The Presentation Functor and Weierstrass Theory for Families of Local Complete Intersection Curves

by

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Abstract

The thesis consists of two independent parts. In the first part, we use the presentation functor to construct a stratification for the compactified Picard scheme of a family of geometrically integral varieties, given a partial normalization of the family. For a family of curves we show in a special case that each stratum is an open subscheme of a certain Grassmanian over the compactified Jacobian of the partial normalization. We also use a variant of the presentation functor to find a natural projective compactification for the Picard scheme of a smooth in codimension 1 variety, partially answering a question raised by Altman and Kleiman. Finally, we extend the notion of the Theta divisor to the compactified Jacobian of any family of curves, and show that the Theta divisor is ample. In the second part, we replace the sheaves of principal parts on a flat family of reduced, local complete intersection curves by sheaves of algebras that behave like the sheaves of principal parts on a smooth family. Then we associate to each linear system on the family a Wronski system, as defined by Laksov and Thorup. By applying their general theory of Wronski systems we obtain in particular a Weierstrass divisor on the family if there are no degenerate components on a general fibre.

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CONTENTS

PART I: The Presentation Functor 9

1. Introduction 9
2. The presentation functor 13
3. The stratification 18
4. Curves 22
5. Higher dimensional varieties 27
6. The Theta divisor 30

PART II: Weierstrass Theory for Families of Local Complete Intersection Curves 43

1. Introduction 43
2. The Wronski algebra system 48
3. A local criterion 56
4. The Wronski algebra system for "general" families 63
5. Some lemmas 69
6. Existence and uniqueness of the Wronski algebra system 74
7. Wronski systems and Wronskians 80

BIBLIOGRAPHY 85

PART I

The Presentation Functor

1. Introduction.

This part of the thesis partially generalizes the theory of the presentation functor, as developed by Altman and Kleiman [**6**], to the case of families of geometrically integral varieties. In addition, and most importantly, we apply the theory to obtain new results about the compactified Picard scheme.

The presentation functor allows us to understand the compactified Picard scheme of a singular variety in terms of the Picard scheme of its normalization. In other words, it allows us to understand torsion-free, rank 1 sheaves on a singular variety in terms of line bundles on a substantially less singular variety (even smooth, in case of curves.)

The presentation functor was introduced by Oda and Seshadri [**30**] in the context of nodal, possibly reducible curves, even though the local picture was already considered by Rego [**31**], whose article was nevertheless published after [**30**]. The results about the presentation functor obtained by Oda and Seshadri were later generalized by Kleppe [**17**], who worked with nodal, integral curves.

Altman and Kleiman [**6**] treated the more general case of families of nodal and cuspidal integral curves. In fact, they did considerably more. They started with a relatively birational morphism $\pi\colon X' \longrightarrow X$ between families X/S and X'/S of geometrically integral curves. They defined the presentation functor associated to π (see (2.3)) and proved that its étale associated sheaf is represented by a scheme P. Assuming that on every fibre $X(s)$ at most a node or a cusp $Q(s)$ is blown up under $\pi(s)$, they showed that P is a $\mathbf{P}^1$-bundle over the compactified Jacobian $\bar{J}'$ of X'/S. On the other hand, they considered the canonical map $\kappa\colon P \longrightarrow \bar{J}$ onto the compactified Jacobian $\bar{J}$ of X/S, and proved that κ is an isomorphism over the open subset of $\bar{J}$ parametrizing torsion-free, rank 1 sheaves invertible at Q, and that κ is a 2-to-1 contraction over the complement (which is isomorphic to $\bar{J}'$.)

(1.1) Assume that S is the spectrum of an algebraically closed field for the following description. Let Q be the conductor of π, namely,

$$Q := \mathrm{Supp}\,(\frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X}),$$

where "Supp" denotes the scheme-theoretic support.

Our point of view is to think of Altman's and Kleiman's results as giving primarily a stratification of $\bar{J}$ in two subschemes: an open subscheme that is also an open subscheme of a $\mathbf{P}^1$-bundle over $\bar{J}'$, and a closed subscheme that is isomorphic to $\bar{J}'$. In general, the key idea to obtain a stratification is to notice that any presentation $h: I \longrightarrow \pi_* I'$ (see (2.3) for the definition of a presentation) can be realized as a composition of two "smaller" presentations. First, one considers the torsion-free, rank 1 sheaf I^π generated by I on X' and the canonical associated homomorphism $h_1: I \longrightarrow \pi_* I^\pi$. Then $h = h_2 \circ h_1$, where $h_2: I^\pi \longrightarrow I'$ is the induced embedding on X'. Since h_1 is given by the sheaf I already, then the additional information given by a presentation h is only h_2. Since the cokernel of h is supported on the conductor subscheme Q of π, then the cokernel of h_2 is supported on $Q' := f^{-1}(Q)$. If Q' is contained in the smooth locus of X', then h_2 is obtained simply by picking an appropriate number of points inside Q', with possible multiplicities. In the case Q is a node or a cusp, and the cokernel of h has length 1, then either I^π is equal to I' if I is invertible at Q, and hence the fibre of κ over the point $[I]$ in $\bar{J}$ corresponding to I consists of just one point, or $I = \pi_* I^\pi$, in which case the fibre of κ over $[I]$ consists of two points (one point counted with multiplicity 2, if Q is a cusp), corresponding to the two points of Q'.

In general, let δ be the genus change of π. Then one obtains a stratification (Theorem 4.3) of $\bar{J}$ in $\delta + 1$ subschemes A^i for $i = 0, 1, \ldots, \delta$, where A^i parametrizes torsion-free, rank 1 sheaves I such that

$$\text{length}\,\left(\frac{\pi_* I^\pi}{I}\right) = i.$$

Furthermore, each A^i is an open subscheme of the Grassmannian of rank i quotient bundles of a certain rank $\delta + 1$ vector bundle on $\bar{J}'$ if the conductor Q is a reduced point of X (Proposition 4.5 and Corollary 4.6.)

(1.2) We also consider the case of higher dimensional varieties. Let X be an integral variety defined over an algebraically closed field k. In [**5**] Altman and Kleiman proved the representability of the étale sheaf associated to the compactified Picard functor $Pic_X^{=\alpha}$, as defined in (2.3). They also showed that the subset C_X^α of points of $\mathrm{Pic}_X^{=\alpha}$ parametrizing the torsion-free rank 1 sheaves which are invertible on the smooth locus of X is closed [**5**, (3.2,ii), p. 28]. But the important question of whether there is a natural subscheme structure for C_X^α still remains. Altman and Kleiman also asked for a good intrinsic description of the points in the closure of the Picard scheme Pic_X^α in C_X^α.

By means of a variant of the presentation functor (see (5.1)) we throw some light into the latter question. More specifically we show that the boundary points represent torsion-free, rank 1 sheaves invertible on the normal locus of X (Proposition 5.4.) As for the former question, in case X is smooth in codimension 1 we show that there is actually a closed subset of C_X^α containing the Picard scheme Pic_X^α and with a natural subscheme structure. In fact, the above closed subscheme represents the variant of the presentation functor mentioned above (Theorem 5.5.)

(1.3) If X is a smooth, complete, connected curve over an algebraically closed field, then its Jacobian J is projective and admits a canonical, rigidified along the identity, ample invertible sheaf. This sheaf is the symmetrization of a Theta divisor. For each $p \in X$, the Theta divisor is the scheme-theoretic image of the morphism $X^{g-1} \longrightarrow J$, given by

$$(p_1, \ldots, p_{g-1}) \mapsto \mathcal{O}_X(p_1 + \cdots + p_{g-1} - (g-1)p).$$

The Theta divisor is itself ample, a consequence of the autoduality of the Jacobian [**35**, no. 62, Cor. 2] by applying [**29**, Application 1, p. 60]. These classical notions and results can be easily extended to families of smooth, complete, connected curves, as shown in [**8**, Prop. 4, p. 260] for instance.

In a more general situation, Deligne has shown that if $f \colon X \longrightarrow S$ is a proper, flat family of Deligne-Mumford stable curves, then the functor parametrizing invertible sheaves of degree 0 on each irreducible component of each fibre of f is represented by a smooth, separated S-scheme $\mathrm{Pic}^0_{X/S}$, and there is a canonical, rigidified along the structure sheaf, S-ample invertible sheaf on $\mathrm{Pic}^0_{X/S}$ [**10**, Prop's. 4.2, 4.3]. A power of this S-ample sheaf gives rise to a quasi-projective embedding of $\mathrm{Pic}^0_{X/S}$ over S, and consequently a natural compactification for $\mathrm{Pic}^0_{X/S}$.

On the other hand, Altman and Kleiman have already constructed a natural compactification $\bar{J}$ for the relative Jacobian J of a family of projective, geometrically integral curves [**4**], [**5**]. A natural question in this context is whether there is a canonical S-ample invertible sheaf on $\bar{J}$ whose restriction to J is the canonical S-ample sheaf whose existence was proved by Deligne, if the fibres of X/S are integral, stable curves. Our purpose is to give an affirmative answer to the above question.

As a matter of fact, we do considerably more than answering to the above question. For any projective family X/S of geometrically integral curves and any section of X/S we construct the Theta line bundle Θ on $\bar{J}/S$, a natural generalization of the Theta line bundle associated to the Theta divisor on the Jacobian of a smooth curve. By using the theory of the presentation functor, we are able to show that Θ is S-ample from the ampleness of the Theta line bundle on the Jacobian of a smooth curve (Theorem 6.20.) In the case the fibres of X/S are Gorenstein, then the inverse map on J/S extends to $\bar{J}/S$. Hence, one can consider the symmetrization of Θ, which gives a canonical, rigidified along the structure sheaf of X/S, relatively ample line bundle (Theorem 6.32.) In addition, the latter sheaf extends Deligne's canonical ample sheaf to the compactified Jacobian if the fibres of X/S are integral, stable curves (see (6.34).)

Associated to Θ there is a canonical global section, the Theta function θ. By using a result of Altman's, Iarrobino's and Kleiman's [**1**], we remark that the zero scheme of θ is a relative Cartier divisor on $\bar{J}/S$ whose geometric fibres are integral if the fibres of X/S can be embedded into a smooth surface (Proposition 6.8.) In this case, one can also prove easily Poincaré's formula, which says that the self-intersection of Θ with itself g times is $g!$, where g is the arithmetic genus of the fibres of X/S (Proposition 6.9.)

(1.4) We give now a summary of the contents of this part of the thesis.

In Section 2 we introduce the presentation functor and review related concepts. Since the extra effort is small, we consider the general case of projective families of geometrically integral varieties. Most of this section is a straightforward generalization of [**6**].

In Section 3 we construct the stratification induced by an S-birational map $f\colon X' \longrightarrow X$ on the compactified Picard scheme of a projective family X/S of geometrically integral varieties.

In Section 4 we restrict ourselves to families of curves and prove the results mentioned in (1.1).

In Section 5 we show that it can pay off to introduce the presentation functor for higher dimensional varieties. We prove the results mentioned in (1.2).

In section 6 we show how to construct the Theta line bundle on the compactified Jacobian of a family of pointed curves, and prove its ampleness.

All schemes involved are assumed noetherian. All families are assumed flat.

2. The presentation functor.

(2.1) Fix a connected base scheme S. By an *S-variety* it will be meant a flat, projective morphism $f\colon X \to S$, whose geometric fibres are integral. An *S-curve* is an S-variety whose fibres have dimension 1. If T is any S-scheme we will denote by $f_T\colon X_T \to T$ the base extension of f to T.

Let $f\colon X \to S$, and $f'\colon X' \to S$ be S-varieties. An S-morphism $\pi\colon X' \to X$ is called *S-birational* if the induced morphisms on the fibers over S are birational. Obviously, if T is any S-scheme, then π_T is also T-birational.

Fix now a finite S-birational morphism $\pi\colon X' \to X$ of S-varieties. By [**EGA IV-3**, 8.11.1, p. 41], if X and X' were S-curves, then the finiteness would follow from the S-birationality. Since $\mathcal{O}_{X'}$ is S-flat and π is S-birational, the comorphism $\pi^c\colon \mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is an injective homomorphism of S-flat sheaves with S-flat cokernel, whose formation commutes with base change [**EGA IV-3**, 11.3.7, p. 135]. Fix an S-ample line bundle $\mathcal{O}_X(1)$ on X. Then $\mathcal{O}_{X'}(1) := \pi^*\mathcal{O}_X(1)$ is also S-ample, since π is finite. We compute Hilbert polynomials on X and X' with respect to these ample sheaves. If $\mathcal{F}$ is a sheaf on X (or on X'), then we denote by $\chi_t(\mathcal{F}(s))$ the Hilbert polynomial of the restriction $\mathcal{F}(s)$ of $\mathcal{F}$ to $X(s)$ (or to $X'(s)$) for each point $s \in S$. Let

$$Q := \mathrm{Supp}(\frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X}) \quad \text{and} \quad Q' := \pi^{-1}(Q),$$

where by "Supp" it is meant the scheme-theoretic support.

PROPOSITION 2.2. *If Q is S-flat, then so is Q'. In addition,*

$$\chi_t(Q'(s)) = \chi_t(Q(s)) + \chi_t(\frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X}(s))$$

for every $s \in S$.

PROOF: Let $\mathcal{M}_Q$ (resp. $\mathcal{M}_{Q'}$) be the ideal sheaf of $Q \subset X$ (resp. $Q' \subset X'$). Of course $\mathcal{M}_{Q'} = \mathcal{M}_Q\mathcal{O}_{X'}$. Since

$$\mathcal{M}_Q = \mathrm{ann}_X(\frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X}),$$

then

$$\mathcal{M}_Q = \mathcal{M}_Q\pi_*\mathcal{O}_{X'} = \pi_*(\mathcal{M}_Q\mathcal{O}_{X'}).$$

From the exact sequence

$$0 \longrightarrow \frac{\mathcal{O}_X}{\mathcal{M}_Q} \longrightarrow \frac{\pi_*\mathcal{O}_{X'}}{\mathcal{M}_Q} \longrightarrow \frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X} \longrightarrow 0$$

and the S-flatness of Q, the above middle sheaf is S-flat. But

$$\frac{\pi_* \mathcal{O}_{X'}}{\mathcal{M}_Q} = \frac{\pi_* \mathcal{O}_{X'}}{\pi_*(\mathcal{M}_Q \mathcal{O}_{X'})} = \pi_*(\frac{\mathcal{O}_{X'}}{\mathcal{M}_{Q'}}).$$

Since π is finite, then Q' is S-flat. The equation in the statement is obvious from the above exact sequence and the above equality. ∎

(2.3) If S is the spectrum of a field, then a coherent $\mathcal{O}_X$-module I is called *torsion-free, rank 1* if I satisfies S_1 and is generically isomorphic to $\mathcal{O}_X$. In the more general setting, a coherent $\mathcal{O}_X$-module $\mathcal{I}$ is called *torsion-free, rank 1* if it is S-flat, and for each s in S the fibre $\mathcal{I}(s)$ is a torsion-free, rank 1 sheaf on $X(s)$.

Let T be an S-scheme. Let $\mathcal{I}$ be a torsion-free, rank 1 sheaf on X_T. A *presentation* of $\mathcal{I}$ is an injective $\mathcal{O}_{X_T}$-homomorphism $h\colon \mathcal{I} \to \pi_{T*}\mathcal{I}'$, where $\mathcal{I}'$ is a torsion-free, rank 1 sheaf on X'_T, and the cokernel of h is T-flat with scheme-theoretic support on Q_T. Two presentations

$$h_1\colon \mathcal{I}_1 \longrightarrow \pi_{T*}\mathcal{I}'_1 \quad \text{and} \quad h_2\colon \mathcal{I}_2 \longrightarrow \pi_{T*}\mathcal{I}'_2$$

are *equivalent* if there exists an invertible sheaf $\mathcal{N}$ on T and a commutative diagram,

$$\begin{array}{ccc}
\mathcal{I}_1 & \xrightarrow{h_1} & \pi_{T*}\mathcal{I}'_1 \\
\downarrow & & \downarrow \\
\mathcal{I}_2 \otimes \mathcal{N} & \xrightarrow{h_2 \otimes \mathrm{id}_{\mathcal{N}}} & \pi_{T*}\mathcal{I}'_2 \otimes \mathcal{N},
\end{array}$$

in which the vertical maps are isomorphisms.

Let $\alpha(t)$ and $\beta(t)$ be numerical polynomials on t. The *compactified Picard functor* $Pic^{=\alpha}_{X/S}$ of X/S is defined on S-schemes T by:

$$Pic^{=\alpha}_{X/S}(T) := \left\{ \begin{array}{c} \text{isomorphism classes of torsion-free, rank 1 sheaves} \\ \mathcal{I} \text{ on } X_T \text{ such that } \chi_t(\mathcal{I}(s)) = \alpha(t) \text{ for all } s \in T \end{array} \right\}$$

modulo the equivalence relation given by tensoring $\mathcal{I}$ with an invertible sheaf $\mathcal{N}$ on T. Denote by $Pic^{=\beta}_{X'/S}$ the compactified Picard functor of X'/S. Denote by $Pic^{\alpha}_{X/S}$ (resp. $Pic^{\beta}_{X'/S}$) the open subfunctor of $Pic^{=\alpha}_{X/S}$ (resp. $Pic^{=\beta}_{X'/S}$) parametrizing invertible sheaves.

Define the *presentation functor* $Pres_{\alpha,\beta}$ on the category of S-schemes by

$$Pres_{\alpha,\beta}(T) := \left\{ \begin{array}{l} h\colon \mathcal{I} \to \pi_{T*}\mathcal{I}';\ \ \mathcal{I} \text{ represents an element} \\ \text{of } Pic^{=\alpha}_{X/S}(T),\ \ \mathcal{I}' \text{ represents an element} \\ \text{of } Pic^{=\beta}_{X'/S}(T), \text{ and } h \text{ is a presentation} \end{array} \right\}$$

modulo the equivalence relation between presentations defined above.

There are natural maps of functors

$$\kappa\colon Pres_{\alpha,\beta} \longrightarrow Pic^{=\alpha}_{X/S} \quad \text{and} \quad \kappa'\colon Pres_{\alpha,\beta} \longrightarrow Pic^{=\beta}_{X'/S}$$

defined by mapping a presentation to its source and target, respectively. Note that κ' is defined because of the following lemma.

LEMMA 2.4. *If $\mathcal{I}_1'$ and $\mathcal{I}_2'$ are torsion-free, rank 1 sheaves on X', then for any homomorphism $h\colon \pi_*\mathcal{I}_1' \longrightarrow \pi_*\mathcal{I}_2'$ there is a unique homomorphism $h'\colon \mathcal{I}_1' \longrightarrow \mathcal{I}_2'$ such that $h = \pi_*h'$.*

PROOF: Consider the following diagram,

$$\begin{array}{ccc} \pi^*\pi_*\mathcal{I}_1' & \xrightarrow{\pi^* h} & \pi^*\pi_*\mathcal{I}_2' \\ \big\downarrow & & \big\downarrow \\ \mathcal{I}_1' & & \mathcal{I}_2' \end{array} \tag{2.5}$$

where the vertical maps are canonical. Since π is finite, the vertical maps are surjective. Since π is S-birational, the support of the kernel of $\pi^*\pi_*\mathcal{I}_1' \longrightarrow \mathcal{I}_1'$ does not include the generic point of any fibre of X'/S. Hence, since $\mathcal{I}_2'$ is torsion-free, there is a homomorphism $h'\colon \mathcal{I}_1' \longrightarrow \mathcal{I}_2'$ completing (2.5) to a commutative diagram. It is clear by adjunction that $h = \pi_*h'$. The uniqueness follows from the S-birationality of π. ∎

(2.6) If F is any functor on the category of S-schemes, we let F_{et} denote the associated sheaf in the étale topology.

Altman and Kleiman proved that the sheaves $Pic^{=\alpha}_{X/S,et}$ and $Pic^{=\beta}_{X'/S,et}$ are represented by proper S-schemes [**5**, (3.2,i), p.28]. Denote by $\mathrm{Pic}^{=\alpha}_{X/S}$ and $\mathrm{Pic}^{=\beta}_{X'/S}$, respectively, the representing schemes. The proof that $Pres_{\alpha,\beta,et}$ is represented by a finite $\mathrm{Pic}^{=\alpha}_{X/S} \times_S \mathrm{Pic}^{=\beta}_{X'/S}$-scheme can be easily adapted from the proof carried out in [**6**, (8), (9), (12)] for the case where X/S is a family of curves. Nevertheless, we present a somewhat different approach here to replace [**6**, (8), (9)]. To this purpose, we introduce the $\mathcal{O}_S$-module $H(\mathcal{F},\mathcal{G})$ below.

(2.7) Let $g: Y \longrightarrow S$ be a proper morphism of schemes. Let $\mathcal{F}, \mathcal{G}$ be two coherent $\mathcal{O}_Y$-modules, with $\mathcal{G}$ flat over S. Then there are a coherent $\mathcal{O}_S$-module $H(\mathcal{F},\mathcal{G})$ and a universal element $h \in \mathrm{Hom}_Y(\mathcal{F}, \mathcal{G} \otimes_S H(\mathcal{F},\mathcal{G}))$ representing the functor

$$M \mapsto \mathrm{Hom}_Y(\mathcal{F}, \mathcal{G} \otimes_S M)$$

on the category of quasi-coherent $\mathcal{O}_S$-modules M [**EGA III-2**, 7.7.8,9]. In other words, the map defined by h,

$$\mathrm{Hom}_T(H(\mathcal{F},\mathcal{G})_T, M) \longrightarrow \mathrm{Hom}_{Y_T}(\mathcal{F}_T, \mathcal{G}_T \otimes_T M),$$

is an isomorphism for any S-scheme T and any quasi-coherent $\mathcal{O}_T$-module M.

(2.8) Assume that X/S admits a section $s: S \longrightarrow X$ through its smooth locus. Consequently, there is clearly a unique section $s': S \longrightarrow X'$ of X'/S through its smooth locus such that $\pi \circ s' = s$. In this case, Altman and Kleiman have proved that X (resp. X') admits a universal sheaf $\mathbf{I}$ (resp. $\mathbf{I}'$) on $X \times_S \mathrm{Pic}^{=\alpha}_{X/S}$ (resp. $X' \times_S \mathrm{Pic}^{=\beta}_{X'/S}$) rigidified along $s_{\mathrm{Pic}^{=\alpha}_{X/S}}$ (resp. $s'_{\mathrm{Pic}^{=\beta}_{X'/S}}$).

PROPOSITION 2.9. *Assuming the set-up of (2.8), the functor $Pres_{\alpha,\beta,et}$ is represented by a closed subscheme, denoted by $\mathrm{Pres}_{\alpha,\beta}$, of*

$$\mathbf{P} := \mathbf{P}_Y(H(\mathbf{I}_Y, \pi_{Y*}\mathbf{I}'_Y)), \ \textit{where} \ \ Y := \mathrm{Pic}^{=\alpha}_{X/S} \times_S \mathrm{Pic}^{=\beta}_{X'/S},$$

and the universal presentation is the restriction to $\mathrm{Pres}_{\alpha,\beta}$ of the composition of the universal quotient

$$H(\mathbf{I}_Y, f_{Y*}\mathbf{I}'_Y)_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}}(1)$$

on $\mathbf{P}$ with the pull-back to $\mathbf{P}$ of the universal element

$$h: \mathbf{I}_Y \longrightarrow \pi_{Y*}\mathbf{I}'_Y \otimes H(\mathbf{I}_Y, \pi_{Y*}\mathbf{I}'_Y)$$

on Y. Moreover, $\mathrm{Pres}_{\alpha,\beta}$ is finite over Y.

PROOF: We first claim that $\mathbf{P}$ represents the functor P that associates to an S-scheme T the set of equivalence classes of injective homomorphisms $h: \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ on X_T satisfying the following properties:

(1) $\mathcal{I}$ (resp. $\mathcal{I}'$) is a torsion-free, rank 1 sheaf on X_T (resp. X'_T) such that $\chi_t(\mathcal{I}(s)) = \alpha(t)$ and $\chi_t(\mathcal{I}'(s)) = \beta(t)$ for all $s \in T$;
(2) the cokernel of h is T-flat;
(3) $h(s)$ is non-zero for every $s \in S$.

The equivalence relation between such homomorphisms is the same as the one for presentations, defined in (2.3). To prove the above claim, let T be any S-scheme. Let $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ be an injective homomorphism on X_T satisfying (1), (2) and (3) above. Then h gives rise to a map $T \longrightarrow Y$ such that $\mathcal{I} = \mathbf{I}_T \otimes N_1$ and $\mathcal{I}' = \mathbf{I}'_T \otimes N_2$, where N_1 and N_2 are invertible $\mathcal{O}_T$-modules. By (2.7), the homomorphism h gives rise to a map

$$q_h\colon H(\mathbf{I}_Y, \pi_{Y*}\mathbf{I}'_Y)_T \longrightarrow N_2 \otimes_T N_1^{-1}.$$

The homomorphism q_h is non-zero at every $s \in T$, since h is fibrewise non-zero. Since $N_2 \otimes N_1^{-1}$ is invertible, then q_h is surjective. Therefore, q_h corresponds to a T-point of $\mathbf{P}$ over Y. Conversely, one can follow the above steps backwards to show that a T-point of $\mathbf{P}$ corresponds to an equivalence class of homomorphisms $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ satisfying (1), (2) and (3) above.

Since $Pres_{\alpha,\beta}$ is the subfunctor of P parametrizing injective homomorphisms whose cokernels have support on a fixed closed subscheme of X, then $Pres_{\alpha,\beta,et}$ is represented as a closed subscheme, $\mathrm{Pres}_{\alpha,\beta}$, of $\mathbf{P}$.

To prove that $\mathrm{Pres}_{\alpha,\beta}$ is finite over Y, consider the universal presentation

$$h\colon \mathbf{I}_{\mathrm{Pres}_{\alpha,\beta}} \longrightarrow \pi_{\mathrm{Pres}_{\alpha,\beta}*}\mathbf{I}'_{\mathrm{Pres}_{\alpha,\beta}} \otimes \mathcal{A}$$

on $X \times_S \mathrm{Pres}_{\alpha,\beta}$, where

$$\mathcal{A} := \mathcal{O}_{\mathbf{P}}(1)|_{\mathrm{Pres}_{\alpha,\beta}}.$$

Since the cokernel of h is supported on $Q \times_S \mathrm{Pres}_{\alpha,\beta}$, then $s^*_{\mathrm{Pres}_{\alpha,\beta}}h$ is an isomorphism. Since the sheaf $\mathbf{I}$ (resp. $\mathbf{I}'$) is rigidified at s (resp. s'), then $\mathcal{A} \cong \mathcal{O}_{\mathrm{Pres}_{\alpha,\beta}}$. Since $\mathcal{A}$ is Y-ample, and $\mathrm{Pres}_{\alpha,\beta}$ is proper over Y, then $\mathrm{Pres}_{\alpha,\beta}$ must be finite over Y. The proof is complete. ∎

THEOREM 2.10. *The etale sheaf $Pres_{\alpha,\beta,et}$ is represented by a finite scheme $\mathrm{Pres}_{\alpha,\beta}$ over $\mathrm{Pic}^{=\alpha}_{X/S} \times_S \mathrm{Pic}^{=\beta}_{X'/S}$.*

PROOF: The same proof given in [**6**, (12)] applies here, since that proof uses only that $Pres_{\alpha,\beta,et}$ is represented by a finite scheme over the product $\mathrm{Pic}^{=\alpha}_{X/S} \times_S \mathrm{Pic}^{=\beta}_{X'/S}$ when there is a section of X/S through its smooth locus, which is our Proposition 2.9.

3. The stratification.

(3.1) An open subset U of X is called *S-dense* if $U(s)$ is dense in $X(s)$ for every $s \in S$. An S-dense open subset U of X is obviously dense in X. If $\mathcal{I}$ is a torsion-free, rank 1 sheaf on X, and $U \subset X$ is S-dense, with inclusion map denoted by $i\colon U \to X$, then the natural homomorphism $\mathcal{I} \longrightarrow i_*i^*\mathcal{I}$ is injective, as it follows easily from [**EGA IV-2**, 6.3.1, p. 138].

Let U be an S-dense open subset of X over which π is an isomorphism, and let $i\colon U \to X$ be the inclusion map. Put $U' := \pi^{-1}(U)$, and let $i'\colon U' \to X'$ denote the inclusion map. Of course $U' \subset X'$ is S-dense. Let $\mathcal{I}$ be a torsion-free, rank 1 sheaf on X. Denote by $\mathcal{I}^\pi$ the image of the canonical homomorphism $\pi^*\mathcal{I} \to i'_*i'^*\pi^*\mathcal{I}$. Since $\pi^*\mathcal{I}$ is not necessarily even S-flat, the canonical homomorphism is not necessarily injective. Let $h^\pi_\mathcal{I}\colon \mathcal{I} \to \pi_*\mathcal{I}^\pi$ be the canonical homomorphism adjoint to the surjection $\pi^*\mathcal{I} \to \mathcal{I}^\pi$. Note that, since $\pi_*i'_*i'^*\pi^* = i_*i^*$, the homomorphism $h^\pi_\mathcal{I}$ is injective.

Proposition 3.2. *$\mathcal{I}^\pi$ and $h^\pi_\mathcal{I}$ do not depend on the choice of the S-dense open subset U of X.*

Proof: Let $V \subset X$ be another S-dense open subset over which π is an isomorphism. Since the intersection of two S-dense open subsets of X is S-dense, we are reduced to consider the case where $V \subset U$. Let $j\colon V \to U$ be the inclusion map. Let $V' := \pi^{-1}(V)$, and $j'\colon V' \to U'$ be the inclusion map. The canonical homomorphism

$$i'_*i'^*\pi^*\mathcal{I} \longrightarrow i'_*j'_*j'^*i'^*\pi^*\mathcal{I} = (i'j')_*(i'j')^*\pi^*\mathcal{I}$$

is injective, since $i'^*\pi^*\mathcal{I}$ is torsion-free, rank 1 over S. From the sequence of canonical homomorphisms

$$\pi^*\mathcal{I} \to i'_*i'^*\pi^*\mathcal{I} \to (i'j')_{(}i'j')^*\pi^*\mathcal{I},$$

and the injectivity of the second map, the proposition follows. ∎

Let $\mathcal{I}'$ be a torsion-free, rank 1 sheaf on X'. Let $h\colon \mathcal{I} \to \pi_*\mathcal{I}'$ be an injective homomorphism.

Lemma 3.3. *There is an inclusion $\mathcal{I}^\pi \to \mathcal{I}'$ making the diagram*

$$\begin{array}{ccc}
\mathcal{I} & \xrightarrow{h^\pi_\mathcal{I}} & \pi_*\mathcal{I}^\pi \\
\| & & \downarrow \\
\mathcal{I} & \xrightarrow{h} & \pi_*\mathcal{I}'
\end{array}$$

commutative.

PROOF: Let $U \subset X$ be an S-dense open subset over which π is an isomorphism. Let $U' := \pi^{-1}(U) \subset X'$, and denote by i (resp. i') the inclusion map of U into X (resp. U' into X'.) Since $h\colon \mathcal{I} \to \pi_*\mathcal{I}'$ is injective, then

$$i'_*i'^*(h^a)\colon i'_*i'^*\pi^*\mathcal{I} \to i'_*i'^*\mathcal{I}'$$

is also injective, where h^a is the adjoint map to h. From the commutative diagram

$$\begin{tikzcd} \pi^*\mathcal{I} \arrow[r] \arrow[d, "h^a"'] & i'_*i'^*\pi^*\mathcal{I} \arrow[d] \\ \mathcal{I}' \arrow[r] & i'_*i'^*\mathcal{I}', \end{tikzcd}$$

and the injectivity of the bottom and right hand maps, the lemma follows. ∎

The following lemma is an easy generalization of [**4**, (3.3), p. 74].

LEMMA 3.4. *Let $\mathcal{I}$ be a torsion-free, rank 1 sheaf on X. Then, locally on S, there is an embedding $\mathcal{I} \to \mathcal{O}_X(m)$ with flat cokernel for m sufficiently large.*

PROOF: Put $\mathcal{G}_m := \underline{\mathrm{Hom}}_X(\mathcal{I}, \mathcal{O}_X(m))$. Pick m large enough that the map $f^*f_*\mathcal{G}_m \to \mathcal{G}_m$ is surjective. Let $s \in S$, and $\xi \in X$ be the generic point of $X(s)$. Since $\mathcal{I}(s)$ is free at ξ of rank 1 and $\mathcal{I}$ is S-flat, then $\mathcal{I}$ is free at ξ of rank 1. Since

$$(f_*\mathcal{G}_m)_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,\xi} \to \mathcal{G}_{m,\xi} \cong \mathcal{O}_{X,\xi} \qquad (3.5)$$

is surjective, there is an element of $(f_*\mathcal{G}_m)_s$ mapping to a generator of $\mathcal{G}_{m,\xi}$ via (3.7). By shrinking S around s, one may lift the former element to a global section of $\mathcal{G}_m$, that is, to a map $h\colon \mathcal{I} \to \mathcal{O}_X(m)$, which is necessarily an isomorphism at ξ. Since $h(s)_\xi$ is an isomorphism and $\mathcal{I}(s)$ is torsion-free, then $h(s)$ is injective. Hence, h is injective with S-flat cokernel in a neighbourhood of $X(s)$ in X [**EGA IV-3**, 11.1.2, p. 118]. Since f is proper, then h is injective with S-flat cokernel on a neighbourhood of s in S. ∎

(3.6) Define the subfunctor A^α_β of $Pres_{\alpha,\beta}$ by

$$A^\alpha_\beta(T) := \{h \in \mathrm{Pres}_{\alpha,\alpha+\beta}(T);\ h^a \text{ is surjective}\}$$

for every S-scheme T, where h^a is the adjoint map to h.

It is clear that $A^\alpha_{\beta,et}$ is represented by an open subscheme A^α_β of $\mathrm{Pres}_{\alpha,\alpha+\beta}$. Denote the restriction of κ to A^α_β by λ_β.

THEOREM 3.7. *The morphisms $\lambda_\beta \colon \mathrm{A}^\alpha_\beta \to \mathrm{Pic}^{=\alpha}_{X/S}$ are embeddings. In addition, the subschemes $\lambda_\beta(\mathrm{A}^\alpha_\beta)$ of $\mathrm{Pic}^{=\alpha}_{X/S}$ are disjoint, and only a finite number of them are non-empty.*

PROOF: Let $R \to S$ be an etale covering such that X_R admits a universal sheaf $\mathbf{I}$ on $X_R \times_R \mathrm{Pic}^{=\alpha}_{X_R/R}$. By Lemma 3.4, one can also assume that there is an embedding $\mathbf{I} \to \mathcal{O}_{X_R}(m)$ with flat cokernel. By composing with the canonical presentation $\mathcal{O}_{X_R}(m) \to \pi_{R*}\pi_R^*\mathcal{O}_{X_R}(m)$ one obtains an embedding $h \colon \mathcal{I} \to \pi_{R*}\mathcal{O}_{X'_R}(m)$ with flat cokernel. Let $\mathcal{N}$ be the cokernel of the adjoint homomorphism $h^a \colon \pi_R^*\mathbf{I} \to \mathcal{O}_{X'_R}(m)$. Let $\{Y_\theta\}$ be the flattening stratification of $\mathcal{N}$ in $\mathrm{Pic}^{=\alpha}_{X_R/R}$, where θ denotes the Hilbert polynomial of the <u>kernel</u> of the quotient map $\mathcal{O}_{X'_R}(m) \to \mathcal{N}$ over Y_θ relative to $\mathcal{O}_{X'}(1)$.

We claim that $\lambda_{\beta,R}$ is an isomorphism onto $Y_{\alpha+\beta}$. In fact, let T be a $Y_{\alpha+\beta}$-scheme. Since

$$\pi_T^*\mathbf{I}_T \xrightarrow{h_T^a} \mathcal{O}_{X'_T}(m) \to \mathcal{N}_T \to 0$$

is exact and $\mathcal{N}_T$ is T-flat, then the image $\mathbf{I}_T^{\pi_T}$ of the adjoint map h_T^a is T-flat of Hilbert polynomial $\alpha+\beta$. Hence, by definition, T must factor through $\mathrm{A}^\alpha_\beta \times_S R$. On the other hand, let T be an $\mathrm{A}^\alpha_\beta \times_S R$-scheme. By Lemma 3.3, the image of the adjoint map $h_T^a \colon \pi_T^*\mathbf{I}_T \to \mathcal{O}_{X'_T}(m)$ is the sheaf $\mathbf{I}_T^{\pi_T}$, which is torsion-free, rank 1 by hypothesis. Moreover, the embedding $\mathbf{I}_T^{\pi_T} \to \mathcal{O}_{X'_T}(m)$ is injective on the fibres of X'_T/T, since the sheaves involved are torsion-free, rank 1 and h_T is injective on the fibres over X'_T/T. Hence, the cokernel $\mathcal{N}_T$ must be T-flat, implying that T factors through $Y_{\alpha+\beta}$.

Since $\lambda_{\beta,R}$ is an embedding, then so is λ_β by [**EGA IV-2**, 2.7.1,xi].

It is obvious from Lemma 3.3 that the $\lambda_\beta(\mathrm{A}^\alpha_\beta)$ are disjoint. Since S was assumed noetherian, then so is $\mathrm{Pic}^{=\alpha}_{X_R/R}$. Hence, the flattening stratification of $\mathcal{N}$ is finite, which implies that only finitely many A^α_β are non-empty. The proof is complete. ∎

Because of the embedding λ_β, we will often say that A^α_β is a subscheme of $\mathrm{Pic}^{=\alpha}_{X/S}$.

(3.8) Define the map of functors (see [**6**, (5)])

$$\epsilon \colon Pic^{=\alpha}_{X'/S} \to Pic^{=\alpha}_{X/S}$$

by $\epsilon(\mathcal{I}') := \pi_{T*}\mathcal{I}'$ for any S-scheme T and any torsion-free, rank 1 sheaf $\mathcal{I}'$ on X'_T.

PROPOSITION 3.9. *ϵ is an isomorphism onto A_0^α.*

PROOF: Let $\mathcal{I}'$ represent an element of $Pic^{=\alpha}_{X'/S}(T)$ for an S-scheme T. The identity id: $\pi_{T*}\mathcal{I}' \to \pi_{T*}\mathcal{I}'$ represents an element of $A_0^\alpha(T)$. Since $\mathcal{I}' = (\pi_{T*}\mathcal{I}')^{\pi_T}$, then ϵ is a monomorphism and maps into A_0^α. On the other hand, if $\mathcal{I}$ represents an element of $A_0^\alpha(T)$, then

$$h_{\mathcal{I}}^{\pi_T} : \mathcal{I} \to \pi_{T*}\mathcal{I}^{\pi_T}$$

is an isomorphism. Hence, ϵ is an epimorphism. The proof is complete. ∎

4. Curves.

(4.1) Assume now that X is an S-curve. If $\mathcal{I}$ is a torsion-free rank 1 sheaf on X, then $\chi_n(\mathcal{I}(s)) = dn + \chi(\mathcal{I}(s))$, where d is the degree of $X(s)$ with respect to $\mathcal{O}_{X(s)}(1)$, for all $d \in S$ [4, 3.4, p. 74]. Hence, we can and we will simplify our notation. We make the following substitutions:

$$\begin{aligned}
\mathrm{Pic}^{=dn+r}_{X/S} &\Longrightarrow \bar{J}_r;\\
\mathrm{Pic}^{=dn+r}_{X'/S} &\Longrightarrow \bar{J}'_r;\\
\mathrm{Pres}_{dn+r,dn+s} &\Longrightarrow P_{r,s};\\
\mathrm{A}^{dn+r}_{dn+s} &\Longrightarrow A^s_r.
\end{aligned}$$

(4.2) Let

$$\delta := \chi(\frac{\pi_*\mathcal{O}_{X'}}{\mathcal{O}_X}(s)).$$

for any $s \in S$. Note that δ does not depend on $s \in S$, because S is connected and the cokernel of the comorphism $\pi^c\colon \mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_{X'}$ is S-flat.

Assume until the end of this section that Q' is contained in the smooth locus of X'/S.

The following proof was inspired by [6, (16,iii)].

THEOREM 4.3.

$$\bar{J}_n = \bigcup_{i=0}^{\delta} A^i_n.$$

In addition, A^δ_n is equal to the open subscheme of $\bar{J}_n$ parametrizing torsion-free, rank 1 sheaves invertible along Q.

PROOF: Because of the set-theoretical nature of the statements, we may assume S is the spectrum of an algebraically closed field. Let I be a torsion-free, rank 1 sheaf on X. Then I can be embedded in an invertible sheaf on X by Lemma 3.6 or [4, (3.3), p. 74]. We may even assume that I is an ideal sheaf of X. Then $I^\pi = I\mathcal{O}_{X'}$.

Let q be a point in Q. Since Q' is contained in the regular locus of X', then

$$(\pi_*I^\pi)_q = I_q(\pi_*\mathcal{O}_{X'})_q = a_q(\pi_*\mathcal{O}_{X'})_q$$

for some $a_q \in I_q$. As a consequence, we have the following exact sequence:

$$0 \longrightarrow \frac{I_q}{a_q \mathcal{O}_{X,q}} \longrightarrow \frac{a_q(\pi_* \mathcal{O}_{X'})_q}{a_q \mathcal{O}_{X,q}} \longrightarrow \frac{(\pi_* I^\pi)_q}{I_q} \longrightarrow 0.$$

Since a_q is a non-zero divisor of $\mathcal{O}_{X,q}$, then

$$\delta_q := \text{length}_{\mathcal{O}_{X,q}}(\frac{(\pi_* \mathcal{O}_{X'})_q}{\mathcal{O}_{X,q}}) = \text{length}_{\mathcal{O}_{X,q}}(\frac{a_q(\pi_* \mathcal{O}_{X'})_q}{a_q \mathcal{O}_{X,q}}).$$

Therefore, from the above exact sequence,

$$\chi(\frac{\pi_* I^\pi}{I}) = \sum_{q \in Q} \text{length}_{\mathcal{O}_{X,q}}(\frac{(\pi_* I^\pi)_q}{I_q}) \leqslant \sum_{q \in Q} \delta_q = \delta. \tag{4.4}$$

So $A_n^i = \emptyset$ for $i > \delta$. Moreover, $I \in A_n^\delta$ if and only if equality holds in (4.4), that is, if and only if $I_q = a_q \mathcal{O}_{X,q}$ for every q in Q. Hence, $I \in A_n^\delta$ if and only if I is invertible along Q. The proof is complete. ∎

PROPOSITION 4.5. *Assume X' admits a universal sheaf $\mathbf{I}'$ on $X' \times_S \bar{J}'_s$. If $f|_Q : Q \longrightarrow S$ is an isomorphism, then*

$$P_{r,s} = \text{Grass}^{s-r}_{\bar{J}'_s}(\mathcal{E}),$$

where $\mathcal{E} := (\pi_{\bar{J}'_s *} \mathbf{I}')\Big|_{Q \times_S \bar{J}'_s}$ is a vector bundle of rank $\delta + 1$ on $\bar{J}'_s$.

PROOF: Since Q' is contained in the S-smooth locus of X', then the restriction $\mathbf{I}'|_{Q' \times_S \bar{J}'_s}$ is invertible. Since Q' is flat and finite over S of degree $\delta + 1$ by Proposition 2.2, and

$$\mathcal{E} = \pi_{\bar{J}'_s *}(\mathbf{I}'|_{Q' \times_S \bar{J}'_s}),$$

then $\mathcal{E}$ is locally free of rank $\delta + 1$ on $Q \times_S \bar{J}'_s$. Now, by [**6**, (9)] we have that

$$P_{r,s} = \text{Quot}^{s-r}_{\mathcal{E}/Q \times_S \bar{J}'_s / \bar{J}'_s}.$$

Since $Q \cong S$, then the above Quot-scheme is in fact the Grassmannian of rank $s - r$ quotient bundles of $\mathcal{E}$. The proof is complete. ∎

COROLLARY 4.6. *If $f|_Q : Q \longrightarrow S$ is an isomorphism, then A_n^i is smooth over $\bar{J}'_{n+i}$ under κ'.*

PROOF: Obvious from Proposition 4.5 and descent [**EGA IV-4**, 17.7.4, p. 72], since A_n^i is an open subscheme of $P_{n,n+i}$. ∎

(4.7) We now proceed to present and generalize Theorem 16 of [**6**], which analyzes the map $\kappa\colon P_{r,r+\delta} \longrightarrow \bar{J}_r$ in terms of the stratification of $\bar{J}_r$ when $\delta = 1$. In our more general setting, to obtain similar results one must replace the presentation scheme by a larger scheme, which is described below.

Let T be an S-scheme. Let $\mathcal{I}$ be a torsion-free, rank 1 sheaf on X_T. A *set-theoretic presentation* of $\mathcal{I}$ is an injective $\mathcal{O}_{X_T}$-module homomorphism $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$, where $\mathcal{I}'$ is a torsion-free, rank 1 sheaf on X'_T, and the cokernel of h is T-flat with set-theoretic support on Q_T. Define the *set-theoretic presentation functor* $\bar{P}_{r,s}$ on the category of S-schemes by

$$\bar{P}_{r,s}(T) := \left\{ \begin{array}{l} h : \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}';\ \mathcal{I} \text{ represents a class in} \\ \bar{J}_r(T),\ \ \mathcal{I}' \text{ represents a class in } \bar{J}'_s(T), \\ \text{and } h \text{ is a set-theoretic presentation} \end{array} \right\}$$

modulo the equivalence relation between presentations defined in (2.3). It is clear that if $s - r = 1$, then $\bar{P}_{r,s} = P_{r,s}$. In general, it is easy to see that $\bar{P}_{r,s,et}$ is represented by a projective $\bar{J}'_s$-scheme, also denoted by $\bar{P}_{r,s}$. Since every presentation is a set-theoretic presentation, $P_{r,s}$ is a closed S-subscheme of $\bar{P}_{r,s}$. Let $\bar{\kappa}$ denote the obvious extension of κ to $\bar{P}_{r,s}$.

(4.8) Define the functor Z^i on the category of S-schemes by

$$Z^i(T) := \{Y \in \mathrm{Hilb}^i_{X'/S}(T);\ \mathrm{supp}(Y) \subset Q'_T\},$$

where by "supp" it is meant the set-theoretic support. It is easy to see that the functor Z^i is represented by the S-projective scheme

$$Z^i := \mathrm{Quot}^i_{\frac{\mathcal{O}_{X'}}{\mathcal{M}^i_{Q'}}/X'/S}.$$

(4.9) Define a map of functors

$$\gamma^i\colon A^i_n \times_S Z^{\delta-i} \to \bar{P}_{n,n+\delta}$$

as follows. Let T be an S-scheme. Let $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ represent an element of $A^i_n(T)$, and the subscheme $Y \subset X'_T$ represent an element of $Z^{\delta-i}(T)$. If $\mathcal{L}$ denotes the ideal sheaf of Y in X'_T, then, since Q' is contained in the S-smooth locus of X', the sheaf $\mathcal{L}$ is invertible. Let $\mathcal{O}_{X'_T} \longrightarrow \mathcal{L}^{-1}$ be the global section of $\mathcal{L}^{-1}$ corresponding to Y. Tensoring by $\mathcal{I}'$ gives an embedding $h'\colon \mathcal{I}' \longrightarrow \mathcal{L}^{-1}\mathcal{I}'$ with flat cokernel of length $\delta - i$ over S, since $\mathcal{I}'$ is invertible along Q'_T. Finally, $\gamma^i(T)(h, Y)$ is defined as the equivalence class represented by the following set-theoretic presentation:

$$\mathcal{I} \xrightarrow{h} \pi_{T*}\mathcal{I}' \xrightarrow{\pi_{T*}h'} \pi_{T*}(\mathcal{L}^{-1}\mathcal{I}'). \tag{4.10}$$

LEMMA 4.11. *γ^i is a monomorphism.*

PROOF: From (4.10) we obtain the adjoint homomorphism:

$$\pi_T^*\mathcal{I} \to \mathcal{I}' \to \mathcal{L}^{-1}\mathcal{I}'. \tag{4.12}$$

A priori, one would obtain only the composition, but since $\mathcal{I}'$ is the image of (4.12), one actually obtains the above factorization. One recovers h by considering the adjoint to $\pi_T^*\mathcal{I} \to \mathcal{I}'$. One also obtains the embedding h'. Since $\mathcal{L}^{-1}\mathcal{I}'$ is torsion-free, rank 1, it is invertible on $X_T'^{sm}$, where X'^{sm} denotes the smooth locus of X'/S. By tensoring (4.12) with the inverse of $\mathcal{L}^{-1}\mathcal{I}'$ along $X_T'^{sm}$, one recovers the embedding

$$\mathcal{L}|_{X_T'^{sm}} \to \mathcal{O}_{X_T'^{sm}},$$

whose corresponding subscheme of $X_T'^{sm}$ is $Y \cap X_T'^{sm}$. Since Y is supported on Q_T' and $Q_T' \subset X_T'^{sm}$, then $Y = Y \cap X_T'^{sm}$. The proof is complete. ∎

(4.13) Define the morphism

$$\bar{\kappa}^i \colon A_n^i \times_S Z^{\delta-i} \to A_n^i$$

as the projection onto the first factor. Let λ^i denote the embedding of A_n^i in $\bar{J}_n$. Then the following diagram is clearly commutative.

$$\begin{array}{ccc} \bar{P}_{n,n+\delta} & \xrightarrow{\bar{\kappa}} & \bar{J}_n \\ {\scriptstyle\gamma^i}\uparrow & & \uparrow{\scriptstyle\lambda^i} \\ A_n^i \times_S Z^{\delta-i} & \xrightarrow[\bar{\kappa}^i]{} & A_n^i. \end{array} \tag{4.14}$$

THEOREM 4.15. *Diagram (4.14) is Cartesian.*

PROOF: Since (4.14) is commutative, we obtain a map of functors,

$$A_n^i \times_S Z^{\delta-i} \to A_n^i \times_{\bar{J}_n} \bar{P}_{n,n+\delta},$$

that is a monomorphism by Lemma 4.11. Let T be an S-scheme. Let $\mathcal{I}$ represent an element of $A_n^i(T)$, and $h\colon \mathcal{I} \to \pi_{T*}\mathcal{I}'$ represent an element of $\bar{P}_{n,n+\delta}(T)$. Let $\mathcal{I}_1'$ be the image of the adjoint map $h^a\colon \pi_T^*\mathcal{I} \to \mathcal{I}'$.

By lemma 3.3, since $\mathcal{I} \in A_n^i(T)$, then $\mathcal{I}'_1$ is torsion-free, rank 1 and the induced homomorphism, $h' \colon \mathcal{I} \to \pi_{T*}\mathcal{I}'_1$, is injective with flat cokernel. Since $\mathcal{I}'$ is invertible on $X_T'^{sm}$, by tensoring the embedding $\mathcal{I}'_1 \to \mathcal{I}'$ with the inverse of $\mathcal{I}'$ along $X_T'^{sm}$ one gets an embedding,

$$\mathcal{I}'_1|_{X_T'^{sm}} \otimes (\mathcal{I}'|_{X_T'^{sm}})^{-1} \to \mathcal{O}_{X_T'^{sm}}, \tag{4.16}$$

which is an isomorphism outside Q'_T, since the cokernel of h is supported on Q'_T. So one can extend (4.16) to X'_T. Let $Y \subset X'_T$ be the subscheme corresponding to the extension of (4.16) to X'_T. Since $\mathcal{I}'_1 \to \mathcal{I}$ has flat cokernel of length $\delta - i$ supported on Q'_T, then Y represents an element of $Z^{\delta-i}(T)$. It is then clear from the definition of γ^i that $\gamma^i(T)(h', Y) = h$. The proof is complete. ∎

COROLLARY 4.17. *The morphism*

$$\bar{\kappa} \colon \bar{P}_{n,n+\delta} \to \bar{J}_n$$

is finite, surjective, and an isomorphism over the open subscheme of $\bar{J}_n$ parametrizing torsion-free, rank 1 sheaves invertible along Q.

PROOF: The surjectivity comes from Theorem 4.15 and the fact that $\bar{\kappa}^i$ is trivially surjective. The finiteness follows from [**EGA IV-3**, 8.11.1, p. 41], since $\bar{\kappa}$ is proper and Z^i is finite over S for every i. Since $Z^0 = S$, it follows from Theorem 4.15 that $\bar{\kappa}$ is an isomorphism over A_n^δ, which is the open subscheme of $\bar{J}_n$ parametrizing torsion-free, rank 1 sheaves invertible along Q by Theorem 4.3. The proof is complete. ∎

We remark that Theorem 4.15 and Corollary 4.17 generalize [**6**, (16), p. 25].

(4.18) Assume that S is the spectrum of an algebraically closed field for the following remark. In view of Corollary 4.17, it is important to remark that κ is not necessarily surjective, in contrast to the surjectivity of $\bar{\kappa}$. In fact, if Q is a reduced point of X, then, by Proposition 4.5, $P_{n,n+\delta}$ is irreducible if $\bar{J}'_{n+\delta}$ is. However, if $\delta > 1$, then X is not Gorenstein, hence $\bar{J}_n$ is reducible [**16**, (1), p. 277]. So, if $\bar{J}'_{n+\delta}$ is irreducible and $\delta > 1$, then κ is not surjective. More precisely, since A_n^δ is an open subscheme of $P_{n,n+\delta}$, and A_n^δ contains the Jacobian J_n of X as an open subscheme, then the image of κ is the closure of J_n in $\bar{J}_n$ if $P_{n,n+\delta}$ is irreducible.

5. Higher dimensional varieties.

(5.1) Let X be an integral variety defined over an algebraically closed field k. The presentation functor allows us to bridge the gap between the compactified Picard scheme of X and the compactified Picard scheme of its normalization $\tilde{X}$, paraphrasing [**6**, p. 15]. However, the Picard scheme Pic_Y^α of a normal variety Y defined over an algebraically closed field is already complete [**2**, 19, p. 138]. Hence, one can restrict ourselves to the following variant of the presentation functor defined in (2.3). Let $\pi\colon \tilde{X} \longrightarrow X$ be the normalization map. Fix an ample line bundle $\mathcal{O}_X(1)$ on X, and let $\mathcal{O}_{\tilde{X}}(1) := \pi^*\mathcal{O}_X(1)$. Let $\alpha(t)$ and $\beta(t)$ be numerical polynomials on t. Define the *restricted presentation functor* $Pres^0_{\alpha,\beta}$ on the category of S-schemes by

$$Pres^0_{\alpha,\beta}(T) := \left\{ \begin{array}{c} h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}';\ \ \mathcal{I} \text{ represents an element} \\ \text{of } Pic_X^{=\alpha}(T),\ \ \mathcal{I}' \text{ represents an element} \\ \text{of } Pic_{\tilde{X}}^{\beta}(T), \text{ and } h \text{ is a presentation} \end{array} \right\}$$

modulo the equivalence relation between presentations defined in (2.3). It is clear that $Pres^0_{\alpha,\beta}$ is represented by a closed subscheme, denoted by $\mathrm{Pres}^0_{\alpha,\beta}$, of $\mathrm{Pres}_{\alpha,\beta}$. Since $\mathrm{Pres}_{\alpha,\beta}$ is projective over $\mathrm{Pic}_{\tilde{X}}^{=\beta}$ by [**6**, (9)], then $\mathrm{Pres}^0_{\alpha,\beta}$ is projective. As before, there are natural maps of functors,

$$\kappa_0\colon Pres^0_{\alpha,\beta} \longrightarrow Pic_X^{=\alpha} \quad \text{and} \quad \kappa_0'\colon Pres^0_{\alpha,\beta} \longrightarrow Pic_{\tilde{X}}^{\beta},$$

defined by mapping a presentation to its source and target, respectively.

(5.2) One clearly has a decomposition,

$$\mathrm{Pic}_X^\alpha = \bigcup_\beta \mathrm{Pic}_X^{\alpha,\beta}, \tag{5.3}$$

of the Picard scheme Pic_X^α of X as the disjoint union of open and closed subschemes $\mathrm{Pic}_X^{\alpha,\beta}$, where $\mathrm{Pic}_X^{\alpha,\beta}$ parametrizes the invertible sheaves L on X with Hilbert polynomial $\alpha(t)$ such that π^*L has Hilbert polynomial $\beta(t)$ on $\tilde{X}$. It is clear that κ_0 is an isomorphism over $\mathrm{Pic}_X^{\alpha,\beta}$, because if $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ represents an element of $Pres^0_{\alpha,\beta}(T)$ for some k-scheme T, and $\mathcal{I}$ represents an element of $Pic_X^{\alpha,\beta}(T)$, then the adjoint map $h^a\colon \pi_T^*\mathcal{I} \longrightarrow \mathcal{I}'$ must be an isomorphism by [**4**, (3.4,ii), p. 74]. Since $\mathrm{Pres}^0_{\alpha,\beta}$ is projective, then the scheme-theoretic image of κ_0 is a closed subscheme $D_X^{\alpha,\beta}$ of $\mathrm{Pic}_X^{=\alpha}$ containing $\mathrm{Pic}_X^{\alpha,\beta}$. Note that if I is a torsion-free, rank 1 sheaf on X representing a point in $D_X^{\alpha,\beta}$, then I is invertible on the normal locus of X. Hence, $D_X^{\alpha,\beta} \subset C_X^{\alpha}$ set-theoretically, where C_X^{α} is the closed subset of $\mathrm{Pic}_X^{=\alpha}$ parametrizing torsion-free, rank 1 sheaves which are invertible on the smooth locus of X. Since C_X^{α} is contained in a quasi-projective open subscheme of $\mathrm{Pic}_X^{=\alpha}$ by [**5**, (3.2,ii)], then $D_X^{\alpha,\beta}$ is projective for every numerical polynomials $\alpha(t)$ and $\beta(t)$. By applying the above reasoning to all the numerical polynomials $\beta(t)$ involved in the union (5.3), one obtains the following result.

PROPOSITION 5.4. *The closure of* Pic_X^{α} *in* $\mathrm{Pic}_X^{=\alpha}$ *is contained in the subset* D_X^{α} *of* $\mathrm{Pic}_X^{=\alpha}$ *parametrizing torsion-free, rank 1 sheaves that are invertible on the normal locus of* X.

We do not know whether $D_X^{\alpha,\beta}$ represents a naturally defined subfunctor of $Pic_X^{=\alpha}$ in general. Nevertheless, one has the following result.

THEOREM 5.5. *If* X *is smooth in codimension 1, then* κ_0 *is a closed embedding. In particular,* $D_X^{\alpha,\beta}$ *is a projective scheme, with scheme structure given by* $\mathrm{Pres}^0_{\alpha,\beta}$. *Moreover, the subschemes* $D_X^{\alpha,\beta}$ *are disjoint.*

PROOF: Since κ_0 is proper, it is enough by [**EGA IV-3**, 8.11.5, p. 42] to show that κ_0 is a monomorphism of functors. Let T be any k-scheme, and let

$$h_1 \colon \mathcal{I}_1 \longrightarrow \pi_{T*}\mathcal{L}_1 \quad \text{and} \quad h_2 \colon \mathcal{I}_2 \longrightarrow \pi_{T*}\mathcal{L}_2$$

be restricted presentations such that $\mathcal{I}_1, \mathcal{I}_2 \in Pic_X^{=\alpha}(T)$. Assume their images under κ_0 are equivalent, that is, assume there is an invertible $\mathcal{O}_T$-module $\mathcal{N}$ such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes_T \mathcal{N}$. We will show that h_1 and h_2 are equivalent. To this purpose, one can actually assume that there is an isomorphism

$$\phi \colon \mathcal{I}_1 \xrightarrow{\sim} \mathcal{I}_2.$$

Denote by X^n the normal locus of X. Let $j \colon X^n \longrightarrow \tilde{X}$ be the inclusion of X^n in $\tilde{X}$. By definition, the restrictions of h_1 and h_2 to X_T^n are isomorphisms. Therefore, one obtains the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{I}_1 & \xrightarrow{h_1} & \pi_{T*}\mathcal{L}_1 & \longrightarrow & \pi_{T*}j_{T*}j_T^*\mathcal{L}_1 \\
\phi\downarrow & & & & \downarrow\pi_{T*}\psi \\
\mathcal{I}_2 & \xrightarrow{h_2} & \pi_{T*}\mathcal{L}_2 & \longrightarrow & \pi_{T*}j_{T*}j_T^*\mathcal{L}_2,
\end{array}$$

where

$$\psi := j_{T*}(h_2|_{X_T^n} \circ \phi|_{X_T^n} \circ (h_1|_{X_T^n})^{-1}).$$

To prove that h_1 and h_2 are equivalent one needs only verify that if $\mathcal{L}$ is an invertible sheaf on $\tilde{X}_T$, then $\mathcal{L} = j_{T*}j_T^*\mathcal{L}$. Since $\mathcal{L}$ is invertible, we are reduced by [**EGA IV-2**, 5.10.5, p. 115] to showing that the depth of any point of $\tilde{X}_T$ outside X_T^n is at least 2. But this follows from [**EGA IV-2**, 6.3.5, p. 140], since $\tilde{X}$ is normal and X^n contains all points of codimension at most 1 of $\tilde{X}$ by hypothesis.

The last statement of the theorem follows from the above argument, since nowhere did we specify the Hilbert polynomial of $\mathcal{L}_1$ or $\mathcal{L}_2$. ∎

6. The Theta divisor.

Let $f\colon X \longrightarrow S$ be an S-curve. Assume S is connected. Denote by g the arithmetic genus of the fibres of X/S. Denote by ω the relative dualizing sheaf of X/S. By [**AK2**, p. 96], the sheaf ω is torsion-free, rank 1.

DEFINITION 6.1. *If M is a coherent sheaf on X which is flat over S, let $\mathcal{D}_f(M)$ denote the determinant of cohomology of M.*

The sheaf $\mathcal{D}_f(M)$ is constructed as follows. Locally on S, there is a complex

$$E^0 \longrightarrow E^1$$

of free, coherent sheaves such that for every coherent sheaf F on S the cohomology groups of $E^{\cdot} \otimes F$ are equal to the higher direct images of $M \otimes F$. The complex $E^{\cdot}$ is unique up to unique quasi-isomorphism. Hence, its determinant,

$$\Big(\bigwedge^{\operatorname{rank} E^0} E^0\Big) \otimes \Big(\bigwedge^{\operatorname{rank} E^1} E^1\Big)^{-1},$$

is unique. The uniqueness allows us to glue together the local determinants to get an invertible sheaf, denoted by $\mathcal{D}_f(M)$, on S.

The main properties of the determinant of cohomology functor are the following three:

(1) If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of sheaves on X which are flat over S, then

$$\mathcal{D}_f(M) \cong \mathcal{D}_f(M') \otimes \mathcal{D}_f(M'');$$

(2) If E is a locally free sheaf on S of constant rank r and M is a sheaf on X which is flat over S, then

$$\mathcal{D}_f(M \otimes E) \cong \mathcal{D}_f(M) \otimes \Big(\bigwedge^{r} E\Big)^{\otimes \chi(M)},$$

where $\chi(M)$ denotes the relative Euler characteristic of M, that is, the Euler characteristic $\chi(M(s))$ of the restriction $M(s)$ of M to the fibre $X(s)$ for any $s \in S$;

(3) If

$$\begin{array}{ccc} Y & \xrightarrow{h'} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{h} & S \end{array}$$

is a Cartesian diagram, then

$$h^*\mathcal{D}_f(M) = \mathcal{D}_{f'}(h'^*(M)).$$

For a more systematic development of the theory of determinants, see [**18**]. It is also possible to avoid mentioning determinants in what follows by adopting the approach used in [**7**, Chapter IV, §3].

(6.2) Let $s\colon S \longrightarrow X$ be a section of f factoring through the smooth locus of X/S. Let D denote the effective relative Cartier divisor on X/S given by s. Let $\bar{J}_d$ denote the compactified Jacobian of degree d on X/S; it parametrizes relative torsion-free, rank 1 sheaves of degree d on X/S (note the numerical difference to the definition in (4.1).) Let J_d denote the open subscheme of $\bar{J}_d$ parametrizing invertible sheaves. Let $\mathbf{I}_d$ be the universal torsion-free, rank 1 sheaf on $X \times_S \bar{J}_d$ which is rigidified along the section s. For simplicity, we will denote by $\mathcal{D}_d$ the determinant of cohomology functor associated with the projection $X \times_S \bar{J}_d \longrightarrow \bar{J}_d$.

DEFINITION 6.3. *The invertible sheaf* $\Theta_d := \mathcal{D}_d(\mathbf{I}_d)^{-1}$ *is called the Theta line bundle on* $\bar{J}_d/S$ *for every* $d \in \mathbf{Z}$.

(6.4) Since $\chi(\mathbf{I}_{g-1}(s)) = 0$ for every $s \in \bar{J}_{g-1}$, the line bundle Θ_{g-1} does not depend on the section s of X/S chosen because of property (2) of the determinant of cohomology functor. In addition, there is a canonical global section θ_{g-1} of Θ_{g-1}, which is constructed as follows. Since $\chi(\mathbf{I}_{g-1}(s)) = 0$ for every $s \in \bar{J}_{g-1}$, then, locally on S, the complex

$$E^0 \xrightarrow{\lambda} E^1,$$

whose cohomology groups are universally equal to the higher direct images of $\mathbf{I}_{g-1}$, is such that the ranks of E^0 and E^1 are equal. By taking the determinant of λ one obtains a local section of Θ_{g-1}. The local sections can be glued together to yield a global section θ_{g-1} of Θ_{g-1}. By the above local description, it is clear that the zero-scheme Z of θ_{g-1} consists of the torsion-free, rank 1 sheaves possessing a non-trivial global section. Equivalently, by Serre's duality, Z consists of the torsion-free, rank 1 sheaves that can be embedded into the dualizing sheaf ω of X/S. In other words, Z is the image of the Abel-Jacobi map of degree $g-1$:

$$\mathcal{A}\colon \mathrm{Quot}^{g-1}_{\omega/X/S} \longrightarrow \bar{J}_{g-1}.$$

If X/S is smooth, then

$$\mathrm{Quot}^{g-1}_{\omega/X/S} \cong \mathrm{Hilb}^{g-1}_{X/S} = \mathrm{Symm}^{g-1}_S(X),$$

where $\mathrm{Symm}^{g-1}_S(X)$ denotes the symmetric product of $g-1$ copies of X over S. Hence, the geometric fibres of $\mathrm{Quot}^{g-1}_{\omega/X/S}$ are integral, and so are the geometric fibres of Z/S.

(6.5) More generally, if the embedding dimension of each point of each geometric fibre of X/S is at most 2, then $\mathrm{Quot}^{g-1}_{\omega/X/S}$ and $\bar{J}_{g-1}$ are S-flat, and their geometric fibres are integral, local complete intersections of dimensions $g-1$ and g, respectively [**1**, (7), (9), pp. 7, 8]. In this case, Z is clearly an effective relative Cartier divisor on $\bar{J}_{g-1}/S$ whose geometric fibres over S are irreducible, local complete intersections. Moreover, it is easy to see that $\mathcal{A}$ is an isomorphism over the open subscheme of Z parametrizing torsion-free, rank 1 sheaves I with $\dim \mathrm{Hom}(I, \omega) = 1$. This open subscheme is S-dense by [**4**, (3.5,d)]. Since the geometric fibres of Z/S are Cohen-Macaulay and irreducible, and the geometric fibres of $\mathrm{Quot}^{g-1}_{\omega/X/S}$ are integral, then the geometric fibres of Z/S are also integral. We summarize these results in the following definitions and proposition.

DEFINITION 6.6. *X/S is a locally planar S-curve if the embedding dimension of each point of each geometric fibre of X/S is at most 2.*

Equivalently, the family X/S is locally planar if, locally on S, it can be embedded into a quasi-projective, smooth family Y/S whose fibres are surfaces [**3**].

DEFINITION 6.7. *If X/S is a locally planar curve S-curve, then Z is called the Theta divisor of $\bar{J}_{g-1}/S$.*

PROPOSITION 6.8. *If X/S is a locally planar S-curve, then the geometric fibres of the Theta divisor are integral and local complete intersections.*

PROPOSITION 6.9. *If S is the spectrum of an algebraically closed field and X is a locally planar curve, then the self-intersection Θ^g_{g-1} is equal to $g!$.*

PROOF: If X is a smooth curve, then Poincaré's formula [**27**, §2] says that the self-intersection Θ^g_{g-1} is equal to $g!$. On the other hand, the principle of conservation of number tells us that if X/S is a locally planar curve and $D_1, \ldots, D_g$ are relative effective Cartier divisors on $\bar{J}_{g-1}/S$, then the intersection number

$$D_1(s) \cdot \ldots \cdot D_g(s)$$

is independent of the choice of $s \in S$. Since every locally planar curve is part of a family whose general fibre is a smooth curve, then the proposition follows from the above two observations. ∎

(6.10) If X/S is smooth, then it is a classical result that Θ_{g-1} is S-ample on $\bar{J}_{g-1}$, as we already remarked in the introduction. It is then natural to ask whether Θ_{g-1} is S-ample on $\bar{J}_{g-1}$ in general. This section of the thesis is focused on answering affirmatively this question.

LEMMA 6.11. *If Θ_e is S-ample for some integer e, then so are Θ_d for all integers d.*

PROOF: Let $\psi\colon \bar{J}_d \longrightarrow \bar{J}_e$ be the isomorphism given by tensoring a family of torsion-free, rank 1 sheaves on X/S with $\mathcal{O}_X((e-d)D)$, where D is the Cartier divisor defined in (6.2). Then

$$\psi^*\Theta_e = \mathcal{D}_d(\mathbf{I}_d((e-d)D))^{-1}$$

by property (3) of the determinant of cohomology functor. By tensoring the canonical exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with $\mathbf{I}_d(nD)$ for $n \in \mathbf{Z}$, and by applying the determinant of cohomology functor $\mathcal{D}_d(\cdot)$, we get

$$\mathcal{D}_d(\mathbf{I}_d((n+1)D)) \cong \mathcal{D}_d(\mathbf{I}_d(nD)) \otimes \mathbf{I}_d|_{D\times \bar{J}_d} \otimes \mathcal{O}_X(nD)|_D$$

by property (1) of $\mathcal{D}_d(\cdot)$. But $\mathbf{I}_d|_{D\times\bar{J}_d} = \mathcal{O}_{\bar{J}_d}$, since $\mathbf{I}_d$ is rigidified along s. Then

$$\mathcal{D}_d(\mathbf{I}_d((n+1)D)) \cong \mathcal{D}_d(\mathbf{I}_d(nD)) \otimes \mathcal{O}_X(nD)|_D$$

for every $n \in \mathbf{Z}$. It is obvious then that $\psi^*\Theta_e \cong \Theta_d \otimes N$ for a certain invertible sheaf N on S. Hence, if Θ_e is S-ample, then so is Θ_d. The proof is complete. ∎

COROLLARY 6.12. *If X/S is smooth, then Θ_d is S-ample for every d.*

PROOF: Immediate from Lemma 6.11 and the fact that Θ_{g-1} is S-ample. ∎

(6.13) We will first restrict ourselves to the case where S is the spectrum of an algebraically closed field. In other words, assume that X is an integral curve defined over an algebraically closed field k. Let $\pi\colon X' \longrightarrow X$ be a partial normalization of X. Let $Q \subset X$ be the conductor subscheme of π, and let $Q' := \pi^{-1}(Q)$. Let $f' := f \circ \pi$. Let ω' be the dualizing sheaf of X'. Let

$$\delta := \chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_X).$$

The section $s\colon S \longrightarrow X$ lifts uniquely to a section $s'\colon S \longrightarrow X'$ such that $\pi \circ s' = s$. Let $\bar{J}'_e$ denote the compactified Jacobian of degree e on X' for every integer e. Let $\mathbf{I}'_e$ be the universal torsion-free, rank 1 sheaf on $X' \times \bar{J}'_e$ which is rigidified along s', for every integer e. Let $\Theta'_e := \mathcal{D}'_e(\mathbf{I}'_e)^{-1}$, where $\mathcal{D}'_e(\cdot)$ denotes the determinant of cohomology functor associated with the projection $X' \times \bar{J}'_e \longrightarrow \bar{J}'_e$.

Let P^ρ_d denote the presentation scheme, representing the functor that associates to an S-scheme T the set of equivalence classes of homomorphisms $h\colon \mathcal{I} \longrightarrow \pi_{T*}\mathcal{I}'$ on X_T such that:

(1) $\mathcal{I} \in \bar{J}_d(T)$;
(2) $\mathcal{I}' \in \bar{J}'_{d'}(T)$, where $d' := d + \rho - \delta$;
(3) h is injective with flat cokernel of length ρ over T, supported on $Q \times T$.

We recall that the obvious morphisms,

$$\kappa\colon P^\rho_d \longrightarrow \bar{J}_d \quad \text{and} \quad \kappa'\colon P^\rho_d \longrightarrow \bar{J}'_{d'},$$

are finite and projective, respectively. Let

$$h\colon \kappa^*\mathbf{I}_d \longrightarrow \pi_{P^\rho_d *}\kappa'^*\mathbf{I}'_{d'}$$

be the universal presentation on $X \times P^\rho_d$. Let N^ρ_d denote the universal cokernel, and let

$$0 \longrightarrow \kappa^*\mathbf{I}_d \xrightarrow{h} \pi_{P^\rho_d *}\kappa'^*\mathbf{I}'_{d'} \longrightarrow N^\rho_d \longrightarrow 0 \tag{6.14}$$

be the associated short exact sequence. The presentation scheme allows us to compare $\mathcal{A}_d$ with $\mathcal{A}'_{d'}$. This is the content of the following lemma.

LEMMA 6.15.

$$\kappa^*\Theta_d = \kappa'^*\Theta'_{d'} \otimes \bigwedge^\rho f_{P^\rho_d *}N^\rho_d.$$

PROOF: From (6.14) and property (1) of the determinant of cohomology functor,

$$\mathcal{D}_{f_{P_d^\rho}}(\kappa^*\mathbf{I}_d) = \mathcal{D}_{f_{P_d^\rho}}(\pi_{P_d^\rho *}\kappa'^*\mathbf{I}'_{d'}) \otimes \mathcal{D}_{f_{P_d^\rho}}(N_d^\rho)^{-1}.$$

Since π is finite, it is clear that $\mathcal{D}_{f_{P_d^\rho}}(\pi_{P_d^\rho *}\kappa'^*\mathbf{I}'_{d'}) = \mathcal{D}_{f'_{P_d^\rho}}(\kappa'^*\mathbf{I}'_{d'})$. In addition, $\mathcal{D}_{f_{P_d^\rho}}(N_d^\rho) = \bigwedge^\rho f_{P_d^\rho *}N_d^\rho$, since the higher direct images of N_d^ρ under $f_{P_d^\rho}$ vanish. By property (3) of the determinant of cohomology functor, the proof is complete. ∎

(6.16) By [**6**, (9)],

$$P_d^\rho = \mathrm{Quot}^\rho_{\mathcal{F}/Q\times J'_{d'}/J'_{d'}}, \quad \text{where} \quad \mathcal{F} := (\pi_{J'_{d'}*}\mathbf{I}'_{d'})\Big|_{Q\times J'_{d'}}.$$

Since Q is a finite scheme, it is clear that

$$P_d^\rho \subset \mathrm{Grass}^\rho_{J'_{d'}}(\mathcal{E}), \quad \text{where} \quad \mathcal{E} := f'_{J'_{d'}*}(\mathbf{I}'_{d'}|_{Q'\times J'_{d'}}),$$

and the pullback to P_d^ρ of the universal rank ρ quotient on the above Grassmannian is $f_{P_d^\rho *}N_d^\rho$.

The following result is needed in the proof that Θ_d is ample.

PROPOSITION 6.17. *Let Y be a k-scheme. Let $L_1, \ldots, L_t$ be ample sheaves on Y and F be a coherent sheaf on Y. Then*

$$L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F$$

is generated by its global sections and

$$H^i(Y, L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F) = 0 \quad \text{for} \quad i > 0$$

if the n_i are non-negative integers and $\sum_{i=1}^t n_i$ is sufficiently large.

PROOF: First, by replacing each L_i by a high enough power, one can assume that $L_1, \ldots, L_t$ are very ample. In particular, each L_i is generated by global sections.

Let $M \in \mathbf{Z}$ be such that for $m > M$ the sheaf $L_i^m \otimes F$ is generated by global sections for every $i = 1, \ldots, t$. Since each L_i is already generated by global sections, then

$$L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F$$

is generated by global sections for every t-uple of non-negative integers $(n_1, \ldots, n_t)$ such that $\sum n_i > Mt$. The first statement of the proposition is thereby proved.

The proof of the second statement is by double induction; first, on the dimension of Y; second, on the number t of ample sheaves. It is clear that if the dimension of Y is zero, then there is nothing to prove. In addition, if $t = 1$ then the statement is already well-known [**14**, Thm. III-5.2, p. 228].

For each $i = 1, \ldots, t$, since L_i is very ample, there is a section of L_i on Y such that its zero-scheme, Z_i, has lower dimension than Y, and the induced sequence,

$$0 \longrightarrow F \longrightarrow L_i \otimes F \longrightarrow L_i \otimes F|_{Z_i} \longrightarrow 0, \tag{6.18}$$

is exact.

By induction hypothesis, there is an integer M such that for every $i = 1, \ldots, t$ and $k > 0$:

(1) $H^k(Y, \otimes_{j \neq i} L_j^{n_j} \otimes F) = 0$ for all $(t-1)$-uples of non-negative integers $(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_t)$ with $\sum_{j \neq i} n_j > M$;

(2) $H^k(Z_i, \otimes L_j^{n_j} \otimes F)$ for all t-uples $(n_1, \ldots, n_t)$ of non-negative integers with $\sum n_j > M$.

Let $N := M(t/(t-1))$. Let $(n_1, \ldots, n_t)$ be a t-uple of non-negative integers such that $\sum n_j > N$. If n_i is the smallest integer in the t-uple, then $\sum_{j \neq i} n_j > M$. By tensoring (6.18) with

$$L_1^{n_1} \otimes \cdots \otimes L_{i-1}^{n_{i-1}} \otimes L_i^l \otimes L_{i+1}^{n_{i+1}} \otimes \cdots \otimes L_t^{n_t}$$

for each $l \geq 0$, and considering the associated long exact sequence in cohomology, one obtains exact sequences

$$\begin{aligned} H^k(Y, L_1^{n_1} \otimes \ldots L_i^l \cdots \otimes L_t^{n_t} \otimes F) \longrightarrow H^k(Y, L_1^{n_1} \otimes \ldots L_i^{l+1} \cdots \otimes L_t^{n_t} \otimes F) \longrightarrow \\ \longrightarrow H^k(Z_i, L_1^{n_1} \otimes \ldots L_i^{l+1} \cdots \otimes L_t^{n_t} \otimes F) \end{aligned} \tag{6.19}$$

for each $l \geq 0$ and $k > 0$. Since $\sum_{j \neq i} n_j > M$, then

$$H^k(Z_i, L_1^{n_1} \otimes \cdots \otimes L_i^{l+1} \otimes \cdots \otimes L_t^{n_t} \otimes F) = 0$$

for each $l \geq 0$ and $k > 0$. By the same token,

$$H^k(Y, L_1^{n_1} \otimes \cdots \otimes L_{i-1}^{n_{i-1}} \otimes L_{i+1}^{n_{i+1}} \otimes \cdots \otimes L_t^{n_t} \otimes F) = 0$$

for each $k > 0$. From these vanishings and induction on the exact sequences (6.19), one concludes that

$$H^k(Y, L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F) = 0$$

for all $k > 0$. The proof is complete. ∎

THEOREM 6.20. *Θ_d is ample on $\bar{J}_d$.*

PROOF: Assume that $\pi\colon X' \longrightarrow X$ is the normalization map of X. Since $\mathbf{I}'_{d'}$ is invertible, then $\mathcal{E}$ is locally free of constant rank equal to the length of Q'. In fact, if

$$Q' = \sum_{i=1}^{t} n_i q_i$$

as a divisor on X', where $q_i \in X'$ and $n_i > 0$, then

$$\mathcal{E} = \bigoplus_{i=1}^{t} f'_{\bar{J}'_{d'}*}\, \mathbf{I}'_{d'}|_{n_i q_i \times \bar{J}'_{d'}}\,.$$

For convenience, let $g := f'_{\bar{J}'_{d'}}$. From the above equality, one obtains that

$$\bigwedge^{\rho} \mathcal{E} = \bigoplus_{|\bar{\lambda}|=\rho} E_{\bar{\lambda}},$$

where

$$\begin{aligned}\bar{\lambda} :=& (\lambda_1, \ldots, \lambda_t) \text{ with } \; 0 \le \lambda_i \le n_i;\\ |\bar{\lambda}| :=& \lambda_1 + \cdots + \lambda_t;\end{aligned}$$

and

$$E_{\bar{\lambda}} := \bigwedge^{\lambda_1} g_*\, \mathbf{I}'_d|_{n_1 q_1 \times \bar{J}'_{d'}} \otimes \cdots \otimes \bigwedge^{\lambda_t} g_*\, \mathbf{I}'_d|_{n_t q_t \times \bar{J}'_{d'}}\,. \tag{6.21}$$

Hence, for all positive integers n and l with $n > l$,

$$(\bigwedge^{\rho} \mathcal{E})^n \otimes (\Theta'_{d'})^{n-l} = (\bigoplus_{|\bar{h}|=n} \binom{n}{\bar{h}} \bigotimes_{|\bar{\lambda}|=\rho} E_{\bar{\lambda}}^{h_{\bar{\lambda}}}) \otimes (\Theta'_{d'})^{n-l} =$$

$$(\bigoplus_{|\bar{h}|=n} \binom{n}{\bar{h}} \bigotimes_{|\bar{\lambda}|=\rho} (E_{\bar{\lambda}} \otimes \Theta'_{d'})^{h_{\bar{\lambda}}}) \otimes (\Theta'_{d'})^{-l}, \tag{6.22}$$

where

$$\begin{aligned}\bar{h} :=& (h_{\bar{\lambda}})_{\bar{\lambda}} \text{ with } \; h_{\bar{\lambda}} \in \mathbf{Z}_{\ge 0};\\ |\bar{h}| :=& \sum_{\bar{\lambda}} h_{\bar{\lambda}};\\ \binom{n}{\bar{h}} :=& \frac{n!}{\prod_{\bar{\lambda}} h_{\bar{\lambda}}!}.\end{aligned}$$

We claim that (6.22) is generated by its global sections if l is fixed and n is sufficiently large. In fact, let $\psi_{\bar\lambda}\colon \bar J'_{d'} \longrightarrow \bar J'_{d-\delta}$ be the isomorphism defined by tensoring an invertible sheaf on X' by $\mathcal{O}_{X'}(-\lambda_1 q_1 - \cdots - \lambda_t q_t)$. It is then easy to prove, by the same argument used in the proof of Proposition 6.11, that

$$\psi_{\bar\lambda}^*(\Theta'_{d-\delta}) \cong \Theta'_{d'} \otimes (\mathbf{I}'_{d'}|_{q_1\times \bar J'_{d'}})^{\lambda_1} \otimes \cdots \otimes (\mathbf{I}'_{d'}|_{q_t\times \bar J'_{d'}})^{\lambda_t}. \tag{6.23}$$

On the other hand, for each $i = 1, \ldots, t$ and each positive integer m one has the following canonical exact sequence on $\bar J'_{d'}$:

$$0 \longrightarrow \mathbf{I}'_{d'}(-mq_i)|_{q_i\times \bar J'_{d'}} \longrightarrow g_*\, \mathbf{I}'_{d'}|_{(m+1)q_i\times \bar J'_{d'}} \longrightarrow g_*\, \mathbf{I}'_{d'}|_{mq_i\times \bar J'_{d'}} \longrightarrow 0.$$

For every positive integer $k \le m+1$, the above exact sequence induces the following canonical short exact sequence:

$$0 \longrightarrow \bigwedge^{k-1} g_*\, \mathbf{I}'_{d'}|_{mq_i\times \bar J'_{d'}} \otimes \mathbf{I}'_{d'}(-mq_i)|_{q_i\times \bar J'_{d'}} \longrightarrow \bigwedge^{k} g_*\, \mathbf{I}'_{d'}|_{(m+1)q_i\times \bar J'_{d'}} \longrightarrow$$
$$\longrightarrow \bigwedge^{k} g_*\, \mathbf{I}'_{d'}|_{mq_i\times \bar J'_{d'}} \longrightarrow 0. \tag{6.24}$$

From (6.21), (6.23) and (6.24), it is not difficult to see by induction that if

$$\Big(\bigotimes_{|\bar\lambda|=\rho} \psi_{\bar\lambda}^*(\Theta'_{d-\delta})^{h_{\bar\lambda}}\Big) \otimes (\Theta'_{d'})^{-l} \tag{6.25}$$

is generated by global sections and its first cohomology vanishes, then

$$\Big(\bigotimes_{|\bar\lambda|=\rho} (E_\lambda \otimes \Theta'_{d'})^{h_\lambda}\Big) \otimes (\Theta'_{d'})^{-l}$$

is also generated by global sections and its first cohomology vanishes. But, since $\Theta'_{d-\delta}$ is ample by Corollary 6.12, then Proposition 6.17 shows that, for fixed l and sufficiently large n, the sheaf (6.25) is generated by global sections and its first cohomology vanishes. By (6.22),

$$\Big(\bigwedge^{\rho} \mathcal{E}\Big)^n \otimes (\Theta'_{d'})^{n-l}$$

is thus generated by global sections for fixed l and sufficiently large n, as claimed.

Therefore, for fixed l and sufficiently large n,

$$\kappa'^*(\Theta'_{d'})^{n-l} \otimes (\bigwedge^{\rho} f_{P^\rho_d *} N^\rho_d)^n$$

is very ample on P^ρ_d relative to $\bar{J}'_{d'}$, as defined in [**14**, p. 120]. Fix l such that $(\Theta'_{d'})^l$ is very ample on $\bar{J}'_{d'}$. By means of the Segre embedding and Lemma 6.15, one gets that $(\kappa^*\Theta_d)^n$ is very ample on P^ρ_d for n sufficiently large. Hence, $\kappa^*\Theta_d$ is after all ample. Since κ is finite, by [**EGA III-1**, 2.6.2] the restriction of Θ_d to the image of κ is ample. The image of κ certainly contains the closure $\bar{A}^\rho_d$ of A^ρ_d in $\bar{J}_d$. By Theorem 4.3, the compactified Jacobian $\bar{J}_d$ of X is the union of the A^ρ_d for $\rho = 0, \ldots, \delta$. Since the restriction of Θ_d to each component of the closed covering

$$\bar{J}_d = \bigcup_{\rho=0}^{\delta} \bar{A}^\rho_d$$

of $\bar{J}_d$ is ample, then Θ_d is itself ample on $\bar{J}_d$. The proof is complete. ∎

(6.26) Assume now that $f\colon X \longrightarrow S$ is an S-curve

COROLLARY 6.27. *The sheaf Θ_d is S-ample on $\bar{J}_d$.*

PROOF: Showing that the invertible sheaf Θ_d is S-ample is equivalent to showing that the restriction of Θ_d to each geometric fibre of $\bar{J}_d/S$ is ample [**EGA III-1**, 4.7.1]. Since the formation of Θ_d commutes with base change by property (3) of the determinant of cohomology functor, then we may restrict ourselves to the case where S is the spectrum of an algebraically closed field. But this case has already been adressed by Theorem 6.20. The proof is complete. ∎

(6.28) We remark that the construction of Θ_d depends on having a section of X/S, so that there is a universal, torsion-free, rank 1 sheaf on $X \times \bar{J}_d$. Nevertheless, one can prove using descent theory that Θ_{g-1} can be defined without assuming the existence of a section.

In fact, by [**EGA IV-4**, 17.16.3, p. 106], there exists an étale covering $S' \longrightarrow S$ such that $X \times_S S'/S'$ admits a section through its smooth locus. As a consequence, there is a universal torsion-free, rank 1 sheaf $\mathbf{I}$ on $X \times_S \bar{J}_{g-1} \times_S S'$. Let $\Theta' := \mathcal{D}(\mathbf{I})^{-1}$, where $\mathcal{D}(\cdot)$ denotes the determinant of cohomology functor associated to the projection

$$X \times_S \bar{J}_{g-1} \times_S S' \longrightarrow \bar{J}_{g-1} \times_S S'.$$

By Corollary 6.27, the sheaf Θ' is S'-ample. We will show that Θ' descends to a sheaf on $\bar{J}_{g-1}$. Let $S'' := S' \times_S S'$ and $S''' := S' \times_S S' \times_S S'$. Let

$$\begin{aligned} q_1, q_2 &: S'' \longrightarrow S'; \\ p_1, p_2, p_3 &: S''' \longrightarrow S'; \\ r_1, r_2, r_3 &: S''' \longrightarrow S'' \end{aligned}$$

denote the projection maps satisfying

$$\begin{aligned} p_1 &= q_1 \circ r_2 = q_1 \circ r_3; \\ p_2 &= q_1 \circ r_1 = q_2 \circ r_3; \\ p_3 &= q_2 \circ r_1 = q_2 \circ r_2. \end{aligned}$$

Since $\mathbf{I}$ is universal, there are an invertible sheaf L on $\bar{J}_{g-1} \times_S S''$ and an isomorphism

$$\phi\colon q_1^*\mathbf{I} \longrightarrow q_2^*\mathbf{I} \otimes L$$

on $X \times_S \bar{J}_{g-1} \times_S S''$. Since

$$\mathcal{D}(q_2^*\mathbf{I} \otimes L) = q_2^*\mathcal{D}(\mathbf{I}) \tag{6.29}$$

by properties (2) and (3) of $\mathcal{D}(\cdot)$, then we obtain an isomorphism,

$$\psi := \mathcal{D}(\phi)\colon q_1^*\Theta' \longrightarrow q_2^*\Theta'.$$

Let

$$\begin{aligned} L_i &:= r_i^* L, \\ \phi_i &:= r_i^* \phi, \\ \psi_i &:= r_i^* \psi \end{aligned}$$

for $i = 1, 2, 3$. By [**13**, Exp. VIII, Cor. 1.3], in order to prove that Θ' descends to a sheaf on $\bar{J}_{g-1}$, one must show that

$$\psi_1 \circ \psi_3 = \psi_2.$$

However, ψ_2 and $\psi_1 \circ \psi_3$ are obtained by applying $\mathcal{D}(\cdot)$ to the isomorphisms

$$\phi_2\colon p_1^*\mathbf{I} \longrightarrow p_3^*\mathbf{I} \otimes L_2 \quad \text{and} \quad (\phi_1 \otimes \mathrm{id}_{L_3}) \circ \phi_3\colon p_1^*\mathbf{I} \longrightarrow p_3^*\mathbf{I} \otimes L_1 \otimes L_3,$$

respectively. Hence, to prove that $\psi_1 \circ \psi_3 = \psi_2$ we need only check that the isomorphism

$$(\phi_2 \otimes \mathrm{id}_{L_1 \otimes L_2^{-1} \otimes L_3})^{-1} \circ (\phi_1 \otimes \mathrm{id}_{L_3}) \circ \phi_3 \colon p_1^* \mathbf{I} \longrightarrow p_1^* \mathbf{I} \otimes L_1 \otimes L_2^{-1} \otimes L_3 \quad (6.30)$$

gives the identity map on $p_1^*\Theta'$ via $\mathcal{D}(\cdot)$. But since $p_1^*\mathbf{I}$ is simple [4, (5.4), p. 83], then (6.30) is given by an isomorphism between the structure sheaf on $\bar{J}_{g-1} \times_S S'''$ and $L_1 \otimes L_2^{-1} \otimes L_3$ [2, (5), p.119]. Because of (6.29), the map (6.30) induces the identity on $p_1^*\Theta'$. Hence, Θ' descends to a unique sheaf Θ on $\bar{J}_{g-1}$. The sheaf Θ is ample since Θ' is ample and $S' \longrightarrow S$ is an étale covering.

(6.31) As already hinted in (6.4), the sheaf Θ_d depends on the choice of a section s of X/S if $d \neq g-1$. Nevertheless, it is a classical result that, if X/S is smooth, then the symmetrization of Θ_0 does not depend on s. Assume that X/S is a family of Gorenstein curves. Then the inverse map (-1) on J_0 extends naturally to an involution of $\bar{J}_0$. In fact, if $\mathcal{I}$ is a family of torsion-free, rank 1 sheaves on X/S, then so is $\underline{Hom}(\mathcal{I}, \mathcal{O}_X)$ by local base change theory [4, (1.9), p. 59], since $\underline{Ext}^1(\mathcal{I}, \mathcal{O}_X) = 0$ by [4, (6.5.3), p. 96]. Hence, it is natural to call an invertible sheaf L on $\bar{J}_0/S$ *symmetric* if $(-1)^*L = L$. The symmetrization

$$\Theta_0 \otimes (-1)^*\Theta_0$$

of Θ_0 does not depend on the choice of a section s of X/S as it can be easily seen from property (2) of the determinant of cohomology functor.

THEOREM 6.32. *If X/S is a family of Gorenstein curves, then*

$$\mathcal{D}_0(\mathbf{I}_0)^{-1} \otimes \mathcal{D}_0(\underline{Hom}(\mathbf{I}_0, \mathcal{O}_X))^{-1} \otimes \mathcal{D}_0(\mathcal{O}_X)^2 \qquad (6.33)$$

is S-ample, symmetric and rigidified along the section of $\bar{J}_0/S$ given by the structure sheaf of X/S.

PROOF: Because of Corollary 6.27, we just need remark that by tensoring with $\mathcal{D}_0(\mathcal{O}_X)^2$ the sheaf $\Theta_0 \otimes (-1)^*\Theta_0$ becomes rigidified. ∎

(6.34) If X/S is a family of integral, stable curves, then (6.33) extends Deligne's canonical ample sheaf on J_0 to $\bar{J}_0$. In fact, the restriction L_0 of (6.33) to J_0 is certainly S-ample and rigidified along the section of J_0/S given by the structure sheaf of X/S for every family X/S of Gorenstein, integral curves. Moreover, the formation of L_0 commutes with base change by property (3) of the determinant of cohomology functor. So the sheaves L_0 satisfy the same defining properties as Deligne's canonical ample sheaves. By [10, 4.2, p. 142], for every family X/S there would be a sheaf with the above properties, and such sheaf would be unique if we considered all families X/S of stable curves, <u>including</u> the reducible ones. But the integral, stable curves form an open subscheme of the moduli of stable curves, and such subscheme contains the moduli of smooth curves. Hence, it is easy to see that Deligne's proof in [**10**, 4.2, p. 142], which makes use of the moduli space of stable curves, works as well if we carry it out for the open subscheme of integral, stable curves instead.

PART II

Weierstrass Theory for Families of Local Complete Intersection Curves

1. Introduction.

Linear systems on smooth curves in characteristic 0 have been extensively studied classically, with strong results on the projective geometry of smooth curves being discovered by the Italian school of Castelnuovo and others. A good part of their results involved the analysis of the ramification points of a linear system, sometimes called Weierstrass points, especially if the linear system is the canonical system.

In positive characteristic, the study of Weierstrass points began with F. K. Schmidt ([**32**] and [**33**]), who nevertheless considered only the canonical system. The study of Weierstrass points of a general linear system began only relatively recently, basically initiated by Matzat [**28**] and Laksov ([**19**] and [**20**]). Much work has been done since in trying to understand the peculiarities of the positive characteristic case.

Generalizing the theory in another direction, in 1984 Widland defined Weierstrass points for the canonical system on any Gorenstein, irreducible curve [**36**]. His definition was later extended to any linear system by Lax [**24**]. Around 1986 Eisenbud and Harris considered the question of Weierstrass points on a curve of compact type [**11**]. Very recently Garcia and Lax [**12**] and Laksov and Thorup [**22**] extended Widland's and Lax's definition to the positive characteristic case.

One of the main goals in extending the notion of Weierstrass points to the singular case is to improve the understanding of smooth curves. Analysing smooth curves by analysing their degenerations to singular curves has always been a very fruitful idea, as shown for instance by the recent work of Eisenbud and Harris (see a summary and references in [**11**].) Hence the need of not only a theory of Weierstrass points on a singular curve, but more generally a theory of Weierstrass points on families of curves.

(1.1) The theory of Weierstrass points on families of smooth curves in characteristic 0 has been developed classically. Recently, Laksov and Thorup developed a framework for understanding Weierstrass points on families of smooth curves in arbitrary characteristic ([**21**] and [**22**]). They begin by defining the notion of a Wronski system of modules on a scheme Y. A *Wronski system of modules* is the data consisting of a locally free sheaf $\mathcal{W}$, a sequence of locally free sheaves $\mathcal{E}^i$ of rank $i+1$ for $i = 0, 1, \ldots$, a sequence of surjective $\mathcal{O}_Y$-linear maps $q^i \colon \mathcal{E}^i \longrightarrow \mathcal{E}^{i-1}$ for $i = 1, 2, \ldots$, and a sequence of $\mathcal{O}_Y$-linear maps $v^i \colon \mathcal{W} \longrightarrow \mathcal{E}^i$ for $i = 0, 1, \ldots$ such that $v^{i-1} = q^i \circ v^i$ for $i = 1, 2, \ldots$. From a Wronski system of modules it is possible to define orders at points and hence the concept of a Weierstrass point. Moreover, Laksov and Thorup show how to associate to a Wronski system a Wronskian determinant, whose zero locus provides a subscheme structure for the set of Weierstrass points. After developing a general theory of Wronski systems, they show how to use the sheaves of principal parts P^i for $i = 0, 1, \ldots$ on a smooth family $f \colon X \longrightarrow S$ of curves to obtain a Wronski system of modules on X for any linear system on X/S. More precisely, if L is an invertible sheaf on X, and $\gamma \colon \mathcal{W} \longrightarrow f_*L$ is an $\mathcal{O}_S$-linear map whose source $\mathcal{W}$ is a locally free sheaf, then one obtains surjections $q^i \colon P^i(L) \longrightarrow P^{i-1}(L)$ for $i = 1, 2, \ldots$ and homomorphisms $v^i \colon f^*\mathcal{W} \longrightarrow P^i(L)$ for $i = 0, 1, \ldots$ such that the data

$$(f^*\mathcal{W}, P^i(L), q^i, v^i, i \geq 0) \tag{1.2}$$

is a Wronski system of modules [**21**, 2.1, 2.2, p. 137, 138]. By definition,

$$P^i := \frac{\mathcal{O}_{X\times_S X}}{I^{i+1}} \quad \text{and} \quad P^i(L) := p_*\Big(\frac{\mathcal{O}_{X\times_S X}}{I^{i+1}} \otimes q^*L\Big),$$

where p and q are the projections of $X \times_S X$ on its factors, and I is the ideal sheaf of the diagonal in $\mathcal{O}_{X\times_S X}$. The data (1.2) is a Wronski system of modules because the sheaves of principal parts on a smooth family of curves are locally free, which is not true in the singular case.

(1.3) The purpose of this part of the thesis is to overcome the "deficiency" of the sheaves of principal parts in being locally free. On a singular, Gorenstein curve the invertible sheaf replacing the sheaf of Kähler differentials Ω^1 is the dualizing sheaf ω. Therefore, a natural idea is to use the canonical map $\eta^1 \colon \Omega^1 \longrightarrow \omega$ on a family X/S to replace the sheaves of principal parts P^i by sheaves of P^i-algebras Q^i that are locally free. More precisely, one wishes to obtain sheaves of algebras Q^i for $i = 0, 1, \ldots$ fitting into commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \Omega^i \arrow[r] \arrow[d, "\eta^i"'] & P^i \arrow[r] \arrow[d, "\psi^i"'] & P^{i-1} \arrow[r] \arrow[d, "\psi^{i-1}"'] & 0 \\ 0 \arrow[r] & \omega^{\otimes i} \arrow[r, "\alpha^i"] & Q^i \arrow[r, "q^i"] & Q^{i-1} \arrow[r] & 0 \end{tikzcd} \tag{1.4}$$

for $i = 1, 2, \ldots$, where the first row is canonical, both rows are exact, the right hand side square is a diagram of algebra homomorphisms, and η^i is induced from η^1 in a canonical way (see Section 2). The data given by the sequence of sheaves Q^i for $i = 0, 1, \ldots$ and the maps in diagram (1.4) is called a *Wronski algebra system* on X/S. In a sense, the goal of constructing a Wronski algebra system on X/S forces us to be a little more ambitious. There are several Wronski algebra system on a single curve (see (4.11)), but if one requires the formation of the sheaves Q^i to be natural, that is, to commute under base change in a fairly large class of families X/S, then there is only one Wronski algebra system. The method developed in the present article produces a canonical and natural Wronski algebra system for a general family of reduced, local complete intersection curves, not necessarily irreducible or complete, in arbitrary characteristic. By replacing the sheaves of principal parts P^i by the sheaves Q^i one can readily apply the method described in (1.1) to associate to each linear system on X/S a canonical Wronski system of modules on X.

(1.5) There are a few novelties introduced in this part of the thesis. First, we are able to consider reducible curves in any characteristic, as long as they are local complete intersections. In particular, we are able to consider families of Deligne-Mumford stable curves. However, we are faced with the same problem Eisenbud and Harris pointed out in [**11**, p. 339], namely, some components of the curve might be degenerate with respect to the linear system in consideration without the whole curve being degenerate. In our set-up, every point in a degenerate component is a Weierstrass point. By contrast, there are at most a finite number of Weierstrass points on an irreducible curve. Excluding the case of degenerate components, the theory developed here adequately defines Weierstrass points and weights on a family of reducible curves, as shown in Section 7.

Second, we are able to consider families of singular curves in any characteristic, instead of a single curve as in the previous literature (see however [**25**] in characteristic 0). In [**22**] Laksov and Thorup independently introduce a replacement of the sheaves of principal parts on an integral curve defined over a perfect field. Nevertheless, it is not clear at all whether their method would extend to families, since they made use of the normalization map of the curve and of Rosenlicht's local characterization of the dualizing sheaf on the curve [**34**]. It is not clear either (even though it is plausible) that their replacement coincides with ours in the case of an integral, local complete intersection curve defined over a perfect field.

Third, a Wronski system of modules gives a priori more information than the Wronskian determinant obtained from it. For instance, one is able to give a structure of determinantal subscheme to a subset of the family defined by a condition of Weierstrass type. More precisely, given a Wronski system of modules $(\mathcal{W}, Q^i, q^i, v^i, i \geq 0)$ on a scheme Y, the k-th degeneracy locus of the map of vector bundles v^i gives a subscheme structure for the locus of points $y \in Y$ whose k-th order is greater than i. Hence it is desirable to obtain a Wronski system of modules for each linear system, as done in the present article, instead of just a Weierstrass divisor.

(1.6) There are important questions still open. First, is it possible to construct a Wronski algebra system on a general family of reduced, Gorenstein curves? If so, is it possible to construct it in a natural way? Is the Wronski algebra system unique in some sense?

Secondly, how can one explain limits of Weierstrass points when an irreducible curve approaches a reducible curve with degenerate components? Eisenbud and Harris have developed the technique of limit linear series when the reducible curve is of compact type [**11**]. Is there a way to handle the problem at least for stable curves? If so, valuable information could be obtained about the moduli of smooth curves, since it has a compactification by stable curves.

(1.7) We now give a brief summary of the contents of this part of the thesis.

In Section 2 we define the notion of a Wronski algebra system, and state the main (and only) theorem of the article, Theorem 2.16. We also start an induction argument for the proof of the theorem, which will be completed in the next four sections.

In Section 3 we provide a local description of a Wronski algebra system, and a local criterion for its existence (Criterion 3.13.) The criterion applies to any family of reduced, Gorenstein curves.

In Section 4 we restrict our attention to families of reduced, local complete intersection curves. We prove the existence and uniqueness of a Wronski algebra system on a "general" family, as defined in (4.9).

In Section 5 we introduce the necessary tools to induce locally on any family a Wronski algebra system from a larger "general" family, and then to patch the induced local systems together.

In Section 6 we use the existence and uniqueness of a Wronski algebra system on a "general" family, proved in Section 4, and the tools developed in Section 5 to wrap up the proof of Theorem 2.16.

In Section 7 we show how the theory developed by Laksov and Thorup in [**21**] and [**22**] can be readily applied, once one has good substitutes for the sheaves of principal parts. Since we also allow for reducible curves, we make the necessary modifications to their set-up.

All schemes considered will be assumed locally noetherian, and all morphisms locally of finite type.

2. The Wronski algebra system.

Let $f\colon X \to S$ be a flat morphism whose geometric fibres are reduced, Gorenstein curves. We will often refer to f as the family X/S. Let $\Omega^1_{X/S}$ denote the sheaf of relative Kähler differentials of f, and $P^n_{X/S}$ the sheaf of relative n-th order principal parts for each $n \geq 0$. If $I_{X/S}$ denotes the ideal sheaf of the diagonal $X \to X \times_S X$, then

$$\Omega^1_{X/S} = \frac{I_{X/S}}{I^2_{X/S}} \quad \text{and} \quad P^n_{X/S} = \frac{\mathcal{O}_{X\times_S X}}{I^{n+1}_{X/S}}.$$

Denote by $\Omega^n_{X/S}$ the $\mathcal{O}_X$-module $\frac{I^n_{X/S}}{I^{n+1}_{X/S}}$ for every $n \geq 0$. There is a canonical exact sequence

$$0 \to \Omega^n_{X/S} \to P^n_{X/S} \xrightarrow{p^n_{X/S}} P^{n-1}_{X/S} \to 0 \tag{2.1}$$

for every $n > 0$. In addition, the formation of $P^n_{X/S}$ and $p^n_{X/S}$ commute with base change and open embeddings. Note that $P^n_{X/S}$ is a sheaf of $\mathcal{O}_X$-algebras in two ways, induced by the two $\mathcal{O}_X$-algebra structures of $\mathcal{O}_{X\times_S X}$. We will distinguish between the two by calling one the left structure and the other the right structure. For general information on the sheaves of principal parts we refer the reader to [**EGA IV-4**, 16.7, p. 36].

(2.2) Assume that the fibres of f are local complete intersections. In addition, assume for the moment that f is quasi-projective. Denote by $\iota\colon X \hookrightarrow Y$ an S-embedding of X into an S-smooth scheme Y with pure relative dimension m over S (for instance, one could take Y to be a projective space over S). Since the geometric fibres of X/S are local complete intersections and Y is S-smooth, the embedding ι is transversally regular relative to S [**EGA IV-4**, 19.3.7, p. 196]. Let J_Y be the ideal sheaf of X in a neighbourhood of X in Y. Then one has a canonical exact sequence of sheaves:

$$\frac{J_Y}{J_Y^2} \xrightarrow{d_Y} \Omega^1_{Y/S} \otimes \mathcal{O}_X \xrightarrow{\pi_Y} \Omega^1_{X/S} \to 0.$$

From the above exact sequence, one constructs the map

$$\mu_Y : \Omega^1_{X/S} \otimes \bigwedge^{m-1} \frac{J_Y}{J_Y^2} \longrightarrow \bigwedge^{m} \Omega^1_{Y/S} \otimes \mathcal{O}_X$$

locally defined on an affine open subset U of X by

$$\pi_Y(\lambda) \otimes g_1 \wedge \cdots \wedge g_{m-1} \longmapsto \lambda \wedge d_Y g_1 \wedge \cdots \wedge d_Y g_{m-1},$$

where λ is a section of $\Omega^1_{Y/S} \otimes \mathcal{O}_X$ on U and $g_1, \ldots, g_{m-1}$ are sections of $\frac{J_Y}{J_Y^2}$ on U. The map μ_Y is well-defined since $\frac{J_Y}{J_Y^2}$ is a locally free $\mathcal{O}_X$-module of rank $m-1$. Let

$$\omega_{X/S} := \bigwedge^m \Omega^1_{Y/S} \otimes \Big(\bigwedge^{m-1} \frac{J_Y}{J_Y^2}\Big)^{-1}. \tag{2.3}$$

By tensoring μ_Y with $(\bigwedge^{m-1} \frac{J_Y}{J_Y^2})^{-1}$ we obtain a map

$$\eta^1_{X/Y/S} \colon \Omega^1_{X/S} \longrightarrow \omega_{X/S}.$$

It is clear from (2.3) that the formation of $\omega_{X/S}$ commutes with base change and open embeddings. Moreover, the above description shows that $\eta^1_{X/Y/S}$ is also natural, that is, for any S-scheme S_1 and any open subscheme X_1 of $X \times_S S_1$, the diagram

$$\begin{tikzcd}
\Omega^1_{X/S} \otimes \mathcal{O}_{X_1} \arrow[r, "\eta^1_{X/Y/S} \otimes \mathcal{O}_{X_1}"] \arrow[d, "\cong"'] & \omega_{X/S} \otimes \mathcal{O}_{X_1} \arrow[d, "\cong"'] \\
\Omega^1_{X_1/S_1} \arrow[r, "\eta^1_{X_1/Y_1/S_1}"] & \omega_{X_1/S_1}
\end{tikzcd}$$

commutes, where Y_1 is an open neighbourhood of X_1 in $Y \times_S S_1$.

(2.4) The homomorphism $\eta^1_{X/Y/S}$ does not depend on a particular embedding of X into an S-smooth scheme Y. In fact, let $\nu \colon X \hookrightarrow Z$ be an S-embedding into an S-smooth scheme Z of pure relative dimension over S. Note that

$$\mu := (\iota, \nu) \colon X \longrightarrow Y \times_S Z$$

is also an S-embedding into an S-smooth scheme of pure dimension over S. We need only prove that

$$\eta^1_{X/Y \times_S Z/S} = \eta^1_{X/Y/S} \quad \text{and} \quad \eta^1_{X/Y \times_S Z/S} = \eta^1_{X/Z/S}.$$

Hence one can assume that there is a smooth morphism $p \colon Y \longrightarrow Z$ of pure relative dimension l such that $\nu = p \circ \iota$. Let J_Z be the ideal sheaf of X in a neighbourhood of X in Z. Then one has the following canonical diagram of homomorphisms on X:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \frac{J_Z}{J_Z^2} \arrow[r, "d_Z"] \arrow[d] & \Omega^1_{Z/S} \otimes \mathcal{O}_X \arrow[r, "\pi_Z"] \arrow[d] & \Omega^1_{X/S} \arrow[r] \arrow[d, equal] & 0 \\
 & \frac{J_Y}{J_Y^2} \arrow[r, "d_Y"] \arrow[d] & \Omega^1_{Y/S} \otimes \mathcal{O}_X \arrow[r, "\pi_Y"] \arrow[d] & \Omega^1_{X/S} \arrow[r] & 0 \\
 & \Omega^1_{Y/Z} \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d] & \Omega^1_{Y/Z} \otimes \mathcal{O}_X \arrow[d] & & \\
 & 0 & 0, & &
\end{tikzcd}$$

where the rows and columns are exact. The above diagram shows that

$$\mu_Y = \mu_Z \otimes \bigwedge^{l} \Omega^1_{Y/Z} \otimes \mathcal{O}_X.$$

Since

$$\eta^1_{X/Z/S} = \mu_Z \otimes (\bigwedge^{m-l} \frac{J_Z}{J_Z^2})^{-1},$$

$$\eta^1_{X/Y/S} = \mu_Y \otimes (\bigwedge^{m} \frac{J_Y}{J_Y^2})^{-1},$$

and

$$\bigwedge^{m} \frac{J_Y}{J_Y^2} = \bigwedge^{m-l} \frac{J_Z}{J_Z^2} \otimes \bigwedge^{l} \Omega^1_{Y/Z} \otimes \mathcal{O}_X,$$

then

$$\eta^1_{X/Y/S} = \eta^1_{X/Z/S}.$$

Hence $\eta^1_{X/S} := \eta^1_{X/Y/S}$ is uniquely defined. We remark that the above argument is certainly not new. It was used in [**23**], for instance, to show that $\omega_{X/S}$, as defined in (2.3), is independent of the embedding ι.

If $f\colon X \longrightarrow S$ is a general flat morphism whose geometric fibres are reduced, local complete intersection curves, then one can cover X with open subschemes X_λ in such a way that X_λ is quasi-projective over S. Because $\eta^1_{X_\lambda/S}$ does not depend on a particular S-embedding of X_λ into an S-smooth scheme of pure relative dimension over S, then one can glue the homomorphisms $\eta^1_{X_\lambda/S}$ together to obtain a global homomorphism $\eta^1_{X/S}$.

We remark that by [**15**, Corollary 23, p. 56] the sheaf $\omega_{X/S}$ is a dualizing sheaf for the family X/S. However, no dualizing property of $\omega_{X/S}$ will be used in the remaining of the article. For our purposes all we need is that $\omega_{X/S}$ is defined by (2.3).

It is worth mentioning that it would actually be possible to obtain a comparison homomorphism between the sheaf of Kähler differentials of f and a certain dualizing sheaf of f without the assumption that the geometric fibres be local complete intersections. As a matter of fact, we will only need the latter assumption in Section 4.

Let $\omega^n_{X/S} := \omega^{\otimes n}_{X/S}$.

PROPOSITION 2.5. *$\eta^1_{X/S}$ induces canonical and natural homomorphisms*

$$\eta^i_{X/S}\colon \Omega^i_{X/S} \to \omega^i_{X/S}$$

for every $i \geq 1$, which are isomorphisms on the smooth locus of X/S.

PROOF: $\eta^1_{X/S}$ induces

$$(\eta^1_{X/S})^2\colon \frac{I_{X/S}}{I^2_{X/S}} \otimes \frac{I_{X/S}}{I^2_{X/S}} \to \omega^2_{X/S}.$$

We will show that $(\eta^1_{X/S})^2$ factors through the multiplication homomorphism

$$m\colon \frac{I_{X/S}}{I^2_{X/S}} \otimes \frac{I_{X/S}}{I^2_{X/S}} \to \frac{I^2_{X/S}}{I^3_{X/S}}.$$

For this we just need to show that the support of the kernel of m does not include any associated points of X, since $\omega_{X/S}$ is invertible. But an associated point of X is an associated point of the fibre over S where it lies [**EGA IV-2**, 6.3.1, p. 138]. Since the geometric fibres of f are reduced, then any associated point of X lies on the smooth locus of X/S, where m is an isomorphism. The construction of $\eta^2_{X/S}$ is thereby completed. The construction of the remaining homomorphisms is analogous. The naturality is obvious from the construction and the naturality of $\eta^1_{X/S}$. ∎

DEFINITION 2.6. *A* **Wronski algebra system** *on X/S is a collection $\{Q^n_{X/S}; n \geq 0\}$ of sheaves of algebras on X together with algebra homomorphisms*

$$\psi^n_{X/S}\colon P^n_{X/S} \to Q^n_{X/S},$$

$$q^n_{X/S}\colon Q^n_{X/S} \to Q^{n-1}_{X/S},$$

and an $\mathcal{O}_X$-module homomorphism

$$\alpha^n_{X/S}\colon \omega^n_{X/S} \to Q^n_{X/S}$$

for every $n \geq 0$ satisfying the following properties:

(1) $Q^0_{X/S} = P^0_{X/S}$;

(2) *the diagram of maps*

$$\begin{tikzcd} 0 \arrow[r] & \Omega^n_{X/S} \arrow[r] \arrow[d, "\eta^n_{X/S}"'] & P^n_{X/S} \arrow[r, "p^n_{X/S}"] \arrow[d, "\psi^n_{X/S}"'] & P^{n-1}_{X/S} \arrow[r] \arrow[d, "\psi^{n-1}_{X/S}"'] & 0 \\ 0 \arrow[r] & \omega^n_{X/S} \arrow[r, "\alpha^n_{X/S}"] & Q^n_{X/S} \arrow[r, "q^n_{X/S}"] & Q^{n-1}_{X/S} \arrow[r] & 0 \end{tikzcd} \tag{2.7}$$

is commutative with exact rows for every $n > 0$.

(2.8) The homomorphism $\psi^n_{X/S}$ induces left and right $\mathcal{O}_X$-algebra structures on $Q^n_{X/S}$ for every $n \geq 0$. By definition, the homomorphism $\alpha^n_{X/S}$ is $\mathcal{O}_X$-linear with respect to both $\mathcal{O}_X$-algebra structures on $Q^n_{X/S}$. Because of the invertibility of $\omega_{X/S}$, the sheaf $Q^n_{X/S}$ is locally free of rank $n+1$ for each of its $\mathcal{O}_X$-algebra structures. Note also that Proposition 2.5 implies that $\psi^n_{X/S}$ is an isomorphism on the smooth locus of X/S for each $n \geq 0$.

We will denote by

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

a Wronski algebra system on X/S. For simplicity we'll sometimes denote a Wronski algebra system on X/S by $(Q^n_{X/S}, n \geq 0)$, leaving the homomorphisms implicit.

DEFINITION 2.9. *Two Wronski algebra systems,*

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0),$$

$$(\tilde{Q}^n_{X/S}, \tilde{\psi}^n_{X/S}, \tilde{q}^n_{X/S}, \tilde{\alpha}^n_{X/S}, n \geq 0)$$

on X/S are equivalent if there are isomorphisms

$$\nu^n_{X/S}\colon Q^n_{X/S} \longrightarrow \tilde{Q}^n_{X/S}$$

for all $n \geq 0$ such that $\tilde{\psi}^n_{X/S} = \nu^n_{X/S} \circ \psi^n_{X/S}$ for $n \geq 0$ and the diagram of maps

$$\begin{tikzcd}
\omega^n_{X/S} \arrow[r, "\alpha^n_{X/S}"] \arrow[d, equal] & Q^n_{X/S} \arrow[r, "q^n_{X/S}"] \arrow[d, "\nu^n_{X/S}"'] & Q^{n-1}_{X/S} \arrow[d, "\nu^{n-1}_{X/S}"'] \\
\omega^n_{X/S} \arrow[r, "\tilde{\alpha}^n_{X/S}"] & \tilde{Q}^n_{X/S} \arrow[r, "\tilde{q}^n_{X/S}"] & \tilde{Q}^{n-1}_{X/S}
\end{tikzcd}$$

commutes for $n > 0$. The systems will be called uniquely equivalent if the $\nu^n_{X/S}$ are unique.

(2.10) Let $h_1\colon S_1 \longrightarrow S$ be any morphism of schemes, and let X_1 denote an open subscheme of $X \times_S S_1$. Let $h\colon X_1 \longrightarrow X$ denote the induced morphism. If

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

is a Wronski algebra system on X/S, then

$$(h^*Q^n_{X/S}, h^*\psi^n_{X/S}, h^*q^n_{X/S}, h^*\alpha^n_{X/S}, n \geq 0)$$

is a Wronski algebra system on X_1/S_1 by the naturality of $p^n_{X/S}$ and $\eta^n_{X/S}$ for $n \geq 1$.

DEFINITION 2.11. *The Wronski algebra system $(h^*Q^n_{X/S}, n \geq 0)$ on X_1/S_1 will be called the* **restriction** *of $(Q^n_{X/S}, n \geq 0)$ to X_1/S_1.*

Let $(Q^n_{X_1/S_1}, n \geq 0)$ be a Wronski algebra system on X_1/S_1.

DEFINITION 2.12. *$(Q^n_{X_1/S_1}, n \geq 0)$ is said to be* **induced** *from the Wronski algebra system $(Q^n_{X/S}, n \geq 0)$ if $(Q^n_{X_1/S_1}, n \geq 0)$ is equivalent to the restriction of $(Q^n_{X/S}, n \geq 0)$ to X_1/S_1.*

(2.13) Let $\mathcal{C}$ be any class consisting of families X/S, whose geometric fibres are reduced, Gorenstein curves, and such that $\mathcal{C}$ is closed under base change and open embeddings. In particular, we can consider the class $\mathcal{C}_{l.c.i.}$ consisting of families whose geometric fibres are reduced, local complete intersection curves.

DEFINITION 2.14. *A* **Wronski algebra system** *on $\mathcal{C}$ consists of a Wronski algebra system*

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

for every family X/S in $\mathcal{C}$ such that if $h\colon S_1 \to S$ is any morphism and X_1 is an open subscheme of $X \times_S S_1$, then $(Q^n_{X_1/S_1}, n \geq 0)$ is induced from $(Q^n_{X/S}, n \geq 0)$.

Denote by $(Q^n, \psi^n, q^n, \alpha^n, n \geq 0)$, or simply by $(Q^n, n \geq 0)$ leaving the homomorphisms implicit, a Wronski algebra system on $\mathcal{C}$.

DEFINITION 2.15. *Two Wronski algebra systems, say $(Q^n, n \geq 0)$ and $(\tilde{Q}^n, n \geq 0)$ on $\mathcal{C}$ are (uniquely) equivalent if for every family X/S in $\mathcal{C}$ the Wronski algebra systems $(Q^n_{X/S}, n \geq 0)$ and $(\tilde{Q}^n_{X/S}, n \geq 0)$ are (uniquely) equivalent.*

The goal of the present article is to show the following theorem.

THEOREM 2.16. *There is a Wronski algebra system on $\mathcal{C}_{l.c.i.}$. Moreover, any two Wronski algebra systems are uniquely equivalent.*

(2.17) Note that it makes perfect sense to talk about a *truncated in order N* Wronski algebra system on a family X/S as being the data

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, 0 \leq n \leq N)$$

satisfying the conditions in Definition 2.6 up to order N. Likewise, all the concepts introduced so far make perfect sense for truncated Wronski algebra systems.

(2.18) As the first step in proving Theorem 2.16, we define the truncated in order 0 Wronski algebra system as

$$(Q^0, \psi^0) := (P^0, \mathrm{id}_{P^0}).$$

We can also easily define the truncated in order 1 Wronski algebra system as the pushout of the infinitesimal $\mathcal{O}_X$-algebra extension (2.1) under $\eta^1_{X/S}$, namely,

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{X/S} \arrow[r] \arrow[d, "\eta^1_{X/S}"'] & P^1_{X/S} \arrow[r, "p^1_{X/S}"] \arrow[d, "\psi^1_{X/S}"'] & P^0_{X/S} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega^1_{X/S} \arrow[r, "\alpha^1_{X/S}"] & Q^1_{X/S} \arrow[r, "q^1_{X/S}"] & Q^0_{X/S} \arrow[r] & 0,
\end{tikzcd}$$

for any family X/S. Because of the categorical nature of the pushout construction, one can easily check that the above data satisfies the conditions in Definition 2.14 and Definition 2.15.

However, the pushout construction will not produce a truncated in higher order Wronski algebra system. The actual proof of Theorem 2.16 will be completed in the next four sections. We will often use in the proof the following trivial lemma and its corollary.

LEMMA 2.19. *If two $\mathcal{O}_X$-linear maps $\beta_1, \beta_2 \colon E \longrightarrow F$, where F is locally free, are equal on the smooth locus of X over S, then they are equal.*

PROOF: The lemma is a trivial consequence of the fact that the associated points of X lie on the smooth locus of X/S, as pointed out in the proof of Proposition 2.5. ∎

COROLLARY 2.20. *If two Wronski algebra systems are equivalent, then they are uniquely equivalent.*

3. A local criterion.

We first describe the local structure of a Wronski algebra system. Let $(Q^n_{X/S}, 0 \leq n < N)$ be a truncated in order $(N-1)$ Wronski algebra system on a family X/S.

(3.1) Assume that X is an affine scheme and $\omega_{X/S}$ is free, generated by τ. Of course $\omega^n_{X/S}$ is free, generated by τ^n for every $n \geq 0$. Pick a global section ζ_{N-1} of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$ under

$$q^2_{X/S} \circ \cdots \circ q^{N-1}_{X/S}.$$

Let ζ_n be the image of ζ_{N-1} in $Q^n_{X/S}$ under

$$q^{n+1}_{X/S} \circ \cdots \circ q^{N-1}_{X/S}$$

for each positive $n < N$.

PROPOSITION 3.2. *The homomorphisms of left $\mathcal{O}_X$-algebras*

$$\mu^n\colon \frac{\mathcal{O}_X[T]}{(T^{n+1})} \longrightarrow Q^n_{X/S}$$

sending T to ζ_n are isomorphisms, making

$$\begin{array}{ccccc}
\mathcal{O}_X & \xrightarrow{\cdot T^n} & \frac{\mathcal{O}_X[T]}{(T^{n+1})} & \xrightarrow{r^n} & \frac{\mathcal{O}_X[T]}{(T^n)} \\
{\scriptstyle \cdot\tau^n}\downarrow & & {\scriptstyle \mu^n}\downarrow & & {\scriptstyle \mu^{n-1}}\downarrow \\
\omega^n_{X/S} & \xrightarrow{\alpha^n_{X/S}} & Q^n_{X/S} & \xrightarrow{q^n_{X/S}} & Q^{n-1}_{X/S},
\end{array}$$

where r^n is the canonical quotient, into a commutative diagram of maps for all $n < N$.

PROOF: We first observe that the above local description is known for the sheaves of principal parts $P^n_{X/S}$ in the case the family X/S is smooth [**21**, 2.4, p. 139]. So it is natural to expect the same description to hold for a good replacement of the sheaves of principal parts.

Let Λ^i_n be the kernel of the composition

$$q^i_{X/S} \circ \cdots \circ q^n_{X/S}\colon Q^n_{X/S} \longrightarrow Q^{i-1}_{X/S}$$

if $i \leq n$ and $\Lambda^i_n = 0$ otherwise. The sheaves Λ^i_n provide a filtration by invertible quotients for $Q^n_{X/S}$,

$$0 = \cdots = \Lambda_n^{n+1} \subset \Lambda_n^n \subset \cdots \subset \Lambda_n^2 \subset \Lambda_n^1 \subset Q_{X/S}^n.$$

In fact, the invertible quotients are powers of $\omega_{X/S}$, the induced map

$$\frac{\Lambda_n^i}{\Lambda_n^{i+1}} \to \omega_{X/S}^i \subset Q_{X/S}^i \tag{3.3}$$

being an isomorphism for every $i \leq n$.

We claim that $\Lambda_n^i \Lambda_n^j = \Lambda_n^{i+j}$. To prove the inclusion "$\subset$" it is enough to show that if $i + j \geq n + 1$, then $\Lambda_n^i \Lambda_n^j = 0$. But the latter equality is true on the smooth locus of X/S, because there the sheaves $Q_{X/S}^n$ are isomorphic to $P_{X/S}^n$, and the proposition holds for the sheaves of principal parts on a smooth family of curves, as observed in the start of the proof. Since the sheaves $Q_{X/S}^n$ are locally free and the smooth locus of X/S contains all the associated points of X, then the equality must be true everywhere. From the inclusion "$\subset$" we obtain a well-defined map, induced by multiplication,

$$m\colon \frac{\Lambda_n^i}{\Lambda_n^{i+1}} \otimes \frac{\Lambda_n^j}{\Lambda_n^{j+1}} \longrightarrow \frac{\Lambda_n^{i+j}}{\Lambda_n^{i+j+1}}. \tag{3.4}$$

By combining (3.3) and (3.4) we obtain the following diagram of maps

$$\begin{tikzcd} \frac{\Lambda_n^i}{\Lambda_n^{i+1}} \otimes \frac{\Lambda_n^j}{\Lambda_n^{j+1}} \arrow[r, "m"] \arrow[d] & \frac{\Lambda_n^{i+j}}{\Lambda_n^{i+j+1}} \arrow[d] \\ \omega_{X/S}^i \otimes \omega_{X/S}^j \arrow[r, equal] & \omega_{X/S}^{i+j}, \end{tikzcd} \tag{3.5}$$

which is commutative, since (3.5) is commutative on the smooth locus of X/S and $\omega_{X/S}$ is invertible. In particular, m is an isomorphism, implying that $\Lambda_n^i \Lambda_n^j = \Lambda_n^{i+j}$ as we wished to prove.

From the claim we obtain that $\Lambda_n^i = (\Lambda_n^1)^i$ for every $i > 0$, which shows that ζ_n^i generates the free rank 1 module

$$\frac{\Lambda_n^i(X)}{\Lambda_n^{i+1}(X)}.$$

Consequently we obtain an isomorphism

$$\mu^n\colon \frac{\mathcal{O}_X[T]}{(T^{n+1})} \to Q_{X/S}^n$$

for every n by sending T to ζ_n. The compatibility in the choice of the ζ_n assures us that $q^n_{X/S}$ is thereby identified with the canonical quotient map

$$r^n: \frac{\mathcal{O}_X[T]}{(T^{n+1})} \longrightarrow \frac{\mathcal{O}_X[T]}{(T^n)}$$

for every $n < N$. Moreover, from (3.5) the diagram of isomorphisms

$$\begin{tikzcd}
(\frac{\Lambda^1_n}{\Lambda^2_n})^n \arrow[r, "m"] \arrow[d] & \Lambda^n_n \arrow[d, "(\alpha^n_{X/S})^{-1}"] \\
\omega^n_{X/S} \arrow[r, equal] & \omega^n_{X/S}
\end{tikzcd}$$

commutes. Hence

$$\alpha^n_{X/S}(\tau^n) = \zeta^n_n = \mu^n(T^n)$$

for every $n < N$. The proof of the proposition is complete. ∎

(3.6) We are going to give a criterion for the local existence of a truncated in order N Wronski algebra system extending $(Q^n_{X/S}, 0 \leq n < N)$. To this purpose we can assume that S and X are affine, and X is a closed subscheme of an S-smooth affine scheme Y of pure relative dimension m over S. Let J be the ideal sheaf of X in Y. Let $f_1, \ldots, f_t$ be regular functions on Y generating J globally. As a convention, we will denote by $\bar{b}$ the restriction to X of a regular function b on Y. We will also assume that there are regular functions $u_1, \ldots, u_m$ on Y such that $du_1, \ldots, du_m$ form a basis for $\Omega^1_{Y/S}$. In particular, their respective restrictions $\bar{u}_1, \ldots, \bar{u}_m$ to X are such that $d\bar{u}_1, \ldots, d\bar{u}_m$ generate $\Omega^1_{X/S}$. For convenience, we let $v_i := \bar{u}_i$ for each $i = 1, \ldots, m$. In addition, assume that $\omega_{X/S}$ is free, generated by τ, and pick a global section ζ of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$. For convenience, we will use the same notation ζ for its images in $Q^n_{X/S}$ for $0 < n < N$.

If c is a section of $\mathcal{O}_Y$ (resp. of $\mathcal{O}_X$), then we will denote again by c its image in $P^n_{Y/S}$ (resp. $P^n_{X/S}$) under the left $\mathcal{O}_Y$-algebra (resp. $\mathcal{O}_X$-algebra) structure of $P^n_{Y/S}$ (resp. $P^n_{X/S}$), and by $\tilde{c}$ its image in $P^n_{Y/S}$ (resp. $P^n_{X/S}$) under the right algebra structure. Note the abuse of notation we commit in not distinguishing between the several sheaves of principal parts.

Let $a_{i,j}$ be regular functions on X defined by

$$\psi^n_{X/S}(\tilde{v}_j) := v_j + a_{1,j}\zeta + \cdots + a_{n,j}\zeta^n$$

for every $n < N$ and each $j = 1, \ldots, m$. Note that

$$\eta^1_{X/S}(dv_j) = a_{1,j}\tau$$

for every $j = 1, \ldots, m$.

Inspired by Proposition 3.2, we want a criterion for the existence of a left $\mathcal{O}_X$-algebra homomorphism

$$\psi^N_{X/S}\colon P^N_{X/S} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^N_{X/S} \arrow[r] \arrow[d, "\eta^N_{X/S}"'] & P^N_{X/S} \arrow[r, "p^N_{X/S}"] \arrow[d, "\psi^N_{X/S}"'] & P^{N-1}_{X/S} \arrow[r] \arrow[d, "\psi^{N-1}_{X/S}"'] & 0 \\
0 \arrow[r] & \omega^N_{X/S} \arrow[r, "\alpha^N_{X/S}"] & \frac{\mathcal{O}_X[T]}{(T^{N+1})} \arrow[r, "q^N_{X/S}"] & Q^{N-1}_{X/S} \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

commutative, where $q^N_{X/S}(T) = \zeta$ and $\alpha^N_{X/S}(\tau^N) = T^N$.

For each $l \geq 0$, let

$$\Gamma_l := \{\gamma := (\gamma_1, \ldots, \gamma_m); \gamma_i \in \mathbf{Z}_{\geq 0} \text{ and } \gamma_1 + \cdots + \gamma_m = l\}.$$

Let

$$\mathcal{O}_Y[Z_1, \ldots, Z_m] \to P^N_{Y/S} \tag{3.8}$$

be the left $\mathcal{O}_Y$-algebra homomorphism sending the variable Z_i to $\tilde{u}_i$ for every $i = 1, \ldots, m$. The map (3.8) is surjective with kernel generated by

$$(Z_1 - u_1)^{\gamma_1} \ldots (Z_m - u_m)^{\gamma_m}$$

for all $\gamma \in \Gamma_{N+1}$. On the other hand, the kernel of the surjective map $P^N_{Y/S} \to P^N_{X/S}$ (induced by the quotient map $\mathcal{O}_Y \to \mathcal{O}_X$) is generated by f_k and $\tilde{f}_k$ for all $k = 1, \ldots, t$. The element $\tilde{f}_k$ can be expressed as

$$\tilde{f}_k = f_k + \sum_{i=1}^{m} D_i f_k(\tilde{u}_i - u_i) + \cdots +$$

$$+\sum_{\gamma\in\Gamma_N} D_1^{\gamma_1}\dots D_m^{\gamma_m} f_k(\tilde{u}_1-u_1)^{\gamma_1}\dots(\tilde{u}_m-u_m)^{\gamma_m}$$

in $P^N_{Y/S}$ for every $k=1,\dots,t$, where D_i^l is the Hasse derivation on Y/S of order l associated to u_i.

To construct a left $\mathcal{O}_X$-algebra homomorphism

$$\psi^N_{X/S}\colon P^N_{X/S}\to\frac{\mathcal{O}_X[T]}{(T^{N+1})},$$

we need only construct a left $\mathcal{O}_Y$-algebra homomorphism

$$\phi^N\colon \mathcal{O}_Y[Z_1,\dots,Z_m]\to\frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

factoring through $P^N_{X/S}$. Since

$$\begin{array}{ccc}
P^N_{X/S} & \xrightarrow{p^N_{X/S}} & P^{N-1}_{X/S} \\
\psi^N_{X/S}\big\downarrow & & \psi^{N-1}_{X/S}\big\downarrow \\
\frac{\mathcal{O}_X[T]}{(T^{N+1})} & \xrightarrow{q^N_{X/S}} & Q^{N-1}_{X/S}
\end{array}$$

must be commutative, one must have

$$\phi^N(Z_i)=v_i+a_{1,i}T+\dots+a_{N-1,i}T^{N-1}-c_iT^N \tag{3.9}$$

for some regular function c_i on X for every $i=1,\dots,m$. As a consequence of (3.9), the commutativity of

$$\begin{array}{ccc}
\Omega^N_{X/S} & \longrightarrow & P^N_{X/S} \\
\eta^N_{X/S}\big\downarrow & & \psi^N_{X/S}\big\downarrow \\
\omega^N_{X/S} & \xrightarrow{\alpha^N_{X/S}} & \frac{\mathcal{O}_X[T]}{(T^{N+1})}
\end{array}$$

is guaranteed for any choice of c_i, because $\Omega^N_{X/S}$ is generated as a submodule of $P^N_{X/S}$ by

$$(\tilde{v}_1-v_1)^{\gamma_1}\dots(\tilde{v}_m-v_m)^{\gamma_m}$$

for all $\gamma\in\Gamma_N$, and

$$\alpha^N_{X/S}\circ\eta^N_{X/S}((\tilde{v}_1-v_1)^{\gamma_1}\dots(\tilde{v}_m-v_m)^{\gamma_m})=\alpha^N_{X/S}(\prod_{j=1}^m\eta^1_{X/S}(dv_j)^{\gamma_j})=$$

$$= \prod_{j=1}^{m} a_{1,j}^{\gamma_j} T^N = \phi^N((Z_1 - u_1)^{\gamma_1} \ldots (Z_m - u_m)^{\gamma_m}),$$

as it can easily be seen from (3.9) and the construction of $\eta^N_{X/S}$ via the multiplication map $m\colon (\Omega^1_{X/S})^{\otimes N} \to \Omega^N_{X/S}$, carried out in the proof of Proposition 2.5.

In order that ϕ^N, as defined by (3.9), factor through $P^N_{X/S}$ it is necessary and sufficient that

$$f_k^N(v_1 + a_{1,1}T + \cdots - c_1 T^N, \ldots, v_m + a_{1,m}T + \cdots - c_m T^N) \tag{3.10}$$

be divisible by T^{N+1} for all $k = 1, \ldots, t$, where

$$f_k^N(Z_1, \ldots, Z_m) := \sum_{i=1}^{m} (D_i f_k)^-(Z_i - v_i) + \cdots +$$

$$+ \sum_{\gamma \in \Gamma_N} (D_1^{\gamma_1} \ldots D_m^{\gamma_m} f_k)^-(Z_1 - v_1)^{\gamma_1} \ldots (Z_m - v_m)^{\gamma_m} \tag{3.11}$$

for each $k = 1, \ldots, t$. Note that (3.10) is already divisible by T^N since $\psi^{N-1}_{X/S}$ is defined. Hence, one may define the following regular function on X:

$$d_k^N := \left. \frac{f_k^N(\ldots, v_i + \cdots + a_{N-1,i}T^{N-1}, \ldots)}{T^N} \right|_{T=0} \tag{3.12}$$

for every $k = 1, \ldots, t$. Let also

$$M := \begin{bmatrix} D_1 f_1 & D_2 f_1 & \ldots & D_m f_1 \\ D_1 f_2 & D_2 f_2 & \ldots & D_m f_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_t & D_2 f_t & \ldots & D_m f_t \end{bmatrix}.$$

Then we have the following criterion.

CRITERION 3.13. *There exists a homomorphism $\psi^N_{X/S}$ making diagram (3.7) commutative if and only if the linear system*

$$\bar{M} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} d_1^N \\ \vdots \\ d_t^N \end{bmatrix}$$

is solvable by regular functions $c_1, \dots, c_m$ *on* X.

PROOF: By combining (3.11) and (3.12) we obtain that

$$f_k^N(\dots, v_i + a_{1,i}T + \cdots + a_{N-1,i}T^{N-1} - c_iT^N, \dots) =$$
$$= (d_k^N - \sum_{i=1}^{m}(D_i f_k)^- c_i)T^N + \dots$$

for every $k = 1, \dots, t$. The criterion follows immediately. ∎

4. The Wronski algebra system for "general" families.

Continuing the set-up of last section, S is an affine scheme, Y is an affine S-smooth scheme whose sheaf of differentials $\Omega^1_{Y/S}$ is free of rank m with basis $du_1, \ldots, du_m$, where $u_1, \ldots, u_m$ are regular functions on Y, and $X \subset Y$ is the closed subscheme defined by regular functions $f_1, \ldots, f_t$ on Y. Assume now that $t = m-1$, and $f_1, \ldots, f_{m-1}$ form a regular sequence on Y relative to S. In other words, we assume that for every fibre of Y/S the restrictions of $f_1, \ldots, f_{m-1}$ to the fibre form a regular sequence. As a consequence of the above assumptions, the sheaf $\omega_{X/S}$ is free, generated by $du_1 \wedge \cdots \wedge du_m \otimes \sigma$, where σ is the dual to $f_1 \wedge \cdots \wedge f_{m-1}$, as it can be immediately seen from (2.3). Hence, we can assume that $\tau = du_1 \wedge \cdots \wedge du_m \otimes \sigma$.

DEFINITION 4.1. *The data*

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

is called a **local data**.

(4.2) Let M_j denote the maximal minor of M obtained by deleting the j-th column of M. Let

$$\delta_j := (-1)^{j+1} \det M_j \quad \text{for} \quad j = 1, \ldots, m.$$

From the construction of $\eta^1_{X/S}$ shown in (2.2) and the above choice of τ it is easy to see that $\bar{\delta}_j = a_{1,j}$ for $j = 1, \ldots, m$. Let

$$\Delta := (\bar{\delta}_1, \ldots, \bar{\delta}_m)\mathcal{O}_X.$$

As already remarked, $\Delta\tau$ is the image of $\eta^1_{X/S}$ in $\omega_{X/S}$; hence $\Delta\tau$ is an intrinsic object.

LEMMA 4.3. *If depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, then there is a homomorphism $\psi^N_{X/S}$ making diagram (3.7) commutative. Moreover, $\psi^N_{X/S}$ is unique in the following sense: if*

$$\theta^N : P^N_{X/S} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

is another homomorphism making diagram (3.7) commutative, then there is a unique isomorphism

$$\lambda^N : \frac{\mathcal{O}_X[T]}{(T^{N+1})} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})},$$

with $\lambda^N(T^N) = T^N$ *and* $q^N_{X/S} \circ \lambda^N = q^N_{X/S}$, *such that* $\lambda^N \circ \psi^N_{X/S} = \theta^N$.

PROOF: We first claim that $a_{i,j} \in \Delta(X)$ for all i, j. The proof will be by induction. We have already remarked that $a_{1,j} \in \Delta(X)$. Assume that $a_{i,j} \in \Delta(X)$ for $i < s$, where $2 \leq s < N$. For each $k = 1, \dots, m-1$, one has

$$f^N_k(\dots, v_j + a_{1,j}T + \dots + a_{N-1,j}T^{N-1}, \dots) = (\sum_{j=1}^{m}(D_j f_k)^- a_{1,j})T + \dots +$$

$$+(\sum_{l=1}^{r}\sum_{\gamma\in\Gamma_l}(D_1^{\gamma_1}\dots D_m^{\gamma_m} f_k)^- \sum_{\substack{|\alpha_j|=\gamma_j \\ s(\alpha)=r}} \prod_{j=1}^{m}\binom{\gamma_j}{\alpha_j}\prod_{i=1}^{N-1} a_{i,j}^{\alpha_{i,j}})T^r + \dots, \quad (4.4)$$

where

$$\begin{aligned}
\alpha :=& (\alpha_{i,j})_{\substack{1\leq i<N \\ 1\leq j\leq m}};\ \alpha_{i,j} \in \mathbf{Z}_{\geq 0},\\
\alpha_j :=& (\alpha_{1,j}, \dots, \alpha_{N-1,j}),\\
|\alpha_j| :=& \alpha_{1,j} + \dots + \alpha_{N-1,j},\\
s(\alpha) :=& \sum_{i,j} i\alpha_{i,j},\\
\binom{\gamma_j}{\alpha_j} =& \frac{\gamma_j!}{\prod_{i=1}^{N-1}\alpha_{i,j}!}.
\end{aligned}$$

Since T^N divides (4.4), the coefficient of T^s in the expression (4.4) must be 0. But this coefficient may be expressed as

$$\sum_{j=1}^{m}(D_j f_k)^- a_{s,j} + E_{k,s},$$

where $E_{k,s}$ does not involve any $a_{i,j}$ with $i \geq s$. Hence, from the induction hypothesis we obtain that $E_{k,s} \in \Delta^2(X)$, and therefore

$$\sum_{j=1}^{m}(D_j f_k)^- a_{s,j} \in \Delta^2(X)$$

for every $k = 1, \dots, m-1$.

Since depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, by the Buchsbaum-Eisenbud criterion [**9**, Theorem, p. 260] the complex

$$0 \longrightarrow \mathcal{O}_X^{\oplus m-1} \xrightarrow{\bar{M}^t} \mathcal{O}_X^{\oplus m} \xrightarrow{(\bar{\delta}_1, \dots, \bar{\delta}_m)} \mathcal{O}_X \longrightarrow \frac{\mathcal{O}_X}{\Delta} \longrightarrow 0 \qquad (4.5)$$

is exact. By applying $\underline{\mathrm{Hom}}(\,\cdot\,, \mathcal{O}_X)$ to (4.5), we obtain the sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{(\bar{\delta}_1, \dots, \bar{\delta}_m)} \mathcal{O}_X^{\oplus m} \xrightarrow{\bar{M}} \mathcal{O}_X^{\oplus m-1} \longrightarrow \underline{\mathrm{Ext}}^2_{\mathcal{O}_X}(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X) \longrightarrow 0, \quad (4.6)$$

which is exact since

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X) = \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X) = 0$$

by the condition on the depth [**26**, 16.7, p. 130]. Since the $\mathcal{O}_X$-module $\underline{\mathrm{Ext}}^2_{\mathcal{O}_X}(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X)$ is annihilated by Δ, then the image of $\bar{M}$ contains $\Delta^{\oplus m-1}$. Since the kernel of $\bar{M}$ lies inside $\Delta^{\oplus m}$, then any section of $\mathcal{O}_X^{\oplus m}$ mapping into $(\Delta^2)^{\oplus m-1}$ under $\bar{M}$ must be in $\Delta^{\oplus m}$. Hence

$$a_{s,1}, \dots, a_{s,m} \in \Delta(X),$$

finishing the proof of the claim.

It is easy to see from (4.4) and the claim just proved that $d_k^N \in \Delta^2(X)$ for every $k = 1, \dots, m-1$. Since the image of $\bar{M}$ contains $\Delta^{\oplus m-1}$, one can find $c_1, \dots, c_m \in \Delta(X)$ such that

$$\sum_{j=1}^{m} (D_j f_k)^{-} c_j = d_k^N \quad \text{for} \quad k = 1, \dots, m-1.$$

By Criterion 3.13, the existence of $\psi^N_{X/S}$ follows.

As for uniqueness, from sequence (4.6) any two solutions $(c_1, \dots, c_m)$ and $(c'_1, \dots, c'_m)$ to the linear system of Criterion 3.13, corresponding to homomorphisms $\psi^N_{X/S}$ and θ^N, respectively, differ by a multiple of $(\bar{\delta}_1, \dots, \bar{\delta}_m)$, say

$$(c'_1, \dots, c'_m) = (c_1, \dots, c_m) + e(\bar{\delta}_1, \dots, \bar{\delta}_m),$$

where e is a regular function on X. If we let

$$\lambda^N : \frac{\mathcal{O}_X[T]}{(T^{N+1})} \longrightarrow \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

be the $\mathcal{O}_X$-algebra homomorphism defined by $\lambda^N(T) := T - eT^N$, then

$$\lambda^N(T^N) = T^N,$$
$$q^N_{X/S} \circ \lambda^N(T) = q^N_{X/S}(T - eT^N) = \zeta = q^N_{X/S}(T),$$

and

$$\begin{aligned}\lambda^N \circ \psi^N_{X/S}(\tilde{v}_i) &= \lambda^N(v_i + a_{1,i}T + \cdots + a_{N-1,i}T^{N-1} - c_iT^N)\\ &= v_i + a_{1,i}T + \cdots + a_{N-1,i}T^{N-1} - (c_i + ea_{1,i})T^N\\ &= v_i + a_{1,i}T + \cdots + a_{N-1,i}T^{N-1} - c_i'T^N\\ &= \theta^N(\tilde{v}_i)\end{aligned}$$

for $i = 1, \ldots, m$. In addition, $\lambda^N_{X/S}$ is unique by Lemma 2.19 (alternatively, because the homomorphism $(\bar{\delta}_1, \ldots, \bar{\delta}_m)$ is injective). The proof of the lemma is complete. ∎

COROLLARY 4.7. *If depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, then there is a Wronski algebra system on X/S. Moreover, any two Wronski algebra systems are uniquely equivalent.*

PROOF: Assume by induction that there is a unique truncated in order $N-1$ Wronski algebra system $(Q^n_{X/S}, 0 \leq n < N)$ on X/S. By Lemma 4.3 there is a truncated in order N Wronski algebra system extending $(Q^n_{X/S}, 0 \leq n < N)$. The first statement of the corollary follows by induction. As for the second statement, we can assume by induction hypothesis that any two Wronski algebra systems on X/S agree up to order $N-1$, say with $(Q^n_{X/S}, 0 \leq n < N)$. Moreover, since X is affine and $\omega_{X/S}$ is free, by Proposition 3.2 any extension $(Q^N_{X/S}, \psi^N_{X/S}, q^N_{X/S}, \alpha^N_{X/S})$ of $(Q^n_{X/S}, 0 \leq n < N)$ is equivalent to an extension of the form

$$\left(\frac{\mathcal{O}_X[T]}{(T^{N+1})}, \psi^N_{X/S}, q^N_{X/S}, \alpha^N_{X/S}\right),$$

where $q^N_{X/S}(T) = \zeta$ and $\alpha^N_{X/S}(\tau^N) = T^N$, with $\tau := du_1 \wedge \cdots \wedge du_m \otimes \sigma$, and ζ a chosen global section of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$. Therefore, by Lemma 4.3 any two extensions are equivalent. The uniqueness of the equivalence follows from Corollary 2.20. ∎

(4.8) Let $f: X \longrightarrow S$ denote a flat morphism whose fibres are reduced, local complete intersection curves.

DEFINITION 4.9. *X/S is said to satisfy* **the depth condition** *if*

$$\underline{Ext}^i_{\mathcal{O}_X}(Coker\ \eta^1_{X/S}, \mathcal{O}_X) = 0$$

for $i = 0, 1$.

PROPOSITION 4.10. *If X/S satisfies the depth condition, then there is a Wronski algebra system on X/S. Moreover, any two Wronski algebra systems on X/S are uniquely equivalent.*

PROOF: One can cover S with affine open subschemes S_λ and $X \times_S S_\lambda$ with affine open subschemes X_μ in such a way that there is a closed S_λ-embedding $\iota_\mu \colon X_\mu \longrightarrow Y_\mu$ into an affine S_λ-smooth scheme Y_μ of pure relative dimension m_μ over S_λ satisfying the following properties:

(1) $\Omega^1_{Y_\mu/S_\lambda}$ is free of rank m_μ;
(2) The sheaf of ideals of X_μ in Y_μ is generated by a regular sequence of length $m_\mu - 1$ on Y_μ relative to S_λ.

Since X/S satisfies the depth condition, so does X_μ/S_λ. By Corollary 4.7, there is a Wronski algebra system $(Q^n_{X_\mu/S_\lambda}, n \geq 0)$ on X_μ/S_λ for every λ and μ. Moreover, by the second statement of Corollary 4.7 one can patch the local Wronski algebra systems $(Q^n_{X_\mu/S_\lambda}, n \geq 0)$ to obtain a global Wronski algebra system on X/S. The first statement of the proposition has been proved. To prove the second statement, one needs only notice that the second statement of Corollary 4.7 shows that any two Wronski algebra systems on X/S are uniquely equivalent locally. The uniqueness allows us to patch the local equivalences together to obtain a global one. The proof is complete. ∎

(4.11) By proving Theorem 2.16 we will show that the depth condition is not necessary for the existence of a Wronski algebra system. Hence the depth condition is not necessary for the existence statement in Lemma 4.3. But the condition is necessary for uniqueness. In fact, consider the affine planar curve $X \subset \mathbf{A}^2_k$ defined by $f := y^3 - x^4$, where k is an algebraically closed field of characteristic different from 2. The idea for this example comes from [**12**, 2.1, p. 6], but the next development is independent. We have

$$a_{1,1} = (D_y f)^- = 3y^2 \quad \text{and} \quad a_{1,2} = -(D_x f)^- = 4x^3$$

on X. Hence

$$d^2 = \left.\frac{(y + 4x^3T)^3 - (x + 3y^2T)^4}{T^2}\right|_{T=0} = -6x^2y^4 = -6x^6y$$

on X. Since

$$[\,-4x^3 \quad 3y^2\,]\begin{bmatrix} 0 \\ -2x^2y^2 \end{bmatrix} = -6x^2y^4,$$

then by Criterion 3.13 there is a homomorphism ψ^2 making diagram (3.7) commute, namely,

$$\psi^2(\tilde{x}) := x + 3y^2T \quad \text{and} \quad \psi^2(\tilde{y}) := y + 4x^3T + 2x^2y^2T^2.$$

However, the homomorphism

$$\theta^2 \colon P_X^2 \longrightarrow \frac{\mathcal{O}_X[T]}{(T^3)}$$

given by

$$\theta^2(\tilde{x}) := x + 3y^2T - \frac{3}{2}x^3yT^2 \quad \text{and} \quad \theta^2(\tilde{y}) := y + 4x^3T$$

is well-defined and also makes diagram (3.7) commute, since

$$[\,-4x^3 \quad 3y^2\,]\begin{bmatrix} \frac{3}{2}x^3y \\ 0 \end{bmatrix} = -6x^6y$$

on X. If λ^2 is an automorphism of $\frac{\mathcal{O}_X[T]}{(T^3)}$ such that $\lambda^2(T^2) = T^2$, then $\lambda^2(T) = T + eT^2$ for a regular function e on X. If $\lambda^2 \circ \psi^2 = \theta^2$, then

$$x + 3y^2T + 3y^2eT^2 = \lambda^2 \circ \psi^2(\tilde{x}) = \theta^2(\tilde{x}) = x + 3y^2T - \frac{3}{2}x^3yT^2.$$

But there is no polynomial $p(x,y)$ such that $3y^2p(x,y) + \frac{3}{2}x^3y$ is divisible by $y^3 - x^4$. Hence, the truncated in order 2 Wronski algebra systems defined by ψ^2 and θ^2 are not equivalent.

5. Some lemmas.

Of course the hypothesis on the depth cannot be satisfied if the family X/S consists simply of a curve over a field. The idea to overcome this problem is to "enlarge" locally the family X/S in such a way that the larger family satisfies the depth condition, apply Proposition 4.10, and then restrict the Wronski algebra system in the larger family to the family X/S. One must also take care that two different "enlargements" do not yield two different Wronski algebra systems, if we are to glue the local systems together. The purpose of this section is to build enough tools to handle the above process.

(5.1) Assume there is a commutative diagram of morphisms of schemes,

$$\begin{array}{ccc} S_1 & \xrightarrow{h_2} & S_3 \\ {\scriptstyle h_0}\uparrow & & \uparrow{\scriptstyle h_3} \\ S_0 & \xrightarrow{h_1} & S_2. \end{array} \tag{5.2}$$

Let X_3 be a flat scheme over S_3 whose geometric fibres are reduced, local complete intersection curves. Let $X_i \subset X \times_{S_3} S_i$ be any open subscheme for $i = 1, 2$. Let X_0 be an open subscheme of

$$X_1 \times_{S_1} S_0 \cap X_2 \times_{S_2} S_0.$$

LEMMA 5.3. *Let $(Q^n_{X_i/S_i}, n \geq 0)$ be a Wronski algebra system on X_i/S_i for $i = 2, 3$. Let $(Q^n_{X_i/S_i}, n \geq 0)$ be a Wronski algebra system on X_i/S_i induced from $(Q^n_{X_{i+2}/S_{i+2}}, n \geq 0)$ for $i = 0, 1$. If X_2/S_2 satisfies the depth condition, then $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_1/S_1}, n \geq 0)$.*

PROOF: Since X_2/S_2 satisfies the depth condition, then by Proposition 4.10 the system $(Q^n_{X_2/S_2}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$. Hence $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$ via $h_3 \circ h_1$. Since $h_2 \circ h_0 = h_3 \circ h_1$ and $(Q^n_{X_1/S_1}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$, then $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_1/S_1}, n \geq 0)$. The proof is complete.

(5.4) We now return to the local case, that is, assume that

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

is a local data. We will provide an explicit "enlargement" for X/S satisfying the depth condition. Let $S' := S \times \operatorname{Spec} \mathbf{Z}[T_{i,j}]$, where $\mathbf{Z}[T_{i,j}]$ is the polynomial ring over $\mathbf{Z}$ in the variables $T_{i,j}$, with $1 \leq i \leq m-1$

and $1 \leq j \leq m+1$. Let $Y' := Y \times_S S'$. We will commit an abuse of notation in not distinguishing between a regular function on Y and its pull-back to Y'. Then Y' is smooth over S', and $\Omega^1_{Y'/S'}$ is free of rank m, with basis $du_1, \ldots, du_m$. Let $Z' \subset Y'$ be the closed subscheme whose sheaf of ideals J' is generated by the regular functions

$$f'_k := f_k + \sum_{j=1}^{m} u_j T_{k,j} + T_{k,m+1} \quad \text{for} \quad k = 1, \ldots, m-1$$

on Y'. If we let $h_S \colon S \longrightarrow S'$ be the closed embedding obtained by making $T_{i,j} = 0$ for all i, j, then it is clear that

$$Y = Y' \times_{S'} S \quad \text{and} \quad X = Z' \times_{S'} S.$$

Let $U' \subset Y'$ be the open subscheme of Y' where $f'_1, \ldots, f'_{m-1}$ is a regular sequence relative to S' [**EGA IV-3**, 11.1.4, p. 118]. Since $f_1, \ldots . f_{m-1}$ is a regular sequence on Y relative to S, then $Y = U' \times_{S'} S$. Let $X' \subset Z' \cap U'$ be the open subscheme of points which are reduced in their geometric fibres [**EGA IV-3**, 12.1.1, p. 174]. Since the geometric fibres of X/S are reduced, then $X = X' \times_{S'} S$. It is clear that the embedding $X' \subset U'$ is transversally regular relative to S'. Since U' is smooth over S', then X'/S' is a flat family whose geometric fibres are reduced, local complete intersection curves.

Note that

$$Z' \cong Y \times \operatorname{Spec} \mathbf{Z}[T_{i,j}]_{\substack{1 \leq i \leq m-1 \\ 1 \leq j \leq m}} \tag{5.5}$$

and

$$D_j f'_k = D_j f_k + T_{k,j} \tag{5.6}$$

for every $k = 1, \ldots, m-1$ and $j = 1, \ldots, m$. Let

$$M' := \begin{bmatrix} D_1 f'_1 & D_2 f'_1 & \ldots & D_m f'_1 \\ D_1 f'_2 & D_2 f'_2 & \ldots & D_m f'_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f'_{m-1} & D_2 f'_{m-1} & \ldots & D_m f'_{m-1} \end{bmatrix}$$

and $\delta'_j := \det M'_j$, where M'_j is the maximal minor of M' obtained by deleting the j-th column for every $j = 1, \ldots, m$. Let

$$\Delta' := (\bar{\delta}'_1, \ldots, \bar{\delta}'_m)\mathcal{O}_{Z'}.$$

Because of (5.5) and (5.6), the matrix M' is "generic." As a consequence,

$$\operatorname{depth}\,(\Delta'(Z'), \mathcal{O}_{Z'}(Z')) = 2,$$

or equivalently

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{Z'}}(\frac{\mathcal{O}_{Z'}}{\Delta'}, \mathcal{O}_{Z'}) = 0$$

for $i = 0, 1$. Hence, X'/S' satisfies the depth condition. By Proposition 4.10, there is a Wronski algebra system $(Q^n_{X'/S'}, n \geq 0)$ on X'/S'. If we consider its restriction to X/S, then we have the following proposition.

PROPOSITION 5.7. *There is a Wronski algebra system* $(Q^n_{X/S}, n \geq 0)$ *on* X/S.

Since we will often refer to the above construction we make the following definition.

DEFINITION 5.8. *The data*

$$(S', Y', Z', X', f'_1, \ldots, f'_{m-1})$$

will be called the **enlargement** *of the local data*

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1}).$$

(5.9) The above construction is functorial in the following sense. Let

$$(S_1, Y_1, X_1, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

be a local data. Let $h\colon S_2 \longrightarrow S_1$ be any morphism of affine schemes, and let $Y_2 \subset Y_1 \times_{S_1} S_2$ be an affine open subscheme. Let $X_2 := Y_2 \cap X_1 \times_{S_1} S_2$. We will commit an abuse of notation in not distinguishing between a regular function on Y_1 and its pull-back to Y_2. The sheaf $\Omega^1_{Y_2/S_2}$ is free, with basis $du_1, \ldots, du_m$, and the ideal sheaf of X_2 in Y_2 is generated by $f_1, \ldots, f_{m-1}$. Let

$$(S'_i, Y'_i, Z'_i, X'_i, f'_1, \ldots, f'_{m-1})$$

be the enlargement of the local data

$$(S_i, Y_i, X_i, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

for $i = 1, 2$.

It is easy to see from construction (5.4) that h lifts to a morphism h' making the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{h_{S_1}} & S'_1 \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h'} \\ S_2 & \xrightarrow{h_{S_2}} & S'_2 \end{array}$$

Cartesian, Y'_2 is an affine open subscheme of $Y'_1 \times_{S'_1} S'_2$, and

$$Z'_2 = Y'_2 \cap Z'_1 \times_{S'_1} S'_2.$$

In addition, we have

$$X'_2 = Y'_2 \cap X'_1 \times_{S'_1} S'_2.$$

Let $(Q^n_{X'_i/S'_i}, n \geq 0)$ be a Wronski algebra system on X'_i/S'_i for $i = 1, 2$. Let $(Q^n_{X_i/S_i}, n \geq 0)$ be a Wronski algebra system on X_i/S_i induced from $(Q^n_{X'_i/S'_i}, n \geq 0)$ for $i = 1, 2$. Since X'_2/S'_2 satisfies the depth condition, by Lemma 5.3 one has the following lemma.

LEMMA 5.10. *The Wronski algebra system* $(Q^n_{X_2/S_2}, n \geq 0)$ *is induced from* $(Q^n_{X_1/S_1}, n \geq 0)$.

(5.11) We now remark that two "enlargements" of X/S can be obtained by restricting from a bigger "enlargement." Let $Y_0 \to S_0$ be any morphism of finite type of affine schemes. Let S_1 and S_2 be affine spaces over S_0 together with sections

$$S_0 \to S_1 \quad \text{and} \quad S_0 \to S_2.$$

Let $Y_i := Y_0 \times_{S_0} S_i$ for $i = 1, 2$. Let $f^i_1, \ldots, f^i_t$ be a sequence of regular functions on Y_i for $i = 1, 2$, such that the sequences restrict to the same sequence $f^0_1, \ldots, f^0_t$ on Y_0.

LEMMA 5.12. *There are an affine space* S_3 *over* $S_1 \times_{S_0} S_2$ *together with sections*

$$S_1 \to S_3 \quad \textit{and} \quad S_2 \to S_3$$

making the diagram of maps

$$\begin{array}{ccc} S_1 & \longrightarrow & S_3 \\ \uparrow & & \uparrow \\ S_0 & \longrightarrow & S_2 \end{array} \tag{5.13}$$

commutative, and regular functions $f_1^3, \dots, f_t^3$ *on* $Y_3 := Y_0 \times_{S_0} S_3$ *restricting to* $f_1^i, \dots, f_t^i$ *on* Y_i *for* $i = 1, 2$.

PROOF: Since Y_0 is of finite type over S_0, then Y_0 can be viewed as a closed subscheme of $\mathbf{A}^k_{S_0}$ for some k, where $\mathbf{A}^k_{S_0}$ denotes the affine space of dimension k over S_0. Hence $f_1^0, \dots, f_t^0$ are restrictions to Y_0 of polynomials $p_1, \dots, p_t$ in k variables with regular functions on S_0 as coefficients. One can obtain "general extensions" of the polynomials $p_1, \dots, p_t$ by adding an independent variable T_λ to each coefficient of each polynomial p_i. Let $f_1^3, \dots, f_t^3$ be the restrictions to $Y_3 := Y_0 \times_{S_0} S_3$, where $S_3 := S_1 \times_{S_0} S_2 \times \operatorname{Spec} \mathbf{Z}[T_\lambda]$, of the "general extensions" of $p_1, \dots, p_t$. By specializing appropriately the variables T_λ for $i = 1, 2$, the sequence $f_1^3, \dots, f_t^3$ specializes to the sequence $f_1^i, \dots, f_t^i$. The proof is complete. ∎

(5.14) Let Z_i be the closed subscheme of Y_i defined by $f_1^i, \dots, f_t^i$ for $i = 0, 1, 2, 3$. Of course,

$$\begin{tikzcd} Z_1 \arrow[r] & Z_3 \\ Z_0 \arrow[r] \arrow[u] & Z_2 \arrow[u] \end{tikzcd}$$

is a commutative diagram which is Cartesian over (5.13), that is,

$$\begin{tikzcd} Z_i \arrow[r] \arrow[d] & Z_j \arrow[d] \\ S_i \arrow[r] & S_j \end{tikzcd}$$

is Cartesian whenever defined. Let U_i denote the open subscheme of Y_i where $f_1^i, \dots, f_t^i$ is a regular sequence relative to S_i for $i = 0, 1, 2, 3$. Let $X_i \subset Z_i \cap U_i$ be the open subscheme of points which are reduced in their geometric fibres for $i = 0, 1, 2, 3$. Then one obtains a commutative diagram, Cartesian over (5.13),

$$\begin{tikzcd} X_1 \arrow[r] & X_3 \\ X_0 \arrow[r] \arrow[u] & X_2. \arrow[u] \end{tikzcd}$$

6. Existence and uniqueness of the Wronski algebra system.

Let $f: X \longrightarrow S$ be a flat, quasi-projective morphism, whose geometric fibres are reduced, local complete intersection curves. Fix an S-embedding $\iota: X \longrightarrow Y$ into an S-smooth scheme Y of pure relative dimension m over S.

PROPOSITION 6.1. *There is a Wronski algebra system on X/S.*

PROOF: Since ι is transversally regular over S [**EGA IV-4**, 19.3.7], one can cover S with affine open subscheme S_λ, and then $X \times_S S_\lambda$ with affine open subschemes $Y_\mu \subset Y \times_S S_\lambda$ in such a way that for every λ, μ:

(1) there are regular functions $u_{\mu,1}, \ldots, u_{\mu,m}$ on Y_μ such that their differentials $du_{\mu,1}, \ldots, du_{\mu,m}$ form a basis for $\Omega^1_{Y_\mu/S_\lambda}$;

(2) if one lets $X_\mu := X \cap Y_\mu$, then X_μ is the closed subscheme of Y_μ given by a regular sequence $f_{\mu,1}, \ldots, f_{\mu,m-1}$ on Y_μ relative to S_λ.

Let

$$(S'_\lambda, Y'_\mu, Z'_\mu, X'_\mu, f'_{\mu,1}, \ldots, f'_{\mu,m-1})$$

be the enlargement of the local data

$$(S_\lambda, Y_\mu, X_\mu, u_{\mu,1}, \ldots, u_{\mu,m}, f_{\mu,1}, \ldots, f_{\mu,m-1}) \tag{6.2}$$

for every λ, μ. Let $(Q^n_{X'_\mu/S'_\lambda}, n \geq 0)$ be a Wronski algebra system on X'_μ/S'_λ, and let $(Q^n_{X_\mu/S_\lambda}, n \geq 0)$ be a Wronski algebra system on X_μ/S_λ induced from $(Q^n_{X'_\mu/S'_\lambda}, n \geq 0)$ for every λ, μ.

Let

$$(S_i, Y_i, X_i, u_{i,1}, \ldots, u_{i,m}, f_{i,1}, \ldots, f_{i,m-1})$$

be one of the local data (6.2) for $i = 1, 2$. One can cover $S_1 \cap S_2$ with affine open subschemes S_0 and

$$X \cap Y_1 \times_{S_1} S_0 \cap Y_2 \times_{S_2} S_0$$

with affine open subschemes

$$Y_0 \subset Y_1 \times_{S_1} S_0 \cap Y_2 \times_{S_2} S_0$$

such that there is an $(m-1) \times (m-1)$ invertible matrix C_0 with regular functions on Y_0 as entries and

$$\begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m-1} \end{bmatrix} = C_0 \begin{bmatrix} f_{2,1} \\ \vdots \\ f_{2,m-1} \end{bmatrix}. \tag{6.3}$$

Let $X_0 := X \cap Y_0$. We will commit an abuse of notation in not distinguishing between the regular functions

$$u_{i,1}, \ldots, u_{i,m}, f_{i,1}, \ldots, f_{i,m-1}$$

on Y_i and their restrictions to Y_0 for $i = 1, 2$. Let

$$(S_0^i, Y_0^i, Z_0^i, X_0^i, f'_{i,1}, \ldots, f'_{i,m-1})$$

be the enlargement of the local data

$$(S_0, Y_0, X_0, u_{i,1}, \ldots, u_{i,m}, f_{i,1}, \ldots, f_{i,m-1})$$

for $i = 1, 2$. Let $(Q^n_{X_0^i/S_0^i}, n \geq 0)$ be a Wronski algebra system on X_0^i/S_0^i, and let $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ be a Wronski algebra system on X_0/S_0 induced from $(Q^n_{X_0^i/S_0^i}, n \geq 0)$ for $i = 1, 2$. By Lemma 5.10, the Wronski algebra system $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_i/S_i}, n \geq 0)$ for $i = 1, 2$.

Since C_0 is invertible, Z_0^2 is defined in Y_0^2 by the components of

$$C_0 \begin{bmatrix} f'_{2,1} \\ \vdots \\ f'_{2,m-1} \end{bmatrix}.$$

By (6.3), the restriction of the above vector to Y_0 is

$$\begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m-1} \end{bmatrix}.$$

Therefore, we may apply Lemma 5.12, (5.14) and construction (5.4) to find a family X_3/S_3 satisfying the depth condition and morphisms

$$S_0^1 \longrightarrow S_3 \quad \text{and} \quad S_0^2 \longrightarrow S_3$$

making

$$\begin{tikzcd} S_0^1 \arrow[r] & S_3 \\ S_0 \arrow[u] \arrow[r] & S_0^2 \arrow[u] \end{tikzcd}$$

commutative, and such that

$$X_0^1 = X_3 \times_{S_3} S_0^1 \quad \text{and} \quad X_0^2 = X_3 \times_{S_3} S_0^2.$$

Let $(Q^n_{X_3/S_3}, n \geq 0)$ be a Wronski algebra system on X_3/S_3. Since X_0^i/S_0^i satisfies the depth condition, then $(Q^n_{X_0^i/S_0^i}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$ for $i = 1, 2$. By Lemma 5.3, the Wronski algebra system $(Q^{2,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_0^1/S_0^1}, n \geq 0)$. Since $(Q^{1,n}_{X_0/S_0}, n \geq 0)$ is also induced from $(Q^n_{X_0^1/S_0^1}, n \geq 0)$ by construction, then

$$(Q^{1,n}_{X_0/S_0}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X_0/S_0}, n \geq 0)$$

are equivalent. Since $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_i/S_i}, n \geq 0)$ for $i = 1, 2$, then the restrictions of

$$(Q^n_{X_1/S_1}, n \geq 0) \quad \text{and} \quad (Q^n_{X_2/S_2}, n \geq 0)$$

to X_0/S_0 are (uniquely) equivalent. The uniqueness of the equivalence allows one to patch together the local Wronski algebra systems $(Q^n_{X_\mu/S_\lambda}, n \geq 0)$ to obtain a global Wronski algebra system on X/S. The proof is complete. ∎

(6.4) It is worth remarking that the above construction does not depend on the covering of S, X, Y by local data. In fact, one could have taken a covering consisting of all possible local data in the above proof. However, we still need to check that the Wronski algebra system constructed in Proposition 6.1 does not depend on the choice of Y and the embedding $\iota\colon X \longrightarrow Y$. For the moment, denote by $(Q^n_{X/Y/S}, n \geq 0)$ the Wronski algebra system constructed in Proposition 6.1.

(6.5) Let X_1/S_1 be any flat, quasi-projective family whose geometric fibres are reduced, local complete intersection curves. Pick any morphism of schemes $h : S_2 \longrightarrow S_1$, and let X_2 be an open subscheme of $X_1 \times_{S_1} S_2$. Let $\iota\colon X_1 \longrightarrow Y_1$ be an S_1-embedding into an S_1-smooth scheme Y_1 of pure relative dimension over S_1. Let Y_2 be any neighbourhood of X_2 in $Y_1 \times_{S_1} S_2$. Let $(Q^n_{X_i/Y_i/S_i}, n \geq 0)$ be the Wronski algebra system on X_i/S_i constructed in Proposition 6.1 for $i = 1, 2$.

PROPOSITION 6.6. *The Wronski algebra system* $(Q^n_{X_2/Y_2/S_2}, n \geq 0)$ *is induced from* $(Q^n_{X_1/Y_1/S_1}, n \geq 0)$.

PROOF: By Corollary 2.20, to prove the equivalence between the system $(Q^n_{X_2/Y_2/S_2}, n \geq 0)$ and the restriction of $(Q^n_{X_1/Y_1/S_1}, n \geq 0)$ to X_2/S_2 one needs only work locally. Hence we can assume there is a local data

$$(S_1, X_1, Y_1, u_1, \dots, u_m, f_1, \dots, f_m),$$

that S_2 is affine and Y_2 is an affine open subscheme of $Y_1 \times_{S_1} S_2$ such that $X_2 = Y_2 \cap X_1 \times_{S_1} S_2$. We have now the same setup of (5.9). Note that by the construction in the proof of Proposition 6.1 and remark (6.4), the Wronski algebra system $(Q^n_{X_i/Y_i/S_i}, n \geq 0)$ is induced from a Wronski algebra system on the enlargement of the local data

$$(S_i, X_i, Y_i, u_1, \dots, u_m, f_1, \dots, f_m)$$

for $i = 1, 2$. By applying Lemma 5.10 the proof becomes complete. ∎

(6.7) We now go back to the set-up of Proposition 6.1 and prove that $(Q^n_{X/Y/S}, n \geq 0)$ does not depend on Y. To this purpose, let

$$\nu : X \hookrightarrow Z$$

be another S-embedding into an S-smooth scheme Z of pure relative dimension over S. Let $(Q^n_{X/Z/S}, n \geq 0)$ be the associated Wronski algebra system.

LEMMA 6.8. *Assume there is an* S*-embedding* $\epsilon : Y \longrightarrow Z$ *such that* $\nu = \epsilon \circ \iota$. *Then* $(Q^n_{X/Y/S}, n \geq 0)$ *is equivalent to* $(Q^n_{X/Z/S}, n \geq 0)$.

PROOF: Let S' be any S-scheme with a section $S \hookrightarrow S'$. Let

$$Y' := Y \times_S S' \quad \text{and} \quad Z' := Z \times_S S',$$

and let $\epsilon' : Y' \longrightarrow Z'$ be the extension of ϵ by S'. Assume there is an S'-subscheme X' of Y' such that $X := X' \times_{S'} S$. Then X' is also an S'-subscheme of Z' via ϵ'. Assume that X'/S' satisfies the depth condition. Then $(Q^n_{X'/Y'/S'}, n \geq 0)$ and $(Q^n_{X'/Z'/S'}, n \geq 0)$ are equivalent. On the other hand, by Proposition 6.6 the system $(Q^n_{X/Y/S}, n \geq 0)$ (resp. $(Q^n_{X/Z/S}, n \geq 0)$) is induced from $(Q^n_{X'/Y'/S'}, n \geq 0)$ (resp. $(Q^n_{X'/Z'/S'}, n \geq 0)$). Hence, $(Q^n_{X/Y/S}, n \geq 0)$ and $(Q^n_{X/Z/S}, n \geq 0)$ are equivalent.

By Corollary 2.20, to prove Lemma 6.8 we need only work locally. But locally one can make the above assumptions, as shown in (5.4). The proof is complete. ∎

LEMMA 6.9. *Assume that* $Y = \mathbf{A}^k_Z$, *where* $\mathbf{A}^k_Z$ *is the affine space over* Z *of dimension* k. *Let* $p\colon Y \longrightarrow Z$ *be the structure map of* Y *over* Z. *Assume that* $\nu = p \circ \iota$. *Then* $(Q^n_{X/Y/S}, n \geq 0)$ *is equivalent to* $(Q^n_{X/Z/S}, n \geq 0)$.

PROOF: One can assume that ι and ν are closed embeddings. One can also assume that Y is affine by Corollary 2.20. Then it is easy to see that there is a section $s\colon Z \longrightarrow Y$ of p such that $s \circ \iota = \nu$. By applying Lemma 6.8 the proof becomes complete. ∎

PROPOSITION 6.10. *The systems* $(Q^n_{X/Y/S}, n \geq 0)$ *and* $(Q^n_{X/Z/S}, n \geq 0)$ *are equivalent.*

PROOF: By Corollary 2.20, one can assume there are S-embeddings of Y and Z into affine spaces $\mathbf{A}^k_S$ and $\mathbf{A}^l_S$, respectively. By Lemma 6.8, one can assume that $Y = \mathbf{A}^k_S$ and $Z = \mathbf{A}^l_S$. Let

$$\mu := (\iota, \nu)\colon X \longrightarrow Y \times_S Z.$$

By Lemma 6.9 applied twice, one has that

$$(Q^n_{X/Y \times_S Z/S}, n \geq 0)$$

is equivalent to both $(Q^n_{X/Y/S}, n \geq 0)$ and $(Q^n_{X/Z/S}, n \geq 0)$. The proof is complete. ∎

COROLLARY 6.11. *There is a Wronski algebra system on* $\mathcal{C}_{l.c.i.}$.

PROOF: Let X/S be a family in $\mathcal{C}_{l.c.i.}$. One can cover X with open subschemes X_λ in such a way that for every λ there is an S-embedding $X_\lambda \hookrightarrow Y_\lambda$ into an S-smooth scheme Y_λ of pure relative dimension over S. Let

$$(Q^n_{X_\lambda/S}, n \geq 0) := (Q^n_{X_\lambda/Y_\lambda/S}, n \geq 0)$$

be the Wronski algebra system constructed in Proposition 6.1. By Proposition 6.10, one can glue the above local systems together to obtain a Wronski algebra system $(Q^n_{X/S}, n \geq 0)$ on X/S. By Proposition 6.6 and Proposition 6.10, there are equivalences relating Wronski algebra systems under base change and open embeddings. Hence, the conditions in Definition 2.14 are met. ∎

PROPOSITION 6.12. *Two Wronski algebra systems on* $\mathcal{C}_{l.c.i.}$ *are uniquely equivalent.*

PROOF: Let $(Q^{i,n}, n \geq 0)$ be a Wronski algebra system on $\mathcal{C}_{l.c.i.}$ for $i = 1, 2$. Let X/S be any family in $\mathcal{C}_{l.c.i.}$. By Corollary 2.20, one can assume there is a local data

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1}).$$

Let

$$(S', Y', Z', X', f_1', \ldots, f_{m-1}')$$

be the enlargement of the above local data. By Proposition 4.10, the Wronski algebra systems

$$(Q^{1,n}_{X'/S'}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X'/S'}, n \geq 0)$$

are equivalent. Since by Definition 2.14 the system $(Q^{i,n}_{X/S}, n \geq 0)$ is induced from $(Q^{i,n}_{X'/S'}, n \geq 0)$ for $i = 1, 2$, then the Wronski algebra systems

$$(Q^{1,n}_{X/S}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X/S}, n \geq 0)$$

are (uniquely) equivalent. The proof is complete. ∎

Proof of Theorem 2.16: By combining Corollary 6.11 with Proposition 6.12, the proof of Theorem 2.16 is complete. ∎

7. Wronski systems and Wronskians

Assume $f\colon X \longrightarrow S$ is a flat morphism whose geometric fibres are reduced, Gorenstein curves. Assume there is a Wronski algebra system

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

on X/S. If no confusion is likely, then we will ommit the subscript X/S. It is important to emphasize though that the existence of a Wronski algebra system has been proved only in the case the fibres of X/S are local complete intersections.

(7.1) Let L be any sheaf of $\mathcal{O}_X$-modules. Denote by $P^n(L)$ (resp. $Q^n(L)$) the tensor product of L by P^n (resp. Q^n) with respect to the right $\mathcal{O}_X$-algebra structure of P^n (resp. Q^n). The sheaf $P^n(L)$ (resp. $Q^n(L)$) will be regarded as an $\mathcal{O}_X$-module via the left $\mathcal{O}_X$-algebra structure of P^n (resp. Q^n).

Tensoring diagram (2.7) on the right by L one obtains a commutative diagram of left $\mathcal{O}_X$-modules, namely

$$\begin{array}{ccc} P^n(L) & \xrightarrow{p^n(L)} & P^{n-1}(L) \\ {\scriptstyle \psi^n(L)}\downarrow & & {\scriptstyle \psi^{n-1}(L)}\downarrow \\ Q^n(L) & \xrightarrow{q^n(L)} & Q^{n-1}(L) \end{array} \tag{7.2}$$

for each $n > 0$. On the other hand, there are canonical homomorphisms

$$\nu^n\colon f^*f_*L \longrightarrow P^n(L)$$

for each $n \geq 0$ such that

$$p^n(L) \circ \nu^n = \nu^{n-1} \tag{7.3}$$

for every $n > 0$.

Assume from now on that L is invertible. Then $Q^n(L)$ is locally free of rank $n+1$ for every $n \geq 0$. Let W be a locally free $\mathcal{O}_S$-module of constant rank $r+1$. Let

$$\gamma\colon W \longrightarrow f_*L$$

be an $\mathcal{O}_S$-linear map. By composing $f^*\gamma$ with ν^n and $\psi^n(L)$ one obtains a map

$$v^n\colon f^*W \longrightarrow Q^n(L)$$

for each $n \geq 0$ such that

$$q^n(L) \circ v^n = v^{n-1} \tag{7.4}$$

for every $n > 0$, because of (7.3) and the commutativity of (7.2). Expression (7.4) shows that $(f^*W, Q^n(L), q^n(L), v^n, n \geq 0)$ is a Wronski system, as defined in [**21**] and [**22**]. For convenience, we will use the following shorter notation,

$$\mathcal{W}_{X/S}(L,\gamma) := (f^*W, Q^n(L), q^n(L), v^n, n \geq 0),$$

or simply $\mathcal{W}(L,\gamma)$, when no confusion is likely.

Associated to a Wronski system one has the concepts of sequence of gaps at a point and of a Weierstrass point, for which we refer the reader to [**22**, Section 2]. The *sequence of orders* at a point is the sequence of gaps shifted by -1.

(7.5) Laksov and Thorup have also shown how to associate to a Wronski system certain maps, called Wronskians, whose zero schemes consist of Weierstrass points of the Wronski system. Let $n_0, n_1, \ldots$ be the sequence of integers defined inductively by

$$\text{rk } v^i := n_0 + \cdots + n_i$$

for $i = 0, 1, \ldots$, where rk v^i denotes the rank of the map of vector bundles v^i. Note that either n_i is 0 or 1 for all $i \geq 0$ [**21**, 1.4, p. 134]. Denote by $\epsilon_0, \epsilon_1, \ldots$ the increasing sequence of indices ϵ for which $n_\epsilon = 1$, in other words, the generic order sequence of $\mathcal{W}(L,\gamma)$. Then, for every non-negative integer h there is a canonical homomorphism

$$w_h \colon f^* \bigwedge^{r_h+1} W \longrightarrow L^{r_h+1} \otimes \omega^{\epsilon_0+\cdots+\epsilon_{r_h}}, \tag{7.6}$$

where $\epsilon_0, \ldots, \epsilon_{r_h}$ is the increasing sequence of orders ϵ less than or equal to h [**21**, 1.5.2, p. 135].

DEFINITION 7.7. *w_h is called the* **Wronskian** *of rank $r_h + 1$ of the Wronski system $\mathcal{W}(L,\gamma)$.*

The homomorphism w_h is in fact defined by r_h rather than by h. The importance of w_h lies on the fact that its zero scheme Z_h parametrizes Weierstrass points of rank at most r_h of $\mathcal{W}(L,\gamma)$ [**21**, 1.7, p. 136].

(7.8) If $U \subset X$ is an open subscheme, then the restriction of the system $\mathcal{W}_{X/S}(L,\gamma)$ to U is a Wronski system; more precisely the restriction is equal to $\mathcal{W}_{U/S}(L_U, \gamma_U)$, where L_U is the restriction of L to U and γ_U is the composition of γ with the push-forward by f of the canonical map

$L \longrightarrow L_U$ on X. If U contains all the associated points of X, then the rank of the restriction v_U^n of v^n to U is equal to the rank of v^n for each $n \geq 0$. Therefore, the Wronskians of $\mathcal{W}_{U/S}(L_U, \gamma_U)$ are the restrictions to U of the Wronskians of $\mathcal{W}_{X/S}(L, \gamma)$ [**21**, 1.5, p. 135]. In particular, if we let $U := X^{sm}$, where X^{sm} is the S-smooth locus of X, then the restrictions of the Wronskians of $\mathcal{W}_{X/S}(L, \gamma)$ to X^{sm} are equal to the Wronskians of $\mathcal{W}_{X^{sm}/S}(L_{X^{sm}}, \gamma_{X^{sm}})$. The latter can be constructed within the set-up of [**21**]. In addition, by Lemma 2.19 the maps w_h are determined by their restrictions to the smooth locus of X/S, what permits us to compare the maps w_h and the Wronskians obtained in the previous literature.

(7.9) Assume that γ is injective. If X/S is smooth, then Laksov and Thorup have shown that the map v^i is injective for sufficiently large i [**21**, 4.6, p. 146]. Their result carries over immediately to the case where the fibres of X/S are geometrically integral. Moreover, since in any case v^i is injective if and only if $v^i(\xi)$ is injective for every associated point ξ of X [**21**, 4.2, p. 144], then one can even claim the following proposition.

PROPOSITION 7.10. *If the fibres of X/S over associated points of S are geometrically integral, then v^i is injective for sufficiently large i.*

(7.11) It is not true in general that v^i is injective for i sufficiently large. The reason is that although γ may be injective for a reducible curve X over a field, there might be linear dependence relations among the sections of W when restricted to an irreducible component of X. In more geometrical terms, the rational map $X \longrightarrow \mathbf{P}(W)$ defined by (L, γ) will map X to a non-degenerate curve in $\mathbf{P}(W)$ if γ is injective, but may map some of the components of X into proper subspaces of $\mathbf{P}(W)$. Easy examples of the non-injectivity of v^i for all i can thus be found by considering non-degenerate reducible curves in projective space with some degenerate components, together with the linear system given by the hyperplane sections. We will see later on that this is in fact the only way that v^i may not be injective for sufficiently large i.

(7.12) If v^i is injective for sufficiently large i, then there must be $r+1$ generic gaps, $\epsilon_0, \ldots, \epsilon_r$. In this case one can consider the Wronskian of rank $r+1$,

$$w := w_h \colon f^* \bigwedge^{r+1} W \longrightarrow L^{r+1} \otimes \omega^{\epsilon_0 + \cdots + \epsilon_r},$$

for h sufficiently large. The map w will be called simply the *Wronskian* of (L, γ). As remarked in (7.8), the Wronskian is in fact determined by its restriction to the smooth locus of X/S. Hence we obtain the following result.

PROPOSITION 7.13. *If $S = \text{Spec } k$, where k is an algebraically closed field, and X is an irreducible curve, then the Wronskian obtained above coincides with the one defined by Lax and Widland in characteristic* 0 *[**24**] or by Garcia and Lax in arbitrary characteristic [**12**].*

DEFINITION 7.14. *The zero scheme Z of the Wronskian w is called the* **Weierstrass subscheme** *of X associated to (L, γ).*

(7.15) The most natural question, given the subscheme Z of X, is whether Z is a Cartier divisor on X. Since Z is the zero scheme of w, then Z is a Cartier divisor if and only if w is not zero at the associated points of X. Equivalently, Z is a Cartier divisor if and only if Z does not contain any irreducible component of any fibre of X/S over an associated point of S. In order for Z to be a relative Cartier divisor over S, then one needs to impose the above condition on every geometric fibre of X/S. More precisely, Z is a relative Cartier divisor if and only if Z does not contain any irreducible component of any fibre of X/S.

(7.16) For each $s \in S$, let $\gamma'(s)$ be the composition

$$\gamma'(s) \colon W(s) \xrightarrow{\gamma(s)} (f_*L)(s) \longrightarrow H^0(X(s), L(s)),$$

where the second homomorphism is given by base change. Laksov and Thorup [**21**, 4.7, p. 147] called (L, γ) a linear system on X/S if $\gamma'(s)$ is injective for each $s \in S$. In our more general situation we have to modify the definition to take into account all the irreducible components of the fibres, in order to prevent a situation like the one described in (7.11).

Let $s \in S$, and let $Y \subset X(s)$ be an irreducible component. Let

$$\gamma'(s)_Y \colon W(s) \xrightarrow{\gamma'(s)} H^0(X(s), L(s)) \longrightarrow H^0(Y, L(s)_Y),$$

where the second homomorphism is given by restriction to Y.

PROPOSITION 7.17. *If $\gamma'(s)_Y$ is injective at a point $s \in S$, for each irreducible component $Y \subset X(s)$, then $v^i(s)$ is injective for sufficiently large i.*

PROOF: The map $v^i(s)$ is injective if $v^i(\xi)$ is injective for all generic points $\xi \in X(s)$. Therefore, the proof of [**21**, 4.5, p. 145] may be easily adapted to yield the proof of the proposition, since the stronger hypothesis that $\gamma'(s)_Y$ be injective for every irreducible component Y of $X(s)$ takes care of all generic points of $X(s)$. The proof is complete. ∎

COROLLARY 7.18. *If $\gamma'(s)_Y$ is injective at every associated point s of S, for each irreducible component $Y \subset X(s)$, then v^i is injective for sufficiently large i. In particular, the Weierstrass subscheme Z of X*

associated to (L, γ) is defined. If in addition the characteristic of the residue field $k(s)$ is 0 for all associated points $s \in S$, then Z is a Cartier divisor.

PROOF: The injectivity of v^i may be checked at the associated points ξ of X, which are the generic points of the fibres $X(s)$ over associated points $s \in S$. By Proposition 7.17, the proof of the first statement is finished. As for the last statement, since v^i is injective for i sufficiently large, then the number of gaps at every associated point ξ of X is $r + 1$. Since the characteristic of the residue field $k(s)$ of the point $s \in S$ lying under ξ is 0, then the sequence of orders at ξ is classical, that is, the sequence is $0, 1, \ldots, r$. Hence ξ is not contained in Z. By (7.15), the subscheme Z is a Cartier divisor. ∎

If one wants Z to be a relative Cartier divisor over S, then one must take into account all fibres, as remarked in (7.15).

DEFINITION 7.19. *(L, γ) is called a* **linear system** *on X/S if $\gamma'(s)_Y$ is injective at each point $s \in S$, for each irreducible component Y of $X(s)$.*

PROPOSITION 7.20. *If (L, γ) is a linear system on X/S and the characteristic of $k(s)$ is 0 for all $s \in S$, then the Weierstrass divisor Z is a relative Cartier divisor.*

PROOF: Since (L, γ) is a linear system on X/S, then one can apply Proposition 7.17 to all fibres of X/S. The proof is then analogous to the one given for Corollary 7.18. ∎

(7.21) The hypothesis on the characteristics of the residue fields of points in S is necessary, even when X/S is a smooth family. The reason is that the sequence of generic orders of a non-singular curve in positive characteristic need not be classical, as several examples show. Hence, it might be the case that the general fibre of a smooth family is classical but a special fibre is not. In this case the whole special fibre is contained in Z. In order to obtain a Cartier divisor (resp. a relative Cartier divisor), one must thus impose conditions on the sequence of orders at every generic point of every fibre of X/S over an associated point of S (resp. of every fibre of X/S).

PROPOSITION 7.22. *If the sequence of orders of (L, γ) at all generic points of all fibres of X/S (resp. of all the fibres over the associated points of S) are equal, then the associated Weierstrass subscheme of X is a relative Cartier divisor (resp. a Cartier divisor.)*

PROOF: As in Corollary 7.18. ∎

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