.11}$$

and the equation systems governing f (3.3.10) and $\langle \phi_{31} \rangle$ (3.3.8),

$$\int_{-H}^{0} \langle F_{31} \rangle \frac{\cosh k(z+H)}{\cosh kH} dz = \frac{\omega^2}{g} \langle \phi_{31} \rangle + \frac{\langle G_{31} \rangle}{g} - \langle \phi_{31} \rangle k \tanh kH - \langle I_{31} \rangle \tag{3.3.12}$$

where the right-hand-side can be simplified further using the dispersion relation.

$$\int_{-H}^{0}\langle F_{31}\rangle\frac{\cosh k(z+H)}{\cosh kH}dz=\frac{\langle G_{31}\rangle}{g}-\langle I_{31}\rangle \tag{3.3.13}$$

The integral on the left-hand-side of (3.3.13) must be calculated.

$\langle F_{31}\rangle$ is obtained from (3.2.55).

$$\begin{aligned}\langle F_{31}\rangle &= w(z+H)\frac{\sinh k(z+H)}{\sinh kH}i\frac{\partial^2 A}{\partial x_1^2}-\frac{g}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{k\cosh k(z+H)}{\cosh kH}\right)A\\ &\quad -\frac{gk}{2\omega}\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)A-\frac{gk}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\frac{\partial A}{\partial x_2}\end{aligned} \tag{3.3.14}$$

All the terms are integrated separately, to do so, the following integrals are needed.

$$\int_{-H}^{0}\frac{\cosh^2 k(z+H)}{\cosh^2 kH}dz=\frac{\omega}{gk}C_g \tag{3.3.15}$$

$$\int_{-H}^{0}\frac{(z+H)\sinh k(z+H)}{\sinh kH\cosh kH}dz=\frac{H}{2k}\frac{\cosh 2kH}{\sinh 2kH}-\frac{1}{4k^2} \tag{3.3.16}$$

and

$$\begin{aligned}&\int_{-H}^{0}\left[\frac{\partial}{\partial x_2}\left(\frac{k\cosh k(z+H)}{\cosh kH}\right)+k\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)\right]\frac{\cosh k(z+H)}{\cosh kH}dz\\ &= \int_{-H}^{0}\frac{dk}{dx_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)^2dz+\int_{-H}^{0}2k\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)\frac{\cosh k(z+H)}{\cosh kH}dz\\ &= \frac{dk}{dx_2}\frac{\omega}{gk}C_g+2k\int_{-H}^{0}\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)^2dz\end{aligned} \tag{3.3.17}$$

Using Leibniz' rule on the last integral, it is possible to get the $\frac{\partial}{\partial x_2}$ operator "outside" the integral. An additional term appears because the depth average H is a function of the variable x_2.

$$\begin{aligned}&\int_{-H}^{0}\left[\frac{\partial}{\partial x_2}\left(\frac{k\cosh k(z+H)}{\cosh kH}\right)+k\frac{\partial}{\partial x_2}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)\right]\frac{\cosh k(z+H)}{\cosh kH}dz\\ &= \frac{dk}{dx_2}\frac{\omega}{gk}C_g+k\frac{\partial}{\partial x_2}\int_{-H}^{0}\left(\frac{\cosh k(z+H)}{\cosh kH}\right)^2dz-k\frac{\partial H}{\partial x_2}\frac{1}{\cosh^2 kH}\\ &= \frac{dk}{dx_2}\frac{\omega}{gk}C_g+k\frac{\partial}{\partial x_2}\left(\frac{\omega}{gk}C_g\right)-\frac{k}{\cosh^2 kH}\frac{\partial H}{\partial x_2}\\ &= \frac{dk}{dx_2}\frac{\omega}{gk}C_g+\frac{k\omega}{gk}\frac{\partial C_g}{\partial x_2}-kC_g\frac{\omega}{g}\left(-\frac{1}{k^2}\right)\frac{dk}{dx_2}-\frac{k}{\cosh^2 kH}\frac{\partial H}{\partial x_2}\\ &= \frac{\omega}{g}\frac{\partial C_g}{\partial x_2}-\frac{k}{\cosh^2 kH}\frac{\partial H}{\partial x_2}\end{aligned} \tag{3.3.18}$$

Adding terms, the integral on the left-hand-side of (3.3.13) is:

$$\int_{-H}^{0}\langle F_{31}\rangle\frac{\cosh k(z+H)}{\cosh kH}dz = \left(\frac{\omega H}{2k}\frac{\cosh 2kH}{\sinh 2kH}-\frac{\omega}{4k^2}+\frac{C_g}{2k}\right)i\frac{\partial^2 A}{\partial x_1^2} -\frac{1}{2}\left(\frac{\partial C_g}{\partial x_2}-\frac{2\omega}{\sinh 2kH}\frac{\partial H}{\partial x_2}\right)A \quad (3.3.19)$$

$\langle G_{31}\rangle$ and $\langle I_{31}\rangle$ are easily obtained from equations (3.2.57) and (3.2.65)

$$\langle G_{31}\rangle = \left(\frac{g}{\omega}C_g^2+\frac{\omega^2 H}{k}C_g\right)i\frac{\partial^2 A}{\partial x_1^2}+g\frac{\partial A}{\partial t_2}+gk\left(\frac{\partial\phi_{10}}{\partial x_1}-\frac{k}{2\omega\cosh^2 kH}\frac{\partial\phi_{10}}{\partial t_1}\right)iA +\frac{\omega^3 k}{16}\frac{\cosh kH}{\sinh^5 kH}\left(\cosh 4kH+8-2\tanh^2 kH\right)|A|^2 iA \quad (3.3.20)$$

$$\langle I_{31}\rangle = \frac{i}{2}\beta\cosh kHA-\frac{gk}{2\omega\cosh kH}\frac{\partial H}{\partial x_2}A \quad (3.3.21)$$

Equation (3.3.13) can now be used to obtain the solvability condition at this order.

$$\left(\frac{\omega H}{2k}\frac{\cosh 2kH}{\sinh 2kH}-\frac{\omega}{4k^2}+\frac{C_g}{2k}\right)i\frac{\partial^2 A}{\partial x_1^2}-\frac{1}{2}\left(\frac{\partial C_g}{\partial x_2}-\frac{2\omega}{\sinh 2kH}\frac{\partial H}{\partial x_2}\right)A$$
$$= \left(\frac{1}{\omega}C_g^2+H\tanh kHC_g\right)i\frac{\partial^2 A}{\partial x_1^2}+\frac{\partial A}{\partial t_2}+k\left(\frac{\partial\phi_{10}}{\partial x_1}-\frac{k}{2\omega\cosh^2 kH}\frac{\partial\phi_{10}}{\partial t_1}\right)iA +\frac{\omega k^2}{16\sinh^4 kH}\left(\cosh 4kH+8-2\tanh^2 kH\right)|A|^2 iA -\frac{i}{2}\beta A+\frac{\omega}{\sinh 2kH}\frac{\partial H}{\partial x_2}A \quad (3.3.22)$$

The derivative ω" of the group velocity:

$$\frac{\partial C_g}{\partial k}=\omega"(k)=\frac{\partial^2\omega}{\partial k^2}=-2\left(\frac{C_g^2}{\omega}+C_g\frac{H\sin kH}{\cosh kH}-\frac{\omega H\cosh^2 kH}{2\sinh 2kH}\right) \quad (3.3.23)$$

is derived from the definition of the group velocity. After grouping terms, a modified Schrödinger equation is obtained.

$$\frac{\partial A}{\partial t_2}+C_g\frac{\partial A}{\partial x_2}-\frac{\omega"}{2}i\frac{\partial^2 A}{\partial x_1^2}+ik\left(\frac{\partial\phi_{10}}{\partial x_1}-\frac{k}{2\omega\cosh^2 kH}\frac{\partial\phi_{10}}{\partial t_1}\right)A +i\frac{\omega k^2|A|^2}{16\sinh^4 kH}\left(\cosh 4kH+8-2\tanh^2 kH\right)A \underbrace{-\frac{i}{2}\beta A}_{(1)}+\underbrace{\frac{1}{2}\frac{\partial C_g}{\partial x_2}A}_{(2)}=0 \quad (3.3.24)$$

where (1) is the only term due to randomness, derived by Mei & Hancock [22] and (2) is the only term due to the slowly varying bottom, derived by Lo in his PhD thesis [19]. The equation for a perfectly smooth bottom of constant average depth (omitting terms (1) and (2)) was derived by Mei [20].

Equations (3.2.20) and (3.3.24) are combined into a single equation using the definition of the multiple scales.

$$\frac{\partial A}{\partial t_1} + C_g \frac{\partial A}{\partial x_1} + \epsilon \left[-\frac{\omega"}{2} i \frac{\partial^2 A}{\partial x_1^2} + ik \left(\frac{\partial \phi_{10}}{\partial x_1} - \frac{k}{2\omega \cosh^2 kH} \frac{\partial \phi_{10}}{\partial t_1} \right) A - \frac{i}{2} \beta A \right.$$
$$\left. + i \frac{\omega k^2 |A|^2}{16 \sinh^4 kH} \left(\cosh 4kH + 8 - 2\tanh^2 kH \right) A + \frac{1}{2} \frac{\partial C_g}{\partial x_2} A \right] = 0 \quad (3.3.25)$$

3.4 Solving for the amplitude in the case of a steady train of attenuated stokes waves

For the particular case of a steady train of attenuated Stokes waves, the dependence on time and the fast coordinate x_1 disappears, only the very fast variable x_2 remains. Equation (3.3.25) is therefore simplified into.

$$C_g \frac{dA}{dx_2} - \frac{1}{2} i\beta A + i \frac{\omega k^2 |A|^2}{16 \sinh^4 kH} \left(\cosh 4kH + 8 - 2\tanh^2 kH \right) A + \frac{1}{2} \frac{dC_g}{dx_2} A = 0 \quad (3.4.26)$$

For clarity purposes, let us define α as the following function of x_2

$$\alpha(x_2) = \frac{\omega k^2}{16 \sinh^4 kH} \left(\cosh 4kH + 8 - 2\tanh^2 kH \right) \quad (3.4.27)$$

and note that α is a real function of x_2.

A and β are complex, they can therefore be decomposed into amplitude and phase or real and imaginary parts: $A = ae^{i\theta}$ and $\beta = \beta_r + i\beta_i$. Equation (3.4.26) is then separated into real and imaginary parts.

$$C_g \frac{da}{dx_2} + \frac{1}{2} \frac{dC_g}{dx_2} a + \frac{1}{2} \beta_i a = 0 \quad (3.4.28)$$

$$C_g \frac{d\theta}{dx_2} + \alpha(x_2) a^2 - \frac{1}{2} \beta_r = 0 \quad (3.4.29)$$

This is solved for a and θ separately by noticing that

$$\frac{d\left(a\sqrt{C_g}\right)}{dx_2} = \frac{1}{\sqrt{C_g}} \left(C_g \frac{da}{dx_2} + \frac{1}{2} \frac{dC_g}{dx_2} a \right) = -\frac{1}{2} \frac{\beta_i}{C_g} a \sqrt{C_g} \quad (3.4.30)$$

The amplitude a is found to be:

$$a = a_0 \sqrt{\frac{C_{g0}}{C_g}} \exp\left[-\frac{1}{2}\int_0^{x_2} \frac{\beta_i(\ell)}{C_g(\ell)}\, d\ell\right] \tag{3.4.31}$$

where a_0 and C_{g_0} are the values of the parameters at $x = 0$.

The equation for θ can now be solved by using the result for a into (3.4.29).

$$\frac{d\theta}{dx_2} = \frac{\beta_r}{2C_g} - C_{g0} a_0^2 \frac{\alpha(x_2)}{C_g^2} \exp\left[-\int_0^{x_2} \frac{\beta_i(\ell)}{C_g(\ell)}\, d\ell\right] \tag{3.4.32}$$

This first order differential equation is integrated once.

$$\theta = \int_0^{x_2} \frac{\beta_r(m)}{2C_g(m)} dm - C_{g0} a_0^2 \int_0^{x_2} \frac{\alpha(m)}{C_g^2(m)} \exp\left[-\int_0^{m} \frac{\beta_i(\ell)}{C_g(\ell)}\, d\ell\right] dm \tag{3.4.33}$$

The final amplitude is therefore:

$$A\sqrt{C_g} = a_0 \sqrt{C_{g0}} \exp\left(-\frac{1}{2}\int_0^{x_2} \frac{\beta_i(\ell)}{C_g(\ell)}\, d\ell\right) \exp\left\{ i \left[\int_0^{x_2} \frac{\beta_r(m)}{2C_g(m)} dm \right.\right.$$
$$\left.\left. - C_{g0} a_0^2 \int_0^{x_2} \frac{\alpha(m)}{C_g^2(m)} \exp\left(-\int_0^{m} \frac{\beta_i(\ell)}{C_g(\ell)}\, d\ell\right) dm \right]\right\} \tag{3.4.34}$$

An analytical expression for the amplitude in the case of a bottom of constant average depth was derived by Mei & Hancock [22]. Let us now look at two special cases for which amplitudes have been previously derived in order to verify this equation.

3.4.1 Special case of a bottom of constant average depth

For a bottom of essentially constant depth, H is independent of the variable x_2 so the wavenumber k, the random coefficient β and the group velocity C_g are also independent of the the horizontal coordinate and the integrals can be calculated easily for constant integrands. The result had been derived by [22].

$$A = a_0 \exp\left(-\frac{\beta_i x_2}{2C_g}\right) \exp\left\{ i\frac{\beta_r}{2C_g} x_2 + i\frac{a_0^2 \alpha}{\beta_i}\left[\exp\left(-\frac{\beta_i}{C_g} x_2\right) - 1\right]\right\} \tag{3.4.35}$$

3.4.2 Special case of no randomness

For a perfectly smooth bottom, $\beta = 0$, so that only the first exponential term in (3.4.34) remains.

$$A\sqrt{C_g} = a_0 \sqrt{C_{g0}} \exp\left(-iC_{g0} a_0^2 \int_0^{x_2} \frac{\alpha(m)}{C_g^2(m)}\, dm\right) \tag{3.4.36}$$

Let us not here that by taking the amplitude, a relation is obtained, it can be interpreted as energy conservation:

$$|A|^2 C_g = a_0^2 C_{g_0} = \text{constant} \tag{3.4.37}$$

For a perfectly smooth bottom of constant depth, the result is a propagating Stokes wave, as expected.

$$A = a_0 \exp\left(-i\frac{\alpha a_0^2}{C_g} x_2\right) \tag{3.4.38}$$

Appendix A

Matlab files

A.1 cef.m

```
function w = cef(z,N)

%  Computes the function w(z) = exp(-z^2) erfc(-iz) using a rational
%  series with N terms.  It is assumed that Im(z) > 0 or Im(z) = 0.
%
%                               Andre Weideman, 1995

M = 2*N;  M2 = 2*M;  k = [-M+1:1:M-1]';     % M2 = no. of sampling points.
L = sqrt(N/sqrt(2));                         % Optimal choice of L.
theta = k*pi/M; t = L*tan(theta/2);          % Define variables theta and t.
f = exp(-t.^2).*(L^2+t.^2); f = [0; f];      % Function to be transformed.
a = real(fft(fftshift(f)))/M2;               % Coefficients of transform.
a = flipud(a(2:N+1));                        % Reorder coefficients.
Z = (L+i*z)./(L-i*z); p = polyval(a,Z);      % Polynomial evaluation.
w = 2*p./(L-i*z).^2+(1/sqrt(pi))./(L-i*z); % Evaluate w(z).


%The Matlab function cef.m computes the complex error function w(z),
%also known as the plasma dispersion or Fadeeva function. The algorithm
```

```
%is discussed in J.A.C. Weideman, "Computation of the Complex Error
% Function", SIAM J. Numer. Anal., Vol. 31, pp. 1497-1518 (1994).
```

A.2 plotbeta.m

```
function[]=plotbeta(sigma,el,m_min,m_step,m_max)
%This function plots beta for different values of the variable
% coefficients.

%INPUTS
%sigma = magnitude of the randomness
%el=correlation length l
%m_min, m_max are the minimum and maximum values of m
% m_step is the step chosen for m for the plots

m=m_min:m_step:m_max; %creating vector m
beta=(el*sqrt(2*pi).*(m*sigma).^2./8).*(1-4.*i./(sqrt(2*pi).*m.*el)...
    +cef(sqrt(2).*(m.*el),100)); %calculating beta for every value of m

figure(1) %plotting the real part of beta vs lm=lk_m
plot(el.*m,real(beta))

figure(2) %plotting the imaginary part of beta vs lm=lk_m
plot(el.*m,imag(beta))
```

A.3 problemnh.m

```
function[p]=problemnh(n,k1,nu,el,xminsig,xmaxsig,sigma,Xmax)
%solves the system for the n amplitudes a_m

%INPUTS
%n=number of harmonics
```

```
%k1=1 for all simulations
%nu is the ratio mu^2/epsilon
%el =the length scale for the randomness 'l'
%xminsig = x-coordinate at which the randomness starts
%xmaxsig= x-coordinate at which the randomness ends
%sigma = amplitude of the randomness, sigma^2 is taken into account
%Xmax = maximum X, slow coordinate

%OUTPUTS
%p is the number of points, or the number of intervals +1

%Creating vectors k, beta, A and B
k=zeros(1,n);
beta=zeros(1,n);
A=zeros(1,n);
B=zeros(1,n);

%looping to enter values in the vectors.
for m=1:n %looping over the n harmonics
    k(m)=m*k1; %defining the n wavenumbers
    A(m)=i*nu*(k(m)^3)/6;
    B(m)=-3*i*k(m)/4;
    beta(m)=-(el*sqrt(2*pi)*(k(m)*sigma)^2/8)...
        *(1-4*i/(sqrt(2*pi)*k(m)*el)+cef(sqrt(2)*(k(m)*el),100));
end

%calling the ODE solver
AT=1e-3*ones(1,n); %defining the absolute tolerance for the solver
options = odeset('RelTol',1e-3,'AbsTol',AT,'MaxStep',1e-3);

%defining the initial conditions
Initharm=zeros(1,n); %All the harmonics have a zero amplitude at X=0
```

```
Initharm(1)=1; %except for the first harmonic

%Calling the ODE solver for n harmonics
if n==6 %Calling the solver for 6 harmonics
    [X,a] = ode45(@equation6h,[0 Xmax],Initharm,options,A,B,beta,...
        xminsig,xmaxsig);
elseif n==10 % Calling the solver for 10 harmonics
    [X,a] = ode45(@equation10h,[0 Xmax],Initharm,options,A,B,beta,...
        xminsig,xmaxsig);
else  % Calling it quits
    sprintf('Simulations can only be run for n=6 or 10, sorry!')
    p=0;
    break
end

%calculating both sides of the energy relation
p=max(size(X));

%RHS is the right hand side of the energy relation,
%involving beta terms
RHS=zeros(p,1);
a2=zeros(p,n);
arhs=zeros(p,n);
for m=1:n %looping for all the harmonics
    a2(:,m)=abs(a(:,m)).^2;
    arhs(:,m)=beta(m).*((xminsig<X(:))&(X(:)<xmaxsig)).*a2(:,m);
end
RHS=2.*real(transpose(sum(transpose(arhs))));

%LHS is the left hand side of the energy relation
%involving derivatives
LHS=zeros(p,1);
```

```
Deriv=zeros(p,n);
for m=1:n
    for l=2:p-1
        Deriv(l,m)=(a2(l+1,m)-a2(l-1,m))/(X(l+1)-X(l-1));
    end
end
LHS=transpose(sum(transpose(Deriv)));
Error=LHS-RHS;

%To check Energy relation, add the plot for
figure(1)
plot(X,Error)

%plotting the six (or ten) harmonics
aplot=abs(a);
figure(2)
plot(X,aplot(:,1),'-',X,aplot(:,2),'-',X,aplot(:,3),'-'...
    ,X,aplot(:,4),'-',X,aplot(:,5),'-',X,aplot(:,6),'-')...
    %,X,aplot(:,7),'-',X,aplot(:,8),'-.',X,aplot(:,9),'-',...
    %X,aplot(:,10),'-.')
```

A.4 equation6h

```
function [da]=equation6h(x,a,A,B,beta,xminsig,xmaxsig)
%ODE solver for 6 harmonics
%called by problemnh(6,.....)

%INPUTS
%A=vector of  i nu k(m)^3/6
%B=vector of coefficients for coupled terms
%beta=vector of beta(m)
%xminsig = x-coordinate at which the randomness starts
```

```
%xmaxsig= x-coordinate at which the randomness ends

%OUTPUTS
%da = solution, in the form of six vector colums da(1) ... da(6)

%defining differential equations
da = zeros(6,1);    % a column vector
da(1) = A(1)*a(1)+beta(1)*a(1)*((xminsig<x)&(x<xmaxsig))...
    +B(1)*(conj(a(1))*a(2)+conj(a(2))*a(3)+conj(a(3))*a(4)...
    +conj(a(4))*a(5)+conj(a(5))*a(6));
da(2) = A(2)*a(2)+beta(2)*a(2)*((xminsig<x)&(x<xmaxsig))...
    +B(2)*(conj(a(1))*a(3)+conj(a(2))*a(4)+conj(a(3))*a(5)...
    +conj(a(4))*a(6)+0.5*a(1)^2);
da(3) = A(3)*a(3)+beta(3)*a(3)*((xminsig<x)&(x<xmaxsig))...
    +B(3)*(conj(a(1))*a(4)+conj(a(2))*a(5)+conj(a(3))*a(6)...
    +a(1)*a(2));
da(4) = A(4)*a(4)+beta(4)*a(4)*((xminsig<x)&(x<xmaxsig))...
    +B(4)*(conj(a(1))*a(5)+conj(a(2))*a(6)+a(1)*a(3)+0.5*a(2)^2);
da(5) = A(5)*a(5)+beta(5)*a(5)*((xminsig<x)&(x<xmaxsig))...
    +B(5)*(conj(a(1))*a(6)+a(1)*a(4)+a(2)*a(3));
da(6) = A(6)*a(6)+beta(6)*a(6)*((xminsig<x)&(x<xmaxsig))...
    +B(6)*(a(1)*a(5)+a(2)*a(4)+0.5*a(3)*a(3));
```

A.5 equation10h.m

```
function [da]=equation10h(x,a,A,B,beta,xminsig,xmaxsig)
%ODE solver for 10 harmonics
%called by problemnh(10,.....)

%INPUTS
%A=vector of  i nu k(m)^3/6
%B=vector of coefficients for coupled terms
```

```
%beta=vector of beta(m)
%xminsig = x-coordinate at which the randomness starts
%xmaxsig= x-coordinate at which the randomness ends

%OUTPUTS
%da = solution, in the form of ten vector colums da(1) ... da(10)

%defining differential equations
da = zeros(10,1);    % a column vector
da(1) = A(1)*a(1)+beta(1)*a(1)*((xminsig<x)&(x<xmaxsig))...
    +B(1)*(conj(a(1))*a(2)+conj(a(2))*a(3)+conj(a(3))*a(4)...
    +conj(a(4))*a(5)+conj(a(5))*a(6)+conj(a(6))*a(7)+conj(a(7))*a(8)...
    +conj(a(8))*a(9)+conj(a(9))*a(10));
da(2) = A(2)*a(2)+beta(2)*a(2)*((xminsig<x)&(x<xmaxsig))...
    +B(2)*(conj(a(1))*a(3)+conj(a(2))*a(4)+conj(a(3))*a(5)...
    +conj(a(4))*a(6)+conj(a(5))*a(7)+conj(a(6))*a(8)+conj(a(7))*a(9)...
    +conj(a(8))*a(10)+0.5*a(1)^2);
da(3) = A(3)*a(3)+beta(3)*a(3)*((xminsig<x)&(x<xmaxsig))...
    +B(3)*(conj(a(1))*a(4)+conj(a(2))*a(5)+conj(a(3))*a(6)...
    +conj(a(4))*a(7)+conj(a(5))*a(8)+conj(a(6))*a(9)+conj(a(7))*a(10)...
    +a(1)*a(2));
da(4) = A(4)*a(4)+beta(4)*a(4)*((xminsig<x)&(x<xmaxsig))...
    +B(4)*(conj(a(1))*a(5)+conj(a(2))*a(6)+conj(a(3))*a(7)...
    +conj(a(4))*a(8)+conj(a(5))*a(9)+conj(a(6))*a(10)+a(1)*a(3)...
    +0.5*a(2)^2);
da(5) = A(5)*a(5)+beta(5)*a(5)*((xminsig<x)&(x<xmaxsig))...
    +B(5)*(conj(a(1))*a(6)+conj(a(2))*a(7)+conj(a(3))*a(8)...
    +conj(a(4))*a(9)+conj(a(5))*a(10)+a(1)*a(4)+a(2)*a(3));
da(6) = A(6)*a(6)+beta(6)*a(6)*((xminsig<x)&(x<xmaxsig))...
    +B(6)*(conj(a(1))*a(7)+conj(a(2))*a(8)+conj(a(3))*a(9)...
    +conj(a(4))*a(10)+a(1)*a(5)+a(2)*a(4)+0.5*a(3)*a(3));
da(7) = A(7)*a(7)+beta(7)*a(7)*((xminsig<x)&(x<xmaxsig))...
```

```
    +B(7)*(conj(a(1))*a(8)+conj(a(2))*a(9)+conj(a(3))*a(10)...
    +a(1)*a(6)+a(2)*a(5)+a(3)*a(4));
da(8) = A(8)*a(8)+beta(8)*a(8)*((xminsig<x)&(x<xmaxsig))...
    +B(8)*(conj(a(1))*a(9)+conj(a(2))*a(10)+a(1)*a(7)+a(2)*a(6)...
    +a(3)*a(5)+0.5*a(4)*a(4));
da(9) = A(9)*a(9)+beta(9)*a(9)*((xminsig<x)&(x<xmaxsig))...
    +B(9)*(conj(a(1))*a(10)+a(1)*a(8)+a(2)*a(7)+a(3)*a(6)+a(4)*a(5));
da(10) = A(10)*a(10)+beta(10)*a(10)*((xminsig<x)&(x<xmaxsig))...
    +B(10)*(a(1)*a(9)+a(2)*a(8)+a(3)*a(7)+a(4)*a(6)+0.5*a(5)*a(5));
```

Appendix B

Residue theorem to evaluate Green's function at the bottom

Green's identity is used determine Green's function as

$$G\left(|x-x'|, z\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha|x-x'|}\frac{\alpha\cosh\alpha z + \frac{\omega^2}{g}\sinh\alpha z}{\alpha\left(\frac{\omega^2}{g}\cosh\alpha H - \alpha\sinh\alpha H\right)}\, d\alpha \tag{B.0.1}$$

At depth $z=-H$, G becomes:

$$G\left(|x-x'|, -H\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha|x-x'|}\frac{\alpha\cosh\alpha H - \frac{\omega^2}{g}\sinh\alpha H}{\alpha\left(\frac{\omega^2}{g}\cosh\alpha H - \alpha\sinh\alpha H\right)}\, d\alpha \tag{B.0.2}$$

The denominator has roots as $\alpha = 0$, $\alpha = \pm k$ and $\alpha = \pm ik_n$, where the k_n are the solutions to the complex dispersion relation $\omega^2 = -gk_n \tanh kH$. The residue theorem is therefore used to evaluate the integral.

The integral of a function with no singularities over a closed contour is zero, if there are singularities included within that closed contour, it is necessary to calculate the residue at each singularity. The integral is then equal to the sum of the residues. The integral is between $-\infty$ and ∞, the contour must first be 'closed' before performing the integration. There are many possibilities to close the contour requiring an 'imaginary link' between $-\infty$ and ∞, such as an upper (or lower) semi-circle at infinity.

In the complex plane where the x-axis is real and the y-axis is imaginary, all the singularities are located. There are three on the real axis: $-k$, 0 and $+k$, and an infinite number on the imaginary axis: $\cdots, -ik_n, -ik_{n-1}, \cdots, -ik_1, ik_1, \cdots, ik_{n-1}, ik_n \cdots$. The contour is closed from $-\infty$ to ∞ with the upper semi-circle because the positive complex roots are necessary for the solutions to vanish at large values of x. The wave is an outgoing wave, so that only $+k$ is included in the semi-circle.

B.1 Singularity around $\alpha = 0$

For clarity purposes, the following function is defined

$$f(\alpha) = e^{i\alpha|x-x'|} \frac{\alpha \cosh \alpha H - \frac{\omega^2}{g} \sinh \alpha H}{\frac{\omega^2}{g} \cosh \alpha H - \alpha \sinh \alpha H} \quad \text{and } g(\alpha) = \alpha \tag{B.1.3}$$

where

$$f(0) = 0 \quad \text{and } \frac{dg}{d\alpha} = 1 = \left.\frac{dg}{d\alpha}\right|_{\alpha=0} \tag{B.1.4}$$

can be evaluated. The residue theorem gives a residue of zero at $\alpha = 0$.

B.2 Residue around $\alpha = k$

f and g are defined differently here as the singularity is no longer at $\alpha = 0$, but at $\alpha = k$.

$$f(\alpha) = e^{i\alpha|x-x'|} \left(\cosh \alpha H - \frac{\omega^2}{g\alpha} \sinh \alpha H\right) \quad \text{and } g(\alpha) = \frac{\omega^2}{g} \cosh \alpha H - \alpha \sinh \alpha H \tag{B.2.5}$$

It is possible to calculate.

$$f(k) = \frac{\omega^2}{gk \sinh kH} e^{ik|x-x'|} \quad \text{and } \frac{dg}{d\alpha} = \frac{\omega^2 H}{g} \sinh \alpha H - \sinh \alpha H - \alpha H \cosh \alpha H \tag{B.2.6}$$

The dispersion relation can be used to evaluate the derivative of g at $\alpha = k$.

$$\left.\frac{dg}{d\alpha}\right|_{\alpha=k} = -\frac{1}{\sinh kH} \left(\frac{\omega^2 H}{g} + \sinh^2 kH\right) \tag{B.2.7}$$

The residue theorem yields a residue for $\alpha = k$

$$\text{Residue}_{\alpha=k} = \frac{-i\frac{\omega^2}{gk} e^{ik|x-x'|}}{\frac{\omega^2 H}{g} + \sinh^2 kH} \tag{B.2.8}$$

B.3 Residue around $\alpha = ik_n$

f and g are the same as for $\alpha = k$, but they are evaluated at $\alpha = ik_n$. Let us note that $\cosh ix = \cos x$ and $\sinh ix = i \sin x$.

$$f(ik_n) = e^{-k_n|x-x'|}\left(\cos k_n H - \frac{\omega^2}{gk_n}\sin k_n H\right) = \frac{\omega^2}{gk_n \sin k_n H}e^{-k_n|x-x'|} \tag{B.3.9}$$

obtained using the complex dispersion relation. And

$$g'(ik_n) = \frac{i}{\sin k_n H}\left[\frac{\omega^2 H}{g}\left(\cos^2 k_n H + \sin^2 k_n H\right) - \sin^2 k_n H\right] \tag{B.3.10}$$

After simplification, the residue theorem yields.

$$\text{Residue}_{\ \alpha=ik_n} = \frac{\frac{\omega^2}{gk_n}e^{-k_n|x-x'|}}{\sin^2 kH - \frac{\omega^2 H}{g}+} \tag{B.3.11}$$

B.4 Final result for $G(|x-x'|, -H)$

Adding all the residues together, and summing over the positive integer values of n, the result for $G(|x-x'|, -H)$ is

$$G\left(|x-x'|, -H\right) = -\frac{i\omega^2}{gk}\frac{e^{ik|x-x'|}}{\frac{\omega^2 H}{g} + \sinh^2 kH} - \sum_{n=1}^{\infty}\frac{\omega^2}{gk_n}\frac{e^{-k_n|x-x'|}}{\frac{\omega^2 H}{g} + \sinh^2 k_n H} \tag{B.4.12}$$

Appendix C

Details of $\langle G_3 \rangle$

$\langle G_3 \rangle$ had been defined as

$$\begin{aligned}\langle G_3 \rangle &= -2\langle\phi_2\rangle_{tt_1} - 2\phi_{1tt_2} - \phi_{1t_1t_1} - \zeta_1 \left(\langle\phi_2\rangle_{tt} + g\langle\phi_2\rangle_z\right)_z \\ &\quad -\langle\zeta_2\rangle \left(\phi_{1tt} + g\phi_{1z}\right)_z - \frac{\zeta_1^2}{2}\left(\phi_{1tt} + g\phi_{1z}\right)_{zz} \\ &\quad -2\left(\phi_{1x}\phi_{1x_1} - \phi_{1x}\phi_{2x} - \phi_{1z}\phi_{2z}\right)_t - \left(\phi_{1x}^2 + \phi_{1z}^2\right)_{t_1} \\ &\quad -\zeta_1\left(\phi_{1x}^2 + \phi_{1z}^2\right)_{tz} - \frac{1}{2}\left(\phi_{1x}\frac{\partial}{\partial x} + \phi_{1z}\frac{\partial}{\partial z}\right)\left(\phi_{1x}^2 + \phi_{1z}^2\right) \qquad \text{(C.0.1)}\end{aligned}$$

Let us calculate each term separately for more clarity.

$$\begin{aligned}-2\langle\phi_2\rangle_{tt_1}\big|_{z=0} &= -\frac{\omega^2 H}{k}\left(i\frac{\partial^2 A}{\partial x_1 \partial t_1}e^{iS-i\omega t} + *\right) \\ &\qquad + \frac{3}{2}\omega^2\frac{\cosh 2kH}{\sinh^4 kH}\left(A\frac{\partial A}{\partial t_1}e^{2iS-2i\omega t} + *\right) \qquad \text{(C.0.2)}\end{aligned}$$

$$-2\phi_{1tt_2}\big|_{z=0} = g\left(\frac{\partial A}{\partial t_2}e^{iS-i\omega t} + *\right) \qquad \text{(C.0.3)}$$

$$-\phi_{1t_1t_1}\big|_{z=0} = -\phi_{10t_1^2} + \frac{g}{2\omega}\left(i\frac{\partial^2 A}{\partial t_1^2}e^{iS-i\omega t} + *\right) \qquad \text{(C.0.4)}$$

$$\begin{aligned}&-\zeta_1\left(\langle\phi_2\rangle_{tt} + g\langle\phi_2\rangle_z\right)_z\big|_{z=0} \\ &= \frac{\omega^3}{4k}\frac{1+\cosh^2 kH}{\sinh^2 kH}\left(A\frac{\partial A}{\partial x_1}e^{2iS-2i\omega t} + * + \frac{\partial AA^*}{\partial x_1}\right) \\ &\quad + \frac{3}{8}\frac{\omega^3 k\cosh kH}{\sinh^5 kH}\left(1 - 2\sinh^2 kH\right)\left(iA^3 e^{3iS-3i\omega t} + iA^2A^* e^{iS-i\omega t} + *\right) \qquad \text{(C.0.5)}\end{aligned}$$

$$
\begin{aligned}
&-\langle\zeta_2\rangle\left(\phi_{1tt}+g\phi_{1z}\right)_z\big|_{z=0} \\
&= \; -\frac{\omega k}{\sinh 2kH}\frac{\partial\phi_{10}}{\partial t_1}\left(iAe^{iS-i\omega t}+*\right) \\
&\quad -\frac{\omega^3 k|A|^2\cosh kH}{16\sinh^5 kH}\left(2\tanh^2 kH+1+2\cosh^2 kH\right)\left(iAe^{iS-i\omega t}+*\right) \\
&\quad +\frac{\omega^3 H}{2\sinh 2kH}\left(A\frac{\partial A}{\partial x_1}e^{2iS-2i\omega t}+*-\frac{\partial AA^*}{\partial x_1}\right) \\
&\quad +\frac{\omega^2}{4\sinh^2 kH}\left(-A\frac{\partial A}{\partial t_1}e^{2iS-2i\omega t}+*+\frac{\partial AA^*}{\partial x_1}\right) \\
&\quad +\frac{\omega^3 k}{16}\frac{\cosh kH\left(1+2\cosh^2 kH\right)}{\sinh^5 kH}\left(iA^3e^{3iS-3i\omega t}+*\right) \qquad \text{(C.0.6)}
\end{aligned}
$$

$$
-\frac{\zeta_1^2}{2}\left(\phi_{1tt}+g\phi_{1z}\right)_{zz}\Big|_{z=0}=0 \qquad \text{(C.0.7)}
$$

$$
\begin{aligned}
&-2\left(\phi_{1x}\phi_{1x_1}-\phi_{1x}\phi_{2x}-\phi_{1z}\phi_{2z}\right)_t\big|_{z=0} \\
&= \frac{\omega^2}{\tanh kH}\frac{\partial\phi_{10}}{\partial x_1}\left(iAe^{iS-i\omega t}+*\right)-\frac{2\omega^3 H}{\tanh kH}\left(A\frac{\partial A}{\partial x_1}e^{2iS-2i\omega t}+*\right) \\
&\quad +\frac{3\omega^3 k\cosh kH}{8\sinh^5 kH}\left(\frac{\sinh^4 kH}{\cosh^2 kH}+\cosh 2kH\right) \\
&\qquad \left(3iA^3e^{3iS-3i\omega t}+i|A|^2Ae^{iS-i\omega t}+*\right) \qquad \text{(C.0.8)}
\end{aligned}
$$

$$
\begin{aligned}
-\left(\phi_{1x}^2+\phi_{1z}^2\right)_{t_1}\Big|_{z=0} &= -\frac{\omega^2}{2}\left(\frac{1}{\tanh^2 kH}+1\right)\frac{\partial AA^*}{\partial t_1} \\
&\quad -\frac{\omega^2}{2\cosh^2 kH}\left(A\frac{\partial A}{\partial t_1}e^{2iS-2i\omega t}+*\right) \qquad \text{(C.0.9)}
\end{aligned}
$$

$$
-\zeta_1\left(\phi_{1x}^2+\phi_{1z}^2\right)_{tz}\Big|_{z=0}=0 \qquad \text{(C.0.10)}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\phi_{1x}\frac{\partial}{\partial x}+\phi_{1z}\frac{\partial}{\partial z}\right)\left(\phi_{1x}^2+\phi_{1z}^2\right)\Big|_{z=0} \\
&= \; -\frac{\omega^3 k}{8}\frac{\cosh kH}{\sin^3 kH}\left(iA^3e^{3iS-3i\omega t}+*\right) \\
&\quad +\frac{\omega^3 k}{8}|A|^2\frac{\cosh kH}{\sin^3 kH}\left(4\sinh^2 kH-1\right)\left(iAe^{iS-i\omega t}+*\right) \qquad \text{(C.0.11)}
\end{aligned}
$$

All the above terms can now be added together to obtain:

$$
\langle G_3\rangle \;=\; -\frac{\partial^2\phi_{10}}{\partial t_1^2}+\frac{\omega^3}{2k}\frac{\cosh^2 kH}{\sinh^2 kH}\frac{\partial AA^*}{\partial x_1}-\frac{\omega^2}{4\sinh^2 kH}\frac{\partial AA^*}{\partial t_1}
$$

$$
\begin{aligned}
&+\left(\frac{g}{\omega}C_g^2+\frac{\omega^2 H}{k}C_g\right)\left(i\frac{\partial^2 A}{\partial x_1^2}e^{iS-i\omega t}+*\right)+g\left(\frac{\partial A}{\partial t_2}e^{iS-i\omega t}+*\right)\\
&+gk\left(\frac{\partial \phi_{10}}{\partial x_1}-\frac{k}{2\omega\cosh^2 kH}\frac{\partial \phi_{10}}{\partial t_1}\right)\left(iAe^{iS-i\omega t}+*\right)\\
&+\frac{\omega^3 k}{16}\frac{\cosh kH}{\sinh^5 kH}\left(\cosh 4kH+8-2\tanh^2 kH\right)|A|^2\left(iAe^{iS-i\omega t}+*\right)\\
&+\langle G_{32}\rangle+\langle G_{33}\rangle \qquad \text{(C.0.12)}
\end{aligned}
$$

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