CODING THEOREMS FOR DISCRETE SOURCE-CHANNEL PAIRS

by

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ABSTRACT

The transmission of a discrete, independent letter information source over a discrete channel is studied. A distortion function is defined between source output letters and decoder output letters and is used to measure the performance of the system for each transmission. The encoders used are block encoders, mapping blocks of source output letters into blocks of channel input letters. This problem has been studied by Shannon who summarized, with a rate-distortion function, R(d), the information transmission requirement of the system which is necessary and sufficient to allow an average per letter transmission distortion equal to d.

In this thesis, the average transmission distortion, which is a function of the coding block length, is both upper and lower bounded. Both bounds converge, with increasing block length, to the same limit, the distortion d_C on the rate-distortion curve for the source at the information rate equal to the capacity of the channel. For noisy channels, the convergence of the lower bound is as a/n and that of the upper bound as $b(\ln n/ n)^{\frac{1}{2}}$. For noiseless channels, both the lower and upper bounds converge to d_C as $c(\ln n/ n)$.

The coefficient a in the lower bound (for noisy channels) interrelates the statistics of the source and channel in such a way as to suggest its utility as a measure of mismatch between source and channel.

A lower bound has also been found to the average transmission distortion when the signaling set is restricted to contain at most e^{nC} members. This bound converges to d_C only as $n^{-1/2}$ indicating that a larger signaling set would be required to obtain the 1/n rate of approach to d_C given by the more general lower bound.

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Chapter 1

INTRODUCTION

By now the results originally obtained by Shannon(1) relating reliability and channel capacity are well known. Roughly speaking, they state that perfect transmission can be achieved if, and only if, the capacity of the channel in the transmission link is greater than the information content of the source. For amplitude and time discrete sources the information content is the entropy of the source, but for amplitude continuous sources the entropy and the information content are not the same, the latter being infinite. This, of course, implies that perfect transmission of amplitude continuous sources, or discrete sources with an entropy that is "too large", is impossible with a given finite capacity channel. Yet this is just the situation that often is presented to the communication engineer who must then try to reduce the average distortion to the lowest possible, or practical, level.

For communication systems in which the capacity of the channel is not sufficient to allow perfect transmission, there are two obvious questions to ask: 1) how small can the average distortion be made if any transmission strategy at all is allowed, and 2) how much does the system complexity, or cost, increase when you are required to get "closer" to this minimum. To answer the first question, Shannon has generalized his original results in a later paper(2) in which the channel requirements are found that are necessary and sufficient to allow transmission at a given level of distortion, or a given error rate. Our work in this thesis is addressed to the second of the two questions. We will use the coding block length to measure the complexity of the system, and will study the behavior of the minimum attainable transmission distortion as the block length is increased.

In the transmission system in Figure 1.1, we assume that both the source and channel are amplitude and time discrete, and that both are constant and memoryless, that is, successive events of each are independent and are governed by the same probability distributions. The encoder is a block encoder that is described in more detail later in this chapter. To measure the distortion in the system, we introduce a non-negative function $d(w,z)$ which gives the distortion in the event letter z is reproduced at the output when letter w was transmitted. Normally, this function would be specified by the user of the system to reflect how undesirable any particular misinterpretation of the source output is to him. We will assume that the distortion between two sequences of messages is the averaged sum of the composing letter distortions.

Shannon's theory associates with each source and distortion function a rate-distortion curve which expresses the minimum attainable transmission distortion in terms of the channel capacity per source output. Associated with each point (R_o,d_o) on the rate-distortion curve is a particular set of transition probabilities, called the "test channel", which has the significance that among all channels that transmit the given source with distortion d_o or less, it operates at the lowest transmission rate, R_o. Equivalently, the test channel is that channel which yields the lowest distortion d_o among those that transmit information from the source

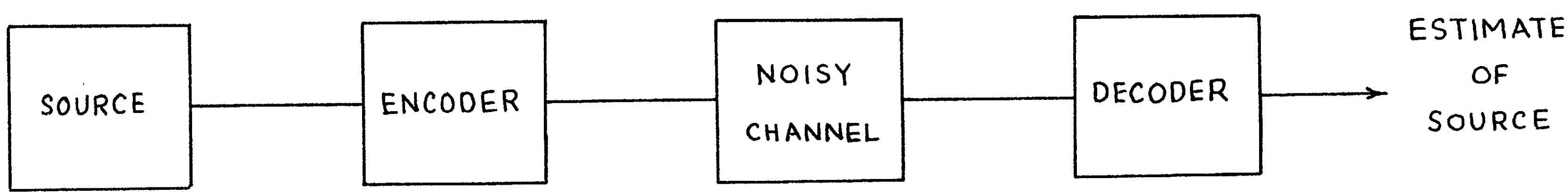

FIGURE 1.1: THE BLOCK DIAGRAM OF THE TRANSMISSION SYSTEM

at a rate R_o or less. It is in this sense the cheapest channel one could use and meet a distortion criterion. The rate $R_o(d_o)$ can be interpreted as the equivalent information content of the source when a distortion d_o is tolerable.

That the rate distortion curve gives the channel capacity sufficient to allow a prescribed performance is shown by Shannon through the intermediate step of proving that the rate-distortion curve actually expresses the entropy and resultant distortion in the "best" discrete representation of an output sequence from the original source. This discrete representation can then be transmitted with no further distortion, if its entropy is less than the channel capacity, by the use of suitable channel coding techniques.

Shannon has found the rate-distortion curves for many discrete sources and an explicit expression for the rate-distortion curve for time discrete gaussian sources. These results, together with Shannon's work with vector sources, were used to get rate-distortion curves for gaussian random processes(3). Bounds to the rate-distortion curves for non-gaussian sources have also been obtained(4,5). However, all of the rate-distortion results derived for both continuous and discrete sources are limiting results, that is, they can be approached in general only when arbitrarily complex operations on very long sequences of source output are allowed before transmitting the "message" through a correspondingly large use of the channel. T. Goblick(6) was the first to study the rate of approach to these limiting results as the source output block length increases, but limited his work to source representation or source encoding, with a deterministic map between the source and its representation. Our work includes a noisy channel, or probabilistic function, between the source and the user.

A performance curve d(n) will be defined for each source-channel pair as the minimum possible average distortion obtainable using a modulator that encodes a string of n successive source outputs into an input signal acceptable by a channel composed of n uses of the original channel. For a source with the rate-distortion curve of Figure 1.2 and a channel with capacity C, the performance curve might look like the one shown in Figure 1.3. From Shannon's theory it is known that the performance curve starts at d_0, the zero-rate distortion, and decreases to asymptotically approach d_C, the distortion corresponding to the information rate C on the rate-distortion curve. The curve, of course, has meaning only for integral values of n. Not all modulators and decoders yield a distortion curve that approaches d_C for large n. For example, a direct connection between a binary source and a binary symmetric channel yields the same distortion for all n which is as small as d_C only when the outputs of the source are equally likely. Of course, all distortion curves must lie above the performance curve which could have been defined as the lower envelope to the set of distortion curves corresponding to all encoder-decoder pairs.

Of particular interest is the rate at which the performance curve approaches its asymptote. If the channel has capacity C and is, in fact, the test channel for the source corresponding to the point (C, d_C) on the rate-distortion, the performance curve reaches its final value at $n = 1$. In this situation we shall say that the source and channel are

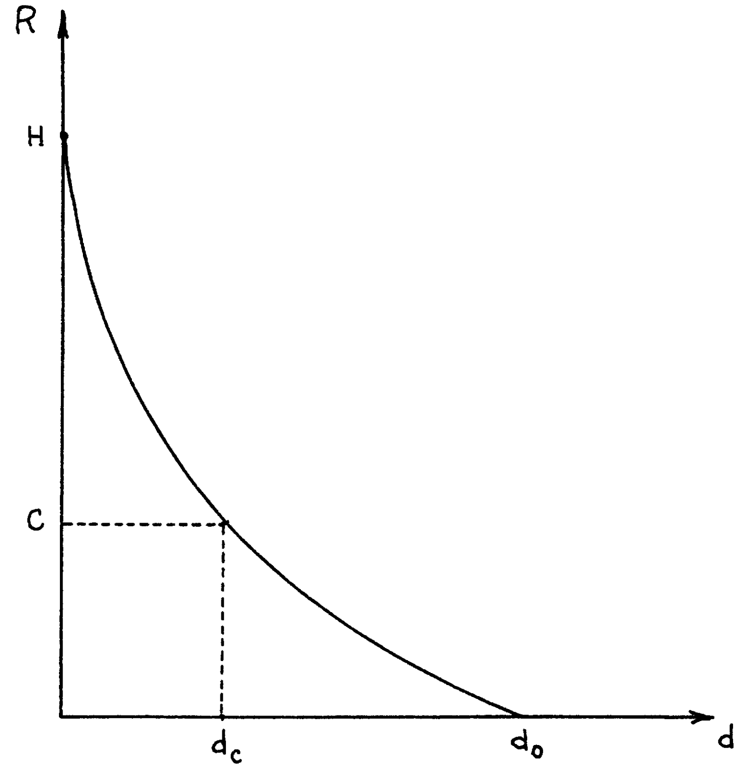

FIGURE 1.2: THE RATE DISTORTION CURVE FOR $\mathcal{S}$

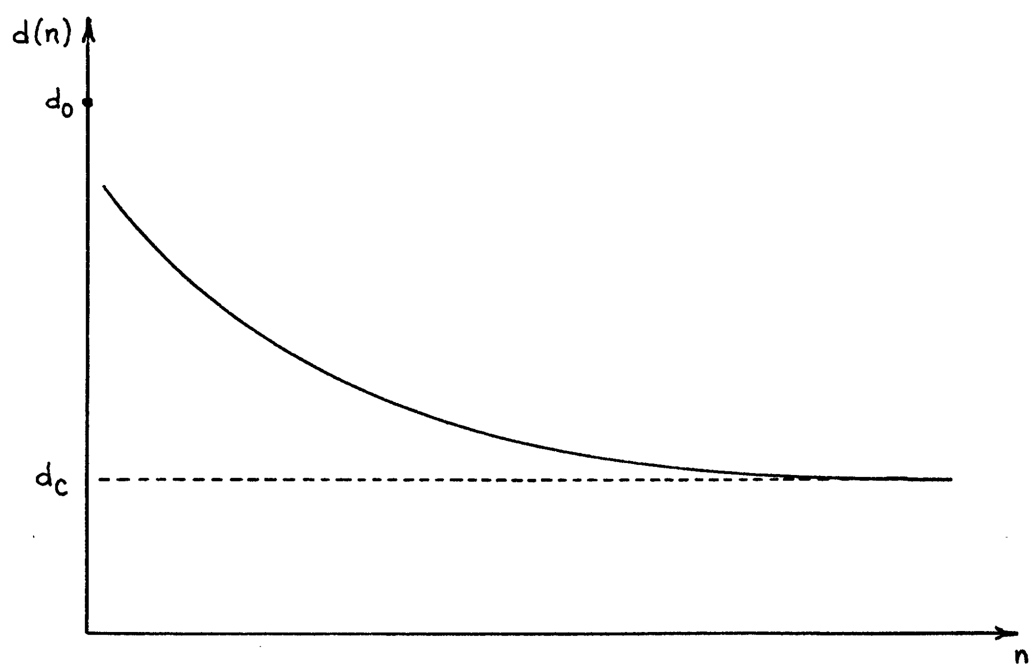

FIGURE 1.3: THE PERFORMANCE CURVE FOR $\mathcal{S}$ AND $\mathcal{C}$

matched. The test channel at the point (C,d_C) could, though, have a capacity $C' > C$. In this case, direct connection to the source would result in a distortion d_C, but the distortion could be made to approach $d_{C'} < d_C$ with increasing n if the optimum modulator were used. Our results, which are summarized below, indicate that the rate of approach of the performance curve to its asymptote is algebraic.

1.1 Summary of Results

In Chapters 2 and 3, lower and upper bounds to the performance curve are found for any discrete source and channel, and their asymptotic behavior is studied in detail. The lower bound is derived from a generalization of the sphere-packing concept and has the asymptotic form

$$d(n) \geq d_c + \frac{a}{n} \; . \tag{1.1}$$

The coefficient a is a non-negative function of the source and channel statistics that interrelates these statistics in such a way that the particular channel (among those of capacity C) for which a has its minimum value depends upon the source being used. The reverse is also true. Among those sources that have a common point (C,d_C) on their rate-distortion curves, the particular source that minimizes a is different for different channels. In addition, the coefficient a is precisely zero when the source and channel are matched. These properties of a suggest that it might be used to define a measure of "mismatch" between the source and channel; the larger the mismatch, the slower is the approach of the lower bound to its asymptote.

The upper bound to the performance curve is derived from a random coding argument and has the asymptotic form

$$d(n) \le d_c + b\left(\frac{\ln n}{n}\right)^{1/2} \tag{1.2}$$

in which b is a positive function of the source and channel statistics. In this derivation, we were forced to use a coding ensemble in which the signal set in each ensemble member is limited to $M < e^{nC}$ points since no more general code could be found that provided the correct asymptote d_C. The restriction to such a signal set in effect introduces an interface between the source and channel. This causes the coefficient b to not reveal the mismatch properties that the coefficient a does in the lower bound since the set of source and channel statistics that minimize b are each independent of the other. We can, though, interpret b as (the reciprocal of) a type of stretch factor similar to those studied by Shannon(7) and Wozencraft(8).

With the restriction to a signal set with $M < e^{nC}$, we have also found a lower bound to distortion that has the form $d(n) \ge d_C + b_1 n^{-\frac{1}{2}}$ (for noisy channels). Thus we conclude that it is necessary to have a signaling set larger than e^{nC} if one is to attain the $\frac{1}{n}$ rate of approach to d_C that appears in the lower bound in Equation 1.1. Although we cannot exhibit such a coding scheme, it is the author's opinion that one does exist and that the lower bound in Equation 1.1 more correctly expresses the behavior of the performance curve. Our reasons for this belief are discussed in Section 3.3.

For the special case of a noiseless channel, we have found upper and lower bound to the performance curve that agree asymptotically. They have the form

$$d_c + \frac{1}{2}\frac{\ln n}{|s|\,n} \leq d(n) \leq d_c + \left(\frac{1}{2} + \epsilon\right)\frac{\ln n}{|s|\,n} \tag{1.3}$$

in which s is equal to the slope of the rate-distortion curve at (C, d_C) and ϵ is an arbitrarily small positive constant.

1.2 System Model

Figure 1.4 provides a more detailed illustration of the transmission system that we will work with. The source $\mathcal{S}$ produces a sequence of letters $\underline{\omega} = \omega_1, \omega_2, \ldots, \omega_n$, each a member of the alphabet $W = \{w_1, \ldots, w_H\}$. This sequence, or source word, is encoded by the modulator into a sequence of channel input letters $\underline{\xi} = \xi_1, \xi_2, \ldots, \xi_n$, each chosen from the alphabet $X = \{x_1, \ldots, x_K\}$. The channel then transforms the channel input word $\underline{\xi}$ into a sequence of channel output letters $\underline{\eta} = \eta_1, \eta_2, \ldots, \eta_n$, which are letters of the alphabet $Y = \{y_1, \ldots, y_L\}$, and $\underline{\eta}$ in turn is decoded by the receiver into a sequence $\underline{\varsigma} = \varsigma_1, \varsigma_2, \ldots, \varsigma_n$ of letters from the decoding space $Z = \{z_1, \ldots, z_J\}$.

Because the source has been assumed constant, successive outputs w^m, $1 \leq m \leq n$, are controlled by equal probability distributions $P_{\omega_m}(w^m) = P_\omega(w)$. The same assumption has been made for the channel statistics, therefore successive transitions between x^m and y^m, $1 \leq m \leq n$, are also controlled by equal probability distributions, $P_{\eta_m|\xi_m}(y^m|x^m) = P_{\eta|\xi}(y|x)$. The superscript on w^m, x^m, and y^m is used

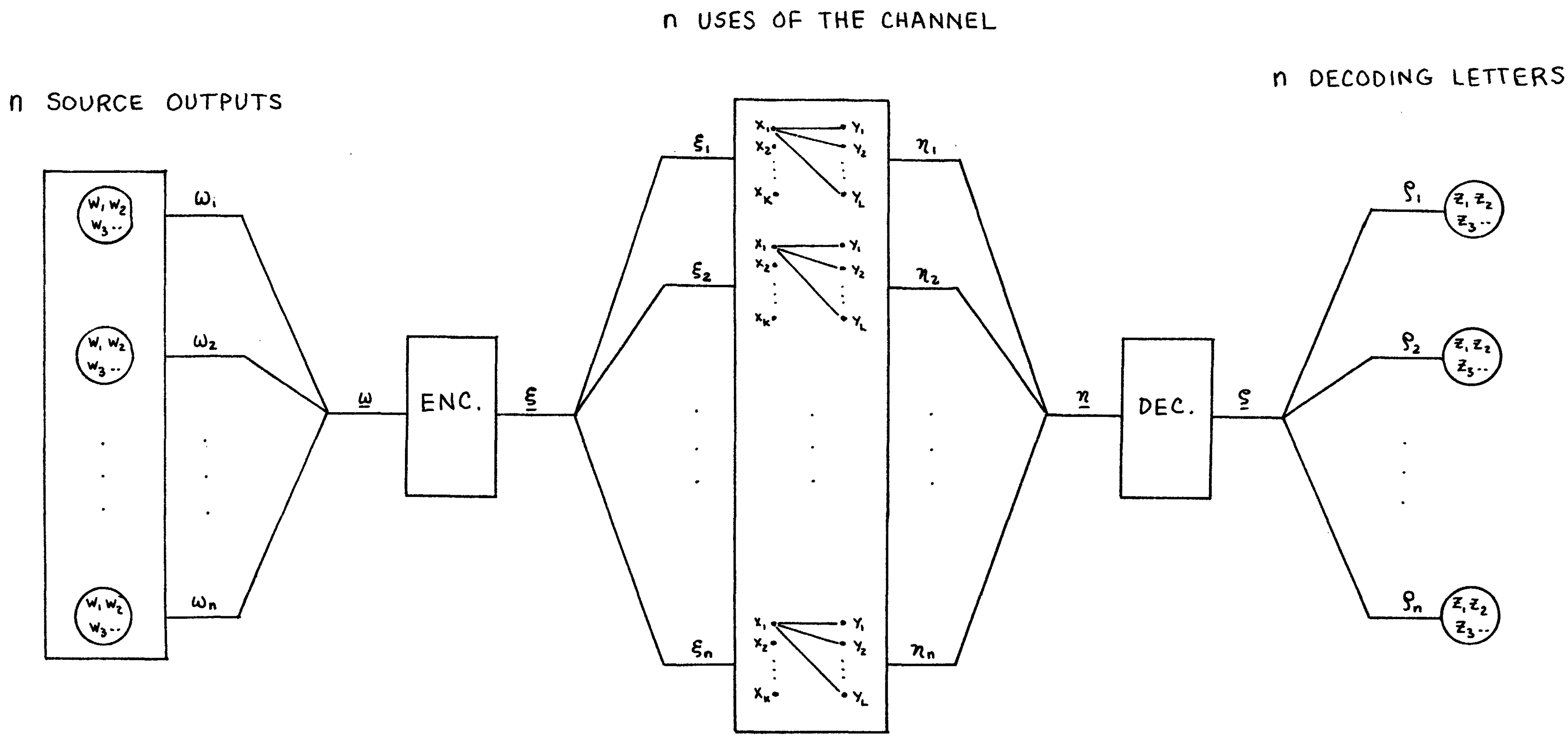

FIGURE 1.4 : A FUNCTIONAL BLOCK DIAGRAM OF THE ENCODING AND DECODING

here to denote the m'th letter in the n-letter words $\underline{w}$, $\underline{x}$, and $\underline{y}$ respectively, and is not to be confused with the particular letters w_m, x_m, and y_m in the alphabets W, X, and Y. Next, since the source and channel have also been assumed memoryless, we can write the probability distributions of sequences of these events as

$$P_{\underline{\omega}}(\underline{w}) = \prod_{m=1}^{n} P_{\omega_m}(w^m) \tag{1.4}$$

$$P_{\underline{\eta}|\underline{\xi}}(\underline{y}|\underline{x}) = \prod_{m=1}^{n} P_{\eta_m|\xi_m}(y^m|x^m) \; . \tag{1.5}$$

Where no confusion will occur, we will hereafter omit the subscripts on these probability distributions.

The distortion in the system is always taken, in our work, to be normalized to a per letter basis. Thus, if n-letter block encoders and decoders are used, the distortion between the source word $\underline{w}$ and the received word $\underline{z}$ is

$$d(\underline{w}, \underline{z}) = \frac{1}{n} \sum_{m=1}^{n} d(w^m, z^m) \; , \tag{1.6}$$

the normalized sum of the n letter distortions.

Finally, although we have set up the problem so that a sequence of n source letters is transmitted as a sequence of n channel letters, different block lengths at the source output and channel input can be allowed by considering a new source and channel that are products of the original ones, with the order of each product adjusted to obtain the desired block length ratio n_s/n_c.

Chapter 2

LOWER BOUND TO AVERAGE DISTORTION

In this chapter, we derive a lower bound to the performance curve of a transmission system that includes a discrete source $\mathcal{S}$ and a discrete channel $\mathcal{C}$. An immediate lower bound could be found from Shannon's original source-encoding results. This would be the distortion d_C on the rate-distortion curve for $\mathcal{S}$ at the information rate equal to the channel capacity of $\mathcal{C}$. The lower bound developed here, though, will include the coding block length as a variable and will be used to obtain information about the rate of approach of the performance curve to its limiting value d_C with increasing block length.

The first step in the derivation of the lower bound is to obtain, in terms of the coding block length n, a bound to distortion which is conditioned on the event that a particular source word $\underline{w}$ has occurred at the source output. To do this, we use a generalization of sphere packing ideas that is described in the first section of the chapter. It is assumed in this step that a particular channel input word $\underline{x}$ is used to transmit $\underline{w}$, but the selection of $\underline{x}$ is delayed until the end of the derivation when we optimize the result over all possible choices of $\underline{x}$. Then we average this conditioned lower bound over the set of all source words W^n to obtain the lower bound to distortion for the total source $\mathcal{S}$. This is our lower bound to the performance curve. Also developed in this chapter are asymptotic expressions for the lower bound which are used to define a measure of mismatch between the source and channel. And, in the last part of the chapter, we derive lower bounds to the transmission distortion for the important special case of a noiseless channel.

In every communication system of the type shown in Figure 1.4 the optimum decoder divides the channel output space Y^n into J^n disjoint decoding sets, corresponding to each of the decoded words, according to a decision rule derived from the source and channel statistics, the particular encoder used, and the distortion function in use. (The symbol Y^n is used here for the set of all possible n-letter received words $\underline{y}$ from the channel.) We denote the set of received words in Y^n that are decoded into $\underline{z}$ by

$$Y(\underline{z}) = \left\{ \underline{y}: \text{ decoded into } \underline{z} \right\}, \tag{2.1}$$

thus the resulting distortion is $d(\underline{w},\underline{z})$ whenever a source message $\underline{w}$ is transmitted and a member of $Y(\underline{z})$ is received from the channel. Some of these sets could be empty, but their union is assumed to include all members of Y^n.

For all but the very simplest encoders, the precise specification of the sets $Y(\underline{z})$ is nearly impossible and therefore so also is the calculation of the actual transmission distortion. Fortunately, we can derive a lower bound to distortion that depends on the decoder structure in only a weak way. For each decoding set $Y(\underline{z})$ we require only its "size" $g(\underline{z})$, which we define in Section 2.1. To be sure, the function $g(\underline{z})$ is also unknown for the optimum decoder, but we are able to carry it along until the end of the development when we replace $g(\underline{z})$ with another function that almost surely decreases the final lower bound expression and therefore continues the inequality.

2.1 Lower Bound for a Single Source Word

We assume in this section that the source output word $\underline{w}$ has occurred and that the channel input word $\underline{x}$ is being used for transmission. For the lower bound to distortion that is developed here we require a measure of the sizes of the decoding sets in Y^n. The simplest way to define the size of the decoding set $Y(\underline{z})$ would be to let it equal the fractional number of channel output words in $Y(\underline{z})$. If we let $N(\underline{z})$ be the number of received words $\underline{y}$ that are decoded into $\underline{z}$, the size of $Y(\underline{z})$ can then be written as

$$g_1(\underline{z}) = \frac{N(\underline{z})}{\sum_{\underline{z}'} N(\underline{z}')} . \tag{2.2}$$

It will later become necessary to use a more general measure of the size of $Y(\underline{z})$, but we first obtain a lower bound using the definition of size in Equation 2.2 since the arguments in the derivation are more easily explained.

We construct two lists: list 1 contains all possible decoded words $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_{J^n}$, ordered in increasing distortion $d(\underline{w},\underline{z})$ from $\underline{w}$, and list 2 contains all possible received words $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_{L^n}$, ordered in decreasing probability $P(y|x)$ conditioned on $\underline{x}$. Next, a pairing function is constructed between members of the two lists. Associated with the first word $\underline{z}_1$ on list 1 are the first $N(\underline{z}_1)$ received words on list 2, with the second word $\underline{z}_2$ on list 1 the next $N(\underline{z}_2)$ words of list 2, etc. (Some of the $N(\underline{z})$ could be zero.) This new association between the $\underline{y}$'s and $\underline{z}$'s (different from Equation 2.1) defines a new collection of sets

$$Y_o(\underline{z}) = \left\{ \underline{y}: \ \underline{y} \text{ on list 2 paired with } \underline{z} \text{ on list 1} \right\} \qquad (2.3)$$

in which $Y_o(\underline{z})$ contains $N_o(\underline{z}) = N(\underline{z})$ channel output words. It can be seen that this pairing function creates the same association between the $\underline{y}$'s and $\underline{z}$'s that would result if all the received words in Y^n were listed as in list 2 with each accompanied by the word in Z^n to which it is decoded, and then the set of $\underline{z}$ words were rearranged to be in decreasing distortion when measured from $\underline{w}$.

The average distortion from a source word $\underline{w}$ is

$$d(\underline{w}) = \sum_{Z^n} d(\underline{w},\underline{z}) \sum_{Y(\underline{z})} P(\underline{y}|\underline{x}) \,, \qquad (2.4)$$

which is a finite sum of two-term products of non-negative numbers. It therefore is reduced whenever the elements of the two sequences are rearranged to form monotone sequences of the opposite type. Since this is precisely the rearrangement we have made with the pairing function on the two lists, we have a lower bound to distortion given by

$$d(\underline{w}) \geq \sum_{Z^n} d(\underline{w},\underline{z}) \sum_{Y_o(\underline{z})} P(\underline{y}|\underline{x}) \,. \qquad (2.5)$$

In general the bound that results when Equation 2.5 is evaluated is not tight (remember we already have the lower bound provided by Shannon: $d(n) \geq d_C$), but it can be improved if a probability function $f(\underline{y})$, defined over Y^n, is included in the ordering of channel output words, and subsequently varied to obtain the best possible bound. In this work we will always take this function to be factorable into a product of letter probabilities as

$$f(\underline{y}) = \prod_{m=1}^{n} f(y^m) . \tag{2.6}$$

Before we proceed with this new derivation, it is necessary, as we have mentioned earlier, to use a more general measure for the size of sets in Y^n. Any subset Y' of Y^n is now defined to have a size equal to the sum of $f(\underline{y})$ over all $\underline{y}$ in Y'. Therefore, we have for the size of the decoding sets $Y(\underline{z})$

$$g(\underline{z}) = \sum_{Y(\underline{z})} f(\underline{y}) . \tag{2.7}$$

This can be seen to agree with the previous definition (Equation 2.2) if the probability function $f(\underline{y})$ is assumed uniform on Y^n. Finally, we see that $g(\underline{z})$ too is a probability function since it is non-negative and

$$\sum_{Z^n} g(\underline{z}) = \sum_{Y^n} f(\underline{y}) = 1 .$$

The improved lower bound is again derived using two lists. List 1 is the same list we used before; it contains the set of decoded words $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_{J^n}$ ordered in increasing distortion from $\underline{w}$. But we replace list 2 with list 2' in which the channel output words $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_{L^n}$ are now ordered in increasing values of the normalized Information Difference

$$I(\underline{x},\underline{y}) = \frac{1}{n} \ln \frac{f(\underline{y})}{P(\underline{y}|\underline{x})} \quad . \tag{2.8}$$

We shall denote the first a $\underline{z}$'s on the top of list 1 by the set Z_a^n and the first b $\underline{y}$'s on the top of list 2' by the set Y_b^n.

Two functions are now constructed from these lists. From list 1, we construct a function d(G) that gives, for any $0 \le G \le 1$, the distortion $d(\underline{w},\underline{z}_a)$ on this list at that word $\underline{z}_a$ for which

$$\sum_{Z_{a-1}^n} g(\underline{z}) \quad < \quad G \quad \le \quad \sum_{Z_a^n} g(\underline{z}) \; . \tag{2.9}$$

And from list 2', a function I(F) is constructed that gives, for any $0 \le F \le 1$, the Information Difference $I(\underline{x},\underline{y}_b)$ at that word $\underline{y}_b$ for which

$$\sum_{Y_{b-1}^n} f(\underline{y}) \quad < \quad F \quad \le \quad \sum_{Y_b^n} f(\underline{y}) \; . \tag{2.10}$$

Once again a pairing function is defined, this time not between members $\underline{z}$ and $\underline{y}$ on the two lists, but between ordinate values of the two functions d(G) and I(F). The definition is: for every number F_o in the interval $[0,1]$, we pair those ordinate values for which $d(F_o) = I(F_o)$. This establishes roughly the same pairings among distortions and

Information Differences as would be induced if we associated: with $\underline{z}_1$ on list 1, the set of $\underline{y}$'s at the top of list 2', $Y_{of}(\underline{z}_1)$, which has a total size as near equal to $g(\underline{z}_1)$ as possible; with $\underline{z}_2$, those $\underline{y}$'s next highest on list 2', $Y_{of}(\underline{z}_2)$, which have a total size as near equal to $g(\underline{z}_2)$ as possible, etc. The only differences would be caused by the fact that in general no set of $\underline{y}$'s on the top of list 2' has a total size precisely equal to $g(\underline{z}_1)$, and similarly down the lists, no set of $\underline{y}$'s in the appropriate position on list 2' would have a size precisely equal to $g(\underline{z}_k)$. Thus the first "borderline" $\underline{y}_i$ would have to be associated with either $\underline{z}_1$, making the size of $Y_{of}(\underline{z}_1)$ greater than $g(\underline{z}_1)$, or $\underline{z}_2$, making the size of $Y_{of}(\underline{z}_1)$ less than $g(\underline{z}_1)$. If one could visualize "splitting" the borderline $\underline{y}_i$ (and all successive borderline $\underline{y}_j$'s), associating part of it with $\underline{z}_1$, part with $\underline{z}_2$, in such a way as to establish the precise equality $Y_{of}(\underline{z}_1) = g(\underline{z}_1)$ (and all other equalities $Y_{of}(\underline{z}_k) = g(\underline{z}_k)$), then the pairings induced among the distortions and Information Differences on the two lists would agree precisely with the pairs $d(F_o)$, $I(F_o)$.

We can now state our basic lower bound to distortion in terms of the distortions and Information Differences that are paired through the equalities: $d(F_o) = I(F_o)$.

Theorem 2.1

The single word transmission distortion $d(\underline{w})$ that exists when the channel input word $\underline{x}$ is used to transmit the source message $\underline{w}$ satisfies

$$d(\underline{w}) \geq \int_0^1 d(F_o)\, e^{-n I(F_o)}\, dF_o \ . \tag{2.11}$$

Proof: Figure 2.1 will be used to help prove the inequality. The distortion using the decoding sets $Y(\underline{z})$ of the optimum decoder is

$$d(\underline{w}) = \sum_{\underline{z}^n} d(\underline{w},\underline{z}) \sum_{Y(\underline{z})} \left[\frac{P(\underline{y}|\underline{x})}{f(\underline{y})} \right] f(\underline{y}), \tag{2.12}$$

or a double sum of three-term products equal to the "volume" in Figure 2.1A enclosed by the two "amplitude functions" d and $\frac{P}{f}$ and the "width measure" f. If the width measure is preserved for both amplitude functions while they are ordered to be monotone of the opposite type, the "volume" must be decreased. To see this, we consider one amplitude function, d, already ordered to be monotone increasing as shown in Figure 2.1A. The changing of the order of the other amplitude function can be accomplished by a sequence of interchanges of the following type. We consider any two points, u and v, along the "width function" f for which the values of the amplitude function d are such that $d(u) \leq d(v)$ and those of the second amplitude function $\frac{P}{f}$ are such that $\frac{P}{f}(u) \leq \frac{P}{f}(v)$. If discontinuities in d and $\frac{P}{f}$ are avoided, there is an interval Δf around each point in which both amplitude functions are single valued. If we interchange the amplitude values of $\frac{P}{f}$ in these two intervals we effect a volume transformation that decreases (or leaves unchanged) the total volume since

initial volume - final volume

$$= \left[d(v)\frac{P}{f}(v) + d(u)\frac{P}{f}(u) \right] \Delta f - \left[d(v)\frac{P}{f}(u) + d(u)\frac{P}{f}(v) \right]$$

$$= \left[d(v) - d(u) \right] \left[\frac{P}{f}(v) - \frac{P}{f}(u) \right]$$

$$\geq 0$$

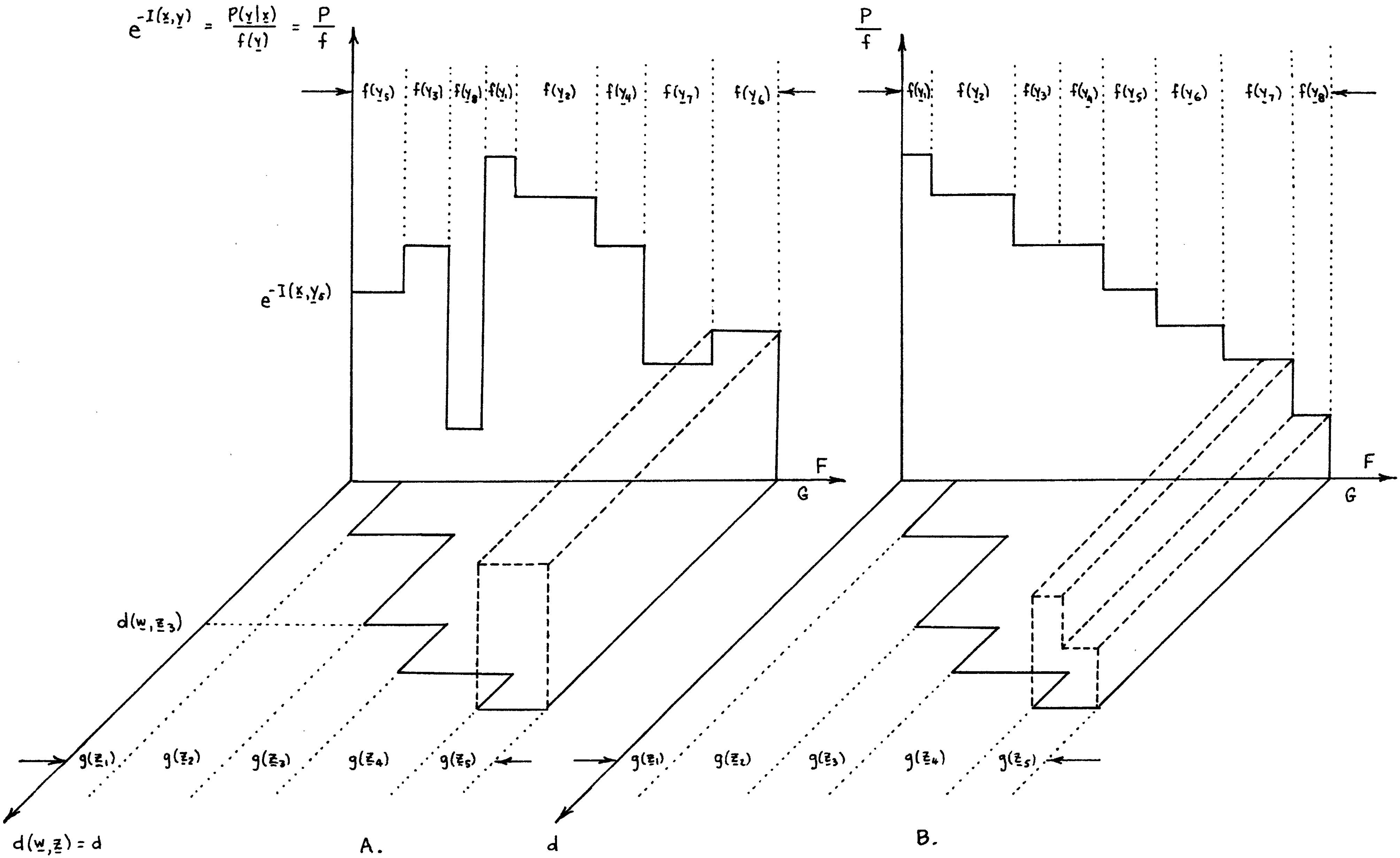

FIGURE 2.1: THE GEOMETRY FOR THEOREM 2.1

when both $d(v) \geq d(u)$ and $\frac{P}{f}(v) \geq \frac{P}{f}(u)$. We continue making volume transformations of this type until the function $\frac{P}{f}$ is monotone decreasing as shown in Figure 2.1B. Since each transformation in the process decreases the volume, or leaves it unchanged, the total volume in Figure 2.1B is no larger than that in Figure 2.1A. If we now label the axis along the width measure f as F_o, and consider the amplitude functions d and $\frac{P}{f}$ as the functions of F_o: $d(F_o)$ and $\frac{P}{f}(F_o)$, we can recognize the integral in Equation 2.11 as giving the total volume of the solid in Figure 2.1B. We need only note that $e^{-nI(F_o)} = \frac{P}{f}(F_o)$ from the definition of I in Equation 2.8. This establishes the inequality in the theorem.

We now wish to focus more closely on the pairings $d(F_o)$, $I(F_o)$ that were established before this last theorem. Because neither d nor I uniquely determines the other, the set of pairs $\left[d(F_o), I(F_o)\right]$ do not constitute a well defined function. However, we will use the properties that exist among these pairs to define a distortion function d(I) which has the property that for any I, the dependent variable d is at least as small as the smallest $d(F_o)$ among the pairs that have $I(F_o) = I$.

To do this we will consider the distortion $d(\underline{w},\underline{z})$ and the Information Difference $I(\underline{x},\underline{y})$ as random variables on Z^n and Y^n respectively, governed by the probability distribution functions $g(\underline{z})$ and $f(\underline{y})$. Their cumulative distribution functions are defined in the usual way by

$$G(d) = \sum_{\substack{\underline{z} \ni \\ d(\underline{w},\underline{z}) \leq d}} g(\underline{z}) \tag{2.13}$$

and

$$F_1(I) = \sum_{\substack{\underline{y} \ni \\ I(\underline{x},\underline{y}) \le I}} f(\underline{y}) . \tag{2.14}$$

For any distortion $d(\underline{w},\underline{z}_a)$ that is paired with an Information Difference $I(\underline{x},\underline{y}_b)$, we know from Equations 2.9 and 2.10 that

$$\sum_{Y_{b-1}^n} f(\underline{y}) < \sum_{Z_a^n} g(\underline{z}) .$$

The sum on the right includes all $\underline{z}$ for which $d(\underline{w},\underline{z}) < d(\underline{w},\underline{z}_a)$ and some $\underline{z}$ for which $d(\underline{w},\underline{z}) = d(\underline{w},\underline{z}_a)$. Therefore the sum is surely upper bounded by $G(d(\underline{w},\underline{z}_a))$. If $I(\underline{x},\underline{y}_{b-1}) = I(\underline{x},\underline{y}_b)$, the sum on the left includes all $\underline{y}$ for which $I(\underline{x},\underline{y}) < I(\underline{x},\underline{y}_b)$ and some $\underline{y}$ for which $I(\underline{x},\underline{y}) = I(\underline{x},\underline{y}_b)$. On the other hand if $I(\underline{x},\underline{y}_{b-1}) < I(\underline{x},\underline{y}_b)$, the sum includes only those $\underline{y}$'s for which $I(\underline{x},\underline{y}) < I(\underline{x},\underline{y}_b)$. In either case the sum is equal to or larger than $F_1(I(\underline{x},\underline{y}_b)^-)$ and we can write

$$F_1\left[I(\underline{x},\underline{y}_b)^-\right] < G\left[d(\underline{w},\underline{z}_a)\right] . \tag{2.15}$$

Therefore if for any I, we equate

$$F_1(I^-) = G(d) \tag{2.16}$$

and solve for d, we must have the inequality $d \le d(\underline{w},\underline{z}_a)$ since $G(d) < G(d(\underline{w},\underline{z}_a))$.

Thus Equation 2.16 provides us with an implicit definition of the desired distortion function $d(I)$.

The following geometric interpretation of $d(I)$ might be helpful. If each size, or "volume", $g(\underline{z})$ of the decoded words is successively placed about the volume $g(\underline{z}_1)$ of the minimum distortion word according to list 1, and the same thing is done with the "volume" $f(\underline{y})$ of the received words on list 2', the volume included by a point in the first construction at a distortion "radius" d is $G(d)$ and that included by a point in the second construction at an Information Difference "radius" I is $F_1(I)$. The set of pairs (d,I) is then (except for edge effects) the correspondence between the "radii" that include the same volume in both geometrical constructions. Figure 2.2 illustrates the construction of the distortion function $d(I)$ through the chain $I \rightarrow F_1(I^-) = G(d) \rightarrow d$.

It is convenient at this point to introduce a second random variable of Information Difference which is described by the probability distribution $P(\underline{y}|\underline{x})$ rather than $f(\underline{y})$. This variable has the cumulative distribution function

$$F_2(I) = \sum_{\substack{\underline{y}\ni \\ I(\underline{x},\underline{y}) \leq I}} P(\underline{y}|\underline{x}) . \tag{2.17}$$

To distinguish the two Information Difference variables, we will denote by I_1 the variable that has the distribution function in Equation 2.14 and by I_2 the variable that has the distribution function in Equation 2.17.

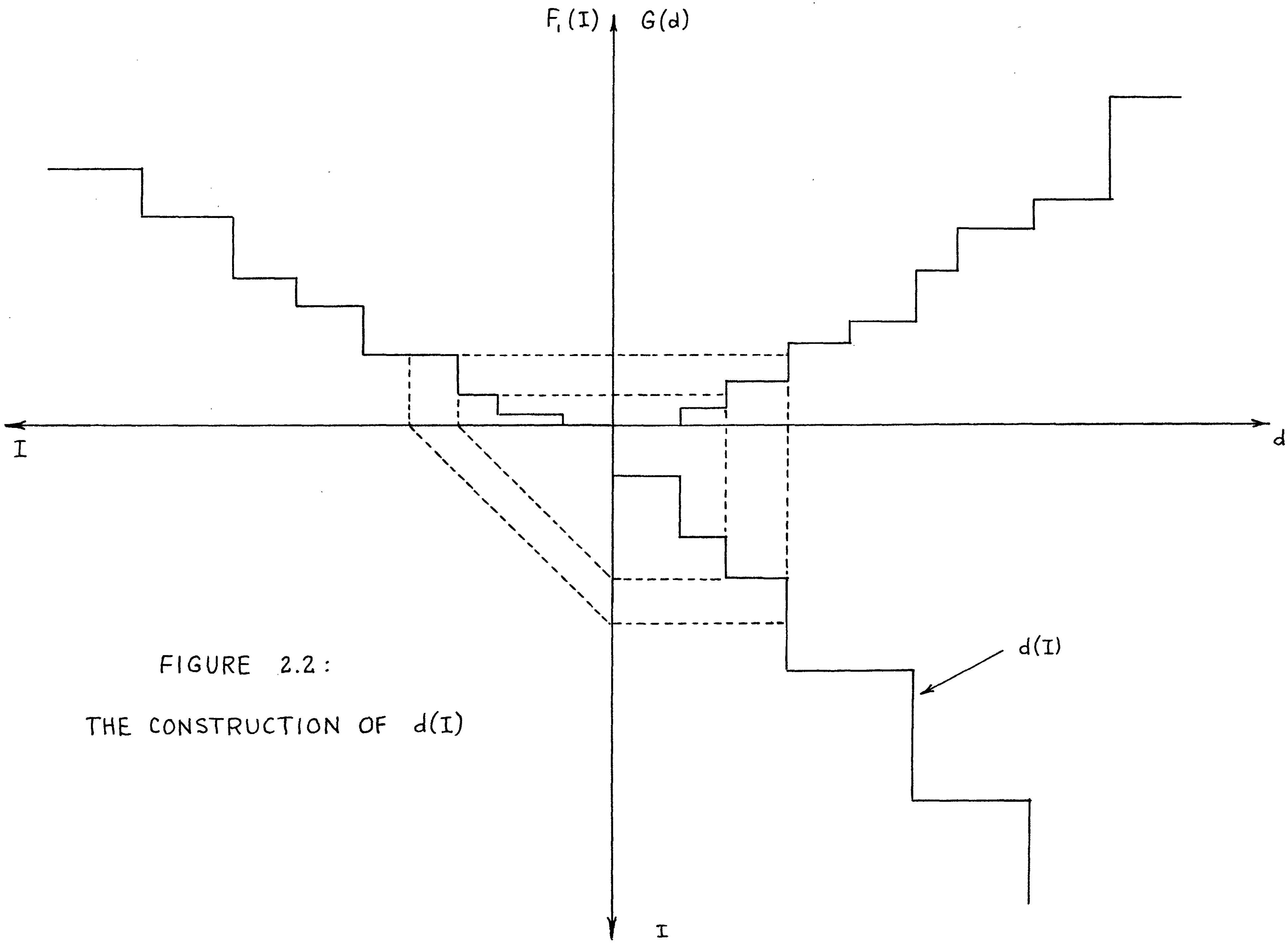

FIGURE 2.2:

THE CONSTRUCTION OF $d(I)$

Since the distortion $d(I)$ has been shown to lower bound all $d(F_o)$ that are paired with $I(F_o)$, we can replace $d(F_o)$ in Equation 2.11 with $d(I(F_o))$. We can also change the integral over F_o to an average over the random variable I_2 to obtain the lower bound in the following theorem.

Theorem 2.2

The single word transmission distortion $d(\underline{w})$ that exists when the channel input word $\underline{x}$ is used to transmit the source message $\underline{w}$ satisfies

$$d(\underline{w}) \geq \int d(I) \; dF_2(I) \; . \tag{2.18}$$

The usefulness of Theorem 2.2 depends upon our being able to evaluate $F_1(I)$, $F_2(I)$, and $G(d)$, and from these, $d(I)$. This is the subject of the next several subsections.

2.1.1 The Random Variable d

In the work so far, the function $g(\underline{z})$ is that particular function of $\underline{z}$ which gives the decoding set sizes in the optimum decoder. It therefore cannot be freely chosen as was the probability function $f(\underline{y})$, which was taken to have the particular product form in Equation 2.6. On the other hand, it would be completely impractical to evaluate the function $g(\underline{z})$ from the optimum decoder since the structure of this decoder is usually highly complex. (Indeed if one could calculate $g(\underline{z})$, one could also calculate the precise distortion and would not have to bound it.) The only alternative is to retain the unknown function $g(\underline{z})$ in the lower bound expressions and to minimize the final lower bound to distortion over all possible probability functions on Z^n. Since $g(\underline{z})$ is one such probability function the inequality in the lower bound is continued. Unfortunately, when this is done, it cannot in general be shown that the function which minimizes the lower bound factors into a product of n letter probabilities. We will, however, approximate this $g(\underline{z})$ by such a product, as in

$$g(\underline{z}) = \prod_{m=1}^{n} q(z^m) . \tag{2.19}$$

We realize that the approximation in Equation 2.19 is not entirely satisfactory as it eliminates non-product probability functions from the minimization of the lower bound and, as far as we know, one of these functions could provide the minimization. However, to further develop the lower bounds in Theorems 2.1 and 2.2 we are required to bound functions that involve the cumulative distribution function G(d) of the random variable $d(\underline{w},\underline{z})$. This in turn requires the use of estimates of the function G(d). The independence assumption in Equation 2.19 (or, as we will show, an independence assumption between blocks of decoding letters of length $r > 1$) will enable us to use the work of Chernov(9), Shannon(10), Gallager(11), and Fano(13) to bound this distribution function.

More specifically, the assumed product form in Equation 2.19 allows us to cast the word distortion random variable $d(\underline{w},\underline{z})$ as a sum of independent letter probabilities in the following way. Among the letter distortions $d(w,z)$ that sum to the total word distortion there are H different types, corresponding to each of the different letters w_i, $1 \leq i \leq H$, that appear in the source word $\underline{w}$. The number of each type of letter distortion in the sum is equal to the number of times each letter of the alphabet W appears in the word $\underline{w}$. This information is usually summarized by a "composition vector" $\underline{q}$, which is equal to $q_1, q_2, \ldots, q_H$ when there are nq_1 appearances of the letter w_1 in the word $\underline{w}$, nq_2 appearances of the letter w_2, etc. If we define D_{im} as the distortion between the m'th appearance of the letter w_i in $\underline{w}$ and the corresponding letter in $\underline{z}$, the normalized word distortion can be written as

$$d = \frac{1}{n} \sum_{i=1}^{H} \sum_{m=1}^{nq_i} D_{im} \, . \tag{2.20}$$

For a given source word $\underline{w}$, we have viewed the word distortion $d(\underline{w},\underline{z})$, or d, as a random variable on Z^n governed by $g(\underline{z})$. If we use the product form in Equation 2.19 for $g(\underline{z})$, we can also view the D_{im}'s in Equation 2.20 as random variables which are independent, and which are described by the distributions

$$P_{D_{im}}(d_{ij}) = g_j \quad ; \qquad 1 \leq m \leq nq_i \quad , \quad 1 \leq i \leq H \, . \tag{2.21}$$

In this equation we have used the abbreviations $d(w_i, z_j) = d_{ij}$ and $g(z_j) = g_j$. Because the random variable $d(\underline{w},\underline{z})$, or d, in Equation 2.20 is now a sum of n independent random variables, its distribution function $G(d)$ is an n-fold convolution of elementary distribution functions which can be estimated from the bounds referred to in the previous paragraph.

Actually, although the consequences of the assumption in Equation 2.19 cannot be firmly established, there is good reason to believe that this approximation does not significantly affect the bound when n is reasonably large. For example, in the next several sections we derive a lower bound to distortion that uses the product form in Equation 2.19. For this bound the required minimization over all probability functions $g(\underline{z})$ is reduced to one over all J dimensional vectors $\underline{g}$. It can be shown that if in the limit as n becomes large, the product form requirement for $g(\underline{z})$ is relaxed, and the minimization of this lower bound is again made over all probability functions $g(\underline{z})$, then the optimizing function $g_o(\underline{z})$ still has the product form.

Even more significant is the result that obtains when the approximation in Equation 2.19 is again made and Shannon's asymptotic formula[10,11] is used to bound the distribution function G(d). For this bound we will show (Section 2.3.2) that it is <u>only</u> the final value of the minimizing decoder set size vector $\underline{g}_o(n = \infty)$ that affects <u>both</u> the asymptote of the lower bound, d_C, and the next lowest order term, which is one proportional to $\frac{1}{n}$. Values of the minimizing vector for finite n, $\underline{g}_o(n < \infty)$, affect only terms of $o(\frac{1}{n})$.

Further, it can be shown that a similar conclusion is reached even if the independence property assumed over letters in Equation 2.19 is generalized to be over blocks of length r. That is

$$g(\underline{z}) = \prod_{m=1}^{n/r} g(\underline{z}'^m) \tag{2.19a}$$

$$\underline{z}'^m = z_j, z_{j+1}, \ldots, z_{j+r-1} \quad ; \quad j = mr - r + 1 .$$

When $g(\underline{z})$ is assumed to have this form, the minimization of the lower bound over all decoder set sizes is a minimization over all probability functions $g(\underline{z}')$ on Z^r. The conclusion that can be made from the bound derived using this assumption is that it is again only the value of the minimizing decoder set size function at $n = \infty$, $g_o(\underline{z}', \infty)$, that influences both the asymptote d_C and the term proportional to $\frac{1}{n}$. But, as we have already mentioned, at $n = \infty$ the minimizing decoder set size function $g_o(\underline{z}', \infty)$ factors into a product of single letter probabilities as in Equation 2.19. When this solution is substituted in the bound (using $r \geq 1$) both the asymptote and the term proportional to $\frac{1}{n}$ are the same for every choice of the constant r. Only lower order terms differ for different values of r.

The form of the lower bound that results when $g(\underline{z})$ is not assumed to have any independence property will be given in Appendix 1F. It will be seen to be greatly more complicated than the corresponding bound that will be derived (in the remainder of this chapter) using the independence property in Equation 2.19, particularly in the terms that will be shown to be unimportant in the latter bound. Using a recently derived property of that particular decoder set size function which minimizes the lower bound (this is that $g_o(\underline{z}_1)$ must equal $g_o(\underline{z}_2)$ when the compositions of $\underline{z}_1$ and $\underline{z}_2$ are equal - the proof is in Appendix 1E), we will show that some of the corresponding terms in the bound in Appendix 1F again cannot behave in a way that significantly affects the bound when n is large. But other terms remain whose behavior for large n cannot yet be established.

It is felt that the property of the minimizing decoder set size function $g_o(\underline{z})$ being only a function of the composition of $\underline{z}$ could be more fully exploited to strengthen the approximate lower bound found in this chapter. However, it is not clear at this point just how to use it, so we will in this thesis continue to use the approximation in Equation 2.19.

2.1.2 The Random Variables I_1 and I_2

From the assumption in Equation 2.6, and because the channel is memoryless, the Information Difference can be written as the sum of letter Information Differences in

$$I(\underline{x},\underline{y}) = \frac{1}{n}\sum_{m'=1}^{n} \ln \frac{f(y^{m'})}{P(y^{m'}|x^{m'})} \quad . \tag{2.22}$$

If the composition vector $\underline{c}$ of the channel input word $\underline{x}$ is equal to $c_1, c_2, \ldots, c_K$, that is, there are nc_1 appearances of the letter x_1 in $\underline{x}$, etc., and if we define by I_{km} the Information Difference between the m'th appearance of the letter x_k in $\underline{x}$ and the corresponding letter in $\underline{y}$, the Information Difference $I(\underline{x},\underline{y})$ can be rewritten as

$$I(\underline{x},\underline{y}) = \frac{1}{n}\sum_{k=1}^{K}\sum_{m=1}^{nc_k} I_{km} \tag{2.23}$$

We have in the previous sections considered two word Information Difference random variables, one, I_1, described by the probability function $f(\underline{y})$, and the other, I_2, described by $P(\underline{y}|\underline{x})$. We wish to cast I_1 and I_2 as sums of n independent random variables since this would again allow us to use Large Number Laws to estimate the distribution functions $F_1(I)$ and $F_2(I)$.

To form I_1 as such a sum, we associate with I_{km} the single letter probability function $f(y)$, which defines I_{km} as a random variable on Y having the distribution function

$$P_{1, I_{km}}\left[\ln \frac{f_\ell}{p_{k\ell}}\right] = f_\ell ; \qquad 1 \le m \le nc_k , \quad 1 \le k \le K . \tag{2.24}$$

In a similar way, I_2 can be made equal to a sum of n independent variables if the single letter probability function $P(y|x)$ is associated with I_{km}. The random variable I_{km} now has the distribution function

$$P_{2, I_{km}}\left[\ln \frac{f_\ell}{p_{k\ell}}\right] = p_{k\ell} ; \qquad 1 \le m \le nc_k , \quad 1 \le k \le K . \tag{2.25}$$

In the above equations we have used $p_{k\ell}$ for $P(y_\ell | x_k)$ and f_ℓ for $f(y_\ell)$.

2.1.3 Evaluation of the Lower Bound in Theorem 2.2

We now focus on the function d(I) in the integrand in Equation 2.18. This function has been implicitly defined by equating G(d) and $F_1(I^-)$, however, since these two distributions can only be approximated, d(I) in turn can also only be approximated. A safe approximation to d(I), that is, one that preserves the inequality of Equation 2.18, can be found by equating an upper bound to G(d) with a lower bound to $F_1(I^-)$. Figure 2.3 illustrates this construction. The result is another distortion function $d_L(I)$ that satisfies

$$d_L(I) \leq d(I) \tag{2.26}$$

which can be used in Equation 2.18 to obtain

$$d(\underset{\sim}{w}) \geq \int d_L(I)\ dF_2(I). \tag{2.27}$$

Because the random variable I_2 is a normalized sum of n independent random variables, its variance is proportional to $\frac{1}{n}$. It follows that when n becomes large, the distribution function $F_2(I)$ has almost all of its "rise" around the mean of I_2, which we will denote by $\bar{I}$. In this region — $I_1 \cong \bar{I}$ and $d \cong d(\bar{I})$ — the values of the two distributions G(d) and $F_1(I)$ are both exponentially small. Fortunately much work has been done to find bounds to the tails of distribution functions, but all the results[9,10,11] are parametric and allow only a parametric representation for $d_L(I)$.

Since it is easier to differentiate rather than integrate a quantity expressed by a set of parametric equations, $d_L(I)$ in Equation 2.27 is expanded in a Taylor Series at $I = \bar{I}$ that is truncated after three terms

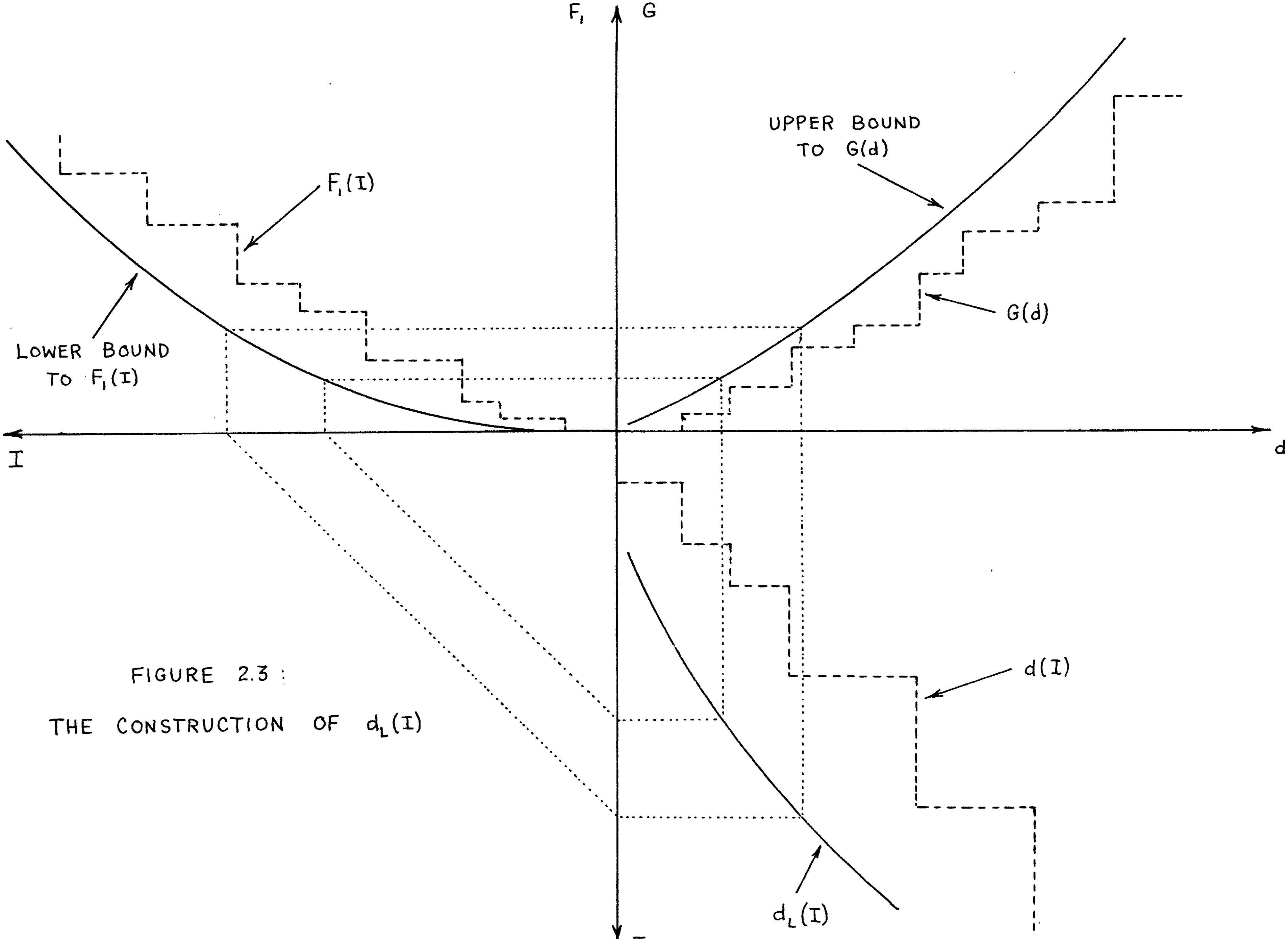

FIGURE 2.3 :

THE CONSTRUCTION OF $d_L(I)$

with a Lagrange Remainder term. Normally we could then write the integral in Equation 2.27 as a sum of central moments of I_2 with the Taylor Series derivatives as coefficients, however, in Appendix 1A we show that the expansion can be used to describe $d_L(I)$ only over a sub-interval $\left[I_a, I_b\right]$ of $\left[I_{min}, I_{max}\right]$. The form of answer that we want, involving the moments of I_2, can be restored if low order correction terms are included in the expression. The details are all in the appendix where we find the result in the next theorem.

Theorem 2.3

The single word transmission distortion $d(\underline{w})$ satisfies

$$d(\underline{w}) \geq d_L(\bar{I}) + \frac{1}{2} d_L''(\bar{I}) \operatorname{var}(I_2) + c_1(n) \tag{2.28}$$

where $c_1(n)$ is a correction term given in Equation A1.9 of Appendix 1A and is $o(\frac{1}{n})$. By comparison the variance of I_2 is proportional to $\frac{1}{n}$.

From this theorem we see that not only has the use of the Taylor Series expansion enabled us to evaluate a difficult integral, but it has also provided a natural way of separating the two important terms in the bound to the distortion $d(\underline{w})$. These terms, $d_L(I)$ and $d_L''(I)$, are evaluated in the next section.

2.1.4 The Function $d_L(I)$

The form of the function $d_L(I)$ depends upon which particular functions are used to bound $G(d)$ and $F_1(I)$. All of the bounds, though, are in terms of the semi-invariant moment generating functions of the random variables involved; therefore, some of the important properties are mentioned here.

For any random variable u_1 which takes the values $u_1(i)$ with probability $P_1(i)$, the function

$$\Phi_1(s) = \sum_i P_1(i)\, e^{s u_1(i)}$$

is defined as the moment-generating function of u_1, so named because successive derivatives evaluated at $s = 0$ provide the moments of u_1. The logarithm of this function

$$\phi_1(s) = \ln \Phi_1(s)$$

is defined as the semi-invariant moment generating function of u_1. The most important property of these functions concerns their form for a random variable that is the sum of two independent random variables. If u_2 is a second random variable, independent of u_1, that takes the values $u_2(i)$ with probability $P_2(i)$ and has a moment generating function $\Phi_2(s)$, then the moment generating function $\Phi(s)$ of $u = u_1 + u_2$ can be shown to be

$$\Phi(s) = \Phi_1(s)\,\Phi_2(s) ,$$

or the product of the moment generating functions of u_1 and u_2. And if

$$\phi_2(s) = \ln \Phi_2(s)$$

the semi-invariant moment generating function $\phi(s)$ of u is

$$\phi(s) = \ln \Phi(s) = \phi_1(s) + \phi_2(s)$$

or the sum of those for u_1 and u_2. By successive application of these equalities, it follows directly that the moment generating function $\Phi(s)$ of $u = u_1 + u_2 + \cdots + u_n$ is

$$\Phi(s) = \prod_{m=1}^{n} \Phi_m(s)$$

and that the semi-invariant moment generating function $\phi(s)$ of u is

$$\phi(s) = \sum_{m=1}^{n} \phi_m(s) \; . \tag{2.29}$$

The random variable nd in Equation 2.20 is the sum of n independent variables (because of the approximation made in Equation 2.19), thus we can use Equation 2.29 to write its semi-invariant moment generating function as

$$M(s) = n\mu(s)$$

$$= \sum_{i=1}^{H} n q_i \, \mu_i(s) \tag{2.30}$$

with

$$\mu_i(s) = \ln \sum_{j=1}^{J} g_j \, e^{s d_{ij}} \; . \tag{2.31}$$

Similarly, the random variable nI_1, which is equal to the sum of random variables given by Equations 2.23 and 2.24, has the semi-invariant moment generating function

$$\Gamma(t) = n\gamma(t)$$

$$= \sum_{k=1}^{K} nc_k \gamma_k(t) \tag{2.32}$$

with

$$\gamma_k(t) = \ln \sum_{\ell=1}^{L} f_\ell^{\,1+t} \, p_{k\ell}^{\,-t} . \tag{2.33}$$

To guarantee that $\gamma_k(t)$ is bounded, the vector $\underline{f}$ is restricted to those which contain only non-zero components. We show later that this does not affect the derived bound.

Returning to the general properties of generating functions, the first derivative of $\phi_1(s)$,

$$\phi_1'(s) = \sum_i u_1(i) \left[\frac{P_1(i)\, e^{su_1(i)}}{\sum_j P_1(j)\, e^{su_1(j)}} \right] \tag{2.34}$$

could be viewed as the expected value of a random variable u_{1s} that assumes the values $u_1(i)$ with the probability given by the bracketed term in Equation 2.34. The distribution of this variable in turn could be thought derived from that of u_1 by "tilting" $P_1(i)$ by the exponential factor $e^{su_1(i)}$ and renormalizing. This "tilting" process is important in all the derivations of estimates to the tails of cumulative distribution functions. Each derivation includes a step that tilts the distribution $P_1(i)$ of u_1 by an amount which sets the expected value of u_{1s} equal to a value u_o, where the estimate of the cumulative distribution of u_1 is desired. Then some estimate is made for the integral of probability of u_{1s} above (for the upper tail) or below (for the lower tail) the point u_o. The second derivative of $\phi_1(s)$, which will appear many times in the derivations to follow, can also be seen to be the variance of u_{1s}. It therefore is non-negative for all s.

We will first use the simple Chernov bound to upper bound G(d), which is the distribution of the normalized variable d. If we restrict d to satisfy

$$0 < d \leq E(d|\underline{q}),$$

the form of this bound is

$$G(d) \leq \exp n\left[\mu(s) - s\mu'(s)\right] \tag{2.35}$$

where

$$\mu'(s) = d.$$

The restriction on the values of d guarantees a solution for s that is finite and that satisfies $s \leq 0$.

For the lower bound to $F_1(I^-)$, the bound developed by Fano[13] is used:

$$F_1(I^-) \geq K(n,t) \exp n\left[\gamma(t) - t\gamma'(t)\right] \tag{2.36}$$

where

$$\gamma'(t) = I^- .$$

The value of I^- is restricted to $I_{min} < I^- \le E(I_1)$ which guarantees a finite non-positive solution for t. The coefficient $K(n,t)$ is given by Equation A1.11 in Appendix 1B.

The bounds in Equations 2.35 and 2.36 are equated

$$\exp n\left[\mu(s) - s\mu'(s)\right] = K(n,t) \exp n\left[\gamma(t) - t\gamma'(t)\right]$$

to obtain the following parametric representation for $d_L(I)$:

$$\mu(s) - s\mu'(s) = \gamma(t) - t\gamma'(t) + \frac{1}{n} \ln K(n,t) \tag{2.37a}$$

with

$$\mu'(s) = d \tag{2.37b}$$

$$\gamma'(t) = I^- . \tag{2.37c}$$

Since the lower bound to $F_1(I^-)$ is continuous in I, we can hereafter drop the "minus" in Equation 2.37c.

The quantities we must obtain from Equations 2.37abc are $d_L(\bar{I})$ and $d_L''(\bar{I})$. Successive differentiation of $d_L(I)$ can be greatly simplified if the coefficient $K(n,t)$ in Equation 2.36 is replaced by a function $K(n)$ that is independent of t and which satisfies

$$K(n,t) \geq K(n) \tag{2.38}$$

over the region of interest. One such function is found in Appendix 1B and is used in Equation 2.37a to obtain

$$\mu(s) - s\mu'(s) = \gamma(t) - t\gamma'(t) + \frac{1}{n} \ln K(n). \tag{2.37d}$$

It is important to note that the function $d_L(I)$, now given by Equations 2.37bcd, changes with n so that <u>each</u> term in the expression for $d(\underline{w})$ in Equation 2.28 is a function of n. At $n = n_o$, the value of d that corresponds to a value of I must be found by first solving for t in Equation 2.37c, then s from Equation 2.37d, and finally d from Equation 2.37b. When I assumes the particular value $\bar{I}$, the values of the other variables are labeled s_o, t_o, and of course $d_L(\bar{I})$.

We already have the first term of Equation 2.28. It is

$$d_L(\bar{I}) = \mu'(s_o). \tag{2.39}$$

To find the second term $d_L''(\bar{I})$, we apply the chain rule several times to differentiate $d_L(I)$:

$$\begin{aligned} d_L'(I) &= \frac{dd}{dI} = \frac{dd}{ds}\frac{ds}{dt}\frac{dt}{dI} \\ &= \mu''(s)\left[\frac{t\,\gamma''(t)}{s\,\mu''(s)}\right]\frac{1}{\gamma''(t)} \\ &= \frac{t}{s} \end{aligned} \tag{2.40}$$

and

$$d_L''(I) = \frac{\partial d'}{\partial s}\frac{ds}{dt}\frac{dt}{dI} + \frac{\partial d'}{\partial t}\frac{dt}{dI}$$

$$= -\frac{t}{s^2}\left[\frac{t\,\gamma''(t)}{s\,\mu''(s)}\right]\frac{1}{\gamma''(t)} + \frac{1}{s}\frac{1}{\gamma''(t)}$$

$$= \frac{1}{s}\left[\frac{1}{\gamma''(t)} - \frac{t^2}{s^2\mu''(s)}\right] . \qquad (2.41)$$

The last derivative must be evaluated at the point $I = \bar{I} = E(I_2)$, $s = s_o$, and $t = t_o$.

If $\gamma(t)$ in Equation 2.32 is differentiated and evaluated at $t = -1$, we have

$$\gamma'(-1) = \sum_{k=1}^{K} c_k \sum_{\ell=1}^{L} p_{k\ell} \ln \frac{f_\ell}{p_{k\ell}}$$

$$= E(I_2) = \bar{I} \qquad (2.42)$$

and can therefore conclude that

$$t_o = -1 . \qquad (2.43)$$

The function $\gamma(t)$ is itself zero at $t = -1$ since each $\gamma_k(-1)$ in Equation 2.33 is zero,

$$\gamma(-1) = 0 , \qquad (2.44)$$

and the second derivative $\gamma''(t)$ at $t = -1$ is

$$\gamma''(-1) = \sum_{k=1}^{K} c_k \left[\sum_{\ell=1}^{L} p_{k\ell} \left(\ln \frac{f_\ell}{p_{k\ell}} \right)^2 - \left(\sum_{\ell=1}^{L} p_{k\ell} \ln \frac{f_\ell}{p_{k\ell}} \right)^2 \right]$$

$$= \sum_{k=1}^{K} c_k \, \mu_2 (I_{km})$$

$$= n \ \mathrm{var}(I_2) \tag{2.45}$$

where the last equality is established using Equation A1.6.

We have now evaluated all the terms in Theorem 2.3, except the correction term $c_1(n)$ which we do in Appendix 1C. If we collect all the results (and remember that the approximation in Equation 2.19 has been made), we have the following lower bound to distortion.

Theorem 2.4a

The single word transmission distortion $d(\underline{w})$ satisfies

$$d(\underline{w}) \geq \mu'(s_0) - \frac{1}{2ns_0} \left[\frac{\gamma''(-1)}{s_0^2 \, \mu''(s_0)} - 1 \right] + c_1(n) \tag{2.46}$$

in which s_0 is given by

$$\mu(s_0) - s_0 \, \mu'(s_0) = \bar{I} + \frac{1}{n} \ln K(n) \, . \tag{2.47}$$

The correction term $c_1(n)$, which is $o(\frac{1}{n})$, is given by Equations A1.9 and A1.14 in Appendices 1A and 1C.

While the bound in Theorem 2.4a, including even the correction terms, is fairly easy to calculate, it is possible with some source-channel pairs, particularly for large n, for this bound (and the bound that results after we average over the source) to not be as tight as we would like. For example, when the source and channel are matched, or "nearly" matched, this bound could give a distortion below the already known lower bound d_C. The use of tighter strict upper and lower bounds to G(d) and $F_1(I)$, like those found by Shannon(10), would considerably improve upon this situation but with their use the correction terms $c_1(n)$ in Equation 2.46 would be all but unmanageable. We can, though, derive a lower bound to distortion that avoids both of these difficulties if one is willing to leave the correction terms unspecified, with only the assurance that they are of a lower order of magnitude than terms proportional to $\frac{1}{n}$. These correction terms, although unknown, would thus be guaranteed to contribute nothing significant to the lower bound expression when n becomes large. Of course, the region of applicability would in turn be limited to large values of n.

The development is precisely the same as that used to obtain the bound in Theorem 2.4a; we only use different bounds to G(d) and $F_1(I)$. These are the following bounds to distribution functions derived by Shannon(10) and Gallager(11):

$$G(d) \le \left[\frac{1}{\sqrt{2\pi n s^2 \mu''(s)}} + A_\upsilon(n,s)\right] \exp n\left[\mu(s) - s\mu'(s)\right] \qquad (2.48)$$

with

$$\mu'(s) = d, \qquad 0 < d \le E(d|\underline{q}),$$

and

$$F_1(I) \geq \left[\frac{1}{\sqrt{2\pi n t^2 \gamma''(t)}} + A_L(n,t) \right] \exp n\left[\gamma(t) - t\gamma'(t)\right] \qquad (2.49)$$

with

$$\gamma'(t) = I, \qquad I_{min} < I \leq E(I_1).$$

The functions $A_U(n,s)$ and $A_L(n,t)$ are both sums of rather difficult integrals but have been shown by Shannon and Gallager to be $o(\sqrt{\frac{1}{n}})$. Actually, these bounds strictly apply only when the variables d and I are non-lattice. For lattice variables the corresponding bounds[10,11] have in their coefficient a quantity Δ which does not change continuously with the argument of the distribution function (here d or I), hence they cannot be used within our derivation. One alternative, which we take, is to decrease one assigned letter distortion $d(w,z)$ by an arbitrarily small irrational number, and similarly, to change two transition probabilities on the channel in a way consistant with a lower bound to distortion. The new variables d' and I' would then be non-lattice.

We now follow the same sequence of steps used to derive Theorem 2.4a, including the continued use of the approximation in Equation 2.19. The bounds to $G(d)$ and $F_1(I)$ in Equations 2.48 and 2.49 are equated to establish the distortion function $d_L(I)$ which is differentiated, evaluated at $I = \bar{I}$, and substituted in Equation 2.28. The details are all in Appendix 1D where we find the result in the next theorem.

Theorem 2.4b

The single word transmission distortion $d(\underline{w})$ satisfies

$$d(\underline{w}) \geq \mu'(s_o) - \frac{1}{2ns_o}\left[\frac{\gamma''(-1)}{s_o^2\,\mu''(s_o)} - 1\right] + o\left(\frac{1}{n}\right) \qquad (2.50)$$

in which s_o is given by

$$\mu(s_o) - s_o\mu'(s_o) = \bar{I} - \frac{1}{2n}\ln\frac{\gamma''(-1)}{s_o^2\,\mu''(s_o)} + o\left(\frac{1}{n}\right). \qquad (2.51)$$

The only property of the source word $\underline{w}$ and the channel input word $\underline{x}$ that appears in Theorems 2.4a and b is their compositions; therefore, if two different source words with the same composition are transmitted with channel input words of the same composition, the same bound results for both distortions. This allows us to replace $d(\underline{w})$ with $d(\underline{q})$ in all the previous inequalities if the composition of $\underline{w}$ is $\underline{q}$. We will use this property in the next section when we average the single word transmission distortion $d(\underline{w})$ over the source W^n.

2.2 Lower Bound to Average Distortion

In the previous section a double sphere-packing technique was used to find a lower bound to distortion for a single transmission. In particular, the source message $\underline{w}$ was transmitted using the channel input word $\underline{x}$. The distortion that resulted was bounded by an expression that included a vector $\underline{f}$, which introduced a measure on the channel output space Y^n, and a vector $\underline{g}$, which specified the decoding set sizes in terms of this measure. Although it would not be inconsistant with a lower bound procedure to allow these vectors to change when the source message changes, the resulting bound would be very poor since changing the decoder structure for different transmitted words is clearly impossible in any actual decoder. Therefore, the vectors $\underline{f}$ and $\underline{g}$ are considered constants that are chosen only at the end of the development to optimize the result.

For the vector $\underline{c}$, which is also contained in the bound to $d(\underline{w})$, the situation is different. The encoder naturally is allowed full knowledge of the source events and could (and usually would) change the channel input word composition for different source outputs. However to simplify the following derivation we will impose a fixed composition constraint at the channel input. It will be shown later in the chapter that this constraint does not affect the asymptotic behavior of the lower bound.

2.2.1 Average over the Source

We now wish to bound the average distortion for the total source by averaging the lower bound to the single word distortion $d(\underline{w})$ over the source space W^n. Since the bound to $d(\underline{w})$ is only a function of the source word composition $\underline{q}$, we can do this by averaging the bound over the set of all possible compositions. It was noted earlier that $\underline{q}$ is a probability vector, therefore this set of compositions is contained in an H-1 dimensional hyperplane, termed the composition space Q^H, which is the "first quadrant" of R^H and intersects each axis q_i at one. Not all points in Q^H are possible word compositions for any particular n; for example with H = 2 and n = 2 there are only three possible compositions. But as n increases the points in Q^H that are source word compositions become quite dense.

The probability that any particular composition $\underline{q}$ occurs at the source output is

$$P_r(\underline{q}) = N(\underline{q}) \prod_{i=1}^{H} p_i^{nq_i} \tag{2.52}$$

in which $N(\underline{q})$ is the number of distinct source sequences with the composition $\underline{q}$ and the product is the probability of each. The number $N(\underline{q})$ is given by

$$N(\underline{q}) = \frac{n!}{\prod_{i=1}^{H} (nq_i)!} .$$

We can now write the average source distortion $d(\mathbb{S})$ as

$$d(\mathbb{S}) = \sum_{\substack{\text{ALL SOURCE} \\ \text{COMPOSITIONS}}} d(\underline{q})\, P(\underline{q}) . \tag{2.53}$$

The lower bound to $d(\mathbb{S})$ is found by using for $d(\underline{q})$ in Equation 2.53 the lower bound found in Theorem 2.4. Rather than write out the entire expression each time we want to use it, we let $d_L(\underline{q})$ denote the right side of Equation 2.46 (Equation 2.50 for the asymptotic bound). Therefore we have

$$d(S) \geq \sum_{\substack{\text{ALL SOURCE} \\ \text{COMPOSITIONS}}} d_L(\underline{q})\, P(\underline{q}) \;. \tag{2.54}$$

Viewed as a function over Q^H, $Pr(\underline{q})$ is a set of impulses. This allows us to consider the distortion function $d_L(\underline{q})$ a continuous function over all Q^H, rather than a function defined only at composition points, and to write

$$d(S) \geq \int \cdots \int_{Q^H} d_L(\underline{q})\, P(\underline{q})\, d\underline{q} \,. \tag{2.55}$$

Again because the expression for $d_L(\underline{q})$ in Equation 2.46 is parametric in s, we will use a Taylor Series expansion of the distortion function to evaluate the integral in Equation 2.55. This is done in Appendix 2A. The point chosen for the Taylor Series is $\underline{p}$, the probability vector characterizing the source. We choose this particular point for the following reason. If $\underline{q}$ is considered an auxiliary random variable (more precisely H <u>dependent</u> random variables) then the point $\underline{p}$ is actually the expected value of $\underline{q}$ as

$$\begin{aligned} E(\underline{q}) &\triangleq E(q_1, q_2, \ldots, q_H) \\ &= E(q_1), E(q_2), \ldots, E(q_H) \\ &= p_1, p_2, \ldots, p_H \\ &= \underline{p}. \end{aligned}$$

The Taylor Series expansion about $\underline{p}$ then contains terms like $(q_i - p_i)$, $(q_i - p_i)(q_j - p_j)$, etc. which, when averaged by $\Pr(\underline{q})$, are the central moments of the components of $\underline{q}$. Since these in turn are proportional to powers of $(\frac{1}{n})$, this procedure again separates the important terms from others that are of lower order. The low order terms are grouped by truncating the series after three terms with a Lagrange Remainder term. These steps are completed in Appendices 2A,B,C where the following lower bound is found (Equation A2.10):

$$d(S) \geq \mu'(s_o, \underline{p}) - \frac{1}{2ns_o}\left[\frac{\gamma''(-1)}{s_o^2 \mu''(s_o, \underline{p})} - 1 - s_o \sum_{ij} \mu'''_{sij}(s_o, \underline{p})\left(p_i \delta_{ij} - p_i p_j\right)\right] + c_2(n) \tag{2.56}$$

in which $c_2(n)$ is a correction term given by Equation A2.11 and s_o satisfies Equation 2.47

It remains now only to calculate the derivative $\mu'''_{sij}(s_o, \underline{p})$ which is the second partial derivative of $\mu'(s_o, \underline{q})$ with respect to the components q_i and q_j, evaluated at $\underline{q} = \underline{p}$. This is included here, rather than the appendix, since the result will be an important term in the final lower bound to average distortion. First we have

$$\mu'(s_o, \underline{q}) = \sum_i q_i \mu_i'(s_o) \, .$$

When the first partial derivative is taken with respect to q_i, it is intended that only q_k, $k \neq i$, and <u>not</u> s_o, are to be held constant, therefore we write

$$\left(\frac{\partial \mu'}{\partial q_i}\right)_{q_{k \neq i}} = \left(\frac{\partial \mu'}{\partial q_i}\right)_{\substack{q_{k \neq i} \\ s_o}} - \left(\frac{\partial \mu'}{\partial s_o}\right)_{\underline{q}} \left(\frac{\partial s_o}{\partial q_i}\right)_{q_{k \neq i}}$$

$$= \mu_i'(s_o) + \sum_j q_j \mu_j''(s_o)\left[\frac{\mu_i(s_o) - s_o \mu_i'(s_o)}{s_o \sum_k q_k \mu_k''(s_o)}\right]$$

$$= \frac{\mu_i(s_o)}{s_o} . \tag{2.57}$$

With another use of the chain rule we find the second derivative $\mu_{s ij}'''(s_o, \underline{p})$ or

$$\left.\frac{\partial^2 \mu'}{\partial q_i \partial q_j}\right|_{\underline{p}} = \left.\frac{\partial}{\partial s_o}\left[\frac{\mu_i(s_o)}{s_o}\right]\frac{\partial s_o}{\partial q_j}\right|_{\underline{p}}$$

$$= - \frac{\left[\mu_i(s_o) - s_o \mu_i'(s_o)\right]\left[\mu_j(s_o) - s_o \mu_j'(s_o)\right]}{s_o^3 \mu''(s_o, \underline{p})} \tag{2.58}$$

which we abbreviate by

$$\mu_{s ij}'''(s_o, \underline{p}) = - \frac{\theta_i \theta_j}{s_o^3 \mu''(s_o, \underline{p})} . \tag{2.59}$$

Because this expression factors into terms involving only i and terms involving only j, we can reduce the sum in Equation 2.56 to

$$- s_o \sum_{ij} \mu_{s ij}'''(s_o, \underline{p})\left[p_i \delta_{ij} - p_i p_j\right] = \frac{1}{s_o^2 \mu''(s_o, \underline{p})}\left[\sum_i p_i \theta_i^2 - \sum_{ij} p_i p_j \theta_i \theta_j\right]$$

$$= \frac{\mathrm{var}(\theta)}{s_o^2 \mu''(s_o, \underline{p})} \triangleq \frac{\sigma^2(\theta)}{s_o^2 \mu''(s_o, \underline{p})} \tag{2.60}$$

if θ is considered a random variable that takes the value $\theta_i = \mu_i(s_o) - s_o \mu_i'(s_o)$ with probability p_i. We now have the following result:

Theorem 2.5a

The average transmission distortion of the source $\mathcal{S}$, when used with the channel $\mathcal{C}$, satisfies

$$d(\mathcal{S}) \geq \mu'(s_0,\underline{p}) - \frac{1}{2ns_0}\left[\frac{\gamma''(-1)+\sigma^2(\theta)}{s_0^2\mu''(s_0,\underline{p})} - 1\right] + c_2(n) \tag{2.61}$$

in which s_o is given by

$$\mu(s_0,\underline{p}) - s_0\mu'(s_0,\underline{p}) = \bar{I} + \frac{1}{n}\ln K(n) \tag{2.62}$$

and $c_2(n)$ is a correction term provided by Equation A2.11 that is of order $(\frac{1}{n})^2$.

We see by comparing this result to that in Theorem 2.4a that the only significant difference is an additional term with a $\frac{1}{n}$ coefficient. The similarity was really expected since for high n "almost all" source outputs have a composition approximately $\underline{p}$. More precisely, for $\Delta > 0$

$$Pr\left(|\underline{q}-\underline{p}| > \Delta\right) \xrightarrow{n} 0$$

and therefore a source output with composition $\underline{p}$ becomes more and more typical. The $\sigma^2(\theta)$ term could be thought of as reflecting how fast the distortion for a $\underline{p}$ composition source word becomes the typical source distortion.

Next we wish to obtain a lower bound to the average transmission distortion of the source using the single word lower bound to distortion in Theorem 2.4b. Except for starting with $d_L(\underline{q})$ from Equation 2.50, the same sequence of steps used to establish the lower bound result in Equation A2.10 can be used to find the lower bound in Equation A2.14. The only differences are a different parametric equation for s_o and correction terms that are now unspecified, only being guaranteed to be $o(\frac{1}{n})$. The derivative $\mu'''_{sij}(s_o,\underline{p})$ is the same as calculated in Equation 2.59 except with an additional term of $o_n(1)$ which, when multiplied by the $\frac{1}{n}$ coefficient, has $o(\frac{1}{n})$ and is therefore grouped with the other correction terms.

Theorem 2.5b

The average transmission distortion of the source $\mathcal{S}$, when used with the channel $\mathcal{C}$, is lower bounded by

$$d(\mathcal{S}) \geq \mu'(s_o,\underline{p}) - \frac{1}{2ns_o}\left[\frac{\gamma''(-1)+\sigma^2(\theta)}{s_o^2\,\mu''(s_o,\underline{p})} - 1\right] + o\left(\frac{1}{n}\right) \qquad (2.63)$$

in which s_o is given by

$$\mu(s_o) - s_o\,\mu'(s_o) = \bar{I} - \frac{1}{2n}\ln\frac{\gamma''(-1)}{s_o^2\,\mu''(s_o,\underline{p})} + o\left(\frac{1}{n}\right). \qquad (2.64)$$

It should be mentioned that these results do not apply when $\gamma''(-1) = 0$, which is a situation that occurs when the channel $\mathcal{C}$ is noiseless, for the reason that we have divided by and cancelled factors equal to $\gamma''(-1)$. The result for the noiseless channel is derived in Section 2.4.

2.2.2 The Optimum Choice of $\underline{g}$, $\underline{f}$, $\underline{c}$

The bounds in Theorems 2.3, 2.4, and 2.5 include vectors $\underline{g}$, $\underline{f}$, and $\underline{c}$ that have not yet been specified. Using the approximation in Equation 2.19, the vector $\underline{g}$ has been used to specify the sizes of the decoding sets in the optimum decoder in terms of the measure $f(\underline{y})$ used on Y^n. This specification was introduced in Equation 2.7. As the decoding structure in the optimum decoder is unknown, we must use in the lower bound expressions that $\underline{g}$ which minimizes them. In general, the solution for the minimizing $\underline{g}$ would be different for different measures $f(\underline{y})$ that might be used.

Next, the measure $f(\underline{y})$ or, using the assumed product form in Equation 2.6, the vector $\underline{f}$ must be chosen. It has been noted earlier that the sphere-packing result in the very first lower bound, Theorem 2.1, is valid for any choice of $\underline{f}$. Therefore any convenient choice of $\underline{f}$ could be used, but, if we want the tightest bound, we should maximize the lower bound over all probability vectors $\underline{f}$ on Y.

Finally, to use the best composition for the channel input words, we minimize the lower bound over all composition vectors $\underline{c}$. Thus we have

Theorem 2.6

The average transmission distortion of the source $\mathcal{S}$, when used with the channel $\mathcal{C}$, is lower bounded by

$$d(\mathcal{S}) \geq \min_{\underline{c}} \max_{\underline{f}} \min_{\underline{g}} d_L(\mathcal{S}) \tag{2.65}$$

in which $d_L(\mathcal{S})$ is used to abbreviate the right side of Equation 2.61 if Theorem 2.5a is used or the right side of Equation 2.63 if Theorem 2.5b is used.

2.3 The Asymptote and the Rate of Approach to the Asymptote

2.3.1 The Asymptote

When n becomes large, the lower bound given by Theorem 2.5 has the limiting form

$$d_\infty(S) \geq \mu'(s_o, \underline{p}) \tag{2.66}$$

in which s_o satisfies

$$\mu(s_o, \underline{p}) - s_o \mu'(s_o, \underline{p}) = \bar{I} \tag{2.67}$$

and, from Equation 2.42,

$$\bar{I} = \sum_{k=1}^{K} c_k \sum_{\ell=1}^{L} p_{k\ell} \ln \frac{f_\ell}{p_{k\ell}} \quad . \tag{2.68}$$

The vectors $\underline{g}$, $\underline{c}$, and $\underline{f}$ that yield the extremum indicated in Theorem 2.6 must now be found.

We first minimize $d_\infty(S)$ over $\underline{g}$ for a fixed $\underline{f}$ and $\underline{c}$. Since the vectors $\underline{f}$ and $\underline{c}$ only enter in the expression for $\bar{I}$, we can minimize $d_\infty(S)$ for a constant $\bar{I}$. But this minimization,

$$d_\infty(S) \geq \min_{\underline{g}} \mu'(s_o, \underline{p})$$

with

$$\mu(s_o, \underline{p}) - s_o \mu'(s_o, \underline{p}) = \bar{I} ,$$

is precisely the expression(6) for the rate-distortion curve for S at the information rate $\bar{I}$. It is further shown in the same reference that the value of $\underline{g}$ which provides the minimization is the vector that describes the output statistics on the test channel for $\mathcal{S}$ at the point $(d_{\bar{I}},\bar{I})$ on the rate-distortion curve.

Next, we turn to the respective maximization and minimization of $d_\infty(\mathcal{S})$ with respect to $\underline{f}$ and $\underline{c}$. Because μ' increases monotonically with $\mu - s\mu'$ when $s < 0$, we can maximize $d_\infty(\mathcal{S})$ with respect to $\underline{f}$ by maximizing $\bar{I}$ in Equation 2.68. Lagrange multiplier theory and the known convexity of $\bar{I}$ provide the immediate solution:

$$f_\ell = \sum_{k=1}^{K} c_k p_{k\ell} \quad , \tag{2.69}$$

which when substituted in Equation 2.68 yields

$$\bar{I} = -\sum_{k=1}^{K} \sum_{\ell=1}^{L} c_k p_{k\ell} \ln \frac{p_{k\ell}}{\sum_m c_m p_{m\ell}} \quad . \tag{2.70}$$

We must now choose the probability vector $\underline{c}$ to minimize $\bar{I}$. But, from Equation 2.70, we see that this is the same as maximizing the mutual information on a channel by varying the input probability density, the result of which, by definition, is the capacity of the channel. Therefore the solution for $\underline{c}$ is that vector which allows the channel to be used to capacity, and the value of $\bar{I}$ at this point is $-C$.

When these results are substituted in Equations 2.66 and 2.67 we have for the asymptotic value at the lower bound in Theorem 2.5

$$d_\infty(S) \geq \mu'(s_0) = d_C \tag{2.71}$$

in which s_o satisfies

$$\mu(s_o) - s_o \mu'(s_o) = - C \tag{2.72}$$

and in which the semi-invariant moment generating function μ has $\underline{q}$ equal to the source output probability $\underline{p}$, and has $\underline{g}$ equal to the output probability on the test channel for $\mathcal{S}$ at R = C.

If we write out Equation 2.71 as

$$d_c = \mu'(s_o) = \sum_i p_i \sum_j \left[\frac{g_j e^{s_o d_{ij}}}{\sum_\ell g_\ell e^{s_o d_{i\ell}}} \right] d_{ij} \quad ,$$

the form suggests that we identify the bracketed expression as the test channel transition probability $q_i(j)$,

$$q_i(j) = \frac{g_j e^{s_o d_{ij}}}{\sum_\ell g_\ell e^{s_o d_{i\ell}}} \tag{2.73}$$

and write

$$d_c = \sum_i p_i \sum_j q_i(j)\, d_{ij} \quad . \tag{2.74}$$

Further, this identification is consistant with Equation 2.72 since we have

$$s_o d_{ij} = \ln \frac{q_i(j)}{g_j} + \ln \sum_\ell g_\ell e^{s_o d_{i\ell}}$$

which can be used in

$$-R_c = \mu'(s_0) - s_0\mu'(s_0)$$

$$= \sum_i p_i \left[\ln \sum_j g_j e^{s_0 d_{ij}} - \sum_j \left[\frac{g_j e^{s_0 d_{ij}}}{\sum_\ell g_\ell e^{s_0 d_{i\ell}}} \right] s_0 d_{ij} \right]$$

to obtain

$$R_c = \sum_i p_i \sum_j q_i(j) \ln \frac{q_i(j)}{g_j} \tag{2.75}$$

or, the information rate on the test channel. That the identity in Equation 2.73 is valid can be verified by comparing it to the solution one gets when the function R_C in Equation 2.75 is minimized with respect to the transition probabilities $q_i(j)$. Because the function is convex down, Lagrange multiplier theory can be used with the constraint Equation 2.74, and, of course the probability constraints $\sum_j q_i(j) = 1$ for each i. The solution for $q_i(j)$ that results is

$$q_i(j) = \eta_i g_j e^{\lambda d_{ij}}$$

in which the Lagrange multiplier η_i^{-1} can be found equal to the denominator of Equation 2.73, and in which the multiplier λ can be identified with s_o.

Equation 2.73 provides an interesting duality between the random variables d and I_1 and their "tilted" versions at $n = \infty$, d_s and I_{1t}. We see in Equation 2.24 that each variable I_{km} has, at the value of Information Difference $\ln(f_\ell/p_{k\ell})$, the probability f_ℓ which is the output probability on the channel $\mathcal{C}$ at ℓ. Tilting I_1 by $t = -1$ tilts each I_{km} by -1 and changes the probability at $\ln(f_\ell/p_{k\ell})$ to $p_{k\ell}$, the transition probability on $\mathcal{C}$ between k and ℓ. (Note that I_{1t} equals I_2 when $t = -1$.) The same thing happens when the variable d is tilted by $s = s_o$ (given by Equation 2.72). In Equation 2.21 we see that at the value of distortion d_{ij}, each variable D_{im} has a probability equal to g_j, the output probability on the test channel for $\mathcal{S}$ at j. Tilting d by s_o tilts each D_{im} by s_o which (after normalization) changes the probability at d_{ij} to $q_i(j)$ (given by Equation 2.73), or the transition probability between i and j on the test channel for $\mathcal{S}$.

2.3.2 The Rate of Approach to the Asymptote

In this section we will be able to simplify the lower bounds in Theorems 2.5b and 2.6 by showing that the variational problems in Equation 2.65 are not required. In addition, we will eliminate n from the parametric equation and therefore have the bound more directly exhibit its dependence upon n. And, we will be able to strengthen the bound slightly by dropping the fixed composition constraint at the channel input.

In this entire section, we use our freedom to choose $\underline{f}$ by setting this vector equal to its value at $n = \infty$, $\underline{f}(\infty)$, thus eliminating it as a variable in Theorem 2.6. We can write the bound in Theorem 2.5b in the general functional form:

$$d(\mathcal{S}) \geq d(n, s, \underline{g}, \underline{c}) = a_0(s, \underline{g}, \underline{c}) + \frac{1}{n} a_1(s, \underline{g}, \underline{c}) \qquad (2.76)$$

with s given implicity by

$$b_0(s,\underline{g},\underline{c}) + \frac{1}{n} b_1(s,\underline{g},\underline{c}) = \text{constant}. \tag{2.77}$$

The low order terms have been omitted because they will not affect the behavior of the bound for large n.

For each pair of vectors $\underline{g}$ and $\underline{c}$, we have in Equations 2.76 and 2.77 a different function of n. The result in Theorem 2.6 states that, at every n, those vectors $\underline{g}$ and $\underline{c}$ are chosen which minimize the distortion or, equivalently, Theorem 2.6 specifies the lower bound as the lower envelope to all the curves in Equations 2.76 and 2.77 corresponding to all choices for $\underline{g}$ and $\underline{c}$. We have seen in the previous section that at $n = \infty$ the optimum choice for $\underline{c}$ is the probability vector that uses the channel $\mathcal{C}$ to capacity. Gallager has shown that no component of this vector is zero. We shall now restrict our attention to sources for which the output probabilities on their test channels are also non-zero. Therefore, the minimizing equations for $\underline{g}$ and $\underline{c}$ at $n = n_0$ are

$$\left(\frac{\partial d_L(\mathcal{S})}{\partial g_j}\right)_{\substack{g_{k \neq j} \\ \underline{c},\, n}} + \lambda = 0 \quad ; \qquad 1 \leq j \leq J \tag{2.78}$$

$$\left(\frac{\partial d_L(\mathcal{S})}{\partial c_k}\right)_{\substack{c_{\ell \neq k} \\ \underline{g},\, n}} + \nu = 0 \quad ; \qquad 1 \leq k \leq K \tag{2.79}$$

in which we have used $d_L(\mathcal{S})$ to abbreviate the right side of Equation 2.76. Together with the constraint equations,

$$\sum_j g_j = 1 \quad , \quad \sum_k c_k = 1 \ ,$$

and the parametric equation in Equation 2.77, Equations 2.78 and 2.79 can be used to determine the minimizing vectors $\underline{g}_o(n)$ and $\underline{c}_o(n)$ which, when substituted in Equation 2.76, result in the single function of n

$$d(S) \geq d_L(S) = d\left(n, s(n), \underline{g}_o(n), \underline{c}_o(n)\right) \tag{2.80}$$

that is the desired lower envelope.

Because $\underline{g}$, $\underline{c}$ and even s are functions of n, it is difficult to see directly from the parametric form of the lower bound in Equations 2.76 and 2.77 how this bound behaves as a function of n. We do know, from Section 2.3.1 that it has the limit d_C. To find the rate of approach to d_C, we obtain the full derivative of $d_L(S)$ with respect to n, including all the implicit functions of n in $\underline{g}_o(n)$, $\underline{c}_o(n)$, and s(n). The chain rule is used to write

$$\frac{d d_L(S)}{dn} = \left(\frac{\partial d_L}{\partial n}\right)_{\substack{\underline{c},\underline{g},\\ s}} + \left(\frac{\partial d_L}{\partial s}\right)_{\substack{\underline{c},\underline{g},\\ n}} \frac{ds}{dn} + \sum_j \left(\frac{\partial d_L}{\partial g_j}\right)_{\substack{g_{k\neq j}\\ \underline{c},n,s}} \frac{dg_j}{dn} + \sum_k \left(\frac{\partial d_L}{\partial c_k}\right)_{\substack{c_{\ell\neq k}\\ \underline{g},n,s}} \frac{dc_k}{dn}$$

with

$$\frac{ds}{dn} = \left(\frac{\partial s}{\partial n}\right)_{\underline{g},\underline{c}} + \sum_j \left(\frac{\partial s}{\partial g_j}\right)_{\substack{g_{k\neq j}\\ \underline{c},n}} \frac{dg_j}{dn} + \sum_k \left(\frac{\partial s}{\partial c_k}\right)_{\substack{c_{\ell\neq k}\\ \underline{g},n}} \frac{dc_k}{dn} .$$

When these equations are combined, we have

$$\frac{dd_L(\underline{S})}{dn} = \left(\frac{\partial d_L}{\partial n}\right)_{\underline{c},\underline{g},s} + \left(\frac{\partial d_L}{\partial s}\right)_{\underline{c},\underline{g},n}\left(\frac{\partial s}{\partial n}\right)_{\underline{g},\underline{c}} + \sum_j \left[\left(\frac{\partial d_L}{\partial s}\right)_{\underline{c},\underline{g},n}\left(\frac{\partial s}{\partial g_j}\right)_{g_{k\neq j},\underline{c},n} + \left(\frac{\partial d_L}{\partial g_j}\right)_{g_{k\neq j},\underline{c},s,n}\right]\frac{dg_j}{dn}$$

$$+ \sum_k \left[\left(\frac{\partial d_L}{\partial s}\right)_{\underline{c},\underline{g},n}\left(\frac{\partial s}{\partial c_k}\right)_{c_{\ell\neq k},\underline{g},n} + \left(\frac{\partial d_L}{\partial c_k}\right)_{c_{\ell\neq k},\underline{g},n,s}\right]\frac{dc_k}{dn} .$$

Since the sum in the first set of brackets equals $\left(\frac{\partial d_L(\underline{S})}{\partial g_j}\right)_{g_{k\neq i},\underline{c},n}$ and that in the second set of brackets equals $\left(\frac{\partial d_L(\underline{S})}{\partial c_k}\right)_{c_{\ell\neq k},\underline{g},n}$, we may use Equations 2.78 and 2.79 to write

$$\frac{dd_L(\underline{S})}{dn} = \left(\frac{\partial d_L}{\partial n}\right)_{\underline{c},\underline{g},s} + \left(\frac{\partial d_L}{\partial s}\right)_{\underline{c},\underline{g},n}\left(\frac{\partial s}{\partial n}\right)_{\underline{g},\underline{c}} - \lambda\sum_j \frac{dg_j}{dn} - \nu\sum \frac{dc_k}{dn} . \quad (2.81)$$

Finally, since both $\underline{g}$ and $\underline{c}$ are probability vectors, the last two sums are equal to zero.

If we continue to use for $d_L(\underline{S})$ the functional forms of Equations 2.76 and 2.77, the non-zero factors in Equation 2.81 are

$$\left(\frac{\partial d_L}{\partial n}\right) = -\frac{a_1}{n^2}$$

$$\left(\frac{\partial d_L}{\partial s}\right) = \frac{\partial a_0}{\partial s} + \frac{1}{n}\frac{\partial a_1}{\partial s}$$

and

$$\left(\frac{\partial s}{\partial n}\right) = \frac{b_1}{n^2\left[\frac{\partial b_0}{\partial s} + \frac{1}{n}\frac{\partial b_1}{\partial s}\right]}$$

which can be used in Equation 2.81 to obtain

$$\frac{dd_L(S)}{dn} = -\frac{1}{n^2}\left[a_1 + b_1\frac{\partial a_0/\partial s}{\partial b_0/\partial s}\right] + o\left(\frac{1}{n^2}\right). \qquad (2.82)$$

Next we compare Theorem 2.5b to Equations 2.76 and 2.77 to make the identifications

$$a_1 = -\frac{1}{2s}\left[\frac{\gamma''+\sigma^2}{s^2\mu''} - 1\right]$$

$$b_1 = \frac{1}{2}\ln\frac{\gamma''}{s^2\mu''}$$

$$a_0 = \mu', \qquad \frac{\partial a_0}{\partial s} = \mu''$$

$$b_0 = \mu - s\mu', \qquad \frac{\partial b_0}{\partial s} = -s\mu''.$$

We use these in Equations 2.82 to find for the full derivative of $d_L(S)$:

$$\frac{dd_L(S)}{dn} = -\frac{1}{n^2}\frac{1}{2|s|}\left[\left(\frac{\gamma''}{s^2\mu''} - 1\right) - \ln\frac{\gamma''}{s^2\mu''} + \frac{\sigma^2}{s^2\mu''}\right] + o\left(\frac{1}{n^2}\right). \qquad (2.83)$$

At this point we still have the vectors $\underline{g}$ and $\underline{c}$ and the parameter s as the functions of n prescribed by the minimization Equations 2.78 and 2.79 and the parametric Equation 2.77. The vectors $\underline{g}_0(n)$ and $\underline{c}_0(n)$ approach, with increasing n, their limiting values $\underline{g}_0(\infty)$ and $\underline{c}_0(\infty)$; therefore the differences $\Delta\underline{g}_0(n) = \underline{g}_0(n) - \underline{g}_0(\infty)$ and $\Delta\underline{c}_0(n) = \underline{c}_0(n) - \underline{c}_0(\infty)$ are both functions of n that approach zero with increasing n. If we use $\underline{g}_0(n) = \underline{g}_0(\infty) + \Delta\underline{g}_0(n)$ and $\underline{c}_0(n) = \underline{c}_0(\infty) + \Delta\underline{c}_0(n)$ in Equation 2.83, and make an expansion to extract the terms involving $\Delta \underline{g}_0(n)$ and $\Delta \underline{c}_0(n)$, these terms would all become $o(\frac{1}{n^2})$ when multiplied by the $(\frac{1}{n})^2$ coefficient. They can therefore be included with the correction terms of this order. A similar argument can be made for the function $s(n)$. As a result, we can use in Equation 2.83 the limiting values $\underline{g}_0(\infty)$, $\underline{c}_0(\infty)$, and $s(\infty)$.

We can now see from Equation 2.83 that our lower bound $d_L(\mathcal{S})$ decreases monotonically (at least for n greater than some N) to approach its limit d_C from above. To prove this, we need only use the inequality

$$\ln w \leq w - 1$$

and the fact that σ^2 and μ'' are variances to show that the sum in the brackets is non-negative.

The function $d_L(\mathcal{S})$ can now be found as a function of n' by integrating $\frac{d d_L(S)}{dn}$ between n' and infinity. The result is

$$d_L(S) = d_c + \frac{1}{n'}\frac{1}{2|S|}\left[\left(\frac{\gamma''}{s^2\mu''} - 1\right) - \ln\frac{\gamma''}{s^2\mu''} + \frac{\sigma^2}{s^2\mu''}\right] + o\left(\frac{1}{n'}\right) \qquad (2.84)$$

Finally, we show in Appendix 3A that the result in Equation 2.84 is also valid when the fixed composition constraint at the channel input is dropped. That is, we can allow the composition $\underline{c}(\underline{q})$ of the channel input word to be a function of the composition $\underline{q}$ of the source word.

The previous results are collected in the following theorem.

Theorem 2.7

A lower bound to the minimum attainable distortion in a system that includes the source $\mathcal{S}$ and the channel $\mathcal{C}$ is given by

$$d(S) \geq d_c + \frac{1}{n}\frac{1}{2|s|}\left[\left(\frac{\gamma''}{s^2\mu''} - 1\right) - \ln\frac{\gamma''}{s^2\mu''} + \frac{\sigma^2}{s^2\mu''}\right] + o\left(\frac{1}{n}\right) \qquad (2.85)$$

in which

C = capacity of $\mathcal{C}$

d_C = the distortion at $R = C$ on the rate-distortion curve for $\mathcal{S}$

$\mu(s) = \sum_i q_i \ln \sum_j g_j e^{sd_{ij}}$

$\gamma(t) = \sum_k c_k \ln \sum_\ell f_\ell^{1+t} p_{k\ell}^{-t}$

$\underline{q}$ = $\underline{p}$, the source output probabilities

$\underline{g}$ = the output probability on the test channel for $\mathcal{S}$ at (d_C, C)

$\underline{c},\underline{f}$ = the input and output probabilities on $\mathcal{C}$ when it is used to capacity

t = -1, and

s satisfies: $\mu - s\mu' = -C$.

The lower bound in Theorem 2.7 approaches its limit algebraically as $\frac{a}{n}$. The coefficient a is never negative, but can in special cases be zero. In general, though, it is a positive constant determined by the statistics of both the source $\mathcal{S}$ and channel $\mathcal{C}$. The lower bound suggests that the larger the value of a, the longer the coding block length must be to obtain a tolerable level of distortion $d_C + \Delta$. In turn, the more complex the modulator and demodulator must become. This further suggests that the coefficient $a(\mathcal{S},\mathcal{C})$ might be used as a measure of "mismatch" between the source $\mathcal{S}$ and channel $\mathcal{C}$. Figure 2.4 illustrates the performance curves for several pairs $(\mathcal{S},\mathcal{C})$ which have the same minimum distortion d_C, but which have different values of the coefficient a.

An interesting situation occurs when a choice exists between using channel $\mathcal{C}_1$ or channel $\mathcal{C}_2$ to transmit the source $\mathcal{S}$. Suppose the capacities C_1 and C_2 satisfy $C_1 > C_2$. It follows, by the monotonicity of the rate-distortion curve for $\mathcal{S}$, that $d_{C_1} < d_{C_2}$. However, the choice of channel $\mathcal{C}_1$ is still not obvious since it is possible that $\mathcal{S}$ is more nearly matched to $\mathcal{C}_2$ than $\mathcal{C}_1$. That is, the coefficient $a(\mathcal{S},\mathcal{C}_2)$ might be less than $a(\mathcal{S},\mathcal{C}_1)$. Further, we will show in Chapter 4 that the difference in coefficients can be quite large. Therefore, the lower bounds to the two performance curves could look like those shown in Figure 2.5 suggesting that the lower capacity channel $\mathcal{C}_2$ might be the better choice, at least for block lengths less than N.

It is also possible for a source $\mathcal{S}$ and a channel $\mathcal{C}$ to have a performance curve which falls to its final value d_C at $n = 1$. This is a situation in which $\mathcal{S}$ and $\mathcal{C}$ are said to be matched, and for such a case it necessarily follows that

$$a(\mathcal{S},\mathcal{C}) = 0 \qquad (2.86)$$

Strictly, Equation 2.86 is not a sufficient condition for matching since one would have to show that the correction terms are all zero as well. Rather than do this, it is easier in any particular example that has $a(\mathcal{S},\mathcal{C}) = 0$, to try and find the particular encoder and decoder that provide the distortion d_C.

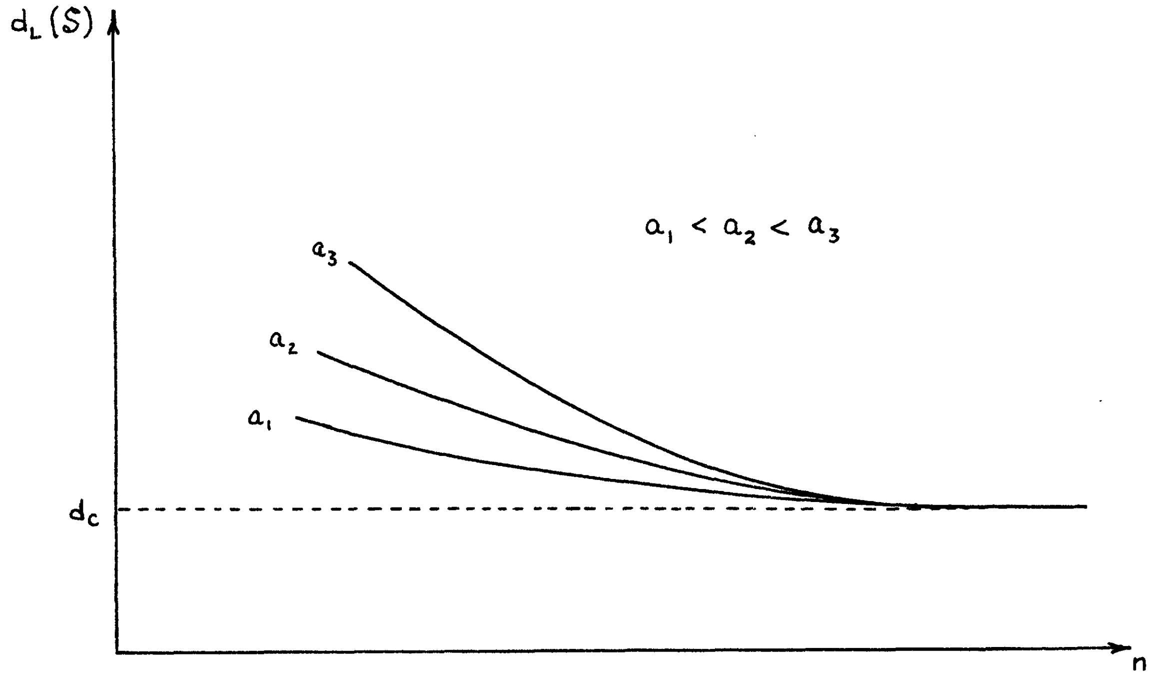

FIGURE 2.4: THE LOWER BOUND TO DISTORTION FOR PAIRS $\mathcal{S}_i, \mathcal{F}_i$ THAT HAVE DIFFERENT MISMATCH CONSTANTS — a_i

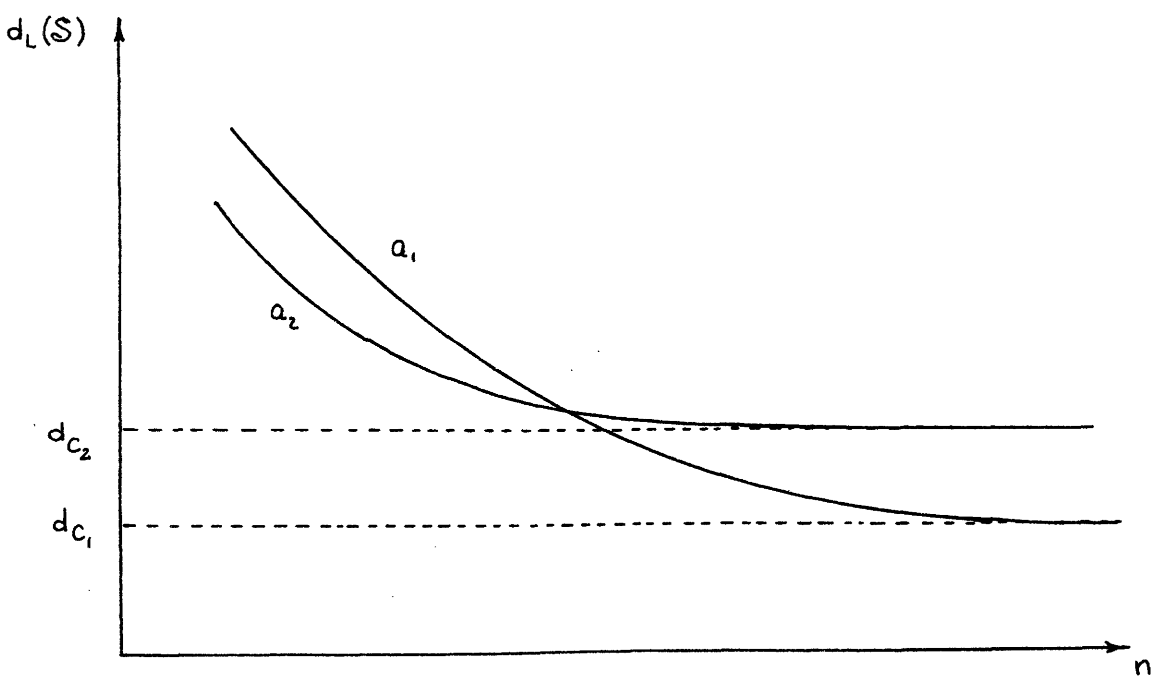

FIGURE 2.5: THE LOWER BOUND TO DISTORTION FOR $\mathcal{S}, \mathcal{F}_1$ AND $\mathcal{S}, \mathcal{F}_2$ WHEN $d_{c_1} < d_{c_2}$ BUT $a_1 > a_2$

If precise matching does exist, these should be easy to find since the distortion d_C must be obtainable for all n, even n = 1. It is not, for example, possible for $\mathcal{S}$ and ζ to be matched only for n > N.

For $a(\mathcal{S},\zeta)$ to be zero, we see from Theorem 2.7, that it is in turn necessary that

$$\sigma^2 = 0$$

and that

$$s^2\mu'' = \gamma''.$$

The first equality holds whenever the source is doubly-uniform, but it is not known that this is a necessary condition. The second equality reflects the requirement that $d_L''(\bar{I})$ in Equation 2.28 be zero. Although not a guarantee for matching, whenever these two equations are both true, the evidence in favor of possible matching must be considered very strong.

We can state one sufficient condition for the lower bound, including all the correction terms, to be independent of n. If the random variables d and I_1 are equal, except possibly for additive and multiplicative constants, that is, if

$$d = k_1 I_1 + k_2$$

we then would have

$$\begin{aligned} G(d_o) &= Pr\,(d < d_o) \\ &= Pr\left(I_1 < \frac{d_o - k_2}{k_1}\right) \\ &= F_1\left(I = \frac{d_o - k_2}{K_1}\right) \end{aligned} ,$$

with the result that the distortion function d(I) would be linear, and Equation 2.18 would be independent of n.

In Chapter 4 we will do several examples illustrating different types of mismatch and their effect on $a(\mathcal{S},\mathcal{C})$.

2.4 Special Cases

Doubly-Uniform Channel

A doubly-uniform channel is defined as one for which each row in the transition probability matrix $\left[p_{k\ell}\right]$ is a permutation of every other row and at the same time each column is a permutation of every other column. For such a channel, if we use for $\underline{f}$, as we are free to do, a uniform vector over Y, then $\gamma_k(t)$ in Equation 2.33 becomes

$$\gamma_k(t) = \ln \left(\frac{1}{L}\right)^{1+t} \sum_{\ell=1}^{L} p_{k\ell}^{\ -t}$$

in which the sum is independent of k. We then have for $\gamma(t)$ in Equation 2.32

$$\gamma(t) = \gamma_k(t)$$

which no longer includes the vector $\underline{c}$. Consequently $\underline{c}$ does not appear in any lower bound expression and the minimization over $\underline{c}$ as well as the maximization over $\underline{f}$ in Theorem 2.6 can be omitted.

Doubly-Uniform Source

We define a source as doubly-uniform if the probability vector $\underline{p}$ which characterizes its output is uniform over W and if, in addition, in the distortion matrix $\left[d_{ij}\right]$ each row is a permutation of every other row and each column is a permutation of every other column. For such a source, it can be shown that the vector $\underline{g}$ which satisfies Equation 2.65 is uniform on Z for all n. It can further be shown that even when the approximation in Equation 2.19 is not made (the form of the resulting lower bound is discussed in Appendix 1F), the optimum choice for the probability function $g(\underline{z})$ on Z^n is also uniform. Thus the function $g(\underline{z})$ is factorable into the product form in Equation 2.19 anyway and this equation can be used without approximation.

The double-uniformity property also allows us to use

$$\mu(s) = \mu_i(s)$$

in Theorems 2.5 and 2.6 since

$$\mu_i(s) = \ln\left(\frac{1}{J}\right)\sum_{j=1}^{J} e^{sd_{ij}}$$

is independent of i. From this we conclude that $\theta_i = \mu_i(s_o) - s_o\mu_i'(s_o)$ is also independent of i and that

$$\sigma^2(\theta) = 0 \tag{2.87}$$

in these theorems.

Product Sources and Channels

A source $\mathcal{S}$ that contains a number of separate independent component sources is usually termed a vector or product source. The component sources do not have to be identical, but we will assume that the output events of each have the same time rate. An output event, or output "letter", of $\mathcal{S}$ is now really a vector that has for its components the output letter of each component source of $\mathcal{S}$. Each source has defined between it and its decoding space a distortion matrix, and the total source distortion is defined here as the sum of the distortions of the component sources.

Similarly a channel $\mathcal{C}$ which consists of a number of separate independent channels is said to be a product channel. Usually such a channel is used to model a more complicated noise process on an actual channel. Again we allow the component channels to be non-identical but restrict the time rate of events on them to be the same. The channel input and output "letters" are also vectors having as components the input and output letters of the channels composing $\mathcal{C}$. We can apply all the results of this chapter to any transmission system that includes a product source and channel if we make the following changes in notation.

The distortion random variable d is the sum of the distortion variables for each component of the source

$$d = d^{(1)} + d^{(2)} + \cdots + d^{(A)}$$

in which $d^{(a)}$ can be written as the sum (see Equation 2.20)

$$d^{(a)} = \frac{1}{n} \sum_{i=1}^{H^{(a)}} \sum_{m=1}^{nq_i^{(a)}} D_{im}^{(a)} \quad .$$

In this sum, all the quantities have been previously defined except for the superscript which is used to indicate the a'th source. With the same superscript notation, the random variables I_1 and I_2 are written

$$I_i = I_i^{(1)} + I_i^{(2)} + \cdots + I_i^{(B)} , \qquad i = 1,2$$

with

$$I_i^{(b)} = \frac{1}{n} \sum_{k=1}^{K^{(b)}} \sum_{m=1}^{nc_k^{(b)}} I_{km}^{(b)} .$$

If $\mu(s)$ and $\gamma(t)$ remain defined as the semi-invariant moment generating functions of d and I_1, they are now equal to

$$\mu(s) = \sum_{a=1}^{A} \mu^{(a)}(s) = \sum_{a=1}^{A} \sum_{i=1}^{H^{(a)}} q_i^{(a)} \mu_i^{(a)}(s)$$

and

$$\gamma(t) = \sum_{b=1}^{B} \gamma^{(b)}(t) = \sum_{b=1}^{B} \sum_{k=1}^{K^{(b)}} c_k^{(b)} \gamma_k^{(b)}(t)$$

and can be substituted directly in Theorems 2.4 through 2.6.

2.5 The Noiseless Channel

A noiseless channel of capacity C has $L = e^C$ noiseless paths, or a transition probability matrix equal to an $e^C \times e^C$ identity matrix. Therefore, available to the modulator that encodes n-letter words from the source is a channel which has as its transition matrix an $e^{nC} \times e^{nC}$ identity matrix. It has been noted earlier that direct application of Theorems 2.5a and 2.5b is not permissable since the variance of I_2 in Equation 2.28 was used to cancel a $\gamma''(-1)$ factor in $d_L''(I)$ and, for a noiseless channel, these factors are equal to zero. Thus a separate development is required.

We return to the lower bound given by Equation 2.11 in Theorem 2.1. If the vector $\underline{f}$ is chosen uniform over Y, we see from the definition of a noiseless channel (L^n outputs) and the definition of Information Difference in Equation 2.8 that $I(\underline{x},\underline{y})$ is equal to $\ln(\frac{1}{L})$ for the output $\underline{y}_1$ on list 2' that has $P(\underline{y}_1 | \underline{x}) = 1$, and is infinite for all other outputs. Since $f(\underline{y}_1) = (\frac{1}{L})^n$, $e^{-nI(F_o)}$ is non-zero only for $0 \le F_o \le (\frac{1}{L})^n$ where it is equal to L^n. Therefore, we can write Equation 2.11 as

$$d(\underline{w}) \ge \int_0^1 d(F_o)\, e^{-nI(F_o)}\, dF_o$$

$$= L^n \int_0^{L^{-n}} d(F_o)\, dF_o \,. \tag{2.88}$$

Further bounding of the integral in Equation 2.88 is a little involved and is left to Appendix 3B where we find the result in Equations A3.9 and A3.10:

$$d(\underline{w}) \ge \mu'(s_a) - \frac{D}{|s_a|\, n}\left(1 + o_n(1)\right) \; ; \quad D \cong 1 \tag{2.89}$$

with s_a given by

$$\mu(s_a) - s_a \mu'(s_a) + \frac{1}{n} \ln \sqrt{n} \left(1 + o_n(1)\right) = -C. \qquad (2.90)$$

The remaining steps, averaging over the source, and minimizing the resulting bound over all choices of the vector $\underline{g}$ (the optimization over $\underline{c}$ and $\underline{f}$ is not necessary - see Section 2.4), are identical in procedure to those employed previously. We state only the result.

Theorem 2.8

The minimum attainable transmission distortion of the source $\mathcal{S}$, when used with a noiseless channel of capacity C, satisfies

$$d(\mathcal{S}) \geq d_c + \frac{1}{2} \frac{\ln n}{|s_o| n} \left(1 + o_n(1)\right) \qquad (2.91)$$

in which s_o is given by

$$\mu(s_o, \underline{p}) - s_o \mu'(s_o, \underline{p}) = -C. \qquad (2.92)$$

We see by comparing Equations 2.85 and 2.91 that while the lower bound to distortion with a noisy channel approaches its asymptote, d_C, as $\frac{1}{n}$, the lower bound to distortion with a noiseless channel approaches d_C only as $\frac{\ln n}{n}$. These bounds are not inconsistant since for a noiseless channel the variance γ'' is zero and therefore the coefficient of $\frac{1}{n}$ in Equation 2.85 is infinite. A similar limiting statement is also true. If a noisy channel is made to approach a noiseless one by reducing the noisy transition probabilities toward zero, and at the same time keeping the channel capacity constant by appropriately reducing either the channel input alphabet size or the channel dimensionality (see Example 4.1), the coefficient of the $\frac{1}{n}$ term increases and is unbounded. These results therefore suggest that when a choice exists between using either a noiseless channel or a noisy one <u>of equal capacity</u>, the noisy channel is always the better choice. And, in as much as we are using the coefficient of the $\frac{1}{n}$ term to measure source-channel mismatch, the noiseless channel represents the worst possible match to any source.

Chapter 3

UPPER BOUND TO AVERAGE DISTORTION

In this chapter we derive an upper bound to the average distortion that must exist when a coding block length n is used to transmit the source $\mathcal{S}$ over the channel $\mathcal{C}$. The upper bound that we find also approaches the limit d_C algebraically from above with increasing block length, but, for noisy channels, the rate of approach to d_C is only as $\sqrt{\frac{\ln n}{n}}$ compared with the $\frac{1}{n}$ rate of approach in the lower bound. We discuss in the chapter what we believe to be the reason for this difference, which is that the number of signal points was restricted to be less than e^{nC}. Some problems encountered in enlarging the signal set are also mentioned. An upper bound to the transmission distortion with a noiseless channel is also derived, and this does agree, asymptotically, with the corresponding lower bound in Chapter 2.

3.1 Upper Bound to Average Distortion

The derivations in this chapter all employ random coding arguments. That is, we shall not explicitly find the encoder and decoder which, when used with $\mathcal{S}$ and $\mathcal{C}$, provide the distortion in the upper bound, but will show that one pair does exist. More specifically, we construct a set of encoder-decoder pairs with a probabilistic rule according to which each system is selected to be used. This defines an ensemble of transmission systems, each with its own distortion, corresponding to all possible coding selections. What we calculate is a bound to the average distortion of this ensemble. Clearly, this provides an upper bound to the minimum distortion in the ensemble, and hence to the minimum attainable distortion in any system that includes $\mathcal{S}$ and $\mathcal{C}$.

3.1.1 The Ensemble, the Encoding and Decoding Method

We denote the set of points on the rate distortion curve for $\mathcal{S}$ by (d_R, R) and assume the capacity of $\mathcal{C}$ to be C. Shannon(2) has already proved that the transmission distortion can be made arbitrarily close to d_C if infinite coding block lengths are allowed; consequently, we insist that our upper bound have the limit d_C with increasing block length. Anything higher would be a poorer result than Shannon's. We cannot, however, immediately start with a source representation that has an information content R precisely equal to the information transmission capability of the system C since the random coding arguments used here only work in situations in which there is a strict inequality between these two quantities. This is characteristic of random coding arguments used for channel coding problems as well which require an inequality between rate and capacity in order to exhibit a non-zero exponent in the probability of error expression.

What we do, therefore, is to choose any point (d^*, R^*) on the rate-distortion curve below (d_C, C) and to code in such a way that the distortion approaches the level d^* with increasing block length. We know this to be possible from Shannon's results, moreover we expect, since the situation is somewhat analogous to a channel coding problem with $R^* < C$, that the distortion can be made to approach d^* exponentially fast. The point (d^*, R^*) is subsequently varied to obtain the best result at any particular block length of interest.

For any selection of (d^*,R^*), we then choose the number of signal points $M = e^{nR}$ used to transmit the source $\mathcal{S}$. To attain a transmission distortion level d^*, we certainly must have the number of signal points large enough to represent the source to at least within d^*, and this requires that R be greater than R^*. We also require that R be less than C to guarantee correct decoding, among the signal points, at the receiver. Thus, we have

$$R^* < R < C \tag{3.1}$$

and, for the corresponding distortions on the rate-distortion curve of $\mathcal{S}$,

$$d_{max} \geq d^* > d_R > d_C \quad . \tag{3.2}$$

The value of R can also later be chosen to optimize the result.

An ensemble of codes of length n is constructed for each selection of R and R^*. We use the probability distribution $p(\underline{x},\underline{z})$ to generate the ensemble by picking, according to $p(\underline{x},\underline{z})$, M independent pairs $(\underline{x},\underline{z})$ from $X^n Z^n$. Thus we have a set of codes containing all possible mappings of the integers 1 through M into pairs of n-letter words $(\underline{x},\underline{z})$, or $(JK)^{nM}$ codes in total, each with the associated probability

$$P_r(\text{code}) = \prod_{i=1}^{M} p(\underline{x}_i, \underline{z}_i) \tag{3.3}$$

Any probability function $p(\underline{x},\underline{z})$ could be used to obtain an upper bound, but we shall use a distribution that factors into $p(\underline{x})\ g(\underline{z})$; therefore, in the ensemble, each set Θ_1 of M decoded words is independent of each set Θ_2 of M channel input words. Equation 3.3 can thus be rewritten as

$$Pr(\text{code}) = P(\theta_1,\theta_2) = P(\theta_1)P(\theta_2) = \prod_{i=1}^{M} p(\underline{x}_i)\prod_{i=1}^{M} g(\underline{z}_i) \tag{3.4}$$

In particular, we shall use for $p(\underline{x})$ and $g(\underline{z})$ the product forms $\prod_{m=1}^{n} p(x^m)$ and $\prod_{m=1}^{n} g(z^m)$ in which the letter probability distribution $p(x)$ is that which yields a mutual information C on $\mathcal{C}$ and the letter probability distribution $g(z)$ is that which gives the output statistics on the test channel for $\mathcal{S}$ at the point (d^*,R^*) on the rate-distortion curve.

The encoding and decoding is done as follows: In every ensemble member there is a list θ_1 of allowed decoded words and a list θ_2 of useable channel input words. When a source output $\underline{w}$ occurs, the encoder scans θ_1 and chooses <u>any</u> member $\underline{z}_o$ in this list for which

$$d(\underline{w},\underline{z}_o) \leq d^* . \tag{3.5}$$

If there are none, the encoder chooses any member at all on the list θ_1, say $\underline{z}_1$. Since the lists are chosen together, there corresponds to $\underline{z}_o$ or $\underline{z}_1$ a particular $\underline{x}$ in θ_2, and this word is used to transmit $\underline{w}$. The decoder uses a maximum likelihood decision rule to decode $\underline{y}$ into a member of θ_2, which is then associated, through the pairings among the two lists, with a member $\underline{z}$ in θ_1. The resulting distortion, by definition, is $d(\underline{w},\underline{z})$.

We observe that neither the source encoder, which is a threshhold device, nor the channel decoder, which does not use the probabilistic information induced on θ_2 by $p(\underline{w})$ through θ_1 to make a minimum cost decision, is an optimum device. These matters are brought up for examination later in the chapter, after the upper bound is further developed.

3.1.2 The Ensemble Average Distortion

Each ensemble member θ is a complete transmission system in itself and has an average distortion that is a function of the particular codes, θ_1 and θ_2, that are used. This average distortion, which is an average over all possible source and channel events, is equal to

$$d(\theta) = d(\theta_1,\theta_2) = \sum_{W^n} P(\underline{w}) \sum_{Y^n} P(\underline{y}|\underline{x})\, d(\underline{w},\underline{z})\,. \qquad (3.6)$$

In Equation 3.6, we remember that $\underline{x}$ is a function of $\underline{w}$, and $\underline{z}$ is a function of $\underline{y}$ through the codes θ_1 and θ_2.

The ensemble average distortion is the average of $d(\theta_1,\theta_2)$ using the probability distribution in Equation 3.4,

$$\overline{d(\theta)} = \sum_{\theta_1}\sum_{\theta_2} d(\theta_1,\theta_2)\, P(\theta_1) P(\theta_2)$$

$$= \sum_{\theta_1}\sum_{\theta_2}\sum_{W^n} P(\underline{w}) \sum_{Y^n} P(\underline{y}|\underline{x})\, d(\underline{w},\underline{z})\, P(\theta_1) P(\theta_2)\,. \qquad (3.7)$$

This summation is more easily evaluated if the ensemble averaging process is interchanged with that over the source and noise events; therefore we write Equation 3.7 as

$$\overline{d(\theta)} = \sum_{W^n} P(\underline{w}) \sum_{Y^n} \left[\sum_{\theta_1}\sum_{\theta_2} P(\underline{y}|\underline{x})\, d(\underline{w},\underline{z})\, P(\theta_1) P(\theta_2) \right]. \qquad (3.8)$$

Next we separate the events $\underline{w}$, θ_1, θ_2, $\underline{y}$ into two sets: (1) those quadruples for which <u>either</u> there does not exist a $\underline{z}$ in θ_1 satisfying

Equation 3.5 or the received word $\underline{y}$ is decoded into a member of θ_2 different from the transmitted word $\underline{x}(\underline{w})$, and (2) its complement. For quadruples in set one, the distortion $d(\underline{w},\underline{z})$ is surely upper bounded by d_{max}. For those in the second set, we use Equation 3.5 and the fact that the decoder returns us through $\underline{x}(\underline{w})$ to $\underline{z}_0$ to upper bound the distortion by d^*. Therefore, if the characteristic function Φ is used to indicate the quadruples in set one, we can upper bound the ensemble average with

$$\overline{d(\theta)} \leq \sum_{W^n} P(\underline{w}) \sum_{Y^n} \sum_{\theta_1} \sum_{\theta_2} P(\underline{y}|\underline{x})\, P(\theta_1) P(\theta_2) \left[d^*(1-\Phi) + d_{max}\, \Phi \right]$$

$$= d^* + (d_{max} - d^*) \sum_{W^n} P(\underline{w}) \sum_{Y^n} \sum_{\theta_1} \sum_{\theta_2} P(\underline{y}|\underline{x})\, P(\theta_1)\, P(\theta_2)\, \Phi$$

$$= d^* + (d_{max} - d^*)\ Pr(\Phi)\,. \tag{3.9}$$

Finally, we use the union bound to upper bound $Pr(\Phi)$, and to write for the ensemble average distortion

$$\overline{d(\theta)} \leq d^* + (d_{max} - d^*) \left[Pr(\nexists\ \underline{z}_0 \text{ in } \theta_1) + Pr(\text{channel error}) \right] \tag{3.10}$$

in which the symbol $\exists'$ is used for "there does not exist".

Since the ensemble average $\overline{d(\theta)}$ can be used to upper bound the minimum attainable transmission distortion $d(\mathcal{S})$, we have the basic form of our upper bound.

Theorem 3.1

The minimum attainable transmission distortion that can exist when the source $\mathcal{S}$ is used with the channel $\mathcal{C}$ satisfies

$$d(\mathcal{S}) \leq d^* + (d_{max} - d^*) \left[Pr(\exists' \underline{z}_0 \text{ in } \theta_1) + Pr(\text{channel error}) \right] \tag{3.11}$$

in which d^* is any distortion level greater than d_C, and R (a variable in the bracketed terms) is any rate in the interval $R^* < R < C$. The bound is a function of n through the quantity in the brackets.

The next section is devoted to the evaluation of the first term in the brackets in Equation 3.11, which is the probability that the source word $\underline{w}$ and the list θ_1 are such that Equation 3.5 is not satisfied by any $\underline{z}$ in θ_1. We will show this term to be a decreasing exponential in n, with a positive exponent that monotonically increases with the difference $R - R^*$. This is analogous to the second term in the brackets, which is the probability of error on the channel $\mathcal{C}$. This quantity has been approximated by many people, but we will use Gallager's bound(14)

$$P_r(e) \leq e^{-nE(R)} \tag{3.12}$$

in which E(R) is a positive monotonically increasing function of the difference C - R.

3.1.3 The Probability of Failure at the Encoder

We say that failure occurs at the encoder, for the source output $\underline{w}$, when each of the M allowed decoded words on list θ_1 are at a distortion $d(\underline{w},\underline{z})$ from $\underline{w}$ greater than d^*. The total probability of this event is the first term in the brackets in Equation 3.11 which, because each of the M words in θ_1 is selected independently, can be written as

$$\Pr(\exists' \underline{z}_o \text{ in } \theta_1) = \sum_{W^n} P(\underline{w}) \Pr(\exists' \underline{z}_o \text{ in } \theta_1 | \underline{w})$$

$$= \sum_{W^n} P(\underline{w}) \left[1 - \Pr_{g(\underline{z})}\left(\underline{z} \ni d(\underline{w},\underline{z}) \le d^* | \underline{w}\right)\right]^M . \qquad (3.13)$$

The last probability is equal to the same cumulative distribution function $G(d^*)$ used in Chapter 2 for the random variable d (except now the probability distribution $g(\underline{z})$ has already been specified). We noted there that the function G(d) was dependent only upon the composition $\underline{q}$ of $\underline{w}$; therefore, in Equation 3.13, we can instead average the bracketed term over the composition space Q^H using the probability distribution in Equation 2.52:

$$\Pr(\exists' z_o \text{ in } \theta_1) = \int_{Q^H} \cdots \int P(\underline{q}) \left[1 - \Pr\left(\underline{z} \ni d(\underline{w},\underline{z}) \le d^* | \text{comp. } \underline{w} = \underline{q}\right)\right]^M d\underline{q}$$

$$= \int_{Q^H} \cdots \int P(\underline{q}) \left[1 - G_{d|\underline{q}}(d^*)\right]^M d\underline{q} . \qquad (3.14)$$

To continue the inequality in Theorem 3.1, we require an upper bound to $\Pr(\exists' \underline{z}_o \text{ in } \theta_1)$ which in turn requires the use of a lower bound to $G_{d|\underline{q}}(d^*)$ in Equation 3.14. For points $\underline{q}$ in Q^H that satisfy $0 < d^* \le E(d|\underline{q})$, we can use Fano's lower bound:

$$G_{d|\underline{q}}(d^*) \geq K(n,\underline{q}) \ \exp n\left[\mu(s,\underline{q}) - s\mu'(s,\underline{q})\right]$$

$$\triangleq K(n,\underline{q}) \ \exp -nR(d^*,\underline{q}) \tag{3.15a}$$

in which s satisfies

$$\mu'(s,\underline{q}) = d^* \tag{3.15b}$$

and $K(n,\underline{q})$ is a coefficient given by Equation A4.1. The bound in Equation 3.15a can also be used for those $\underline{q}$ that make $E(d|\underline{q}) > d^*$ if in this equation the value $s = 0$ is used in place of the solution to Equation 3.15b. The semi-invariant moment generating function $\mu(s,\underline{q})$ is the same as that in Equations 2.30 and 2.31:

$$\mu(s,\underline{q}) = \sum_i q_i \ln \sum_j q_j e^{sd_{ij}} .$$

Because d^* is a threshhold that is constant for all $\underline{q}$, the exponent $R(d^*,\underline{q})$ in Equation 3.15a can be considered a function of $\underline{q}$ over Q^H. So also can $K(n,\underline{q})$. Thus, we have the following bound to the probability of failure at the encoder:

$$\Pr\left(\exists' \underline{z}_o \text{ in } \theta_1\right) \leq \int_{Q^H}\cdots\int P(\underline{q})\left[1 - K(n,\underline{q}) \exp -nR(d^*,\underline{q})\right]^{e^{nR}} d\underline{q} \tag{3.16}$$

A simple upper bound can, in turn, be found for this last integral if we divide the composition space Q^H into two disjoint subspaces, Q and Q', that are defined by

$$Q \triangleq \left\{\underline{q} : \ R(d^*,q) < R - \delta\right\} \tag{3.17}$$

and

$$Q' \triangleq \left\{ \underline{q} : R(d^*, \underline{q}) \geq R - \delta \right\} \tag{3.18}$$

where δ is any positive number satisfying $R^* < R - \delta$. The idea behind this separation is illustrated in Figure 3.1. The bracketed term in the integrand of Equation 3.16 has the form $(1 - e^{-nA})^{e^{nB}}$ which approaches zero with increasing n when A < B, and one when A > B. In the first region, which, except for the δ, corresponds to the set Q, we shall use the upper bound

$$\left(1 - e^{-nA}\right)^{e^{nB}} \leq e^{-e^{n(B-A)}} \tag{3.19}$$

and in the second region, corresponding to Q', the (poorer) bound

$$\left(1 - e^{-nA}\right)^{e^{nB}} \leq 1 . \tag{3.20}$$

We can therefore upper bound the probability of failure at the encoder by

$$\Pr\left(\exists' \underline{z}_0 \text{ in } \theta_1\right) \leq \int_Q \cdots \int P(\underline{q})\, e^{-K(n,\underline{q}) \exp n\left(R - R(d^*,\underline{q})\right)} d\underline{q}$$

$$+ \int_{Q'} \cdots \int P(\underline{q})\,(1)\, d\underline{q} . \tag{3.21}$$

If in the first integral we replace $K(n,\underline{q})$ by

$$K(n) \triangleq \min_{\underline{q}} K(n,\underline{q}) ,$$

we can use Equation 3.17 to further upper bound this integral, and, the

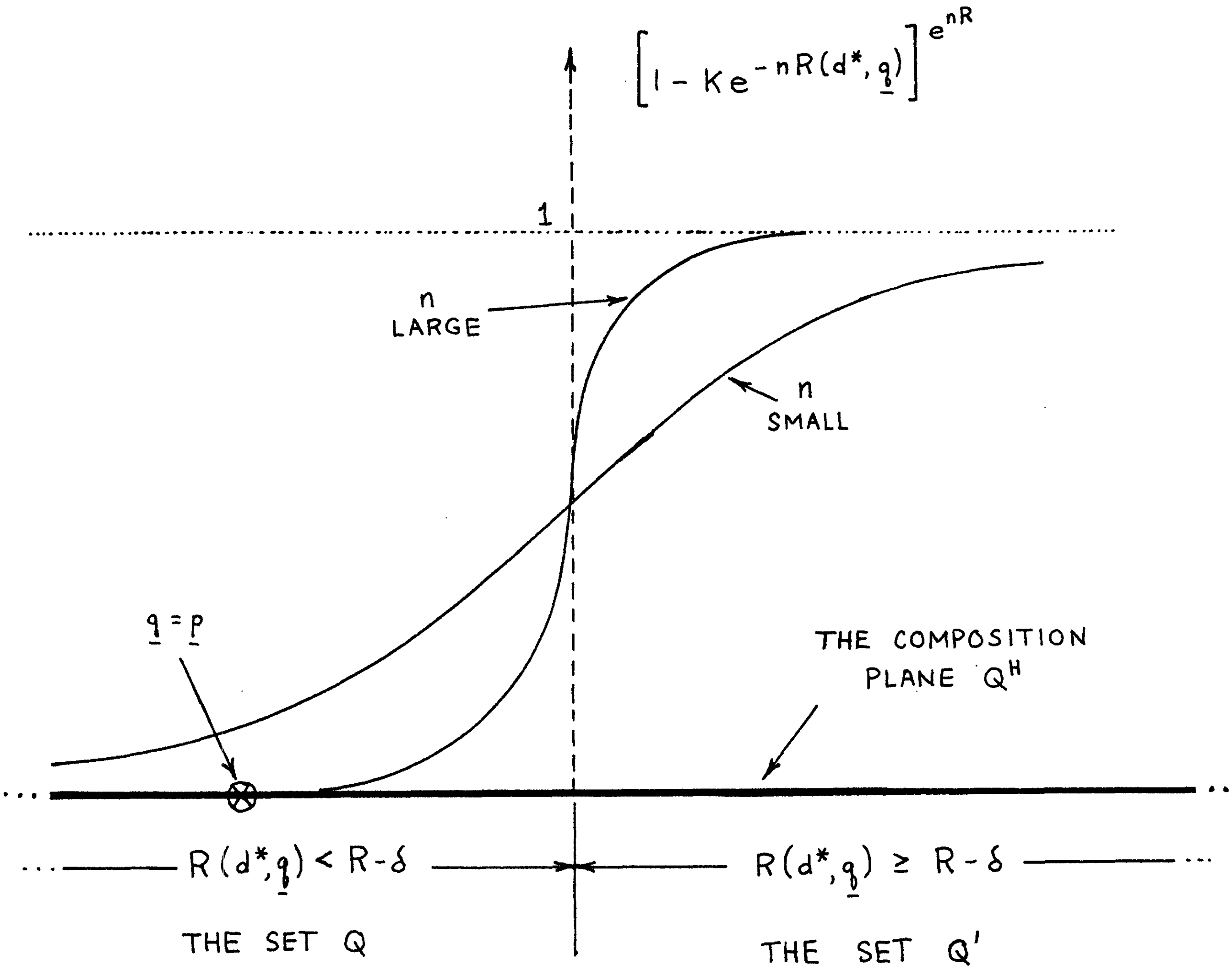

FIGURE 3.1: THE DIVISION OF THE COMPOSITION PLANE Q^H INTO THE SETS Q AND Q'

total probability of failure at the encoder, by

$$\Pr(\exists' \underline{z}_o \text{ in } \theta_1) \leq \int_Q \cdots \int P(\underline{q})\, e^{-K(n)e^{n\delta}}\, d\underline{q} + \Pr(Q')$$

$$\leq e^{-K(n)e^{n\delta}} + \Pr(Q'). \tag{3.22}$$

From the parametric form of the rate-distortion curve found in Chapter 2 (and previously by Goblick)

$$\mu(s,\underline{p}) - s\mu'(s,\underline{p}) = -R^* \tag{3.23}$$

$$\mu'(s,\underline{p}) = d^*$$

we can equate $R(d,\underline{p}) = R^*$ and, from $R^* < R - \delta$, conclude that the point $\underline{q} = \underline{p}$ is in region Q. Consequently, the region Q' includes only the tails of the probability distribution $P(\underline{q})$ and has a total probability that is an exponentially decreasing function of n. This establishes the exponentially decreasing behavior of the probability of failure at the encoder. It was only to obtain this type of behavior that the inequality $R^* < R$ was imposed earlier in the chapter.

In the next two subsections we derive two upper bounds to the probability Pr(Q'). The first derivation provides a crude bound that is fairly simple to calculate and the second an exponentially correct bound that is in turn much more difficult to evaluate. The reader may bypass these derivations and skip to Theorem 3.2 in Section 3.1.4 in which all the results are collected. The important properties of the two bounds to Pr(Q') are listed after this theorem.

A. The Hypercube Method

With this method we enclose the set Q' by another set Q_1' that has a relatively simple configuration, and upper bound $Pr(Q')$ by $Pr(Q_1')$. While the result is not exponentially tight, it is easy to evaluate numerically.

We construct in R^H a hypercube of dimension $2u$ centered at $\underline{q} = \underline{p}$,

$$K^H = \left\{ \underline{q} : p_i - u \leq q_i \leq p_i + u \right\}$$

and intersect with it the composition space Q^H, which is the $H - 1$ dimensional hyperplane in the first quadrant of R^H that intersects each axis q_i at one. The intersection forms a "solid" Q_1,

$$Q_1 = Q^H \cap K^H$$

which contains vertices of the form

$$\underline{q}_v = q_{1v}, q_{2v}, \ldots, q_{Hv} \tag{3.24}$$

$$\sum_{i=1}^{H} q_{iv} = 1$$

$$q_{iv} = p_i + u \quad \text{or} \quad p_i - u \qquad \text{if } H \text{ is even}$$

$$= p_i + u \quad \text{or} \quad p_i - u, \quad \text{with one additional component equal to } p_i, \quad \text{if } H \text{ is odd.}$$

that are joined by straight lines. At this point, we use a result derived in Appendix 4B which is that the set Q is convex. That is, if $\underline{q}_1$ and

$\underline{q}_2$ are both in Q, so also are all the points on the line $\lambda\underline{q}_1 + (1-\lambda)\underline{q}_2$. This property of Q insures us that whenever the vertices of Q_1 are in the set Q, the entire set Q_1 is in Q,

$$Q_1 \subseteq Q,$$

with the consequence that

$$\Pr(Q') \leq \Pr(Q_1') \ . \qquad (3.25)$$

To use the bound in Equation 3.25 we adjust u to be the largest positive number (to get the tightest bound) for which all the vertices $\underline{q}_v$ in Equation 3.24 are in Q. This requires the inequality

$$\max_{\underline{q}_v} \left[-\sum_i q_{iv} \left(\mu_i(s) - s\mu_i'(s) \right) \right] \overset{\text{just}}{<} R - \delta \qquad (3.26a)$$

in which (when $d^* \leq E(d|\underline{q}_v)$) s is the solution of

$$\sum_i q_{iv}\, \mu_i'(s) = d^*. \qquad (3.26b)$$

Those $\underline{q}_v$ for which $d^* > E(d|\underline{q}_v)$ are not critical with regard to the maximization in Equation 3.26.

The remaining step is to bound the total probability of the set Q_1. Because this probability equals the probability that any of the dependent events $q_i \ \varepsilon' \left[p_i - u, p_i + u \right]$ occurs, we can use the union bound to upper bound $\Pr(Q_1')$ by the sum of the individual probabilities. Thus

$$Pr(Q_1') \leq \sum_{i=1}^{H} Pr\left(q_i \,\epsilon' \left[p_i - u, p_i + u\right]\right)$$

$$= \sum_{i=1}^{H} Pr\left(q_i < p_i - u\right) + Pr\left(q_i > p_i + u\right).$$

These quantities can be further upper bounded by a simple application of Chernov bounds (this is done for us in Wozencraft and Jacobs[8], page 102) which provides the bound

$$Pr\left(Q_1'\right) \leq \sum_{i=1}^{H} e^{-nX_i} + e^{-nY_i} \tag{3.27}$$

in which

$$\left.\begin{array}{l} X_i \\ Y_i \end{array}\right\} = -\ln\left[\left(\frac{p_i}{d_i}\right)^{d_i}\left(\frac{1-p_i}{1-d_i}\right)^{1-d_i}\right] \tag{3.28}$$

where

$d_i = p_i - u$ for X_i

$= p_i + u$ for Y_i .

Finally, we show in Appendix 4C that if the bound in Equation 3.27 is replaced by

$$Pr\left(Q_1'\right) \leq 2H\, e^{-n \min\left(X_1, X_2, \ldots, Y_1, Y_2, \ldots Y_H\right)} \tag{3.29}$$

there are only two candidates for the minimizing quantity in the exponent.

We can therefore use for the probability Pr(Q') in Equation 3.22 the upper bound :

$$Pr(Q') \leq c_1 e^{-nE_{s1}(R)} \tag{3.30}$$

in which

$$E_{s1}(R) = \min_i (X_i, Y_i) \tag{3.31}$$

$$c_1 = 2H .$$

B. The Maximum Probability Point Method

This method can be used to derive an upper bound to Pr(Q') that is exponentially correct but with the penalty that the result is more difficult to evaluate. Roughly, what is done is to find the particular composition $\underline{q}^o$ in Q' for which the individual probability is greatest and to upper bound all other probabilities in Q' by $P(\underline{q}^o)$.

First, we use Fano's bounds to a factorial[13] to upper bound the probability $P(\underline{q})$ in Equation 2.52,

$$P(\underline{q}) = \frac{n!}{\prod_{i=1}^{H} (nq_i)!} \prod_{i=1}^{H} p_i^{nq_i}$$

$$\leq \frac{e^{\frac{1}{12n}} \sqrt{2\pi n} \; n^n \, e^{-n}}{\prod_{i=1}^{H} e^{\frac{1}{12nq_i+1}} \sqrt{2\pi n q_i} \; (nq_i)^{nq_i} \, e^{-nq_i}} \prod_{i=1}^{H} p_i^{nq_i}$$

$$\leq e^{\frac{1}{12n}} \sqrt{2\pi n} \prod_{i=1}^{H} \left(\frac{p_i}{q_i}\right)^{nq_i} \qquad \text{for } n > N$$

$$P(\underline{q}) \leq C(n) \prod_{i=1}^{H} \left(\frac{p_i}{q_i}\right)^{nq_i}, \tag{3.32}$$

and note that the right side of Equation 3.32 could be considered a continuous function of $\underline{q}$ over Q^H, rather than the point function at the composition points in Q^H. In Appendix 4D we show that the maximum value of this function in the (closed) set Q' occurs on the boundary of Q'. Therefore, to obtain an upper bound to the probability of every composition point in Q', we use the boundary constraint (equality in Equation 3.33)

$$\sum_i q_i \left(\mu_i(s) - s\mu_i'(s)\right) = -(R - \delta) \tag{3.33}$$

$$\sum_i q_i \, \mu_i'(s) = d^* \tag{3.34}$$

and the probability constraint

$$\sum_i q_i = 1$$

to maximize the function in Equation 3.32 over the boundary of Q'. Actually, the monotonicity of ln x and x^n is used to maximize instead the sum $\sum_i q_i \ln(p_i/q_i)$. When H > 2, we use the Lagrange multiplier technique to find the stationary points:

$$\frac{\partial}{\partial q_i}\left[\sum_j q_j \ln\frac{p_j}{q_j} + \lambda \sum_j q_j + \nu \sum_j q_j\left[\mu_j(s) - s\mu_j'(s)\right]\right]\Bigg|_{\underline{P}} = 0$$

which becomes, if we remember that s is the implicit function of $\underline{q}$ in Equation 3.34,

$$\ln \frac{p_i}{q_i} - 1 + \lambda + \nu\left[\mu_i(s) - s\mu_i'(s) + s\sum_j q_j\mu_j''(s)\left(\frac{\mu_i'(s)}{\sum_j q_j\mu_j''(s)}\right)\right] = 0$$

or

$$\ln \frac{p_i}{q_i} + \lambda' + \nu\mu_i(s) = 0.$$

If more than one solution exists, the function $\sum_i q_i \ln(p_i/q_i)$ must be evaluated for each to find the maximum value. And, of course, it should be compared with the values of this function on the boundary. The solution for the non-zero q_i^{o}, including the evaluation of the multiplier λ', is

$$q_i^{o} = \frac{p_i e^{\nu\mu_i(s)}}{\sum_\ell p_\ell e^{\nu\mu_\ell(s)}} = \frac{p_i\left[\sum_j q_j e^{sd_{ij}}\right]^{\nu}}{\sum_\ell p_\ell\left[\sum_j q_j e^{sd_{\ell j}}\right]^{\nu}} \tag{3.35}$$

in which s and ν must be considered parameters specified by Equations 3.33 and 3.34. For H = 2 the boundary contains only 2 points, and each must be evaluated separately.

We use this solution in Equation 3.32 to obtain the upper bound to the probability at every composition point in Q',

$$P(\underline{q}) \leq c(n)\left[\prod_{i=1}^{H}\left(\frac{p_i}{q_i^{o}}\right)^{q_i^{o}}\right]^n$$

and from this, the upper bound to the total probability of Q',

$$\Pr(Q') \leq (\text{no. of comp. in } Q')\ C(n) \left[\prod_{i=1}^{H} \left(\frac{p_i}{q_i^\circ} \right)^{q_i^\circ} \right]^n .$$

To further upper bound Pr(Q'), we upper bound the number of compositions in Q' by the total number in $Q^H = Q \cup Q'$, which is found in the Appendix 4E to equal

$$(\text{no. of comp. in } Q^H) = \frac{(n+H-1)!}{n!\,(H-1)!}$$

The resulting upper bound to Pr(Q') is

$$P_r(Q') \leq c_2(n)\, e^{-nE_{s2}(R)} \qquad (3.36)$$

in which

$$E_{s2}(R) = -\sum_i q_i^\circ \ln \frac{p_i}{q_i^\circ} \qquad (3.37)$$

$$c_2(n) = e^{\frac{1}{12n}} \sqrt{2\pi n}\ \frac{(n+H-1)!}{n!\,(H-1)!}$$

.

3.1.4 The Set of Upper Bounds

The results in the previous sections establish the following upper bound to the transmission distortion.

Theorem 3.2

The minimum attainable transmission distortion is upper bounded by

$$d(S) \leq d^* + (d_{max} - d^*) \left[e^{-K(n)e^{n\delta}} + c_i(n)\, e^{-n E_{si}(R)} + e^{-nE(R)} \right] \qquad (3.38)$$

$i = 1,2$

for every d^* and R that satisfy

$$d_{max} \geq d^* > d_R > d_C \tag{3.39}$$

$$R^* < R < C \quad . \tag{3.40}$$

In Equation 3.38 the first two terms in the brackets are a bound to the probability of failure at the encoder. The second of these two terms is the bound (i = 1 for the hypercube method, i = 2 for the maximum probability point method) to the probability Pr(Q') that was derived on the preceding pages. The coefficient $c_i(n)$, which is at most algebraic in n, and the exponent $E_{si}(R)$, which is a positive function of R that is independent of n, are given by Equation 3.31 for i = 1 and by Equation 3.37 for i = 2. The first term in the brackets is a double exponential in n and contributes nothing significant to the sum when n is even moderately large. The last term is the upper bound to the probability of error on the channel.

Actually, the freedom provided by Equations 3.39 and 3.40 can be used to generate a set of upper bounds corresponding to all possible choices of d^* and R. The properties of these bounds depend upon those of the two exponential functions in Equation 3.38. In Appendix 4F we show that

1. $E_{s1}(R)$ and $E_{s2}(R)$ are positive monotone increasing functions of the difference $R - R^*$ (3.41)
2. $E_{s1}(R^*) = E_{s2}(R^*) = 0$
3. $E'_{s1}(R^*) = E'_{s2}(R^*) = 0$
4. $E''_{s1}(R^*) \neq 0 \neq E''_{s2}(R^*)$,

which, when compared with the corresponding properties of the reliability function $E(R)$[14]

1. $E(R)$ is a positive monotone increasing function of the difference $C - R$ (3.42)
2. $E(C) = 0$
3. $E'(C) = 0$
4. $E''(C) \neq 0$,

reveal that the two exponential functions are very similar. Typically, their curves would look like those in Figure 3.2.

With these curves, we now examine the behavior of the bounds in Theorem 3.2. When d^* is chosen much larger than d_C, the non-zero slope of the rate-distortion curve (Figure 3.3) allows a choice of R that can make both the differences $C - R$ and $R - R^*$ large. In turn, the exponents $E_{si}(R)$ and $E(R)$ in Equation 3.38 are large and the exponential terms decay very rapidly with n. But for this choice, the asymptote d^* is much greater than the level d_C, which we know can be approached. On the other hand, if we choose d^* only slightly greater than d_C, we have an upper bound with an asymptote that is nearly d_C. But now the differences $C - R$ and $R - R^*$, and therefore the exponents $E_{si}(R)$ and $E(R)$, are much smaller, and the rate of approach to the asymptote d^* is correspondingly slower. Thus, in the selection of d^* and R, there is a tradeoff between a small asymptotic value and a fast rate of approach. This is illustrated in Figure 3.4, in which we show a set of curves obtained from a set of upper bound expressions in Theorem 3.2. The best compromise for any value of n is given by the lower

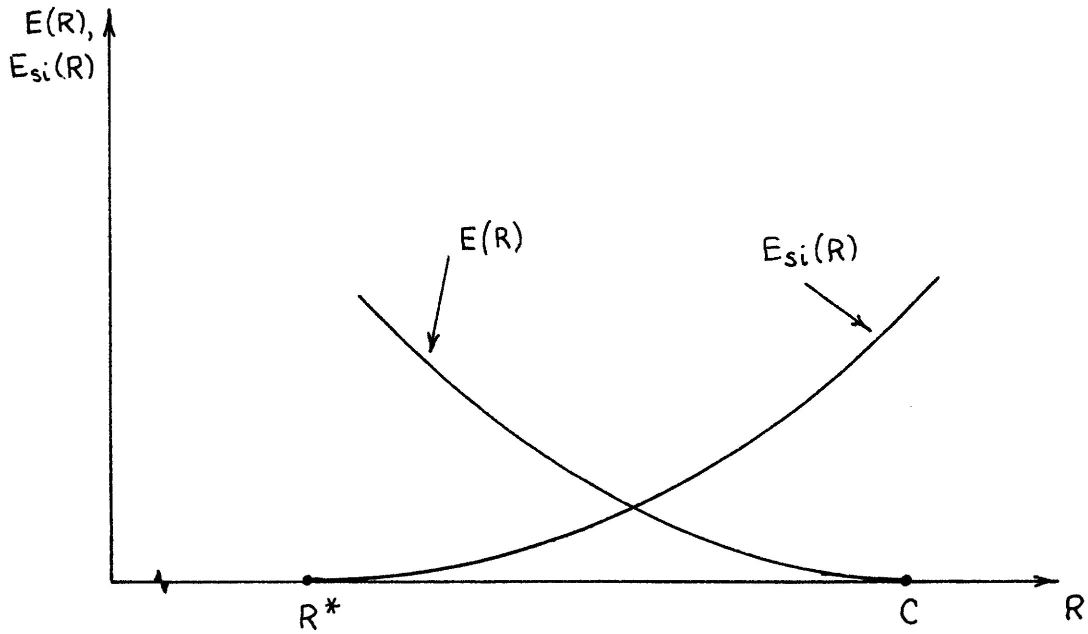

FIGURE 3.2 : TYPICAL BEHAVIOR OF $E_{si}(R)$ AND $E(R)$ NEAR THEIR ZERO VALUE

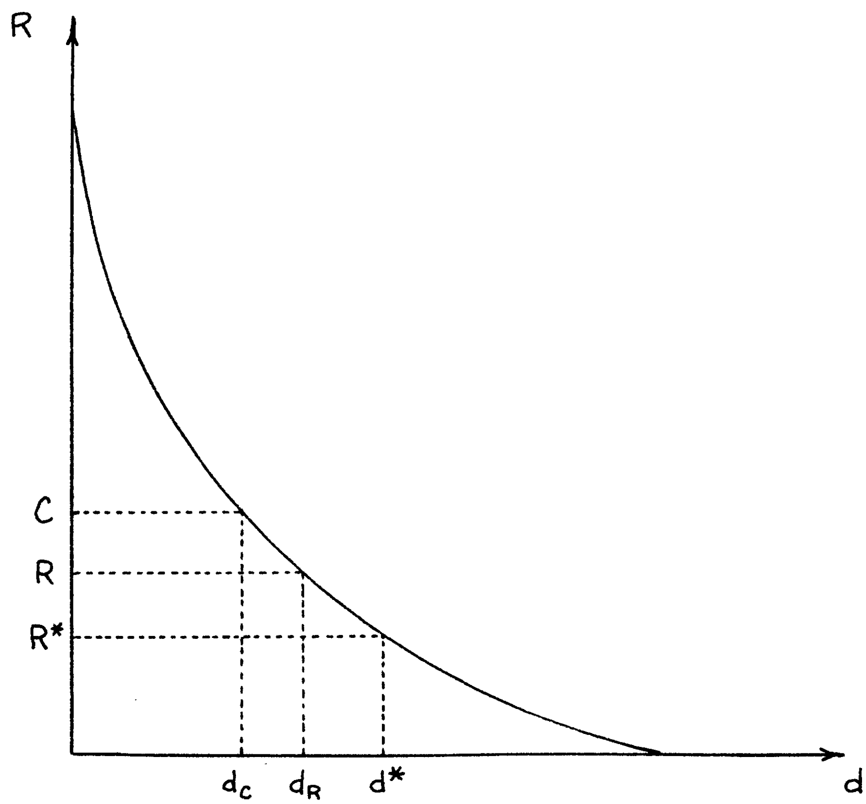

FIGURE 3.3 : THE RATE-DISTORTION CURVE FOR $\mathcal{S}$ ILLUSTRATING THE RELATIONS AMONG THE PARAMETERS IN THEOREM 3.2

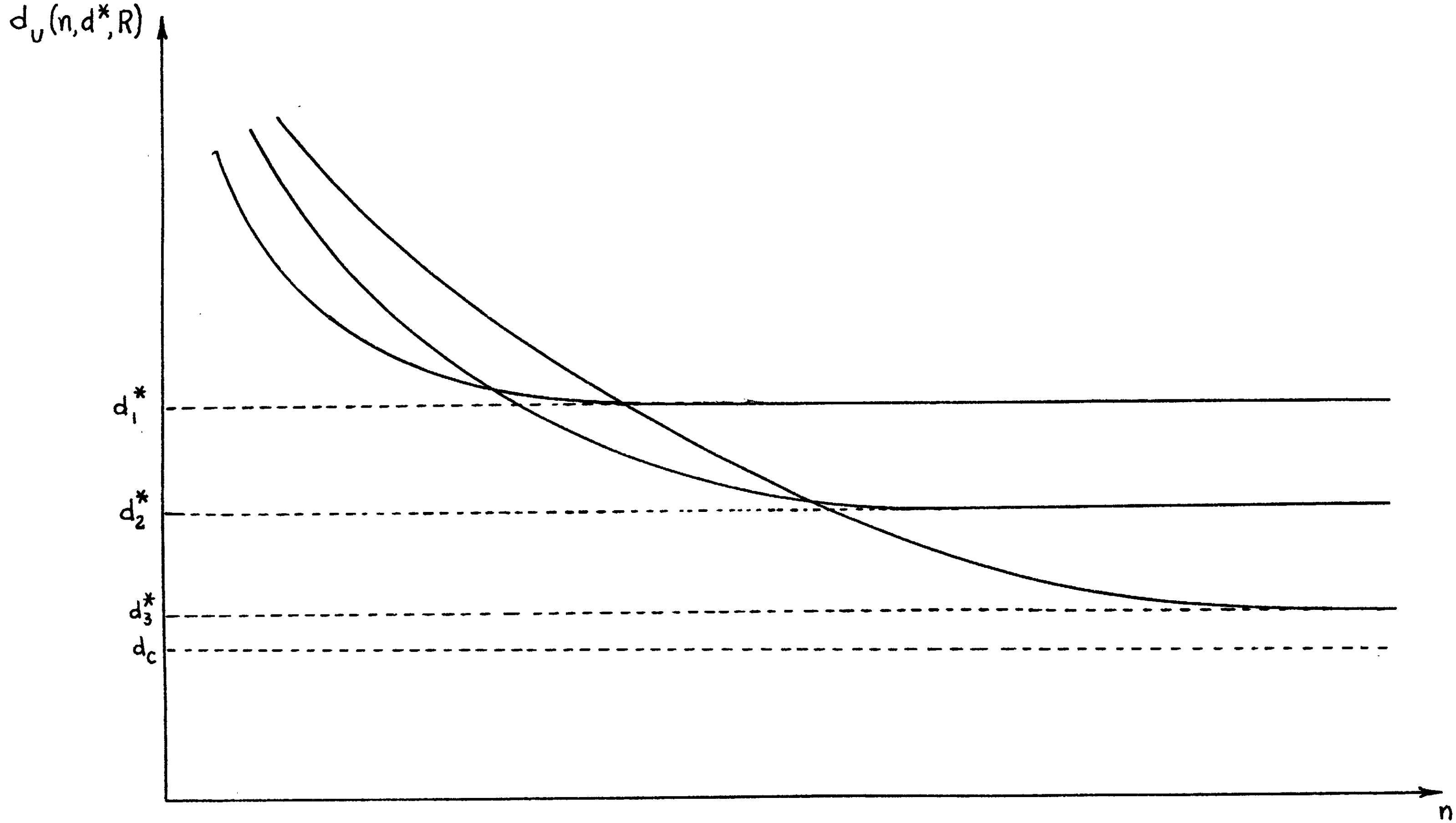

FIGURE 3.4: THE UPPER BOUND IN THEOREM 3.2 WITH THREE DIFFERENT VALUES FOR d^* AND R

envelope to the set of bounds in Theorem 3.2. If we abbreviate the right side of Equation 3.38 by $d_U(n,d^*,R)$, the lower envelope is given by

$$\text{lower envelope} = \min_{d^*,R} d_U(n,d^*,R) \ . \tag{3.43}$$

and we can rewrite Theorem 3.2 as

Theorem 3.2a

The minimum attainable transmission distortion is upper bounded by

$$d(\mathcal{S}) \leq \min_{d^*,R} d_U(n,d^*,R) \ . \tag{3.44}$$

It is emphasized that the minimization in Equation 3.44 is only included to provide the tightest upper bound. As Theorem 3.2 indicates, any d^* or R could be used to obtain a correct bound. In the next section we study the asymptotic behavior of the lower envelope in Equation 3.44.

At this point, though, we wish to include an important conclusion that can be established from the set of upper bounds in Equation 3.38. Each individual bound indicates that, in a system where the distortion level d_C is attainable, if one would tolerate a distortion $d^* = d_C + \Delta$, this level could be approached exponentially fast as the coding block length is increased. Actually, a much stronger statement is possible. Since the distortion curve for $d^* = d_C + \frac{1}{2}\Delta$ approaches this level in the limit it must cross, at some finite n, the level $d_C + \Delta$. Because both curves are for the same pair $\mathcal{S}$ and $\mathcal{C}$, this proves the distortion level $d_C + \Delta$ is not only approachable exponentially fast, it is in fact attainable with a finite coding block length. This is true for any $\Delta > 0$, no matter how small.

3.1.5 The Asymptotic Behavior of the Lower Envelope

If in Equation 3.44, the lower envelope is to approach d_C with increasing n, we see that the choice of d^* must also approach d_C with increasing n. The continuity of the rate-distortion curve in turn requires that R^* approach C, and therefore that R approach C, with increasing n. Consequently, the exponents $E_{si}(R)$ and $E(R)$ in Equation 3.38 are both functions of n that approach zero as n becomes large. This suggests that we study the asymptotic behavior of the lower envelope by using Taylor Series expansions for $E_{si}(R)$ at R^* and for $E(R)$ at C. Thus, we write

$$E_{si}(R) = E_{si}(R^*) + E'_{si}(R^*)(R-R^*) + \tfrac{1}{2} E''_{si}(R^*)(R-R^*)^2 + \tfrac{1}{6} E'''_{si}(R_1)(R-R^*)^3$$

in which we use the properties of $E_{si}(R)$ listed in Equation 3.41 and the boundedness of the third derivative (see Appendix 4F) to obtain

$$E_{si}(R) = \tfrac{1}{2} E''_{si}(R^*)(R-R^*)^2 + o\left((R-R^*)^2\right) \tag{3.45}$$

in the region just above R^*. In the same way we use the properties of $E(R)$ in Equation 3.42 and the boundedness of the third derivative to write

$$E(R) = E(C) + E'(C)(C-R) + \tfrac{1}{2} E''(C)(C-R)^2 + \tfrac{1}{6} E'''(R_2)(C-R)^3$$

$$= \tfrac{1}{2} E''(C)\,(C-R)^2 + o\left((C-R)^2\right) \tag{3.46}$$

in the region just below C.

When Equations 3.45 and 3.46 are used in Equations 3.38 and 3.44 to establish the asymptotic behavior of the lower envelope to the set of bounds, the high order remainder terms in the Taylor formula can be dropped.

So also can the double exponential involving δ since we have required that $\delta > 0$. Therefore, we have for the asymptotic form of Equation 3.44

$$d(S) \leq \min_{d^*, R} \left[d^* + (d_{max} - d^*) \left[c_i(n) e^{-nc_{3i}(R-R^*)^2} + e^{-nc_4(C-R)^2} \right] \right] \quad (3.47)$$

in which we have used c_{3i} for $\frac{1}{2}E''_{si}(R^*)$ and c_4 for $\frac{1}{2}E''(C)$.

We avoid the minimization on R by choosing the value of R that equates the two exponents,

$$c_{3i}(R-R^*)^2 = c_4 (C-R)^2. \quad (3.48)$$

While this selection of R is non-optimum for finite n, it can be shown that it asymptotically approaches R_{opt} from Equation 3.47 and does not change the asymptotic behavior of the upper bound. The reason for this choice is that we are now allowed to combine the exponential terms in this equation. If we start with Equation 3.48 and the obvious equality

$$(C-R) + (R-R^*) = C-R^* ,$$

a little algebra can be used to establish

$$(C-R)^2 = (C-R^*)^2 \frac{c_{3i}}{\left(\sqrt{c_{3i}} + \sqrt{c_4}\right)^2} \quad (3.49)$$

and

$$(R-R^*)^2 = (C-R^*)^2 \frac{c_4}{\left(\sqrt{c_{3i}} + \sqrt{c_4}\right)^2} \quad (3.50)$$

which we use to write the exponents in terms of the common difference $C-R^*$.

Next, we wish to express the difference $C-R^*$ in terms of the difference d_C-d^*. Taylor's Formula with remainder is again used, now for the rate-distortion curve $R(d^*)$ at the point (C,d_C), to write

$$R(d^*) = R(d_c) + R'(d_c)(d^*-d_c) + o(d^*-d_c)$$

or

$$C-R^* = -R'(d_c)(d^*-d_c) - o(d^*-d_c)$$

$$= -s_0(d^*-d_c) - o(d^*-d_c). \qquad (3.51)$$

In this last equation, we have used the fact that the slope of the rate-distortion curve at the point (C,d_C) is equal to the value of s that satisfies Equation 3.23 when $R^* = C$ and $d^* = d_C$. If we are interested only in small differences (which is the case as $n\to\infty$), the high order remainder term can be dropped.

We now substitute Equations 3.49, 3.50 and 3.51 in Equation 3.47, subtract d_C from both sides of this last equation, and change the minimizing variable to d^*-d_C to obtain

$$d(S)-d_c \le \min_{d^*-d_c}\left[(d^*-d_c)+\left[(d_{max}-d_c)-(d^*-d_c)\right](c_i(n)+1)\exp\left[-n\frac{C_{3i}C_4 S_0^2}{\left(\sqrt{C_{3i}}+\sqrt{C_4}\right)^2}(d^*-d_c)^2\right]\right]. \qquad (3.52)$$

The basic form of this upper bound is

$$d(S)-d_c \le \min_x\left[x + (A-x)D_i(n)e^{-Bnx^2}\right] \quad ; \quad i=1,2$$

$$\triangleq \min_x g_i(x,n)$$

$$\triangleq g_i(x_0(n),n) \qquad (3.53)$$

which is the lower envelope of a family of curves that are studied in Appendix 4G. The results found there are

$$g_1\left(x_0(n), n\right) \le \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{nB}}\left(1 + o_n(1)\right)$$

$$g_2\left(x_0(n), n\right) \le \sqrt{H}\sqrt{\frac{\ln n}{nB}}\left(1 + o_n(1)\right)$$

which can be used in Equations 3.52 and 3.53 to obtain our asymptotic upper bound to distortion.

Theorem 3.3

The minimum attainable transmission distortion of the source $\mathcal{S}$, when used with the channel $\mathcal{C}$, asymptotically satisfies

$$d(\mathcal{S}) \le d_c + b_i\sqrt{\frac{\ln n}{n}}\left(1 + o_n(1)\right) \tag{3.54}$$

in which

$$b_1 = \frac{1}{\sqrt{2}}\frac{1}{|S_0|}\left(\frac{1}{\sqrt{c_{31}}} + \frac{1}{\sqrt{c_4}}\right)$$

$$b_2 = \sqrt{H}\frac{1}{|S_0|}\left(\frac{1}{\sqrt{c_{32}}} + \frac{1}{\sqrt{c_4}}\right)$$

$$c_{3i} = \tfrac{1}{2}E''_{si}(R^*)$$

$$c_4 = \tfrac{1}{2}E''(C)$$

For a fixed source $\mathcal{S}$, we see from this theorem that the coefficient b_i is smallest when $\mathcal{S}$ is used with that channel (among those of equal capacity) for which the constant c_4 is highest. In the same way, the coefficient b_i is seen to be a decreasing function of c_{3i} when the channel is fixed. Since the constant c_4 is independent of the source and c_{3i} independent of the channel, our upper bound does not provide an indicator of matching between the source and channel as we obtained in the lower bound. This was actually expected since here we were forced to separate the source and channel with an interface containing at most e^{nC} points.

The coefficients c_{3i}, though, have an interesting significance. They are, for i = 1 and 2, equal to one-half the derivatives $E''_{S1}(R^*)$ and $E''_{S2}(R^*)$ which are found in Appendix 4F to be indicators of how fast the boundary of Q' initially moves away from $\underline{p}$ with increasing R. In turn, this indicates, in a reciprocal manner, the necessary rate of change of the rate required to handle source words with compositions just around $\underline{p}$ which are just less than typical (the rate at $\underline{p}$ equals R^* which is made to approach C as n is increased). Thus, we can think of the coefficients c_{3i} as a type of "stretch factor" for the source.

When the result in Equation 3.54 is compared with those of the previous chapter, we see that the $\sqrt{\frac{\ln n}{n}}$ rate of approach to d_C is slower than the $\frac{1}{n}$ rate of approach of the lower bound. Mathematically at least, the reason for the upper bound decreasing more slowly than $\sqrt{\frac{1}{n}}$ is that, for small arguments, the lowest order term in the two exponents E(R) and $E_{Si}(R)$ is quadratic. Their form for large n, $e^{-n(\Delta R)^2}$, shows that values of (ΔR) larger than $\sqrt{\frac{1}{n}}$ are required to have these exponents go to zero with increasing n. From this inequality, and because the slope of the rate-distortion curve is non-zero, the corresponding values of distortion difference (Δd) must also be larger than $\sqrt{\frac{1}{n}}$.

In the sections to follow we will see that this type of exponential term is present in the upper bound because we have used threshhold devices in the transmission system. One at the encoder leads to the first exponential term in Equation 3.38. It uses the rule in Equation 3.5 to choose, for each source word $\underline{w}$, any decoder word $\underline{z}$ on list θ_1 at a distortion less than d^*. When list θ_1 is lacking such an entry, any $\underline{z}$ at all on the list is chosen which, since the members of θ_1 are chosen independently, is then independent of $\underline{w}$. The resulting distortion in this circumstance is usually much greater than d^*. In the next section, we compare the performance of this encoder with another that does not employ such a threshhold.

A second threshhold operation in our system is at the channel decoder, but is really dependent upon the coding in the entire system. It leads to the second exponential term in Equation 3.38. To isolate its effect on the system performance, we assume that failure has not occurred at the encoder, that is, there does exist a $\underline{z}$ on θ_1 with $d(\underline{w},\underline{z}) \leq d^*$. Now, if the channel decoder makes no error, we are assured that the resulting distortion is less than d^*. However, if an error is made, the believed channel input word $\underline{x}_1$ is statistically independent of the actual word $\underline{x}$, because the entries on θ_2 are chosen independently. Since the two lists θ_1 and θ_2 are also independent of each other, the result is again a decoded word $\underline{z}_1$ that is independent of the source word $\underline{w}$ with a distortion $d(\underline{w},\underline{z}_1)$ that is usually much greater than d^*. This type of threshhold is examined in more detail in Section 3.3.

3.2 Improved Upper Bound to Distortion for a Noiseless Channel (Source Representation)

The special case of a noiseless channel is considered in this section. Since such a channel contains e^C noiseless transitions, or "direct" paths, transmission of the encoder output is trivial, and the communication problem is only one of source representation. For this representation you are allowed to choose, from an e^C letter representation alphabet, one representation letter for every source output letter. Just as one is allowed n uses of the channel to transmit an n-letter source output, one is allowed an n-letter representation word to approximate an n-letter source word.

If the threshhold source encoder defined by Equation 3.5 is used in the ensemble of representation codes θ_1 of Section 3.1, the ensemble average representation error is very similar to the ensemble average transmission error derived in that section. The only difference in the derivation is that the Pr(channel error) term is no longer present in Equation 3.10, nor in any succeeding equation, with the only result being that the constant b_i in Theorem 3.3 is different. The result corresponding to Theorem 3.3 for a noiseless channel is

Theorem 3.3a

If a threshhold encoder and a noiseless channel are used, the minimum attainable transmission distortion of the source $\mathcal{S}$ asymptotically satisfies

$$d(\mathcal{S}) \le d_c + b_i \sqrt{\frac{\ln n}{n}} \left(1 + o_n(1)\right) \tag{3.55}$$

in which

$$b_1 = \frac{1}{\sqrt{2}} \frac{1}{|s_0|} \frac{1}{\sqrt{c_{31}}} , \qquad b_2 = \sqrt{H} \frac{1}{|s_0|} \frac{1}{\sqrt{c_{32}}}$$

$$c_{3i} = \tfrac{1}{2} E_{si}''(R^*) .$$

This result applies only to sources that are not doubly-uniform. For doubly-uniform sources see Section 3.2.1.

In this section, we show that the use of an optimum source encoder, rather than the threshhold encoder used to get the result in Theorem 3.3a, can improve the asymptotic form of the upper bound to one which has a $\frac{\ln n}{n}$ rate of approach to d_C. Thus, eliminating one of the two threshhold devices of the transmission system in Section 3.1 eliminates one of the two causes of the $\sqrt{\frac{\ln n}{n}}$ rate of approach to d_C. (The other threshhold operation is examined in Section 3.3). The upper bound derived in this section also represents an improvement upon the best previously known upper bound to distortion for a source representation limited in information content to C(6). This result approached the asymptote d_C essentially as $n^{-\frac{1}{2}}$.

The coding ensemble used here is the same set of codes, θ_1, used in Section 3.1. It contains all possible lists of $M = e^{nC}$ representation words chosen from Z^n and has associated with each ensemble member a probability of use

$$\Pr(\text{code}) = P(\theta_1) = \prod_{i=1}^{M} g(\underline{z}_i). \tag{3.56}$$

We also use for $g(\underline{z})$ the same distribution function used in Section 3.1, that is, $g(\underline{z}) = \prod_{m=1}^{n} g(z^m)$ in which the letter probability distortion $g(z)$ is the set of output probabilities on the test channel for $\mathcal{S}$ at the point (d_C, C) on the rate-distortion curve.

Each ensemble member consists of the source $\mathcal{S}$, an encoder, the channel $\mathcal{C}$ with M noiseless paths, and a list Θ_1 of M allowed decoded words which are associated, in any way, with the paths of the channel. But it is now assumed that the encoder in each ensemble member is an optimum device, that is, it maps each source output $\underline{w}$ into that $\underline{z}$ on Θ_1 for which $d(\underline{w},\underline{z})$ is smallest, and uses the path on $\mathcal{C}$ associated with this $\underline{z}$ for "transmission".

For the ensemble member using list Θ_1, the average distortion over all possible source events is

$$d(\theta_1) = \sum_{W^n} P(\underline{w}) \left[\min_{\substack{1 \le i \le M \\ \underline{z}_i \in \theta_1}} d(\underline{w}, \underline{z}_i) \right]. \tag{3.57}$$

This is averaged over all possible codes, using the distribution in Equation 3.56 to obtain the ensemble average distortion

$$\overline{d(\theta_1)} = \sum_{\theta_1} P(\theta_1) \sum_{W^n} P(\underline{w}) \left[\min_{\substack{1 \le i \le M \\ \underline{z}_i \in \theta_1}} d(\underline{w}, \underline{z}_i) \right]$$

$$= \sum_{W^n} P(\underline{w}) \sum_{\theta_1} P(\theta_1) \left[\min_{\substack{1 \le i \le M \\ \underline{z}_i \in \theta_1}} d(\underline{w}, \underline{z}_i) \right]. \tag{3.58}$$

In Equation 3.58, the quantities $d(\underline{w},\underline{z}_i)$ could be thought of as distortion random variables, conditioned on $\underline{w}$, with the probability distribution $P_{d|\underline{w}}(d)$ generated from the elementary event probabilities $g(\underline{z})$ by

$$P_{d|\underline{w}}(d) = \sum_{\substack{\underline{z} \ni \\ d(\underline{w},\underline{z}) = d}} g(\underline{z}) .$$

(This is the same distortion variable used in Chapter 2.) The set of M independent and identically distributed variables $d(\underline{w},\underline{z}_i)$ has a minimum $d_{min}(\underline{w})$ that can also be considered a random variable conditioned on $\underline{w}$. Since

$$P_{d_{min}|\underline{w}}(d) = \sum_{\substack{\theta_1 \ni \\ \min d(\underline{w},\underline{z}_i) = d \\ \underline{z}_i \in \theta_1}} P(\theta_1) \quad ,$$

the last sum in Equation 3.58 is the average of this variable, and we can write

$$\overline{d(\theta_1)} = \sum_{W^n} P(\underline{w}) \sum_{d} d \, P_{d_{min}|\underline{w}}(d) \, . \tag{3.59}$$

At this point we once again change the average over W^n to one over the composition space Q^H:

$$\begin{aligned} \overline{d(\theta_1)} &= \int \cdots \int_{Q^H} P(\underline{q}) \, d\underline{q} \int_0^{d_{max}} d \; dF_{d_{min}|\underline{q}}(d) \\ &\triangleq \int \cdots \int_{Q^H} P(\underline{q}) \, d\underline{q} \; \overline{d_{min}(\underline{q})} \, . \end{aligned} \tag{3.60}$$

This is possible here since each variable $d(\underline{w},\underline{z}_i)$ is dependent only upon the composition $\underline{q}$ of $\underline{w}$ and, therefore, so also is $d_{min}(\underline{w})$.

We next upper bound the integral on d in Equation 3.60. This is easier to do if we first integrate by parts to obtain

$$\int_0^{d_{max}} d \; dF_{d_{min}|\underline{q}}(d) = \int_0^{d_{max}} \left[1 - F_{d_{min}|\underline{q}}(d) \right] dd \, ,$$

or

$$\overline{d_{min}(\underline{q})} = \int_0^{d_{max}} \left[1 - F_{d_{min}|\underline{q}}(d)\right] dd. \tag{3.61}$$

The integrand in Equation 3.61 is the probability that all M points on θ_1 have a distortion $d(\underline{w},\underline{z})$ from $\underline{w}$ greater than d. The independence property of the members of θ_1 allows us to write this probability as

$$\left[1 - F_{d_{min}|\underline{q}}(d)\right] = \left[1 - G_{d|\underline{q}}(d)\right]^{e^{nC}}$$

in which G(d) is the same distribution function used in Chapter 2 and in Equation 3.14 of this chapter.

We have seen in Chapter 2 that the variable d (Equation 2.20) has a variance proportional to $\frac{1}{n}$ for every $\underline{q}$. Therefore, the function $\left[1-G_{d|\underline{q}}(d)\right]$, which for every n decreases monotonically from one to zero, approaches with increasing n a negative step at the value of distortion $d = E(d|\underline{q})$. The same is also true of $\left[1-G_{d|\underline{q}}(d)\right]^{e^{nC}}$ which approaches a negative step at some (lower) value of distortion, $d_C(\underline{q})$. For this reason, we divide the integral in Equation 3.61 into two parts,

$$\overline{d_{min}(\underline{q})} = \int_0^{d_c(\underline{q})+\Delta} \left[1 - G_{d|\underline{q}}(d)\right]^{e^{nC}} dd + \int_{d_c(\underline{q})+\Delta}^{d_{max}} \left[1 - G_{d|\underline{q}}(d)\right]^{e^{nC}} dd, \tag{3.62}$$

with $\Delta > 0$, and separately bound each integrand, with the result that

$$\overline{d_{min}(\underline{q})} \leq \int_0^{d_c(\underline{q})+\Delta} (1)\, dd + \int_{d_c(\underline{q})+\Delta}^{d_{max}} \left[1 - G_{d|\underline{q}}\left(d_c(\underline{q})+\Delta\right)\right]^{e^{nC}} dd$$

$$= d_c(\underline{q}) + \Delta + \left[d_{max} - d_c(\underline{q}) - \Delta\right]\left[1 - G_{d|\underline{q}}\left(d_c(\underline{q})+\Delta\right)\right]^{e^{nC}}. \tag{3.63}$$

Fano's lower bound to distribution functions can be used to bound $G_{d|\underline{q}}[d_C(\underline{q})+\Delta]$ by

$$G_{d|\underline{q}}\left[d_c(\underline{q})+\Delta\right] \geq K(n,\underline{q}) \exp n\left[\mu(s,\underline{q}) - s\mu'(s,\underline{q})\right]$$

$$\triangleq K(n,\underline{q}) \exp -n R\left[d_c(\underline{q})+\Delta, \underline{q}\right] \tag{3.64a}$$

in which s satisfies

$$\mu'(s,\underline{q}) = d_c(\underline{q}) + \Delta < E(d|\underline{q}) \tag{3.64b}$$

and $K(n,\underline{q})$ is the coefficient given by Equation A4.1. With this bound substituted in Equation 3.63 we obtain

$$\overline{d_{min}(\underline{q})} \leq d_c(\underline{q}) + \Delta + \left[d_{max} - d_c(\underline{q}) - \Delta\right]\left[1 - K(n,\underline{q})e^{-nR\left(d_c(\underline{q})+\Delta,\underline{q}\right)}\right]^{e^{nC}}. \tag{3.65}$$

We want the last term to vanish with increasing n; therefore we choose for $d_C(\underline{q})$ that value of distortion which sets

$$R(d_C(\underline{q}), \underline{q}) = C . \tag{3.66}$$

Because $R(d,\underline{q})$ is monotone decreasing in d, we have

$$R\left(d_c(\underline{q})+\Delta, \underline{q}\right) < R\left(d_c(\underline{q}), \underline{q}\right) = C,$$

thus, we can use Equation 3.19 to continue the inequality in Equation 3.65 as

$$\overline{d_{min}(\underline{q})} \leq d_c(\underline{q}) + \Delta + \left[d_{max} - d_c(\underline{q}) - \Delta\right] e^{-K(n,\underline{q}) \exp n\left[C - R\left(d_c(\underline{q}) + \Delta\right)\right]}. \tag{3.67}$$

At this point, Equation 3.67 could be used in Equation 3.60 to obtain a set of upper bounds corresponding to all positive choices of Δ, the lower envelope of which would provide the final upper bound to distortion. However, our objective in this section is to obtain the asymptotic form of the upper bound and compare it with the expression obtained using the threshhold encoder of Section 3.1. For this purpose, it is both more convenient and more accurate to use Shannon's lower bound[10,11] to $G_{d|\underline{q}}\left[d_C(\underline{q})+\Delta\right]$ (rather than Equation 3.64):

$$G_{d|\underline{q}}\left[d_c(\underline{q})+\Delta\right] \geq \left[\frac{1}{\sqrt{2\pi n s^2 \mu''(s,\underline{q})}} + A_L(s,n)\right] \exp n\left[\mu(s,\underline{q}) - s\mu'(s,\underline{q})\right]$$

$$\triangleq L(n,\underline{q}) \exp -nR\left[d_c(\underline{q})+\Delta, \underline{q}\right] \tag{3.68}$$

with

$$\mu'(s,\underline{q}) = d_c(\underline{q}) + \Delta < E(d|\underline{q})$$

and $A_L(s,n) = o(\sqrt{\frac{1}{n}})$. The only change so far would be to replace $K(n,\underline{q})$ by $L(n,\underline{q})$ in Equation 3.67:

$$\overline{d_{min}(\underline{q})} \leq d_c(\underline{q}) + \Delta + \left[d_{max} - d_c(\underline{q}) - \Delta\right] e^{-L(n,\underline{q}) \exp n\left[C - R(d_c(\underline{q})+\Delta)\right]} \tag{3.69}$$

(When it applies, the lower bound for lattice variables[10,11] should be used in place of Equation 3.68. Our results are not changed in this case as they depend only upon the factor $\sqrt{\frac{1}{n}}$ in the coefficient and this factor is common to both forms.)

Equation 3.69 provides, for each $\underline{q}$, a set of upper bounds to $\overline{d_{min}(\underline{q})}$ very similar to the family of curves studied in Appendix 4G for the noisy channel. In the choice of the parameter Δ there is once again a tradeoff between a small asymptote, $d_C(\underline{q}) + \Delta$, and a fast rate of approach. For each n, this parameter should be chosen to optimize the bound. Since we want an upper bound to $\overline{d_{min}(\underline{q})}$ that approaches $d_C(\underline{q})$ with increasing n, clearly the optimizing parameter $\Delta_o(n)$ must approach zero as n increases. Because $R(d,\underline{q})$ is continuous in d, it follows that the last exponent in the double exponential term must also approach zero with increasing n. As an asymptotic bound is our goal, we extract the essential behavior of the exponent for small Δ by forming a Taylor Series for $R(d,\underline{q})$ at $d_C(\underline{q})$:

$$R(d,\underline{q}) = R(d_c(\underline{q}),\underline{q}) + R'(d_c(\underline{q}),\underline{q})\left[d - d_c(\underline{q})\right] + \text{rem. term}.$$

This, together with Equation 3.66, is used to write the exponent as

$$C - R\left[d_c(\underline{q}) + \Delta, \underline{q}\right] = -\Delta R'\left[d_c(\underline{q}), \underline{q}\right] + \text{rem. term} \qquad (3.70)$$

in which the derivative has the simple form

$$\left.\frac{dR}{dd}\right|_{d_c(\underline{q})} = \left.\frac{dR}{ds}\frac{ds}{dd}\right|_{d_c(\underline{q})} = -s\mu''(s,\underline{q})\left.\frac{1}{\mu''(s,\underline{q})}\right|_{d_c(\underline{q})} = -s\Big|_{d_c(\underline{q})}.$$

We can use these simplifications in the set of bounds in Equation 3.69 and upper bound $\overline{d_{min}(\underline{q})}$ by the lower envelope to this set:

$$\overline{d_{min}(\underline{q})} \leq \min_{\Delta}\left[d_c(\underline{q}) + \Delta + \left[d_{max} - d_c(\underline{q}) - \Delta\right] e^{-L(n,\underline{q})\exp - sn\Delta}\right]. \qquad (3.71)$$

The optimization on Δ is performed in Appendix 5A from which we use Equation A5.3 to write for our final upper bound to $\overline{d_{min}(\underline{q})}$:

$$\overline{d_{min}(\underline{q})} \leq d_c(\underline{q}) + (\tfrac{1}{2} + \epsilon_1) \frac{\ln n}{-sn} \left(1 + o_n(1)\right) . \tag{3.72}$$

It remains to substitute this bound to $\overline{d_{min}(\underline{q})}$ in Equation 3.60 and to average it over the composition space Q^H. We have

$$\overline{d(\theta_1)} \leq \int_{Q^H} \cdots \int P(\underline{q}) \left[d_c(\underline{q}) + (\tfrac{1}{2} + \epsilon_1) \frac{\ln n}{-sn} \right] d\underline{q} \tag{3.73}$$

in which we recall that $d_C(\underline{q})$ is that distortion which sets $R\left[d_C(\underline{q}), \underline{q}\right]$ equal to C in Equations 3.64a and b, and s is the parameter value in these equations when they are so satisfied. Both functions of $\underline{q}$ are well behaved, being continuous and possessing (at least the first several) derivatives on all of Q^H. Therefore, to evaluate the integral in Equation 3.73, we use the now familiar technique of representing the bracketed function in this integral with a Taylor Series expansion at $\underline{q} = \underline{p}$. Because all terms in the series beyond the third term involve central moments of the components of $\underline{q}$ of order at least $(\frac{1}{n})^2$, the series is truncated with a Lagrange Remainder term at that point. The details are included in Appendix 5B where we find the result

$$\overline{d(\theta_1)} \leq d_c + \left(\tfrac{1}{2} + \epsilon_1\right) \frac{\ln n}{-s_0 n} \left(1 + o_n(1)\right) . \tag{3.74}$$

Since the ensemble average distortion provides a bound to the attainable distortion, we have the upper bound in the next theorem.

Theorem 3.4

The minimum attainable transmission distortion of the source $\mathcal{S}$, when used with a noiseless channel of capacity C, asymptotically satisfies

$$d(\mathcal{S}) \le d_c + \left(\tfrac{1}{2} + \epsilon_1\right) \frac{\ln n}{-s_0 n} \left(1 + o_n(1)\right) \tag{3.75}$$

in which s_0 satisfies

$$\mu(s_0, \underline{p}) - s_0 \mu'(s_0, \underline{p}) = -C.$$

Except for the ϵ_1, Equation 3.75 agrees precisely with the asymptotic lower bound found in Equation 2.91.

3.2.1 Comparison of the Upper Bounds to Distortion for Threshhold and Optimum Source Encoders

In an attempt to understand the difference in the form of the asymptotic bounds in Theorems 3.3a and 3.4, let us compare Equations 3.15a,b with 3.64a,b. In the first set the distortion d is fixed at d^* for all $\underline{q}$ (the threshhold in the encoder) which makes the exponent $R(d^*,\underline{q})$ a function of $\underline{q}$. In the second set (used for the optimum encoder) we reverse this by fixing $R(d,\underline{q})$ equal to C which makes the distortion $d_C(\underline{q})$ a function of $\underline{q}$. For the threshhold encoder we had

$$\overline{d(\theta_1)} \le \int_{Q^H}\!\!\cdots\!\int P(\underline{q})\, d\underline{q} \left[d^* + d_{max} \Pr_{\theta_1}\left(\exists' \underline{z} \text{ in } \theta_1 \middle| \underline{q}\right)\right]$$

$$\le \int_{Q^H}\!\!\cdots\!\int P(\underline{q})\, d\underline{q} \left[d^* + d_{max} \left(1 - K e^{-nR(d^*,\underline{q})}\right)^{e^{nC}} \right].$$

As n becomes large, the bracketed term in the integrand approaches a multi-dimensional step function along the locus of $\underline{q}$ for which $R(d^*,\underline{q}) = C$. Since d^* is chosen closer to d_C as n increases, $R(d^*,\underline{p})$ approaches C; therefore, in the limit as $n \to \infty$, the locus of the step function includes the point $\underline{q} = \underline{p}$. Thus all derivatives of the bracketed term at $\underline{q} = \underline{p}$ are unbounded as n becomes large. Since the variance of $P(\underline{q})$ around $\underline{p}$ is proportional to $\frac{1}{n}$, this situation is roughly analogous to integrating a bounded ramp function of increasing slope (in n) against a gaussian distribution $N(0,\frac{1}{n})$, which we know does not go to zero any faster than $\sqrt{\frac{1}{n}}$.

In contrast, we had for the optimum encoder

$$\overline{d(\theta_1)} \le \int_{Q^H} \cdots \int P(\underline{q})\, d\underline{q}\; \overline{d_{min}(\underline{q})}$$

$$\le \int_{Q^H} \cdots \int P(\underline{q})\, d\underline{q} \left[d_c(\underline{q}) + \Delta + d_{max}\, e^{-L \exp(-sn\Delta)} \right]$$

in which the bracketed term is now a well behaved function of $\underline{q}$ through the point $\underline{q} = \underline{p}$ for all n. It is because this function and its (first several) derivatives remain bounded with increasing n that the integral approaches its limit as $\frac{1}{n}$.

We might think of this encoder as a threshhold encoder, <u>but</u> with a threshhold that varies depending on the particular source output. In particular, for any source output word with composition $\underline{q}$, it uses a threshhold $d_C(\underline{q}) + \Delta$ just large enough so that for large n there is almost surely a representation word on θ_1 that is acceptable. It does not require, as does the fixed threshhold encoder, that the set of source words not meeting a fixed distortion level d^* have a total probability that goes to zero with n(by choice of $d^*(n)$). This restriction is really more severe than one would think you need, since some of the source words $\underline{w}$ discarded by the threshhold encoder are just outside $\underline{p}$, having characteristics just less than typical, for which the distortion $d(\underline{w},\underline{z}_i)$ might be only marginally greater than any fixed d^*.

There is one situation, though, for which the source encoders both provide a representation distortion that approaches the limit d_C as $\frac{\ln n}{n}$. This is when the source S is doubly-uniform (see Section 2.5). Because $\mu(s,\underline{q})$ is independent of $\underline{q}$ for such a source, $R(d^*,\underline{q})$ in Equation 3.15a is also independent of $\underline{q}$, with the result that the set Q' in Equation 3.18 is always empty. Therefore $Pr(Q') = 0$ in Equation 3.22, and we have for the set of upper bounds to representation distortion, using threshhold encoders:

$$d(S) \leq d^* + \left(d_{max} - d^*\right) e^{-L(n,\underline{q})\, e^{-n\delta}}$$

in which we have used the lower bound in Equation 3.68 rather than that in Equation 3.15. It can now be shown, using precisely the same procedure as in Appendix 5A, that the lower envelope to this set of bounds approaches the limit d_C as $\frac{\ln n}{n}$.

3.3 Lower Bound to Average Distortion for an M-Point System when $M = e^{nC}$

The result in Equation 3.75 indicates that if we replace the threshhold source encoders in the ensemble of transmission systems of Section 3.1 with optimum source encoders, we could eliminate one of the reasons for the rate of approach of the upper bound being limited to $\sqrt{\frac{\ln n}{n}}$. This replacement alone, though, would not significantly decrease the upper bound as there remains in the mathematics the second cause of the $\sqrt{\frac{\ln n}{n}}$ factor. This comes from the term involving the error event on the channel. Again there is a threshhold effect associated with this event. Whenever the channel noise is just large enough to cause the decoder to mistake the channel input word $\underline{x}_1$ for the actual input word $\underline{x}$, the resulting distortion jumps from the no-error distortion $d(\underline{w},\underline{z})$ to $d(\underline{w},\underline{z}_1)$. Since $\underline{z}$ and $\underline{z}_1$ are chosen independently when Θ_1 is constructed, $\underline{w}$ and $\underline{z}_1$ are independent, with the result that the distortion $d(\underline{w},\underline{z}_1)$ is usually much greater than $d(\underline{w},\underline{z})$.

Although the introduction of dependencies among the $\underline{z}$'s and the $\underline{x}$'s on lists Θ_1 and Θ_2 can reduce the effect of the discontinuity, we show in this section that if the signaling set is restricted to contain only $M = e^{nC}$ points, one cannot avoid a threshhold effect similar to the one described, and that this threshhold in turn keeps the average distortion from approaching its asymptote any faster than as $\sqrt{\frac{1}{n}}$. We do this by finding a (crude) lower bound to the average distortion that behaves essentially as

$$d_M(S) \geq d_c + \frac{a}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right). \tag{3.76}$$

We start with the distortion function d(I) defined by Equations 2.13, 2.14, and 2.16 of Chapter 2 and again lower bound it by the function $d_L(I)$ that is given by Equations A1.15a,b,c. To emphasize that the distortion function $d_L(I)$ is a function of the source word $\underline{w}$ as well as the channel event I, we now write it as $d_L(I,\underline{q})$. In Chapter 2, we chose to first average $d_L(I,\underline{q})$ over I and then over $\underline{q}$, but it is easier to include the system constraint of M signal points if we reverse the averaging processes. Therefore, we average first over the source, keeping I fixed, to obtain a function only of the channel events:

$$D_L(I) = \int \cdots \int_{Q^M} d_L(I,\underline{q})\, Pr(\underline{q})\, d\underline{q}\ . \tag{3.77}$$

If a Taylor Series is used for $d_L(I,\underline{q})$ at $\underline{q} = \underline{p}$, we can write for this integral:

$$D_L(I) = d_L(I,\underline{p}) + \frac{1}{2n|s|}\ \frac{\sigma^2(\theta)}{s^2\mu''(s,\underline{p})} + o\left(\frac{1}{n}\right) \tag{3.78}$$

with s given by Equation A1.16 in which t satisfies Equation A1.15c.

The derivative of $D_L(I)$ is

$$D_L'(I) \doteq \frac{d}{ds} D_L(I) \frac{ds}{dt}\ \frac{dt}{dI}$$

$$= \frac{\mu''}{\gamma''}\ \frac{t\gamma'' + \frac{1}{2n}\left(\frac{\gamma'''}{\gamma''} + \frac{2}{t}\right)}{s\mu'' + \frac{1}{2n}\left(\frac{\mu'''}{\mu''} + \frac{2}{s}\right)} + \frac{1}{2n}\ \frac{d}{ds}\left(\frac{\sigma^2}{-s^3\mu''}\right) \frac{ds}{dt}\ \frac{dt}{dI} + o\left(\frac{1}{n}\right)$$

$$\triangleq \frac{t}{s} + \frac{B}{n} + o\left(\frac{1}{n}\right) \tag{3.79}$$

in which we have used some of the terms evaluated in Appendix 1D. From Equation 3.79 we see that this derivative is positive for all I in $[I_a, I_b]$ (see Appendix 1A). Consequently, the function $D_L(I)$, which starts at $d_{min} = 0$ and ends at d_{max}, crosses the level d_C once with a positive slope. But, we know that the average source distortion, regardless of the channel event, must be at least as great as d_C since at most M signals are being used to represent and transmit the source. Therefore, we can use the minimum of the two functions

$$D_M(I) = \min\left(d_c, D_L(I)\right) \tag{3.80}$$

in

$$d_M(\mathcal{S}) \geq \int D_M(I)\, dF_2(I) \tag{3.81}$$

to obtain our lower bound to the total transmission distortion subject to an M-point signal constraint. Because the slope of $D_L(I)$ is positive, the function $D_M(I)$ in Equation 3.80 has a discontinuity in all its derivatives at the point $I = I_n$, where $D_L(I)$ crosses d_C.

In Appendix 6A we lower bound the integral in Equation 3.81 and show that the discontinuity in the slope of the integrand is responsible for the integral value approaching its limit no faster than as $\sqrt{\frac{1}{n}}$. The result in Equation A6.4 proves the lower bound to distortion claimed in Equation 3.76.

The discontinuity in the slope of $d_M(I)$ reflects the following property of a coding system that contains only M transmission signals. For $I < I_n$, which, when $I_n \cong C$, roughly corresponds to noise transitions that are less severe than those which are typical, the system with M points in its signal set can still only do the best it can ever do -- which is to transmit and correctly decode one of the M signals. And for this situation, the average distortion can be no smaller than the average source representation distortion. Even if one is lucky enough to get a nearly noiseless transmission, the system cannot respond by presenting to the user a very low distortion reproduction of the source. It would still be limited to the average representation distortion. In order to improve upon this situation, one would have to enlarge the signal set to include more than M points, say to $M_1 = e^{nC_1}$, $C_1 > C$. This would enable the system to respond to a low noise event, at least those with $I_{n1} \cong C_1 < I < I_n \cong C$, and to provide a lower distortion transmission, now limited only to the smaller average representation distortion possible using a rate C_1. Such an increase in the signal set would also move the discontinuity in $d_M'(I)$ down to $I_{n1} \cong C_1 < C$. Therefore in the region around $I = C$, where in the limit the distribution function $F_2(I)$ has all of its rise, the function $d_M(I)$ would be smooth. An analysis similar to that in Appendix 6A would now reveal a lower bound that approaches d_C not as $\sqrt{\frac{1}{n}}$ but as $\frac{1}{n}$.

Although the increase in the signal set necessary to achieve this need only have C_1 marginally greater than C, the correct relative positioning of these signal points in the channel input space, and their association with the source words that they represent, is very critical since now a channel error is more the rule than the exception. This placement and association must be done in such a way as to have the channel errors that occur more frequently associated with incorrect estimates of the source that have lower distortion. Such a code is extremely difficult to construct. Indeed, we were not ever able to find one that was satisfactory, in the sense of having an average distortion that approached d_C with increasing n. However, the fact that the requirements of this code "seem rather weak", only requiring that the resulting average distortion be a smooth function of the Information Difference through the neighborhood, $I \cong C$, of highly likely noise transitions, leads the author to believe that such codes do exist, and that the $\frac{1}{n}$ behavior of the lower bound of Chapter 2 more accurately describes the performance curve when a signal set larger than e^{nC} is allowed.

3.4 An External View of the Coding Ensemble in Section 3.1

Shannon's test channel $\mathcal{C}_t$ for the source $\mathcal{S}$ at the point (d_C, C) on the R(d) curve is defined as that set of transition probabilities between W and Z which minimizes the average distortion while transmitting information at a rate (no greater than) C. Therefore, if one wanted to transmit $\mathcal{S}$ and was allowed only a transmission rate C, the distortion d_C could be attained only if you could form the precise set of transition probabilities given by $\mathcal{C}_t$. From the definition we know that the channel $\mathcal{C}_t$ can support a rate C, but, because the source statistics $p(\underline{w})$ do not in general maximize the mutual information on $\mathcal{C}_t$, it is usually the case that the capacity C_t of $\mathcal{C}_t$ is strictly greater than C.

On the other hand, the channel $\mathcal{C}$ provided for our use in the transmission system has only a capacity C. And, we know that the capacity of the cascade - encoder, channel $\mathcal{C}$, decoder - cannot be increased above C. Consequently, it is impossible to design encoders and decoders to form a cascade, or "coded channel"

$$\mathcal{C}_{\text{coded}} \triangleq [\text{Encoder}][\text{Channel } \mathcal{C}][\text{Decoder}]$$

$$\triangleq \mathcal{E}\mathcal{C}\mathcal{D}$$

that is precisely equal to $\mathcal{C}_t$. But the theory states that with a communication channel $\mathcal{C}$ of capacity C, there do exist coders and decoders, and therefore a coded channel $\mathcal{C}_{\text{coded}}$, that can transmit $\mathcal{S}$ with a distortion arbitrarily close to d_C. (It necessarily follows that the transmission rate is arbitrarily close to, but less than, C.) We know that this usually requires long coding block lengths and transmission between W^n and Z^n over the n-th product channel $\mathcal{C}^n$. We conclude that, as n becomes large, there must exist a set of transitions between W^n and Z^n, given by

$$\mathcal{C}_{\text{coded}}^{[n]} = \mathcal{E}^{[n]}\, \mathcal{C}^{n}\, \mathcal{D}^{[n]}$$

that provides a distortion arbitrarily close to d_C, yet has a capacity limited to nC. This compares with the n-th product of the test channel $\mathcal{C}_t^{\,n}$ which provides a distortion precisely equal to d_C, but which has a channel capacity $nC_t > nC$.

One such set of transitions is given by the following example: Consider the high probability set of source words $W_H^{\,n}$ which have a composition within some small fixed radius of $\underline{p}$. For each of these, consider the high (conditional) probability set of output words $Z_H^n|\underline{w}$ (as governed by the test channel statistics) for which the distortion $d(\underline{w},\underline{z})$ and the mutual information $I(\underline{w},\underline{z})$ are within some small interval around the means d_C and C. Pairs $\underline{w}$ and $\underline{z}$ in these sets are joined by their test channel transition probabilities $q_{tc}(\underline{z}|\underline{w})$. Since for large n the set W_H^n and the sets $Z_H^n|\underline{w}$ can be made to include all possible events except a set of arbitrarily small probability, the average distortion and information rate on this channel must be correspondingly close to d_C and C regardless of the transitions assigned to $\underline{w},\underline{z}$ pairs outside W_H^n and $Z_H^n|\underline{w}$. For the same reason, the output statistics on the channel we are constructing will be approximately equal to $g(\underline{z})$, the output probabilities on the test channel $\mathcal{C}_t^n$. We complete the definition of the channel by minimizing the channel capacity. We assign to the pairs $\underline{w} \in W_H^n$, $\underline{z} \in' Z_H^n|\underline{w}$ a transition probability proportional to $g(\underline{z})$ (with the proportionality adjusted to make $\sum_{z^n} q(\underline{z}|\underline{w}) = 1$). And, to pairs $\underline{w} \in' W_H^n$, $\underline{z}$, we assign the transition probability $g(\underline{z})$.

To see that the capacity of this channel is approximately C, we join all the inputs not in W_H^n. This doesn't change anything since they are indistinguishable at the decoder anyway. Except for this one input and its transitions, we can see that the channel is approximately uniform from the input and further that it is approximately doubly uniform if only the $\underline{z}$'s in the sets $Z_H^n|\underline{w}$ are considered. Since the $\underline{w},\underline{z}$ pairs not in the sets $\underline{w}$, $Z_H^n|\underline{w}$ do not contribute any appreciable probability, the double uniformity approximation is used to conclude that a uniform probability is needed over W_H^n to approximately maximize the information transmission. But over W_H^n, the probability distribution $p(\underline{w})$ is itself approximately uniform and therefore it approximately maximizes the mutual information. The conclusion is that our constructed channel, which is being used close to the rate C by $p(\underline{w})$, is also being used approximately to capacity.

It is just this type of channel that is formed in the coding ensemble of Section 3.1. While each ensemble member uses only M points in Z^n and has an equivocation nearly zero, the ensemble viewed as a whole uses all members of Z^n and has a positive equivocation bounded away from zero. In this ensemble, typical source words like those in W_H^n, are almost always transmitted through the system without error to a decoded word at a distortion from $\underline{w}$ about d_C, that is, to a $\underline{z}$ in $Z_H^n|\underline{w}$. In the unlikely event there was not a member of θ_1 within d_C from $\underline{w}$, any $\underline{z}$ at all on θ_1 was used to represent (and hence decode) $\underline{w}$. Since the members of θ_1 were chosen according to $g(\underline{z})$, there results possible transitions between $\underline{w} \in W_H^n$ and $\underline{z} \in' Z_H^n|\underline{w}$ of probability $g(\underline{z})$. This is the reason we assigned such transitions in the channel that we have just constructed. Finally, for the non-typical $\underline{w} \in' W_H^n$, there very well might not be a satisfactory $\underline{z}$ on θ_1, the result of which would again be transition probabilities between $W_H^{n\prime}$ and Z^n equal to $g(\underline{z})$. These transitions also agree with those in the channel we have constructed.

Thus the coding ensemble constructs a set of transition probabilities between W^n and Z^n that have all the important properties of the test channel $\mathcal{C}_t^n$, at <u>least as far as the given source $\mathcal{S}$ is concerned.</u> The distortion and information transmission rate are essentially the same, as are the important transition and output probabilities. The only difference is that the transitions in the coding ensemble do not have as high a capacity as does $\mathcal{C}_t^n$, and this is completely irrelevant to a user of the system who cannot control the probabilities of his source $\mathcal{S}$.

Chapter 4

ILLUSTRATIVE EXAMPLES

In the first three examples of this chapter, we illustrate different types of source-channel mismatch and calculate the effect of each upon the coefficient a in the lower bound of Equation 2.85. Each of these examples tends to strengthen the suggestion in the lower bound result that this coefficient is a measure of source-channel mismatch since it increases monotonically as the channel is perturbed away from the matching channel. Because the channel statistics influence only the first two terms of a, we use in these examples a doubly uniform source for which the $\sigma^2(\theta)$ term is zero (see Equation 2.87). To further isolate the relative matching properties of the source-channel pairs, we keep constant the channel capacity per source output, C, as the channel is varied. Thus the distortion per source component has the same asymptote d_C for all source-channel pairs and the only difference in the lower bound curves, at least asymptotically, is in the coefficient a.

In Example 4.4 we include in the system a continuous channel which is to be used by a discrete source with a discrete modulator. Now, as the modulator changes the discrete channel extracted from the actual channel changes and <u>both</u> its capacity and its matching characteristics change. We will see that both properties are not necessarily optimized for the same modulator structure and, therefore, that one must strike a compromise (influenced by the block length of interest) between a modulator design that minimizes the asymptote d_C and maximizes the rate of approach to d_C.

In the last example, we make a sample calculation of all the constants in both the lower and upper bounds to distortion for two particular transmission systems.

Example 4.1

This example illustrates a dimensionality, or coding block length, mismatch between a source and channel. We take for the source $\mathcal{S}$ the m_s'th product of a binary symmetric source, defined by $\underline{p} = (\frac{1}{2},\frac{1}{2})$ and $d_{11} = d_{22} = 0$, $d_{12} = d_{21} = 1$. For the channel $\mathcal{C}$ we take the m_c'th product of a binary symmetric channel, each component $\mathcal{C}_i$ having a crossover probability p. The channel capacity per source component is (m_c/m_s) times the capacity of $\mathcal{C}_i$ and is kept constant as m_c/m_s is varied by appropriately changing the crossover probabilities p. Figure 4.1 shows the dependence of a upon m_c/m_s. When comparing the two curves in this figure, it must be noted that the ordinate has been normalized by d_C. We know that for $m_c/m_s = 1$ the source and channel are precisely matched and this is indicated in the figure by a value a = 0 at that point. Above this point, a increases monotonically in m_c/m_s and can be shown to have the asymptotic form $a \simeq k\sqrt{m_c/m_s}$. Below $m_c/m_s = 1$, a also becomes unbounded as m_c/m_s approaches the ratio that requires each component channel $\mathcal{C}_i$ be noiseless. This is consistant with the noiseless channel result (Equation 2.91) which indicated that the rate of approach of the distortion to d_C was not as $\frac{a}{n}$ but as $\frac{\ln n}{n}$.

Example 4.2

Here we do not change the relative dimensionality, only the form of the channel. The source is a binary symmetric source and the channel a binary

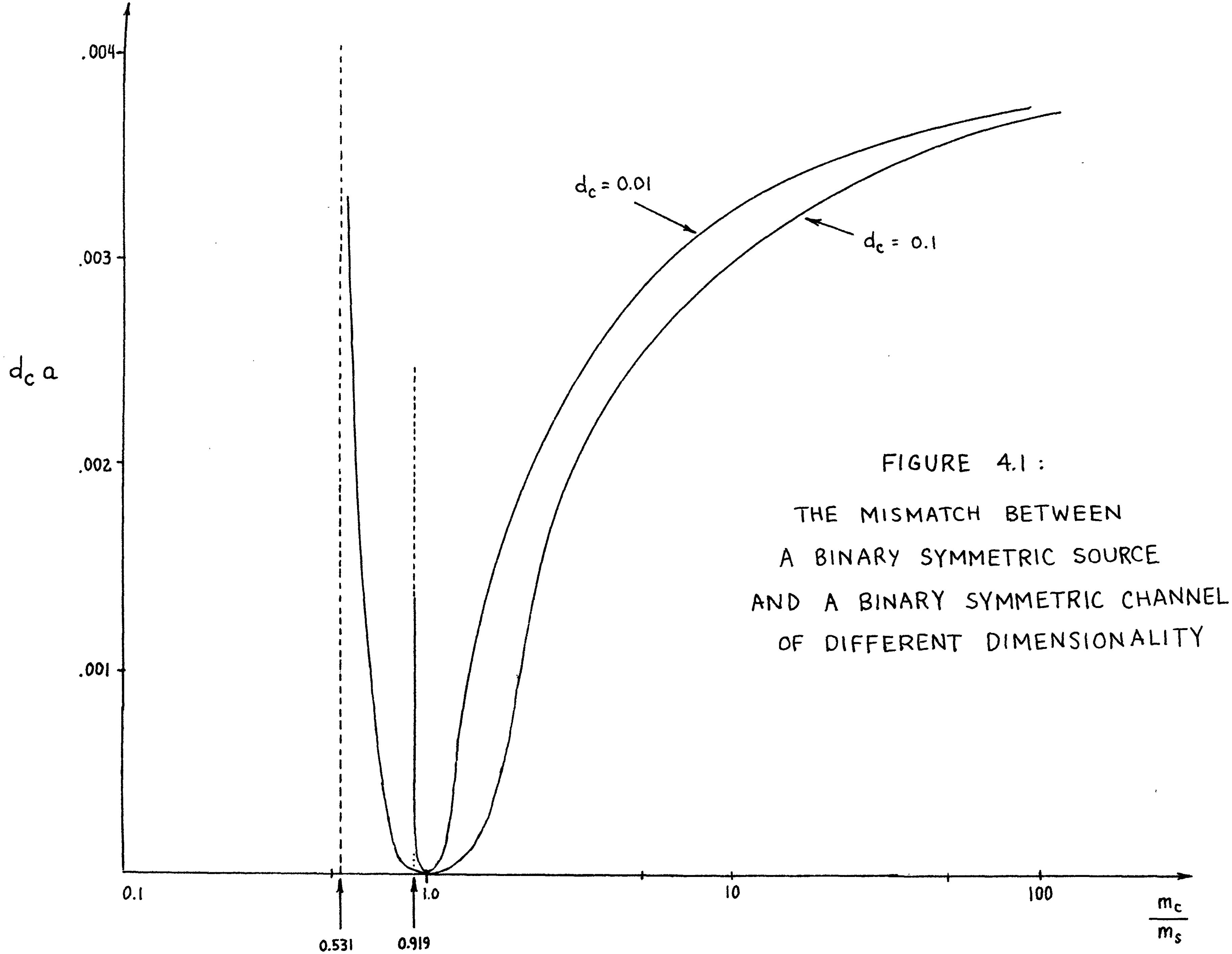

FIGURE 4.1 :

THE MISMATCH BETWEEN
A BINARY SYMMETRIC SOURCE
AND A BINARY SYMMETRIC CHANNEL
OF DIFFERENT DIMENSIONALITY

nonsymmetric channel of varying asymmetry. The crossover probabilities are again changed in a way that does not vary the capacity. We see in Figure 4.2 that a is rather insensitive to small perturbations from a BSC and in any case is affected less by this type of mismatch than a dimensionality mismatch. A similar result obtains if the source is also allowed to be nonsymmetric.

Example 4.3

The last example illustrating a source-channel mismatch includes a binary symmetric source and a discrete channel that models the m orthogonal signal modulator used in the next example. This channel has m inputs and m outputs and has from each input one transition of probability $1-(m-1)p$ and $m-1$ transitions of probability p. The numbers m and p are varied together in such a way that the capacity of the channel remains constant. We see in Figure 4.3 that the mismatch coefficient a is much higher when the binary symmetric source is used with this channel than when it is used with that product binary symmetric channel of Example 4.1 which has available an input alphabet of equal size. The comparison can be made on Figures 4.1 and 4.3 at points for which $m_c/m_s = \log_2 m$.

Example 4.4

For this example we assume the channel in the transmission system to be a band limited channel with additive white gaussian noise in the allowed bandwidth. During the interval (0,T), the discrete modulator is constrained to transmit one of m orthogonal signals in each of B bauds and altogether an energy no greater than E. To model the bandwidth constraint

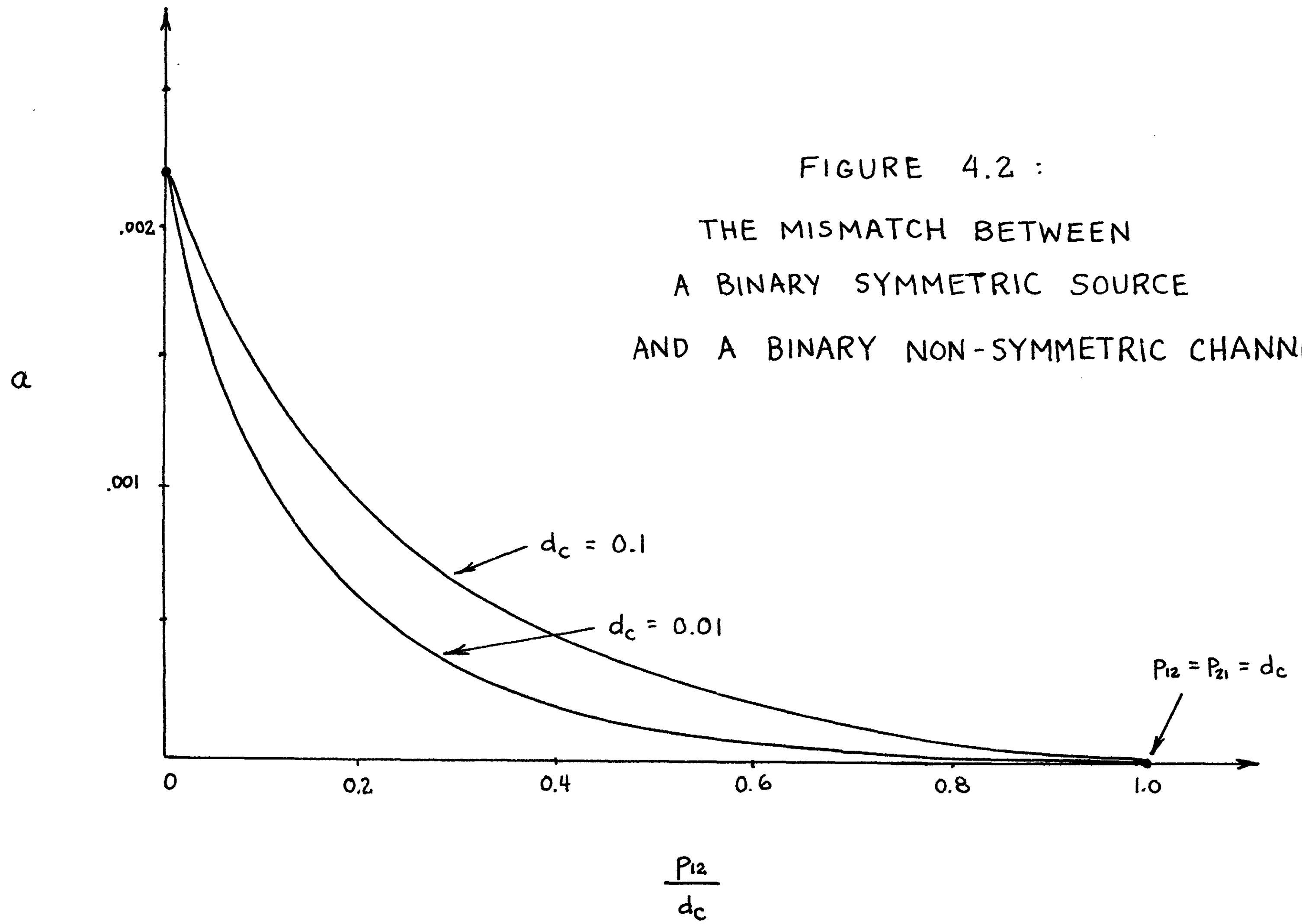

FIGURE 4.2 :

THE MISMATCH BETWEEN
A BINARY SYMMETRIC SOURCE
AND A BINARY NON-SYMMETRIC CHANNEL

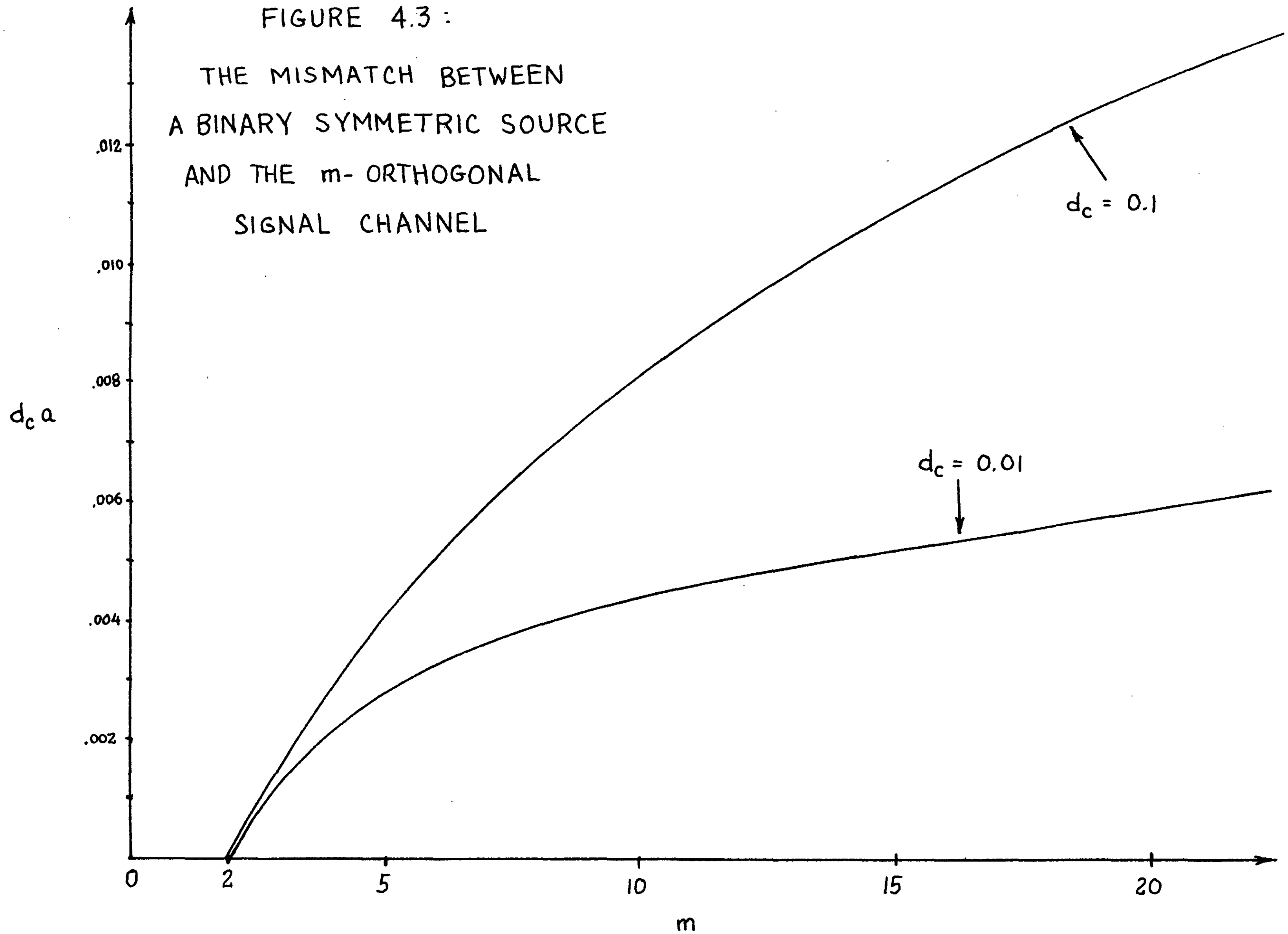

FIGURE 4.3 :

THE MISMATCH BETWEEN A BINARY SYMMETRIC SOURCE AND THE m- ORTHOGONAL SIGNAL CHANNEL the mB product is assumed constant, but m and B can otherwise be varied to optimize the system. Thus the equivalent discrete channel is the B'th product of the m input doubly uniform channel of Example 4.3. The source to be transmitted is a binary symmetric source with an output rate of M_S digits every T seconds.

In Figure 4.4 we show the minimum attainable distortion d_C (determined through the channel capacity) and the mismatch coefficient a as a function of m. For the values shown in the figure, we see that while d_C is minimized at m = 15, the coefficient a is then quite large. And, around m = 22 where a = 0, the minimum distortion d_C is higher than that which can be realized with smaller m. The conclusion from this is that the modulator should be designed with m = 15 (to maximize capacity and minimize the asymptote d_C) only when one is willing to use very long coding block lengths. For shorter block lengths, a larger value of m, and a corresponding smaller value of a, could cause the average distortion to be lower in spite of the higher asymptote d_C. For this example a compromise design with m about 19 would probably be best over a range of intermediate block lengths.

It is interesting to note in this example that the coefficient a can be zero even when the source and channel are not matched. This is consistant with the interpretation of a given in Chapter 2 as a necessary but not sufficient condition for matching. We remember that the coefficient a being zero does not imply that the lower bound $d_L(S)$ in Equation 2.85 is precisely d_C for all n. There are several other terms of $o(\frac{1}{n})$ in this equation, that have not been specified, which are not necessarily zero when a = 0.

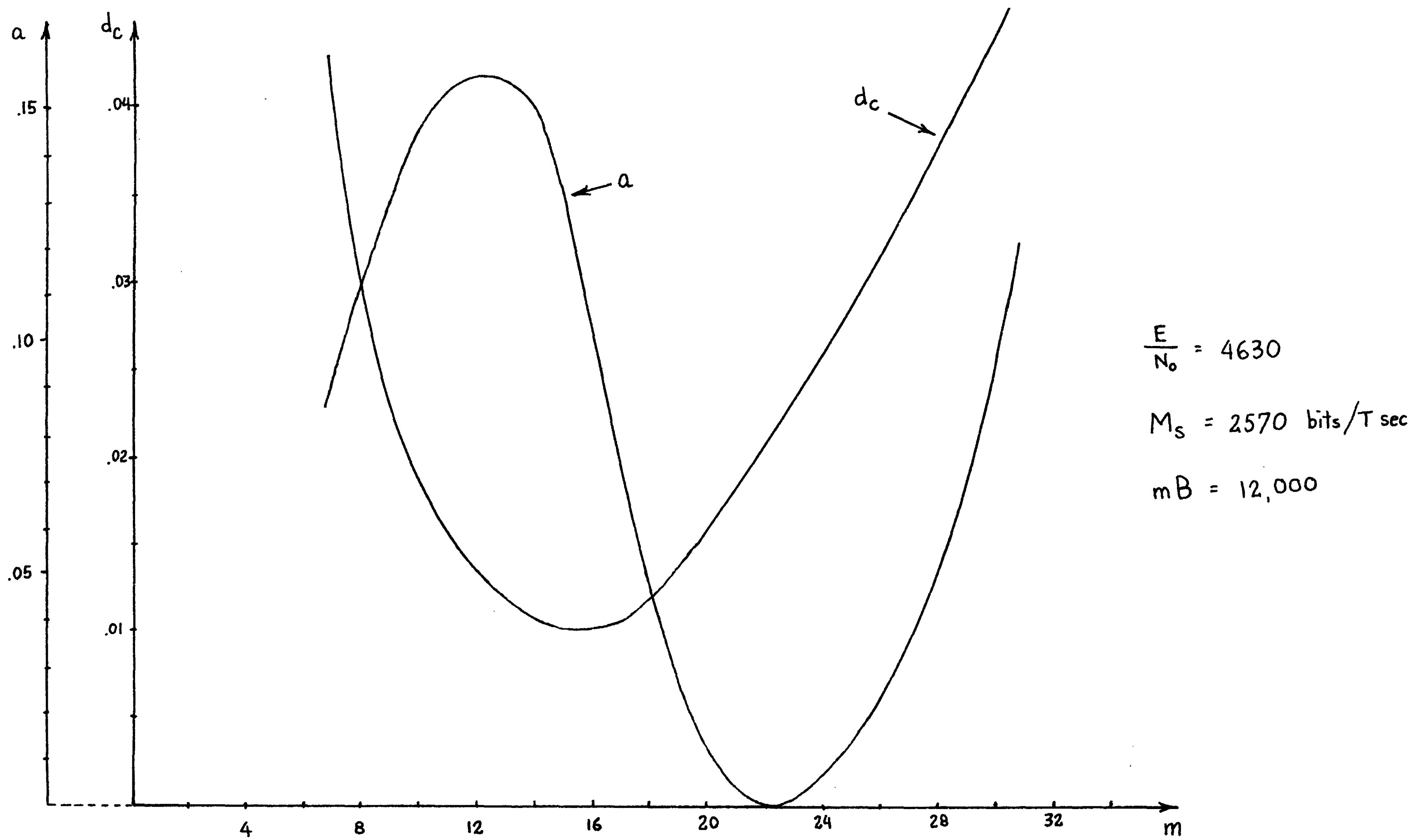

FIGURE 4.4 : THE INFLUENCE OF THE MODULATOR DESIGN IN EXAMPLE 4.4 UPON THE MINIMUM ATTAINABLE DISTORTION AND THE MISMATCH COEFFICIENT

Example 4.5

In this final example, we go through all the calculations, in both the upper and lower bounds, for a given source and channel. For $\mathcal{S}$, we take a binary nonsymmetric source with $\underline{p} = (.75, .25)$ and $d_{11} = d_{22} = 0$, $d_{12} = 1$, $d_{21} = 2$. Two calculations are made. The first includes in the system a binary symmetric channel of sufficient capacity to obtain a distortion level $d_C = 0.02$, and the second includes an m-input doubly uniform channel representing the transmission of m orthogonal signals. For this last channel we choose $m = 20$ and adjust the capacity to allow a distortion $d_C = 0.1$.

To find the channel capacities necessary to obtain distortions of $d_C = 0.02$ and 0.1, we must first find the test channels for $\mathcal{S}$ at these points on the rate-distortion curve. A straightforward minimization of the mutual information by varying the transition probabilities, under a distortion and probability constraint, yields the test channels shown in Figure 4.5. The corresponding mutual information rates, which are the required capacities, are also shown in the figure.

In the lower bounds, the evaluation of the coefficient a requires that of the functions $\mu''(s_0)$, $\gamma''(-1)$, and $\sigma^2(\theta)$. From Equations 2.30 and 2.31 we have

$$\mu(s) = p_1 \mu_1(s) + p_2 \mu_2(s)$$

$$= p_1 \ln\left(g_1 + g_2 e^{s}\right) + p_2 \ln\left(g_1 e^{2s} + g_2\right)$$

$$\mu'(s) = \frac{p_1 g_2 e^{s}}{g_1 + g_2 e^{s}} + \frac{2 p_2 g_1 e^{2s}}{g_1 e^{2s} + g_2}$$

$$\mu''(s) = \frac{p_1 g_1 g_2 e^{s}}{(g_1 + g_2 e^{s})^2} + \frac{4 p_2 g_1 g_2 e^{2s}}{(g_1 e^{2s} + g_2)^2}$$

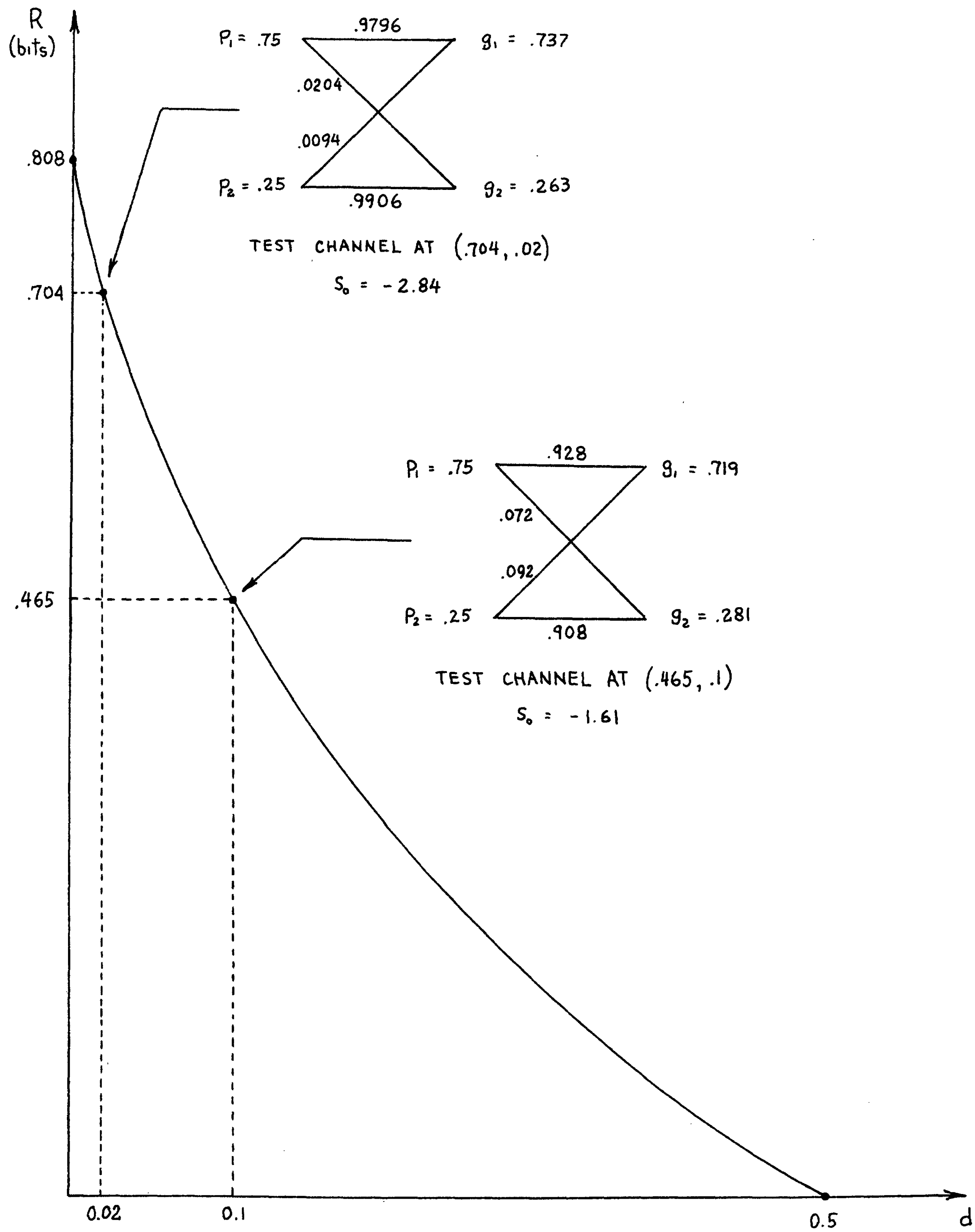

FIGURE 4.5 : THE RATE DISTORTION CURVE FOR THE SOURCE IN EXAMPLE 4.5; THE TEST CHANNEL FOR TWO DIFFERENT DISTORTIONS

and from Equation 2.60 we have

$$\sigma^2(\theta) = \sum_{i=1}^{2} p_i \left[\mu_i(s) - s\mu_i'(s)\right]^2 - \left[\sum_{i=1}^{2} p_i\left[\mu_i(s) - s\mu_i'(s)\right]\right]^2$$

which can all be evaluated using the values of $\underline{p}$, $\underline{g}$, and s_o given in Figure 4.5.

For the binary symmetric channel (used to obtain $d_C = 0.02$), we use Equations 2.32 and 2.33 to write

$$\gamma_{BSC}(t) = c_1 \ln\left[f_1^{\,1+t}(1-p)^{-t} + f_2^{\,1+t}\, p^{-t}\right]$$

$$+ c_2 \ln\left[f_1^{\,1+t}\, p^{-t} + f_2^{\,1+t}(1-p)^{-t}\right]$$

for which the second derivative, evaluated at $t = -1$ (see Equation 2.43) and at $\underline{c} = \underline{f} = (\frac{1}{2},\frac{1}{2})$ (the probabilities on $\mathcal{C}$ when it is used to capacity), equals

$$\gamma_{BSC}''(-1) = (1-p)\, p \left[\ln \frac{1-p}{p}\right]^2 .$$

In this example, the value of p that provides the required capacity of .704 bits is $p = 0.0523$.

The corresponding quantities for the m-orthogonal signal channel (used to obtain $d_C = 0.1$) are

$$\gamma_m(t) = \sum_{i=1}^{m} c_i \ln\left[f_i^{\,1+t}\left(1-(m-1)\,p\right)^{-t} + \sum_{j\neq i} f_j^{\,1+t}\, p^{-t}\right]$$

and

$$\gamma_m''(-1) = p(m-1)\left[1-(m-1)p\right]\left[\ln \frac{p}{1-(m-1)p}\right]^2$$

in which we have substituted the vector values

$$\underline{c} = \underline{f} = \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}$$

The value of p that yields the required channel capacity of .465 bits when m = 20 is p = .0368.

The data for the two lower bounds is summarized in the following table:

Table 4.1

	BSC	m $\perp$ Signal
d_C	0.02	0.10
C	0.704 b.	0.465 b.
p	0.0523	0.0368 (m = 20)
$\gamma''(-1)$	0.413	0.928
$s_o^2\mu''(s_o)$	0.196	0.350
$\sigma^2(\theta)$	0.201	0.100
s	-2.84	-1.61
a	0.244	0.299

To evaluate the coefficient b in the upper bound, we are required to calculate the constants c_{3i} and c_4 in Equation 3.54. The source constant c_{3i} was defined as one-half the second derivative of the function $E_{si}(R)$ at $R = R^*$ when the parameter R^* is chosen very close to C (and d^* close to d_C). The pertinent geometry for the calculation of c_{3i} by both the hypercube method (i = 1) and the maximum probability point method (i = 2) is shown in Figure 4.6. In the figure we show the two sets Q and Q' which separate Q^2 into those compositions for which $R(d_C,\underline{q}) < R-\delta$ and $\geq R-\delta$ when $R > C = R(d_C,\underline{p})$ (see Equations 3.15ab, 3.17 and 3.18). Around the point $\underline{q} = \underline{p}$, it can be shown that $R(d_C,\underline{q})$ increases with increasing q_2, therefore the critical vertex of the hypercube (here the square K^2) is the upper one as shown in the figure. This is also the point of maximum probability $P(\underline{q})$ for $\underline{q} \in Q'$.

In Equation A4.8 we found

$$E_{s1}''(R^*) = \left[\frac{1}{a_1\mu_1(s) + a_2\mu_2(s)}\right]^2 \min_i \frac{1}{P_i(1-P_i)} \tag{4.1}$$

in which $\underline{a} = (a_1,a_2)$ is a vector, with $|a_i| = 1$ and $a_1 + a_2 = 0$, that indicates the critical vertex when u is close to zero. Here $a_1 = -1$ and $a_2 = +1$. Using the maximum probability point method, we found in Equation A4.9

$$E_{s2}''(R^*) = \frac{1}{P_1}\left(\frac{dq_1^\circ}{dR}\right)^2 + \frac{1}{P_2}\left(\frac{dq_2^\circ}{dR}\right)^2$$

which for this example becomes

$$E_{s2}''(R^*) = \frac{1}{P_1}\frac{1}{\left(-\frac{dR}{du}\right)^2} + \frac{1}{P_2}\frac{1}{\left(\frac{dR}{du}\right)^2}$$

$$= \frac{1}{P_1(1-P_1)}\left[\frac{1}{\mu_1(s) - \mu_2(s)}\right]^2. \tag{4.2}$$

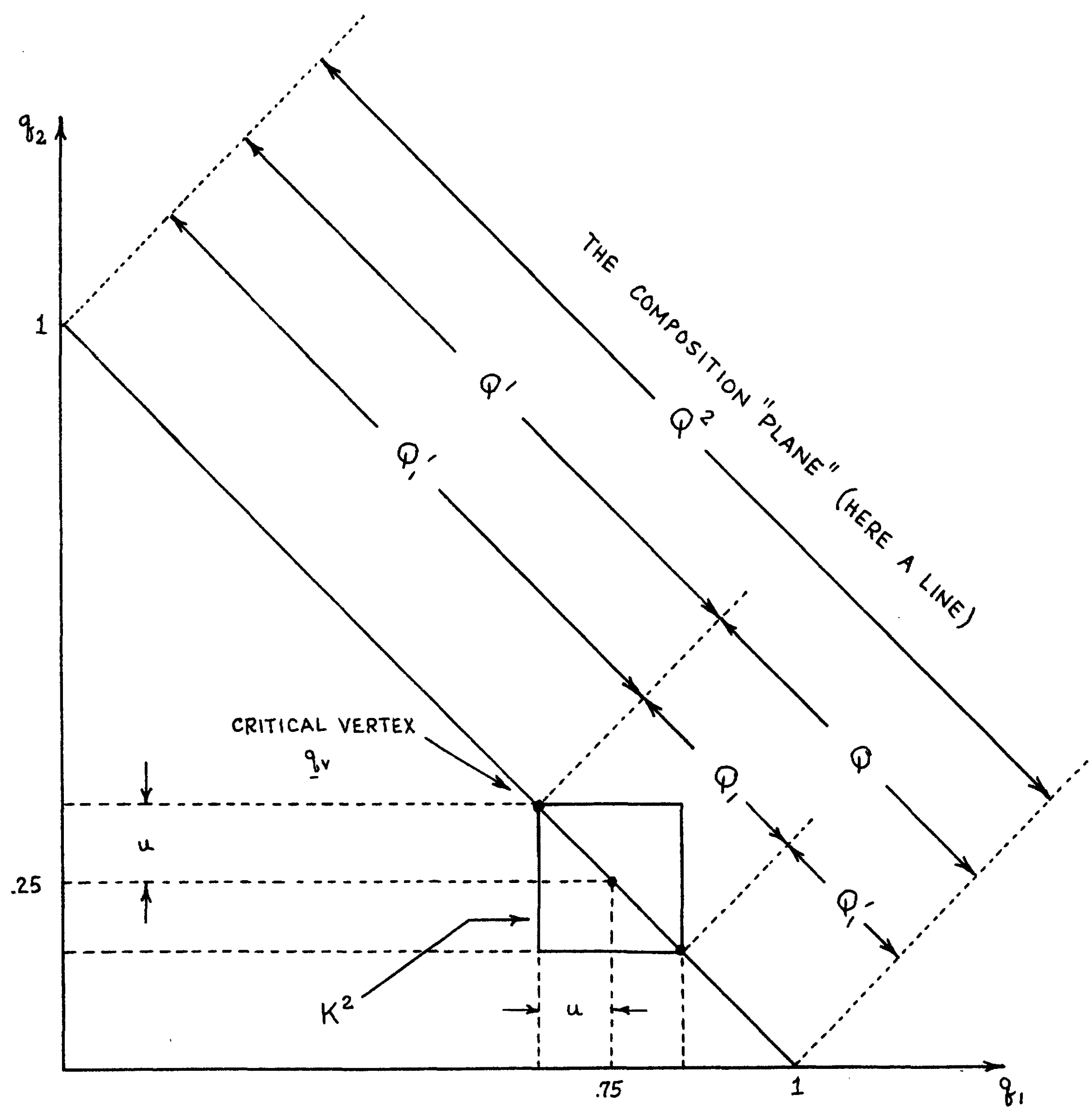

FIGURE 4.6: THE COMPOSITION SPACE FOR THE SOURCE IN EXAMPLE 4.5 ILLUSTRATING THE SETS Q' AND Q_1'

We see, by comparing Equations 4.1 and 4.2, that $E''_{s1}(R^*) = E''_{s2}(R^*)$ for the source in this example.

The channel constant c_4 was defined as one-half the second derivative of the channel exponent function $E(R)$ at the point $R = C$. We use the expression for $E(R)$ found by Gallager[14]

$$E(R) = E_o(\rho) - \rho E_o'(\rho)$$

$$R = E_o'(\rho)$$

from which the required second derivative can be found equal to

$$E''(R)\Big|_C = -\frac{1}{E_o''(\rho)\Big|_{\rho=0}} .$$

Direct differentiation of Equation 20 in Gallager's paper (keeping the channel input probabilities p_k fixed) yields, in our notation

$$E''(C)^{-1} = \sum_j \sum_k c_k p_{jk} \left(\ln \frac{p_{jk}}{f_j} \right)^2 - \left[\sum_j \sum_k c_k p_{jk} \ln \frac{p_{jk}}{f_j} \right]^2 . \qquad (4.3)$$

We see the expression in Equation 4.3 is very similar to our function $\gamma''(-1)$ given by Equation 2.45. Although they are different in general, they can be shown to agree when the channel is doubly uniform, therefore

$$c_4 = \frac{1}{2} E''(C) = \frac{1}{2\gamma''(-1)} .$$

All of the constants in both upper bounds can now be found. They are given in the following table:

Table 4.2

	BSC	m ⊥ Signal
d_C	0.02	0.10
C	0.704 b.	0.405 b.
p	0.0523	0.0368 (m = 20)
c_{31}, c_{32}	2.45	3.18
c_4	1.21	0.54
s	-2.84	-1.61
b	0.385	0.840

This completes our final example.

We have provided in this chapter several illustrations of source-channel mismatch and have calculated the effect of each type of mismatch on the coefficient a in the lower bound to distortion. We have also illustrated, with an example, how the mismatch coefficient, a, might influence the design of a modulator that is to be used between a given source and channel. In addition we have included a sample calculation of all the important constants in both the lower and upper bounds. Although the upper and lower bounds do not agree in form (thus preventing a meaningful comparison between the coefficients a and b), these bounds are both valuable in that there had not previously existed any result relating the minimum attainable transmission distortion to the coding block length in the type of communication system we have considered. And, as one can see from the last example, the calculations required to obtain the coefficients a and b, as well as the asymptote d_C, are not particularly difficult. It was to obtain easily calculated bounds of this type that much of our effort was directed toward establishing the asymptotic behavior of the lower and upper bounds in Chapters 2 and 3. Even though their use for block lengths that are not very large would necessarily introduce approximations, the simplicity of these bounds would enable one to obtain quick first order information about the minimum attainable transmission distortion. In any situation where this approximation is undesireable, the strict lower and upper bounds to distortion in Chapters 2 and 3 could be used.

Appendix 1

APPENDIX FOR SECTION 2.1

A. Derivation of the Lower Bound in Theorem 2.3

Equation 2.27 reads

$$d(\underline{w}) \geq \int_{I_{min}}^{I_{max}} d_L(I)\, dF_2(I) \,. \tag{A1.1}$$

Before the Taylor Series expansion is used for $d_L(I)$ at $I = \bar{I}$, we divide the region of integration into two sets: an interval $\mathcal{J} = \left[I_a, I_b\right]$ that includes $\bar{I}$, and its complement $\mathcal{J}'$. I_b is chosen to be less than both the expected value of I_1, $E(I_1)$, and the value of I corresponding to the expected value of d, $I\left[E(d)\right]$. (We note that these expected values are conditioned on the vectors $\underline{x}$ and $\underline{w}$ respectively.) This excludes both of these points from $\mathcal{J}$ which is necessary since the bounds to $G(d)$ and $F_1(I)$ that are used to define $d_L(I)$ are valid only when $I < E(I_1)$ and $d < E(d)$. Looking ahead just a little (Equations 2.40, 2.41 and A2.5) we see that the point I_{min} (where $t = -\infty$) should also be excluded from $\mathcal{J}$ because the derivatives in the Taylor Series for $d_L(I)$ are not necessarily bounded at this endpoint. (This is also true at $I\left[E(d)\right]$ where $s = 0$.) Thus we have

$$I_{min} < I_a < \bar{I} < I_b < E(I_1),\ I\left[E(d)\right] \tag{A1.2}$$

These are the only requirements in the selection of $\mathcal{J}$, which could otherwise be chosen to optimize the answer. However, it will be seen that the choice of I_a and I_b is not critical since it affects only low order terms which are not significant when n becomes large.

A further lower bound to $d(\underline{w})$ results if we retain only the integral over $\mathcal{J}$, or

$$d(\underline{w}) \geq \int_{I_a}^{I_b} d_L(I)\, dF_2(I)\,. \tag{A1.3}$$

In this equation we can use the Taylor Formula with Remainder

$$T(I-\bar{I}) = d_L(\bar{I}) + d_L'(\bar{I})(I-\bar{I}) + \tfrac{1}{2} d_L''(\bar{I})(I-\bar{I})^2 + \tfrac{1}{6} d_L'''(I')(I-\bar{I})^3$$

with $I_a \leq I' \leq I_b$, to obtain

$$d(\underline{w}) \geq \int_{I_a}^{I_b} T(I-\bar{I})\, dF_2(I)$$

$$= \int_{I_{min}}^{I_{max}} T(I-\bar{I})\, dF_2(I) - \int_{I_{min}}^{I_a} T(I-\bar{I})\, dF_2(I) - \int_{I_b}^{I_{max}} T(I-\bar{I})\, dF_2(I). \tag{A1.4}$$

The first integral can be written in terms of the central moments μ_i of I_2,

$$\int_{I_{min}}^{I_{max}} \cdots = d_L(\bar{I}) + d_L'(\bar{I})\mu_1(I_2) + \tfrac{1}{2} d_L''(\bar{I})\mu_2(I_2) + \tfrac{1}{6} d_L'''(I')\mu_3(I_2). \tag{A1.5}$$

This in turn can be written in terms of the central moments $\mu_i(I_{km})$ of the variables that sum to I_2 (Equation 2.23) if we use the following identities:

$$\mu_1(I_2) = 0$$

$$\mu_2(I_2) = \frac{1}{n} \sum_{k=1}^{K} c_k\, \mu_2(I_{km})$$

$$\mu_3(I_2) = \left(\frac{1}{n}\right)^2 \sum_{k=1}^{K} c_k\, \mu_3(I_{km})\,. \tag{A1.6}$$

Finally, if we define the function $|T|(I-\bar{I})$ as $T(I-\bar{I})$ with the absolute value taken of each term, the last two integrals in Equation A1.4 can be bounded by

$$\int_{I_{min}}^{I_a} \cdots \leq |T|(\bar{I}-I_{min}) \; Pr(I_2 < I_a) \qquad (A1.7)$$

and

$$\int_{I_b}^{I_{max}} \cdots \leq |T|(I_{max}-\bar{I}) \; Pr(I_2 > I_b). \qquad (A1.8)$$

In these expressions, I_{min} is finite if $f_\ell \neq 0$ for all ℓ (and this is a restriction that will be imposed in Section 2.1.4) and I_{max} can be taken as the largest finite value of $\ln \frac{f_\ell}{P_{k\ell}}$ since this is the largest value of I for which the random variable I_2 has non-zero probability.

Since the third central moment of I_2 is proportional to $(\frac{1}{n})^2$ and the probabilities in Equations A1.7 and A1.8 are both exponentially small in n, we will consider these terms only as correction terms to the first three terms in Equation A1.5. We summarize these results:

$$d(\underline{w}) \geq d_L(\bar{I}) + \frac{1}{2} d_L''(\bar{I})\mu_2(I_2) + c_1(n), \qquad (A1.9)$$

$$c_1(n) = \min_{I' \in \mathcal{I}} \left[\frac{1}{6} d_L'''(I')\mu_3(I_2)\right] - |T|(\bar{I}-I_{min}) \; Pr(I_2 < I_a)$$

$$- |T|(I_{max}-\bar{I}) \; Pr(I_2 > I_b),$$

$$\mu_2(I_2) \sim \frac{1}{n}, \qquad c_1(n) = o\left(\frac{1}{n}\right).$$

The minimization is included in the first correction term since in any particular example this is easier than finding I'.

B. Lower Bound to $K(n,t)$ over $[I_a, I_b]$

We have used in Equation 2.36 Fano's lower bound to the tail of a distribution:

$$F_1(I^-) \geq K(n,t) \exp n\left[\gamma(t) - t\gamma'(t)\right] \quad \text{(A1.10)}$$

in which

$$K(n,t) = \left(\frac{1}{2\pi n}\right)^{\frac{K(L-1)}{2}} \exp\left[-|t|\Delta - \frac{KL}{12} - \frac{1}{n}\sum_k \sum_\ell \frac{1}{c_K Q_{K\ell}(t)}\right] \quad \text{(A1.11)}$$

$$Q_{K\ell}(t) = \frac{f_\ell^{1+t} \; p_{K\ell}^{-t}}{\sum_m f_m^{1+t} \; p_{Km}^{-t}}$$

and

$$\gamma'(t) = I^-.$$

It is intended to bound the function $F_1(I)$ over the entire interval $[I_a, I_b]$. In this interval t is negative and increases monotonically from t_a to t_b according to Equation A1.11. Therefore we can lower bound $K(n,t)$ by

$$K(n) = \left(\frac{1}{2\pi n}\right)^{\frac{K(L-1)}{2}} \exp\left[-|t_a|\Delta + \frac{KL}{12} - \frac{1}{n}\max_{t_a \leq t \leq t_b} \sum_k \sum_\ell \frac{1}{c_K Q_{K\ell}(t)}\right] \quad \text{(A1.12)}$$

and use this to further lower bound $F_1(I)$ by

$$F_1(I) \geq K(n) \exp n\left[\gamma(t) - t\gamma'(t)\right]. \quad \text{(A1.13)}$$

Now when the bound to $F_1(I)$ is differentiated with respect to t, the coefficient can be treated as a constant over the entire interval $\left[I_a, I_b\right]$.

C. The Correction Term in Theorem 2.4a

The correction term $c_1(n)$ in Theorem 2.3 (Equation A1.9) includes a term with $d_L'''(I')$. With a little algebra, the third derivative of $d_L(I)$ can be found to be

$$d_L'''(I) = -\frac{3t}{s^3\mu''}\left[\frac{1}{\gamma''} - \frac{t^2}{s^2\mu''}\right] - \frac{1}{s}\left[\frac{\gamma'''}{\gamma''^3} - \frac{t^3\mu'''}{s^3\mu''^3}\right]. \tag{A1.14}$$

We have, in Appendix 1A, defined the interval $\left[I_a, I_b\right]$ to include only points for which s is non-zero and t is finite. And, since it has been assumed (following 2.33) that the components of $\underline{f}$ are non-zero, γ'' is also non-zero. Further, it can be shown that the vector $\underline{g}$, which must later be chosen to minimize the bound, cannot have only one non-zero component (since the decoder would then be allowed only a single decoding letter). This eliminates the possibility of μ'' being zero. It can also be shown that γ''' and μ''' are finite. Therefore, the total third derivative is bounded over $\left[I_a, I_b\right]$ and the correction term $c_1(n)$ in Equation A1.9 is $o(\frac{1}{n})$.

D. Proof of Theorem 2.4b

The following parametric representation for $d_L(I)$ in the interval $\left[I_a, I_b\right]$ is found by equating the bounds in Equations 2.48 and 2.49:

$$\mu(s) - s\mu'(s) + \frac{1}{n}\ln\left[\frac{1}{\sqrt{2\pi n s^2\mu''(s)}} + A_U\right]$$

$$= \gamma(t) - t\gamma'(t) + \frac{1}{n}\ln\left[\frac{1}{\sqrt{2\pi n t^2\gamma''(t)}} + A_L\right] \tag{A1.15a}$$

with

$$\mu'(s) = d \tag{A1.15b}$$

and

$$\gamma'(t) = I \ . \tag{A1.15c}$$

Equation A1.15a can be rewritten as

$$\mu(s) - s\mu'(s) = \gamma(t) - t\gamma'(t) - \frac{1}{2n} \ln \frac{t^2\gamma''(t)}{s^2\mu''(s)}$$

$$+ \frac{1}{n} \ln \left[\frac{1 + \sqrt{2\pi n\, t^2 \gamma''(t)}\ A_L}{1 + \sqrt{2\pi n\, s^2 \mu''(s)}\ A_U} \right] . \tag{A1.16}$$

Because the functions A_L and A_U are $o(\frac{1}{\sqrt{n}})$, the term in brackets approaches one as n becomes large, making the logarithm approach zero. Therefore, the entire last term is $o(\frac{1}{n})$ over the interval $[I_a, I_b]$.

At the value $I = \bar{I}$, we have $s = s_o$ and $t = t_o$, therefore

$$d_L(\bar{I}) = \mu'(s_o) \tag{A1.17}$$

as in Equation 2.39. The derivatives though are slightly different. If Equation A1.16 is abbreviated

$$\mu(s) - s\mu'(s) = \gamma(t) - t\gamma'(t) - \frac{1}{2n} B(s,t)$$

we find

$$\frac{ds}{dt} = \frac{t\gamma''(t) + \frac{1}{2n} B_t'(s,t)}{s\mu''(s) - \frac{1}{2n} B_s'(s,t)} \tag{A1.18}$$

which can be used in

$$d_L'(I) = \frac{dd}{dI} = \frac{dd}{ds}\,\frac{ds}{dt}\,\frac{dt}{dI}$$

to establish

$$\begin{aligned} d_L'(I) &= \frac{t + \frac{1}{2n}(B_t'/\gamma'')}{s - \frac{1}{2n}(B_s'/\mu'')} \\ &= \frac{t}{s} + \frac{1}{2n} C(s,t,n) \\ &= \frac{t}{s} + o_n(1) \, . \end{aligned} \tag{A1.19}$$

The symbol $o_n(1)$ is used for any function $h(n)$ that has the limit zero when n becomes large and can be used here for C/n since C is bounded over $[I_a, I_b]$. In the same way we find

$$d_L''(I) = \frac{1}{s}\left[\frac{1}{\gamma''(t)} - \frac{t^2}{s^2\mu''(s)}\right] + \frac{1}{2n}\frac{dC}{dI}$$

or, since dC/dI can be shown to also be bounded in $[I_a, I_b]$,

$$d_L''(I) = \frac{1}{s}\left[\frac{1}{\gamma''(t)} - \frac{t^2}{s^2\mu''(s)}\right] + o_n(1) \, .$$

At the point $I = \bar{I}$, this derivative equals

$$d_L''(\bar{I}) = \frac{1}{s_0}\left[\frac{1}{\gamma''(-1)} - \frac{1}{s_0^2\mu''(s_0)}\right] + o_n(1) \, . \tag{A1.20}$$

The third derivative, which is included in the correction term $c_1(n)$, is given by Equation A1.14, but again with an additional term of $o_n(1)$, therefore it is still bounded and the entire correction term is $o(\frac{1}{n})$. The terms of $o_n(1)$ in Equation A1.20 can be grouped with the correction terms because the $\frac{1}{n}$ factor in the $\text{var}(I_2)$ reduces these terms to also be $o(\frac{1}{n})$.

When we collect these results and use Equation 2.45 in Equation 2.28 we have

$$d(\underline{w}) \geq \mu'(s_o) - \frac{1}{2ns_o}\left[\frac{\gamma''(-1)}{s_o^2\mu''(s_o)} - 1\right] + o\left(\frac{1}{n}\right) \qquad \text{(A1.21)}$$

with

$$\mu(s_o) - s_o\mu'(s_o) = \bar{I} - \frac{1}{2n}\ln\frac{\gamma''(-1)}{s_o^2\mu''(s_o)} + o\left(\frac{1}{n}\right).$$

E. A Property of that Function $g_o(\underline{z})$ which Minimizes the Lower Bound in Theorem 2.1

In Theorem 2.1 we found a lower bound to the single word transmission distortion $d(\underline{w})$. It was given by Equation 2.11. If we abbreviate the right side of this equation by $d_L(\underline{w})$ and average both $d(\underline{w})$ and $d_L(\underline{w})$ over all source words $\underline{w}$, we obtain a lower bound to average distortion for the total source,

$$d(S) \geq \sum_{W^n} P(\underline{w})\, d_L(\underline{w}) \,, \qquad \text{(A1.22)}$$

which is a function of both the probability functions $f(\underline{y})$ and $g(\underline{z})$. We show in this appendix that for any choice of $f(\underline{y})$, there exists a function $g_o(\underline{z})$, that is only a function of the composition of $\underline{z}$, for which the right side of Equation A1.22 is at least as small as it is for any other probability function on Z^n. What is meant by $g_o(\underline{z})$ being only a function of the composition of $\underline{z}$ is that $g_o(\underline{z}_1) = g_o(\underline{z}_2)$ for all pairs of decoder words, $\underline{z}_1$ and $\underline{z}_2$, that have the same composition.

The idea of the proof is to show that whenever the probability function $g(\underline{z})$ has $g(\underline{z}_1) \neq g(\underline{z}_2)$ for any pair of decoder words with equal composition, it is possible to change $g(\underline{z})$ to a probability function that is constant over each set of $\underline{z}$'s with equal composition and at the same time reduce (or at worst leave unchanged) the sum in Equation A1.22. We first separate the source words into sets with equal composition and write

$$d(S) \geq \sum_{\substack{\text{ALL} \\ \text{COMPOSITIONS} \\ \underline{q}}} P(\underline{w} \mid \text{comp.}\, \underline{w} = \underline{q}) \sum_{\substack{\forall \underline{w} \ni \\ \text{comp.}\, \underline{w} = \underline{q}}} d_L(\underline{w}) . \tag{A1.23}$$

What we show is that for every composition $\underline{q}$, the inner sum is never increased by the change in $g(\underline{z})$ described above.

We start with any source word $\underline{w}_o$ with composition $\underline{q}$ and construct the set of all permutations $p(\underline{w}_o)$ of $\underline{w}_o$. Although this set will include many duplications, every source word with composition $\underline{q}$ will be present, and each of these will be present in the same number, say $n(\underline{w}_o)$. Therefore, the inner sum in Equation A1.23 may be written as

$$\sum_{\substack{\forall \underline{w} \ni \\ \text{comp.}\, \underline{w} = \underline{q}}} d_L(\underline{w}) = \frac{1}{n(w_o)} \sum_{\substack{\forall \underline{w} \ni \\ \underline{w} = \text{perm.}(\underline{w}_o)}} d_L(\underline{w}) . \tag{A1.24}$$

Each permutation of $\underline{w}_o$ is placed at the top of a column in which we list all decoder words $\underline{z}$ ordered in increasing distortion $d(p(\underline{w}_o),\underline{z})$ from $p(\underline{w}_o)$. But these lists are constructed in a specific way. Under $\underline{w}_o$ we list the decoder words in any way at all, as long as they have the prescribed ordering. But, under the permutation $p(\underline{w}_o)$ of $\underline{w}_o$ we place the list of $\underline{z}$ words under $\underline{w}_o$, after having permuted each $\underline{z}$ word in the same way $p(\underline{w}_o)$ is permuted from $\underline{w}_o$. It can be verified that each list of permuted decoder words has the prescribed ordering in distortion. In fact, along every row of permutations of $\underline{z}$, the distortions are equal as $d(\underline{w}_o,\underline{z}) = d(p(\underline{w}_o),p(\underline{z}))$. It can also be shown that in any row that contains a decoder word $\underline{z}$ of composition $\underline{h}$, all decoder words of composition $\underline{h}$ also appear and in equal numbers.

We now examine the terms $d_L(\underline{w})$ in Equation A1.24 and recall from Theorem 2.1 that this bound to distortion is equal to the volume shown in Figure 2.1B. When $g(\underline{z})$ is allowed to be any probability function whatever, the "amplitude function" $d(G)$ in Figure 2.1B will in general be different for different permutations $p(\underline{w}_o)$ of $\underline{w}_o$ (and therefore so also will the distortion $d_L\left[p(\underline{w}_o)\right]$ be different for different $p(\underline{w}_o)$). Different functions $d(G,p(\underline{w}_o))$ might look like those in Figure A.1A. The distance along G to the first jump of $d(G,p(\underline{w}_o))$ is equal to $g(\underline{z}_1)$, where $\underline{z}_1$ is the first decoder word on the list under $p(\underline{w}_o)$. As $g(\underline{z})$ is not as yet assumed to be constant over sets of equal composition decoder words, there might be several different distances to the first jump among the functions $d(G,p(\underline{w}_o))$, but every jump is to a value of distortion d_1 equal to the distortion between $p(\underline{w}_o)$ and the second decoder word on the list (we assume there is only one decoder word with distortion zero). In a similar way, the second jump of the functions

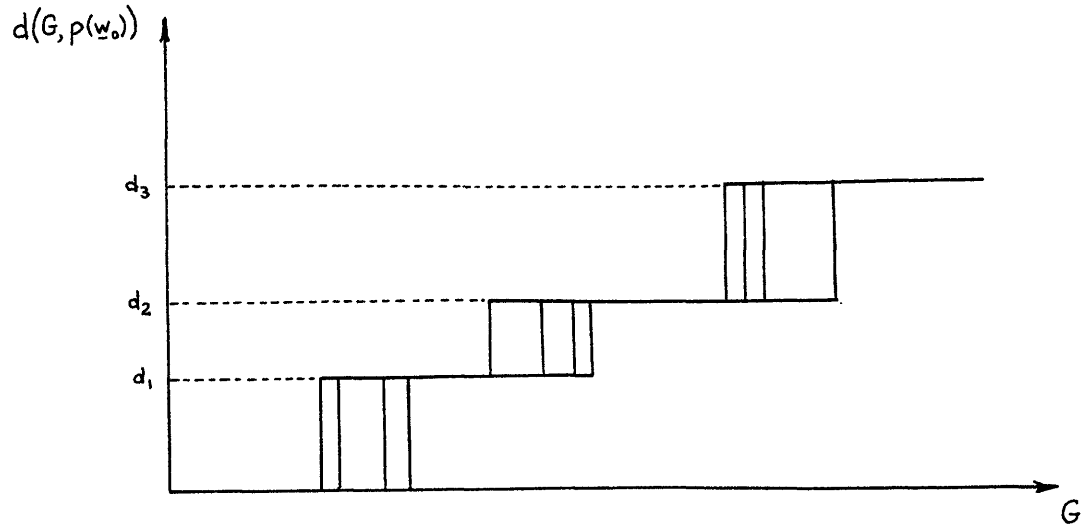

FIGURE A.1A : TYPICAL DISTORTION FUNCTIONS $d(G, p(\underline{w}_0))$

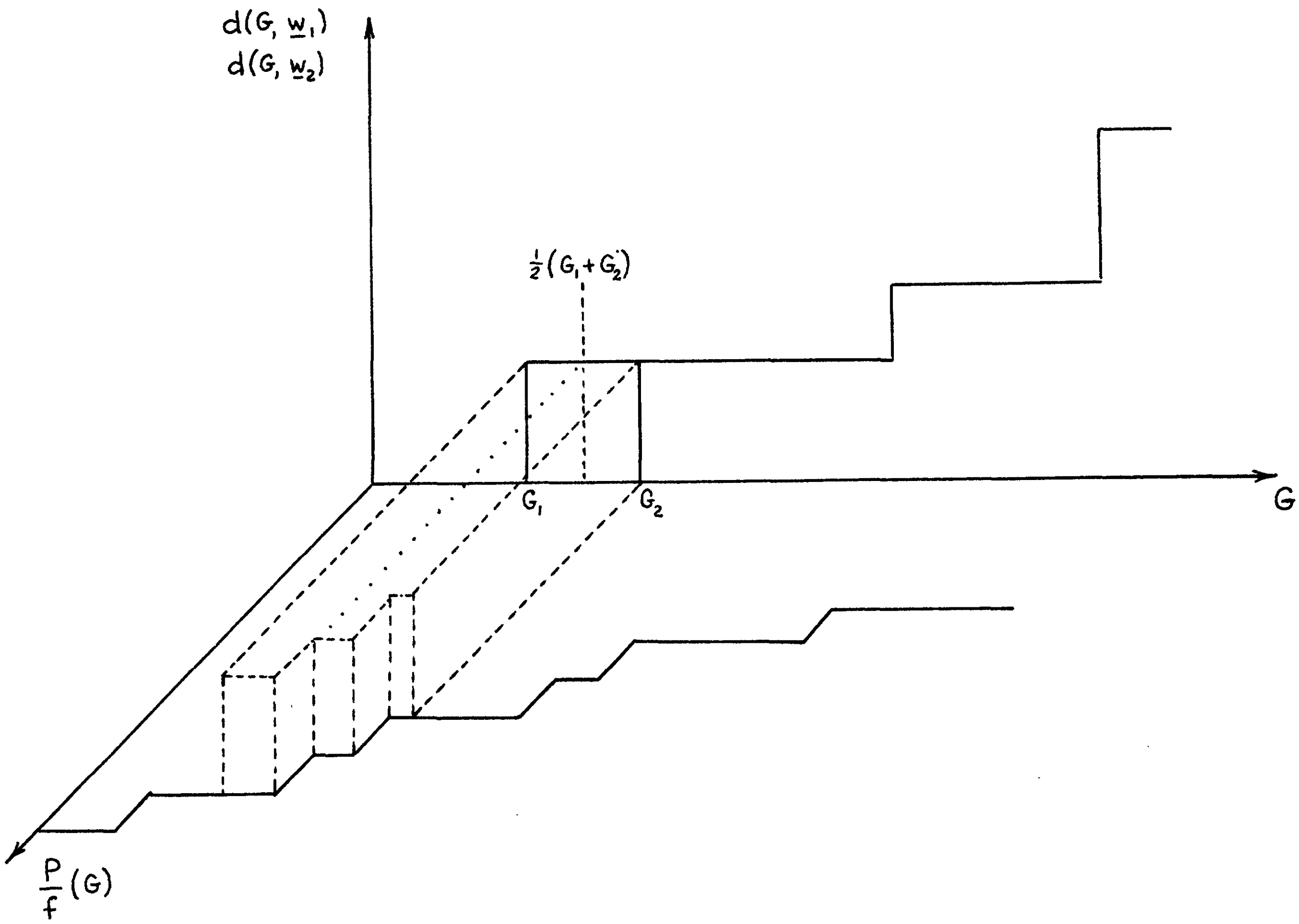

FIGURE A.1B : THE CHANGE IN VOLUME THAT RESULTS WHEN THE RISE POSITIONS OF THE FUNCTIONS $d(G, p(\underline{w}_0))$ ARE AVERAGED

$d(G,p(\underline{w}_o))$ would not all occur at the same value of G. (A typical one would occur at $g(\underline{z}_i) + g(\underline{z}_j) + g(\underline{z}_k) + g(\underline{z}_\ell)$ where $\underline{z}_i$, $\underline{z}_j$, $\underline{z}_k$, and $\underline{z}_\ell$ are all the decoder words at a distortion from $p(\underline{w}_o)$ less than or equal to d_1.) But, the second jumps would again all be of equal height, going from d_1 to d_2, the next highest distortion on the list. The same sort of geometry exists for all other jumps of the $d(G,p(\underline{w}_o))$ functions.

We now define a new probability function $g_1(\underline{z})$ which, over every set of equal composition $\underline{z}$ words, has a value equal to the average of $g(\underline{z})$ over that set. Thus $g_1(\underline{z})$ is a function only of the composition of $\underline{z}$. Because each row of decoder words across the set of columns constructed earlier contains only $\underline{z}$ words of equal composition, we now have the property that $g_1(\underline{z})$ is constant over each row. Further, it is also equal to the arithmetic average of $g(\underline{z})$ on the row since in any row all words of a given composition appear an equal number of times. We can therefore conclude that the functions $d_1(G,p(\underline{w}_o))$ which exist with the new probability function $g_1(\underline{z})$ are all identical. And, all jumps occur at a position along G which is the arithmetic average of the positions along G of the corresponding set of jumps of $d(G,p(\underline{w}_o))$. That is, the first jump of $d_1(G,p(\underline{w}_o)) \triangleq d_1(G)$ occurs at a value of G equal to $G_1 = \frac{1}{n!}\sum G_{1i}$ where $\{G_{1i}\}$ is the set of first jump positions of the $d(G,p(\underline{w}_o))$; the second jump of $d_1(G)$ occurs at $G = G_2 = \frac{1}{n!}\sum G_{2i}$ where $\{G_{2i}\}$ is the set of second jump positions of the $d(G,p(\underline{w}_o))$, etc. This is the property we will use to prove the desired result.

To simplify the final argument, we first consider only two functions, $d(G,\underline{w}_1)$ and $d(G,\underline{w}_2)$, that differ in only the position of the first jump (see Figure A.1B). The corresponding values of $d_L(\underline{w}_1)$ and $d_L(\underline{w}_2)$ are the volumes in Figure 2.1B enclosed by these "amplitude functions", the second "amplitude function" $\frac{p}{f}(G)$, and the "width function" G. If we change the position of the first jump of $d(G,\underline{w}_1)$ from G_1 to the arithmetic mean $\frac{1}{2}(G_1 + G_2)$, we decrease the volume $d_L(\underline{w}_1)$. And if we change the position of the first jump of $d(G,\underline{w}_2)$ from G_2 to $\frac{1}{2}(G_1 + G_2)$, we increase the volume $d_L(\underline{w}_2)$. However the net change of volume can never be positive since the function $\frac{p}{f}(G)$ is monotone non-increasing, hence is always at least as large in $\left[G_1, \frac{1}{2}(G_1 + G_2)\right]$ as it is in the equal width interval $\left[\frac{1}{2}(G_1 + G_2), G_2\right]$.

The same argument can be employed several times when there are more than two functions with different jump positions. One need only move pairs of jumps on opposite sides of the arithmetic mean jump position toward that position by equal amounts ΔG. Likewise, this argument can also be used for the set of jumps between any two successive distortion levels of the staircase functions $d(G,p(\underline{w}_o))$. Therefore, when all jumps are moved to the arithmetic mean positions, and all functions $d(G,p(\underline{w}_o))$ are changed to $d_1(G)$, the net volume change must be negative or at most zero. This proves that both sums in Equation A1.24 are no larger when $g_1(\underline{z})$ is used than when $g(\underline{z})$ is used. Since this is true for any composition vector $\underline{q}$, the same is true of the double sum in Equation A1.23. This establishes the claim that the minimum value of the sum in Equation A1.22 can be attained with a function $g_o(\underline{z})$ that is only a function of the composition of $\underline{z}$.

F. The Form of the Lower Bound without the Approximation in Equation 2.19

When the product form for $g(\underline{z})$ in Equation 2.19 is not assumed, we still can use a Chernov bound to upper bound G(d). If the distortion variable d is conditioned on the occurrence of the source word $\underline{w}_o$, we have

$$G(d) \leq e^{M(s) - sM'(s)} \tag{A1.25}$$

in which the semi-invariant moment generating function

$$M(s) = \ln \sum_{z^n} g(\underline{z})\, e^{sd(\underline{w}_o, \underline{z})} \tag{A1.26}$$

is now that of the complete word random variable $d(\underline{w}_o, \underline{z})$. As we can write Equation A1.25 as

$$G(d) \leq e^{n\left[\frac{M(s)}{n} - s\frac{M'(s)}{n}\right]} \tag{A1.27}$$

all of the results derived in Chapter 2 using the Chernov upper bound to G(d) in Equation 2.35 (or any result derived using Shannon's tighter strict upper bound to $G(d)^{(10)}$) can still be applied if we change $\mu(s)$ to $\frac{1}{n}M(s)$, and similarly $\mu^{(i)}(s)$ to $\frac{1}{n}M^{(i)}(s)$, in all the results. The derivatives $M'(s)$, $M''(s)$, and $M'''(s)$ can be shown from Equation A1.26 to be respectively the mean, variance, and third moment of the random variable $d(\underline{w}_o, \underline{z})$ according to the tilted probability distribution $K\, g(\underline{z})\, e^{s\, d(\underline{w}_o, \underline{z})}$.

When the changes prescribed above are made, we see from Equation 2.46 that there appears in the denominator of what was the coefficient of the $\frac{1}{n}$ term a term $\frac{1}{n}M''(s)$ and from Equation A1.14 that in addition there appears in the numerator of what was the coefficient of the $(\frac{1}{n})^2$ term, a term equal to $\frac{1}{n}M'''(s)$. The first term in Equation 2.46 is also changed to $\frac{1}{n}M'(s)$. The concern now is that when this bound is minimized over the probability function $g(\underline{z})$ there might exist such probability functions for which $M''(s)$ would increase more slowly than as n (and therefore the denominator terms $\frac{1}{n}M''(s)$ would decrease toward zero) and/or for which $M'''(s)$ would increase faster than n (and therefore the numerator term $\frac{1}{n}M'''(s)$ would become unbounded). Such a circumstance would make unclear the total n dependence of the terms in the lower bound that previously decreased as $\frac{1}{n}$ and $(\frac{1}{n})^2$. In addition, the first term, $\frac{1}{n}M'(s)$, might include low order terms which could introduce additional factors in the lower order terms of the bound.

The remainder of this appendix presents those properties of the derivatives of M(s) that we have been able to find. These are derived using the important result in Appendix 1E which allows us to restrict our attention to probability functions $g(\underline{z})$ that are uniform over each set of equal composition $\underline{z}$ words. Thus, over the set of decoder words that have composition $\underline{h}$, the tilted probability distribution $K\, g(\underline{z})\, e^{s\, d(\underline{w}_o,\underline{z})}$ can be written as $K(\underline{h})\, e^{s\, d(\underline{w}_o,\underline{z})}$. We focus on any such set of equal composition $\underline{z}$ words, say the one with composition $\underline{h}$. The conditional tilted probability distribution of the $\underline{z}$ words in this set is $K'_{\underline{h}}\, e^{s\, d(\underline{w}_o,\underline{z})}$ where $K'_{\underline{h}}$ is another normalizing constant. The letters of the $\underline{z}$ words in this set are not independent, not only because of the equal composition of the $\underline{z}$ words, but in addition because of the tilting of the $\underline{z}$ word probabilities by $e^{s\, d(\underline{w}_o,\underline{z})}$. Further, the single letter statistics change with n, again because the word probabilities are tilted by $e^{s\, d(\underline{w}_o,\underline{z})}$ which is a function of the word distortions. However, as will be shown in the next subsection, the (conditional) expected value of $d(\underline{w}_o,\underline{z})$ still increases with n as n.

(The word distortion $d(\underline{w}_o,\underline{z})$ in Equation A1.26 is considered here to be unnormalized as without Equation 2.19 this is the elementary variable. The resulting bound to distortion remains normalized however since we have included the factor $\frac{1}{n}$ in the brackets in Equation A1.27.) From this result, one can conclude that the (unconditioned) expected value of the word distortion over Z^n also increases as n and therefore so also does M'(s).

To establish the possible behavior of M''(s), we again first find the conditional variance of the word distortions on the set of $\underline{z}$ words of composition $\underline{h}$ using the probability distribution $K_{\underline{h}}' \, e^{s\, d(\underline{w}_o,\underline{z})}$. This is done in the next subsection where we find that this variance is at least proportional to n. From this result, it follows that the unconditional variance of the word distortion on Z^n (using the probability distribution $K(\underline{h})\, e^{s\, d(\underline{w}_o,\underline{z})}$) is also at least proportional to n because

$$\mathrm{Var}(d) = \sum_{Z^n} \left[d(\underline{w}_o,\underline{z}) - \bar{d} \right]^2 K(\underline{h})\, e^{s\, d(\underline{w}_o,\underline{z})}$$

$$= \sum_{\substack{\text{ALL COMP.}\\ \text{OF } \underline{z}}} \; \sum_{\substack{\text{ALL } \underline{z} \ni \\ \text{COMP. } \underline{z} = \underline{h}}} \left[d(\underline{w}_o,\underline{z}) - \bar{d} \right]^2 K(\underline{h})\, e^{s\, d(\underline{w}_o,\underline{z})}$$

$$\geq \sum_{\substack{\text{ALL COMP.}\\ \text{OF } \underline{z}}} \; \sum_{\substack{\text{ALL } \underline{z} \ni \\ \text{COMP. } \underline{z} = \underline{h}}} \left[d(\underline{w}_o,\underline{z}) - E(d \mid \text{comp } \underline{z} = \underline{h}) \right]^2 K(\underline{h})\, e^{s\, d(\underline{w}_o,\underline{z})} .$$

Therefore the term $\frac{1}{n}M''(s)$ does not go to zero with n. Actually to firmly establish this last statement, it is additionally required that the probability function $g(\underline{z})$, which in general would be allowed to change with n, does not approach, with increasing n, an impulse function at the words $\underline{z}$ that contain only one letter from the alphabet Z. (For example, there is only one $\underline{z}$ word with compostion $\underline{h} = (1,0,\ldots,0)$ which makes the conditional variance precisely zero for all n.) However this type of $g(\underline{z})$ function can be eliminated since in this case the decoder, in the limit, would be allowed only decoding words in which every letter was the same, and with this restriction the resulting distortion obviously must be very poor.

The allowed behavior of the third derivative $M'''(s)$ with probability functions $g(\underline{z})$ that are only a function of the composition of $\underline{z}$ cannot yet be established. Although examples can be given for which $\frac{1}{n}M'''(s)$ is unbounded, no function $g(\underline{z})$ can be found for which $\frac{1}{n}M'''(s)/(\frac{1}{n}M''(s))^3$ is unbounded and it is in this form that $M'''(s)$ always appears in the lower bound (see Equation A1.14). We shall not, in this thesis, pursue this point further.

The Expected Value of $d(\underline{w}_o,\underline{z})$ and the Variance of $d(\underline{w}_o,\underline{z})$ Conditioned on the Composition of $\underline{z}$ being $\underline{h}$

We divide the set of decoder words of composition $\underline{h}$ into sets which have the same joint composition with the given source word $\underline{w}_o$. Any joint composition can be specified by an $H \times J$ matrix Λ whose elements $n(w_i,z_j) \triangleq n_{ij}$ specify the number of pairings between the letters w_i and z_j that occur among the n pairs of letters in $\underline{w}_o$ and $\underline{z}$. If the composition of $\underline{w}_o$ is $\underline{q}$ and that of $\underline{z}$ is $\underline{h}$, we have

$$\sum_{i=1}^{H}\sum_{j=1}^{J} n_{ij} = n \tag{A1.28a}$$

$$\sum_{i=1}^{H} n_{ij} = nh_j \tag{A1.28b}$$

$$\sum_{j=1}^{J} n_{ij} = nq_i \tag{A1.28c}$$

It can be verified that the number of different decoder words that have a joint composition with $\underline{w}_o$ given by the matrix Λ is equal to

$$N(\Lambda) = \frac{\prod_i (nq_i)!}{\prod_{ij} n_{ij}!} \tag{A1.29}$$

and that the (unnormalized) distortion between $\underline{w}_o$ and each of these $\underline{z}$ words is equal to

$$d(\Lambda) \triangleq d(\underline{w}_o, \underline{z} \,|\, \Lambda) = \sum_{ij} n_{ij}\, d_{ij} \;. \tag{A1.30}$$

We can now write the desired conditional mean and variance in terms of the set of matrices Λ_k that describe all of the possible joint compositions between $\underline{w}_o$ and those $\underline{z}$ with composition $\underline{h}$:

$$\mathrm{Var}\left[d(\underline{w}_o, \underline{z}) \,\middle|\, \text{comp. } \underline{z} = \underline{h},\ \text{comp. } \underline{w}_o = \underline{q} \right]$$

$$= \sum_{\substack{\text{ALL } \underline{z} \ni \\ \text{COMP. } \underline{z} = \underline{h}}} K'_{\underline{h}}\, e^{s\,d(\underline{w}_o, \underline{z})} \left[d(\underline{w}_o, \underline{z}) - E(d \,|\, \text{comp } \underline{z} = \underline{h}) \right]^2$$

$$= \sum_k \left[\frac{N(\Lambda_k)\, e^{s d(\Lambda_k)}}{\sum_\ell N(\Lambda_\ell)\, e^{s d(\Lambda_\ell)}} \right] \left[d(\Lambda_k) - \overline{d(\Lambda_k)} \right]^2$$

$$\triangleq \sum_k \Pr(\Lambda_k) \left[d(\Lambda_k) - \overline{d(\Lambda_k)} \right]^2 \tag{A1.31}$$

and

$$\overline{d(\Lambda_k)} = \sum_k Pr(\Lambda_k)\, d(\Lambda_k)$$

$$= \sum_k Pr(\Lambda_k) \sum_{ij} n_{ij}^{(k)} d_{ij}$$

$$\triangleq \sum_{ij} \overline{n_{ij}}\, d_{ij} \quad . \tag{A1.32}$$

These can be combined to further write:

$$Var\left[d(\underline{w}_o, \underline{z}) \middle| \text{comp}\cdot \underline{z} = \underline{h},\ \text{comp}\ \underline{w}_o = \underline{q} \right] = \sum_k Pr(\Lambda_k) \left[\sum_{ij} \left(n_{ij}^{(k)} - \overline{n_{ij}} \right) d_{ij} \right]^2 . \tag{A1.33}$$

Our intent is to show that $\overline{d(\Lambda_k)}$ increases asymptotically as n and that the conditional variance increases asymptotically no more slowly than as n.

We first look at the probabilities $Pr(\Lambda_k)$ of the joint composition matrices. With $K_1(n)$ defined as the appropriate normalizing factor, we have from Equations A1.29 through A1.31:

$$Pr(\Lambda_k) = K_1(n)\, N(\Lambda_k)\, e^{sd(\Lambda_k)}$$

$$= K_1(n) \frac{\prod_i (nq_i)!}{\prod_{ij} n_{ij}^{(k)}!} \prod_{ij} e^{s n_{ij}^{(k)} d_{ij}} \quad .$$

If Stirling's formula(12),

$$x! \sim \sqrt{2\pi}\ x^{x+\frac{1}{2}}\ e^{-x} ,$$

which is asymptotically correct, is used for the factorial expressions, we can write, for large n,

$$\Pr(\Lambda_k) \sim K_1(n) \frac{\prod_i \sqrt{2\pi}\,(nq_i)^{nq_i+\frac{1}{2}}\; e^{-nq_i}}{\prod_{ij} \sqrt{2\pi}\,\left(n_{ij}^{(k)}\right)^{n_{ij}^{(k)}+\frac{1}{2}}\; e^{-n_{ij}^{(k)}}} \prod_{ij} e^{s n_{ij}^{(k)} d_{ij}}$$

which becomes, after some algebra

$$\Pr(\Lambda_k) \sim K_1(n)\,(2\pi n)^{\frac{N}{2}(1-J)} \frac{\prod_i q_i^{\frac{1}{2}}}{\prod_{ij}\left(\frac{n_{ij}^{(k)}}{n}\right)^{\frac{1}{2}}} \prod_{ij}\left[\frac{nq_i\, e^{sd_{ij}}}{n_{ij}^{(k)}}\right]^{n_{ij}^{(k)}} . \tag{A1.34}$$

We let $E_k(n)$ denote the last product of exponentials and note that it is this term which is critical with regard to the behavior of $\Pr(\Lambda_k)$, at least for large n.

Because the normalizing factor $K_1(n)$ is difficult to calculate, we compare the probability of a joint composition matrix Λ_k to that of the maximum probability matrix Λ_o. To find the set $\{n_{ij}^{o}\}$ for this matrix we use Lagrange multiplier theory to maximize $\ln E_k(n)$ (it is convex) over $\{n_{ij}^{(k)}\}$ with the constraints in Equations A1.28abc. The result is:

$$n_{ij}^{o} = n g_j q_i \frac{e^{sd(w_i, z_j)}\; e^{\nu_i}}{\sum_k e^{\nu_k}\, e^{sd(w_k, z_j)}\, q_k}$$

$$\triangleq n r_{ij}^{o}$$

in which r_{ij}^{o} can be shown to be independent of n. Comparing the factor $E_k(n)$ using the set $\{n_{ij}^{(k)}\}$ of the matrix Λ_k to $E_o(n)$ using the set $\{n_{ij}^{o}\}$ of the matrix Λ_o, we find

$$\ln \frac{E_\kappa(n)}{E_o(n)} = -\sum_{ij} nr_{ij}^o \left[1 + \frac{\Delta_{ij}^{(\kappa)}}{nr_{ij}^o}\right] \ln\left[1 + \frac{\Delta_{ij}^{(\kappa)}}{nr_{ij}^o}\right] \tag{A1.35}$$

where the quantity $\Delta_{ij}^{(\kappa)}$ is defined as the deviation of $n_{ij}^{(\kappa)}$ from n_{ij}^o:

$$n_{ij}^{(\kappa)} \triangleq n_{ij}^o + \Delta_{ij}^{(\kappa)} = nr_{ij}^o + \Delta_{ij}^{(\kappa)} .$$

As our interest will be on matrices Λ_κ which for large n have $\Delta_{ij}^{(\kappa)}/nr_{ij}^o$ much smaller than one, we expand the logarithm in Equation A1.35 and obtain

$$\ln \frac{E_\kappa(n)}{E_o(n)} = -\sum_{ij} \frac{\Delta_{ij}^{(\kappa)2}}{nr_{ij}^o} \left(1 + o_n(1)\right) . \tag{A1.36}$$

If the entries $n_{ij}^{(\kappa)}$ in the joint composition matrices Λ_κ are taken as the coordinates of the space R^{HJ}, each matrix Λ_κ can be represented by one point in that space. Because of the constraints among the $n_{ij}^{(\kappa)}$ in Equations A1.28abc, these points all lie on an HJ-H-J+1 dimensional hyperplane in this space (the unconstrained dimensionality HJ is reduced by one by each of the H+J-1 independent constraints). Using this geometry we can establish the desired property of $\overline{d(\Lambda_\kappa)}$. We have

$$\begin{aligned}
\overline{d(\Lambda_\kappa)} &= \sum_\kappa P_r(\Lambda_\kappa) \sum_{ij} n_{ij}^{(\kappa)} d_{ij} \\
&= \sum_{ij} n_{ij}^o d_{ij} + \sum_\kappa P_r(\Lambda_\kappa) \sum_{ij} \Delta_{ij}^{(\kappa)} d_{ij} \\
&= n\left[\sum_{ij} r_{ij}^o d_{ij} + \sum_\kappa P_r(\Lambda_\kappa) \sum_{ij} \frac{\Delta_{ij}^{(\kappa)}}{n} d_{ij}\right] .
\end{aligned} \tag{A1.37}$$

If $\overline{d(\Lambda_\kappa)}$ is not to increase as n, the last term in the brackets cannot approach zero with increasing n. But we will show that this term must go to zero with increasing n. We construct in R^{HJ} a sequence of spheres S_n centered at the point with coordinate values n^o_{ij}, with radii $R(n) = an^{3/4}$, and divide at each n the sum over k into those matrices within the spheres and those outside. (The sequence of spheres S_n has S_n increasing in size with n less rapidly than is the total size of the hyperplane, any dimension of which increases with n as n, thus on a normalized basis these spheres would be shrinking in size around the point with coordinates n^o_{ij}/n.) For any sequence of matrices $\Lambda_\kappa(n)$ outside the spheres, $\Lambda_\kappa(n)$ must possess, for at least one i,j, a $\frac{\Delta_{ij}^{(\kappa)2}}{n} \gtrsim n^{\frac{1}{2}}$, therefore from Equation A1.36, a ratio $E_k(n)/E_o(n)$ that is exponentially small in n. For these matrices $\Lambda_\kappa(n)$, the ratio of $Pr\left[\Lambda_\kappa(n)\right]/Pr\left[\Lambda_o(n)\right]$ is therefore also exponentially small in n (see Equation A1.34). As there is only an algebraically large number of terms in the last sum of Equation A1.37, matrices outside the spheres S_n can therefore be discounted from contributing anything significant to this sum. On the other hand, any sequence of matrices $\Lambda_\kappa(n)$ inside the spheres S_n have $\frac{\Delta_{ij}^{(\kappa)}}{n}$ at most proportional to $n^{-\frac{1}{4}}$. It follows that over these matrices the double sum in Equation A1.37 also approaches zero with increasing n. This establishes the result that $\overline{d(\Lambda_\kappa)}$ increases with n as n.

To find the properties of the conditional variance, we construct in R^{HJ} two sequences of spheres, S_{1n} and S_{2n}, centered at the point with coordinate values n^o_{ij}. The sequence of radii of S_{1n} is $R_1(n)$ with $R_1(n) = o(\sqrt{n})$, and the sequence of radii of S_{2n} is $R_2(n)$, with $R_2(n) = b\sqrt{n}$. (The same paranthetical comment applies to these sequences of spheres as to those in the previous paragraph.) Because any sequence of matrices $\Lambda_{1k}(n)$ within the spheres S_{1n} has $\Delta_{ij}^{(k)} = o(\sqrt{n})$ for all pairs i,j, we see from Equation A1.36 that the ratio $E_k(n)/E_o(n)$ approaches unity for these sequences. It follows, from Equation A1.34, that the ratio between $Pr\left[\Lambda_{1k}(n)\right]$ and $Pr\left[\Lambda_o(n)\right]$ also approaches unity. In contrast a sequence of matrices $\Lambda_{2k}(n)$ constrained only to be in S_{2n} has $E_k(n)/E_o(n)$ approach C_1 and therefore the ratio of $Pr\left[\Lambda_{2k}(n)\right]$ to $Pr\left[\Lambda_o(n)\right]$ approach C_1, where C_1 is a positive constant in (0,1] that is independent of n. These relations are used to find the asymptotic behavior, with n, of the total probability of all matrices within the two sequences of spheres S_{1n} and S_{2n}. We have

$$\sum_{\Lambda_k(n)\,\varepsilon\, S_{1n}} Pr\left[\Lambda_k(n)\right] \sim Pr\left[\Lambda_o(n)\right] \cdot N\left[R_1(n)\right] \tag{A1.38}$$

$$\sum_{\Lambda_k(n)\,\varepsilon\, S_{2n}} Pr\left[\Lambda_k(n)\right] \sim C\, Pr\left[\Lambda_o(n)\right] \cdot N\left[R_2(n)\right] \tag{A1.39}$$

$N\left[R\right] \triangleq$ the number of joint composition matrices within a sphere of radius R.

If we take the ratio of these probabilities we find

$$\frac{\displaystyle\sum_{\Lambda_k(n)\,\varepsilon\, S_{1n}} Pr\left[\Lambda_k(n)\right]}{\displaystyle\sum_{\Lambda_k(n)\,\varepsilon\, S_{2n}} Pr\left[\Lambda_k(n)\right]} \sim \frac{1}{C}\,\frac{N\left[R_1(n)\right]}{N\left[R_2(n)\right]} \,. \tag{A1.40}$$

We can now use the result in Appendix 4E, plus the fact that the points in the composition plane which represent allowed joint composition matrices are uniformly spaced to write

$$N\left[R\right] \sim C_2\, R^{HJ-H-J+1} \,. \tag{A1.41}$$

Therefore we have

$$\frac{\sum_{\Lambda_k(n) \in S_{1n}} Pr\left[\Lambda_k(n)\right]}{\sum_{\Lambda_k(n) \in S_{2n}} Pr\left[\Lambda_k(n)\right]} \sim \frac{1}{C}\left[\frac{R_1(n)}{R_2(n)}\right]^{HJ-H-J+1}$$

which, because of the assumed form of $R_1(n)$ and $R_2(n)$, approaches zero with increasing n. The conclusion is that within <u>any</u> sequence of spheres S_{1n} centered at n_{ij}^{o}, with $R_1(n) = o(\sqrt{n})$, the total probability of the $\Lambda_k(n)$ matrices vanishes with increasing n, hence almost all probability becomes outside such spheres.

Returning now to Equation A1.33, we see that if a similar result could be proven for spheres S_{3n} of radii $R_3(n)$, centered at the point with coordinates $\overline{n_{ij}}$, we would establish the last desired result, which is that the conditional variance in this equation increases no more slowly than as n. We consider two cases. First, if the differences $\overline{n_{ij}} - n_{ij}^{o}$ are such that the distance between the center of the S_{3n} spheres and the center of the S_{1n} spheres is $o(\sqrt{n})$, we can always adjust the radii $R_1(n)$ so that $R_1(n) = o(\sqrt{n})$ and that S_{1n} includes S_{3n}. The result is that the total probability of the matrices Λ_k within S_{3n} is upper bounded by that within the S_{1n} and therefore approaches zero with increasing n. Finally, if the difference between the centers of the spheres S_{3n} and the spheres S_{1n} is proportional to $\sqrt{n}$ (or larger), we can set $R_1(n)$ equal to $R_3(n)$, which is $o(\sqrt{n})$, and use the fact that the maximum probability of Λ_k is at the center of the spheres S_{1n}, to again upper bound the total probability of the matrices within S_{3n} by that of the matrices within S_{1n}. Thus we conclude that for every sequence of spheres S_{3n} centered at $\overline{n_{ij}}$, with radii $R_3(n) = o(\sqrt{n})$, the total probability of the matrices Λ_k within these spheres approaches zero with increasing n. Hence the only significant contribution to the sum in Equation A1.33 is outside all such sequences where for each k (at least one of) the differences $n_{ij}^{(k)} - \overline{n_{ij}}$ increases at least as fast as $\sqrt{n}$.

Appendix 2

APPENDIX FOR SECTION 2.2

A. Taylor's Formula for $d_L(\underline{q})$ in Equation 2.55

We start with Equation 2.55:

$$d(S) \geq \int_{Q^H}\cdots\int d_L(\underline{q})\, P_r(\underline{q})\, d\underline{q}\,.$$

The function $d_L(\underline{q})$ in Equation 2.46 is abbreviated

$$d_L(\underline{q}) = \mu'(s_o, \underline{q}) + \frac{1}{2n}\, d_1(s_o, \underline{q}) + c_1(n)$$

and since we do not wish to differentiate $c_1(n)$, which is a function of $\underline{q}$, we replace this term by that in

$$d_L(\underline{q}) = \mu'(s_o, \underline{q}) + \frac{1}{2n}\, d_1(s_o, \underline{q}) + \min_{\underline{q}}\, c_1(n) \qquad \text{(A2.1)}$$

We will use the following subscript notation:

$$d'_{L_i}(\underline{p}) = \left(\frac{\partial d_L(\underline{q})}{\partial q_i}\right)_{q_{k\neq i}}\Bigg|_{\underline{p}} \,,$$

$$d''_{L_{ij}}(\underline{p}) = \left(\frac{\partial}{\partial q_j}\left(\frac{\partial d_L(\underline{q})}{\partial q_i}\right)_{q_{k\neq i}}\right)_{q_{k\neq j}}\Bigg|_{\underline{p}} \,, \qquad \text{etc.}$$

(Note that s_o is an implicit function of $\underline{q}$ given by Equation 2.47 and is <u>not</u> to be held constant in these derivatives.) Thus we can write the Taylor Formula with Remainder for $d_L(\underline{q})$ as

$$d_L(\underline{q}) = d_L(\underline{p}) + \sum_i d'_{Li}(\underline{p})(q_i - p_i)$$

$$+ \frac{1}{2}\sum_{ij} d''_{Lij}(\underline{p})(q_i - p_i)(q_j - p_j)$$

$$+ \frac{1}{6}\sum_{ijk} d'''_{Lijk}(\underline{\phi})(q_i - p_i)(q_j - p_j)(q_k - p_k),$$

which can be used in Equation 2.55 to obtain

$$d(S) \geq d_L(\underline{p}) + \sum_i d'_{Li}(\underline{p})\, E(q_i - p_i)$$

$$+ \frac{1}{2}\sum_{ij} d''_{Lij}(\underline{p})\, E\Big[(q_i - p_i)(q_j - p_j)\Big]$$

$$+ \frac{1}{6}\sum_{ijk} d'''_{Lijk}(\underline{\phi})\, E\Big[(q_i - p_i)(q_j - p_j)(q_k - p_k)\Big]. \qquad \text{(A2.2)}$$

Next we find

B. The Central Moments of $\underline{q}$

To model the dependence among q_i, q_j, q_k, we introduce a set of auxiliary variables θ which have the sums

$$q_i = \frac{1}{n}\sum_{a=1}^{n} \theta_{ia} \;; \qquad q_j = \frac{1}{n}\sum_{b=1}^{n} \theta_{jb} \;; \qquad q_k = \frac{1}{n}\sum_{c=1}^{n} \theta_{kc} \;.$$

The variable θ_{ia} is defined to be one or zero depending upon whether the a'th letter of the word $\underline{w}$ is or is not w_i. The probabilities of these events are p_i and $1-p_i$ respectively. The variables θ_{ia} and θ_{jb} are obviously dependent when a = b since the letters w_i and w_j cannot both appear as the a'th letter of the word, but are independent when $a \neq b$ since the source is memoryless. Similarly, θ_{ia}, θ_{jb}, and θ_{kc} are jointly dependent when a = b = c, for which case their joint distribution is

$$
\begin{aligned}
\Pr(\theta_{ia},\theta_{ja},\theta_{ka}) &= p_i && \text{at } (1,0,0) \\
&= p_j && \text{at } (0,1,0) \\
&= p_k && \text{at } (0,0,1) \\
&= 1-p_i-p_j-p_k && \text{at } (0,0,0) \\
&= 0 && \text{all other points.}
\end{aligned}
$$

The joint distribution for θ_{ia} and θ_{ja} can be found in the usual way by summing over the values of θ_{ka}.

The first central moment of q_i is

$$E(q_i-p_i) = 0.$$

The second central moment is

$$E\left[(q_i-p_i)(q_j-p_j)\right] = E(q_iq_j) - p_ip_j$$

in which we substitute

$$
\begin{aligned}
E(q_iq_j) &= \left(\tfrac{1}{n}\right)^2 E\left[\sum_{a=1}^{n}\theta_{ia}\sum_{b=1}^{n}\theta_{jb}\right] \\
&= \left(\tfrac{1}{n}\right)^2 E\left[\sum_{a=1}^{n}\theta_{ia}\theta_{ja} + \sum_{\substack{a,b\\a\neq b}}\theta_{ia}\theta_{jb}\right]
\end{aligned}
$$

$$E\left(q_i q_j\right) = \left(\frac{1}{n}\right)^2 \left[n\delta_{ij} p_i + n(n-1) p_i p_j \right] \tag{A2.3}$$

to obtain

$$E\left[(q_i - p_i)(q_j - p_j) \right] = \frac{1}{n} \left(p_i \delta_{ij} - p_i p_j \right). \tag{A2.4}$$

Finally, the third central moment is

$$E\left[(q_i - p_i)(q_j - p_j)(q_k - p_k) \right] = E\left(q_i q_j q_k\right) - p_i E\left(q_j q_k\right)$$

$$-p_j E\left(q_i q_k\right) - p_k E\left(q_i q_j\right) + 2\, p_i p_j p_k \tag{A2.5}$$

in which

$$E\left(q_i q_j q_k\right) = \left(\frac{1}{n}\right)^3 E\left[\sum_{a=1}^{n} \theta_{ia} \sum_{b=1}^{n} \theta_{jb} \sum_{c=1}^{n} \theta_{kc} \right]$$

$$= \left(\frac{1}{n}\right)^3 E\left[\sum_{a=1}^{n} \theta_{ia}\theta_{ja}\theta_{ka} + \sum_{\substack{a,b \\ a\neq b}} \left(\theta_{ia}\theta_{ja}\theta_{kb} + \theta_{ia}\theta_{jb}\theta_{ka} + \theta_{ib}\theta_{ja}\theta_{ka} \right) \right.$$

$$\left. + \sum_{\substack{a,b,c \\ a\neq b\neq c}} \theta_{ia}\theta_{jb}\theta_{kc} \right]$$

$$= \left(\frac{1}{n}\right)^3 \left[n p_i \delta_{ijk} + n(n-1)\left(p_j p_k \delta_{ij} + p_i p_j \delta_{ik} + p_i p_k \delta_{jk} \right) \right.$$

$$\left. + n(n-1)(n-2)\, p_i p_j p_k \right]. \tag{A2.6}$$

If we use Equations A2.3 and A2.6 in A2.5 we have

$$E\left[(q_i - p_i)(q_j - p_j)(q_k - p_k) \right]$$

$$= \left(\frac{1}{n}\right)^2 \left[p_i \delta_{ijk} - p_i p_j \delta_{ki} - p_j p_k \delta_{ij} - p_k p_i \delta_{jk} + 2 p_i p_j p_k \right]. \tag{A2.7}$$

C. Proof of Theorems 2.5a and 2.5b

Equation A2.2 can now be written as

$$d(S) \geq d_L(\underline{p}) + \frac{1}{2n}\sum_{ij} d''_{Lij}(\underline{p})\left(p_i\delta_{ij} - p_ip_j\right) + \frac{1}{6n^2}\sum_{ijk} d'''_{Lijk}(\underline{\phi})\,\nu_{ijk} \tag{A2.8}$$

where we have used ν_{ijk} for the bracketed expression in Equation A2.7. We have been able to verify that the sum in the last term is finite so the $(\frac{1}{n})^2$ coefficient allows this term to be considered only as a correction term. In the second derivative term in Equation A2.8, the functions d''_{Lij} each contain two terms (see Equation A2.1):

$$d''_{Lij}(\underline{p}) = \mu'''_{sij}(s_o, \underline{p}) + \frac{1}{2n}\, d''_{1ij}(s_o, \underline{p})\,. \tag{A2.9}$$

[Here we are using the admittedly awkward notation

$$\mu'''_{sij}(s_o, \underline{p}) = \left(\frac{\partial}{\partial q_j}\left(\frac{\partial \mu'(s_o, \underline{q})}{\partial q_i}\right)_{q_{k\neq i}}\right)_{q_{k\neq j}}\Bigg|_{\underline{p}}$$

in which

$$\mu'(s_o, \underline{q}) = \left(\frac{\partial \mu(s, \underline{q})}{\partial s}\right)_{\underline{q}}\Bigg|_{s_o(\underline{q})}$$

with $s_o(\underline{q})$ meant to represent the implicit relation in Equation 2.47.] The last term in Equation A2.9 has, when used in Equation A2.8, a total coefficient of $(\frac{1}{n})^2$ so it too will be grouped with the correction terms. Thus we have

$$d(S) \geq \mu'(s_o, \underline{p}) + \frac{1}{2n}\left[d_1(s_o, \underline{p}) + \sum_{ij} \mu'''_{sij}(s_o, \underline{p})\left(p_i\delta_{ij} - p_ip_j\right)\right] + c_2(n) \tag{A2.10}$$

with

$$C_2(n) = \left(\frac{1}{n}\right)^2 \left[\frac{1}{4}\sum_{ij} d_{1ij}''(s_0,\underline{p}) + \frac{1}{6}\sum_{ijk} d_{Lijk}'''(\underline{\phi})\,\nu_{ijk}\right] + \min_{\underline{q}} C_1(n)$$

or since $\underline{\phi}$ is unknown

$$C_2(n) = \left(\frac{1}{n}\right)^2 \left[\frac{1}{4}\sum_{ij} d_{1ij}''(s_0,\underline{p}) + \min_{\underline{q}} \frac{1}{6}\sum_{ijk} d_{Lijk}'''(\underline{q})\,\nu_{ijk}\right] + \min_{\underline{q}} C_1(n). \qquad \text{(A2.11)}$$

In these equations s_o satisfies

$$\mu(s_0,\underline{p}) - s_0\mu'(s_0,\underline{p}) = \bar{I} + \frac{1}{n}\ln K(n). \qquad \text{(A2.12)}$$

This result is used in Chapter 2 to obtain Theorem 2.5a.

In Equation A2.1 of Appendix 2A we used the expression for $d_L(\underline{q})$ from Theorem 2.4a. If instead we use the asymptotic bound in Theorem 2.4b,

$$d_L(\underline{q}) = \mu'(s_0,\underline{q}) + \frac{1}{2n} d_1(s_0,\underline{q}) + o\left(\frac{1}{n}\right), \qquad \text{(A2.13)}$$

and follow the same sequence of steps we find

$$d(S) \geq \mu'(s_0,\underline{p}) + \frac{1}{2n}\left[d_1(s_0,\underline{p}) + \sum_{ij} \mu_{sij}'''(s_0,\underline{p})\left(p_i\delta_{ij} - p_ip_j\right)\right] + o\left(\frac{1}{n}\right) \qquad \text{(A2.14)}$$

in which s_o now satisfies

$$\mu(s_0,\underline{p}) - s_0\mu'(s_0,\underline{p}) = \bar{I} - \frac{1}{2n}\ln\frac{\gamma''(-1)}{s_0^2\mu''(s_0,\underline{p})} + o\left(\frac{1}{n}\right). \qquad \text{(A2.15)}$$

Once again, for this bound we do not specify the (greatly more complicated) correction terms, only their order. Theorem 2.5b follows from this result.

Appendix 3

APPENDIX FOR SECTIONS 2.3 AND 2.5

A. Elimination of the Fixed Composition Constraint at the Channel Input

We start with Theorem 2.4b:

$$d(\underline{q}) \geq \mu'(s_0) - \frac{1}{2n s_0}\left[\frac{\gamma''(-1)}{s_0^2 \mu''(s_0)} - 1\right] + o\left(\frac{1}{n}\right) \tag{2.50}$$

$$\mu(s_0) - s_0 \mu'(s_0) = \bar{I} - \frac{1}{2n} \ln \frac{\gamma''(-1)}{s_0^2 \mu''(s_0)} + o\left(\frac{1}{n}\right) \tag{2.51}$$

and set the vector $\underline{f}$ equal to its value at $n = \infty$, $\underline{f}(\infty)$. We drop the fixed composition constraint by allowing the vector $\underline{c}$ in $\gamma(t)$ to be a function of $\underline{q}$. For each vector $\underline{c}(\underline{q})$, we have in Equations 2.50 and 2.51 a different function of n. At this point, before $d(\underline{q})$ is averaged over the source, we choose, at each n, that vector $\underline{c}_0(\underline{q},n)$ which minimizes the bound.

We see from Equations 2.50 and 2.51 that at $n = \infty$, the vector $\underline{c}_0(\underline{q},\infty)$ is that vector which minimizes $\bar{I}$. And, in Section 2.3.1, this vector was found to be the one that uses the channel $\mathcal{C}$ to capacity. It follows that $\underline{c}_0(\underline{q},\infty)$ has only non-zero components and therefore so also does $\underline{c}_0(\underline{q},n)$, at least for sufficiently large n. Consequently the components of $\underline{c}_0(\underline{q},n)$ can be found by solving the Lagrange equations,

$$\left(\frac{\partial d_L(\underline{q})}{\partial c_k}\right)_{\substack{c_{\ell\neq k},\\ n}} + \lambda = 0 , \tag{A3.1}$$

together with the constraint equation $\sum_k c_k = 1$, and the parametric Equation 2.51. In Equation A3.1 we have used $d_L(\underline{q})$ for the right side of Equation 2.50.

The minimizing vector $\underline{c}_0(\underline{q},n)$ is used in Equations 2.50 and 2.51, and $d_L(\underline{q})$ is then averaged over the source. The procedure is the same as in Appendices 2ABC and we again get the result in Equations A2.14 and A2.15. The difference comes when we evaluate the derivatives $\mu'''_{sij}(s_0,\underline{p})$ in Equation A2.14. In particular, the term that is different is $\left(\frac{\partial s_0}{\partial q_i}\right)_{q_{k\neq i}}$ in Equation 2.57. We let h represent the parametric Equation 2.51:

$$h = \mu(s_0) - s_0\mu'(s_0) - \gamma'(-1) + \frac{1}{2n}\ln\frac{\gamma''(-1)}{s_0^2\mu''(s_0)} = 0 .$$

Then

$$\left(\frac{\partial s_0}{\partial q_i}\right)_{q_{k\neq i}} = -\frac{\left(\partial h/\partial q_i\right)_{q_{k\neq i},\,s_0}}{\left(\partial h/\partial s_0\right)_{\underline{q}}}$$

$$= \frac{\mu_i(s_0) - s_0\mu_i'(s_0) - \sum_k \left(\partial c_k(\underline{q},n)/\partial q_i\right)\gamma_k'(-1)}{s_0\mu''(s_0)} + o_n(1)$$

which when used in Equation 2.57 changes the result to read

$$\left(\frac{\partial\mu'}{\partial q_i}\right)_{q_{j\neq i}} = \frac{\mu_i(s_0)}{s_0} - \frac{1}{s_0}\sum_k\left(\frac{\partial c_k(\underline{q},n)}{\partial q_i}\right)\gamma_k'(-1) + o_n(1) .$$

The second derivative, after a little algebra, can be found to be

$$\mu'''_{s_{ij}}(s_o,\underline{p}) = -\frac{\theta_i\theta_j}{s_o^3\mu''(s_o)}\Bigg|_{\underline{p}} + \frac{\theta_i}{s_o^3\mu''(s_o)}\sum_k\left(\frac{\partial c_k(\underline{q},n)}{\partial q_j}\right)\gamma_k'\Bigg|_{\underline{p}}$$

$$+\frac{\theta_j}{s_o^3\mu''(s_o)}\sum_k\left(\frac{\partial c_k(\underline{q},n)}{\partial q_i}\right)\gamma_k'\Bigg|_{\underline{p}} - \frac{1}{s_o}\sum_k\left(\frac{\partial^2 c_k(\underline{q},n)}{\partial q_i\partial q_j}\right)\gamma_k'\Bigg|_{\underline{p}}$$

$$-\frac{1}{s_o^3\mu''(s_o)}\sum_k\left(\frac{\partial c_k(\underline{q},n)}{\partial q_i}\right)\gamma_k'\sum_k\left(\frac{\partial c_k(\underline{q},n)}{\partial q_j}\right)\gamma_k'\Bigg|_{\underline{p}} + o_n(1). \tag{A3.2}$$

As we would expect, for a doubly-uniform channel, the sums are all zero as $\gamma_k(t)$ is independent of k, and the variable channel input composition produces no change. In general though, when Equation A3.2 is used in Equation A2.14, we get in place of Equation 2.63:

$$d(S) \geq \mu'(s_o,\underline{p}) - \frac{1}{2ns_o}\left[\frac{\gamma''(-1)+\sigma^2(\theta)}{s_o^2\mu''(s_o)} - 1 - R\left(\frac{\partial c_k(\underline{q},n)}{\partial q_i}\Bigg|_{\underline{p}}, \frac{\partial^2 c_k(\underline{q},n)}{\partial q_i\partial q_j}\Bigg|_{\underline{p}}\right)\right] + o\left(\frac{1}{n}\right) \tag{A3.3}$$

In Equation A3.3, $R(\cdot,\cdot)$ is just notational for the summation (in Equation A2.14) of the last four terms in Equation A3.2. If the evaluated derivatives, $\frac{\partial c_k(\underline{q},n)}{\partial q_i}\Big|_{\underline{p}}$ and $\frac{\partial^2 c_k(\underline{q},n)}{\partial q_i\partial q_j}\Big|_{\underline{p}}$ for all i,j are considered the variables of R, we can see from Equation A3.2 that R is continuous in all of its variables and equals zero when all of its variables equal zero.

We have already observed that at $n = \infty$ the vector $\underline{c}_o(\underline{q}, \infty)$ which minimizes the distortion is that vector which minimizes $\bar{I}$. As $\bar{I}$ is independent of $\underline{q}$, so also is $\underline{c}_o(\underline{q}, \infty)$. It follows that as $n \to \infty$, the derivatives $\frac{\partial c_k(q,n)}{\partial q_i}$, $\frac{\partial^2 c_k(q,n)}{\partial q_i \partial q_j}$ must approach zero. Because $R(\cdot,\cdot)$ equals zero when all its variables equal zero, and is continuous in this region, it also must approach zero as $n \to \infty$. Therefore the term $\frac{1}{n}R(\cdot,\cdot)$ in Equation A3.3 is $o(\frac{1}{n})$ and can be included with the other low order terms. The procedure used in Section 2.3.2 to obtain Equation 2.84 can now be used again, omitting the minimization with respect to $\underline{c}$, to obtain the result in Theorem 2.7.

B. The Lower Bound to $d(\underline{w})$ for a Noiseless Channel

We start with the lower bound in Equation 2.88:

$$d(\underline{w}) \geq L^n \int_0^{L^{-n}} d(F_o)\, dF_o \; .$$

By considering d rather than F_o the variable of integration, we can rewrite this integral as

$$d(\underline{w}) \geq L^n \int_0^{d(L^{-n})} \left[L^{-n} - F_o(d) \right] dd \, . \qquad \text{(A3.4)}$$

(The functions $d(F_o)$ and $F_o(d)$ are not strictly inverses as both are staircase functions which have no inverse. For the purposes of this integration though the "rises" of the staircase functions could be drawn in and the axes simply reversed to obtain $F_o(d)$ from $d(F_o)$.)

We can replace the upper limit of the integral in Equation A3.4 by any $d_a \leq d(L^{-n})$ and write

$$d(\underline{w}) \geq L^n \int_0^{d_a} \left[L^{-n} - F_o(d) \right] dd$$

$$= d_a - L^n \int_0^{d_a} F_o(d)\, dd .$$

Next, we divide the region of integration into two parts

$$d(\underline{w}) \geq d_a - L^n \int_0^{d_b} F_o(d)\, dd \quad - L^n \int_{d_b}^{d_a} F_o(d)\, dd$$

with $0 < d_b < d_a$, and use the monotonicity of $F_o(d)$ to further bound $d(\underline{w})$ by

$$d(\underline{w}) \geq d_a - L^n d_b F_o(d_b) - L^n \int_{d_b}^{d_a} F_o(d)\, dd . \tag{A3.5}$$

If we recognize that $F_o(d)$ is exactly equal to the distribution function $G(d)$ in Equation 2.13, and if we continue to use the approximation in Equation 2.19, we can use Shannon's upper bound,

$$F_o(d) = G(d) \leq \left[\frac{1}{\sqrt{2\pi n s^2 \mu''(s)}} + A_u(n,s) \right] e^{n\left(\mu(s) - s\mu'(s)\right)} \tag{A3.6a}$$

$$\triangleq L(n,s)\; e^{n\left(\mu(s) - s\mu'(s)\right)}$$

with

$$\mu'(s) = d \;, \qquad 0 \leq d \leq E(d|\underline{w}) \tag{A3.6b}$$

in the last two terms of Equation A3.5 to continue the inequality by

$$d(\underline{w}) \geq d_a - e^{nC} d_b L(n,s_b)\, e^{n(\mu(s_b) - s_b\mu'(s_b))}$$

$$- e^{nC} \int_{d_b}^{d_a} L(n,s)\, e^{n(\mu(s) - s\mu'(s))}\, dd . \tag{A3.7}$$

In this equation s_b satisfies $\mu'(s_b) = d_b$, and s satisfies $\mu'(s) = d$. Solutions are guaranteed since $0 < d_b < d \leq d_a \leq E(d|\underline{w})$.

We now choose d_a equal to $\mu'(s_a)$ where s_a is given by

$$L(n,s_a)\, e^{n(\mu(s_a) - s_a\mu'(s_a))} = e^{-nC} , \tag{A3.8}$$

or

$$\mu(s_a) - s_a\mu'(s_a) + \frac{1}{n} \ln L(n,s_a) = -C .$$

Since $d(L^{-n})$ is the distortion for which the distribution function G(d) equals e^{-nC}, and d_a is the distortion for which the upper bound to G(d) equals e^{-nC}, we must have $d_a \leq d(L^{-n})$ as we have already used.

If d_b is next chosen strictly less than d_a, we have

$$\mu(s_b) - s_b\mu'(s_b) < \mu(s_a) - s_a\mu'(s_a),$$

with the result that, for all n greater than some N,

$$e^{nC} L(n,s_b)\, e^{n(\mu(s_b) - s_b\mu'(s_b))}$$

$$= \frac{L(n,s_b)}{L(n,s_a)} \frac{e^{n(\mu(s_b) - s_b\mu'(s_b))}}{e^{n(\mu(s_a) - s_a\mu'(s_a))}}$$

is exponentially small in n. Therefore the second term in Equation A3.7 is surely $o(\frac{1}{n})$.

We can upper bound the last integral in Equation A3.7 if we use the well known inequality from Chernov bounds:

$$e^{n(\mu(s) - s\mu'(s))} < e^{n(\mu(s_a) - s_a d)} \qquad ; \qquad s_a \leq 0$$

when

$$\mu'(s) = d.$$

In addition, we multiply this integral by the factor $\exp n\left[s_a d_a - s_a \mu'(s_a)\right]$, which is equal to unity, to obtain

$$e^{nC} \int_{d_b}^{d_a} L(n,s)\, e^{n(\mu(s) - s\mu'(s))}\, dd$$

$$\leq e^{nC} L(n,s_a)\, e^{n(\mu(s_a) - s_a\mu'(s_a))} \left[\max_{s_b \leq s \leq s_a} \frac{L(n,s)}{L(n,s_a)}\right] e^{ns_a d_a} \int_{d_b}^{d_a} e^{-ns_a d}\, dd.$$

If we use Equation A3.8 and

$$D \triangleq \left[\max_{s_b \leq s \leq s_a} \frac{L(n,s)}{L(n,s_a)}\right]$$

we find

$$e^{nC} \int_{d_b}^{d_a} L(n,s)\, e^{n(\mu(s) - s\mu'(s))} \leq \frac{D}{-s_a n}\, e^{ns_a d_a}\left[e^{-ns_a d_a} - e^{-ns_a d_b}\right]$$

$$= \frac{D}{-s_a n}\left[1 - e^{ns_a(d_a - d_b)}\right]$$

or

$$e^{nC}\int L(n,s)\, e^{n(\mu(s) - s\mu'(s))}\, dd \;\le\; \frac{D}{-s_a n}\left[1 + o_n(1)\right].$$

Since d_b can be chosen arbitrarily close to d_a, the constant D can be made approximately equal to one.

When we collect these results, we can further lower bound $d(\underline{w})$ in Equation A3.7 by

$$d(\underline{w}) \ge \mu'(s_a) - \frac{D}{|s_a|n}\left[1 + o_n(1)\right] \tag{A3.9}$$

in which s_a is given by Equation A3.8:

$$\mu(s_a) - s_a\mu'(s_a) + \frac{1}{n}\ln\sqrt{n}\left[1 + o_n(1)\right] = -C. \tag{A3.10}$$

Appendix 4

APPENDIX FOR SECTION 3.1

A. The Coefficient $K(n,\underline{q})$ in Equation 3.15

Fano's coefficient[13] in Equation 3.15 is

$$K(n,\underline{q}) = \left(\frac{1}{2\pi n}\right)^{\frac{H(J-1)}{2}} \exp\left[-|s|\Delta - \frac{JH}{12} - \frac{1}{n}\sum_i \sum_j \frac{1}{q_i Q_{ij}(s)}\right] \tag{A4.1}$$

with

$$Q_{ij}(s) = \frac{q_j e^{sd_{ij}}}{\sum_\ell q_\ell e^{sd_{i\ell}}} .$$

B. The Convexity of the Set Q

Gallager has shown[11] that for $s < 0$, $\mu(s) - s\mu'(s)$ is a negative monotone increasing function of $\mu'(s)$ that has slope $-s$, and that ends with $s = 0$ at $\mu'(s) = E(d)$ (using our variable). Because $\mu'(s)$ is itself a monotone increasing function of s, we also know that $\mu(s) - s\mu'(s)$ is a convex up function of $\mu'(s)$. Since we wish to refer to this function many times in the appendix, we abbreviate it with F, and to indicate that the function is different for different composition vectors $\underline{q}$, we further write $F(\underline{q})$.

For a fixed s, both the functions $\mu(s)$ and $\mu'(s)$ are linear in $\underline{q}$; therefore, if $\underline{q}_1$ and $\underline{q}_2$ are two compositions in Q^H and $\underline{q}_3$ a third given by $\lambda\underline{q}_1 + (1-\lambda)\underline{q}_2$, we can write

$$\mu(s,\underline{q}_3) = \lambda\mu(s,\underline{q}_1) + (1-\lambda)\mu(s,\underline{q}_2)$$

and

$$\mu(s,\underline{q}_3) - s\mu'(s,\underline{q}_3) = \lambda\left[\mu(s,\underline{q}_1) - s\mu'(s,\underline{q}_1)\right] + (1-\lambda)\left[\mu(s,\underline{q}_2) - s\mu'(s,\underline{q}_2)\right].$$

These equations provide the following relation among $F(\underline{q}_1)$, $F(\underline{q}_2)$, and $F(\underline{q}_3)$ which we illustrate in Figure A.2. If points of equal slope, say s_1, on $F(\underline{q}_1)$ and $F(\underline{q}_2)$ are connected by a straight line Λ, the point on $F(\underline{q}_3)$ at $s = s_1$ is on Λ at a fractional distance λ from $F(\underline{q}_1)$ to $F(\underline{q}_2)$.

We wish to use this fact, and the convexity of $F(\underline{q})$ to prove that for equal values of the abscissa, the ordinate on $F(\underline{q}_3)$ is at least as large as the minimum of the ordinates on $F(\underline{q}_1)$ and $F(\underline{q}_2)$ (though not necessarily between them). If the two points of equal slope on $F(\underline{q}_1)$ and $F(\underline{q}_2)$ are again connected by the straight line Λ, and the two (parallel) tangents drawn, then surely Λ must lie between the tangents. Therefore, in the interval along the abscissa on which Λ exists, Λ must have ordinate values greater than one of the two tangents (say the tangent to $F(\underline{q}_2)$). But, since $F(\underline{q}_2)$ is convex up, its tangent must be greater everywhere than the function itself; consequently Λ has ordinate values greater than $F(\underline{q}_2)$ and we have established the desired result of this paragraph.

Now assume both $\underline{q}_1$ and $\underline{q}_2$ are in Q. That is

$$\mu(s,\underline{q}_1) - s\mu'(s,\underline{q}_1) = -R(d^*,\underline{q}_1) > -(R-\delta)$$

$$\mu(s,\underline{q}_2) - s\mu'(s,\underline{q}_2) = -R(d^*,\underline{q}_2) > -(R-\delta)$$

with

$$\mu'(s,\underline{q}_1) = \mu'(s,\underline{q}_2) = d^*.$$

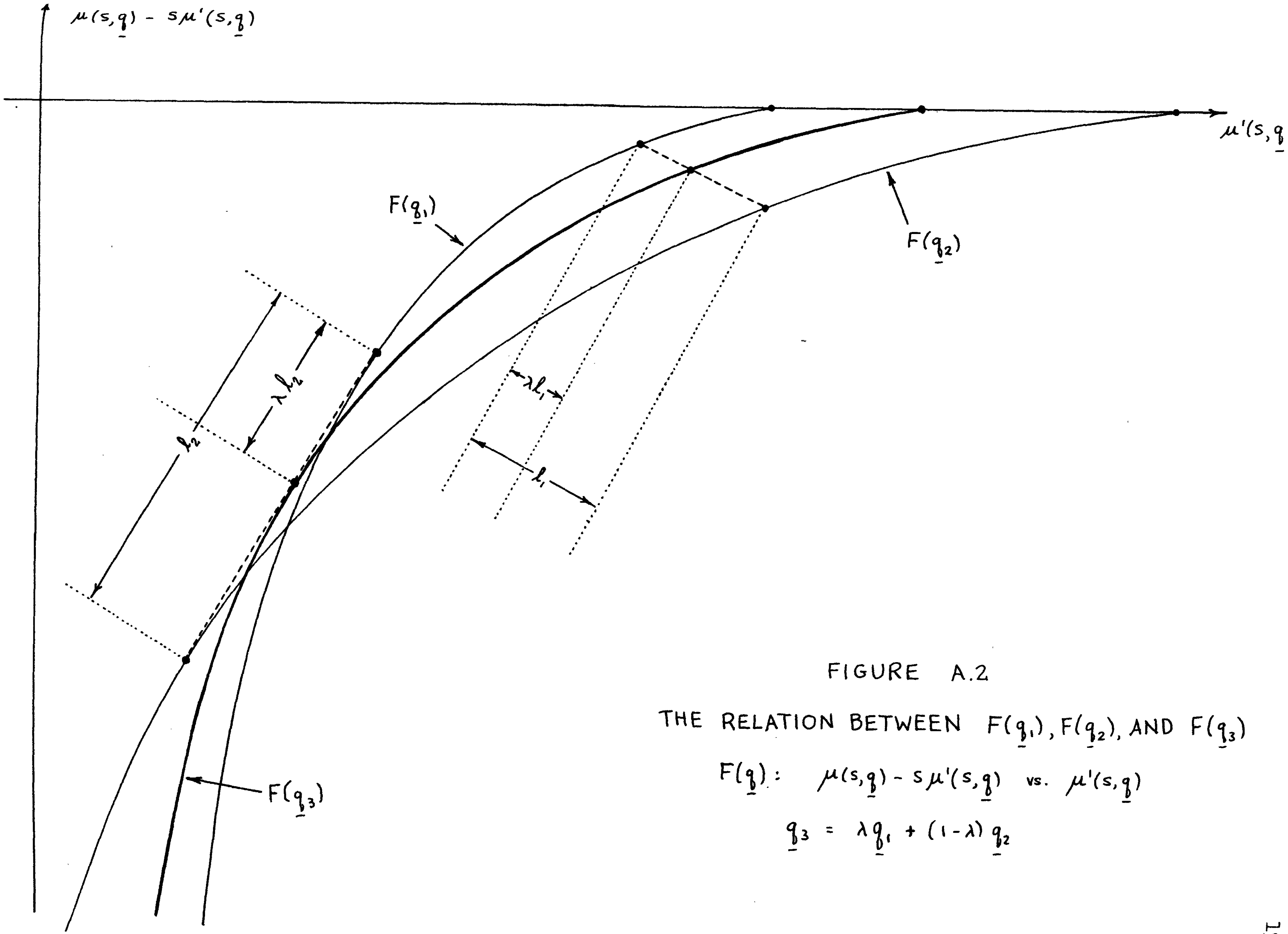

FIGURE A.2

THE RELATION BETWEEN $F(\underline{q}_1)$, $F(\underline{q}_2)$, AND $F(\underline{q}_3)$

$F(\underline{q})$: $\mu(s,\underline{q}) - s\mu'(s,\underline{q})$ vs. $\mu'(s,\underline{q})$

$\underline{q}_3 = \lambda \underline{q}_1 + (1-\lambda)\,\underline{q}_2$

The result in the previous paragraph can now be used to state that if

$$\mu'(s, \underline{q}_3) = d^*$$

also, then

$$\begin{aligned} -R(d^*, \underline{q}_3) &= \mu(s, \underline{q}_3) - s\mu'(s, \underline{q}_3) \\ &> \min\left[\mu(s, \underline{q}_1) - s\mu'(s, \underline{q}_1) \;,\; \mu(s, \underline{q}_2) - s\mu'(s, \underline{q}_2) \right] \\ &> -(R - \delta) \end{aligned}$$

and therefore that

$$\underline{q}_3 \;\varepsilon\; Q,$$

which establishes the convexity of Q.

C. Simplification of Equation 3.27

For every $u > 0$, the continuous function $X(p)$ in Equation 3.28 is positive and convex down in the interval $(0,1)$, hence has only one minimum, say at $p_{ox}(u)$. Because $Y(p) = X(1 - p)$, it too has only one minimum, $p_{oy}(u)$, which is symmetric with $p_{ox}(u)$ about $p = \frac{1}{2}$. These properties of $X(p)$ and $Y(p)$ can be used to show that the function $\min(X(p), Y(p))$ is symmetric about $p = \frac{1}{2}$, has two equal minima at p_{ox} and p_{oy}, and is convex down in the two subintervals $(0,\frac{1}{2})$, $(\frac{1}{2},1)$. Consequently, there are only two candidates among the components of $\underline{p}$ to provide the minimization of $X(p)$ and $Y(p)$:

1) that component p_i closest to but not in (p_{ox}, p_{oy})

2) that component p_j closest to but not in $(p_{ox}, p_{oy})'$.

Therefore, there are only the two corresponding possibilities for the minimizing quantity in Equation 3.29.

D. The Maximization of $\prod_{i=1}^{H} \left(\frac{p_i}{q_i}\right)^{nq_i}$ in Q'

Equation 3.32 upper bounds the composition point probabilities in Q' by

$$P(\underline{q}) \leq C(n) \left[\prod_i \left(\frac{p_i}{q_i}\right)^{q_i} \right]^n .$$

In this appendix, we show that the maximum value of this function in Q' is on the boundary of Q'. Let us form, in Q^H, the directed line

$$\underline{p} + \lambda \underline{a}, \qquad \lambda \geq 0$$

with

$$\sum_i a_i = 0$$

Along this line, the distance from $\underline{p}$ increases monotonically with λ. If we examine the behavior of $\sum_i q_i \ln(p_i/q_i)$ along the line, we have a function of the single variable

$$f(\lambda) = \sum_i (p_i + \lambda a_i) \ln \frac{p_i}{p_i + \lambda a_i}$$

that is zero at $\lambda = 0$ and, since

$$f'(\lambda) = \sum_i a_i \ln \frac{p_i}{p_i + \lambda a_i} \quad \begin{array}{ll} = 0 , & \lambda = 0 \\ < 0 , & \lambda > 0 \end{array}$$

is monotone decreasing with increasing λ. Therefore, any possibility of an interior point $\underline{q}_I$ of Q' being the maximizing point of the function in Equation 3.32 can be discounted, since the function value is higher at all points in Q' on the straight line joining $\underline{q}_I$ with $\underline{p}$.

E. The Number of Compositions in Q^H

We consider in this section the unnormalized composition space

$$\sum_{i=1}^{H} Q_i = n \tag{A4.2}$$

which is the H-1 dimensional hyperplane that intersects each axis Q_i at n. What we wish to find is the number c(n,H) of different vectors $\underline{Q} = Q_1, Q_2, \ldots, Q_H$ there are with nonnegative integer components that satisfy Equation A4.2.

If the hyperplane is bounded by the $Q_i = 0$ planes, the resulting solid is an H-1 dimensional simplex with dimension $\sqrt{2}\,n$. One typical vertex is $\underline{Q} = n,0,\ldots,0$ which is opposite the base $Q_1 = 0$. We intersect the simplex with planes $Q_1 = n$, $Q_1 = n-1$, ..., $Q_1 = 0$ and count the number of composition points on each level. On the first level we have only one composition point:

$$n,0,0,\ldots,0.$$

On the second level the composition points are

$$n-1,\ 1,0,0,\ldots,0\ ,$$

$$n-1,\ 0,1,0,\ldots,0\ ;\ \text{etc.}$$

of which there are H-1 in all. On the third level, typical composition points are

$$n-2,\ 1,0,1,0,\ldots,0$$

$$n-2,\ 0,2,0,0,\ldots,0\ .$$

If we continue in this way, we see that at level i, the number of composition points are equal to the total number of possible compositions of i-letter words from an H-1 letter alphabet. Therefore, we have the recursion relation

$$c(n,H) = \sum_{i=0}^{n} c(i,H-1)\ . \tag{A4.3}$$

Starting with the easily verified equality

$$c(i,2) = i+1,$$

we can generate:

$$c(n,2) = n+1$$

$$c(n,3) = \sum_{i=0}^{n} (i+1) = \frac{(n+2)(n+1)}{2}$$

$$c(n,4) = \sum_{i=0}^{n} \frac{(i+2)(i+1)}{2} = \frac{(n+3)(n+2)(n+1)}{(3)(2)}$$

$$\vdots$$

$$c(n,H) = \sum_{i=0}^{n} \frac{(i+H-2)(i+H-3)\cdots(i+1)}{(H-2)!}$$

$$= \frac{(n+H-1)(n+H-2)(n+H-3)\cdots(n+1)}{(H-1)\,(H-2)!}$$

$$= \frac{(n+H-1)!}{n!\,(H-1)!} \,.$$

In $c(n,H)$ the highest power of n is n^{H-1} and has the coefficient $1/(H-1)!$; therefore, to study the asymptotic behavior of the upper bound we can use

$$c(n,H) = \frac{1}{(H-1)!}\; n^{H-1}\left[1 + o_n(1)\right] .$$

F. Properties of the Exponent $E_{s1}(R)$

We have already established, by comparing Equations 3.15 and 3.23, that $R(d^*,\underline{p}) = R^*$. It follows, using the definition of Q in Equation 3.17, that when $R = R^*$ (and $\delta = 0$), the point $\underline{p}$ is on the boundary of Q'. Hence the maximum hypercube K^H has size zero, or $u = 0$ in Equation 3.28, with the result that $X_i = Y_i = 0$ and

$$E_{s1}(R^*) = 0 .$$

For $R > R^*$ and $R - \delta > R(d^*,\underline{p}) = R^*$, the definition of Q establishes that $\underline{p}$ is in Q and not on the boundary of Q'. Therefore, we have the successive inequalities: $u > 0$, $X_i, Y_i > 0$, and

$$E_{s1}(R) > 0 .$$

The definition of Q further establishes the monotone increasing property of $E_{s1}(R)$ above R^*. Equation 3.17 implies that the set $Q(R_1)$ for

$R = R_1$ includes the set $Q(R_2)$ for $R = R_2$ whenever $R_2 < R_1$, from which we conclude that the critical vertex $\underline{q}_{v,crit}(R_1)$ at $R = R_1$ must be at a greater distance from $\underline{p}$ than that between $\underline{q}_{v,crit}(R_2)$ and $\underline{p}$. The proportionality between this distance and u, and the monotonicity between $E_{s1}(R)$ and u proves the claimed monotonicity between $E_{s1}(R)$ and R.

To obtain the first derivative at R^*, we assume that just above R^*(u just > 0) some particular X_i provides the minimization indicated in Equation 3.31. Then

$$E'_{s1}(R^*) = \left.\frac{dX_i}{dR}\right|_{R^*} = \left.\frac{dX_i}{du}\right|_0 \left.\frac{du}{dR}\right|_{R^*} .$$

But

$$\frac{dX_i}{du} = \ln\frac{d_i}{p_i} - \ln\frac{1-d_i}{1-p_i} ,$$

hence

$$\left.\frac{dX_i}{du}\right|_0 = 0 , \qquad (A4.4)$$

and, if we also assume some particular vertex, $\underline{q}_v = \underline{p} + \underline{a}u$, $a_i = \pm 1$ (possibly one $a_i = 0$), restricts the magnitude of u just above R^*,

$$\frac{dR}{du} = \frac{\partial R}{\partial s}\frac{ds}{du} + \frac{\partial R}{\partial u}$$

$$= \sum_j (p_j + a_j u)\left[-s\mu_j''(s)\right]\frac{\sum_j a_j \mu_j'(s)}{\sum_j (p_j + a_j u)\mu_j''(s)} - \sum_j a_j\left[\mu_j(s) - s\mu_j'(s)\right]$$

$$= \sum_j a_j \mu_j(s)$$

with

$$\left.\frac{dR}{du}\right|_{R^*} = -\sum_j a_j \mu_j(s_o) \tag{A4.5}$$

and s_o given by Equation 3.26b with $\underline{q}_v = \underline{p}$. Therefore, if

$$\sum_j a_j \mu_j(s_o) \neq 0 , \tag{A4.6}$$

we have

$$E'_{s1}(R^*) = 0 . \tag{A4.7}$$

(If the unusual situation occurs with Equation A4.6 not true, the hypercube K^H can be altered slightly, $|a_i| \neq 1$, to again obtain an inequality in Equation A4.6 and the equality in Equation A4.7.)

The second derivative equals

$$E''_{s1}(R^*) = \left.\frac{dX_i}{du}\right|_o \left.\frac{d^2u}{dR^2}\right|_{R^*} + \left.\left(\frac{du}{dR}\right)^2\right|_{R^*} \left.\frac{d^2X_i}{du^2}\right|_o$$

in which we substitute

$$\left.\frac{d^2u}{dR^2}\right|_{R^*} = \left.\frac{\left[\sum_j a_j \mu_j'\right]^2}{\left[\sum_j a_j \mu_j\right]^3 \sum_j (p_j + a_j u)\mu_j''}\right|_{R^*,\, u=0} ,$$

$$\left.\frac{d^2X_i}{du^2}\right|_o = \left.\frac{1}{d_i} + \frac{1}{1-d_i}\right|_{u=0} = \frac{1}{p_i(1-p_i)} ,$$

Equations A4.4 and A4.5 to obtain

$$E''_{s1}(R^*) = \frac{1}{\left[\sum_j a_j \mu_j(s_o)\right]^2} \frac{1}{p_i(1-p_i)} \begin{array}{l} \neq 0 \\ < \infty \end{array} .$$

Since that component which provides the minimization in Equation 3.31 is the one which results in the smallest value of $E''_{s1}(R^*)$, we can write

$$E''_{s1}(R^*) = \frac{1}{\left[\sum_j a_j \mu_j(s_o)\right]^2} \min_i \frac{1}{p_i(1-p_i)} \quad . \tag{A4.8}$$

It can also be verified that

$$E'''_{s1}(R) = \frac{d\chi_i}{du}\frac{d^3u}{dR^3} + 3\frac{d^2u}{dR^2}\frac{d^2\chi_i}{du^2}\frac{du}{dR} + \left(\frac{du}{dR}\right)^3 \frac{d^3\chi_i}{du^3} < \infty \,.$$

Properties of the Exponent $E_{s2}(R)$

We have already shown that $\underline{p}$ is on the boundary of Q' when $R = R^*$. Since

$$\sum_j q_j \ln \frac{p_j}{q_j} < \sum_j q_j\left[\frac{p_j}{q_j} - 1\right] = 0$$

when $\underline{q} \neq \underline{p}$, but is identically zero when $\underline{q} = \underline{p}$, $\underline{p}$ is also the point on the boundary that maximizes Equation 3.32. Therefore, $\underline{q}^o = \underline{p}$ and

$$E_{s2}(R^*) = 0 \;.$$

We have also seen previously that if $R > R^*$, $\underline{p}$ is not on the boundary of Q'; therefore $\underline{q}^o \neq \underline{p}$, and

$$E_{s2}(R) > 0 \;.$$

In Appendix 4D, when we maximized the function $H(\underline{q}) \triangleq \sum_i q_i \ln(p_i/q_i)$ in Q', we showed that this function decreases monotonically with increasing λ

along every directed line from $\underline{p}$. Let the maximizing vector of this function at $R = R_1$ be $\underline{q}^o(R_1)$ and the line joining $\underline{p}$ to this point be Λ. The inclusion property used earlier, $Q(R_1) \supset Q(R_2)$ if $R_1 > R_2$, proves that the boundary of Q' for $R = R_2$ intersects Λ. Hence at the point of intersection $\underline{q}_\lambda$, $H(\underline{q}^o(R_1)) < H(\underline{q}_\lambda)$, which in turn cannot be greater than $H(\underline{q}^o(R_2))$. This establishes the monotone increasing property of $E_{s2}(R)$ with R.

Next, we find the first and second derivatives of $E_{s2}(R)$ at $R = R^*$. They are:

$$
\begin{aligned}
E'_{s2}(R^*) &= -\sum_i \frac{\partial}{\partial q_i^o}\left(q_i^o \ln \frac{p_i}{q_i^o}\right) \frac{dq_i^o}{dR}\Bigg|_{\underline{p}} \\
&= -\sum_i \left(1 + \ln \frac{p_i}{q_i^o}\right)\Bigg|_{\underline{p}} \frac{dq_i^o}{dR}\Bigg|_{\underline{p}} \\
&= -\sum_i \frac{dq_i^o}{dR}\Bigg|_{\underline{p}} \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
E''_{s2}(R^*) &= -\sum_i \left[\left(1 + \ln \frac{p_i}{q_i^o}\right) \frac{d^2 q_i^o}{dR^2}\Bigg|_{\underline{p}} - \frac{1}{q_i^o}\left(\frac{dq_i^o}{dR}\right)^2\Bigg|_{\underline{p}}\right] \\
&= \sum_i \frac{1}{p_i}\left(\frac{dq_i^o}{dR}\right)^2\Bigg|_{\underline{p}} \quad \begin{matrix} \neq 0 \\ < \infty \end{matrix} .
\end{aligned}
\qquad \text{(A4.9)}
$$

Finally, it can be shown that

$$
E'''_{s2}(R^*) = \sum_i \left[-\frac{1}{p_i^2}\left(\frac{dq_i^o}{dR}\right)^3\Bigg|_{\underline{p}} + \frac{3}{p_i}\,\frac{dq_i^o}{dR}\,\frac{d^2 q_i^o}{dR^2}\Bigg|_{\underline{p}}\right] < \infty .
$$

G. The Family of Curves $f(x,n) = x + (1-x)e^{-nx^2}$; the Lower Envelope

If x is considered the parameter, each function of n starts at $f(x,0) = 1$ and decreases exponentially to $f(x,\infty) = x$. The larger is x, the more negative is the initial slope; therefore, any two curves must cross as in Figure A.3. Consequently, the parameter x_o, that identifies at any $n = n_o$ the minimum of $f(x,n_o)$, must change with n. Since this parameter $x_o(n)$ is the solution of

$$f'_x(x,n) = 0 \,, \tag{A4.10}$$

we have

$$1 - e^{-nx_o^2}\left[1 + (1-x_o)\,2nx_o\right] = 0$$

or

$$e^{nx_o^2} - 1 = 2nx_o(1-x_o) \,. \tag{A4.11}$$

Figure A4 shows the required graphical solution which clearly always exists. The substitution of $x_o(n)$ in $f(x,n)$ specifies the single function of n, $f(x_o(n),n)$, which is the desired lower envelope. Unfortunately, an explicit solution is not possible for $x_o(n)$ nor $f(x_o(n),n)$, but we can obtain bounds to both that are adequate for our purposes.

From the graphical solution in Figure A.4, we see that any conjectured solution, x_o?, must be too large if in Equation A4.11 the left side exceeds the right side and too small if the reverse is true. This criterion could also be used on a trial functional solution $x_o(n)$?. Now, if the left side of Equation A4.11 is functionally stronger in n than the right,

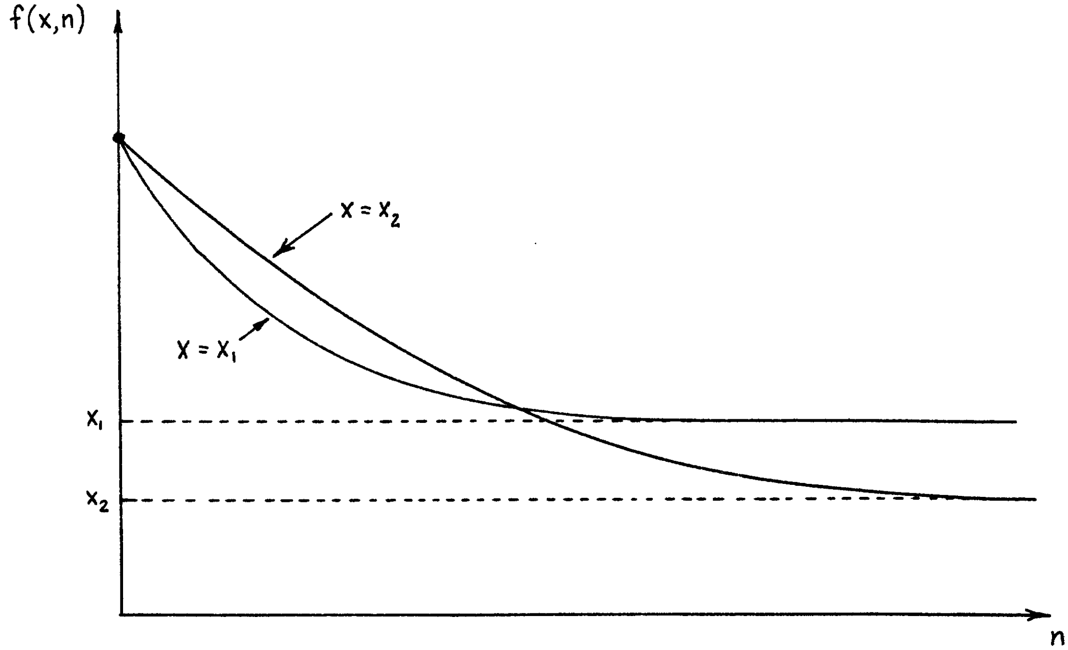

FIGURE A.3 : THE FUNCTION $f(x,n) = x + (1-x)\,e^{-nx^2}$

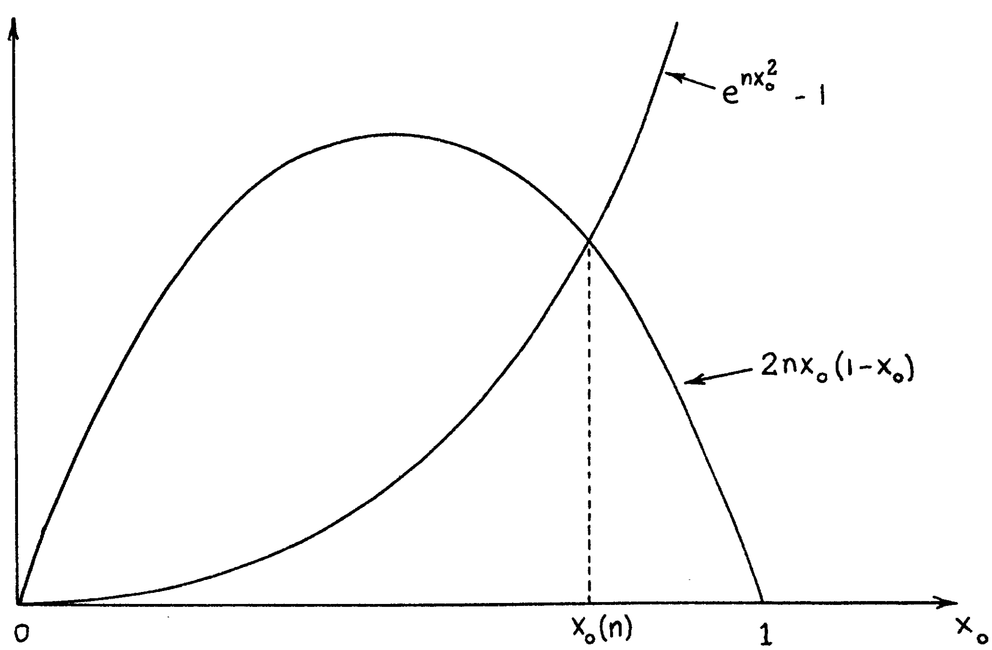

FIGURE A.4 : THE GRAPHICAL SOLUTION OF EQUATION A4.11

we know that our trial solution $x_o(n)$? is too strong in n. In a like manner, if the right side is stronger in n than the left, the solution $x_o(n)$? must be made stronger in n. For example, $x_o(n)? = \sqrt{\frac{1}{n}}$ yields

$$e^1 - 1 \overset{?}{=} 2\sqrt{n}\left(1 - \frac{1}{\sqrt{n}}\right).$$

Since the right side is larger, the actual functional solution $x_o(n)$ must be larger in n than $\sqrt{\frac{1}{n}}$. Another guess, $x_o(n)? = \sqrt{\frac{\ln n}{n}}$, yields

$$n - 1 \overset{?}{=} 2\sqrt{n \ln n}\left(1 - \sqrt{\frac{\ln n}{n}}\right)$$

in which the left side is now larger, hence $x_o(n)$ is smaller in n than $\sqrt{\frac{\ln n}{n}}$. Continuing in this way, we are led to the trial solution $x_o(n)? = \sqrt{\frac{\ln n^a}{n}}$. We now have

$$n^a - 1 \overset{?}{=} 2\sqrt{an \ln n}\left(1 - \sqrt{\frac{\ln n^a}{n}}\right)$$

in which the right side is larger for $a \le \frac{1}{2}$, but the left side is for $a > \frac{1}{2}$. This pins down the functional form of <u>the highest order term</u> in $x_o(n)$ as $\sqrt{\frac{\ln n}{2n}}$, and we can write

$$\sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}\left(1 + o_n(1)\right) \le x_o(n) \le \sqrt{\tfrac{1}{2} + \epsilon_1}\sqrt{\frac{\ln n}{n}}\left(1 + o_n(1)\right).$$

It follows that

$$f(x_0(n), n) \geq \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}\left(1+o_n(1)\right) + \left[1 + \sqrt{\tfrac{1}{2}+\epsilon_1}\sqrt{\frac{\ln n}{n}}\left(1+o_n(1)\right)\right] e^{-n\left(\frac{1}{2}+\epsilon_1\right)\frac{\ln n}{n}\left(1+o_n(1)\right)}$$

$$= \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}\left(1+o_n(1)\right)$$

and, since the lower envelope is smaller than any individual f(x,n), that

$$f(x_0(n),n) \leq f\left(\sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}, n\right)$$

$$= \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}} + \left(1 - \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}\right) e^{-\frac{1}{2}\ln n}$$

$$= \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{n}}\left(1+o_n(1)\right).$$

The lower envelope to the family of curves

$$g_i(x,n) = x + (A-x)\,D_i(n)\,e^{-Bnx^2}, \qquad i = 1,2$$

in Equation 3.53 is found in the same way. Corresponding to Equations A4.10 and A4.11, we have

$$g'_x(x,n) = 0$$

and

$$\frac{1}{D_i(n)}\,e^{Bnx_0^2} - 1 = 2Bnx_0\left(A - x_0\right). \qquad \text{(A4.12)}$$

The solution $x_o(n)$ to Equation A4.12, and the resulting lower envelope $g(x_o(n),n)$ can be bounded if we repeat the argument used previously. The results are

i = 1:

$$x_o(n)? = \sqrt{\frac{\ln n^a}{Bn}}$$

$$a_{crit} = \frac{1}{2}$$

$$\sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right) \le x_o(n) \le \sqrt{\tfrac{1}{2}+\epsilon_1}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right)$$

$$\sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right) \le g_1(x_o(n),n) \le \sqrt{\tfrac{1}{2}}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right),$$

i = 2:

$$x_o(n)? = \sqrt{\frac{\ln n^{H-1/2+a}}{Bn}}$$

$$a_{crit} = 1/2$$

$$\sqrt{H}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right) \le x_o(n) \le \sqrt{H+\epsilon_1}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right)$$

$$\sqrt{H}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right) \le g_2(x_o(n),n) \le \sqrt{H}\sqrt{\frac{\ln n}{Bn}}\left(1 + o_n(1)\right)$$

in which ϵ_1 is an arbitrarily small positive constant.

Appendix 5

APPENDIX FOR SECTION 3.2

A. The Family of Curves $f(\Delta,n) = \Delta + (D-\Delta)\exp\left(-Le^{-ns\Delta}\right)$; The Lower Envelope

In Equation 3.71 the set of bounds are

$$d(\Delta,n) = d_c(\underline{q}) + \Delta + \left(d_{max} - d_c(\underline{q}) - \Delta\right) e^{-L(n,\underline{q}) \exp -ns\Delta} .$$

To find the optimizing $\Delta_o(n)$, we consider first the simplified problem

$$\min_{\Delta} f(\Delta,n) = \min_{\Delta}\left[\Delta + (D-\Delta)\, e^{-Le^{-sn\Delta}}\right] \triangleq f\left(\Delta_o(n), n\right)$$

for which $\Delta_o(n)$ is given by

$$f'_{\Delta}(\Delta,n) = 0 ,$$

or

$$e^{Le^{-sn\Delta_o}} - 1 = -sn\left(D-\Delta_o\right) e^{-sn\Delta_o} L . \qquad (A5.1)$$

The graphical solution for $\Delta_o(n)$ in Figure A.5 reveals a single solution in $0 < \Delta_o < D$, which we bound using the same procedure as in Appendix 4G. If the trial solution

$$\Delta_o(n)? = \frac{\ln n^a}{-sn}$$

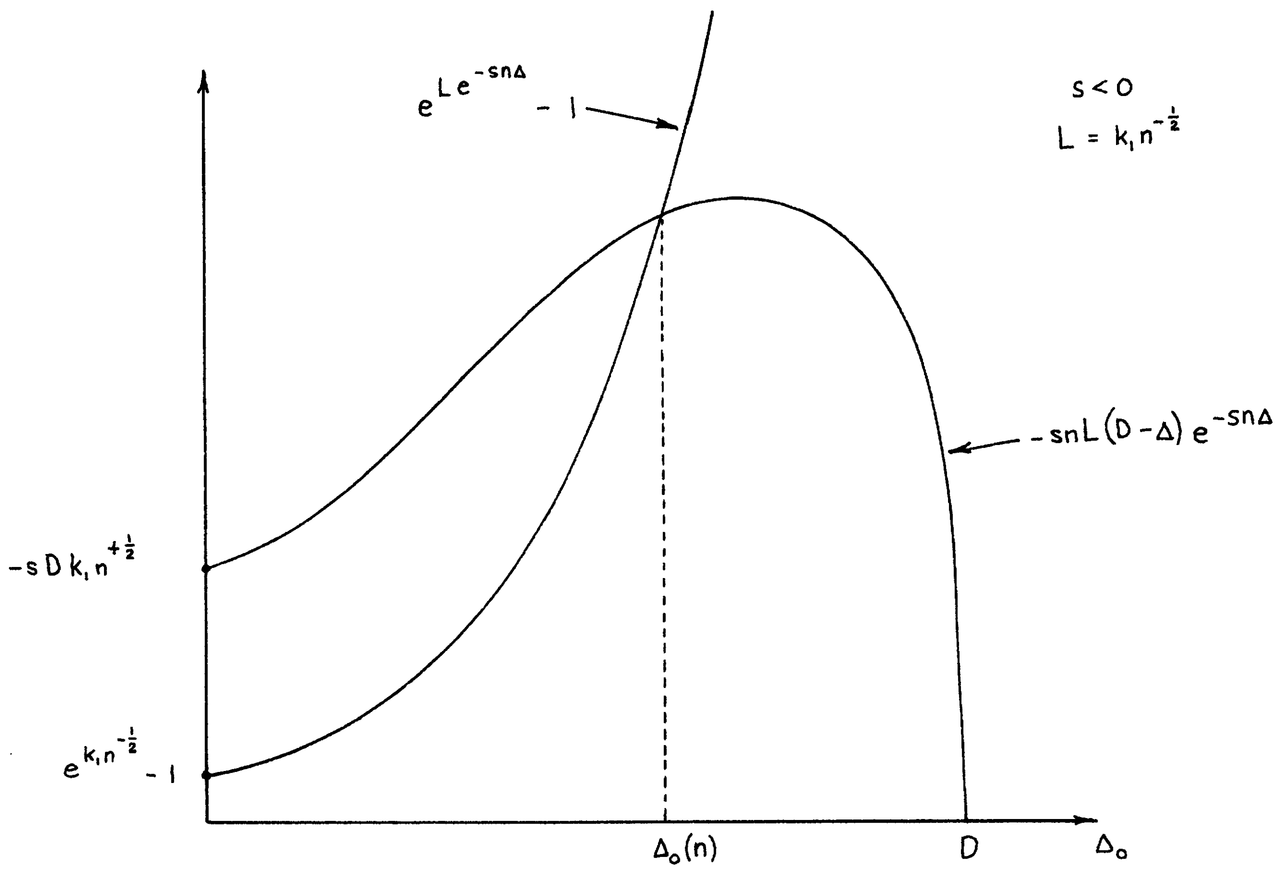

FIGURE A.5: THE GRAPHICAL SOLUTION OF EQUATION A5.1

is used in Equation A5.1, we have

$$e^{Le^{a \ln n}} - 1 \overset{?}{\doteq} -sn\left(D + \frac{\ln n^a}{sn}\right) L\, e^{\ln n^a}$$

or, using Equation 3.68,

$$e^{k_1 n^{a-\frac{1}{2}}} - 1 \overset{?}{\doteq} -sk_1\left(D + \frac{\ln n^a}{sn}\right) n^{a-\frac{1}{2}}.$$

It can be shown that the left side is larger for $a > \frac{1}{2}$, but the right side is for $a \leq \frac{1}{2}$. Thus we have

$$\frac{1}{2}\frac{\ln n}{-sn}\left(1 + o_n(1)\right) \leq \Delta_o(n) \leq \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\left(1 + o_n(1)\right) \qquad \text{(A5.2)}$$

in which ϵ_1 is an arbitrarily small positive constant. The function $f(\Delta_o(n),n)$ can in turn be lower bounded by

$$f\left(\Delta_o(n),n\right) \geq \frac{1}{2}\frac{\ln n}{-sn} + \left[D - \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\left(1+o_n(1)\right)\right] e^{-k_1 n^{-\frac{1}{2}} e^{\left(\frac{1}{2}+\epsilon_1\right)\ln n\left(1+o_n(1)\right)}}$$

$$= \frac{1}{2}\frac{\ln n}{-sn}\left(1 + o_n(1)\right)$$

and, since the lower envelope is smaller than any individual curve, $f(\Delta_o(n),n)$ can be upper bounded by

$$f\left(\Delta_o(n),n\right) \leq f\left(\left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn},\ n\right)$$

$$= \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn} + \left[D - \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\right] e^{-k_1 n^{\epsilon_1}}$$

$$= \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\left(1 + o_n(1)\right).$$

When this result is used in the distortion functions $d(\Delta,n)$, we have

$$d(\Delta_o(n),n) \leq d_c(\underline{q}) + \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\left(1 + o_n(1)\right). \tag{A5.3}$$

B. The Average of $\overline{d_{min}(\underline{q})}$ over Q^H

We start with Equation 3.73,

$$\overline{d(\theta_1)} \leq \int_{Q^H}\!\!\cdots\!\int P(\underline{q})\left[d_c(\underline{q}) + \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\right]d\underline{q},$$

and use Taylor's Formula with Remainder for the bracketed term to write

$$\begin{aligned}\overline{d(\theta_1)} \leq\; & d_c(\underline{p}) + \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn} \\ & + \sum_i \frac{\partial}{\partial q_i}\left[d_c(\underline{q}) + \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\right]\Bigg|_{\underline{p}} E(q_i - p_i) \\ & + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial q_i \partial q_j}\left[d_c(\underline{q}) + \left(\tfrac{1}{2}+\epsilon_1\right)\frac{\ln n}{-sn}\right]\Bigg|_{\underline{\phi}} E\left[(q_i-p_i)(q_j-p_j)\right]\end{aligned} \tag{A5.4}$$

where $s_o \triangleq s_o(\underline{p})$ and $\underline{\phi} \in Q^H$.

When the parametric form of the rate-distortion curve in Equations 3.23 is compared with Equations 3.64 and 3.66, we recognize that the distortion term $d_C(\underline{p})$ in Equation A5.4 is equal to d_C, the value of distortion on the R(d) curve at R = C.

Finally, because the derivatives

$$\frac{\partial^2 d_c(\underline{q})}{\partial q_i \partial q_j} = -\frac{\theta_i \theta_j}{s^3 \mu''},$$

$$\frac{\partial}{\partial q_i}\left(\frac{1}{s}\right) = -\frac{1}{s^2}\,\frac{\theta_i}{s\mu''}\ ,$$

and

$$\frac{\partial^2}{\partial q_i \partial q_j}\left(\frac{1}{s}\right) = \frac{\theta_i \theta_j}{s^5 \mu''^3}\left(3 + s\mu''' - \mu''\right) + \frac{\theta_i \mu_j'' + \theta_j \mu_i''}{s^3 \mu''}$$

are all finite, the $\frac{1}{n}$ coefficient in

$$E\left[(q_i - p_i)(q_j - p_j)\right] = \frac{1}{n}\left(p_i \delta_{ij} - p_i p_j\right) \tag{A5.5}$$

makes this term in Equation A5.4 asymptotically unimportant compared with the $\frac{\ln n}{n}$ term. Consequently, we have

$$\overline{d(\theta_1)} \le d_c + \left(\tfrac{1}{2} + \epsilon_1\right)\frac{\ln n}{-sn}\left(1 + o_n(1)\right). \tag{A5.6}$$

Appendix 6

APPENDIX FOR SECTION 3.3

A. Lower Bound to Average Distortion for an M-Point System

If we use the same procedure used in Section 2.3.2 to remove the block length variable n from the parametric equation, Equation 3.78 becomes, at I = -C,

$$D_L(-C) = d_L(-C, \underline{p}) + \frac{1}{2n|s_0|} \frac{\sigma^2}{s_0^2 \mu''(s_0, \underline{p})} + o\left(\frac{1}{n}\right)$$

$$= d_C + \frac{1}{2n|s_0|}\left[\frac{\sigma^2}{s_0^2 \mu''(s_0, \underline{p})} + \ln \frac{\gamma''(-1)}{s_0^2 \mu''(s_0, \underline{p})}\right] + o\left(\frac{1}{n}\right)$$

$$\triangleq d_C + \frac{A}{n} + o\left(\frac{1}{n}\right). \tag{A6.1}$$

Together with Equation 3.79 and the boundedness of $D_L''(I)$, Equation A6.1 can be used to show

$$D_L(I) = D_L(-C) + D_L'(-C)(I+C) + \frac{1}{2} D_L''(I')(I+C)^2$$

$$= \left[d_C + \frac{A}{n} + o\left(\frac{1}{n}\right)\right] + \left[\frac{1}{|s|} + \frac{B}{n} + o\left(\frac{1}{n}\right)\right](I+C)$$

$$+ \frac{1}{2} D_L''(I')(I+C)^2$$

or

$$D_L(I_n) - d_C = 0$$

$$= \frac{A}{n} + \frac{1}{|s|}(I_n + C) + \frac{B}{n}(I_n + C) + \frac{1}{2} D_L''(I')(I+C)^2 + o\left(\frac{1}{n}\right).$$

The last equation requires that

$$I_n + C = -\frac{A|s|}{n} + o\left(\frac{1}{n}\right), \tag{A6.2}$$

or that I_n approach $-C$ at the rate $\frac{1}{n}$ (from above or below since A can be positive or negative). Thus in Equation 3.81 the discontinuity in the derivatives of $D_M(I)$ approaches $I = -C$ faster than does the standard derivation of $F_2(I)$ approach zero (which is only as $\frac{1}{\sqrt{n}}$). It is this property of $D_M(I)$ that will enable us to prove that the integral in Equation 3.81 approaches d_C only as fast as $\frac{1}{\sqrt{n}}$.

In Equation 3.81, the function $D_M(I)$ is a monotone non-decreasing function of I that has a value d_C below $I = I_n$, a value given by $D_L(I)$ above $I = I_n$, and at $I = I_n^+$ a derivative $D_M'(I_n^+) = D_L'(I_n^+)$ that is strictly greater than zero, even in the limit as $n \to \infty$. Therefore, it is possible to choose a $\Delta > 0$ and an $m > 0$ such that the function

$$D_{ML}(I) = \begin{cases} d_c & I \le I_n \\ m(I - I_n) & I_n < I < I_n + \Delta \\ d_c + m\Delta & I_n + \Delta \le I \end{cases}$$

satisfies

$$D_{ML}(I) \le D_M(I)$$

for all I. We can use $D_{ML}(I)$ in Equation 3.81 to continue the inequality as

$$d_M(S) \ge \int_{I_{min}}^{I_{max}} D_{ML}(I)\, dF_2(I) = d_c + \int_{I_n}^{I_n+\Delta} m(I - I_n)\, dF_2(I) + m\Delta \Pr(I_2 > I_n + \Delta)$$

from which the last term can be dropped since it is positive.

Next, the random variable I_2, which has

$$E(I_2) = -C = \gamma'(-1)$$

$$Var(I_2) = \frac{1}{n}\gamma''(-1)$$

is normalized by setting

$$I' = \sqrt{\frac{n}{\gamma''}}\,(I_2 + C)\,.$$

In terms of the new variable I', the lower bound to $d_M(S)$ is

$$d_M(S) \geq d_c + \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} m\left[\sqrt{\frac{\gamma''}{n}}\,I' - C - I_n\right] dH_{I'}(I')$$

which, after integration by parts, becomes

$$d_M(S) \geq d_c + m\Delta H_{I'}\left(\sqrt{\frac{n}{\gamma''}}\,(I_n + C + \Delta)\right)$$

$$- m\sqrt{\frac{\gamma''}{n}}\int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} H(I')\,dI' \,. \qquad (A6.3)$$

To approximate H(I') we use the theorem due to Berry[15] that states:

$$\left|H(z) - \Phi(z)\right| \leq \frac{C\rho_3}{\sqrt{n}}$$

with

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,dx\,,$$

$$\rho_3 = \frac{\sum_{k=1}^{K} c_k \beta_{3k}}{\gamma''(-1)^{3/2}}\,,$$

and

$$\beta_{3k} = \int_{-\infty}^{+\infty} \left| I - \gamma_k'(-1) \right|^3 dF_{I_k}(I)$$

in which I_k is the Information Difference variable that has the distribution given by Equation 2.25.

When this substitution is made in Equation A6.3, we find

$$d_M(S) \geq d_c + m\Delta\, \Phi\left(\sqrt{\frac{n}{\gamma''}}\,(I_n + C + \Delta)\right) - m\Delta \frac{C\rho_3}{\sqrt{n}}$$

$$- m\sqrt{\frac{\gamma''}{n}} \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} \Phi(I')\, dI' - m\sqrt{\frac{\gamma''}{n}} \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} \frac{C\rho_3}{\sqrt{n}}\, dI',$$

or, after again using integration by parts on the second and fourth terms, this time in reverse:

$$d_M(S) \geq d_c + \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} m\left(\sqrt{\frac{\gamma''}{n}}\, I' - C - I_n\right) d\Phi(I') - \frac{2m\Delta C\rho_3}{\sqrt{n}}$$

$$= d_c + m\sqrt{\frac{\gamma''}{2\pi n}} \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} I' e^{-\frac{1}{2}I'^2}\, dI' - m(I_n+C) \int_{\sqrt{\frac{n}{\gamma''}}(I_n+C)}^{\sqrt{\frac{n}{\gamma''}}(I_n+C+\Delta)} d\Phi(I') - \frac{2m\Delta C\rho_3}{\sqrt{n}}.$$

The second last term is $o(\frac{1}{\sqrt{n}})$ by Equation A6.2. And, if Δ is allowed to approach zero as $n^{-\frac{1}{4}}$, the last term is also $o(\frac{1}{\sqrt{n}})$. This leaves for the lower bound

$$d_M(S) \geq d_c + m\sqrt{\frac{\gamma''}{2\pi n}} \left[e^{-\frac{1}{2}\frac{n}{\gamma''}(I_n+C)^2} - e^{-\frac{1}{2}\frac{n}{\gamma''}(I_n+C+\Delta)^2} \right] + o\left(\frac{1}{\sqrt{n}}\right)$$

which, using Equation A6.2 and the assumed behavior of Δ, can be shown to equal

$$d_M(S) \geq d_c + m\sqrt{\frac{\gamma''}{2\pi n}} + o\left(\frac{1}{\sqrt{n}}\right). \qquad (A6.4)$$

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BIOGRAPHICAL NOTE

Randolph John Pilc was born on July 20, 1937 in Long Island City, New York. From 1955 to 1960 he attended the College of the City of New York where he was awarded, in February 1960, the degree of Bachelor of Electrical Engineering, magna cum laude. Upon graduation he joined the technical staff of the Bell Telephone Laboratories and through their Communications Development Training Program was awarded, in May 1962, the degree of Master of Electrical Engineering from New York University. At that time, Mr. Pilc was also awarded the Bell Telephone Laboratories Doctoral Fellowship which he chose to use at the Massachusetts Institute of Technology. Mr. Pilc is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi.

In 1961 he married the former Elaine Ethelanne Adam of Short Hills, New Jersey. They have a son, Gary, born in 1963 in Malden, Massachusetts.