High-Reliability Architectures for Networks under Stress

by

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Abstract

In this thesis, we develop a methodology for architecting high-reliability communication networks. Previous results in the network reliability field are mostly theoretical in nature with little immediate applicability to the design of real networks. We bring together these contributions and develop new results and insights which are of value in designing networks that meet prescribed levels of reliability. Furthermore, most existing results assume that component failures are statistically independent in nature. We take initial steps in developing a methodology for the design of networks with statistically dependent link failures. We also study the architectures of networks under extreme stress.

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Chapter 1

Introduction

Network reliability — the notion of connectedness of network nodes in the face of component failures — is an important consideration in network design for obvious reasons. Network reliability has become an especially important issue in optical networking, as optical networks are increasingly being used to sense and control vehicular functions, such as aircraft flight and engine control.

The network reliability synthesis problem is the design of a network which achieves a prescribed level of "reliability", while minimizing the number of components used. While the concept of network reliability is somewhat vague and therefore subject to interpretation, all reliability metrics are strongly influenced by factors such as topology and component quality. In this thesis, we therefore examine the impact of topology on network reliability under different regimes of component vulnerability.

1.1 Problem motivation

The importance of reliability in network design has been recognized for some time. Network reliability as a research area began in 1956 with Moore and Shannon's seminal paper [54], which addressed the design of arbitrarily reliable circuits from unreliable relay components. Since then, numerous researchers have studied network reliability in various forms, often using graph theory as a framework for modelling and analysis.

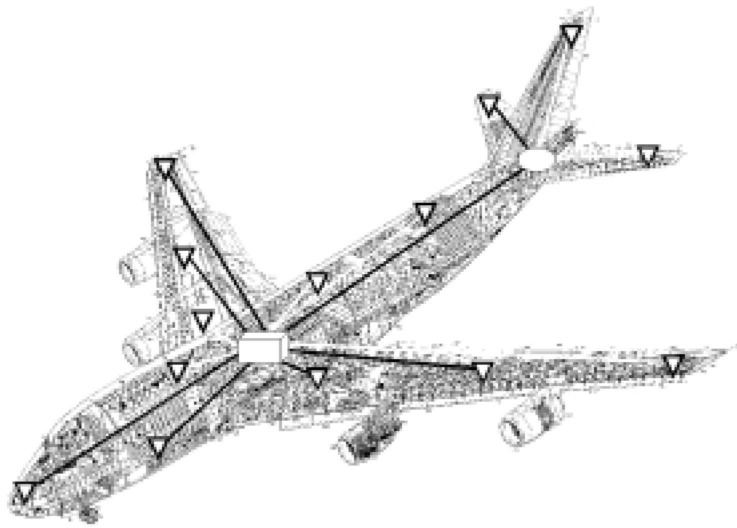


Figure 1-1: Local-area network responsible for transporting control signals in an aircraft.

A large portion of previous contributions to the research area of network reliability are of theoretical nature with little immediate applicability to the design of real communication networks. Studies in the field have focused on different degrees of component vulnerability and have employed various reliability metrics with little "real-world" justification. In addition, existing results in the field are generally fragmented and a cohesive methodology for planning a network, based on different reliability metrics, has yet to emerge. This work attempts to bridge the gap between theory and practice by providing design guidelines, substantiated by analytical results and simulations, which are of immediate value in the planning of networks. The results presented in this work are applicable to local-area network (LAN) and metropolitan-area network (MAN) design – especially optical versions of these networks – where communication link costs are inexpensive enough to permit a rich set of possible topologies to be imposed on a set of network nodes. The model we will be using in this work, where links are vulnerable and nodes are invulnerable, is particularly relevant to optical networks where the electronics in nodes are significantly more reliable than the optics in communication links, and lightpaths between sources and destinations are routed without any intervening electronics.

Most reliability studies to date have focused on the analysis and design of networks when links are very reliable. However, the design of networks when links are unreliable, which is addressed in this paper, should not be overlooked for several reasons. In situations where the probability that a network is connected is quite small, some degree of connectedness in the network could still allow for important functions to be carried out, such as relaying emergency signals in times of distress. Another reason is that even small probabilities of connectedness could allow for acceptable expected times to failure for emergency functions or procedures to be carried out should a network come under stress. The key is multiple connectedness used in some form of path diversity, the exact architecture of which can take a number of forms.

Modelling assumptions in the vast majority of previous work reflect only a small fraction of real network failure scenarios. Specifically, existing work in the field deals almost exclusively with component failures which are statistically independent and small enough so that the probability of more than a single failure is small. These assumptions, while sometimes permitting tractable analysis, are not always accurate depictions of real networks. They are appropriate when modelling benign component failures, such as those due to normal wear of components. However, they are clearly inappropriate in situations where, for example, LANs found in automobiles and aircrafts are subjected to environmental stresses that cause localized, correlated failures; or when network components share a common piece of equipment. This thesis thus explores more realistic, and more complex, reliability models which permit statistical dependency among component failures. While the results obtained for such models are preliminary, they do develop intuition for the critical factors in reliable network design, and represent a first step towards the formulation of a general design methodology for networks.

A limitation that this work shares with the majority of existing work in the field is that only link failures are considered. Link failures in our model, however, can include any failures at the node (e.g. line card failures) which result in the failure of a single link. Analyses of networks which are prone to node failures have been attempted by several researchers and have proven to be more difficult. This work addresses link failures with low or high probability of occurrence through closed-form analyses. The intermediate region of failure probability is less conducive to analysis and algorithmic approaches, references to which are provided in §2.3.3, may be employed therein.

Recently, network reliability metrics have been broadened to include some measure of performance, such as throughput or delay, since for many networks a more meaningful measure than connectedness is the degree to which network performance is degraded [18]. Connectedness measures, however, remain useful in situations where network performance is considered satisfactory as long as the network remains connected, or when the network's ability to provide a minimal level of service is of interest. In addition, connectedness is the relevant metric in many military applications, where capacities of network elements are over-designed, such that connectedness of nodes ensures acceptable network performance.

1.2 Network reliability in practice

In this section, we touch upon the most common practices in network reliability. The design of mechanisms and protocols to restore networks in the event of component failure is an issue which is intimately coupled with the physical topology of a network. We, therefore, conduct a brief survey of the network topologies and techniques used in practice to protect networks.

The mechanisms used to restore networks are generally based on $1+1$ or $1:1$ (or more generally, $1:n$) point-to-point link protection schemes. In $1+1$ protection, data from the source node is sent simultaneously on two communication links. The destination node then chooses either of the links for reception. If the chosen link fails at some point, then the destination node simply switches over to the second link. In $1:1$ protection, each communication link is still backed up by an additional link, but at any given time, data is transmitted on only one of the links. In the more general $1:n$ protection scheme, n primary communication links are protected by 1 backup link. This scheme clearly has the potential to be more capacity efficient than both $1+1$ and $1:1$ protection schemes.

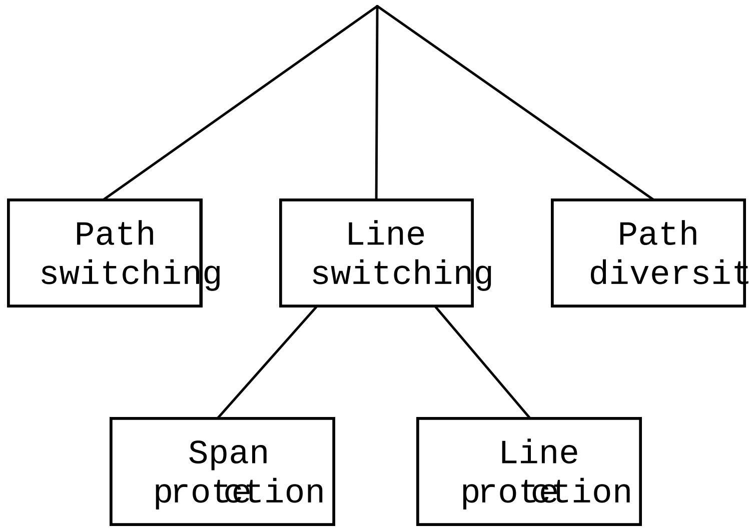


Figure 1-2: Relationship among reliability schemes.

In networks, as opposed to point-to-point links, protection is commonly achieved through path switching or line switching (see Figure 1-3). In *path switching,* restored data is routed along a path which is disjoint from the original path, along which a link or node has failed. In *line switching,* restoration only occurs locally at the failed link. Line switching is achieved either through span protection or line protection. In *span protection,* data on a failed link is switched to a second link existing between the same two end-nodes. *Line protection,* on the other hand, involves rerouting data through the network between the end-nodes of the failed link. See Figure 1-3 for an illustration of these schemes. For high-reliability applications, path diversity can be used. Multiple disjoint paths are used to carry information, either coded or uncoded. These multiple paths provide an additional reliability over that of a single path. The relationship among the above reliability schemes is illustrated in Figure 1-2.

In practice, ring-based architectures are popular choices for carrier and enterprise applications mainly because of the simplicity of the underlying ring topology. For example, most carrier networks comprise SONET/SDH[1] self-healing fiber rings. Among these ring-based architectures, the most common are: two-fiber unidirectional

[1]SONET (Synchronous Optical Network) is the current transmission and multiplexing standard for high-speed signals within the carrier infrastructure in North America. A closely related standard, SDH (Synchronous Digital Hierarchy), has been adopted in Europe and Japan [65].

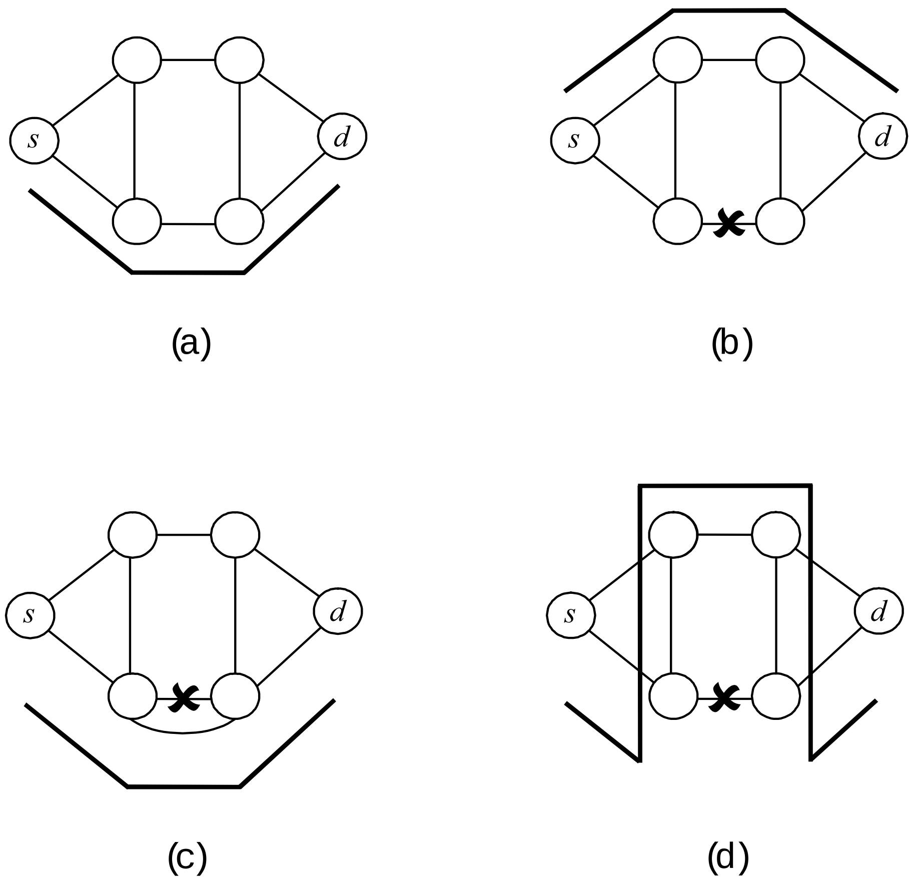


Figure 1-3: Path and line switching in a mesh network. (a) Normal operation. (b) Path-switched restoration after a link failure. (c) Span protection, a form of line switching. (d) Line protection, another form of line switching [65].

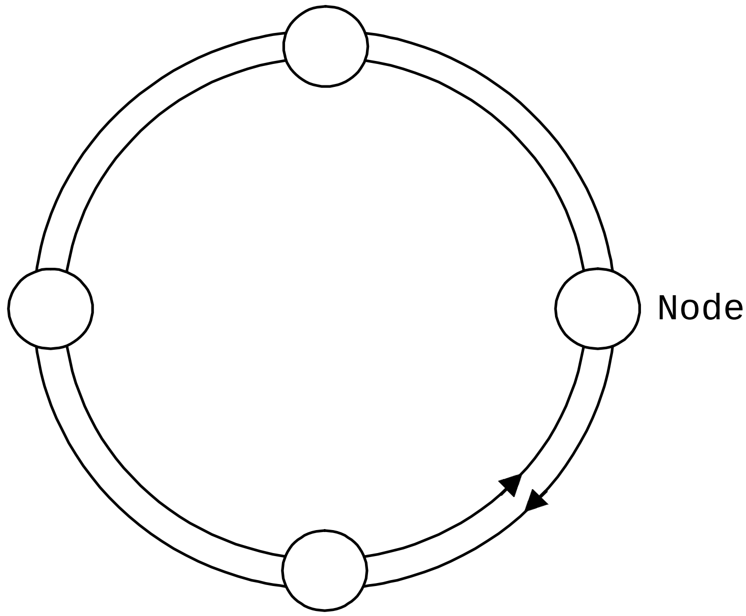


Figure 1-4: Unidirectional path-switched ring (UPSR) [65].

path-switched rings (UPSR), four-fiber bidirectional line-switched rings (BLSR/4), and two-fiber bidirectional line-switched rings (BLSR/2). Sample UPSR and BLSR topologies are illustrated in Figures 1-4 and 1-5, respectively. In UPSRs, data is sent from a source to a destination node contra-directionally on both fibers, although reception occurs from only one fiber, in a manner akin to $1 + 1$ protection. On the other hand, the restoration schemes employed by BLSRs are analogous to $1 + 1$ protection. In BLSR architectures, data is normally exchanged between two nodes contra-directionally along the *shortest path* on one fiber pair. The BLSR/4 architecture consists of two primary contra-directional fibers and two secondary contra-directional fibers. In the event of a fiber failure in a BLSR/4, restoration is effected either through span protection, or if the span protecting fiber is also cut, through line protection. The BLSR/2 architecture can be thought of as a BLSR/4 architecture with the secondary fiber pair embedded in the primary fiber, as half the capacity on each of the fibers is allocated to protection purposes. In the event of a fiber failure in a BLSR/2 network, only line protection can be employed. A reliability analysis is carried out for BLSR networks in §4.2. It should be noted that for both the UPSR

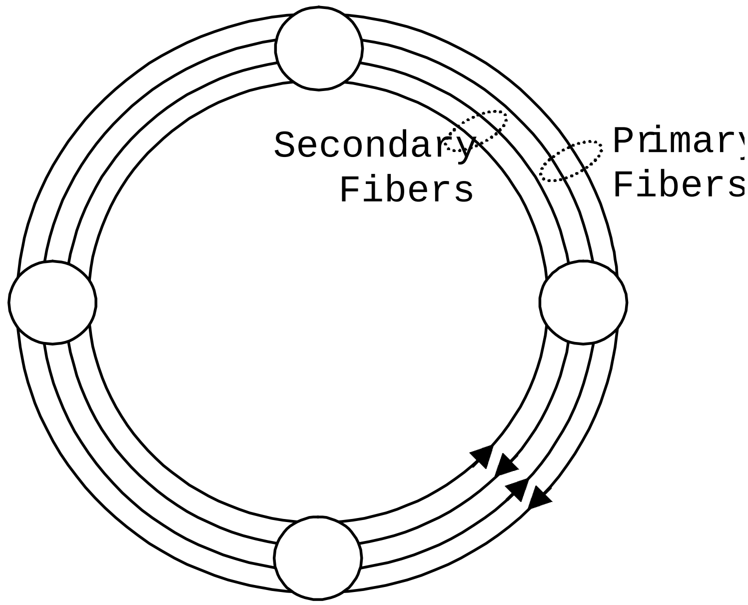


Figure 1-5: Four-fiber bidirectional line-switched ring (BLSR/4) [65].

and BLSR architectures, at least 100% redundant capacity is required. Furthermore, in such architectures, the primary fibers are typically not fully utilized, yielding an overall installed-working capacity ratio as high as 200% to 300%. For an excellent detailed discussion of protection in ring-based networks, see [91].

In mesh networks, protection is generally achieved by overlaying rings or cycles atop the original network. One particular approach involves covering the network nodes with cycles such that primary or important routes are backed up. Apart from having unprotected traffic in the network, this scheme suffers from potentially large cycles, which result in reduced wavelength-assignment efficiency and excessive jitter [53]. In addition, the reach of most practical optical networks, and hence the size of their largest cycle, is limited by signal regeneration requirements. If all links are to carry protected traffic, then the mesh network can be covered in such a way that each link is part of at least one cycle. However, in such schemes, network management complications arise at links which are part of more than one cycle [53]. A solution to these network management issues is to cover the network so that every link is covered by exactly two rings, each with two fibers [22, 23]. The problem with this scheme,

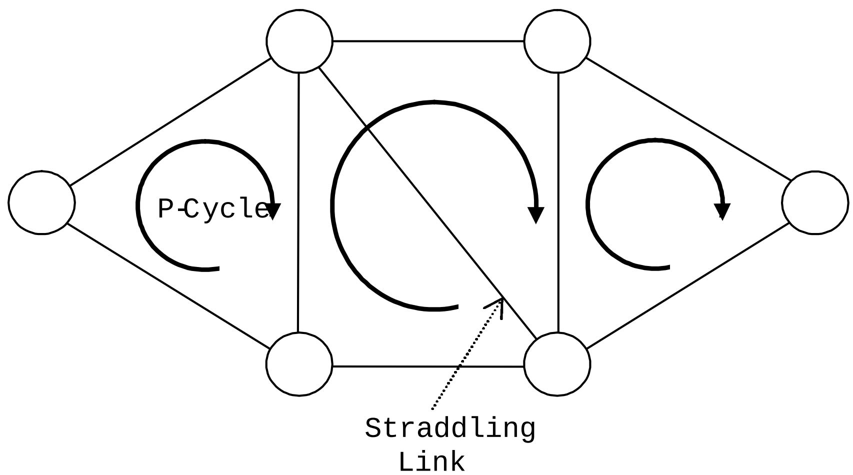


Figure 1-6: Protection cycles (p-cycles) in a mesh network.

known as *double cycle covers*, is that the links and nodes used to recover traffic on a given link depend on the direction of the traffic, leading to asymmetric restoration times for bidirectional connections[2].

More recently, protection cycles, or p-cycles, have been proposed as a method to carry out protection in mesh networks. The *p-cycle* method involves the formation of cycles in the spare capacity of a mesh restorable network prior to any failure event. Each node in the network is required to lie on at least one such cycle (see Figure 1-6). Hence, the p-cycles form two disjoint paths between any two nodes, and the network is thus survivable to a single link failure. The p-cycle scheme is capacity efficient because of the method's ability to support protection of *straddling* link failures — failures of links with end-nodes on a p-cycle, without the links themselves being part of a cycle. In the limiting case where all possible straddling links exist in an n node p-cycle, $n(n-2)$ links can be protected, which is $n-2$ times more efficient than the ring architecture. Still, the p-cycle method suffers from the same network management issues as other schemes at the junction of multiple cycles. See [31, 32, 76] for a more detailed discussion of p-cycles. Yet another scheme is *generalized loop-back*, a mesh

[2]In general, restoration times are a function of propagation and switching delays. Even when restoration paths are of comparable lengths, leading to similar propagation delays, switching times can vary for different restoration paths, owing to different switching technologies and slot synchronization issues.

restoration method where two directed graphs embedded in the network topology, the primary and the secondary, are used to carry protected traffic [51, 52]. If a failure occurs along the primary graph then traffic is looped-back on the secondary in a manner similar to BLSRs. Unlike the covering schemes described above, links corresponding to the secondary graph may carry additional traffic, yielding better capacity efficiency.

Lightpath diversity is a recently proposed reliability scheme which is relevant to high-reliability optical network applications [85]. In lightpath diversity, a power-limited optical transmitter splits its transmitted data along multiple disjoint optical paths. The signals from the multiple paths are then recombined at the receiver and decoded. Wen and Chan have determined in [85] the number of disjoint lightpaths which minimizes the probability of decoding error, and have shown that in the limit of high SNR, the probability of decoding error is equal to the probability that the source and destination nodes become disconnected – a figure of merit addressed in this work.

1.3 Thesis outline

Most of the necessary background, including definitions, notation and relevant work completed in the field, are covered in the following chapter. Additional background will be provided throughout the work when necessary. In Chapter 3, the design of reliable networks is addressed when statistically independent link failures are assumed. In Chapter 4, the techniques of Chapter 3 are specialized in a series of case studies of special network topologies. Chapter 5 is a generalization of Chapter 3, where statistically dependent link failures are treated. Finally, Chapter 6 highlights the contributions of this work and outlines further areas for research.

Chapter 2

Graph theory and network reliability background

As discussed in the introductory chapter, graph theory is generally used as a framework for modelling and analysis in network reliability studies. By exploiting the richness of graph theory, previous researchers have identified a myriad of metrics to define the vague notion of network reliability. These criteria can be broadly categorized as either deterministic or probabilistic reliability metrics.

This chapter introduces graph theoretic concepts which are fundamental to an understanding of network reliability, and surveys the relevant existing work in the field. In the next section, basic graph theoretic definitions and notation are presented. Deterministic reliability metrics are then surveyed in §2.2, including a discussion of Harary graphs, circulants, Moore graphs and cages. Probabilistic reliability metrics are next addressed in §2.3. This section also includes a brief discussion on the complexity of computing probabilistic reliability metrics, and mentions techniques used to determine or bound these metrics. We conclude the chapter in §2.4.

2.1 Definitions and notation

In this work, networks will be modelled as undirected graphs. An *undirected graph* G is an ordered pair of sets (N, E), where the elements of N are nodes and the elements of E are edges. Edges in a graph will correspond to links in a network, and nodes in a graph will correspond to nodes in a network. An *edge* is an unordered pair of distinct nodes. The sizes of sets N and E are denoted by n and e, respectively. Two nodes are *adjacent* if they are elements of an edge. An edge is *incident* at its end nodes. The *incidence matrix* $\mathbf{A}$ of an undirected graph is the $n \times e$ matrix (each row corresponds to a node and each column to an edge) with the (i, j)th entry defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if edge } j \text{ is incident at node } i, \\ 0, & \text{otherwise.} \end{cases}$$

The size of the set of edges incident at node i is its *degree* and is denoted by d_i. The smallest degree of all nodes in a graph is denoted by δ, and the largest degree is denoted by Δ. If $\Delta = \delta$ then the graph is *regular* of degree Δ. If a graph is not regular but $\delta = \lfloor 2e/n \rfloor$, then the graph is *almost-regular*. Graph $G' = (N', E')$ is a *subgraph* of $G = (N, E)$ if $N' \subseteq N$ and $E' \subseteq E$ and if the endpoints of all edges in E' lie in N'.

A *path* is a sequence of distinct nodes such that consecutive nodes share an edge. Any two paths are *edge-disjoint* if they have no edges in common and *node-disjoint* if they have no nodes in common apart from the end nodes. The maximum length of any shortest path between any two nodes in a graph G is the *diameter* $k(G)$ of the graph. If we modify the definition of a path such that the first and last nodes in the sequence are identical, then we have the definition of a *cycle*. The minimum length of any cycle in a graph G is the *girth* $g(G)$ of the graph. For any regular graph, $g \leq 2k + 1$. See Figure 2-1 for an illustration of the above definitions.

Two distinct nodes are *connected* if there exists a path between the nodes. An undirected graph is *connected* if there exists a path between every pair of distinct nodes. A (minimal) set of edges in a graph whose removal disconnects the graph is a *(prime) edge cutset*. A (minimal) set of nodes which has the same property is a *(prime) node cutset*. The minimum cardinality of an edge cutset is the *edge connectivity* or *cohesion* $\lambda(G)$. The minimum cardinality of a node cutset is the *node connectivity* or *connectivity* $\chi(G)$. See Figure 2-2 for an illustration of the above

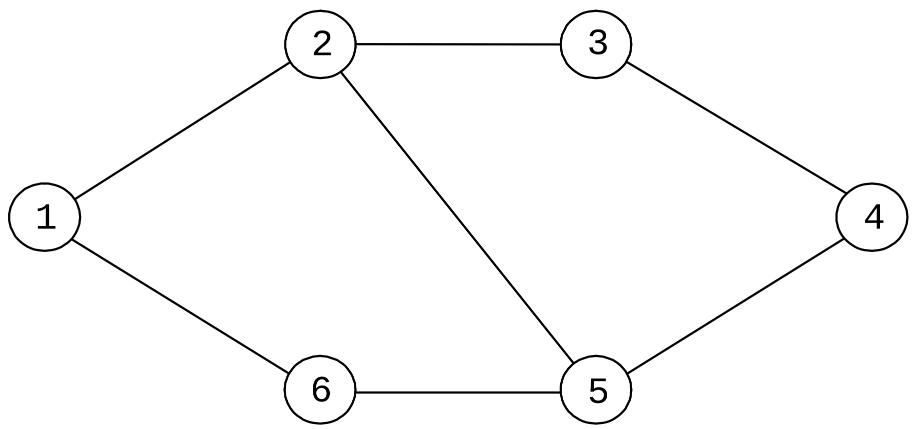


Figure 2-1: Basic graph theoretic definitions. The two sequences of edges $1, 2, 3, 4$ and $1, 6, 5, 4$ represent paths which are both edge- and node-disjoint. The diameter of the graph is three, as the shortest path between the most distant nodes 1 and 4 (or 3 and 6) is length three. The sequence of edges $1, 2, 3, 4, 5, 6, 1$ forms a cycle; in fact, it is what is known as a *Hamilton cycle*, since it includes all possible nodes of the graph. The girth of the graph is four, since cycle $1, 2, 5, 6, 1$ is a minimum length cycle of length four.

definitions.

An undirected graph G is a *tree* if it is connected and has no cycles. Another property of a tree is that it has $n - 1$ edges. Given a connected, undirected graph $G = (N, E)$, let E' be a subset of E such that $T = (N, E')$ is a tree. T is called a *spanning tree* of G. We denote the number of spanning trees in G by $t(G)$. Clearly, the deletion of any edge in a tree results in the disconnection of the tree. As a result, prime edge cutsets of the graph G can be formed from one of the $n - 1$ edges of a spanning tree of G and some of the edges outside this spanning tree. These prime cutsets can be represented as binary vectors, and can be shown to form a space of dimension $n - 1$ over the two element field $\{0, 1\}$ in which $1 + 1 = 0$.

The n-node graph which has all of its nodes adjacent is the *complete graph* K_n. A *bipartite graph* is a graph G whose node set is partitioned into two subsets N_1 and N_2, such that all edges of G join N_1 and N_2. If a bipartite graph contains every possible edge joining N_1 and N_2 it is a *complete bipartite graph* K_{n_1,n_2}. See Figure 2-3 for an example of a bipartite graph. Generalizing the concept of a bipartite graph to a partitioning of a graph's nodes into several disjoint subsets, where nodes within the

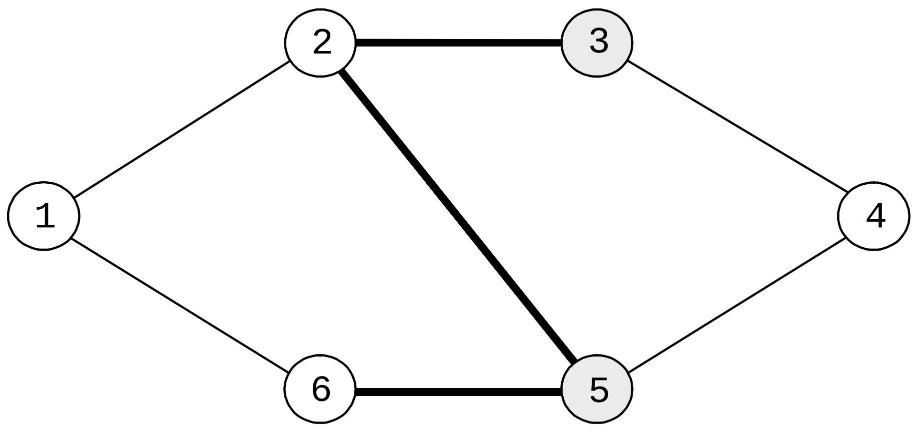


Figure 2-2: Illustration of cutsets. The set of heavy-lined edges represents a prime edge cutset, as its removal disconnects the graph and no proper subset of this set could disconnect the graph. Non-prime edge cutsets can be formed by adding to this set any arbitrary edge. Similarly, the set of shaded nodes represents a prime node cutset. A non-prime node cutset could be formed by adding node 2, for example, to this set.

same subset cannot not be adjacent, yields the definition of a *multipartite graph.* See Figure 2-4 for an example of such a graph.

2.2 Deterministic reliability metrics

Deterministic reliability metrics refer to non-probabilistic, topological properties of networks which indicate the relative difficulty in disrupting some form of communication among network nodes. Let $K \subseteq N$ be the set of nodes in the graph underlying a network among which communication is of interest. Then, a *k-terminal* deterministic reliability metric reflects the difficulty in disrupting communication among any two nodes $s, d \in K$. When $|K| = n$, this is called an *all-terminal metric*, and when $|K| = 2$, it is called a *two-terminal metric.*

2.2.1 All-terminal metrics

Two rudimentary, all-terminal reliability criteria are the cohesion and connectivity of the graph underlying a network. An n-node, e-edge graph having maximum cohesion

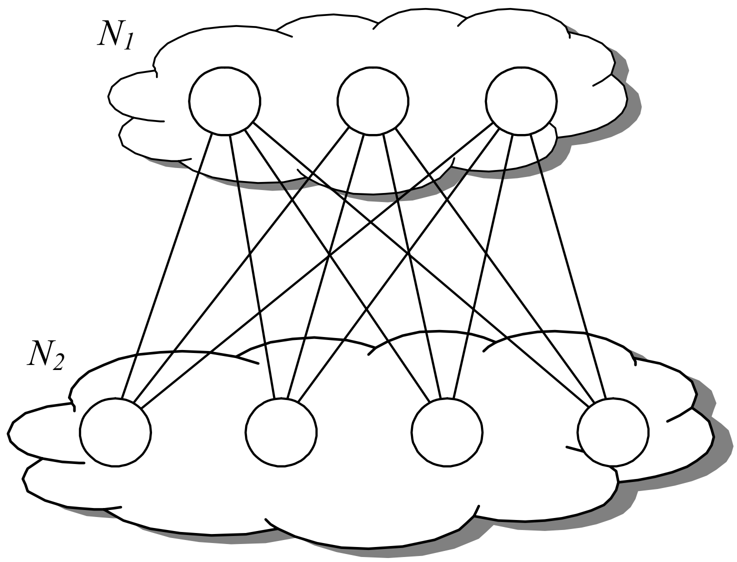


Figure 2-3: The complete bipartite graph $K_{3,4}$.

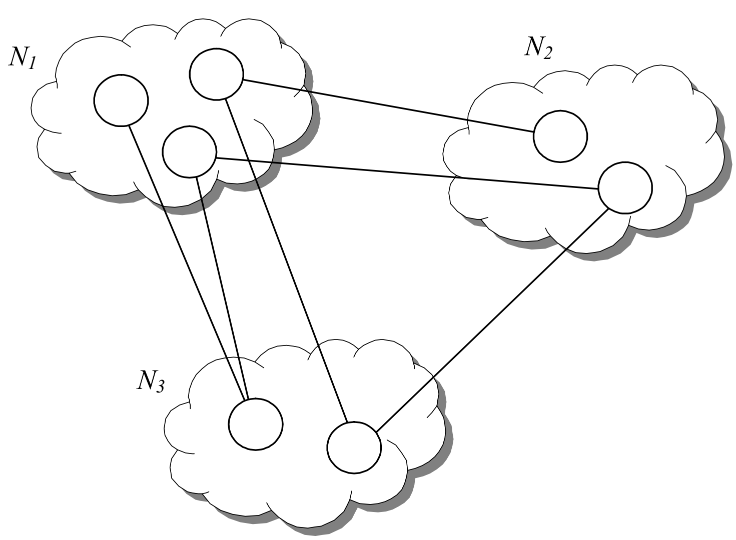


Figure 2-4: An incomplete 3-partite graph.

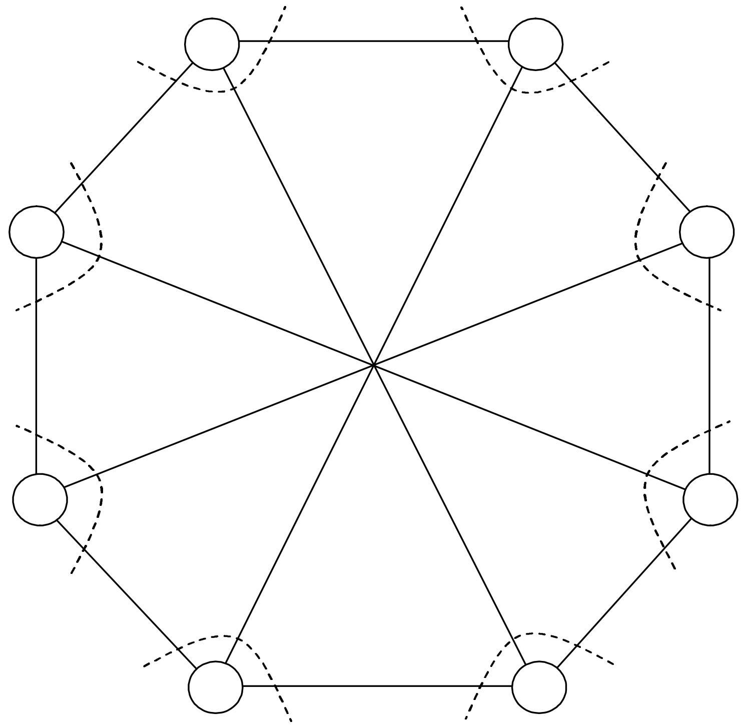


Figure 2-5: Illustration of the super-λ concept. Each dashed line intersects three edges that form a cutset isolating exactly one node. Conversely, any cutset of order three can be seen to isolate a node. Hence, this graph is super-λ.

is a *max*-λ graph. Similarly, an n-node, e-edge graph having maximum connectivity is a *max*-χ graph. More refined deterministic criteria for network reliability can also be defined, such as the number of edge or node cutsets of order λ or χ in a *max*-λ or *max*-χ graph, respectively. A graph is *super*-λ if it is max-λ and every edge disconnecting set of order λ isolates a point of degree λ. See Figure 2-5 for an illustration of this definition. Similarly, a graph is *super*-χ if it is max-χ and every node disconnecting set of order χ isolates a point of degree χ. Boesch [9] has shown that if either n or λ (respectively, χ) is even, then finding a max-λ (max-χ) graph is equivalent to finding regular graphs of degree λ (χ). The case of n and λ (or χ) both odd is slightly more complex.

The following result relates connectivity and cohesion to the basic parameters of a graph [37]:

Theorem 2.1

$$\chi \leq \lambda \leq \delta \leq \frac{1}{n}\sum_{i=1}^{n} d_i = 2e/n. \tag{2.1}$$

Harary has shown [36] that the bounds in Theorem 2.1 can be achieved, through the construction of *Harary graphs*. Harary graphs are discussed below, in addition to the more general family of *circulant* graphs to which they belong. Since these bounds can be achieved, we see that any max-χ graph is necessarily max-λ, although the converse is not true in general.

A more refined measure of network reliability than cohesion is *generalized cohesion* — the minimum number of edges that must be removed from a graph in order to isolate any subgraph with a fixed number of nodes from the rest of the graph [12][1]. In the same work, upper and lower bounds for generalized cohesion are derived. It has further been shown [10, 86] that a necessary condition for a graph to be optimal with respect to generalized cohesion — where optimality is defined as maximization of generalized cohesion for subgraphs with all possible numbers of nodes — is that the girth of the graph be a maximum.

An alternative measure of a graph's ability to remain connected is the number of spanning trees it possesses. The characterization of graphs with a maximum number of trees has been solved for sparse graphs when the number of edges is at most $n + 3$, and for dense graphs when the number of edges is at most $n/2$ less than that of the complete graph K_n (more on these graphs in §2.3.1). In addition, for the remaining cases where at most n edges are missing from K_n, Kel'mans, Petingi, Boesch, Suffel, Gilbert and Myrvold have described graphs with a maximum number of trees, assuming that they are almost regular [41, 58, 59, 30]. If ρ_k denotes a path on k nodes and ζ_k denotes a cycle on k nodes, their results can be summarized as follows:

- For $e = n(n-1)/2 - n$, the *complement*[2] of an optimal graph is the union of at most one of ζ_4 or ζ_5, and ζ_3's;

[1] *Generalized connectivity* can be defined analogously.

[2] The complement of a graph $G(N, E)$ is the graph $\overline{G}$ with N as its node set but two nodes are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

- For $e = n(n-1)/2 - (n-1)$, the complement of an optimal graph is the union of ζ_3's, and ρ_2's and either ρ_2, ρ_3 or ρ_4;

- For $n(n-1)/2-(n-2) \leq e \leq n(n-1)/2-n/2$, the complement of an optimal graph is the union of ζ_3's, ρ_2's and either ρ_2 or ρ_3 or two ρ_3's.

Petingi and Rodriguez characterized graphs with the maximum number of trees when $n(n-1)/2 - 3n/2 \leq e \leq n(n-1)/2 - n$ and $n \geq n_0$, where n_0 can be explicitly determined [60]. In addition, the family of regular, complete multipartite graphs have been shown by Cheng [17] to possess the maximum number of spanning trees. Incidentally, these regular, complete multipartite graphs are also super-λ. Cheng's result was extended by Petingi and Rodriguez in [60] to show that almost-regular, complete, multipartite graphs have the maximum number of spanning trees.

It is reasonable for network diameter to play a role in some reliability criteria since the quality of information transmitted in many networks is degraded as the diameter of the network increases. Furthermore, in present-day optical networks where node switching costs are a limiting factor, it has been shown [33] that minimum diameter is generally required to minimize network cost. The diameter of a network has been incorporated into reliability criteria in several different ways. Bollobás [16], for example, investigated in special cases the minimum number of edges required for a class of graphs with constrained diameter and number of nodes to increase in diameter. Wilkov [86] has shown that Moore and Singleton graphs, to be discussed in greater detail in §2.2.2, are max-χ graphs with minimum diameter. Unfortunately, these graphs are defined for a small set of parameters and thus constitute only a small portion of all max-χ graphs. Trufanov [78] suggested a method to reduce the diameter of Harary graphs. Several others have developed families of max-χ graphs with minimum diameter [29, 42, 64]. In [5], the concepts of *persistence* and *line persistence* are introduced as the minimum number of nodes and edges, respectively, required to increase the diameter of a graph.

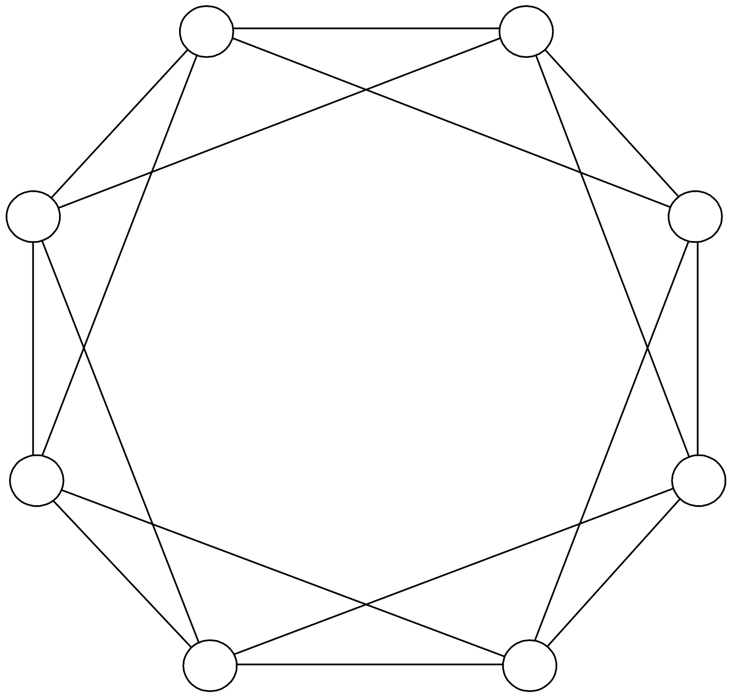


Figure 2-6: The $H(8,4)$ Harary graph.

Harary graphs and circulants

As previously mentioned, Harary graphs, first presented in [36], achieve the bounds presented in Theorem 2.1. In a $H(n,\Delta)$ Harary graph where Δ is even, each node i, $0 \le i \le n-1$, is adjacent to nodes $i \pm 1, i \pm 2, \ldots, i \pm \lfloor \Delta/2 \rfloor (\mathrm{mod}\ n)$; and if Δ is odd, then each node $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$ is also adjacent to node $i + \lfloor n/2 \rfloor$. See Figure 2-6 for an example of a Harary graph. Harary graphs have the following properties [9]:

- $H(n,\Delta)$ has $e = \lceil nk/2 \rceil$, $\chi = \lambda = \Delta$;
- $H(n,\Delta)$ is regular of degree Δ, unless n and Δ are both odd;
- $H(n,\Delta)$ has one node of degree $\Delta + 1$ and $n-1$ nodes of degree Δ if n and Δ are both odd.

Harary graphs belong to a more general family of graphs known as *circulants*. The circulant graph $C_n\langle a_1, a_2, \ldots, a_h \rangle$, or more compactly, $C_n\langle a_i \rangle$, where $0 < a_1 <$

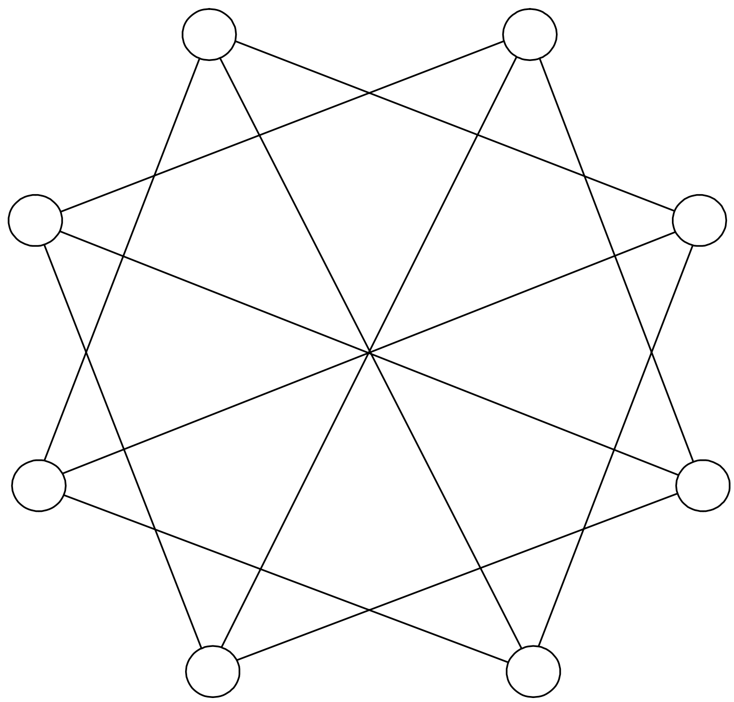


Figure 2-7: The $C_8\langle 2, 4\rangle$ circulant graph. Note that the graph is not connected.

$a_2 < \ldots < a_h < (n+1)/2$, has $i \pm a_1, i \pm a_2, \ldots, i \pm a_h (\text{mod } n)$ adjacent to each node i. See Figure 2-7 for a sample circulant graph. Owing to a theorem by Mader [50], which proves that every connected node-symmetric[3] graph has $\lambda = \Delta$, all connected circulants are max-λ. Furthermore, in [14], Boesch and Wang prove the following result:

Theorem 2.2 *The only circulants which are not super-λ are the cycles and the graphs $C_{2m}\langle 2, 4, \ldots, m-1, m\rangle$ with $m \geq 3$, and m an odd integer.*

In [13], Boesch and Tindell derive complex conditions for circulants to be max-χ.

In [83], Wang and Yang derive the following useful result for the number of spanning trees in circulant graphs[4]:

[3]Two nodes u and v in a graph are *similar* if there is an automorphism which maps u onto v. A graph in which all nodes are similar is *node-symmetric*.

[4]The form of $t(G)$ in Theorem 2.3 and Lemma 4.1 may seem surprising. However, as discussed in §3.2.3, $t(G)$ is equal to the cofactor of a matrix, which for circulant graphs is a *circulant matrix*. Furthermore, it is a well-known fact that the eigenvalues of circulant matrices are the roots of unity, from which the form of $t(G)$ emerges.

Theorem 2.3 *The number of spanning trees in the degree Δ circulant graph $G = C_n\langle a_1, a_2, \ldots, a_h\rangle$ is:*

$$t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2(a_j i\pi/n)\right], & \textit{if } \Delta \textit{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2(a_j i\pi/n) - (-1)^i + 1\right], & \textit{if } \Delta \textit{ is odd.} \end{cases} \tag{2.2}$$

The above result can be specialized to the case of even degree Harary graphs:

Lemma 2.1 *The number of spanning trees in the degree Δ Harary graph $G = C_n\langle 1, 2, \ldots, h\rangle$ is:*

$$t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[\Delta - \frac{\sin((2h+1)i\pi/n)}{\sin(i\pi/n)} + 1\right], & \textit{if } G \textit{ is not complete,} \\ n^{n-2}, & \textit{if } G \textit{ is complete.} \end{cases} \tag{2.3}$$

In [14], Boesch and Wang examine the diameter properties of circulants and derive lower diameter bounds for the family of graphs. Their results are discussed in depth in §3.4.4. In [15], the same authors determined that even degree Harary graphs possess the fewest number of edge cutsets of cutset cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. Each cutset in the above range of cardinalities was shown to isolate a single node in the Harary graph.

Other maximally connected graphs

In [28], Frank and Frisch showed max-χ (and hence, max-λ) to be a property of complete bipartite graphs when the number of nodes in the graph is odd. The case of an even number of nodes requires the bipartite graph to be formed by a superposition of Hamilton cycles. Furthermore, any edge-disjoint union of Hamilton cycles is max-λ. In [27], Frank shows a certain class of complete bipartite graphs to be super-χ. In [10], Boesch and Felzer prove that certain complete k-partite graphs are super-λ and super-χ, as well as optimal with respect to generalized cohesion. Hakimi and Amin develop families of graphs in [34] which come close to being super-χ for a wide range of graph parameters.

2.2.2 Two-terminal metrics

Wilkov [87] was among the first to consider the two-terminal reliability of graphs. In his study, he defined the deterministic parameters $X^e(m)$ and $X^n(m)$ as the maximum number of prime edge and node cutsets, respectively, of cardinality m with respect to any pair of nodes in a graph. The following lower bounds were derived for $X^e(m)$ for a Δ regular graph of maximum connectivity and of girth g:

$$X^e(m) \geq \begin{cases} 2\binom{\Delta-1}{i}, & \text{for } g = 3,\ m = \Delta + i(\Delta - 2) \text{ and } 0 \leq i \leq \Delta - 1, \\ 2\binom{\Delta}{i}, & \text{for } g > 3,\ m = \Delta + i(\Delta - 2) \text{ and } 0 \leq i \leq \Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Graphs which achieve these bounds were shown to necessarily possess maximum connectivity, maximum girth, and minimum number of small cycles. Moore and Singleton graphs, defined in the next subsection, were shown to be optimal as they achieve these bounds [89]. Realizing that Moore graphs only exist for a handful of configurations, Wilkov introduced in [88] a heuristic procedure for constructing regular and almost-regular graphs which come close to achieving the bounds derived in [87], given the desired number of nodes and edges in the graph.

Cages and Moore graphs

In this subsection, we discuss regular graphs which, for a given number of nodes and edges, achieve maximum girth. The problem of finding such graphs is equivalent to the well-studied *cage* problem — finding regular graphs of degree Δ and girth g with the minimum number of nodes $n(\Delta, g)$. The study of cages began with Tutte's pioneering work [79], with significant contributions subsequently made by Erdös and Sachs [24]. The search for cages with degrees exceeding three and girths exceeding five has proven to be very difficult with few results obtained.

The following lower bound for $n(\Delta, g)$:

$$n(\Delta, g) \geq \begin{cases} \frac{\Delta(\Delta-1)^k-2}{\Delta-2}, & \text{if } g = 2k+1, \\ \frac{2(\Delta-1)^k-2}{\Delta-2}, & \text{if } g = 2k \end{cases} \tag{2.5}$$

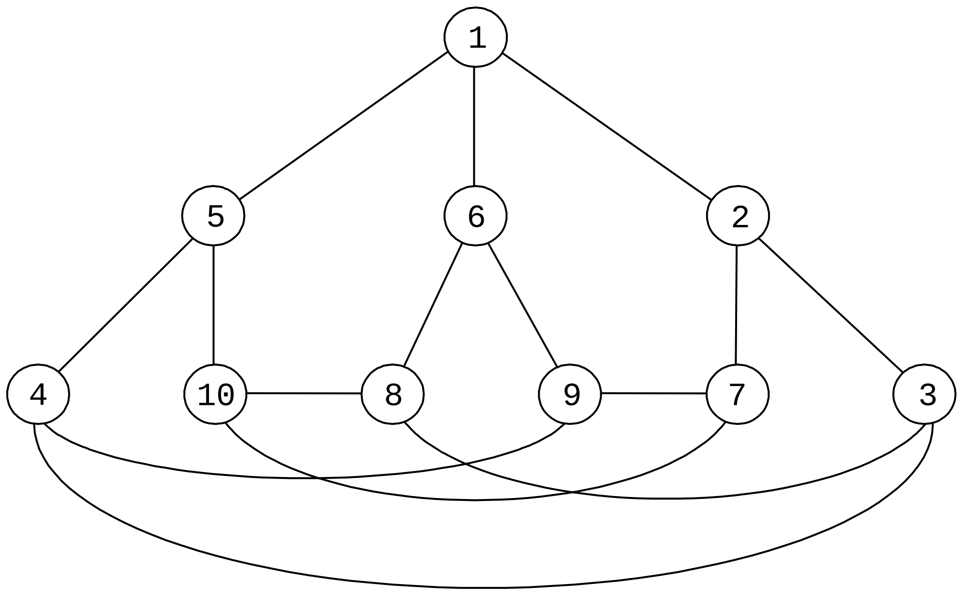
known as the *Moore bound*, can be obtained by inspecting a tree with number of nodes equal to the above lower bound, in which all internal nodes have degree Δ. Any graph which achieves the Moore bound for $g = 2k + 1$ is known as a *Moore graph*, and for $g = 2k$ is a *Singleton graph*. Moore and Singleton graphs are, by definition, cages. From this point on, we shall refer to Moore and Singleton graphs jointly as Moore graphs. A well-known property of Moore graphs is that they have minimum diameter k over all regular graphs of the same degree having the same number of nodes. Moore graphs exist for graphs of girth $g = 3$ (complete graphs), $g = 4$ (complete bipartite graphs), $g = 6, 8$ or 12 with $\Delta - 1$ a prime power (generalized polygons), and $g = 5$ with $\Delta = 2, 3, 7$ and possibly 57 [25]. See Figure 2-8 for a diagram of the Moore graph with $g = 5$ and $\Delta = 3$, also known as the Petersen graph. Simple instances of Moore graphs are degree two Hamiltonian cycles of even girth. For a more detailed discussion of Moore graphs, see [38, 72, 25].

For values of Δ and g for which Moore graphs do not exist, only a handful of cage constructions have been made — see [90] for a survey.

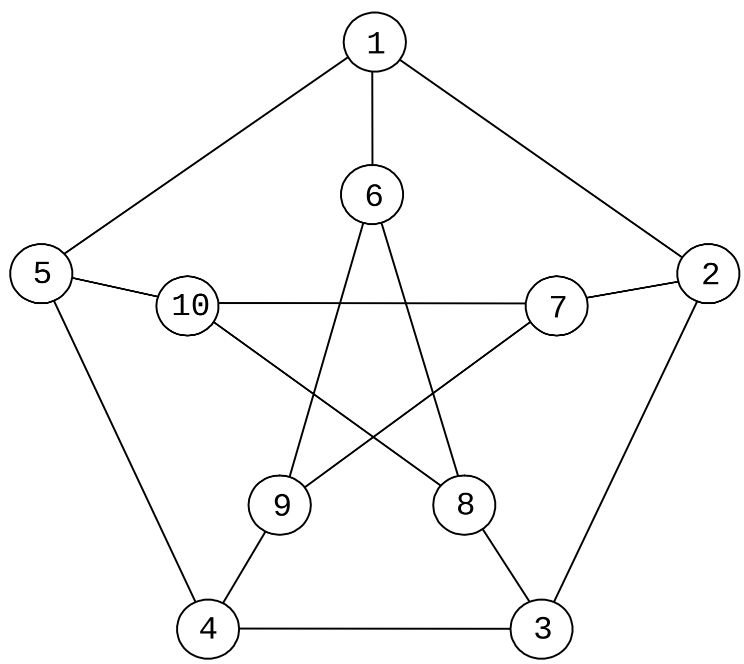
2.3 Probabilistic reliability metrics

Deterministic reliability metrics do not provide an adequate measure of the susceptibility of networks to disconnection because these metrics do not account for the reliability of network components. Probabilistic reliability criteria, on the other hand, require knowledge of deterministic network properties, in addition to the reliability of network components, and thus yield a more meaningful measure of network reliability. For this reason, this thesis will primarily be concerned with probabilistic reliability criteria.

Probabilistic reliability metrics require the concept of a probabilistic graph. A



(a)



(b)

Figure 2-8: Two representations of the $g = 5, \Delta = 3$ Moore graph, also known as the Petersen graph. The upper diagram (a) is the full tree representation using node 1 as the root node. For any Moore graph, a full-tree representation using any node as the root is possible.

probabilistic graph is an undirected graph $G = (N, E)$ where each node in N has an associated probability of being in an operational state and likewise for each edge in E. In probabilistic reliability analyses, networks under stress are modelled as probabilistic graphs.

Baran and Frank [4, 26] developed one of the first approaches to probabilistic reliability analysis by measuring network reliability as the expected fraction of nodes that are able to communicate after the network has been subjected to stress. Most other approaches to probabilistic reliability analysis have focused on the probability that a subset of nodes in a network are connected. As in §2.2, let $K \subseteq N$ be the set of nodes in the probabilistic graph underlying a network among which communication is of critical interest. Then, the *k-terminal reliability* of the probabilistic graph is the probability that any two nodes $s, d \in K$ have an operating path connecting them. When $|K| = n$, this is called the *all-terminal reliability*, and when $|K| = 2$, it is called *two-terminal reliability*.

2.3.1 All-terminal reliability

For a probabilistic graph $G = (N, E)$ with perfectly reliable nodes and links which fail independently with probability p, the probability that G is connected is given by:

$$P_c(G, p) = \sum_{i=n-1}^{e} A_i (1-p)^i p^{e-i} \tag{2.6}$$

where A_i denotes the number of connected subgraphs with i edges. The probability of connection can also be expressed as:

$$P_c(G, p) = 1 - \sum_{i=\lambda}^{e} C_i p^i (1-p)^{e-i} \tag{2.7}$$

where C_i denotes the number of edge cutsets of cardinality i.

For values of p sufficiently close to zero, $P_c(G, p)$ can be accurately approximated by $1 - C_\lambda p^\lambda (1-p)^{e-\lambda}$. In this case, an optimally reliable graph — one that achieves the maximum $P_c(G, p)$ over all graphs with the same number of nodes and edges —

has a minimum number of cutsets of size $\lambda = \lfloor 2e/n \rfloor$. Therefore, in this regime of p, optimally reliable graphs are super-λ graphs. For values of p sufficiently close to unity, $P_c(G, p)$ can be accurately approximated by the first term in (2.6), $A_{n-1}(1 - p)^{n-1} p^{e-n+1}$, where $A_{n-1} = t(G)$. Therefore, for values of p sufficiently close to unity, an optimally reliable graph has a maximum number of spanning trees. See §2.2.1 and the following discussion for a characterization of graphs with a maximum number of trees.

An interesting result due to Kel'mans [40] is that there exist two graphs G_a and G_b such that $P_c(G_a, p_0) < P_c(G_b, p_0)$ for p_0 sufficiently close to zero and $P_c(G_a, p_1) > P_c(G_b, p_1)$ for p_1 sufficiently close to unity. This implies that the relative reliability of networks may not only depend on the network topologies, but also on the reliability of the links from which they are constructed. This prompts the question: does there exist an n-node, e-edge graph which has the largest probability of connection for all values of p (i.e. $0 \leq p \leq 1$)?

If such uniformly optimally reliable graphs exist, then they must have a maximum number of spanning trees to ensure optimality when $p \approx 1$, and a lexicographically minimum cutset vector[5] to ensure optimality when $p \approx 0$. Uniformly optimally reliable graphs have been obtained by Boesch, Li and Suffel [11] for graphs having up to $n + 2$ edges, and the $n + 3$ case has been solved by Wang [82]. When the number of edges in a graph is at most $n/2$ less than that of the complete graph K_n, then a uniformly optimally reliable graph is one whose complement consists of *independent* edges – edges that are not incident at any of the same nodes [69]. These findings are summarized in Table 2.1 [55]. Contrary to Boesch's conjecture in [8] that uniformly optimally reliable graphs always exist[6], Myrvold et. al. have proven the nonexistence of such graphs in special cases [56].

[5] We define i^{th} component of the cutset vector of a graph to be the number of cutsets of cardinality i in the graph.

[6] Leggett, in his Ph.D. dissertation [47], also erroneously implied (through his Theorem II.2) the certain existence of uniformly optimally reliable graphs.

e	Uniformly Optimally Reliable Graph
$n-1$	Any tree
n	n-cycle
$n+1$	θ-graph with path lengths as even as possible
$n+2$	Particular subdivision of K_4
$n+3$	Particular subdivision of $K_{3,3}$
$\frac{n(n-1)}{2} - \frac{n}{2} \leq e \leq \frac{n(n-1)}{2}$	K_n with at most $\frac{n}{2}$ independent edges missing

Table 2.1: Uniformly optimally reliable graphs. A description of the optimal graph or its complement graph is given in the second column.

2.3.2 Two-terminal reliability

Expressions similar to (2.6) and (2.7) can be written down for the two-terminal reliability of $G = (N, E)$, the probability that nodes $s, d \in N$ are connected $P_c^{sd}(G, p)$:

$$P_c^{sd}(G,p) = \sum_{i=w_{sd}}^{e} A_i^{sd}(1-p)^i p^{e-i} \quad (2.8)$$

$$P_c^{sd}(G,p) = 1 - \sum_{i=\lambda_{sd}}^{e} C_i^{sd} p^i (1-p)^{e-i} \quad (2.9)$$

where w_{sd} is the shortest path length between nodes s and d, A_i^{sd} is the number of subgraphs with i edges that connect nodes s and d, λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d, and C_i^{sd} is the number of cutsets with respect to nodes s and d of cardinality i.

If we wish to maximize $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when p is small, then it is apparent from (2.9) that the property of super-λ is a necessary condition. Furthermore, from our previous discussion of deterministic node pair metrics in §2.2.2, we know that graphs with maximum girth and minimum number of small cycles are good candidates among the class of super-λ graphs. Notice that we cannot conclude that these properties are necessary conditions for optimal two-terminal reliability performance when p is small, where we define optimality as maximization of $\min_{s,d}\left[P_c^{sd}(G,p)\right]$. The reason is that the deterministic metric $X^e(m)$, on which such a conclusion would be based, does not capture the degree to which prime edge cutsets of the same cardinality overlap. This degree of overlap would clearly influence the values of C_i^{sd} in (2.9). Hence,

	Probabilistic metric
ATR $p \approx 0$	$P_c(G,p) = 1 - \sum_{i=\lambda}^{e} C_i p^i (1-p)^{e-i}$ $P_c(G,p) \to 1 - C_\lambda p^\lambda, \text{ as } p \to 0$
TTR $p \approx 0$	$P_c^{sd}(G,p) = 1 - \sum_{i=\lambda_{sd}}^{e} C_i^{sd} p^i (1-p)^{e-i}$ $P_c^{sd}(G,p) \to 1 - C_{\lambda_{sd}}^{sd} p^{\lambda_{sd}}, \text{ as } p \to 0$
ATR $p \approx 1$	$P_c(G,p) = \sum_{i=n-1}^{e} A_i (1-p)^i p^{e-i}$ $P_c(G,p) \to t(G)(1-p)^{n-1}, \text{ as } p \to 1$
TTR $p \approx 1$	$P_c^{sd}(G,p) = \sum_{i=w_{sd}}^{e} A_i^{sd} (1-p)^i p^{e-i}$ $P_c^{sd}(G,p) \to A_{w_{sd}}^{sd} (1-p)^{w_{sd}}, \text{ as } p \to 1$

Table 2.2: Summary of probabilistic reliability metrics and asymptotes.

an assessment of a graph's probabilistic two-terminal reliability performance requires knowledge of the structure of the cutsets. Nonetheless, we believe that Moore graphs and other cages, when they exist, possess good (although not necessarily optimal) performance with respect to $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when p is small. On the other hand, when $p \approx 1$, (2.8) indicates that minimum diameter is required for optimality, and Moore graphs and other cages are optimal topologies.

2.3.3 Determining probabilities of connection

Provan and Ball [62] have shown that the calculation of the probability of connection of a graph $P_c(G,p)$ belongs to the NP-Hard class of intractable problems. An analogous result was proven for two-terminal reliability by Valiant [80]. In order to avoid the combinatorial difficulty of these problems, one must resort to Monte Carlo sampling techniques, finding efficient algorithms for special classes of graphs [66, 67, 68], or using bounding techniques. Algorithmic bounding techniques were first introduced in [81] by Van Slyke and Frank, with tighter bounds subsequently obtained by researchers, such as Ball and Provan, Lomonosov and Polesskii, and Colbourn and Harms [3, 48, 19, 20].

Simple analytic bounds for $P_c(G,p)$ have been derived by several researchers. One of the first such results is due to Jacobs [39]:

$$1-\sum_{i=\lambda}^{e}\binom{e}{i}p^i(1-p)^{e-i} \le P_c(G,p) \le 1-\sum_{i=e-n+2}^{e}\binom{e}{i}p^i(1-p)^{e-i}. \qquad (2.10)$$

The lower bound follows by upper-bounding the number of cutsets of cardinality i by $\binom{e}{i}$. The upper bound follows from the fact that any subgraph of $G=(N,E)$ with n nodes and fewer than $n-1$ edges will be disconnected. The lower bound is tight for $p \approx 0$ and the upper bound is tight for $p \approx 1$. Van Slyke and Frank [81] improved upon these bounds slightly by changing the combinatorial coefficients of the first terms in both summations. In [49], Lomonosov and Polesskii derived the following lower bound for $P_c(G,p)$:

$$P_c(G,p) \ge n\left(1-p^{t(G)/2}\right)^{n-1} - (n-1)\left(1-p^{t(G)/2}\right)^{n}. \qquad (2.11)$$

A cumbersome lower bound for $P_c(G,p)$, derived by Leggett [47] is given by:

$$P_c(G,p) \ge 1-\sum_{i=\lambda}^{e} X(i)p^i(1-p)^{e-i} \qquad (2.12)$$

where

$$X(\lambda) = \binom{e}{n-1} - t(G)$$

and

$$X(i+1) = \frac{(1+1/i)^i(m-\lambda-i)X(i)}{(1+1/(i-n+3))^{i-n+3}(i+1)}.$$

An analogous upper bound for $P_c(G,p)$ is also derived in the same work. For large graphs, the lower bound reduces to the following closed form expression:

$$P_c(G,p) \geq B(n-1;e,1-p) - \left(\frac{e-\lambda}{e-n-\lambda+1}\right)^{\lambda} \sqrt{n/2}p^{\lambda}B(n-1;e-\lambda,1-p) \quad (2.13)$$

and the upper bound to:

$$P_c(G,p) \leq B(n-1;e,1-p) - \sqrt{2n}p^{\lambda}B(n-1;e-\lambda,1-p) \quad (2.14)$$

where $B(x;y,z) = \sum_{i=o}^{x} \binom{y}{i} z^i(1-z)^{y-i}$ is the cumulative binomial distribution.

In the following chapters, alternative simple bounding techniques to the above are presented. It should be emphasized that these techniques are *not* the central contribution of this work[7]. Rather, these techniques serve to support network design conclusions developed herein, which *are* the main thrust of this work.

2.4 Summary

In this chapter, we carried out a thorough survey of the research area of network reliability. We brought many previous contributions to the field into a cohesive framework, in which we broadly categorized reliability metrics as being either deterministic or probabilistic. In the course of our survey, we also introduced Harary graphs, circulants, Moore graphs and cages. These families of graphs were shown to possess special reliability properties.

[7]In fact, some of the referenced techniques in this chapter are more computationally efficient and/or lead to tighter bounds.

Chapter 3

Network design with statistically independent link failures

In this chapter, we consider networks which have invulnerable nodes and links which fail in a statistically independent fashion with the same probability. Aided by new and simple techniques to bound the probability of connection of a network and the probability of connection of a node pair in a network, we introduce a methodology for the design of reliable networks.

In the next section, we outline the modelling assumptions employed in this chapter. §3.2 develops simple bounding techniques which will be of assistance to us in our design approach. In §3.3, we motivate our methodology for the design of reliable networks. We detail our methodology in §3.4. In §3.5, we present and interpret a series of simulation results. We then conclude with a summary of the chapter.

3.1 Network model

As mentioned in the introductory chapter, networks will be modelled as probabilistic graphs. In addition, unless otherwise stated, we assume the following about the graphs underlying the networks considered in this chapter:

- Nodes are invulnerable;

- Edges fail in a statistically independent fashion with probability p;
- Edge capacities are assumed to be sufficiently large to carry any possible network flow;
- Once an edge fails it cannot be repaired.

Since we will be dealing exclusively with edge failures, for the sake of brevity, we will refer to edge failures simply as failures and to edge cutsets simply as cutsets.

Analyses in this work will be carried out separately for the $p \leq 1/2$ and the $p \geq 1/2$ regimes. In the $p \leq 1/2$ regime, the bounds and techniques developed are most effective when p is less than approximately 0.2, and we will denote this range by $p \approx 0$. Similarly, $p \approx 1$ will be used to denote the range of p from approximately 0.8 to 1, where the techniques and bounds developed for $p \geq 1/2$ regime are most effective.

3.2 Bounding techniques

In this section, we develop bounding techniques which will be of assistance in executing our design methodology in §3.4. The next two subsections deal with bounding graph and node pair connectedness when $p \approx 0$. Graph and node pair connectedness in the $p \approx 1$ regime is treated in the two subsequent subsections. In the discussion that follows, we assume that all graphs are Δ regular and have maximum connectivity. The bounds derived in this section are summarized in Table 3.3. The quality of these bounds is illustrated in the following chapter in our case study of Harary graphs.

3.2.1 All-terminal reliability when $p \approx 0$

In this subsection, we derive upper and lower bounds for the probability that graph G is connected $P_c(G, p)$. The general approach we follow is based on enumeration of prime failure events. We define a *prime failure event* as an event in which a subset of nodes become disconnected from the rest of the graph through the failure of the minimal number of edges. Clearly, prime failure events are only a subset of all possible graph disconnection events, since graph disconnection can also occur when more than the minimal number of edges fail. Therefore, in order to obtain an upper bound for $P_c(G,p)$, we subtract from unity the probabilities of the mutually exclusive prime failure events:

$$P_c(G,p) \leq 1 - \sum_{i=\lambda}^{e} B_i p^i (1-p)^{e-i} \tag{3.1}$$

where B_i is the number of prime failure events of cardinality i. To obtain a lower bound for $P_c(G,p)$, we note that any failure scenario requires that at least one of the prime failure events occur. Therefore, we obtain a lower bound for $P_c(G,p)$ by subtracting from unity the union bound of the prime failure events:

$$P_c(G,p) \geq 1 - \sum_{i=\lambda}^{e} B_i p^i. \tag{3.2}$$

It now remains to determine the coefficients B_i. If the graph under consideration is either trivially small, or simple and symmetric as is the case with Ethernet, ring or Harary networks, then closed form, analytic solutions or bounds are obtainable (see §4.1, §4.2 and §4.3); otherwise, one must resort to more general techniques.

We now introduce a technique to determine the coefficients B_i for general graphs. It is known [77] that a vector representation of the prime failure events of a graph can be expressed in two ways as the modulo two sum of a subset of rows of a graph's incidence matrix. Specifically, a prime failure event partitions a network into two subsets of nodes. Therefore, we can obtain a prime failure event by adding modulo two the rows that correspond to each of the nodes in one of the partitions. Conversely, it can be shown that the modulo two sum of any proper subset of rows of a graph's incidence matrix yields a prime failure event. Therefore, we can find all prime failure events of a graph by summing modulo two the rows of the $2^{n-1}-1$ subsets of the rows the incidence matrix which yield distinct partitions of the network[1]. The B_i coefficients

[1]Note that if we sum modulo two the rows of all 2^n possible subsets, then we are counting every partitioning scenario twice, including the null and complete partitions.

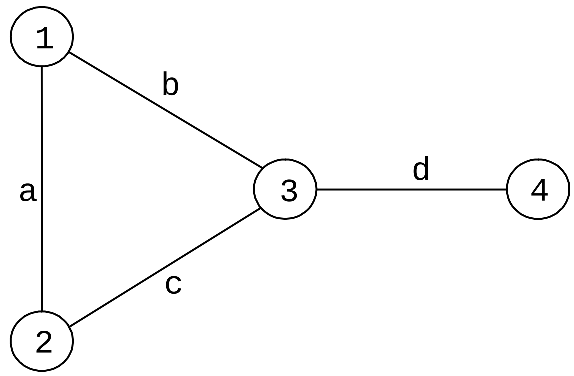


Figure 3-1: Graph considered in Example 3-1.

are determined by simply counting the number of prime failure events obtained which have cardinality i. This technique is illustrated in the following example.

Example 3-1. Consider the graph depicted in Figure 3-1. The incidence matrix for the graph is:

$$\mathbf{I} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We first determine the $2^{n-1} - 1 = 7$ subsets which give us distinct prime failure events. Then, for each of these seven prime failure events we sum modulo two the corresponding rows of $\mathbf{I}$ to obtain the binary prime failure event vectors. These seven partitions and their corresponding vectors are listed in Table 3.1. We see that there is one prime failure event of cardinality one, three prime failure events of cardinality two, and three prime failure events of cardinality three. Thus, $P_c(G,p)$ can be bounded as follows:

$$1 - \left[p + 3p^2 + 3p^3\right] \leq P_c(G,p) \leq 1 - \left[p(1-p)^3 + 3p^2(1-p)^2 + 3p^3(1-p)\right].$$

Another approach to upper bounding $P_c(G,p)$ when $p \approx 0$ is to compute a lower

Partition	Prime failure event vector
1	[1 1 0 0]
1,2	[0 1 1 0]
1,3	[1 0 1 1]
1,4	[1 1 0 1]
1,2,3	[0 0 0 1]
1,2,4	[0 1 1 1]
1,3,4	[1 0 1 0]

Table 3.1: Prime failure events for Example 3-1. Entries in the left column are nodes contained in one possible partition of the graph. The corresponding entries in the right column are the event vectors formed by summing modulo two the rows corresponding to the nodes in the left column.

bound on the first few terms of the summation in (2.7) and to then subtract these terms from unity. As discussed in §2.2.1, Boesch and Wang demonstrated in [15] that even degree Harary graphs possess the fewest number of edge cutsets of cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. The number of cutsets of cardinality i achieved by Harary graphs in this range is $n\binom{e-\Delta}{i-\Delta}$. This expression is in fact a lower bound achievable by any Δ regular graph with n nodes. Using this result, we obtain the following upper bound for $P_c(G,p)$ for any Δ regular graph with n nodes:

$$P_c(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-3} n\binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}. \tag{3.3}$$

In fact, it is easy to see that this bound can be tightened slightly by extending the range of the summation to $2\Delta - 1$.

3.2.2 Two-terminal reliability when $p \approx 0$

If instead of the probability that graph $G = (N, E)$ is connected $P_c(G,p)$, we desire the probability that nodes $s, d \in N$ are connected $P_c^{sd}(G,p)$, we can use an approach similar to that of §3.2.1 to obtain the following bounds:

$$1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i \leq P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i (1-p)^{e-i} \tag{3.4}$$

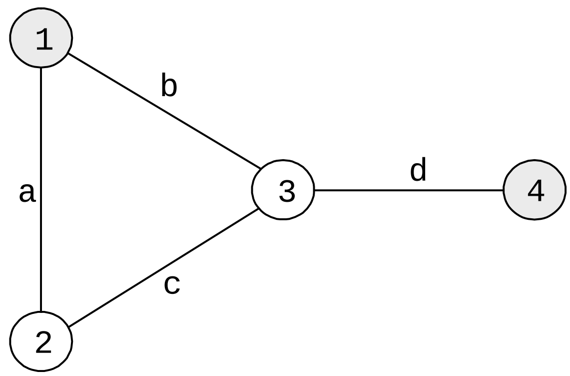


Figure 3-2: Graph considered in Example 3-2. Nodes 1 and 4 serve as the source-destination node pair.

where B_i^{sd} is the number of prime failure events with respect to nodes s and d of cardinality i, and λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d.

In order to determine the coefficients B_i^{sd}, we use an approach similar to that of §3.2.1. Since we are only interested in prime failure events of G which disconnect nodes s and d, we add modulo two to the row corresponding to s all possible subsets of the remaining rows of the incidence matrix, except for the row corresponding to d. Clearly, there are 2^{n-2} such possible subsets. This will provide us with a binary vector representation of all possible prime failure events which disconnect s and d. We illustrate the technique with a simple example.

Example 3-2. We again consider the graph of Figure 3-1, which is redrawn in Figure 3-2 with nodes 1 and 4 shaded to indicate their significance as the source-destination node pair. Once again, the incidence matrix for the graph is:

$$\mathbf{I} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Partition	Prime failure event vector
1	[1 1 0 0]
1,2	[0 1 1 0]
1,3	[1 0 1 1]
1,2,3	[0 0 0 1]

Table 3.2: Prime failure events for Example 3-2. Entries in the left column are nodes contained in one possible partition of the graph. The corresponding entries in the right column are the event vectors formed by summing modulo two the rows corresponding to the nodes in the left column.

We first determine the $2^{n-2} = 4$ subsets which give us distinct prime failure events in which nodes 1 and 4 reside in distinct partitions. Then, for each of these four prime failure events we sum modulo two the corresponding rows of $\mathbf{I}$ to obtain the binary prime failure event vectors. These four partitions and their corresponding vectors are listed in Table 3.2. We see that there is one prime failure event of cardinality one, two prime failure events of cardinality two, and one prime failure event of cardinality three. Thus, $P_c^{1,4}(G,p)$ can be bounded as follows:

$$1 - \left[p + 2p^2 + p^3\right] \leq P_c^{1,4}(G,p) \leq 1 - \left[p(1-p)^3 + 2p^2(1-p)^2 + p^3(1-p)\right].$$

In a similar manner to §3.2.1, we can upper bound $P_c^{sd}(G,p)$ by lower bounding the first few terms in the summation of (2.9). As discussed in §2.2.2, Wilkov [87] obtained lower bounds for $X^e(m)$ when m is small. We restate these bounds for a Δ regular graph of maximum connectivity and of girth g:

$$X^e(m) \geq \begin{cases} 2\binom{\Delta-1}{i}, & \text{for } g = 3,\ m = \Delta + i(\Delta-2) \text{ and } 0 \leq i \leq \Delta - 1, \\ 2\binom{\Delta}{i}, & \text{for } g > 3,\ m = \Delta + i(\Delta-2) \text{ and } 0 \leq i \leq \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Graphs which achieve these lower bounds necessarily possess maximum connectivity, maximum girth, and minimum number of small cycles. Furthermore, Moore graphs, when they exist, were shown to be optimal in this respect. If the requirement that the above bounds be met for all associated values of m is relaxed to the first $2\Delta-3$ values, then we equivalently require that any (not necessarily prime) cutset of cardinality i for for $\lambda \leq i \leq 2\Delta-3$ isolates either s or d alone. Therefore, in an analogous fashion to §3.2.1, we can lower bound C_i^{sd} for $\lambda \leq i \leq 2\Delta-3$ for any Δ regular graph with n nodes, and obtain the following upper bound for $P_c^{sd}(G,p)$:

$$P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-3} 2\binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}. \tag{3.5}$$

As in the case of all-terminal reliability, it is easy to see that this bound can be tightened slightly by extending the range of the summation to $2\Delta-1$.

3.2.3 All-terminal reliability when $p \approx 1$

We approach the task of bounding $P_c(G,p)$ in the regime of $p \approx 1$ in an analogous fashion to §3.2.1. The events of interest here, however, are the existence of spanning trees rather than prime failure events. As discussed in §2.3.1, a lower bound for $P_c(G,p)$ is obtained by summing the events that correspond to a spanning tree existing *and* the remaining links in the network being inoperative:

$$P_c(G,p) \geq t(G)(1-p)^{n-1}p^{e-n+1}. \tag{3.6}$$

An alternative lower bound is Lomonosov and Polesskii's bound (2.11):

$$P_c(G,p) \geq n\left(1-p^{t(G)/2}\right)^{n-1} - (n-1)\left(1-p^{t(G)/2}\right)^{n}.$$

An upper bound for $P_c(G,p)$ can be obtained be invoking the union bound on the spanning tree events:

$$P_c(G,p) \leq t(G)(1-p)^{n-1}. \tag{3.7}$$

It now remains to determine $t(G)$. Fortunately, this is a well studied problem, and $t(G)$ is known [84] to be the determinant of an $(n-1)\times(n-1)$ matrix $\mathbf{T(G)}$

whose $(i,j)^{\text{th}}$ entry is defined as follows:

$$t_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Bounding $t(G)$

Since determining $t(G)$ requires the sometimes tedious computation of the determinant of an $(n-1) \times (n-1)$ matrix $\mathbf{T(G)}$, it is sometimes helpful to have simple bounds on $t(G)$. More importantly, determining bounds for $t(G)$ is useful when it is of interest how close a graph $G = (N, E)$ comes to achieving the maximum number of spanning trees for any graph with n nodes and e edges. A trivial upper bound for $t(G)$ is $\binom{e}{n-1}$, which is simply the number of subgraphs of G with $n-1$ edges.

A tighter upper bound for $t(G)$ can be obtained by examining the structure of the matrix $\mathbf{T(G)}$. If $\mathbf{T(G)}$ is converted to upper triangular form, then $t(G)$ is equal to the product of the diagonal entries of this upper triangular matrix. Recall that we have assumed that G is Δ regular. We convert $\mathbf{T(G)}$ to upper triangular form by successive pivoting operations — adding multiples of rows to rows below to create zeros underneath the pivoting element. Notice that, with the exception of the first diagonal element, each diagonal element of the resulting upper triangular matrix can be written as Δ less a variable number of positive terms, each of which is at least $1/\Delta$. Furthermore, the number of such terms, summed over all diagonal elements, is equal to the number of row operations performed which is $e - \Delta$. We now bound the determinant of the upper triangular matrix as the product of Δ multiplied by $n-2$ product terms, each of which is equal to Δ less a number of $1/\Delta$ terms. In order to maximize this product, we distribute the $e - \Delta$ terms as evenly as possible among the $n-2$ product terms[2]. Hence, we have the following upper bound for $t(G)$:

$$t(G) \leq \Delta \left(\Delta - 1/2\right)^{n-2}. \tag{3.8}$$

[2]This can be shown by a simple Lagrangian multiplier argument.

It is interesting to compare this bound with Cayley's expression of n^{n-2} for the number of spanning trees in the complete graph K_n. The bound provided in (3.8) becomes $(n-1)\,(n-3/2)^{n-2}$, which is approximately a factor of n larger than Cayley's expression.

3.2.4 Two-terminal reliability when $p \approx 1$

When $p \approx 1$, most of the links in a network have failed and the underlying graph has relatively few edges. In such sparsely connected graphs, the disconnection of nodes s and d is nearly equivalent to a set of edge-disjoint paths between s and d all having failed. This is because any edges defined between nodes on different edge-disjoint paths would probably have failed, leaving the edge-disjoint paths unconnected. To be precise, the disconnection of nodes s and d actually implies the failure of a set of Δ edge-disjoint paths between s and d. Hence, we can lower bound $P_c^{sd}(G,p)$ as follows:

$$\begin{aligned} P_c^{sd}(G,p) &\geq 1 - \Pr(\Delta \text{ edge-disjoint paths fail}) \\ &= 1 - \prod_{i=1}^{\Delta} \Pr(\text{path } i \text{ fails}) \\ &= 1 - \prod_{i=1}^{\Delta} \left[1-(1-p)^{l_i}\right] \end{aligned} \tag{3.9}$$

where l_i is the length of the ith edge-disjoint path, and the second and third lines follow from the independence of edge failures.

The value of $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when $p \approx 1$ corresponds to a node pair with shortest path length equal to the graph diameter $k(G)$. A simple lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ is:

$$\min_{s,d}\left[P_c^{sd}(G,p)\right] \geq (1-p)^{k(G)}, \tag{3.10}$$

which is just the probability that the shortest path between the most distant node pair is available. A tighter lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ can be derived using (3.9) if the lengths or an upper bound on the lengths of the edge-disjoint paths joining the most distant node pair is available. This idea is exploited in §3.4.4 where we design for two-terminal reliability in the $p \approx 1$ regime.

	Bounds
ATR $p \approx 0$	$1 - \sum_{i=\lambda}^{e} B_i p^i \leq P_c(G,p) \leq 1 - \sum_{i=\lambda}^{e} B_i p^i (1-p)^{e-i}$ $P_c(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-1} n \binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}$
TTR $p \approx 0$	$1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i \leq P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i (1-p)^{e-i}$ $P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-1} 2 \binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}$
ATR $p \approx 1$	$t(G)(1-p)^{n-1} p^{e-n+1} \leq P_c(G,p) \leq t(G)(1-p)^{n-1}$ $t(G) \leq \Delta \left(\Delta - 1/2\right)^{n-2}$
TTR $p \approx 1$	$P_c^{sd}(G,p) \geq 1 - \prod_{i=1}^{\Delta} \left[1 - (1-p)^{l_i}\right]$ $\min_{s,d} \left[P_c^{sd}(G,p)\right] \geq (1-p)^{k(G)}$

Table 3.3: Summary of reliability bounds. Bounds for all-terminal reliability (ATR) and two-terminal reliability (TTR) in the $p \approx 0$ and $p \approx 1$ regimes are listed.

3.3 Motivation for an approach to reliable network design

In this section, we first develop "real-world" context for our network reliability studies. We then motivate our approach to reliable network design, to be carried out in §3.4, by interpreting the results surveyed in Chapter 2. We conclude this section by describing the general guiding principles of our design methodology.

3.3.1 Interpretation of metrics and regimes of vulnerability

In our survey of network reliability in Chapter 2, we introduced a myriad of metrics by which the subjective notion of network reliability can be measured. The choice of metric should be intimately related to the application at hand. In this work, we will mainly consider the all- and two-terminal probabilistic metrics discussed in §2.3.1 and §2.3.2, respectively. All-terminal reliability is a useful metric in situations where it is critical that all nodes in a network remain connected. An example would be a LAN which transports signals for control surfaces and engine management controls for jets. As discussed in Chapter 1, simple connectedness is a sufficient metric for such high-performance applications, since the quality and capacity of these network components are typically over-designed such that connectedness of nodes ensures almost perfect network performance. Two-terminal reliability is an appropriate reliability metric when we are mainly concerned with the connectedness of a particular node pair in a network, or when the connectedness of any node pair must exceed a prescribed level. In addition, all-terminal reliability can be used in conjunction with two-terminal reliability in situations where there is a prescribed level of connectedness for both the entire network and particular node pairs in the network.

Having discussed the appropriateness of our chosen reliability metrics, it is now necessary to consider the stresses under which these metrics will be evaluated. That is, we now provide "real-world" justification for different network component vulnerability regimes. When networks are subjected to relatively benign stresses, such as normal wear of components and rare stresses which are localized to individual network components, we can apply the model described in §3.1 with the additional assumption that $p \approx 0$. On the other hand, conditioned on the event that a network has been subjected to a catastrophic stress which is of a large physical scale compared with that of the network, we can model the network as described in §3.1 with the additional assumption that $p \approx 1$. An example of such a scenario is an aircraft subjected to violent turbulence, with the result that most components in the aircraft LAN have failed. The $p \approx 1$ model could also be useful in *destructive* military applications where, for example, it may be of interest whether a subset of nodes in an adversary's network is connected with given probability after being attacked. We remark that there exist a multitude of "real-world" situations where none of the above statistically independent failure models are applicable. Such scenarios are addressed in Chapter 5.

3.3.2 Interpretation of network reliability studies

In our discussion of reliability metrics in Chapter 2, we surveyed existing results on optimal graphs as well as some optimality conditions for several reliability metrics. As discussed in the previous subsection, the two measures of network reliability that we will be focusing on in this chapter are the all- and two-terminal probabilistic metrics. We choose these metrics because they are among the simplest and most useful of the metrics considered.

With respect to all-terminal reliability, uniformly optimally reliable graphs are the best graphs that one can hope for, as they maximize $P_c(G, p)$ for all values of p. Unfortunately, uniformly optimally reliable graphs have only been characterized for either very sparse or very dense graphs. Thus, in situations where these optimal graphs are inappropriate, one must turn to alternative topologies. The results presented in §2.3.1 indicate that when $p \approx 0$, we seek graphs which are super-λ. Among super-λ graphs, even degree Harary graphs were shown to be especially good when p is small, since they achieve the fewest number of cutsets of cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. When $p \approx 1$, optimal graphs — those possessing the maximum number of spanning trees — have only been characterized for either very sparse or very dense graphs. Unfortunately, owing to the combinatorial nature of computing $P_c(G, p)$, similar results are not available for p outside these two extreme regions. Hence, depending upon the application at hand, when uniformly optimally reliable graphs are inappropriate, one should seek graphs which strike a sensible balance between optimality at these two extreme regions of p.

A consideration of two-terminal reliability for $p \approx 0$ indicates that optimal graphs are super-λ. Among these super-λ graphs, graphs with maximum girth[3] and minimum number of small cycles fair particularly well, as they achieve the bounds for $X^e(m)$ presented by Wilkov in [87]. Moore graphs, and cages in general, are optimal in this respect, although these graphs exist for only a handful of configurations. For $p \approx 1$, minimum diameter is required for a graph to be optimal. Cages are again good candidate topologies, since they achieve the minimum diameter over all regular graphs of the same degree and the same number of nodes. However, in situations where cages are inapplicable, alternative topologies must be employed.

Fortunately, synergies exist among some of the properties required for graphs to be optimal with respect to all- and two-terminal reliability. For example, optimality with respect all- and two-terminal reliability when $p \approx 0$ requires that a graph be super-λ. Unfortunately, the relationship between diameter and number of spanning trees, the two key properties of graphs when $p \approx 1$, is less clear. In most instances, regular graphs with small diameters have a large number of spanning trees. However, in general, a smaller diameter does not imply a larger number of spanning trees, or vice versa. Consider, for example, the two circulants $C_{50}\langle 5, 6\rangle$ and $C_{50}\langle 1, 21\rangle$. The circulant $C_{50}\langle 5, 6\rangle$ was shown by Boesch and Wang [14] to possess the minimum diameter of five over all degree four circulants with 50 nodes, and possesses 7.2392×10^{24} spanning trees. On the other hand, $C_{50}\langle 1, 21\rangle$ has diameter eight and 7.4884×10^{24} spanning trees. We can further compare these two circulants to the Harary graph $C_{50}\langle 1, 2\rangle$ which has a larger diameter of 13 and three orders of magnitude fewer spanning trees: 7.9207×10^{21}. Thus, although a smaller diameter does not

[3] Furthermore, the property of maximum girth is known to be a necessary condition for optimality with respect to generalized cohesion.

necessarily imply a larger number of spanning trees, or vice versa, there does seem to exist an inverse correlation between these properties, as illustrated by this example. The intuition behind this trend is that for the same number of nodes and edges, the nodes of a graph with a larger diameter are generally more distant from one another. The result is that there are fewer combinations of edges of the graph that could form spanning trees since there are more constraints on the edges in order that more distant nodes be connected. Hence, the number of spanning trees generally decreases with diameter when the number of nodes and edges is held constant.

3.3.3 Principles of the design methodology

When the network application at hand lends itself to a configuration for which an optimal graph exists, then the design process is essentially complete. However, since truly optimal graphs are rare, we unfortunately must resort to alternative topologies which are not quite optimal. However, if we choose these alternative topologies wisely – by bearing in mind relevant optimality conditions and general trends – we can design networks with excellent reliability properties.

Ideally, a network design methodology would appeal to a single, simple family of graphs for all possible network configurations. The family of circulant graphs is the ideal candidate for such a reliability methodology for a number of reasons. The circulant family of graphs is rich — a circulant graph can be defined for most combinations of number of nodes and degree. In addition, circulants inherently possess good reliability properties. For example, in our discussion of circulants in §2.2.1, Theorem 2.2 indicated that nearly all circulants are super-λ. In addition, in a recent work by Sawionek, Wojciechowski and Arabas [70], the family of circulant graphs were shown to *most probably* contain a uniformly optimally reliable graph when such a graph exists, except for when $e \leq n+3$. The authors used local, discrete, approximate optimization techniques to show this result. While this result does not conclusively prove that circulants are optimal, it does verify the belief that the circulant family includes graphs with excellent reliability properties. Another benefit of using circulant graphs in a design methodology is that their properties are well-studied and that many of the results can be used to formulate powerful design guidelines. Thus, for all of the aforementioned reasons, in §3.4 when we design reliable networks, we will be focusing on circulants as candidate graphs.

3.4 Design of reliable networks

Having summarized and interpreted the relationships among notions of optimality and the properties of graphs, we are now in a position to tackle the task of synthesizing reliable networks. In §3.4, we outline procedures for the design of reliable networks.

In the following subsections, we outline network design procedures which minimize the the number of links in a network when the number of nodes n and the level of reliability, with respect to some metric, is constrained. As previously discussed, we focus on all- and two-terminal reliability in the two extreme regions of edge failure probability.

3.4.1 Designing for all-terminal reliability when $p \approx 0$

When $p \approx 0$ and we would like to design a network for a prescribed level of all-terminal reliability, then we know that the class of optimal graphs is restricted to those that are super-λ. In [7], Bauer et. al. derive an explicit bound on p for which super-λ graphs are optimal. The following result gives the range of p for which any super-λ graph has a larger value of $P_c(G,p)$ than any non super-λ graph. The details of the proof can be found in [6].

Theorem 3.1 *For given n, e, $(e \geq n \geq 7)$, let*

1. *$\widehat{\lambda}$ denote the maximum value of λ among all n node, e edge graphs;*

2. *$\widehat{m}_\lambda$ denote the minimum value of m_λ among all such graphs having maximum λ, namely $\widehat{\lambda}$;*

3. *G denote any n node, e edge graph having either $\lambda \neq \widehat{\lambda}$, or $\lambda(G) = \widehat{\lambda}$ and $m_\lambda \neq \widehat{\lambda}$;*

4. $\widehat{G}$ *denote any* n *node,* e *edge graph having* $\lambda = \widehat{\lambda}$ *and* $m_\lambda = \widehat{\lambda}$*;*

5.

$$\widehat{p} = \frac{1}{\frac{e+1}{\widehat{\lambda}+2} + \frac{e-\widehat{\lambda}}{\widehat{\lambda}+1}\left[\binom{e}{\widehat{\lambda}} - \widehat{m}_\lambda - 1\right]}.$$

Then,

$$P_c(G,p) > P_c(\widehat{G},p), \text{ if } p \le \widehat{p}.$$

In [7], Bauer et. al. also derive somewhat complicated conditions which ensure that $P_c(G,p) > (1+\epsilon)P_c(\widehat{G},p)$.

Of the class of super-λ graphs, even degree Harary graphs were shown to be especially good when p is small, since they achieve the fewest number of cutsets of cardinality i, when $\lambda \le i \le 2\Delta - 3$. Thus, if we are principally concerned with all-terminal reliability in the $p \approx 0$ regime, then we should design networks as Harary graphs.

In order to obtain a good estimate on the required degree Δ of a Harary graph to suit a particular application, we can use the following bounds on $P_c(G,p)$, derived in §4.3.1, which are tight when p is small:

$$\begin{aligned} P_c(G,p) &\ge 1 - \left(np^\Delta + \sum_{i=2}^{\lfloor \Delta/2 \rfloor + 1} \binom{n}{i} p^{i\Delta - 2\binom{i}{2}} + \sum_{i=\lfloor \Delta/2 \rfloor + 2}^{\lfloor n/2 \rfloor} \binom{n}{i} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\ &\ge 1 - \left(np^\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \binom{n}{\left\lfloor \frac{\Delta}{2} \right\rfloor + 1} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) + p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \left[2^{n-1} - n \right] \right). \end{aligned} \tag{3.11}$$

If considerations other than all-terminal reliability, such as network diameter, are important, then a circulant other than a Harary graph may be more appropriate. In this case, we would still use (3.11) to obtain an estimate of the required Δ. However, since even degree Harary graphs are optimal for all-terminal reliability when p is small, it may be necessary to increment Δ to meet the prescribed level of all-terminal reliability. Again, (3.1) and (3.2) could be used to determine this.

We conclude this subsection with an example of the above procedure. Suppose that a 20 node network is to be designed which possesses a maximum probability of graph disconnection of 10^{-6} when the probability that a link is functional is 10^{-2}. Using (3.11), we observe that a value of Δ of four or more would guarantee the desired level of reliability, as $\Delta = 4$ yields an upper bound of 8.1665×10^{-7} for the probability of graph disconnection. Furthermore, using $np^{\Delta}(1-p)^{e-\Delta}$ as a simple lower bound for the probability of graph disconnection, we see that a degree three Harary graph is inadequate as this lower bound corresponds to a value of 1.3789×10^{-5}. This conclusion could have also been reached by appealing to the tighter, more complex bounds given in (3.1) and (3.2).

3.4.2 Designing for two-terminal reliability when $p \approx 0$

We herein consider the task of designing a network with n nodes which meets an objective value of $\min_{s,d}\left[P_c^{sd}(G,p)\right]$. As discussed in §2.3.2, good candidate graphs are those that achieve the bounds (2.4) in §2.2.2. Cages are the graphs which do so, although they only exist for a handful of configurations. In §3.2.2, we introduced the idea of considering the bounds in (2.4) for only $\lambda \leq m \leq 2\Delta - 3$. The interpretation of the bounds in this range is that any (not necessarily prime) cutset of cardinality i for for $\lambda \leq i \leq 2\Delta - 3$ isolates either s or d alone. Incidentally, even degree Harary graphs were shown to possess this property, as discussed in §2.2.1. Hence, Harary graphs are a good design choice when two-terminal reliability is of principal interest.

Our procedure for designing appropriate Harary graphs mirrors that of the previous subsection. In order to obtain a good estimate on the required degree Δ of the Harary graph to suit a particular application, we can use the following bounds on

$P_c^{sd}(G,p)$, derived in §4.3.2, which are tight when p is small:

$$\begin{aligned} P_c^{sd}(G,p) &\geq 1 - \left(2p^{\Delta} + 2 \sum_{i=2}^{\lfloor \Delta/2 \rfloor + 1} \binom{n-2}{i-1} p^{i\Delta - 2\binom{i}{2}} + 2 \sum_{i=\lfloor \Delta/2 \rfloor + 2}^{\lfloor n/2 \rfloor} \binom{n-2}{i-1} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\ &\geq 1 - \left(2p^{\Delta} + 2 \left\lfloor \frac{\Delta}{2} \right\rfloor \binom{n-2}{\lfloor \frac{\Delta}{2} \rfloor} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) + p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} 2^{n-2} \right). \end{aligned} \tag{3.12}$$

If considerations other than all-terminal reliability, such as network diameter, are important, however, then a circulant other than a Harary graph may be more appropriate. In this case, we would still use (3.12) to obtain an estimate of the required Δ. However, since even degree Harary graphs are particularly good for two-terminal reliability when p is small, it may be necessary to increment Δ to meet the prescribed level of all-terminal reliability. Again, (3.4) could be used to determine this.

3.4.3 Designing for all-terminal reliability when $p \approx 1$

In this subsection, our task is to design a network with n nodes which adheres to a prescribed level of all-terminal reliability when $p \approx 1$. In order to obtain a *rough* estimate of the required degree Δ, we employ (3.6) in conjunction with (3.8) to obtain:

$$P_c(G,p) \sim \Delta \left(\Delta - 1/2\right)^{n-2} (1-p)^{n-1} p^{e-n+1}. \tag{3.13}$$

After determining an estimate for Δ from the above expression, we now search the finite space of n node circulants with degree Δ for the configuration with the largest number of spanning trees. Recall that the number of spanning trees of a circulant is easily computed using (2.2). Alternatively, we noted in §3.3.2 that there seems to exist an inverse correlation between the number of spanning trees and the diameter of a graph. Therefore, if we wish to design a network with a large number of spanning trees, it is reasonable to alternatively design a network with a small diameter. Thus, if a configuration for a minimum diameter circulant is readily available, as in the case of degree four circulants, an exhaustive search over all candidate circulant graphs could be avoided. After deciding upon a configuration of an n node, degree Δ circulant G and determining its number of spanning trees $t(G)$, we compute $t(G)(1-p)^{n-1}p^{e-n+1}$. If this value exceeds the prescribed level of reliability, then by (3.6), G achieves the prescribed level of reliability.

To illustrate this procedure, suppose that a 20 node network is to be designed which possesses a probability of graph connection of at least 10^{-4} when the probability p that a link fails is 0.9. Using (3.13), we decide upon $\Delta = 12$ as our starting point. The 20 node, degree 12 circulant with the maximum number of spanning trees is $C_{20}\langle 1, 3, 5, 6, 7, 9\rangle$ with 3.3317×10^{19} spanning trees. Using this value, we compute $t(G)(1-p)^{n-1}p^{e-n+1}$ to be 7.9645×10^{-5}, which is slightly worse than our required level of reliability. We thus increase Δ to 13, and consider the circulant $C_{20}\langle 1, 2, 4, 5, 7, 8, 10\rangle$, which has a maximum 1.5889×10^{20} spanning trees. This yields a value of 1.3244×10^{-4} for $t(G)(1-p)^{n-1}p^{e-n+1}$, which implies that $C_{20}\langle 1, 2, 4, 5, 7, 8, 10\rangle$ meets our requirements.

3.4.4 Designing for two-terminal reliability when $p \approx 1$

In this subsection, we consider the task of designing a network with a constraint on the two-terminal reliability metric $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when $p \approx 1$. We recall from §3.2.4 that $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ can be bounded as follows:

$$(1-p)^{k(G)} \leq \min_{s,d}\left[P_c^{sd}(G,p)\right]. \tag{3.14}$$

Using this inequality, we first determine a value for the diameter k. The value chosen for Δ should be as small as possible, while still sufficiently large to ensure that a circulant with the specified values of n, k and Δ can be constructed. As mentioned in §2.2.1, the relationship among n, k and Δ for circulant graphs was investigated in [14] by Boesch and Wang[4]. Table 3.4 summarizes their results. With knowledge of n and k one can obtain a lower bound for Δ by consulting Table 3.4. The design task is now reduced to constructing a circulant graph with the specified parameters

[4]At the end of this section, we interpret some of the results contained in that work.

$\Delta \backslash k$	1	2	3	4	5	6
2	$3,3$	$5,5$	$7,7$	$9,9$	$11,11$	$13,13$
3	$4,4$	$8,10$	$12,22$	$16,46$	$20,94$	$24,190$
4	$5,5$	$13,17$	$25,53$	$41,161$	$61,485$	$85,1457$
5	$6,6$	$18,24$	$38,106$	$66,426$	$102,2230$	$146,6826$
6	$7,7$	$25,37$	$63,187$	$129,937$	$231,4687$	$377,24437$
7	$8,8$	$32,50$	$88,302$	$192,1814$	$360,10886$	$608,65318$
8	$9,9$	$41,65$	$129,457$	$321,3201$	$681,22409$	$1289,156865$
9	$10,10$	$50,82$	$170,658$	$450,5266$	$1002,42130$	$1970,337042$
10	$11,11$	$61,101$	$231,911$	$681,8201$	$1683,73811$	$3653,664301$
11	$12,12$	$72,122$	$292,1222$	$912,12222$	$2364,122222$	$5336,1222222$
12	$13,13$	$85,145$	$377,1597$	$1289,17569$	$3653,193261$	$8989,2125873$

Table 3.4: Relationship among n, k and Δ for circulant graphs. The vertical axis represents degree Δ and the horizontal axis represents diameter k. The first element in each ordered pair entry is an upper bound on the number of nodes possible in any circulant with the associated degree and diameter; the second element is the corresponding Moore bound for the number of supportable nodes.

n, k and Δ. Since there are only a finite number of circulant graphs with n nodes and degree Δ, it is a simple matter to search for a configuration which meets the required diameter k. If no such graph exists, we increment Δ and again search for a configuration which meets the required diameter k.

As an example of the above procedure, suppose that a 20 node network is to be designed which possesses a minimum probability of node pair connection of at least 10^{-4} when the probability that a link fails p is 0.9. Using the inequality in (3.14), we conclude that a diameter of four is required for the network. Referring to Table 3.4, we observe that for a circulant graph of 20 nodes and diameter four, a degree of at least four is required. However, upon closer examination of the table, we notice that, for the same degree, it may be possible to find a 20 node circulant with diameter three instead, which would be preferable. We, therefore, first search the space of 20 node, degree four circulants for a graph which possesses diameter three. Fortunately, such a graph exists — it is the circulant $C_{20}\langle 3,4\rangle$. We would therefore design our network as $C_{20}\langle 3,4\rangle$, although an inferior circulant with diameter four, such as $C_{20}\langle 4,5\rangle$, would suffice as well. These two candidate topologies are illustrated in Figures 3-3 and 3-4.

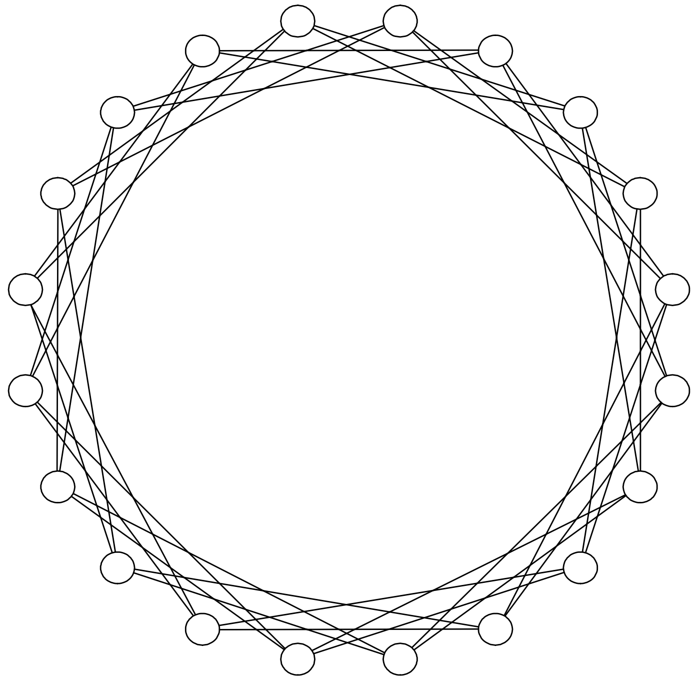


Figure 3-3: The $C_{20}\langle 3, 4\rangle$ circulant graph. This graph possesses the minimum diameter of three among all 20 node, degree four circulants.

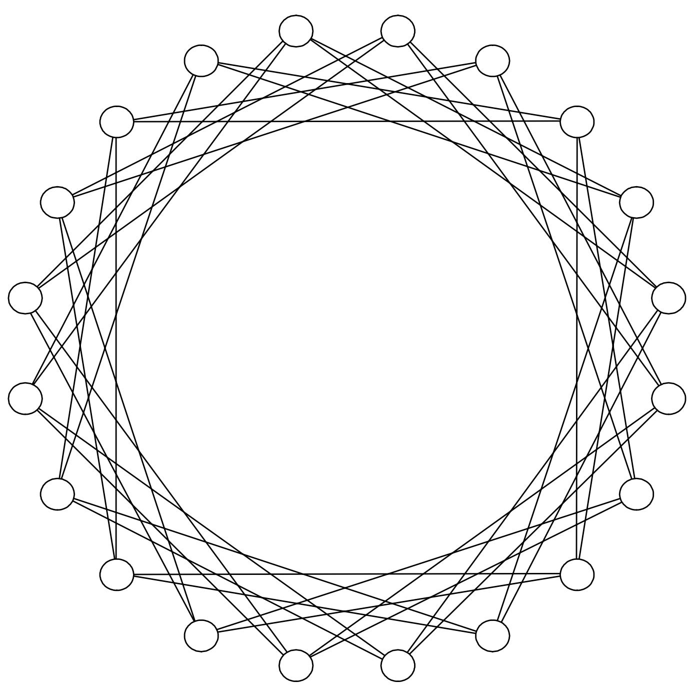


Figure 3-4: The $C_{20}\langle 4, 5\rangle$ circulant graph. This graph possesses a diameter of four.

Diameters of circulant graphs

We now present and interpret the central results of Boesch and Wang's work [14] on the diameters of circulants.

Theorem 3.2 *Let* $G = C_n\langle a_1, a_2, \ldots, a_h\rangle$, $a_h < n/2$. *If* $X_m(h) \geq n > X_{m-1}(h)$, *then* $k(G) \geq m$, *where:*

$$\begin{aligned} X_m(h) &= 1 + \sum_{i=1}^{m} Y_i \\ Y_i &= \sum_{j=1}^{min(h,i)} \binom{h}{j}\binom{i-1}{j-1} 2^j. \end{aligned} \tag{3.15}$$

Theorem 3.3 *Let* $G = C_n\langle a_1, a_2, \ldots, a_h, n/2\rangle$, n *even. If* $Z_m(h) \geq n-1 > Z_{m-1}(h)$, *then* $k(G) \geq m$, *where:*

$$\begin{aligned} Z_m(h) &= 1 + \sum_{i=1}^{m} (Y_i + Y_{i-1}) \\ &\textit{with} \\ Y_0 &= 1, \ \textit{for } i \geq 1 \end{aligned} \tag{3.16}$$

and Y_i *is as defined in Theorem 3.2.*

In Theorem 3.2, (3.15) can be rewritten as:

$$\begin{aligned} X_m(h) &= 1 + \sum_{i=1}^{m} \sum_{j=1}^{\min(h,i)} \binom{h}{j}\binom{i-1}{j-1} 2^j \\ &= 1 + \sum_{j=1}^{\min(h,m)} \sum_{i=j}^{m} \binom{h}{j}\binom{i-1}{j-1} 2^j \\ &= 1 + \sum_{j=1}^{\min(h,m)} \binom{h}{j}\binom{m}{j} 2^j. \end{aligned} \tag{3.17}$$

Let us first examine the relationship between $X_m(h)$ and m, while holding h fixed. Owing to the fact that $\binom{m}{j}$ can be expressed as an m^{th} order polynomial in m divided by an $(m-j)^{\text{th}}$ order polynomial in m, we observe that each term of the form $\binom{h}{j}\binom{m}{j}2^j$

can be expressed as a j^{th} order polynomial in m. Hence, $X_m(h)$ can be expressed as a polynomial in m with leading term m^h. Moreover, it is not difficult to show that the leading coefficient in this polynomial is $\frac{2^h}{h!}$. For example:

$$X_m(1) = 2m + 1$$
$$X_m(2) = 2m^2 + 2m + 1$$
$$X_m(3) = \frac{4}{3}m^3 + 2m^2 + \frac{8}{3}m + 1.$$

We therefore conclude that in the best case, the diameters of even degree Δ circulants grow as the $\left(\frac{\Delta}{2}\right)^{\text{th}}$ root of the number of nodes n. In a similar manner to the even degree circulant case, for odd degree circulants we can show that $Z_m(h)$ in (3.16) can be expressed as a polynomial in m of order h, with leading coefficient $\frac{2^{h+1}}{h!}$. For example:

$$Z_m(1) = 4m - 1$$
$$Z_m(2) = 4m^2 + 1$$
$$Z_m(3) = \frac{8}{3}m^3 + \frac{16}{3}m - 1.$$

Hence, in the best case, the diameters of odd degree Δ circulants grow as the $\left(\frac{\Delta-1}{2}\right)^{\text{th}}$ root of the number of nodes n.

For the special case of degree four circulants, Boesch and Wang provided a construction of minimum diameter graphs, which are also super-λ and sometimes super-χ. These graphs have the form $C_n\langle m, m+1\rangle$, where $m = \left\lceil \left(-1 + \sqrt{2n-1}\right)/2 \right\rceil$, and m is further shown to be the graph diameter. Table 3.5 compares the diameters of these graphs with those of Harary graphs for several values of n. Recall from our analysis in §4.3.4 that the diameters of Harary graphs grow linearly with the number of nodes n. On the other hand, we recall from our discussion in §2.2.2 that the diameters of Moore graphs grow with the logarithm of the number of nodes n.

We now turn our attention to the relationship between $X_m(h)$ and h while holding m fixed. Examining the structure of (3.17), we observe that $X_m(h)$ is symmetric in

n	$k\left(C_n\langle 1,2\rangle\right)$	$m = k\left(C_n\langle m, m+1\rangle\right)$
10	3	2
20	5	3
30	8	4
40	10	4
50	13	5
60	15	5

Table 3.5: Diameter comparison of degree four circulants. Degree four Harary graphs (first column) are compared with minimum diameter degree four circulants (second column). Note that $m = \lceil \left(-1 + \sqrt{2n-1}\right)/2 \rceil$.

h and m. Therefore, $X_m(h)$ can be expressed as an m^{th} order polynomial in h with the same coefficients as the h^{th} order polynomial in m. Thus, for example:

$$X_1(h) = 2h + 1$$
$$X_2(h) = 2h^2 + 2h + 1$$
$$X_3(h) = \frac{4}{3}h^3 + 2h^2 + \frac{8}{3}h + 1.$$

Using this observation for even degree circulants, we conclude that for odd degree circulants $Z_m(h)$ in (3.16) can be expressed as an m^{th} order polynomial in h. The leading coefficient of this polynomial can be easily shown to be $\frac{2^m}{m!}$. For example:

$$Z_1(h) = 2h + 1$$
$$Z_2(h) = 2h^2 + 4h + 1$$
$$Z_3(h) = \frac{4}{3}h^3 + 4h^2 + \frac{14}{3}h + 1.$$

Hence, we conclude that for minimum diameter circulants of degree Δ and diameter k, the maximum number of supported nodes is bounded by a k^{th} order polynomial in Δ with leading coefficient $\frac{1}{k!}$. Compare this to Moore graphs, where the number of supported nodes is given by a k^{th} order polynomial in Δ with leading coefficient 1. On the other hand, for Harary graphs the number of supported nodes grows linearly with Δ and the constant of proportionality is roughly k for even Δ and $2k$ for odd Δ.

It is interesting to compare the graphs corresponding to the circulant and Moore bounds in Table 3.4. Examining the table entries, it is apparent that for networks of 50 nodes or less the difference in the minimum degree required, when the diameter is held constant, is usually zero or one and occasionally two. Furthermore, it is worth recalling that, with the exception of a few configurations, Moore graphs are not realizable. We, therefore, conclude that circulant graphs which achieve the bounds in Table 3.4 are optimal or nearly optimal with respect to two-terminal reliability when the number of nodes is on the order of tens, which is the case for most networks of interest.

3.5 Simulation results

In this section, we present simulation results for several network designs. These results verify our previous insights, and also shed considerable light on the relative performance of different network configurations. In the following discussion, the two-terminal reliability of a graph refers to the worst-case two-terminal reliability over all node pairs in the graph.

In our first set of simulations, we consider the Petersen graph (the Moore graph with $g = 5$ and $\Delta = 3$)[5], and the Harary graphs, $H(10, 3)$ and $H(10, 4)$. When $p \approx 0$, Figure 3-5 indicates the expected result that $H(10, 4)$ possesses a lower probability of disconnection by a factor of approximately p relative to the Petersen graph and $H(10, 3)$. Perhaps an unexpected finding is the closeness of the performance of the Petersen graph and $H(10, 3)$ when $p \approx 0$. With respect to all-terminal reliability, we expect that odd-degree Harary graphs achieve close to the minimum number of cutsets of small cardinality, while it is only known that the Petersen graph is super-λ. On the other hand, the Petersen graph achieves the minimum bound on the two-terminal, deterministic parameter $X^e(m)$ when m is small, as discussed in §2.2.2. Thus, we expect the Petersen graph to attain close to optimal performance with respect to two-terminal reliability. On the other hand, it is only known that $H(10, 3)$

[5]See Figure 2-8 for an illustration of the graph.

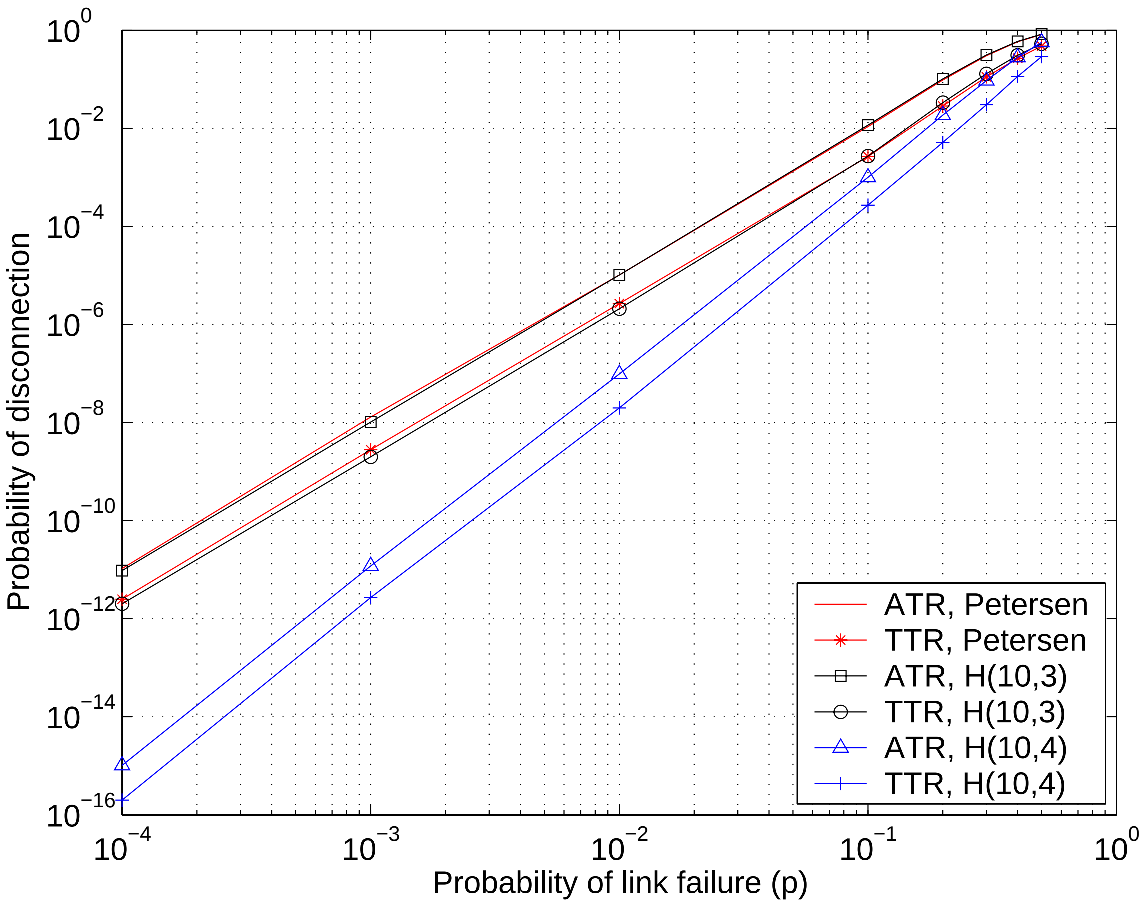


Figure 3-5: Probability of disconnection versus p for the Petersen graph, $H(10,3)$ and $H(10,4)$ when $p \leq 1/2$.

achieves the bound on $X^e(\lambda)$. In spite of this, the all- and two-terminal reliability performances of the Petersen graph and $H(10,3)$ are remarkably similar. In fact, all- and two-terminal reliability can be well-approximated by np^Δ and $2p^\Delta$, respectively, when $p \approx 0$. Thus, with respect to all- and two-terminal reliability when $p \approx 0$, the small family of Moore graphs offer little or no benefit over the richer family of super-λ graphs.

When $p \geq 1/2$, the Petersen graph and $H(10,3)$ exhibit very similar all-terminal reliability performance, as shown in Figure 3-6. This can be attributed to the fact that the graphs possess a similar number of spanning trees – 2000 and 1815, respectively. On the other hand, $H(10,4)$ performs considerably better than these two graphs, owing to its greater number of spanning trees (30250). More interesting is the relative

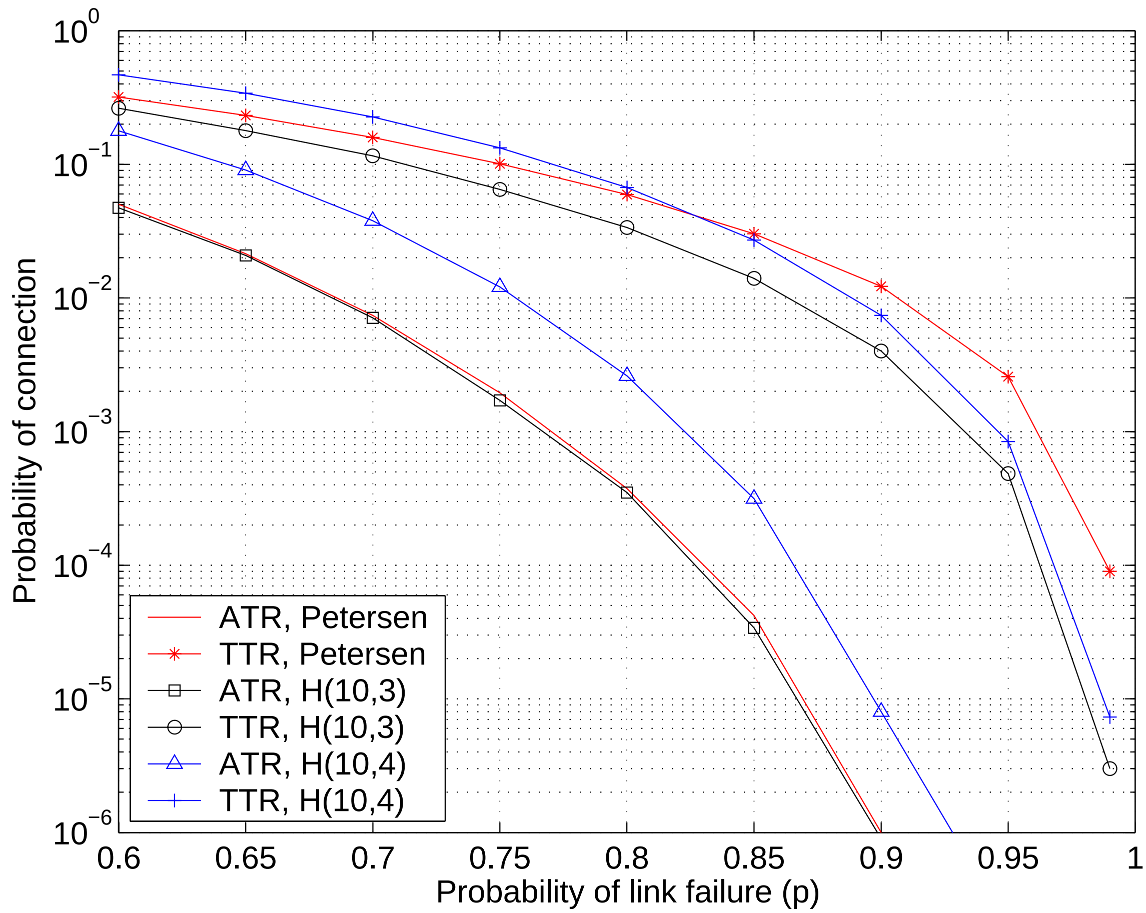


Figure 3-6: Probability of connection versus p for the Petersen graph, $H(10,3)$ and $H(10,4)$ when $p \geq 1/2$.

two-terminal performance of these three graphs. When $0.6 \lessapprox p \lessapprox 0.8$, $H(10,4)$ performs better than the Petersen graph and $H(10,3)$, owing to its richer number of initial edges. However, once p becomes sufficiently large, the connectedness of a node pair is governed mostly by the existence of a shortest path between the nodes. Since the Petersen graph has a diameter of two, whereas $H(10,3)$ and $H(10,4)$ have diameters of four, we expect that for p sufficiently large the Petersen graph has the best two-terminal performance among all three graphs. This is indeed the case, as illustrated in Figure 3-6. Therefore, it is only when two-terminal reliability in the $p \approx 1$ regime is of interest that Moore graphs present a significant advantage over other competing topologies.

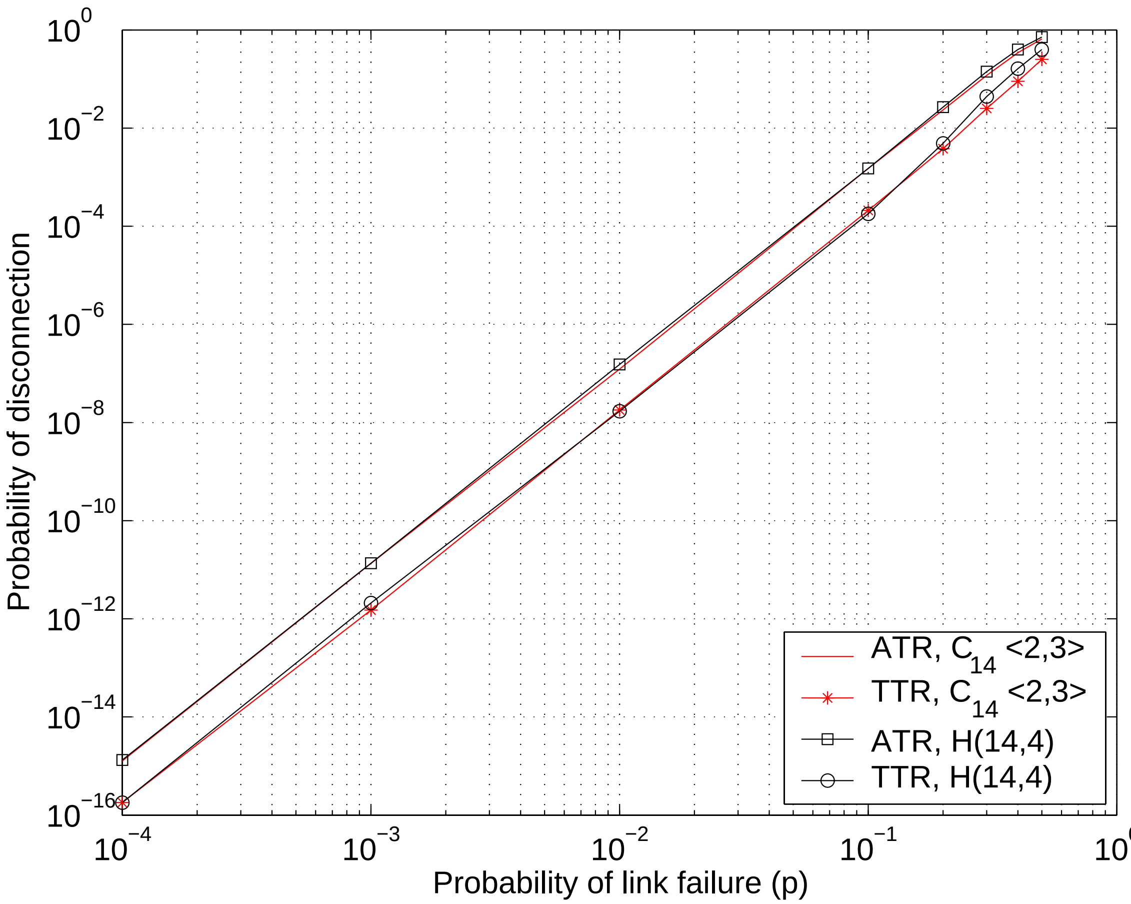


Figure 3-7: Probability of disconnection versus p for $C_{14}\langle 2, 3\rangle$ and $H(14, 4)$ when $p \leq 1/2$.

In our second set of simulations, we consider the circulant $C_{14}\langle 2, 3\rangle$, which has minimum diameter of three among all circulants with the same number of nodes and edges, and the Harary graph $H(14, 4)$. When $p \leq 1/2$, the graphs perform as illustrated in Figure 3-7. The most noteworthy observation is that, as in the previous set of simulations, Harary graphs perform very similarly to other super-λ graphs. Thus, if another performance metric is of importance besides all- or two-terminal reliability in the $p \approx 0$ regime, then very little reliability is lost in resorting to an alternate super-λ topology.

When $p \geq 1/2$, Figure 3-8 illustrates that $C_{14}\langle 2, 3\rangle$ and $H(14, 4)$ exhibit very similar all-terminal reliability performances. This can be explained by the fact that the two graphs possess a similar number of spanning trees (4.3747×10^6 and 1.9898×10^6,

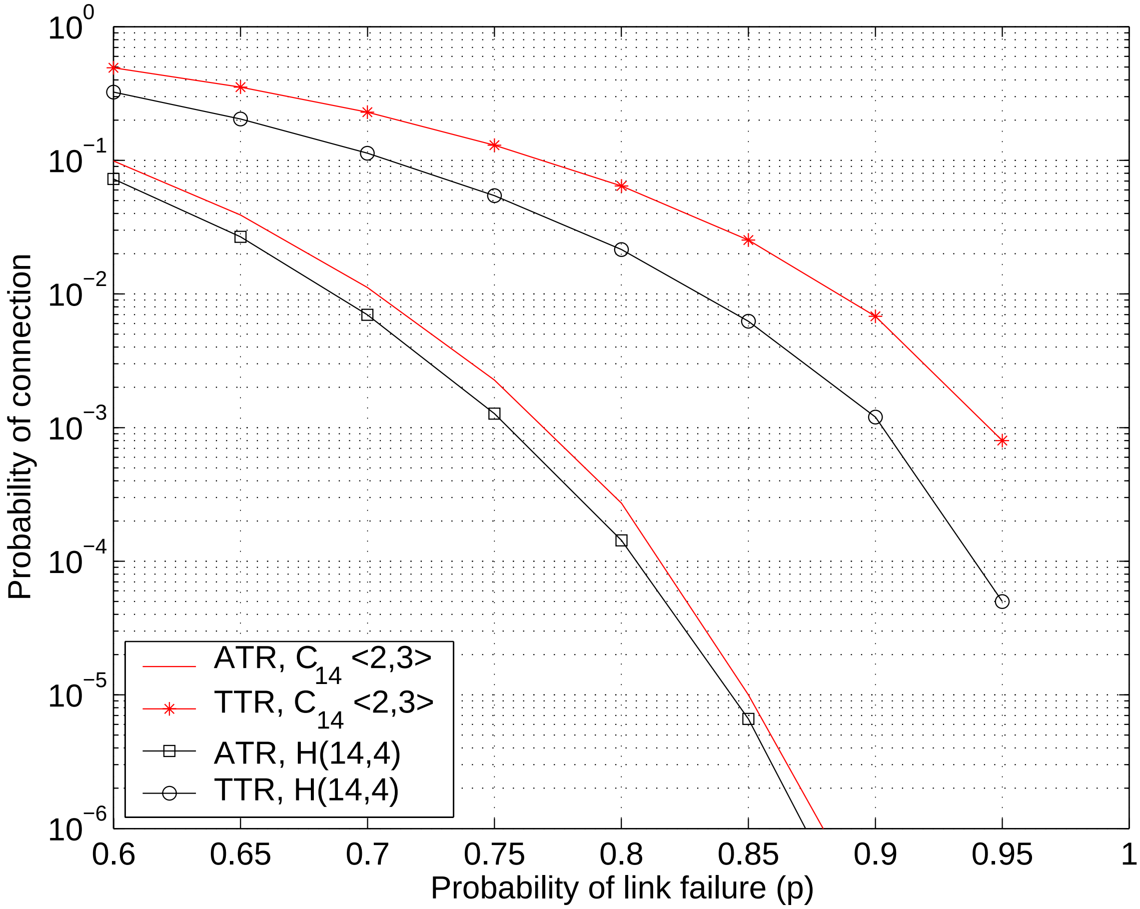


Figure 3-8: Probability of connection versus p for $C_{14}\langle 2,3\rangle$ and $H(14,4)$ when $p \geq 1/2$.

respectively). The two-terminal performance difference between these two graphs, however, is significant when p becomes sufficiently large. As previously discussed, in the $p \approx 1$ regime, graph diameter is the figure of merit when considering two-terminal reliability. Hence, we expect $C_{14}\langle 2,3\rangle$ with a diameter of three to perform better than $H(14,4)$ with a diameter of four, which is indeed the case.

In our third set of simulations, we consider the circulants $C_{14}\langle 2,4,6,7\rangle$, and $C_{14}\langle 1,3,5,7\rangle$[6], as well as the Harary graph $H(14,7)$. Figure 3-9 depicts the performance of the three graphs when $p \leq 1/2$. The graph $C_{14}\langle 2,4,6,7\rangle$ is one of the rare circulants which is not super-λ. It possesses 15 cutsets of order $\lambda = 7$, whereas the other two super-λ graphs have $n = 14$ cutsets of order seven. As far as all-terminal

[6]An n node circulant in which node i is adjacent to nodes $i+1$, $i+1+\alpha$, $i+1+2\alpha, \ldots, i-1 (\bmod n)$ is known as a *symmetric Hamilton graph*.

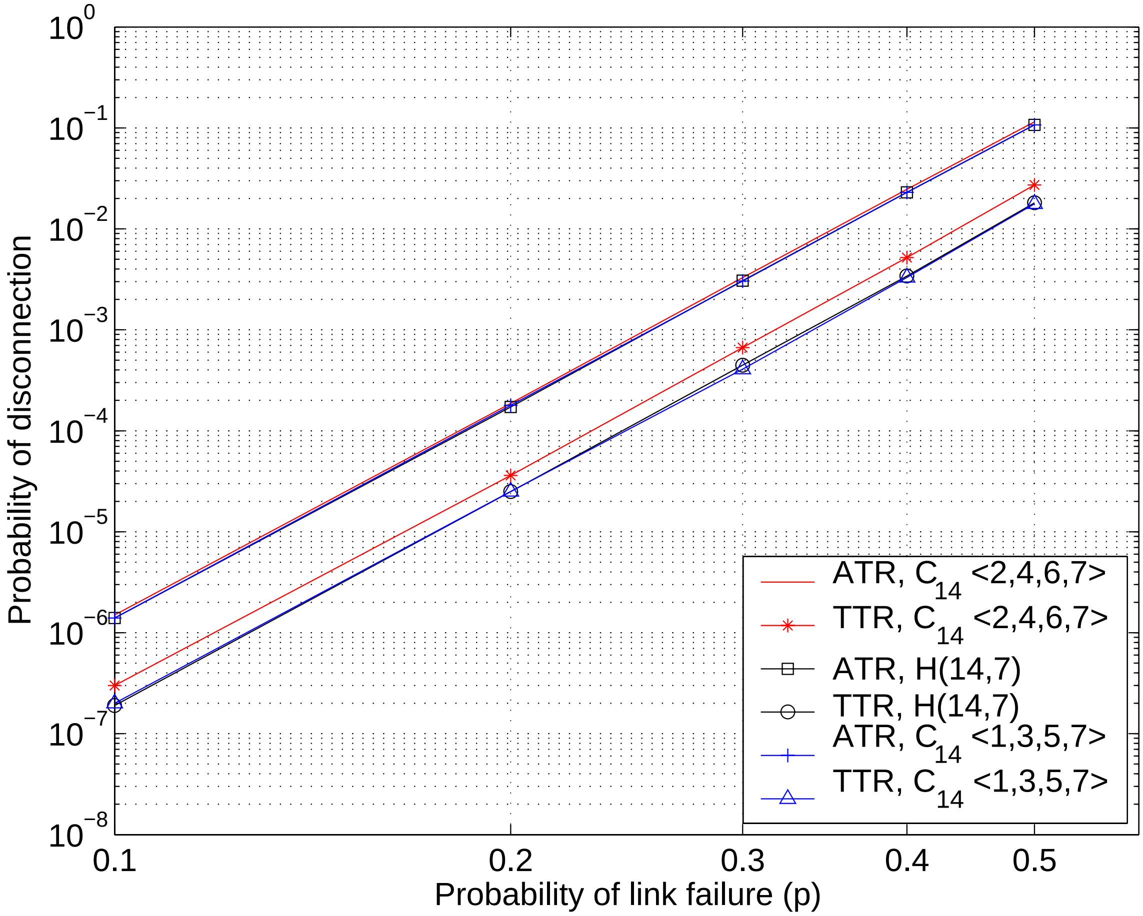


Figure 3-9: Probability of disconnection versus p for $C_{14}\langle 2,4,6,7\rangle$, $C_{14}\langle 1,3,5,7\rangle$ and $H(14,7)$ when $p \leq 1/2$.

reliability, the performance of $C_{14}\langle 2,4,6,7\rangle$ is nearly identical to that of $C_{14}\langle 1,3,5,7\rangle$ and $H(14,7)$. The two-terminal performance difference is more appreciable, however. Recall that we define two-terminal reliability as $\min_{s,d}\left[P_c^{sd}(G,p)\right]$. Thus, the worst-case probability of node pair disconnection of $C_{14}\langle 2,4,6,7\rangle$ is approximately 1.5 times larger than that of the other two graphs since it is possible to find a node pair in the graph for which there are three cutsets of order seven, whereas in the other two graphs all node pairs have only two such cutsets.

When $p \geq 1/2$, the performance of the three graphs is illustrated in Figure 3-10. The circulant $C_{14}\langle 1,3,5,7\rangle$ possesses the maximum number of spanning trees (1.3841×10^{10}) among all 14 node, degree seven circulants. By comparison, $C_{14}\langle 2,4,6,7\rangle$

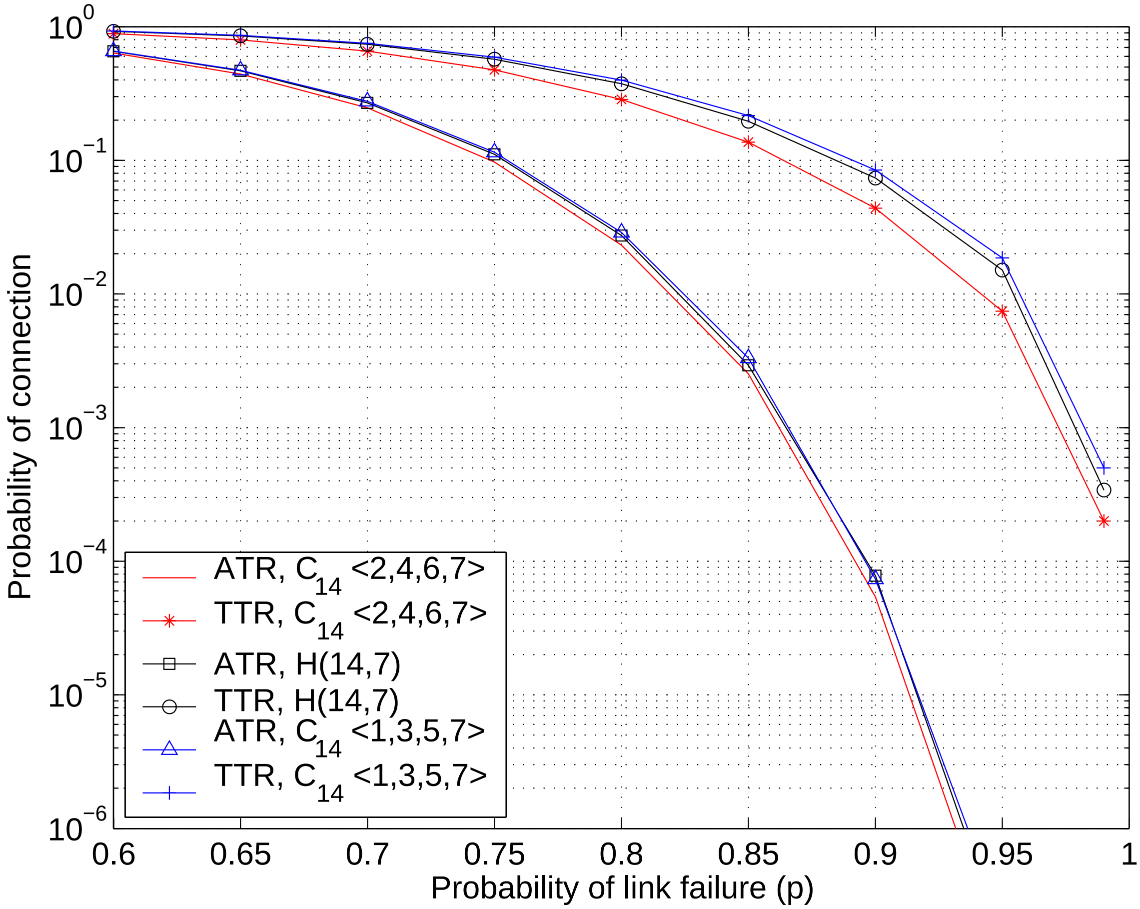


Figure 3-10: Probability of connection versus p for $C_{14}\langle 2,4,6,7\rangle$, $C_{14}\langle 1,3,5,7\rangle$ and $H(14,7)$ when $p \geq 1/2$.

and $H(14,7)$ possess 8.9319×10^{9} and 1.1944×10^{10} spanning trees, respectively. These values explain the very similar all-terminal performance of the three graphs. The three graphs also exhibit similar two-terminal performances, as can be seen in the same figure. This trend is a result of the fact that these three richly connected graphs all have diameter two. In general, we expect Harary graphs to have significantly worse two-terminal (and to a lesser extent, all-terminal) reliability performance relative to other circulants owing to its large diameter. This trend was not present in this set of simulations because the degree of the circulants was large enough to ensure that the diameters were comparable.

In our final set of simulations, we investigate the effect of node degree on reliability in the $p \geq 1/2$ regime. Specifically, we are interested in determining the node degrees required to achieve all- and two-terminal reliabilities in the useful range of 0.1 to 1. For our simulations, we consider a variety of 14 node circulants. Two graphs, the Harary graph $H(14, 4)$ and the minimum diameter circulant $C_{14}\langle 2, 3\rangle$, are of degree four; two graphs, $H(14, 7)$ and $C_{14}\langle 1, 3, 5, 7\rangle$, are of degree seven; two graphs, $H(14, 10)$ and $C_{14}\langle 1, 2, 4, 5, 6\rangle$, are of degree ten; and the complete graph K_{14} is of degree 13.

Figure 3-11 depicts the all-terminal performance of these graphs in the $p \geq 1/2$ regime. As expected, the all-terminal reliability increases monotonically with node degree. Another observation is that the performance difference among graphs with the same node degree is more pronounced at lower node degrees than at higher node degrees. Intuitively, this is because structural changes in sparser graphs can more dramatically affect the reliability properties of graphs than in denser graphs. This also explains why in our previous simulations in the $p \geq 1/2$ regime the performance difference among graphs becomes more pronounced as p approaches one. Unfortunately, these simulations also indicate that to achieve all-terminal reliabilities in the range of 0.1 to 1 when $p \geq 1/2$, very high node degrees are required. In fact, when p exceeds roughly 0.85, even the complete graph K_{14} cannot achieve a reliability above 0.1. Furthermore, in line with our previous observation, once we realize that a high node degree is required to achieve a reliability in the range of 0.1 to 1, the graph's actual structure is not very important.

Our simulation results for the two-terminal reliability of the same seven graphs in the $p \geq 1/2$ regime are illustrated in Figure 3-11. The trends observed in this figure are similar to those discussed above. In fact, for two terminal-reliability, these trends are even more apparent. For example, the performance difference of graphs with the same node degree is quite significant at lower degrees, while the performance of the graphs is virtually indistinguishable at higher degrees. Intuitively, this is because topological idiosyncrasies of graphs can be magnified in a graph's two-terminal reliability figure since the connectedness of the worst node pair is only considered; whereas in all-terminal reliability, the average connectedness of node pairs, in a sense, is evaluated. The simulation results also indicate that two-terminal reliabilities above

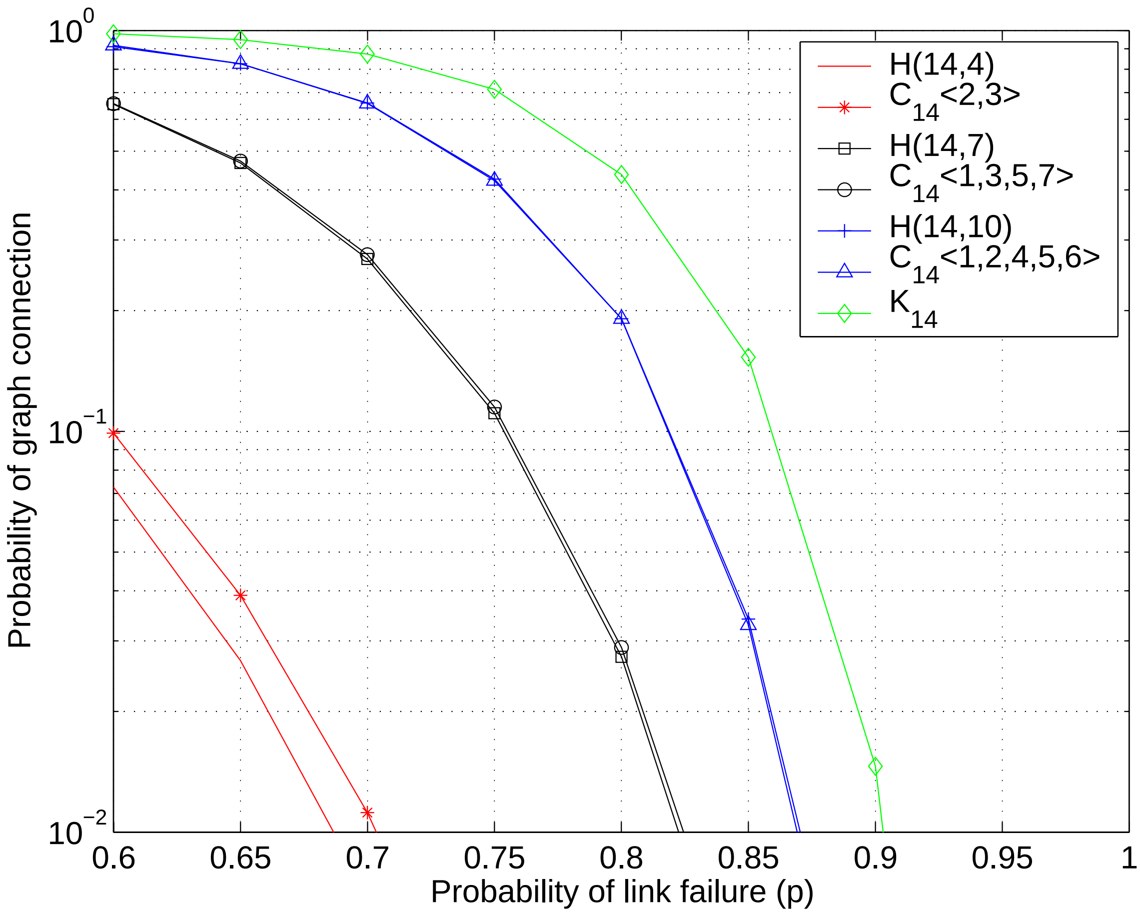


Figure 3-11: Probability of graph connection versus p for $H(14,4)$, $C_{14}\langle 2,3\rangle$, $H(14,7)$, $C_{14}\langle 1,3,5,7\rangle$, $H(14,10)$, $C_{14}\langle 1,2,4,5,6\rangle$ and the complete graph K_{14}, when $p \geq 1/2$.

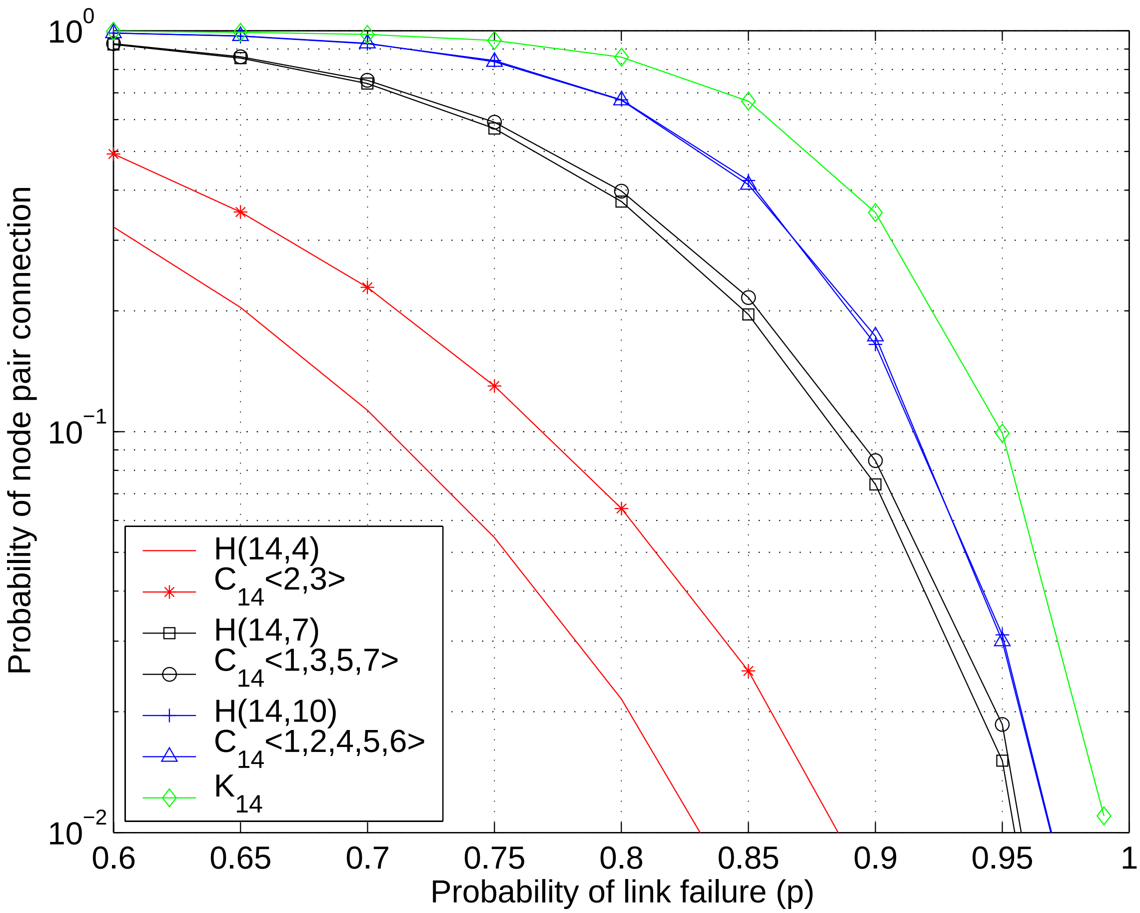


Figure 3-12: Worst-case probability of node pair connection versus p for $H(14,4)$, $C_{14}\langle 2,3\rangle$, $H(14,7)$, $C_{14}\langle 1,3,5,7\rangle$, $H(14,10)$, $C_{14}\langle 1,2,4,5,6\rangle$ and the complete graph K_{14}, when $p \geq 1/2$.

0.1 can be achieved at low to moderate node degrees. For example, the minimum diameter circulant $C_{14}\langle 2,3\rangle$ of degree four, achieves two-terminal reliabilities above 0.1 when p is approximately less than 0.75.

3.6 Summary

In this chapter, we developed an approach to reliable network design assuming that nodes are invulnerable and that links fail in a statistically independent fashion. Our methodology provides excellent solutions to the network synthesis problem with respect to all- and two-terminal reliability over the two extreme regions of link failure.

The networks proposed by our methodology belong to the rich family of circulant graphs, which were previously shown to possess desirable reliability properties. See Table 3.6 for a summary of our design results. As a byproduct of developing our network design methodology, we introduced several new techniques to compute and bound the reliability metrics of networks.

In addition, we carried out an extensive set of simulations to evaluate our methodology and gain new insights into the reliability properties of different graphs. In the $p \approx 0$ regime, we observed that all super-λ graphs with the same number of nodes and edges perform virtually identically. This is a desirable result, since it does not limit our design of networks to Harary graphs which have undesirable reliability properties when $p \approx 1$, or Moore graphs which seldom exist. Furthermore, for the class of super-λ graphs, all- and two-terminal reliability was well-approximated by np^{Δ} and $2p^{\Delta}$, respectively, when $p \approx 0$.

When $p \approx 1$, the relevant figures of merit for all- and two-terminal reliability are number of spanning trees and diameter, respectively. With respect to all-terminal reliability, the variation in performance among graphs with the same number of nodes and node degree was shown to be relatively small, as the graphs possess a comparable number of spanning trees. Two-terminal reliability performance, on the other hand, was shown to be more sensitive to graph structure, particularly at lower node degrees. We attributed this to the definition of two-terminal reliability as the worst-case probability of node pair connection, which allows local topological differences of graphs to be magnified. Finally, we observed that in order to obtain all- and two-terminal reliabilities in the 0.1 to 1 range, very large node degrees are required and that for such high node degree graphs, the actual graph structure is not very important.

	All-terminal reliability $\max_G \ P_c(G,p)$	Two-terminal reliability $\max_G \ \min_{s,d} \left[P_c^{sd}(G,p)\right]$
$p \approx 0$	Super-λ *Harary graphs,* *other super-λ graphs*	Super-λ *Harary, Moore graphs,* *other super-λ graphs*
$p \approx 1$	Maximum number of trees *Maximum tree circulants*	Minimum diameter *Moore graphs,* *minimum diameter circulants*

Table 3.6: Summary of design results. The top line in each quadrant is a necessary condition for optimality with respect to the corresponding vulnerability region and reliability metric. The lines below are the types of graphs suggested by our methodology.

Chapter 4

Case studies of special network topologies

In this chapter, we carry out analyses of special families of graphs. When the topology is simple, as in the case of the Ethernet and ring graphs, exact analytical expressions for the probability of connection is possible. For more complex topologies, we employ bounding techniques.

4.1 Ethernet graph

Ethernet local-area networks (LANs), which represent the majority of LANs implemented today, are architected in a redundant star topology. In such a topology, each node is connected to a primary and secondary switch through a dedicated link. In addition, the two switches are bridged. This architecture is illustrated in Figure 4-1. Communication between a node pair, although normally first attempted through the primary switch, can be carried out via any available path. There are four such paths:

1. s →Primary Switch→ d;

2. s →Primary Switch→Backup Switch→ d;

3. s →Backup Switch→ d;

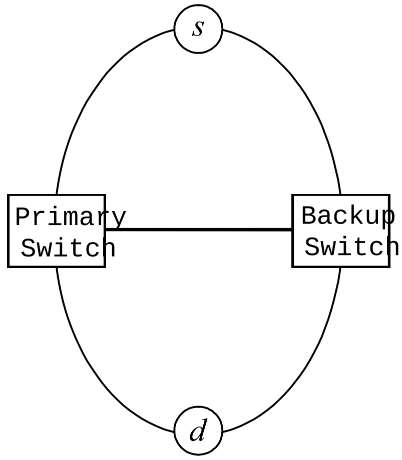


Figure 4-1: Ethernet LAN topology.

4. s →Backup Switch→Primary Switch→ d.

The simplicity of the topology permits us to consider node failures without much additional effort. We therefore assume that the switches fail independently with probability q. In addition, we denote by p_b the probability of failure of the bridge joining the switches. The results derived in this section are summarized in Table 4.1.

4.1.1 All-terminal reliability

In calculating the probability that the Ethernet graph remains connected, we assume that non-switch nodes cannot act as intermediate nodes in relaying traffic to other nodes. The probability of connection is given by the following expression:

$$P_c(G,p,p_b,q) = \Pr(G \text{ connected} \mid \text{both switches up})\Pr(\text{both switches up}) + \Pr(G \text{ connected} \mid \text{exactly one switch up})\Pr(\text{exactly one switch up}).$$

The probability that both switches are up is simply $(1-q)^2$. The probability of connection, given that both switches are up can be further decomposed as:

$$\begin{aligned}\Pr(G \text{ connected} \mid \text{both switches up}) = \\ &\Pr(G \text{ connected} \mid \text{both switches up, bridge up})\Pr(\text{bridge up}) \\ &+ \Pr(G \text{ connected} \mid \text{both switches up, bridge down})\Pr(\text{bridge down}).\end{aligned}$$

Here, the probability of connection, given that both switches are up and the bridge is up is the probability that at least one link incident at each node is up, which is $(1-p^2)^n$. The probability of connection, given that both switches are up and the bridge is down is the probability that all of the links incident at the primary switch or the backup switch are operative, which is $2(1-p)^n - (1-p)^{2n}$. Now, the probability that exactly one switch is up is $2q(1-q)$. Finally, the probability of connection, given that exactly one switch is up is the probability that all links incident at that switch are operative, which is $(1-p)^n$. Putting all of this together, we have:

$$\begin{aligned}P_c(G,p,p_b,q) = \left(1-p^2\right)^n (1-p_b)(1-q)^2 + p_b\left[2(1-p)^n - (1-p)^{2n}\right](1-q)^2 \\ + 2q(1-q)(1-p)^n. \quad (4.1)\end{aligned}$$

If q is small relative to p and p_b, this expression simplifies to:

$$P_c(G,p,p_b,q) \approx \left(1-p^2\right)^n (1-p_b) + p_b\left[2(1-p)^n - (1-p)^{2n}\right].$$

If $p \approx 0$, $p_b \approx 0$ and n is large, this simplifies further to $P_c(G,p,p_b,q) \approx 1-np^2$, which would be a good approximation to the all-terminal reliability of a super-λ graph of degree two, if one existed[1]. If, on the other hand, $p \approx 1$ and $p_b \approx 1$, this expression simplifies to $(1-p)^n\left[(1+p)^n(1-p_b)+2\right]$.

[1]As discussed in Chapter 2, a super-λ graph of degree two does not exist. The Ethernet topology appears to be such a graph owing to the existence of the two switches.

4.1.2 Two-terminal reliability

The probability of connection of a node pair $P_c^{sd}(G, p, q)$ is the probability that at least one of the four paths is operative:

$$\begin{aligned}
P_c^{sd}(G, p, p_b, q) = \Pr(\text{path 1 up}) + \Pr(\text{path 2 up}) + \Pr(\text{path 3 up}) + \Pr(\text{path 4up}) \\
- \Pr(\text{paths 1 and 2 up}) - \Pr(\text{paths 1 and 3 up}) - \Pr(\text{paths 1 and 4 up}) \\
- \Pr(\text{paths 2 and 3 up}) - \Pr(\text{paths 2 and 4 up}) - \Pr(\text{paths 3 and 4 up}) \\
+ \Pr(\text{paths 1, 2 and 3 up}) + \Pr(\text{paths 1, 2 and 4 up}) + \Pr(\text{paths 1, 3 and 4 up}) \\
+ \Pr(\text{paths 2, 3 and 4 up}) - \Pr(\text{paths 1, 2, 3 and 4 up}).
\end{aligned}$$

Hence, the above expression becomes:

$$\begin{aligned}
P_c^{sd}(G, p, p_b, q) = 2(1-p)^2(1-q) + 2(1-p)^2(1-p_b)(1-q)^2 - 4(1-p)^3(1-p_b)(1-q)^2 \\
- (1-p)^4(1-q)^2 + 2(1-p)^4(1-p_b)(1-q)^2. \quad (4.2)
\end{aligned}$$

When q is small relative to p and p_b, this expression simplifies to:

$$P_c^{sd}(G, p, p_b, q) \approx 2(1-p)^2 + 2(1-p)^2(1-p_b) - 4(1-p)^3(1-p_b) - (1-p)^4 + 2(1-p)^4(1-p_b).$$

When $p \approx 1$ and $p_b \approx 1$, this expression simplifies further to $2(1-p)^2$, which is approximately equal to the probability that either one of the two-hop paths (path 1 or path 3) between the node pair exists.

An attractive feature of the connection probabilities derived for the Ethernet graph is that they possess a weak dependence on the total number of nodes in the network. This dependence is manifested only through the switch reliability parameter q. Specifically, as the number of nodes in the network increases, the switches become more complex and hence, generally less reliable.

	Probability of Connection
ATR	$P_c(G,p,p_b,q) = \left(1-p^2\right)^n (1-p_b)(1-q)^2 + p_b\left[2(1-p)^n - (1-p)^{2n}\right](1-q)^2 + 2q(1-q)(1-p)^n$ $P_c(G,p,p_b,q) \to 1 - np^2,\ \text{if } q \ll p, p_b \text{ and } p, p_b \to 0$ $P_c(G,p,p_b,q) \to (1-p)^n\left[(1+p)^n(1-p_b)+2\right],\ \text{if } q \approx 0 \text{ and } p, p_b \to 1$
TTR	$P_c^{sd}(G,p,p_b,q) = 2(1-p)^2(1-q) + 2(1-p)^2(1-p_b)(1-q)^2 - 4(1-p)^3(1-p_b)(1-q)^2 - (1-p)^4(1-q)^2 + 2(1-p)^4(1-p_b)(1-q)^2$ $P_c^{sd}(G,p,p_b,q) \to 2(1-p)^2,\ \text{if } q \approx 0 \text{ and } p, p_b \to 1$

Table 4.1: Summary of probability of connection expressions for the Ethernet graph.

4.2 Ring graph

As in the Ethernet graph, the simplicity of the ring topology allows us to consider node failures without much additional effort. We therefore assume that nodes fail independently with probability q. The results derived in this section are summarized in Tables 4.2 and 4.3.

4.2.1 All-terminal reliability

The probability that a ring graph remains connected $P_c(G, p, q)$ is equal to the sum of the probabilities of the following mutually exclusive events:

- None of the network elements (edges or nodes) fail;
- Exactly one of the network elements fails;
- Exactly one node and one or both of the edges incident to it fail;
- More than one but no more than $n-1$ consecutive nodes fail and the edges not incident to any one of these failed nodes remain operative.

Hence, $P_c(G, p, q)$ is given by:

$$P_c(G,p,q) = [(1-p)(1-q)]^n + \left[np(1-p)^{n-1}(1-q)^n + nq(1-q)^{n-1}(1-p)^n\right] \\ + \left[nq(1-q)^{n-1}\left(1-(1-p)^2\right)(1-p)^{n-2}\right] + \sum_{i=2}^{n-1} nq^i(1-q)^{n-i}(1-p)^{n-i-1}. \quad (4.3)$$

When q is small relative to p the above expression simplifies to:

$$P_c(G,p,q) \approx (1-p)^n + np(1-p)^{n-1}.$$

If, in addition, $p \approx 0$, this expression simplifies to $P_c(G, p, q) \approx 1 - n^2p^2$. If, on the other hand, $p \approx 1$, the expression simplifies to $n(1-p)^{n-1}$, which is the union bound on the events that at least $n-1$ links are operational.

	Probability of Connection
ATR	$P_c(G,p,q) = [(1-p)(1-q)]^n + \left[np(1-p)^{n-1}(1-q)^n + nq(1-q)^{n-1}(1-p)^n\right] + \left[nq(1-q)^{n-1}\left(1-(1-p)^2\right)(1-p)^{n-2}\right] + \sum_{i=2}^{n-1} nq^i(1-q)^{n-i}(1-p)^{n-i-1}$
TTR	$P_c^{s,s+k}(G,p,q) = (1-p)^k(1-q)^{k-1} + (1-p)^{n-k}(1-q)^{n-k-1} - (1-p)^n(1-q)^{n-2}$ $\min_k\left[P_c^{s,s+k}(G,p,q)\right] = \frac{[(1-p)(1-q)]^{\lfloor n/2 \rfloor}}{1-q} + \frac{[(1-p)(1-q)]^{\lceil n/2 \rceil}}{1-q} - \frac{[(1-p)(1-q)]^n}{(1-q)^2}$ $P_c^{av}(G,p,q) = \frac{2\left([(1-p)(1-q)]^{(n+1)/2} + (1-p)(1-q)\right)\left(1-[(1-p)(1-q)]^{(n-1)/2}\right)}{(n-1)(1-q)\left[1-(1-p)(1-q)\right]} - \frac{[(1-p)(1-q)]^n}{(1-q)^2}, n \text{ odd}$ $P_c^{av}(G,p,q) = \frac{1}{n-1}\left(\frac{2\left[(1-p)(1-q)\right]^{n/2}}{1-q} - \frac{[(1-p)(1-q)]^n}{(1-q)^2}\right) + \frac{2}{n-1}\left[(1-n/2)\frac{[(1-p)(1-q)]^n}{(1-q)^2} + \frac{\left([(1-p)(1-q)]^{n/2+1} + (1-p)(1-q)\right)\left(1-[(1-p)(1-q)]^{n/2-1}\right)}{(1-q)\left[1-(1-p)(1-q)\right]}\right], n \text{ even}$

Table 4.2: Summary of probability of connection expressions for the ring graph.

	Probability of Connection
ATR	$P_c(G,p,q) \to 1 - n^2p^2,$ when $q \ll p$ and $p \to 0$ $P_c(G,p,q) \to n(1-p)^{n-1},$ when $q \approx 0$ and $p \to 1$
TTR	$\min_k \left[P_c^{s,s+k}(G,p,q)\right] \to 1 - n^2p/4,$ when $q \ll p$ and $p \to 0$ $\min_k \left[P_c^{s,s+k}(G,p,q)\right] \to (1-p)^{\lfloor n/2 \rfloor},$ when $q \approx 0$ and $p \to 1$ $P_c^{av}(G,p,q) \to \dfrac{2(1-p)}{n\left[1-(1-p)(1-q)\right]},$ as $n \to \infty$

Table 4.3: Summary of asymptotic probability of connection expressions for the ring graph.

4.2.2 Two-terminal reliability

In a ring graph, $n = e$, and there are two node-disjoint (and hence, edge-disjoint) paths between any pair of nodes. Consider nodes s and $s+k$ which are k nodes apart. The probability of connection of the node pair $P_c^{s,s+k}(G,p,q)$ is the probability that at least one of the two disjoint paths is operative:

$$\begin{aligned} P_c^{s,s+k}(G,p,q) &= \Pr(\text{path 1 up}) + \Pr(\text{path 2 up}) - \Pr(\text{paths 1 and 2 up}) \\ &= (1-p)^k(1-q)^{k-1} + (1-p)^{n-k}(1-q)^{n-k-1} - (1-p)^n(1-q)^{n-2}. \end{aligned} \tag{4.4}$$

By setting the first derivative of the above expression to zero and noting that the second derivative is positive, we can show that the worst-case probability of node pair connection occurs when the node separation k equals $n/2$. Therefore:

$$\min_k \left[P_c^{s,s+k}(G,p,q)\right] = \frac{\left[(1-p)(1-q)\right]^{\lfloor n/2 \rfloor}}{1-q} + \frac{\left[(1-p)(1-q)\right]^{\lceil n/2 \rceil}}{1-q} - \frac{\left[(1-p)(1-q)\right]^n}{(1-q)^2}. \tag{4.5}$$

When q is small relative to p, $p \approx 0$ and n is large, the above expression simplifies to

$$\begin{aligned} \min_k \left[P_c^{s,s+k}(G,p,q)\right] &\approx (1-p)^{\lfloor n/2 \rfloor} + (1-p)^{\lceil n/2 \rceil} - (1-p)^n \\ &\approx 1 - n^2/4p \end{aligned}$$

which is one less the probability that any one of the approximately $n^2/4$ cutsets of order two fail. When $q \approx 0$ and $p \approx 1$, (4.5) simplifies to $(1-p)^{\lfloor n/2 \rfloor}$, which is the probability that the shortest path between the diameterically-spaced node pair is operational.

The average probability that a node pair is connected $P_c^{av}(G,p,q)$ can be expressed in terms our above result:

$$P_c^{av}(G,p,q) = \sum_{i=1}^{\lfloor n/2 \rfloor} P_c^{s,s+i}(G,p,q)\Pr(d=i) \tag{4.6}$$

where $\Pr(d=i)$ denotes the probability that the node pair separation is i.

When n is odd, there exist n node pair possibilities corresponding to each node pair separation value of $1, 2, \ldots, (n-1)/2$. When n is even, there exist n node pair possibilities corresponding to each node pair separation value of $1, 2, \ldots, n/2-1$, and $n/2$ node pair possibilities corresponding to a node pair separation of $n/2$. Therefore, assuming that each node pair possibility is equiprobable, (4.3) becomes, for the case of n odd:

$$P_c^{av}(G,p,q) = \frac{2}{n-1} \sum_{i=1}^{\frac{n-1}{2}} P_c^{s,s+i}(G,p,q).$$

After substituting in (4.11) and performing some tedious algebra, we obtain:

$$P_c^{av}(G,p,q) = \frac{2\left([(1-p)(1-q)]^{(n+1)/2} + (1-p)(1-q)\right)\left(1 - [(1-p)(1-q)]^{(n-1)/2}\right)}{(n-1)(1-q)\left[1-(1-p)(1-q)\right]} - \frac{[(1-p)(1-q)]^n}{(1-q)^2}. \tag{4.7}$$

Assuming now that n is even, (4.3) becomes:

$$P_c^{av}(G,p,q) = \frac{2}{n-1}\sum_{i=1}^{n/2-1} P_c^{s,s+i}(G,p,q) + \frac{1}{n-1}P_c^{s,s+n/2}(G,p,q).$$

After substituting in (4.11) and performing some tedious algebra, we obtain:

$$\begin{aligned} P_c^{av}(G,p,q) = \frac{1}{n-1}\left(\frac{2\left[(1-p)(1-q)\right]^{n/2}}{1-q} - \frac{\left[(1-p)(1-q)\right]^{n}}{(1-q)^2}\right) \\ + \frac{2}{n-1}\Bigg[(1-n/2)\,\frac{\left[(1-p)(1-q)\right]^{n}}{(1-q)^2} \\ + \frac{\left(\left[(1-p)(1-q)\right]^{n/2+1} + (1-p)(1-q)\right)\left(1-\left[(1-p)(1-q)\right]^{n/2-1}\right)}{(1-q)\left[1-(1-p)(1-q)\right]}\Bigg]. \end{aligned} \tag{4.8}$$

As the number of nodes in the graph n gets very large $P_c^{av}(G,p,q)$ is given by:

$$\lim_{n\to\infty} P_c^{av}(G,p,q) = \frac{2(1-p)}{n\left[1-(1-p)(1-q)\right]}, \tag{4.9}$$

which simplifies to $2(1-p)/np$ when q is small relative to p.

4.2.3 Generalization to multi-ring graphs

In this section, we extend the above analysis to multi-ring graphs. In an m multi-ring graph, there are m undirected edges between nodes that would otherwise have one undirected edge in a regular ring graph. The multi-ring graph model is a fairly accurate representation of the UPSR ($m=1$) and BLSR ($m=1,2$) network architectures discussed in §1.2. These architectures, although presented with underlying directed graphs, can, for the purpose of reliability analysis, be thought of as networks with undirected underlying graphs. This is because the two contra-directional fibers within a primary or secondary ring are often placed in close physical proximity, such that the failure of one fiber generally implies the failure of its complementary fiber. Thus, as far as reliability modelling is concerned, this is indistinguishable from a network with true bidirectional links.

	Probability of Connection
ATR	$P_c(G,p,q) = [(1-p^m)(1-q)]^n + \left[np^m(1-p^m)^{n-1}(1-q)^n + nq(1-q)^{n-1}(1-p^m)^n\right] + \left[nq(1-q)^{n-1}\left(1-(1-p^m)^2\right)(1-p^m)^{n-2}\right] + \sum_{i=2}^{n-1} nq^i(1-q)^{n-i}(1-p^m)^{n-i-1}$
TTR	$P_c^{s,s+k}(G,p,q) = (1-p^m)^k(1-q)^{k-1} + (1-p^m)^{n-k}(1-q)^{n-k-1} - (1-p^m)^n(1-q)^{n-2}$ $\min_k\left[P_c^{s,s+k}(G,p,q)\right] = \frac{[(1-p^m)(1-q)]^{\lfloor n/2\rfloor}}{1-q} + \frac{[(1-p^m)(1-q)]^{\lceil n/2\rceil}}{1-q} - \frac{[(1-p^m)(1-q)]^n}{(1-q)^2}$ $P_c^{av}(G,p,q) = \frac{2\left([(1-p^m)(1-q)]^{(n+1)/2} + (1-p^m)(1-q)\right)\left(1-[(1-p^m)(1-q)]^{(n-1)/2}\right)}{(n-1)(1-q)[1-(1-p^m)(1-q)]} - \frac{[(1-p^m)(1-q)]^n}{(1-q)^2}, n \text{ odd}$ $P_c^{av}(G,p,q) = \frac{1}{n-1}\left(\frac{2[(1-p^m)(1-q)]^{n/2}}{1-q} - \frac{[(1-p^m)(1-q)]^n}{(1-q)^2}\right) + \frac{2}{n-1}\left[(1-n/2)\frac{[(1-p^m)(1-q)]^n}{(1-q)^2} + \frac{\left([(1-p^m)(1-q)]^{n/2+1} + (1-p^m)(1-q)\right)\left(1-[(1-p^m)(1-q)]^{n/2-1}\right)}{(1-q)[1-(1-p^m)(1-q)]}\right], n \text{ even}$

Table 4.4: Summary of probability of connection expressions for multi-ring graphs.

In extending the above analysis to an m multi-ring graph all that is required is replacing the edge failure probability p with p^m. This is because an m multi-ring graph can be thought of as a resilient regular ring graph with probability of adjacent nodes becoming disconnected of p^m. The probability that an m multi-ring graph is connected is thus given by:

$$P_c(G,p,q) = [(1-p^m)(1-q)]^n + \left[np^m(1-p^m)^{n-1}(1-q)^n + nq(1-q)^{n-1}(1-p^m)^n\right] + \left[nq(1-q)^{n-1}\left(1-(1-p^m)^2\right)(1-p^m)^{n-2}\right] + \sum_{i=2}^{n-1} nq^i(1-q)^{n-i}(1-p^m)^{n-i-1}. \tag{4.10}$$

The probability of connection of a node pair $P_c^{s,s+k}(G,p,q)$ is given by:

$$P_c^{s,s+k}(G,p,q) = (1-p^m)^k(1-q)^{k-1} + (1-p^m)^{n-k}(1-q)^{n-k-1} - (1-p^m)^n(1-q)^{n-2}. \tag{4.11}$$

The worst-case probability of node pair connection $\min_k\left[P_c^{s,s+k}(G,p,q)\right]$ is given by:

$$\min_k\left[P_c^{s,s+k}(G,p,q)\right] = \frac{[(1-p^m)(1-q)]^{\lfloor n/2\rfloor}}{1-q} + \frac{[(1-p^m)(1-q)]^{\lceil n/2\rceil}}{1-q} - \frac{[(1-p^m)(1-q)]^n}{(1-q)^2}. \tag{4.12}$$

The average probability that a node pair is connected $P_c^{av}(G,p,q)$ for the case of n odd becomes:

$$P_c^{av}(G,p,q) = \frac{2\left([(1-p^m)(1-q)]^{(n+1)/2} + (1-p^m)(1-q)\right)\left(1-[(1-p^m)(1-q)]^{(n-1)/2}\right)}{(n-1)(1-q)\left[1-(1-p^m)(1-q)\right]} - \frac{[(1-p^m)(1-q)]^n}{(1-q)^2}. \tag{4.13}$$

	Probability of Connection
ATR	$P_c(G,p,q) \to 1 - n^2 p^{2m}$, when $q \ll p$ and $p \to 0$ $P_c(G,p,q) \to n\,(1-p^m)^{n-1}$, when $q \approx 0$ and $p \to 1$
TTR	$\min_k \left[P_c^{s,s+k}(G,p,q)\right] \to 1 - n^2 p^m/4$, when $q \ll p$ and $p \to 0$ $\min_k \left[P_c^{s,s+k}(G,p,q)\right] \to (1-p^m)^{\lfloor n/2 \rfloor}$, when $q \approx 0$ and $p \to 1$ $P_c^{av}(G,p,q) \to \dfrac{2\,(1-p^m)}{n\,[1-(1-p^m)\,(1-q)]}$, as $n \to \infty$

Table 4.5: Summary of asymptotic probability of connection expressions for the multi-ring graph.

For the case of n even, $P_c^{av}(G,p,q)$ becomes:

$$P_c^{av}(G,p,q) = \frac{1}{n-1}\left(\frac{2\,[(1-p^m)\,(1-q)]^{n/2}}{1-q} - \frac{[(1-p^m)\,(1-q)]^n}{(1-q)^2}\right) + \frac{2}{n-1}\left[(1-n/2)\,\frac{[(1-p^m)\,(1-q)]^n}{(1-q)^2} + \frac{\left([(1-p^m)\,(1-q)]^{n/2+1} + (1-p^m)\,(1-q)\right)\left(1-[(1-p^m)\,(1-q)]^{n/2-1}\right)}{(1-q)\,[1-(1-p^m)\,(1-q)]}\right]. \tag{4.14}$$

As the number of nodes in the graph n gets very large $P_c^{av}(G,p,q)$ is given by:

$$\lim_{n\to\infty} P_c^{av}(G,p,q) = \frac{2\,(1-p^m)}{n\,[1-(1-p^m)\,(1-q)]}. \tag{4.15}$$

The results derived in this subsection are summarized in Tables 4.4 and 4.5.

4.3 Harary graphs

In the following analysis, we consider the reliability of Harary graphs. The results derived in this section are summarized in Tables 4.6 and 4.7. Before beginning the analysis, we will prove an intuitive and useful result on this family of graphs:

Theorem 4.1 *Consider a Harary graph $H(n,\Delta)$, where Δ is even. Partition the n nodes into a subset of j nodes S_j and a subset of $n-j$ nodes S_{n-j}, where we assume that $j \leq n-j$. Then, the minimum number of edges joining S_j to S_{n-j} occurs when the j nodes in S_j (and hence, the $n-j$ nodes in S_{n-j}) are consecutively numbered (modulo n).*

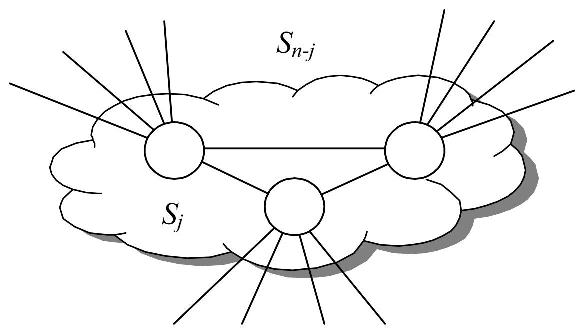
To prove the theorem, we need the following lemma:

Lemma 4.1 *Partition the n nodes of the $H(n,\Delta)$ Harary graph into a subset of $j \leq n-j$ nodes S_j, and a subset of $n-j$ nodes S_{n-j}, such that the nodes in S_j (and hence, the $n-j$ nodes in S_{n-j}) are consecutively numbered (modulo n). Then, the number of edges joining S_j to S_{n-j} is:*

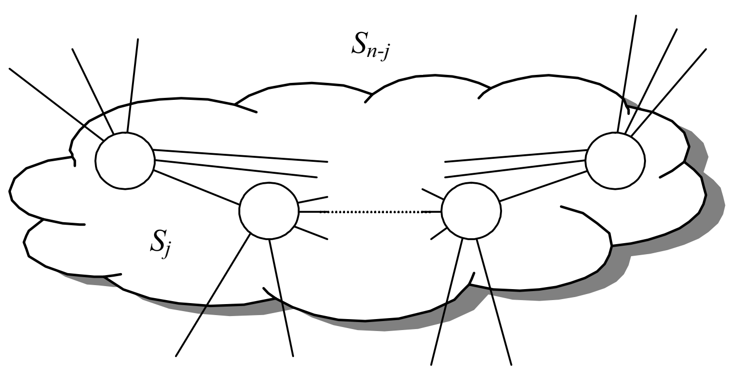
$$\begin{array}{ll} \Delta, & \text{if } j = 1, \\ j\Delta - 2\binom{j}{2}, & \text{if } 2 \leq j \leq \lfloor \Delta/2 \rfloor + 1, \\ \lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil, & \text{otherwise.} \end{array} \qquad (4.16)$$

Proof. The case of $j=1$ is trivial. When $2 \leq j \leq \lfloor \Delta/2 \rfloor + 1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected. In this case, the number of edges joining S_j to S_{n-j} follows from the fact that the total number of edge endpoints incident at S_j's nodes is $j\Delta$ and that the total number of edge endpoints in a fully connected subgraph of j nodes is $2\binom{j}{2}$. For the remaining case, when the nodes are consecutively arranged, the nodes at either end of the S_j partition possess $\lceil \Delta/2 \rceil$ connections to S_{n-j}, the nodes which are second from either end of the partition possess $\lceil \Delta/2 \rceil - 1$ connections to S_{n-j}, and so on. Hence, the total number of edges joining S_j to S_{n-j} is the constant $2\sum_{i=1}^{\lceil \Delta/2 \rceil} i = (\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil)$, as required. □

We are now ready to prove Theorem 4.1:



(a)



(b)

Figure 4-2: Illustration of Lemma 4.1 for a Harary graph with $\Delta = 6$. The top diagram (a) corresponds to the $2 \le j \le \lfloor \Delta/2 \rfloor + 1$ case, where the nodes in S_j are fully connected. In this particular diagram, $|S_j| = 3$. Note that each node in S_j contributes the same number of external links — 4 in this case. The bottom diagram (b) corresponds to the $\lfloor \Delta/2 \rfloor + 2 \le j \le \lfloor n/2 \rfloor$ case, where the nodes in S_j are not fully connected. Starting from the outer edges, each successive node contributes $3, 2, 1, 0, 0, \ldots$ external links.

Proof of Theorem 4.1. The case of $j = 1$ is trivial. Consider now the case of $2 \leq j \leq \Delta/2+1$. Note that minimizing the number of edges joining S_j to S_{n-j} is equivalent to maximizing the number of internal edges shared by the nodes of one of the partitions. When $2 \leq j \leq \Delta/2 + 1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected, yielding the maximum number of internal connections, and hence the minimum number of external edges.

For the remaining case where $\Delta/2 + 2 \leq j \leq n/2$, we carry out the proof by induction. We may use our result for $j = \Delta/2 + 1$ as our base case. Now, assume that a consecutive arrangement of j nodes achieves the minimum number of external edges. Let us now proceed by contradiction by assuming the existence of a partition S'_{j+1} of $j+1$ nodes which achieves a smaller number of external edges than the number achieved by a consecutive arrangement of $j + 1$ nodes in Lemma 4.1.

If we can find a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}, then we move this node to S'_{n-j-1}. This creates a partitioning of the graph into j and $n - j$ nodes which achieves fewer edges joining the two partitions than a consecutive arrangement. This would contradict our induction hypothesis, implying that a consecutive arrangement of nodes is optimal.

Now, let us consider the case where there does not exist a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}. We proceed by finding a pair of consecutive nodes in the graph such that one of the nodes u belongs to S'_{j+1} and the other node v belongs to S'_{n-j-1}. Examining the window of $\Delta + 1$ consecutive nodes centered at u, our assumption that there does not exist a node in S'_{j+1} which has at least $\Delta/2$ edges to S'_{n-j-1} requires that at least $\Delta/2 + 2$ nodes in this window belong to S'_{j+1}. We now consider the window of $\Delta + 1$ consecutive nodes centered at v. Since the window formed by the union of u and v's windows of length $\Delta + 1$ has size $\Delta + 2$ nodes, there can be at most $\Delta/2$ nodes in this larger window that belong to S'_{n-j-1}. By moving v to S'_{j+1}, we create a partitioning of the graph into $j + 2$ and $n - j - 2$ nodes which achieves fewer edges joining the two partitions than that of the S'_{j+1} and S'_{n-j-1} partitioning, and hence, fewer than that of a consecutive arrangement of j and $n - j$ nodes. Note that by moving v to S'_{j+1}, we have not created a node in S'_{j+1} which possesses at least $\Delta/2$ edges to the other partition. This is because the $j+1$ nodes initially in S'_{j+1} only gain internal edges by moving v to S'_{j+1}, and v now possesses fewer than $\Delta/2$ edges to the other partition. Thus, we can continue in this way – finding a pair of consecutive nodes in different partitions and moving one node to the other partition, always decreasing the number of edges connecting the partitions, until we have increased the size of our initial partition of j nodes to $n-j$ nodes. At this point, we have created a partitioning of the graph into j and $n-j$ nodes which achieves fewer edges joining the partitions than the partitioning of the graph in our induction hypothesis, which was assumed to be optimal. This is a contradiction, implying that a consecutive arrangement of nodes is optimal. □

4.3.1 All-terminal reliability when $p \approx 0$

Every graph disconnection scenario can be viewed as a partitioning of the graph into two subsets of nodes which are disconnected. Now, since a partition of j consecutive nodes minimizes the number of edges joining S_j to S_{n-j} in an even degree Harary graph, the probability that a partition of j nodes becomes disconnected from a partition of S_{n-j} nodes is maximized when the partition of j nodes are consecutive. We can therefore form an upper bound for the probability of graph disconnection (and hence, a lower bound for the probability of graph connection) by upper bounding the probability of S_j and S_{n-j} becoming disconnected by the consecutive case, and then employing a union bound on these events. Furthermore, since the $H(n, 2\lfloor\frac{\Delta}{2}\rfloor)$ Harary graph is a subgraph of the $H(n,\Delta)$ Harary graph, the all-terminal reliability of an odd degree Harary graphs is lower bounded by the all-terminal reliability of the Harary graph with degree one less. Thus, a lower bound for $P_c(G,p)$ for a Harary graph $H(n,\Delta)$ is:

$$P_c(G,p) \geq 1 - \left(np^{\Delta} + \sum_{i=2}^{\lfloor\Delta/2\rfloor+1} \binom{n}{i} p^{i\Delta - 2\binom{i}{2}} + \sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor} \binom{n}{i} p^{\lceil\Delta/2\rceil^2 + \lceil\Delta/2\rceil} \right). \quad (4.17)$$

Because cutset failure events were used to derive (4.17), the bound is tight for p close to zero and looser for p close to unity. We can derive a slightly looser lower bound for $P_c(G,p)$ by bounding some of the terms in (4.17) as follows:

$$\begin{aligned}
P_c(G,p) &\geq 1 - \left(np^{\Delta} + \sum_{i=2}^{\lfloor \Delta/2 \rfloor + 1} \binom{n}{i} p^{i\Delta - 2\binom{i}{2}} + \sum_{i=\lfloor \Delta/2 \rfloor + 2}^{\lfloor n/2 \rfloor} \binom{n}{i} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\
&\geq 1 - \left(np^{\Delta} + \sum_{i=2}^{\lfloor \Delta/2 \rfloor + 1} \binom{n}{i} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) + \sum_{i=2}^{\lfloor n/2 \rfloor} \binom{n}{i} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\
&\geq 1 - \left(np^{\Delta} + \left\lfloor \frac{\Delta}{2} \right\rfloor \binom{n}{\lfloor \frac{\Delta}{2} \rfloor + 1} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \right. \\
&\left. + p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \left[2^{n-1} + \frac{1}{2} \binom{n}{\lfloor n/2 \rfloor} - n - 1 \right] \right)
\end{aligned} \tag{4.18}$$

where the second line follows from the fact that $p^{2\Delta-2}$ maximizes $p^{i\Delta - 2\binom{i}{2}}$ over $i \in [2, \lfloor \Delta/2 \rfloor + 1]$; and the third line follows from a simple bounding argument on $\binom{n}{i}$ over $i \in [2, \lfloor \Delta/2 \rfloor + 1]$ and the relationship between the sum of combinations and the number of subsets of a set.

Recall from §3.2.1 that the probability of graph disconnection $P_d(G,p)$ for the $H(n,\Delta)$ Harary graph can be bounded as follows:

$$\begin{aligned}
P_d(G,p) &= \sum_{i=\lambda}^{e} C_i p^i (1-p)^{e-i} \\
&= \sum_{i=\lambda}^{2\Delta-1} n \binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i} + \sum_{i=2\Delta}^{e} C_i p^i (1-p)^{e-i} \\
&\geq \left(\frac{p}{1-p} \right)^{\Delta} n(1-p)^e \sum_{i=0}^{\Delta-1} \binom{e-\Delta}{i} \left(\frac{p}{1-p} \right)^i .
\end{aligned}$$

This bound can be rewritten as:

$$\log\left[P_d(G,p)\right] \geq \Delta \log\left[\frac{p}{1-p}\right] + \log\left[n(1-p)^e\right] + \log\left[\sum_{i=0}^{\Delta-1} \binom{e-\Delta}{i} \left(\frac{p}{1-p}\right)^i\right]. \tag{4.19}$$

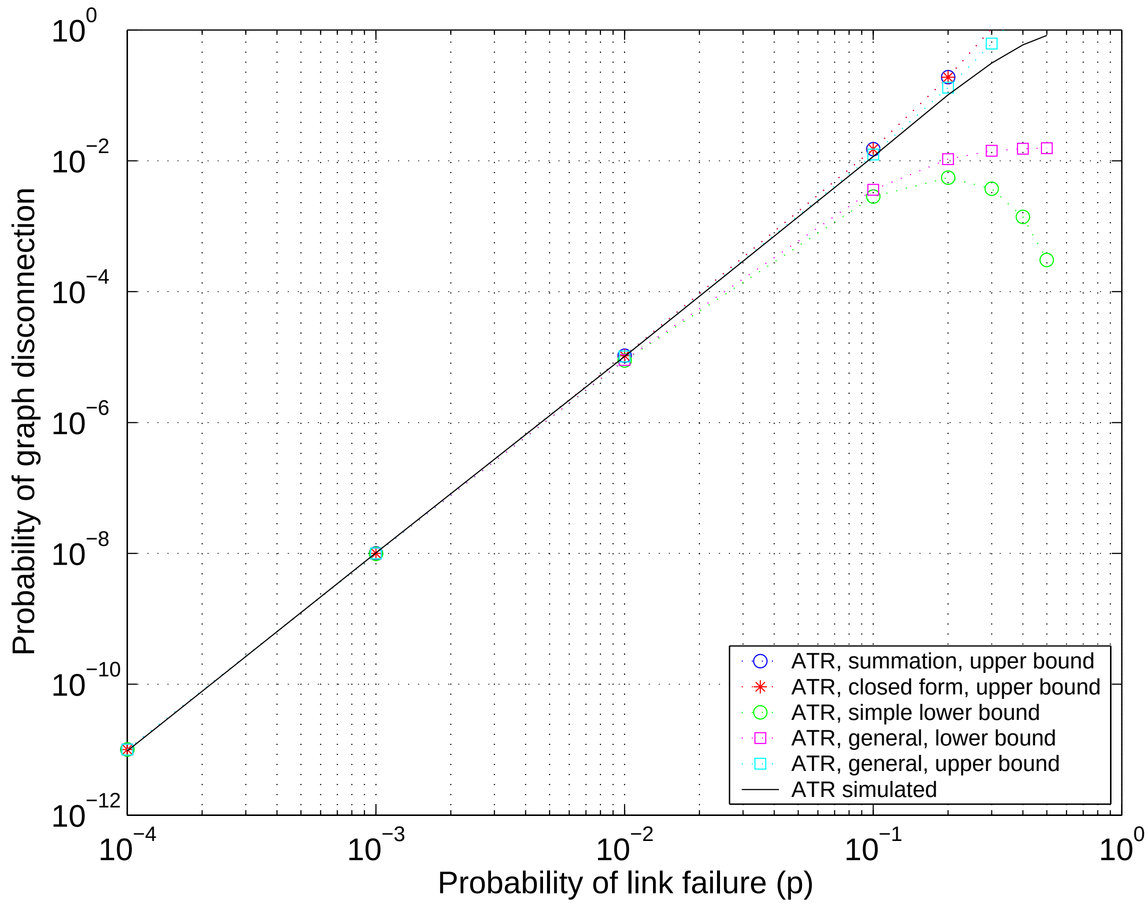


Figure 4-3: Probability of graph disconnection versus p for $H(10, 3)$ when $p \leq 1/2$. "ATR, simple lower bound" refers to $np^{\Delta}(1-p)^{e-\Delta}$, "ATR, general, lower bound" refers to (3.1), "ATR, general, upper bound" refers to (3.2), "ATR, summation, upper bound" refers to (4.17), and "ATR, closed form, upper bound" refers to (4.18).

For small p, the above bounds are tight. Furthermore, for small p, the logarithm of $P_d(G, p)$ is approximately linear in $\log p$ with slope Δ. Figure 4-3 illustrates this trend for the ten node, degree three Harary graph. Plotted in this figure are the bounds derived in this subsection for Harary graphs as well as some more general bounds. The bounds plotted are quite tight for values of p less than approximately 0.1. Furthermore, the more useful upper bounds on the probability of disconnection are tighter than the lower bounds. The bounds derived here thus useful tools for the design of Harary networks in the $p \approx 0$ regime.

Re-examining (4.19), we note that when p is small the logarithm of $P_d(G, p)$ is

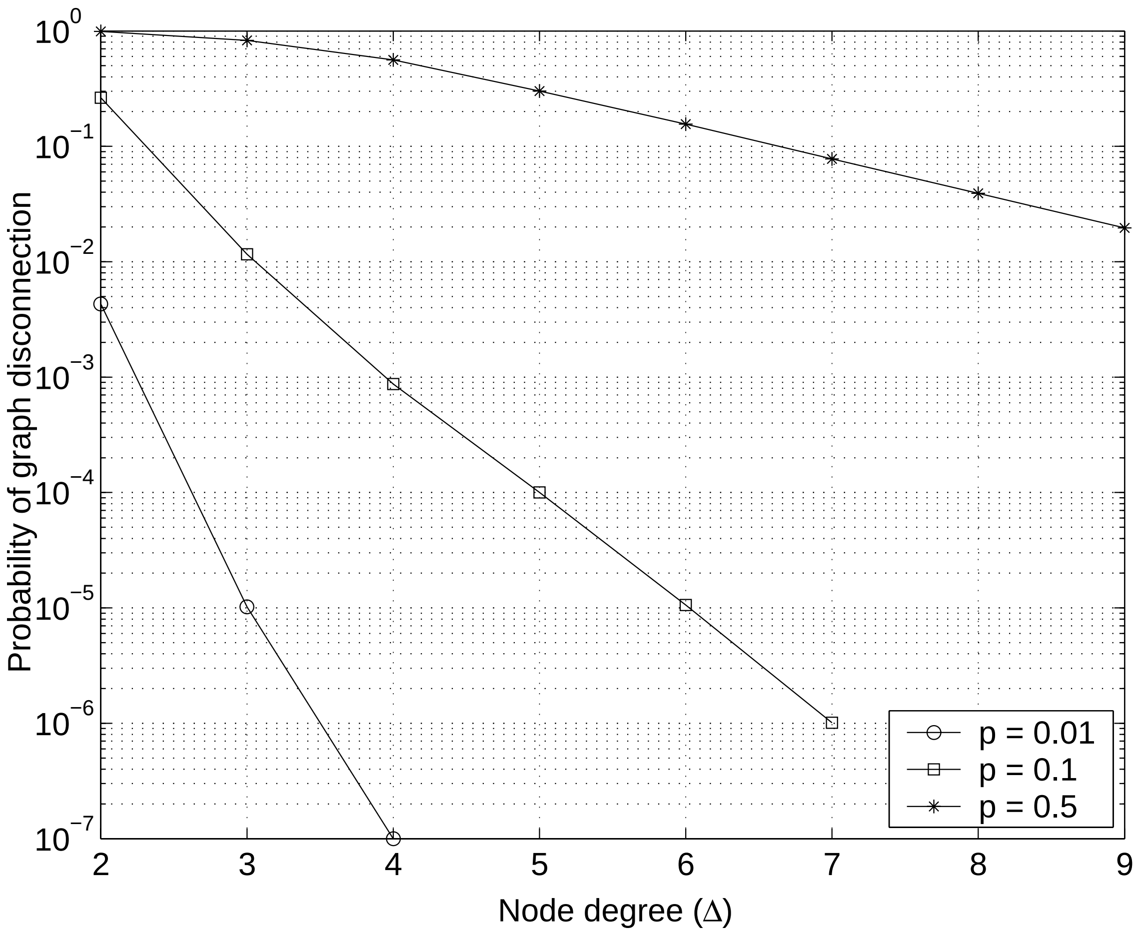


Figure 4-4: Probability of graph disconnection versus node degree for ten node Harary graphs. Three link failure probabilities were studied: $p = 0.01$, 0.1, and 0.5.

approximately linear in Δ with slope $\log p$. Figure 4-4 illustrates this trend for three different values of p for ten node Harary graphs.

4.3.2 Two-terminal reliability when $p \approx 0$

The derivation of a lower bound for the node pair connection probability $P_c^{sd}(G, p)$ is virtually identical to that of $P_c(G, p)$ for p close to zero in §4.3.1. The difference is that we are only interested in partitions of the network nodes that result in nodes s

and d residing in different partitions. Hence, we modify (4.17) to obtain:

$$P_c^{sd}(G,p) \geq 1 - \left(2p^{\Delta} + 2\sum_{i=2}^{\Delta/2+1} \binom{n-2}{i-1} p^{i\Delta - 2\binom{i}{2}} + 2\sum_{i=\Delta/2+2}^{\lfloor n/2 \rfloor} \binom{n-2}{i-1} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right). \tag{4.20}$$

In a manner similar to §4.3.1, we can derive a slightly looser upper bound for $P_c^{sd}(G,p)$:

$$\begin{aligned} P_c^{sd}(G,p) &\geq 1 - \left(2p^{\Delta} + 2\sum_{i=2}^{\lfloor \Delta/2 \rfloor+1} \binom{n-2}{i-1} p^{i\Delta - 2\binom{i}{2}} + 2\sum_{i=\lfloor \Delta/2 \rfloor+2}^{\lfloor n/2 \rfloor} \binom{n-2}{i-1} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\ &\geq 1 - \left(2p^{\Delta} + 2\sum_{i=2}^{\lfloor \Delta/2 \rfloor+1} \binom{n-2}{i-1} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) + 2\sum_{i=2}^{\lfloor n/2 \rfloor} \binom{n-2}{i-1} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \\ &\geq 1 - \left(2p^{\Delta} + 2\left\lfloor \frac{\Delta}{2} \right\rfloor \binom{n-2}{\lfloor \frac{\Delta}{2} \rfloor} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) \right. \\ &\quad \left. + p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \left[2^{n-2} + \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} - 2 \right] \right). \end{aligned} \tag{4.21}$$

The quality of these bounds is illustrated in Figure 4-5 for the ten node, degree three Harary graph. As in the all-terminal case, for small p, the logarithm of the probability of node-pair disconnection $P_d^{sd}(G,p)$ is approximately linear in $\log p$ with slope Δ. However, the two-terminal values of $P_d^{sd}(G,p)$ are smaller than $P_d(G,p)$ by a factor of approximately $n/2$. Furthermore, the two-terminal bounds plotted are quite tight for values of p less than approximately 0.1, and the upper bounds on the probability of disconnection are tighter than the lower bounds.

Figure 4-6 compares the all- and two-terminal reliability performance of ten node Harary graphs when $p \approx 0$.

4.3.3 All-terminal reliability when $p \approx 1$

For p close to unity, we bound $P_c(G,p)$ using the approach outlined in §3.2.3, which requires knowledge of the number of spanning trees in a graph. We recall that the

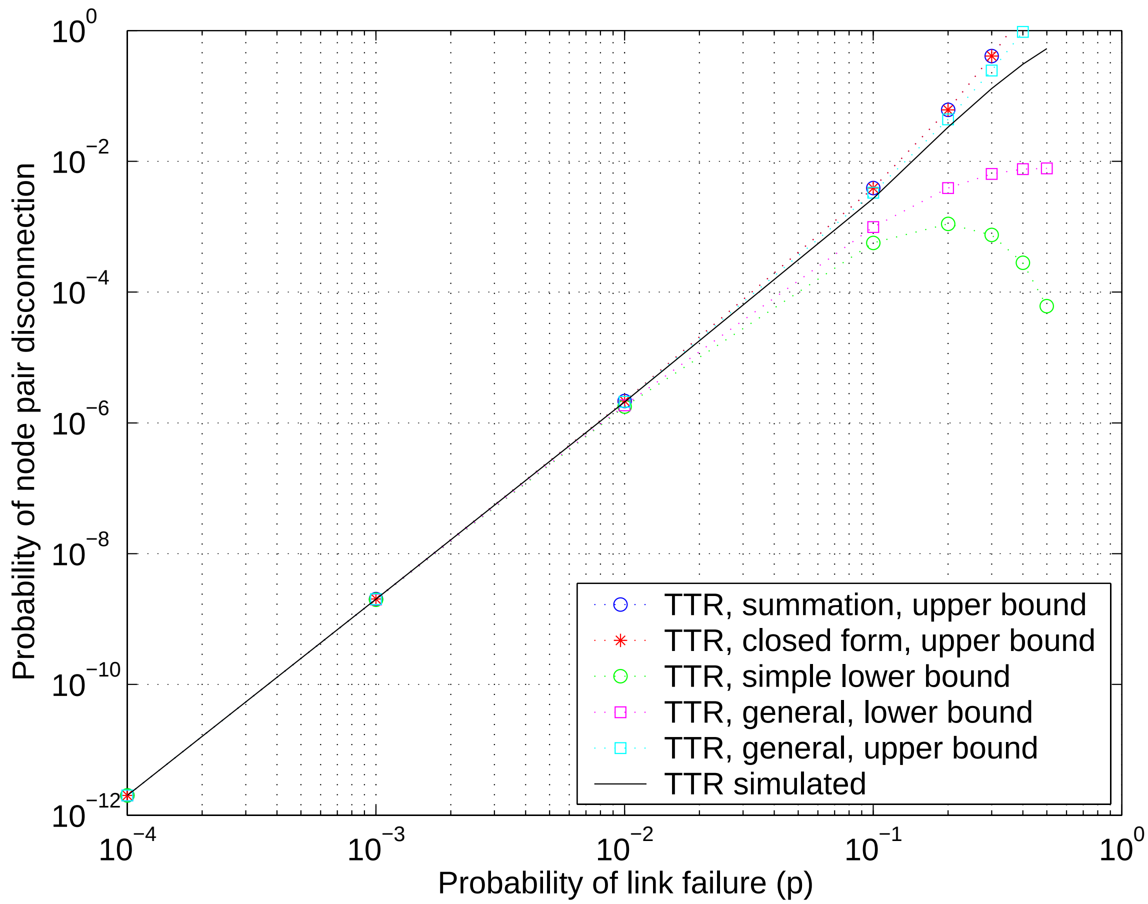


Figure 4-5: Worst-case probability of node pair disconnection versus p for $H(10, 3)$ when $p \leq 1/2$. "TTR, simple lower bound" refers to $2p^{\Delta}(1-p)^{e-\Delta}$, "TTR, general, lower bound" refers to the right inequality of (3.4), "TTR, general, upper bound" refers to the left inequality of (3.4), "TTR, summation, upper bound" refers to (4.20), and "TTR, closed form, upper bound" refers to (4.21).

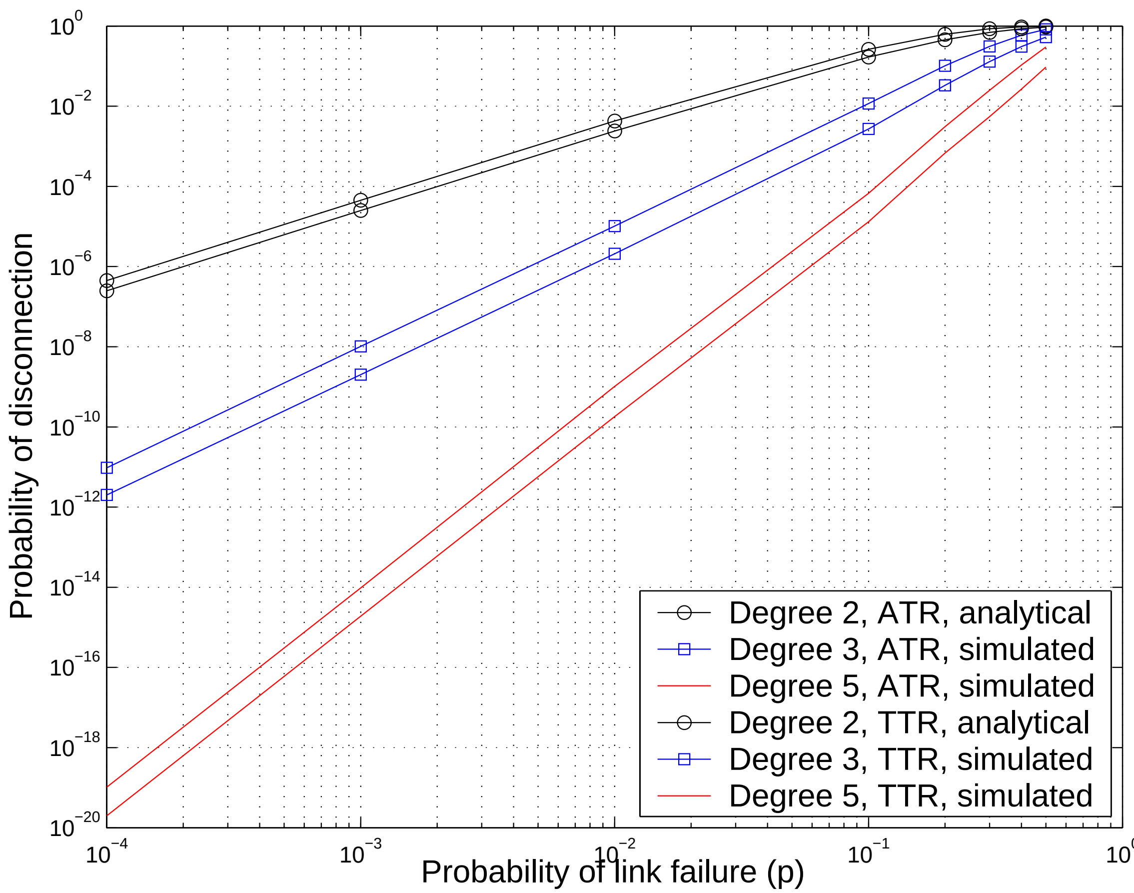


Figure 4-6: Probability of disconnection versus p for $H(10,2)$, $H(10,3)$ and $H(10,5)$ when $p \leq 1/2$. "Degree 2, ATR, analytical" refers to (4.3), and "Degree 2, TTR, analytical" refers to (4.5).

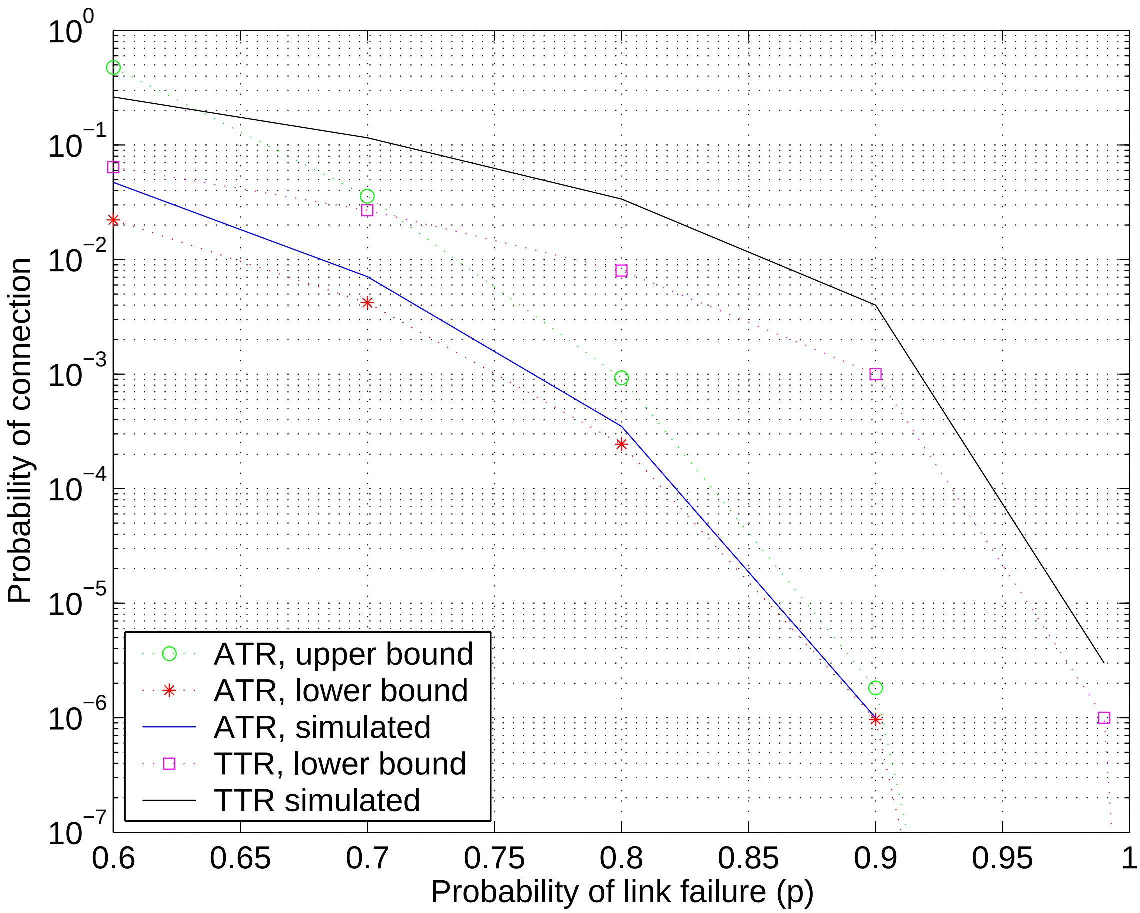


Figure 4-7: Probability of graph connection and worst-case probability of node pair connection versus p for $H(10,3)$ when $p \geq 1/2$. "ATR, lower bound" refers to (3.6), "ATR, upper bound" refers to (3.7), and "TTR, lower bound" refers to (4.27).

number of spanning trees in an even degree Δ Harary graph $G = C_n\langle 1, 2, \ldots, h\rangle$ is given in Lemma 4.1. For a more general result which applies to all Harary graphs, we specialize Theorem 2.3:

Corollary 4.1 *The number of spanning trees in the degree Δ Harary graph is:*

$$t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2\left(ji\pi/n\right)\right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2\left(ji\pi/n\right) - (-1)^i + 1\right], & \text{if } \Delta \text{ is odd.} \end{cases} \tag{4.22}$$

The quality of these bounds is illustrated in Figure 4-7 for the ten node, degree three Harary graph.

It is also worth mentioning that Polesskii obtained the following lower bound for $P_c(G,p)$ for Harary graphs in [61]:

$$P_c(G,p) \geq 1 - \left[1 - (1-p)^{n-1}\right]^{\frac{e}{n-1}} \tag{4.23}$$

which is tightest when $p \approx 1$.

4.3.4 Two-terminal reliability when $p \approx 1$

When the probability of link failure p is close to unity, we bound the probability of node pair connection using the technique outlined in §3.2.4. This technique requires knowledge of the edge-disjoint path lengths between nodes s and d. We consider Harary graphs of even degree only, as the case of odd degree is significantly more complex. Let d_{sd} denote the node separation of s and d. Define the parameter h as $\min(d_{sd}, N - d_{sd})$. By inspecting the structure of even degree Harary graphs, the length of path i for $i = 1, \ldots, \min(h, \Delta/2)$ is found to be:

$$l_i = \left\lceil \frac{h - i + 1}{\Delta/2} \right\rceil + 1 - \delta_1(i) \tag{4.24}$$

where the function $\delta_x(i)$ equals unity when its argument i equals x and is otherwise equal to zero. If $\Delta/2 > h$, then the length of path i for $i = h+1, \ldots, \Delta/2$ is given by:

$$l_i = \left\lceil \frac{i - h}{\Delta/2} \right\rceil + 1. \tag{4.25}$$

Finally, the length of path i for $i = \Delta/2 + 1, \ldots, \Delta$ is given by:

$$l_i = \left\lceil \frac{n - h - i + 1}{\Delta/2} \right\rceil + 1 - \delta_{\Delta/2+1}(i). \tag{4.26}$$

These path lengths can now be substituted into (3.9) to obtain a lower bound for $P_c^{sd}(G,p)$.

When $p \approx 1$, $P_c^{sd}(G,p)$ is minimized for node pairs which are most distantly placed in G. For even degree Harary graphs, such node pairs have indices which differ by

$\lceil (n-1)/2 \rceil$. The diameter of even degree Harary graphs is thus $\frac{2}{\Delta}\lceil\frac{n-1}{2}\rceil$. For odd degree Harary graphs[2], most distantly placed nodes can be shown to have indices which differ by $\lceil (n+\Delta-3)/4 \rceil$, with a resulting graph diameter of $\frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil$. Thus, using (3.10), we have the following lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ for Harary graphs:

$$(1-p)^{k(G)} \leq \min_{s,d}\left[P_c^{sd}(G,p)\right] \tag{4.27}$$

where

$$k(G) = \begin{cases} \frac{2}{\Delta}\lceil\frac{n-1}{2}\rceil, & \text{if } \Delta \text{ is even,} \\ \frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil, & \text{if } \Delta \text{ is odd.} \end{cases} \tag{4.28}$$

The quality of this bound is illustrated in Figure 4-7 for the ten node, degree three Harary graph. Note that as the number of nodes n increases relative to the degree Δ, odd degree Harary graphs possess diameters which are approximately half as large as even degree Harary graphs.

Figure 4-8 compares the all- and two-terminal reliability performance of ten node Harary graphs when $p \approx 1$.

4.4 Comparison of topologies

We conclude this chapter with a comparison of the Ethernet, ring, multi-ring and Harary topologies. We assume that each graph supports 14 nodes, and that the degree of the multi-ring and the Harary graph is four. We further assume that nodes, including the two switches in the Ethernet topology, are invulnerable, and that the Ethernet bridge reliability is identical to that of the other links.

Figure 4-9 depicts the performance of the topologies when $p \leq 1/2$. Between Ethernet and the ring, which are the degree two topologies, Ethernet exhibits better all- and two-terminal reliability. Ethernet's superior performance can be attributed to the fact that it scales weakly with the number of nodes in the graph. For example, for all-terminal reliability, the number of cutsets of order two is $n = 14$ in Ethernet,

[2] We restrict our attention to odd degree Harary graphs which are *strictly* regular. These graphs therefore have an even number of nodes.

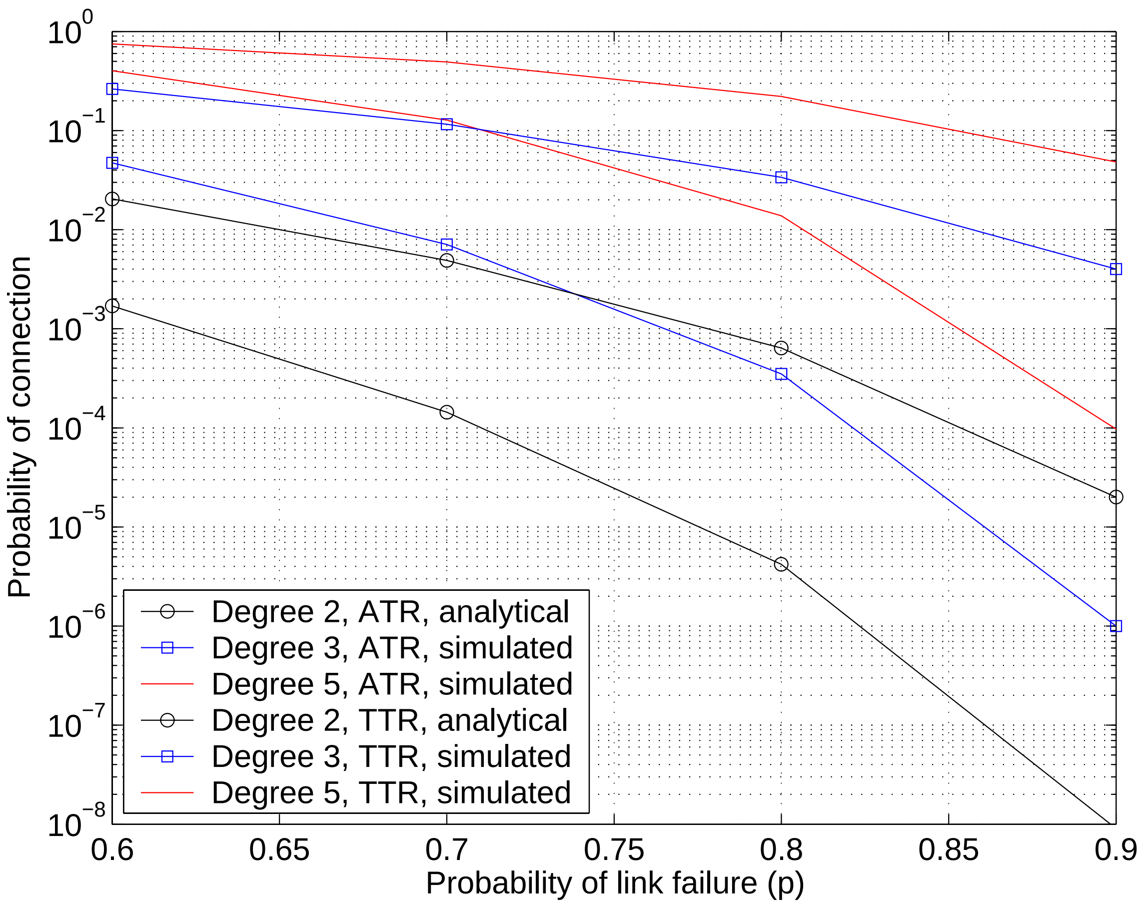


Figure 4-8: Probability of connection versus p for $H(10, 2)$, $H(10, 3)$ and $H(10, 5)$ when $p \geq 1/2$. "Degree 2, ATR, analytical" refers to (4.3), and "Degree 2, TTR, analytical" refers to (4.11).

	Probability of Connection
ATR $p \approx 0$	$P_c(G,p) \geq 1-\left(np^{\Delta}+\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n}{i}p^{i\Delta-2\binom{i}{2}}+\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n}{i}p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)$ $\geq 1-\left(np^{\Delta}+\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n}{\lfloor\frac{\Delta}{2}\rfloor+1}\left(p^{2\Delta-2}-p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)+p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-1}+\frac{1}{2}\binom{n}{\lfloor n/2\rfloor}-n-1\right]\right)$
TTR $p \approx 0$	$P_c^{sd}(G,p) \geq 1-\left(2p^{\Delta}+2\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n-2}{i-1}p^{i\Delta-2\binom{i}{2}}+2\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n-2}{i-1}p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)$ $\geq 1-\left(2p^{\Delta}+2\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n-2}{\lfloor\frac{\Delta}{2}\rfloor}\left(p^{2\Delta-2}-p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)+p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-2}+\binom{n-2}{\lfloor\frac{n-2}{2}\rfloor}-2\right]\right)$
ATR $p \approx 1$	$t(G)(1-p)^{n-1}p^{e-n+1} \leq P_c(G,p) \leq t(G)(1-p)^{n-1}$ $t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2(ji\pi/n)\right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2(ji\pi/n)-(-1)^i+1\right], & \text{if } \Delta \text{ is odd.} \end{cases}$
TTR $p \approx 1$	$P_c^{sd}(G,p) \geq 1-\prod_{i=1}^{\Delta}\left[1-(1-p)^{l_i}\right]$ When Δ is even: $l_i = \begin{cases} \left\lceil\frac{h-i+1}{\Delta/2}\right\rceil+1-\delta_1(i), & \text{if } 1 \leq i \leq \min(h,\Delta/2), \\ \left\lceil\frac{i-h}{\Delta/2}\right\rceil+1, & \text{if } h+1 \leq i \leq \Delta/2, \\ \left\lceil\frac{n-h-i+1}{\Delta/2}\right\rceil+1-\delta_{\Delta/2+1}(i), & \text{if } \Delta/2+1 \leq i \leq \Delta. \end{cases}$ $(1-p)^{k(G)} \leq \min_{s,d}\left[P_c^{sd}(G,p)\right]$ $k(G) = \begin{cases} \frac{2}{\Delta}\left\lceil\frac{n-1}{2}\right\rceil, & \text{if } \Delta \text{ is even,} \\ \frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil, & \text{if } \Delta \text{ is odd.} \end{cases}$

Table 4.6: Summary of probability of connection expressions for Harary graphs.

	Probability of Connection
ATR $p \approx 0$	$P_c(G,p) \to 1 - np^{\Delta}, \text{ as } p \to 0$
TTR $p \approx 0$	$P_c^{sd}(G,p) \to 1 - 2p^{\Delta}, \text{ as } p \to 0$
ATR $p \approx 1$	$P_c(G,p) \to t(G)(1-p)^{n-1}, \text{ as } p \to 1$ $t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2(ji\pi/n)\right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2(ji\pi/n) - (-1)^i + 1\right], & \text{if } \Delta \text{ is odd.} \end{cases}$
TTR $p \approx 1$	$\min_{s,d}\left[P_c^{sd}(G,p)\right] \to (1-p)^{k(G)}, \text{ as } p \to 1$ $k(G) \to \begin{cases} n/\Delta, & \text{if } \Delta \text{ is even,} \\ n/2\Delta, & \text{if } \Delta \text{ is odd.} \end{cases} \quad \text{as } n \to \infty$

Table 4.7: Summary of asymptotic probability of connection expressions for Harary graphs.

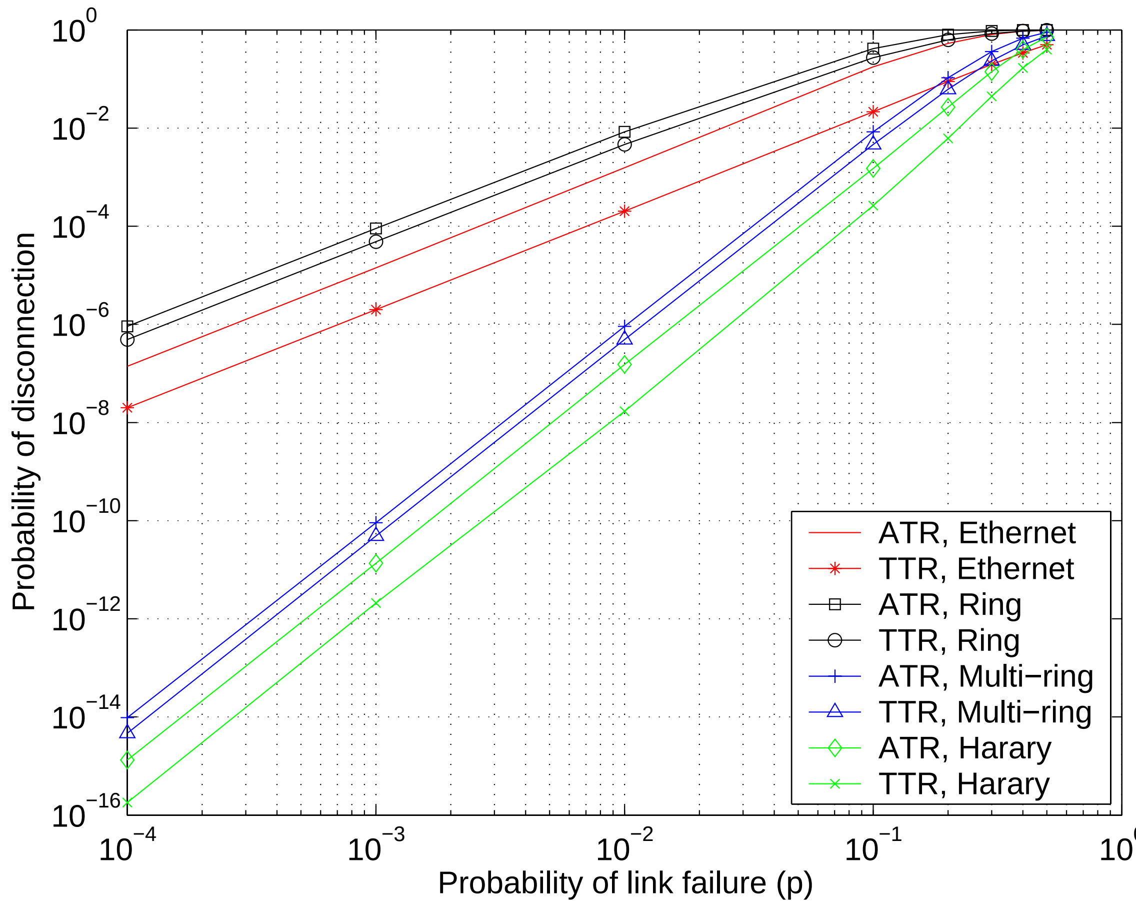


Figure 4-9: Probability of disconnection versus p for the 14 node Ethernet, ring, double-ring and $H(14,4)$ graphs when $p \leq 1/2$.

whereas it is $\binom{n}{2} = 91$ in the ring. Similarly, for two-terminal reliability, the number of cutsets of order two is two in Ethernet, whereas it is $n^2/4 = 49$ in the ring. The same scalability explanation also applies when accounting for the superior performance of $H(14,4)$ relative to the double ring, which is also a degree four graph. With respect to all-terminal reliability, $H(14,4)$, since it is super-λ, possesses $n = 14$ cutsets of order four, whereas the double ring possesses $\binom{n}{2} = 91$ cutsets of order four. For two-terminal reliability, the number of cutsets of order two is two in $H(14,4)$, whereas it is $n^2/4 = 49$ for the double ring.

In Figure 4-10, the performance of the topologies is plotted when $p \geq 1/2$. With respect to all-terminal reliability, it is easy to see that Ethernet has far more spanning trees than the ring, which only has $n = 14$, thus accounting for its superior reliability

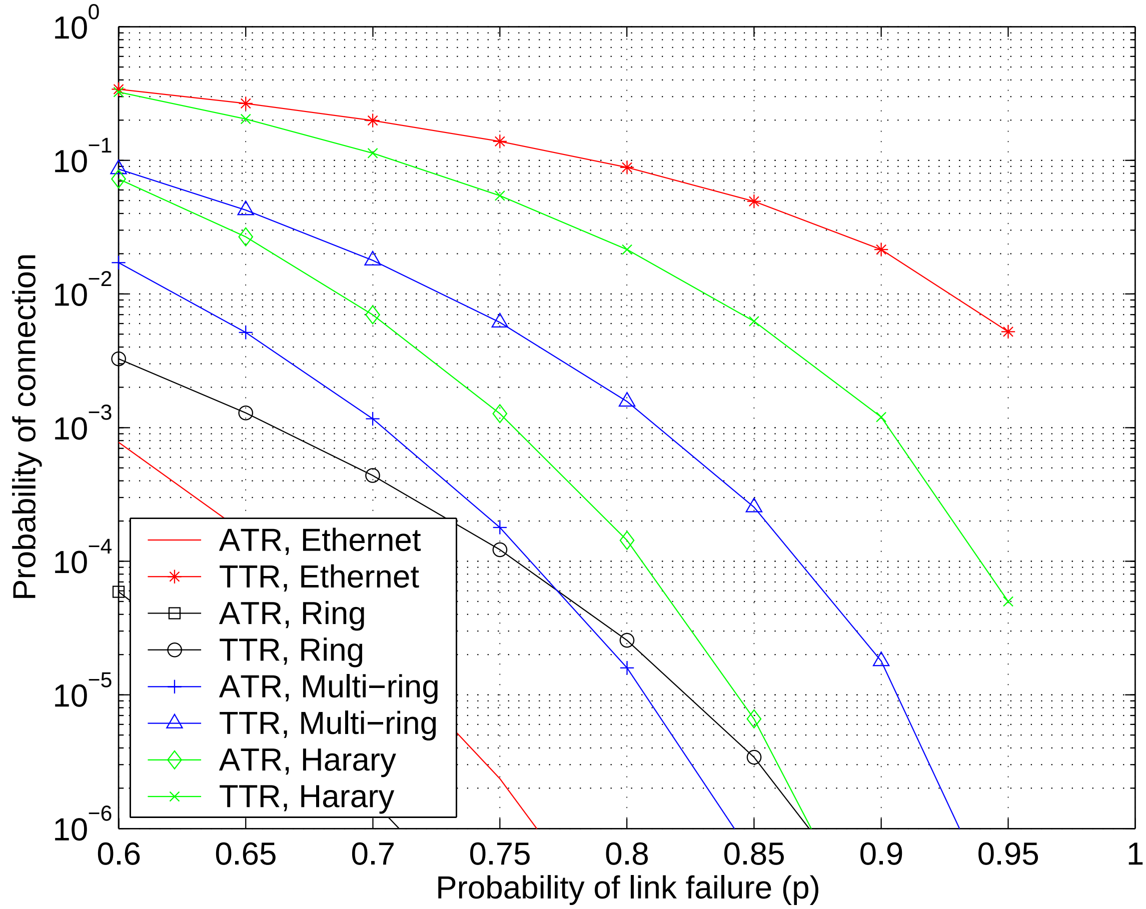


Figure 4-10: Probability of connection versus p for the 14 node Ethernet, ring, double-ring and $H(14,4)$ graphs when $p \geq 1/2$.

performance. Similarly, $H(14,4)$ has 1.9898×10^6 spanning trees, whereas the double ring has $n2^{n-1} = 1.1469 \times 10^5$ spanning trees. Hence, we expect $H(14,4)$ to perform better than the double ring, which is indeed the case. With respect to two-terminal reliability, the performance difference between Ethernet and the ring is enormous. This is because Ethernet has a diameter of two, whereas the ring has a diameter of $\lfloor n/2 \rfloor = 7$. The two-terminal reliability difference between $H(14,4)$ and the double ring is also significant, owing to the fact that $H(14,4)$ has a diameter of four, whereas the double ring has a diameter of $\lfloor n/2 \rfloor = 7$.

We conclude that the reliability of rings is consistently poorer than that of the Ethernet topology. Of course, the price paid for this superior reliability is the cost of the switches. We also conclude that multi-rings have poor reliability performance relative to super-λ circulants of the same degree, such as Harary graphs.

4.5 Summary

In this chapter, we carried out reliability case studies of special topologies – Ethernet, ring, multi-ring, and Harary graphs. Exact expressions for all- and two-terminal reliability were derived for the Ethernet, ring and multi-ring topologies. In addition, the simplicity of the topologies permitted the consideration of node failures.

In our analysis of Harary graphs, we began by proving an intuitive and useful theorem for this family of graphs. This theorem allowed us to develop tight, closed form reliability bounds for Harary graphs when p is small. When $p \approx 1$, we employed the techniques of Chapter 3 in conjunction with the some new results on the diameter and path lengths of Harary graphs.

We concluded the chapter with a comparison of the Ethernet, ring, multi-ring and Harary topologies. Between Ethernet and the ring, which are the degree two topologies, Ethernet exhibited better all- and two-terminal reliability, owing to Ethernet's weak scalability with the number of nodes in the graph. Between the multi-ring and a same degree Harary graph, Harary graphs perform better, indicating the advantage of more strategic positioning of link capacity rather than adding redundant backup links.

Chapter 5

Network design with statistically dependent link failures

As discussed in the introductory chapter, many situations arise for which the modelling assumption that network links fail in a statistically independent fashion is inappropriate. Dependencies among component failures in networks exists for a number of reasons. For example, the use of common equipment by more than one network component creates failure dependencies among these components. In addition, network components which are in close physical proximity are likely to be affected by common environmental stresses.

As illustrated by the following example, modelling link failures in a statistically independent fashion can lead to dangerously optimistic conclusions regarding the reliability of a system. Thus, there is a need to be able to model the statistical dependencies among component failures. This need is particularly important for systems which demand high levels of reliability.

Example 5-1. Consider two nodes s and d wishing to communicate. Let there exist m parallel links, $l_1, l_2, \ldots, l_m$, between these two nodes, as illustrated in Figure 5-1. Let us assume that there is a Markovian failure dependency among these m links; that is, conditioned on the state of link $j-1$, link j is independent of the states of links $1, 2, \ldots, j-2$. Let l_j denote the event that link j is operational, $\bar{l}_j$ that link j

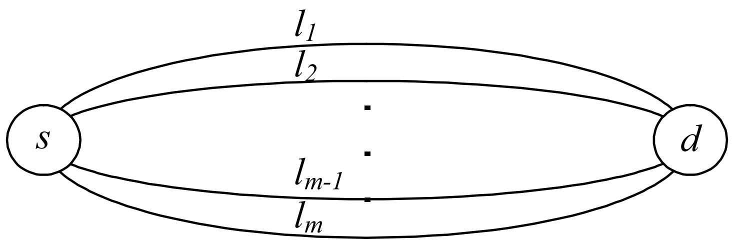


Figure 5-1: Illustration of the network studied in Example 5-1.

is not operational, and let $\dot{l}$ denote either of the previous two events. Let us further assume that the marginal probability distributions of the states of each of the links is identical (i.e. $\Pr\left(\bar{l}_j\right) = p$), and that $\Pr\left(\bar{l}_j|\bar{l}_{j-1}\right)$ is also identical[1] for all j. Thus, the probability that nodes s and d are disconnected is given by:

$$\begin{aligned}\Pr\left(s, d \text{ disconnected}\right) &= \Pr\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_m\right)\\ &= \Pr\left(\bar{l}_1\right)\Pr\left(\bar{l}_2|\bar{l}_1\right)\Pr\left(\bar{l}_3|\bar{l}_1\bar{l}_2\right)\ldots\Pr\left(\bar{l}_m|\bar{l}_1\bar{l}_2\ldots\bar{l}_{m-1}\right)\\ &= \Pr\left(\bar{l}_1\right)\Pr\left(\bar{l}_2|\bar{l}_1\right)\Pr\left(\bar{l}_3|\bar{l}_2\right)\ldots\Pr\left(\bar{l}_m|\bar{l}_{m-1}\right)\\ &= p\left[\Pr\left(\bar{l}_j|\bar{l}_{j-1}\right)\right]^{m-1}.\end{aligned} \tag{5.1}$$

If we know the conditional distribution of the state of a link given the state of an adjacent link, then we simply substitute this into (5.1). Alternatively, if only the correlation coefficient ρ of the states of adjacent links is available, then we can derive the conditional distribution. Let I_j be the indicator function for the event that link j is down; that is,

$$I_j = \begin{cases} 0, & \text{if link } j \text{ is up,}\\ 1, & \text{if link } j \text{ is down.}\end{cases}$$

By the definition of the correlation coefficient, we have the following relation:

$$\rho = \frac{E\left[I_{j-1}I_j\right] - E\left[I_{j-1}\right]E\left[I_j\right]}{\sqrt{\left(E\left[I_{j-1}^2\right] - E\left[I_{j-1}\right]^2\right)\left(E\left[I_j^2\right] - E\left[I_j\right]^2\right)}}$$

[1]Note that a consequence of our Markovian model is the expected result that: $\Pr\left(\bar{l}_j|\bar{l}_{j-1}\right) = \Pr\left(\bar{l}_{j-1}|\bar{l}_j\right)\Pr\left(\bar{l}_j\right)/\Pr\left(\bar{l}_{j-1}\right) = \Pr\left(\bar{l}_{j-1}|\bar{l}_j\right)$.

which, after solving for $E[I_{j-1}I_j]$ and substituting in the marginal distributions of the links, becomes:

$$E[I_{j-1}I_j] = \rho p(1-p) + p^2.$$

In addition, by the nature of the indicator function:

$$E[I_{j-1}I_j] = \Pr(I_{j-1}=1, I_j=1).$$

Using these two relations, we arrive at the following expression for the desired conditional distribution:

$$\begin{aligned}\Pr\left(\bar{l}_j|\bar{l}_{j-1}\right) &= \Pr(I_j=1|I_{j-1}=1)\\ &= \frac{\Pr(I_{j-1}=1, I_j=1)}{\Pr(I_{j-1}=1)}\\ &= \rho(1-p)+p.\end{aligned} \tag{5.2}$$

Hence, the probability that nodes s and d are disconnected is obtained by substituting (5.2) into (5.1):

$$\Pr(s, d \text{ disconnected}) = p\left[\rho(1-p)+p\right]^{m-1}. \tag{5.3}$$

This expression is plotted for several values of p in Figure 5-2. If $p \approx 0$ and $\rho \gg p$, then the probability that s and d are disconnected is approximately equal to $p\rho^{m-1}$. This asymptote is also illustrated in Figure 5-2. For even moderate values of ρ, this expression can be significantly larger than p^m, which is the probability that s and d are disconnected if the link failures are modelled as independent. Take, for example, $p = 0.001$, $\rho = 0.1$ and $m = 3$. Then, the probability that s and d are disconnected is approximately 10^{-5}, whereas an independence analysis would yield an overly optimistic value of 10^{-9}. Some researchers have even argued that results such as those provided by our Markov assumptions, are overly optimistic. They suspect that in many real systems, the correlations among link failures compound quickly such that the probability of disconnection may not decay to zero exponentially with

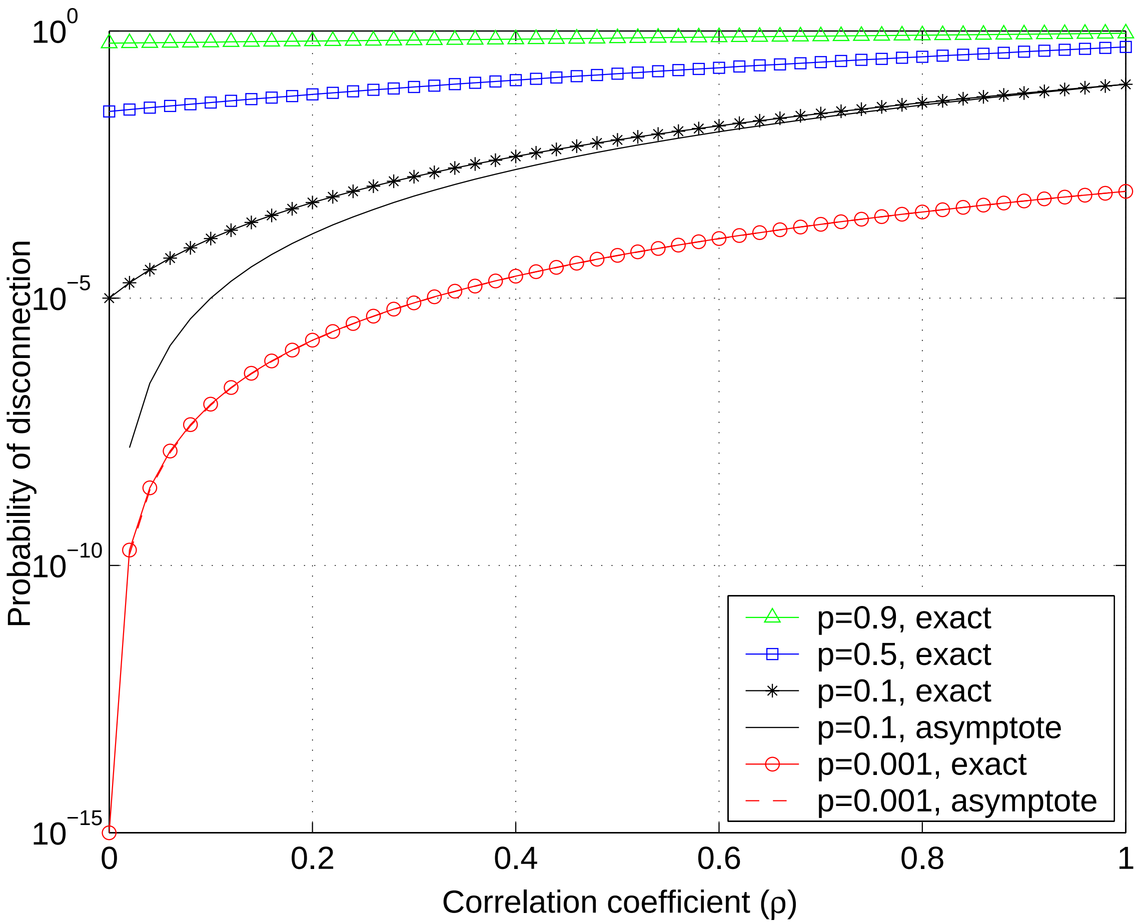


Figure 5-2: Probability of disconnection versus correlation coefficient ρ for the network studied in Example 5-1 with $m = 5$ parallel links. The exact probability of disconnection expression (5.3) is plotted for $p = 0.9$, 0.5, 0.1 and 0.001; the asymptote $p\rho^{m-1}$ is plotted for $p = 0.1$ and 0.001.

the number of parallel links as it does in our example. Regardless, there is a clear danger in relying on a reliability model in which failures are statistically independent.

Having motivated the need for a network reliability model which accounts for failure dependencies, we will next present a survey of existing models which have been proposed to meet this need. This survey is prefaced by a brief discussion on the desirable features of a network reliability model which incorporates statistical dependencies.

5.1 Survey of existing models

In the independent failure model previously considered, there were few required input parameters – apart from the underlying graph, only the probability of link failure p was required. In contrast, a model which incorporates statistical dependencies also requires knowledge of how the different network components interact. Since the number of parameters required to describe such dependencies can be quite large, it is important that the form of these inputs be intuitive and readily available to the network designer. In fact, if the joint probability distribution of all vulnerable network components is derived by means of conditional distributions, then the number of conditional probability terms grows exponentially with the number of vulnerable components. As we shall see, this exponential growth is also characteristic of cause-based models.

In addition to possessing intuitive inputs, the model must, of course, be a good approximation of reality. As we shall see shortly, in making simplifying assumptions regarding the nature of the dependencies among components, it is possible to create an *inconsistent* model. In calculating the probability of a joint event, the chain rule for probability is often used. In a *consistent* model, different chain rule expansions for the same joint event should not yield different results. Thus, in an *inconsistent* probabilistic model, different chain rule expansions for the same joint event result in different values for the probability of the joint event.

We now embark on a survey of existing reliability models which incorporate failure dependencies. Throughout the course of this survey, we comment on the merits and limitations of these models.

5.1.1 $\alpha-$Model

The $\alpha-$model, proposed by Spragins in [73], was a first attempt at modelling the dependencies among link failures in a network. Spragins developed the $\alpha-$model in order to provide a theoretical fit to the careful experimental study of dependent communication links carried out by Proverto [63]. In contrast to the modelling assumptions introduced in §3.1 which imply that we are considering a network over a brief enough window of time such that edges cannot be repaired, the $\alpha-$model allows for an extended notion of time in which edges can repeatedly fail and be repaired.

The model is a variation of the basic birth-death Markov chain. The system being modelled is a collection of network communication links whose reliability parameters are all equal. The states in the Markov chain represent the number of components which are operational, and in keeping with the standard birth-death assumptions, failure and repair times are assumed to be exponential. However, the parameters of these distributions are, in general, dependent upon the number of components which are operational. The name of the $\alpha-$model is derived from Spragins' simplifying assumption that the failure rate when at least one component is down is α times greater than when all components are up.

By referring to Proverto's data, it is obvious that the $\alpha-$model is superior to the independent failure model in modelling actual system behavior. Furthermore, the input parameters to the model, the failure and repair rates, are relatively simple to determine experimentally. In fact, in [73], Spragins derives expressions for obtaining these parameters from Proverto's data. Perhaps the most significant contribution of Spragins' $\alpha-$model is that it mathematically justified the conjecture that the independent failure model can lead to dangerous conclusions regarding the reliability of systems, and that more work in the research area was therefore needed.

There are limitations to the $\alpha-$model; the model is not flexible enough to handle more complex systems where there can be variation in the level of dependency among link failures. Furthermore, the $\alpha-$model only provides probabilities that a certain number of components are not operational, but does not give any indication regarding the connectedness of the network.

5.1.2 $q - \psi$ Model

In the $q-\psi$ model, presented by Spragins and Assiri in [1, 75], the authors developed a framework which could handle more complex networks than the $\alpha-$model and is amenable to the calculation of probabilities of connection. The $q - \psi$ model is based on the following assumptions:

1. The unconditional probability that any given link is down[2] is q;

2. The conditional probability that a link is down, given that at least one other link sharing one node with this link is down is $\psi \geq q$;

3. The probabilistic coupling between a cascade of link sets is Markovian in nature[3]; that is, conditioned on the state of an adjacent link set, the state of a link set is independent of more distant link sets. Spragins and Assiri extended this assumption by assuming that probabilities conditioned on the states of several links are only influenced by the closest links.

Using the above assumptions and the probabilistic chain rule, it is possible to obtain tractable expressions for the probability of joint failure events in a network. In order to compute the two-terminal probability of connection, Spragins and Assiri employed an algorithm devised in [35] by Hänsler, McAuliffe and Wilkov for generating cutsets in a graph which are mutually exclusive and collectively exhaustive. According to Lam and Li [43], a problem with the $q - \psi$ model is that the above assumptions, aimed at reducing the number of input parameters, render the model inconsistent. That is, the probability of a joint event can have different values when different orderings of the chain rule expansion are used.

[2]Note that q in the $q - \psi$ model is identical to the parameter p used throughout this thesis.

[3]This Markovian assumption is an extension of the assumption used in Example 5-1.

5.1.3 $\epsilon-$Model

In response to the limitations of the $q-\psi$ model as a result of its simplifying assumptions, Pan and Spragins introduced in [57] the $\epsilon-$model to allow for the existence of general failure dependencies in a consistent manner. The $\epsilon-$model is so named because of the use of the symbol ϵ to denote the perturbations from the independent failure model caused by the presence of failure dependencies. As with the $q-\psi$ model, the $\epsilon-$model can be used in conjunction with the algorithm devised in [35] by Hänsler, McAuliffe and Wilkov in order to compute two-terminal reliabilities.

Employing the notation introduced in Example 5-1, the probability of a joint event can be expressed as:

$$\Pr\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_m\right) = \Pr\left(\dot{l}_1\right)\Pr\left(\dot{l}_2\right)\ldots\Pr\left(\dot{l}_m\right)\epsilon\left(\dot{l}_1;\dot{l}_2\right)\epsilon\left(\dot{l}_1\dot{l}_2;\dot{l}_3\right)\ldots\epsilon\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1};\dot{l}_m\right)$$

where

$$\epsilon\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1};\dot{l}_m\right) \equiv \frac{\Pr\left(\dot{l}_m|\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1}\right)}{\Pr\left(\dot{l}_m\right)}$$

is the ratio of the a posteriori probability of link m being in a certain state given the states of links 1 through $m-1$, to the a priori probability of link m being in that state. The ϵ's, which are the inputs to the $\epsilon-$model, are very intuitive quantities for the network designer to deal with. Furthermore, the most natural ϵ's are those that reflect the increase in probability that a given link is down, given that other links are down. Pan and Spragins provide expressions in [57] which allow the iterative computation of any ϵ from ϵ's of this type. Other properties of the ϵ's are that they are symmetric in their arguments, and that they are restricted to the range:

$$0 \leq \epsilon\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1};\dot{l}_m\right) \leq \frac{1}{\max\left[\Pr\left(\dot{l}_m\right),\Pr\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1}\right)\right]}$$

by the definition of ϵ and the fact that both $\Pr\left(\dot{l}_m|\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1}\right)$ and $\Pr\left(\dot{l}_1\dot{l}_2\ldots\dot{l}_{m-1}|\dot{l}_m\right)$ must be less than or equal to 1. Additional properties of the ϵ's are discussed in [57, 71, 74]. Using the properties and the interrelationships among the ϵ's, it can be shown that $2^m - 1$ of the "natural" ϵ parameters suffice to describe a system of m vulnerable links in the most general case. Of course, this number can be reduced considerably if symmetries exist in the network.

5.1.4 Colored Network Model

In an effort to circumvent the specification of an exponential number of conditional probability distributions which must be consistent, Lam and Li developed the Colored Network Model (CNM) in [43]. The CNM transforms a network with link failure dependencies into a network with invulnerable links and nodes which fail independently. The CNM possesses the following properties:

1. Edges are colored;
2. Edges adjacent to a node can be of different colors;
3. All edges adjacent to the same node of the same color are considered being controlled by a colored subnode within that node. Thus, every node is made up of one or more subnodes;
4. Only these colored subnodes can fail. All subnodes fail in a statistically independent fashion with the same probability;
5. An edge may be incident to subnodes of different colors, so that it may have different colors on its two ends.

Using the CNM, the original network is transformed into another network in which each node is replaced by an invulnerable master node connected to one or more colored subnodes. Every colored subnode terminates edges of the same color as itself. Thus, the resulting probabilistic graph is one in which links are invulnerable and subnodes fail in a statistically independent fashion. Figure 5-3 shows the transformation of a network using the CNM. Algorithms to compute the reliability of such networks have been studied in [2, 21, 35, 44]. The major limitation of the CNM is that links incident to nodes must fail in mutually exclusive groups, which is often not a realistic assumption.

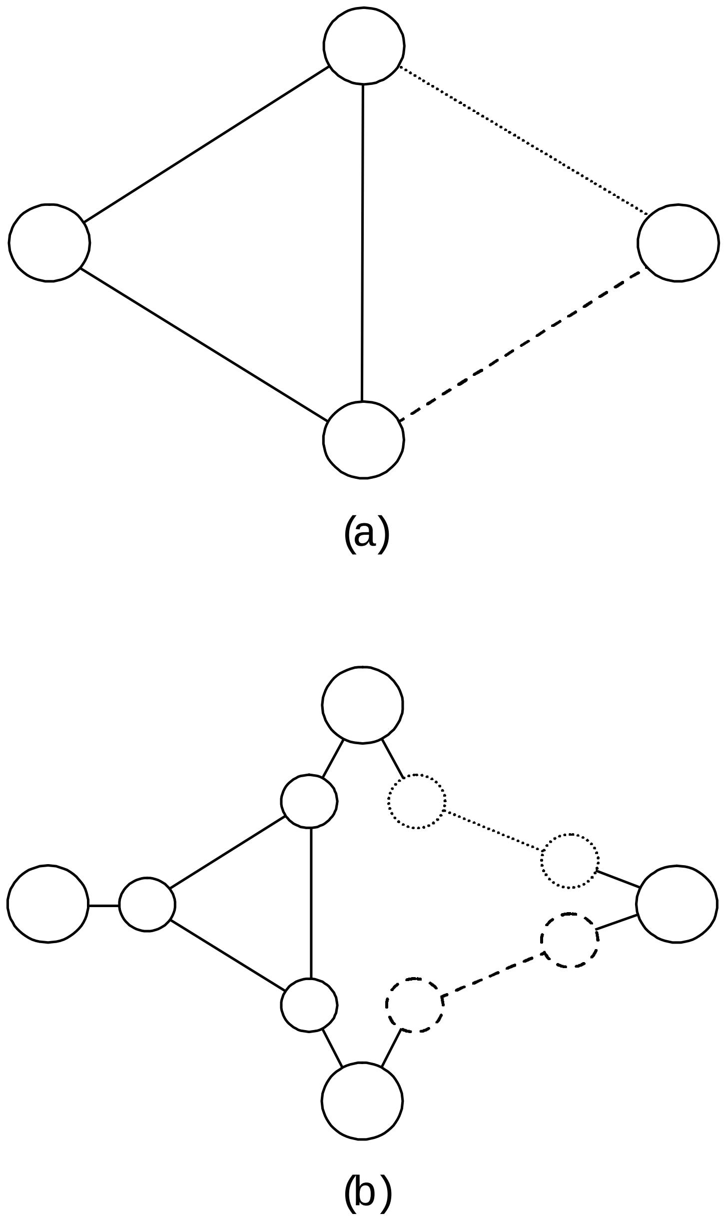


Figure 5-3: Example of the CNM. The original network in (a) is transformed into the network in (b). Different line types represent different colors.

5.1.5 Event-Based Network Model

The Event-Based Network Model (EBRM), proposed by Lam and Li in [45], was an attempt to retain the attractive features of the CNM – the avoidance of conditional probabilities and consistency requirements – while allowing the ability to model more general failure dependencies than the CNM.

The EBRM explicitly models the mechanisms causing dependency among link failures. In the model, a network is still modelled as a probabilistic graph. However, there are now additional elements in the graph, known as *event elements*, which are added to the affected links. An event element is said to be "down" when the failure-causing event occurs, and is said to be "up" otherwise. Furthermore, all failure-causing events are assumed to be independent and to occur with known probabilities. Figure 5-4 illustrates the EBRM concept for a simple graph. In this example, the link joining nodes 1 and 2 is affected by event elements a and b. Thus, the link is operational if and only if these two event elements are "up". On the other hand, if an event element is "down", then all links affected by that element will fail.

In order to compute the connectedness of a graph with the EBRM, an algorithm initially proposed in [68, 66] is employed. Basically, the algorithm outputs a list of subsets of network links corresponding to connected acyclic subgraphs of the networks. For each subset, the probability that all of its components are working is calculated as a simple function of the underlying event elements. The EBRM concept was later extended to the Cause-Based Multimode Model (CBMM) by Le and Li in [46]. The CBMM is a generalization of the EBRM in that event elements can cause degraded states for network elements, not just simple binary up/down states as in the EBRM.

On the surface, the attractive feature of the EBRM is that it circumvents the difficulties inherent in dealing with conditional probabilities – the exponential number of distributions and the consistency requirements. However, conditional probabilities are eliminated in the EBRM at the expense of introducing many event elements. In fact, for a network with m vulnerable links there can be $2^m - 1$ different event elements. Furthermore, each of these distinct event elements can represent an aggregate of

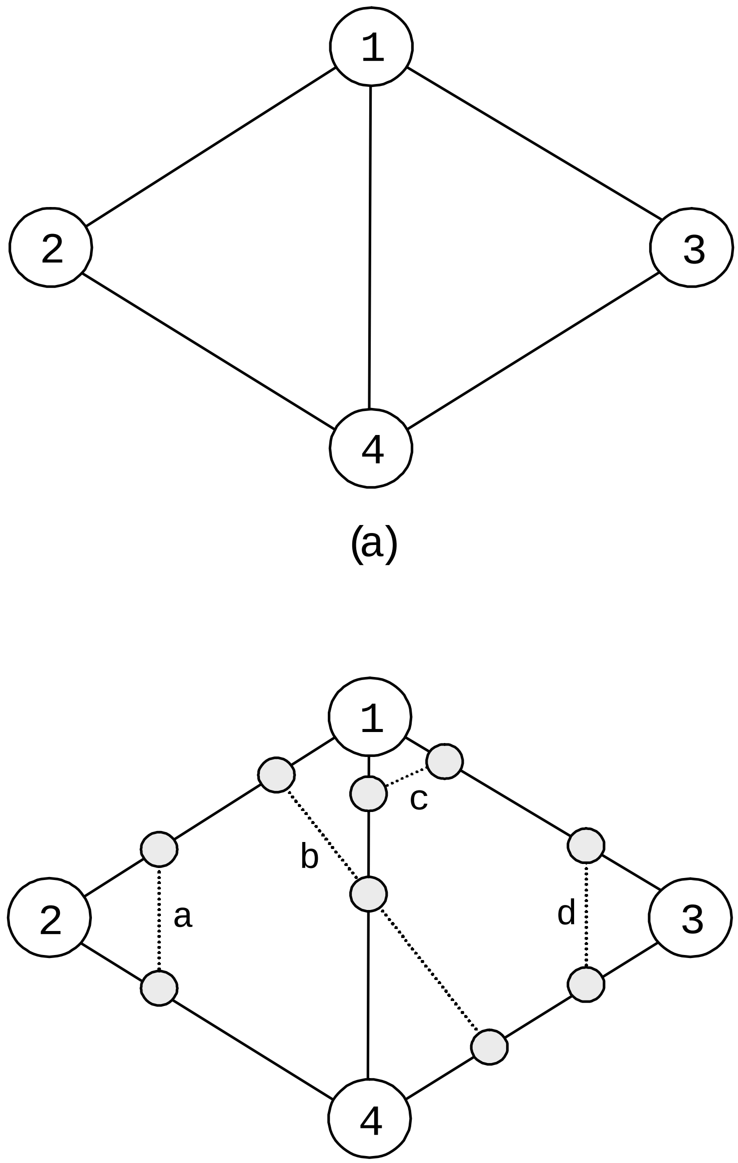


Figure 5-4: Example of the EBRM. The original network is illustrated in (a). After incorporating event elements, the resulting network in (b) is obtained.

smaller real events. Thus, it seems that the exponential growth of input parameters with the size of the network is an inescapable reality.

A major difference between the EBRM and the $\epsilon-$model is the nature of the inputs to the models. The EBRM is cause-based and accepts as inputs the underlying failure-causing events. For example, in an optical networking context, event elements could correspond to optical fibers in the same bundle being cut as a result of an environmental stress. The $\epsilon-$model, on the other hand, is a convenient mathematical construct that does not require detailed knowledge of how network components fail. Arguments regarding which type of input is more intuitive can be made either way, and therefore, the ultimate decision as to which model to use should depend upon the nature of the data available to the network designer.

5.2 Simple dependent failure reliability analyses

In this section, we carry out simple, approximate reliability analyses of special network topologies based on the dependent link failure models presented in the previous section.

5.2.1 Reliability of the Ethernet graph

In this subsection, we investigate the reliability of the Ethernet graph (see Figure 5-5) when dependence among link failures is present. The results derived in this subsection are summarized in Table 5.1. We assume that nodes, including switches, are invulnerable, and that link failures are correlated only if the corresponding links are incident at the same non-switch node.

Conditioning on the state of the bridge between the two switches, the all-terminal reliability of the Ethernet graph is thus given by:

$$P_c(G,p,p_b) = \Pr(G \text{ connected} \mid \text{bridge up})\Pr(\text{bridge up}) \\ + \Pr(G \text{ connected} \mid \text{bridge down})\Pr(\text{bridge down}).$$

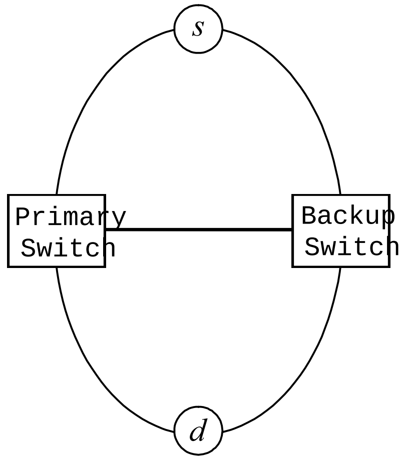


Figure 5-5: Ethernet LAN topology.

Here, the probability of connection given that the bridge is up is the probability that at least one link incident at each node is up, which is $[2(1-p)-(1-p)\left(\Pr\left(l_j|l_{j-1}\right)\right)]^n$. The probability of connection given that the bridge is down is the probability that all of the links incident at the primary switch or the backup switch are operative, which is $2(1-p)^n - [(1-p)\left(\Pr\left(l_j|l_{j-1}\right)\right)]^n$. Putting all of this together, we have:

$$\begin{aligned} P_c(G,p,p_b) = [2(1-p)-(1-p)\left(\Pr\left(l_j|l_{j-1}\right)\right)]^n (1-p_b) \\ + \left(2(1-p)^n - [(1-p)\left(\Pr\left(l_j|l_{j-1}\right)\right)]^n\right) p_b. \end{aligned}$$

To compute the two-terminal probability of connection, we recall that there exist four paths between each source-destination node pair:

1. $s \rightarrow$Primary Switch$\rightarrow d$;
2. $s \rightarrow$Primary Switch$\rightarrow$Backup Switch$\rightarrow d$;
3. $s \rightarrow$Backup Switch$\rightarrow d$;
4. $s \rightarrow$Backup Switch$\rightarrow$Primary Switch$\rightarrow d$.

The probability of connection of a node pair $P_c^{sd}(G,p,p_b)$ is the probability that at least one of the four paths is operative:

$$\begin{aligned}
P_c^{sd}(G,p,p_b) = \Pr(\text{path 1 up}) + \Pr(\text{path 2 up}) + \Pr(\text{path 3 up}) + \Pr(\text{path 4up}) \\
- \Pr(\text{paths 1 and 2 up}) - \Pr(\text{paths 1 and 3 up}) - \Pr(\text{paths 1 and 4 up}) \\
- \Pr(\text{paths 2 and 3 up}) - \Pr(\text{paths 2 and 4 up}) - \Pr(\text{paths 3 and 4 up}) \\
+ \Pr(\text{paths 1, 2 and 3 up}) + \Pr(\text{paths 1, 2 and 4 up}) + \Pr(\text{paths 1, 3 and 4 up}) \\
+ \Pr(\text{paths 2, 3 and 4 up}) - \Pr(\text{paths 1, 2, 3 and 4 up}).
\end{aligned}$$

Using our dependent failure model, the above expression becomes:

$$\begin{aligned}
P_c^{sd}(G,p,p_b) = 2(1-p)^2 + 2(1-p)^2(1-p_b) - 4(1-p)^2(1-p_b)\left[\Pr\left(l_j|l_{j-1}\right)\right] \\
- (1-p)^2\left[\Pr\left(l_j|l_{j-1}\right)\right]^2 + 2(1-p)^2(1-p_b)\left[\Pr\left(l_j|l_{j-1}\right)\right]^2.
\end{aligned}$$

The all- and two-terminal reliabilities of the Ethernet graph are plotted and discussed further in §5.3.

5.2.2 Reliability of ring and multi-ring graphs

In this subsection, we we develop closed-form expressions for the all- and two-terminal reliability of rings and multi-rings, assuming that nodes are invulnerable and that links fail in a statistically dependent fashion in accordance with the Markov model of Example 5-1. The results derived in this subsection are summarized in Table 5.2.

Ring graph

Assuming that nodes are invulnerable, the probability that a ring remains connected is the probability that zero links or exactly one link fails in the ring. These two probabilities can be computed using a chain rule expansion along consecutive links around the ring in a similar manner as in Example 5-1. The only difference is that the state of the final link in the expansion is influenced by its neighboring links on either side, rather than by just one link as in Example 5-1. This last probability term

	Probability of Connection
ATR	$$P_c(G,p,p_b) = \left[2(1-p) - (1-p)\left(\Pr\left(l_j \vert l_{j-1}\right)\right)\right]^n (1-p_b) + \left(2(1-p)^n - \left[(1-p)\left(\Pr\left(l_j \vert l_{j-1}\right)\right)\right]^n\right) p_b$$ $$P_c(G,p,p_b) \to 1 - np^2,\ \text{as } \rho, p, p_b \to 0$$ $$P_c(G,p,p_b) \to (1-p)^n \left[2^n(1-p_b) + 2\right],\ \text{as } \rho \to 0 \text{ and } p, p_b \to 1$$ $$P_c(G,p,p_b) \to (1-p)^n,\ \text{as } \rho \to 1$$
TTR	$$P_c^{sd}(G,p,p_b) = 2(1-p)^2 + 2(1-p)^2(1-p_b) - 4(1-p)^2(1-p_b)\left[\Pr\left(l_j \vert l_{j-1}\right)\right] - (1-p)^2\left[\Pr\left(l_j \vert l_{j-1}\right)\right]^2 + 2(1-p)^2(1-p_b)\left[\Pr\left(l_j \vert l_{j-1}\right)\right]^2$$ $$P_c^{sd}(G,p,p_b) \to 2(1-p)^2,\ \text{as } \rho \to 0 \text{ and } p, p_b \to 1$$ $$P_c^{sd}(G,p,p_b) \to (1-p)^2,\ \text{as } \rho \to 1$$

Table 5.1: Summary of probability of connection expressions for the Ethernet graph.

must therefore be specified in order to complete the model. For an n node ring, the probability of graph connection can thus be expressed as:

$$\begin{aligned} P_c(G,p) &= \Pr(\text{no links fail}) + \Pr(\text{one link fails}) \\ &= (1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2}\Pr\left(l_j|l_{j-1},l_{j+1}\right) + n(1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2}\Pr\left(\bar{l}_j|l_{j-1},l_{j+1}\right) \\ &= (1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2}\left[1+(n-1)\Pr\left(\bar{l}_j|l_{j-1},l_{j+1}\right)\right]. \end{aligned} \quad (5.4)$$

To compute the two-terminal reliability, we note that the probability that a node pair remains is connected is equal to the probability that all of the links on at least one of the two disjoint paths between the nodes remain operational. Hence, for a diameterically-spaced pair of nodes on an n ring graph, the two-terminal probability of connection is:

$$\begin{aligned} P_c^{s,s+\lfloor n/2\rfloor}(G,p) = &\quad \Pr(\text{path 1 up}) + \Pr(\text{path 2 up}) - \Pr(\text{paths 1 and 2 up}) \\ = &\quad (1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{\lfloor n/2\rfloor-1} + (1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{\lceil n/2\rceil-1} \\ &\quad -(1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2}\Pr\left(l_j|l_{j-1},l_{j+1}\right). \end{aligned} \quad (5.5)$$

The all- and two-terminal reliabilities of the ring topology are plotted and discussed further in §5.3.

Multi-ring graph

We now generalize the above analysis to multi-rings. As in the independent failure model, we only need to replace the parameter p in the above equations with a parameter which reflects the probability of the m parallel links failing in an m multi-ring. We may incorporate statistical dependence into this parameter by using the method of Example 5-1. Specifically, the all- and two-terminal reliabilities are given by (5.4) and (5.5) with p replaced by $p\left[\Pr\left(\bar{l}_j|\bar{l}_{j-1}\right)\right]^{m-1}$. Note that the conditional probability in this expression for parallel links is different from the previous conditional probability for consecutive links in the ring. The all- and two-terminal reliabilities of the multi-ring topology are plotted and discussed further in §5.3.

	Probability of Connection
ATR	$P_c(G,p) = (1-p)\left[\Pr(l_j \mid l_{j-1})\right]^{n-2}\left[1 + (n-1)\Pr\left(\bar{l}_j \mid l_{j-1}, l_{j+1}\right)\right]$ $P_c(G,p) \to 1 - n^2p^2,\ \text{as } \rho, p \to 0$ $P_c(G,p) \to n(1-p)^{n-1},\ \text{as } \rho \to 0 \text{ and } p \to 1$ $P_c(G,p) \to (1-p),\ \text{as } \rho \to 1$
TTR	$P_c^{s,s+\lfloor n/2 \rfloor}(G,p) = (1-p)\left[\Pr(l_j \mid l_{j-1})\right]^{\lfloor n/2 \rfloor - 1} + (1-p)\left[\Pr(l_j \mid l_{j-1})\right]^{\lceil n/2 \rceil - 1} - (1-p)\left[\Pr(l_j \mid l_{j-1})\right]^{n-2}\Pr(l_j \mid l_{j-1}, l_{j+1})$ $P_c^{s,s+\lfloor n/2 \rfloor}(G,p) \to 1 - n^2p/4,\ \text{as } \rho, p \to 0$ $P_c^{s,s+\lfloor n/2 \rfloor}(G,p) \to (1-p)^{\lfloor n/2 \rfloor},\ \text{as } \rho \to 0 \text{ and } p \to 1$ $P_c^{s,s+\lfloor n/2 \rfloor}(G,p) \to (1-p),\ \text{as } \rho \to 1$

Table 5.2: Summary of probability of connection expressions for the ring graph.

5.2.3 Reliability of Harary graphs when $p \approx 0$

In this subsection, we develop approximate expressions for the all- and two-terminal reliability of Harary graphs when link failure dependencies are present. The results of this section are summarized in Table 5.3. We use the basic idea of the $\epsilon-$model in conjunction with the Harary graph analysis of Chapter 4. As noted in §4.3.1, every graph disconnection scenario can be viewed as a partitioning of the graph into two subsets of nodes which are disconnected. Since by Theorem 4.1 a partition of j consecutive nodes minimizes the number of edges joining S_j to S_{n-j}, and since the edges joining S_j to S_{n-j} are therefore in "closest" proximity when the nodes in S_j are consecutive, we reason that a conservative estimate for the reliability of Harary graphs can be obtained by treating each possible S_j as a consecutive partition of nodes. We cannot rigorously state that such an estimate would be a lower bound for the probability of graph connection because in order to do so, we would need a complete probability distribution for the states of all links in the graph.

We now derive an expression for the probability of the joint failure of the edges joining S_j to S_{n-j} when S_j is a consecutive partition of nodes and $p \approx 0$. Recall from our discussion of the $\epsilon-$model in §5.1.3 that the probability that a collection of links $l_1, l_2, \ldots, l_m$ with identical marginal distributions fails can be expressed as:

$$\begin{aligned}\Pr\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_m\right) &= \Pr\left(\bar{l}_1\right)\Pr\left(\bar{l}_2\right)\ldots\Pr\left(\bar{l}_m\right)\epsilon\left(\bar{l}_1;\bar{l}_2\right)\epsilon\left(\bar{l}_1\bar{l}_2;\bar{l}_3\right)\ldots\epsilon\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_{m-1};\bar{l}_m\right)\\ &= p^m\epsilon\left(\bar{l}_1;\bar{l}_2\right)\epsilon\left(\bar{l}_1\bar{l}_2;\bar{l}_3\right)\ldots\epsilon\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_{m-1};\bar{l}_m\right).\end{aligned}$$

In order to simplify our analysis, we assume that when S_j is a consecutive partition of nodes:

$$\epsilon\left(\bar{l}_1;\bar{l}_2\right) = \epsilon\left(\bar{l}_1\bar{l}_2;\bar{l}_3\right) = \ldots = \epsilon\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_{m-1};\bar{l}_m\right) \equiv \hat{\epsilon}.$$

That is, a link is $\hat{\epsilon}$ more likely to fail when one or more of the remaining links in the set $l_1, l_2, \ldots, l_m$ have failed. This assumption is much like the second assumption of the $q-\psi$ model, in that we are viewing S_j as a "supernode". This is clearly an approximation, since we would expect a link to be more likely to fail when more links around it have failed. Our expression for the probability that the m edges joining S_j

to S_{n-j} have failed is thus given by:

$$\begin{aligned}\Pr\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_m\right) &= p^m\epsilon\left(\bar{l}_1;\bar{l}_2\right)\epsilon\left(\bar{l}_1\bar{l}_2;\bar{l}_3\right)\ldots\epsilon\left(\bar{l}_1\bar{l}_2\ldots\bar{l}_{m-1};\bar{l}_m\right)\\ &= p^m\hat{\epsilon}^{m-1}\equiv\frac{p_{\hat{\epsilon}}^m}{\hat{\epsilon}},\end{aligned}\tag{5.6}$$

where $p_{\hat{\epsilon}}\equiv p\hat{\epsilon}$.

In the notation of the $q-\psi$-model, q would be represented by p and ψ by $p_{\hat{\epsilon}}$.

A trivial upper bound for both all- and two-terminal probabilities of connection is $1-p_{\hat{\epsilon}}^{\Delta}/\hat{\epsilon}$, which is simply the probability that not all the links incident at a particular node fail. For two-terminal reliability, we can tighten this bound without much additional effort. Assume that the node pair of interest is sufficiently spaced in the network such that the sets of links incident at each node fail independently. Then, a simple upper bound of $1-\left(2p_{\hat{\epsilon}}^{\Delta}/\hat{\epsilon}-p_{\hat{\epsilon}}^{2\Delta}/\hat{\epsilon}^2\right)$ is obtained by computing the probability of the event that either node's set of incident links fails. An analogous simple upper bound for all-terminal reliability would not be as forthcoming, since the independence assumption of the sets of incident links clearly does not hold for closely spaced nodes.

In order to obtain conservative estimates of the all- and two-terminal probabilities of connection, we substitute (5.6) into (4.17), (4.18), (4.20) and (4.21), which are good reliability bounds for Harary graphs when $p\approx 0$. All-terminal reliability is then given by:

$$\begin{aligned}P_c(G,p) &\approx 1-\frac{1}{\hat{\epsilon}}\left(np_{\hat{\epsilon}}^{\Delta}+\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n}{i}p_{\hat{\epsilon}}^{i\Delta-2\binom{i}{2}}+\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n}{i}p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)\\ &\approx 1-\frac{1}{\hat{\epsilon}}\left(np_{\hat{\epsilon}}^{\Delta}+\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n}{\left\lfloor\frac{\Delta}{2}\right\rfloor+1}\left(p_{\hat{\epsilon}}^{2\Delta-2}-p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)\right.\\ &\qquad\left.+p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-1}+\frac{1}{2}\binom{n}{\lfloor n/2\rfloor}-n-1\right]\right)\end{aligned}\tag{5.7}$$

and two-terminal reliability is given by:

$$
\begin{aligned}
P_c^{sd}(G,p) &\approx 1-\frac{1}{\hat{\epsilon}}\left(2p_{\hat{\epsilon}}^{\Delta}+2\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n-2}{i-1}p_{\hat{\epsilon}}^{i\Delta-2\binom{i}{2}}+2\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n-2}{i-1}p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)\\
&\approx 1-\frac{1}{\hat{\epsilon}}\left(2p_{\hat{\epsilon}}^{\Delta}+2\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n-2}{\lfloor\frac{\Delta}{2}\rfloor}\left(p_{\hat{\epsilon}}^{2\Delta-2}-p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)\right.\\
&\qquad\left.+p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-2}+\binom{n-2}{\lfloor\frac{n-2}{2}\rfloor}-2\right]\right).
\end{aligned}
\tag{5.8}
$$

Thus, when $p_{\hat{\epsilon}} = p\hat{\epsilon} \approx 0$, the probability of graph and node pair disconnection for Harary graphs is approximately $\frac{n}{\hat{\epsilon}}p_{\hat{\epsilon}}^{\Delta}$ and $\frac{2}{\hat{\epsilon}}p_{\hat{\epsilon}}^{\Delta}$, respectively. The analogous expressions for the independent failure model are np^{Δ} and $2p^{\Delta}$, respectively. In order to get a feeling for the difference in these two sets of expressions, let us consider a 20 node, degree four Harary graph with probability of link failure 10^{-2} and $\hat{\epsilon} = 5$. Our dependency model yields the values 2.5×10^{-5} and 2.5×10^{-6} for the all- and two-terminal probabilities of disconnection, respectively. On the other hand, the independence model yields the values 2×10^{-7} and 2.5×2^{-8}, respectively. Figures 5-6 and 5-7 illustrate the relationship between the probability of disconnection and $\hat{\epsilon}$ for the Harary graph $H(10,3)$ when $p = 10^{-4}$ and $p = 0.1$. Equations (5.7) and (5.8) indicate that when $p_{\hat{\epsilon}} = p\hat{\epsilon} \approx 0$ the probability of disconnection is linear in $\log\hat{\epsilon}$ with slope $\Delta - 1$. This expected trend is verified in the figures. Furthermore, the breakpoints of $\hat{\epsilon}$ at which the probabilities of disconnection become nonlinear are inversely proportional to p. This is a simple consequence of the fact that the breakpoints are largely determined by $p_{\hat{\epsilon}} = p\hat{\epsilon}$.

For large values of $\hat{\epsilon}$, we observe from Figures 5-6 and 5-7 that (5.7) and (5.8) are no longer good estimates of the all- and two-terminal reliabilities. In the limit of $\hat{\epsilon} = p^{-1}$ (equivalently, $\rho = 1$), these estimates do not approach p, which is the expected probability of disconnection. We attribute the diminishing accuracy of (5.7) and (5.8) to the fact that these estimates are union bounds on prime failure events. As $\hat{\epsilon}$ increases, the probability that multiple prime failure events simultaneously increases,

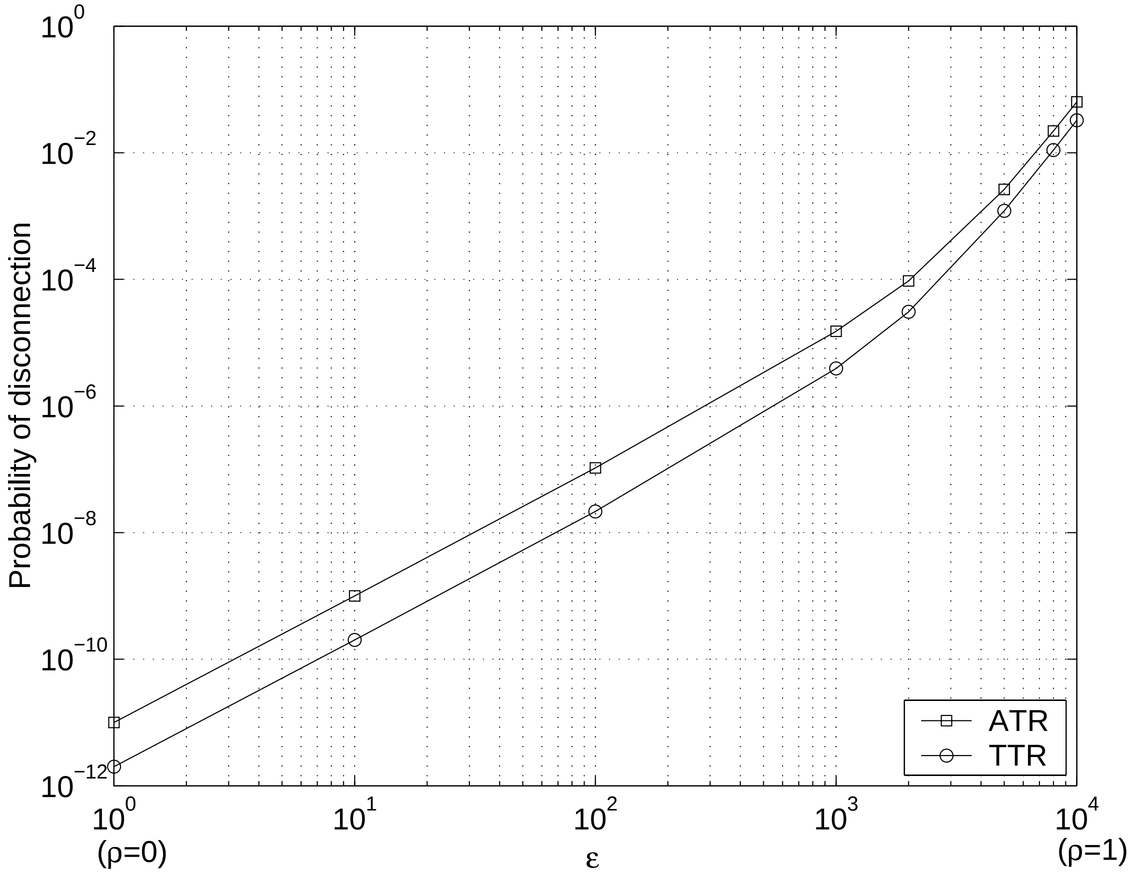


Figure 5-6: Probability of disconnection versus $\hat{\epsilon}$ for $H(10, 3)$ when $p = 10^{-4}$.

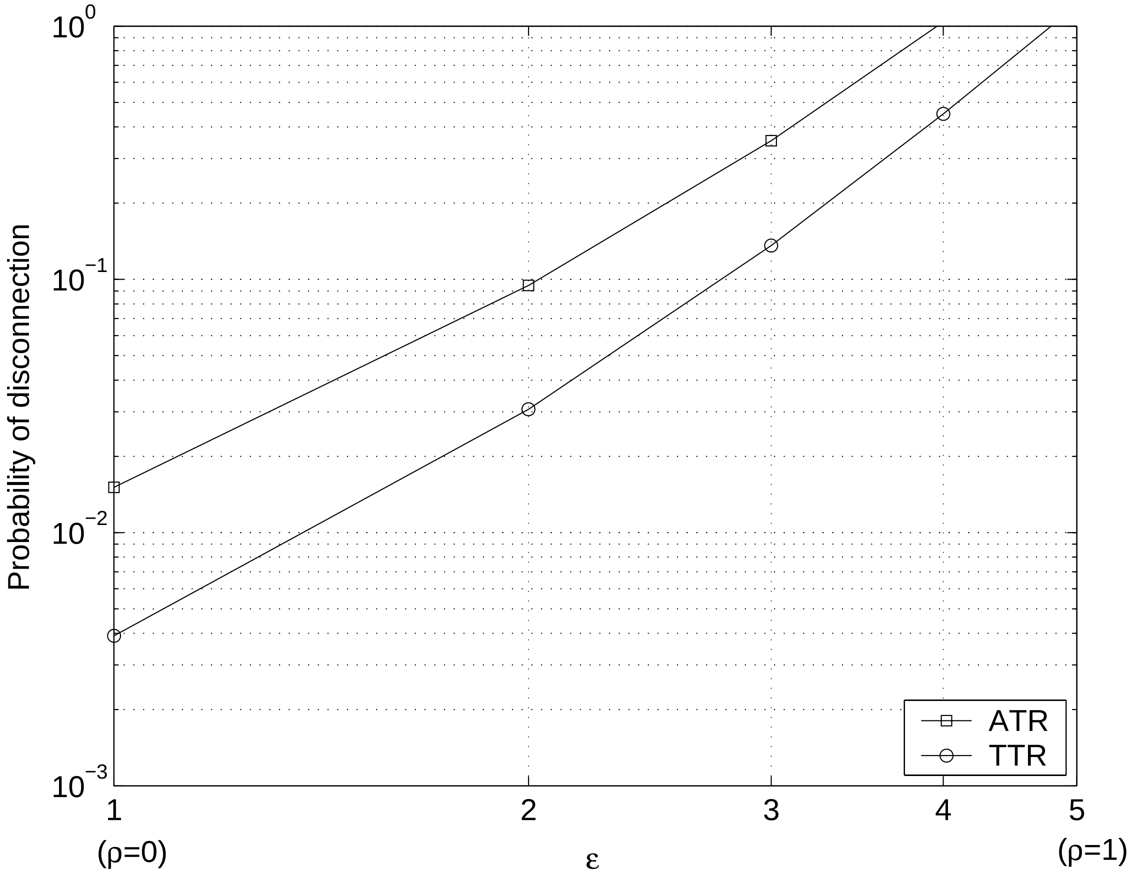


Figure 5-7: Probability of disconnection versus $\hat{\epsilon}$ for $H(10, 3)$ when $p = 0.1$.

thereby making the union bound loose.

5.2.4 All-terminal reliability when $p \approx 1$

In this subsection, we state a simple, approximate expression for the all-terminal reliability of a graph when $p \approx 1$. Recall that when $p \approx 1$, the probability that a graph remains connected is approximately equal to the probability that the operational links in the graph form a spanning tree. In the independent failure model, each of the $t(G)$ spanning trees of a graph have a probability of $(1-p)^{n-1}p^{e-n+1} \approx (1-p)^{n-1}$ of occurring. However, when statistical dependence among link failures is present, the probabilities of occurrence of the different spanning trees are in general not the same, as they depend upon the exact structure of the spanning trees. Nevertheless, we can approximate the all-terminal reliability by assuming that links remain operational in a similar manner as assumed for Harary graphs in §5.2.3. That is, the probability of a spanning tree occurring is given by $(1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2}$. Hence, the all-terminal reliability of a graph can be approximated by:

$$P_c(G,p) \approx t(G)(1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{n-2} \tag{5.9}$$

where we recall that $\Pr\left(l_j|l_{j-1}\right)$ denotes the probability that link j is operational given that an adjacent link $j-1$ is operational.

Figure 5-8 illustrates the relationship between the probability of graph connection and $\epsilon \equiv \Pr\left(l_j|l_{j-1}\right)/(1-p)$ for the the ten node, degree three Petersen graph and the Harary graph $H(10,3)$, when $p = 0.9$. The figure depicts the linear relationship between the logarithm of the probability of graph connection and the logarithm of ϵ. Note that as ϵ increases to values near $(1-p)^{-1}$, the all-terminal reliability estimate exceeds unity, whereas it should approach $(1-p)$. Thus, (5.9) is a reasonable all-terminal reliability estimate for only small values of ρ. The diminishing accuracy of the estimate as ρ increases is expected, since (5.9) is a union bound on the spanning tree events. As ρ increases, we are increasing the probability of occurrence of each spanning tree event, and the union bound becomes looser because the probability of

	Probability of Connection
ATR	$P_c(G,p) \approx 1-\frac{1}{\hat{\epsilon}}\left(np_{\hat{\epsilon}}^{\Delta}+\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n}{i}p_{\hat{\epsilon}}^{i\Delta-2\binom{i}{2}}+\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n}{i}p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)$ $\approx 1-\frac{1}{\hat{\epsilon}}\left(np_{\hat{\epsilon}}^{\Delta}+\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n}{\lfloor\frac{\Delta}{2}\rfloor+1}\left(p_{\hat{\epsilon}}^{2\Delta-2}-p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)+p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-1}+\frac{1}{2}\binom{n}{\lfloor n/2\rfloor}-n-1\right]\right)$ $P_c(G,p)\to\left(1-\frac{np_{\hat{\epsilon}}^{\Delta}}{\hat{\epsilon}}\right),\ \text{as } p_{\hat{\epsilon}}\to 0$
TTR	$P_c^{sd}(G,p) \approx 1-\frac{1}{\hat{\epsilon}}\left(2p_{\hat{\epsilon}}^{\Delta}+2\sum_{i=2}^{\lfloor\Delta/2\rfloor+1}\binom{n-2}{i-1}p_{\hat{\epsilon}}^{i\Delta-2\binom{i}{2}}+2\sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor}\binom{n-2}{i-1}p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)$ $\approx 1-\frac{1}{\hat{\epsilon}}\left(2p_{\hat{\epsilon}}^{\Delta}+2\left\lfloor\frac{\Delta}{2}\right\rfloor\binom{n-2}{\lfloor\frac{\Delta}{2}\rfloor}\left(p_{\hat{\epsilon}}^{2\Delta-2}-p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right)+p_{\hat{\epsilon}}^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\left[2^{n-2}+\binom{n-2}{\lfloor\frac{n-2}{2}\rfloor}-2\right]\right)$ $P_c(G,p)\to\left(1-\frac{2p_{\hat{\epsilon}}^{\Delta}}{\hat{\epsilon}}\right),\ \text{as } p_{\hat{\epsilon}}\to 0$

Table 5.3: Summary of probability of connection expressions for Harary graphs when $p \approx 0$.

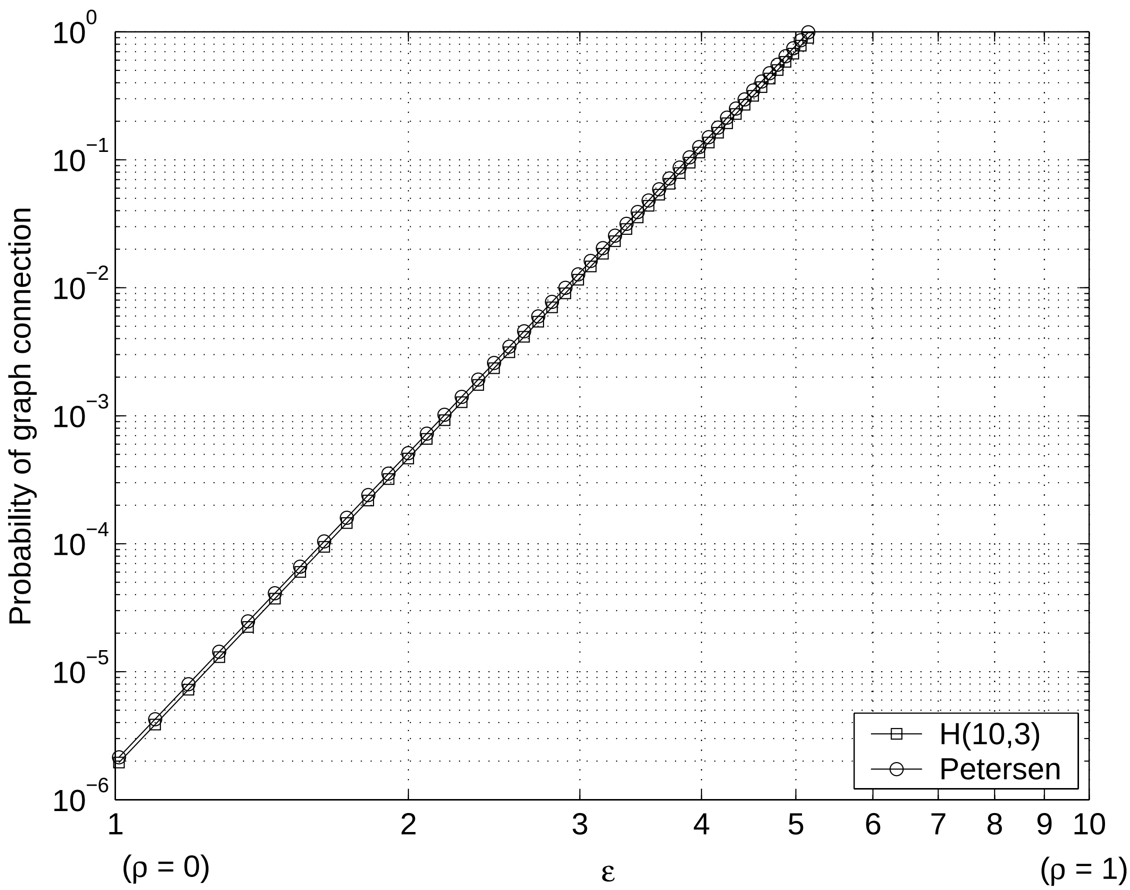


Figure 5-8: Probability of graph connection versus $\epsilon \equiv \Pr(l_j|l_{j-1})/(1-p)$ for the Petersen graph and $H(10,3)$ when $p = 0.9$.

occurrence of multiple spanning tree events is no longer insignificant.

The approximated performances of the Petersen and Harary graphs plotted in Figure 5-8 are seen to be quite similar. In fact, (5.9) indicates that the ratio of the approximate all-terminal reliabilities of two graphs with the same number of nodes is given by the ratio of their number of spanning trees. In our example, the Petersen graph has 2000 spanning trees and the Harary graph has 1815 spanning trees, which amounts to a small all-terminal reliability performance difference.

5.2.5 Two-terminal reliability when $p \approx 1$

When $p \approx 1$ and we are interested in the two-terminal reliability of a graph, we use a variation of the simple bound derived in §3.2.4:

$$\min_{s,d}\left[P_c^{sd}(G,p)\right] \geq (1-p)^{k(G)}$$

Using the Markov model of Example 5-1 along the shortest path between the worst-case node pair, the above expression becomes:

$$\min_{s,d}\left[P_c^{sd}(G,p)\right] \approx (1-p)\left[\Pr\left(l_j|l_{j-1}\right)\right]^{k(G)-1}$$

where, again, $\Pr\left(l_j|l_{j-1}\right)$ denotes the probability that link j is operational given that link $j-1$ is operational.

As an example, let us consider a 20 node, degree four Harary graph with probability of link operation 10^{-2} and conditional probability $\Pr\left(l_j|l_{j-1}\right) = 5 \times 10^{-2}$. Recall that for Harary graphs, the diameter grows linearly with the number of nodes in the graph in the following fashion:

$$k(G) = \begin{cases} \frac{2}{\Delta}\left\lceil\frac{n-1}{2}\right\rceil, & \text{if } \Delta \text{ is even,} \\ \frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Thus, the network diameter is $k = 5$. Our dependency model yields a lower bound of 6.25×10^{-8}, whereas the independence model yields a lower bound of 1×10^{-10}.

	Probability of Connection
ATR	$P_c(G,p) \approx t(G)(1-p)\left[\Pr\left(l_j \mid l_{j-1}\right)\right]^{n-2}$
TTR	$\min_{s,d}\left[P_c^{sd}(G,p)\right] \approx (1-p)\left[\Pr\left(l_j \mid l_{j-1}\right)\right]^{k(G)-1}$

Table 5.4: Summary of probability of connection expressions when $p \approx 1$.

Figures 5-9 and 5-10 illustrate the relationship between the probability of node pair connection and $\epsilon \equiv \Pr\left(l_j \mid l_{j-1}\right)/(1-p)$ for the the ten node, degree three Petersen graph and the Harary graph $H(10,3)$, when $p = 0.9$ and $p = 0.9999$, respectively. The figures depict the linear relationship between the logarithm of the probability of node pair connection and the logarithm of ϵ. As expected, as ϵ increases, the performance difference between the diameter two Petersen graph and the diameter three Harary graph $H(10,3)$ diminishes until it is zero in the limit that $\epsilon = (1-p)^{-1}$ and the probability of node pair connection is $1-p$.

5.3 Comparison of models and topologies

We conclude this chapter with a comparison of the models and topologies studied in the previous section. As we shall see, it is difficult to make a fair reliability comparison among the Ethernet, ring, multi-ring and Harary graphs studied in this chapter because the underlying dependent failure model is different in some of these cases.

5.3.1 All- and two-terminal reliability when $p \approx 0$

In Figure 5-11, we plot the all- and two-terminal reliability performance of the ten node Ethernet, ring, double-ring and $H(10,3)$ graphs as a function of the correlation

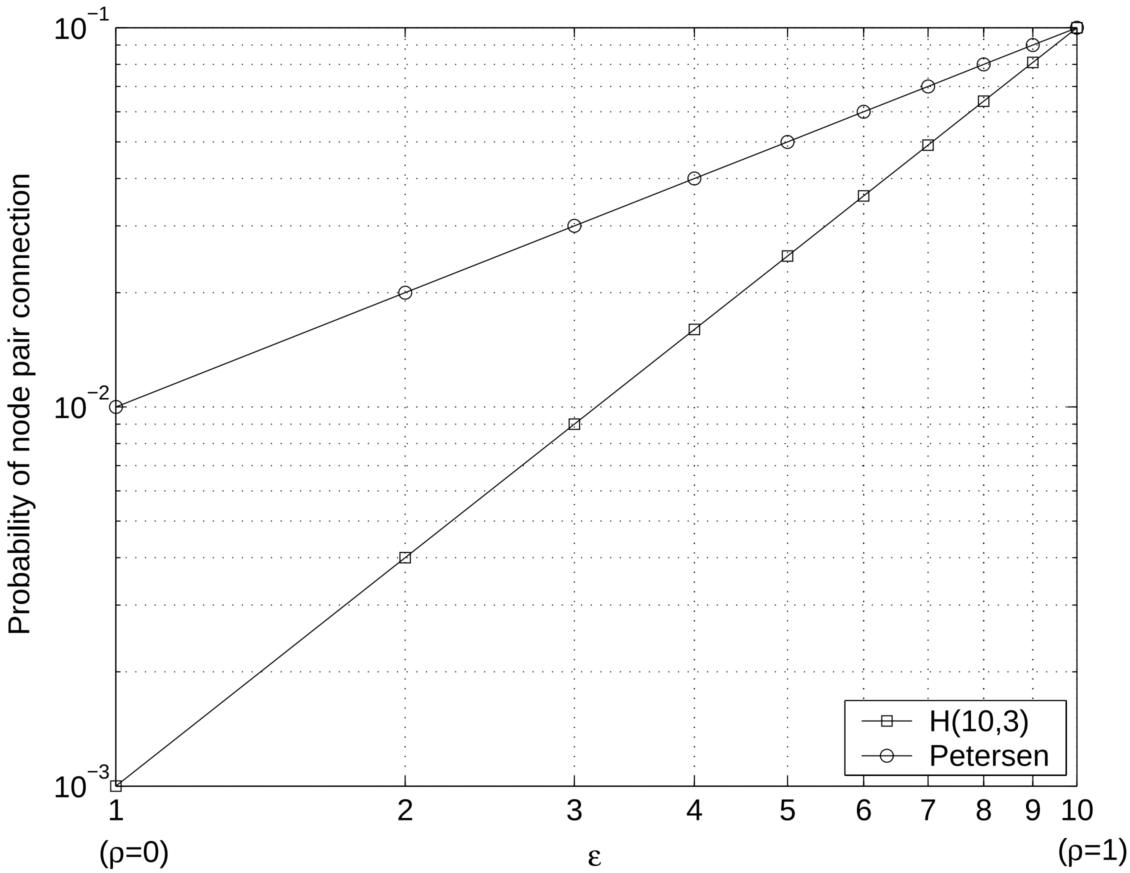


Figure 5-9: Probability of node pair connection versus $\epsilon \equiv \Pr(l_j|l_{j-1})/(1-p)$ for the Petersen graph and $H(10,3)$ when $p = 0.9$.

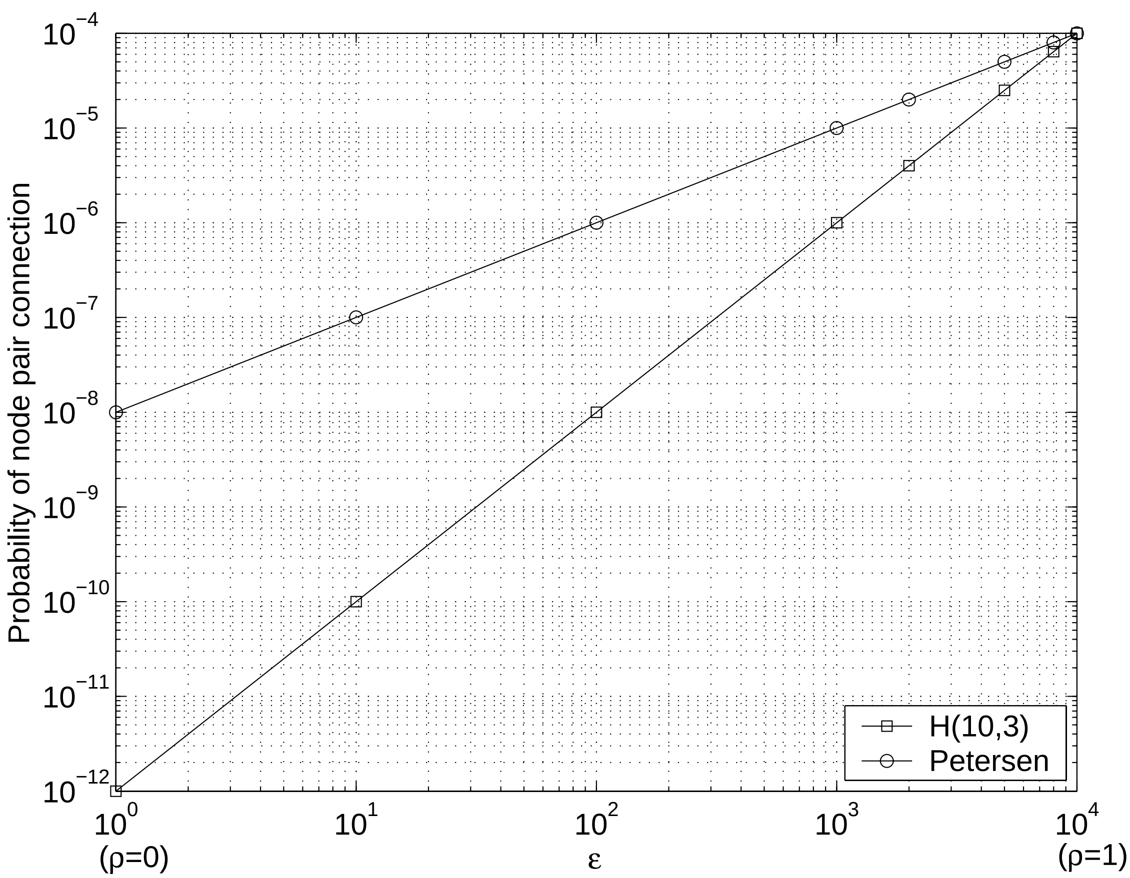


Figure 5-10: Probability of node pair connection versus $\epsilon \equiv \Pr\left(l_j | l_{j-1}\right) / (1-p)$ for the Petersen graph and $H(10,3)$ when $p = 0.9999$.

coefficient ρ, when $p = 10^{-4}$. The relationship between ρ and the conditional probabilities governing the states of adjacent links is given by (5.2). When the correlation coefficient ρ is small – that is, when link failures are almost independent – the relative performance of the topologies is what we would expect from the independent failure model. See §4.4 for a comparison when independent link failures are assumed.

As ρ increases, the different assumptions in the different dependent failure models manifest themselves. For example, $H(10, 3)$, which possesses the best reliability performance among all graphs when $\rho \approx 0$, exhibits increasingly poor performance relative to the other graphs as ρ increases to one. We attribute this to the conservative model developed for Harary graphs in §5.2.3 for $p \approx 0$. In this model, we first made the pessimistic assumption that every graph disconnection scenario is a partitioning of the graph into two consecutive subsets, in accordance with Theorem 4.1. We then made the additional pessimistic assumption that the links joining these two partitions are equally correlated. Thus, as ρ increases we expect the accuracy of our model to diminish. In fact, in the extreme scenario where $\rho = 1$, we require the all- and two-terminal reliabilities to reduce to p. However, as illustrated in Figure 5-11, our model yields probabilities of disconnection greater than p. Similarly, our dependent failure model for the Ethernet graph in §5.2.1 becomes increasingly inaccurate as ρ increases. Again, in the extreme scenario where $\rho = 1$, we require the all- and two-terminal reliabilities to reduce to p. However, owing to our assumption that correlation only exists among the two links incident at each non-switch node, we obtain probabilities of disconnection greater than p in this extreme case. On the other hand, our model for the ring and multi-ring graphs in §5.2.2 yields correct asymptotic reliabilities when $\rho \approx 1$, as illustrated in Figure 5-11.

5.3.2 All- and two-terminal reliability when $p \approx 1$

Figure 5-12 illustrates the all-terminal reliability as a function of the correlation coefficient ρ for the ten node Ethernet, ring, double-ring, $H(10, 3)$ and Petersen graphs when $p = 0.9$. The analysis underlying the performance of the $H(10, 3)$ and Petersen graphs is that of §5.2.4, and for the ring and multi-ring topologies we follow §5.2.2.

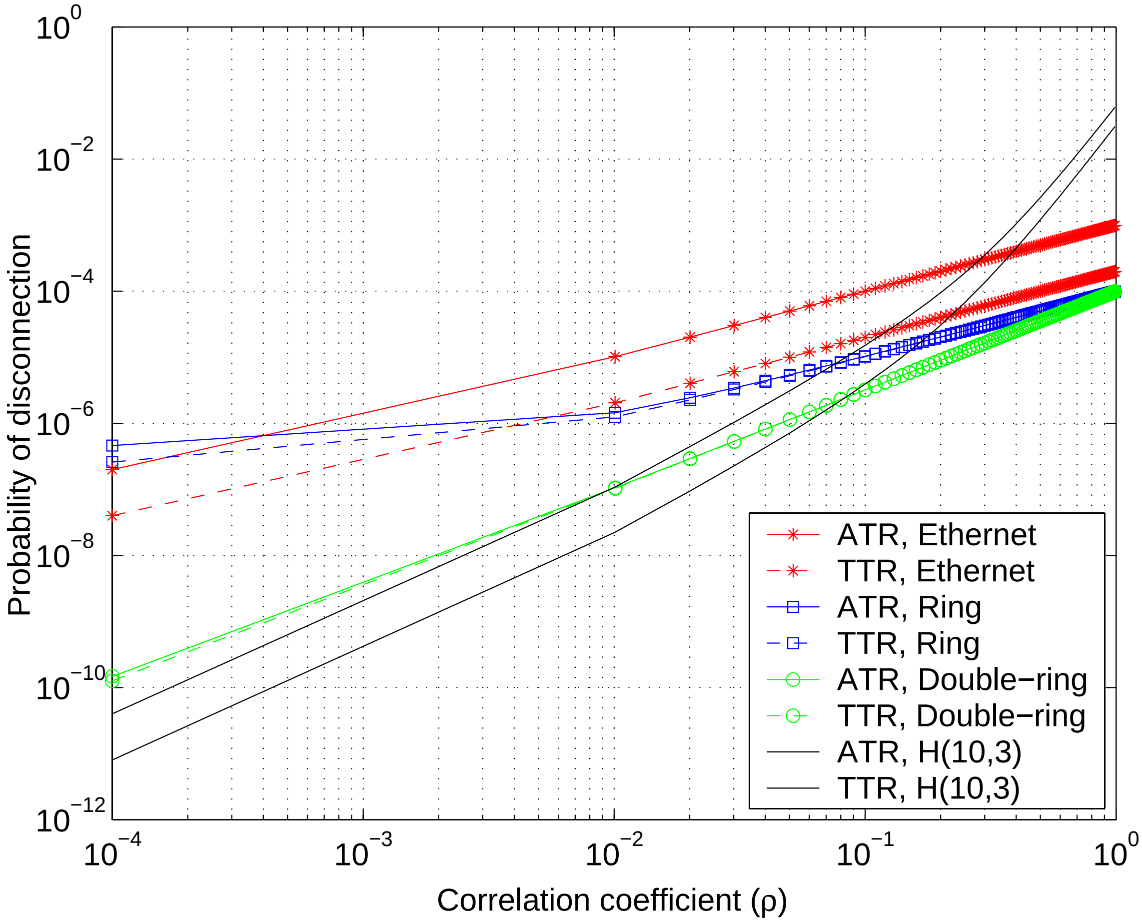


Figure 5-11: Probability of disconnection versus correlation coefficient ρ for the ten node Ethernet, ring, double-ring and $H(10, 3)$ graphs when $p = 10^{-4}$. For the double-ring, the correlation coefficient for the two parallel links between any two adjacent nodes was assumed to be $\sqrt{\rho}$.

Lastly, for the Ethernet graph, the model used is that of §5.2.1. When $\rho \approx 0$, the trends depicted in Figure 5-12 are what we expect from the independent failure model. As ρ increases, however, the modelling assumptions underlying the different topologies take effect. The ring and multi-ring graphs' all-terminal reliabilities converge to the correct value of $(1-p)$ as ρ approaches unity. The all-terminal reliability of the Harary and Petersen graphs exceeds unity as ρ approaches unity for the reasons discussed in §5.2.4.

The all-terminal reliability of Ethernet exhibits a peculiar downward trend as ρ increases. When $\rho \approx 0$, the all-terminal reliability of Ethernet is approximately $(1-p)^n\left[2+2^n(1-p)\right]$. If $2 \ll 2^n(1-p)$, then the all-terminal reliability is dominated by the probability of graph connection given that the bridge is operational. Conversely, if $2 \gg 2^n(1-p)$, then the all-terminal reliability is dominated by the probability of graph connection given that the bridge has failed. When $\rho \approx 1$, the all-terminal reliability of Ethernet is dominated by the probability of graph connection given that the bridge has failed, and is approximately $(1-p)^n$, which is at least a factor of two worse than the all-terminal reliability when $\rho \approx 0$. On the other hand, if all link failures in the Ethernet topology were correlated, then the all-terminal reliability would converge to $(1-p)$ as ρ approaches one. However, since our Ethernet link failure model assumes independence among different sets of link failures, the all-terminal probability converges to the probability that the two links incident at each non-switch node are operational, which is $(1-p)^n$.

Figure 5-13 depicts the two-terminal reliability as a function of the correlation coefficient ρ for the ten node Ethernet, ring, double-ring, $H(10,3)$ and Petersen graphs, when $p = 0.9$. The analysis underlying the performance of the $H(10,3)$ and Petersen graphs is that of §5.2.5, in which we conservatively only account for the probability that the shortest path between the node pair exists. The model underlying the ring and multi-ring topologies is that of §5.2.2. Lastly, for the Ethernet graph, the model used is that of §5.2.1, which implies that link failures along the shortest path between the node pair are statistically independent.

When $\rho \approx 0$, the trends depicted in Figure 5-13 are what we expect from the

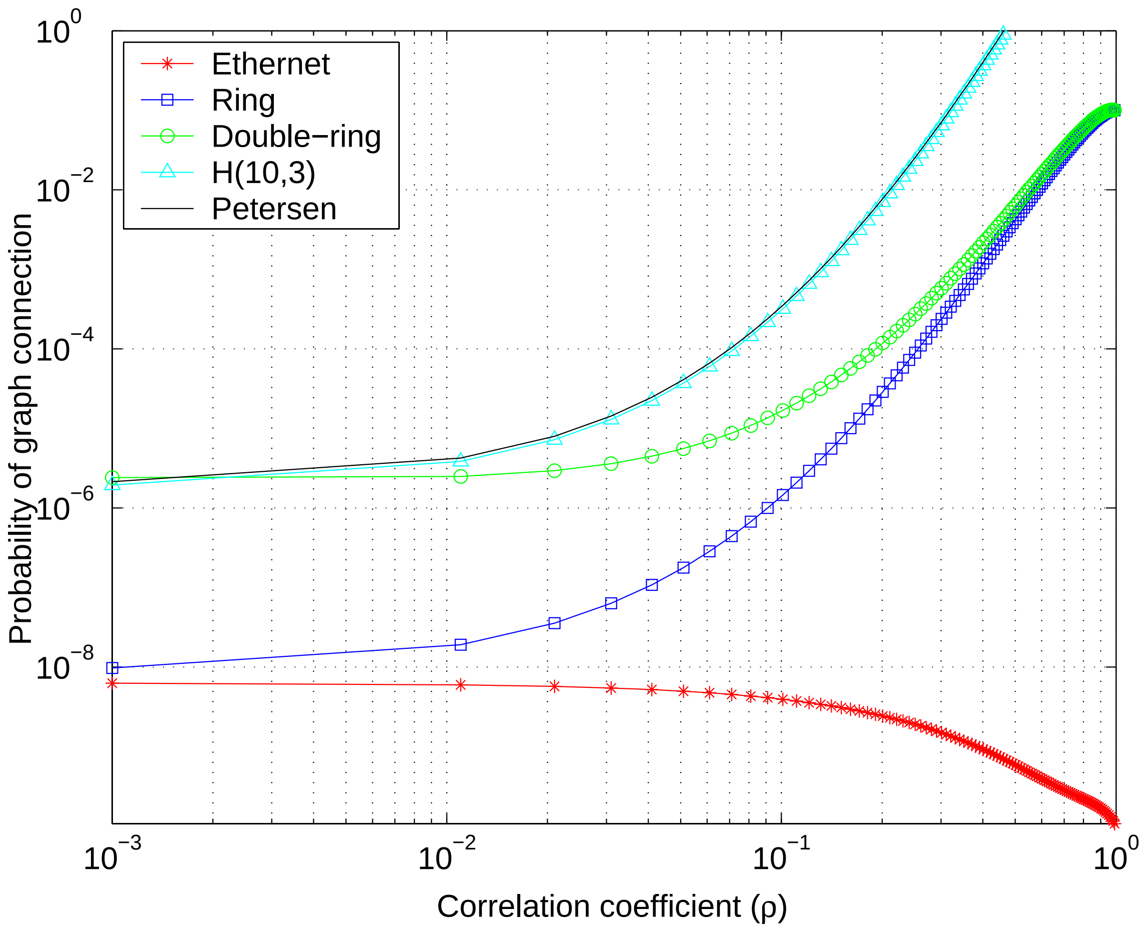


Figure 5-12: Probability of graph connection versus correlation coefficient ρ for the ten node Ethernet, ring, double-ring, $H(10,3)$ and Petersen graphs when $p = 0.9$. For the double-ring, the correlation coefficient for the two parallel links between any two adjacent nodes was assumed to be $\sqrt{\rho}$.

independent failure model. Specifically, the relative performance of the topologies is largely governed by their respective diameters. As ρ increases, however, the effect of these different network diameters diminishes. As can be seen from Figure 5-13, the reliability performances of the $H(10,3)$, Petersen, ring and multi-ring graphs converge to the expected value of $(1-p)$. Ethernet, however, owing to the assumptions of its model, exhibits a similar downward trend as in the case of all-terminal reliability. When $\rho \approx 0$, the two-terminal reliability is approximately $2(1-p)^2$, which is approximately equal to the probability of one of the two-hop paths between the source and destination being operational. On the other hand, when $\rho \approx 1$, the two links from each non-switch node act as one link and there is effectively only one two-hop path between the source and destination. In this case, the two-terminal reliability is approximately $(1-p)^2$.

5.4 Summary

In this chapter, we considered the reliability of networks when statistical dependence exists among link failures. We began by motivating the need for such dependent failure models, and then reviewed the models previously proposed by researchers. We then carried out simple, approximate dependent failure analyses for several special network topologies. Unfortunately, the different assumptions used in each of these models precluded a detailed comparison among these topologies, except when little correlation among link failures was present. These models, however, may find use in comparisons among graphs belonging to the same family.

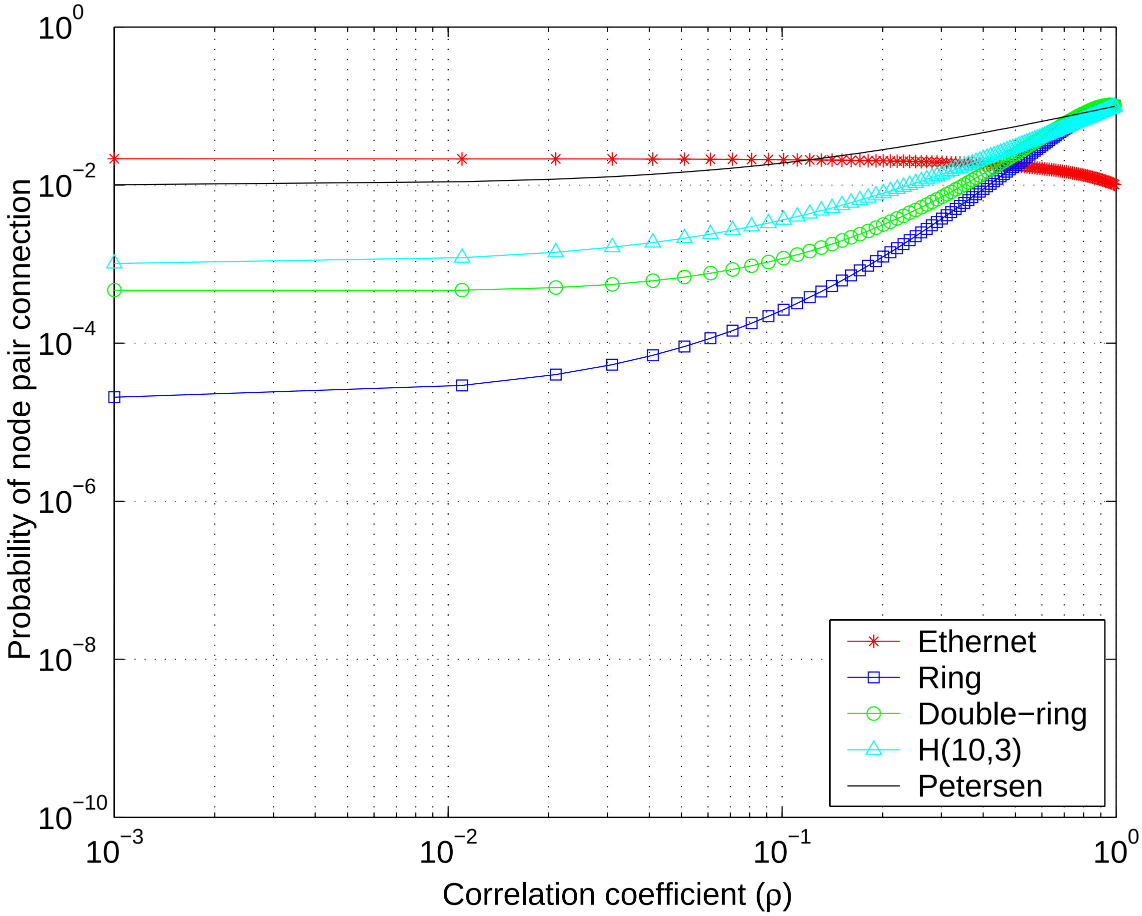


Figure 5-13: Worst-case probability of node pair connection versus correlation coefficient ρ for the ten node Ethernet, ring, double-ring, $H(10,3)$ and Petersen graphs when $p = 0.9$. For the double-ring, the correlation coefficient for the two parallel links between any two adjacent nodes was assumed to be $\sqrt{\rho}$.

Chapter 6

Conclusion

In this final chapter, we summarize the work presented in this thesis and highlight the novel contributions made to the field of network reliability. We then proceed to discuss avenues for further research.

6.1 Summary of work and contributions

This work began with a motivating chapter in which the importance of network reliability was addressed, as well as some of the most common reliability practices in real networks. The particular relevance of this thesis to optical networking was justified as well. Optical networks are increasingly being used in applications that demand high-reliability, such as controlling vital vehicular functions. Furthermore, the network model adopted in this work, in which links are vulnerable but nodes are not, is quite realistic for optical networks. This is because in optical networks the electronics in the nodes are significantly more reliable than the optics in the communication links, and lightpaths between sources and destinations are routed without any intervening electronics. Furthermore, the idea of lightpath diversity, currently being considered for high-reliability optical network applications, is closely related to the idea of maintaining a level of connectedness in a network, which is addressed in this work.

In Chapter 2, we carried out a thorough survey of the research area of network reliability, in which we brought many previous contributions in the field into a cohesive framework. We categorized reliability metrics as being either deterministic or probabilistic, and introduced the Harary, circulant, and Moore families of graphs which possess special reliability properties.

After setting the stage with our survey of the network reliability field, we considered in Chapter 3 the design of networks with statistically independent link failures and invulnerable nodes. In the first part of the chapter, we introduced new bounding techniques for all- and two-terminal reliability which yielded tight bounds for p less than approximately 0.2 and p larger than approximately 0.8. We then developed real-world context for our reliability metrics, and outlined and justified a design methodology in which circulant graphs were the principal candidate topologies. The next part of the chapter was devoted to using the bounding techniques and insights developed earlier to specify configurations of graphs which meet prescribed reliability levels. The chapter concluded with a discussion on a series of simulation results. Among the insights gained from the simulations were that in the $p \approx 0$ regime, all super-λ graphs with the same number of nodes and edges perform almost identically. This is a positive result as it enables the network designer to then optimize the choice of topology with respect to other network metrics apart from reliability in the $p \approx 0$ regime. With respect to all-terminal reliability when $p \approx 1$, the variation in performance among super-λ graphs with the same number of nodes and edges was shown to be relatively small, as the graphs possess a comparable number of spanning trees. Two-terminal reliability performance, on the other hand, was shown to be more sensitive to graph structure, particularly at lower node degrees. We attributed this to the definition of two-terminal reliability as the worst-case probability of node pair connection, which allows local topological differences of graphs to be magnified. Finally, we observed that in order to obtain all- and two-terminal reliabilities in the 0.1 to 1 range when $p \geq 1/2$, very large node degrees are required and that for such high node degree graphs, the actual graph structure is not very important.

In Chapter 4, we carried out reliability case studies of special topologies – Ethernet, ring, multi-ring, and Harary graphs. Exact expressions for all- and two-terminal reliability were derived for the Ethernet, ring and multi-ring topologies. In addition, the simplicity of the topologies permitted the consideration of node failures. In our analysis of Harary graphs, we began by proving an intuitive and useful theorem for this family of graphs. This theorem allowed us to develop tight, closed form reliability bounds for Harary graphs when p is small. When $p \approx 1$, we employed the techniques of Chapter 3 in conjunction with the some new results on the diameter and path lengths of Harary graphs. We concluded the chapter with a comparison of the Ethernet, ring, multi-ring and Harary topologies. Between Ethernet and the ring, which are the degree two topologies, Ethernet exhibited better all- and two-terminal reliability, owing to Ethernet's weak scalability with the number of nodes in the graph. Between the multi-ring and a same degree Harary graph, Harary graphs perform better, indicating the advantage of more strategic positioning of link capacity rather than adding redundant backup links.

Chapter 5 broadened the scope of this thesis by allowing for the possibility of statistical dependency among link failures. The chapter began with an example which motivated the need for models which permit statistical dependency among link failures. The example also served to introduce a simple Markov dependent failure model. We next surveyed existing dependent failure models in the literature. The attractive features and limitations of each model were discussed. We then conducted approximate dependent failure analyses of several special topologies – Ethernet, ring, multi-ring and Harary graphs. Unfortunately, the models developed for these topologies rested on different assumptions, thereby making detailed comparisons among families of graphs difficult. On the other hand, these models may find use in comparisons among graphs belonging to the same family.

It should be noted that the work presented in this thesis only brings the network architect part way to the goal of designing a highly reliable network. The value provided by the models and techniques of this work is determined by the network designer's ability to set up a model which accurately depicts a real network. As we saw in Chapter 5, it is not enough to simply consider a network structure and a set of marginal component failure probabilities, as the interactions among network components significantly influence the reliability function of the network. Thus, to obtain meaningful results from a network reliability model, great care should be taken in determining the set of inputs to the model.

6.2 Avenues for further research

Clearly, more more work needs to be done with respect to dependent component failure models. Consistent, approximate models which strike a good balance between simplicity and applicability to a variety of topologies need to be developed. In addition, these models should possess intuitive inputs which are readily available to the network designer. Subsequently, optimality conditions for different regions of component vulnerability and dependency, akin to those developed for the independent failure model, need to be pursued.

Furthermore, as discussed in the introductory chapter, reliability metrics have recently been broadened to include some measure of performance, such as throughput or delay, rather than the simple connectedness measures adopted in this work. Applying these broader *performability* metrics to dependent component failure models would yield useful insights regarding the design of practical communication networks.

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