DYNAMICS OF ELASTIC

TAUT INCLINED CABLES

by

Laurent Grinfogel

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Signature of Author_____
Department of Ocean Engineering
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Certified by_____
Michael S. Triantafyllou
Thesis Supervisor

Accepted by_____
A.D. Carmichael
Chairman, Ocean Engineering Department Committee

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ABSTRACT

Thesis Supervisor: Dr. Michael S. Triantafyllou

Title   : Associate Professor of Ocean Engineering

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Asymptotic, analytic expressions for the natural frequencies, the natural modes and the dynamic tension of a taut, elastic, inclined cable are derived for small ratios of cable weight to end forces. It is explained in detail how, starting from the exact solution of the linearized dynamic equations, perturbation expansions can provide simple, accurate results. The results for horizontal cable can be obtained as a particular case by letting the inclination angle of the cable tend towards zero. Indeed, all the characteristics of the dynamics of an inclined cable, such as the existence of hybrid modes, an amplification of the dynamic tension for a particular value of the elastic stiffness, and no cross-over occurrence are captured by our approximations. All the results compare extremely well numerically with the respective exact quantities calculated by computer codes.

The natural modes, as calculated here, proved not to be orthogonal. Consequently our analytic formulations might be less convenient to combine into more complex solutions, to solve, for instance, for the non linear dynamics of cables.

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Introduction

Surprisingly enough, however simple cables may appear, when considered as structural elements, a thorough understanding of their dynamic behavior has been achieved fairly recently, especially for the more complex cases, such as inclined cables and large sag cables in a strong current. A complete summary of these solutions can be found in [1]. Nonetheless, man has been able to use cables for centuries because the structural properties of cables come only from their static behaviors, and as long as one makes sure that no excitation will induce vibrations in the cable, a study of the statics is sufficient. Any motion induced in the cable will deteriorate its static behavior by generating additional stresses, which may rupture the cable. The role of a proper design consists in either avoiding that the cable be excited, or in accounting properly for the excess of tension generated by the motion.

Of particular interest to the structural designer is the taut elastic inclined cable. Its linear dynamics are now precisely modelled ([2]), but at the price of a greater complexity. In particular, no easy-to-use analytic formulae for the natural modes, or the natural frequencies exist, and computers must be used. This justifies that we try to simplify the exact formulation when the ratio of sag over span is small. The economy in computer time is not so important; the principal advantage of having analytic expressions is that we can study in depth the influence of the principal parameters. Also, closed form expressions for the natural modes can help build non linear solutions as demonstrated in [3].

In the sequel, we will show how to fully describe analytically a taut inclined cable. We will first explain its linear dynamics (Chapter 1),

then successively derive analytic expressions for the natural frequency equation (chapter 2) and the natural modes (chapter 3). At this step, the governing equations of a cable are entirely solved and the dynamic tension is easily obtained using these solutions (chapter 4). At each step, our approximate results are carefully checked against the exact ones, which are numerically computed.

Chapter 1

## Solution of the Linear Dynamics of Inclined Marine Cables

1.1 Governing Equations

1.1.1 Notation

In the sequel, we will consider a perfectly elastic cable with the following geometric and dynamic properties:

L   unstretched cable length

A   stretched cable cross-section

m (W)   stretched cable mass (weight) per unit length

M   stretched cable mass plus added mass per unit length

E   Young's modulus

Also, we will denote by s the unstretched Lagrangian coordinate along the cable (s=0 at the bottom, s=L at the top), ϕ_0 (ϕ) the static (dynamic) part of the angle measured between the horizontal and the tangent to the cable at point s and by H the horizontal component of the force at the top end of the cable (see figures 1.1.1 and 1.1.2). Also, $\alpha(s)$ stands for the local static curvature $d\phi_0(s)/ds$, and T (T_0) for the dynamic (static) part of the effective tension. (We recall that the effective tension T_e is the sum of the tension in its usual sense plus the hydrostatic pressure times the cross-section at point s. T_e is a scalar which introduces itself
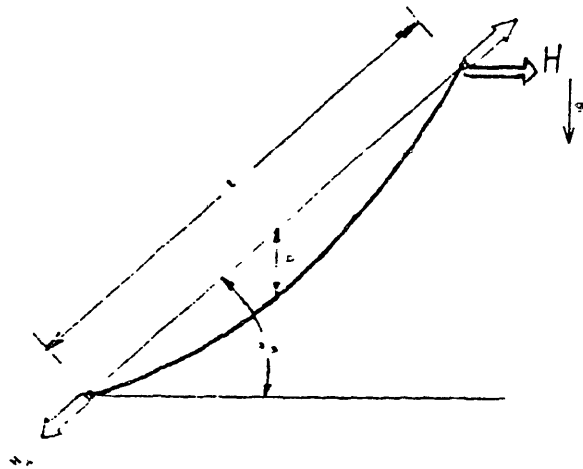
1.1.1. Taut inclined cable
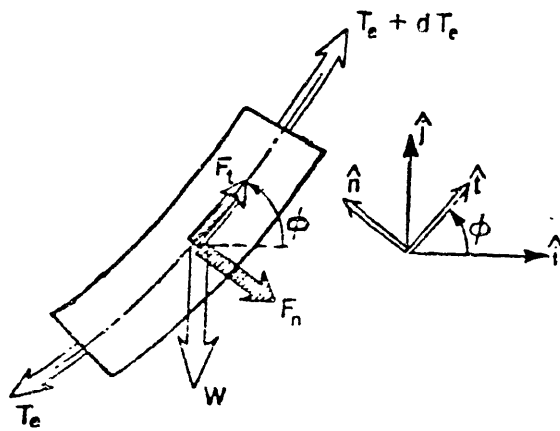
1.1.2. Coordinate system and forces acting
        on a segment of cable

naturally in the static equations. For further detail on the static equations, see [5].)

1.1.2 Governing Equations

As shown in [5] (chapter 14), the governing equations for the dynamics of a cable are highly non-linear, which makes it difficult to solve them without further assumptions. We focus on those dynamic cable configurations which remain close to the static equilibrium, which means that we do not want the dynamic behavior to change significantly the static performance. By retaining first order terms, a set of linearized equations for the cable dynamics is derived:

$$M \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial s} [T_0 \phi] + \frac{T \partial \phi_0}{\partial s} \tag{1}$$

$$m \frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial s} - W_0 \cos \phi_0 \, \phi \tag{2}$$

$$\frac{\partial u}{\partial s} - w \frac{\partial \phi_0}{\partial s} = e \tag{3}$$

$$\frac{\partial w}{\partial s} + u \frac{\partial \phi_0}{\partial s} = \phi \tag{4}$$

$$T = EA \, e \tag{5}$$

where u and w are the dynamic displacements along, respectively, the static tangent t_0 and normal n_0 directions (see figure 1.1.2), W_0 is the weight minus the buoyancy, e is the dynamic stretching, ϕ the dynamic angle and T the dynamic tension. This set of equations assumes that the static configuration is two dimensional (lying in a vertical plane), in which case, within linear theory the out-of-plane motion is uncoupled from the

in-plane motion. In the sequel, we will not pay any more attention to the out-of-plane motion which is quite simple to study.

1.2 Solution of the Linearized Equations

1.2.1 Methodology

Even this linearized set of equations can not be solved for in the general case and an approximate solution requires a lot of physical insight. The study of two limiting cases of cable vibration, the taut string and the inelastic chain can help us gain this insight. Since the properties of a taut cable lie between those of a taut string and of an inelastic chain, it seems reasonable to build our solution for the taut cable using arguments from both these limiting cases.

Since, equations 1 to 5 are linear, we can speak of modes for the tangential and transverse displacements u and w (by taking a Fourier transform with respect to the time variable). We will see in 1.2.2 in the case of the taut string that u and w have wavenumbers (that is, rate of spatial change) with different orders of magnitude. The recognition of these different orders of magnitude is the basic step towards applying the perturbation method principles to distinguish two different kinds of solution for our problem. Thus it is important to understand the physical characteristics that these solutions represent.

1.2.2 Taut String

A taut string is a cable without sag, i.e. with zero curvature, or, alternatively a cable whose ratio $(W_0 L/H)$ of the cable weight to the end forces tends towards 0. The static tension T_0 is constant along the string. The dynamics equations in that case simplify into (by setting $\dfrac{\partial \phi_0}{\partial s}$ and W_0 to 0 in (1, 2, 3, 4)):

$$\frac{\partial^2 u}{\partial t^2} = \frac{EA}{m} \frac{\partial^2 u}{\partial s^2} \tag{5}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{T_0}{M} \frac{\partial^2 w}{\partial s^2} \tag{6}$$

We see that the transverse (w) and longitudinal (u) modes are uncoupled. If we call natural modes the solutions that cancel out at both extremeties (this happens for a set of particular frequencies called the natural frequencies: these particular solutions are especially useful as it is easy to build other solutions by combining them), then the natural modes of the taut strings are:

$$u(s,t) \sim \sin(k_n s)\, \sin(\omega_n t)$$

with $k_n = \omega_n / \sqrt{EA/m}$ and ω_n given by $\sin(k_n L) = 0$

$$w(s,t) \sim \sin(k_n' s)\, \sin(\omega_n' t)$$

with $k_n' = \omega_n'/\sqrt{T_0/M}$ and ω_n' given by $\sin(k_n' L) = 0$

If we now induce forced vibrations in the taut string at a frequency ω_0, then the wavenumbers μ and k of the longitudinal and transverse modes will be in the ratio:

$$\frac{k}{\mu} \sim \sqrt{\frac{E}{T_0/A}}$$

Since T_0/A is the static stress and we design the cable so that T_0/A is much less than the yield stress, which in turn is much less than Young's modulus, then $k/\mu \gg 1$, i.e. the transverse waves travel much faster than the longitudinal ones or alternatively, the spatial rate of change of w is much greater than that of u. Indeed, μ can be very low, so that u can be set as:

$$u \sim \sin \mu s \sin \omega t \approx \mu s \sin \omega t$$

u is similar to a distributed string: this is called quasi-static stretching.

The dynamic tension is generated only by the longitudinal mode (cf. 5 with $e = \frac{\partial u}{\partial s}$) and is thus proportional to s. The transverse mode does not generate any tension, to first order.

1.2.3 Inextensible Catenary

Inextensible cables, hanging between two ends reconfigure themselves as they vibrate around a higher, and thus shorter average position than the static configuration, so that the excess lenght is used to accomodate the sinusoidal like arcs of vibration.

If we distinguish now between symmetric modes (even number of nodes) and antisymmetric modes (odd number of nodes) we can expect that those latter will generate less tension than the former since intuitively a symmetric mode seems to require more stretching than an antisymmetric one.

Since the case of an elastic cable lies between the case of a taut string and the case of a catenary, we can expect a taut cable to behave

with a mixture of taut string and inextensible catenary properties. This is what we are going to study now.

1.2.4 Fast and Slow Solutions

Potential solutions of equations 1, 2, 3, 4 might be as in the case of a taut string, fast travelling transverse waves and slowly varying longitudinal waves (in short, respectively "fast solution" and "slow solutions").

We denote $\dfrac{W_0 L}{H}$ by ε, which in the cases we want to address is a small parameter. It provides us with a scale to give sense to the expressions "fast" and "slow" solutions. A "fast" solution will be a function of s whereas a slow solution will be a function of εs. More precisely, if logically we assume that the solution of the static equations is of zeroth order in ε, the dynamic quantities for the slow solution will be set as:

dynamic tension	$\varepsilon T(\varepsilon s)$
longitudinal displacement	$\varepsilon u(\varepsilon s)$
transverse displacement	$\varepsilon w(\varepsilon s)$
dynamic angle	$\varepsilon \phi(\varepsilon s)$

whereas for the fast solution:

dynamic tension	$\varepsilon^2 T(s)$
longitudinal component	$\varepsilon^2 u(s)$
transverse component	$\varepsilon w(s)$
dynamic angle	$\varepsilon \phi(s)$

The orders of magnitude have been chosen in accordance to what was observed in the study of the taut string.

By using a perturbation method a second time, we derive two groups of equations from 1, 2, 3, 4:

First group: $\frac{d}{ds} [T_0 \frac{d\tilde{w}}{ds}] + M\omega^2\tilde{w} = 0$ (7)

$\frac{d\tilde{u}}{ds} = \frac{d\phi_0}{ds} \cdot \tilde{w}$ (8)

Second group: $\frac{d}{ds} [\frac{1}{Q} \frac{d\tilde{u}}{ds}] + \tilde{u} = 0$ (9)

$\frac{d}{ds} [\tilde{w}/(d\phi_0/ds)] = h \tilde{u}$ (10)

with $\quad Q = \frac{m\omega^2}{EA} - h (\frac{d\phi_0}{ds})^2, \quad h = \frac{m}{M}$

and $\quad w(s,t) = \tilde{w}(s) \, e^{i\omega t}$

$\quad u(s,t) = \tilde{u}(s) \, e^{i\omega t}$

The solution is the sum of the four independent solutions derived from these two groups of equations. The first group provides the fast solution and the second group the slow one.

The solution of the first group is:

$$\tilde{w}_1(s) = \frac{1}{\sqrt[4]{T_0(s)/M}} \exp \{\pm \int_0^s \frac{i\omega \, ds}{\sqrt{T_0(s)/M}}\}$$

$$\tilde{u}_1(s) = \frac{\sqrt[4]{T_0(s)/M}}{\pm i\omega} (\frac{d\phi_0}{ds}) \exp \{\pm \int_0^s \frac{i \, \omega ds}{\sqrt{T_0(s)/M}}\}$$

and indeed it looks very much like the transverse solution for the taut string. The main difference with the taut string is that u and w are

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coupled and therefore each mode will be a mixture of the fast and slow solutions which will be derived in the sequel. This very mixture captures our intuition that a taut cable may behave both as taut string and as an inelastic chain.

Unfortunately, the second group has no general solution and we can solve it only for particular cases. [1] contains a summary of various solutions. We will pay attention to only two of them: the horizontal taut cable and the inclined taut cable.

1.2.5 Horizontal Cable

This case will interest us in so far as it makes possible checks of our results for the inclined cable, in the asymptotic case of a zero inclination angle.

The slow solution is:

$$\tilde{u}_2 (s) = \exp \left\{ \pm \sqrt[i]{Q} \, s \right\}$$

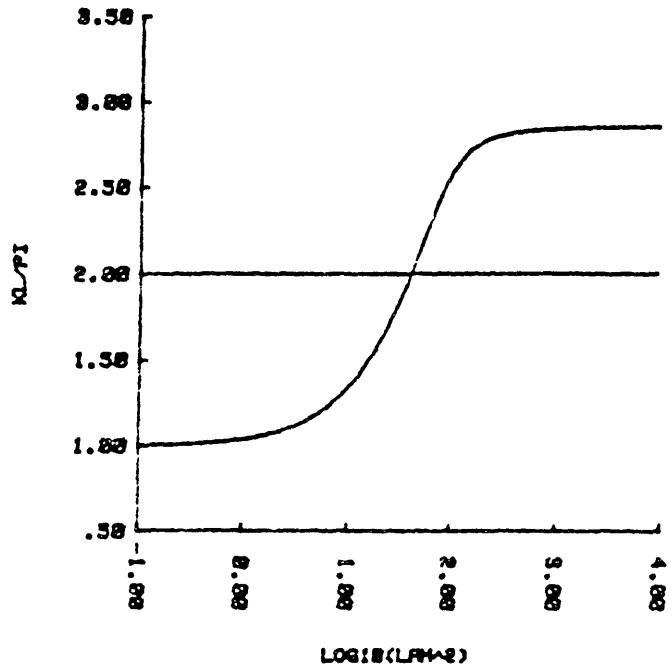$$\tilde{w}_2 (s) = \pm h \frac{W_0}{H} \frac{1}{i\sqrt{Q}} \exp \left\{ \pm i\sqrt{Q}s \right\}$$

By expressing that the end displacements are null, we find the equation for the natural frequencies:

$$\sin (kL/2) \left\{ \tan(kL/2) - kL/2 + \frac{4}{\lambda^2} (kL/2)^3 \right\} = 0$$

(11)

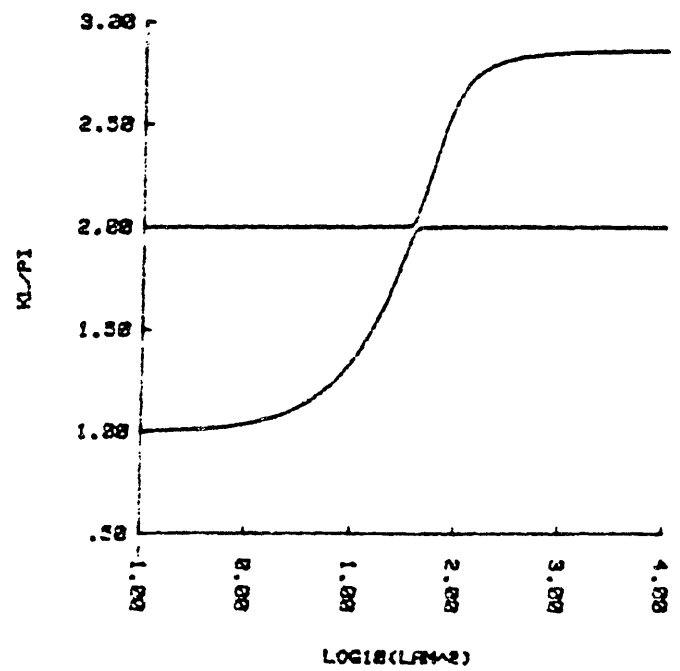$$\text{with } k = \omega/\sqrt{H/M} \text{ and } \lambda^2 = (EA/H) \left(\frac{W_0 L}{H} \right)^2$$

λ is the fundamental parameter which expresses this mixture of taut string and caterary effects.

Figure 1.2.5 shows a plot of equation 11. The solutions of

1.2.5. Natural frequencies for the first two
modes of a horizontal cable

1.2.6.2. Natural frequencies for the first two
modes of an inclined cable

$\sin \left(\frac{kL}{2}\right) = 0$ provide the natural frequencies of the antisymmetric modes while the solutions of the second factor in 11 provide the natural frequencies of the symmetric modes, which generate most of the dynamic tension. It is worth noticing that at a specific point, the cross-over point, the curve that gives the symmetric mode frequencies crosses the one giving the antisymmetric mode frequencies. This is the main characteristic of the dynamics of taut horizontal cables. At this point, Q is zero all along the cable (since the curvature is constant) and the slow solution undergoes a change from sinusoidal type shape (taut string) to exponential type shape (chain dynamics). These results are slightly modified in the case of the taut inclined cable.

1.2.6 Taut Inclined Cable

The solution to the second group of equations is

$$\tilde{u}_2 (s) = - Q_0^{1/3} \{ C_3 A_i' (-z) + C_4 B_i' (-z) \}$$

$$\tilde{w}_2 (s) = \frac{d\phi_0}{ds} \{ C_3 A_i (-z) + C_4 B_i (-z) \}$$

with $\qquad Q = \frac{m\omega^2}{EA} - h \left(\frac{d\phi_0}{ds}\right)^2 = Q_0 (s-s_0)$

$$z = Q_0^{1/3} (s-s_0)$$

A_i, B_i, A_i', B_i' are the Airy, Bairy functions and their derivatives, C_3 and C_4 are two constants. This solution bears one slight analogy with the horizontal cable solution: Q(s) can become zero at one point on the cable (depending on the frequency), but at only one. At that point, the Airy and Bairy functions have a transition from exponential-like shape (at the

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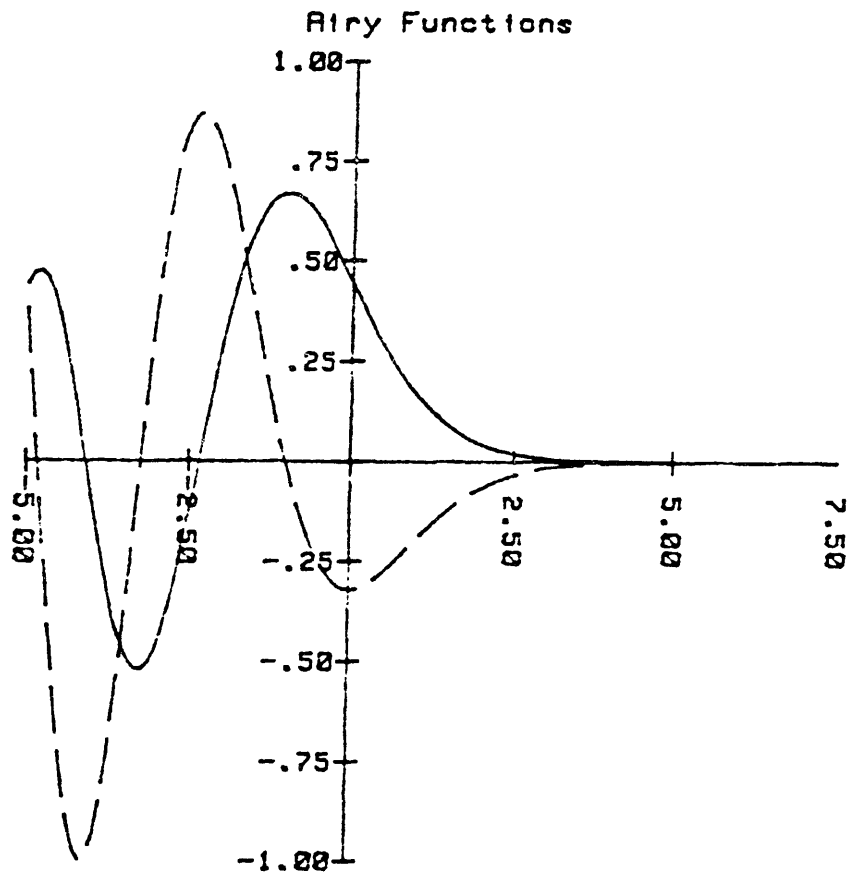
Airy Functions

1.2.6.1. Airy function and its derivative

bottom) to sinusoidal-like shape (at the top) (see figure 1.2.6.1 for the shape of the Airy function, [4] for the Bairy functions). The static tension is greater at the top of the cable than at the bottom, due to the weight of the cable. Hence the cable behaves closer to a taut string in its upper part. On the contrary, it behaves closer to a catenary in its lower part. This creates hydrid modes, which are neither symmetric nor antisymmetric.

As regards the natural frequencies, we can see on figure 1.2.6.2 that a mode cross-over never occurs, but that the curves are very close in the hybrid mode region.

1.3 Conclusion

One will notice that no equation for the natural frequencies has been given for an inclined cable because an analytic form would be too complicated to be of any use. The same holds true for the mode shapes. Therefore, starting from the exact analytic solutions for the displacements:

$$\tilde{u}(s) = \frac{1}{\sqrt[4]{T_0(s)/M}} \{C_1 R(s) + C_2/R(s)\} + C_3 h \alpha(s) A_i[-z] + C_4 h B_i[-z] \tag{12}$$

$$\tilde{w}(s) = \frac{\alpha(s)}{i\omega} \sqrt[4]{T_0(s)/M} \{C_1 R(s) - C_2/R(s)\}$$

$$+ C_3 (-Q_0^{1/3}) A_i'[-z] + C_4(-Q_0^{1/3}) B_i'[-z] \tag{13}$$

$$z = Q_0^{1/3}(s-s_0)$$

$$Q(s) = \frac{m\omega^2}{EA} - h [\alpha(s)]^2 = Q_0(s-s_0) \tag{14}$$

our objective is to derive an approximated version of the natural frequency equation, the modes, and the dynamic tension when $\varepsilon = \dfrac{W_o L}{H}$ is small, in order to not only economize in computer time, but especially to have analytic forms of the natural modes that can be used to study complex dynamic motions.

Approximation of the Equation for the Natural Frequencies

2.1 Derivation of an Approximate Equation for the Natural Frequencies

2.1.1 Principle

Finding the natural modes of our cable consists in adjusting the constants C_1, C_2, C_3, C_4 in (12) and (13) so that

$$\tilde{u}(0) = \tilde{u}(L) = 0 \qquad \tilde{w}(0) = \tilde{w}(L) = 0 \tag{15}$$

This is done by solving the systems of linear equations represented by (15) with unknowns C_1 C_2 C_3 C_4. We will find nor·zero solutions only if the system is of rank lower than four. This condition is expressed mathematically by setting the determinant of system (15) to zero. This is our natural frequency equation

2.1.2 Obtention of the Exact Natural Frequency Equation

Here is the full determinant:

$$
\begin{vmatrix}
\exp[W(0)] & \exp[-W(0)] & h\alpha(0)\sqrt[4]{T_0(0)/M}\ A_i[-z(0)] & h\alpha(0)\sqrt[4]{T_0(0)/M}\ B_i[-z(0)] \\[2em]
\exp[W(L)] & \exp[-W(L)] & h\alpha(L)\sqrt[4]{T_0(L)/M}\ A_i[-z(L)] & h\alpha(L)\sqrt[4]{T_0(L)/M}\ B_i[-z(L)] \\[2em]
\exp[W(0)] & -\exp-(W(0)) & \dfrac{i\omega}{\alpha(0)}\dfrac{\frac{dA_i[-z(0)]}{ds}}{\sqrt[4]{T_0(0)/M}} & \dfrac{i\omega}{\alpha(0)}\dfrac{\frac{dB_i[-z(0)]}{ds}}{\sqrt[4]{T_0(0)/M}} \\[2em]
\exp[W(L)] & -\exp[W(L)] & \dfrac{i\omega}{\alpha(L)}\dfrac{\frac{dA_i[-z(L)]}{ds}}{\sqrt[4]{T_0(L)/M}} & \dfrac{i\omega}{\alpha(L)}\dfrac{\frac{dB_i[-z(L)]}{ds}}{\sqrt[4]{T_0(L)/M}}
\end{vmatrix}
$$

with $W(s) = \int_0^S \dfrac{\omega\, ds}{\sqrt{T_0(s)/M}}$

The full development yields (equation 16):

$$\dfrac{\omega^2\ 2i\ \sin\ [W(L)]}{\alpha(0)\alpha(L)\ \sqrt[4]{T(0)/M}\ \sqrt[4]{T(L)/M}}\ \{\ \dfrac{dA_i}{ds}\ [-z(0)]\ \dfrac{dB_i}{ds}\ [-z(L)]\ -\ \dfrac{dA_i}{ds}\ [-z(L)]\ \dfrac{dP_i}{ds}\ [-z(0)]\}$$

(Term 1)

$$+\ 2i\omega\ h\ \{A_i[-z(L)]\ \dfrac{dB_i}{ds}\ [-z(L)]\ -\ \dfrac{dA_i}{ds}\ [-z(L)]\ B_i\ [-z(L)]\}$$

(Term 2)

$$+\ 2i\omega h\ \{A_i[-z(0)]\ \dfrac{dB_i}{ds}\ [-z(0)]\ -\ \dfrac{dA_i}{ds}\ [-z(0)]\ B_i[-z(0)]\}$$

(Term 3)

$$+i\omega h\ \dfrac{\alpha(L)}{\alpha(0)}\ \dfrac{\sqrt[4]{T(L)/M}}{\sqrt[4]{T(0)/M}}\ (-2\cos[W(L)])\{A_i[-z(L)]\ \dfrac{dB_i}{ds}\ [-z(0)]\ -\ \dfrac{dA_i}{ds}\ [-z(L)]B_i[-z(0)]\}$$

(Term 4)

$$+\ i\omega h\ \dfrac{\alpha(0)}{\alpha(L)}\ \dfrac{\sqrt[4]{T(L)/M}}{\sqrt[4]{T(0)/M}}\ (-2\cos[W(L)])\ \{A_i[-z(0)]\ \dfrac{dB_i}{ds}\ [-z(L)]\ -\ \dfrac{dA_i}{ds}\ [-z(L)]B_i[-z(0)]\}$$

(Term 5)

$$+\ 2i\ \sin\ [W(L)]\ h^2\ \alpha(0)\ \alpha(L)\ \sqrt[4]{T(0)/M}\ \sqrt[4]{T(L)/M}$$

$$\{A_i[-z(0)]\ B_i[-z(L)]\ -\ A_i[-z(L)]\ B_i[-z(0)]\ =\ 0$$

(Term 6)

Term 2 and 3 can be computed exactly without approximation as the Wronskian of the Airy and Bairy functions at any point is $\dfrac{1}{\Pi}$ ([4]):

$$\text{term } 2\ =\ \text{term } 3\ =\ 2\omega hi\ (-Q_0^{1/3})\ \dfrac{1}{\Pi}$$

For the other terms, perturbation expansions must be made, but the computational effort can be reduced using the symmetry of the expressions.

2.1.3 Principle of the Expansions

Equation (16) shows that one must expand two different kinds of expressions:

- products of static quantities and $\sin[W(L)]$ or $\cos[W(L)]$. Each of these quantities has an expansion readily available from former results on the static solution or can be easily calculated, like $\cos[W(L)]$ or $\sin[W(L)]$

- expressions containing Airy, Bairy functions and their derivatives. These functions have an argument $-z(s)$ whose order of magnitude is unknown, since $z(s)$ is an involved expression in ω and the elastic stiffness, quantities which vary over a wide range when we study the influence of these two parameters on the dynamic response of our cable. A perturbation expansion requires that we choose a point and an order for the expansion. We will devote a special paragraph to this question and we will see that two domains of elastic stiffness must be distinguished where two different expansions will be performed.

As regards the terms involving only static quantities, although the order of expansion is still unknown (it must be consistent with the order of expansion of the terms in $z(s)$), we choose to expand them up to second order in ε. We may have to reduce the expansion to the zeroth or first order later, but since we will encounter most of these expressions in the section dealing with the modes, it is wiser to compute them here once and for all.

An important simplication can be made in their calculations,though. Let us have a look at some of these static quantities. We call T_0 the tension at the point where the static angle is exactly equal to the inclination angle of the cable ϕ_a (i.e. the angle of the cord joining the

two extremities). This point always exists. This should not be confused with the point with a Lagrangrian coordinate s_0, which appears in the variable $z(s)$ and which does not always exist, that is, depending on the frequency, s_0 will or will not be comprised between 0 and L. We choose as definition of ε:

$$\varepsilon = \frac{W_0 L}{T_0}$$

(The definition of ε was in the case of a horizontal cable $\varepsilon = \frac{W_0 L}{H}$, H = horizontal force at the top, which is consistent with our definition here).

Whenever we have to non-dimensionalize quantities by a reference force, we will use T_0. Therefore using T_0, the static tension and curvature are:

$$\alpha(s) = \frac{\cos\phi_a}{L}\varepsilon \left\{1 - 2\varepsilon \sin\phi_a\left(\frac{s}{L} - \frac{1}{2}\right) + \varepsilon^2\left[\left(\frac{s}{L} - \frac{1}{2}\right)^2 (4\sin^2\phi_a - 1) - \frac{\sin^2\phi_a}{6}\right]\right\}$$

$$T(s) = T_0 \left\{1 + \varepsilon \sin\phi_a \left(\frac{s}{L} - \frac{1}{2}\right) + \frac{\varepsilon^2}{2}\left[\cos^2\phi_a\left(\frac{s}{L} - \frac{1}{2}\right)^2 + \frac{\sin^2\phi_a}{6}\right]\right\}$$

We notice the term in $\left(\frac{s}{L} - \frac{1}{2}\right)$: it comes from the fact that the Lagrangrian coordinates s_a of the point where $\phi(s_a) = \phi_a$ is:

$$\frac{s_a}{L} = \frac{1}{2} - \frac{\varepsilon \sin\phi_a}{12}$$

Subsequently, all products like $\alpha(0)$ $\alpha(L)$ or $T_0(0)$ $T_0(L)$ will have no first order terms. $W(L) = \int_0^L \frac{ds \, \omega}{\sqrt{T(s)/M}}$ is the integral of an odd function and will have no first order terms as well.

If we add the symmetry of expressions like

$$\frac{\alpha(L)}{\alpha(0)} \frac{\sqrt[4]{T_0(L)/M}}{\sqrt[4]{T_0(0)/M}} \quad \text{and} \quad \frac{\alpha(0)}{\alpha(L)} \frac{\sqrt[4]{T_0(0)/M}}{\sqrt[4]{T_0(L)/M}}$$

where the first order terms disappear any way in the sum, as these terms are the inverses of each other, we may have now a feeling about why in the full expansion of equation (16), terms of order ε disappear, with the consequence that if we want to point out the difference between an inclined cable and a horizontal one, we have to develop (16) up to second order in ε.

Some expansions of compounds of static quantities which we encounter frequently are given in appendix A. We will just have, therefore, to plug these expansions in our equations and we will be able to derive a significant number of other results.

2.1.4 Hybrid Mode and Non Hybrid Mode Domains

Let us focus our attention first on $Q(s)$ and $z(s)$, the argument of the Airy functions.

By definition, we have

$$Q(s) = \frac{m\omega^2}{EA} - h\,\alpha^2(s)$$

If we define $k = \dfrac{\omega}{\sqrt{T_0/M}}$ (analogous definition to the taut string transverse wave number) and plug in the expansion of $\alpha^2(s)$, to the first order we obtain

$$Q(s) = \frac{h\,\varepsilon^2 \cos^2\phi_a}{L^2} \left\{ \frac{T_0\,k^2 L^2}{EA\,\varepsilon^2 \cos^2\phi_a} - 1 + 4\varepsilon \sin\phi_a \left(\frac{s}{L} - \frac{1}{2}\right) \right\}$$

We define $\lambda^2 = \dfrac{EA\,\varepsilon^2\cos^2\phi_a}{T_o}$. This is an analogous quantity to the λ^2 defined for the horizontal cable where

$$\lambda^2 = (EA/H)\,\left(\frac{W_o L}{H}\right)^2 \quad \text{(see [1])}$$

since T_o is equal to H for a horizontal cable. In both cases, λ^2 has the same interpretation and will be one of the fundamental parameters which determine the dynamics of the cable.

Therefore, we end up with:

$$Q(s) = \frac{h\varepsilon^2\cos^2\phi_\alpha}{L^2}\left\{\frac{k^2 L^2}{\lambda^2} - 1 - 2\varepsilon\sin\phi_a + 4\varepsilon\sin\phi_a\,\frac{s}{L}\right\}$$

$$= Q_o(s - s_o) \quad \text{by definition of } s_o \text{ and } Q_o$$

We derive:

$$Q_o = 4\,h\,\frac{\varepsilon^3}{L^3}\,\cos^2\phi_a\,\sin\phi_a \tag{17}$$

$$z(s) = \frac{L\,Q_o^{1/3}}{4\,\varepsilon\sin\phi_a}\left(\frac{k^2 L^2}{\lambda^2} - 1\right) + LQ_o^{1/3}\left(\frac{s}{L} - \frac{1}{2}\right) \tag{18}$$

$$= z\!\left(\frac{L}{2}\right) + Q_o^{1/3}L\left(\frac{s}{L} - \frac{1}{2}\right)$$

$z(s)$ is the sum of two terms:

- $LQ_o^{1/3}\left(\dfrac{s}{L} - \dfrac{1}{2}\right)$ which is of order ε for all s.

- $\dfrac{L\,Q_o^{1/3}}{4\varepsilon\sin\phi_a}\left(\dfrac{k^2 L^2}{\lambda^2} - 1\right)$ which is of order $\left(\dfrac{k^2 L^2}{\lambda^2} - 1\right)$

This quantity may be of order 1, or of order ε. More precisely, $\left(\dfrac{k^2 L^2}{\lambda^2} - 1\right)$ is of order ε and becomes equal to zero, at the point where,

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on the diagram ω vs. λ^2 (see figure 1.2.6.2), the curve for the first mode comes closest to the curve ω vs. λ^2 for the second mode, in what we call the hybrid mode region, or what we will refer to as "near cross over", although we recall that no cross over occurs for an inclined cable.

Consequently, we must distinguish two domains for our expansions of the Airy and Bairy functions:

- the hybrid mode domain, where $(\frac{k^2 L^2}{\lambda^2} - 1)$ is of order ε,
- outside the hybrid mode domain, where $(\frac{k^2 L^2}{\lambda^2} - 1)$ is of order 1

Defining these two regions in terms of an "order" of magnitude may not seem very precise. We can define these regions precisely as follows:

$$z(s) = Q_0^{1/3} (s-s_0)$$

with $\frac{s_0}{L} = \frac{1}{4 \, \varepsilon \, \sin \phi_a} \left\{ 1 - \frac{k^2 L^2}{\lambda^2} + 2\varepsilon \, \sin\phi_a \right\}$

Assuming that $0 < \frac{s_0}{L} < 1$, $z(s)$ can become equal to zero at a point on the cable, and thus for $s < s_0$, $[-z(s)]$ is negative and therefore the Airy and Bairy functions have an exponential form (see 1.2.6.1). For $s > s_0$ $[-z(s)]$ is positive and the Airy functions have a sinusoidal form. This means that at the bottom part of the cable, there is no travelling wave, while at the upper part the wave travels as in the taut string case. These two different solutions create hybrid modes. We can define the hybrid mode region as the region $0 < \frac{s_0}{L} < 1$ where

$$-2\varepsilon\sin\phi_a \;\leqslant\; \frac{k^2 L^2}{\lambda^2} - 1 \;\leqslant\; 2\varepsilon\sin\phi_a$$

That is exactly what we mean when we say "$(\frac{k^2 L^2}{\lambda^2} - 1)$ is of order ε"

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Fortunately, except when actually deriving the natural frequency equation and the modes, we will not have to pay attention, after the final expressions are obtained, to whether $(\frac{k^2 L^2}{\lambda^2} - 1)$ is of order ϵ or not. We will derive formulas which hold true throughout the λ^2 domain.

We can now proceed to expand the Airy and Bairy functions.

2.1.5 Expansions of the Airy and Bairy functions

In the previous paragraph, we concluded that $z(s)$ is either of order ϵ, in which case we can expand the Airy and Bairy functions about 0, or of order 1, in which case we expand them at $[-z(\frac{L}{2})]$ (which is of order 1 in that case). We will see in the sequel see the consequences for not distinguishing between these two cases. We will see that by doing the complete expansion of:

$$\{B_i'[-z(0)] \, A_i[-z(L)] - A_i'[-z(0)] \, B_i[-z(L)]\} \tag{19}$$

The reader will be spared the computation of all similar terms in (16) and will find all the expansions, up to second order, collected in appendix B. These expressions will recur in the derivations for the modes.

- outside the hybrid mode region

$$z(s) = z(\tfrac{L}{2}) + Q_0^{1/3} L \, (\tfrac{s}{L} - \tfrac{1}{2})$$

All the derivatives are at the point $[-z(\frac{L}{2})]$

$$(19) = \{B_i' + \frac{Q_0^{1/3}}{2} L \, B_i'' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 B_i''' \} \, \{A_i - \frac{Q_0^{1/3}}{2} L \, A_i' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 A_i''\}$$

$$- \{A_i' + \frac{Q_0^{1/3}}{2} L \, A_i'' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 A_i''' \} \, \{B_i - \frac{Q_0^{1/3}}{2} L \, B_i' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 B_i''\}$$

$$= (A_i B_i' - A_i' B_i) + \frac{Q_0^{2/3}}{2} L^2 [\tfrac{1}{4} (B_i' A_i'' - A_i' B_i'') + \tfrac{1}{2} (-A_i' B_i'' + A_i'' B_i') + \tfrac{1}{4}(-A_i''' B_i + B_i''' A_i)]$$

The terms in $Q_0^{1/3} L$ cancel out.

Since $(B_i' A_i'' - A_i' B_i'') [-z(\frac{L}{2})] = (-A_i' B_i'' + A_i'' B_i')[-z(\frac{L}{2})] = (-A_i''' B_i + B_i''' A_i)[-z(\frac{L}{2})]$

$$= \frac{-z(\frac{L}{2})}{\Pi}$$

$$(19) \quad = \frac{1}{\Pi} - \frac{Q_0^{2/3}}{2\Pi} L^2 z(\frac{L}{2})$$

where the last term is of order ε^2

- in the hybrid mode region

All the expansions are made at the point 0. Since $G_i'' = z G_i$ with G_i the Airy or Bairy function, $G_i''(0) = 0$ and $G_i'''(0) = G_i(0)$

Thus, to second order:

$$(19) \quad = \{B_i' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 B_i\} \; \{A_i - \frac{Q_0^{1/3}}{2} L \; A_i'\}$$

$$- \{A_i' + \frac{Q_0^{2/3}}{2 \cdot 4} L^2 A_i\} \; \{B_i - \frac{Q_0^{1/3}}{2} L \; B_i'\}$$

$$= (A_i B_i' - A_i' B_i) + O(\varepsilon^3) = \frac{1}{\Pi} + O(\varepsilon^3)$$

If we had made only an expansion outside the hybrid mode domain, we would have kept the term $\frac{Q_0^{2/3}}{2\pi} L^2 z(\frac{L}{2})$ which is of order ε^3 near crossover and, thus, must be discarded. However, although mathematically incorrect, the influence of such "false second order terms" is neglible if we keep them inadvertently. One possible strategy could be therefore to expand all terms as if we were outside the cross over area, then realize that near cross over, because $(\frac{k^2 L^2}{\lambda^2} - 1)$ is one order higher in ε, some terms are of order ε^3, then discard them, reorder terms in ascending powers of ε

and have one formula for each domain. Another strategy, less rigorous mathematically, but easy to use in practice could be to keep all terms, even if they are of order ε^3 for some wavenumbers, knowing their influence is negligible ,and thus obtain a unique formula valid for both domains (which is especially useful for plots). The disadvantage is that outside the cross-over domain, we must expand up to the same order as the expansion in the hybrid mode region, which may not be necessary to obtain the same accuracy outside the hybrid mode region and therefore involves a lot of unnecessary algebra. Nonetheless the simplicity of a single final result outweighs such considerations and this technique will be used in this thesis.

For the natural frequency equation, we will use a different methodology, for the following reasons: plugging in (16) the expansion to first order given in Appendices A and B, and after factoring out powers of ε, it is easy to see that the natural frequency equation outside the hybrid mode domain is, to zeroth order[1]:

$$\sin(\tfrac{kL}{2}) \; \{\cos(\tfrac{kL}{2}) \; \tfrac{kL}{2} \; (\tfrac{k^2 L^2}{\lambda^2} - 1) + \sin \tfrac{kL}{2} \} + O(\varepsilon^2) = 0 \qquad (20)$$

To second order and away from the cross-over region, the equation has exactly the same form as for a horizontal cable, except that the definition of λ^2 accounts for the inclination angle. The main point, however, i.e. the existence of symmetric and antisymetric modes remains valid.

[1]
Actually this equation is of first order if sin(kL/2) is or order ε and $(k^2 L^2 / \lambda^2 - 1)$ is of order 1. This is the case for low or high values of λ. This represents a third domain, which we have not dealt with especially, since the two cases discussed above encompass this domain.

This equation is obviously not valid near cross-over since both $(\frac{k^2 L^2}{\lambda^2} - 1)$ and $\sin(\frac{kL}{2})$ are of order ε in this region, thus the first term in (20) is of order ε^2, and the other terms of the expansion which are of order ε^2 must be retained.

From the discussion above, we conclude that (20) is a very good approximation of the natural frequency equation, correct to second order and away from the cross-over region, and there is no need for second order terms which would have a negligible influence. Besides, outside the hybrid mode domain, there are several second order terms, so the advantage of an analytic formula is lost. Near cross-over, the second order terms are not a correction: they are the leading order part of the expansion and must be kept. Fortunately, the second order terms near cross-over are much fewer since, as we can see in Appendix B, the expansions for the Airy and Bairy functions are simpler in that domain.

In brief

- a zeroth order expansion is sufficient away from cross over

- a second order expansion is necessary near cross over, but the algebra is simpler in that region.

Following this strategy, we end up with:

$$\sin(\frac{kL}{2}) \left\{ \cos(\frac{kL}{2}) \frac{kL}{2} (\frac{k^2 L^2}{\lambda^2} - 1) + \sin\frac{kL}{2} \right\} - \varepsilon^2 \sin^2\phi_a \, \cos kL \, \frac{49}{64} = 0$$

If we now notice that outside the cross over region, the term $\varepsilon^2 \sin^2\phi_a \cos kL \frac{49}{64}$ has a negligible influence, we can keep the unique natural frequency equation:

$$\sin(\frac{kL}{2}) \left\{ \cos(\frac{kL}{2}) \frac{kL}{2} (\frac{k^2 L^2}{\lambda^2} - 1) + \sin\frac{kL}{2} \right\} - \varepsilon^2 \sin^2\phi_a \, \cos kL \, \frac{49}{64} = 0 \quad (21)$$

2.2 Verification

First, from a qualitative point of view it is interesting to check whether, if ϕ_a tends toward 0, we recover the equation for a horizontal cable.

Second, the correction of order ε^2 is in agreement with our intuition, since if we look at the plots of ω vs. λ^2, near cross over both $\sin\frac{kL}{2}$ and $(\frac{k^2 L^2}{\lambda^2} - 1)$ are small. From a quantitative point of view, it is not at all obvious that we will obtain any meaningful result by a perturbation expansion of any order. If we look at a plot of the Airy and Bairy functions (see figure 1.2.6.1), we see that even near zero their slopes are large and thus the higher order terms of the expansion may not be negligible. Fortunately enough, approximate formulae, given in Appendix B, provide results which compare well with the exact ones. In particular, the natural frequencies, given by the approximate equation above, agree very well with the exact results.

Figure 2.2. is a plot of the non-dimensional quantity $\frac{kL}{2}$ versus λ^2 It is no use superposing the exact and approximate solution, because the difference can not be detected within the scale of the plot. For this reason we enclosed two tables with numerical values, one for a moderate and one for a high value of ε. We notice that, for $\varepsilon = 0.18$, three digits after the decimal point are captured by our approximation.

2.3 Conclusion

Most of this chapter was devoted to explaining the approximations of the Airy and Bairy functions. We had to distinguish two domains even though we end up with a single equation. The same reasoning and tech-
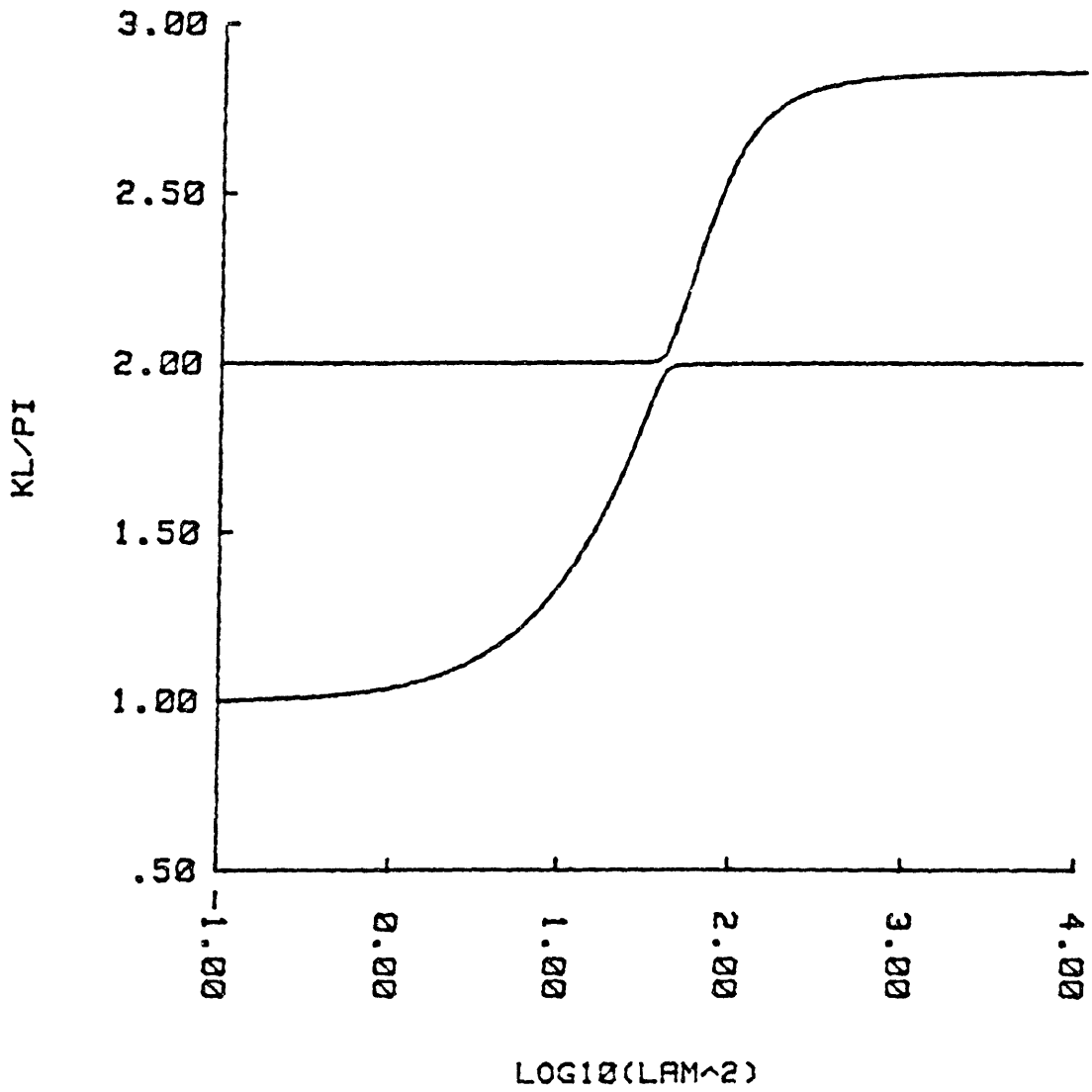
2.2. First two natural modes for =0.18, =23.46

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original.

Lamoda^2	LOG(Lam)	First exact	First appr.
19.0	1.279	1.5837	1.5852
30.0	1.477	1.8230	1.8240
38.0	1.580	1.9602	1.9607
64.0	1.806	1.9978	1.9978

Lamoda^2	LOG(Lam)	Second exact	Second appr.
38.0	1.580	2.0129	2.0129
64.0	1.806	2.3372	2.3367
100.0	2.000	2.5991	2.5972
150.0	2.176	2.7290	2.7261

Comparison of the exact and approximated natural frequencies of the first two modes for ϵ =0.18, φ =23.46

Lambda^2	LOG(Lam.	First exact	First appr.
18.0	1.279	1.5642	1.6706
30.0	1.477	1.6573	1.7620
38.0	1.580	1.8650	1.8750
64.0	1.806	1.9563	1.9553

Lambda^2	LOG(Lam)	Second exact	Second appr.
38.0	1.580	2.0819	2.0950
64.0	1.906	2.3328	2.3557
100.0	2.000	2.5757	2.5384
150.0	2.176	2.7095	2.7114

Comparison of the exact and approximated natural frequencies of the first two modes for $\epsilon = 0.49$, $\varphi = 45$

niques which proved fruitful for the equation providing the natural

frequencies will be used again next for the derivation of the natural

modes.

Chapter 3

Approximation of the Natural Modes

3.1 Derivation of the Natural Modes

3.1.1 Principle

The natural modes are the non-zero solutions of the governing

equations which cancel out at both ends. This, as we saw in 2.1.1,

implies that the determinant of system (15) must be zero and, consequently,

the modes are obtained by solving (15) for three out of the four constants

C_1, C_2, C_3, C_4 in terms of the fourth constant for the values of ω that

satisfy the natural frequency equation.

We rewrite our solutions as:

$$\tilde{w}(s) = C_1 \alpha(s)\, A_i[-z] + C_2 \alpha(s) B_i[-z] + \frac{1}{\sqrt[4]{T_0(s)/M}}\, \{C_3 \cos[W(s)] - C_4 \sin[W(s)]\}$$

$$\tilde{u}(s) = C_1 \left(\frac{Q_0}{h}\right)^{1/3} A_i'[-z] + C_2 \left(\frac{Q_0}{h}\right)^{1/3} B_i'[-z]$$

$$+ \frac{\alpha(s)}{\omega} \sqrt[4]{T_0(s)/M}\, \{C_3 \sin[W(s)] + C_4 \cos[W(s)]\}$$

Through an appropriate choice of the constants C_3, C_4 we have
expressed the modes as real functions of s, which makes manipulations

easier, at the expense of losing symmetry. For this reason, in chapter 2

we kept complex solution in order to retain a perfectly symmetric 4x4

matrix.

The matrix of the system becomes:

$$\begin{vmatrix} (-\dfrac{Q_o^{1/3}}{h}) \, A_i'[-z(0)] & (-\dfrac{Q_o^{1/3}}{h}) \, B_i'[-z(0)] & 0 & \dfrac{\alpha(0)}{\omega} \sqrt[4]{T(0)/M} \\[3mm] \alpha(0) \, A_i[-z(0)] & \alpha(0) \, B_i[-z(0)] & \dfrac{1}{\sqrt[4]{T(0)/M}} & 0 \\[3mm] (-\dfrac{Q_o^{1/3}}{h}) \, A_i'[-z(L)] & (-\dfrac{Q_o^{1/3}}{h}) \, B_i'[-z(L)] & \dfrac{\alpha(L)}{\omega}\sqrt[4]{\dfrac{T(L)}{M}}\sin[W(L)] & \dfrac{\alpha(L)}{\omega}\sqrt[4]{\dfrac{T(L)}{M}}\cos[W(\\[3mm] \alpha(L) \, A_i[-z(L)] & \alpha(L) B_i[-z(L)] & \dfrac{1}{\sqrt[4]{T(L)/M}}\cos[W(L)] & \dfrac{-1}{\sqrt[4]{T(L)/M}}\sin[W(L)] \end{vmatrix}$$

and we will denote by A the 3x3 upper left corner matrix. We will solve our system for C_1, C_2, C_3.

Each unknown is equal to the ratio of the determinant of its cofactor by the determinant of matrix A. We must compute exactly and analytically all these quantities:

$$\det(A) = \{\dfrac{-1}{\sqrt[4]{T(0)/M}} \, (\dfrac{Q_o^{2/3}}{h^2}) \, [A_i'[-z(0)] \, B_i'[-z(L)] - A_i'[-z(L)] \, B_i'[-z(0)]]$$

$$+ \dfrac{\alpha(0)\alpha(L)}{\pi \, \omega} \, \sqrt[4]{T(L)/M} \, \sin[W(L)] \, (\dfrac{Q_o^{1/3}}{h})\}$$

$$\dfrac{\text{cof}(C_3)}{C_4} = \{\dfrac{\alpha^2(0)}{\omega} \sqrt[4]{T(o)/M} \, (\dfrac{Q_o^{1/3}}{h}) \, [A_i[-z(o)] \, B_i'[-z(L)] - A_i'[-z(L)] \, B_i[-z(0)]]$$

$$- \dfrac{\alpha(0)\alpha(L)}{\pi \, \omega} \, \sqrt[4]{T(L)/M} \, \cos[W(L)] \, (\dfrac{Q_o^{1/3}}{h})\}$$

$$\dfrac{\text{cof}(C_1)}{C_4} = - \{(\dfrac{Q_o^{1/3}}{h}) \, \dfrac{1}{\omega} \, [\alpha(0)B_i'[-z(L)] - \alpha(L) \, \dfrac{\sqrt[4]{T(L)/M}}{\sqrt[4]{T(0)/M}} \, \cos[W(L)] \, B_i'[-z(0)]]$$

$$+ \dfrac{\alpha^2(0)\alpha(L)}{\omega^2} \sqrt[4]{\dfrac{T(L)}{M}} \sqrt[4]{\dfrac{T(0)}{M}} \, \sin[W(L)] \, B_i[-z(0)]\}$$

$$\frac{\text{cof}(C_2)}{C_4} = - \{(\frac{Q_o^{1/3}}{h}) \frac{1}{\omega} [\alpha(L) \frac{\sqrt[4]{\frac{T(L)}{M}}}{\sqrt[4]{T(0)/M}} \cos[W(L)] A_i'[-z(0)] - \alpha(0) A_i'[-z(L)]]$$

$$- \frac{\alpha^2(0)\alpha(L)}{\omega^2} \sqrt[4]{T(L)/M} \sqrt[4]{T(0)/M} \sin[W(L)] A_i[-z(0)]\}$$

We see that det(A) and cof(C_3) have retained their symmetry regarding the Airy and Bairy functions and can be easily expanded using Appendix B. As for Cof(C_1) and Cof(C_2), if we attempt to expand them, we would get into a lot of trouble, because, of their due to a lack of symmetry, the Wronskian of the Airy functions does not appear, so the expressions can not be simplified. We choose C_4 = det A and calculate the complete analytic form for the slow solution. This yields

$$w^s(s) = \alpha(s) (\frac{Q_o^{1/3}}{h}) \frac{1}{\omega} \alpha(0) \{A_i'[-z(L)]B_i'[-z] - B_i'[-z(L)]A_i'[-z]\}$$

$$+ \alpha(s)\alpha(L) \frac{\sqrt[4]{\frac{T(L)}{M}}}{\sqrt[4]{\frac{T(0)}{M}}} \cos [W(L)] (\frac{Q_o^{1/3}}{h}) \frac{1}{\omega} \{B_i'[-z(0)]A_i[-z] - A_i'[-z(0)] B_i[-z]\}$$

$$+ \alpha(s) \frac{\alpha^2(0)\alpha(L)}{\omega^2} \sqrt[4]{T(L)/M} \sqrt[4]{T(0)/M} \sin[W(L)] \{A_i[-z(0)]B_i[-z] - B_i[-z(0)] A_i[-z]\}$$

$$(22)$$

$$u^s(s) = - \frac{Q_o^{2/3}}{h^2} \frac{1}{\omega} \alpha(0) \{A_i'[-z(L)]B_i'[-z] - B_i'[-z(L)]A_i'[-z]\}$$

$$- \frac{Q_o^{2/3}}{h^2} \frac{1}{\omega} \alpha(L) \frac{\sqrt[4]{\frac{T(L)}{M}}}{\sqrt[4]{T(0)/M}} \cos[W(L)] \{B_i'[-z(0)]A_i'[-z] - A_i'[-z(0)]B_i'[-z]\}$$

$$\frac{-Q_o^{1/3}}{h} \alpha^2(0)\alpha(L) \sqrt[4]{T(L)/M} \sqrt[4]{T(L)/M} \sin[W(L)] \{A_i[-z(0)]B_i'[-z] - B_i[-z(0)]A_i'[-z]\}$$

$$(23)$$

Now we retrieve perfectly symmetric expressions as regards the Airy functions, that we are able to expand. Cof (C_3) and det(A) can be expanded separately and then plugged in the fast solution. We are going to apply those two techniques successively first to the transverse modes and then to the longitudinal modes.

3.2 Transverse Natural Modes

3.2.1 Transverse Modes Outside the Cross Over Region

As in chapter two, we have again to distinguish between two domains. Using Appendix A and B and plugging the expansions up to the appropriate

order in (22) with $Sf = \dfrac{\cos^2\phi_a \, \varepsilon^2}{\pi\sqrt{T_0/M} \, kL} \dfrac{Q_0^{1/3}}{L} \dfrac{}{h}$:

$$\frac{w^s(s)\det(A)}{C_4 \quad Sf} = \left\{-2 \sin^2 \frac{kL}{2} + \varepsilon \sin \phi_a \left(- 1 - \frac{3}{4} \cos kL + 4 \sin^2 \frac{kL}{2} \left(\frac{s}{L} - \frac{1}{2}\right)\right)\right\} \tag{24}$$

For the fast solution

$$\frac{\text{cof}(C_3)}{C_4} = \frac{1}{\pi \sqrt[4]{T_0/M}} \frac{\cos^2\phi_a \varepsilon^2}{kL \quad L} \left(\frac{Q_0^{1/3}}{h}\right) \left\{2 \sin^2\left(\frac{kL}{2}\right) + \frac{\varepsilon\sin\phi_a}{8} (-\cos kL + 15)\right\} \tag{25}$$

$$\det A = \frac{1}{\pi \sqrt[4]{T_0/M}} \frac{\cos^2\phi_a \, \varepsilon^2}{kL \quad L} \left(\frac{Q_0^{1/3}}{h}\right) \left\{\left[\sin kL + kL\left(\frac{k^2L^2}{\lambda^2} - 1\right)\right]\left(1 + \frac{\varepsilon\sin\phi_a}{8}\right)\right\} \tag{26}$$

which yields the following fast solution:

$$\frac{w^f(s)\det A}{C_4 \quad Sf} = \left\{\left[2 \sin^2 \frac{kL}{2} \cos\left(\frac{kLs}{L}\right) - \sin\left(\frac{kLs}{L}\right)\left[\sin kL + kL\left(\frac{k^2L^2}{\lambda^2} - 1\right)\right]\right]\right.$$

$$+ \frac{\varepsilon \sin\phi_a}{4} \left[2 \sin^2 \frac{kL}{2} \left[\left(\frac{kLs}{L}\right)\left(\frac{s}{L} - 1\right) \sin\left(\frac{kLs}{L}\right) - \left(\frac{s}{L} - \frac{1}{2}\right) \cos\left(\frac{kLs}{L}\right)\right]\right.$$

$$+ \frac{1}{2} \cos \left(\frac{kLs}{L}\right)(-\cos kL + 15)$$

$$+ \left[- \sin\left(\frac{kLs}{L}\right) \frac{1}{2} + \left(\frac{kLs}{L}\right)\left(\frac{s}{L} - 1\right) \cos \left(\frac{kLs}{L}\right) + \left(\frac{s}{L} - \frac{1}{2}\right) \sin \left(\frac{kLs}{L}\right) \right]$$

$$\left[\sin kL + kL\left(\frac{k^2 L^2}{\lambda^2} - 1\right) \right] \} \tag{27}$$

The reader will understand why we have expanded only to first order.

We will not rewrite the complete transverse displacement, which is easily done by summing 24 and 27.

$$\frac{w(s)detA}{C_4 \quad Sf} = \{2 \sin^2\left(\frac{kL}{2}\right)\left[\cos \frac{kLs}{L} - 1\right] - \sin \left(\frac{kLs}{L}\right) \left[\sin kL + kL \left(\frac{k^2 L^2}{\lambda^2} - 1\right)\right]\} \tag{28}$$

We notice that $w(0) = 0$ exactly and $w(L) = 0$ to second order.

3.2.2 Transverse mode near cross over

Plugging in (22) the appropriate expansions, we get:

$$\frac{w^s(s)detA}{C_4 \quad Sf} = \{\frac{-7}{4} \epsilon \sin\phi_a + \epsilon^2 \sin^2\phi_a \left[\frac{-7}{32} + \frac{7}{2} \left(\frac{s}{L} - \frac{1}{2}\right)\right] - 2 \sin^2 \frac{kL}{2}\} \tag{29}$$

For the fast solution

$$detA = \frac{1}{\pi \sqrt[4]{T_0/M}} \frac{\cos^2\phi_a}{kL} \frac{\epsilon^2}{L} \frac{Q_0^{1/3}}{h} \{kL \left(\frac{k^2 L^2}{\lambda^2} - 1\right) + \sin kL$$

$$+ \frac{\epsilon \sin\phi_a}{8} \left[kL\left(\frac{k^2 L^2}{\lambda^2} - 1\right) + \sin kL\right]\} \tag{30}$$

$$\frac{cof(C_3)}{C_4} = \frac{1}{\pi \sqrt[4]{T_0/M}} \frac{\cos^2\phi_a}{kL} \frac{\epsilon^2}{L} \frac{Q_0^{1/3}}{h} \{\frac{7}{4} \epsilon \sin\phi_a + \frac{7}{4} \epsilon^2 \sin^2\phi_a + 2 \sin^2 \frac{kL}{2}\} \tag{31}$$

which yields the fast solution

$$\frac{w^f(s)detA}{C_4 \; Sf} = \{\frac{7}{4} \; \varepsilon \; \sin\phi_a \; \cos \; (\frac{kLs}{L}) \; - \; \sin \; (\frac{kLs}{L}) \; [kL(\frac{k^2L^2}{\lambda^2} - 1) + \sin \; kL]\}$$

$$+ \; \varepsilon^2 \; \sin^2\phi_a \; \{\frac{7}{16} \; [(\frac{kLs}{L})(\frac{s}{L} - 1) \; \sin \; (\frac{kLs}{L}) - (\frac{s}{L} - \frac{1}{2}) \; \cos \; (\frac{kLs}{L})] + \frac{7}{4} \; \cos \; (\frac{kLs}{L})\}$$

$$+ \; \{2 \; \sin^2 \; \frac{kL}{2} \; \cos \; (\frac{kLs}{L})\}$$

$$+ \; \frac{\varepsilon \; \sin\phi_a}{4} \{[(\frac{kLs}{L})(\frac{s}{L} - 1) \; \cos \; (\frac{kLs}{L}) + (\frac{s}{L} - \frac{1}{2}) \; \sin \; (\frac{kLs}{L})][kL(\frac{k^2L^2}{\lambda^2} - 1) + \sin \; kL]$$

$$- \; \sin \; (\frac{kLs}{L}) \; \frac{1}{2} \; [kL \; (\frac{k^2L^2}{\lambda^2} - 1) + \sin \; kL]\}$$

$$(32)$$

To zeroth order:

$$\frac{w(s)detA}{C_4 \; Sf} = \{\frac{7}{4} \; \varepsilon \; \sin\phi_a \; [\cos \; (\frac{kLs}{L}) - 1] - \sin \; (\frac{kLs}{L})[kL(\frac{k^2L^2}{\lambda^2} - 1) + \sin \; kL]\}$$

At $s = 0$, $w(0) = 0$ exactly whereas at $s = L$, $w(L) = 0$ correct to order ε^3.

3.2.3 Comparison of the transverse mode equation in the two domains

Because of the importance of the transverse displacement, it would be interesting to have one unique equation valid for both domains.

If we take the expression of w outside the hybrid mode region and have $\sin \; \frac{kL}{2}$ and $(\frac{k^2L^2}{\lambda^2} - 1)$ tend to a value of order ε, after reordering we obtain:

$$\frac{w(s)detA}{C_4 \; Sf} = \frac{7}{4} \; \varepsilon \; \sin\phi_a \; [\cos(\frac{kLs}{L}) - 1] - \sin \; (\frac{kLs}{L})[\sin \; kL + kL(\frac{k^2L^2}{\lambda^2} - 1)]$$

$$+ \; 2 \; \sin^2 \; (\frac{kL}{2})[\cos(\frac{kLs}{L}) - 1] + \frac{\varepsilon \; \sin\phi_a}{4} \{[- \frac{1}{2} \; \sin(\frac{kLs}{L}) + \frac{kLs}{L}(\frac{s}{L} - 1) \; \cos(\frac{kLs}{L})$$

$$+ \; (\frac{s}{L} - \frac{1}{2}) \; \sin \; (\frac{kLs}{L})][\sin \; kL + kL \; (\frac{k^2L^2}{\lambda^2} - 1)]\}$$

$$(33)$$

If we compare this expression to the expression for the transverse mode at cross-over, we notice that the principal parts are similar, but in the correction term some terms are missing and indeed (33) does not vanish at both ends up to second order.

This is due to the fact that, outside the hybrid mode region, the principal part of w is of order 0 and its correction term of order 1, whereas near cross-over, the retained terms are shifted by one order (the principal part is of order ϵ, the correction term is of order ϵ^2). Consequently, terms of order ϵ^2 outside the cross-over region, which we have ommitted and which would remain of order ϵ^2 near cross-over, should also be retained.

- Our formula for w outside the cross over region, which is similar in its principal part with w in the cross over area, is already a first step towards a unified formula. However, we will see that a correction term near cross-over is absolutely necessary. Possible remedies include:

- including the second order terms outside the hybrid mode domain. It is possible to do so with the expansions we have provided in Appendices A and B. However, the algebra is quite involved.

- using the technique we have already resorted to for the natural frequency equation: that is, adding the second order terms which appear at cross-over and which are absent from equation (33), to the expression of w outside the hybrid mode region. These terms have no influence outside the cross-over region, but they allow us to obtain accurate expressions for w within the cross-over region.

Therefore, an acceptable formula valid throughout the λ^2 domain is:

$$\frac{w(s)\det A}{C_4 \; Sf} = \{2\sin^2(\tfrac{kL}{2})\,[\cos(\tfrac{kLs}{L}) - 1] - \sin(\tfrac{kLs}{L})[\sin kL + kL\,(\tfrac{k^2 L^2}{\lambda^2} - 1)]\}$$

$$+ \frac{\varepsilon\sin\phi_a}{4}\,\{2\sin^2\tfrac{kL}{2}\,[(\tfrac{kLs}{L})(\tfrac{s}{L} - 1)\sin(\tfrac{kLs}{L}) - (\tfrac{s}{L} - \tfrac{1}{2})\cos(\tfrac{kLs}{L})]$$

$$+ \tfrac{1}{2}\cos(\tfrac{kLs}{L})(-\cos kL + 15)$$

$$+ [-\sin(\tfrac{kLs}{L})\tfrac{1}{2} + (\tfrac{s}{L} - 1)(\tfrac{kLs}{L})\cos(\tfrac{kLs}{L}) + (\tfrac{s}{L} - \tfrac{1}{2})\sin(\tfrac{kLs}{L})]$$

$$[\sin kL + kL\,(\tfrac{k^2 L^2}{\lambda^2} - 1)] + 4\,[-1 - \tfrac{3}{4}\cos kL + 4\sin^2(\tfrac{kL}{2})(\tfrac{s}{L} - \tfrac{1}{2})]\}$$

$$+ \varepsilon^2\sin\phi_a\,\{\tfrac{7}{2}(\tfrac{s}{L} - \tfrac{1}{2}) + \tfrac{7}{4}\cos(\tfrac{kLs}{L}) - \tfrac{7}{16}(\tfrac{s}{L} - \tfrac{1}{2})\cos(\tfrac{kLs}{L}) - \tfrac{7}{32}$$

$$+ \tfrac{7}{16}(\tfrac{kLs}{L})(\tfrac{s}{L} - 1)\sin(\tfrac{kLs}{L})\} \tag{34}$$

3.3 Longitudinal Modes

3.3.1 Longitudinal component outside the hybrid mode region

As we will see in the sequel soon, the longitudinal component is of order ε relative to the transverse component. For this reason, we will only compute its leading order part without including any higher order terms.

Outside the hybrid mode region, we define $Sf' = \dfrac{\cos^3\phi_a\; \varepsilon^3}{\pi\sqrt{T_o/M}\,(kL)^2\,L}\,(\tfrac{Q_o}{h})^{1/3}$,

We can re-use (25) and (26) for C_3 and $\det A$

This yields the following fast solution:

$$\frac{u^f(s)\det A}{C_4 \; Sf'} = \{2\sin^2\tfrac{kL}{2}\sin(\tfrac{kLs}{L}) + \cos(\tfrac{kLs}{L})[\sin kL + (\tfrac{k^2 L^2}{\lambda^2} - 1)\,kL]\}$$

As for the slow solution

$$\frac{u^s(s)detA}{C_4 \, S_f'} = - \{kL \, (\frac{k^2L^2}{\lambda^2} - 1) \, [- 2 \sin^2(\frac{kL}{2}) \, \frac{s}{L} + 1] + \sin kL\}$$

so that

$$\frac{u(s)detA}{C_4 \, S_f'} = \{2 \sin^2 \frac{KL}{2} \, [kL \, (\frac{k^2L^2}{\lambda^2} - 1) \, \frac{s}{L} + \sin (\frac{kLs}{L})] +$$

$$[\cos (\frac{kLs}{L}) - 1] \, [\sin kL + (\frac{k^2L^2}{\lambda^2} - 1) \, kL]\} \tag{35}$$

3.3.2 Longitudinal component near cross over

We use again (30) and (31) for C_3 and det(A) and we get:

$$\frac{u^f(s)detA}{C_4 \, S_f'} = \{\cos \frac{kLs}{L} \, [\sin kL + (\frac{k^2L^2}{\lambda^2} - 1) \, kL] + \frac{7}{4} \, \varepsilon \, \sin\phi_a \, \sin (\frac{kLs}{L})\}$$

$$\frac{u^s(s)detA}{C_4 \, S_f'} = - [kL(\frac{k^2L^2}{\lambda^2} - 1) + \sin kL]$$

Thus

$$\frac{u(s)detA}{C_4 \, S_f'} = \{[kL \, (\frac{k^2L^2}{\lambda^2} - 1) + \sin kL][\cos (\frac{kLs}{L}) - 1] + \frac{7}{4} \, \varepsilon \, \sin \phi_a \, \sin (\frac{kLs}{L})\}$$

$u(0) = 0$ exactly

$u(L) = 0$ to the second order $\tag{36}$

3.3.3 Uniformly valid formulation

From (35) and (36), an approximate formulation valid for any λ is:

$$\frac{u(s)detA}{C_4 \, S_f'} = \{[kL \, (\frac{k^2L^2}{\lambda^2} - 1) + \sin kL][\cos (\frac{kLs}{L}) - 1]$$

$$+ \ 2 \ \sin^2 \frac{kL}{2} \ [kL \ (\frac{k^2L^2}{\lambda^2} - 1) \ \frac{s}{L} + \sin \ (\frac{kLs}{L})] + \frac{7}{4} \ \varepsilon \ \sin\phi_a \ \sin \ (\frac{kLs}{L})\} \qquad (37)$$

3.4 Verification

3.4.1 Transverse component

Figures 3.4.1.1 and 3.4.1.2 show a superposition of the exact transverse mode and w, derived in paragraph 3.2.1, both with its correction term and without it according to equation (26) for kL/π = 1.58 (outside the cross-over region). We can see that for low values of ε and ϕ_a the superposition is perfect, while figure 3.4.1.2 demonstrates that the correction is hardly necessary.

Figures 3.4.1.3 and 3.4.1.4 point out to the same facts for kL/π = 1.96 (cross over). There, the correction term is absolutely necessary. Notice that at cross-over, the program which computes the exact solution has to invert matrices with very small determinants. Therefore, depending on which 3x3 matrix (see 3.1) we choose to invert the superposition might not be accurate due to numerical errors (Figure 3.4.1.5).

In figure 3.4.1.6 w is shown for kL/π = 1.96 using the general formula (34). Very good agreement with the plot shown in figure 3.4.1.4 is obtained, demonstrating the validity of the approximation.
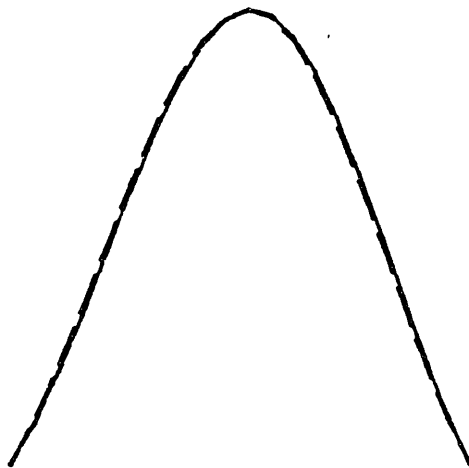
For illustration purposes, we have also shown:

- the approximation of the second mode at cross over (kL/π = 2.01) (Figure 3.4.1.7)
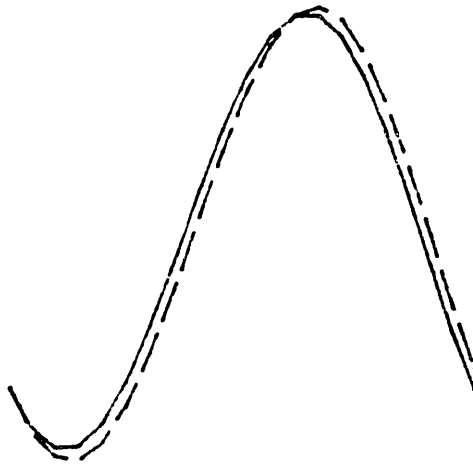
- the very good approximation of the second mode far from cross over (Figure 3.4.1.8 and 3.4.1.9)
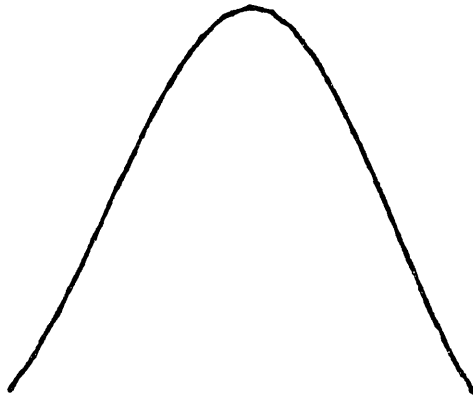
3.4.1.1. Superposition of the exact transverse mode and its
approximation correct to first order, for KL/PI=1.58,
ϵ=0.18,φ_a=23.46°

3.4.1.2. Superposition of the exact transverse mode and its
approximation, correct to zeroth order (dotted),
for KL/PI=1.58,ϵ=0.18,φ_a=23.46°

3.4.1.3. Superposition of the exact transverse mode and
its approximation, correct to first order (dotted),
for KL/PI=1.96, $\mathcal{E}$ =0.18, φ_a=23.46°

3.4.1.4. Superposition of the exact transverse mode and
its approximation, correct to second order (dotted),
for KL/PI=1.96, $\mathcal{E}$ =0.18, φ_a=23.46°
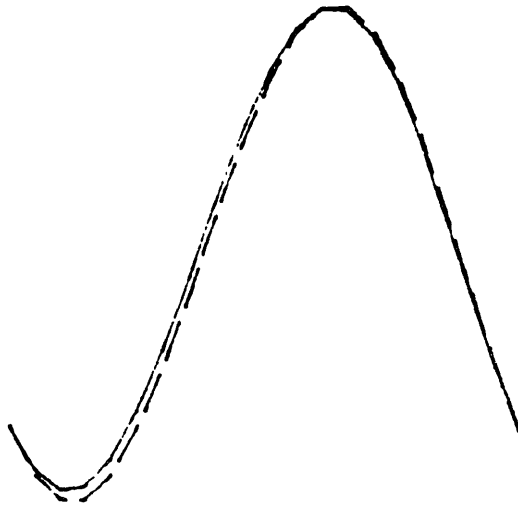
3.4.1.5. Same as 3.4.1.4., pointing out potential
numerical errors in the computation of
the exact solution
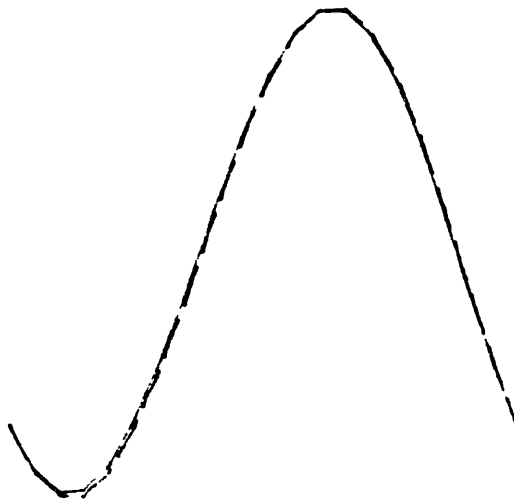
3.4.1.6. Same as 3.4.1.4., but with the approximation
valid for all values of the elastic stiffness
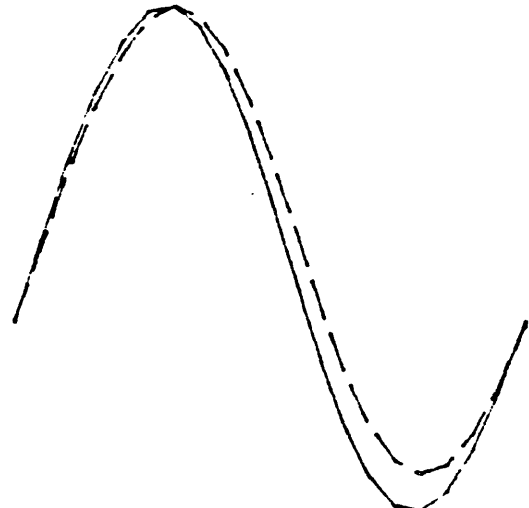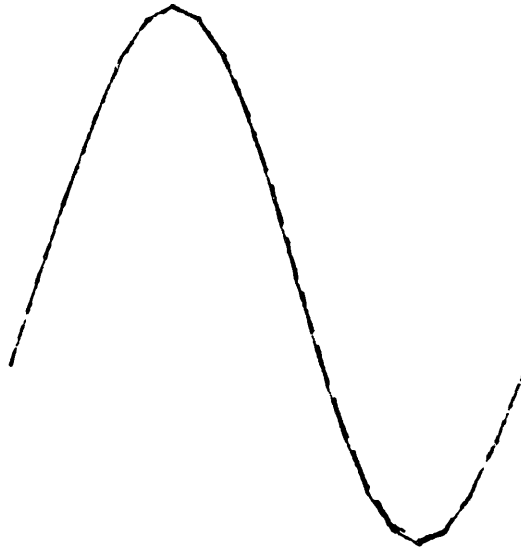
3.4.1.7. Superposition of the exact second transverse mode
with its approximation (dotted) for KL/PI=2.01
with (bottom) and without (top) numerical errors
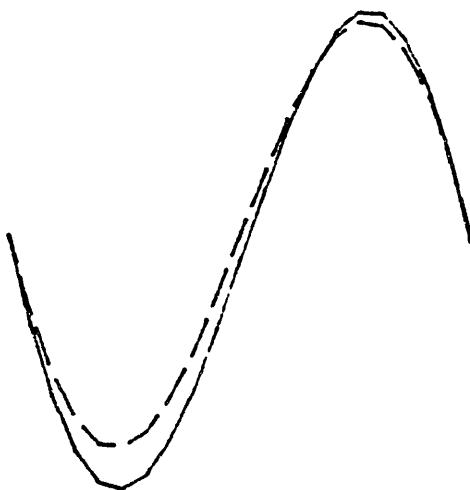occurring in the computation of the exact solution
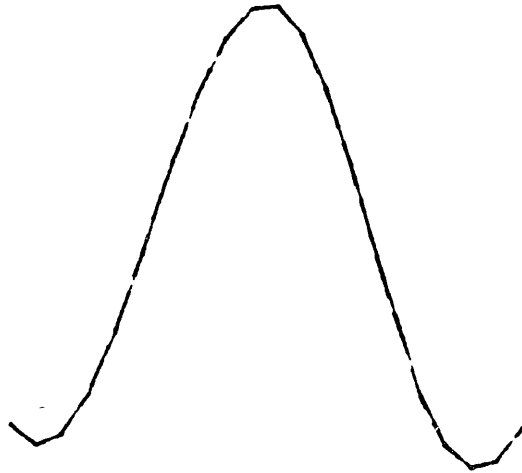
3.4.1.8. Superposition of the exact second transverse mode
and its approximation for KL/PI=2.33
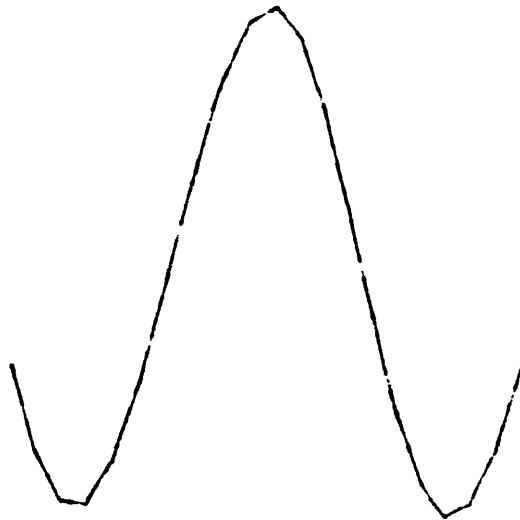
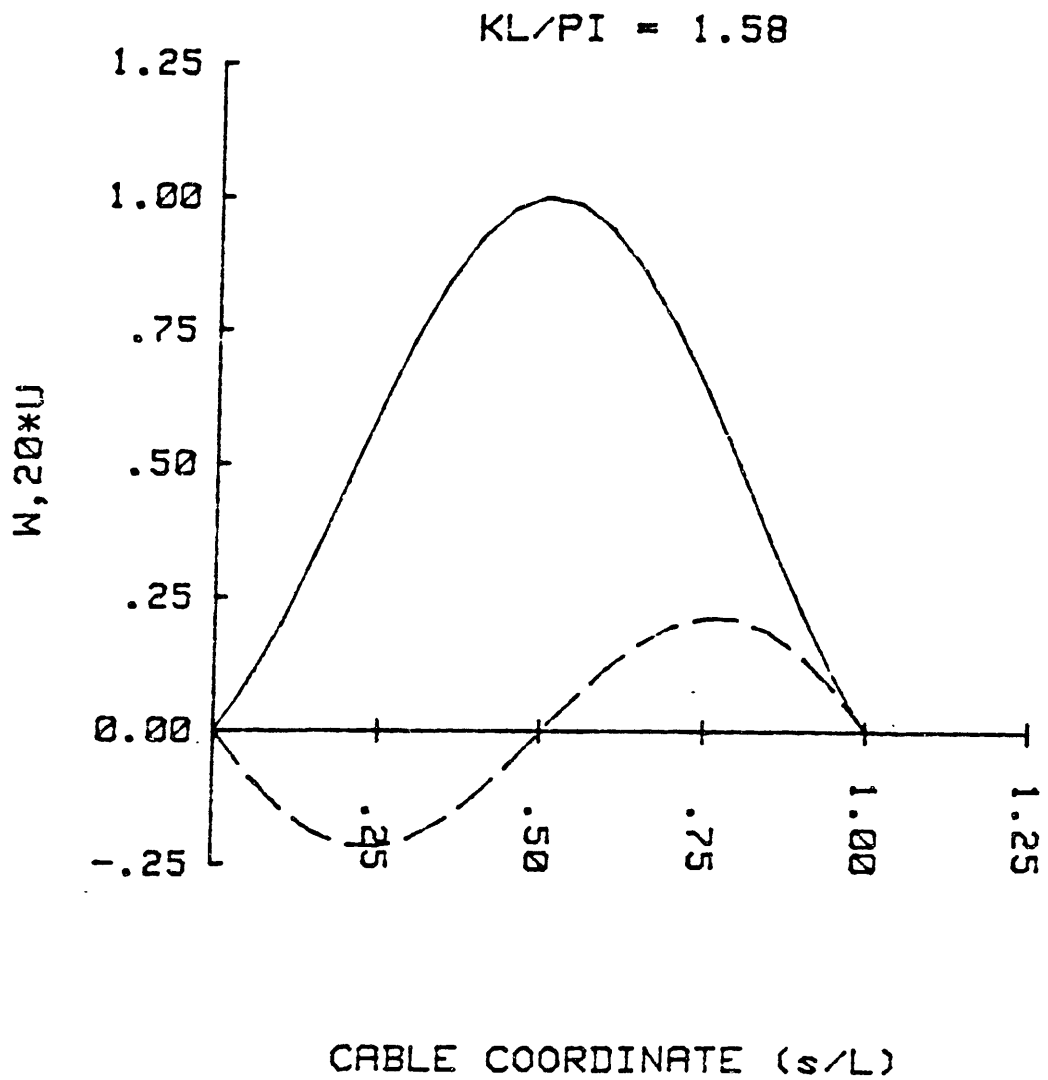

3.4.1.9. Superposition of the exact second transverse mode
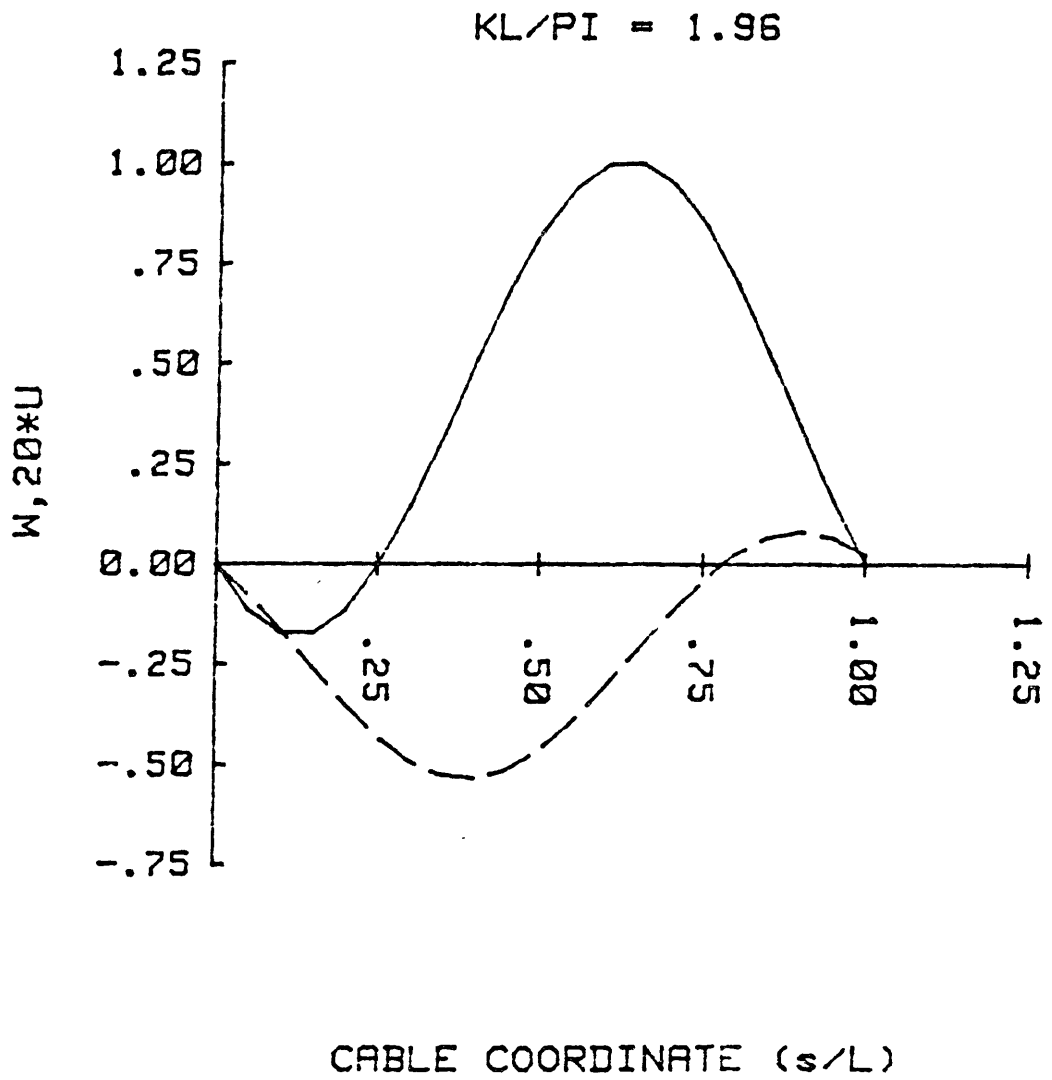and its approximation for KL/PI=2.73
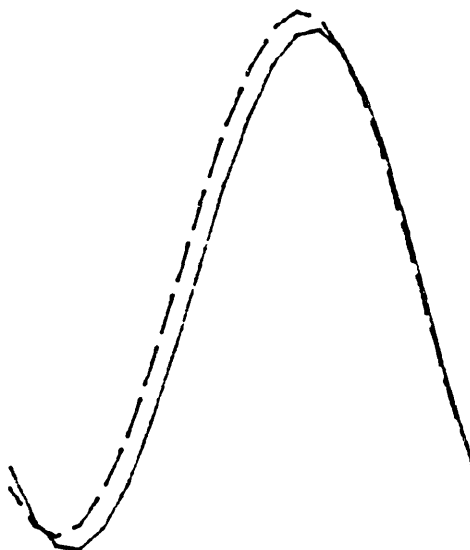
3.4.2.1. Superposition of the exact longitudinal mode
and its approximation (dotted) for KL/PI=1.96,
ϵ=0.18, φ_a=23.46

3.4.2.2. Superposition of the exact longitudinal mode
and its approximation (dotted) for KL/PI=1.58,
ϵ=0.18, φ_a=23.46

KL/PI = 1.58

3.4.2.3. Transverse (solid) and longitudinal (dotted) modes for KL/PI=1.58

KL/PI = 1.96

CABLE COORDINATE (s/L)

3.4.2.4. Transverse (solid) and longitudinal (dotted) modes for KL/PI=1.96

3.4.2 Longitudinal component

Figures 3.4.2.1 and 3.4.2.2 show the exact longitudinal mode and the approximate longitudinal mode given by equation (37) for $\frac{kL}{\pi}$ = 1.58 and 1.96.

We also included a plot of u and w, superimposed on the same graph, for two different values of λ.

3.5 Orthogonality of the modes

Neither w nor u are orthogonal in the scalar product form $\int_0^L \psi_n \psi_m \, ds$, where ψ_n (ψ_{mn}) represent the nth (m^{th}) transverse or longitudinal mode.

Another scalar product, which is useful when computing the kinetic energy of a cable whose displacement is a linear combination of natural modes, is

$$SP = \int_0^L (m \, u_i \, u_j + M \, w_i \, w_j) \, ds$$

where subscript i(j) refers to the ith (j^{th}) natural mode.

Since the longitudinal component is of order ε relative to the transverse mode, $u_i \, u_j$ is of order ε^2 in the product $w_i \, w_j$. Therefore, u_i, u_j, w_i, w_j are orthogonal, in the generalized sense, correct to first order, if

$$\int_0^L w_i \, w_j \, ds = 0$$

which is the first scalar product we contemplated. Therefore our natural modes are not orthogonal.

3.6 Conclusion

Based on the same assumptions we derived in chapter 2, we were able to approximate both the transverse and longitudinal modes by a uniformly valid formula, whose precision is demonstrated by figures 3.4.1.1 to 3.4.2.2.

The approximate modes are not orthogonal for any of the scalar products we have contemplated.

Now using these approximate formulae for the modes, we can proceed to derive the ultimate objective of the study: the tension generated by the motion of the cable.

Chapter 4

Dynamic Tension

4.1 Derivation

The dynamic tension is the ultimate objective of our study. In the three stage process we have described in this thesis, i.e. the computation of the natural frequencies, natural modes and tensions, the natural modes can be considered as only a preliminary step in our calculations. They are important to build solutions or check our hypotheses regarding the magnitude of the displacements, however the designer is interested in the spectral response of the cable in tension, so it is important to establish the effect of λ^2, on the dynamic tension.

Since, by finding the natural modes, we have solved the two groups of equations (7) through (10), the dynamic tension is simply given by the equation:

$$- M\omega^2 \tilde{w}^S(s) = T(s)\ \alpha(s)\ +\ O(\varepsilon)$$

where $\tilde{w}^S$ stands for the slow transverse mode. As a matter of fact the slow solution generates most of the tension (which agrees with the hypotheses made in the perturbation method, paragraph 1.2.4).

Figure 4.1.1 and 4.1.2 illustrates this fact for two different values of λ^2, one near and one away from cross-over.

4.2 Approximation

We will derive only the leading order part of the dynamic tension

Dotted lines represent the approximations of the tensions
Solid lines represent the exact tensions

KL/PI = 1.58

KL/PI = 1.96
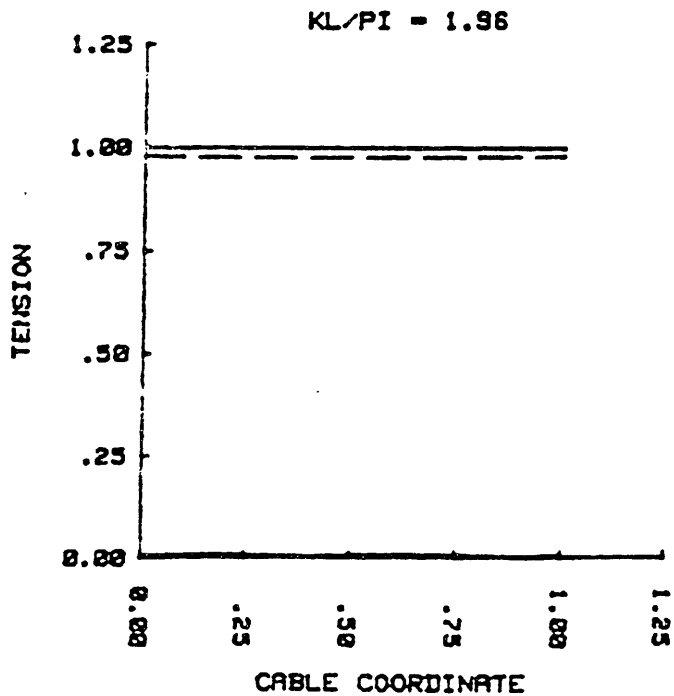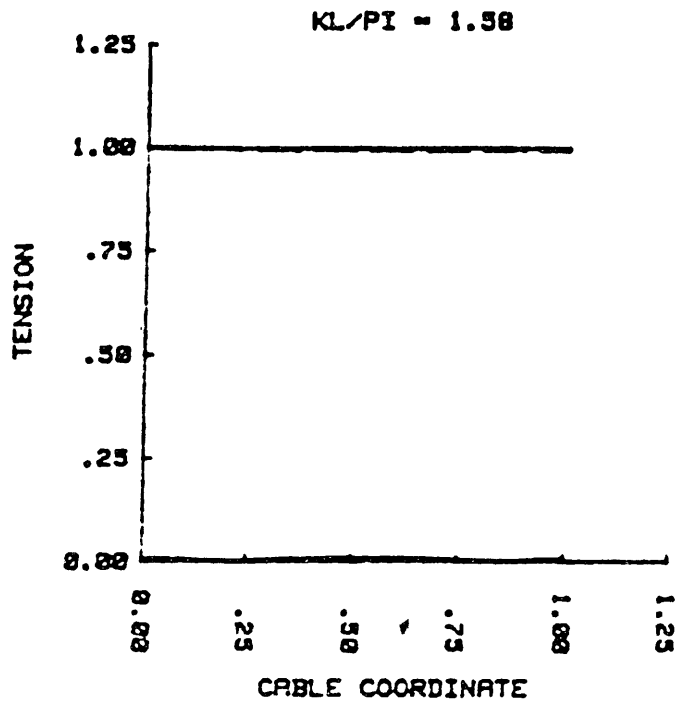
4.1.1. Comparison of the tension generated by bhe slow and
fast solution: the "fast" tension is confounded with
the X axis.

since due to the quasi-stretching approximation, the dynamic tension is, almost constant throughout the cable

From equation (29), w(s) near cross over is:

$$\frac{w(s)}{C_4 S_f} \, detA = -\frac{7}{4} \, \varepsilon \, \sin\phi_a$$

Outside the hybrid mode region, from equation (24)

$$\frac{w^s}{C_4 \, S_f} \, detA = -2 \sin^2 \frac{kL}{2}$$

with $\frac{S_f}{detA}$ in both cases

$$\frac{S_f}{detA} = \frac{1}{4 \; T_0/M \; [\; \sin kL \; + \; kL \; (\frac{k^2 L^2}{\lambda^2} - 1)]}$$
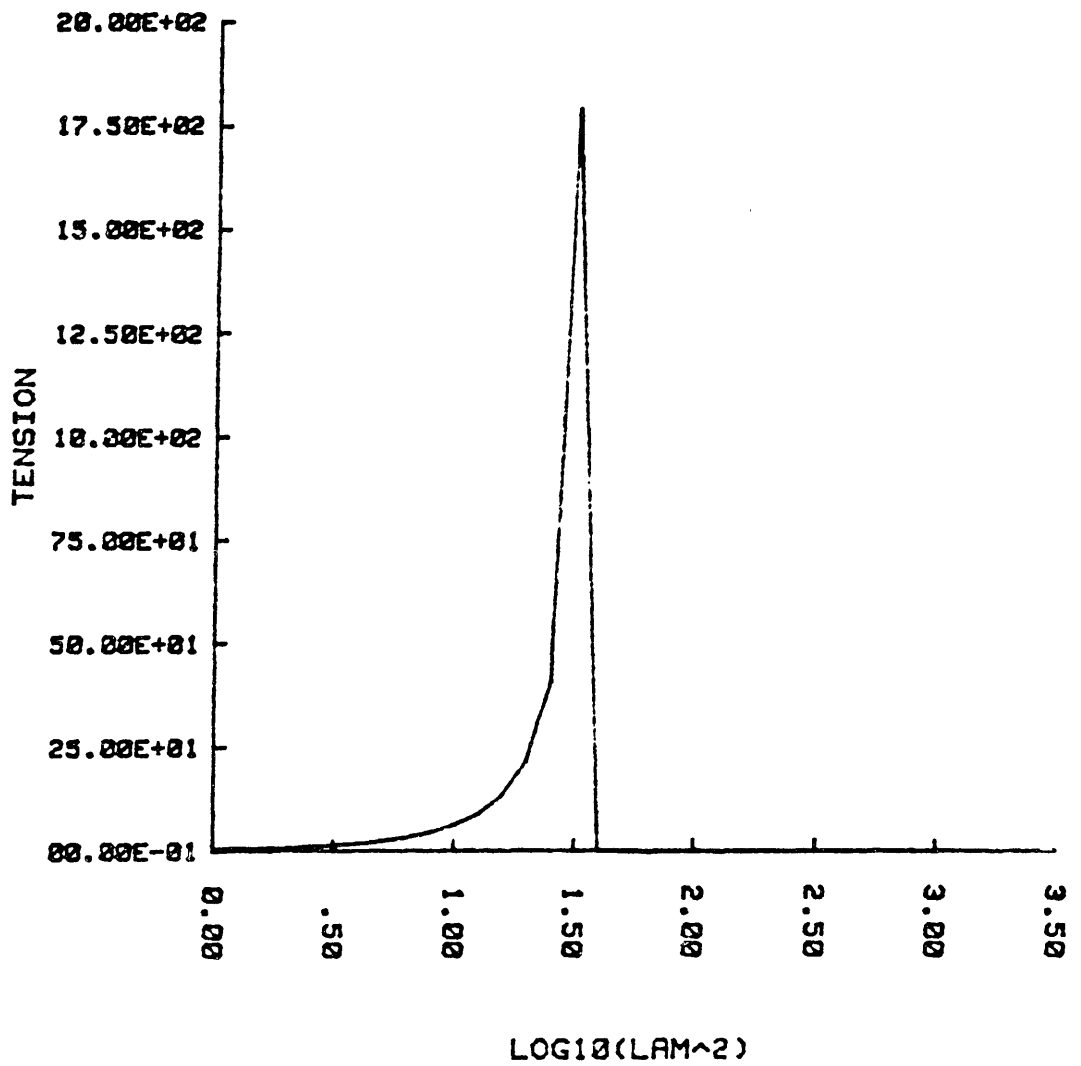
Therefore, an approximation for the dynamic tension for all frequencies is

$$\frac{T}{C_4} = \frac{M\omega^2 \; [+ \; 2 \sin^2 \frac{kL}{2} + \frac{7}{4} \, \varepsilon \, \sin\phi_a]}{4 \; T_0/M \; [\sin kL \; + \; kL \; (\frac{k^2 L^2}{\lambda^2} - 1)] \, \frac{\cos\phi_a \; \varepsilon}{L}} \tag{38}$$

4.3 Results and Conclusions

Figure 4.3.1 represents the dynamic tension vs. λ^2 for $C_4 = 1$. We see that at cross-over a phenomenon of dynamic amplification occurs, i.e. the dynamic tension is very large in the hybrid mode region, whereas for low or high values of λ^2 the dynamic tension is relatively small.

A well designed mooring cable should have those natural frequencies that are likely to be excited away from the cross-over region.

4.3.2. Amplification of the dynamic tension at
the cross-over of the first mode
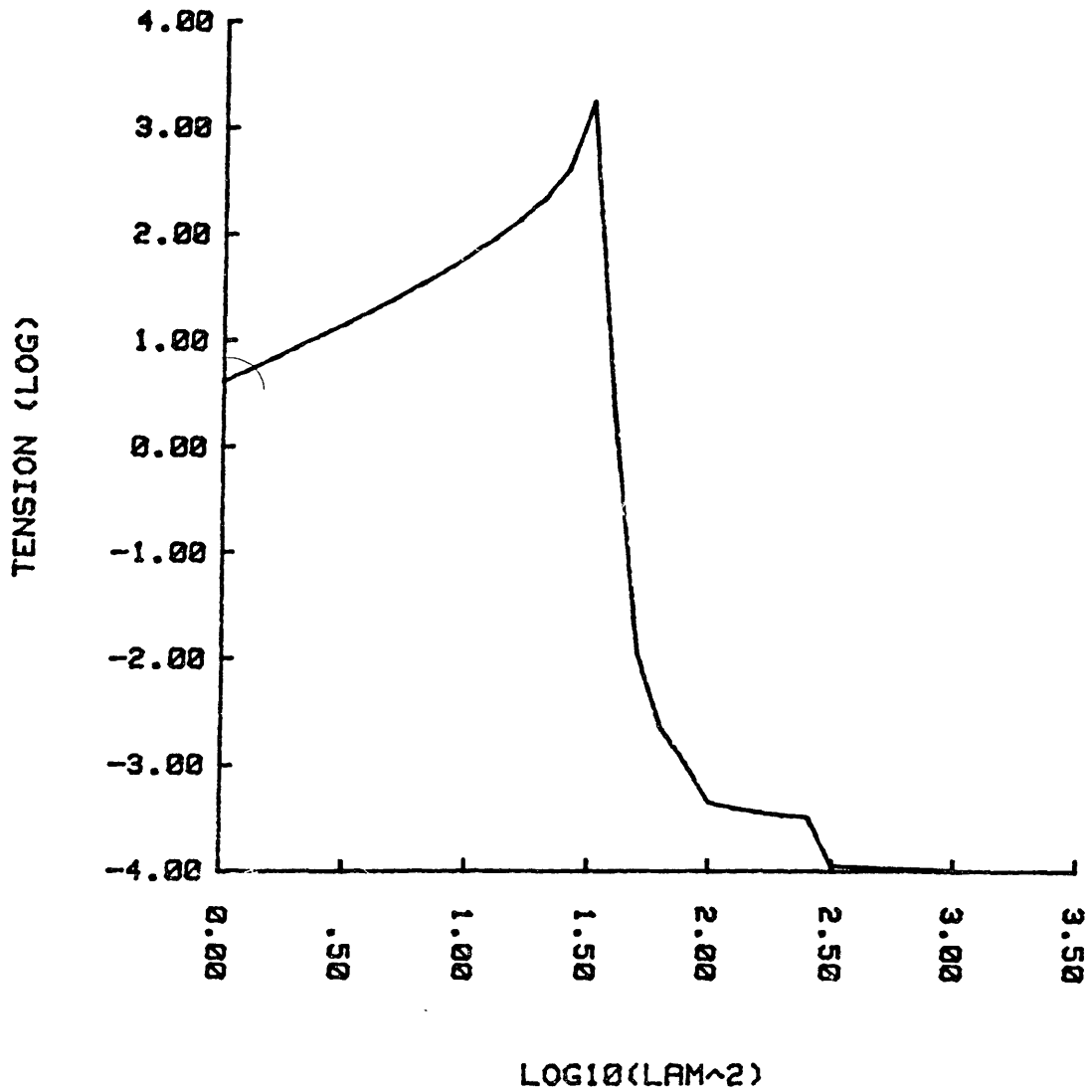
4.3.3. Same as 4.3.2., with a logarithmic
scale for the tension

Chapter 5

Conclusion

We have successively derived approximations to the equations for the natural frequencies, the modes and the dynamic tension of a taut inclined marine cable, by distinguishing two regions, one near and one away from cross-over. Subsequently, we were able to provide uniformly valid expressions by retaining terms of second order with respect to the sag to span ratio. Particular attention was paid to the derivation of the transverse modes, which in turn allow accurate predictions for the dynamic tension.

How these formulae should be used depends on the application. If one is interested in what happens in a specific domain, i.e. near or away from cross-over one should use the specific formulation for this domain, so that fewer terms will be necessary; otherwise, one has to use the general formulations and hence a larger number of terms.

Finally, we hope that with the careful outline of the methodology provided in the text and the appendices one will find it easy to improve the accuracy of some expressions if necessary, as for example in the case of the longitudinal modes.

Appendix A

This appendix deals with the expansion of expressions containing static quantities only.

$$\alpha(s) = \frac{\cos\phi_a \; \varepsilon}{L} \left\{ 1 - 2\varepsilon \sin\phi_a \left(\frac{s}{L} - \frac{1}{2}\right) + \varepsilon^2 \left[\left(\frac{s}{L} - \frac{1}{2}\right)^2 (4\sin^2\phi_a - 1) - \frac{\sin^2\phi_a}{6} \right] \right\}$$

$$T(s) = T_0 \left\{ 1 + \varepsilon \sin\phi_a \left(\frac{s}{L} - \frac{1}{2}\right) + \frac{\varepsilon^2}{2} \left[\cos^2\phi_a \left(\frac{s}{L} - \frac{1}{2}\right)^2 + \frac{\sin^2\phi_a}{6} \right] \right\}$$

$$\alpha(0) = \frac{\cos\phi_a \; \varepsilon}{L} \left\{ 1 + \varepsilon \sin\phi_a + \varepsilon^2 \left[\frac{5}{6}\sin^2\phi_a - \frac{1}{4} \right] \right\}$$

$$\alpha(L) = \frac{\cos\phi_a \; \varepsilon}{L} \left\{ 1 - \varepsilon \sin\phi_a + \varepsilon^2 \left[\frac{5}{6}\sin^2\phi_a - \frac{1}{4} \right] \right\}$$

$$\sqrt[4]{\frac{T(s)}{M}} = \sqrt[4]{T_0/M} \left\{ 1 + \frac{\varepsilon \sin\phi_a}{4} \left(\frac{s}{L} - \frac{1}{2}\right) + \frac{\varepsilon^2}{4} \left[\left(\frac{s}{L} - \frac{1}{2}\right)^2 \left(\frac{\cos^2\phi_a}{2} - \frac{3\sin^2\phi_a}{8} \right) \right. \right.$$

$$\left. \left. + \frac{\sin^2\phi_a}{12} \right] \right\}$$

$$\sqrt[4]{\frac{T(0)}{M}} = \sqrt[4]{T_0/M} \left\{ 1 - \frac{\varepsilon \sin\phi_a}{8} + \frac{\varepsilon^2}{32} \left[\cos^2\phi_a + \sin^2\phi_a \frac{5}{12} \right] \right\}$$

$$\frac{\sqrt[4]{\frac{T(L)}{M}}}{\sqrt[4]{\frac{T(0)}{M}}} = \left\{ 1 + \frac{\varepsilon^2 \sin\phi_a}{4} + \frac{\varepsilon^2 \sin^2\phi_a}{32} \right\}$$

$$\sqrt[4]{\frac{T(L)}{M}} \; \sqrt[4]{\frac{T(0)}{M}} = \sqrt{T_0/M} \left\{ 1 + \frac{\varepsilon^2}{16} \left[\cos^2\phi_a + \frac{\sin^2\phi_a}{6} \right] \right\}$$

$$\alpha(0)\alpha(L) = \frac{\cos^2\phi_a \; \varepsilon^2}{L^2} \left\{ 1 + \varepsilon^2 \left(\frac{2}{3}\sin^2\phi_a - \frac{1}{2} \right) \right\}$$

$$\alpha^2(0) = \frac{\cos^2\phi \; \varepsilon^2}{L^2} \; \{1 + 2 \; \varepsilon \; \sin\phi_a + \varepsilon^2 \; [\frac{-1}{2} + \frac{8}{3} \sin^2\phi_a]\}$$

$$\cos \; [W(s)] = \cos \; (\frac{kLs}{L}) + \frac{\varepsilon \; \sin\phi_a}{4} \; \sin \; (\frac{kLs}{L})(\frac{kLs}{L})(\frac{s}{L} - 1)$$

$$\sin \; [W(s)] = \sin \; (\frac{kLs}{L}) - \frac{\varepsilon \; \sin\phi_a}{4} \; \cos \; (\frac{kLs}{L})(\frac{kLs}{L})(\frac{s}{L} - 1)$$

Appendix B

In this appendix, the reader will find expansions of complex expressions of Airy and Bairy functions in the two domains considered in chapter 2.

- outside the hybrid mode region.

$$B_i'[-z(L)] \, A_i[-z(s)] - A_i'[-z(L)] \, B_i[-z(s)] = \frac{1}{\pi} - \frac{Q_0^{2/3}}{2\pi} \, L^2 \, [z(\tfrac{L}{2})](\tfrac{s}{L} - 1)^2$$

$$B_i'[-z(0)] \, A_i[-z(s)] - A_i'[-z(0)] \, B_i[-z(s)] = \frac{1}{\pi} - \frac{Q_0^{2/3}}{2\pi} \, L^2 [z(\tfrac{L}{2})](\tfrac{s}{L})^2$$

$$B_i[-z(0)] \, A_i[-z(s)] - A_i[-z(0)] \, B_i[-z(s)] = \frac{Q_0^{1/3}}{\pi} \, L(\tfrac{s}{L}) + 0(\varepsilon^3)$$

$$B_i'[-z(L)] \, A_i'[-z(s)] - A_i'[-z(L)] \, B_i'[-z(s)] = \frac{Q_0^{1/3}}{\pi} \, L \, [z(\tfrac{L}{2})](\tfrac{s}{L} - 1)$$
$$+ \frac{Q_0^{2/3}}{2\pi} \, L^2 \, (\tfrac{s}{L})(\tfrac{s}{L} - 1)$$

$$B_i'[-z(0)] \, A_i'[-z(s)] - A_i'[-z(0)] \, B_i'[-z(s)] = \frac{Q_0^{1/3}}{\pi} \, [z(\tfrac{L}{2})](\tfrac{s}{L}) + \frac{Q_0^{2/3}L^2}{2\pi}(\tfrac{s}{L})(\tfrac{s}{L} - 1)$$

$$B_i[-z(0)] \, A_i'[-z(s)] - A_i[-z(0)] \, B_i'[-z(s)] = \frac{-1}{\pi} + \frac{Q_0^{2/3}L^2}{2\pi} \, [z(\tfrac{L}{2})](\tfrac{s}{L})^2$$

- In the hybrid mode region

$$B_i'[-z(L)] \, A_i[-z(s)] - A_i'[-z(L)] \, B_i[-z(s)] = \frac{1}{\pi} + 0 \, (\varepsilon^3)$$

$$B_i'[-z(0)] \, A_i[-z(s)] - A_i'[-z(0)] \, B_i[-z(s)] = \frac{1}{\pi} + 0 \, (\varepsilon^3)$$

$$B_i[-z(0)] \, A_i[-z(s)] - A_i[-z(0)] \, B_i[-z(s)] = \frac{1}{\pi} Q_0^{1/3} L \, (\tfrac{s}{L}) + 0 \, (\varepsilon^3)$$

$$B_i'[-z(L)] \; A_i'[-z(s)] \; - \; A_i'[-z(L)] \; B_i'[-z(s)]$$

$$= \frac{Q_o^{1/3}L}{\pi} \; [z(\tfrac{L}{2})](\tfrac{s}{L} - 1) + \frac{Q_o^{2/3}L^2}{2\pi} \; (\tfrac{s}{L})(\tfrac{s}{L} - 1)$$

$$B_i'[-z(0)] \; A_i'[-z(s)] \; - \; A_i'[-z(0)] \; B_i'[-z(s)]$$

$$= \frac{Q_o^{1/3}L}{\pi} \; [z(\tfrac{L}{2})](\tfrac{s}{L} - 1) + \frac{Q_o^{2/3}L^2}{2\pi} \; (\tfrac{s}{L})(\tfrac{s}{L} - 1)$$

$$B_i[-z(0)] \; A_i'[-z(s)] \; - \; A_i[-z(0)] \; B_i'[-z(s) = \frac{-1}{\pi} + 0 \; (\varepsilon^3)$$

Bibliography

[1] M.S. Triantafyllou, A. Bliek, "The Dynamics of Inclined Taut and Slack Marine Cables", OTC 4498, 15th Annual OTC, May 1-5, Houston, Texas 1983.

[2] M.S. Triantafyllou, "The Dynamics of Taut Inclined Cables", Department of Ocean Engineering, M.I.T.

[3] H.M. Irvine, "Cable Structures", M.I.T. Press, Cambridge, 1981.

[4] M. Abramowitz, I. Stegun, "Handbook of Mathematical Functions", Dover, New York, 1981.

[5] M.S. Triantafyllou, C. Chryssostomidis, "Environment description, force prediction and statistics for design applications in offshore engineering", class notes, M.I.T., Cambridge, MA, 1981.