Two Topics in Online Auctions

by

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Abstract

This thesis studies two operations management topics in online auctions, and is divided into two parts.

Motivated by the increasing use of ShopBots to scan Internet auctions, the first part of the thesis analytically examines whether or not two competing auctioneers selling the same commodity should share, or pool, some or all of their bidders. Under pooling, the bidding population is represented by three compartments: bidders dedicated to auction 1, bidders dedicated to auction 2, and pooled bidders participating in both auctions simultaneously. Under a bidder strategy shown to induce a Bayesian equilibrium, a closed form expression for the auctioneers' expected revenue under pooling is found, and pooling is recommended where it produces a greater expected revenue than no pooling (i.e., our objective is revenue maximization). Pooling is generally found to be beneficial as long as the two auctions are not too asymmetric and the underlying valuation distribution has certain concavity characteristics. Asymptotic order statistic arguments are used where explicit characterizations are intractable.

The second part of the thesis considers a manufacturer who uses a reverse, or procurement, auction to determine which supplier will be awarded a contract. Each bid consists of a price and a set of non-price attributes (e.g., quality, lead time). The manufacturer is assumed to know the suppliers' cost functions (in terms of the non-price attributes). We analyze how the manufacturer chooses a scoring rule (i.e., a function that ranks the bids in terms of the price and non-price attributes) that attempts to maximize his own utility. Under the assumption that suppliers submit their myopic best-response bids (i.e., they choose their minimum-cost bid to achieve any given score), our proposed scoring rule indeed maximizes the manufacturer's utility within the open-ascending format. The analysis reveals connections between the manufacturer's utility maximization problem and various geometric aspects of the manufacturer's utility and the suppliers' cost functions.

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A Pooling Analysis of Two Simultaneous Online Auctions

Chapter 1

Introduction

In the past several years an extraordinary number of online markets – estimated as of Fall 2001 to be on the order of 1000 [7] – have been created to bring together buyers and sellers. With the existence and growing use of such markets, purveyors of commodities that used to compete for revenue on the basis of a published price can now compete for revenue based on dynamic pricing. Far and away the prevalent online dynamic pricing mechanism, auctions are used by sellers to make transactions based directly on bidders' willingness-to-pay.

For an auction to function properly, the number of bidders needs to exceed the number of items available for sale. Bidders who are not allocated an item at the end of an auction walk away empty-handed, but may have been useful to the seller by way of driving upwards the eventual winning price of the auction. That is, all but a handful of bidders can be viewed as means to drive up price, not as inevitable buyers. Furthermore, price is increasing with the actual number of bidders available to drive it up. These simple auction fundamentals suggest that competing auctioneers selling the same commodity might benefit from in some way sharing their bidders, so as to drive up the prices of both auctions.

Rather than analyzing the effects of totally merging online auctioneers' markets, such as the automotive consortium Covisint, in this study we investigate less formal, more moderate methods of market sharing in which some, but not necessarily all, bidders participate in both auctions. Our motivation for considering a mitigated case is twofold: First, merging competing auctions involves many complicated factors such as allocating the cost savings, agreeing on standards, et cetera; in fact, it took three months for the founding members of Covisint just to come up with a name [23]. Secondly, sophisticated bidders currently effect exactly this type of mitigated market sharing by virtue of simultaneously participating in multiple auctions. Increasingly consumer buyers are employing intelligent software, or *ShopBots*, to scan the Internet for auctions selling the particular item they want; one such tool, BidXS, currently searches about 30 consumer auctions. For B2B auctions – although participation in the auction is more complicated – a ShopBot, or simply having two browser windows open at the same time, is possible.

The benefits of sharing a scarce, but not necessarily used, resource has long been known in the area of inventory sharing, or inventory pooling [11], and we reference this operations management motivation by naming this mitigated market sharing in auctions *bidder pooling*. we analyze the decision faced by two auctioneers deciding whether or not to pool with each other. The agreement to pool requires that the auctioneers synchronize their auctions, and, as we discuss, the auctioneers may additionally find it beneficial to encourage the use of ShopBots. On the other hand, an auctioneer's decision not to pool could be executed simply by timing his auction differently, or, if this is not possible, by discouraging or disabling the use of ShopBots by his bidders. The crux of the pooling decision problem is summarized by the observation that "pooling is a two-way street." If an auctioneer pools his bidders, his item will have access to the competing auctioneer's bidders; conversely, his bidders will have access to the items of the competing auctioneer. The analysis attempts to discover when such a two-way street is advantageous to the auctioneer.

In this study we compare auction outcomes solely on the basis of auctioneer revenue. There are two main approaches to analyzing auctions in the literature: (i) revenue maximization and (ii) efficiency maximization. Revenue maximization, in which the relevant metric is the auctioneer's welfare, in concerned with extracting as high a price as possible from the bidders. Efficiency maximization, on the other hand, attempts to boost the total social welfare brought about by the auction, and is the traditional approach used in the public sector and the Economics literature. We feel that revenue maximization is the appropriate metric in our private sector setting; although businesses may eventually alienate customers by such an approach [18], revenue maximization seems the most likely course, at least in the medium to short term [25].

A handful of papers from the Economics field have studied competing auctioneers. Peters endogenizes the selling mechanism that will emerge in equilibrium among a large number of sellers and buyers, and provides conditions under which this mechanism is a Vickrey (sealed, second-price) auction [21]. More recently, Ellison and Fudenberg [10] look at what equilibrium characteristics emerge when sellers and buyers can unilaterally choose between participating in either of two uniform price (e.g., open ascending) auction houses.

This study appears to be among the first study to treat the bidder pooling problem, which is considerably more complex than assuming each bidder participates in at most one auction (Peters and Severinov [22], who studied equilibrium bidding strategies and prices concurrently with but independently from this study, is the only other example of which we are aware). In our model, the pooling decision is not taken unilaterally (pooling requires cooperation to execute), and as only two agents (the two competing auctioneers) perform decision analysis, this study focuses less on the inherent elements of cooperative equilibrium and more on the analysis of the strategic pooling decision itself, taking a more operational cut at the problem. The analysis shows that pooling can increase revenues, and where possible makes mathematically precise the conditions for which this is true. The qualitative take away is that, provided some conditions on the valuation distributions, pooling is mutually beneficial if the sizes of the original auctions are not too asymmetric.

This part of the thesis is organized as follows. Chapter 2 provides a brief description of the model. Chapter 3.1 begins with a game-theoretic justification of the proposed bidder behavior assumed in the sequel. Section 3.2 introduces a closed form solution for the pooling tradeoff under generally distributed bidder valuations (willingness-to-pay). Section 3.3 restricts to the case in which both auctioneers sell a single item, and under some conditions of symmetry characterizes which valuation distributions make pooling beneficial, and indicates how this benefit is tied to hedging against low prices. Section 3.4 develops further analytical results towards characterizing when pooling is beneficial in both single and multi-item auctions, but restricts to the two valuation distributions most often assumed in the auction literature, the Uniform and Exponential, as well as the Pareto. Section 3.5 provides insights into the multi-item case under general valuations, and we provide concluding remarks for Part I in Chapter 4.

Chapter 2

The Model

We assume an English auction format: if m items are for sale they go to the m highest bidders at the $(m+1)^{st}$ highest submitted bid. We ignore each auctioneer's reservation price, or equivalently assume that the reservation price is zero; while this simplifying assumption helps focus the analysis on the value of additional bidding competition via pooling, for English auctions the addition of even a single bidder has been shown to increase revenue more than an optimally set reservation price (Bulow and Klemperer [5]). For readability, discussion of bidder behavior is delayed until the next section. Because continuous valuations simplify the work of deriving and manipulating order statistics (in particular, continuity facilitates tractable expressions for order statistic PDFs), for the analyses of the pooling decision problem in Sections 3.2-3.5 we assume in all that bidder valuations are continuously distributed; however, the game theoretic justification of the bidder behavior given in the next section assumes discrete bidder valuations to yield a tractable game.

To preserve tractability we also limit our analysis to a single auction event, rather than addressing the ramifications of periodic auctions; see [17] for a discussion of the issues and literature of periodic auctions. Our approach can be thought of in two ways. First, it can be construed as assuming that we only are concerned about a single auction event in isolation. Second, it treats the perhaps more appealing situation in which bidder valuations between periods are independent, and auctioneers have some degree of control over a relatively static number of bidders (so that the auctioneer can choose to aid or discourage his bidders' participation in other competing auctions), making what each auctioneer can offer in terms of bidder sharing similar from period to period. In this case, the decision of whether or not to pool takes on a more long-term strategic implication for the auctioneers. We also point out that our setup assumes a degree of temporal overlap among the two auctions sufficient to allow all those wishing to compete in both or either auction(s) to do so before either auction ends. While this may not hold in current practice, it could be easily enforced by two auctioneers who decide to pool. One additional assumption, commonality (across auctions) of the fraction of pooling to dedicated bidders, is introduced in equation (3.4) at the end of Section 3.2, and is applied to simplify the analyses thereafter (i.e., Sections 3.3-3.5).

Our analysis approach is as follows. We compute the expected revenue under mutual pooling among auctioneers, compute the expected revenue under no pooling (simply the revenue in the auctioneer's original auction), and recommend pooling if the former is greater than the latter. To formalize the analysis of the pooling decision we adopt the following notation.

Auctioneer i sells m_i items and has n_i original bidders (in the absence of pooling), of which (during pooling) d_i (mnemonic for dedicated) participate solely in auction i, and s_i participate simultaneously in both auctions (are "shared" by auctioneer i). That is, $d_i + s_i = n_i$. We let $s = s_1 + s_2$ be the total number of bidders participating in both auctions simultaneously, and $d = d_1 + d_2$ denote the total number of dedicated bidders; therefore, the total number of bidders, participating in either one auction exclusively or both auctions simultaneously, is $d + s = n_1 + n_2$ (see Figure 2-1). To avoid introducing new notation, we will also describe bidder compartments using the variables just defined; for instance, where no confusion will likely arise, we refer to the d_i compartment as being the bidders dedicated to auctioneer i. We refer to the situation $s_1 = n_1$, $s_2 = n_2$ (equivalently $d_1 = d_2 = 0$) as *full pooling*, and otherwise

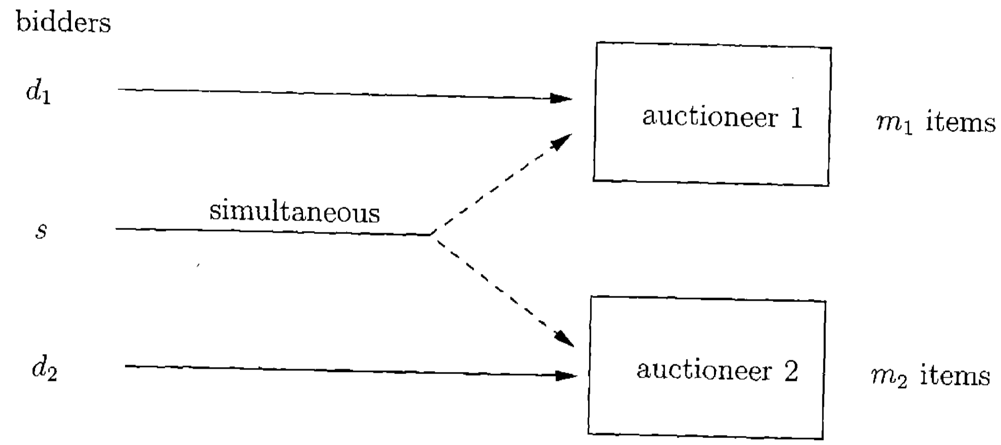

Figure 2-1: The pooling model.

use the term *partial pooling*. Throughout, we let $X_{b:a}$ denote the $(b+1)^{st}$ largest order statistic out of a draws (i.e., the a draws are ordered as $X_{a-1:a} \leq X_{a-2:a} \leq \cdots X_{0:a}$), and adopt the convention that $\binom{a}{b} = 0$ if $a < b$ and $= 1$ if $a = b$.

Because the expected price in both auctions need not be the same, for convenience we formalize the expected price discussion – the main analysis theme of this part of the thesis – by adopting the viewpoint of auctioneer 1. We denote the price of auction 1 by $\Pi(d_1, s, d_2, m_1, m_2)$; note that this value does not depend on the individual s_i's (which when pooled comprise an indistinguishable cohort), simply their sum s.

Chapter 3

Analysis

3.1 Bidder Behavior

In this section we present equilibrium results regarding strategic behavior of rational dedicated and pooled bidders facing two simultaneous English auctions. We show that the dynamic bidding strategy of "cherry picking" the best available prices forms a Bayesian equilibrium in the formalized simultaneous auction game introduced and analyzed in Section A.1 of the Appendix. Detailed discussion of the game format, information structure, equilibrium strategy and assumptions are relegated to Section A.1; the salient features are presented below, framing the Proposition that concludes this section. The basic game setup we adopt is roughly that used by Peters and Severinov in [22]. There are three notable differences between their game setup and that given here: the present study includes dedicated bidders, auctioneers can sell multiple items instead of just single items, and we consider only two simultaneous auctions instead of many.

Game theoretic analyses of English auctions most often assume that a dominance argument implies rational bidders need only employ a myopic, maximum bid-level strategy by which a player not currently winning an item submits a bid just large enough to take the lead, provided this bid does not exceed his valuation (his maximum bid-level). While appealingly straightforward, dominance analyses do not address the English auction's dynamic nature, and thus ignore the attending possibility of dynamic bidding strategies (Kamecke [16] provides a formal discussion of this observation). Kamecke proposes a maximin bidder payoff structure for which the myopic strategy for English auctions remains dominant within the wider set of dynamic strategies. While in our simultaneous English auction game the myopic strategy we propose is not dominant, we do retain the traditional, post-auction surplus maximizing bidder payoff structure. It is easy to see why a myopic strategy fails a dominance argument under traditional payoffs: e.g., if unexpected bids (such as exceeding the current price by twice the minimum bid increment) causes competing bidders to respond by dropping out of the auction.

We define our bidder strategy, σ^*, as follows. (1) If it is the buyer's turn to bid, then: if the buyer is the current high bidder in an auction, then the buyer submits no bid; otherwise, (2) the buyer should bid the minimum amount required to take the lead in whichever auction she has access to and is currently cheapest, provided such a bid would not exceed her valuation. If neither auction is currently cheaper than the other, a pooled bidder should bid in the auction for which the bid will be a leading bid; if the bid would lead in either auction, and if during the last round of turns no one bid and no one entered the market, the bidder bids in either auction with equal probability.

For now we brush past the precise meaning of the terms entry and turn as used in the definition above, but they have a natural interpretation and simply formalize the game's play. In our game model, the standard Bayesian (unknown bidder valuations) framework holds. In addition, we impose the assumption that auctioneers announce the identity of the winning bidders (this is done on auction sites such as eBay); in the context of the strategy given above, this assumption allows bidders to predict whether or not a new bid will become a leading bid, and to detect when bidders currently in the market stop bidding.

The appropriate equilibrium concept for such a dynamic game and strategy is that of Perfect Bayesian Equilibrium [12]. In particular, assuming all players $j \neq i$ play σ^*, the strategy σ^* must be optimal for player i for all stages and information sets, not simply those lying on the equilibrium path. We conclude this section with a proposition establishing this for the bidding strategy σ^*, which we take to be the bidders' strategy in Part I's subsequent analyses of the pooling decision problem.

Proposition 1. *σ^* is a Perfect Bayesian Nash Equilibrium strategy for the simultaneous auction game described in Section A.1.*

As mentioned earlier when discussing the modelling assumptions, unlike subsequent Part I's results the game-theoretic analysis requires discrete bidder valuations; this makes the game tractable by avoiding complicated bid-ordering issues.

3.2 Price in Pooled Multi-Item Auctions

Despite its straightforward structure, the pooling decision analysis is quickly complicated by the difficulty of computing the expected price (which is tantamount to computing revenue) under pooling; to see this, notice that the price in a pooled auction depends not only on the valuations of its bidders, but – via pooling bidders – can depend indirectly on the valuations of bidders in competing auctions. This section presents Part I's key building block, a closed form solution for the PDF and moments of the price in auction 1 under pooling with auction 2.

Proposition 2 (PDF of Price). *Let $\Pi(d_1, s, d_2, m_1, m_2)$ denote the price of auction 1 when auctioneer i sells m_i items, has d_i dedicated bidders, and the number of bidders participating simultaneously in both auctions is s. Then*

$$\begin{aligned}
&P(\Pi(d_1, s, d_2, m_1, m_2) = \pi) \\
&= P(X_{m_1+m_2\,:\,d+s} = \pi)\Big[1 - P(X_{m_2\,:\,d_2+s} < X_{m_1\,:\,d_1}) - P(X_{m_1\,:\,d_1+s} < X_{m_2\,:\,d_2})\Big] \\
&\quad + P(X_{m_1\,:\,d_1} = \pi\,, X_{m_2\,:\,d_2+s} < \pi) + P(X_{m_1\,:\,d_1+s} = \pi\,, \pi < X_{m_2\,:\,d_2}), \qquad (3.1)
\end{aligned}$$

where $X_{b:a}$ denotes the $(b+1)^{st}$ largest order statistic out of a draws.

Proposition 2 characterizes auctioneer 1's price Π by applying conditional probability in a clever manner, namely, conditioning on whether or not auctions 1 and 2 have the same price. Because it avoids an extremely detailed case-by-case analysis of possible bid values, orderings, and compartmental locations (the ingredients of a realization π), this approach saves time at the expense of (perhaps) immediate transparency; more straightforward, lengthy proofs are available from the author (for example, conditioning on the number and location of items won by the s compartment bidders).

Proof. Proposition 2 is proved by reasoning out when the winning prices in auctions 1 and 2 will be the same, and conversely when they will be different. The latter will occur when circumstances preclude s compartment bidders from setting the prices of both auctions simultaneously. Such circumstances arise when the valuations among one auction's dedicated bidders are sufficiently high relative to all other bidders' valuations. More specifically, if the $(m_1+1)^{\text{st}}$ highest valuation among the d_1 bidders is higher than the $(m_2+1)^{\text{st}}$ valuation among all the d_1 *and* s bidders, then the winning price in auction 1 will exceed that of auction 2 – there is no rational incentive for s bidders to bid up auction 2's price towards auction 1's higher price. Call this event U_1, for unequal prices with auction 1's price higher. The price in event U_1 is captured by the second term on the right hand side of (3.1), while the third term captures this situation in reverse (call this event U_2).

The key to equation (3.1)'s simplicity lies in the fact that when neither event U_1 or U_2 occurs (call this event E), the resulting prices of the auctions are equal. Essentially, both auctions' dedicated bidders' valuations are competitive with each other, the s compartment valuations are not so small that the higher-priced dedicated auction could overwhelm all the s and the other auction's dedicated bidders (as in U_1 and U_2), and consequently some s compartment bidder is compelled to set prices in both auctions simultaneously, by either winning an item or submitting a price-setting drop-out bid. Note that the probability of event E is independent of the actual $(m_1 + m_2 + 1)^{\text{st}}$ highest value among the $d + s$ valuations (it is simply the fraction of bidder valuation orderings which yield E), and hence conditioning on E does not change the distribution of the $(m_1 + m_2 + 1)^{\text{st}}$ highest value. Since this is exactly the price π when both auctions share the same price (event E), the distribution of π under event E is consistent with $m_1 + m_2$ items being allocated to the highest of the $d + s$ bidders, which is captured by the first term on the right hand side of equation (3.1). □

Corollary 1 (Moments of Price, Probabilistic Expression).

$$
\begin{aligned}
&E[\Pi^k(d_1, s, d_2, m_1, m_2)] \\
&\quad = E[X^k_{m_1+m_2\,:\,d+s}]\underbrace{\Big(1 - P(X_{m_2\,:\,d_2+s} < X_{m_1\,:\,d_1}) - P(X_{m_1\,:\,d_1+s} < X_{m_2\,:\,d_2})\Big)}_{P(E)} \\
&\qquad + E[X^k_{m_1\,:\,d_1} \mid \underbrace{X_{m_2\,:\,d_2+s} < X_{m_1\,:\,d_1}}_{U_1}]\,\underbrace{P(X_{m_2\,:\,d_2+s} < X_{m_1\,:\,d_1})}_{P(U_1)} \\
&\qquad + E[X^k_{m_1\,:\,d_1+s} \mid \underbrace{X_{m_1\,:\,d_1+s} < X_{m_2\,:\,d_2}}_{U_2}]\,\underbrace{P(X_{m_1\,:\,d_1+s} < X_{m_2\,:\,d_2})}_{P(U_2)},
\end{aligned}
\tag{3.2}
$$

Corollary 1 simply rewrites (3.1) with conditioning, and integrates after multiplying by π^k. The labels below the terms in (3.2) correspond to the U_1, U_2 and E shorthand from the proof of Proposition 2; for readability, we apply this shorthand throughout the sequel. Equation (3.2) makes explicit the fundamental tradeoffs of pooling. The second term captures the probability that auction 1's price is determined by the d_1 dedicated auction 1 bidders, without competitive influence from the s or d_2 bidders; in the context of the rational incentive to pool, this "worst case" situation for auctioneer 1, in which his s_1 shared bidders are effectively coopted by the low price of auction 2, is tempered in auctioneer 1's favor by the probability of such a low price occurring among the d_2 and s bidders – note that $P(X_{m_2\,:\,d_2+s} < X_{m_1\,:\,d_1})$ will be small if m_2 is large compared to $d_2 + s$ (conversely, this worst case might not seem so bad when viewed simply as the likelihood of very strong competitiveness within the d_1 compartment). On the other hand, the "best case" scenario from an incentive-to-pool perspective is captured in the third term, in which the price of auction 1 is determined by competitive bidding among $d_1 + s$, that is, auction 1 experiences all the competitiveness it could hope for under pooling (essentially, auction 1 co-ops the s_2 bidders shared by auction 2). Here, pooling acts as an insurance policy against abnormally low valuations among the d_1 compartment. A middle ground is tread by the first term, in which both auctions experience the same price, equal to that which would be realized if the auctions were formally merged, that is, if $n_1 = s_1$, $n_2 = s_2$.

Equation (3.2) is computed using primitive order statistics in the following corollary.

Corollary 2 (Moments of Price as a Weighted Sum of Primitive Order Statistic Moments).

$$\begin{aligned}
E[X^k_{m_1:d_1+s}|U_1] &= \sum_{i=0}^{m_2} \frac{d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}} E[X^k_{m_1+i:d+s}],\\
P(U_1) &= \sum_{i=0}^{m_2} \frac{d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}},\\
E[X^k_{m_1:d_1}|U_2] &= E[X^k_{m_1:d_1+s}] - \sum_{i=0}^{m_2} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i}}{(d+s)\binom{d+s-1}{m_1+i}} E[X^k_{m_1+i:d+s}],\\
P(U_2) &= 1 - \sum_{i=0}^{m_2} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i}}{(d+s)\binom{d+s-1}{m_1+i}}, \qquad \textit{and}
\end{aligned}$$

$$E[\Pi^k(d_1, s, d_2, m_1, m_2)] =$$

$$E[X^k_{m_1:d_1+s}] - \sum_{i=0}^{m_2-1} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i} - d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}} E[X^k_{m_1+i:d+s} - X^k_{m_1+m_2:d+s}]. \tag{3.3}$$

Corollary 2 just rewrites equation (3.2) using basic probabilistic arguments; see Section A.2 for details.

The first term on the right hand side of (3.3), $E[X^k_{m_1:d_1+s}]$, represents an unrealistically high price for auction 1 (one that would be realized if auctioneer 1 got sole access to the s shared bidders), and the second term can be interpreted as a corrective adjustment for this overstatement. As a check, note that when $s = 0$ (no pooling), this adjustment becomes zero, and $E[X^k_{m_1:d_1+s}]$ equals $E[X^k_{m_1:n1}]$, the k^{th} moment of auction 1's price in the absence of pooling.

Equation (3.3) is a closed form expression capturing the tradeoffs of pooling as discussed following Corollary 1, but exact analysis of the pooling decision is nonetheless difficult for several reasons: Π depends on five variables; the summed terms in the simplified expression for Π in (3.3) are fractions of binomial coefficients; and even the primitive order statistics in (3.3) are generally intractable except for a few special cases.

To help focus the analysis, in the remainder of Part I we remove one of Π's five degrees of freedom by creating a relationship between d_1, d_2, s_1 and s_2. In particular, ignoring integrality for the time being, we assume that

$$\frac{s_1}{d_1 + s_1} = \frac{s_2}{d_2 + s_2}. \tag{3.4}$$

That is, (3.4) assumes that the proportion of shared to total bidders is the same for both auctioneers. This is equivalent to assuming that, among the bidder population at large, a constant proportion of bidders are "savvy" bidders that pool when simultaneous auctions are held for the same item. For instance, if 20% of all bidders on the Internet today use ShopBots, then both sides of (3.4) would equal 0.2. We call the fraction of pooling to dedicated bidders the *pooling proportion.* To account for integrality, we formalize this approach by assuming that for pooling proportion $p \in [0, 1]$, of auctioneer i's n_i total bidders, $\lfloor pn_i \rfloor$ are shared (pool) and $n_i - \lfloor pn_i \rfloor$ are dedicated.

The unwieldy terms in (3.3) essentially preclude a general exact analysis of the pooling decision. However, (3.3) simplifies significantly for single item, equally sized auctions, and in the next section we exploit this fact in deriving exact analytical results.

3.3 Pooling Single Item Auctions

In this section, price and revenue are interchangeable, since each auctioneer sells only one item. Also, for improved readability we suppress the arguments m_1 and m_2 from Π.

Plugging in $m_1 = m_2 = 1$ into (3.3) and applying

$$E[X^k_{b:a}] = \sum_{i=b}^{a} \binom{i-1}{b-1} \binom{a}{i} (-1)^{i-b} E[X^k_{0:i}]$$

from [8], p38 yields

$$\begin{aligned} &E[\Pi^k(d_1, s, d_2)] \\ &= E[X^k_{1:d_1+s}] + s(2d_1 + s - 1) \cdot \left(\frac{1}{2} E[X^k_{0:d+s}] - E[X^k_{0:d+s-1}] + \frac{1}{2} E[X^k_{0:d+s-2}] \right). \end{aligned} \tag{3.5}$$

If we assume that auctions 1 and 2 have equal size, namely $n_1 = n_2 = n$, (3.5) simplifies enough that exact analytical results are possible; this is the focus of the remainder of this next section.

Proposition 3 (Expected Revenue Increases with Pooling Proportion). *Suppose that auctioneers 1 and 2 each sell a single item and have $n_1 = n_2 = n$ total bidders. For both auctions, let $\frac{t}{n}$ denote the pooling proportion, where $t \in \{0, 1, \ldots, n\}$, and furthermore let*

$$\delta_t \triangleq E[X_{1:n+t} - X_{1:n+t-1}].$$

Then, if δ_t is non-increasing in t, (equivalently, if the expected revenue of non-pooling *single-item auctions is concave in the number of bidders), then*

$$E[X_{1:n}] \equiv E[\Pi(n,0,n)] \leq E[\Pi(n-1,2,n-1)] \leq \cdots \leq E[\Pi(2,2(n-2),2)]$$
$$\leq E[\Pi(1,2(n-1),1)] = E[\Pi(0,2n,0)] \equiv E[X_{2:2n}];$$

that is, expected revenue can only increase or stay constant when increasing the pooling proportion $\frac{t}{n}$*, with strict equality from* $t = n-1$ *to* $t = n$*. If the additional condition* $\delta_{t'} > \delta_{t'+1}$ *holds at some* $t' < n$*, that is, if for* $n+t'$ *bidders the revenue of a single item auction without pooling is strictly concave in the number of bidders, then increasing the pooling fraction within the range* 0 *to* $\frac{t'}{n}$ *will strictly increase expected revenue.*

Although not intuitive, the proof hinges on showing that, as $t-1$ increases to t , the resulting difference in expected revenue equals δ_t minus a constant term, C. By then showing that this constant C equals δ_n, the conclusion of the proposition follows immediately.

The condition of Proposition 3 (concavity of single item auction expected revenue in number of bidders) is equivalent to requiring diminishing expected revenue returns with each increase in the number of bidders (or more to the point, returns to scale becoming no worse with each increase in bidders). It can be proven to hold for U[a,b], exp(λ) and Pareto(k, α) with $\alpha > 1$, distributions (cf. §3.4.1), and numerical experiments suggest it holds for Weibull, normal, and chi-squared distributions, to list a few. However, the complex nature of order statistics make it difficult to rephrase this condition in terms of the underlying distribution of valuations. Further, the condition can be shown to fail: for example, (pathological) valuations distributed as a Bernoulli random variable with small probability of equaling 1 ("success") (e.g. $p = 0.2$) yields $\delta_1 < \delta_2$.

A perhaps more fruitful, albeit heuristic approach to characterizing which valuation CDFs F give rise to the condition rests on the rough approximation

$$E[X_{i:j}] \approx F^{-1}\left(\frac{j-i}{j}\right) \qquad \text{for } j-i, j \text{ large, } 0<i<j. \tag{3.6}$$

(Although we will not use specifics here, the full result, including a characterization of the asymptotically normal distribution of $X_{i:j}$ as $j-i, j \to \infty$, is proved in Theorem 5.8 of [2].) By this approximation, the condition holds as long as

$$\begin{aligned} F^{-1}\left(\frac{n+t-1}{n+t}\right) - F^{-1}\left(\frac{n+(t-1)-1}{n+t-1}\right) \\ \geq F^{-1}\left(\frac{n+(t+1)-1}{n+t+1}\right) - F^{-1}\left(\frac{n+t-1}{n+t}\right) \end{aligned} \tag{3.7}$$

for $t = 1, \ldots, n-1$. By a first order Taylor expansion with the higher order terms dropped, (3.7) is equivalent to

$$\frac{f\left(F^{-1}\left(\frac{n+t-1}{n+t}\right)\right)}{f\left(F^{-1}\left(\frac{n+(t-1)-1}{n+t-1}\right)\right)} \geq \frac{n+t-1}{n+t+1}; \tag{3.8}$$

see Section A.4 for details. In words, the inequality (3.8) implies that the pdf f cannot be increasing too slowly along the values $\frac{n+t-1}{n+t}$; roughly speaking, this allows only a certain amount of concavity in F (conversely, (3.8) is automatically satisfied for F convex). Intuitively, because ratcheting up the pooling proportion t/n effects a small possible increase in available bidders in return for a small possible decrease, the revenue effects of such increases must be sufficiently large – as captured by the inequality (3.8) – for such ratcheting to be beneficial. $U[a, b]$ and $\exp(\lambda)$ distributions can both be shown to strictly satisfy (3.8) (we omit the calculations), corroborating Claim 1 and suggesting that (3.8) indeed has merit as a proxy for the condition of Proposition 3. The approximation (3.6) will also be employed in Section 3.5, where exact analysis is intractable.

Taking a step back, we see that if Proposition 3 holds, then ShopBots actually *help* auctioneers' revenue. Indeed, when Proposition 3 holds, auctioneers are happiest if all bidders use ShopBots (pool) (and are just as happy if very close to all – $n-1$ out of n bidders – use ShopBots).

Furthermore, using (A.37) recursively, we can write

$$E[\Pi(n-s, 2s, n-s)] = \sum_{j=1}^{s}(-C+\delta_j) - E[\Pi(n,0,n)] = \sum_{j=1}^{s}\delta_j - sC - E[X_{1:n}]$$

to quantify the revenue effect of changing the proportion of pooling bidders. The above equation holds regardless of the behavior of the δ functions (regardless of whether or not Proposition 3 holds). For the cases in which δ_s is non-increasing (increasing the pooling proportion is favorable), we see that the greatest gains come early on in the process of increasing the pooling proportion. This could be helpful in deciding where to use a marketing budget to promote the use of ShopBots among bidders: All other things being equal, it would be better to target ShopBot campaigns (e.g., free ShopBot software or banner ads on the auctioneer's website) to those bidding populations with the lowest proportion of poolers. (When pooling is not favorable [most likely when F is far from concave near $F^{-1}\left(\frac{n-1}{n}\right)$], the reverse problem of discouraging ShopBots should be uninteresting, since the effects can be achieved by just eliminating temporal overlap between auctions.)

The remainder of this subsection is devoted to exploring the reasons why Proposition 3 holds. The conditioning events E (equal prices) and U (unequal prices, $U \triangleq U_1 \cup U_2$) (cf. Section 3.2) provide a natural framework for our analysis. We begin by noting that event E is beneficial for both auctioneers: $E[\Pi(n-t, 2t, n-t)|E] = E[X_{2:2n}]$ by the proof of Proposition 2, and Proposition 3's assumptions imply further that the expected full pooling price $E[X_{2:2n}]$ is at least as large as the non-pooling expected price $E[X_{1:n}]$. This leaves open the possibility that event U is either beneficial, or non-beneficial, for the auctioneers. The Proposition below states conditions under which $E[\Pi(n-t, 2t, n-t)|E] \geq E[\Pi(n-t, 2t, n-t)|U] \geq E[X_{1:n}]$, that is, both events E and U are beneficial for the auctioneers, the former more so than the latter.

Proposition 4 (Expected Split Price Lies Between Expected Prices of Original and Merged Auctions). *Under the assumptions of Proposition 3, define further* $\hat{\delta}_t \triangleq \delta_t - \delta_{t-1}$, *and let* U *be the event that the prices in auctions 1 and 2 are unequal. If* δ_t *is non-increasing in* t, $\hat{\delta}_t$ *is non-decreasing in* t, *and* $E[\Pi(n-1,2,n-1)|U]$ *and* $E[\Pi(2,2n-4,2)|U]$ *are both greater than or equal to the original auction's price* $E[X_{1:n}]$, *then*

$$E[X_{1:n}] \leq E[\Pi(n-t,2t,n-t)|U] \leq E[X_{2:2n}] \qquad \text{for all } t \in [0,\ldots,n-2]. \quad (3.9)$$

Note that $t = 0$ corresponds to no pooling, in which case Proposition 4 holds by $E[X_{1:n}] \leq E[X_{2:n}]$ (Proposition 3). Conversely, $t = n$ or $n-1$ correspond to full pooling (for the latter, recall that each auctioneer sells one item), meaning $P(U) = 0$ and $E[\Pi|U]$ is undefined. Proof of Proposition 4 for $t \in [1,\ldots,n-2]$ directly utilizes Proposition 3 for the LHS inequality, and proves the RHS inequality utilizing the expressions for $E[\Pi|U]P(U)$ and $P(U)$ found in (3.3) (see Section A.5 for details). In Claim 2 of Section 3.4.1 we prove that Proposition 4 holds for $U[a,b]$ and exponential(λ) valuations. Numerical experiments suggest that (3.9) also holds for Weibull, normal, and chi-squared distributions (among others), as does the natural multi-item analogue of (3.9) in which $n_1 = n_2 = n$, $m_1 = m_2 = m > 1$.

Proposition 4 may seem intuitive given Proposition 3, but in fact it reveals the somewhat surprising importance of conditioning when computing $E[\Pi(n-t,2t,n-t)|U]$. Using Corollary 2 we have

$$\begin{aligned}
&E[\Pi(n-t,2t,n-t)|U]P(U) \\
&\quad= E[\Pi(n-t,2t,n-t)|U_1]P(U_1) + E[\Pi(n-t,2t,n-t)|U_2]P(U_2), \\
&\quad= E[X_{1:n+t}|X_{1:n+t} < X_{1:n-t}]P(X_{1:n+t} < X_{1:n-t}) \\
&\qquad\quad + E[X_{1:n-t}|X_{1:n-t} > X_{1:n+t}]P(X_{1:n-t} > X_{1:n+t}),
\end{aligned}$$

which implies

$$\begin{aligned} E[\Pi(n-t,2t,n-t)|U] &= E[X_{1:n+t}|X_{1:n+t} < X_{1:n-t}]\frac{1}{2} \\ &\quad + E[X_{1:n-t}|X_{1:n-t} > X_{1:n+t}]\frac{1}{2} \qquad (3.10) \\ &\quad \text{by symmetry,} \qquad (3.11) \\ &\geq E[X_{1:n}] \qquad \text{under the conditions of Proposition 4.} \end{aligned}$$

On the other hand, if $E[X_{1:j}]$ is concave in the number of bidders j (which, per the discussion following Proposition 3, is heuristically equivalent to F^{-1} concave over $\frac{2-1}{2}, \frac{3-1}{3}, \ldots, \frac{2n-1}{2n}$), then ignoring the conditioning in (3.11) yields

$$E[X_{1:n+t}]\frac{1}{2} + E[X_{1:n-t}]\frac{1}{2} \;<\; E[X_{1:n}].$$

That is, "fair gambling" for the pooling cohort would be non-beneficial for the auctioneers.

In contrast, the mechanics of pooling act to hedge against the pure winner-loser scenario of fair gambling: as discussed in the proof of Proposition 2, under event U the pooling cohort goes to the auctioneer with the lower price. Proposition 4 shows that this hedging effect – captured by the conditioning – is powerful enough to make the U event beneficial for the auctioneers. Generally speaking, the pooling decision is difficult to analyze in closed form because the unwieldy conditioned expectations in Corollary 2 cannot be innocuously unconditioned for tractability; without conditioning, the hedging effects are lost.

3.4 Pooling Under Uniform, Exponential, and Pareto Valuations

In this section we restrict our attention to three valuation distribution families: uniform[a,b], exponential rate λ, and Pareto(k, α) with shape parameter $\alpha > 1$ (Pareto with finite mean). Tractability of the pooling analysis in this section owes to the fact that these three distributions' order statistics have known closed-form solutions. Thanks to their relative tractability and their respective treatment of bounded and unbounded valuation distributions, the uniform and exponential are the most common valuation distributions assumed in the auction literature. While less tractable than the uniform or exponential, the Pareto distribution in included here to help illuminate the pooling effects of heavy-tailed valuation distributions.

3.4.1 Pooling equal size single item auctions

Although the closed forms of $E[X_{b:a}]$ for $U[a,b]$, $\exp(\lambda)$ and Pareto$(k, \alpha > 1)$ (see proof of Claim 1 below) enable a tedious ($E[\Pi(d_1, s, d_2)]$ for the $U[a,b]$ valuations contains 32 different terms when simplified), direct analysis of pooling when assuming $n_1 = n_2 = n$, we instead can directly apply the machinery of Proposition 3. With the (almost trivial) proof below, we prove that for both distributions, when $n_1 = n_2 = n$, the auctioneers' revenue increases strictly with the common pooling proportion p.

Claim 1. *Proposition 3 holds for $U[a,b]$, Exponential(λ), and Pareto(α, k) with $\alpha > 1$ (i.e., Pareto with finite mean).*

Proof. For the $U[0,1]$ case, $E[X_{i:j}] = \frac{j-i}{j+1}$, $i = 0, \ldots, j-1$ ([8], p.27), hence

$$\begin{aligned} \delta_t \equiv E\left[X_{1:n+t} - X_{1:n+t-1}\right] &= \frac{n+t-1}{n+t+1} - \frac{n+t-2}{n+t}, \\ &= \frac{2}{(n+t+1)(n+t)}, \end{aligned} \tag{3.12}$$

which decreases with t; with the observation

$$E_{U[a,b]}[X_{i:j}] \;=\; b + (b-a)\cdot E_{U[0,1]}[X_{i:j}], \tag{3.13}$$

we see that Claim 1 holds for $U[a,b]$ valuations as well.

For the exponential case,

$$E[X_{i:j}] = \frac{1}{\lambda}\sum_{l=i+1}^{j}\frac{1}{l}, \qquad i = 0,\dots,j-1 \tag{3.14}$$

(see [8], p. 17), so

$$\delta_t \;\equiv\; E\big[X_{1:n+t} - X_{1:n+t-1}\big] \;=\; \frac{1}{\lambda}\sum_{l=2}^{n+t}\frac{1}{l} - \frac{1}{\lambda}\sum_{l=2}^{n+t-1}\frac{1}{l} \;=\; \frac{1}{\lambda(n+t)},$$

which also decreases with t.

Finally, for the Pareto distributions $F(x) = 1-(k/x)^{\alpha}$, $x > k$, we have for $\alpha > 1$

$$E[X_{i:j}] = \frac{\Gamma(j+1)\Gamma(i+1-\alpha^{-1})}{\Gamma(j+1-\alpha^{-1})\Gamma(i+1)}k, \qquad i = 0,\dots,j-1 \tag{3.15}$$

(see [14], p. 241). Using the fact that equation (3.15) implies

$$E[X_{i:j}] = \frac{\alpha j}{\alpha j - 1}E[X_{i:j-1}],$$

we get

$$\delta_t \;\equiv\; E\big[X_{1:n+t} - X_{1:n+t-1}\big] \;=\; E\big[X_{1:n+t-1}\big]\frac{1}{\alpha(n+t)-1},$$

and

$$\begin{aligned}
\delta_{t+1} - \delta_t &= E\big[X_{1:n+t}\big]\frac{1}{\alpha(n+t+1)-1} - E\big[X_{1:n+t-1}\big]\frac{1}{\alpha(n+t)-1}, \\
&= E\big[X_{1:n+t-1}\big]\left(\frac{\alpha(n+t)}{\alpha(n+t)-1}\cdot\frac{1}{\alpha(n+t+1)-1} - \frac{1}{\alpha(n+t)-1}\right) \\
&= E\big[X_{1:n+t-1}\big]\left(\frac{1-\alpha}{(\alpha(n+t+1)-1)(\alpha(n+t)-1)}\right),
\end{aligned}$$

which is negative for $\alpha > 1$.

Hence, under the $U[a,b]$, $\exp(\lambda)$, and Pareto$(k, \alpha > 1)$ distributions, Proposition 3 holds and revenue strictly increases with pooling.

□

To assess the maximum possible benefit from pooling, we look at the relative price increase from full pooling relative to no pooling. This quantity is given by

$$\frac{X_{2:2n} - X_{1:n}}{X_{1:n}} = \frac{1}{2n+1} \quad \text{for} \quad U[a,b],$$

$$\begin{aligned}
\frac{X_{2:2n} - X_{1:n}}{X_{1:n}} &= \frac{\sum_{i=n+1}^{2n}\frac{1}{i} - \frac{1}{2}}{\sum_{i=2}^{n}\frac{1}{i}} \quad \text{for} \quad \exp(\lambda), \\
&\in \left[\frac{\ln(2n+1) - \ln(n+1) - \frac{1}{2}}{\ln n}, \frac{\ln 2 - \frac{1}{2}}{\ln(n+1) - \ln 2}\right], \qquad (3.16)
\end{aligned}$$

$$\text{and} \quad \frac{X_{2:2n} - X_{1:n}}{X_{1:n}} = \frac{2\alpha - 1}{2\alpha}\prod_{i=n+1}^{2n}\left(1 + \frac{1}{\alpha i - 1}\right) - 1 \quad \text{for Pareto}(k,\alpha),\ \alpha > 1, \qquad (3.17)$$

where (3.16) follows from equation (A.57). Hence, the relative price increase from full pooling grows as $O(\frac{1}{n})$ in the $U[a,b]$ case and grown no faster than $O\Big(\frac{1}{\ln(\frac{n}{2})}\Big)$ in the $\exp(\lambda)$ case. The faster growth in the $\exp(\lambda)$ case is due to the fatter tail of the exponential distribution. In both these cases, the diminishing returns to full pooling as the number of bidders grows is caused by price being pushed toward the far right of the tail in the unpooled case. In contrast, full pooling actually brings increasing returns as the number of bidders grows in the Pareto case due to this distribution's heavy tail. The right hand side of equation (3.17) approaches $\frac{2\alpha-1}{2\alpha}2^{1/\alpha}-1$ as $n \to \infty$, and approaches zero as the tail grows ($\alpha \to 1$, the distribution's mean becomes unbounded) and as the tail shrinks ($\alpha \to \infty$, the distribution's mean approaches the lower bound k). Using equation (A.57), the right hand side of equation (3.17) can be bounded above by

$$\frac{2\alpha-1}{2\alpha}\left(\frac{2n+1}{n+1}\right)^{1/\alpha}-1 \;<\; \frac{2\alpha-1}{2\alpha}2^{1/\alpha}-1 \;\leq\; \max_{\alpha>1}\frac{2\alpha-1}{2\alpha}2^{1/\alpha}-1 \;\approx\; .0614;$$

hence, for any n, α and k, the maximum benefit of full pooling for the Pareto case is approximately 6%.

Figure 3-1(a) illustrates the points made thusfar in this section: The percentage price increase is increasing and concave in the pooling proportion, and for uniform and exponential valuations this percentage increase decreases in the market size n. Note that while the uniform and exponential curves are independent of their respective parameters, the Pareto curve is insensitive to k but shifts with α, the Pareto shape parameter, which we fix at $\alpha = 2.2$.

Using the moments derived in Corollary 2, Figure 3-1(b) computes for the uniform, exponential, and Pareto with $\alpha = 2.2$ ($\alpha > 2$ ensures finite variance) distributions the coefficient of variation (standard deviation divided by the mean) of the revenue, and shows that this quantity is decreasing in the pooling proportion, decreasing in the market size. Hence, there is no risk-reward tradeoff with pooling: It simultaneously increases the mean and decreases the coefficient of variation of the revenue.

For tractability, in the remainder of Section 3.4 we restrict attention to the uniform and exponential cases, save a brief resumption of the Pareto analysis within a narrow scope to conclude Section 3.4.2.

As mentioned in Section 3.3, the insights into the hedging effects of pooling apply to the $U[a, b]$ and $\exp(\lambda)$ cases:

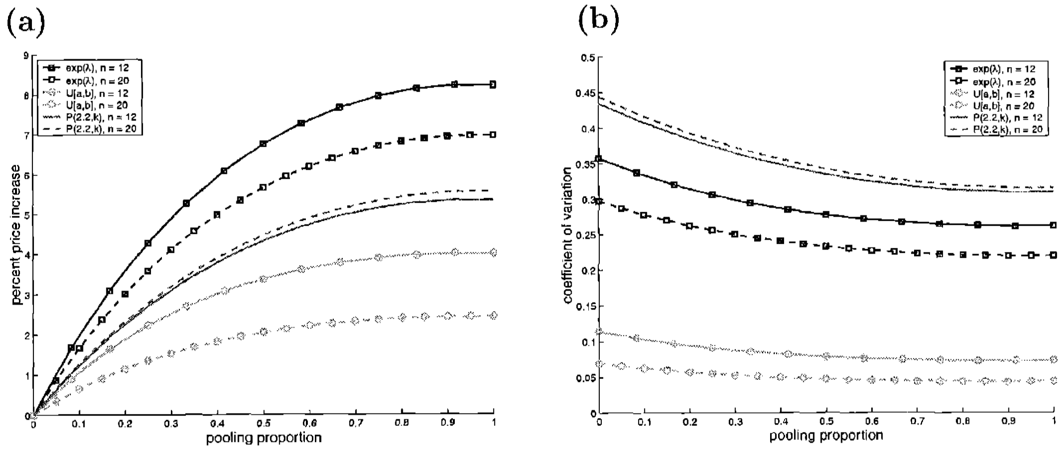

Figure 3-1: (a) The percentage price increase relative to no pooling and (b) the coefficient of variation of price, as a function of the pooling proportion p, for $n = 12$ and 20 (i.e., 24 and 40 total bidders among the two auctions), and for $U[a, b]$, $\exp(\lambda)$, and Pareto$(2.2, k)$ valuations.

Claim 2. *Proposition 4 holds for $U[a, b]$ and Exponential(λ) cases.*

(See Section A.6 for proof.)

3.4.2 Pooling asymmetric single item auctions

We now allow $n_1 \neq n_2$, that is, we allow auctioneers 1 and 2 to differ in the number of bidders they bring to the table. For a given p we are interested in the minimum number of bidders auctioneer 2 must have before auctioneer 1 finds pooling with auctioneer 2 profitable. We will denote this value by $n_2^*(p, n_1)$.

Definition of $n_2^*(p, n_1)$: Given pooling proportion p, and n_1 bidders in auction 1, $n_2^*(p, n_1)$ is the minimum number of bidders that auctioneer 2 must have to make auctioneer 1 favor pooling with auctioneer 2.

Correspondingly, given pooling proportion p, $\lfloor pn_2^*(p, n_1) \rfloor$ is the minimum number of bidders that auctioneer 2 must share to make auctioneer 1 favor pooling with auctioneer 2.

By symmetry, we can also apply the exact same idea from the perspective of auctioneer 2; taking n_1^* to be the analogous operator on a fraction and a positive integer, we call the set of integers (n_1, n_2) where $n_2 > n_2^*(p, n_1)$ and $n_1 > n_1^*(p, n_2)$ the *mutually feasible pooling set.* The following two claims are aimed at characterizing the mutual feasible pooling set for the uniform and exponential distributions, respectively.

Claim 3 (Characterizing Mutually Feasible Pooling Set: U[a,b]). *For the U[a,b] distribution, if $n_1 p \in \{1, 2, \ldots, n_1\}$, then*

$$\left\lfloor p\left(\frac{n_1}{2.3} + \frac{1}{2}\right)\right\rfloor \leq \lfloor p n_2^*(p, n_1)\rfloor \leq \left\lceil p\left(\frac{n_1}{2} + \frac{1}{2}\right)\right\rceil \tag{3.18}$$

that is, auctioneer 2 needs to have enough bidders such that he shares at least about half as many bidders as auctioneer 1.

Claim 4 (Characterizing Mutually Feasible Pooling Set: Exp(λ)). *For the $exp(\lambda)$ distribution, if $n_1 p \in \{1, 2, \ldots, n_1\}$, then*

$$\lfloor n_1 \theta_l(p)\rfloor \leq \lfloor p n_2^*(p, n_1)\rfloor \leq \lceil (n_1 + 1)\theta_u(p, n_1)\rceil$$

where

$$\theta_l(p) = \exp\left(\frac{p}{2}\right) - 1, \qquad \textit{and} \qquad \theta_u(p, n_1) = \exp\frac{p(4n_1 - n_1 p - 2)}{6n_1 - 4} - 1.$$

The bounds in Claims 3 and 4 are expressed in terms of $\lfloor p n_2^*(p, n_1)\rfloor$ for ease of analysis; of the two factors that impact auctioneer 1's revenue – the number of bidders shared by auctioneer 2 and the number of bidders dedicated to auctioneer 2 – it is the number of shared bidders that permits a fairly nice analysis (this is perhaps not surprising, since intuitively the number of shared bidders is the greater driver of revenue for auctioneer 1). However, it is more intuitive to discuss these results in terms of the minimum market share, $\frac{n_2^*(p,n_1)}{n_1 + n_2^*(p,n_1)}$, that the smaller auctioneer (auctioneer 2) must possess in order for the larger auctioneer to want to pool with him. This minimum required market share is approximately 1/3 for the $U[a, b]$ case in Claim 3. To assess the market share bounds arising from Claim 4, we look at $\frac{\theta_l(p)}{p+\theta_l(p)}$ and $\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)}$:

$$\frac{\theta_l(p)}{p+\theta_l(p)} \approx \frac{\frac{p}{2}+\frac{p^2}{8}+\frac{p^3}{48}}{\frac{3p}{2}+\frac{p^2}{8}+\frac{p^3}{48}}, \quad \lim_{p\to 0}\frac{\theta_l(p)}{p+\theta_l(p)} = \frac{1}{3}, \quad \lim_{p\to 1}\frac{\theta_l(p)}{p+\theta_l(p)} = 1-e^{-1/2} \approx 0.393,$$

$$\lim_{p\to 0}\lim_{n_1\to\infty}\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)} = \lim_{n_1\to\infty}\lim_{p\to 0}\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)} = 0.4,$$

$$\lim_{p\to 1}\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)} = 1-e^{-1/2} \approx 0.393, \quad \text{and}$$

$$\arg\max_{p\in[0,1]}\lim_{n_1\to\infty}\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)} = \lim_{n_1\to\infty}\arg\max_{p\in[0,1]}\frac{\theta_u(p,n_1)}{p+\theta_u(p,n_1)} = 0.404.$$

Hence, the bounds on the market share in the $\exp(\lambda)$ case range from 1/3 to 0.404 when n_1 is large, and the lower and upper bounds equal 0.393 under full pooling. That is, the hurdle for pooling appears to be somewhat higher in the $\exp(\lambda)$ case than the $U[a, b]$ case.

To help characterize the impact a distribution's tail has on the hurdle for pooling, we present the following result for the Pareto distribution. While the following claim addresses only the full pooling ($p = 1$) case, and hence is more limited in scope than Claims 3 and 4, the claim does show how the pooling hurdle is impacted by both heavier and lighter tails within the Pareto distribution family.

Claim 5 (Characterizing Mutually Feasible Pooling Set: Pareto). *For the Pareto(α, k) distribution with $\alpha > 1$ (finite mean),*

$$\lim_{\alpha\to 1^+} n^*_{\text{Pareto}(\alpha,k)}(1, n_1) = n_1, \quad \text{and}$$

$$\lim_{\alpha\to\infty} n^*_{\text{Pareto}(\alpha,k)}(1, n_1) = n^*_{\exp(\lambda)}(1, n_1).$$

In words, Claim 5 shows that as the tail size increases (i.e., Pareto shape parameter $\alpha \to 1$), the market must be equally split among the two auctioneers for pooling to be mutually beneficial, i.e., the minimum market share for auctioneer 2 approaches 1/2. Conversely, as the tail gets lighter ($\alpha \to \infty$) the minimum market share for auctioneer 2 shrinks to that required for the exponential case, which we showed above is approximately in the range [1/3, 0.4]. The latter result is not surprising given the fact that the minimum market share analysis is insensitive to linear transformations of the underlying valuations, and if X is distributed according to Pareto(α, k), and $Y = \alpha X - \alpha k$ is a linear transformation of X, then

$$F_Y(y) = F_X\left(\frac{y + \alpha k}{\alpha}\right) = 1 - \left(\frac{k\alpha}{y + \alpha k}\right)^{\alpha} = 1 - \left(1 + \frac{y}{\alpha k}\right)^{-\alpha},$$

so

$$\lim_{\alpha \to \infty} F_Y(y) = \lim_{\alpha \to \infty}\left[1 - \left(1 + \frac{y}{\alpha k}\right)^{-\alpha}\right] = 1 - \exp\left(\frac{-y}{k}\right),$$

that is, Y, a linear transformation of X, converges in distribution to a exponential random variable with rate $\lambda = 1/k$.

Figure 3-2 plots the exact values of the minimum required market share, $\frac{n_2^*(p,n_1)}{n_1 + n_2^*(p,n_1)}$, versus n_1 for $U[a, b]$ and $\exp(\lambda)$ valuations, for both $p = 0.2$ and $p = 1$. These curves indicate that $n_2^*(p, n_1) \approx n_2^*(1, n_1)$ for $0 < p < 1$, suggesting that the mutually-feasible pooling set is well described by analyzing the (much more tractable) full-pooling case. That is, the plots in Figures 3-2(a) and 3-2(b) are roughly horizontal lines with values 1/3 and 0.393, respectively. However, the nature of single item auctions limits the variation that can exist between auctioneers – only the sizes of n_1 and n_2 can differ. In Section 3.5 we find that, for the multi-item case in which the numbers of bidders and items can both differ between auctioneers, a full pooling analysis has clear limitations when searching for insight into the mutually feasible pooling set.

3.4.3 Competition Ratio versus market size

Towards insight into the multi-item mutually feasible pooling set, this section investigates the pooling price effects of *competition ratio*, the ratio of bidders to items, and

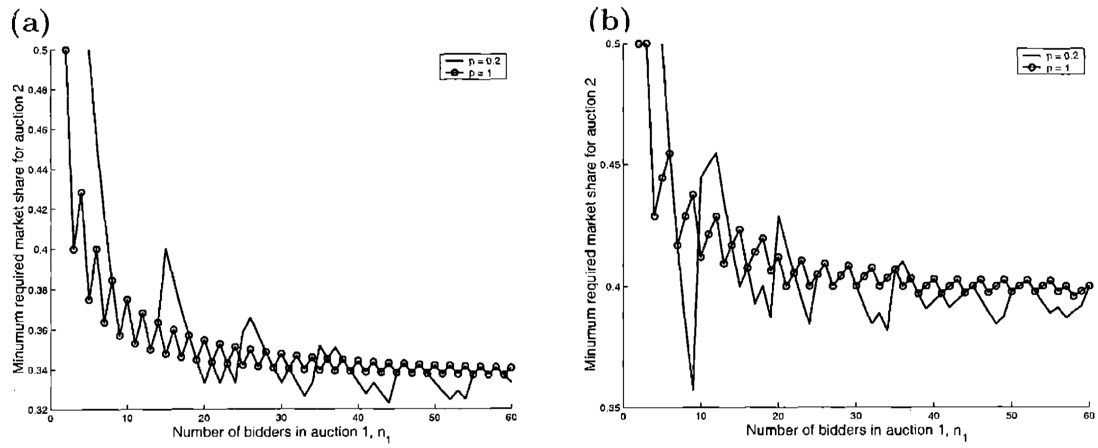

Figure 3-2: The minimum market share, $\frac{n_2^*(p,n_1)}{n_1+n_2^*(p,n_1)}$, of auction 2 such that auctioneer 1 will want to pool, as a function of the number of bidders in auction 1, for pooling proportions $p = 0.2$ and $p = 1$, for (a) $U[a, b]$ and (b) $\exp(\lambda)$ valuations.

market size, the number of bidders. In this terminology, an auction with 6 bidders and 2 items has the same competition ratio but twice the market size as an auction with 3 bidders and 1 item. For $U[0, 1]$ valuations, the expression $E[X_{m:n}] = \frac{n-m}{n+1}$ reveals that, for a non-pooled auction, the expected price is increasing in both competition ratio and market size (albeit more significantly in the former).

Under pooling, however, the impacts of competition ratio and market size are mixed. Choosing $n_1 = n$, $m_1 = m$, $n_2 = kn$, and $m_2 = k(m + 1)$, we make the competition ratio of auction 1 exceed that of auction 2, with the market size of auction 2 increasing in k. Continuing with the $U[0, 1]$ case, the expected price for auctioneer 1 under full pooling is $\frac{(k+1)n-(k+1)m-k}{(k+1)n+1}$; subtracting this price from the expected non-pooling price yields

$$\frac{n-m}{n+1} - \frac{(k+1)n - (k+1)m - k}{(k+1)n+1} = -\frac{k(m+1)}{(n(k+1)+1)(n+1)} < 0.$$

In words, auctioneer 1 is not compelled to pool fully with auctioneer 2: the latter's lesser competition ratio cannot be overcome by greater market size (increasing k). In fact,

$$\frac{d}{dk}\left(-\frac{k(m+1)}{(n(k+1)+1)(n+1)}\right) = -\frac{m+1}{((k+1)n+1)^2},$$

so increasing k actually makes auctioneer 1 worse off! While the above illustrates that market size can be a disadvantage under full pooling ($p = 1$), it is reasonable to suspect that perhaps this trend is reversed for when $p << 1$. As described in the proof of Proposition 2, when prices are unequal (event U), the pooling cohort is essentially confined to one auction; realizing that auctioneer 2 shares k bidders for every one bidder shared by auctioneer 1, it is conceivable that, for small p (equivalently, large $P(U)$), increasing k might actually help auctioneer 1. The Claim below formalizes these insights for a tractable instance under $U[a,b]$ and $\exp(\lambda)$ valuations.

Claim 6 (Market Size Helps in Spite of Competition Ratio Disparity if Pooling Proportion Sufficiently Small). *Let*

$$\beta_t \triangleq E[\Pi(n-t, 3t, 2(n-t), 0, 2)] - E[\Pi(n-t, 2t, n-t, 0, 1)]$$

and suppose $n \geq 4$. For both $U[a,b]$ and exponential(λ) valuations there exists $\bar{t} \in [1, \ldots, n]$ such that $\beta_t > 0$ for $t \in [1, \ldots, \bar{t}]$, $\beta_t \leq 0$ for $t \in [\bar{t}+1, \ldots, n]$. That is, when $n_1 = n_2 = n$, $m_1 = 0$, $m_1 = 1$, doubling the market size of auction 2 helps the pooling price of auction 1 only if the pooling proportion t/n is below $\bar{t}/n$. More precisely,

$$\text{for } U[a,b], \quad \begin{cases} \frac{3}{5} \leq \frac{\bar{t}}{n} & \text{if } n \geq 4; \\ \frac{4}{5} \leq \frac{\bar{t}}{n} & \text{if } n \geq 9, \end{cases} \quad \text{and for } \exp(\lambda), \quad \frac{\bar{t}}{n} \leq \frac{1}{2} \quad \text{if } n \geq 4.$$

Because equation (3.3) becomes increasingly unwieldy as m_2 grows above 1, in Claim 6 we create disparate competition ratios and market sizes by taking auction 1 to be a first price auction ($m_1 = 0$), and auction 2 to have either one or two items. While auction 1 in this case is not interpretable as English, the insights of the Claim are based on the mathematics of equation (3.3) (an expression of order statistics), and are indicative of the competition ratio versus market size relationship when $m_1,\ m_2 \geq 1$; see Figure 3-4.

From Claim 6 see that when auction 1 has greater competition ratio than auction 2, $E[\Pi]$ is positively or negatively impacted by increased market size in auction 2, depending respectively on whether the pooling proportion is small or large. Hence, we see that the partial pooling can behave fundamentally differently than full pooling. Further, we see that, relative to the $\exp(\lambda)$ case, the impacts of market size under $U[a,b]$ valuations are positive in a larger interval of pooling proportions. This indication that market size's impact is more pronounced in the $U[a,b]$ case follows from the fact that, roughly speaking, β_t boils down to a tradeoff between adding bidders and adding items (cf. equation (A.73) in the proof of Claim 6). While the impacts of bidders relative to items is difficult to quantify on intuitive grounds (mainly due to the relativity), a simple calculation yields

$$\frac{E_{U[a,b]}[X_{m:n+1} - X_{m:n}]}{E_{U[a,b]}[X_{m-1:n} - X_{m:n}]} = \frac{m+1}{n+2} > \frac{m}{n+1} = \frac{E_{\exp(\lambda)}[X_{m:n+1} - X_{m:n}]}{E_{\exp(\lambda)}[X_{m-1:n} - X_{m:n}]},$$

suggesting the bidders versus items tradeoff favors auctions having $U[a,b]$ valuations more than $\exp(\lambda)$ valuations.

3.5 Pooling Multi-Item Auctions

While the complex expression for $E[\Pi]$ in multi-item pooled auctions (cf. equation (3.3)) precludes direct extension of the single item results of Sections 3.3 and 3.4, in this section we introduce some simple observations that help characterize multi-item auctions' mutually feasible pooling sets. In particular, in Sections 3.5.1 and 3.5.2 we investigate $n_2^*(p, n_1, m_1, m_2)$ and $\left(\frac{n_2}{m_2}\right)^*(p, n_1, m_1, n_2)$ (respectively), natural generalizations of $n_2^*(p, n_1)$ from Section 3.4.2.

3.5.1 Full pooling analysis

For the U[0,1] case in which $p = 1$ (full pooling), it is easy to compute auction 1's price under pooling; in particular, $E[\Pi(d_1, s, d_2, m_1, m_2)] = E[X_{m_1+m_2\,:\,n_1+n_2}] = \frac{n_1+n_2-m_1-m_2}{n_1+n_2+1}$. Because this equals $E[X_{m_1\,:\,n_1}] = \frac{n_1-m_1}{n_1+1}$ if $\frac{n_2}{m_2} = \frac{n_1+1}{m_1+1}$, using this and equation 3.13 we conclude that

$$n_2^*(1, n_1, n_2, m_1, m_2) = \left\lceil (n_1+1)\frac{m_2}{m_1+1} \right\rceil \qquad \text{for } U[a,b]. \tag{3.19}$$

An analogous approach for $\exp(\lambda)$, using (A.57) to bound summations in the difference $E[X_{m_1+m_2\,:\,n_1+n_2}] - E[X_{m_1\,:\,n_1}]$, yields

$$\left\lfloor n_1 \cdot \frac{m_2}{m_1+1} \right\rfloor \;\le\; n_2^*(1, n_1, n_2, m_1, m_2) \;\le\; \left\lceil (n_1+1) \cdot \frac{m_2}{m_1} \right\rceil \qquad \text{for } \exp(\lambda). \tag{3.20}$$

Similarly, the analysis of Section A.9 shows that

$$\lim_{\alpha \to 1} n^*_{2,\text{Pareto}(\alpha,k)}(1, n_1, n_2, m_1, m_2) \;=\; n_1 \frac{m_2}{m_1}, \quad \text{and} \tag{3.21}$$

$$\lim_{\alpha \to \infty} n^*_{2,\text{Pareto}(\alpha,k)}(1, n_1, n_2, m_1, m_2) = n^*_{2,\exp(\lambda)}(1, n_1, n_2, m_1, m_2) \tag{3.22}$$

(that is, as α becomes large the n_2^* for the Pareto case approaches n_2^* for the exponential case). Together, equations (3.19)-(3.22) suggest that perhaps $n_2^*(p) \approx n_1 \cdot \frac{m_2}{m_1}$ (where we have suppressed the remaining arguments of n_2^* for readability).

Furthermore, for m_i, n_i large, we can use the approximation (3.6) to conclude that when $n_2 \approx n_1 \frac{m_2}{m_1}$, $E[X_{m_1+m_2\,:\,n_1+n_2}] \approx E[X_{m_1\,:\,n_1}]$. In words, full pooling has approximately the same expected price as no pooling when m_i and n_i are large. Applying the insight gleaned for single item auctions via Proposition 3, it appears reasonable to expect that if $p = 1$ is no better than $p = 0$, then $0 < p < 1$ is probably no better either; if this is true, then for large m_i, n_i, $n_2^*(p) \approx n_1 \frac{m_2}{m_1}$. Furthermore, a similar analysis from auctioneer 2's viewpoint suggests that $n_1^*(p, n_2, m_1, m_2) \approx \frac{m_1}{m_2} n_2$; consequently, we predict that for large m_i and n_i, the mutually-feasible pooling region

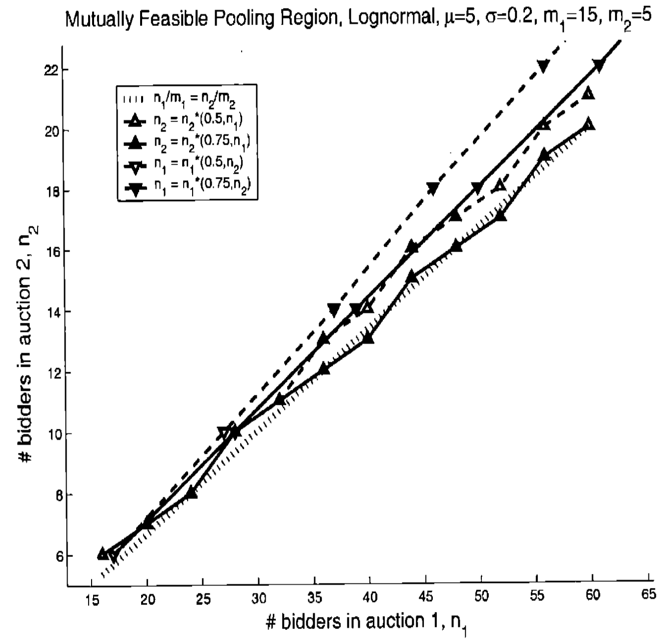

Figure 3-3: The mutually-feasible pooling region for pooling proportions $p = 0.5$ ($\square$) and $p = 0.75$ ($\cdot$). Also displayed is the line $n_2 = \frac{m_2}{m_1} n_1$. The valuations are log-normal with $\mu = 5$, $\sigma = 0.2$, and there are $m_1 = 15$ and $m_2 = 5$ items for sale.

consists of only a very small area in (n_1, n_2) space surrounding the line $n_2 = \frac{m_2}{m_1} n_1$. That is, both auctions need almost the same bidder-to-item ratio, a quantity we refer to as the *competition ratio*, for pooling to be mutually beneficial. Numerical experiments depicted in Figure 3-3 corroborate these claims, even for moderate m_i, n_i.

The approximation (3.6) also affords insight into the general benefit of pooling, even if $n_2 \neq n_1 \frac{m_2}{m_1}$. In particular, it is rather straightforward to argue (heuristically) that, if F^{-1} is concave, then *full pooling* increases the *total* expected revenue (the sum of auction 1 and auction 2's expected revenues). (It is another matter to show whether or not a particular auctioneer's piece of the larger pie is bigger than his original, non-pooling piece.) In particular, if we let $p_i = \frac{n_i - m_i}{n_i}$, and $p_0 = \frac{n_1 + n_2 - m_1 - m_2}{n_1 + n_2}$, it can be shown (via algebra) that $p_0 \geq \frac{m_1}{m_1+m_2}p_1 + \frac{m_2}{m_1+m_2}p_2$. Then, since F^{-1} is increasing,

$$
\begin{aligned}
(m_1+m_2)F^{-1}(p_0) &\geq (m_1+m_2)F^{-1}\left(\frac{m_1}{m_1+m_2}p_1 + \frac{m_2}{m_1+m_2}p_2\right) \\
&\geq m_1F^{-1}(p_1) + m_2F^{-1}(p_2),
\end{aligned}
$$

where the second inequality holds if F^{-1} is concave (equivalently F is convex). The far left term is the approximate total expected revenue under full pooling, and the far right the total expected revenue under no pooling.

3.5.2 Numerical study: competition ratio versus market size

We now use numerical plots of mutually feasible pooling regions (Figure 3-4) to delve into aspects of partial pooling that are too subtle to characterize by the full pooling analysis above. The plots in Figure 3-4 corroborate the insight of Claim 6: market size can compensate for competition ratio for small pooling proportions (plots (a) and (c) versus (b) and (d)), and, relative to $\exp(\lambda)$ valuations, impacts of market size are greater for $U[a, b]$ (plot (a) versus (c)). Furthermore, while Claim 6 considered only the perspective of auctioneer 1, Figure 3-4 considers both auctioneers' perspectives (recall the definition of mutually feasible pooling set); the net result is a mutually feasible pooling set tilted slightly clockwise for larger auction 2 market size. In general, the Figure 3-4 plots' closeness to $n_1\frac{m_2}{m_1}$ indicates that, while limited, the rule of thumb $n_2^*(p) \approx n_1\frac{m_2}{m_1}$ has substantial merit.

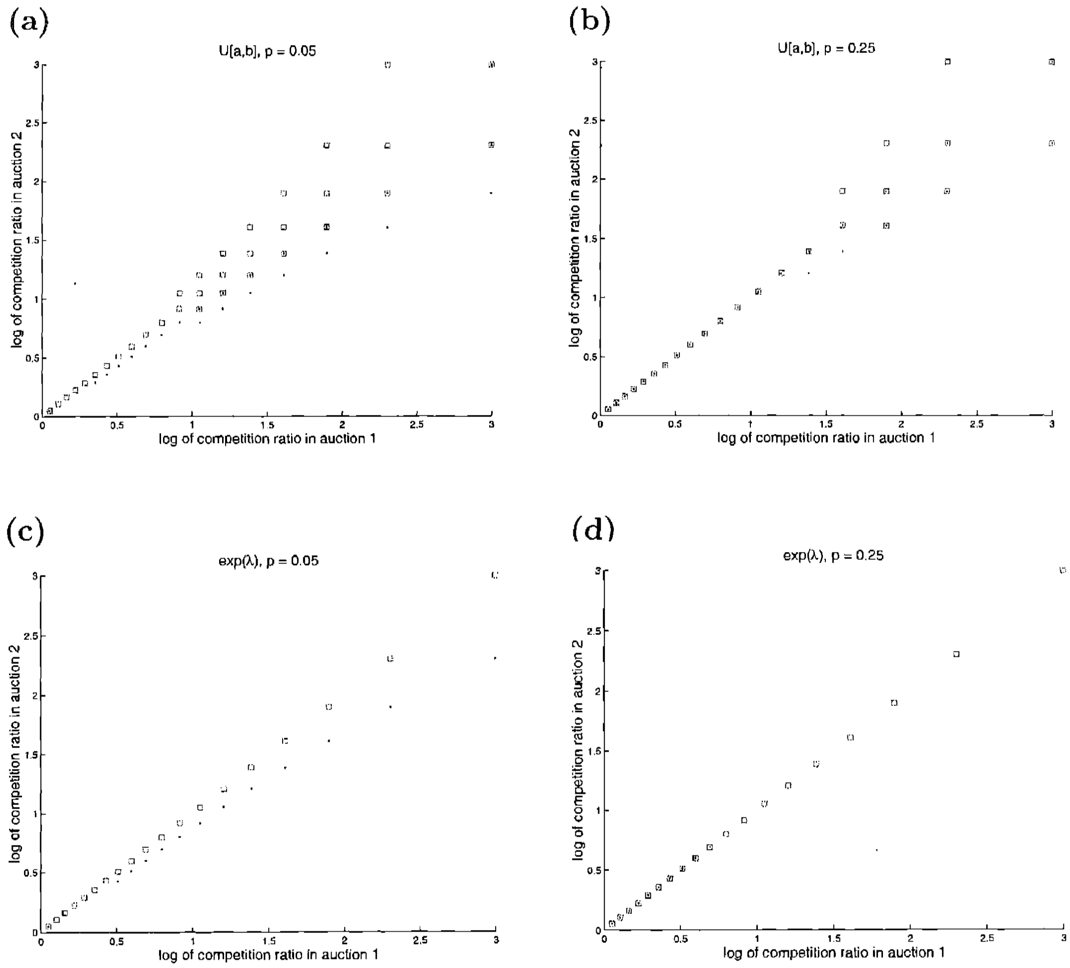

Figure 3-4: Mutually feasible pooling region in terms of competition ratios (number of bidders to number of items), when auction 1 has market size (number of bidders) 20, auction 2 has market sizes 20 (□) and 60 (·); for ease of presentation, a log-scale is used. The four figures consider two pooling proportions ($p = 0.05$ and $p = 0.25$) and two valuation distributions ($U[a, b]$ and $\exp(\lambda)$).

Chapter 4

Part I Conclusions

Arguably the most crucial aspect of any auction is the number and quality of the customers that participate, or bid, in the auction. Despite the importance of this issue, the existing literature pertinent to private sector auctions treats it as an exogenous factor, and focuses attention primarily on developing new and optimal auction formats. In this part of the thesis we investigate endogenizing operationally the number and type (e.g., dedicated or shared) of bidders by considering the issue of bidder sharing among competing auctioneers selling the same commodity. Not only is this sharing plausible, it is also proven: on the Internet today defacto bidder sharing is a result of bidders employing smart software agents – so called ShopBots – to bid in multiple auctions simultaneously. To our knowledge, this is the first work to consider bidder sharing from an operational perspective.

This part of the thesis uses purely analytical methods to explore the tradeoffs involved in auction pooling, which are particularly complicated in the case of partially pooled auctions (involve orderings of order statistics). We show that, for single item auctions, auctioneers having the same number of bidders actually benefit as the proportion of pooling (or ShopBot-using) bidders in the overall population grows, provided that the revenue from a second-price, single item auction would be concave in the number of bidders. This latter condition relies on only the underlying distribution of bidder valuations; in essence, it states that, on average, the increase in revenue from adding one additional bidder to an auction with 100 bidders is less than the increase in revenue from adding one additional bidder to an auction with 99 bidders.(Furthermore, this condition is heuristically equivalent to "limited" concavity of the underlying valuation distribution.)

For the separate cases of uniform and exponential bidder valuations, we also characterize how much two (single item) auctioneers must share with one another before pooling is mutually beneficial. While in the uniform valuations case the smaller auction needs to have at least about a one-third market share, the analogous proportion is larger in the exponential valuations case – between one-third and 0.4 as the total market size (number of bidders) gets large. For auctions selling many items to many bidders, heuristic arguments and numerical results imply that, in general, the auctioneers' ratios of items to bidders need to be approximately the same for pooling to be mutually beneficial. Though ignored by this latter rule of thumb, the other relevant metric of auction efficiency – the absolute numbers of items and bidders – is found to have what amounts to second order effects on pooling.

Although necessarily limited in scope (since we assumed an English auction format, and ignored reservation price), the results of Part I suggest competing auctioneers can indeed benefit from bidder sharing as long as the sharing is not too lopsided, and that the use of ShopBots should actually be mutually encouraged by the auctioneers in this case. Although the important issue of what impact pooling has on bidders is not directly addressed, we feel it suffices for now to note that, if the strategies of all other bidders are fixed, a particular bidder can only do better by pooling. That is, in a game-theoretic sense (the most appropriate setting for such a discussion), the case of all bidders wishing to pool is a unique Nash equilibrium (this intuitive conclusion is not formalized in the current study).

Part II

Optimal Scoring Rule Analysis for a Multi-Attribute Online Auction

Chapter 5

Introduction

The market for online business-to-business auctions is enormous (estimated at $746B in 2004 by Kafka *et al.* 2000 [15]). Within the industrial procurement setting, high-value complex items have traditionally been procured via Request for Quotes (RFQ) processes. An RFQ process allows the sale to be determined by a variety of attributes, involving not only price, but quality, lead time, contract terms, supplier reputation, and incumbent switching costs. It also lets the manufacturer reveal his preferences and permits the suppliers to compete on their own specialized dimensions. Consequently, eMarketplaces are currently being developed to partially automate (to varying degrees) the RFQ process (see Kafka *et al.* for examples). This part of the thesis studies multi-attribute auctions, an emerging mechanism for fully automated RFQs. For further discussion of multi-attribute auctions in this setting see [3], the companion paper to this part of the thesis.

There are two primary objectives in the auction theory literature, revenue maximization (on the part of the auctioneer) and (allocative) efficiency; in multi-attribute auctions, it is appropriate to speak in terms of utility maximization rather than revenue maximization. Efficiency is often the goal in public-sector auctions, whereas utility maximization is typically strived for in private-sector auctions. An efficient auction mechanism maximizes the total surplus, without concerning itself with how this surplus is divided among the bidders and the auctioneer. We assume an utility-maximizing auctioneer; see Section 2.5 of the companion paper for a brief discussion of efficiency.

Adopting the procurement motivation from the companion paper, we often refer to the manufacturer as the auctioneer (or bid-taker) and the suppliers as bidders. To engage bidders in a multi-attribute auction, an auctioneer needs to provide the bidders with some information pertaining to how he values the non-price attributes. While several rather obtuse approaches are possible (e.g., the auctioneer could provide shadow prices from a mathematical program without revealing the mathematical program), the predominant approach in RFQ practice – and the one favored by most bidders because of its straightforward nature – is for the auctioneer to announce a *scoring rule* in terms of the bid price and various attributes. This scoring rule may, or may not, be identical to the auctioneer's true utility function; indeed, this is the crux of the strategic problem from the auctioneer's viewpoint.

There are two key papers on multi-attribute auctions with scoring rules. In a white paper for the eMarketplace perfect.com, Milgrom [18] has recently shown that efficiency is achieved if the auctioneer announces his true utility function as the scoring rule, and conducts a Vickrey (i.e., second-price, sealed-bid) auction based on the resulting scores. The other important paper – and the one most relevant for our purposes – is by Che [6], who considers a two-dimensional price-quality procurement auction in a sealed-bid setting. He assumes that the quality costs of his two symmetric suppliers are a function of a single parameter that is private information and independent across suppliers (Branco [4] generalizes this research to the case of correlated supplier costs). Che assumes a positive cross partial of cost with respect to quality and type, implying that the producer with the lower cost for producing a given quantity is also the producer with the lower marginal cost of producing quality (this is not the case in our Figures 7-2 and 7-3 in Section 7.2). He shows that to maximize utility, the auctioneer – who only knows the probability distribution of the cost parameter – announces a scoring rule that understates the value of quality, so as to limit the informational rents collected by the low-cost suppliers.

In this part of the thesis, the auctioneer maximizes his utility but does so with knowledge of the suppliers' costs; with this informational assumption and assumptions on bidder behavior (discussed in the next section), the auctioneer can predict the auction outcome (his ex-post utility) for a given scoring rule. Under these assumptions, we derive the optimal scoring rule for an auctioneer running an open-ascending multi-attribute auction. We assume there is no minimum score required for the auctioneer to transact, which is used in price-only optimal auctions assuming bidders with private valuation information [19].

The informational assumption and optimal scoring rule analysis in this part of the thesis can be thought of in two ways: First, the auctioneer could use some method to learn the suppliers' costs before applying this knowledge in selecting an optimal scoring rule; this is the approach studied in the companion paper [3], which proposes a multi-round auction mechanism (not unlike traditional request for quotes processes) in which the auctioneer observes suppliers' bids and solves an inverse problem to deduce their costs. In such a situation, an obvious alternative to conducting an auction under an optimal scoring rule is simply for the buyer to identify the most able supplier and "cut a deal" with him, leaving the buyer with the entire surplus. Despite the fact that committing to an auction and selecting an optimal scoring rule is considerably more difficult than simply cutting a deal, and does not guarantee full surplus extraction for the buyer, we feel that other considerations tip the scale in favor of the optimal scoring rule approach. In particular, while a buyer might prefer utility-maximization to the goal of efficiency, arguments in favor of the latter by Milgrom [18] and Wise and Morrison [25] cannot be dismissed out of hand. They and others point out that blatant attempts to extract the entire surplus would likely be viewed by suppliers as unfair, and could hence lead to damaging reputation effects under repetition of such a strategy. (See the companion paper [3], which expounds further on these points and draws on discussions with a chief technology officer of a company that sells software for online procurement auctions.) The optimal scoring rule strategy, on the other hand, has the quality of tangible competition: the winning supplier's take-home surplus is exactly equal to the difference between his score and the score of the next best bidder, which is not unlike the outcome in many RFQs, contracts and auctions. To summarize the points made in [3], the optimal scoring rule approach attempts to walk a fine line of maximizing the buyer's utility while generating the perception that the suppliers' disappointing profits are a result of a highly competitive marketplace.

Second, we view the informational assumption and optimal scoring rule analysis as a tractable framework within which to study the value of competition; by analyzing the auction outcome under the complete cost knowledge hypothesis, the cost to the auctioneer of incomplete knowledge of suppliers' costs can be better understood, and, even more generally, the optimal scoring rule under the complete cost knowledge framework could be used to analyze the sensitivity of competition to various factors, e.g., estimating the value of five suppliers instead of four, and the effects of different supplier cost structure combinations.

The remainder of this part of the thesis is organized as follows: The model is briefly introduced in Chapter 6, the analysis of the optimal scoring rule is presented in Chapter 7, some practical considerations are discussed in Chapter 8, and finally brief Part II conclusions are provided in Chapter 9.

Chapter 6

The Model

6.1 Notation

We assume that the auctioneer is buying a single item. Although this item may represent six months of production for a subassembly, we assume that it is sold to a single bidder, and we model it as a single item. We use mnemonic subscripts, where $a = 1, \ldots, A$ indexes the attributes, and s indexes the S suppliers.

For multi-attribute auctions, it is important to distinguish between attributes that are endogenous (i.e., bidder-controllable), such as lead time and quality, versus attributes that are exogenous, such as a bidder's reputation at the time of the auction. For expositional purposes, we assume that all non-price attributes are endogenous, and defer a discussion of exogenous attributes to Section 8.2. Each bid is of the form $(p, x_1, \ldots, x_A)$, where p denotes price and x_a is the magnitude of non-price attribute a for $a = 1, \ldots, A$. To simplify the presentation and analysis, we assume that the non-price attributes are continuous, nonnegative variables (thus the domains of the cost, utility and scoring functions are the nonnegative real numbers) and that larger values of x_a are more desirable from the auctioneer's point of view and more costly from the suppliers' point of view. Hence, attributes such as tolerance or lead time, which are desirable and costly when low in magnitude, need to be defined relative to a worst-case upper bound. Supplier s's cost function is additive across attributes, and is given by $\sum_{a=1}^{A} c_{as}(x_a)$, where c_{as} is increasing, convex (convexity and concavity are strict in Part II of the thesis) and twice continuously differentiable in x_a.

The auctioneer's true utility function and scoring rules are assumed to be additive across non-price attributes. While the separability across attributes of the cost and utility functions and scoring rules makes the optimal scoring rule selection problem more tractable and may not hold in practice, most existing multi-attribute auction software packages also make these assumptions (one exception is CombineNets ClearBox 2.0, www.combinenet.com). The true utility function is given by $\sum_{a=1}^{A} v_a(x_a) - p$, where v_a (mnemonic for value) is increasing, concave and twice continuously differentiable in x_a. The scoring rule is denoted $\sum_{a=1}^{A} \hat{v}_a(x_a) - p$; however, to avoid introducing new notation, we refer to $\hat{v}_1, \ldots, \hat{v}_A$, collectively and individually, as scoring rules. We require that for fixed a, the scoring rule $\hat{v}_{ar}$ must be increasing and concave in x_a. We also assume that $\frac{\partial c_{as}(x_a)}{\partial x_a} < \frac{\partial \hat{v}_{ar}(x_a)}{\partial x_a}$ as $x_a \to 0^+ \ \ \forall s$, and $\frac{\partial \hat{v}_{ar}(x_a)}{\partial x_a} \to 0$ as $x_a \to \infty$, to guarantee that the solution to the optimization problem in (6.1)-(6.2) possesses a finite positive solution.

6.2 Mechanism

At the beginning of the auction the auctioneer announces the scoring rule $\sum_{a=1}^{A} \hat{v}_a(x_a) - p$ to all suppliers. Suppliers then submit bids of the form $(p, x_1, \ldots, x_A)$ in an open-ascending manner. We envision this mechanism taking place electronically (e.g., over the Internet). Within the auction, suppliers have ample opportunity to bid. The auctioneer ranks the bids according to the scoring rule and displays the ranked scores, but does not reveal the bidders' identities or detailed bids. In contrast to a traditional open-ascending auction, submitted bids need not exceed the current best bid. However, there is a minimum bid increment (with respect to the scoring rule) to take the lead (thereby speeding up the auction), and we assume that the highest bidder at the end of the auction wins the contract, at his proposed bid.

6.3 Supplier Behavior

Suppliers are assumed to bid according to their *myopic best-response* (MBR). A supplier using MBR chooses his next bid to maximize his current profit, assuming no other suppliers change their bids; i.e., he behaves as if the auction was ending after his bid. More specifically, if during the auction the current top score is S (we use S to denote the score and the number of suppliers, but this should cause no confusion) and the minimum bid increment is ϵ, then supplier s solves the following optimization problem (note that the subscript s is suppressed in the decision variables x_{as} and p_s):

$$\max_{p, x_1, \ldots, x_A} \quad p - \sum_{a=1}^{A} c_{as}(x_a) \tag{6.1}$$

$$\text{subject to} \quad \sum_{a=1}^{A} \hat{v}_a(x_a) - p = S + \epsilon. \tag{6.2}$$

If the optimal objective function value in (6.1) is nonnegative then the corresponding optimal solution is the MBR bid. If the optimal objective function value is negative, then the MBR is to not submit a new bid.

As pointed out in the companion paper [3], the MBR assumption has been used in a variety of recent auction studies (e.g., [9], [24], [20], [13], although the first two studies refer to MBR as "straightforward bidding"). In contrast to the fully rational behavior model of Part I, the MBR model given above asserts a middle ground regarding bidders' rationality. On the one hand, bidders are assumed to be sophisticated enough to formulate and solve (6.1)-(6.2). On the other hand, a more astute bidder would formulate prior distributions on the other bidders' cost functions and the auctioneer's utility function, and would account for the fact that these other players would be solving their own game-theoretic problems. Although we view the MBR assumption as an attractive way to include a degree of supplier rationality while maintaining tractability of the game, note that some may consider the inclusion of any rationality to be controversial, as there may be bidders who do not even use an optimization-based mental (such as (6.1)-(6.2)) for bidding.

Our earlier assumptions about the cost function and scoring rules imply that the solution to (6.1)-(6.2) can be found by solving the first-order conditions

$$\frac{\partial \hat{v}_a(x_a)}{\partial x_a} = \frac{\partial c_{as}(x_a)}{\partial x_a} \quad \text{for} \quad a = 1, \ldots, A. \tag{6.3}$$

If we let x_a^* denote the solution to (6.3), then the corresponding bid price is

$$p^* = \sum_{a=1}^{A} \hat{v}_{ar}(x_a^*) - S - \epsilon. \tag{6.4}$$

If $p^* \geq \sum_{a=1}^{A} c_{as}(x_a^*)$ then $(p^*, x_1^*, \ldots, x_A^*)$ is the MBR bid; otherwise, the MBR is to not submit a new bid.

Chapter 7

Analysis

7.1 The Optimal Scoring Rule

We now turn to our analysis. With the true cost functions in hand, the auctioneer can determine his optimal scoring rule. By the MBR assumption, supplier s will submit bids that solve (6.1), (6.2). In Section 6.1 we imposed sufficient, but not necessary, conditions on $\hat{v}_1, \ldots, \hat{v}_A$ for problem (6.1), (6.2) to possess a unique finite positive solution; if $\hat{v}_1, \ldots, \hat{v}_A$ satisfies these conditions, we refer to $\hat{v}_1, \ldots, \hat{v}_A$ as *feasible*.

Notice that supplier s will drop out of the auction's open-ascending competition no later than when his profit $p - \sum_{a=1}^{A} c_{as}(x_a^*)$ equals zero (where x_a^* is the solution to (6.3)), which occurs at the *maximum drop-out score*

$$S_s = \sum_{a=1}^{A} \hat{v}_a(x_a^*) - \sum_{a=1}^{A} c_{as}(x_a^*). \tag{7.1}$$

Our analysis below, culminating in (7.7), depends only on the top two bidders, and we index the suppliers so that their maximum drop-out scores satisfy $S_1 \geq S_2 \geq \ldots \geq S_S$; note that this ranking depends upon our choice of scoring rule. To guarantee that these two suppliers can actually bid, we impose the constraint

$$S_2 > \epsilon. \tag{7.2}$$

To keep our analysis simple, we ignore the effect of the minimum bid increment on the detailed sequence of bids, and exclude the possibility that $S_1 \leq S_2 + \epsilon$. That is, we require the scoring rule $\hat{v}_1, \ldots, \hat{v}_A$ to satisfy

$$S_1 > S_2 + \epsilon. \tag{7.3}$$

Depending on the detailed sequence of bids, supplier 1's winning score may lie anywhere in the interval $(S_2 - \epsilon, S_2 + \epsilon]$. To make the analysis cleaner, we assume that the winning bidder wins with a score equal to S_2; this assumption miscalculates the auctioneer's final utility by at most ϵ, which is dwarfed by the magnitude of the bids. With this assumption, the winning score in the open-ascending auction will be submitted by supplier 1 and will equal S_2; supplier 1's winning bid is the solution to

$$\max_{p, x_1, \ldots, x_A} \quad p - \sum_{a=1}^{A} c_{a1}(x_a) \tag{7.4}$$

$$\text{subject to} \quad \sum_{a=1}^{A} \hat{v}_a(x_a) - p = S_2. \tag{7.5}$$

By (6.4) and (7.1), this solution is x^*_{a1} (we now include the supplier subscript in the bids), which solves (6.3) for $s = 1$, and

$$\begin{aligned} p^*_1 &= \sum_{a=1}^{A} \hat{v}_a(x^*_{a1}) - S_2 \\ &= \sum_{a=1}^{A} \hat{v}_a(x^*_{a1}) - \sum_{a=1}^{A} \hat{v}_a(x^*_{a2}) + \sum_{a=1}^{A} c_{a2}(x^*_{a2}). \end{aligned} \tag{7.6}$$

Recall that the auctioneer's true utility function is $\sum_{a=1}^{A} v_a(x_a) - p$. Using equation (7.6), the auctioneer's *optimal* scoring rule is the feasible scoring rule that solves

$$\max_{\hat{v}_a} \quad \sum_{a=1}^{A} v_a(x^*_{a1}) - \sum_{a=1}^{A} \hat{v}_a(x^*_{a1}) + \sum_{a=1}^{A} \hat{v}_a(x^*_{a2}) - \sum_{a=1}^{A} c_{a2}(x^*_{a2}), \tag{7.7}$$

subject to constraints (7.1)-(7.3). It is possible that the auctioneer's true valuation function violates equation (7.3); in spite of – or rather, because of – such "near ties," the auctioneer extracts nearly all the surplus in the auction by revealing his true valuation function as the scoring rule. In this case, solving (7.7) subject to (7.1)-(7.3) can do no better than simply announcing the true valuation function as the scoring rule (see the proof of Proposition 6), and – despite its violation of (7.3) – we consider v to be optimal.

We re-emphasize that although equations (7.1)-(7.3), (7.7) depend on only the top two of the S suppliers, the rankings of the suppliers in this optimization problem is a function of the decision variables (i.e., scoring rule). A brute force approach to the problem is to solve (7.1)-(7.3), (7.7) for all $2\binom{S}{2}$ ordered pairs of suppliers, and the ordered pair that generates the highest utility in (7.7) provides the optimal scoring rule. With the aid of Propositions 5 and 6 below, we derive a more efficient approach to this problem. In the discussion below we do not refer to a specific scoring rule, and therefore drop the assumption that the suppliers are ordered such that $S_1 \geq S_2 \geq \ldots \geq S_S$.

To structure the presentation, we call a bid $(p, x_1, \ldots, x_A)$ *enforceable* if there exists a feasible scoring rule $\hat{v}_1, \ldots, \hat{v}_A$ satisfying (7.2)-(7.3) that causes the auction to be won with attribute levels $x_1, \ldots, x_A$ at price p. We say that such a rule *enforces* bid $(p, x_1, \ldots, x_A)$, and utility $v(x_1, \ldots, x_A) - p$, where v denotes the auctioneer's true valuation function. For ease of presentation, we let $c_s(\vec{x}) = \sum_{a=1}^{A} c_{as}(x_a)$ and $v(\vec{x}) = \sum_{a=1}^{A} v_a(x_a)$, where $\vec{x} = (x_1, \ldots, x_A)$. Proposition 5 and all subsequent nonobvious results are proved in Appendix B.

Proposition 5 (Enforceability). *Let supplier i be the low-cost supplier at $\vec{x} = (x_1, \ldots, x_A)$, i.e., $c_i(\vec{x}) < c_s(\vec{x})$, $s \neq i$. Let T_i be the hyperplane tangent to supplier i's cost surface c_i at $\vec{x}$, and let supplier i's profit π satisfy $\pi > \epsilon$, where ϵ is the minimum bid increment. Then $(c_i(\vec{x}) + \pi, \vec{x})$ is enforceable if and only if $c_s(\vec{x}) > c_i(\vec{x}) + \pi$ for all $s \neq i$, and for some $j \neq i$ $T_i + \pi$ intersects supplier j's cost surface c_j (i.e., if* *there exists a $\vec{z}$ such that $c_j(\vec{z}) < T_i(\vec{z}) + \pi$).*

Intuition for Proposition 5 is provided in Figure 7-1. The four graphs of Figure 7-1 depict the geometry of enforcing the rightmost circle in graph (a). Graph (b) shows a tangent of c_2 that permits a "half rainbow"-shaped function to pass through both it and $T_1 + \pi$ in graph (c); this function, when shifted to intersect the origin, acts as an enforcing scoring rule in (d). Enforcement of the rightmost circle in (a) is achieved because of perfect competition between this bid and a dropout bid for supplier 2, namely the leftmost circle in (a). Generally speaking, for two given bids, graphs (d) and (c) visually equate perfect induced competition with the existence of a half rainbow-shaped curve passing through the bids' corresponding point-slope pairs; this geometric observation governs how the auctioneer may induce competition through choice of scoring rule, leading to the necessary and sufficient conditions for enforcement in Proposition 5, which recasts the half rainbow requirement in terms of tangent hyperplanes. The general proof of Proposition 5 requires an attribute-by-attribute construction of an enforcing scoring rule as done for a single attribute in Figure 7-1, and proper modification of this rule to account for the presence of other (i.e., beyond the two pictured in Figure 7-1) suppliers.

Referring to graph (a) in Figure 7-1, Proposition 5 would hold even if π is lowered until $T_1 + \pi$ barely intersects c_2; this observation leads to the following corollary.

Corollary 3 (Minimum Prices). *Let supplier i be the low-cost supplier at $\vec{x}$ such that $c_i(\vec{x}) + \epsilon < c_s(\vec{x})$ for all $s \neq i$. If for some $j \neq i$, $T_i + \epsilon$ intersects supplier j's cost surface c_j, then enforceable prices at $\vec{x}$ approach $c_i(\vec{x}) + \epsilon$ from above; otherwise, enforceable prices at $\vec{x}$ approach $c_i(\vec{x}) + \min_{\vec{z}, s \neq i}\{c_s(\vec{z}) - T_i(\vec{z})\}$ from above.*

When applying Corollary 3 to cases in which $(p + \delta, \vec{x})$ is enforceable for $\delta \to 0^+$, we will ignore the arbitrarily small δ and for practical purposes consider $(p, \vec{x})$ enforceable. In words, Corollary 3 states that supplier i's profit, π, will be ϵ if $T_i + \epsilon$ intersects some c_j, and is $\min_{\vec{z}, s \neq i}\{c_s(\vec{z}) - T_i(\vec{z})\}$ otherwise. Hence, this result allows us to put the price tag $c_i(\vec{x}) + \pi$ on any A-tuple of non-price attribute levels, which

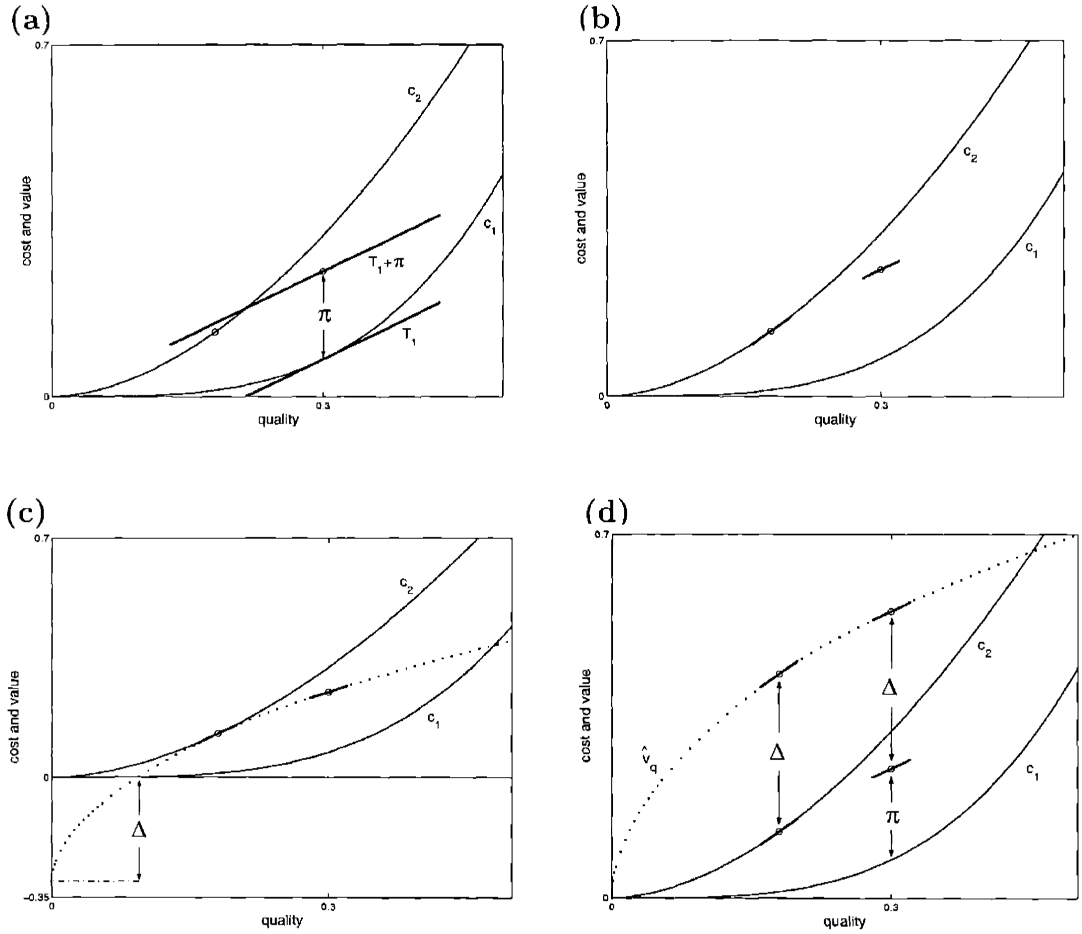

Figure 7-1: Graphical intuition behind the proof of Proposition 5 for a two-supplier, two-dimensional (price and quality) auction.

quantifies how effectively competition can, or cannot, be exploited via a properly chosen scoring rule. This result transforms the problem from a "what if" we choose scoring rule $\hat{v}$ approach, to an "informed shopper" approach of utility maximization given prices $c_i(\vec{x}) + \pi$. Notice that the prices themselves are not market prices in the traditional sense, but rather the prices of idiosyncratic markets distorted for hypercompetition (lowering supplier i's profit π).

Illustrations of enforceability with two suppliers are provided in Figures 7-2 and 7-3 in Section 7.2 for a two-dimensional (price, quality) problem. Near $q = 0.86$, the prices that are enforceable in Figure 7-3 are lower than those of Figure 7-2; the tangent lines (one-dimensional hyperplanes) to c_1 near $q = 0.86$ are much closer to c_2 in Figure 7-3, and by Corollary 3 this permits prices much closer to supplier 1's true cost curve.

We now broaden our view and present a result that incorporates the auctioneer's utility maximization problem in (7.1)-(7.3), (7.7). If supplier s wins the auction under an optimal scoring rule, we say that supplier s is *optimal.*

Proposition 6 (Restricting Search Over Suppliers). *For suppliers $s = 1, \ldots, S$, let $M_s = \max_{\vec{x}}\{v(\vec{x}) - c_s(\vec{x})\}$, where v is the auctioneer's true valuation function and c_s is supplier s's cost surface. Suppose (without loss of generality) that $M_i \geq M_s$, for all s. Then supplier i is optimal.*

Note that M_s is supplier s's maximum drop-out score if the auctioneer reveals his true valuation in the scoring rule. To describe the main intuition behind the proof of Proposition 6, we note that the maximization problem that defines M_i has its optimal solution at some $\vec{x}^*$ at which v and c_i's tangent hyperplanes (call them T_v and T_i) are parallel. By Corollary 3, supplier i's profit is $\pi = \min_{\vec{z}, s \neq i}\{c_s(\vec{z}) - T_i(\vec{z})\}$ if $T_i + \epsilon$ does not intersect any c_s, $s \neq i$, and $\pi = \epsilon$ if for some $s \neq i$ $T_i + \epsilon$ intersects c_s. In the former case, the auctioneer can enforce $(c_i(\vec{x}^*) + \pi, \vec{x}^*)$ and receive $M_i - \pi$ in utility. Since T_v and $T_i + \pi$ sandwich v and c_s (for $s \neq i$), respectively, and are $M_i - \pi$ units apart, this utility bounds from above any utility possible with supplier $s \neq i$

winning. In the latter case where for some $s \neq i$, $T_i + \epsilon$ intersects c_s, we can enforce price $c_i(\vec{x}^*) + \epsilon$ at $\vec{x}^*$, in which case the auctioneer walks away with utility $M_i - \epsilon$; since any winning supplier must receive profit of at least ϵ, the result follows. In the above we tacitly assume that $M_i - \epsilon > M_s$ for all $s \neq i$; if not, we have the trivial case in which announcing v leads to a "near tie." To see this, note that the payoff if $s \neq i$ wins (where $M_i - \epsilon \leq M_s$) is M_s, while the payoff if i wins is no smaller than $M_i - (M_i - M_s)$; in both cases the auctioneer's utility is at least $M_i - \epsilon$, which bounds any solution to (7.1)-(7.3), (7.7) from above. The function v is considered an optimal scoring rule and i and s are both taken to be optimal suppliers.

In the remainder of this subsection, we apply these results to construct a three-step method for finding the optimal scoring. Proposition 5 and its Corollary reduce the problem to one of utility maximization given prices, but the prices are determined with respect to low-cost supplier i and competing supplier j. The crucial idea behind the method's first step is that Proposition 6 allows us to fix the identity of the low-cost supplier when computing prices.

Step 1 (Choose an Optimal Supplier): For $s = 1, \ldots, S$, let

$$M_s = \max_{\vec{x}} \left\{ \sum_{a=1}^{A} v_a(x_a) - \sum_{a=1}^{A} c_{as}(x_a) \right\}. \tag{7.8}$$

Set $i = \arg\max_{s=1,\ldots,S} M_s$; i is the optimal supplier. If there exists an $s \neq i$ such that $M_s \geq M_i - \epsilon$, announce the true valuation function in the scoring rule and exit the three-step method. Otherwise, proceed to step 2.

Step 2 (Find a Best Competitor): Maximize utility given price, which by

Corollary 3 is given by

$$\max_{\vec{x},\pi} \quad \sum_{a=1}^{A} v_a(x_a) - \sum_{a=1}^{A} c_{ai}(x_a) - \pi \tag{7.9}$$

$$\text{subject to} \quad \sum_{a=1}^{A} c_{as}(x_a) > \sum_{a=1}^{A} c_{ai}(x_a) + \epsilon, \ s \neq i, \tag{7.10}$$

$$\pi \geq \epsilon, \tag{7.11}$$

$$\pi \geq \min_{s \neq i} \min_{\vec{z}} \left\{ \sum_{a=1}^{A} c_{as}(z_a) - T_i(\vec{z}) \right\}, \tag{7.12}$$

where T_i is the hyperplane tangent to supplier i's cost surface at $\vec{x}$ and π is supplier i's variable profit. In Section B.4, we simplify (7.9)-(7.12) by finding a closed-form solution (equation (B.23) in Section B.4) to the innermost minimization in (7.12). We will call any supplier j who achieves the minimization in (7.12) in the optimal solution (and thereby enables the optimal solution to be enforced) a "best competitor" to supplier i.

Step 3 (Choose Optimal Scoring Rule): The derivation of this scoring rule is given in Section B.5. Let $\vec{x}^*$, π^* denote an optimal solution to (7.9)-(7.11), (B.23). There are three cases to consider. If equation (B.23) is tight at the optimal solution, then the scoring rule is the function $\hat{v}$ constructed in Claim 16 of the "$\Rightarrow$" direction proof of Proposition 5 in Section B.2. For fixed dimension a, this optimal scoring function $\hat{v}_a$ has the form

$$\hat{v}_a(x_a) = \begin{cases} \omega_{a1} x_a^{\omega_{a2}} & \text{if} \quad x_a \leq z_{a1}; \\ \omega_{a3}(z_{a1} - x_a)^{\omega_{a4}} + \omega_{a5} x_a + \omega_{a6} & \text{if} \quad z_{a1} < x_a \leq z_{a2}; \\ \omega_{a7} x_a^{\omega_{a8}} & \text{if} \quad x_a > z_{a2}, \end{cases} \tag{7.13}$$

where three of the ten parameters $\omega_{a1}, \ldots, \omega_{a8}, z_{a1}, z_{a2}$ are actually redundant, but are included here to preserve readability. This function enforces optimal bid

$$\left(\sum_{a=1}^{A} c_{ai}(x_a^*) + \pi^*,\ \vec{x}^*\right) \tag{7.14}$$

in a two-supplier auction between i and j. If equation (B.23) is not tight at the optimal solution, then – as explained in Section B.2 – the optimal scoring rule must buffer against supplier i actually losing when $\hat{v}$ is announced to all S suppliers. The choice of optimal scoring rule in this case depends on whether the true valuation v, if used as a scoring rule, satisfies or violates constraint (7.2). If v satisfies (7.2) then the optimal scoring rule is $\lambda^*\hat{v} + (1-\lambda^*)v$, where λ^* is given in (B.21) at the end of Section B.2. This scoring rule enforces (7.14), using for each attribute the number of parameters in v_a plus 8 (seven from $\hat{v}_a$, plus the parameter λ^*). Otherwise, if (B.23) is non-binding but v violates (7.2), then the optimal scoring rule is $\lambda^*\hat{v} + (1-\lambda^*)g$, where $\hat{v}$ is given in (7.13), λ^* is given in (B.21), and g is defined in Section B.2. The optimal scoring function in this pathological case – in which at most one supplier can bid below the true valuation – requires 18 parameters per attribute to enforce (7.14): the function g_a has a form similar to (7.13), but with the middle case repeated, for a total of ten non-redundant parameters.

7.2 Numerical Example

We illustrate our mechanism with a simple numerical example that has $S = 2$ suppliers and $A = 1$ non-price attribute, which we call quality. The cost of quality is of the form $\theta_{s1}q + \theta_{s2}q^3 + \theta_{s3}q^{15}$, where we suppress the subscript a for the attribute and use q in place of x_1. More specifically, supplier 1's cost is $q^3 + 4q^{15}$ and supplier 2's cost is $3q + q^{15}$; i.e., $\theta_{11} = 0$, $\theta_{12} = 1$, $\theta_{13} = 4$, $\theta_{11} = 3$, $\theta_{12} = 0$, and $\theta_{13} = 1$. The true value function is $5q^{0.9}$; see Figure 7-2 for a plot of the cost and value functions. We also assume that the minimum bid increment is $\epsilon = 0.1$.

With knowledge of the cost parameters θ_{as}, the auctioneer chooses the optimal scoring rule $\lambda\hat{v} + (1-\lambda)v$ for the auction (v satisfies (7.2)). In the first step, the auctioneer finds $M_2 = 1.6823$, which is smaller than $M_1 = 3.4375$; hence, supplier 1 is optimal. Because M_1 is more than ϵ units greater than M_2, we solve (7.9)-(7.11), (B.23) to find q^*, π^*, the optimal quality level and profit at which supplier 1 wins the auction. Since $(c_2^{'})^{-1}(x) = \left(\frac{x-3}{15}\right)^{1/14}$, and $c_2^{'}(0) = 3$ and $c_1^{'}(0.7593) = 3$, we solve

$$\begin{aligned}
\max_{q\geq 0,\pi} \quad & \psi_1 q^{\psi_2} - \theta_{11}q - \theta_{12}q^3 - \theta_{13}q^{15} - \pi \\
\text{subject to} \quad & \theta_{11}q + \theta_{12}q^3 + \theta_{13}q^{15} \;>\; \theta_{21}q + \theta_{22}q^3 + \theta_{23}q^{15} + \epsilon, \\
& \pi \geq \epsilon, \\
& \pi \geq \theta_{21}\hat{q} - \theta_{22}\hat{q}^3 - \theta_{23}\hat{q}^{15} - [\theta_{11}q + \theta_{12}q^3 + \theta_{13}q^{15}] \\
& \quad + \hat{q}[\theta_{11} + 3\theta_{12}q^2 + 15\theta_{13}q^{14}], \\
& \hat{q} = \begin{cases} 0 - q & \text{if } q \leq 0.7593 \\ \left(\frac{\theta_{11}+\theta_{12}\cdot 3q^2+\theta_{13}\cdot 15q^{14}}{15} - 3\right)^{1/14} - q & \text{if } q > 0.7593 \end{cases}.
\end{aligned}$$

The minimum enforceable prices (per Corollary 3) are shown in Figure 7-2. An exhaustive search (with a discretization grid of 0.0001) yields $q^* = 0.6238$ and $\pi^* = 0.5327$. In the third and final step, the construction in Section B.2 produces parameter values $\lambda^* = 1$, $w_1 = 0.4194$, $w_2 = 0.1304$, $w_3 = 47.7807$, $w_4 = 13.0341$, $w_5 = 1.2484$, $w_6 = 0.3000$, $w_7 = 1.5167$, $w_8 = 0.7219$, $z_1 = 0.0100$, and $z_2 = 0.6238$; see Figure 7-2. (The rule constructed above enforces $(c_1(q^*) + \pi^* + \epsilon, q^*)$, though actually prices arbitrarily close to (but greater than) $c_1(q^*) + \pi^*$ can be enforced.) Under this scoring strategy, the losing supplier's (supplier 2's) quality level is 0.01, the auctioneer's utility is $\psi_1(q^*)^{\psi_2} - \theta_{11}q^* - \theta_{12}(q^*)^3 - \theta_{a3}(q^*)^{15} - \pi^* - \epsilon = 2.3909$, the winning supplier's profit is $\pi^* + \epsilon = 0.6327$, and the total surplus is 3.0236.

To illustrate the effects of inducible competition, we next consider a "high-competition" example in which the parameters for supplier 1's cost and the true value functions are unchanged, but we set $\theta_{21} = 2$, $\theta_{22} = 1$, and $\theta_{23} = 0$; the cost curves, the true value function, and the resulting minimum enforceable prices are

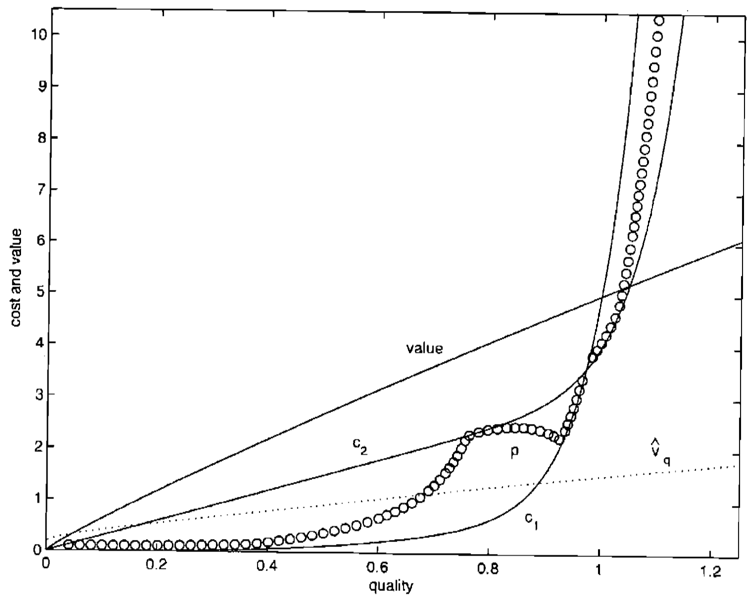

Figure 7-2: True value, supplier 1's cost c_1, supplier 2's cost c_2, minimum enforceable price p, and the optimal scoring rule $(\cdots)$ versus quality.

shown in Figure 7-3. Running the optimization (7.9)-(7.11), (B.23) under the new parameters for supplier 2 results in $q^* = 0.8482$ and $\pi^* = 0.1000 = \epsilon$. If we enforce $(c_1(q^*) + \pi^* + \epsilon, q^*)$ (see Figure 7-3), the payoff to the auctioneer is 3.2627, which is roughly 35% greater than in the previous example. Referring to Figure 7-3, notice that c_2 crosses the line tangent to c_1 at c_1's "elbow," which allows the auctioneer to enforce near-cost prices just past where the elbow starts.

We see that, depending on the situation, the optimal scoring rule can downplay (Figure 7-2) or overstate (Figure 7-3) the true valuation of quality. In summary, our scoring rule can operate in a fundamentally different manner than Che's rule [6]. Che's optimal scoring rule understates the true value of quality to reduce the information rents received by the more cost-efficient suppliers. In contrast, the auctioneer in our mechanism knows the suppliers' cost functions before the auction, and the optimal scoring rule might even exaggerate the value function. Some practical implications of such a scoring rule are discussed in 8.1.

To put the optimal scoring rule into perspective, we also consider a "straw" mech-

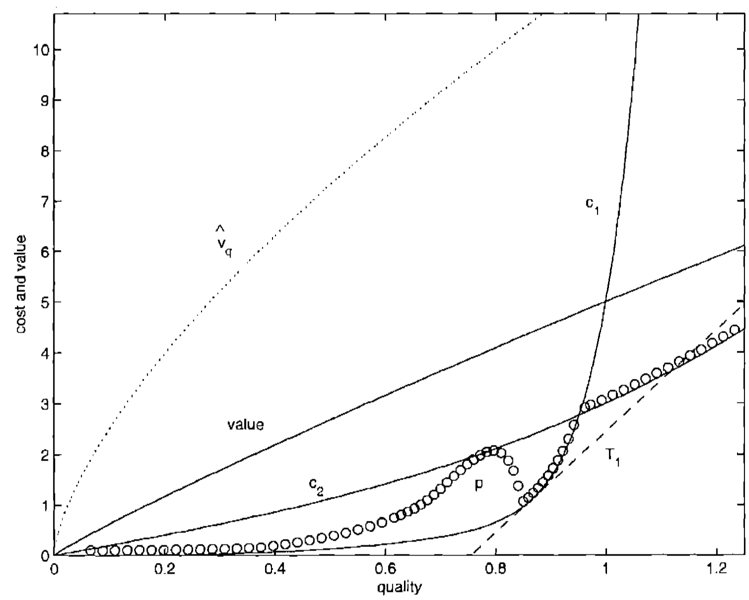

Figure 7-3: True value, supplier 1's cost c_1, supplier 2's cost c_2, minimum enforceable price p, and optimal scoring rule $(\cdots)$ versus quality for the high-competition example. T_1, the line tangent to supplier 1's cost curve at $q = 0.8640$, intersects the cost curve of supplier 2.

anism where the auctioneer announces his true utility function as the scoring rule, and assume that the suppliers submit their MBR bids. This scenario adheres to equations (7.1)-(7.3), (7.7), but with $5q^{0.9}$ in place of $\hat{v}$; we consider the first, "low-competition" example. Supplier s's maximum drop out score is M_s, so supplier 1 wins the auction bidding quality $q^* = 0.8008$ (the solution to the right hand side of (7.8) for $s = 1$) at price $c_1(q^*) + M_1 - M_2$. (By Step 1 of Section 7.1, we note that the winning supplier under the optimal scoring rule will be the same as the winning supplier under the true utility function, but the winning bids will likely be different.) The auctioneer's utility is $v(q^*) - c_1(q^*) - (M_1 - M_2) = M_2 = 1.6823$, supplier 1's profit is 1.7552, and the total surplus is 3.4375. Hence, as expected (see Milgrom [18]), the optimal scoring rule leads to an increase in the auctioneer's utility, and a decrease in efficiency relative to the straw mechanism. However, an industrial data set would be required to assess the magnitude of utility enhancement that might be achieved in practice; such an assessment is beyond the scope of this study.

Chapter 8

Practical Considerations

8.1 Scoring Rule Analysis

There are three potential problems with the analysis of 7.1: the determination of the optimal enforceable bid may require solving a nonconvex math program; the resulting scoring rule may be too complex (for each attribute, 8 plus the number of parameters in v_a if v satisfies (7.3), 18 otherwise) for practical implementation; and the scoring rule may force the losing supplier to submit bids with negligible non-price attribute levels. To deal with the first problem, Step 2 of our method can be replaced by restricting (7.9)-(7.11), (B.23) to $\vec{x} = \vec{\hat{x}}$, where $\vec{\hat{x}}$ maximizes the right side of (7.8) for $s = i$. This alternative Step 2 searches for the lowest possible enforceable price at $\vec{\hat{x}}$, the bid level with the potential to yield the largest utility for the auctioneer. While this alternative Step 2 is not guaranteed to find the optimal scoring rule, it will do so if any supplier's cost surface intersects the hyperplane tangent to supplier i's cost surface at $\vec{\hat{x}}$. Furthermore, the proof of Proposition 6 shows that, though not necessarily optimal, the rule generated using this alternative Step 2 yields an auctioneer's utility that is greater than or equal to the utility level from any auction in which supplier i is not the top bidder; i.e., even with this simplified approach, we are still sure to do better than is possible via full optimization over generic scoring rules in which supplier $s \neq i$ wins.

The complexity of the scoring rule (see (7.13)) can perhaps be finessed in practice by providing the scoring rule in graphical form (one graph per non-price attribute), together with a calculation device that converts uncommitted bids into scores. An alternative approach is to employ a parametric scoring rule in Step 3. With the identity of suppliers i and j (a best competitor to i) in hand from the method's first two steps, the auctioneer's scoring rule selection problem becomes a maximization over the scoring rule parameters, which we denote $\vec{\phi}_1, \ldots, \vec{\phi}_A$:

$$\max_{\vec{\phi}_a} \quad \sum_{a=1}^{A} v_a(x^*_{ai}) - \sum_{a=1}^{A} \hat{v}_a(x^*_{ai}; \vec{\phi}_a) + \sum_{a=1}^{A} \hat{v}_a(x^*_{aj}; \vec{\phi}_a) - \sum_{a=1}^{A} c_{aj}(x^*_{aj}) \quad (8.1)$$

$$\text{subject to} \quad S_s = \sum_{a=1}^{A} \hat{v}_a(x^*_{as}; \vec{\phi}_a) - \sum_{a=1}^{A} c_{as}(x^*_{as}), \quad s = i, j, \quad (8.2)$$

(7.2)-(7.3), and the scoring rule constraints at the end of 6.1. Since i and j were selected with respect to a generic scoring rule, they are not guaranteed to be optimal for the parameterized case. However, this drawback – though difficult to quantify *a priori* – compensates for the need to solve $2\binom{S}{2}$ mathematical programs (a version of (7.2)-(7.3), (8.1)-(8.2) for every ordered supplier pair), which may not be practical if S is large. In practice, some approach midway between these two extremes could be used – for instance, examining all pairs from the top 10% of candidates in Steps 1 and 2.

In addressing the third potential problem, we first note that our scoring-rule restrictions in Section 6.1 (i.e., the scoring rules are strictly concave and satisfy the conditions on the derivatives at zero and infinity to ensure a unique, interior MBR bid response by suppliers) are not innocuous. In the absence of these constraints, we have constructed nonpathological cases in which the optimal scoring rule is convex or closely mimics step 1's supplier i's cost surface everywhere except near $\vec{\hat{x}}$, where it is precisely ϵ units higher. While ignoring these constraints may increase the auctioneer's utility over the short run, in the longer run cost-mimicking can eliminate meaningful bids from the non-low-cost suppliers, compromising competition in – and consequently the credibility of – the auction. Furthermore, a non-concave scoring rule may make transparent the strategic nature of our proposed mechanism. While we have been careful to avoid cost-mimicking and convex scoring rules, we note that the optimal scoring rule can force the best competitor (i.e., supplier j in step 2) to submit bids with negligible non-price attribute levels; e.g., in our first numerical example, supplier 2's quality level is 0.01. This phenomenon may arouse bidder suspicion, and it may be shrewd for the auctioneer to either add a lower-bound constraint on the MBR attribute levels that result from his scoring rule or impose a reservation level for all non-price attributes.

8.2 Exogenous Attributes

In addition to bid price and endogenous attributes such as quality and lead time, exogenous attributes such as a supplier's reputation and his past history with the manufacturer typically play a vital role in the allocation decision. These factors are easily incorporated into our model. Let e_s represent the auctioneer's total utility derived from supplier s's exogenous attributes; for an incumbent supplier, this utility might incorporate the fixed cost to switch to a different supplier. Then the auctioneer's true utility function (with the supplier notation suppressed) becomes $\sum_{a=1}^{A} v_a(x_a) + e - p$.

We recommend that the auctioneer reveals to supplier s his truthful exogenous value e_s, but not the other suppliers' exogenous values. While we have not attempted to prove that truthful revelation is optimal on the auctioneer's part, withholding all information about e_s would be unsatisfactory to the suppliers because they would only possess a partial scoring function. Moreover, a large portion of the exogenous value is likely to be based on standardized supplier ratings, which are readily available in many industries.

Under the assumption that the true e_s is revealed to supplier s, the analysis extends in a straightforward manner. The supplier's cost surface c_s is simply shifted vertically by $-e_s$ units; this shift is allocated to the costs over individual attributes by taking c_{as} to be shifted $-\lambda_{as}e_s$ units vertically, where $\lambda_{as} \geq 0$ and $\sum_{a=1}^{A} \lambda_{as} = 1$. Although by strategically assigning exogenous attribute levels the auctioneer can contrive to enforce any $A-$tuple at ϵ profit, generating competition in this way is likely to be much more obvious to suppliers than relying on scoring rules as described in Section 7.1.

Chapter 9

Part II Conclusions

Aside from transaction cost savings, the prospect for competition is what makes a procurement auction compelling for the buyer. Our (largely geometric) analysis in Chapter 7 shows how the auctioneer, via the choice of the scoring rule, can manipulate the rules of the competition so as to maximize his own utility within the open-ascending auction format. In particular, it is optimal to first identify the winning supplier, which is the one with the largest drop-out score (see equation (7.1)) if the auctioneer revealed his true valuation in the announced scoring rule, and then to identify his best competitor, i.e., the one that will minimize the winning supplier's profit, thereby leaving more utility for the auctioneer. Ideally, all suppliers exit the auction with a renewed sense of respect and fear for the opposing suppliers, rather than feeling as if they have been manipulated by the auctioneer. While our numerical examples consider only one non-price attribute, our analysis has the potential to provide nonobvious insights about which attributes and competing suppliers provide the most fruitful focus of competition.

Bibliography

[1] N. Balakrishnan and Cohen A. C. *Order Statistics and Inference: Estimation Methods*. Academic Press, San Diego, California, 1991.

[2] A. A. Balkema and L. de Haan. Limit distributions for order statistics II. *Theory of Probability and its Applications*, 23:341–358, 1978.

[3] Damian R. Beil and Lawrence M. Wein. An inverse-optimization-based auction mechanism to support a multi-attribute RFQ process. *To appear in Management Science*, 2002.

[4] F. Branco. The design of multidimensional auctions. *RAND Journal of Economics*, 28:63–81, 1997.

[5] Jeremy Bulow and Paul Klemperer. Auctions versus negotiations. *The American Economic Review*, 86(1):180–194, 1996.

[6] Y.-K. Che. Design competition through multidimensional auctions. *RAND Journal of Economics*, 24:668–680, 1993.

[7] C. Cronin. Five success factors for private trading exchnages. *e-Business Advisor Magazine*, October 2001.

[8] H. A. David. *Order Statistics*. Wiley, New York, 1970.

[9] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94:863–872, 1986.

[10] Glenn Ellison and Drew Fudenberg. Competing auctions. February 2002.

[11] G. D. Eppen, K. Martin, and L. Schrage. A scenario approach to capacity planning. *Operations Research*, 37:517–527, 1989.

[12] D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, 1991.

[13] Jérémie Gallien and Lawrence M. Wein. Design and analysis of a smart market for industrial procurement. *to appear in Management Science*, 2000.

[14] Normal L. Johnson and Samuel Kotz. *Continuous Univariate Distributions*, volume I. John Wiley and Sons, New York, 1 edition, 1970.

[15] S. J. Kafka, B. D. Temkin, and L. Wegner. B2b auctions go beyond price. Technical report, Forrester Research, Inc., Cambridge, Massachusetts, 2000.

[16] Ulrich Kamecke. Dominance or maximin: How to solve and english auction. *International Journal of Game Theory*, 27(3):407–426, 1998.

[17] P. D. Klemperer. Auction theory: A guide to the literature. *Journal of Economic Surveys*, 13:227–286, 1999.

[18] Paul Milgrom. An economist's vision of the b-to-b marketplace. Executive White Paper, www.perfect.com, 2000.

[19] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6:58–73, 1981.

[20] David C. Parkes and L. H. Ungar. Iterative combinatorial auctions: Theory and practice. *Proceedings 17th National Conference on Artificial Intelligence (AAAI-00)*, pages 74–81, 2000.

[21] Michael Peters. A competitive distribution of auctions. *Review of Economic Studies*, 64:97–123, 1997.

[22] Michael Peters and Sergei Severinov. Internet auctions with many traders. Working paper, April 2002.

[23] David Welch. Can covisint pull itself out of a ditch? *Business Week*, pages 128B–D, May 2001.

[24] M. P. Wellman, W. E. Walsh, P. R. Wurman, and J. K. MacKie-Mason. Auction protocols for decentralized scheduling. *Games and Economic Behavior*, 35:271–303, 2001.

[25] R. Wise and D. Morrison. Beyond the exchange: The future of B2B. *Harvard Business Review*, pages 86–96, November-December 2000.

Appendix A

Part I Proofs

A.1 Proof of Proposition 1

This section provides theoretical support for the strategy outlines in Section 3.1b. The analysis requires a game-theoretic framework, to which for tractability we add the assumption that bidder's valuations and bids lie in a discrete grid of mesh size δ (δ can be thought of as the minimum bid increment). This imposed discreteness of valuations and bids avoids complicated bid-ordering issues and allows all bidders to bid up to their valuations if they wish.

Game procedure. In formalizing the auction game, we say that a buyer *enters the market* the moment she places her first bid in whichever auction she has access to and likes best. When auctioneer a receives a bid, he (i) posts the current *standing bid* of his auction equal to the $m_a + 1$-st highest submitted bid he has received or zero (or the auctioneer's reserve price) if he has received fewer than $m_a + 1$ bids, and (ii) announces the identity of the winning bidders (but not value of the winning bids). All bids exceeding the current standing bid are winning bids; if there are only $m_a - m < m_a$ such bids, the remaining m winning bids are chosen to be the earliest m bids that equal the current standing bid. We call a buyer's bid *successful* if the buyer is a winner after the bid is submitted; otherwise, we say the bid is *unsuccessful*.

All bids must exceed the standing bid by at least the minimum bid increment δ. The standing bid is assumed to be the $m_a + 1$ highest bid submitted by $m_a + 1$ unique bidders; therefore, if a high bidder revises his bid, the standing bid does not change.

After each entry to the market, every previously entered buyer is given, in order of entry into the market (earliest first, latest last), the opportunity to submit a new bid (not necessarily with the same auctioneer) or pass. Following this round-robin bidding procedure, a new buyer enters the market, provided there is a buyer willing to do so. The process of buyer entry can be thought of in terms of the following procedure involving nature: All buyers who have not yet entered the market gather around a single door leading into the market; then, from among all buyers who wish to enter the market at a given time, nature randomly selects a single buyer and allows her to enter through the door. The auction proceeds as described above until there are no new buyers who wish to enter the market and all entered bidders pass on their turn to bid. At this point, auctioneer a awards m_a items to the m_a bidders who are winning at auction a's final standing bid; each such bidder pays auctioneer a an amount equal to this final standing bid. For convenience, we allow this terminology to subsume instances in which auctioneer a's reservation price is not met, we imagine there are m_a "reservation" dedicated bidders installed in auction a at the market's outset, where each such bidder's valuation and bid equals auctioneer a's reservation price.

Information. State of the game Γ at time t is taken to be the standing bids at time t, the winning bids at time t, the identities of the winning bidders at time t, and the entire history of bids and winner identities prior to time t, the vector of buyer valuations, and the order of moves (i.e., entry).

Naturally, not all of Γ is known to the players in the game. We assume the public information at time t to be the history of standing bids and winner identities up to and including time t, and the order of moves among bidders in the market at time t. Each buyer's valuation is private information, but all buyer's have an identical payoff function structure equal to their valuation minus the price they transact at (or equal to zero, if they do not transact). Conversely, it is assumed that the auctioneer's payoff structure is known to be his transaction price.

Strategy σ^*. For a dedicated bidder we define the strategy σ_d^* as follows: If it is the buyer's turn to bid, then

(a) if the buyer is the current high bidder in an auction with a bid exceeding the auction's current standing bid by the minimum bid increment δ, then the buyer should pass;

(b) otherwise, if the buyer is the current high bidder in an auction with bid not exceeding the auction's current standing bid by δ, the buyer should submit a bid equal to the standing bid plus δ, provided such a bid does not exceed the buyer's valuation. If such a bid would exceed the buyer's valuation, the buyer should pass.

For a shared bidder we define the strategy σ_s^* as follows: If it is the buyer's turn to bid, then

(c) if the buyer is the current high bidder in an auction, or if both the current standing bids equal the buyer's valuation, then the buyer should pass;

(d) otherwise, if auction a has the unique lowest standing bid, submit in auction a a bid that exceeds this standing bid by δ;

(e) otherwise, if both standing bids are equal, if auction a is the unique auction in which the most recent m_a bids have been successful, submit a bid according to (d) in auction $-a$. If the most recent m_a bids in both auctions $a = 1, 2$ have been successful, and if in the time since the bidder's previous turn to her current turn every bidder passed on his turn and no new bidder entered the market, then submit a bid according to in (d) in either auction with equal probability. Otherwise, the buyer should pass.

Note that the second sentence of item (e) implies that shared bidders wait for all dedicated bidders to enter and finish their bidding before bidding in a situation in which both auctions have the same standing bid.

Solution concept. To show σ^* is a Bayesian Nash equilibrium (Proposition 1), we need to show that, from any state of the game Γ, if all players $j \neq i$ play σ^* in the continuation of the game, then in the continuation player i cannot gain by deviating from σ^*. The argument has two major steps: first we bound from above i's transaction price when playing σ^* (Lemma 4), then prove that this is indeed a lower bound for any strategy i might choose to play (Lemma 2). Along the way, virtually every proof is by contradiction.

Additional assumptions. Before embarking on the proof of Proposition 1, we impose two, natural assumptions on the play of the game.

A1: We assume that at every stage of the game, buyers do not bid above their true valuations.

A2: Buyer i wins no more than a single item when playing σ^* in the continuation.

By reducing the strategy space available to the buyers, assumption A1 makes the analysis tractable; given the inherent riskiness associated to bidding above one's true valuation, we predict that very little realism is sacrificed by this assumption. Assumption A2 serves to make the payoff to buyer i known under the existing unit-demand model. This assumption could be relaxed, for example, by assigning payoff of $-\infty$ to any buyer winning multiple items, then carefully showing that if i wins multiple items under σ^*, then she must win multiple items under any strategy $\bar{\sigma} \neq \sigma^*$; while it would not be hard to prove such an extension, we leave this to future work.

The formalization of the game thus far is roughly the same as that used in Peters and Severinov [22]. There are three notable differences between their game setup and that given here: the present study includes dedicated bidders, auctioneers can sell multiple items instead of just single items, and we consider only two simultaneous auctions instead of many.

Notation: For state of the game Γ let the stage of the game be denoted τ. Further, let

$$
\begin{aligned}
Z_{aj} &\triangleq \text{set of winning bids held by buyer } j \text{ in auction } a \text{ at stage } \tau, \\
B_a &\triangleq \cup_{j \in d_a \cup S} Z_{aj}, \\
W_{at} &\triangleq \text{set of wining bidders in auction } a \text{ at time } t, \\
W_a &\triangleq \text{final set of winning bidders in auction } a, \\
W^*_{1t} &\triangleq \text{set of winning bidders in auction } a \text{ at stage } t \text{ when } \sigma^* \text{ is played} \\
&\qquad \text{in the continuation,}
\end{aligned}
$$

$$
\begin{aligned}
g_a^t &\triangleq \text{standing bid in auction } a \text{ at stage } t, \\
w_a^t &\triangleq \text{lowest winning bid in auction } a \text{ at stage } t, \\
v_j &\triangleq \text{the valuation of bidder } j, \\
d_a(p) &\triangleq \{j \in d_a \cup B_a \mid v_l < p \; \forall \, l \in Z_{aj}\} \\
&= \text{dedicated bidders in auction } a \text{ with no stage } \tau \text{ bids} \\
&\qquad \text{winning at standing bid } p, \\
S(p_1', p_2') &\triangleq \{j \in S \mid v_l < p_a' \; \forall \, l \in Z_{aj}, \; a = 1, 2\} \\
&= \text{shared bidders with no stage } \tau \text{ bids winning in either auction} \\
&\qquad \text{given standing bids } p_1, \; p_2, \\
A(H, p) &\triangleq \{j \in H \mid v_l > p\}, \\
AE(H, p) &\triangleq \{j \in H \mid v_l \geq p\}.
\end{aligned}
$$

For each bid in B_a, we replace the bid by a "phantom bidder" with valuation and submitted bid equal to the bid value itself. Thus, W_{at} and W_a may contain both real and phantom bidders.

Next, for $a = 1, 2$ define p_a to be the largest value in $\{0, \delta, 2\delta, \ldots\}$ such that

$$|AE\left(d_a(p_a) \cup B_a, p_a\right)| \geq m_a + 1, \tag{A.1}$$

p_{1s} the largest value in $\{0, \delta, 2\delta, \ldots\}$ such that

$$|AE\left(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s}\right)| \geq m_1 + 1 \tag{A.2}$$

(similarly for p_{2s}), and p_{12s} the largest value in $\{0, \delta, 2\delta, \ldots\}$ such that

$$|AE\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{12s}\right)| \geq m_1 + m_2 + 1. \tag{A.3}$$

Note that, from these definitions,

$$\begin{aligned} |A\left(d_a(p_a) \cup B_a, p_a\right)| &\leq m_a, & \text{(A.4)} \\ |A\left(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s}\right)| &\leq m_1, & \text{(A.5)} \\ \text{and} \quad |A\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{1s}\right)| &\leq m_1 + m_2. & \text{(A.6)} \end{aligned}$$

To get a feel for the definitions above, note that, e.g., if the final standing bid in auction a were p, the set $AE\left(d_a(p) \cup B_a, p\right)$ would consist of bidders (original and phantom) who are competitive in auction a in the following sense: either they are a phantom bidder (in set $AE\left(B_a, p\right)$) who has locked in a winning bid at final standing bid p, or they are an original dedicated bidder with no such winning phantom bidders (hence is in set $AE\left(d_a(p), p\right)$) who are themselves capable of being a winner (their valuation equals or exceeds the final standing bid p). The competitiveness of shared bidders follows analogously, albeit requiring that the shared bidder has no winning phantom bidders in either auction (this is true of those bidders in the set $S(p', p'')$ when p', p'' are the standing bids in auctions 1 and 2, respectively.

First, some Claims that will be useful in the proof of Lemma 4.

Claim 7. *Let t_1 be the smallest t such that $g_1^t \geq p_{1s} + \delta$. Then there exists a $t_2 < t_1$ such that $g_2^{t_2} = p_2 + \delta$.*

Proof. $p_1^* > p_{1s}$ implies $p_1^* \geq p_{1s} + \delta$. At some stage t_1, g_1 increased from p_{1s} to $p_{1s} + \delta$. For this to happen, we must have had

$$\begin{aligned} g_1^{t_1-1} &= p_{1s}, \\ w_1^{t_1-1} &= p_{1s} + \delta, \end{aligned} \tag{A.7}$$

and the existence of a bidder j compelled by σ^* to bid $p_{1s}+\delta$ unsuccessfully in auction 1. In particular,

$$j \notin W_{1t_1-1}^* \cup W_{2t_1-1}^* \quad \text{(by parts (a) and (c) of } \sigma^* \text{'s definition)}, \tag{A.8}$$

$$l \in Z_{aj} \Rightarrow v_l < g_a^{t_1-1}, \; a = 1, 2 \quad \text{(again by parts (a) and (c)), and} \tag{A.9}$$

$$v_j \geq p_{1s} + \delta > p_{1s} \quad \text{(since } j \text{ doesn't bid beyond his true valuation)}. \tag{A.10}$$

Let

$$\bar{A}_1 \triangleq A\left(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s}\right).$$

Note $l \in W_{1t_1-1}^*$ implies $l \in B_1$, or $l \in d_1(g_1^{t_1-1} = p_{1s})$ (either l corresponds to a stage τ bid that is still winning at stage $t_1 - 1 \geq \tau$, or l is a bidder who bid after stage $t' > \tau$, hence certainly had no stage τ bids still winning at stage t', hence neither does she have any at $t_1 - 1 > t'$). Either way, by (A.7) l also satisfies $v_l \geq p_{1s} + \delta > p_{1s}$. Hence, $l \in \bar{A}_1$.

Note that since j bids at time $t > \tau$, we must have $j \notin B_1$, that is, $j \in d_1 \cup S$. If $j \in d_1$, then $g_1^{t_1-1} = p_{1s}$ together with (A.9)-(A.10) imply that $j \in A\left(d_1(p_{1s}), p_{1s}\right)$, i.e., $j \in \bar{A}_1$. But,

$$\begin{aligned} j \cup W_{1t_1-1}^* &\subseteq \bar{A}_1 \\ m_1 + 1 &\leq |\bar{A}_1| \quad \text{(by (A.8))}, \end{aligned}$$

which contradicts (A.5). Hence, $j \in S$. Then suppose that the Claim is false, i.e., $g_2^{t_1-1} \leq p_2$. Then $g_1^{t_1-1} = p_{1s}$, (A.9)-(A.10) imply that $j \in A\left(S(p_{1s}, p_2), p_{1s}\right)$, which implies $j \in \bar{A}_1$, leading again to the contradiction $j \cup W^*_{1t_1-1} \subseteq \bar{A}_1 \Rightarrow |\bar{A}_1| \geq m_1 + 1$. Therefore we must have $g_2^{t_1} \geq p_2 + \delta$, which proves Claim 7. □

Claim 8. *Let t_2 be the smallest t such that $g_2^t \geq p_2 + \delta$. Then there exists a $t_1 < t_2$ such that $g_1^{t_1} = p_{1s} + \delta$.*

Proof. The proof of Claim 8 is analogous the proof of Claim 7. First, note that for g_2 to increase from p_2 to $p_2 + \delta$ at stage t_2, we need

$$\begin{aligned} g_2^{t_2-1} &= p_2, \\ w_2^{t_2-1} &= p_2 + \delta, \end{aligned} \tag{A.11}$$

and the existence of a bidder $j \in d_2 \cup S$ compelled by σ^* to bid $p_2 + \delta$ unsuccessfully in auction 2. In particular,

$$j \notin W^*_{1t_2-1} \cup W^*_{2t_2-1}, \tag{A.12}$$

$$l \in Z_{aj} \Rightarrow v_l < g_a^{t_2-1}, \; a = 1, 2 \quad \text{and,} \tag{A.13}$$

$$v_j \geq p_2 + \delta > p_2. \tag{A.14}$$

First, note that $j \in \cap S$ implies $g_1^{t_2-1} \geq p_2 \geq p_{1s} + \delta$, and taking $t \equiv t_2 - 1$ we are done. In what remains we consider $j \in d_2$.

Suppose $W^*_{2t_2-1} \cap S = \emptyset$. Then for any $l \in W^*_{2t_2-1}$, we have $l \in d_2 \cup B_2$. Further, (A.11) implies $v_l \geq p_2 + \delta > p_2$ (l satisfies (A.14)), and l plays σ^* implies l satisfies (A.13). Since l satisfies (A.13)-(A.14), we have $l \in A\left(d_2(p_2) \cup B_2, p_2\right) \triangleq \bar{A}_2$. But, $j \in d_2$ satisfies (A.13)-(A.14) implies $j \in \bar{A}_2$ as well. Then, we have

$$\begin{aligned} j \cup W^*_{2t_2-1} &\subseteq \bar{A}_2 \\ m_2 + 1 &\leq |\bar{A}_2| \quad \text{(by (A.12))}, \end{aligned}$$

which is a contradiction to equation (A.4).

Now, suppose there exists $l \in W^*_{2t_2-1} \cap S$. By (A.11) we have l bid $p_2+\delta$ in auction 2 at some stage $t < t_2$. Since l follows σ^* we must have had $g_1^t \geq p_2 \geq p_{1s} + \delta$, and taking $t_1 \equiv t$ we are done. Hence, Claim 8 is proved. $\square$

We now state and prove the first of two Lemmas used to prove Proposition 1.

Lemma 1 (Upper bound on p_a^*). *Suppose all bidders play σ^* in the continuation, and let p_a^* be the final price in auction a. Then*

$$(p_1^*, p_2^*) \leq \begin{cases} (p_1, p_{2s}) & \text{if } p_1 > p_{2s}; \\ (p_{1s}, p_2) & \text{if } p_{1s} < p_2; \\ (p_{12s}, p_{12s}) & \text{otherwise.} \end{cases} \tag{A.15}$$

Proof. Note that the three cases in equation (A.15) are the analogues to U_1, U_2, and E from the main text; however, here the cases pertain to the auction outcome beginning from any arbitrary state Γ, where Γ need not lie on the equilibrium path of σ^*. We break up the proof of Lemma 4 along the three cases, for each combining bidder behavior dictated by σ^* with the formal definitions of p_1, p_2, p_{1s}, p_{2s}, and p_{12s} to yield a contradiction if Lemma 4 doesn't hold.

Case $p_{1s} < p_2$. We apply Claims 7 and 8 to prove Lemma 4 for the case $p_{1s} < p_2$. More precisely, we suppose $p_{1s} < p_1^*$ and produce a contradiction. Let t_1 be the smallest t such that $g_1^t = p_{1s} + \delta$. From Claim 7, there exists $t_2 < t_1$ such that $g_2^{t_2} = p_2 + \delta$. Let $\underline{t}_2$ be the smallest such t_2. From Claim 8, there exists $\underline{t}_1 < \underline{t}_2$ such that $g_1^{\underline{t}_1} = p_{1s} + \delta$. But,

$$\underline{t}_1 < \underline{t}_2 \leq t_2 < t_1 \quad \Rightarrow \quad \underline{t}_1 < t_1,$$

which contradicts the fact that t_1 was chosen to be the smallest t such that $g_1^t = p_{1s}+\delta$. Hence, $p_1^* \leq p_{1s}$. A similar argument shows $p_2^* \leq p_2$, and we are done with case $p_{1s} < p_2$.

Case $p_{2s} < p_1$. By to symmetry to the case $p_{1s} < p_2$, we have $p_1^* \leq p_1$ and $p^* \leq p_{2s}$.

Case $p_{1s} \geq p_2$ **and** $p_{2s} \geq p_1$. Suppose $p_1^* \geq p_{12s} + \delta$. Then there exists some bidder j who at stage t sets the standing bid in auction 1 to $p_{12s} + \delta$. This implies that

$$w_1^{t-1} = p_{12s} + \delta.$$

First, suppose there exists $l \in W_{1t-1}^* \cap S$. Then $l \in W_{1t}^*$ implies l bid $p_{12s} + \delta$ at some stage $t' < t$. Since l plays σ^*, $l \in S$, we must have $g_2^{t'} \geq p_{12s} + \delta$ (σ^* (d)), or $g_2^{t'} = p_{12s}$ and all dedicated bidders have entered the market and chosen to pass on their most recent turn. We deal first with the latter case.

If prior to stage t' all the bidders in d_1 have passed on their turn to bid $p_{12s} + \delta$ in auction 1, and $g_1^t = p_{12s}$, then $k \in d_1 \setminus W_{1t}^* \Rightarrow v_k \leq p_{12s}$. Since $v_j > p_{12s}$ and $j \notin W_{1t}^*$, we have $j \notin d_1$, so $j \in S$. Bidder j knows his stage t bid of $p_{12s} + \delta$ in auction 1 will be unsuccessful. Hence, for $j \in S$ playing σ^* to nonetheless bid $p_{12s} + \delta$ unsuccessfully in auction 1, a bid of $p_{12s} + \delta$ must also be unsuccessful in auction 2. But, this implies

$$\begin{aligned} w_1^t &= p_{12s} + \delta, \quad \text{and} \\ w_2^t &= p_{12s} + \delta. \end{aligned}$$

But, $w_a^t > p_{12s}$ for a=1,2 implies that $l \in W_{at}^* \Rightarrow v_l > p_{12s}$. Since l plays σ^* and hence also satisfies $k \in Z_{al} < g_a^t = p_{12s}$ for $a = 1, 2$, we have

$$l \in A\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{1s}\right) \triangleq A.$$

Furthermore, analogous to the proofs of Claims 7 and 8, we have $j \in A$ and $j \notin W_{1t}^* \cup W_{2t}^*$. Hence,

$$j \cup W_{1t}^* \cup W_{2t}^* \subseteq A \Rightarrow m_1 + m_2 + 1 \leq |A|,$$

which contradicts (A.6).

The case $g_2^{t'} \geq p_{12s} + \delta$ is dealt with similarly, as it implies $g_2^t \geq g_2^{t'} > p_{12s}$, which again implies that $j \cup W_{1t}^* \cup W_{2t}^* \subseteq A$.

Now, it remains only to check the case in which $W_{1t}^* \cap S = \emptyset$. In this case, one possibility is $j \in S$, yielding the same contradiction as above for $j \in S$. The other possibility is $j \in d_1$, which together with $w_1^t > p_{12s}$ implies that $j \cup W_{1t}^* \subseteq A\left(d_1(p_{12s}) \cup B_1, p_{12s}\right)$, so $m_1 + 1 \leq \left|A\left(d_1(p_{12s}) \cup B_1, p_{12s}\right)\right|$. This in turn implies that $p_1 > p_{12s}$, which contradicts the assumption that $p_1 \leq p_{12s}$.

Hence, when $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$, we have $p_a^* \leq p_{12s}$. This completes the proof of Lemma 4. □

We now prove three additional Claims that will be useful in proving Lemma 2. Let

$$\overline{W}_{at}, \overline{W}_a$$

be the analogues to W_{at}^* and W_a when i plays $\bar{\sigma} \neq \sigma^*$ in the continuation.

Claim 9. *Suppose that in the continuation $j \neq i$ plays σ^* and i plays $\bar{\sigma} \neq \sigma^*$, $i \in \overline{W}_1$, auction a ends with price $\bar{p}_a$, and p', p'' are values in $\{0, \delta, 2\delta, \ldots\}$ such that $\bar{p}_1 < p'$ and $\bar{p}_2 \geq p''$. Then*

$$j \in AE\left(d_1(p') \cup B_1 \cup (S(p', p'') \setminus \overline{W}_2), p'\right)$$
$$\Rightarrow \quad j \in \overline{W}_1, \quad or \quad \exists\, l \in Z_{1j} \quad such\ that \quad l \in \overline{W}_1.$$

Proof. We break the proof into five cases:

$$\begin{aligned} j &\in d_1(p') \setminus d_1(\bar{p}_1) \quad \Rightarrow \quad \exists\, l \in Z_{1j} \quad \text{such that} \quad p' > v_l \geq \bar{p}_1, \\ &\quad \Rightarrow \quad l \in \overline{W}_1 \quad \text{since} \quad l \in AE\,(B_1, \bar{p}_1) \quad \text{and} \quad l \in \overline{W}_{1\tau}; \qquad \text{(A.16)} \\ j &\in d_1(p') \cap d_1(\bar{p}_1). \quad j \in d_1(p') \quad \Rightarrow \quad v_j \geq p' > \bar{p}_1 \\ &\quad \Rightarrow \quad j \in A\,(d_1(\bar{p}_1), \bar{p}_1) \quad \Rightarrow \quad j \in \overline{W}_1; \qquad \text{(A.17)} \\ j &\in B_1 \quad \Rightarrow \quad v_j \geq p' > \bar{p}_1, \quad \Rightarrow \quad j \in \overline{W}_1 \\ &\quad \text{since} \quad j \in \overline{W}_{1\tau} \quad \text{and} \quad j \in AE\,(B_1, \bar{p}_1). \qquad \text{(A.18)} \end{aligned}$$

The last two cases concern $j \in S(p', p'')$. Note that

$$j \in S(p', p'') \quad \Rightarrow \quad j \in S(p', \bar{p}_2) \quad \text{since} \quad v_l < p'' \leq \bar{p}_2 \; \forall\, l \in Z_{2j}.$$

The final two cases are dealt with as follows:

$$\begin{aligned} j &\in \left(S(p', \bar{p}_2) \setminus \overline{W}_2\right) \setminus S(\bar{p}_1, \bar{p}_2) \Rightarrow \quad \exists \quad l \in Z_{1j} \quad \text{such that} \quad p' > v_l \geq \bar{p}_1, \\ &\quad \Rightarrow \quad l \in \overline{W}_1 \quad \text{since} \quad l \in AE\,(B_1, \bar{p}_1) \quad \text{and} \quad l \in \overline{W}_{1\tau}; \\ j &\in \left(S(p', \bar{p}_2) \setminus \overline{W}_2\right) \cap S(\bar{p}_1, \bar{p}_2). \quad j \in S(p', \bar{p}_2) \quad \Rightarrow \quad v_j \geq p' > \bar{p}_1, \quad \text{so} \\ &\quad j \in A\,(S(\bar{p}_1, \bar{p}_2), \bar{p}_1). \quad \text{Since } j \in S(p', \bar{p}_2) \setminus \overline{W}_2 \;\Rightarrow\; j \notin \overline{W}_2, \text{ we have } j \in \overline{W}_1. \end{aligned}$$

Equations (A.16), (A.18) and (A.19) arise from the fact that $l \in Z_{1j}$ is by definition a bidder who is among those winning at stage τ with bid v_l; if the eventual price in auction 1, $\bar{p}_1$, does not exceed v_l, then no bid could have displaced l from auction 1's winning set of bidders, i.e., $l \in \overline{W}_1$. The fact that auction 1 ends with price $\bar{p}_1$ also implies why equations (A.17) and (A.19) hold: if there exists a bidder j with access to auction 1 such that j is neither winning in auction 2 nor has standing bids from stage τ winning in an auction, then $v_j > \bar{p}_1$ implies $j \in \overline{W}_1$; otherwise, $j \neq i$ (since $i \in \overline{W}_1$ by assumption) would be compelled by σ^* to place a new bid (perhaps in auction 1, depending on the price in auction 2 if j is a shared bidder), contradicting the fact that the auction ended (bidding ceased) with final standing bids $\bar{p}_1,\ \bar{p}_2$. $\square$

Claim 10. *If* $p_{1s} \ \geq\ p_2$ *and* $p_{2s} \ \geq\ p_1$*, then* $p_2 \leq p_{12s}$.

Proof. First, suppose that $p_2 > p_{12s}$. Since p_{12s} is the largest value in $\{0, \delta, 2\delta, \ldots\}$ satisfying equation (A.3), we have

$$|AE\,(d_1(p_2) \cup B_1 \cup d_2(p_2) \cup B_2 \cup S(p_2, p_2), p_2)\,| \leq m_1 + m_2.$$

But, p_2 satisfies equation (A.1), i.e.,

$$|AE\,(d_2(p_2) \cup B_2, p_2)\,| \geq m_2 + 1,$$

so then

$$|AE\,(d_1(p_2) \cup B_1 \cup S(p_2, p_2), p_2)\,| \leq m_1 - 1. \tag{A.19}$$

But, because p_{1s} satisfies equation (A.2), we have

$$|AE\,(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s})\,| \geq m_1 + 1. \tag{A.20}$$

Next, we show that the Claim's assumption $p_{1s} > p_2$ implies

$$\begin{aligned} &j \in AE\,(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s}) \\ &\qquad\qquad \Rightarrow \quad (j \cup Z_{1j}) \cap AE\,(d_1(p_2) \cup B_1 \cup S(p_2, p_2), p_2) \neq \emptyset, \end{aligned} \tag{A.21}$$

which, taken with equation (A.21), contradicts equation (A.19).

Let $j \in AE\,(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s})$; we break the proof of (A.21) into five cases:

$$
\begin{aligned}
j &\in d_1(p_{1s}) \setminus d_1(p_2) \quad \Rightarrow \quad \exists\, l \in Z_{1j} \quad \text{such that} \quad p_{1s} > v_l \geq p_2, \\
&\Rightarrow \quad l \in AE\,(B_1, p_2)\,; \\
j &\in d_1(p_{1s}) \cap d_1(p_2). \quad j \in d_1(p_{1s}) \quad \Rightarrow \quad v_j \geq p_{1s} > p_2 \\
&\Rightarrow \quad j \in AE\,(d_1(p_2), p_2)\,; \\
j &\in B_1 \quad \Rightarrow \quad v_j \geq p_{1s} > p_2 \\
&\Rightarrow \quad j \in AE\,(B_1, p_2)\,; \\
j &\in S(p_{1s}, p_2) \setminus S(p_2, p_2) \quad \Rightarrow \quad \exists\, l \in Z_{1j} \quad \text{such that} \quad p_{1s} > v_l \geq p_2, \\
&\Rightarrow \quad l \in AE\,(B_1, p_2)\,; \\
j &\in S(p_{1s}, p_2) \cap S(p_2, p_2). \quad j \in S(p_{1s}, p_2) \quad \Rightarrow \quad v_j \geq p_{1s} > p_2 \\
&\Rightarrow \quad j \in AE\,(S(p_2, p_2), p_2)\,.
\end{aligned}
$$

Hence, $p_2 \leq p_{12s}$ and Claim 10 is proved. □

Claim 11. *If* $p_{1s} \geq p_2$ *and* $p_{2s} \geq p_1$, *then*

$$|AE\,(d_1(p_{12s}) \cup B_1 \cup S(p_{12s}, p_{12s}), p_{12s})\,| \geq m_1 + 1. \tag{A.22}$$

Proof. By Claim 10, we have $p_2 \leq p_{12s}$. Now, suppose $p_2 = p_{12s}$. Then since $p_{1s} > p_2 = p_{12s}$, equations (A.20)–(A.21) with p_{12s} in the place of p_2 proves that equation (A.22) holds and hence the Claim.

Finally, suppose $p_2 < p_{12s}$. Then p_2 the largest value in $\{0, \delta, 2\delta, \ldots\}$ satisfying (A.1) implies

$$|AE\,(d_2(p_{12s}) \cup B_2, p_{12s})\,| \leq m_2,$$

which together with equation (A.3) yields

$$|AE\,(d_1(p_{12s}) \cup B_1 \cup S(p_{12s}, p_{12s}), p_{12s})\,| \geq m_1 + 1,$$

i.e., equation (A.22) holds and Claim 11 is proved. □

Lemma 2. *Suppose $j \neq i$ plays σ^* in the continuation, and i plays $\bar{\sigma} \neq \sigma^*$ and transacts at price $\bar{p}$. Then $\bar{p} \geq p^*$, the price i would have transacted at had he played σ^*.*

Proof. We begin by proving the Lemma for the case in which $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$. The remaining two cases, $p_{1s} < p_2$, and $p_{2s} < p_1$, are proved subsequently with an argument that conditions on whether or not i is a dedicated bidder.

Case $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$. Note that Claim 11 is specific to case $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$, and allows us to make use of Claim 9.

Since with assumption (A2) we have that i wins an item under σ^*, when i instead plays $\bar{\sigma}$ the only case we need consider is that in which i also wins an item (when i is not a winner, his payoff (zero) can be no worse than a payoff associated to winning, since assumption A1 bounds winning payoffs below by zero). We assume without loss of generality that $i \in \overline{W}_1$; to yield a contradiction if Lemma 2 fails, suppose $\bar{p}_1 < p_{12s}$.

Subcase (a). First, suppose

$$\bar{p}_2 \geq p_{12s} > \bar{p}_1. \tag{A.23}$$

First, note that $\overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2) \neq \emptyset$; if not,

$$\begin{aligned}
|\overline{W}_1| &\geq |AE\left(d_1(p_{12s}) \cup B_1 \cup \left(S(p_{12s}, p_{12s}) \setminus \overline{W}_2\right), p_{12s}\right)| \\
&\quad \text{by applying Claim 9 with } p' = p'' = p_{12s}, \\
&= |AE\left(d_1(p_{12s}) \cup B_1 \cup S(p_{12s}, p_{12s}), p_{12s}\right)| \\
&\quad \text{by } \overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2) \neq \emptyset, \\
&\geq m_1 + 1 \quad \text{by Claim 11,}
\end{aligned}$$

which contradicts the fact that $|\overline{W}_1| = m_1$.

Let j be such that $j \in \overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2)$. $j \in \overline{W}_2$ implies $j \neq i$, since we assumed $i \in \overline{W}_1$ and i wins only one item.

Subsubcase (a1) $\bar{p}_2 > p_{12s}$. We show a contradiction. $\bar{p}_2 > p_{12s} > \bar{p}_1$ implies

$$\bar{p}_2 - \delta > \bar{p}_1. \tag{A.24}$$

Let t be the stage in which j submits his winning bid in auction 2. Since $j \neq i$ follows σ^*, we have $g_2^t \in \{\bar{p}_2 - \delta, \bar{p}_2\}$. But, since j follows σ^*, we must have

$$\begin{aligned} g_1^t &\geq g_2^t, \\ &\geq \bar{p}_2 - \delta, \\ &> \bar{p}_1 \quad \text{by (A.24)}, \end{aligned}$$

which contradicts the fact that g_1^t, the standing bid in auction 1 at stage t, cannot be greater than the ultimate final bid in auction 1, $\bar{p}_1$. This shows that $\bar{p}_2$ cannot exceed p_{12s}; given that case (a) assumes $\bar{p}_2 \geq p_{12s}$, we finish case (a) by treating the case when $\bar{p}_2$ equals p_{12s}.

Subsubcase (a2) $\bar{p}_2 = p_{12s}$. We again show a contradiction. Since j plays σ^*, we have $g_1^t \geq g_2^t$, which implies $g_2^t \leq \bar{p}_1 < p_{12s}$, so we must have $g_2^t = p_{12s} - \delta \equiv \bar{p}_1$. Also, by parts (b) and (e) of the definition of σ^* (the former corresponding to the play of $h \in d_2$, where $i \in \overline{W}_1 \quad \Rightarrow \quad h \neq i$, the latter corresponding to j's play) we have if $h \in AE\left(d_2(p_{12s}), p_{12s}\right)$, either $\exists\ k \in Z_{2h} \cap \overline{W}_{2t}$ such that $v_k \geq p_{12s}$ (recall that such a v_k would correspond to an actual bid already placed), or h submitted a winning bid (possibly a revised bid) equal to p_{12s} in auction 2. (To see this, recall also that, by part (b) of σ^*, h will outbid her phantom bidders if necessary to have a winning bid of p_{12s} in auction 2.) Since j's bid of p_{12s} was successful, this implies that $AE\left(d_2(p_{12s}) \cup B_2, p_{12s}\right) < m_2$; because p_{12s} is the ending price in auction 2, this in turn implies $AE\left(d_2(p_{12s}) \cup B_2, p_{12s}\right) \subset \overline{W}_2$, from which we conclude

$$\begin{aligned} &AE\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{12s}\right) \setminus \overline{W}_2 \\ &\qquad\qquad = AE\left(d_1(p_{12s}) \cup B_1 \cup \left(S(p_{12s}, p_{12s}) \setminus \overline{W}_2\right), p_{12s}\right). \end{aligned}$$

Since no bidder in $d_1 \cup d_2 \cup S \cup B_1 \cup B_2$ wins more than one item, $\overline{W}_2$ corresponds to exactly m_2 unique bidders in $AE\left(d_2(p_{12s}) \cup B_2, p_{12s}\right)$. In particular, this fact when combined with

$$|AE\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{12s}\right)| \geq m_1 + m_2 + 1$$

from equation (A.3) yields

$$|AE\left(d_1(p_{12s}) \cup B_1 \cup \left(S(p_{12s}, p_{12s}) \setminus \overline{W}_2\right), p_{12s}\right)| \geq m_1 + 1. \qquad \text{(A.25)}$$

But, the assumption $\bar{p}_2 = p_{12s} > \bar{p}_1$ allows us to use Claim 9 to conclude that the cardinality of $AE\left(d_1(p_{12s}) \cup B_1 \cup (S(p_{12s}, p_{12s}) \setminus \overline{W}_2), p_{12s}\right)$ must not exceed the cardinality of $\overline{W}_1$; since the latter equals m_1 (again, no bidder in $d_1 \cup d_2 \cup S \cup B_1 \cup B_2$ wins more than one item), and the former is at least $m_1 + 1$ by equation (A.25), we have a contradiction.

Subcase (b). Finally, suppose $\bar{p}_2 < p_{12s}$. We use the inequality $\bar{p}_1, \bar{p}_2 < p_{12s}$ to show

$$\begin{aligned} j \in AE\left(d_1(p_{12s}) \cup B_1 \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{12s}\right) & \\ \Rightarrow \quad (j \cup Z_{1j} \cup Z_{2j}) \cap (\overline{W}_1 \cup \overline{W}_2) \neq \emptyset. & \quad \text{(A.26)} \end{aligned}$$

The proof of (A.26) is analogous to that of Claim 9, involving five cases:

$$\begin{aligned}
j &\in d_a(p_{12s}) \setminus d_a(\bar{p}_a) \quad \Rightarrow \exists\, l \in Z_{aj} \text{ such that } p_{12s} > v_l \geq \bar{p}_a \quad \Rightarrow \quad l \in \overline{W}_{a;} \\
j &\in d_a(p_{12s}) \cap d_a(\bar{p}_a). \quad j \in d_a(p_{12s}) \quad \Rightarrow \quad v_j \geq p_{12s} > \bar{p}_a, \text{ and hence } j \in \overline{W}_{a;} \\
j &\in B_1 \cup B_2. \quad v_j \geq p_{12s} > \bar{p}_1, \bar{p}_2 \quad \Rightarrow \quad j \in \overline{W}_1 \cup \overline{W}_2; \\
j &\in S(p_{12s}, p_{12s}) \setminus S(\bar{p}_1, \bar{p}_2) \quad \Rightarrow \quad \exists\, l \in Z_{aj} \text{ such that } p_{12s} > v_l \geq \bar{p}_a, \\
& \Rightarrow l \in \overline{W}_{a;} \\
j &\in S(p_{12s}, p_{12s}) \cap S(\bar{p}_1, \bar{p}_2). \quad v_j \geq p_{12s} > \bar{p}_1, \bar{p}_2 \quad \Rightarrow \quad j \in \overline{W}_1 \cup \overline{W}_2.
\end{aligned}$$

With (A.26) in hand, we quickly get a contradiction:

$$\begin{aligned} m_1 + m_2 &= |\overline{W}_1 \cup \overline{W}_2| \quad \text{by } m_1 + m_2 \text{ items for sale,} \\ &\geq |AE\,(d_1(p_{12s}) \cup d_2(p_{12s}) \cup B_2 \cup S(p_{12s}, p_{12s}), p_{12s})\,| \\ &\quad \text{by equation (A.26),} \\ &\geq m_1 + m_2 + 1 \quad \text{by equation (A.3).} \end{aligned}$$

Since we have produced a contradiction to all cases in which $\bar{p}_1 < p_{12s}$, we have proved Lemma 2 holds when $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$. In the remainder of Lemma 2's proof, we condition on $i \in d_1$ and $i \in S$ (note that by symmetry with $i \in d_1$, case $i \in d_2$ need not be addressed).

Case $i \in d_1$ and $p_{1s} < p_2$. Suppose $\bar{p}_1 < p_{1s}$; we produce a contradiction. For all $j \in S \cup d_2$, $j \neq i$ implies j plays σ^*. Clearly, for $p < p_2$, we have

$$|AE\,(d_2(p_2) \cup B_2, p_2)\,| \geq m_2 + 1 \quad \text{by equation (A.1),} \tag{A.27}$$

$$\Rightarrow \quad |A\,(d_2(p) \cup B_2, p)\,| \geq m_2 + 1 \quad \text{by } p < p_2 \tag{A.28}$$

implies that if a standing bid of p in auction 2 survives a round in which all bidders in d_2 choose to pass (for example, if p is the final price in auction 2), then some d_2 bidder violated σ^*. Note that such a bidder is guaranteed to exist, since $j \in A\,(B_2, p)$ implies $j \in \overline{W}_1$, which would imply $|A\,(B_2, p)\,| \leq m_2$. Hence, we have $\bar{p}_2 \geq p_2$.

We next show that $\overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2) = \emptyset$. If not, for $j \in \overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2)$, let t be the stage that j submitted his winning bid in auction 2. Then $g_2^t \geq \bar{p}_2 - \delta \geq p_2 - \delta$. But, since

$$p_2 - \delta \;\geq\; p_{1s} \;> \bar{p}_1 \geq \; g_1^t,$$

we have $g_1^t \leq p_2 - 2\delta$, which implies j violates part (c) of σ^* if she bids in auction 2 at stage t.

Summarizing the above two paragraphs, we have $\bar{p}_1 < p_{1s} < p_2 \leq \bar{p}_2$, and $\overline{W}_2 \cap$

$S(\bar{p}_1, \bar{p}_2) = \emptyset$. Then

$$\begin{aligned} m_1 &= |\overline{W}_1| && \text{(A.29)} \\ &\geq |AE\,(d_1(p_{1s}) \cup B_1 \cup S(p_{1s}, p_2), p_{1s})\,| && \text{(A.30)} \\ &\quad \text{(by Claim 9 applied with } p' = p_{1s},\ p'' = p_2), \\ &\geq m_1 + 1 \quad \text{(by (A.2))}, && \text{(A.31)} \end{aligned}$$

which is a contradiction. Hence, $\bar{p}_1 \geq p_{1s}$.

Case $i \in d_1$, $p_1 > p_{2s}$. Suppose $\bar{p}_1 < p_1$; a contradiction follows from

$$\begin{aligned} |AE\,(d_1(p_1) \cup B_2, p_1)\,| &\geq m_1 + 1 \quad \text{by equation (A.1)}, \\ \Rightarrow \quad |A\,(d_2(\bar{p}_1) \cup B_2, \bar{p}_1)\,| &\geq m_1 + 1 \quad \text{by } \bar{p}_1 < p_1. && \text{(A.32)} \end{aligned}$$

In particular, $i \in \overline{W}_1$ and $A\,(B_1, \bar{p}_1) \subset \overline{W}_1$ imply that there exists a $j \in A\,(d_1(\bar{p}_1), \bar{p}_1) \setminus \overline{W}_1$, which implies that j violated σ^* if auction 1 ends with price $\bar{p}_1$. Hence, $\bar{p}_1 \geq p_1$.

Case $i \in S$ and $p_{1s} < p_2$. By equations (A.27)-(A.28) we have $\bar{p}_2 \geq p_2 > p_{1s}$. It remains only to show that if i wins in auction 1, then $\bar{p}_1 \geq p_{1s}$. First, $\bar{p}_2 \geq p_2 > p_{1s} > \bar{p}_1$ implies $\overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2) = \emptyset$; otherwise, we get that $j \in \overline{W}_2 \cap S(\bar{p}_1, \bar{p}_2)$ violates part (c) of σ^* (note $j \in \overline{W}_2$ implies $j \neq i$, since we assumed $i \in \overline{W}_1$ and i wins only one item). We can then apply (A.29)-(A.31) to produce a contradiction to $\bar{p}_1 < p_{1s}$.

Noting that case $i \in S$ and $p_{2s} < p_1$ follows by symmetry from the case just treated above, we conclude the proof Lemma 2. □

Combining Lemmas 4 and 2 immediately yields Proposition 1; in particular, no bidder can increase her payoff by unilaterally deviating from strategy σ^*, no matter what the state of the game and what beliefs she may hold about that state.

A.2 Proof of Corollary 2

Proof. Corollary 2 is easy to see when $P(\text{II})$ is written out using

$$\begin{aligned} P(X_{b:a} = x) &= a\binom{a-1}{b} f(x) F^{a-b-1}(x)(1-F(x))^b, \quad \text{and} \\ P(X_{b:a} < x) &= \sum_{i=0}^{b} \binom{a}{i} F^{a-i}(x)(1-F(x))^i \end{aligned} \tag{A.33}$$

(see Chapter 2 of [1]), where f and F are the PDF and CDF of the underlying valuation distribution. Substituting in the expressions of (A.33), we get

$$\begin{aligned} & P(X_{m_1:d_1} = \pi \mid X_{m_2:d_2+s} < X_{m_1:d_1}) P(X_{m_2:d_2+s} < X_{m_1:d_1}) \\ &= P(X_{m_1:d_1} = \pi,\, X_{m_2:d_2+s} < \pi) \\ &= d_1 \binom{d_1-1}{m_1} f(\pi) F^{d_1-m_1-1}(\pi)(1-F(\pi))^{m_1} \\ &\quad \cdot \sum_{i=0}^{m_2} \binom{d_2+s}{i} F^{d_2+s-i}(\pi)(1-F(\pi))^i. \end{aligned} \tag{A.34}$$

The exact value for $P(X_{m_2:d_2+s} < X_{m_1:d_1})$, namely

$$P(X_{m_2:d_2+s} < X_{m_1:d_1}) = \sum_{i=0}^{m_2} \frac{d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}},$$

follows by integrating the far left hand side and far right hand side of (3.3) over the domain of valuations. Furthermore, the second term on the right hand side of (3.3) is yielded when (A.34) is multiplied by π^k and integrated. Analogous arguments yield the exact value

$$P(X_{m_1:d_1+s} < X_{m_2:d_2}) = \sum_{i=m_2+1}^{d_2} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i}}{(d+s)\binom{d+s-1}{m_1+i}},$$

as well as the third term on the right hand side of (3.3). □

A.3 Proof of Proposition 3

Proof. Using equation (3.5) we find that

$$\begin{aligned}
&E[\Pi(n-t,2t,n-t)] - E[\Pi(n-(t-1),2(t-1),n-(t-1))] \\
&= 2t\Big(2(n-t)+2t-1\Big)\left[\frac{1}{2}E[X_{0:2n}] - E[X_{0:2n-1}] + \frac{1}{2}E[X_{0:2n-2}]\right] + E[X_{1:n+t}] \\
&\quad -(2t-2)\Big(2(n-t+1)+2(t-1)-1\Big) \qquad \text{(A.35)}\\
&\quad \cdot\left[\frac{1}{2}E[X_{0:2n}] - E[X_{0:2n-1}] + \frac{1}{2}E[X_{0:2n-2}]\right] \\
&\quad - E[X_{1:n+t-1}], \\
&= 2(2n-1)\left[\frac{1}{2}E[X_{0:2n}] - E[X_{0:2n-1}] + \frac{1}{2}E[X_{0:2n-2}]\right] \\
&\quad + E[X_{1:n+t}] - E[X_{1:n+t-1}]. \qquad \text{(A.36)}
\end{aligned}$$

Notice that on the left hand side of equation (A.36) the first term is independent of t, and can be written as a constant $-C$, $C \in \mathbb{R}^+$. Here we have used the observation that the bracketed portion of the first term is also the bracketed portion of the term that adjusts $E[\Pi(n-t,2t,n-t)]$ $(E[\Pi(n-(t-1),2(t-1),n-(t-1))])$ away from $E[X_{1:n+t}]$ $(E[X_{1:n+t-1}])$, so must be negative. Letting

$$\delta_t \triangleq E[X_{1:n+t}] - E[X_{1:n+t-1}],$$

we write

$$E[\Pi(n-t,2t,n-t)] - E[\Pi(n-(t-1),2(t-1),n-(t-1))] = -C + \delta_t. \quad \text{(A.37)}$$

Therefore, for $t \in \{0,1,\ldots,n-1\}$,

$$E[\Pi(n-(t-1),2(t-1),n-(t-1))] \le E[\Pi(n-t,2t,n-t)] \iff \delta_t \ge C, \quad \text{(A.38)}$$

and

$$E[\Pi(1,2(n-1),1)] \;=\; E[\Pi(0,2n,0)] \quad\Longleftrightarrow\quad \delta_n = C.$$

We begin by showing $\delta_n = C$. This result follows by expressing the expectations of second-highest valuations in terms of expectations of highest valuations. In particular, we use the fact that $E[X_{1:j}] = jE[X_{0:j-1}] - (j-1)E[X_{0:j}]$ ([8], p38) to write

$$\begin{aligned}
-C+\delta_n &= -C + \big(2nE[X_{0:2n-1}] - (2n-1)E[X_{0:2n}]\big) \\
&\quad -\big((2n-1)E[X_{0:2n-2}] - (2n-2)E[X_{0:2n-1}]\big) \\
&= 2(2n-1)\left[\frac{1}{2}E[X_{0:2n}] - E[X_{0:2n-1}] + \frac{1}{2}E[X_{0:2n-2}]\right] \\
&\quad +2nE[X_{0:2n-1}] - (2n-1)E[X_{0:2n}] - (2n-1)E[X_{0:2n-2}] \\
&\quad +(2n-2)E[X_{0:2n-1}] \qquad \text{by substituting in for } -C, \\
&= (2n-1)E[X_{0:2n}] - 2(2n-1)E[X_{0:2n-1}] + (2n-1)E[X_{0:2n-2}] \\
&\quad +2nE[X_{0:2n-1}] - (2n-1)E[X_{0:2n}] - (2n-1)E[X_{0:2n-2}] \\
&\quad +(2n-2)E[X_{0:2n-1}] \\
&= 0 \qquad \text{by cancellation.}
\end{aligned}$$

Showing $\delta_t \geq C$ is now easy. Since for $t \in \{0,1,\ldots,n-1\}$,

$$\delta_t \geq \delta_{t-1} \geq \cdots \geq \delta_{n-1} \geq \delta_n \qquad \text{by assumption,}$$

and we just showed that $\delta_n = C$, we have that $\delta_t \geq C$ for all $t \in \{0,1,\ldots,n-1\}$. If for some $t' < n$ we have $\delta_{t'} > \delta_{t'+1}$, then, since $\delta_{t'+1} \geq \delta_n = C$, we have that for $j \leq t'$, $\delta_{t'} > C$, that is, the inequalities in (A.38) are strict. □

A.4 Derivation of Equation (3.8)

To ease the presentation, let $x_t \triangleq \frac{n+t-1}{n+t}$. Then (3.7) is equivalent to

$$F^{-1}(x_t) - F^{-1}(x_{t-1}) \;\geq\; F^{-1}(x_{t+1}) - F^{-1}(x_t),$$

which by Taylor expansion is equivalent to

$$\frac{dF^{-1}}{dx}(x_{t-1})(x_t - x_{t-1}) + o(x_t - x_{t-1}) \;\geq\; \frac{dF^{-1}}{dx}(x_t)(x_{t+1} - x_t) + o(x_{t+1} - x_t), \quad \text{(A.39)}$$

where $\lim_{z\to 0} o(z)/z = 0$. Since $x_{t+1} - x_t < x_t - x_{t-1}$, equation (A.39) implies

$$\frac{\frac{dF^{-1}}{dx}(x_{t-1})}{\frac{dF^{-1}}{dx}(x_t)} \;\geq\; \frac{x_{t+1} - x_t}{x_t - x_{t-1}} + o(x_t - x_{t-1}),$$

that is,

$$\frac{f(F^{-1}(x_t))}{f(F^{-1}(x_{t-1}))} \;\geq\; \frac{n+t-1}{n+t+1} + o\left(\frac{1}{(n+t)(n+t-1)}\right),$$

where f is the PDF of the valuation distribution F.

A.5 Proof of Proposition 4

Proof. The inequality $E[\Pi(n-t, 2t, n-t)|U] \;\leq\; E[X_{2:2n}]$ follows directly from Proposition 3, which holds thanks to the assumption that δ_t is non-increasing. In particular, if this inequality fails for some $t \in [0, \ldots, n-2]$, then Proposition 1 is contradicted by

$$
\begin{aligned}
E[\Pi(n-t,2t,n-t)] &= E[\Pi(n-t,2t,n-t)|U]P(U) \\
&\quad + E[\Pi(n-t,2t,n-t)|E]P(E), \\
&= E[\Pi(n-t,2t,n-t)|U]P(U) + E[X_{2:2n}]P(E), \\
&> E[X_{2:2n}]P(U) + E[X_{2:2n}|E]P(E), \\
&= E[X_{2:2n}] = E[\Pi(0,2n,0)],
\end{aligned}
$$

where E is the event that the auctions' prices are equal.

To complete the proof of the Proposition (prove the righthand inequality in (3.9)), we let

$$\Delta_t \triangleq E[\Pi(n-t,2t,n-t)|U] - E[X_{1:n}]$$

and show $\Delta_t > 0$ for $t \in [1,\ldots,n-2]$. To make a clean argument below, we adopt the convention $\Delta_{n-1} = 0$. Using equation (A.42) from the proof of Claim 2 we write

$$\Delta_t = E[X_{1:n+t} - X_{1:n}] - \frac{t}{n}E[X_{1:2n} - X_{1:n}] - \frac{t(n^2-t^2)}{n(2n-1)(n-1)}E[X_{2:2n} - X_{1:n}]. \tag{A.40}$$

Letting $\hat{\Delta}_t \triangleq \Delta_t - \Delta_{t-1}$ and $\hat{\hat{\Delta}}_t \triangleq \hat{\Delta}_t - \hat{\Delta}_{t-1}$, we get

$$
\begin{aligned}
\hat{\Delta}_t &= \delta_t - \frac{E[X_{1:2n} - X_{1:n}]}{n} + \frac{3t^2 - 3t - n^2 + 1}{n(2n-1)(n-1)}E[X_{2:2n} - X_{1:n}], \qquad \text{and} \\
\hat{\hat{\Delta}}_t &= \hat{\delta}_t + \frac{6t-6}{n(2n-1)(n-1)}E[X_{2:2n} - X_{1:n}].
\end{aligned}
$$

Since $E[X_{2:2n} - X_{1:n}] > 0$ by Proposition 3, and $\hat{\delta}_t$ is non-decreasing by assumption, it follows that $\hat{\hat{\Delta}}_t$ is increasing in t. This implies that $\hat{\hat{\Delta}}_t$ switches sign at most once in the range $t \in [3,\ldots,n-2]$. This in turn implies $\hat{\Delta}_t$ switches sign at most twice in $[2,\ldots,n-2]$; this fact can be refined by realizing $\hat{\Delta}_1 \equiv \Delta_1 - \Delta_0 = \Delta_1$, and $\hat{\Delta}_{n-1} \equiv \Delta_{n-1} - \Delta_{n-2} = -\Delta_{n-2}$ are (respectively) positive and negative by our assumptions that $\Delta_1,\ \Delta_{n-2} > 0$, implying that $\hat{\Delta}_t$ switches sign exactly once in

$[2, \ldots, n-2]$. In other words, there exists a unique $\bar{t} \in [1, \ldots, n-2]$ such that Δ_t is increasing in $[1, \ldots, \bar{t}]$ and decreasing in $[\bar{t}+1, \ldots, n-2]$. Since $\Delta_1,\ \Delta_{n-2} \geq 0$ by assumption, Δ_t must be non-negative for all $t \in [1, \ldots, n-2]$ and we are done. □

A.6 Proof of Claim 2

Proof. We begin with the $U[0,1]$ case (the $U[a,b]$ case is treated the same way, per equation (3.13)). Equation (3.12) and the definition of $\hat{\delta}_t$ yield

$$\hat{\delta}_t = \frac{-4}{(n+t)(n+t+1)(n+t-1)},$$

which clearly increases in t. Next, we set

$$\Delta_t \triangleq E[\Pi(n-t, 2t, n-t)|U]P(U) - E[X_{1:n}]P(U),$$

and use Corollary 2 to write

$$\begin{aligned}
\Delta_t &\equiv E[\Pi(n-t, 2t, n-t)|U_1]P(U_1) + E[\Pi(n-t, 2t, n-t)|U_2]P(U_2) \\
&\quad - E[X_{1:n}](P(U_1) + P(U_2)) \\
&= E[X_{1:n+t} - X_{1:n}] - \frac{(n-t)(n-t-1) - (n+t)(n+t-1)}{2n(2n-1)} E[X_{1:2n} - X_{1:n}] \\
&\quad - \frac{(n-t)(n-t-1)(n+t) - (n+t)(n+t-1)(n-t)}{2n\binom{2n-1}{2}} E[X_{2:2n} - X_{1:n}] \\
&= E[X_{1:n+t} - X_{1:n}] - \frac{t}{n} E[X_{1:2n} - X_{1:n}] \qquad \text{(A.41)} \\
&\quad - \frac{t(n^2 - t^2)}{n(2n-1)(n-1)} E[X_{2:2n} - X_{1:n}]. \qquad \text{(A.42)}
\end{aligned}$$

Substituting $E[X_{i:j}] = \frac{j-i}{j+1}$ into (A.42) and simplifying yields

$$\begin{aligned}
\Delta_1 &= \frac{3n^3 - 8n^2 + 3n + 2}{n(n+1)(n+2)(4n^2-1)}, \qquad \text{and} \\
\Delta_{n-2} &= \frac{4n-8}{n(n+1)(4n^2-1)},
\end{aligned}$$

both of which are non-negative for $n \geq 2$.

Analogously, for the Exponential(λ) case

$$\hat{\delta}_t = \frac{-1}{\lambda(n+t)(n+t-1)},$$

which is clearly increasing in t. Further,

$$\Delta_1 = \frac{5n^2+1}{2\lambda(n+1)n(n-1)} - \frac{3}{\lambda(2n-1)}\sum_{l=n+1}^{2n}\frac{1}{l}, \quad \text{(A.43)}$$

$$\geq \frac{5n^2+1}{2\lambda(n+1)n(n-1)} - \frac{3}{\lambda(2n-1)}\ln(2). \quad \text{(A.44)}$$

The righthand side of (A.44), an upper bound on Δ_1, has roots at $n = 4.69$ and .25. Because (A.43) equals 0, 1/75, 17/980, and 137/7560 for $n = 2, 3, 4,$ and 5 (respectively), we conclude $\Delta_1 \geq 0$ for all $n \geq 2$.

To show $\Delta_{n-2} \geq 0$ for $n \geq 2$, we compute

$$\Delta_{n-2} = \frac{6}{\lambda(2n-1)n}\sum_{l=n+1}^{2n}\frac{1}{l} - \frac{7}{2(2n-1)n}, \qquad \text{which is positive if} \qquad \sum_{l=n+1}^{2n}\frac{1}{l} \geq \frac{7}{12}.$$

Since $\sum_{l=n+1}^{2n}\frac{1}{l}$ is increasing in n and equals 7/12 when $n = 2$, proof of the claim is complete. □

A.7 Proof of Claim 3

Proof. We prove Claim 3 below for $U[0,1]$ valuations; by equation 3.13, the arguments which follow apply to the general, $U[a,b]$ case as well (as in the proof of Claim 2, the proof of Claim 3 relies on proving the negativity or nonnegativity of differences of uniform order statistics). While the Claim assumes $pn \in \mathbb{Z}_+$, there is no guarantee that $pn_2^*(p, n_1) \in \mathbb{Z}_+$. Finding $n_2^*(p, n_1)$ amounts to solving for n_2 such that

$$\begin{aligned} E[\Pi(n_1(1-p), n_1 p + \lfloor n_2 p \rfloor, n_2 - \lfloor n_2 p \rfloor)] - E[X_{1:n_1}] &\geq 0 \quad \text{and,} \\ E[\Pi(n_1(1-p), n_1 p + \lfloor (n_2-1)p \rfloor, (n_2-1) - \lfloor (n_2-1)p \rfloor)] - E[X_{1:n_1}] &< 0. \end{aligned}$$

To consider an equation that we can solve, we for the moment ignore the $\lfloor \cdot \rfloor$ operator and instead find the $n_2'(p, n_1)$ which solves

$$E[\Pi(n_1(1-p), n_1 p + n_2 p, n_2(1-p))] - E[X_{1:n_1}] = 0. \tag{A.45}$$

Note that we are abusing notation in that the revenue Π has no realistic interpretation as an operator on non-integers. Therefore, for the purposes of this proof it may help to think of Π simply as a function over $\mathbb{R}_+$.

We prove Claim 3 using two Lemmas. The first Lemma ignores integrality (looks at Π over $\mathbb{R}$) and lays the groundwork for the bounds in (3.18); the second Lemma permits these bounds to be modified for integer arguments.

Lemma 3.

$$\frac{n_1}{2.3} + \frac{1}{2} \leq n_2'(p, n_1) \leq \frac{n_1}{2} + \frac{1}{2}, \tag{A.46}$$

where $n_2'(p, n_1)$ is the real, positive root of equation (A.45).

Proof. First, a few preliminary calculations.

$$\begin{aligned} &E[\Pi(n_1(1-p), n_1 p + p n_2, n_2(1-p))] - E[X_{1:n_1}] = \\ &\quad p\big(2 n_2^3 + (-p^2 n_1 + 4 n_1 - p^2) n_2^2 + (2 n_1^2 - 3 p n_1 - 3 n_1^2 p + p^2 n_1 + p^2 n_1^2 - 2) n_2 \\ &\quad + p n_1 + 1 - 3 n_1^2 + p n_1 + 1 - 3 n_1^2 - 2 n_1^3 \\ &\quad + 2 n_1^2 p + n_1^3 p\ \big) \,/\, \big((n_1 + p n_2 + 1)(n_1 + n_2 - 1)(n_1 + n_2 + 1)(n_1 + 1)\big), \qquad \text{(A.47)} \\ &\triangleq \frac{p \cdot h(n_2)}{(n_1 + p n_2 + 1)(n_1 + n_2 - 1)(n_1 + n_2 + 1)(n_1 + 1)}, \end{aligned}$$

where $h(n_2)$ is defined to be (A.47)'s right hand side's numerator divided by p. Equation (A.47) is zero where the numerator is zero; since by assumption $p \geq 1/n_1 > 0$, this occurs when $h(n_2)$ is zero. As h is a third degree polynomial in n_2, its real positive root $n_2'(p, n_1)$ is complicated and difficult to obtain insight from. For this reason we resort to proving bounds on $n_2'(p, n_1)$ (cf. equation (A.46)). Even this is a little tricky, since the bounds we derive are independent of p, and so we must take care to show that the bounds are valid for all p such that $n_1 p \in \{1, 2 \ldots, n_1\}$.

We begin by showing the upper bound, which is easier. The inequalities

$$\begin{aligned}
\frac{dh}{dn_2}(n2 = 1) &= (1-p)(2-p)n_1^2 + (8 - p^2 - 3p)n_1 + 4 - 2p^2 \; > \; 0 \\
& \qquad \text{for } n_1 \; \geq \; 2, \\
\frac{d^2h}{dn_2^2} &= (8 - 2p^2)n_1 + 12n_2 - 2p^2 \; > \; 0 \qquad \text{for } n_1, \; n_2 \; \geq \; 2, \text{ and} \\
h\left(n_2 = \frac{n_1}{2} + \frac{1}{2}\right) &= \frac{1}{4}(1-p)(n_1+1)^2((1-p)n_1 + 1 + p) \; \geq \; 0 \qquad \text{for } n_1 \geq 2,
\end{aligned}$$

show that h is nonnegative for $n_2 \geq \frac{n_1}{2} + \frac{1}{2} > 1$. We conclude that $\frac{n_1}{2} + \frac{1}{2}$ bounds $n_2'(p, n_1)$ from above for all p such that $n_1 p \in \{1, 2 \ldots, n_1\}$ (note that the bound is tight when $p = 1$).

Proving the lower bound is more involved. Substituting $n_2 = \frac{10n_1}{23} + \frac{1}{2}$ into h and dividing out the common denominator $4 \cdot 23^3$ yields

$$\begin{aligned}
(11960\, n_1^3 + 15134\, n_1^2 - 12167 - 8993\, n_1)\, p^2 - (14812\, n_1^3 + 39146\, n_1^2 + 24334\, n_1)\, p \\
- 10216\, n_1^3 + 12167 + 38088\, n_1 + 14904\, n_1^2. \quad \text{(A.48)}
\end{aligned}$$

To show that (A.48) is non-positive for $n_1 \geq 2$ and p such that $n_1 p \in \{1, 2 \ldots, n_1\}$, we begin by using inspection for the case $n_1 = 2$. When $n_1 = 2$, for $p = 1/2$ and $p = 1$, equation (A.48) evaluates to $-256509/4$ and -131454, respectively. To show that (A.48) is also non-positive for $n_1 > 2$, we find it convenient to first argue that the terms involving p and p^2 sum to a negative value, and then finish by showing that the terms involving p^0 also sum to something negative.

Isolating the and rearranging the p and p^2 terms of equation (A.48), and using the fact that $p \in (0, 1]$, we can easily see that

$$\begin{aligned}
&(11960\, n_1^3 + 15134\, n_1^2 - 12167 - 8993\, n_1)\, p^2 + (-14812\, n_1^3 - 39146\, n_1^2 - 24334\, n_1)\, p \\
&= (11960\, p^2 - 14812\, p)\, n_1^3 + (15134\, p^2 - 39146\, p)\, n_1^2 + (-8993\, p^2 - 24334\, p)\, n_1 - 12167\, p^2 \\
&< \; 0.
\end{aligned}$$

It remains to show that the p^0 term of (A.48) is negative. To see this, note that

$$\frac{d}{dn_1}\left(-10216\, n_1^3 + 12167 + 38088\, n_1 + 14904\, n_1^2\right) \;=\; 38088 + 29808 n_1 - 30648 n_1^2$$

implies that the p^0 term is decreasing for $n_1 > 2$. Thus, it only remains to show that the p^0 term is negative for $n_1 = 3$; by direct computation,

$$-10216\, n_1^3 + 12167 + 38088\, n_1 + 14904\, n_1^2\big|_{n_1=3} = -15265,$$

completing the argument that (A.48) is negative for all $n_1 \geq 2$ and p such that $n_1 p \in \{1, 2 \ldots, n_1\}$. This allows us to conclude that $\frac{n_1}{2.3} + \frac{1}{2} \leq n_2'(p, n_1)$. □

To put Lemma 1 to work in the context of integer arguments for Π, we now prove the following.

Lemma 4. *Let $a, b, c \in \mathbb{N}$, and $g < c$. Then, for u[0,1] valuations,*

$$E[\Pi(a, b+g, c-g)] \geq E[\Pi(a, b, c)]. \tag{A.49}$$

In other words, relative to the revenue of auctioneer 1, it is more valuable to add to the pooling compartment and take from the compartment of bidders dedicated to auctioneer 2 than vice-versa.

Proof.

$$
\begin{aligned}
&E[\Pi(a, b+g, c-g)] - E[\Pi(a,b,c)] = \\
&-g\big(-1-12\,a\,c\,b+3\,b^2+3\,b\,g+3\,b^2\,g+b\,g^2+3\,a^2\,g+a\,g^2+6\,a\,b\,g-6\,a^2\,c-6\,a\,c^2 \\
&\qquad +3\,a^2+3\,a\,g+2\,a-2\,c^3+2\,c+g^2+2\,b-6\,c\,b^2-6\,c^2\,b+6\,a\,b\big) \\
&\quad /\big((a+c+1+b)\,(a+b+g+1)\,(a+c-1+b)\,(a+c+b)\,(a+b+1)\big). \quad \text{(A.50)}
\end{aligned}
$$

Clearly, the denominator of the right hand side of (A.50) is positive. The numerator is nonnegative if

$$
\begin{aligned}
&-1-12\,a\,c\,b+3\,b^2+3\,b\,g+3\,b^2\,g+b\,g^2+3\,a^2\,g+a\,g^2+6\,a\,b\,g-6\,a^2\,c-6\,a\,c^2 \\
&\qquad +3\,a^2+3\,a\,g+2\,a-2\,c^3+2\,c+g^2+2\,b-6\,c\,b^2-6\,c^2\,b+6\,a\,b \\
&=(-a-b-1)\,g^2+(-6\,a\,b-3\,a-3\,b-3\,b^2-3\,a^2)\,g+1+12\,a\,c\,b-3\,b^2-6\,a\,b-2\,b \\
&\qquad +6\,c\,b^2+6\,c^2\,b-3\,a^2+6\,a^2\,c+6\,a\,c^2-2\,c-2\,a+2\,c^3 \quad \text{(A.51)}
\end{aligned}
$$

is nonnegative. Since clearly (A.51) is decreasing in g, we can prove the inequality (A.49) by directly showing that (A.51) is nonnegative for $g = c$. Substituting $g = c$ into equation (A.51) and simplifying yields

$$(c-1)(a+b+c+1)(3a+3b+2c-1),$$

which is nonnegative since $a, b, c \in \mathbb{N}$. This concludes our proof of the inequality in (A.49). □

We now use Lemmas 1 and 2 to derive the statement of Claim 3. We first note that rephrasing Lemma 1 gives

$$E\left[\Pi\left(n_1(1-p),\; n_1p+p(\frac{n_1}{2}+\frac{1}{2}),\; (\frac{n_1}{2}+\frac{1}{2})(1-p)\right)\right] \geq E[X_{1:n_1}], \qquad \text{(A.52)}$$

and

$$E\left[\Pi\left(n_1(1-p),\ n_1p+p(\frac{n_1}{2.3}+\frac{1}{2}),\ (\frac{n_1}{2.3}+\frac{1}{2})(1-p)\right)\right] \le E[X_{1:n_1}]. \qquad \text{(A.53)}$$

We can now show that if n_2 is large enough such that

$$\lfloor pn_2 \rfloor \ge \left\lceil p(\frac{n_1}{2}+\frac{1}{2}) \right\rceil, \quad \text{then} \quad E[\Pi(n_1(1-p),\ pn_1+\lfloor pn_2\rfloor,\ n_2-\lfloor pn_2\rfloor)] \ge E[X_{1:n_1}]. \qquad \text{(A.54)}$$

By taking $g = \lfloor pn_2 \rfloor - \lceil p(\frac{n_1}{2}+\frac{1}{2}) \rceil$, we get

$$\begin{aligned}
&E\left[\Pi\left(n_1(1-p),\ pn_1+\lfloor pn_2\rfloor,\ n_2-\lfloor pn_2\rfloor\right)\right] \\
&= E\left[\Pi\left(n_1(1-p),\ pn_1+\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil+g,\ n_2-\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil-g\right)\right] \\
&\ge E\left[\Pi\left(n_1(1-p),\ pn_1+\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil,\ n_2-\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil\right)\right],
\end{aligned}$$

where the inequality comes from applying Lemma 1 (where g is as defined above, $a = n_1(1-p)$, $b = pn_1 + \lceil p(\frac{n_1}{2}+\frac{1}{2})\rceil$, and $c = n_2 - \lceil p(\frac{n_1}{2}+\frac{1}{2})\rceil$ is clearly greater than or equal to g). Continuing in a similar fashion,

$$\begin{aligned}
&E\left[\Pi\left(n_1(1-p),\ pn_1+\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil,\ n_2-\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil\right)\right] \\
&\ge E\left[\Pi\left(n_1(1-p),\ pn_1+p(\frac{n_1}{2}+\frac{1}{2}),\ n_2-p(\frac{n_1}{2}+\frac{1}{2})\right)\right] \\
&\qquad \text{by applying Lemma 1 again, this time with } g = \left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil - p(\frac{n_1}{2}+\frac{1}{2}), \\
&\ge E\left[\Pi\left(n_1(1-p),\ pn_1+p(\frac{n_1}{2}+\frac{1}{2}),\ \frac{n_1}{2}+\frac{1}{2}-p(\frac{n_1}{2}+\frac{1}{2})\right)\right], \\
&= E\left[\Pi\left(n_1(1-p),\ pn_1+p(\frac{n_1}{2}+\frac{1}{2}),\ (\frac{n_1}{2}+\frac{1}{2})(1-p)\right)\right] \\
&\ge E[X_{1:n_1}] \quad \text{by equation (A.52)},
\end{aligned}$$

where the second inequality follows from

$$\lfloor pn_2 \rfloor \;\geq\; \left\lceil p(\frac{n_1}{2}+\frac{1}{2}) \right\rceil \;\geq\; p(\frac{n_1}{2}+\frac{1}{2}), \qquad \text{which implies}$$

$$pn_2-(pn_2-\lfloor pn_2 \rfloor) \geq p(\frac{n_1}{2}+\frac{1}{2}), \qquad \Rightarrow \qquad pn_2 \geq p(\frac{n_1}{2}+\frac{1}{2}), \qquad \Rightarrow \qquad n_2 \geq \frac{n_1}{2}+\frac{1}{2},$$

and the intuitive fact that, all other things left fixed, increasing the number of bidders dedicated to auctioneer 2 can only increase the revenue to auctioneer 1.

Following the same steps for the remaining inequality, we now show that if n_2 is so small that

$$\lfloor n_2 p \rfloor \;<\; \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor, \quad \text{then}$$

$$E\left[\Pi\left(n_1(1-p),\; pn_1+\lfloor n_2 p \rfloor,\; n_2-\lfloor n_2 p \rfloor\right)\right] \;\leq\; E[X_{1:n_1}]. \quad \text{(A.55)}$$

To see this,

$$\begin{aligned}
&E\left[\Pi\left(n_1(1-p),\; pn_1+\lfloor pn_2 \rfloor,\; n_2-\lfloor pn_2 \rfloor\right)\right] \\
&\leq\; E\left[\Pi\left(n_1(1-p),\; pn_1+\left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor,\; n_2-\left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor\right)\right] \quad \text{by Lemma 1,} \\
&\leq\; E\left[\Pi\left(n_1(1-p),\; pn_1+\left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor,\; \frac{n_1}{2.3}+\frac{1}{2}-\left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor\right)\right], \\
&\leq\; E\left[\Pi\left(n_1(1-p),\; pn_1+p(\frac{n_1}{2.3}+\frac{1}{2}),\; (\frac{n_1}{2.3}+\frac{1}{2})(1-p)\right)\right] \quad \text{by Lemma 1 again,} \\
&\leq\; E[X_{1:n_1}] \quad \text{by equation (A.53),}
\end{aligned}$$

where, similar to the previous analysis, the second inequality follows from the fact that

$$\begin{aligned}
\lfloor pn_2 \rfloor &< \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor, \\
\Rightarrow \quad \lfloor pn_2 \rfloor + 1 &\leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor, \\
\Rightarrow \quad pn_2 - (pn_2 - \lfloor pn_2 \rfloor) + 1 &\leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor, \\
\Rightarrow \quad pn_2 &\leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor, \\
\Rightarrow \quad pn_2 &\leq p(\frac{n_1}{2.3}+\frac{1}{2}), \\
\Rightarrow \quad n_2 &\leq \frac{n_1}{2.3}+\frac{1}{2}.
\end{aligned}$$

Taking equations (A.54) and (A.55) together proves

$$\left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor \;\leq\; n_2^*(p,n_1) \;\leq\; \left\lceil p(\frac{n_1}{2}+\frac{1}{2}) \right\rceil$$

and the Claim. □

A.8 Proof of Claim 4

Proof. We first prove the case in which $p = 1$. We find $n_2^*(1, n_1)$ by finding the value of n_2 at which auctioneer 1 is indifferent between pooling or not pooling with auctioneer 2, i.e., $n_2 = n_2^*(1, n_1)$ solves

$$E[\Pi(0, n_1 + n_2, 0)] - E[X_{1:n_1}] = 0.$$

As the Claim's result indicates, we don't compute this root directly, but find bounds on this root.

Plugging in (3.14) into (3.5) yields (after simplification)

$$E[\Pi(d_1, s, d_2)] = -\frac{s(2d_1 + s - 1)}{2\lambda(d + s)(d + s - 1)} + \frac{1}{\lambda}\sum_{j=2}^{d_1+s}\frac{1}{j}. \tag{A.56}$$

By equation (A.56),

$$\begin{aligned} E[\Pi(0, n_1 + n_2, 0)] &= -\frac{(n_1 + n_2)(2 \cdot 0 + (n_1 + n_2) - 1)}{2\lambda(0 + 0 + (n_1 + n_2))(0 + 0 + (n_1 + n_2) - 1)} \\ &\quad + \frac{1}{\lambda}\sum_{j=2}^{0+(n_1+n_2)}\frac{1}{j}, \\ &= \frac{-1}{2\lambda} + \frac{1}{\lambda}\sum_{j=2}^{n_1+n_2}\frac{1}{j}. \end{aligned}$$

Combining this with equation (3.14) yields

$$E[\Pi(0, n_1 + n_2, 0)] - E[X_{1:n_1}] = \frac{-1}{2\lambda} + \frac{1}{\lambda}\sum_{j=2}^{n_1+n_2}\frac{1}{j} - \frac{1}{\lambda}\sum_{j=2}^{n_1}\frac{1}{j} = \frac{-1}{2\lambda} + \frac{1}{\lambda}\sum_{j=n_1+1}^{n_1+n_2}\frac{1}{j}.$$

Our bounds originate in the fact that

$$\ln\left(\frac{b+1}{a}\right) = \int_{x=a}^{b+1}\frac{1}{x}dx \le \sum_{j=a}^{b}\frac{1}{j} \le \int_{x=a-1}^{b}\frac{1}{x}dx = \ln\left(\frac{b}{a-1}\right). \tag{A.57}$$

This allows us to write

$$\frac{-1}{2\lambda} + \ln\left(\frac{n_1 + n_2 + 1}{n_1 + 1}\right) \le \frac{-1}{2\lambda} + \frac{1}{\lambda}\sum_{j=n_1+1}^{n_1+n_2}\frac{1}{j} \le \frac{-1}{2\lambda} + \ln\left(\frac{n_1 + n_2}{n_1}\right).$$

The upper bound on $n_2^*(1, n_1)$ follows by $n_2^*(1, n_1)$'s integrality (for the upper ceiling), the definition of $\theta_u(1)$, and the fact that

$$\begin{aligned}
\frac{-1}{2\lambda} + \ln\left(\frac{n_1+n_2+1}{n_1+1}\right) > 0 \iff & \ln\left(\frac{n_1+1}{n_1+n_2+1}\right) < \frac{-1}{2\lambda}, \\
\iff & \frac{n_1+1}{n_1+n_2+1} < \exp(\frac{-1}{2}), \\
\iff & \frac{n_1(1-\exp(\frac{-1}{2})) + (1-\exp(\frac{-1}{2}))}{\exp(\frac{-1}{2})} < n_2.
\end{aligned}$$

The lower bound follows analogously:

$$\begin{aligned}
\frac{-1}{2\lambda} + \ln\left(\frac{n_1+n_2}{n_1}\right) < 0 \iff & \ln\left(\frac{n_1}{n_1+n_2}\right) > \frac{-1}{2\lambda}, \\
\iff & \frac{n_1}{n_1+n_2} > \exp(\frac{-1}{2}), \\
\iff & \frac{n_1(1-\exp(\frac{-1}{2}))}{\exp(\frac{-1}{2})} > n_2.
\end{aligned}$$

This concludes the proof for the case $p = 1$.

For the remainder of the proof below we assume $p < 1$. We begin by showing that

$$\left\lfloor n_1\left(\exp\left(\frac{p}{2}\right) - 1\right)\right\rfloor \leq \lfloor p n_2^*(p, n_1)\rfloor. \tag{A.58}$$

Namely, we show that for n_2 violating (A.58), the difference between the partial pooling revenue and the non-pooling revenue,

$$\begin{aligned}
& E\Big[\Pi\left(\lceil n_1(1-p)\rceil, \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor, \lceil n_2(1-p)\rceil\right)\Big] - E\left[X_{1:n_1}\right] \\
&= \frac{1}{\lambda}\sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor}\frac{1}{j} - \frac{(\lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor)(2\lceil n_1(1-p)\rceil + \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor - 1)}{2\lambda(n_1+n_2)(n_1+n_2-1)},
\end{aligned} \tag{A.59}$$

is negative, meaning that (n_1, n_2) would not be in the mutually feasible pooling set.

Our argument below is summarized as follows. First, for $n_2 \geq n_1/2$ we derive an upper bound on (A.59), and derive (A.58) as a necessary condition for this upper bound to be nonnegative. Next, we prove that this upper bound holds for $n_2^*(p, n_1)$

by showing that $n_2^*(p, n_1) \geq n_1/2$, meaning that $n_2^*(p, n_1)$ must satisfy (A.58).

Because we will need it below, we first note that the following inequality holds. For $p < 1$ and $n_2 \geq n_1/2$,

$$\ln\left(\frac{n_1 + n_2 p}{n_1 + \lfloor n_2 p \rfloor}\right) \geq \frac{(2n_1 - 1)(n_2 p - \lfloor n_2 p \rfloor) + (n_2 p)^2 - \lfloor n_2 p \rfloor^2}{2(n_1 + n_2)(n_1 + n_2 - 1)}. \tag{A.60}$$

Equation (A.60) is established by

$$\begin{aligned} \ln\left(\frac{n_1 + n_2 p}{n_1 + \lfloor n_2 p \rfloor}\right) &= \ln\left(1 + \frac{g}{n_1 + \lfloor n_2 p \rfloor}\right) && \text{where } g \triangleq n_2 p - \lfloor n_2 p \rfloor, \\ &\geq \frac{g}{n_1 + \lfloor n_2 p \rfloor} \cdot \frac{n_1 + \lfloor n_2 p \rfloor}{n_1 + n_2 p} && \text{by } \ln(1+x) \geq \frac{x}{1+x}, \\ &= \frac{g}{n_1 + n_2 p}, \end{aligned}$$

and the fact that for $p < 1$ and $n_2 \geq n_1/2$,

$$\frac{1}{n_1 + n_2 p} \geq \frac{2n_1 - 1 + n_2 p + \lfloor n_2 p \rfloor}{2(n_1 + n_2)(n_1 + n_2 - 1)}. \tag{A.61}$$

For $p < 1$ and $n_2 \geq n_1/2$, to prove (A.61) we show that

$$2(n_1 + n_2)(n_1 + n_2 - 1) \geq (n_1 + n_2 p)(2n_1 + n_2 p + \lfloor n_2 p \rfloor - 1). \tag{A.62}$$

Noticing that $p < 1$ and $n_1 p \in \mathbb{Z}^+$ is equivalent to $p \leq (n_1 - 1)/n_1$, we bound the right hand side of (A.62) by

$$\begin{aligned} &(n_1 + n_2 p)(2n_1 + n_2 p + \lfloor n_2 p \rfloor - 1) \\ &\leq (n_1 + n_2 p)(2n_1 + 2n_2 p - 1) \quad \text{since } n_2 p \geq \lfloor n_2 p \rfloor, \\ &\leq 2\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1}\right)\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1} - \frac{1}{2}\right). \end{aligned}$$

To put this bound to work, we observe that the left hand side of (A.62) minus this bound is increasing in n_2; more precisely,

$$\frac{d}{dn_2}\left[2(n_1+n_2)(n_1+n_2-1) - 2\left(n_1+n_2\frac{(n_1-1)}{n_1}\right)\left(n_1+n_2\frac{(n_1-1)}{n_1}-\frac{1}{2}\right)\right] \\ = \frac{n_1(3n_1-1)+n_2(8n_1-4)}{n_1^2} \geq 0.$$

Simple algebra yields

$$2(n_1+n_2)(n_1+n_2-1) - 2\left(n_1+n_2\frac{(n_1-1)}{n_1}\right)\left(n_1+n_2\frac{(n_1-1)}{n_1}-\frac{1}{2}\right)\Bigg|_{n_2=\frac{n_1}{2}} \\ = \frac{3n_1}{2} - 1 \geq 0,$$

so the inequality in (A.62) holds for $p < 1$ and $n_2 \geq n_1/2$.

We next derive an upper bound on (A.59) for $n_2 \geq n_1/2$; because we are only concerned with the sign of (A.59), we divide out the positive constant λ^{-1}. Dividing

(A.59) by λ^{-1} and using the fact that $n_1 p \in \mathbb{Z}^+$, we get

$$\begin{aligned}
&\sum_{j=n_1+1}^{n_1+\lfloor n_2p\rfloor} \frac{1}{j} - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1 - n_1p + \lfloor n_2p\rfloor - 1)}{2(n_1+n_2)(n_1+n_2-1)} && \text{(A.63)}\\
&\leq \ln_1\left(\frac{n_1+\lfloor n_2p\rfloor}{n_1}\right) - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1 - n_1p + \lfloor n_2p\rfloor - 1)}{2(n_1+n_2)(n_1+n_2-1)} \\
&\qquad \text{since } \ln\left(\frac{a}{b}\right) \geq \sum_{b+1}^{a} \frac{1}{j}, \\
&\leq \ln\left(\frac{n_1+\lfloor n_2p\rfloor}{n_1}\right) + \ln\left(\frac{n_1+n_2p}{n_1+\lfloor n_2p\rfloor}\right) && \text{(A.64)}\\
&\qquad - \frac{(2n_1-1)(n_2p-\lfloor n_2p\rfloor) + (n_2p)^2 - \lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)} \\
&\qquad - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1 - n_1p + \lfloor n_2p\rfloor - 1)}{2(n_1+n_2)(n_1+n_2-1)} \qquad \text{by equation (A.60)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) \\
&\qquad + \frac{-(2n_1-1)(n_2p - \lfloor n_2p\rfloor + n_1p + \lfloor n_2p\rfloor)}{2(n_1+n_2)(n_1+n_2-1)} \\
&\qquad + \frac{-(n_1p+\lfloor n_2p\rfloor)(-n_1p+\lfloor n_2p\rfloor) - (n_2p)^2 + \lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) + \frac{-(2n_1-1)(n_2p+n_1p) + (n_1p)^2 - \lfloor n_2p\rfloor^2 - (n_2p)^2 + \lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) + \frac{-(2n_1-1)p(n_2+n_1) - (n_1p+n_2p)(-n_1p+n_2p)}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p(2n_1 - p(n_1-n_2) - 1)}{2(n_1+n_2-1)}, \\
&\leq \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p(2n_1 - (n_1-n_2) - 1)}{2(n_1+n_2-1)} \qquad \text{since } p < 1, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p}{2}. && \text{(A.65)}
\end{aligned}$$

We can characterize when this upper bound will be nonnegative. In particular,

$$\begin{aligned}
\ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p}{2} \geq 0 \quad &\iff \quad \frac{n_1+n_2p}{n_1} \geq \exp\left(\frac{p}{2}\right), \\
&\iff \quad n_2p \geq n_1\left(\exp\left(\frac{p}{2}\right) - 1\right) > n_1\frac{p}{2},
\end{aligned}$$

where the last inequality follows from expanding the exponential as a first order Taylor series and truncating to yield

$$\exp\left(\frac{p}{2}\right) - 1 \;>\; \frac{p}{2} \qquad \text{for } p > 0.$$

Hence, we get that for $n_2 = n_1/2$, the right hand side of equation (A.65) is less than zero, implying the same for equations (A.63), and hence (A.59). Because the difference between partial pooling and no pooling grows (becomes more positive or less negative) with increasing n_2, (A.59) negative for $n_2 = n_1/2$ implies that $n_2^*(p, n_1) \geq n_1/2$.

Since $n_2^*(p, n_1) \geq n_1/2$, the series of inequalities beginning with (A.63) and ending with (A.65) are valid for $n_2^*(p, n_1)$. Because (A.65) is negative if $n_2 p < n_1(\exp(p/2) - 1)$, we conclude that

$$\lfloor p n_2^*(p, n_1) \rfloor \geq \left\lfloor n_1 \left(\exp\left(\frac{p}{2}\right) - 1 \right) \right\rfloor .$$

To complete the proof of the Claim, it remains only to show that

$$\lfloor p n_2^*(p, n_1) \rfloor \;\leq\; \left\lceil (n_1 + 1) \left(\exp\left(\frac{p(4n_1 - n_1 p - 2)}{6n_1 - 4} \right) - 1 \right) \right\rceil . \qquad \text{(A.66)}$$

We proceed by bounding (A.59) from below, and show that (A.66) is a sufficient for this lower bound to be positive.

Let

$$a(x) \triangleq \frac{-p(2n_1 - n_1 p + xp - 1)}{2(n_1 + x - 1)},$$

and notice that

$$\frac{da}{dx} = \frac{p(1-p)(2n_1 - 1)}{2(n_1 + x - 1)^2} \geq 0 \qquad \text{(A.67)}$$

implies that a is increasing in x.

Once again, dividing (A.59) by λ^{-1} and using the fact that $n_1 p \in \mathbb{Z}^+$, for $n_2 \geq n_1/2$

we get

$$\begin{aligned}
&\sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} - \frac{(n_1 p + \lfloor n_2 p\rfloor)(2n_1 - n_1 p + \lfloor n_2 p\rfloor - 1)}{2(n_1+n_2)(n_1+n_2-1)} \\
\geq\ & \sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} - \frac{(n_1 p + n_2 p)(2n_1 - n_1 p + n_2 p - 1)}{2(n_1+n_2)(n_1+n_2-1)} \qquad \text{since } n_2 p \geq \lfloor n_2 p\rfloor, \\
=\ & \sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} + a(n_2), \\
\geq\ & \ln\left(\frac{n_1 + \lfloor n_2 p\rfloor + 1}{n_1+1}\right) + a(n_2) \qquad \text{since } \sum_b^a \frac{1}{j} \geq \ln\left(\frac{a+1}{b}\right), \\
\geq\ & \ln\left(\frac{n_1 + \lfloor n_2 p\rfloor + 1}{n_1+1}\right) + a\left(\frac{n_1}{2}\right) \qquad \text{by equation (A.67) and } n_2 \geq \frac{n_1}{2}.
\end{aligned}$$

But,

$$\begin{aligned}
&\ln\left(\frac{n_1 + \lfloor n_2 p\rfloor + 1}{n_1+1}\right) + a\left(\frac{n_1}{2}\right) \geq 0 \\
\iff\ & \lfloor n_2 p\rfloor \ \geq\ (n_1+1)\left(\exp\left(-a\left(\frac{n_1}{2}\right)\right) - 1\right). \qquad \text{(A.68)}
\end{aligned}$$

We conclude that $n_2^*(p, n_1)$ is less than or equal to any n_2' satisfying (A.68) and $n_2' \geq n_1/2$ (since any such n_2' makes (A.59) nonnegative). However, only (A.68) is necessary, as it implies that $n_2 \geq n_1/2$:

$$\begin{aligned}
n_2 p \ \geq\ \lfloor n_2 p\rfloor \ &\geq\ n_1\left(\exp\left(-a\left(\frac{n_1}{2}\right)\right) - 1\right) \\
&\geq\ n_1\left(-a\left(\frac{n_1}{2}\right)\right) \qquad \text{by first order Taylor expansion,} \\
&=\ n_1\left(\frac{p(4n_1 - n_1 p - 2)}{6n_1 - 4}\right), \\
\Rightarrow n_2 \ &\geq\ n_1\left(\frac{4n_1 - n_1 p - 2}{6n_1 - 4}\right), \\
&\geq\ n_1\left(\frac{4n_1 - n_1 - 2}{6n_1 - 4}\right) \qquad \text{since } p < 1, \\
&=\ \frac{n_1}{2}.
\end{aligned}$$

Hence, $n_2^*(p, n_1)$ is less than or equal to the smallest n_2 satisfying (A.68); in other words, (A.66) holds. $\square$

A.9 Proof of Claim 5

Proof. We prove Claim 5 by way of characterizing the more general value $n_2^*(p, n_1, m_1, m_2)$, which we write simply as n_2^*. The value n_2^* is defined to be the smallest integer n_2 such that

$$E[X_{m_1 : n_1}] \le E[X_{m_1+m_2 : n_1+n_2}]. \tag{A.69}$$

Equation (A.69) is equivalent to

$$\begin{aligned} 1 &\ge \frac{E[X_{m_1 : n_1}]}{E[X_{m_1+m_2 : n_1+n_2}]}, \\ &= \frac{E[X_{m_1 : n_1}]}{E[X_{m_1+1 : n_1}]} \cdot \frac{E[X_{m_1+1 : n_1}]}{E[X_{m_1+2 : n_1}]} \cdots \frac{E[X_{m_1+m_2-1 : n_1}]}{E[X_{m_1+m_2 : n_1}]} \\ &\quad \cdot \frac{E[X_{m_1+m_2 : n_1}]}{E[X_{m_1+m_2 : n_1+1}]} \cdot \frac{E[X_{m_1+m_2 : n_1+1}]}{E[X_{m_1+m_2 : n_1+2}]} \cdots \frac{E[X_{m_1+m_2 : n_1+n_2-1}]}{E[X_{m_1+m_2 : n_1+n_1}]}, \\ &= \prod_{i=m_1+1}^{m_1+m_2} \left(\frac{\alpha i}{\alpha i - 1} \right) \cdot \prod_{i=n_1+1}^{n_1+n_2} \left(\frac{\alpha i - 1}{\alpha i} \right), \end{aligned} \tag{A.70}$$

where the final equality follows from equation (3.15). As $\alpha \to 1$, terms in equation (A.70) cancel, leaving

$$1 \ge \frac{m_1 + m_2}{m_1} \cdot \frac{n_1}{n_1 + n_2}, \quad \text{which is equivalent to} \quad n_2 \ge n_1 \frac{m_2}{m_1},$$

proving the first limit of Claim 5.

Re-writing equation (A.70) as

$$\prod_{i=n_1+1}^{n_1+n_2} \left(1 - \frac{1}{\alpha i}\right) \le \prod_{i=m_1+1}^{m_1+m_2} \left(1 - \frac{1}{\alpha i}\right),$$

and taking the natural log of both sides yields

$$\sum_{i=n_1+1}^{n_1+n_2} \ln\left(1-\frac{1}{\alpha i}\right) \leq \sum_{i=m_1+1}^{m_1+m_2} \ln\left(1-\frac{1}{\alpha i}\right). \tag{A.71}$$

Sending $\alpha \to \infty$ and applying fact that $\lim_{x\to 0}\ln(1+x) = x$, equation (A.71) becomes

$$\sum_{i=n_1+1}^{n_1+n_2} \frac{1}{i} \geq \sum_{i=m_1+1}^{m_1+m_2} \frac{1}{i}. \tag{A.72}$$

The second limit of Claim 5 then follows from the fact that the formula for $E_{\exp(\lambda)}[X_{i:j}]$ in equation (3.14) implies equation (A.71) can be rewritten as $E_{\exp(\lambda)}[X_{m_1+m_2:n_1+n_2}] \geq E_{\exp(\lambda)}[X_{m_1:n_1}]$. □

A.10 Proof of Claim 6

Proof. Using (3.3) we can write

$$\begin{aligned}\beta_t &= E[X_{0:n+2t} - X_{0:n+t}] - \frac{t}{n}\left(E[X_{0:3n} - X_{2:3n}] - E[X_{0:2n} - X_{1:2n}]\right) \\ &\qquad - \frac{t(n-t)}{n(3n-1)}E[X_{1:3n} - X_{2:3n}]. \end{aligned} \tag{A.73}$$

We will begin with the $U[0,1]$ case. Substituting equation $E[X_{i:j}] = \frac{j-i}{j+1}$ into (A.73) yields

$$\begin{aligned}\beta_t &= \frac{t}{(n+2t+1)(n+t+1)} - \frac{t(5n^2+3n-1-2nt-t)}{n(3n+1)(3n-1)(2n+1)}, \\ &= \frac{t\left[(4n+2)t^3 - (4n^2-3n-5)t^2 - (13n^3+19n^2+2n-4)t\right]}{(n+2t+1)(n+t+1)n(3n+1)(3n-1)(2n+1)} \\ &\quad + \frac{t\left[13n^4 - 4n^3 - 12n^2 - 2n + 1\right]}{(n+2t+1)(n+t+1)n(3n+1)(3n-1)(2n+1)}, \\ &\triangleq \frac{t[g(t)]}{(n+2t+1)(n+t+1)n(3n+1)(3n-1)(2n+1)}.\end{aligned}$$

That is, we define a function $g(t)$ to be the numerator of the second line's righthand side, divided by t. This allows us to prove the sign of β_t by proving the same for the simpler function $g(t)$. First, we see that $\frac{dg}{dt}$ is convex in t; in particular,

$$\frac{dg}{dt}(t) = 3(4n+2)t^2 - 2(4n^2 - 3n - 5)t - 13n^3 - 19n^2 - 2n + 4.$$

Furthermore,

$$\begin{aligned}\frac{dg}{dt}(t=1) &= -13n^3 - 27n^2 + 16n + 20 < 0 \qquad \text{for all } n \geq 1, \text{ and} \\ \frac{dg}{dt}(t=n) &= -9n^3 - 7n^2 + 8n + 4 < 0 \qquad \text{for all } n \geq 1,\end{aligned}$$

that is, $\frac{dg}{dt}$ is negative at both endpoints $t = 1,\ n$, implying that $g(t)$ decreases in the interval $[1, \ldots, n]$. The existence of $\bar{t}$ for $n \geq 3$ can now be proved by showing g is positive at the left endpoint $t = 1$ (the negativity of g at the right endpoint $t = n$ follows from the paragraph immediately preceding the statement of Claim 6).

$$g(t=1) = 13n^4 - 17n^3 - 35n^2 + 3n + 12 > 0 \qquad \text{for all } n \geq 3.$$

The bounds on $\bar{t}$ follow similarly:

$$\begin{aligned}g\left(t = \frac{3n}{5}\right) &= \frac{578}{125}n^4 - \frac{1736}{125}n^3 - \frac{57}{5}n^2 + \frac{2}{5}n + 1 > 0 \quad \text{for } n \geq 4, \quad (A.74) \\ g\left(t = \frac{4n}{5}\right) &= \frac{261}{125}n^4 - \frac{2032}{125}n^3 - \frac{52}{5}n^2 + \frac{6}{5}n + 1 > 0 \quad \text{for } n \geq 9.\end{aligned}$$

We will run through the first bound; the other follows similarly. Coupled with g decreasing over $[1, \ldots, n]$, equation (A.74) implies that, for the largest $t \in [1, \ldots, n]$ such that $t/n \leq 3/5$, we have $\beta_t > 0$. That is, $\bar{t}/n \geq 3/5$.

We next prove the claim for the exp(1) distribution (the exp(λ) case follows analogously). Substituting equation (3.14) into (A.73) and simplifying yields

$$\beta_t = \sum_{l=n+t+1}^{n+2t} \frac{1}{l} - \frac{t}{n}\left(\frac{1}{2}\right) - \frac{t(n-t)}{n(3n-1)}\left(\frac{1}{2}\right). \tag{A.75}$$

The approach to proving the existence of $\bar{t}$ and $\bar{t}/n \leq n/2$ is analogous to that used above for the $U[0,1]$ case, with additional care required by the fact that β_t is only defined over $\mathbb{Z}$, and does not readily extend to $\mathbb{R}$. First, let us define

$$\begin{aligned}
\hat{\beta}_t &\triangleq \beta_t - \beta_{t-1}, \\
&= \frac{4t^4 - 2t^3 - (11n^2 - n)t^2 - (3n^3 - 6n^2 + n)t + n^4 + n^3}{(n+2t)(n+2t-1)(n+t)(3n-1)n} \quad \text{after simplification,} \\
&\triangleq \frac{h(t)}{(n+2t)(n+2t-1)(n+t)(3n-1)n},
\end{aligned}$$

where $h(t)$ is simply shorthand for the second line's righthand side's numerator. Next we show $h(t)$ is decreasing for $t \in [1,\ldots,n]$. The first and second derivatives of h are

$$\begin{aligned}
\frac{dh(t)}{dt} &= 16t^3 - 6t^2 - (22n^2 - 2n)t - 3n^3 + 6n^2 - n, \qquad \text{and} \\
\frac{d^2h(t)}{dt^2} &= -22n^2 + 2n + 48t^2 - 12t.
\end{aligned}$$

Since $\frac{d^2h}{dt^2} = 0$ if and only if $t = 1/8 \pm (9 + 264n^2 - 24n)^{1/2}/24$, when $n \geq 2$ we must have at most one first order condition (FOC) of $\frac{dh}{dt}$ in $\mathbb{R}^+$, and hence at most one in $[1,n] \subset \mathbb{R}^+$. Since

$$\begin{aligned}
\frac{dh(t=1)}{dt} &= -3n^3 - 16n^2 + n + 10 \; < \; 0 \qquad \text{for all } n \geq 2, \\
\frac{dh(t=n)}{dt} &= -9n^3 + 2n^2 - n \; < \; 0 \qquad \text{for all } n \geq 2, \\
\frac{d^2h(t=1)}{dt^2} &= -22n^2 + 2n + 36 \; < \; 0 \qquad \text{for all } n \geq 2, \qquad \text{and} \\
\frac{d^2h(t=n)}{dt^2} &= 26n^2 - 10n \; > 0 \qquad \text{for all } n \geq 2,
\end{aligned}$$

we have that $\frac{dh(t)}{dt}$ is negative at endpoints $t = 1$ and $t = n$, and has negative and positive slopes at the endpoints (respectively). Coupled with the fact that $\frac{dh(t)}{dt}$ has at most one FOC in $[1, n]$, these observations imply $\frac{dh(t)}{dt}$ is negative for all $t \in [1, n]$ when $n \geq 2$. By the definition of h, we have immediately that $\hat{\beta}_t$ must be decreasing for $t \in [1, \ldots, n]$ when $n \geq 2$.

Next, we note that

$$\begin{aligned} h(t=1) &= n^4 - 2n^3 - 5n^2 + 2 > 0 \qquad \text{for all } n \geq 4, \qquad \text{and} \\ h(t=n) &= -9n^4 + 6n^3 - n^2 < 0 \qquad \text{for all } n \geq 2. \end{aligned}$$

For $n \geq 4$, this implies that $\hat{\beta}_t$ is positive and negative at the left and right endpoints (respectively) of the interval $[1, \ldots, n]$. Since $\beta_{t=0} \equiv 0$, and $\beta_{t=n} < 0$ by the paragraph immediately preceding the statement of Claim 6, we have that β_t is positive over some interval $[1, \ldots, \bar{t}]$, then is negative in the complement $[\bar{t}+1, \ldots, n]$. That is, we have proved that $\bar{t}$ exists.

To complete the claim's proof it remains to show that $\bar{t}/n \leq 1/2$. Recalling that β_t is only defined over $t \in \mathbb{Z}$ (cf. equation (A.75)), we argue two cases: n even; and n odd. In the first case, $n/2$ is the largest t such that $t/n \leq 1/2$. Plugging $t = n/2$ into (A.75) yields

$$\begin{aligned} \beta_{\frac{n}{2}} &= \sum_{l=\frac{3n}{2}+1}^{2n} - \frac{7n-2}{24n-8}, \\ &\leq \ln\left(\frac{2n}{\frac{3n}{2}}\right) - \frac{7n-2}{24n-8} \qquad \text{by the inequality in (A.57)}, \\ &\triangleq u(n). \end{aligned}$$

Differentiating u gives $\frac{du(n)}{dn} = (3n-1)^{-2}/8 > 0$, implying that u increases with n. Since $\lim_{n\to\infty} u(n) = \ln(4/3) - 7/12 = -.0039$, we have $u < 0$ for all $n \geq 2$. Hence, for n even we have $\beta_{n/2} < 0$, and hence $\bar{t}/n \leq 1/2$.

For the case n odd, let $n = 2k+1$, $k \in \mathbb{Z}^+$. In this case, $t = k$ is the largest t such that $t/n \leq 1/2$. Substituting $n = 2k+1$ and $t = k$ into (A.75) and applying the same bounding argument as above yields

$$\begin{aligned}\beta_k &\le \ln\left(\frac{4k+1}{3k+1}\right) - \frac{k(7k+3)}{2(2k+1)(6k+2)}, \\ &\triangleq v(k).\end{aligned}$$

Differentiating v gives

$$\frac{dv(k)}{dk} = -\frac{20k^3+9k^2-2k-1}{4(3k+1)^2(4k+1)(2k+1)^2} < 0 \qquad \text{for } k \ge 8.$$

Hence, to show $\beta_k < 0$ for $k \ge 8$, we need only show that $\beta_k|_{k=8} < 0$. For completeness, we show the same for $k = 1, \dots, 7$, thereby showing $\beta_k < 0$ for $k \ge 1$ (i.e., $n \ge 3$ odd):

$$\begin{aligned}&\beta_k|_{k=1} = -.083, \quad \beta_k|_{k=3} = -.006, \quad \beta_k|_{k=5} = -.0053, \quad \beta_k|_{k=7} = -.0049, \\ &\beta_k|_{k=2} = -.0067, \quad \beta_k|_{k=4} = -.0055, \quad \beta_k|_{k=6} = -.0051.\end{aligned}$$

Hence, when $n = 2k+1$, we have $\beta_k < 0$, and hence $\bar{t}/n \le k/n \le 1/2$. □

Appendix B

Part II Proofs

B.1 Proof of Proposition 5 "⇐" Direction

We begin by proving the "⇐" direction, which is easier. **A1.1. Proposition 1 "⇐" Direction.** Suppose $\hat{v}_1, \ldots, \hat{v}_A$ enforces $(c_i(\vec{x}) + \pi, \vec{x})$. For convenience, let $\hat{v}(\vec{x}) = \sum_{a=1}^{A} \hat{v}_a(x_a)$. Clearly, the assumption that supplier i is the winner of the auction implies that $c_s(\vec{x}) > c_i(\vec{x}) + \pi \ \forall \ s \neq i$. Let j be the last supplier to drop out of the auction. We now show that $T_i + \pi$ intersects c_j.

Let $\vec{z}$ denote supplier j's bid induced by $\hat{v}$. The assumption that supplier i wins the auction with profit π implies that

$$\hat{v}(\vec{x}) - c_i(\vec{x}) - (\hat{v}(\vec{z}) - c_j(\vec{z})) = \pi,$$

$$\Rightarrow \hat{v}(\vec{x}) - \hat{v}(\vec{z}) = \pi + c_i(\vec{x}) - c_j(\vec{z}),$$

$$\Rightarrow \nabla\hat{v}(\vec{x})\prime(\vec{x} - \vec{z}) < \pi + c_i(\vec{x}) - c_j(\vec{z}) \qquad \text{since } \hat{v} \text{ is concave},$$

$$\Rightarrow \nabla c_i(\vec{x})\prime(\vec{x} - \vec{z}) < \pi + c_i(\vec{x}) - c_j(\vec{z}) \qquad \text{since supplier } i \text{ bids } \vec{x} \text{ iff } \nabla c_i(\vec{x}) = \nabla\hat{v}(\vec{x}),$$

$$\Rightarrow c_j(\vec{z}) < \pi + c_i(\vec{x}) + \nabla c_i(\vec{x})\prime(\vec{z} - \vec{x}) = T_i(\vec{z}) + \pi.$$

Since c_j is convex, increasing and lies below $T_i + \pi$ at $\vec{z}$, c_j must eventually cross the hyperplane $T_i + \pi$.

We next prove the other direction of Proposition 1, which is more difficult.

B.2 Proof of Proposition 5 "$\Rightarrow$" Direction

The proof proceeds in two stages. First, we prove Proposition 1 for the two-supplier case, where $s = 1, 2$. We index the assumptions as (A1): $c_2(\vec{x}) > c_1(\vec{x}) + \pi$; and (A2): the hyperplane $T_1 + \pi$ intersects c_2. Given (A1) and (A2), we need to find a feasible scoring rule $\hat{v}_1, \ldots, \hat{v}_A$ such that $S_2 > \epsilon$ and supplier 1 wins the auction with bid $(c_1(\vec{x}) + \pi, \vec{x})$. Such a $\hat{v}$ is found via Claims 1-5.

In the second stage, the two-supplier results are extended to S suppliers. Claim 7 (proof of the "$\Rightarrow$" direction of Proposition 1) shows that $(c_i(\vec{x}) + \pi, \vec{x})$ can be enforced by taking a convex combination of the scoring rule that enforces $(c_i(\vec{x}) + \pi, \vec{x})$ in the two-supplier case (with i and j in the roles of 1 and 2), and a second scoring rule constructed in Claim 6.

Stage 1. Claim 1 contains the main construction result for the two-dimensional (e.g., price and a non-price attribute), two supplier case; Claim 1 will be our main tool, as the construction of $\hat{v}$ for the multi-attribute case uses an attribute-by-attribute construction procedure. This procedure is provided in Claim 5, which builds upon the groundwork laid by Claims 1-4.

Claim 12. *For $i = 1,\ 2$, let $g_i : \mathbb{R}^+ \to \mathbb{R}$ be convex, increasing smooth functions. Let l_1 be the line tangent to g_1 at $z_1 > 0$, and let $\alpha \geq 0$. If $g_2(z_1) > g_1(z_1) + \alpha$ and $l_1 + \alpha$ intersects g_2, then there exists $z_2 > 0$ and a concave, increasing smooth function f such that $f'(z) \to 0$ as $z \to \infty$, for $i = 1,\ 2$ $f'(z) > g_i(z)$ as $z \to 0^+$, and*

$$f(z_1) = g_1(z_1) + \alpha + h, \quad f(z_2) = g_2(z_2) + h, \tag{B.1}$$

$$f'(z_1) = g'(z_1), \quad \textit{and} \quad f'(z_2) = g_2'(z_2), \tag{B.2}$$

where h can be chosen arbitrarily large.

Proof. We first find a z_2 such that

$$g_2(z_2) + g_2'(z_2)[z_1 - z_2] > g_1(z_1) + \alpha, \quad \text{and} \tag{B.3}$$

$$g_2(z_2) < l_1(z_2) + \alpha. \tag{B.4}$$

Since g_2 is smooth, convex, and increasing and lies above $l_1 + \alpha$ at z_1, g_2 intersects $l_1 + \alpha$ either to the right or left of z_1 (but not both).

(c1): g_2 intersects $l_1 + \alpha$ to the left of z_1. Let $\bar{z}$ be the closest point to z_1 at which g_2 intersects $l_1 + \alpha$. At $\bar{z}$, g_2 must cross $l_1 + \alpha$ from below; hence, $g_2'(\bar{z}) > g_1'(z_1) =$ slope of $l_1 + \alpha$. Since $(\bar{z}, g_2(\bar{z}))$ and $(z_1, g_1(z_1) + \alpha)$ are points on the line $l_1 + \alpha$, we can write the point slope formula for $l_1 + \alpha$ at the point $(\bar{z}, g_2(\bar{z}))$ and evaluate at z_1, yielding the equation

$$g_2(\bar{z}) + g_1'(z_1)[z_1 - \bar{z}] = g_1(z_1) + \alpha.$$

Replacing $g_1'(z_1)$ by the larger value $g_2'(\bar{z})$ implies

$$g_2(\bar{z}) + g_2'(\bar{z})[z_1 - \bar{z}] > g_1(z_1) + \alpha, \tag{B.5}$$

since $z_1 > \bar{z}$. Noting that the left side of (B.5) can be viewed as a continuous function of $\bar{z}$, we can find a $\eta_1 > 0$ such that (B.5) holds with z in place of $\bar{z}$ as long as $|\bar{z} - z| < \eta_1$. Since g_2 is itself continuous and approaches $l_1 + \alpha$ from below at $\bar{z}$, there exists a $\eta_2 > 0$ such that $g_2(z) < l_1(z) + \alpha$ provided that $z < \bar{z}$ and $|\bar{z} - z| < \eta_2$. An appropriate value for z_2 is then found by taking

$$z_2 = \bar{z} - \eta, \tag{B.6}$$

provided $0 < \eta < \min\{\eta_1, \eta_2\}$.

(c2): g_2 intersects $l_1 + \alpha$ to the right of z_1. Let $\bar{z}$ be the closest point to z_1 at which g_2 intersect $l_1 + \alpha$. At $\bar{z}$ g_2 must cross $l_1 + \alpha$ from above; the remaining arguments to find z_2 are straightforward analogues to those of (c1).

Now that we have shown that a z_2 satisfying (B.3)-(B.4) exists, we now construct f, beginning with the case $z_2 < z_1$; the complement case is essentially the same, and is omitted for brevity.

The proof for the case $z_2 < z_1$ synthesizes f from three concave, increasing functions f_1, f_2, and f_3 (note that the subscripts on f in this proof do not correspond to attributes), defined over respective intervals $[0, z_2]$, $[z_2, z_1]$, and $[z_1, \infty)$. For each f_i we enforce conditions at z_1 and z_2 toward satisfying (B.1)-(B.2):

$$f_i(z_2) = g_2(z_2) + h, \quad \text{and} \quad f_i'(z_2) = g_2'(z_2), \quad \text{for} \quad i = 1, 2, \quad \text{and} \tag{B.7}$$

$$f_i(z_1) = g_1(z_1) + \alpha + h, \quad \text{and} \quad f_i'(z_1) = g_1'(z_1), \quad \text{for} \quad i = 2, 3, \tag{B.8}$$

as well as choose the functions' forms to ensure that the other conditions of the Claim's statement are satisfied. We begin by constructing f_1.

We set $f_1(z) = \gamma_{11} z^{\gamma_{12}}$. Solving (B.7) with $i = 1$ for γ_{11}, γ_{12} yields

$$\gamma_{11} = [g_2(z_2) + h] z_2^{-\gamma_{12}}, \quad \text{and} \quad \gamma_{12} = \frac{g_2'(z_2) z_2}{g_2(z_2) + h}.$$

Over $[0, z_2]$ f_1 is increasing, since $\gamma_{11} > 0$. Choosing $h > g_2'(z_2) z_2 - g_2(z_2)$ ensures that $\gamma_{12} < 1$, and thereby the concavity of f_1 and $f_1'(z) \to \infty$ as $z \to 0^+$.

We find f_2 by first finding what f_2 would be in a new, shifted and rotated coordinate space. We then take the resulting function, $\bar{f}_2$, and apply a series of reflections, translations and rotations to produce the desired f_2. The new coordinate space we find convenient is that in which we view $(z_1, g_1(z_1) + \alpha + h)$ as the origin. We take $l_1 + \alpha + h$ as the horizontal axis, the line perpendicular to $l_1 + \alpha + h$ at $(z_1, g_1(z_1) + \alpha + h)$ as the vertical axis, and as the positive quadrant everything below the former and to the left of the latter. The appropriate equations for $\bar{f}_2$ are

$$\bar{f}_2(z_1 - z_2) = l_1(z_1) + \alpha - g_2(z_2), \quad \text{and}$$

$$\begin{aligned} \bar{f}_2'(z_1 - z_2) &= g_2'(z_2) - \text{ slope of } (l_1 + \alpha + h), \\ &= g_2'(z_2) - g_1'(z_1). \end{aligned}$$

If we let $\bar{f}_2(\bar{z}) = \bar{\gamma}_{21}\bar{z}^{\bar{\gamma}_{22}}$ and solve the above two equations for $\bar{\gamma}_{21}$, $\bar{\gamma}_{22}$, we get

$$\bar{\gamma}_{21} = [l_1(z_2) + \alpha - g_2(z_2)][z_1 - z_2]^{-\gamma_{22}}, \quad \text{and}$$

$$\bar{\gamma}_{22} = [g_2'(z_2) - g_1'(z_1)][l_1(z_2) + \alpha - g_2(z_2)]^{-1}[z_1 - z_2],$$

where $l_1(z_2) = g_1(z_1) + g_1'(z_1)[z_2 - z_1]$. The step-by-step transformations to yield f_2 from $\bar{f}_2$ (to translate the new coordinate space into traditional Euclidean coordinate space) are: 1. Reflect about the origin: $-\bar{f}_2(-\bar{z})$; 2. Shift z_1 units to the right and $g_1(z_1) + \alpha + h$ units up: $-\bar{f}_2(-(\bar{z} - z_1)) + g_1(z_1) + \alpha + h$; 3. Rotate counter-clockwise by $\sin^{-1} g_1'(z_1)$ degrees, pivoting about the point $(z_1, g_1(z_1) + \alpha + h)$:

$$-\bar{f}_2(-(\bar{z} - z_1)) + g_1(z_1) + \alpha + h + g_1'(z_1)[\bar{z} - z_1]. \tag{B.9}$$

Equation (B.9) is a function in the traditional Euclidean space; if we replace $\bar{z}$ by z and set the result equal to f_2 we have

$$f_2(z) = -\bar{\gamma}_{21}(-z + z_1)^{\bar{\gamma}_{22}} + g_1(z_1) + \alpha + h + g_1'(z_1)[z - z_1];$$

it is straightforward to check that the equations (B.7)-(B.8) with $i = 2$ are satisfied. Since $f_2'(z) = \bar{\gamma}_{22}\bar{\gamma}_{21}(-z + z_1)^{\gamma_{22}-1} + g_1'(z_1)$, $f_2''(z) = -\bar{\gamma}_{22}(\bar{\gamma}_{22} - 1)\bar{\gamma}_{21}(-z + z_1)^{\bar{\gamma}_{22}-2}$, for verifying that in $[z_2, z_1]$ f_2 is concave, increasing, it suffices to show that $\bar{\gamma}_{21} > 0$ and $\bar{\gamma}_{22} > 1$. The first inequality is true by (B.4) and our assumption that $z_2 < z_1$.

A series of equivalences prove that the second inequality holds:

$$\begin{aligned}
\bar{\gamma}_{22} = [g_2'(z_2) - g_1'(z_1)][l_1(z_2) + \alpha - g_2(z_2)]^{-1}[z_1 - z_2] &> 1, \\
\iff [g_2'(z_2) - g_1'(z_1)][z_1 - z_2] &> [l_1(z_2) + \alpha - g_2(z_2)] \\
&\quad \text{by (B.4)}, \\
\iff g_2'(z_2)[z_1 - z_2] - g_1'(z_1)[z_1 - z_2] &> g_1(z_1) + g_1'(z_1)[z_2 - z_1] \\
&\quad + \alpha - g_2(z_2), \\
\iff g_2(z_2) + g_2'(z_2)[z_1 - z_2] &> g_1(z_1) + \alpha,
\end{aligned}$$

which is equation (B.3).

If we set $f_3(z) = \gamma_{31} z^{\gamma_{32}}$ and solve (B.8), we get

$$\gamma_{31} = [g_1(z_1) + \alpha + h] z_1^{-\gamma_{32}}, \quad \text{and} \quad \gamma_{32} = \frac{g_1'(z_1) z_1}{g_1(z_1) + \alpha + h}.$$

As for f_1, $\gamma_{31} > 0$ implies that f_3 increases over $[z_2, \infty)$. Concavity and $f_3'(z) \to 0$ as $z \to \infty$ is ensured if $\gamma_{32} < 1$, or equivalently, if $h > g_1'(z_1) z_1 - g_1(z_1) - \alpha$.

To complete the overall construction of f we need only choose h such that

$$h > \max\{g_2'(z_2) z_2 - g_2(z_2), g_1'(z_1) z_1 - g_1(z_1) - \alpha\}.$$

Finally, note that the smoothness of the individual f_i's and the requirements (B.7)-(B.8) ensure that f is smooth.

Taking a step back, notice that in defining f we used only z_1, z_2, $g_1(z_1)$, $g_2(z_2)$, $g_1'(z_1)$, $g_2'(z_2)$, and h – i.e., f is a function of seven parameters. □

Claim 13. *Consider a two-supplier auction. Under assumptions (A1) and (A2), there exists a* $\tilde{\pi} > \pi > \epsilon$ *and a scoring rule* f *that enforces supplier 1 winning at* $(c_1(\vec{x}) + \tilde{\pi}, \vec{x})$.

Proof. Let $d_a = c_{a2}(x_a) - c_{a1}(x_a)$, $a = 1, \ldots, A$. We construct f one dimension at a time. Assume without loss of generality that the attributes are ordered such that for

$a = 1, \ldots, \hat{A}$, $c'_{a1}(x_a) \neq c'_{a2}(x_a)$. Notice that $\hat{A} \geq 1$; otherwise, $T_1 + \pi$ intersecting c_2 implies that

$$\begin{aligned}\pi &> \min_{\vec{v}}\{c_2(\vec{v}) - T_1(\vec{v})\}, \\ &= c_2(\vec{x}) - c_1(\vec{x}) \qquad \text{if } \hat{A} = 0,\end{aligned}$$

which contradicts assumption (A1).

For $a = 1, \ldots, \hat{A}$, for fixed a, consider the curve $\bar{c}_{a1} \triangleq c_{a1} + d_a - \delta_a$. Let $\bar{l}_{a1}$ be the line tangent to $\bar{c}_{a1}$ at x_a. For $\delta_a > 0$ small enough, $\bar{l}_{a1}$ intersects c_{a2}, and we can apply Claim 12 (with $g_1 = \bar{c}_{a1}$, $g_2 = c_{a2}$, and $\alpha = 0$) to find a feasible scoring rule f_a such that, for a $z_a \in \mathbb{R}^+$,

$$f_a(x_a) = \bar{c}_{a1}(x_a) + h_a = c_{a1}(x_a) + d_a - \delta_a + h_a, \qquad f_a(z_a) = c_{a2}(z_a) + h_a,$$

$$f'_a(x_a) = \bar{c}'_{a1}(x_a) = c'_{a1}(x_a), \qquad \text{and} \qquad f'_a(z_a) = c'_{a2}(z_a),$$

where h_a can be made arbitrarily large. Now, if we take the arbitrarily large h_a from Claim 12 to be at least greater than $\max\{-d_a + \delta_a + \epsilon, \epsilon\}$ (to ensure that $f_a(x_a) > c_{a1}(x_a) + \epsilon$, and $f_a(z_a) > c_{a2}(z_a) + \epsilon$), f_a yields a difference in dimension a maximum dropout scores of

$$f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) = d_a - \delta_a.$$

For $a = \hat{A} + 1, \ldots, A$, a fixed, if in auction a we announce a scoring rule $f_a(z) = \mu_{a1} z^{\mu_{a2}}$ such that

$$f_a(x_a) = \max\{c_{a1}(x_a), c_{a2}(x_a)\} + h_a, \qquad \text{and} \qquad f'_a(x_a) = c'_{a1}(x_a), \tag{B.10}$$

(h_a some positive real number greater than ϵ), then supplier 2's bid for dimension a

is $z_a = x_a$ and

$$f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) = c_{a2}(x_a) - c_{a1}(x_a) \equiv d_a.$$

Since f_a is defined by two parameters (has two degrees of freedom) and has two equations in (B.10) to satisfy, such an f_a can be found (i.e., appropriate values of μ_{a1} and μ_{a2} can be found).

After applying the above for $a = 1, \ldots, A$, let $\delta = \min\{\delta_1, \ldots, \delta_{\hat{A}}\}$. Since we can choose the δ_a's as small as we like, for easier bookkeeping set $\delta_a = \delta$ for $a = 1, \ldots, \hat{A}$. We then define

$$\begin{aligned} \tilde{\pi} \triangleq f(\vec{x}) - c_1(\vec{x}) - (f(\vec{z}) - c_2(\vec{z})) &= \sum_{a=1}^{A} [f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a))], \\ &= \sum_{a=1}^{A} d_a - \delta\hat{A}. \end{aligned} \tag{B.11}$$

Assumption (A1) implies that $c_2(\vec{x}) - c_1(\vec{x}) > \pi$, which re-written is $\sum_{a=1}^{A} d_a > \pi$. Hence, if δ is chosen sufficiently small (possible since the earlier δ arguments for Case 1 assumed only that δ was sufficiently small), equation (B.11) is strictly greater than π, meaning that f enforces $(c_1(\vec{x})+\tilde{\pi}, \vec{x})$ when announced to suppliers 1 and 2, where $\tilde{\pi} > \pi$.

Notice that the f we constructed is a feasible scoring rule: for $a = 1, \ldots, \hat{A}$ the feasibility of the scoring rule f_a follows directly from the assumptions on Claim 1's f; and for $a = \hat{A}+1, \ldots, A$, the feasibility of scoring rule f_a follows by simple analysis. For f enforcing $(c_1(\vec{x})+\tilde{\pi}, \vec{x})$, it remains only to check that $S_1, S_2 > \epsilon$, and $S_1 > S_2+\epsilon$. The first condition holds by our choices of h_a; the second condition follows from $\pi > \epsilon$. This concludes the proof of Claim 2. □

Claim 14. *Consider a two-supplier auction. Suppose f enforces $(c_1(\vec{x}) + \tilde{\pi}, \vec{x})$, $\tilde{\pi} >$*

$\pi > \epsilon$. *Let* $\hat{c}_{a1} = c_{a1} - w_a$, *where*

$$w_a = \sum_{\substack{i=1,\ldots,A \\ i \neq a}} \left[f_i(x_i) - c_{i1}(x_i) - (f_i(z_i) - c_{i2}(z_i)) \right], \tag{B.12}$$

and $\vec{z}$ *is supplier 2's bid induced by* f. *Let* $\hat{l}_{a1}$ *be the line tangent to* $\hat{c}_{a1}$ *at* x_a. *If we can pick an* $\alpha \geq 0$ *such that* $\hat{c}_{a1}(x_a) + \pi + \alpha < c_{a2}(x_a)$ *and* $\hat{l}_{a1} + \pi + \alpha$ *intersects* c_{a2}, *then we can find a feasible scoring rule for dimension* a, $\hat{f}_a$, *such that announcing* $f_1, \ldots, f_{a-1}, \hat{f}_a, f_{a+1}, \ldots, f_A$ *enforces* $(c_1(\vec{x}) + \pi + \alpha, \vec{x})$.

To better understand the idea behind Claim 14, notice that if we ran all dimensions except that for attribute a, then checked the difference between the bidding positions of supplier 1 and supplier 2, this difference would be w_a. (Equivalently, if we ran an $A-1$ dimension auction over attributes $1, \ldots, a-1, a+1, \ldots, A$, w_a would be the amount by which supplier 1 won ($w_a > 0$) or lost ($w_a \leq 0$) the auction.) In essence, the shift of c_{a1} to $\hat{c}_{a1}$ summarizes supplier 1's position just before dimension a is run. We now prove Claim 14.

Proof. Suppose that there exists an $\alpha \geq 0$ such that $\hat{c}_{a1}(x_a) + \pi + \alpha < c_{a2}(x_a)$ and $\hat{l}_{a1} + \pi + \alpha$ intersects c_{a2}. Then, using Claim 12 we can find a scoring rule $\hat{f}_a$ and a $\hat{z}_a \in \mathbb{R}^+$ such that, by choosing the arbitrarily large h_a at least greater than $\max\{w_a + \epsilon, \epsilon\}$,

$$\begin{aligned} \hat{f}_a(x_a) = \hat{c}_{a1}(x_a) + \pi + \alpha + h_a &= c_{a1}(x_a) - w_a + \pi + \alpha + h_a, \\ &> c_{a1}(x_a) + \epsilon \quad \text{by } \pi,\ \alpha \geq 0 \text{ and how we chose } h_a, \end{aligned}$$

$$\hat{f}_a(\hat{z}_a) = c_{a2}(\hat{z}_a) + h_a, \quad \hat{f}'_a(x_a) = \hat{c}'_{a1}(x_a) = c'_{a1}(x_a), \quad \text{and} \quad \hat{f}'_a(\hat{z}_a) = c'_{a2}(\hat{z}_a).$$

Announcing scoring rules $f_1, \ldots, f_{a-1}, \hat{f}_a, f_{a+1}, \ldots, f_A$ results in a post-auction profit for supplier 1 of

$$\begin{aligned} &w_a + \hat{f}_a(x_a) - c_{a1}(x_a) - (\hat{f}_a(\hat{z}_a) - c_{a2}(\hat{z}_a)) \\ &\quad = w_a + c_{a1}(x_a) - w_a + \pi + \alpha + h_a - c_{a1}(x_a) - (c_{a2}(\hat{z}_a) + h_a - c_{a2}(\hat{z}_a)) \\ &\quad = \pi + \alpha. \end{aligned} \tag{B.13}$$

To prove the claim, it remains only to check that $f_1, \ldots, f_{a-1}, \hat{f}_a, f_{a+1}, \ldots, f_A$ induces $S_1, S_2 > \epsilon$ and $S_1 > S_2 + \epsilon$. The former holds by how we chose h_a; the latter holds because $\tilde{\pi} > \pi > \epsilon$. □

Claim 15. *Consider a two-supplier auction. Suppose f enforces $(c_1(\vec{x}) + \tilde{\pi}, \vec{x})$, $\tilde{\pi} > \pi > \epsilon$. Let $\hat{c}_{a1} = c_{a1} - w_a$, where w_a is as given in equation* (B.12). *Then*

$$\hat{c}_{a1}(x_a) + \pi < c_{a2}(x_a).$$

Proof. First, note that since f enforces $(c_1(\vec{x}) + \tilde{\pi}, \vec{x})$, we know that after running all dimensions supplier 1 wins the auction by $\tilde{\pi}$; in equations,

$$w_a + f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) = \tilde{\pi}, \tag{B.14}$$

where $\vec{z}$ is the bid by supplier 2 induced by f. Furthermore, using the definition of $\vec{z}$,

$$\begin{aligned} &f_a(x_a) - c_{a2}(x_a) \le \max_y\{f_a(y) - c_{a2}(y)\} = f_a(z_a) - c_{a2}(z_a), \\ &\Rightarrow f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) \le c_{a2}(x_a) - c_{a1}(x_a). \end{aligned} \tag{B.15}$$

Then, combining equations (B.14) and (B.15), we have

$$\pi < \tilde{\pi} \le w_a + c_{a2}(x_a) - c_{a1}(x_a) = c_{a2}(x_a) - \hat{c}_{a1}(x_a),$$

which verifies Claim 15. □

Claim 16. *Consider a two-supplier auction. Suppose* $\pi > \epsilon$ *and assumptions (A1) and (A2) hold. Then there exists a scoring rule* $\hat{v}$ *that enforces* $(c_1(\vec{x}) + \pi, \vec{x})$.

Proof. First, use Claim 2 to construct a feasible scoring rule f that enforces $(c_1(\vec{x}) + \tilde{\pi}, \vec{x})$, $\tilde{\pi} > \pi$. We carry out iterations on f to construct a scoring rule $\hat{v}$ to enforce $(c_1(\vec{x}) + \pi, \vec{x})$. The general idea is to start with f, then dimension-by-dimension decrease the profit of supplier 1 (by revising f_a's) until supplier 1 wins the auction with profit exactly equal to π, at which point we set $\hat{v} \equiv f$.

Let $\vec{z}$ be supplier 2's bid induced by $f_1, \ldots, f_A$. Assume w.o.l.o.g. that the attributes are ordered such that $c'_{A1}(x_A) \neq c'_{A2}(x_A)$. Notice that we cannot have $c'_{a1}(x_a) = c'_{a2}(x_a)$ for all a, since if we did, then T_1 would be parallel to c_2's tangent hyperplane at $\vec{x}$, implying that by (A1) $T_1 + \tilde{\pi}$ cannot not intersect c_2, which contradicts (A2).

Set $a = 1$.

Procedure:

If $c'_{a1}(x_a) = c'_{a2}(x_a)$, set $a = a + 1$.

Otherwise, let $\hat{c}_{a1}$ and w_a be as defined in Claim 14, and let l_{a1} ($\hat{l}_{a1}$) be the line tangent to c_{a1} ($\hat{c}_{a1}$) at x_a. Also, let

$$D_a = \max_y \{l_{a1}(y) - c_{a2}(y)\},$$

and let

$$\hat{D}_a = D_a - w_a + \pi.$$

Now, if $\hat{D}_a > 0$, set $\alpha = 0$. The line $\hat{l}_{a1} + \pi + \alpha$ intersects c_{a2} (since $\hat{D}_a > 0$), and

$$\begin{aligned} \hat{c}_{a1}(x_a) + \pi + \alpha &= \hat{c}_{a1}(x_a) + \pi, \\ &< c_{a2}(x_a) \qquad \text{by Claim 15.} \end{aligned}$$

Hence, we can apply Claim 14, set $f_a = \hat{f}_a$, and by announcing f enforce $(c_1(\vec{x})+\pi, \vec{x})$, and we are done – terminate the procedure.

Otherwise, if $\hat{D}_a \leq 0$, we find $\eta_a > 0$ small enough that $\hat{c}_{a1}(x_a) + \pi - \hat{D}_a + \eta_a < c_{a2}(x_a)$. This is possible since

$$\begin{aligned} c_{a2}(x_a) - (\hat{c}_{a1}(x_a) + \pi) &= c_{a2}(x_a) - (\hat{l}_{a1}(x_a) + \pi), \\ &> \min_y \left\{ c_{a2}(y) - (\hat{l}_{a1}(y) + \pi) \right\}, \\ &= -\hat{D}_a \qquad \text{since } \hat{D}_a \leq 0. \end{aligned}$$

The minimum in the second line will occur either at zero or with first order conditions. The inequality follows since x_a will satisfy neither: $c'_{a1}(x_a) \neq c'_{a2}(x_a)$ by assumption, and since f enforces $\vec{x}$, $\vec{x}$ is an interior solution to (6.1), (6.2), and so $x_a > 0$. For producing a contradiction if the procedure runs through all the indices without terminating (explained below), we take

$$\delta_a < \min\left\{ \eta_a, \; A^{-1}\left(\sum_{i=1}^{A} D_i + \pi \right) \right\},$$

and set $\alpha = -\hat{D}_a + \delta_a$. To check that $\delta_a > 0$, note that assumption (A2) implies

$$\begin{aligned} 0 &< \max_{\vec{y}} \{ T_1(\vec{y}) + \pi - c_2(\vec{y}) \}, \\ &= \sum_{i=1}^{A} D_i + \pi. \end{aligned}$$

Since $\delta_a > 0$, the definitions of $\hat{D}_a$ and α imply that $\hat{l}_{a1} + \pi + \alpha$ intersects c_{a2}. Thus we can apply Claim 14 to construct an $\hat{f}_a$, then set $f_a = \hat{f}_a$, such that announcing f enforces $(c_1(\vec{x}) + \pi - \hat{D}_a + \delta_a, \vec{x})$, which, re-written, is

$$(c_1(\vec{x}) + w_a - D_a + \delta_a, \vec{x}). \tag{B.16}$$

To prepare for the next step in the iterative procedure, let $\hat{z}_a$ be supplier 2's bid in dimension a induced by $\hat{f}_a$, set $z_a = \hat{z}_a$, and set $\tilde{\pi} = w_a - D_a + \delta_a$. Since

$$0 \geq \hat{D}_a = D_a - w_a + \pi \qquad \text{implies that} \qquad w_a - D_a + \delta_a > \pi,$$

we get that $\tilde{\pi} > \pi$.

Finally, to complete the groundwork for the proof of termination, by the definition of w_a, the fact that we have not changed $f_1, \ldots, f_{a-1}, f_{a+1}, \ldots, f_A$, and that f now enforces (B.16), we have that

$$f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) = -D_a + \delta_a. \tag{B.17}$$

Setting $a = a + 1$, we can now repeat the same procedure with our updated f, $\vec{z}$, and $\tilde{\pi}$.

Proof of termination with the desired f: Suppose that the procedure iterates through indices $1, \ldots, A$ without terminating with an $\alpha = 0$. We show that this would produce a contradiction.

After the procedure iterates through all the indices, f enforces $(c_{a1}(\vec{x}) + w_A - D_A + \delta_A, \vec{x})$, with $w_A - D_A + \delta_A > \pi > 0$. Since no index is visited twice, each f_a is updated exactly once, and: If $c'_{a1}(x_a) = c'_{a2}(x_a)$, then $z_a = x_a$ implies

$$f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a)) = c_{a2}(x_a) - c_{a1}(x_a) = -D_a, \tag{B.18}$$

since in this case the maximization that defines D_a will occur with first order conditions; and, if $c'_{a1}(x_a) \neq c'_{a2}(x_a)$, then equation (B.17) holds. Hence,

$$\begin{aligned} w_A - D_A + \delta_A &= \sum_{a=1}^{A-1} [f_a(x_a) - c_{a1}(x_a) - (f_a(z_a) - c_{a2}(z_a))] - D_A + \delta_A \\ & \quad \text{by (B.12)}, \\ &\leq \sum_{a=1}^{A} -D_a + \frac{A}{A}\left(-\sum_{a=1}^{A} D_a + \pi\right) \\ & \quad \text{by (B.17)-(B.18) and our choice of } \delta_a, \\ &= \pi, \end{aligned}$$

which contradicts $w_A - D_A + \delta_A > \pi$. We conclude that the procedure must terminate with $\alpha = 0$ for some index, and an f that enforces $(c_1(\vec{x}) + \pi, \vec{x})$. This proves Claim 16. □

Stage 2: Extension to S-supplier auction. To eliminate a variable, we assume without loss of generality that supplier $i = 1$ in the statement of Proposition 1.

Claim 17. *Let $c_1(\vec{x}) + \pi < c_s(\vec{x})$ $\forall s = 2, \ldots, S$, where $\pi > \epsilon$. There exists a feasible scoring rule f that satisfies (7.2)-(7.3) and causes supplier 1 to win the S-supplier auction with attribute bid $\vec{x}$ and profit at least π.*

Proof. Set $d_{as} = c_{as}(x_a) - c_{a1}(x_a)$, $D_s = \sum_{a=1}^{A} d_{as}$, and $\hat{c}_{as} = c_{as} - d_{as} + D_s/A$. Suppose f is some feasible scoring rule with respect to the cost surface $\hat{c}_s$ (and hence c_s). Then f induces interior attribute bids from the cost functions $\hat{c}_{as}$ and c_{as}; furthermore, since these cost functions differ from each other by a constant, they both must yield the same attribute bid z_{as}. Defining

$$\begin{aligned} \hat{S}_{as} &\triangleq \max_z \{f_a(z) - \hat{c}_{as}(z)\}, \\ &= \max_z \{f_a(z) - c_{as}(z)\} + d_{as} - \frac{D_s}{A}, \\ &= f_a(z_{as}) - c_{as}(z_{as}) + d_{as} - \frac{D_s}{A} \end{aligned}$$

implies that $\sum_{a=1}^{A} \hat{S}_{as} = S_s$. We construct a scoring rule f that is feasible for the $\hat{c}_s$ cost surfaces, with $\hat{S}_{as} < \hat{S}_{a1} - \pi/A \; \forall \; a, s$, and $\vec{z}_1 = \vec{x}$. These conditions imply that when f is announced to all S suppliers (with actual cost surfaces c_s), supplier 1 wins at $\vec{x}$ with profit at least π and (7.3) is satisfied; by also ensuring that $\hat{S}_{as} > \epsilon/A \; \forall \; s$, we guarantee that f satisfies (7.2). Once again, we construct f dimension-by-dimension. We present the construction for fixed a.

Notice that $\hat{c}_{a1} = c_{a1}$, and

$$\begin{aligned}
\hat{c}_{as}(x_a) - \hat{c}_{a1}(x_a) &= c_{as}(x_a) - d_{as} + \frac{D_s}{A} - c_{a1}(x_a), \\
&= c_{as}(x_a) - (c_{as}(x_a) - c_{a1}(x_a)) + \frac{D_s}{A} - c_{a1}(x_a), \\
&= \frac{D_s}{A} > \frac{\pi}{A},
\end{aligned}$$

where the inequality follows because $D_s/A = (c_s(\vec{x}) - c_1(\vec{x}))/A$. Let $\hat{l}_{a1}$ denote the line tangent to $\hat{c}_{a1}$ at x_a. We consider two cases, (c1) and (c2).

(c1): $\hat{l}_{a1} + \pi/A$ does not intersect any curve $\hat{c}_{as}$, $s \neq 1$. In this case, $\hat{l}_{a1} + \pi/A$ lies below $\hat{c}_{as}$, $s \neq 1$. Choose $f_a(z) = \gamma_{a1} z^{\gamma_{a2}}$ such that $f_a(x_a) = \hat{c}_{a1}(x_a) + \pi/A + h_a$, $f_a'(x_a) = \hat{c}_{a1}'(x_a)$, $h_a \in \mathbb{R}^+$. Solving for γ_{a1}, γ_{a2} yields

$$\gamma_{a1} = [\hat{c}_{a1}(x_a) + \frac{\pi}{A} + h_a] x_a^{-\gamma_{a2}}, \qquad \gamma_{a2} = \frac{\hat{c}_{a1}'(x_a) x_a}{\hat{c}_{a1}(x_a) + \frac{\pi}{A} + h_a}.$$

The function f_a is increasing, since $\gamma_{a1} > 0$. Choosing

$$h_a > \max\left\{\hat{c}_{a1}'(x_a)x_a - \hat{c}_{a1}(x_a) - \frac{\pi}{A}, \; \max_s \{\hat{c}_{as}(x_a) - \hat{c}_{a1}(x_a)\}\right\}$$

ensures that $\gamma_{a1} < 1$ (and hence f_a is concave, $f_a' \to \infty$ as $z \to 0^+$, and $f_a' \to 0$ as

$z \to \infty$), as well as ensuring that $\hat{S}_{as} > \pi/A > \epsilon/A$. Furthermore, for $s \neq 1$,

$$\begin{aligned} f_a(z) - \hat{c}_{as}(z) &< f_a(z) - \hat{l}_{a1}(z) - \frac{\pi}{A} \qquad \text{because } \hat{l}_{a1}(z) + \frac{\pi}{A} \text{ lies below } \hat{c}_{as}, \\ &< \hat{l}_{a1}(z) + \hat{S}_{a1} - \hat{l}_{a1}(z) - \frac{\pi}{A}, \\ &= \hat{S}_{a1} - \frac{\pi}{A}, \end{aligned}$$

where the second inequality follows because $\hat{l}_{a1}(z) + \hat{S}_{a1}$ is tangent to f_a at x_a, and f_a is concave. Hence, $\hat{S}_{as} = \max_z\{f_a(z) - \hat{c}_{as}(z)\} < \hat{S}_{a1} - \pi/A$ for $s \neq 1$.

(c2): $\hat{l}_{a1} + \pi/A$ intersects some curve $\hat{c}_{as}$, $s \neq 1$. Let $\Delta_a = \max_{z,s\neq 1}\{\hat{l}_{a1}(z) + \pi/A - \hat{c}_{as}(z)\}$. Set $\underline{z}_a < x_a$ such that $\hat{l}_{a1} + \pi/A$ intersects no curves $\hat{c}_{as}$, $s \neq 1$, in $[\underline{z}_a, x_a]$, and set $\bar{z}_a > x_a$ such that $\hat{l}_{a1} + \pi/A$ intersects no curves $\hat{c}_{as}$, $s \neq 1$, in $[x_a, \bar{z}_a]$.

In Claim 1 we showed that if $(z_1, g_1(z_1))$, $g_1'(z_1)$, and $(z_2, g_2(z_2))$, $g_2'(z_2)$ are a point-slope pair such that (B.3)-(B.4) hold, then we can find a function f such that (B.1)-(B.2) hold for $h > H$, $H \in \mathbb{R}^+$. Notice that if we set

$$z_1 = x_a, \qquad g_1(z_1) = \hat{c}_{a1}(x_a), \qquad g_1'(z_1) = \hat{c}_{a1}'(x_a),$$

$$z_2 = \underline{z}_a, \qquad g_2(z_2) = \hat{c}_{a1}(x_a) + \frac{\pi}{A} - \left(\Delta_a + \hat{c}_{a1}'(x_a)[x_a - \underline{z}_a]\right),$$

$$\text{and} \qquad g_2'(z_2) > \frac{\Delta_a}{x_a - \underline{z}_a} + \hat{c}_{a1}'(x_a), \qquad \text{then}$$

$$\begin{aligned} g_2(z_2) + g_2'(z_2)[z_1 - z_2] &= g_2(z_2) + g_2'(z_2)[x_a - \underline{z}_a], \\ &> g_2(z_2) + \Delta_a + \hat{c}_{a1}'(x_a)[x_a - \underline{z}_a], \\ &= \hat{c}_{a1}(x_a) + \frac{\pi}{A} - \left(\Delta_a + \hat{c}_{a1}'(x_a)[x_a - \underline{z}_a]\right) \\ &\quad + \Delta_a + \hat{c}_{a1}'(x_a)[x_a - \underline{z}_a], \\ &= \hat{c}_{a1}(x_a) + \frac{\pi}{A} = g_1(z_1) + \frac{\pi}{A}, \end{aligned}$$

$$
\begin{aligned}
\text{and} \qquad g_2(z_2) &= \hat{c}_{a1}(x_a) + \frac{\pi}{A} - (\Delta_a + \hat{c}'_{a1}(x_a)[x_a - \underline{z}_a]), \\
&= \hat{c}_{a1}(x_a) + \hat{c}'_{a1}(x_a)[\underline{z}_a - x_a] + \frac{\pi}{A} - \Delta_a, \\
&< \hat{l}_{a1}(\underline{z}_a) + \frac{\pi}{A} \qquad \text{since } \Delta_a > 0.
\end{aligned}
$$

That is, (B.3)-(B.4) hold (with π/A in the role of α and $\hat{l}_{a1}$ in the role of l_1), and we can find an increasing, concave, smooth function $\underline{f}_a$ such that $\underline{f}'_a(z_a) \to \infty$ as $z_a \to 0^+$,

$$\underline{f}_a(\underline{z}_a) = \hat{c}_{a1}(x_a) + \frac{\pi}{A} - (\Delta_a + \hat{c}'_{a1}(x_a)[x_a - \underline{z}_a]) + \underline{h}_a,$$

$$\underline{f}_a(x_a) = \hat{c}_{a1}(x_a) + \frac{\pi}{A} + \underline{h}_a, \qquad \text{and} \qquad \underline{f}'_a(x_a) = \hat{c}'_{a1}(x_a),$$

for $\underline{h}_a > \underline{H}_a$, $\underline{H}_a \in \mathbb{R}^+$. If we leave z_1, $g_1(z_1)$, and $g'_1(z_1)$ the same but take $z_2 = \bar{z}_a$, $g_2(z_2) = \hat{c}_{a1}(x_a) + \pi/A - (\Delta_a + \hat{c}'_{a1}(x_a)[x_a - \bar{z}_a])$, and $g'_2(z_2) < \frac{\Delta_a}{x_a - \bar{z}_a} + \hat{c}'_{a1}(x_a)$, then arguments similar to those above show that (B.3)-(B.4) hold (again with π/A in the role of α and $\hat{l}_{a1}$ in the role of l_1), and hence we can find an increasing, concave, smooth function $\bar{f}_a$ such that $\bar{f}'_a(z_a) \to 0$ as $z_a \to \infty$,

$$\bar{f}_a(x_a) = \hat{c}_{a1}(x_a) + \frac{\pi}{A} + \bar{h}_a,$$

$$\bar{f}_a(\bar{z}_a) = \hat{c}_{a1}(x_a) + \frac{\pi}{A} - (\Delta_a + \hat{c}'_{a1}(x_a)[x_a - \bar{z}_a]) + \bar{h}_a, \qquad \text{and} \qquad \bar{f}'_a(x_a) = \hat{c}'_{a1}(x_a),$$

for $\bar{h}_a > \bar{H}_a$, $\bar{H}_a$ some positive real number. Setting

$$\underline{h}_a = \bar{h}_a = h_a > \max\left\{\underline{H}_a,\ \bar{H}_a,\ \max_s \{\hat{c}_{as}(x_a) - \hat{c}_{a1}(x_a)\}\right\}$$

and

$$f_a(z) = \begin{cases} \underline{f}_a(z) & \text{if } z \le x_a, \\ \bar{f}_a(z) & \text{if } z > x_a \end{cases}$$

ensures that f_a is smooth, concave, increasing, $f'_a \to \infty$ as $z \to 0^+$, $f'_a \to 0$ as $z \to \infty$, and $\hat{S}_{as} > \epsilon/A$.

It remains to show that $\hat{S}_{a1} - \pi/A > \hat{S}_{as}\ \forall\ s \neq 1$. Suppose not, that is, suppose $\hat{S}_{a1} - \pi/A \leq \hat{S}_{as}$ for some $s \neq 1$. Let z_{as} denote supplier s's attribute bid in dimension a induced by f. We first show that

$$\hat{c}_{as}(z_{as}) < \hat{l}_{a1}(z_{as}) + \frac{\pi}{A}. \tag{B.19}$$

Suppose $\hat{c}_{as}(z_{as}) \geq \hat{l}_{a1}(z_{as}) + \pi/A$. Then

$$\begin{aligned} \hat{S}_{as} = f_a(z_{as}) - \hat{c}_{as}(z_{as}) &\leq f_a(z_{as}) - \hat{l}_{a1}(z_{as}) - \frac{\pi}{A}, \\ &< \hat{l}_{a1}(z_{as}) + \hat{S}_{a1} - \hat{l}_{a1}(z_{as}) - \frac{\pi}{A}, \\ &= \hat{S}_{a1} - \frac{\pi}{A}, \end{aligned}$$

which is a contradiction. For the second inequality, we have made use of the fact that $\hat{l}_{a1} + \hat{S}_{a1}$ is tangent to f_a at x_a, and f_a is concave. We now use (B.19) along with the definition of f_a to derive a contradiction if $\hat{S}_{a1} - \pi/A \leq \hat{S}_{as}$. We treat the case $z_{as} \leq x_a$; the complement case is treated similarly, and is omitted for brevity. Let l_{af} denote the line tangent to f_a at $\underline{z}_a$.

$$\begin{aligned} f_a(z_{as}) - \hat{c}_{as}(z_{as}) &< f_a(z_{as}) - (\hat{l}_{a1}(z_{as}) + \frac{\pi}{A} - \Delta_a) \qquad \text{by the definition of } \Delta_a, \\ &< l_{af}(z_{as}) - \hat{l}_{a1}(z_{as}) - \frac{\pi}{A} + \Delta_a, \\ &< l_{af}(\underline{z}_a) - \hat{l}_{a1}(\underline{z}_a) - \frac{\pi}{A} + \Delta_a, \\ &= \hat{c}_{a1}(x_a) + \frac{\pi}{A} - (\Delta_a + \hat{c}'_{a1}(x_a)[x_a - \underline{z}_a]) + h_a \\ &\quad - (\hat{c}_{a1}(x_a) + \hat{c}'_{a1}(x_a)[\underline{z}_a - x_a]) - \frac{\pi}{A} + \Delta_a, \\ &= h_a, \\ &= f_a(x_a) - (\hat{c}_{a1}(x_a) + \frac{\pi}{A}) = \hat{S}_{a1} - \frac{\pi}{A}, \end{aligned}$$

which contradicts $\hat{S}_{a1} - \pi/A \leq \hat{S}_{as}$. The third inequality follows because $l_{af} - \hat{l}_{a1}$ is an increasing function (since l_{af} is steeper than $\hat{l}_{a1}$), and $z_{as} < \underline{z}_a$ by (B.19) together with the assumption that $z_{as} \leq x_a$. Taking a step back, notice that in defining f_a

in case 1 we used only x_a, $c_{a1}(x_a)$ and $c'_{a1}(x_a)$ (three parameters), and in case 2 we used these three in addition to $\underline{z}$, $g_2(\underline{z})$, $g'_2(\underline{z})$ (when defining $\underline{f}$), $\bar{z}$, $g_2(\bar{z})$, $g'_2(\bar{z})$ (when defining $\bar{f}$), and h_a – i.e., f_a is a function of ten non-redundant parameters. □

Claim 18. *Let $c_1(\vec{x}) + \pi < c_s(\vec{x})$ $\forall s = 2, \ldots, S$, where $\pi > \epsilon$. If there exists a $j \neq 1$ such that $T_1 + \pi$ intersects c_j, then $(c_1(\vec{x}) + \pi, \vec{x})$ is enforceable.*

Proof. Let $\hat{v}$ denote the scoring rule for the two-supplier auction constructed in Claim 5 such that $S_1(\hat{v}) - \pi = S_j(\hat{v})$, where $S_s(\hat{v})$ denotes supplier s's maximum dropout score if $\hat{v}$ is announced. Since for all a $\hat{v}'_a(z) \to \infty$ as $z \to 0^+$ and $\hat{v}'_a(z) \to 0$ as $z \to \infty$, $\hat{v}$ is feasible (induces positive, finite attribute bids) in the S-supplier auction.

(c1): $S_1(\hat{v}) - \pi \geq S_s(\hat{v})$ $\forall$ $s \neq 1$. In this case, announcing $\hat{v}$ enforces $(c_1(\vec{x}) + \pi, \vec{x})$ and we are done.

(c2): $\exists$ $k \neq 1$ such that $S_1(\hat{v}) - \pi < S_k(\hat{v})$. Let g be the feasible scoring rule satisfying (7.2)-(7.3) constructed in Claim 6 such that $S_1(g) - \pi > S_s(g)$ for all $s \neq 1$. In the remainder of our proof, we will show that there exists a convex combination of $\hat{v}$ and g that enforces $(c_1(\vec{x}) + \pi, \vec{x})$. Notice that since $\hat{v}$ and g are feasible, the same is true for convex combinations of $\hat{v}$ and g. Define

$$S_s(\lambda) \triangleq \sum_{a=1}^{A} (\lambda \hat{v}_a + (1 - \lambda) g_a)(x_{as}(\lambda)) - \sum_{a=1}^{A} c_{as}(x_{as}(\lambda)), \tag{B.20}$$

where $\vec{x}_s(\lambda)$ is the attribute bid of supplier s when $\lambda\hat{v} + (1 - \lambda)g$ is announced.

Notice that since $\nabla\hat{v}(\vec{x}) = \nabla g(\vec{x}) = \nabla c_1(\vec{x})$, supplier 1's attribute bid $\vec{x}_1(\lambda)$ will equal $\vec{x}$ for all $\lambda \in [0, 1]$. This implies that $S_1(\lambda) = \lambda S_1(\hat{v}) + (1 - \lambda)S_1(g) > \pi + \epsilon$ for all $\lambda \in [0, 1]$, because $S_1(\hat{v}), S_1(g) > \pi + \epsilon$ since $\hat{v}$ and g satisfy (7.2). By these observations, if there exists λ^* for which $S_1(\lambda^*) - \pi \geq S_s(\lambda^*)$ holds for all $s \neq 1$, and holds for some supplier $s \neq 1$ with equality, then $\lambda^*\hat{v} + (1 - \lambda^*)g$ is a feasible scoring rule that satisfies (7.2)-(7.3) and enforces $(c_1(\vec{x}) + \pi, \vec{x})$ in the S-supplier auction. To find such a λ^*, we first show that $S_s(\lambda)$ is a continuous function of λ.

We begin by showing that $x_{as}(\lambda)$ is continuous in λ, where for convenience we drop the subscript s in the general proof that follows. We first suppose that $x_a(0) \leq x_a(1)$, and show that $x_a(0) \leq x_a(\lambda) \leq x_a(1)$ for all $\lambda \in [0, 1]$ (the proof in the case $x_a(1) \leq x_a(0)$ is essentially the same, with the roles of $x_a(0)$ and $x_a(1)$ reversed). We prove $x_a(0) \leq x_a(\lambda)$ for all $\lambda \in [0, 1]$; the other inequality follows similarly.

By the definition of $x_a(0)$, $z_a < x_a(0)$ implies $g_a'(z_a) > c_a'(z_a)$, and similarly the definition of $x_a(1)$ implies that for $z_a < x_a(1)$, $\hat{v}_a'(z_a) > c_a'(z_a)$. Since $x_a(0) \leq x_a(1)$, we have that for $z_a < x_a(0)$, $g_a'(z_a) > c_a'(z_a)$ and $\hat{v}_a'(z_a) > c_a'(z_a)$, and hence $\lambda\hat{v}_a'(z_a) + (1-\lambda)g_a'(z_a) > c_a'(z_a)$ for all $\lambda \in [0, 1]$. By the above analysis and $\hat{v}_a'$, g_a', and $-c_a'$ strictly decreasing, for fixed λ the root $x_a(\lambda)$ of $\lambda\hat{v}_a'(z_a)+(1-\lambda)g_a'(z_a)-c_a'(z_a)$ must lie to the right of $x_a(0)$, which is what we wanted to show.

Next, we consider perturbing λ to $\lambda + \delta$, for $|\delta|$ small (where δ must be positive if $\lambda = 0$, negative if $\lambda = 1$), and show that for any fixed $\eta > 0$, $\exists\ \hat{\delta} > 0$ such that $|x_a(\lambda) - x_a(\lambda + \delta)| < \eta$ for all $|\delta| < \hat{\delta}$.

First, note that for $z_a \in [x_a(0), x_a(1)]$ and δ fixed,

$$\begin{aligned}
& \left|\left((\lambda+\delta)\hat{v}_a' + (1-\lambda-\delta)g_a' - c_a'\right)(z_a) - \left(\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a'\right)(z_a)\right| \\
= \ & \left|\delta(\hat{v}_a' - g_a')(z_a)\right|, \\
< \ & |\delta| \max_{z_a \in [x_a(0), x_a(1)]} \left|(\hat{v}_a' - g_a')(z_a)\right|, \\
\leq \ & |\delta| B,
\end{aligned}$$

where some $B < \infty$ exists since $[x_a(0), x_a(1)]$ is compact and $\hat{v}_a' - g_a'$ is continuous. In other words, over $[x_a(0), x_a(1)]$, the curve $(\lambda+\delta)\hat{v}_a' + (1-\lambda-\delta)g_a' - c_a'$ lies between the two curves $\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a' + \delta B$ and $\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a' - \delta B$. Let $w_a(\delta)$ be the root of the strictly decreasing function $\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a' + \delta B$. By the bound given above, and the fact that $x_a(\lambda+\delta) \in [x_a(0), x_a(1)]$, we have that $w_a(-\delta)$ and $w_a(\delta)$ sandwich $x_a(\lambda+\delta)$. Since the same is true for $x_a(\lambda)$, we have our desired result if we can find $\hat{\delta}$ such that $|\delta| < \hat{\delta}$ implies $|w_a(\delta) - w_a(-\delta)| < \eta$.

Pick $\hat{\delta}$ such that $\hat{\delta}B < -(\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a')(x_a(\lambda) + \eta/2)$. The right side is positive by the fact that $x_a(\lambda)$ is a root of the decreasing function $\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a'$, and $\eta > 0$. Furthermore, choosing $\hat{\delta}$ in this way implies that $(\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a' + \hat{\delta}B)(x_a(\lambda) + \eta/2) < 0$. Since $(\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a' + \hat{\delta}B)(x_a(\lambda)) = \hat{\delta}B > 0$, we have that $w_a(|\delta|) \in (x_a(\lambda), x_a(\lambda) + \eta/2)$ for $|\delta| < \hat{\delta}$. By also ensuring that $\hat{\delta}B < (\lambda\hat{v}_a' + (1-\lambda)g_a' - c_a')(x_a(\lambda) - \eta/2)$, we have $w_a(-|\delta|) \in (x_a(\lambda) - \eta/2, x_a(\lambda))$ for $|\delta| < \hat{\delta}$, and the continuity of $x_a(\lambda)$ follows.

Since $(\lambda\hat{v}_a + (1-\lambda)g_a - c_a)(x_a(\lambda))$ is a composition of continuous functions, and the sum of continuous functions is continuous, the righthand side of (B.20) is continuous in λ – i.e., $S_s(\lambda)$ is continuous in λ for all s.

By our choice of g we have that $S_1(0) - \pi > S_s(0)$ for all $s \neq 1$. Set

$$\lambda^* = \min\{\lambda | S_s(\lambda) = S_1(\lambda) - \pi \quad \text{for some } s \neq 1,\ 0 < \lambda < 1\}. \tag{B.21}$$

By the continuity of the $S_s(\lambda)$'s and the fact that $S_k(1) > S_1(1) - \pi$, such a λ^* exists. As noted below equation (B.20), for such a λ^*, $\lambda^*\hat{v} + (1-\lambda^*)g$ enforces $(c_1(\vec{x}) + \pi, \vec{x})$ in the S-supplier auction. Notice that the scoring rule in this case is a function of 18 parameters: seven come from the function $\hat{v}$, ten come from the function g, and the last is the parameter λ^*.

□

B.3 Proof of Proposition 6

In the statement of the Proposition, we tacitly assume that $M_i > \epsilon$; otherwise, the auctioneer's valuation is everywhere below the suppliers' costs plus the minimum bid increment ϵ, meaning that no auction would be pursued (which is not interesting). We break the proof of Proposition 2 into three cases, and begin with an observation that will be useful in all three.

Recall that the definition of optimality was made in relation to the optimum of (7.1)-(7.3), (7.7). Let u^* be the optimum of the general version of (7.1)-(7.3), (7.7), (i.e., (7.1)-(7.3), (7.7), with c_1 and c_2 instead of their parameterized counterparts).

Towards proving the optimality of supplier i, we first show that $u^* \le M_i - \epsilon$. In contrast to the formalization of (7.1)-(7.3), (7.7), here we make no assumption on how the suppliers' subscripts are ordered for a given scoring rule. For scoring rules $\hat{v}$ such that k and l are the top bidders and $S_k > S_l + \epsilon$, the appropriate mathematical program is

$$\begin{aligned} \max_{\hat{v}} \quad & v(\vec{x}_k^{**}) - \hat{v}(\vec{x}_k^*) + S_l \\ \text{subject to} \quad & S_s = \hat{v}(\vec{x}_s^*) - c_s(\vec{x}_s^*), \quad s = k, l \\ & S_l > \epsilon. \end{aligned} \tag{B.22}$$

Examining the objective function (B.22), we see that

$$\begin{aligned} v(\vec{x}_k^{**}) - \hat{v}(\vec{x}_k^*) + S_l &= v(\vec{x}_k^{**}) - c_k(\vec{x}_k^{**}) - (\hat{v}(\vec{x}_k^*) - c_k(\vec{x}_k^{**})) + S_l, \\ &= v(\vec{x}_k^{**}) - c_k(\vec{x}_k^{**}) - S_k + S_l, \\ &\le v(\vec{x}_k^{**}) - c_k(\vec{x}_k^{**}) - \epsilon \qquad \text{since } S_k > S_l + \epsilon, \\ &\le \max_{\vec{x}}\{v(\vec{x}) - c_k(\vec{x})\} - \epsilon, \\ &= M_k - \epsilon. \end{aligned}$$

Hence, we've shown that $u^* \le M_i - \epsilon$. This observation motivates the first case, which provides conditions in which the auctioneer is guaranteed to do no worse than u^* by simply announcing the true valuation function v as the scoring rule; in this case, we consider v to be optimal (as discussed below equation (7.7)).

(c1): $\exists\, k \ne i$ such that $M_k \ge M_i - \epsilon$. Suppose we announce the true valuation function v as the scoring rule. We need to show that, no matter who (i or k) wins, the post-auction utility to the auctioneer is at least $M_i - \epsilon$ (and hence both suppliers are optimal). Suppose supplier i wins the auction. Since we assume the winning supplier wins at the losing supplier's highest possible score, the profit at which supplier i wins the auction is at most $S_i - S_k < \epsilon$. Because the auctioneer announced his true valuation function as the scoring rule, the utility to the auctioneer equals the winner's maximum dropout score minus the winner's profit, i.e., is at least $S_i - \epsilon = M_i - \epsilon$. Next, suppose that supplier k wins the auction. Supplier i drops out at the highest possible losing score; since $S_i < S_k$, supplier k wins the auction with zero profit, and the post-auction utility of the auctioneer is $S_k \geq M_i - \epsilon$. Here we have assumed that supplier k actually bids, i.e., $S_k \geq \epsilon$. If instead $S_k = M_k < \epsilon$, then the utility for the auctioneer is still at least $M_i - \epsilon$: supplier i's winning score must have magnitude at least ϵ (ϵ is the minimum bid increment), and $M_k < \epsilon$ implies that $M_i \leq 2\epsilon$.

In the remaining two cases of the proof, announcing the true valuation function does not generate at least $M_i - \epsilon$ in utility for the auctioneer, and we show that supplier i wins under an optimal solution to (7.1)-(7.3), (7.7). Let $M_i = v(\vec{x}^*) - c_i(\vec{x}^*)$, and let T_i be the hyperplane tangent to c_i at $\vec{x}^*$. Because v is concave and c_i is convex, notice that we have $T_i + M_i$ tangent to v at $\vec{x}^*$.

(c2): $M_i - \epsilon > M_s$ for all $s \neq i$, and $\exists\, j \neq i$ such that $T_i + \epsilon$ intersects c_j. Corollary 1 implies that $(c_i(\vec{x}^*) + \epsilon, \vec{x}^*)$ can be enforced, yielding a payoff of $M_i - \epsilon$ for the auctioneer. Hence, i is optimal.

(c3): $M_i - \epsilon > M_s$ for all $s \neq i$, and $\nexists\, j \neq i$ such that $T_i + \epsilon$ intersects c_j. Our argument relies on hyperplanes. Let $\pi = \min_{\vec{z}, s \neq i}\{c_s(\vec{z}) - T_i(\vec{z})\}$. The hyperplane $T_i + \pi$ must lie below or tangent to c_s for all $s \neq i$. We use $T_i + M_i$ and $T_i + \pi$ to bound v from above, and c_s from below, respectively. Consider a scoring rule for which supplier $s \neq i$ wins. Let $\vec{w}$ be supplier s's winning attribute bid. The auctioneer's utility is at most $v(\vec{w}) - c_s(\vec{w})$, where

$$\begin{aligned} v(\vec{w}) - c_s(\vec{w}) &< (T_i(\vec{w}) + M_1) - (T_i(\vec{w}) + \pi), \\ &= M_i - \pi, \end{aligned}$$

which implies that

$$\max_{\vec{w}}\{v(\vec{w}) - c_s(\vec{w})\} = M_s < M_i - \pi.$$

Applying Corollary 1, $(c_i(\vec{x}^*) + \pi + \delta, \vec{x}^*)$ is enforceable as $\delta \to 0^+$, yielding for fixed δ a payoff of $M_i - \pi - \delta$ for the auctioneer. Choosing $\delta < M_i - \pi - \max_{s \neq i} M_s$ shows that with supplier i winning we can improve on the best case for which supplier $s \neq i$ wins. Hence, the optimum u^* is achieved for some scoring rule $\hat{v}$ for which supplier i wins (supplier i is optimal). □

B.4 Simplification of (7.9)-(7.12)

We simplify the mathematical program (7.9)-(7.12) by finding a closed-form solution to the innermost minimization in (7.12). Since the c_{as} are convex and increasing in x_a, for fixed s the innermost minimization occurs at z_a^* such that, for $a = 1, \ldots, A$, $\left.\frac{\partial c_{as}(z_a)}{\partial z_a}\right|_{z_a = z_a^*} = \left.\frac{\partial c_{ai}(z_a)}{\partial z_a}\right|_{z_a = x_a}$ if $\left.\frac{\partial c_{as}(z_a)}{\partial z_a}\right|_{z_a = 0} < \left.\frac{\partial c_{ai}(z_a)}{\partial z_a}\right|_{z_a = x_a}$, or at $z_a^* = 0$, otherwise. In words, if a point exists at which the tangent to c_{as} is parallel to c_{ai}'s tangent at x_a, we take this point as z_a^*; if such a point does not exist, then the minimization occurs at the left endpoint of the feasible interval in dimension a, and thus we take $z_a^* = 0$.

By incorporating z_a^*, we get

$$\pi \;\geq\; \min_{s \neq i} \left\{ \sum_{a=1}^{A} c_{as}(z_a^*) - \sum_{a=1}^{A} c_{ai}(x_a) - \sum_{a=1}^{A} (z_a^* - x_a) \left.\frac{\partial c_{ai}(z_a)}{\partial z_a}\right|_{z_a = x_a} \right\} \qquad \text{(B.23)}$$

in place of (7.12). The last two factors within the right side of (B.23) are the expanded version of $-T_i(\vec{z}^*)$. Notice that in solving (7.9)-(7.11), (B.23) we need only search over $\vec{x}$'s for which supplier i's cost is below the auctioneer's true valuation, i.e., $\sum_{a=1}^{A} c_{ai}(x_a) < \sum_{a=1}^{A} v_a(x_a)$. Also, the search can be terminated if and when the objective value reaches its upper bound of $M_i - \epsilon$.

B.5 Optimal Scoring Rule in Step 3

Let $\vec{x}^*, \pi^*$ be the optimal solution to (7.9)-(7.11), (B.23).

(c1): Equation (B.23) (equivalently (7.12)) is tight at the optimal solution to (7.9)-

(7.11), (B.23). Let $\pi_s = \min_{\vec{z}}\{c_s(\vec{z}) - T_i(\vec{z})\}$ and let $j = \arg\min_{s \neq i} \pi_s$, and note that $\pi^* = \pi_j$. For $\hat{\delta} > 0$, small, let $\hat{v}$ be the scoring rule constructed in Claim 5 to enforce $(c_i(\vec{x}^*) + \pi^* + \hat{\delta}, \vec{x}^*)$ in the two-supplier auction between i and j. By our choice of $\hat{v}$, $S_j = S_i - \pi^* - \hat{\delta}$ when $\hat{v}$ is announced. Since $\hat{v}$ induces supplier i to bid attribute level $\vec{x}^*$, we must have that $T_i + S_i$ is tangent to $\hat{v}$ at $\vec{x}^*$. Furthermore, $T_i + \pi^*$ bounds c_s from below for all $s \neq i$. Together these observations imply that, if $\hat{v}$ is announced to all S suppliers, $S_s < S_i - \pi^*$ for all $s \neq i$, and supplier i wins the auction with profit at most $\pi^* + \hat{\delta}$ (by $S_j = S_i - \pi^* - \hat{\delta}$). That is, $\hat{v}$ enforces $(c_i(\vec{x}^*) + \pi^* + \delta, \vec{x}^*)$ $(\delta \leq \hat{\delta})$ in the S-supplier auction (since $\hat{\delta}$ is arbitrarily small, $\hat{v}$ is considered optimal). That is, the optimal scoring rule is $\hat{v}$ in (7.13).

(c2): Equation (B.23) is nonbinding at the optimal solution to (7.9)-(7.11), (B.23), and v satisfies (7.2). In this case, we show that Claim 7 can be applied in a special way. In particular, we show that the true valuation function v satisfies the conditions of Claim 6; this implies that Claim 7 can be applied to construct an optimal scoring rule of the form $\lambda^*\hat{v} + (1 - \lambda^*)v$, where $\hat{v}$ is the scoring rule constructed in Claim 5 to enforce $(c_i(\vec{x}^*) + \pi^*, \vec{x}^*)$ in the two-supplier auction between i and j. To rephrase the conditions of Claim 6, we need to show that v is a feasible scoring rule satisfying (7.2)-(7.3) that causes supplier i to win the auction with attribute bid $\vec{x}^*$ and profit at least π^*.

To see that v induces supplier i to bid $\vec{x}^*$, we suppose not (suppose that v induces bid $\vec{z} \neq \vec{x}^*$) and derive a contradiction by showing that the payoff can be improved (and enforceability maintained) by moving slightly away from $\vec{x}^*$ toward $\vec{z}$. Let a be such that $x_a^* \neq z_a$, and let $d = z_a - x_a^*$. Then, for $0 < \delta \leq 1$,

$$v_a(x_a^* + \delta d) - c_{ai}(x_a^* + \delta d) > v_a(x_a^*) - c_{ai}(x_a^*), \tag{B.24}$$

since $v_a - c_{ai}$ is strictly concave, and is maximized at z_a. Clearly, for δ small, we can ensure that at the new point $(x_1^*, \ldots, x_a^* + \delta d, \ldots, x_A^*)$ that we move to, the hyperplane tangent to c_i will, when shifted up by π^* units, still intersect the surface c_j. This implies that $(c_i(x_1^*, \ldots, x_a^* + \delta d, \ldots, x_A^*) + \pi^*, (x_1^*, \ldots, x_a^* + \delta d, \ldots, x_A^*))$ is enforceable, which together with (B.24) contradicts the fact that $\vec{x}^*$, π^* is optimal. Hence, $\vec{x}^* = \vec{z}$, so announcing v induces i to bid $\vec{x}^*$.

Notice that constraint $\pi \geq \epsilon$ is tight for our optimal solution in this case, so $\pi^* = \epsilon$. Furthermore, because we did not exit the three-step method of §2.5 at step 3, we have $M_i - \epsilon > M_s$ for all $s \neq i$. That is, announcing v yields a profit of at least ϵ for supplier i, and v satisfies equation (7.3). Since equation (7.2) holds by assumption, we are done.

(c3): Equation (B.23) is nonbinding at the optimal solution to (7.9)-(7.11), (B.23), and v does not satisfy (7.2). In this case, the general construction of Proposition 1's "$\Rightarrow$" direction proof (§A1.2) can be applied to enforce $(c_i(\vec{x}^*) + \pi^*, \vec{x}^*)$.