Portfolio Strategies in Supply Contracts

by

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Abstract

Traditionally, industrial buyers have focused on long-term contracts for many of their purchasing needs. Recently, however, some high-tech manufacturers have started looking at more flexible contracts for non-strategic components, which enables them to buy from a variety of suppliers and the spot market. We study this type of strategies in a general framework for supply contracts, in which portfolios of contracts can be analyzed and optimized. We examine a multi-period model where expected profit is optimized, and a single-period model where a mean-variance objective is considered. In addition, we investigate what the consequences of such purchasing behavior might be. For this purpose, we study the game where suppliers compete on price and flexibility for the buyer’s orders. We characterize the suppliers’ Nash equilibria in pure strategies and show that, when demand is log-concave, there exists one or multiple equilibria, and that in any of these, suppliers bid in clusters against other suppliers with similar technologies.

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## Contents

1 Introduction  
  1.1 Motivation ........................................... 15  
  1.2 Contents ................................................ 18

2 A Unified Framework for the Analysis of Supply Contracts  
  2.1 Contract practices ........................................ 23  
  2.2 Literature review .......................................... 25  
  2.3 Mathematical formulation of contracts ................... 25  
  2.4 Spot markets .............................................. 28  
  2.5 Multi-period inventory model ............................. 30  
  2.6 Replenishment strategies ................................. 34  
  2.7 Structural results ......................................... 35  
  2.8 The backlogging model ................................. 39

3 Portfolio Contracts  
  3.1 Contract description ..................................... 43  
  3.2 Optimal replenishment policy ............................ 44  
  3.3 Portfolio selection ....................................... 45  
  3.4 Trade-offs between inventory and capacity ............. 52  
  3.5 Numerical results ....................................... 53  
    3.5.1 Example solved ...................................... 53  
    3.5.2 Sensitivity analysis .................................. 53  
    3.5.3 Profit distribution .................................. 61
3.6 Total-capacity-commitment contracts .................................. 62

4 Risk Modeling ................................................................. 65
  4.1 Motivation ................................................................. 65
  4.2 The mean-variance problem ........................................... 68
    4.2.1 The traditional financial model ................................ 68
    4.2.2 A newsvendor model ............................................ 70
    4.2.3 Properties of supply option portfolios ....................... 73
  4.3 Discussion .............................................................. 84
  4.4 Summary and discussion of correlation cases .................... 87

5 Suppliers' Behavior ........................................................ 93
  5.1 Motivation ............................................................... 93
  5.2 Literature review ..................................................... 95
  5.3 Assumptions and notation ........................................... 97
  5.4 A closed-form procurement strategy ............................... 99
  5.5 The suppliers' profit ............................................... 102
  5.6 Border distributions ............................................... 108

6 Cluster Competition in Equilibrium ..................................... 115
  6.1 Efficiency .............................................................. 116
  6.2 Optimal bids .......................................................... 116
  6.3 Equilibria with efficient suppliers only .......................... 119
    6.3.1 Characteristics of equilibrium ............................... 119
    6.3.2 Existence of equilibria ....................................... 122
    6.3.3 Supply chain profit bounds .................................. 128
  6.4 Equilibria with inefficient suppliers ............................ 131
  6.5 Practical implementation and experiments ....................... 133
  6.6 Insights .............................................................. 140

7 Extensions and Conclusion ............................................... 143
  7.1 Application to disruption management ............................ 143
7.2 Multiple selling channels ........................................ 152
  7.2.1 The revenue model ............................................ 152
  7.2.2 Cost structure with spot market ............................. 153
  7.2.3 Profit computation ............................................ 154
  7.2.4 Example ..................................................... 157

7.3 Conclusion ..................................................... 157

A Proofs .................................................................. 161

A.1 Chapter 2 .......................................................... 161
  A.1.1 Lemma 1 ....................................................... 161
  A.1.2 Proposition 2 .................................................. 162
  A.1.3 Proposition 3 .................................................. 163
  A.1.4 Proposition 4 .................................................. 164

A.2 Chapter 3 .......................................................... 165
  A.2.1 Theorem 1 ....................................................... 165
  A.2.2 Theorem 2 ....................................................... 167
  A.2.3 Theorem 4 ....................................................... 168
  A.2.4 Proposition 5 ................................................... 170
  A.2.5 Proposition 6 ................................................... 171
  A.2.6 Theorem 5 ....................................................... 172

A.3 Chapter 4 .......................................................... 174
  A.3.1 Proposition 7 ................................................... 174
  A.3.2 Proposition 8 ................................................... 179
  A.3.3 Proposition 9 ................................................... 180
  A.3.4 Theorem 9 ....................................................... 182

A.4 Chapter 5 .......................................................... 183
  A.4.1 Proposition 10 .................................................. 183
  A.4.2 Proposition 11 .................................................. 185
  A.4.3 Proposition 13 .................................................. 186
  A.4.4 Lemma 2 ......................................................... 187
List of Figures

2-1 Example of a quantity-flexibility contract, as a combination of a fixed-commitment contract and an option contract. The parameters of the contract are the price 5 per unit, the quantity 100 and the flexibility level of 20%. .................................................. 27

2-2 Illustration of Proposition 3. .................................................. 38

3-1 Instance of the optimal replenishment policy in terms of $I_{t+1}(I_t)$ and $z^*_t(I_t)$. .................................................. 46

3-2 Regions where options are dominated by other options, following the two cases of Proposition 5. The figure on the left illustrates case (i), the figure on the right case (ii). .................................................. 52

3-3 Optimal portfolio for the benchmark case .................................................. 54

3-4 Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the per-period profit, without contract and with the optimal portfolio. .................................................. 56

3-5 Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal fixed-commitment capacity levels. .................................................. 57

3-6 Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal option capacity levels. .................................................. 58

3-7 Comparison of profit distribution for different contracts. .................................................. 62
4-1 Simulation of mean-variance curve for a single supplier as a function of the amount bought from this supplier.

4-2 Comparison of the mean-variance curves for the different models: financial (circled line) and newsvendor (crossed line).

4-3 The lower-level sets of variance are not always convex.

5-1 Illustration of active and inactive bids.

5-2 Division of the bidding strategies in different regions.

6-1 Profit for the different situations, we observe that it is optimal to set $y_+ = y_m$ or $y_- = y_m$ depending on the case.

6-2 Example of equilibrium. There are 6 suppliers that form 3 clusters: 1 with 2, 3 with 4 and 5 with 6. As pointed out in the theorem, the bid within a cluster, e.g., the bid of 3 and 4 (full dot in the center of the figure, with the label PRICE next to it), falls in the segment connecting the costs of the two suppliers (hollow dots with the label COST next to them).

6-3 By moving bid $(w^1, v^1)$ to bid $(w^0, v^0)$ as indicated by the arrow, the optimal behavior of a supplier with cost $(c, f)$ is still to bid $(w^*, v^*) = (w^2, v^2)$.

6-4 Example of the modifications performed in cases 1 (left figure) and 4 (right). The dots represent the costs of 6 (left) and 4 (right) suppliers respectively, and the ones connected by lines fall in the same group. The arrows represent how we modify the bids as a function of the cost: the beginning of the arrow is the cost of a given supplier, and the end of the arrow is where the corresponding bid falls.

6-5 Tool available to each one of the suppliers. A given supplier can examine the pay-offs it receives for any set of strategies. After it is comfortable with its bid, it can submit it by pressing the button SUBMIT.

It is then directed to the screen shown in Figure 6-6.

6-6 Screen where a supplier is asked to confirm submission of a bid.
6-7 Monitoring screen for the buyer. It is free to close the current round at any time, while examining what suppliers have submitted a bid. At the bottom, one can find the graph of cost versus current bids.

6-8 Bids after 3 rounds for an experiment with 7 suppliers from the course ESD.269J [Advanced Logistics and Supply Chain Strategies], in the spring of 2003.

6-9 Bids after 3 rounds for an experiment with 7 suppliers from the course 1.270J/ESD.273J [Logistics and Supply Chain Management], in the fall of 2003.

7-1 Optimal portfolio quantities as a function of the disruption probability $\alpha$.

A-1 Geometric situation of costs $(c^i, f^i)$ and $(c^j, f^j)$ in region $A^{th}$.

A-2 Suppliers $l$ and $h$ are active and supplier $k$ is turned inactive by supplier $i$'s bid.
Chapter 1

Introduction

1.1 Motivation

A recent trend for many industrial manufacturers has been outsourcing; firms are considering outsourcing everything from production and manufacturing to the procurement function itself. Indeed, in the mid 90s, there was a significant increase in purchasing volume as a percentage of the firm's total sales. More recently, between 1998 and 2000, outsourcing in the electronics industry has increased from 15 percent of all components to 40 percent, as noted in [45].

For instance, throughout the 90s, strategic outsourcing, i.e., outsourcing the manufacturing of key components, was used as a tool to rapidly cut costs. In their recent study, Lakenan, Boyd and Frey [31] reviewed the case of eight major Contract Equipment Manufacturers (CEMs) – Solectron, Flextronics, SCI Systems, Jabil Circuit, Celestica, ACT Manufacturing, Plexus and Sanmina – which were the main suppliers to Original Equipment Manufacturers (OEMs) such as Dell, Marconi, NEC Computers, Nortel, and Silicon Graphics. The aggregated revenue for the eight CEMs quadrupled between 1996 and 2000 while their capital expenditure grew 11-fold.

Of course, the increase in the level of outsourcing implies that the procurement function becomes critical for a manufacturer to remain in control of its destiny. As a result, many OEMs focus on closely collaborating with the suppliers of their strategic components. In some cases, this is done using private exchange market-places and/or
effective supply contracts; both of these try to coordinate the supply chain.

A different approach has been applied for *non-strategic components*. In this case, products can be purchased from a variety of suppliers and flexibility to market conditions is perceived as more important than a permanent relationship with the suppliers. Indeed, *commodity products*, e.g., electricity, steel, grain, cotton or computer memory, are typically available from a large number of suppliers and can be purchased in spot markets. Because these are highly standard products, switching from one supplier to another is not considered a major problem.

Despite the non-strategic nature of commodity products, it is critical to identify effective procurement strategies for these components, since manufacturers may be completely dependent on them. For instance, production costs might be very sensitive to the cost of some commodity products, e.g., electricity for automobile manufacturers, or memory for computer manufacturers. At the same time, uncertainty in supply and customer demand raises the question of whether to purchase supply now or wait for better market conditions in the future.

Thus, an effective procurement strategy for commodity products has to focus on both driving costs down and reducing risks. These risks include both *inventory* and *price* risks. By inventory risk we refer to inventory shortages or unsold products while price risk refers to the purchasing price which is uncertain if the procurement strategy depends on spot markets.

A traditional procurement strategy that eliminates price risk is the use of *long-term* contracts, typically *fixed-commitment* contracts. These contracts specify a fixed amount of supply to be delivered at some point in the future; the supplier and the manufacturer agree on both the price and the quantity delivered to the manufacturer. Thus, in this case, the manufacturer bears no price risk while taking huge inventory risk due to uncertainty in demand and the inability to adjust order quantities.

One way to reduce inventory risk is through *option* contracts, in which the buyer pre-pays a relatively small fraction of the product price up-front, in return for a commitment from the supplier to reserve capacity up to a certain level. The initial payment is typically referred to as *reservation price* or *premium*. If the buyer does not
exercise the option, the initial payment is lost. The buyer can purchase any amount of supply up to the option level, by paying an additional price, agreed to at the time the contract is signed, for each unit purchased. This additional price is referred to as execution price or exercise price. Of course, the total price (reservation plus execution price) paid by the manufacturer for each purchased unit is typically higher than the price of a fixed-commitment contract.

Option contracts provide the manufacturer with flexibility to adjust order quantities depending on realized demand and hence these contracts reduce inventory risk. Thus, these contracts shift risk from the manufacturer to the supplier since the supplier is now exposed to customer demand uncertainty. This is in contrast to fixed-commitment contracts in which the manufacturer takes all the risk.

A different strategy used in practice to share risk between suppliers and manufacturers is through quantity-flexibility contracts. In these contracts, a fixed amount of supply is determined when the contract is signed, but the amount to be delivered and paid for can differ by no more than a given percentage determined upon signing the contract.

Interestingly, as we show later on, quantity-flexibility contracts are equivalent to a combination of a fixed-commitment contract plus an option contract. This observation suggests that a procurement strategy that combines several contracts at the same time can be beneficial to both the manufacturer and the supplier. Most relevant to this research is the work done at Hewlett-Packard (HP) on the use of a portfolio approach for the procurement of electricity or memory products, see Billington [7]. Specifically, 50% of HP's procurement cost is invested in long-term contracts, 35% in option contracts and the remaining is left to the spot market, see [15].

The objective of this thesis is to develop a unified framework that allows the analysis of different contractual arrangements between manufacturers and suppliers. In particular, we propose several models that allow the different partners in the supply chain to select their best possible strategy in order to procure or provide commodity products.

In a more general perspective, the purpose of this research is to understand how
different players in the supply chain would behave for buying or selling commodities that require installing capacities long before the selling season takes place. In this context, it is of particular interest to observe what types of production modes or technologies are chosen. That is, if several types of productive capacity exist, in terms of different installation and operational cost, what mix will be chosen by buyers? What pricing structure will be chosen by suppliers?

1.2 Contents

The materials presented in this thesis can be separated into two different categories. The first part of the thesis, Chapters 2 to 4, presents a model in which manufacturers optimize their procurement strategies. The second part of the thesis, Chapters 5 and 6, develops a model in which suppliers optimize their contract pricing strategy, in the presence of competition.

At this point, it is necessary to define the meaning of "best" or "optimized" strategy. Indeed, what is typically best for the manufacturer results in a non-desirable solution for its suppliers, and vice-versa. Thus, the definition of the objectives of the different players in the supply chain is critical to our models. In particular, we consider risk-neutral profit objectives, i.e., each player in the supply chain maximizes its expected profit. This approach seems well adapted to long horizon planning situations for large manufacturers: given the long term perspective, the uncertainty in profits is washed away by the law of large numbers, and becomes irrelevant. For shorter horizons, the impact of risk is an issue that cannot be overlooked. For this reason, in Chapter 4, we present a single-period model where both the profit mean and variance are considered. Specifically, in this chapter we derive the mean-variance efficient frontier for supply contracts.

The structure of the thesis is described below. Chapters 2 and 3 contain materials from Martínez-de-Albéniz and Simchi-Levi [36], Chapter 4 from Martínez-de-Albéniz and Simchi-Levi [38] and Chapters 5 and 6 from Martínez-de-Albéniz and Simchi-Levi [37].
In Chapter 2, we present a general framework for the evaluation of contract opportunities for a single manufacturer. In this setting, the manufacturer can choose a contract within a pool of available contracts. The manufacturer's objective is to select a contract and to identify a replenishment strategy so as to maximize its expected profit. Under some fairly general and realistic assumptions, we are able to characterize the structure of the replenishment policy in a multi-period setting.

The next step, in Chapter 3, is to describe the feasible contracts in a more detailed way, in order to derive stronger properties both of the replenishment and contract selection problems. We choose to define the contracts as *portfolios of option contracts*. That is, a contract is defined as a simultaneous selection of several suppliers (or contracts in a menu of contracts), with different supply characteristics in terms of price and flexibility; the manufacturer has the choice of reserving more or less capacity from each one of these contracts. Given a portfolio contract, we solve the replenishment problem and show that the optimal ordering policy is a *multiple modified base-stock policy*. Then, we provide some characteristics of the optimal portfolio. Namely, we show that the portfolio selection problem is a convex optimization problem; we provide some necessary conditions for "dominating" contracts; we study the substitution effects between capacity and inventory; we present a numerical study to understand the effects of the parameters of the model on the optimal portfolio. Finally, we provide a closed-form expression for the single-period problem, that is used in the following chapters.

In Chapter 4, we focus on a single-period model and evaluate the impact of risk, measured as profit variance, in the optimal portfolio. Of course, it is well-known, from the financial literature, that mean-variance is not a very desirable objective, since it may lead to stochastically dominated solutions, see Håkansson [27]. From a practical point of view, this type of objective is undesirable in the sense that it penalizes profits above average the same way as profits below average. However, this type of objectives has been extremely fruitful in the financial literature. By transposing it into an operational setting, with operational constraints, we hope to change the treatment of supply contracts in the operations management literature
from a simplistic risk-neutral approach into a more complex risk-sensitive perspective. The results presented in this chapter include the characterization of the profit variance structure. Even though the variance is neither convex or concave, we are able to show that it has a unique minimum under some assumptions. This result implies that one can solve mean-variance problems using standard greedy methods, such as steepest descent gradient methods.

After studying the contract selection problem from the manufacturer’s point of view, we develop, in Chapter 5, a model where suppliers now choose their pricing strategy. Thus, Chapter 5 presents a supplier competition model in a setting where the buyer does not select a single sourcing supplier, as is typical in the operations management literature, but purchases a portfolio of contracts from the suppliers. This joint purchasing behavior pushes suppliers to compete through two attributes, price and flexibility, by changing the option parameters. In this chapter, we derive a supplier's expected profit as a function of not only its own offering, but also as a function of the competitors' strategy. In order to derive equilibrium results, we present the class of border demand distributions, which guarantee some regularity properties of the suppliers' pay-offs. We show that this class includes an important class of distributions referred to as log-concave. Interestingly, the log-concave class covers distributions such as uniform, exponential or normal distributions.

After laying out the basic features of the model in Chapter 5, we study in Chapter 6 the behavior of suppliers in equilibrium, in an environment where they compete both on price and flexibility. Specifically, we characterize the Nash equilibria in pure strategies. We show that when the demand distribution is border, the model gives rise to what we call cluster competition. The characteristics of such equilibria provide some insightful observations: mainly, suppliers will co-exist by forming clusters of identical bids, and within every cluster, of two or three suppliers, suppliers will compete among themselves, thus reaching stable equilibria. We show existence of such equilibrium when the demand is log-concave. An interesting question is the impact of competition on supply chain total profit. We show that these competitive equilibria have a limited impact on supply chain profits. Namely, we prove that the total profit of all supply
chain parties, in a competitive equilibrium, cannot be less than half of the optimal total profit in general, and no less than 75% of the optimal total profit when the demand is log-concave. Finally, we describe how to implement in practice such auction in an iterative fashion. We present a web-based game to test our model and discuss some conclusions from our experiments with MBA and PhD students from MIT.

We conclude the thesis in Chapter 7. In this chapter, we propose several extensions of this research for future work. We develop a model for disruption management, where supplier default risk can be handled. Also, we provide an extension of the models to cases where different classes of customers are served. This covers situations when the buyer sells at a premium price and at a discount, or when the buyer can sell back excess inventory to a spot market with bid-ask spreads.

All the proofs are presented in the appendix.
Chapter 2

A Unified Framework for the Analysis of Supply Contracts

2.1 Contract practices

In recent years, many companies have started outsourcing some of the components used in their manufacturing processes. Usually, the contracts implemented in outsourcing arrangements emphasize two important characteristics: price and ordering flexibility. This section reviews some of the most widely used practices in contracting.

Definition 1 A fixed-commitment or forward-buy contract is an agreement where a buyer commits to purchase a certain amount of supply from a supplier, at a pre-specified price, either per-unit price or total price, at a pre-specified delivery date.

Definition 2 A buy-back contract is an agreement where a buyer commits to purchase a certain amount of supply from a supplier, at a pre-specified price, either per-unit price or total price, at a pre-specified delivery date. The buyer has the ability to return any amount of the purchase for a pre-specified per-unit refund.

Definition 3 A quantity-flexibility contract is an agreement where a buyer commits to buy some amount of supply in a pre-specified range \([x^l, x^h]\) from a supplier, at a pre-specified per-unit price, at a pre-specified delivery date.
Definition 4 An option contract is an agreement where a buyer has the opportunity to buy any amount between 0 and a pre-specified capacity upper limit of supply from a supplier, at a pre-specified per-unit execution or exercise price, at a pre-specified delivery date. This contract involves the pre-payment of a reservation fee from the buyer to the supplier, i.e., a total of the reservation per-unit price times the capacity reserved.

It turns out that the contracts presented so far, i.e., fixed-commitment, buy-back and quantity-flexibility, can be expressed through option contracts. This is noted in Barnes-Schuster et al. [3], and illustrated in the next proposition.

Proposition 1 A quantity-flexibility contract with parameters $[x^l, x^h]$ at a price $u$ is equivalent to a combination of a fixed-commitment contract for a capacity $x^l$ at a per-unit price $u$ and an option contract for a capacity $x^h - x^l$ at a zero reservation price and an execution price of $u$.

This proof is presented as an example in Section 2.3.

Finally, we should point out that all the contracts presented so far involve a pre-determined delivery date. The next contract, called total-capacity-commitment contract, allows delivery timing flexibility.

Definition 5 A total-capacity-commitment contract is an agreement where a buyer commits to buy a certain total amount of supply from a supplier, at a pre-specified price, either per-unit price or total price, for a pre-specified time period, with the possibility of requesting deliveries at any point in this time period.

In this contract, a buyer is allocated a certain amount of supply, say $x$, from the supplier, in a certain period, e.g., in the year of 2004. This implies that the buyer can request 20% of $x$ in the first quarter of 2004 (or any of the $x$ units), and request the remaining 80% later on that year.

Of course, managing dynamically the available supply, i.e., determining when to request deliveries in what amounts, is a challenging problem for buyers. This is discussed in more detail in Anupindi and Bassok [1]. We analyze this contract in Section 3.6.
2.2 Literature review

The academic literature on supply contracts is quite recent. For a review see Cachon [11] or Lariviere [32]. As observed in [32], the literature can be classified into two main categories. The first focuses on replenishment policies and detailed contract parameters for a given type of contract. Examples include Anupindi and Bassok [1] for quantity-flexibility contracts, Brown and Lee [9] for option contracts applied in the semiconductor industry setting, or Wu et al. [53], Kleinknecht and Akella [30], Spinler et al. [50] or Golovachkina and Bradley [25] for option contracts in the presence of a spot market. Typically, the objective in this category is to optimize the buyer’s procurement strategy with very little regard to the impact of the decision on the seller. The second category focuses on optimizing the terms of the contract so as to improve supply chain coordination. Examples here include buy-back contracts, see Pasternack [44], revenue sharing contracts, see Cachon and Lariviere [12], or option contracts, see Barnes-Schuster et al. [3]. Unlike the first category, here the objective is to characterize contracts that allow each party to optimize its own profit but lead to a globally optimized supply chain.

The model developed in this chapter belongs to the first category. Specifically, we analyze in a general situation a supply contract taking into account the presence of the spot market. Our objective is to design effective contracts and identify the replenishment policy so as to maximize the buyer’s expected profit.

2.3 Mathematical formulation of contracts

We formalize a contract for some delivery time $t$ as a function $r_t(\cdot)$. The interpretation of this contract is the following: the manufacturer may buy an amount $q_t \geq 0$ at a total cost of $r_t(q_t)$. In other words, choosing a a contract $r_t$ can be seen as designing the cost structure for the manufacturer for period $t$.

**Assumption 1** For $t = 1, \ldots, T$, the function $r_t$ is convex.
To justify the assumption observe that when the manufacturer faces customer demand, it must choose among the different sourcing alternatives. Evidently, to minimize costs, the manufacturer will prefer the cheapest unit-price sources. Thus, given that it executes an amount $q$, the incurred cost will be convex in $q$.

The contracts presented in the previous section satisfy this assumption.

**Example 1** Consider a fixed-commitment contract for an amount $x$ at a per-unit price $u$. Thus, buying an amount of $q$ costs

$$r_{FC}(q) = \begin{cases} \text{ux} & \text{when } q \leq x \\ +\infty & \text{otherwise} \end{cases}$$

**Example 2** Consider an option contract for an amount $x$ at a per-unit reservation price $v$ and a per-unit execution price $w$. Thus, buying an amount of $q$ costs

$$r_{O}(q) = \begin{cases} vx + wq & \text{when } q \leq x \\ +\infty & \text{otherwise} \end{cases}$$

**Example 3** Consider a quantity-flexibility contract for a range $[x', x^h]$ at a per-unit price $u$. Thus, buying an amount of $q$ costs

$$r_{QF}(q) = \begin{cases} ux' & \text{when } q \leq x' \\ wq & \text{when } x' \leq q \leq x^h \\ +\infty & \text{otherwise} \end{cases}$$

It is easy to see that this function is such that

$$r_{QF}(q) = \min_{q_{FC} + q_{O} = q} r_{FC}(q_{FC}) + r_{O}(q_{O}),$$

where $r_{FC}$ is the cost of a fixed-commitment contract with amount $x'$ at a price $u$ and $r_{O}$ the cost of an option contract with capacity $x^h$ at a reservation price $v$ and an execution price $u$.

Figure 2-1 provides an example of the shape of the function $r_{QF}$.
Figure 2-1: Example of a quantity-flexibility contract, as a combination of a fixed-commitment contract and an option contract. The parameters of the contract are the price 5 per unit, the quantity 100 and the flexibility level of 20%.
In some industrial settings, manufacturers may be offered volume discounts. These discounts, in an option contract, typically take the form of a discount on the reservation price and hence are counted as a fixed payment. Once this contract has been determined, the variable payment (as a function of the executed quantity) is typically convex. Therefore, Assumption 1 is often satisfied in this case, since the discounts are counted as a fixed component of \( r \). The next example illustrates that Assumption 1 holds even under volume discount on the reservation price.

**Example 4** Consider a portfolio contract made up of \( n \) options. Each option has a maximum capacity \( x^i \), a total reservation cost (paid at the beginning of the horizon) \( v^i(x^i) \) and an execution cost \( w^i \geq 0 \) per unit delivered (paid when executing part or all of this option), for each \( i = 1, \ldots, n \). Given that we must buy an amount \( q \) of supply, we have

\[
    r(q) = \sum_{i=1}^{n} v^i(x^i) + \min \sum_{i=1}^{n} w^i q^i \quad \text{subject to} \quad \begin{cases} 
        \sum_{i=1}^{n} q^i = q \\
        0 \leq q^i \leq x^i \quad \forall i = 1, \ldots, n.
    \end{cases}
\]

This linear program is convex in \( q \).

This modeling approach is quite general. Moreover, Assumption 1 is not very restrictive, since most practical contracts satisfy it.

### 2.4 Spot markets

Since our focus is on commodity products, we assume that there is a supply market that can be used by the manufacturer to purchase, at any time period, additional components. Thus, at period \( t \), the manufacturer obtains an amount \( q_t \in \mathbb{R} \) of supply at a total cost \( s_t(q_t) \). Notice that, when \( q_t \geq 0 \), \( s_t(q_t) \geq 0 \) will represent the cost of purchasing, while when \( q_t \leq 0 \), \( s_t(q_t) \leq 0 \) will represent minus the revenue obtained by selling \(-q_t \) to the spot market.

This spot market cost function, \( s_t(\cdot) \), is random and will only be known at the beginning of period \( t \). Prior to that time, the manufacturer has only probabilistic
information on the spot market costs $s_t$. Because our model can handle a learning process, the distribution of the spot market cost can be improved as time goes by. More on this is discussed in the next section.

**Assumption 2** For $t = 1, \ldots, T$, the random function $s_t$ is a convex function for all outcomes.

This assumption is equivalent to saying that in the spot market, the marginal unit cost is increasing with quantity. This is a natural assumption since the spot market has limited supply with many competing buyers and hence the more the manufacturer purchases from the spot market the higher the marginal price it has to pay. Moreover, the assumption implies that, if the market is defined by an ask price and a bid price, then the ask price should be smaller than the bid price, i.e., the bid-ask spread is non-negative.

In some industries, as mentioned in Section 2.3, manufacturers may face volume discounts. We argue that these are not relevant for the spot market modeling. Indeed, spot market purchases are typically made to complement shortages in long term planning, and this implies small volumes and fast delivery. On the other hand, volume discounts are available when the orders are made long in advance. In this case, the volumes are important and the production time is long so that economies of scale can be created through better planning and scheduling.

A special case captured by the model is the case in which the spot market unit price at time $t$ is constant, equal to $S_t$, and the market does not accept sales from the manufacturer and has limited supply capacity $\kappa_t$. The vector $(S_t, \kappa_t)$ is a random variable revealed only in period $t$. In particular, if there is no spot market, we can select either $S_t = \infty$ or $\kappa_t = 0$.

In some situations, the manufacturer is not allowed to sell inventory back to the spot market. In these cases, the function is defined for $q_t \geq 0$ and $s_t(q_t) = +\infty$ when $q_t < 0$. This typically occurs for engineered products, that are tailored for the manufacturer, such as cell-phone displays.

In some other situations, the manufacturer may be able to sell back excess in-
ventory to the spot market and thus supply contracts may provide arbitrage opportunities. This is particularly relevant for OEMs, e.g., automotive, PC or electronics manufacturers, since they are natural consumers of commodity components. Thus, manufacturers in these industries could take advantage of their size in order to realize additional profits from pure trading. In this setting, financial models may not be applicable to industrial procurement; this is true since procurement spot markets are far from being efficient, due to the limited number of suppliers and buyers or operational characteristics, e.g., lead times and transaction costs. Our framework provides a good starting point to model capacitated spot markets and hence explore these issues.

2.5 Multi-period inventory model

Consider a manufacturer of a single item who faces demand during several time periods. Future demands are unknown but as time goes by, more precise information on the distribution of demand becomes available. The manufacturer's objective is to optimally manage its supply by buying a well-chosen contract for every time period. Specifically, in all chapters except 4, its objective is to maximize expected profit by effectively managing the supply process.

For this purpose, the manufacturer needs the right trade-off between price and flexibility. That is, the manufacturer needs to find the appropriate mix of low price yet low flexibility contracts (e.g., fixed-commitment), reasonable price but better flexibility contracts (e.g., option) or unknown price and quantity supply but no commitment (i.e., the spot market). Once these decisions are made at the beginning of the horizon, the manufacturer has to manage its inventory effectively, which can be viewed as an extra supply source, carried over from period to period.

At the beginning of the planning horizon, i.e., in period $t = 0$, the manufacturer decides on the type of contracts it will buy from its suppliers for the entire planning horizon. This planning horizon is finite and has $T$ time periods indexed from 1 to $T$ in an increasing order. The contract specifies for every time period the cost of receiving any amount of supply, $r = (r_1, \ldots, r_T)$, as presented in Section 2.3. Of course, $r$ is a
vector of functions.

At the beginning of period \( t, t = 1, \ldots, T \), the customer demand \( d_t \) and the spot market structure, as defined in Section 2.4, \( s_t \) become known. At that time, the manufacturer decides how much of the contract to execute. It is also able to buy supply directly from the spot market. In any event, the inventory carried from previous periods plus the incoming supply can be used to satisfy this period's demand. The remaining inventory is carried over to the next period. Finally, unsatisfied demand is lost to the competition. We allow the manufacturer to reject customers, with no penalty other than the loss of revenue. However, rejection penalties can also be considered in the model, with no changes in the results. Thus, this is a \textit{discretionary sales} model. We denote by \( I \in \mathbb{R}^{T+1} \) the vector of inventory levels. \( I_t \), for \( t = 1, \ldots, T + 1 \), is the inventory level at the beginning of period \( t \). To simplify the presentation, we assume \( I_1 = 0 \), although the model can handle any initial inventory level.

We impose \( I \geq 0 \). Therefore, this is a lost sales model, with discretionary sales. The manufacturer may lose some of the demand, however this loss will not affect future demand. Note that the inventory position is known at all times. Demand is known before stock is replenished, so the system moves from a known inventory position to another known inventory position after adding replenishment and subtracting demand. We can thus make sure that inventory levels remain non-negative.

As is common in traditional inventory models, an inventory holding cost \( h_t(I_t) \) is incurred at the beginning of period \( t \). The family of functions \( h_t(\cdot), t = 1, \ldots, T \), is known in advance.

\textbf{Assumption 3} \textit{For } \( t = 1, \ldots, T \), \textit{the function } \( h_t \) \textit{is convex and non-decreasing.}

This is a standard assumption used in periodic review inventory models. It is typically satisfied in practice, since the per-unit cost of keeping inventory increases due to scarcity of resources.

In addition, in every period, the manufacturer receives the amount ordered at that period and sells products to the end customers at a price \( p_t \), decided exogenously in advance.
Assumption 4 For \( t = 1, \ldots, T + 1 \), the price \( p_t \) at which the manufacturer charges the end customers is given exogenously, in advance of time period 0.

This assumption is used throughout the thesis. However, the same arguments can be used with the following more general assumption, when the prices are set by market forces.

Assumption 5 For \( t = 1, \ldots, T + 1 \), the price \( p_t \) at which the manufacturer charges the end customers is random and is observed at the beginning of period \( t \). The expectation of \( p_t \) at a period \( t' < t \) is finite.

Assumption 4 (and also 5) implies that the pricing decision is not taken jointly with contract negotiation and inventory replenishment decisions. Even though joint optimization could yield important improvement, in most cases, companies manage these two decisions separately, pricing through marketing or sales divisions, and purchasing and inventory management through procurement/purchasing and operations divisions. It is then realistic to take prices as a given input in the problem. The assumption is also commonly satisfied in markets in which the manufacturer is price-taker and the market price depends on external factors such as the state of the economy.

Note that the price \( p_{T+1} \) is the salvage value at the end of the planning horizon. Thus, remaining inventory is sold at salvage unit price \( p_{T+1} > 0 \), i.e., the manufacturer receives a revenue of \( p_{T+1} I_{T+1} \) if \( I_{T+1} \) items are left at the end of the horizon.

Having defined the revenue stream, we need to make sure that the manufacturer makes no infinite profit, in expectation. This is expressed in the following assumption.

Assumption 6 For \( t = 1, \ldots, T \),

\[
\partial r_t(q) \rightarrow +\infty \text{ when } q \rightarrow +\infty,
\]

and

\[
\partial s_t(q) \rightarrow +\infty \text{ when } q \rightarrow +\infty,
\]
\[ \partial s_t(q) \to -\infty \text{ when } q \to -\infty, \]

*with probability 1.*

Of course, this assumption is very strong, but the results that follow still hold under weaker assumptions, requiring that the marginal contract price and the spot bid price are smaller than the selling price, when customer demand is infinite, and the spot ask price.

At every time period, \( t, t = 0, \ldots, T - 1 \), the manufacturer knows the probability distribution of future events such as demand \((D_t)_{t=t+1, \ldots, T}\) and spot market cost structure \((s_t)_{t=t+1, \ldots, T}\).

We denote by \( \Phi_t \) the information state at time \( t \) and we define \( \Phi \) recursively in the following way:

\[ \Phi_t = (\Phi_{t-1}, D_t, s_t) \forall t = 1, \ldots, T. \] \hspace{1cm} (2.1)

**Assumption 7** *The distribution of future customer demand and spot market cost structure depends only on \( \Phi_t \), not on past decisions made by the manufacturer.*

Thus, we assume that future events are independent of the decisions that the manufacturer took in the past. Note that future events do not depend on decisions made by the firm so far but may depend on past demand or past spot market cost structures. In other words, we assume that the manufacturer is small relative to the market and its behavior cannot affect future demand or market prices. Since we make no assumption on the relationship between demand and spot market cost structure, we allow the spot market prices to be correlated with customer demands.

Since the distributions of future demand and spot market price depend only on \( \Phi_t \), and \( \Phi_t \) is determined by past events, this modeling approach allows the manufacturer to improve its forecast as more information is available. For instance, as the manufacturer approaches period \( t \) it may be the case that the standard deviation of period \( t \) demand can be reduced.
2.6 Replenishment strategies

The decisions the manufacturer must take can be analyzed using dynamic programming, formulated from the last period to the first one, i.e., from $T + 1$ to $0$.

At each time period $t = 1, \ldots, T + 1$, the state space is defined by

- the inventory position $I_t$;
- the information vector $\Phi_t$, which contains the current demand to be served $d_t$ and the spot market description $s_t$;
- the supply contract $r$.

At time period $t = 0$, we only have the information vector $\Phi_0$, given a priori. At time period $t = T + 1$, the only relevant information is the remaining inventory, $I_{T+1}$.

For every period $t$, $t = 1, \ldots, T$, the decision space includes the following quantities:

- the quantity $q_t^r \geq 0$ that the manufacturer executes from the supply contract at a cost $r_t(q_t^r)$;
- the quantity $q_t^s \in \mathbb{R}$ that the manufacturer purchases from the spot market at a cost $s_t(q_t^s)$;
- the amount of demand not satisfied $q_t^n$, $0 \leq q_t^n \leq d_t$.

These decisions completely characterize the inventory position at the beginning of period $t + 1$, $I_{t+1}$, where $I_{t+1} = I_t + q_t^r + q_t^s + q_t^n - d_t$. Since this is a lost sales model, we must have $I_{t+1} \geq 0$.

At the end of the horizon, the manufacturer sells the remaining inventory at salvage value, $p_{T+1}$. Therefore, the profit-to-go function at this period is

$$V_{T+1}(I_{T+1}, \Phi_{T+1}, r(\cdot)) = p_{T+1} I_{T+1}.$$
For every $t, t = 1, \ldots, T$, the profit-to-go function at period $t$ can be written as

$$V_t(I_t, \Phi_t, r(\cdot)) = \max_{q_t^r, q_t^s, I_{t+1}} \left( \begin{array}{c}
p_t(I_t - I_{t+1} + q_t^r + q_t^s) \\
-r_t(q_t^r) - s_t(q_t^s) \\
-h_t(I_t) \\
+\mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}, r(\cdot)) \end{array} \right)$$

subject to

$$\begin{align*}
q_t^r & \geq 0 \\
I_{t+1} & \geq 0 \\
I_t - I_{t+1} + q_t^r + q_t^s & \geq 0 \\
I_t - I_{t+1} + q_t^r + q_t^s & \leq d_t
\end{align*}$$

(2.2)

Observe that in this formulation the decision variables are $q_t^r$, $q_t^s$ and $I_{t+1}$ which is equivalent to optimizing on the variables $q_t^r$, $q_t^s$ and $q_t^a$.

Finally, in period $t = 0$, the manufacturer has to choose one contract $r$ among a family of contracts $\mathcal{R}$. The manufacturer’s contract selection problem is thus,

$$V_0(\Phi_0) = \max_{r(\cdot) \in \mathcal{R}} \mathbb{E}_{\Phi_1} V_1(I_1 = 0, \Phi_1, r(\cdot)).$$

We are now ready to analyze the optimal inventory replenishment policy given the current information and a choice of $r$.

### 2.7 Structural results

Before presenting the results for the general model described above, we show the following lemma. This will be applied in various parts of the analysis.

**Lemma 1** Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a concave function. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Define $P(b) = \{x \in \mathbb{R}^n | Ax \leq b\}$ and

$$g(b) = \max_{x \in \mathbb{R}^n} f(x, b)$$

subject to $x \in P(b)$

(2.3)
Let $Q = \{ b \in \mathbb{R}^n | P(b) \neq \emptyset \}$. Then $Q$ is a convex set and $g : Q \to \mathbb{R}$ is a concave function.

This relatively simple result is crucial in proving most of the properties of the dynamic program. All the technical proofs are presented in Appendix A.

In the rest of this section, we drop $r(\cdot)$ from the notation because it is fixed through all the time periods $t = 1, \ldots, T + 1$.

**Proposition 2** Consider any time period $t, t = 1, \ldots, T + 1$. Given $\Phi_t$, the profit-to-go function $V_t(I_t, \Phi_t)$ is concave in $I_t$.

The proposition thus implies that the marginal profit from every additional unit of inventory is non-increasing with the inventory level.

We now characterize the optimal replenishment policy. Given time period $t$ and $\Phi_t$, define function $U_{t+1}$ as follows.

$$ U_{t+1}(I_{t+1}) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}). \quad (2.4) $$

By applying Proposition 2 and taking the expectation on $\Phi_{t+1}$, we see that $U_{t+1}$ is concave in $I_{t+1}$. Observe that the optimization problem defined by Equation (2.2) can be rewritten as

$$ V_t(I_t, \Phi_t) = p_t d_t - h_t(I_t) + \max_{I_{t+1}} \left\{ -C_t(I_{t+1} - I_t + d_t) + U_{t+1}(I_{t+1}) \right\} \quad (2.5) $$

subject to

$$ \begin{cases} 
I_{t+1} \geq 0 \\
I_{t+1} \geq I_t - d_t,
\end{cases} $$

where the function $C_t$ is defined as follows.

$$ C_t(q_t) = \min_{q_t, q_t^n, q_t^u} \left\{ p_t q_t^n + r_t(q_t^n) + s_t(q_t^u) \right\} \text{ subject to } \begin{cases} 
q_t^u \geq 0 \\
q_t^n \geq 0 \\
q_t^n \leq d_t \\
q_t^u + q_t^n + q_t^n = z_t.
\end{cases} \quad (2.6) $$
Lemma 1 shows that $C_t$ is convex in $z_t$. Intuitively, this is explained as follows. The optimal policy that produces a total of $z_t$ units starts producing using cheaper means first and applies more expensive means later.

This property is used to derive the next proposition. This proposition states that the optimal inventory target level as a function of the on-hand inventory is increasing and non-expanding. In other words, the higher the inventory level at the beginning of a given period, $I_t$, the higher the inventory level at the end of the period, $I_{t+1}$; similarly, the higher the inventory level, $I_t$, the smaller the ordering quantity, $z_t = I_{t+1} - I_t + d_t$.

**Proposition 3** Given $\Phi_t$, $\forall t = 1, \ldots, T$, define $I_{t+1}^*(I_t)$ to be the smallest optimal control for the next period inventory level in the optimization problem defined by Equation (2.5) with parameter $I_t$. Then, for every $I_t^1 \geq I_t^0 \geq 0$, we have

$$0 \leq I_{t+1}^*(I_t^1) - I_{t+1}^*(I_t^0) \leq I_t^1 - I_t^0$$

A sketch of the proof of the proposition, in the differentiable case, uses Figure 2-2. In this case, the optimality condition is

$$\frac{dU_{t+1}}{dI_{t+1} | I_{t+1}^*} = \frac{dC_t}{dz_t | z_t^*}$$

subject to the constraint that $I_{t+1} = I_t - d_t + z_t$. To determine the optimal $I_{t+1}$, it is sufficient to find the intersection of the non-increasing function $\frac{dU_{t+1}}{dI_{t+1}}$ and the non-decreasing function $I_{t+1} \rightarrow \frac{dC_t}{dz_t}(-I_t + d_t + I_{t+1})$. Therefore, the parameter $I_t$ is used to determine by how much we shift the graph of $\frac{dC_t}{dz_t}$. When we increase $I_t$ from $I_t^0$ to $I_t^1$, we shift the graph of $z \rightarrow \frac{dC_t}{dz_t}(-I_t + d_t + I_{t+1})$ to the right by $I_t^1 - I_t^0$, and hence, since $\frac{dU_{t+1}}{dI_{t+1}}$ is non-increasing, the optimal inventory position at the next time period $I_{t+1}$ cannot decrease. Moreover, since we have shifted the graph of $\frac{dC_t}{dz_t}$ by $I_t^1 - I_t^0$, the intersection cannot happen after $I_{t+1}^0 + I_t^1 - I_t^0$. 

37
Figure 2-2: Illustration of Proposition 3.
In fact, in the differentiable case, the proposition below characterizes the behavior of $I_{t+1}^*$ as a function of $I_t$.

**Proposition 4** Given $\Phi_t$, assume that the functions $U_{t+1}$ and $C_t$ are twice differentiable at $I_{t+1}^*(I_t)$ and $I_{t+1}^*(I_t) - I_t + d_t$ respectively. Then if

$$\frac{d^2 C_t}{dz^2} \bigg|_{I_{t+1}^*(I_t) - I_t + d_t} - \frac{d^2 U_{t+1}}{dI_{t+1}^2} \bigg|_{I_{t+1}^*(I_t)} \neq 0$$

and $I_{t+1}^*$ is differentiable at $I_t$ we have that

$$\frac{dI_{t+1}^*}{dI_t} \bigg|_{I_t} = \frac{\frac{d^2 C_t}{dz^2} \bigg|_{I_{t+1}^*(I_t) - I_t + d_t}}{\frac{d^2 C_t}{dz^2} \bigg|_{I_{t+1}^*(I_t) - I_t + d_t} - \frac{d^2 U_{t+1}}{dI_{t+1}^2} \bigg|_{I_{t+1}^*(I_t)}}$$

This proposition can be very powerful in practice, since it may reduce computational effort. If, for a given $I_t^0$, we know the optimal next-period level $I_{t+1}^0 = I_{t+1}^*(I_t^0)$, we can use this formula to generate the optimal control for any other $I_t$ by generating the optimal controls $I_{t+1}^*(I)$ for $I \in [I_t^0, I_t]$ (or $[I_t, I_t^0]$ when $I_t < I_t^0$). Specifically, a naive implementation of the dynamic programming would require in each period, and for each possible value of $I_t$, to identify the next period target inventory level by searching on all possible values of $I_{t+1}$. Our proposition reduces the search space for any time period from these two dimensions to one dimension.

### 2.8 The backlogging model

In this chapter, we have assumed that at the beginning of a period, all the information on demand and spot markets is revealed. Of course, in the standard newsvendor model, demand is observed after replenishment decisions are taken.

It turns out that it is easy to extend the framework to a model in which each period's order is made before demand is known and all shortages are backlogged. This is done by replacing each period $t$ by periods $(t^a, t^b)$, $t = 1, \ldots, T$, with the following representations:
• In period $t^a$, the manufacturer chooses the best trade-off between buying from the contract, purchased at period 0, purchasing from the spot market at current market conditions, or depleting inventory carried from period to period.

• In period $t^b$, demand is realized and the manufacturer serves it all with the available inventory. Unsatisfied demand is backlogged.

Notice that, in this formulation, the manufacturer serves demand if it has inventory on hand, and backlogs it otherwise. Decisions are made in periods $t^a$, $t = 1, \ldots, T$. Specifically, in time $t^a$ one needs to decide on order quantities and spot market purchases; these items will be added to inventory at period $t^b$.

This sequence of events implies that we can no longer guarantee that $I \geq 0$. Indeed, customer demand being random, it can always be big enough to bring the inventory position below 0. Therefore, in addition to the new sequence of events, one has to use a backlog model instead of a lost sales model. We thus have to define $h_t(\cdot)$ as both inventory holding and backlogging cost function for each $t = 1, \ldots, T$.

All the results presented so far hold under this alternative modeling approach as well. In addition, the convexity results presented in the next chapter will also hold.
Chapter 3

Portfolio Contracts

So far we have focused on general, convex contracts; i.e., for every time period, total purchasing cost was convex in the amount purchased. A special case of these convex contracts is the situation where suppliers offer different contracts, and the manufacturer selects one or more of these contracts. This gives rise to the notion of portfolio in supply contracts: the manufacturer faces the problem of selecting capacities in each one of the contracts offered by its suppliers.

This concept is similar to the notion of portfolio investments in finance. The firm, by purchasing different stocks and bonds, can obtain higher returns while hedging market risk. In our context, by selecting different contracts simultaneously, the manufacturer can increase expected profits.

Thus, we now focus on the special case of contracts where a set of option contracts (with different attributes) are selected simultaneously. We choose to use the option contract as the building unit of our model, since options and portfolios of options can replicate most contracts used in practice, as pointed out in Section 2.1.

This type of selection problem is faced by manufacturers for many commoditized components. To our knowledge, many companies do use this approach regarding energy purchasing, as described in Example 5, especially in the automotive and chemicals sectors. More sophisticated, engineered, components are also procured using multiple sources simultaneously, such as memory units, displays and other components of high-tech products.
Example 5. Consider a manufacturer that needs to find supply sources for electricity. The manufacturer produces and sells products at a unit price, \( p = 20 \), and we assume that the only contributor to the production cost is the cost of electricity. One way to interpret \( p \) is as the profit margin, i.e. the final price of the product minus the loaded costs from other components.

To simplify the example we assume that a unit of electricity is required to produce a unit of finished good, and that unused electricity is lost. The manufacturer thus has information on the distribution of the potential electricity demand. More precisely, it knows that demand for electricity follows a truncated normal distribution of mean \( \mu = 1000 \) and standard deviation \( \sigma = 300 \).

Three power companies are available for supply:

- Company 1 offers a fixed-commitment contract with the following conditions: power is bought in advance at a price \( v^1 = 10 \) per unit, and there is no price to pay at delivery (\( w^1 = 0 \)).

- Company 2 offers an option contract with payment of \( v^2 = 6 \) per unit paid in advance and then \( w^2 = 6 \) per unit paid upon delivery.

- Company 3 has similar terms to company 2, but with prices \( v^3 = 3 \) per unit paid in advance and \( w^3 = 12 \) per unit paid upon delivery.

The manufacturer problem is to find the right balance between the different contracts: how much to commit from the fixed-commitment contract? how much capacity to buy for each one of the two option contracts? and, how much supply to leave uncommitted?

Assume now that there also exists a spot market for electricity, such that the spot market price is uncorrelated with the specific demand for electricity generated by the manufacturer. Supply from this spot market is unlimited and must be paid at a unit price \( S \), where \( S \) follows a uniform distribution in \([10, 20]\). No sales back to the market are allowed.

How does the optimal solution to the previous problem change when it is possible to use the spot market as an extra source? We solve this problem in Section 3.5.1.
3.1 Contract description

We will formalize the type of contract portfolios in this section. In addition to the contracts specification, we also express a stronger assumption regarding the spot market structure. This formulation is the standard modeling of spot markets used in industry.

**Assumption 8** The spot market unit cost is constant and equal to \( S_t \), and the market only offers a limited supply \( \kappa_t \), for \( t = 1, \ldots, T \). That is, the structure of \( s_t(q) \) is as follows.

\[
s_t(q) = \begin{cases} 
S_t q & \text{for } 0 \leq q \leq \kappa_t \\
+\infty & \text{else}
\end{cases}
\]

Of course, the spot market unit price and capacity are random variables realized at the beginning of each time period. The previous modeling assumption is based on our industry experience where we observe manufacturers using these two variables to describe the spot market.

As we mentioned before, the contract is defined as a portfolio of simple options. Note that the up-front cost of reserving the option capacities (reservation cost) is a fixed cost, and thus has no impact on the replenishment policy applied in periods \( t > 0 \).

**Assumption 9** For \( t = 1, \ldots, T \), the contract \( r_t \) is made up of \( n_t \) options with capacities \( x_t^i \) and execution unit price \( w_t^i \), \( i_t = 1, \ldots, n_t \). Without loss of generality we assume that \( w_t^1 \leq \ldots \leq w_t^{n_t} \). Therefore, the total execution cost associated with these options is determined by solving,

\[
r_t(q) = \sum_{i=1}^{n_t} w_t^i x_t^i + \min \sum_{i=1}^{n_t} w_t^i q_t^i \\
\text{subject to } \begin{cases} 
\sum_{i=1}^{n_t} q_t^i = q \\
0 \leq q_t^i \leq x_t^i \forall i = 1, \ldots, n_t
\end{cases}
\]
3.2 Optimal replenishment policy

With these assumptions, for any period $t = 1, \ldots, T$, the manufacturer faces $n_t + 2$ supply sources, which are:

- $n_t$ simple options of unit cost $w^i_t$ and capacity $x^i_t$ for $i = 1, \ldots, n_t$;

- the spot market, which offers supply at unit cost $S_t$ up to a capacity of $\kappa_t$;

- not to serve demand; the firm can choose not to fulfill demand, which is equivalent to serving all demand by buying supply at a unit cost $p_t$ up to a capacity of $D_t$.

Each of these sources offers supply at fixed marginal cost (execution price) for a given capacity. These supply sources can be represented as a pair (unit cost, capacity level): $(w^i_t, x^i_t)$ for $i = 1, \ldots, n_t$, $(S_t, \kappa_t)$ and $(p_t, D_t)$. Define $\bar{n}_t = n_t + 2$ and sort these pairs by increasing unit cost. Let $(\bar{w}^j_t, \bar{x}^j_t)$ be the pair with $j^{th}$ smallest unit cost. Thus, $\bar{w}^1_t \leq \ldots \leq \bar{w}^{\bar{n}_t}_t$.

The new definition of the supply sources, $(\bar{w}^j_t, \bar{x}^j_t)$, allows us to rewrite Equation (2.6) as follows,

$$C_t(z_t) = \sum_{i=1}^{n_t} v^i_t x^i_t + \min \sum_{i=1}^{\bar{n}_t} \bar{w}^i_t q^i_t$$

subject to

$$\begin{cases}
\sum_{i=1}^{\bar{n}_t} q^i_t &= z_t \\
0 \leq q^i_t &\leq \bar{x}^i_t \ \forall i = 1, \ldots, \bar{n}_t
\end{cases} \tag{3.1}$$

The sorting performed above has the following advantage. It allows each source to be represented by the same notation, where $\bar{x}^i_t$ is the capacity available from the given source, and $\bar{w}^i_t$ the marginal cost of using it.

This special convex piece-wise linear cost function yields important properties of the optimal replenishment policy. This is illustrated in the next theorem.
Theorem 1 Given \( \Phi_t \), for \( t = 1, \ldots, T \), there exist inventory target levels \( b^1_t, \ldots, b^n_t \) such that

(i) for \( i = 1, \ldots, n_t - 1 \), \( b^{i+1}_t \leq b^i_t - \bar{x}^i_t \);

(ii) for \( i = 1, \ldots, n_t \), it is optimal to order from source \( i \)

- 0 if \( I_t \geq b^i_t \);
- \( b^i_t - I_t \) if \( b^i_t - \bar{x}^i_t \leq I_t \leq b^i_t \);
- \( \bar{x}^i_t \) otherwise.

The theorem thus completely characterizes the optimal replenishment policy for a portfolio of option contracts. In particular, the optimal policy is obtained by following a modified base-stock policy for every option, the spot market, and the demand source. The ordering of the contracts implies that contracts are executed based on execution price, starting with the contract with the cheapest price. Of course, a contract with execution price higher than the spot price will only be used once the manufacturer has exhausted the spot market capacity.

The aggregated policy is shown in Figure 3-1. We see that the functions \( I_{t+1}(\cdot) \) and \( z_t(\cdot) \) have slope 0 or 1, or \(-1\) or 0 respectively. This implies that for a given interval of \( I_t \), i.e. the initial inventory level at the beginning of period \( t \), either the order quantity does not change or the target inventory level remains constant.

Note that this result could have been obtained by applying carefully Proposition 4. Indeed, in this case, the second derivative of the cost function \( C_t \) is either 0, when some source is being used without capacity constraint, or \(+\infty\), when some source reaches capacity and the manufacturer faces a jump in marginal procurement cost by switching to a more expensive source.

### 3.3 Portfolio selection

The analysis so far takes as an input the structure of the portfolio contract and derives the replenishment policy effectively. Notice that the replenishment policy does not
Figure 3-1: Instance of the optimal replenishment policy in terms of $I_{t+1}(I_t)$ and $z^*_t(I_t)$.

depend on the cost incurred by reserving capacity from the suppliers, i.e., the terms $v_t^h x_t^h$, which is determined at the beginning at the planning horizon. Thus, the fixed (reservation) cost associated with the contracts could be ignored. It remains to be established, however, the structure of the optimal contract that will minimize both replenishment and reservation cost. This problem is a design problem in which at the beginning of the planning horizon the buyer needs to determine how much capacity to reserve at different suppliers for future time periods.

In order to solve the selection problem, we first need to define formally what contracts the suppliers are offering the manufacturer. We denote by $\mathcal{X}$ the feasible set for capacities

$$ x = \left( x_t^h \right)_{t=1, \ldots, T} $$

that the manufacturer can select at period $t = 0$, for the entire planning horizon.

**Assumption 10** The feasible set for capacities, $\mathcal{X}$, is a convex set.

For instance, this assumption captures the case when $\mathcal{X}$ is linearly constrained.
We motivate this assumption by presenting some examples of industrial constraints that determine the feasible set of capacities. In all of them, the convexity assumption is satisfied.

**Example 6** Consider a single time period and $n$ options. The supplier offers the manufacturer no more that a certain amount $\kappa^i$ of type $i$ option for every $i = 1, \ldots, n$. Then,

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n_+ \mid x^i \leq \kappa^i, i = 1, \ldots, n \right\},$$

and thus $\mathcal{X}$ is convex.

**Example 7** Consider the single period case. Assume that the supplier offers one quantity-flexibility contract with down-ward flexibility $\alpha$ and up-ward flexibility $\beta$ (percentages) and unit cost $u$: if the manufacturer reserves $x \geq 0$ units in period $t = 0$, then it can order in period $t = 1$ an amount $q$ at a cost of:

$$r_{flex}(q) = \begin{cases} 
  u(1 - \alpha)x & \text{for } q \leq (1 - \alpha)x, \\
  uq & \text{for } (1 - \alpha)x \leq q \leq (1 + \beta)x, \\
  +\infty & \text{otherwise}.
\end{cases}$$

On the other hand, consider a portfolio contract made up of 2 options, the first one having terms $v^1 = u$, $w^1 = 0$ (fixed-commitment) and $x^1 = (1 - \alpha)x$, and the second $v^2 = 0$, $w^2 = u$ (pure option) and $x^2 = (\alpha + \beta)x$. The corresponding procurement cost is

$$r_{port}(q) = \begin{cases} 
  ux^1 & \text{for } q \leq x^1, \\
  ux^1 + u(q - x^1) & \text{for } x^1 \leq q \leq x^1 + x^2, \\
  +\infty & \text{otherwise}.
\end{cases}$$

We clearly see that any quantity-flexibility contract can be replicated by a portfolio made up of these two options with respective capacities, $x^1 \geq 0$ and $x^2 \geq 0$, being constrained by

$$\frac{x^1}{1 - \alpha} = \frac{x^2}{2\alpha}.$$ 

Thus, selecting the optimal level for the quantity-flexibility contract is equivalent to
finding the optimal portfolio of options 1 and 2 in the following convex feasible set

$$\mathcal{X} = \left\{ x \in \mathbb{R}_+^2 | \frac{x^1}{1 - \alpha} = \frac{x^2}{2\alpha} \right\}.$$ 

Hence, this transformation shows that including quantity-flexibility contracts in the portfolio framework preserves the convex structure of the problem.

**Example 8** In some industries, manufacturers have the policy to purchase a given total amount of capacity for certain groups of suppliers, such as local or minority-owned suppliers. Denote by $G$ the set of such suppliers and $\gamma$ their target share of business. The policy can then be modeled as the following constraint.

$$\sum_{i \in G} x^i \geq \gamma \sum_{i=1}^n x^i.$$ 

Such constraint is linear and hence

$$\mathcal{X} = \left\{ x \in \mathbb{R}_+^n | \sum_{i \in G} x^i \geq \gamma \sum_{i=1}^n x^i \right\}$$

is convex.

We now define the shape of this up-front cost, that is already counted in the profit-to-go function $V_t$.

**Assumption 11** Selecting capacities $x \in \mathcal{X}$ at period $t = 0$ has a reservation cost $v(x)$, where $v(\cdot)$ is a convex function.

For instance, when considering a portfolio of option contracts, as described in Section 3.1, $v(\cdot)$ is linear,

$$v(x) = \sum_{t=1}^T \sum_{i_t=1}^{n_t} v_{i_t} x_{i_t}.$$ 

Thus, we have defined here a more general reservation cost for which the results presented below still hold. In some sense, formulating this more general reservation cost function will allow us to identify modeling issues that one may find in real
applications, and help us improve the model. The next paragraph presents one of the possible issues arising in practice.

In fact, this assumption is not satisfied when volume discounts are available on the reservation cost. In this case, one can break the non-convex reservation cost into linear parts by introducing integral variables denoting the type of discount obtained. This makes the feasible set non-convex. Yet, the portfolio selection problem can be broken down into several problems, where, for each one of them, the feasible set is convex and the reservation cost \( v(\cdot) \) is linear. However, the number of such subproblems is unfortunately exponential with the number of discount levels.

Below we show that the problem of selecting the appropriate portfolio, which is solved at the beginning of the planning horizon, is a concave maximization problem.

**Theorem 2** Under Assumptions 10 and 11, given \( \Phi_t, \forall t = 1, \ldots, T + 1 \), \( V_t(I_t, \Phi_t, x) \) is concave in \((I_t, x)\).

The previous result directly implies the following.

**Theorem 3** Under Assumptions 10 and 11, at period \( t = 0 \) the problem of choosing the capacities, i.e.,

\[
\max_{x \in X} \left\{ E \Phi, V_1(0, \Phi_1, x) \right\},
\]

is a concave maximization problem.

This result is key from a practical standpoint. Indeed, an extensive class of methods from nonlinear programming can handle this type of concave maximization problems. Among those, gradient methods, Newton’s method or interior point methods, are very well known. It is worth pointing out that in certain cases, e.g., the single period model, we are able to derive closed-form solutions for the derivative of the objective function, in which case the optimization problem is greatly simplified.

Consider the single period model, i.e., \( T = 1 \). This special case is important since it captures the realities of industries where no inventories are kept. This happens for non-storable components such as electricity or perishable products, or also for
systems where inventory storage costs are so high that it is not economically viable to keep inventories.

Assume that the spot market offers unlimited supply, i.e., \( \kappa_1 = \infty \) with probability 1. Assume that the selling price is fixed at \( p \). Assume also that

\[
v(x) = \sum_{i=1}^{n} v^i x^i.
\]

We sort the available options by execution price, i.e., \( w^1 \leq \ldots \leq w^n \), and without loss of generality we assume that \( w^n \leq p \). We define \( v^{n+1} = 0 \) and \( w^{n+1} = p \) and for \( i = 1, \ldots, n \),

\[
y^i = \sum_{k=1}^{i} x^k.
\]

In what follows we call \( D \) the demand and \( S \) the spot market price. Assume that \( ED < \infty \), which implies that all expected profits must be finite. Finally, define

\[
J(y) = \mathbb{E}_{(D,S)} V_1(0, (D, S), x).
\]

**Theorem 4** With this notation, when \( J \) is differentiable, we have that for all \( i = 1, \ldots, n \)

\[
\frac{dJ}{dy^i} = - \left( v^i - v^{i+1} \right) + \mathbb{E} \left\{ 1_{y^i \leq D} \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \right\}.
\]

The theorem thus provides a closed form equation that characterizes the amounts of options to be purchased by the manufacturer in an optimal portfolio. In particular, a similar analysis allows to identify which options are most attractive to the manufacturer and which ones are dominated and should not be considered.

So far we have shown that the portfolio selection problem is a concave maximization problem. Unfortunately, while this result is numerically important, it does not provide insight about the optimal amounts of options to be purchased by the manufacturer, or insight about the options that the manufacturer should not even consider.
Thus, our objective here is to provide conditions and guidelines that help establish whether a specific option is attractive for the manufacturer, i.e. such an option should be part of the optimal portfolio. This is presented in the following propositions.

**Proposition 5** At time period $t$, assume that all suppliers are uncapatitated. Option $i_t$ should not be included in an optimal portfolio when

(i) there is an option $k_t$ such that $v_i^t > v_{k_t}^t$ and $v_i^t + w_{i_t}^t > v_{k_t}^t + w_{k_t}^t$;

(ii) there are options $j_t$ and $k_t$ such that $w_{i_t}^t < w_{j_t}^t < w_{k_t}^t$ and

$$\frac{w_{j_t}^t - w_{i_t}^t}{w_{k_t}^t - w_{j_t}^t} v_i^t + \frac{w_{k_t}^t - w_{j_t}^t}{w_{k_t}^t - w_{i_t}^t} v_{j_t}^t < v_{i_t}^t;$$

The proposition thus presents conditions under which a specific option contract can be ignored from an optimal portfolio. As can be seen in the proof of the proposition, this is done by identifying other option contracts that can replace this specific contract and yield a better expected profit.

So far we have identified option contracts that in some sense are dominated by other contracts. The next result characterizes option contracts that are dominated by the spot market.

**Proposition 6** Assume that $\kappa_t = \infty$ with probability 1. Then, at time period $t$, option $i_t$ should not be included in an optimal portfolio when

$$\mathbb{E}\left[ S_t - w_{i_t}^t \right]^{+} \leq v_{i_t}^t.$$

Of course, $a^+ = \max(0, a)$.

Figure 3-2 illustrates Proposition 5. The figure suggests that all the options that the manufacturer should consider have a reservation price small enough so that the option cannot be dominated by other options. If we were to plot all the option prices in a graph, the non-dominated options would appear in lower envelope of the points plotted.
Figure 3.2: Regions where options are dominated by other options, following the two cases of Proposition 5. The figure on the left illustrates case (i), the figure on the right case (ii).

3.4 Trade-offs between inventory and capacity

The previous sections provide insights on the value of the different options available to the manufacturer. In this section we present the effects of the portfolio strategy on the inventory target levels.

**Theorem 5** For every time period $t$, for every source $i$, $i = 1, \ldots, n_i$, the base-stock level $b_i^t$ is such that for every $t'$ and $k = 1, \ldots, n_{t'}$,

- for $t' < t$, $b_i^t$ is independent with respect to $x_{t'}^k$;
- for $t' = t$, $b_i^t$ is non-decreasing with respect to $x_{t'}^k$;
- for $t' > t$, $b_i^t$ is non-increasing with respect to $x_{t'}^k$.

The intuition behind the theorem is as follows. The first case is straightforward, since past capacities available at $t'$ should not have an impact on present target inventory levels at $t > t'$. The second case can be explained by observing that the higher the available capacities at a given period the smaller the execution cost at that period. Therefore, it is better to increase the current inventory target levels in order to take advantage of present lower purchasing costs. The third case is also intuitive,
it implies that the higher future capacities are at \( t' \), the lower present inventories should be at \( t < t' \). This is not only true because future purchases reduce the need to currently keep inventory, but also because safety stocks, that protect the manufacturer against situations when demand is higher than the available capacity, can be reduced.

3.5 Numerical results

3.5.1 Example solved

Recall the problem posed in Example 5. We start by solving the problem when there is no spot market, e.g., the spot market price is very high. In this case, Theorem 4 is sufficient to determine an optimal solution satisfying \( \hat{\nabla} J = 0 \).

\[
\begin{align*}
-v^2 + v^1 &= 4 = P\{y^1 \leq D\}(w^2 - w^1) = 6P\{y^1 \leq D\} \\
-v^3 + v^2 &= 3 = P\{y^2 \leq D\}(w^3 - w^2) = 6P\{y^2 \leq D\} \\
0 + v^3 &= 3 = P\{y^3 \leq D\}(p - w^3) = 8P\{y^3 \leq D\}
\end{align*}
\]

This implies that it is optimal to purchase \( x_{\text{NoSpot}}^1 = 871, x_{\text{NoSpot}}^2 = 129 \) and \( x_{\text{NoSpot}}^3 = 96 \).

The optimal strategy is different when the spot market becomes competitive, e.g., when the spot market unit price, \( S \), follows a uniform distribution in \([10, 20]\). In this case, it is optimal not to buy any options. That is, the new optimal solution is \( x_{\text{Spot}}^1 = 871 \) and \( x_{\text{Spot}}^2 = x_{\text{Spot}}^3 = 0 \).

This implies that in this example the benefit from buying options is undermined by the existence of a spot market. That is, it is better to leave supply uncommitted in order to take advantage of the spot market.

3.5.2 Sensitivity analysis

In this section, we present computational results that illustrate the sensitivity of the optimal expected profit and corresponding optimal portfolio in terms of the different parameters used in the model. We specifically focus on the impact of the length of
the planning horizon, the inventory holding cost, the volatility (standard deviation) of demand and spot market prices and the option reservation price, also known as premium.

The benchmark for the sensitivity analysis consists of a $T = 5$ period model, where $n_t = 2$ identical contracts are offered at every time period. These two contracts are a fixed-commitment contract (contract 1) of reservation price 40 and an option contract (contract 2) with premium equal to 20% of the execution price 40, i.e.,

$$v^1 = 40, \quad w^1 = 0,$$

$$v^2 = 8, \quad w^2 = 40.$$

The customer selling price is $p_t = 100$ for every time period $t = 1, \ldots, 5$. In addition, we assume that the demand and the spot market price are independent between them and across time periods, and follow truncated normal distributions with means $\mu_D = 1$ and $\mu_S = 80$, and standard variations $\sigma_D = 0.2$ and $\sigma_S = 20$ respectively. Finally, we assume that the inventory holding cost function is linear and has a unit holding cost equal to $h = 5$.

![Graph showing capacity levels over time for long-term contracts and option contracts.](image)

**Figure 3-3:** Optimal portfolio for the benchmark case
We observe, in Figure 3-3, that the optimal portfolio consists of a high capacity level purchased from the fixed-commitment contract for every period, and a small level of options. The figure suggests that options are used in increasing amounts as we approach the end of the horizon. A reason for this is that all unused inventory is lost after period 5, and therefore, it is riskier to hold inventory when we approach the end of the selling season. Before period 5, unused inventory can be carried to the next period at a reasonable cost $h = 5$, and as a consequence, flexibility is not needed in great amounts. Finally, we notice that the level of fixed commitment for earlier periods is higher than for later periods. This occurs because we start with no inventory, and thus the manufacturer needs to build some safety stock in the first period.

We compare this benchmark against situations where the parameters, e.g., the length of the planning horizon, inventory holding cost, demand and spot price volatility, option premium, vary. The results are presented in three figures. Figure 3-4 compares the performance of the optimal portfolio contract to that of relying only on the spot market, i.e., a strategy in which the manufacturer does not have a contract and thus commits no capacity in advance. Figures 3-5 and 3-6 depict the optimal levels of the fixed-commitment and option contracts as a function of the various parameters.

**Length of planning horizon.** We observe in Figure 3-4(a) that the longer the length of the planning horizon, the higher the expected profit per time period, both under the optimal portfolio or without contracting. This is because it is less risky to hold inventory for longer horizons. Eventually, the per-period profit reaches an upper limit, corresponding to the infinite horizon stationary profit.

**Inventory holding cost.** When the unit inventory cost is higher, the profits are smaller, both with and without contracts. This is of course intuitive. However, Figure 3-4(b) also suggests that the portfolio contract is less sensitive to a higher inventory holding cost. This is true since a portfolio strategy allows the manufacturer to rely on future purchases and thus reduces the need for inventory. On the other hand, when
Figure 3.4: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the per-period profit, without contract and with the optimal portfolio.
Figure 3-5: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal fixed-commitment capacity levels.
Figure 3-6: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal option capacity levels.
the manufacturer relies only on the spot market, it needs to build inventory when prices are attractive and as a result keep higher inventory levels. Finally, Figures 3-5(b) and 3-6(b) suggest that, when inventory holding cost is very high, the model can be decoupled into a series of single period models; the level of capacity reserved (fixed-commitment and option) in the case of high inventory holding cost is time independent and equal to the single period optimal level.

**Demand volatility.** In Figure 3-4(c), we observe that demand volatility, measured by the demand standard deviation, decreases profit, in particular for the portfolio contract case. This is indeed intuitive since, unlike spot purchasing, contracting implies commitments that can be quite expensive but will not necessarily be used. Also, Figures 3-5(c) and 3-6(c) suggest that the optimal capacity level is quite sensitive to demand volatility, i.e., the higher the demand uncertainty, the less fixed commitment the manufacturer will make and the higher the option level, exactly what one would expect from a portfolio strategy.

**Spot market volatility.** The impact of spot market volatility is different than the impact of demand volatility. Indeed, Figure 3-4(d) shows that the larger the spot price volatility, the larger the expected profits under both the portfolio contract or without any contract. This is true since high variability in spot prices implies opportunities for the manufacturer to stock at a lower price. The impact of spot market volatility is much higher on the no contract strategy. This is because contracting becomes riskier when spot price uncertainty increases, whereas no commitment allows the manufacturer to buy more often at a low spot price and stock for times when spot price is high. Finally, we notice in Figure 3-6(d) that for higher spot price volatilities, option capacity levels increase dramatically, in contrast to fixed-commitment levels, plotted in Figure 3-5(d). That is, the larger the uncertainty in the spot price, the smaller the level of fixed-commitment contracting.

**Option premium.** We notice that option premium, defined as the ratio of the reservation price and the execution price, does not affect the expected profits of the man-
ufacturer very much, as observed in Figure 3-4(e). However, while varying the option premium does not affect the manufacturer’s profit, it has an impact on the optimal portfolio strategy. Indeed, Figures 3-5(e) and 3-6(e) show that the higher the option premium, the higher fixed-commitment capacity levels and the lower option capacity levels.

A general observation in these experiments is that the optimal fixed-commitment capacity levels are decreasing with time. This is intuitive, since the closer we get to the end of the horizon, the larger the risk of being stuck with worthless inventory, and as a consequence, the manufacturer commits to smaller levels. This is illustrated in Figure 3-5(a). The figure suggests that at the beginning of the horizon, fixed-commitment capacity levels are higher in order to build some safety stocks; in the middle of the horizon, the capacity levels are nearly constant; and at the end of the horizon, they decrease abruptly. By contrast, Figure 3-6(a) suggests reverse dynamics for the option capacity levels. That is, the option level at the beginning of the horizon is relatively small; constant in the middle of the horizon; and much higher at the end of the horizon. This suggests that there is a substitution process between fixed-commitment and option levels. That is, the manufacturer prefers fixed-commitment contracts at the beginning of the horizon since inventory risk is not high. This is not the case at the end of the planning horizon, where high inventory risk forces the use of option contracts.

We also observe, in Figures 3-5(a) and 3-6(a), that the capacity levels in the middle of the horizon are time independent, both for fixed-commitment and option contracts. These capacity levels are probably close to those of the stationary levels associated with the infinite horizon model. Thus, for a reasonably long planning horizon, the problem can be broken into three components: a starting transient process in which capacity allows the manufacturer to build inventory; a stationary behavior; and an ending transient process where inventory is disposed at the end of the planning horizon.
3.5.3 Profit distribution

Our objective in this section is to obtain some insight into the distribution of the manufacturer profit provided by different types of contracts. In this paper, the objective is to maximize expected profit. Decisions based on this objective have an impact not only on the first moment of profit but also on the entire distribution of profit. We investigate here the effects of capacity decisions on the profit distribution.

We show here an instance with two time periods and independent demands $D_1$ and $D_2$, following truncated (non-negative) normal distributions with means $\mu_1 = 100$ and $\mu_2 = 200$ and standard deviations $\sigma_1 = 40$ and $\sigma_2 = 100$ respectively. The end-customer prices are $p_1 = p_2 = 15$.

We present the computation of the distribution of profit for three cases:

1. Optimal (for expected profit) fixed-commitment contract at unit price $v_1 = 9$, $v_2 = 7$ (option with no execution cost: $w^1 = w^2 = 0$).

2. Optimal option contract at execution unit price $w^1 = 9$, $w^2 = 7$ and up-front unit price $v^1 = v^2 = 2$.

3. Optimal portfolio of both.

We first selected the optimal contract in terms of expected profit by using a dynamic program. The optimal capacities were:

1. $x_1 \approx 120$ and $x_2 \approx 180$ for the fixed-commitment only contract, yielding an expected profit of 1,517.

2. $x_1 ^2 \approx 120$ and $x_2 ^2 \approx 260$ for the option only contract, yielding 1,281.

3. $x_1 ^2 \approx 100$, $x_2 ^2 \approx 140$, $x_1 ^2 \approx 20$ and $x_2 ^2 \approx 110$ for the portfolio contract, yielding 1,613.

Then we ran a Monte Carlo simulation to estimate the distribution of profit for each of the contracts.
Figure 3-7 depicts the simulation results. It shows that, as expected, the portfolio contract has the largest expected profit. On the other hand, the coefficient of variation of the profit obtained by the portfolio contract is smaller than that of the fixed-commitment contract but larger than that of the option contract. Computational studies with different data sets provide similar results: portfolio contracts dominate fixed-commitment contracts both from profit and risk points of view.

![Expected profit for different contracts](image)

Figure 3-7: Comparison of profit distribution for different contracts.

3.6 **Total-capacity-commitment contracts**

The model and results described in this section can be extended to include in the portfolio a *total-capacity-commitment* contract, as described in Section 2.1. The manufacturer purchases at the beginning of the planning horizon capacity to be used anytime during the planning horizon. Thus, at each time period, this special contract is a source of supply with a capacity equal to the initial capacity minus what has already
been used. The execution cost in this case might be dependent on the time period. Examples of this type of contracts can be found in Anupindi and Bassok [1] or in Bonser and Wu [8].

The analysis of a portfolio that includes such contracts is similar to what has been presented. We can examine the case in which one of the supply sources has a constraint on the total capacity available across time periods. Let \( I_t \) be the initial capacity available at the beginning of the planning horizon.

Of course, we need to add to the space state the amount of capacity left at the beginning of time period \( t \); we denote this amount by \( I_t \). Let \( u_t \) be the execution unit cost at period \( t \) for any amount \( q_t \) less than the remaining capacity, \( I_t \). Under these assumptions the dynamic program, Equation (2.5), is modified as follows:

\[
V_t(I_t, l_t, \Phi_t, \Phi_t) = p_t d_t - h_t(I_t) + \max_{z_t, q_t} \left\{ -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \right\}
\]

subject to

\[
\begin{align*}
& z_t \geq 0 \\
& q_t \geq 0 \\
& I_{t+1} = I_t - d_t + z_t + q_t \geq 0,
\end{align*}
\]

(3.2)

where \( U_{t+1} \) and \( C_t \) are defined similarly to Equations (2.4) and (3.1). The convexity results presented in Proposition 2 and Theorem 3 still hold by conducting the same induction arguments and using Lemma 1. Of course, the structure of the optimal replenishment policy is slightly different since it depends now not only on \( I_t \) but also on \( l_t \). For this purpose, observe that

\[
\max_{z_t, q_t} -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) = \max_{z_t} \left\{ -C_t(z_t) + \max_{q_t} \left\{ -u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \right\} \right\}
\]

It is straightforward to see that, given \( l_t \), the optimal control for \( z_t \) behaves as de-
scribed in Theorem 1 and illustrated in Figure 3-1. This is true since the function
\[
\max_{q_t} -u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t)
\]
is a concave function of \(I_t + z_t\). Note that in this case, the breakpoints characterizing
the inventory policy depend on \(l_t\). In this sense, the structure is preserved. Differently,
by using now that
\[
\max_{z_t,q_t} -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t)
= \max_{q_t} \left\{ -u_t q_t + \left[ \max_{z_t} -C_t(z_t) + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \right] \right\},
\]
we observe that given \(l_t\), the optimal control for \(q_t\) is not necessarily piecewise linear
in \(I_t\). Indeed, the reason for this is that using the remaining capacity has an influence
not only on future inventory position \(I_{t+1}\) but also on future capacity \(l_{t+1} = l_t - q_t\).
Chapter 4

Risk Modeling

4.1 Motivation

For most manufacturers, effective supply chain strategies require careful consideration of procurement decisions. These decisions need to take into account not only the many aspects of the manufacturing process, but also the balance between overstocking and shortages. Traditionally, the academic literature has modeled this trade-off by introducing, in particular, the newsvendor model. This tool has allowed manufacturers to quantify stocking and purchasing decisions.

As seen in previous chapters, this type of model can successfully determine multi-period multi-source procurement optimal strategies. One of the drawbacks of this approach is that it is based on the assumption that decision makers, the manufacturers, are risk neutral and hence optimize their expected profit. However, recent experiences, such as Cisco’s $2.5 billion inventory write-off in April 2001, suggest that risk matters, as illustrated in [4]. Thus, the challenge for the operations management community is to develop risk management models to complement current purchasing tools. This is especially important when, together with inventory and shortage risk, manufacturers face spot price risk. That is, by committing in advance to a given contract, manufacturers take the risk of not being able to take advantage of a low spot market price; similarly, when they do not secure enough supply in advance, they take the risk of paying a high spot market price.
This is especially obvious when one looks at the way the financial industry addresses similar issues. Since the emergence of the Capital Asset Pricing Model (CAPM), financial theory has developed models that quantify the trade-offs between risky returns with higher average returns and risk-free returns. Thus, the market is risk-averse in the sense that it requests larger average profits when these profits are uncertain. The first financial models in the 1960s were quickly transferred into industry, triggering the development of mutual funds. A company's returns risk could be evaluated through its "beta", i.e., its correlation with the market index variations. With these methods, investors could manage market risk while seeking high returns. This is in contrast with the way managers make purchasing decisions.

The objective of this section is to present a model where risk, measured by the variance of profit, is considered together with expected profit. We assume that a variety of supply options, including fixed-commitment contracts, are available to the manufacturer, with different reservation and execution unit prices, and these can be purchased in advance of knowing demand and spot market price. We define, in a single-period setting, the set of efficient mean-variance profit pairs, similarly to Markowitz [35], as a function of the amounts purchased for every option.

We emphasize the challenges of a newsvendor model compared to traditional financial model. Indeed, in finance, portfolio decisions can be reversed in the sense that investors can sell whatever asset they hold at any time; in manufacturing, however, once inventories are ordered, they are written off at high cost, i.e., sold at a low salvage value, since the components are somehow engineered for a particular manufacturer, see the Cisco case alluded to before.

Our analysis of the trade-off between profit mean and variance highlight two special portfolios: a maximum expected profit portfolio, obtained by the classical risk-neutral newsvendor model, and a minimum profit variance portfolio. The characterization of this second portfolio is new, and is similar to the minimum variance portfolio in a financial portfolio problem without risk-free assets. Between these two "extreme" portfolios, we find a set of mean-variance efficient portfolios, for which we give bounds. The main difference between our model and its financial formulation
is that in the later model the efficient frontier is typically unbounded, while in our newsvendor model we show that the efficient frontier is bounded. In addition, we characterize the upper-level sets of mean-variance utility functions and prove that they are connected. Hence, a greedy method will find the portfolios on the efficient frontier.

The literature on risk in supply contracts is quite limited. An exception is the work of Lau [33], who proposes alternative optimization objectives for the newsboy problem, and in particular the objective of maximizing the probability of achieving a given level of profit. For this purpose, Lau presents formulas for the moments of the profit for a general demand distribution. Also, Chen and Federgruen [17] study the variance of profit in a simple single-period newsvendor setting, where no portfolios or spot market are involved. In their model, they show that the variance is simply increasing with the capacity ordered. In our more complex model, this is not so.

The most common approach to deal with risk in an industrial setting has come from the financial world. The objective in this stream of literature is to analyze a stochastic inventory model together with trading in the market, and to hedge the inventory project with a financial operation. The single period problem has been studied by Anvari [2], followed by Chung [18]. They apply the financial CAPM theory to provide a market valuation of an inventory project, and derive an optimal ordering quantity provided that the market index and the demand follow a bivariate normal distribution. In a more general setting, a lot of research has been done on the so-called mean-variance hedging problem. Duffie and Richardson [20], Schweizer [47], Gouriéroux, Laurent and Pham [26], for instance, have analyzed the problem of dynamically hedging some asset with the available assets in the market. The objective is to minimize the final deviation between the asset and the hedge with respect to some stochastic metric. Caldentey and Haugh [13] have applied this approach to the newsvendor problem, thus generating a dynamic trading strategy in the financial markets together with a single-period inventory decision. We should point out that this research does not consider the intrinsic risk of the inventory project, and uses the correlation between customer demand and the market returns to reduce risk.
In contrast, the purpose of this section is to describe the intrinsic risk of an inventory decision, as a function of inventory and component price risk, without the risk associated with financial markets.

4.2 The mean-variance problem

Consider a firm that sells a product at a predetermined price $p$. To manufacture the product the firm uses a component that can be found in a spot market at a spot price $S$, and production only takes place when $p > S$. The firm’s profit is thus

$$\Pi^0 = (p - S)^+ D,$$

and it depends on a stochastic demand, $D$, and a stochastic spot price, $S$. Notice that this notation is the same as the one used in Assumption 8.

Assume now that this firm is able to sign contracts in advance with suppliers. This enables the firm to become less sensitive to spot price fluctuations. For example, buying supply through an forward contract can reduce the exposure of the firm to the spot price.

We assume that $n$ suppliers are available, each offering a supply option contract for a reservation fee of $v^i$ per unit and an execution fee of $w^i$ per unit, $i = 1, \ldots, n$.

The following two sections illustrate two different modeling approaches of the problem: the first one is the one used in financial markets; the second one uses the framework presented in Chapters 2 and 3 to determine the profit of the firm.

4.2.1 The traditional financial model

Traditionally, in finance, options have been used as a side-bet with no real consequences: they are mechanisms that arrange side-payments as a function of the spot price. That is, in this setting the firm is using the contracts not only to provide raw-material for production, but also to sell back excess capacity to the spot market.

Under this framework, the profit of the buyer, denoted by the superscript $F$ (for
financial), can be written as

$$\Pi^F = (p - S)^+ D + \sum_{i=1}^{n} \left\{(S - w^i)^+ - v^i\right\} x^i,$$  \hspace{1cm} (4.1)

where $x^i \geq 0$ is the amount of options purchased from supplier $i$, $i = 1, \ldots, n$. These options can therefore be treated like assets with returns $(S - w^i)^+ - v^i$ but no up-front cost. This setting is equivalent to the portfolio optimization problem, where $x^i$ is the amount of stock $i$ purchased and $(S - w^i)^+ - v^i$ is the return of the stock.

In Markowitz’s work, see [35], a mean-variance optimal trade-off is found, the so-called efficient frontier. One can similarly determine an optimal mean-variance frontier in our case. Specifically,

$$\mathbb{E}\Pi^F = \mathbb{E}[(p - S)^+ D] + \sum_{i=1}^{n} \mathbb{E}[(S - w^i)^+ - v^i] x^i,$$  \hspace{1cm} (4.2)

and

$$Var\Pi^F = \begin{cases} Var[(p - S)^+ D] \\ + \sum_{i=1}^{n} Var[(S - w^i)^+ - v^i] (x^i)^2 \\ +2 \sum_{i=1}^{n} Covar[(S - w^i)^+ - v^i, (p - S)^+ D] x^i \\ +2 \sum_{i<j} Covar[(S - w^i)^+ - v^i, (S - w^j)^+ - v^j] x^i x^j. \end{cases}$$  \hspace{1cm} (4.3)

The mean-variance optimal trade-off curve is defined by the points $(Var\Pi, \mathbb{E}\Pi)$ found in the optimization programs

$$\min_{x \geq 0} \ Var\Pi \ subject \ to \ \mathbb{E}\Pi \geq \mu,$$  \hspace{1cm} (4.4)

for $\mu \in \mathbb{R}$, or equivalently

$$\max_{x \geq 0} \ \mathbb{E}\Pi \ subject \ to \ Var\Pi \leq \sigma^2,$$  \hspace{1cm} (4.5)
for $\sigma^2 \in \mathbb{R}_+$, or also,

$$\max_{\lambda \geq 0} E\Pi - \lambda Var\Pi,$$

for $\lambda \in \mathbb{R}_+$.

This type of objective has been used extensively in the literature, as a standard for risk-profit trade-offs. Interestingly, this objective function is equivalent to a utility maximization problem, following the seminal work of Von Neumann and Morgenstern [43], under special conditions. Namely, the mean-variance objective falls into the utility framework in two cases:

- The utility function is quadratic, i.e., $U(\Pi) = a\Pi - b\Pi^2$, for $a, b \geq 0$.

- The utility function is CARA (constant absolute risk aversion), i.e., $U(\Pi) = -\frac{1}{\alpha} e^{-\alpha\Pi}$, while the profits are normally distributed. The condition on the profit distribution is reasonable in financial markets, but not, as we show, in our model.

Figure 4-1 shows the trade-off curve (efficient frontier) between expected profit and variance of profit for the model discussed in this section. In this setting, we do not have a risk-free asset, which is often used in the financial literature. The "mutual fund" theorem, essential to the CAPM, described in Sharpe [48], Mossin [41] or Merton [39], does not hold, since there is no riskless bond; there is no market price for systematic risk. Instead, we can work with the efficient frontier, and have the manufacturer (and every other buyer in this market) choose its own efficient portfolio.

### 4.2.2 A newsvendor model

Of course, the financial model described above does not capture some of the constraints associated with real world purchasing practices. These constraints are either contractual, i.e., the supplier does not allow the buyer to resell the component, or operational, i.e., design constraints that are typically associated with product specification requiring specializing the component to the buyer's needs. This type of
constraint implies that it is difficult or costly for the buyer to resell the components back to the market.

To capture these issues, we adopt the model presented in previous chapters. The manufacturer can not sell back excess supply to the spot market, and thus we restrict the use of capacity to serving the demand faced by the manufacturer. Hence, unused capacity is lost, in the sense that the manufacturer is unable to exercise the options and sell the corresponding supply units to the spot market, at the spot price. We thus rule out the possibility of the manufacturer becoming a trader and gaining financial advantage from speculation.

We adopt the same mean-variance approach as presented in Equations (4.4), (4.5) or (4.6). This makes sense when the manufacturer is not risk neutral, and prefers more certainty in profits to less. The risk aversion can be captured by penalizing high variance while rewarding high expected profit. We should however point out that, in our case, the mean-variance problem only falls into the Von Neumann-Morgenstern utility framework when the considered utility is quadratic. This is not so when the
utility is exponential (CARA), since the capacity decisions make the distribution of profit not normal, as we will see, even when the customer demand is normally distributed.

Assuming without loss of generality that \( w^1 \leq \ldots \leq w^n \), and letting \( w^{n+1} = p, v^{n+1} = 0 \), we define for \( i = 1, \ldots, n \),

\[
y^i = \sum_{j=1}^{i} x^j,
\]

and

\[
Z_i = \left\{ \min(S, w^{i+1}) - \min(S, w^i) \right\}, \quad Z_0 = (p - S)^+ = p - \min(S, p).
\]

The manufacturer's profit, denoted now with the superscript \( N \) (for newsvendor), can be written as

\[
\Pi^N = p \min(D, y^n) + (p - S)^+ (D - y^n)^+ - \sum_{i=1}^{n} v^i x^i - \sum_{i=1}^{n} \min(S, w^i) \min \{ x^i, (D - y^{i-1})^+ \}.
\]

Using the relationship \( (x - y)^+ = x - \min(x, y) \) for any \( x, y \), we can reformulate

\[
p \min(D, y^n) + (p - S)^+ (D - y^n)^+ = p \min(D, y^n) + \{ p - \min(S, p) \} \{ D - \min(D, y^n) \}
\]

\[
= Z_0 D + \min(S, p) \min(D, y^n),
\]

and

\[
\min \{ x^i, (D - y^{i-1})^+ \} = \min \{ y^i - y^{i-1}, (D - y^{i-1})^+ \}
\]

\[
= \min(D, y^i) - \min(D, y^{i-1}).
\]

We can plug these expressions into \( \Pi^N \) and observe that \( \min(D, y^n) = \sum_{i=1}^{n} \{ \min(D, y^i) - \min(D, y^{i-1}) \} \).
\( \min(D, y^{i-1}) \). We thus obtain

\[
\Pi^N = Z_0 D + \min(S, p) \min(D, y^n) - \sum_{i=1}^{n} v^i x^i \\
- \sum_{i=1}^{n} \min(S, w^i) \{ \min(D, y^i) - \min(D, y^{i-1}) \} \\
= Z_0 D - \sum_{i=1}^{n} v^i (y^i - y^{i-1}) \\
+ \sum_{i=1}^{n} \{ \min(S, p) - \min(S, w^i) \} \{ \min(D, y^i) - \min(D, y^{i-1}) \} \\
= Z_0 D + \sum_{i=1}^{n} (v^{i+1} - v^i) y^i \\
+ \sum_{i=1}^{n} Z_i \min(D, y^i). \tag{4.7}
\]

This result has actually already been used, in Theorem 4, and the derivation is contained in the appendix.

Of course, a similar transformation shows that

\[
\Pi^F = Z_0 D + (S - p)^+ y^n + \sum_{i=1}^{n} (v^{i+1} - v^i) y^i + \sum_{i=1}^{n} Z_i y^i.
\]

The difference between the two profit functions stems from the assumption that in the newsvendor model the buyer is not able to sell excess supply to the spot market.

### 4.2.3 Properties of supply option portfolios

To proceed with the mean-variance analysis of a portfolio, we make the following assumption.

**Assumption 12** The customer demand \( D \) is independent of the spot market price \( S \).

The assumption thus implies that demand faced by the buyer does not drive spot market prices. This is typically the case when the buyer’s product does not capture
a large portion of the component’s demand. This is a strong assumption, and we provide a discussion on the general case in Section 4.4.

**Assumption 13** \( D \) and \( S \) follow respectively continuous distributions with p.d.f. \( f_D > 0 \) and \( f_S \) (on \( \mathbb{R}_+ \)) that are continuous and c.d.f. \( F_D \) and \( F_S \).

In addition, we define \( \overline{F}_D = 1 - F_D \) and \( \overline{F}_S = 1 - F_S \).

The analysis of mean and variance of profit is similar to Equations (4.2) and (4.3):

\[
\text{EII}^N = \mathbb{E}[Z_0D] + \sum_{i=1}^{n} \left\{ \mathbb{E}[Z_i] \int_{0}^{y^i} \overline{F}_D(q)dq - (v^{i+1} - v^i)y^i \right\}, \quad (4.8)
\]

and

\[
\text{VarII}^N = \begin{cases} 
\text{Var} [Z_0D] + \sum_{i=1}^{n} \text{Var} [Z_i \min(D, y^i)] \\
+ 2 \sum_{i=1}^{n} \text{Covar} [Z_i \min(D, y^i), Z_0D] \\
+ 2 \sum_{i<j} \text{Covar} [Z_i \min(D, y^i), Z_j \min(D, y^j)].
\end{cases} \quad (4.9)
\]

As derived in Chapter 3, the expected profit of the buyer is a concave function of the variables \((x^1, \ldots, x^n)\), or equivalently of a linear transformation of them, \((y^1, \ldots, y^n)\). In particular, using the independence of \( D \) and \( S \),

\[
\frac{d\text{EII}^N}{dy^i} = v^{i+1} - v^i + \mathbb{E}[Z_i]\overline{F}_D(y^i). \quad (4.10)
\]

To understand the behavior of variance, define

\[
A_i = \frac{1}{2} \text{Var} [Z_i \min(D, y^i)], \\
B_i = \text{Covar} [Z_i \min(D, y^i), Z_0D], \\
C_{ij} = \text{Covar} [Z_i \min(D, y^i), Z_j \min(D, y^j)]. \quad (4.11)
\]

Thus, the variance can thus be expressed as

\[
\text{Var} [Z_0D] + 2 \sum_{i=1}^{n} (A_i + B_i) + 2 \sum_{i<j} C_{ij},
\]

74
and hence,
\[
\frac{1}{2} \frac{d\text{Var}W^N}{dy^i} = \frac{dA_i}{dy^i} + \frac{dB_i}{dy^i} + \sum_{j<i} \frac{dC_{ji}}{dy^i} + \sum_{j>i} \frac{dC_{ij}}{dy^i}. 
\]

(4.12)

The formulas for such expressions are presented below.

\[A_i(y) = \frac{1}{2} \mathbb{E}[Z_i^2] \mathbb{E}[\min(D, y^i)^2] - \frac{1}{2} \mathbb{E}[Z_i^2] \mathbb{E}[\min(D, y^i)^2] \]
\[\frac{dA_i}{dy^i} = \mathbb{E}[Z_i^2] y' F_D(y^i) - \mathbb{E}[Z_i^2] \mathbb{E}[\min(D, y^i)] F_D(y^i) \]
\[\frac{dA_i}{dy^j} = 0 \text{ for } j \neq i \]

\[B_i(y) = \mathbb{E}[Z_0 Z_i \mathbb{E}[D \min(D, y^i)] - \mathbb{E}[Z_0] \mathbb{E}[Z_i] \mathbb{E}[D] \mathbb{E}[\min(D, y^i)] \]
\[\frac{dB_i}{dy^i} = \mathbb{E}[Z_0 Z_i] (y' F_D(y^i) + \int_{y^i}^{\infty} F_D(u) du) - \mathbb{E}[Z_0] \mathbb{E}[Z_i] \mathbb{E}[D] F_D(y^i) \]
\[\frac{dB_i}{dy^j} = 0 \text{ for } j \neq i \]

\[C_{ij}(y) = \mathbb{E}[Z_i Z_j \mathbb{E}[\min(D, y^i) \min(D, y^j)] - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \mathbb{E}[\min(D, y^i)] \mathbb{E}[\min(D, y^j)] \]
\[\frac{dC_{ij}}{dy^i} = \mathbb{E}[Z_i Z_j] (y' F_D(y^i) + \int_{y^i}^{y^j} F_D(u) du) - \mathbb{E}[Z_i] \mathbb{E}[Z_j] F_D(y^i) \int_{0}^{y^i} F_D(u) du \]
\[\frac{dC_{ij}}{dy^j} = \mathbb{E}[Z_i Z_j] y' F_D(y^j) - \mathbb{E}[Z_i] \mathbb{E}[Z_j] F_D(y^j) \int_{0}^{y^j} F_D(u) du \]

In summary,

\[\frac{dA_i}{dy^i} = F_D(y^i) \left( \mathbb{E}[Z_i^2] y^i - \mathbb{E}[Z_i]^2 \right) \int_{y^i}^{\infty} F_D(u) du \]
\[\frac{dB_i}{dy^i} = F_D(y^i) \left( \mathbb{E}[Z_0 Z_i] (y^i + \int_{0}^{\infty} \frac{F_D(u) du}{F_D(y^i)}) - \mathbb{E}[Z_0] \mathbb{E}[Z_i] \mathbb{E}[D] \right) \]
\[\frac{dC_{ij}}{dy^i} = F_D(y^i) \left( \mathbb{E}[Z_i Z_j] (y^i + \int_{y^i}^{\infty} \frac{F_D(u) du}{F_D(y^i)}) - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \int_{0}^{y^i} F_D(u) du \right) \]
\[\frac{dC_{ij}}{dy^j} = F_D(y^j) \left( \mathbb{E}[Z_i Z_j] y^j - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \int_{0}^{y^j} F_D(u) du \right) \]

(4.13)

We will start the analysis with single supplier case. There, we define the quantity \( y_E \) as the portfolio maximizing expected profit, obtained by solving the equation
\[ F_D(y_E) = \frac{v}{\mathbb{E}[(S - w)^+]}. \]  

(4.14)

Observe that \( y_E \) is well defined since \( F \) is decreasing. Of course, when \( v = 0 \), we select \( y_E = \infty \), and when \( v \geq \mathbb{E}[(S - w)^+] \), we choose \( y_E = 0 \).

Similarly, \( y_V \) is defined as the portfolio minimizing profit variance. Using Equation (4.13), we observe that the sign of the variance is the same as the right-hand side of the following equation. Thus, \( y_V \) is determined by solving

\[
0 = \mathbb{E}[(S - w)^+ y_V - \mathbb{E}[(S - w)^+]^2 \int_0^{y_V} F_D(u)du \\
+ \mathbb{E}[(p - S)^+(S - w)^+] y_V \int_0^{y_V} F_D(u)du \left( \frac{\int_0^{y_V} F_D(u)du}{F_D(y_V)} \right) \\
- \mathbb{E}[(p - S)^+] \mathbb{E}[(S - w)^+] \mathbb{E}[D].
\]  

(4.15)

The right-hand side of Equation (4.15) is increasing: its derivative is

\[
\mathbb{E}[(S - w)^+ f_D(y_V)] + \mathbb{E}[(p - S)^+(S - w)^+] \int_0^{y_V} F_D(u)du \left( \frac{\int_0^{y_V} F_D(u)du}{F_D(y_V)} \right)^2 \\
> 0.
\]

In addition, we notice that the right-hand side of Equation (4.15) tends to infinity when \( y \to \infty \) and is equal to \( \mathbb{E}[(p - S)^+(S - w)^+] \mathbb{E}[D] - \mathbb{E}[(p - S)^+] \mathbb{E}[(S - w)^+] \mathbb{E}[D] \leq 0 \)
when \( y = 0 \), since \( \text{Covar}[(p - S)^+, (S - w)^+] \leq 0 \). This implies that \( y_V \) is well-defined. Moreover, this shows that the variance is quasi-convex as a function of \( y \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( &lt; y_E )</th>
<th>( \geq y_E )</th>
<th>( &lt; y_V )</th>
<th>( \geq y_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Profit</td>
<td>( \uparrow )</td>
<td>( \downarrow )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance of Profit</td>
<td></td>
<td></td>
<td>( \downarrow )</td>
<td>( \uparrow )</td>
</tr>
</tbody>
</table>

This implies the following theorem.

**Theorem 6** Consider a buyer procuring \( y \) from a single supplier that faces the problem defined by
\[
\max_{y \geq 0} \mathbb{E}\Pi - \lambda \text{Var}\Pi,
\]
for some \( \lambda \in \mathbb{R}_+ \). Then it always purchases an amount \( y^* \) in the interval \([y_E, y_V]\) (when \( y_E \leq y_V \)) or \([y_V, y_E]\) (when \( y_E \geq y_V \)).

This theorem is illustrated by the crossed curve in Figure 4-2. We observe that in this case \( y_V < y_E \). As \( y \) increases, starting from 0, variance decreases and expectation increases at first; then, when \( y > y_V \), variance starts to increase while expectation keeps growing; finally, when \( y > y_E \), variance still increases (but reaches a limit though) as expectation starts to decrease.

When compared to the benchmark, i.e., the financial model, we observe that the expected profit is always smaller for the same level of risk. Also, as the amount \( y \) bought from the supplier increases, variance is bounded from above in the newsvendor model but grows unbounded for the financial version. In other words, using the financial model in a situation where the newsvendor model is more appropriate may lead to a major underestimation of profit risk, for a given level of expected profit.

When multiple suppliers are available, the expected profit behaves in a similar way to the single supplier case. Indeed, in Theorem 3, we have shown the following result. Since here \( f_D > 0 \), we have strict concavity.

**Theorem 7** The expected profit is a strictly concave function of the amounts purchased from each contract. It attains a unique maximum for a portfolio \( y_E \).

Unfortunately, the variance of profit behaves in a more complicated way. The following proposition provides some information about the first and second moments of the variance.

**Proposition 7** For each \( i, i = 1, \ldots, n \), we have that

\[
\frac{d\text{Var}\Pi^N}{dy^i} = F_D(y^i)\Phi_i(y)
\]
Figure 4-2: Comparison of the mean-variance curves for the different models: financial (circled line) and newsvendor (crossed line).

for some functions \( \Phi_t \). Moreover, we have that

\[
H = \left( \frac{d^2 \text{Var} II^N}{dy^j dy^i} \right)_{i,j} \gamma \begin{pmatrix} \ddots & 0 \\ -f_D(y^i) \Phi_t(y) \\ 0 & \ddots \end{pmatrix},
\]

where \( A > B \) means that the matrix \( A - B \) is definite positive.

We use this result to characterize the behavior of the profit variance. Let

\[
F = \{ y | 0 \leq y^1 \leq \ldots \leq y^n \}. \tag{4.16}
\]

Define also the following notation.

**Definition 6** Consider a twice-differentiable function \( f : F \to \mathbb{R} \), where \( F \) is defined
in Equation (4.16). For any $I \subset \{1, \ldots, n\}$, let

$$A_I = F \cap \{y \mid y^i = y^{i-1} \text{ for } i \notin I\}.$$  \hspace{1cm} (4.17)

We can write $I = \{i_1, \ldots, i_m\}$, and thus define, for $0 \leq z_1 \leq \ldots \leq z_m$,

$$y^i(z) = z_j \text{ for every } i \text{ such that } i_j \leq i < i_{j+1}, \text{ for } j = 0, \ldots, m - 1.$$

Let

$$g(z_1, \ldots, z_m) = f\left(y(z)\right).$$

Let $y \in A_I$. $y$ is a $I$-unconstrained critical point of $f$ if and only if for $j = 1, \ldots, m$,

$$\frac{dg}{dz_j}\left(y(z)\right) = 0.$$

Intuitively, an $I$-unconstrained critical point for a function $f$ is a portfolio $y$ such that the function $f$, restricted to $A_I$, has a critical point at $y$.

**Definition 7** Given a function $f : F \rightarrow \mathbb{R}$, the lower-level set at $c$ is

$$\{y \in F \mid f(y) \leq c\},$$

and the upper-level set at $c$ is

$$\{y \in F \mid f(y) \geq c\}.$$

**Proposition 8** For some $I$, let $y \in A_I$ and assume that $y$ is an $I$-unconstrained critical point of the variance. Then, portfolio $y$ is a strict local minimizer of the variance in the set $A_I$.

The previous result implies the following important qualitative result.

**Proposition 9** The lower-level sets of the variance of a portfolio are connected. That is, for any $c$, for any $y_0, y_1$, if $\text{Var}^N(y_0) \leq c$ and $\text{Var}^N(y_1) \leq c$, then there
is a continuous path $y(\cdot)$ such that $y(0) = y_0$, $y(1) = y_1$ and for all $t \in [0, 1]$, $\text{Var} \Pi^N(y(t)) \leq c$.

We provide in the appendix a proof tailored for this specific problem. However, using the mountain-pass lemma, see Struwe [51] pp.74 and following, one can show that whenever a sufficiently well-behaved function is such that critical points are all strict local minima (resp. maxima), then this function has connected lower-level sets (resp. upper-level sets).

As a result of the proposition, we directly obtain the following theorem, that describes the structure of the variance.

**Theorem 8** Any portfolio that minimizes the variance locally is a global minimizer. Moreover, such a portfolio, $y_\mathbf{v}$, is unique. Thus, a greedy search method will lead to the global minimum, $y_\mathbf{v}$.

An interesting by-product of the proof of Proposition 9 is the existence of $M < \infty$ such that the global minimum belongs to a bounded "box" $B$,

$$B = \{y \mid 0 \leq y^1 \leq \ldots \leq y^n \leq M\}.$$ 

We construct this bound $M$ as follows. For every $k = 1, \ldots, n$, the expression

$$\mathbb{E}[(Z_k + \ldots + Z_n)^2|y^{(k)}] - \mathbb{E}[Z_k + \ldots + Z_n]^2 \int_0^{y^{(k)}} F_D(u)du$$

$$+ \mathbb{E}[Z_0(Z_k + \ldots + Z_n)]|y^{(k)} + \int_0^{y^{(k)}} \frac{F_D(u)du}{F_D(y^{(k)})}) - \mathbb{E}[Z_0]\mathbb{E}[Z_k + \ldots + Z_n]\mathbb{E}[D]$$

is strictly increasing and tends to $+\infty$ when $y^{(k)} \to +\infty$. Thus, there is $M_k > 0$ such that the expression is non-negative for $y^{(k)} > M_k$. We then define $M = \max\{M_1, \ldots, M_n\}$.

In general, the lower-level sets of variance are not convex. Figure 4-3 shows this observation for the following data. We consider two different options, with reservation
and execution fees equal to $v^1 = 5$, $w^1 = 0$ and $v^2 = 1$, $w^2 = 6$ respectively. Customer price is $p = 10$ and spot market price follows a truncated normal distribution with mean 8 and standard deviation 1. Finally demand follows a truncated normal distribution with mean 60 and standard deviation 20.

![Variance of profit](image)

**Figure 4-3:** The lower-level sets of variance are not always convex

The non-convexity of the lower-level sets raises the following challenge. How does the variance behave when the feasible portfolio set is not $F$, but a smaller set? In particular, what happens when the feasible portfolio $y$ is constrained within a line, i.e., there are $y_0$ and $\Delta y$ such that all feasible portfolios can be expressed as $y = y_0 + t\Delta y$ for some $t$?

This situation may arise when the manufacturer owns a portfolio $(y^1, \ldots, y^{i-1}, y^{i-1}, y^{i+1}, \ldots, y^n)$ already and is approached by a new supplier, which offers a new contract $i$, with parameters $(w^i, v^i)$. Notice that at this point $x^i = 0$. Increasing $x^i$ to $x^i + t$ implies changing $(y^i, \ldots, y^n)$ to $(y^i + t, \ldots, y^n + t)$. Of course, in this instance, we have $\Delta y^j = 1$ for $j \geq i$ and 0 otherwise.
Using Proposition 7, we have that

\[ \frac{d\text{Var}\Pi^N}{dt} = \sum_{i=1}^{n} \Delta y^i \bar{F}_D(y^i) \Phi_i(y) \]

and

\[ \frac{d^2\text{Var}\Pi^N}{dt^2} \geq - \sum_{i=1}^{n} (\Delta y^i)^2 f_D(y^i) \Phi_i(y). \]

This inequality does not, in general, allow us to characterize the structure of \( \text{Var}\Pi^N(t) \). However, in specific cases, we can show that this function is quasi-convex. This happens when the following two conditions are satisfied.

1. The demand is exponentially distributed, i.e., there is \( \mu > 0 \) such that \( \bar{F}_D(y) = e^{-\mu y} \), and hence \( f_D(y) = \mu \bar{F}_D(y) \).

2. \( \Delta y^i = 0,1 \) for all \( i = 1, \ldots, n \).

Indeed, observe that \( \sum_{i=1}^{n} \Delta y^i \bar{F}(y^i) \Phi_i(y) = 0 \) implies that

\[ \frac{d^2\text{Var}\Pi^N}{dt^2} \geq - \sum_{i=1}^{n} (\Delta y^i)^2 f(y^i) \Phi_i(y) = -\mu \sum_{i=1}^{n} \Delta y^i \bar{F}_D(y^i) \Phi_i(y) = 0. \]

Thus, the variance is a quasi-convex function of \( t \) in this case.

The situation depicted above can be easily treated using the quasi-convexity of the variance. In other words, the problem of adding a contract \( i \) to an existing portfolio is as easy as the single supplier case. This is true since here \( \Delta y^j = 1 \) for \( j \geq i \) and 0 otherwise.

We can now turn to solving the problem posed by Equation (4.6), finding the mean-variance trade-offs where more expectation of profit is preferred to less, and less variance is preferred to more. The efficient frontier, in terms of average profit and profit variance, will clearly be defined between the minimum variance portfolio \( y_V \), defined in Theorem 8, and the maximum expectation portfolio \( y_E \), defined in Theorem 7. Since expectation and variance of profit are continuous, we can find a continuum of mean-variance pairs in the efficient frontier, between these two portfolios, i.e., between
\( y_E \) and \( y_V \). These belong in the bounded set

\[
E = \{ y \mid 0 \leq y^1 \leq \cdots \leq y^n \leq \bar{y} \},
\]

where \( \bar{y} = \max(M, y_{E_n}) \).

The next proposition summarizes the results regarding this problem.

**Theorem 9**  For \( \lambda \geq 0 \), let the manufacturer’s utility function be

\[
U(y) = E\Pi^N - \lambda \text{Var}\Pi^N.
\]

Then, for each \( i, i = 1, \ldots, n \), we have that

\[
\frac{dU}{dy^i} = -a_i + F_D(y^i) \Psi_i(y)
\]

for some functions \( \Psi_i \) and the scalars \( a_i = v^i - v^{i+1} \). Moreover, we have that

\[
H = \begin{pmatrix} \frac{d^2U}{dy^i dy^j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{d^2U}{dy^j dy^n} \end{pmatrix} = \begin{pmatrix} \cdots \\ -f_D(y^i) \Psi_i(y) \\ 0 \end{pmatrix}.
\]

When \( v^1 \geq \ldots \geq v^n \geq 0 \), the utility upper-level sets are connected and the utility has a unique local maximum \( y^* \). A greedy algorithm will lead to the global maximizer of the utility function, \( y^* \).

This first part of the theorem is similar to Proposition 7, and the final part is derived using the same ideas of Proposition 9 and Theorem 8.

Observe that, in the theorem, we assumed that \( v^1 \geq \ldots \geq v^n \geq 0 \). This is not a restrictive assumption, given that \( 0 \leq w^1 \leq \ldots \leq w^n \). Indeed, an option that has a lower execution price should have a higher reservation price. Otherwise, we would be able to identify some option that is dominated by some other, i.e., find \( i, j \) such that \( v^i \leq v^j \) and \( w^i \leq w^j \). We could then eliminate option \( j \) from the pool of ”acceptable” contracts, because it is too expensive.
We must point out that even though the mean-variance frontier is continuous, the corresponding efficient portfolios might not change continuously. Since the variance level sets are connected but not convex, as observed in Figure 4-3, we might find discontinuous jumps in the efficient portfolios.

4.3 Discussion

So far we have discussed a single period model where a manufacturer chooses a portfolio of options based on a profit mean-variance trade-off. As seen in Chapter 3, when the manufacturer maximizes its expected profit, it is possible to extend the analysis to a multi-period case. However, as we will discuss next, we have encountered important difficulties in extending the analysis for mean-variance objectives.

The single period model is nevertheless applicable in itself, in a multi-period environment. This can typically be done when the component is perishable or non-storable. In this case, since inventory is not transferred from one period to the next, one can apply this model for every period independently. For example, the model is applicable for fashion items or other products with short life-cycle, e.g., laser printers.

In the financial literature, the CAPM has been extended to a multi-period setting by Merton [40] and others.

However, extending mean-variance objectives to a multi-period environment has encountered significant problems. For instance, recently, Li and Ng [34] have formulated, in a dynamic programming framework, the problem of maximizing expected return of investment under variance constraints. They provide through a clever relaxation of the problem a way to keep track of the initial variance constraint as we go by in the dynamic program. Unfortunately, the model formulation is not completely intuitive. Specifically, it assumes that once the problem is posed at the beginning of the horizon, the investor, as it moves forward in a scenario tree, keeps record of the variance constraint posed at the root of this tree. This implies that when it moves to a branch with more wealth than average, it will consciously choose less return in order to reduce the variance of returns across all branches of the tree. This point
makes such formulation inappropriate for real situations.

A multi-period extension of the present newsvendor model has some additional difficulties. Indeed, when dealing with financial models, the portfolio holder can liquidate its portfolio at the beginning of every time period and reinvest the corresponding cash into a brand new portfolio. However, this is not possible in the model analyzed in this paper since we focus on components that cannot be sold back to the market since they are tailored for the buyer or due to contractual constraints.

We now present an example that illustrates the challenges with a multi-period model. Consider a two-period model with a single supplier providing a fixed commitment contract, i.e., an option with zero execution price, to be executed in the first period. The sequence of events is the following:

(i) Before period 1, the manufacturer can reserve $x_1$ units of capacity (to be used in period 1) at price $v_1$ per unit.

(ii) At the beginning of period 1, it observes the realization of demand $D_1$ and spot price $S_1$. The manufacturer then uses the available capacity $x_1$ (at zero execution price) and the spot market to purchase supply, serve demand and stock $I$ units of inventory.

(iii) At the end of period 1, $I$ units of inventory are left.

(iv) At the beginning of period 2, $D_2$ and $S_2$ are observed. Demand is satisfied using the on-hand inventory $I$ and any units purchased in the spot market.

The stocking decision $I$ is similar to the single-period mean-variance trade-off discussed in this paper. Since it is likely that $I \geq x_1 - D_1$, because the manufacturer can raise the inventory up to the level $x_1 - D_1$ for free, $I$ can be dependent of $D_1, x_1$ and $S_1$.

Thus,

$$
\Pi = p_1 D_1 - v_1 x_1 - S_1 (D_1 + I - x_1)^+ + p_2 D_2 - S_2 (D_2 - I)^+ .
$$
Once $D_1$ and $S_1$ become known, at the beginning of the first period, the decision on $I$ involves a trade-off between the second period’s expected profit and variance. Observe that when $I < x_1 - D_1$ an additional unit of inventory costs 0, whereas, when $I \geq x_1 - D_1$, it costs $S_1$. It is clear that $S_1$, together with $x_1 - D_1$, is important in determining the level $I^E$ that maximizes expected profit, as described in Theorem 6. We also notice that the variance of the second period’s profit is independent of $S_1$: this implies that $I^V$ is independent of $D_1, x_1$ and $S_1$. In general, the manufacturer’s decision on $I$ will be within $[I^E, I^V]$ (or $[I^V, I^E]$), and thus depend on the values of $D_1, x_1$ and $S_1$.

It is now clear that when we analyze the variance of profit at the beginning of period 1, as a function of $x_1$, most of the complications come from the fact that the decision $I(D_1, x_1, S_1)$ is random. This implies that the function $I$ depends on the level of risk that the manufacturer takes as a function of $D_1, S_1$. For instance, if the manufacturer is risk-averse (i.e., selects $I = I^V$) when $S_1$ is high and risk-neutral (i.e., selects $I = I^E(x_1 - D_1, S_1)$) when $S_1$ is low, then $I(D_1, x_1, S_1)$ will have a high variance. As a result, this variance will influence the total variance of $\Pi$.

Formally, by using the conditional variance formula, and assuming that $(D_2, S_2)$ are independent of $(D_1, S_1)$, we have

$$Var\Pi = E\left[Var\left(p_1D_1 - v_1x_1 - S_1(D_1 + I(D_1, x_1, S_1) - x_1)^+ | S_1, D_1\right)\right] + E\left[Var\left(p_2D_2 - S_2(D_2 - I(D_1, x_1, S_1))^+ | S_1, D_1\right)\right]$$

$$= E\left[Var\left(p_2D_2 - S_2(D_2 - I(D_1, x_1, S_1))^+ | S_1, D_1\right)\right] + E\left[-\mathbb{E}\left(S_2(D_2 - I(D_1, x_1, S_1))^+ | S_1, D_1\right)\right]$$

We see that now the choice of $x_1$ will have an influence on the random inventory decision $I(D_1, x_1, S_1)$ and this implies that controlling the variance is difficult. In the case where $I = I^V$, and thus independent of $x_1$, the problem of minimizing variance
in terms of $x_1 \geq 0$ is equivalent to minimizing the function

$$Var\left[ p_1 D_1 - v_1 x_1 - S_1(D_1 + I^V - x_1)^+ \right].$$  \hspace{1cm} (4.18)

In this particular case, the analysis is similar to the single period case, except that we must work with a modified demand, equal to $D_1 + I^V$. Similarly, we can extend this example to a multiple period case, provided that at every time period we minimize the "variance-to-go", disregarding the expected profit. The manufacturer in this case finds the minimum variance portfolio for a demand $D_t + I^V_{t+1}$.

To summarize, the insight provided here is the following. A single period model allows a manufacturer to consider mean-variance trade-offs, and the problem can be described by some interesting properties, involving a maximum-profit-expected portfolio $y_E$, and a minimum-profit-variance portfolio $y_V$. Between these two portfolios, a continuum of efficient mean-variance pairs exist. However, in a multi-period setting, we face many complications. First, similarly to financial theory, it is difficult to pose an optimization problem since intermediate decisions depend on intermediate mean-variance trade-offs, thus requiring a complete specification of the preferences of the manufacturer in all the states of the world. Second, by using an inventory model that does not allow to sell back inventory to the spot market, past inventory decisions create constraints on present inventory decisions, and this in turn modifies the present mean-variance trade-off.

Hence, we see that multi-period models are significantly more difficult than single-period models. There are two important exceptions to this conclusion: when the manufacturer only cares about expected profit or when it only cares about future variance.

4.4 Summary and discussion of correlation cases

Most inventory decisions imply tremendous risks for a buyer, especially when this stock is limited to in-house production and there is no way to get rid of it after
purchasing, e.g., the recent Cisco case. Thus, this type of decisions should take into account not only expected profit but also the associated risk. For this purpose, we propose, as has been done in the financial literature, to apply a mean-variance analysis to procurement contracts.

Our focus in this chapter has been on a single-period inventory setting where purchasing decisions create both inventory risk, i.e., created by demand uncertainty, and price risk, i.e., created by alternative spot sourcing uncertainty. The contracts used in our model are portfolios of simple option contracts, which can replicate fixed-commitment contracts, quantity-flexibility contracts or buy-back contracts, as seen in Section 2.1.

We have shown that there is an efficient frontier bounded by the maximum expectation portfolio and the minimum variance portfolio, and provide bounds for this frontier. These two portfolios would be selected by a risk-neutral buyer and a risk-obsessed, i.e., with infinite risk aversion, buyer, respectively. We investigate structural properties of mean-variance utility objectives, which, even though they are not concave in general, can be shown to have connected upper-level sets. Such a result provides a theoretical foundation to the use of greedy algorithms to solve these trade-offs to optimality.

The model presented in this chapter relies on a strong assumptions, that limit its applicability: the demand and the spot market are independently distributed. In what follows, we present an extension of the results when spot market and demand are correlated.

**Assumption 14** The demand $D$ and the spot market price $S$ are continuously distributed with a bivariate p.d.f. $f(q,s) > 0$ for $q,s \in \mathbb{R}_+$.

We use the same notation $f_D$ and $f_S$ to denote the marginal distributions with respect to $D$ and $S$.

Using this assumption to replace Assumption 12, we can revise the calculations
involving expected profit and variance of profit. Recall that

\[ \Pi^N = Z_0 D + \sum_{i=1}^{n} \left\{ Z_i \min(D, y^i) - (v^{i+1} - v^i)y^i \right\}. \]

Equation (4.8) becomes,

\[ \mathbb{E}\Pi^N = \mathbb{E}[Z_0 D] + \sum_{i=1}^{n} \left\{ \mathbb{E}[Z_i \min(D, y^i)] - (v^{i+1} - v^i)y^i \right\}, \quad (4.19) \]

It is straightforward to see that correlation between \( Z_i \) and \( D \) does not change the fact that expected profit is the sum of \( n \) concave functions of a single variable \( y^i \). Thus, expected profit is still, without the independence assumption, a strict concave function of \( y \). In fact, this result is implied from Theorem 3, where no distributional assumption was made on demand and spot prices.

Extending the structural results to profit variance, in the case of correlated spot price and demand, is much more challenging. Using the formulation of Equation (4.12) together with Equation (4.11), we need to determine expressions for \( A_i, B_i \).
and $C_{ij}, i < j = 1, \ldots, n$. This is presented in the next equation.

\[
\frac{dA_i}{dy^i} = \mathbb{E}[Z_i^2 1_{y^i \leq D}] y^i - \mathbb{E}[Z_i 1_{y^i \leq D}] \mathbb{E}[Z_i \min(D, y^i)] \\
= \Phi(y^i) \left( \mathbb{E}[Z_i^2 | y^i \leq D] y^i - \mathbb{E}[Z_i | y^i \leq D] \mathbb{E}[\min(Z_i, y^i)] \right)
\]

\[
\frac{dB_i}{dy^i} = \mathbb{E}[Z_0 Z_i D 1_{y^i \leq D}] - \mathbb{E}[Z_0 D] \mathbb{E}[Z_i 1_{y^i \leq D}] \\
= \Phi(y^i) \left( \mathbb{E}[Z_0 Z_i D | y^i \leq D] - \mathbb{E}[Z_0 D] \mathbb{E}[Z_i | y^i \leq D] \right)
\]

\[
\frac{dC_{ij}}{dy^i} = \mathbb{E}[Z_i Z_j 1_{y^i \leq D} \min(D, y^j)] - \mathbb{E}[Z_i 1_{y^i \leq D}] \mathbb{E}[Z_j \min(D, y^j)] \\
= \Phi(y^i) \left( \mathbb{E}[Z_i Z_j \min(D, y^j) | y^i \leq D] - \mathbb{E}[Z_i | y^i \leq D] \mathbb{E}[Z_j \min(D, y^j)] \right)
\]

\[
\frac{dC_{ij}}{dy^j} = \mathbb{E}[Z_i Z_j 1_{y^j \leq D}] y^j - \mathbb{E}[Z_i \min(D, y^j)] \mathbb{E}[Z_j 1_{y^j \leq D}] \\
= \Phi(y^j) \left( \mathbb{E}[Z_i Z_j | y^j \leq D] y^j - \mathbb{E}[Z_i \min(D, y^j)] \mathbb{E}[Z_j | y^j \leq D] \right)
\] (4.20)

Thus,

\[
\frac{d\text{Var} \Pi^N}{dy^i} = \Phi(y^i) \Phi_i(y),
\]

with

\[
\frac{1}{2} \Phi_i(y) = \mathbb{E}[Z_i^2 | y^i \leq D] y^i - \mathbb{E}[Z_i | y^i \leq D] \mathbb{E}[\min(Z_i, y^i)] \\
+ \mathbb{E}[Z_0 Z_i D | y^i \leq D] - \mathbb{E}[Z_0 D] \mathbb{E}[Z_i | y^i \leq D] \\
+ \sum_{j > i} \mathbb{E}[Z_i Z_j \min(D, y^j) | y^i \leq D] - \mathbb{E}[Z_i | y^i \leq D] \mathbb{E}[Z_j \min(D, y^j)] \\
+ \sum_{j < i} \mathbb{E}[Z_i Z_j | y^i \leq D] y^j - \mathbb{E}[Z_i | y^i \leq D] \mathbb{E}[Z_j \min(D, y^j)]
\]

In order to extend the results of Proposition 7 to the correlated case, we need to
evaluate the gradient of the functions $\Phi_i$, $i = 1, \ldots, n$.

\[
\frac{1}{2} \frac{d\Phi_i}{dy^j} = \mathbb{E}[Z_i^2|y^j \leq D] - \overline{F}_D(y^j)\mathbb{E}[Z_i|y^j \leq D]^2 \\
+ \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} y^j - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_i \min(D, y^j)] \\
+ \frac{d\mathbb{E}[Z_0 Z_i|y^j \leq D]}{dy^j} - \mathbb{E}[Z_0 D] \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \\
+ \sum_{j > i} \left\{ \frac{d\mathbb{E}[Z_i Z_j \min(D, y^j)|y^j \leq D]}{dy^j} - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_j \min(D, y^j)] \right\} \\
+ \sum_{j < i} \left\{ \frac{d\mathbb{E}[Z_i Z_j|y^j \leq D]}{dy^j} - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_j \min(D, y^j)] \right\}
\]

and

\[
\frac{1}{2} \frac{d\Phi_i}{dy^j} = \overline{F}_D(y^j) \left\{ \mathbb{E}[Z_i Z_j|y^j \leq D] - \mathbb{E}[Z_i|y^j \leq D] \mathbb{E}[Z_j|y^j \leq D] \right\},
\]

for $j > i$,

\[
\frac{1}{2} \frac{d\Phi_i}{dy^j} = \overline{F}_D(y^j) \left\{ \mathbb{E}[Z_i Z_j|y^j \leq D] - \mathbb{E}[Z_i|y^j \leq D] \mathbb{E}[Z_j|y^j \leq D] \right\},
\]

for $j < i$.

We observe that, besides the additional terms found in the diagonal, the structure of the Hessian in the case of correlation is preserved. Thus, all the results will hold as soon as the following terms in the diagonal are non-negative, i.e.,

\[
\frac{d\mathbb{E}[Z_i^2|y^j \leq D]}{dy^j} y^j - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_i \min(D, y^j)] \\
+ \frac{d\mathbb{E}[Z_0 Z_i|y^j \leq D]}{dy^j} - \mathbb{E}[Z_0 D] \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \\
+ \sum_{j > i} \left\{ \frac{d\mathbb{E}[Z_i Z_j \min(D, y^j)|y^j \leq D]}{dy^j} - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_j \min(D, y^j)] \right\} \\
+ \sum_{j < i} \left\{ \frac{d\mathbb{E}[Z_i Z_j|y^j \leq D]}{dy^j} - \frac{d\mathbb{E}[Z_i|y^j \leq D]}{dy^j} \mathbb{E}[Z_j \min(D, y^j)] \right\} \\
\geq 0
\]

Unfortunately, there is no evidence that this holds, even for simple normal bivariate distributions of $(D, S)$. 

91
Chapter 5

 Suppliers' Behavior

The previous chapters study the behavior of a buyer when it is facing multiple procurement opportunities. We have shown that this problem has a well-behaved structure, both for a multi-period expected profit objective (convex) and a single-period mean-variance objective (connected level sets).

In the present chapter, we develop a model where suppliers behave strategically. The focus is on understanding how the suppliers respond when the buyer starts using multi-sourcing and portfolio purchasing. We consider a single-period model where many suppliers and a single buyer must negotiate the contract terms for a single component. This negotiation may take the form of a sealed-bid auction or an iterative mechanism. We study the sealed-bid negotiation and provide, in the next chapter, an equivalent iterative mechanism.

5.1 Motivation

The development of electronic marketplaces for supply has been a major revolution in the way manufacturers and suppliers interact. Indeed, the success story of start-ups such as Freemarkets in the mid 90s was quickly followed by the blossoming of many competing e-markets. These markets brought together many suppliers and, by forcing competition, reduced the overall buyer's cost.

Of course, in the process, suppliers saw their revenue drop due to the increased
price pressure. A common complaint was that the e-market bidding process put too much emphasis on price, and did not value enough suppliers' characteristics such as flexibility, quality or lead time.

Indeed, manufacturers were unable to quantify the value of these attributes, which led to the destruction of the traditional relationships between manufacturers and suppliers. Specifically, manufacturers were incapable of rewarding flexible or high-quality suppliers versus inflexible or low-quality suppliers. Thus, there is a need for models, tools and bidding mechanisms that capture the multiple dimensions of the suppliers' characteristics, force them to compete and differentiate on these multiple dimensions, and hence allow buyers to value attributes other than price.

Our model of portfolio contracts developed in Chapter 3 provides an opportunity to model this situation. In this framework, a manufacturer can optimally structure its sourcing channels so that it takes advantage of the flexibility of different fixed-commitment or option contracts. As a result, the manufacturer purchases from each competing supplier a share of capacity that reflects the trade-off between price and flexibility offered by that supplier. Thus, flexibility and price are the two attributes that manufacturers care about. Based on this framework, we can analyze the changes in the way suppliers compete in the marketplace.

In this model, we assume that suppliers have a differentiated cost structure. They incur an initial cost for reserving capacity and an additional cost for using the capacity to satisfy the buyer's orders. The differences in the cost parameters are typically due to the different technologies used by various suppliers.

There are a number of fields where this cost specification could be used. For instance, in the electricity industry, different types of power plants exist, from nuclear to coal or gas power plants. Nuclear power plants have a relatively small degree of flexibility in adjusting production level to meet demand and hence all the incurred cost is associated with reserving capacity. On the other hand, gas power plants can adjust the production level rapidly and hence most of the cost is associated with delivering electricity. In manufacturing, and especially in the plastics, chemicals or semi-conductor industries, buyers reserve capacity with suppliers in advance of
production time. Of course, different suppliers may have different costs for reserving capacity and delivering supply, depending on the type of technology (machinery) and their geographical location (labor, transportation). Finally, our model may also be relevant to the travel and tourism industry, where the service providers may have different cost structures in terms of capacity reservation cost (cost of leasing airplanes or hotels) and variable cost (operating cost).

These cost characteristics obviously impact the negotiation process. In our model, each supplier offers an option contract to the buyer characterized by two pricing parameters, a capacity reservation fee and an execution fee. Consequently, suppliers can become more competitive by pushing in two directions: either lowering the reservation price or the execution price. The trade-off is clear. A supplier that charges mainly a reservation fee (and a small execution fee) competes on price but not flexibility. On the other hand, a supplier that charges mainly an execution fee (and a small reservation fee) typically emphasizes flexibility and not price.

We describe the market equilibrium outcomes of such system in this chapter and the next, and in particular the behavior of market prices for existing supply options. Interestingly, this model is an extension of the Bertrand price competition model to two dimensions. An important result in one dimension is that, in equilibrium, there is a unique supplier, the least costly supplier, that captures all the orders at a market price that is between its cost and the cost of the second most competitive supplier. We show that this is not the case when two attributes are important to the buyer. Indeed, we demonstrate that in equilibrium, a variety of suppliers coexists, and these suppliers offer different prices. We call this cluster competition, since suppliers tend to cluster in small groups of two or three suppliers each, such that within the same group all suppliers use similar technologies and offer the same type of contract.

5.2 Literature review

Most relevant to our model are papers that analyze the behavior of suppliers in offering options to a buyer, the prelude to introducing competition between suppliers. The
existing literature usually models a Stackelberg game where a single buyer is the follower and a single supplier is the leader. Typically, competition in such models is introduced by a spot market. This spot market is the buyer's sourcing alternative and a potential client for the supplier. The focus is on finding conditions for which both players are willing to sign a contract and determining option prices as the outcome of the negotiation process.

The first publication in this stream of literature is by Wu et al. [53]. Motivated by electricity markets, they derive option prices as a function of the cost of the system and the elasticity of demand. Later, Spinler et al. [50] and Golovachkina and Bradley [25] analyze models similar to that of Wu et al. Interestingly, there are no papers that directly analyze competition among suppliers since this implies utilizing the notion of portfolio contracts, developed in Chapter 3. In this chapter, we move from the traditional models of competition through dual sourcing, i.e., single supplier offering an option contract versus spot market, to a model of pure competition between suppliers offering different types of options.

A second stream of the literature concentrates on analyzing multi-attribute auctions. This research is quite recent and follows the development of online auctions in B2B markets. Typically, the objective is to find conditions that guarantee that an auction mechanism leads to an optimal outcome. An optimal outcome may be defined as social efficiency or utility maximization from the auctioneer's point of view (similarly to the seminal work of Myerson [42] in a one-dimensional auction).

In this line of research, various authors have studied the winner determination problem, where a single supplier is awarded all the orders. This differs from our formulation where all the suppliers may potentially be selected for part of the procurement. For instance, Beil and Wein [5], following Che [16], present a multi-attribute Request For Quotation (RFQ) process where the buyer declares a scoring rule and chooses a winner among many suppliers, the one that obtains the highest score for the declared rule.

In a different direction, some research has been done on mechanism design where many bidders can be awarded orders at the same time. For instance, Schummer and
Vohra [46] analyze a class of two-dimensional option auction mechanisms for a set of suppliers confronted with a single buyer. Their formulation is similar to our model but focuses on designing an efficient procurement mechanism where the suppliers truthfully reveal their costs. Because suppliers submit their true costs, their paper does not address competition between suppliers. In comparison, we analyze a model where suppliers get paid what they bid, similarly to a first price auction. We focus on the suppliers’ behavior in this context rather than on examining efficient allocation mechanisms.

5.3 Assumptions and notation

We adopt the same notation as in previous chapters. We consider here a single-period situation, where a single manufacturer looking for supply of a component that is used in the manufacturing of the final product. This component may be obtained from a pool of \( n \) suppliers, each of which offers an option contract for the component. Such a contract is defined by two parameters, \( v \geq 0 \), the reservation price, and \( w \geq 0 \), the execution price. These values are determined by the supplier based on its cost structure as well as on whether the supplier emphasizes price or flexibility. Specifically, supplier \( i, i = 1, \ldots, n \), takes position in the market by offering options at a reservation price \( v^i \) and an execution price \( w^i \).

In previous chapters, the option parameters \( (w^i, v^i)_{i=1,\ldots,n} \) are an input, since only the buyer’s problem is considered. In the current chapter, we are interested in knowing how the pricing of the contracts is influenced by competition.

The suppliers’ cost structure is assumed to consist of two parts. Each supplier incurs a fixed unit cost for reserving capacity, \( f^i, i = 1, \ldots, n \), that can be seen as the unit cost of building a factory of the appropriate size, developing the technology required to produce the component, hiring manpower, or signing its own supply contracts with its suppliers, e.g., the energy provider. In addition, the supplier pays a unit cost, \( c^i, i = 1, \ldots, n \), for each unit executed by the buyer. This cost is typically the cost of raw materials and operational costs.

97
These costs differ from supplier to supplier and may be explained by the use of different technologies (e.g., different type of power plants in electricity generation, with much different fixed costs and variable costs) or management practices. Without loss of generality, we assume that $c^1 \leq \ldots \leq c^n$.

Therefore, the profit of supplier $i$, $i = 1, \ldots, n$, is $(v^i - f^i)x^i + (w^i - c^i)q^i$ when a buyer reserves $x^i$ units of capacity and executes $q^i$ units, $q^i \leq x^i$. The objective of the suppliers is to maximize their expected profit by selecting $(w^i, v^i)$ optimally.

On the demand side, we denote by $p$ the selling price to the customer and assume that it is an input, not a decision variable, as pointed out in Section 2.5. The demand distribution satisfies the following assumption.

**Assumption 15** The total customer demand $D$ follows a distribution defined over an interval $[d, \bar{d}] \subset [0, \infty]$. The c.d.f. of the demand $F(\cdot)$ is strictly increasing in $[d, \bar{d}]$. We assume that $F(\cdot)$ is a continuous and differentiable function over $(d, \bar{d})$. Define $f(\cdot) = F'(\cdot) > 0$ and $\bar{F}(\cdot) = 1 - F(\cdot)$.

We analyze a two-stage model. In the first stage all the suppliers submit bids that are defined by $(w^i, v^i)$, $i = 1, \ldots, n$. At the same time, and based on these bids, the manufacturer decides on the amount of capacity to reserve with each supplier. In the second period, demand is realized and the manufacturer decides the amount to execute from each contract. If total capacity is not enough, unsatisfied demand is lost.

This is a game a la Stackelberg in which the suppliers are leaders and the manufacturer is the follower. Thus, there are multiple leaders that compete knowing the reaction of the follower. This type of game, related to backward induction and sub-game perfection concepts, is described in detail in Fudenberg and Tirole [23], pp. 92-96.

Of course, the manufacturer's objective is to maximize expected profit based on the suppliers bids, as in chapters 2 and 3. Suppliers have complete visibility to the manufacturer decision making process. Therefore, given any $n$ pairs $(w^i, v^i)$, $i = 1, \ldots, n$, each supplier can figure out the amount of capacity that the manufacturer
would reserve with each individual supplier as well as the distribution of the amount of supply executed (requested) by the manufacturer. The costs \((c^i, f^i), i = 1, \ldots, n\), are private information, i.e., each supplier knows only its own cost. This is summarized in the following assumption.

**Assumption 16** Supplier \(i, i = 1, \ldots, n\), has information on \((c^i, f^i)\), \(p\) and the distribution of customer demand \(F\).

In addition, we assume that the suppliers submit sealed bids simultaneously, and that they are not allowed to change their bids again. Thus, this is a one-shot game. Every supplier submits a bid that maximizes its expected profit. We are interested in determining the Nash equilibria of this game in pure strategies, i.e., the \(n\)-uples \(\left(w^i, v^i\right)_{i=1, \ldots, n}\) where no supplier has an incentive to unilaterally change its bid.

### 5.4 A closed-form procurement strategy

In this section, we review the results of Chapter 3 in a single-period environment. Define as before \(v^{n+1} = 0\) and \(w^{n+1} = p\) and

\[
C(x, q) = \sum_{i=1}^{n+1} v^i x^i + \max_{i=1}^{n+1} w^i q^i
\]

subject to

\[
\begin{align*}
0 &\leq q^i \leq x^i & i = 1, \ldots, n, \\
0 &\leq q^{n+1}, \\
\sum_{i=1}^{n+1} q_i &= q
\end{align*}
\]

Then the manufacturer’s profit is \(\Pi(x, D) = pD - C(x, D)\). Thus, the expected profit is \(\overline{\Pi}(x) = pE[D] - E[C(x, D)]\).

Let \(y^0 = 0\) and

\[
y^i = x^1 + \ldots + x^i \text{ for } i = 1, \ldots, n. \tag{5.1}
\]

We can thus define \(V(y) = \overline{\Pi}(x)\). Applying Theorem 4, we have that

\[
\frac{dV}{dy^i}(y) = (v^{i+1} - v^i) + (w^{i+1} - w^i) Pr[D \geq y^i]. \tag{5.2}
\]
Equation (5.2) thus provides the structure of the manufacturer's optimal portfolio which is determined by the c.d.f. of customer demand. In particular, under Assumption 15, the profit is a strictly concave function of \((y^1, \ldots, y^n)\) defined over the set

\[
P = \left\{ (y^1, \ldots, y^n) \in \mathbb{R}^n \left| 0 \leq y^1 \leq \ldots \leq y^n \right. \right\}
\]  

(5.3)

Strict concavity implies that the optimal solution is unique, when all suppliers offer different bids. Thus, in the Stackelberg game analyzed here, the leaders, i.e., the suppliers, know exactly how the follower, i.e., the buyer, behaves.

To characterize the optimal portfolio, \((x^1, \ldots, x^n)\), or equivalently, \((y^1, \ldots, y^n)\), we need the following definitions.

**Definition 8** Supplier \(i\) is called active if \(x^i > 0\). Otherwise, it is called inactive.

**Definition 9** Given a set of \(m\) different pairs \(\{(a^1, b^1), \ldots, (a^m, b^m)\}\) with \(a^1 \leq \ldots \leq a^m\), the winning set is the minimal subset \(S = \{i_1, \ldots, i_k\}\) of these points such that:

(a) \(a^{i_1} \leq \ldots \leq a^{i_k}\);

(b) for \(1 \leq i < i_1\), \(b^i - b^{i_1} \geq -(a^i - a^{i_1})\);

(c) for \(j = 2, \ldots, k\), for \(i_{j-1} < i < i_j\), \(b^i - b^{i_j} \geq \frac{b^{i_{j-1}} - b^{i_j}}{a^{i_{j-1}} - a^{i_j}}(a^i - a^{i_j})\);

(d) for \(i_k \leq i \leq m\), \(b^i \geq b^{i_k}\).

\(i_1, \ldots, i_k\) are called winning points among the \(m\) pairs.

**Definition 10** Given a set of \(m\) pairs \(\{(a^1, b^1), \ldots, (a^m, b^m)\}\) with \(a^1 \leq \ldots \leq a^t\), the lower envelope is the curve \(Z(a,b)(\cdot)\) defined as follows

\[
Z(a,b)(u) = \begin{cases} 
  v^1 - (u - w^1) & \text{for } u \leq w^1 \\
  v^2 - \frac{u - v^2}{w^2 - w^1}(u - w^2) & \text{for } w^1 \leq u \leq w^2 \\
  \vdots & \\
  v^k - \frac{u - v^k}{w^k - w^{k-1}}(u - w^k) & \text{for } w^{k-1} \leq u \leq w^k \\
  0 & \text{for } w^k \geq u,
\end{cases}
\]  

(5.4)
where \( \{(w^1, v^1), \ldots, (w^k, v^k)\} \) are the winning points of \( \{(a^1, b^1), \ldots, (a^n, b^n)\} \) after sorting the first coordinates in increasing order.

These definitions, together with Equation (5.2), are used to characterize the optimal portfolio explicitly, as is done in the next proposition.

**Proposition 10** Supplier \( i, i = 1, \ldots, n \), is active if and only if \( i \) is a winning point of \( \{(w^1, v^1), \ldots, (w^{n+1}, v^{n+1})\} \).

Figure 5-1: Illustration of active and inactive bids

The winning points, i.e., all the active suppliers, can be determined graphically, see Figure 5-1. Plot the pairs \((w^i, v^i)\) in a graph with the \( w^i \) in the x coordinate and the \( v^i \) in the y coordinate. Add also the point \((w^{n+1}, v^{n+1}) = (p, 0)\). Determine the convex hull of the points, and in particular find the extreme points on the lower envelope as defined in Definition 10; these are the points \( i_1 < \ldots < i_k \).

Hence, the lower envelope is piecewise linear and convex. The segments have increasing slopes or equivalently decreasing negative slopes, that is,

\[
\frac{v^{i_1} - v^{i_2}}{w^{i_2} - w^{i_1}} > \ldots > \frac{v^{i_{k-1}} - v^{i_k}}{w^{i_k} - w^{i_{k-1}}},
\]
The buyer’s optimal portfolio only includes segments with negative slopes between 0 and 1. To identify these segments, let \( l \geq 1 \) be such that \( \frac{v^{l-1} - v^l}{w^{l-1} - w^l} \geq 1 > \frac{v^{l+1} - v^l}{w^{l+1} - w^l} \) or \( l = 1 \) if all the ratios are below 1. Similarly, find \( h \leq k \) such that \( \frac{v^{h-1} - v^h}{w^{h-1} - w^h} > 0 \geq \frac{v^{h+1} - v^h}{w^{h+1} - w^h} \). Typically \( h = n + 1 \) since all prices \((v^i, w^i)\) are non-negative and \( v^i + w^i \leq p \) usually.

With these definitions, and recalling \( y^0 = 0 \), the optimal portfolio is defined by

\[
y^{i*} = \begin{cases} 
F^{-1}(\frac{v^j - v^{j+1}}{w^j - w^{j+1}}) & \text{if } i = i_j, \ j = l, \ldots, h - 1, \\
y^{i-1*} & \text{for all others.}
\end{cases}
\]

The vector \( x^* \) follows directly from \( y^* \). In particular \( x^{i*} = 0 \) for \( i \) different than \( i_l, \ldots, i_h \).

### 5.5 The suppliers’ profit

Given that the manufacturer uses a portfolio approach as described in the previous section, each supplier will set its reservation and execution price to maximize its expected profit, taking into account the behavior of other suppliers.

Consider the decision of supplier \( i, \ i = 1, \ldots, n \). It is confronted by bids from other suppliers. Let \((w^{(-i)}, v^{(-i)})\) be the vector representing all other bids with the additional point \((w^{n+1} = p, v^{n+1} = 0)\).

Given the bids in the vector \((w^{(-i)}, v^{(-i)})\), we can identify the manufacturer optimal procurement strategy. Without loss of generality, assume that there are \( k \leq n \) active suppliers, indexed from 1 to \( k \), and \( w^1 \leq \ldots \leq w^k \) (one of the suppliers might be the dummy supplier with parameters \((p, 0)\)). The manufacturer’s best procurement strategy is

\[
y^{i \ (-i)} = F^{-1}(\frac{v^j - v^{j+1}}{w^j - w^{j+1}}) \quad j = 1, \ldots, k - 1, \\
y^{k \ (-i)} = F^{-1}(0).
\]

If supplier \( i \) places a bid \((w^i, v^i)\), the manufacturer optimal solution may change
to take this bid into account. Of course, suppliers that were not active before are not going to be active with the new bid from supplier $i$. However, it is entirely possible that some suppliers may become inactive when supplier $i$ enters with the bid $(w^i, v^i)$. Finally, supplier $i$ may capture zero capacity if its bid makes it inactive. Clearly, in this case, if supplier $i$ is inactive, we can withdraw it from the pool of bids and consequently the capacities allocated to the other suppliers remain unchanged. This happens when $(w^i, v^i)$ is above the lower envelope which is described by the function $Z_{(w^{(-i)}, v^{(-i)})}(\cdot)$ in Definition 10. Thus, when $v^i \geq Z_{(w^{(-i)}, v^{(-i)})}(w^i)$, supplier $i$ is inactive and its profit is $\Pi = 0$. We define this bidding region which makes $i$ inactive as

$$A_{(w^{(-i)}, v^{(-i)})}^{OUT} = \{(w, v) \in \mathbb{R}^2_+ | v \leq Z_{(w^{(-i)}, v^{(-i)})}(w)\}.$$  

If supplier $i$’s bid is not in that region, then supplier $i$ becomes active. Adding bid $(w^i, v^i)$ to the rest of the bids may change the convex hull of the points in two different ways:

- **Supplier $i$ becomes the first active supplier**, i.e., there exist $h \in \{1, \ldots, k\}$ such that suppliers $i, h, \ldots, k - 1$ are active and suppliers $1, \ldots, h - 1$ are inactive. We define this region as $A_{(w^{(-i)}, v^{(-i)})}^{0h}$

$$A_{(w^{(-i)}, v^{(-i)})}^{0h} = \left\{(w, v) \in \mathbb{R}^2_+ \left| \begin{array}{l}
v - v^i \leq -(w - w^i) \\
v - v^h \leq -\left(\frac{v^{h-1} - v^h}{w^h - w^{h-1}}\right)(w - w^h) \\
\text{(or nothing if } h = 1) \\
v - v^h \geq -\left(\frac{v^h - v^{h+1}}{w^{h+1} - w^h}\right)(w - w^h)
\end{array} \right\}$$  

(5.6)

- **Supplier $i$ is not the first active supplier**, i.e., there exist $l \in \{1, \ldots, k - 1\}$ and $h \in \{1, \ldots, k\}$, $h > l$, such that suppliers $i, 1, \ldots, l, h, \ldots, k$ are active and $l + 1, \ldots, h - 1$ inactive. We define this region as $A_{(w^{(-i)}, v^{(-i)})}^{lh}$.
\[
A_{(w(-l),v(-l))}^{th} = \begin{cases}
(v - v') \geq -\left(\frac{v^{l-1} - v^l}{w^l - w^{l-1}}\right)(w - w') \\
\text{(or } v - v^l \geq -(w - w^l) \text{ if } l = 1) \\
v - v^l \leq -\left(\frac{v^l - v^{l+1}}{w^{l+1} - w^l}\right)(w - w') \\
v - v^h \leq -\left(\frac{v^h - v^{h+1}}{w^{h+1} - w^h}\right)(w - w^h) \\
v - v^h \geq -\left(\frac{v^h - v^{h+1}}{w^{h+1} - w^h}\right)(w - w^h)
\end{cases}
\] (5.7)

These regions are nicely illustrated in Figure 5-2. Intuitively, a bid in region 
\(A_{(w(-l),v(-l))}^{th}\) implies that supplier \(i\) forces suppliers \(l + 1, \ldots, h - 1\) out of the market, 
i.e., these suppliers receive zero capacity allocation.

The capacity allocated by the buyer to supplier \(i\), \(x^i\), if \(i\) bids in \(A_{(w(-l),v(-l))}^{th}\), 
\(l > 0\), is \(x^i = y^+_i - y^-_i\) where \(y^+_i\) and \(y^-_i\) are given by the following set of equations.

We drop the sub-index \(i\) to simplify notation.

\[
\frac{v^l - v}{w - w^l} = \bar{F}(y_-)
\]

\[
\frac{v - v^h}{w^h - w} = \bar{F}(y_+).
\] (5.8)

The case of \(l = 0\) is special. The buyer allocates then capacity such that

\[
0 = y_-
\]

\[
\frac{v - v^h}{w^h - w} = \bar{F}(y_+).
\]

For \(l > 0\), since \((w, v) \in A_{(w(-l),v(-l))}^{th}\), one must keep in mind that \(y'^{l-1}(-i) \leq y_- \leq y^l(-i)\), \(y^{h-1}(-i) \leq y_+ \leq y^h(-i)\) and therefore \(y_- \leq y_+\) (moreover, if \(y_- = y_+\), supplier \(i\) is in \(A_{(w(-l),v(-l))}^{OUT}\) and is thus inactive).

The profit of supplier \(i\) in this case is

\[
\Pi = (v - f)(y_+ - y_-) + (w - c)\mathbb{E}[\min\{\max(D - y_-, 0), y_+ - y_-\}].
\]

104
Definition of the regions $A_{ij}$

Figure 5-2: Division of the bidding strategies in different regions
Since \( E[\min \{\max(D - y, 0), y^* - y\}] = \int_{y}^{y^*} (u - y) f(u) du + (y^* - y) \bar{F}(y^*) \), integration in parts yields

\[
\Pi = (v - f)(y^* - y) + (w - c) \int_{y}^{y^*} \bar{F}(u) du.
\]

Using Equation (5.8), one can express \((w, v)\) as a function of \(y^*\) and \(y^*\) when \(y^* < y^*\), since \(f(\cdot) > 0\). Specifically,

\[
v = v^h + \bar{F}(y^*) \frac{-(v^l - v^h) + \bar{F}(y)(v^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)} = v^l - \bar{F}(y^*) \frac{v^l - v^h}{\bar{F}(y^*) - \bar{F}(y^*)} = v^l + \frac{(v^l - v^h) - \bar{F}(y^*)(w^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)}.
\]

\[
w = w^h - \frac{-(v^l - v^h) + \bar{F}(y)(w^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)} = w^l + \frac{(v^l - v^h) - \bar{F}(y^*)(w^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)}.
\]

This implies that we can express \(\Pi\) using \(y^*\) and \(y^*\) instead of \(v^*\) and \(w^*\). Within

\[A_{\alpha(y^*), \gamma(y^*)}, l > 0,\]

\[
\Pi(w, v) = J^l(y^*, y^*) = \left\{ \begin{array}{ll}
(v^h - f)(y^* - y^*) + (w^h - c) \int_{y}^{y^*} \bar{F}(u) du & \\
- \left[ \frac{-(v^l - v^h) + \bar{F}(y^*)(w^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)} \right] \int_{y}^{y^*} \left( \bar{F}(u) - \bar{F}(y^*) \right) du & \\
(v^l - f)(y^* - y^*) + (w^l - c) \int_{y}^{y^*} \bar{F}(u) du & \\
- \left[ \frac{(v^l - v^h) - \bar{F}(y^*)(w^h - w^l)}{\bar{F}(y^*) - \bar{F}(y^*)} \right] \int_{y}^{y^*} \left( \bar{F}(u) - \bar{F}(y^*) \right) du & \\
\end{array} \right.
\]

When \(l = 0\), the transformation described in Equation (5.9) is not well defined. Instead, we consider

\[
v + w = v^l + w^l,
\]

\[
\frac{v - v^h}{w^h - w^l} = \bar{F}(y^*).
\]

To see why this transformation holds, we observe that for a given \(y^*, y^* = 0\), the profit with a bid \(w = w^h - t\) and \(v = v^h + \bar{F}(y^*)t\), \(t \geq 0\), is,
\[
\Pi = (v^h + \bar{F}(y_+)_t - f)y_+ + (w^h - t - c) \int_0^{y_+} \bar{F}(u)du \\
= (v^h - f)y_+ + (w^h - c) \int_0^{y_+} \bar{F}(u)du - t \int_0^{y_+} \bar{F}(u)du - \int_0^{y_+} (\bar{F}(u) - \bar{F}(y_+))du.
\] (5.12)

Thus, to maximize \( \Pi \) it is best for the supplier to select \( t \) as small as possible. This justifies the extension of Equation (5.8) for \( l = 0 \). Consequently, we obtain that \( J^{lh}(y_-, y_+) \), as defined in Equation (5.10), is extended, for \( l = 0 \), to

\[
\Pi(w, v) = J^{0h}(y_+) = \begin{cases} 
(v^h - f)y_+ + (w^h - c) \int_0^{y_+} \bar{F}(u)du \\ 
- \left[ \frac{(w^h + w^h) - (v^h + w^h)}{1 - \bar{F}(y_+)} \right] \int_0^{y_+} \left( \bar{F}(u) - \bar{F}(y_+) \right)du
\end{cases}
\]

Finally, the problem faced by supplier \( i \) is:

\[
\sup_{(w, v)} \Pi(w, v) = \max \left\{ 0, \max_{l = 0, \ldots, k - 1} \sup_{(y_-, y_+)} J^{lh}(y_-, y_+) \right\}
\]

The optimization problem is defined as a supremum of profit, in terms of either \((w, v)\) or \((y_-, y_+)\). As we shall see later, when optimizing on \((w, v)\), there does not always exist an optimal solution, and the supremum may be obtained by bidding identically to another supplier. However, when using \((y_-, y_+)\) as decision variables, an optimal solution is always obtained.

Since \( \bar{F}(\cdot) \) is differentiable over \((d, \bar{d})\), the expected profit is differentiable in \((y_-, y_+)\).
\[ \frac{dJ_i}{dy_-} = (f - v^i) + (c - w^i)\bar{F}(y_-) \]
\[ + f(y_-) \left[ \frac{(v^i - v^h) - \bar{F}(y_+) (w^h - w^i)}{(\bar{F}(y_-) - \bar{F}(y_+))^2} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \bar{F}(y_+) \right) du \]
\[ (5.13) \]

\[ \frac{dJ_i}{dy_+} = (v^h - f) + (w^h - c)\bar{F}(y_+) \]
\[ - f(y_+) \left[ \frac{-(v^i - v^h) + \bar{F}(y_-) (w^h - w^i)}{(\bar{F}(y_-) - \bar{F}(y_+))^2} \right] \int_{y_-}^{y_+} \left( \bar{F}(y_-) - \bar{F}(u) \right) du \]

Notice that \((v^i - v^h) - \bar{F}(y_+) (w^h - w^i) \geq 0\) and \(-(v^i - v^h) + \bar{F}(y_-) (w^h - w^i) \geq 0\) hold; we shall use this observation later.

### 5.6 Border distributions

Before proceeding with equilibrium analysis in Chapter 6, we need to provide some technical conditions that are needed in order to prove the results. Indeed, it is clear that the profit of a given supplier as a function of \((y_-, y_+)\) will not be well-behaved for all demand distributions. The following property will guarantee some regularity on the profit functions.

**Definition 11** A demand distribution is a border distribution when for all cost parameters \((c^i, f^i)\), for all regions \(A_{(w^{-1}, v^{-1})}^{i}\), defined by Equations (5.6) or (5.7), there is an optimal bid \((w^i, v^i)\) for supplier \(i\) that belongs to the border of that region.

The property implies that for any supplier bidding in region \(A_{(w^{-1}, v^{-1})}^{i}\), there is an optimal bid on the boundary of that region, for all cost parameters. For instance, for a supplier bidding in region \(A^{12}\) of Figure 5-2, there is an optimal bid on the boundary of \(A^{12}\) with either \(A^{02}\) or \(A^{13}\) or \(A^{OUT}\), for all cost parameters.

Hence, under this property, we may assume that the suppliers will place their bids in the border of some region. The property will allow us to determine their optimal bids, as is done in Theorem 12.
Observe that border distributions may also allow for optimal bids in the interior of the region, but these are unstable in the sense that a small perturbation in the cost parameters will completely change the optimal bid. Therefore, we consider only optimal bids on the borders of the regions defined in Equations (5.7) and (5.6).

The next result provides an equivalent condition for a distribution to be border.

**Proposition 11** The demand distribution is a border distribution if and only if, for all \( y_i \leq \bar{y}_i \leq y_m \leq y_h \leq \bar{y}_h \), and for all \( \alpha_1, \alpha_2 \), the function

\[
\left[ \frac{\alpha_1 - \bar{F}(y_m)}{\alpha_1 - \alpha_2} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \alpha_2 \right) du - \left[ \frac{\bar{F}(y_-) - \bar{F}(y_m)}{\bar{F}(y_-) - \bar{F}(y_+)} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \bar{F}(y_+) \right) du
\]

does not have any interior strict local maximum for \((y_-, y_+)\) in the area

\[
\left\{ (y_- , y_+) \left| y_i \leq y_- \leq \bar{y}_i \leq y_h \leq y_+ \leq \bar{y}_h \right. \right\}.
\]

Using this proposition, it is straightforward to show that the uniform distribution is border.

**Definition 12** The demand distribution is uniform, i.e. there exists \( a, b \in \mathbb{R}_+ \), \( a < b \), such that

\[
F(t) = \begin{cases} 
0 & \text{if } t < a \\
\frac{t - a}{b - a} & \text{if } a \leq t \leq b \\
1 & \text{if } t > b
\end{cases}
\]

For a uniform distribution, we can simplify the profit obtained by a supplier after bidding in \( A^{th} \). By transforming \( \frac{y - a}{b - a} \) for \( y \in [a, b] \) into a new variable, we obtain an equation where \( a \) and \( b \) are replaced by \( a = 0 \) and \( b = 1 \). Therefore, without loss of generality, one can assume that \( a = 0 \) and \( b = 1 \). Thus,

\[
J^{th}(y_- , y_+) = \begin{cases} 
(v^h - f)(y_+ - y_-) + (u^h - c)(y_+ - y_-) \left[ 1 - \frac{y_+ + y_-}{2} \right] \\
- \left[ -(v' - v^h) + (1 - y_-)(u^h - w') \right] \left[ \frac{y_+ - y_-}{2} \right]
\end{cases}
\]
This is a quadratic function on \((y_-, y_+)\). It can be expressed as
\[
J^{th}(y_-, y_+) = \frac{1}{2}(y_- \ y_+)H \begin{pmatrix} y_- \\ y_+ \end{pmatrix} + (g - g) \begin{pmatrix} y_- \\ y_+ \end{pmatrix} + \text{constant}
\]
where \(H\) is the Hessian
\[
H = \begin{pmatrix} -(c - w^l) & (w^h - w^l)/2 \\ (w^h - w^l)/2 & -(w^h - c) \end{pmatrix}
\]
and
\[
g = f + c - \frac{v^l + v^h}{2} - \frac{w^l + w^h}{2}
\]

The determinant of \(H\) is \((c - w^l)(w^h - c) - (w^h - w^l)^2/4\), which is always non-positive and reaches its maximum, 0, when \(c = (w^h + w^l)/2\). Hence, there cannot be a strict local maximum in \(A^{th}\). Thus, we have shown the next proposition.

**Proposition 12** A uniform distribution is a border distribution.

Using Proposition 11, we can also show directly that a truncated exponential distribution is border.

**Definition 13** A distribution is truncated exponential if and only if there are parameters \(\beta, a, b, a < b\), such that
\[
f(t) = Ke^{\beta t}1_{[a,b]}(t),
\]
where \(1/K = \int_a^b e^{\beta t} dt\).

Of course, an exponential distribution of parameter \(\lambda\) is a truncated exponential distribution with \(\beta = \lambda\), \(a = 0\), \(b = \infty\) and \(K = \lambda\).

**Proposition 13** A truncated exponential distribution is a border distribution.

We now know that uniform and truncated exponentials all belong to the class of border distributions. However, the definition of the border distributions class is quite
complicated, and one may wonder if there is a general class of functions that satisfy the property. The following results provide some insight on this question.

**Definition 14** A non-negative function $f$ with convex support $[a, b] \in \mathbb{R}$, is called log-concave if for all $x, y \in [a, b]$, for all $\lambda \in [0, 1]$,

$$f\left(\lambda x + (1 - \lambda)y\right) \geq f(x)^{\lambda} \cdot f(y)^{1-\lambda}.$$

This is a special case of $\rho$-concavity, as defined in Caplin and Nalebuff [14], with $\rho = 0$.

**Definition 15** Consider $\rho \in \mathbb{R}$. The positive function $f$ with convex support $[a, b] \in \mathbb{R}$, is called $\rho$-concave if for all $x, y \in [a, b]$, for all $\lambda \in [0, 1]$,

$$f\left(\lambda x + (1 - \lambda)y\right) \geq \left[\lambda f(x)\rho + (1 - \lambda)f(y)\rho\right]^{1/\rho}.$$

When $\rho = 1$, we obtain the usual concavity condition, and when $\rho = -\infty$, we obtain that

$$f\left(\lambda x + (1 - \lambda)y\right) \geq \min\left[f(x), f(y)\right],$$

implying that $f$ is quasi-concave.

The concept of $\rho$-concavity has been used extensively in the mathematics and economics literature. In particular, it has been applied to Hotelling models, described in Hotelling [29], that deal with price competition with product differentiation. In this setting, a number of firms are located in a multi-dimensional product space, and compete on price. Customers have different multi-attribute preferences, and will become customers of the firm that offers them the highest utility. As D’Aspremont et al [19] show, the existence of price equilibria is not trivial, even in simple systems. However, as shown in Caplin and Nalebuff [14], when the utilities are linear in the attributes, and that the preference distributions are log-concave, then there exists a price equilibrium. $\rho$-concavity proves itself useful due to an extension of the Brunn-Minkowski theorem.

We can now define the class of log-concave distributions, denoted by $\mathcal{F}$. 
Definition 16 A distribution is log-concave when its p.d.f. is log-concave.

We are now ready to present the main sufficient condition for border distributions.

Theorem 10 A log-concave distribution is a border distribution.

As mentioned in Caplin and Nalebuff [14], log-concave distributions include beta, exponential, gamma, Laplace, normal, uniform and Weibull distributions. Therefore, this result proves that the truncated normal distribution, defined below, is border, which is the most widely used characterization of demand distributions in practice.

Definition 17 A distribution is truncated normal when there exists $\mu > 0$, $\sigma^2 \geq 0$ such that

$$f(t) = \kappa e^{-\frac{(t-\mu)^2}{2\sigma^2}} \text{ for } t > 0$$

where $\kappa = \left[ \int_0^\infty e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \right]^{-1}$.

In order to prove the theorem, we use several interesting properties of log-concave distributions.

Lemma 2 Let $f : (a, b) \rightarrow [0, \infty)$, where $(a, b)$ is an interval of $\mathbb{R}$ (possibly unbounded). Assume that $f$ is log-concave. Then, for any $a \leq x_0 \leq b$, the functions

$$F_1 : (x_0, b) \rightarrow [0, \infty) \quad \text{and} \quad F_2 : (a, x_0) \rightarrow [0, \infty)$$

$$x \rightarrow \int_{x_0}^x f(u)du \quad \text{and} \quad x \rightarrow \int_x^{x_0} f(u)du$$

are such that $F_1$ and $F_2$ are log-concave.

Define, for each demand distribution, for $x \leq y$,

$$L(x, y) = \int_x^y \frac{(F(x) - \bar{F}(u))du}{F(x) - \bar{F}(y)}$$

and

$$R(x, y) = \int_x^y \frac{(\bar{F}(u) - \bar{F}(y))du}{F(x) - \bar{F}(y)} = (y - x) - L(x, y).$$

(5.15)
It is easy to see that $L(x, y)$ is the inverse of the log-derivative of the function of $y$

$$
\int_x^y (\bar{F}(x) - \bar{F}(u))du.
$$

Using Lemma 2, we can derive the following result.

**Lemma 3** For all $y$, $x \in (-\infty, y] \rightarrow L(x, y)$ is non-increasing, and for all $x$, $y \in [x, \infty) \rightarrow R(x, y)$ is non-decreasing. Moreover, when $f$ is log-concave, for all $x$, $y \in [x, \infty) \rightarrow L(x, y)$ is non-decreasing, and for all $y$, $x \in (-\infty, y] \rightarrow R(x, y)$ is non-increasing.

This monotonicity condition is crucial to derive the results of Theorem 10 and others. It can be expressed in a different form, presented in the following lemma.

**Lemma 4** Define for $x \leq y$,

$$
A(x, y) = \frac{f(x)}{\bar{F}(x) - \bar{F}(y)} R(x, y) \quad (5.17)
$$

and

$$
B(x, y) = \frac{f(y)}{\bar{F}(x) - \bar{F}(y)} L(x, y). \quad (5.18)
$$

If $f$ is log-concave, then

$$
0 \leq A(x, y) \leq 1 \text{ and } 0 \leq B(x, y) \leq 1.
$$

The proof of the lemma is straightforward, since

$$
\frac{dL}{dy} = 1 - B(x, y) \text{ and } \frac{dR}{dx} = -1 + A(x, y).
$$
Chapter 6

Cluster Competition in Equilibrium

We consider a game in which the suppliers compete for selling capacity. This is a game à la Stackelberg in which the manufacturer is the follower and the suppliers are the leaders. This chapter analyzes the Nash equilibria in pure strategies.

As discussed in the previous chapter, when the strategies are defined only through \((w, v)\), no equilibria might exist since some supplier’s problem may not have an optimal solution. This occurs when two suppliers submit the same bid, and as a consequence, the buyer’s problem has multiple optimal solutions.

To get rid of this technical problem, we may study Nash \(\epsilon\)-equilibria, i.e. a set of strategies such that each supplier obtains at least the maximum possible payment minus \(\epsilon\), and let \(\epsilon\) tend to 0. This would make the analysis cumbersome. To simplify the analysis, we choose instead to define a strategy as a bid \((w, v)\) accompanied by, in case of a tie, an agreement on the allocation of capacity to each supplier. One can formalize this step by defining a bargaining sub-game in case of a tie. Since we are interested in equilibrium characteristics, we only need to the conditions under which the two suppliers in a tie agree on the allocation. That is, when suppliers \(i\) and \(j\) bid the same reservation and execution prices, they will be in equilibrium when the buyer allocates them a slice of capacity \((y_{\text{lower}}, y_{\text{higher}})\), and they agree to split this allocation with supplier \(i\) getting the slice \((y_{\text{lower}}, y_{\text{middle}})\) and supplier \(j\) the slice
\((y_{\text{middle}}, y_{\text{higher}})\). This can be interpreted as a rationing rule implicitly determined by the suppliers. In practice, this is achieved by each supplier bidding very close to each other such that the targeted allocation to each supplier is obtained. In the rest of the chapter, we will obtain equilibria when such agreements in case of ties can be reached.

### 6.1 Efficiency

We start by defining the concept of efficiency which leads to a natural and desirable property of Nash equilibria. Note that no assumptions on the demand distribution are made.

**Definition 18** We say that supplier \(i\) is efficient when \((c^i, f^i)\) is a winning point in the set \(\{(c^1, f^1), \ldots, (c^n, f^n), (p, 0)\}\).

**Theorem 11** Assume that supplier \(i\) is efficient. Then, in every Nash equilibrium, supplier \(i\) must be active.

### 6.2 Optimal bids

As we will soon see, it is of particular interest to examine the situation in \(A^{l_{h}}\) where there is no active supplier between \(l\) and \(h\) and both \(l\) and \(h\) are active. Notice that these are all the regions that share an edge with \(A^{OUT}\).

Consider supplier \(i\) bidding in such a region, \(A^{l_{h}}_{(w(-i),\nu(-i))}\), and define \(y_{m}\) to be the quantity captured by supplier \(l\) when \(i\) is absent, i.e.,

\[
\overline{F}(y_{m}) = \frac{v^l - v^h}{w^h - w^l}.
\] (6.1)

The constraints for \(y_{-}\) and \(y_{+}\) are then \(y^{l-1}(-i) \leq y_{-} \leq y_{m} \leq y_{+} \leq y^{h}(-i)\) where \(y^{l-1}(-i)\) and \(y^{h}(-i)\) are defined in (5.5).

If the bid of supplier \(i\) does not make \(l\) or \(h\) inactive, we can derive useful properties. In this case, the optimal bid cannot be such that \(y_{-} = y^{l-1}(-i)\) (because it makes
$l$ inactive) or $y_+ = y^h$ (inactive). Therefore, since it is optimal to bid on the border of the region, it must be that $y_+ = y_m$ or $y_+ = y_m$ is optimal. These imply that supplier $i$ bids the same pair $(w, v)$ as $l$ or $h$, respectively. These bids are going to be very close to each other, but we can consider these two bids as identical with the line connecting the two bids having a slope $-\bar{F}(y)$ for some $y$.

In the first case, i.e., when $y_+ = y_m$ is optimal, recall that $-(v^l - v^h) + \bar{F}(y_-)(w^h - w^l) = 0$ so from Equation (5.13)

$$\frac{dJ^h}{dy_+} = (v^h - f) + (w^h - c)\bar{F}(y_+)$$

and therefore we must have that $c \leq w^h$ and

$$\bar{F}(y_+) = \frac{f - v^h}{w^h - c}.$$

Similarly, when $y_+ = y_m$ is optimal, then $w^l \leq c$ and

$$\frac{dJ^l}{dy_-} = (f - v^l) + (c - w^l)\bar{F}(y_-),$$

$$\bar{F}(y_-) = \frac{v^l - f}{c - w^l}.$$

We summarize these results in the next theorem.

**Theorem 12** Given a border distribution, assume that, for a supplier with costs $(c, f)$, it is optimal to bid in some unique region $A^h$, $l > 0$, where there is no active supplier between $l$ and $h$. Define $y_m$ as in Equation (6.1) and, for some $y_0, y_3$, we can represent $A^h$ by all the pairs of quantities $(y_-, y_+)$ such that $y_0 \leq y_- \leq y_m \leq y_+ \leq y_3$. Define $y_1$ and $y_2$ as follows,

$$\bar{F}(y_1) = \frac{v^l - f}{c - w^l},$$

(6.2)

$$\bar{F}(y_2) = \frac{f - v^h}{w^h - c}.$$  

(6.3)
Then, one and only one case from the following is true.

- either \( y_0 \leq y_1 \leq y_m \) and \( y_2 > y_3 \), and \((w^*, v^*) = (w^i, v^i)\), \( y^*_+ = y_m \) and \( y^- = y_1 \),
- or \( y_0 > y_1 \) and \( y_m \leq y_2 \leq y_3 \), and \((w^*, v^*) = (w^h, v^h)\), \( y^-_+ = y_m \) and \( y^*_+ = y_2 \),
- or \( y_0 \leq y_1 \leq y_m \leq y_2 \leq y_3 \); \((w^*, v^*) = (w^l, v^l)\), \( y^*_+ = y_m \) and \( y^- = y_1 \), only if

\[
\int_{y_1}^{y_m} \frac{[F(y_1) - F(u)]}{F(y_1) - F(y_m)} \, du \geq \int_{y_m}^{y_2} \frac{[F(u) - F(y_2)]}{F(y_m) - F(y_2)} \, du; \quad (6.4)
\]

\( (w^*, v^*) = (w^h, v^h) \), \( y^*_+ = y_m \) and \( y^*_+ = y_2 \), only if

\[
\int_{y_1}^{y_m} \frac{[F(y_1) - F(u)]}{F(y_1) - F(y_m)} \, du \leq \int_{y_m}^{y_2} \frac{[F(u) - F(y_2)]}{F(y_m) - F(y_2)} \, du. \quad (6.5)
\]

Figure 6-1 illustrates the two optimal bids discussed in the theorem. In the figure we show iso-profit curves as a function of \( y^- \) and \( y^*_+ \). As you can see, the figure on the left illustrates the first case in which the optimal bid is \( y^*_+ = y_m \) and \( y^- = y_1 \) whereas the figure on the right illustrates the case in which the optimal bid is \( y^*_+ = y_m \) and \( y^*_+ = y_2 \). Notice that bidding in the left-side edge of the definition box is equivalent to making supplier \( l \) inactive and bidding on the up-side edge to making \( h \) inactive.

Again, notice that such optimal bids imply that two suppliers end up submitting the same bid \((w, v)\). The buyer thus orders capacity from both indifferently. In equilibrium, after capacity has been ordered by the buyer, the two suppliers share it such that they obtain the capacity they were aiming at by selecting the quantities \((y_-, y^*_+)\). That is, if \( y^*_+ = y_m \) and \( y^- = y_1 \) is optimal for the current supplier, then it obtains the share of capacity between \( y_1 \) and \( y_m \), and if \( y^*_+ = y_m \) and \( y^*_+ = y_2 \) is optimal, then it obtains the share of capacity between \( y_m \) and \( y_2 \). Theorem 12 allows us to derive a general property of any equilibrium.

**Proposition 14** Consider a border distribution. In any Nash equilibrium, if \( i \) and \( j \) are active and \((w^i, v^i) = (w^j, v^j)\) then \((w^i, v^i)\) belongs in the segment \([[(c^i, f^i), (c^j, f^j)]]\).
6.3 Equilibria with efficient suppliers only

In this section, we study the situation where all suppliers are efficient, that is, all the suppliers receive some capacity order from the buyer when they bid their true cost.

6.3.1 Characteristics of equilibrium

**Proposition 15** Given a border distribution, assume that all suppliers are efficient. Then, in every Nash equilibrium, for every pair \((i, j)\), if \(c^i < c^j\) then \(w^i \leq w^j\).

Proposition 15 implies that if all suppliers are efficient, supplier \(i, i = 1, \ldots, n\), bids in region \(A_{(W^{-i}), v^{(-i)}}^{i=1, i+1}\) in every Nash equilibrium. More importantly, this result confirms the intuition on the suppliers’ bidding behavior. No supplier will bid an execution fee, \(w\), lower than a competitor’s execution fee if the competitor’s execution cost is smaller. Put differently, the smaller a supplier’s execution cost, \(c\), the lower this supplier’s execution bid, \(w\).

The following theorem provides a strong necessary condition on the Nash equilibria of the game.

**Theorem 13** For a border distribution, assume that all the suppliers are efficient. Define \(c^{n+1} = p, f^{n+1} = 0\). Then, in every Nash equilibrium, supplier \(i, i = 1, \ldots, n,\)
places its bid \((w^i, v^i)\):

- in the segment \([(c_1^{i-1}, f_1^{i-1}); (c_1^i, f_1^i)], and then \((w^i, v^i) = (w^{i-1}, v^{i-1})\);
- or in the segment \([(c_1^i, f_1^i); (c_1^{i+1}, f_1^{i+1})], and then \((w^i, v^i) = (w^{i+1}, v^{i+1})\).

The theorem thus implies that supplier bids will be clustered in groups of two or three suppliers. This is true since according to the theorem either two suppliers bid somewhere in the segment connecting their true cost parameters, or one supplier bids its true costs and two other suppliers place a similar bid to this one. Thus, in practice, one will observe less bids than the number of suppliers, roughly half of them.

**Equilibria of the game**

![Equilibria of the game](image)

**Figure 6-2:** Example of equilibrium. There are 6 suppliers that form 3 clusters: 1 with 2, 3 with 4 and 5 with 6. As pointed out in the theorem, the bid within a cluster, e.g., the bid of 3 and 4 (full dot in the center of the figure, with the label PRICE next to it), falls in the segment connecting the costs of the two suppliers (hollow dots with the label COST next to them).

The type of competition described in this result has some interesting properties. The most striking feature is that more than one supplier will be offering the same bid.
One may then wonder whether any supplier in that position should instead reduce its bidding a little bit so that it puts its rival out of the market. The answer provided by the theorem is that this is not the case, and that all suppliers offering the same bid are better off by co-existing with someone else. This allows them to capture, among all the orders allocated to the group of suppliers, those which are better suited for their production costs, \((c, f)\). We call this cluster competition, since in equilibrium the market is divided into stable clusters.

The theorem also suggests that every supplier is competing directly with one of its rival suppliers, i.e., with one of the suppliers whose execution cost, \(c\), is either the smallest among all suppliers with higher \(c\), or the highest among all suppliers with lower \(c\).

An important insight from this observation is that the supplier will care only about competing with its closest competitor, and not competing against other suppliers; this implies that in equilibrium, competition is no longer done on a global basis (among all suppliers) but rather locally (between two or three competing suppliers). For instance, if different suppliers use different technologies, and hence incur different costs, each supplier should focus only on competing with those using similar technologies.

Finally, observe that this result does not rule out the existence of multiple Nash equilibria, and in general the set of Nash equilibria contains multiple possibilities. In any case, this result shows that the possible equilibria are restricted to option prices, \((w, v)\), which belong to the lower envelope of the true suppliers’ costs. Such equilibria should satisfy the optimality conditions in Equations (6.4) and (6.5). The following example illustrates the multiplicity of Nash equilibria.

**Example 9** Assume that customer demand is uniformly distributed in \([0, 1]\). It is thus a border distribution, as shown in Theorem 10. Let \(n = 2\) and the true costs be

\[
(c^1, f^1) = (0, 40), \quad (c^2, f^2) = (40, 20), \quad p = 100.
\]

Both suppliers are efficient. For any \(w \in (0, 40)\), the following bids form Nash equi-
(w^1, v^1) = (w^2, v^2) = (w, 40 - w/2), \quad y_1 = 0.5, \quad y_2 = 0.5 + \frac{10}{100 - w}.

So far, we provided necessary conditions for the Nash equilibria of the game between the suppliers. We need now to provide sufficient conditions for the existence of equilibrium. This is done in the next section.

### 6.3.2 Existence of equilibria

In the previous chapter, we established that log-concavity guaranteed the desired border distribution property. Another interesting consequence of log-concavity is that it also guarantees the existence of equilibrium in the game.

**Theorem 14** When $f$ is log-concave and all suppliers are efficient, then there exists a Nash equilibrium in pure strategies of the game.

To prove the existence of an equilibrium, we need the following steps.

**Lemma 5** Consider a log-concave distribution $f \in \mathcal{F}$. Let $(c, f), (w^1, v^1), (w^2, v^2) \in \mathbb{R}^2_+$ such that $0 \leq w^1 < c < w^2$ and

$$0 \leq \frac{f - v^2}{v^2 - c} \leq \frac{v^1 - f}{c - v^1} \leq 1.$$

Let $(w^0, v^0) = (w^1 - t, v^1 + t)$ for some $t \geq 0$. Define

$$
\overline{F}(x^0) = \frac{v^0 - f}{c - w^0} \\
\overline{F}(y^0) = \frac{v^2 - w^0}{w^2 - v^0} \\
\overline{F}(x^1) = \frac{v^1 - f}{c - w^1} \\
\overline{F}(y^1) = \frac{v^1 - v^2}{w^2 - v^1} \\
\overline{F}(z) = \frac{f - v^2}{w^2 - c}.
$$

If $L(x^1, y^1) \leq R(y^1, z)$, then $L(x^0, y^0) \leq R(y^0, z)$. 

122
This result simply implies that if a supplier with cost \((c, f)\) prefers bidding next to bid \((w^2, v^2)\) rather than \((w^1, v^1)\), then it also prefers bidding \((w^2, v^2)\) to \((w^0, v^0)\). This is illustrated in Figure 6-3.

![Diagram illustrating the preference of suppliers](image)

Figure 6-3: By moving bid \((w^1, v^1)\) to bid \((w^0, v^0)\) as indicated by the arrow, the optimal behavior of a supplier with cost \((c, f)\) is still to bid \((w^*, v^*) = (w^2, v^2)\).

**Lemma 6** Consider a log-concave distribution \(f \in \mathcal{F}\). Let \((c, f), (w^1, v^1), (w^2, v^2) \in \mathbb{R}^2_+\) such that \(0 \leq w^1 < c < w^2\) and

\[
0 \leq \frac{f - v^2}{w^2 - c} \leq \frac{v^1 - f}{c - w^1} \leq 1.
\]
Let \((w^3, v^3) = (w^1 + t, v^1)\) for some \(t \geq 0\). Define

\[
\begin{align*}
\overline{F}(x) &= \frac{v^1 - f}{v^1 - w^1} \\
\overline{F}(y^2) &= \frac{w^2 - v^3}{w^2 - w^3} \\
\overline{F}(z^2) &= \frac{f - v^3}{v^1 - v^3} \\
\overline{F}(y^3) &= \frac{w^3 - v^3}{w^3 - c} \\
\overline{F}(z^3) &= \frac{f - v^3}{w^3 - c}.
\end{align*}
\]

If \(L(x, y^2) \geq R(y^2, z^2)\), then \(L(x, y^3) \geq R(y^3, z^3)\).

In what follows, we prove the existence theorem. Unlike usual equilibrium existence proofs, we show the result by explicitly constructing a set of strategies that form an equilibrium.

By definition, a set of strategies \(\left(\bar{w}^i, \bar{v}^i\right)_{i=1,\ldots,n}\) is an equilibrium when each supplier \(i\) has no incentive to change its strategy unilaterally. That is, for each supplier \(i\), it is optimal to bid \((w^i, v^i) = (\bar{w}^i, \bar{v}^i)\).

The equilibrium that we propose is such that \(\bar{w}^1 \leq \ldots \leq \bar{w}^n\), and, whenever \(\bar{w}^i = \bar{w}^j\), then \(c^i < c^j\). To apply Theorem 12, we define

\[
\overline{F}(y^{i(i-1)}) = \frac{\bar{v}^{i-1} - f^i}{c^i - \bar{w}^{i-1}}, \tag{6.6}
\]

\[
\overline{F}(y^{i(i+1)}) = \frac{f^i - \bar{v}^{i+1}}{\bar{w}^{i+1} - c^i}, \tag{6.7}
\]

\[
\overline{F}(y^i) = \frac{\bar{v}^{i-1} - \bar{v}^{i+1}}{\bar{w}^{i+1} - \bar{w}^{i-1}}. \tag{6.8}
\]

Thus, supplier \(i\) bidding \((w^i, v^i) = (\bar{w}^{i-1}, \bar{v}^{i-1})\) is in equilibrium if and only if \(L(y^{i(i-1)}, \bar{y}^i) \geq R(\bar{y}^i, y^{i(i+1)})\); supplier \(i\) bidding \((w^i, v^i) = (\bar{w}^{i+1}, \bar{v}^{i+1})\) is in equilibrium if and only if \(L(y^{i(i-1)}, \bar{y}^i) \leq R(\bar{y}^i, y^{i(i+1)})\).

Consider also the quantities \(y^{i*}\) such that

\[
\overline{F}(y^{i*}) = \frac{\bar{v}^{i-1} - f^i}{c^i - c^{i-1}}. \tag{6.9}
\]

124
Assume for now that every supplier bids its true costs. Each one is of course not optimizing its behavior. For each \( i = 1, \ldots, n \), let \( \text{dir}(i) \) be the supplier against which supplier \( i \) would compete, i.e. \( \text{dir}(i) = i - 1 \) if the best possible bid of \( i \) in this situation is \( (w^i, v^i) = (c^{i-1}, f^{i-1}) \), and \( \text{dir}(i) = i + 1 \) when the best bid is \( (w^i, v^i) = (c^{i+1}, f^{i+1}) \).

We thus can break the suppliers into groups of consecutive suppliers such that the members with lower indices are such that \( \text{dir}(i) = i + 1 \) and the ones with higher indices are such that \( \text{dir}(i) = i - 1 \). For instance, if there are \( n = 6 \) suppliers and

\[
\begin{align*}
\text{dir}(1) &= 2 & \text{dir}(2) &= 3 & \text{dir}(3) &= 2 \\
\text{dir}(4) &= 3 & \text{dir}(5) &= 6 & \text{dir}(6) &= 7 \text{[the dummy supplier]},
\end{align*}
\]

then we form two groups, \{1, 2, 3, 4\} and \{5, 6\}. The idea of the proof is to use this structure in order to form an equilibrium.

Clearly, by applying Theorem 12, \( \text{dir}(i) = i + 1 \) if \( L(y^{i-1*}, \hat{y}^{i**}) \leq R(\hat{y}^{i*}, y^{i+1*}) \) and \( \text{dir}(i) = i - 1 \) otherwise, where

\[
\bar{F}(\hat{y}^{i**}) = \frac{f^{i-1} - f^{i+1}}{c^{i+1} - c^{i-1}}.
\]

(6.10)

In each group, consider the following bidding strategy. Let \( n_+ \) the number of suppliers within the group such that \( \text{dir}(i) = i + 1 \) and \( n_- \) the number of those such that \( \text{dir}(i) = i - 1 \). We can denote the suppliers as \( 1, \ldots, n_- + n_+ \), by calling the first in the group 1, the second 2 and so on. Define the strategy with the 4 different cases.

(i) When \( n_+ \) and \( n_- \) are both even, then for \( k = 1, \ldots, n_- / 2 \),

\[
\begin{pmatrix}
  w^{2k-1} \\ v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
  w^{2k} \\ v^{2k}
\end{pmatrix} = \begin{pmatrix}
  c^{2k} \\ f^{2k}
\end{pmatrix},
\]

and for \( k = n_- / 2 + 1, \ldots, (n_- + n_+) / 2 \),

\[
\begin{pmatrix}
  w^{2k-1} \\ v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
  w^{2k} \\ v^{2k}
\end{pmatrix} = \begin{pmatrix}
  c^{2k-1} \\ f^{2k-1}
\end{pmatrix}.
\]

125
(ii) When \( n_- \) is even and \( n_+ \) is odd, then for \( k = 1, \ldots, n_-/2 \),

\[
\begin{pmatrix}
w^{2k-1} \\
v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
w^{2k} \\
v^{2k}
\end{pmatrix} = \begin{pmatrix}
c^{2k} \\
f^{2k}
\end{pmatrix},
\]

\[
\begin{pmatrix}
w^{n_-+1} \\
v^{n_-+1}
\end{pmatrix} = \begin{pmatrix}
c^{n_-} \\
f^{n_-}
\end{pmatrix},
\]

and for \( k = n_-/2 + 1, \ldots, (n_- + n_+ - 1)/2 \),

\[
\begin{pmatrix}
w^{2k} \\
v^{2k}
\end{pmatrix} = \begin{pmatrix}
w^{2k+1} \\
v^{2k+1}
\end{pmatrix} = \begin{pmatrix}
c^{2k} \\
f^{2k}
\end{pmatrix}.
\]

(iii) When \( n_- \) is odd and \( n_+ \) is even, then for \( k = 1, \ldots, (n_- - 1)/2 \),

\[
\begin{pmatrix}
w^{2k-1} \\
v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
w^{2k} \\
v^{2k}
\end{pmatrix} = \begin{pmatrix}
c^{2k} \\
f^{2k}
\end{pmatrix},
\]

\[
\begin{pmatrix}
w^{n_-} \\
v^{n_-}
\end{pmatrix} = \begin{pmatrix}
c^{n_-+1} \\
f^{n_-+1}
\end{pmatrix},
\]

and for \( k = (n_- - 1)/2 + 1, \ldots, (n_- + n_+ - 1)/2 \),

\[
\begin{pmatrix}
w^{2k} \\
v^{2k}
\end{pmatrix} = \begin{pmatrix}
w^{2k+1} \\
v^{2k+1}
\end{pmatrix} = \begin{pmatrix}
c^{2k} \\
f^{2k}
\end{pmatrix}.
\]

(iv) When \( n_- \) and \( n_+ \) are both odd, then for \( k = 1, \ldots, (n_- - 1)/2 \),

\[
\begin{pmatrix}
w^{2k-1} \\
v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
w^{2k} \\
v^{2k}
\end{pmatrix} = \begin{pmatrix}
c^{2k} \\
f^{2k}
\end{pmatrix},
\]

\[
\begin{pmatrix}
w^{n_-} \\
v^{n_-}
\end{pmatrix} = \begin{pmatrix}
w^{n_-+1} \\
v^{n_-+1}
\end{pmatrix} = \begin{pmatrix}
(c^{n_-} + c^{n_-+1})/2 \\
f^{n_-} + f^{n_-+1}/2
\end{pmatrix},
\]

126
and for \( k = (n_- + 1)/2 + 1, \ldots , (n_- + n_+) / 2, \)

\[
\begin{pmatrix}
    w^{2k-1} \\
    v^{2k-1}
\end{pmatrix} = \begin{pmatrix}
    w^{2k} \\
    v^{2k}
\end{pmatrix} = \begin{pmatrix}
    c^{2k-1} \\
    f^{2k-1}
\end{pmatrix}.
\]

Figure 6-4: Example of the modifications performed in cases 1 (left figure) and 4 (right). The dots represent the costs of 6 (left) and 4 (right) suppliers respectively, and the ones connected by lines fall in the same group. The arrows represent how we modify the bids as a function of the cost: the beginning of the arrow is the cost of a given supplier, and the end of the arrow is where the corresponding bid falls.

Observe that we have made several modifications. We will show that for each one of them, the corresponding supplier is in equilibrium. Thus, we construct a set of strategies that is a Nash equilibrium in pure strategies.

For this purpose, we use that the function \( L \) is non-decreasing in its second argument, and \( R \) is non-increasing in its first argument, as shown in Lemma 3.

- When a supplier is first in a given group, then \( \text{dir}(i) = i + 1 \), and \( y^{i(i+1)} = y^{i*} \).

Moreover, define \( \hat{y}^{iM} \) such that

\[
\overline{F}(\hat{y}^{iM}) = \frac{f^{i-1} - \hat{y}^{i+1}}{w^{i+1} - c^{i-1}}.
\]

In all cases \( \hat{y}^{iM} \leq \hat{y}^{i*} \), as defined in Equation (6.10). Since we had \( L(y^{i-1*}, \hat{y}^{i*}) \leq R(\hat{y}^{i*}, y^{i*}) \), we must have \( L(y^{i-1*}, \hat{y}^{iM}) \leq R(\hat{y}^{iM}, y^{i(i+1)}) \). Consider now the
change in the bid of supplier $i - 1$, that belongs to a different group: we have $\text{dir}(i - 1) = i - 2$ necessarily. Thus, $\bar{w}^{i-1} < c^{i-1}$ and $\tilde{\bar{v}}^{i-1} = f^{i-1} + \alpha(c^{i-1} - \bar{w}^{i-1})$, for some $\alpha \in [\bar{F}(\bar{y}^{i-1}), 1]$. Of course, it is sufficient to consider $\alpha = 1$. By Lemma 5, we obtain that $L(y^{(i-1)}, \tilde{\bar{y}}^i) \leq R(\tilde{\bar{y}}^i, y^{(i+1)})$, i.e., supplier $i$ is in equilibrium by bidding the same as supplier $i + 1$.

- Similarly, when a supplier is last in a given group, the same argument together with Lemma 6 implies that the supplier is in equilibrium.

- Suppose finally that a supplier is not first or last in a group. The first possibility happens when $\text{dir}(i) = i + 1$. Since the supplier is not first, $\text{dir}(i - 1) = i$. Two cases can occur: that $\bar{w}^{i-1} = c^i$, and clearly in that case, supplier $i$ cannot do better than bid its own cost; or $\bar{w}^{i-1} = c^{i-1}$, in which case, bidding with supplier $i + 1$ ensures that $L(y^{(i-1)}, \tilde{\bar{y}}^i) \leq R(\tilde{\bar{y}}^i, y^{(i+1)})$. The second and last possibility is that $\text{dir}(i) = i - 1$, in which case $\text{dir}(i + 1) = i$, and the same argument applies.

### 6.3.3 Supply chain profit bounds

Finally, to conclude this section, we provide a bound on the inefficiencies created by suppliers’ competition. We define the total welfare as follows:

$$U = (\text{PROFIT OF BUYER}) + \sum_{i=1}^{n} (\text{PROFIT OF SUPPLIER } i).$$

The payments between buyer and suppliers will cancel out, and this quantity will only capture the true revenue from customers minus the costs of production. Thus, we can express the total supply chain welfare as

$$U = p \int_0^{y_n} \bar{F}(u) du - \sum_{i=1}^{n} f^i(y^i - y^{i-1}) - \sum_{i=1}^{n} c^i \int_{y^{i-1}}^{y^i} \bar{F}(u) du$$

$$= \sum_{i=1}^{n} \Delta c^i \int_0^{y^i} [\bar{F}(u) - \bar{F}(y^*)] du.$$
where \( \overline{F}(y^{i*}) = \frac{f_i^i - f_i^{i+1}}{c_i^{i+1} - c_i^i} \) and \( \Delta c^i = c^{i+1} - c^i \). These quantities are well-defined when all the suppliers are efficient. The social welfare is maximized when \( y^i = y^{i*}, \ i = 1, \ldots, n \). In this case, the optimal welfare is

\[
U^* = \sum_{i=1}^{n} \Delta c^i \int_{0}^{y^{i*}} [\overline{F}(u) - \overline{F}(y^{i*})] du.
\]

When the suppliers compete, the allocation of capacities, \( y^i, i = 1, \ldots, n \), is not necessarily efficient, in the sense that it is possible that \( y^i \neq y^{i*} \) for some \( i \). The loss in welfare, due to the suppliers’ competition, is equal to

\[
\Delta U = \sum_{i=1}^{n} \Delta c^i \int_{y^i}^{y^{i*}} [\overline{F}(u) - \overline{F}(y^{i*})] du.
\]

**Theorem 15** Given a border demand distribution and efficient suppliers, in every Nash equilibrium, the allocation of capacities obtains at least 50% of the optimal total welfare, i.e.,

\[
\frac{\Delta U}{U^*} \leq \frac{1}{2}.
\]

This theorem shows that the distortion on the optimal decisions created by competition among suppliers is bounded. This bound is obtained over all demand distributions, but for a specific distribution, the bound might be better. A stronger bound is obtained for the subclass of log-concave distributions, i.e., a class that, as shown after Theorem 10, includes many frequently used distributions.

**Theorem 16** Given a log-concave demand distribution and efficient suppliers, in every Nash equilibrium, the allocation of capacities obtains at least 75% of the optimal total welfare, i.e.,

\[
\frac{\Delta U}{U^*} \leq \frac{1}{4}.
\]

This bound is tight, for \( n = 2 \) suppliers and a \([0, 1] \) uniform distribution. Define the following data, with \( \epsilon \) close to 0.

\[
(c^1, f^1) = (0.50 - \epsilon), \quad (c^2, f^2) = (50, 0), \quad p = 100
\]
The optimal allocation is $y^{1*} = \epsilon/50$ and $y^{2*} = 1$. The bids $w^1 = w^2 = 0$, $v^1 = v^2 = 50 - \epsilon$, together with the allocation $y^1 = \epsilon/50$, $y^2 = 1/2 + \epsilon/100$, form a Nash equilibrium. When $\epsilon$ approaches 0, the welfare loss is

$$
\Delta U = \frac{1}{2} \left( y^1 - y^{1*} \right)^2 + \frac{1}{2} \left( y^2 - y^{2*} \right)^2 \\
= \frac{1}{2} \left( \frac{1}{2} - \frac{\epsilon}{100} \right)^2 \\
\rightarrow \frac{1}{8},
$$

while

$$
U^* = \frac{1}{2} \left( y^{1*} \right)^2 + \frac{1}{2} \left( y^{2*} \right)^2 \\
= \frac{1}{2} \left( \frac{\epsilon}{50} \right)^2 + \frac{1}{2} \\
\rightarrow \frac{1}{2}.
$$

Thus, $\Delta U/U^*$ clearly approaches $\frac{1}{4}$.

In order to prove Theorem 16, we start by reducing the proof to a technical condition on the demand distribution.

**Lemma 7** When all suppliers are efficient, every Nash equilibrium is such that

$$
\Delta U \leq 25\%U^*,
$$

provided that for each $0 \leq x \leq y \leq z$, such that $L(x,y) \geq R(y,z)$,

$$
\left[ F(y) - F(z) \right] \left[ 2L(x,y) - R(x,y) \right] - \left[ F(x) - F(y) \right] R(x,y) \leq 0 \quad (6.11)
$$

The theorem is finally proved using the following technical result.

**Lemma 8** When $f$ is log-concave, then for each $0 \leq x \leq y \leq z$, such that $L(x,y) \geq R(y,z)$,

$$
\left[ F(y) - F(z) \right] \left[ 2L(x,y) - R(x,y) \right] - \left[ F(x) - F(y) \right] R(x,y) \leq 0.
$$
6.4 Equilibria with inefficient suppliers

The previous results, characterizing any Nash equilibrium, are obtained under the assumption that all suppliers are efficient. We now investigate the case in which not all suppliers are efficient.

Interestingly, as we demonstrate below, it might happen that a non-efficient supplier is active at a Nash equilibrium. This occurs because bids are only partially linked to the true costs, and a non-efficient supplier may capture market share by positioning itself in a segment of the market with no, or low, competition.

Example 10 Assume that customer demand is uniformly distributed in $[0, 1]$. Let $n = 3$ and the true costs be

$$(c^1, f^1) = (0, 40), \quad (c^2, f^2) = (40, 20), \quad (c^3, f^3) = (70, 11), \quad p = 100.$$

Clearly, supplier 3 is not efficient. If this was a centralized system, in which the true costs are considered, we would have $y^{1*} = 0.5$, $y^{2*} = 0.666$ and $y^{3*} = 0.666$, and so the buyer would purchase capacities $x^{1*} = 0.5$, $x^{2*} = 0.166$ and $x^{3*} = 0$.

The following bids form a Nash equilibrium:

$$(w^1, v^1) = (w^2, v^2) = (20, 30), \quad (w^3, v^3) = (100, 0), \quad y^1 = 0.5, \quad y^2 = 0.625, \quad y^3 = 0.633.$$

Thus, a non-efficient supplier captures capacity.

Unfortunately, even the statement in Proposition 15 may not be true when inefficient suppliers exist. This is demonstrated by the following example.

Example 11 Assume that customer demand is uniformly distributed in $[0, 1]$. Let $n = 3$ and the true costs be

$$(c^1, f^1) = (0, 40), \quad (c^2, f^2) = (40, 20), \quad (c^3, f^3) = (70, 11), \quad p = 100.$$
Again supplier 3 is not efficient. The following bids form a Nash equilibrium:

\[(w^1, v^1) = (w^2, v^2) = (30, 25), \quad (w^3, v^3) = (0, 55), \quad y^1 = 0.5, \quad y^2 = 0.643, \quad y^3 = 0.\]

Thus, we have \(c^2 < c^3\) and still \(w^3 < w^2\).

The two examples suggest that the presence of inefficient suppliers can lead to counter-intuitive situations. This can be explained by referring to the Bertrand model with asymmetric players. Although it is commonly argued that the only Nash equilibrium in pure strategies is such that the most competitive producer captures all the market at a price equal to the second most competitive cost, as in Tirole [52] p.211, this equilibrium is not unique. As noted by Erlei [22], all the prices between the smallest and the second smallest costs are Nash equilibria of the system. This is true since an inefficient player can impact the market price by placing absurd bids knowing that it will not capture any market share.

The next proposition depicts the behavior of the suppliers at equilibrium, whenever an equilibrium exists.

**Proposition 16** For a border distribution, let \(\{(w^1, v^1), \ldots, (w^n, v^n), (p, 0)\}\) be the bids of the suppliers in a Nash equilibrium. Assume that supplier \(i\) is active. Then we must have that:

- either there is \(j = 1, \ldots, n + 1\) such that supplier \(j\) is active, \((w^j, v^j) = (w^i, v^i)\) and moreover \((w^j, v^j)\) belongs in the segment \([ (c^i, f^i); (c^j, f^j) ]\);

- or there are \(j, k = 1, \ldots, n + 1\) such that supplier \(k\) is inactive, supplier \(j\) is active and \((w^j, v^j) = (w^k, v^k) + \theta(w^k - w^j, v^k - v^j)\) for some \(\theta \geq 0\).

This proposition adds a new case to what was presented in Theorem 13. This new situation arises when an inactive supplier sets the price of some active supplier, which is similar to entry deterrence pricing in industrial organization models, see Tirole [52]. This reaction keeps the inefficient supplier out of the market by making its entry non-profitable. This is illustrated by the next example.
Example 12 Assume that customer demand is uniformly distributed in $[0, 1]$. Let $n = 4$ and the true costs be

$$(c^1, f^1) = (0, 40), \quad (c^2, f^2) = (40, 20), \quad (c^3, f^3) = (70, 6), \quad (c^4, f^4) = (80, 6), \quad p = 100.$$

Supplier 4 is not efficient, the rest are. The following bids form a Nash equilibrium

$$(w^1, v^1) = (w^2, v^2) = (20, 30), \quad (w^3, v^3) = (w^4, v^4) = (80, 4)$$

$$y^1 = 0.5, \quad y^2 = 0.567, \quad y^3 = 0.8, \quad y^4 = 0.8.$$

Evidently, supplier 4, by placing a bid with which it would never make a positive profit, sets the price of supplier 3, who is efficient and must react to the threat of supplier 4.

6.5 Practical implementation and experiments

The analysis conducted so far assumes that the suppliers submit a sealed bid based on perfect information on the customer demand distribution. The existence of multiple equilibria makes this specific mechanism very difficult to implement because the suppliers cannot predict which equilibrium will occur, and thus cannot bid accordingly. This can lead to practical bids being different from the predicted Nash equilibria. The natural way to address this issue would be to study Nash equilibria in mixed strategies, but the complexity of those is tremendous, i.e., the bidding space is a probability distribution in two dimensions. Therefore, for practical purposes, one should develop a mechanism that allows suppliers to obtain some additional information in order to reach a bidding equilibrium.

When all the participants are efficient suppliers, we propose an iterative mechanism that conserves the structure of the Nash equilibria demonstrated in this chapter.

1. Each supplier submits a bid $(w, v)$ to the buyer and the set of bids becomes public information.

2. The buyer publicly declares the allocation of capacity to each supplier and therefore each supplier can estimate its expected profit.
3. If a supplier wants to change its bid, GOTO 1, otherwise TERMINATE.

Intuitively, by imposing technical conditions, such as requesting that a change in a bid should be by at least a given number, δ, we can guarantee that the mechanism terminates in a finite time if the buyer and the suppliers optimize their profit as if this was a single-shot game. Such a change in a bid can be made in any direction (either an increase or a decrease) of reservation and execution prices.

Then, when the process terminates, it must be that no supplier has an incentive to change its bid unilaterally. Hence, the final bids should be very close to the predicted Nash equilibria, but may not be equal if we impose a condition of changing bids by at least δ. This process can be implemented even if suppliers have only information about their own cost and no information about other suppliers' costs.

We have developed a web-based version of the game, where the mechanism described above can be tested in practice. We have obtained some empirical results from experiments with classes of MBA and PhD students at MIT that provide partial evidence that the behavior of suppliers confronted to the iterative process may converge to a Nash equilibrium described in Section 6.3.

The game can be obtained in http://supplychain.mit.edu/procurementportal, and can be played in the classroom.

In our experiments, bidders had complete information on both, demand distribution and other suppliers' cost parameters. Figure 6-5 shows the interface of the tool available to each supplier. Of course, the supplier must estimate what other suppliers may do. Based on this, the supplier can either decide to place a bid manually and test the expected profits obtained, by using the button ESTIMATE, or press the button OPTIMIZE in order to compute the best possible bid as a response to its competitors' bids. The crucial part for a supplier is to guess the behavior of others, since the profits are very sensitive on the decisions of its competitors. In order to resolve this uncertainty, the past bid of every supplier is made public. By doing this, suppliers can exchange information so that they can eventually reach an equilibrium.

Several rounds are played. In our class experiments, the suppliers did not know how many rounds would be played, and we did not announce when the last round
Figure 6-5: Tool available to each one of the suppliers. A given supplier can examine the pay-offs it receives for any set of strategies. After it is comfortable with its bid, it can submit it by pressing the button SUBMIT. It is then directed to the screen shown in Figure 6-6.
Submit Confirmation
You have submitted the following bid

| Execution fee | $65.7 |
| Reservation fee | $7.1 |

Do you want to continue?

Yes, submit bid | No, go back

Figure 6-6: Screen where a supplier is asked to confirm submission of a bid.

was going to be played. This allows us to force suppliers to consider each round as if it was the last round, thus mimicking a single-shot behavior.

We obtained mixed results. The most representative result was obtained in the MBA elective ESD.269J [Advanced Logistics and Supply Chain Strategies]. The results are presented in figure 6-8. Groups 1 and 2, and groups 3 and 4 were virtually bidding the same, and, as predicted by Theorem 13, close to the segment connecting its respective costs. In this respect, this suggests that they may have converged to a stable position, where they would remain in following bids. On the other hand, groups 5, 6 and 7 were bidding on their own, thus not optimizing their bids given what others do.

We should point out that during the experiments, we observed that some of the groups identified some other group as direct competitor, and did not pay attention to anything but that competitor's strategy. This is one of the insights from our model: the competitive interaction forces the suppliers to identify their "natural" competitor and bid close to this rival. Such a behavior naturally bids to cluster competition.

We conducted the same experiment in the PhD class 1.270J/ESD.273J [Logistics and Supply Chain Management]. In this experiment, we observed some qualitatively different behavior, from groups 4 and 5. What happened is the following: these two
Figure 6-7: Monitoring screen for the buyer. It is free to close the current round at any time, while examining what suppliers have submitted a bid. At the bottom, one can find the graph of cost versus current bids.
Figure 6-8: Bids after 3 rounds for an experiment with 7 suppliers from the course ESD.269J [Advanced Logistics and Supply Chain Strategies], in the spring of 2003.

groups maintained the bids shown in Figure 6-9 and were therefore suffering losses. Of course, such a behavior may seem irrational, and indeed did not pay off since when the auction closed, they obtained negative pay-offs. However, these groups realized that, in a multi-round auction, by maintaining such a bid, they would force the competitor to bid close to it, thus providing them with big profits. Specifically, group 4, by bidding close to the cost of group 5, realized that the best strategy for 5 would be to bid its cost, i.e., group 5’s cost; this situation would benefit group 4 enormously. Group 5 had the same intuition. In the end, these two groups threatened each other, and ended up not giving up. The outcome of such threats and nobody giving up before the end of the auction was that they both lost money.

It would be interesting to conduct a more comprehensive experiment, with a significant number of games played by the general public. Through our web-based game, we hope to collect more data in order to derive some conclusions on how decision-makers behave under competition.
Figure 6-9: Bids after 3 rounds for an experiment with 7 suppliers from the course 1.270J/ESD.273J [Logistics and Supply Chain Management], in the fall of 2003.
6.6 Insights

In this chapter, we have presented a model on supplier competition through option contracts for capacity and supply delivery allocation from a buyer. The Nash equilibria in pure strategies give rise to what we have called cluster competition. This provides several insights.

1 It pays to be efficient. No matter how the competitors bid, when a supplier is efficient, it will capture orders from the buyer and will have a non-negative expected profit.

In other words, being an efficient supplier means capturing market share, and no other supplier can push an efficient supplier out of business. Thus, efficiency guarantees long term survival. Notice that our definition of efficiency allows having multiple efficient technologies, because the cost space is two-dimensional. This implies that an inefficient supplier may become efficient by reaching the efficient frontier defined by the lower envelope of the true costs of the other suppliers. Hence, this inefficient supplier does not necessarily have to change technology and copy the same exact cost as other suppliers; what is needed is a local improvement of its costs so as to move to the efficient frontier. This encourages technological variety.

2 It is enough to compete against suppliers with similar technologies. When all suppliers are efficient, a supplier will compete against another supplier with similar technology, either the one with next lower or next higher execution cost.

Thus, two suppliers with completely different technologies will never compete against each other. This local competition characteristic leads to our third insight.

3 Competition preserves diversity and segments the market. At a market equilibrium with efficient suppliers, the suppliers are clustered into small groups of no more than three suppliers and no less than two suppliers. All suppliers within each group offer the same option and share the order from the buyer.
The market will thus be segmented by groups of similar technologies. Competition will diminish technological variety but will not eliminate it. This is in contrast to market behavior in the price-only competition. Thus, in our model, if at some point a supplier "kills" its competitors in a given niche, i.e., a given cluster, and its competitors exit the market, this supplier will increase its market share by moving to a different niche.

4 Prices are directly related to true cost. The equilibrium prices of the different options offered by the suppliers lie in the lower envelope of the costs of the system. That is, the reservation and execution equilibrium prices are linked to the true reservation and execution costs and no inflation of prices is stable.

This insight shows the link between the costs of the system and the option prices available in the market. Specifically, if all suppliers are efficient, this implies a range of possible bids, each of which is along the lower envelope of the true suppliers' costs. However, many equilibria are possible, and hence it is not possible to predict the option prices.

5 Competition leads to a loss of supply chain profit. While suppliers' prices are related to their true costs, the allocation of capacity can be quite different from the one achieved in a centralized system. However, our analysis indicates that the loss of system profit is no more than 25% of the maximum possible.

The results derived in this chapter will be incomplete if we do not mention important extensions of our model. One possible direction is to allow buyers to purchase products at a spot market in addition to using the contracts signed with the suppliers. In such a model, suppliers and buyers negotiate contracts knowing that additional supply or demand are available in the spot market. Such a model would generalize not only the model in the current paper but also the models presented in Wu et al. [53], Spinler et al. [50] and Golovachkina and Bradley [25]. Thus, the insights presented in this section would not only take into account the behavior of competitor suppliers, but also equilibrium expectations on spot market prices.
Chapter 7

Extensions and Conclusion

In this final chapter, we present two extensions of the models developed in the thesis. The first one integrates credit risk considerations into the procurement decision-making. The second extension provides a framework for the treatment of multiple classes of customers, with different pricing characteristics and different demand distributions. Finally, in the last section, we conclude the thesis.

7.1 Application to disruption management

When signing option contracts with suppliers, a manufacturer might question the ability of the suppliers to honor deliveries in full. Similar to credit risk in finance, we can define for every time period $t$ and option contract $i_t$, a random variable, that we call credit, $A_{i_t}^{t}$ in $[0,1]$, such that the amount that can be ordered by the manufacturer from contract $i_t$ at time $t$ is no more than $x_{i_t}^{t} A_{i_t}^{t}$. Thus, the random variable $A_{i_t}^{t}$ corresponds to the effective portion of capacity that the manufacturer is able to request.

Typically, $A_{i_t}^{t}$ can be correlated with customer demand and spot market prices, and depends exclusively on the past information. Formally, for $t = 1, \ldots, T$, at the beginning of period $t$, $\{A_t^i\}_{i=1,\ldots,n_t}$ become known (and thus are included in the information vector $\Phi_t$ as defined in Equation (2.1)) and, based on these, the manufacturer decides the amount of supply to purchase from every contract $(t,i_t)$,
for \( i_t = 1, \ldots, n_t \):

For \( i_t = 1, \ldots, n_t \) request amount 0 \( \leq q_t^{i_t} \leq a_t^{i_t} x_t^{i_t} \),

where \( a_t^{i_t} \) is the outcome of the random variable \( A_t^{i_t} \). With this modeling approach, all the analysis from Chapter 3 can be conducted without changes, and therefore, all the results still hold.

One can thus use this framework to model supplier risk in a given manufacturer's procurement strategy. We present below a single-period version of the model that could be developed.

Consider the following sequence of events.

(A) The manufacturer installs capacity \( x^i \) at each one of the contracts \( (w^i, v^i) \) available, \( i = 1, \ldots, n \).

(B) In the beginning of the selling season, the demand, \( D \), the spot market price and capacity, \( S \) and \( \kappa \), and the availabilities of the suppliers, \( A^i, i = 1, \ldots, n \), are observed with certainty.

(C) Given this information, the manufacturer selects the amounts to purchase from every single contract \( i = 1, \ldots, n \), and from the spot market. Finally, it also decides the amount of demand to be lost, if any.

As presented in Chapter 3, the decisions taken in stage (C) above can be described in a simple form. Theorem 1 describes how the replenishment decisions must be executed. The supply sources are ranked by increasing execution cost. Every source, i.e. an option contract, the spot market or losing sales, is sequentially requested, until the total demand is satisfied. Obviously, we may not fulfill all demand since we are using the possibility of rejecting customers as a way to satisfy demand.

As we will argue, this model is similar to the random yield problem. This is a well known problem described in the literature, with different variations. Gerchak et al. [24] present a multi-period single-supplier inventory model where orders are not satisfied in full but only a random fraction of them are received. They assume that
unsatisfied demand is back-ordered and that orders are placed before demand or the fraction of orders is observed. In this sense, their sequence of events is different than ours. As a consequence, the replenishment policy is different: for each time period, it is characterized by a re-order point $\bar{I}$. The optimal policy is defined as follows

$$Q^*(I) = \begin{cases} 0 & \text{if } I \geq \bar{I}, \\ Q(I) & \text{otherwise.} \end{cases}$$

The function $Q(I)$ is decreasing. Moreover, as described in Henig and Gerchak [28], if the fraction to be received is less than 1 with probability one, as often the case in practice, then

$$\frac{dQ}{dI} \leq -1.$$ 

Thus, by reversing the sequence of events, the structure of the optimal policy changes completely. We can modify our model to follow the usual sequence of events described in Gerchak et al. [24], similarly to the process presented in Section 2.8. Of course, the optimal replenishment policy will this time have a different structure, because of the uncertainty on the fraction of orders received. Hopefully, the concavity results hold, as is demonstrated in Henig and Gerchak [28].

As we can observe, our modeling approach yields qualitatively different results compared to the random yield models. We will argue that our model is well adapted for long term capacity planning purposes, when the objective is to quantify the shares of different suppliers in a manufacturing process. In this situation, the manufacturer typically observes when a supplier is incapable of delivering the full amount specified in the contract or goes out of business. This extended model can be used in designing robust supply chains. For instance, Sheffi [49] discusses qualitatively how to design supply relationships while taking terrorism threats into account. Our model can be used for quantifying optimally the design parameters so as to maximize the manufacturer's expected profit.

In contrast, the random yield problem presented above is used to model the effects of quality problems in sourcing, where the full amount is received by the manufacturer.
but a fraction of it needs to be discarded due to quality issues.

Our model, similarly to Theorem 4, provides a closed form expression for the optimality equation of the portfolio selection problem. This will provide interesting insights on the effects of supply default risk in capacity planning.

For this purpose, we assume that the spot market offers unlimited supply, i.e., \( \kappa = \infty \) with probability 1. We sort the available options by execution price, i.e., \( w^1 \leq \ldots \leq w^n \), and without loss of generality we assume that \( w^n \leq p \). We define \( v^{n+1} = 0 \) and \( w^{n+1} = p \). Assume that \( \mathbb{E}D < \infty \), which means that all expected profits must be finite.

So far, these assumptions are identical to those in Theorem 4.

Let \( A^i, i = 1, \ldots, n \), be the fraction of capacities that will effectively be available from each supplier, \( i, i = 1, \ldots, n \).

The profit obtained by the manufacturer is equal to

\[
J(x) = -\sum_{i=1}^{n} v^i x^i + \mathbb{E}_{(D,S)} \left\{ (p - S)^+ D \right\} \\
+ \mathbb{E}_{(D,S,A^1,\ldots,A^n)} \left\{ \sum_{i=1}^{n} \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \min \left( D, \sum_{k=1}^{i} A^k x^k \right) \right\}.
\]

This is obtained in the same way as Theorem 4. Using that for any random variable \( Z \geq 0 \) and any constant \( a \geq 0 \), \( \mathbb{E}[\min(Z,a)] = \int_{0}^{a} \mathbb{P}[Z \geq t] dt \), we can write this expression as

\[
J(x) = -\sum_{i=1}^{n} v^i x^i + \mathbb{E}_{(D,S)} \left\{ (p - S)^+ D \right\} \\
+ \mathbb{E}_{(S,A^1,\ldots,A^n)} \left\{ \sum_{i=1}^{n} \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \times \int_{0}^{\sum_{i=1}^{n} A^k x^k} \mathbb{P}[D \geq t|A^1,\ldots,A^n,S] dt \right\}.
\]

Define

\[
\bar{F}(x,a^1,\ldots,a^n,s) = \mathbb{P} \left[ D \geq x \mid A^k = a^k \ \forall k, S = s \right].
\]

This represents the conditional distribution of customer demand as a function of the credits of every supplier. We assume that this function is differentiable in \( x \) for
all choices of \( a \) and \( s \).

Thus, the function \( J(\cdot) \) is clearly twice differentiable, and

\[
\frac{dJ}{dx^i} = -v^i + E \left\{ A^i \left[ \sum_{l=i}^{n} \left[ \min(S, w^{l+1}) - \min(S, w^l) \right] \hat{F} \left( \sum_{k=1}^{l} A^k x^k, A^1, \ldots, A^n, S \right) \right] \right\},
\]

\[
\frac{d^2 J}{dx^i dx^j} = E \left\{ A^i A^j \left[ \sum_{l=j}^{n} \left[ \min(S, w^{l+1}) - \min(S, w^l) \right] \frac{d\hat{F}}{dx} \left( \sum_{k=1}^{l} A^k x^k, A^1, \ldots, A^n, S \right) \right] \right\},
\]

for \( i \leq j \).

It is clear that the function \( J(\cdot) \) is concave, since \( \frac{d\hat{F}}{dx} \leq 0 \). This guarantees that KKT conditions are sufficient and necessary at optimality.

When the credit is certain, i.e. \( A^i = 1 \) with probability 1 for all \( i \), we obtain the optimality equations from Theorem 4. We can observe that the variable \( y^i = \sum_{k=1}^{i} x^k \), \( i = 1, \ldots, n \), appears naturally. In the case of uncertain credits, this is no longer true.

Some insights can be derived when demand is independent of the yields of the suppliers and the spot market price, as one might expect in practice. Then

\[
\hat{F}(x, a^1, \ldots, a^n, s) = \hat{F}(x) = \mathbb{P}[D \geq x].
\]

We observe that under these assumptions the optimality equation, i.e., the first-order condition, does not only involve the calculation of a percentile, i.e. solve \( q = \hat{F}(x) \) for some \( q \), as in the certain yield case, but utilizes an expectation of such percentiles. This implies that the optimal portfolio will be much more sensitive to the entire demand distribution and not only to some specific quantile values.

In addition, some additional specifications on the demand distribution allow us to derive closed form expressions. If demand is exponentially distributed, the optimality equation can be simplified. Without loss of generality (scaling the problem will scale the optimal portfolio), we assume that the parameter of the exponential is \( \lambda = 1 \). Also, we assume that \( S, A^1, \ldots, A^n \) are pairwise independent. Thus, by writing \( Z^i = \)
\[
\frac{dJ}{dx_i} = -v^i + \mathbb{E}\left\{ A^i \sum_{l=1}^{n} Z^l \left[ \prod_{k=1}^{l} e^{-A^k x^k} \right] \right\}
\]
\[
= -v^i + \prod_{k=1}^{i-1} \mathbb{E}[e^{-A^k x^k}] \mathbb{E}[A^i e^{-A^i x^i}] \left\{ \sum_{l=i}^{n} \mathbb{E}[Z^l] \left[ \prod_{k=i+1}^{l} \mathbb{E}[e^{-A^k x^k}] \right] \right\}
\]
\[
= -v^i + \prod_{k=1}^{i-1} \mathbb{E}[e^{-A^k x^k}] \mathbb{E}[A^i e^{-A^i x^i}] \mathbb{E}[Z^i]
\]
\[
+ \prod_{k=1}^{i-1} \mathbb{E}[e^{-A^k x^k}] \frac{\mathbb{E}[A^i e^{-A^i x^i}] \mathbb{E}[e^{-A^{i+1} x^{i+1}}]}{\mathbb{E}[A^{i+1} e^{-A^{i+1} x^{i+1}}]} (\frac{dJ}{dx_i} + v^{i+1}).
\]

In the case of an interior solution, the first-order conditions are the optimality equations: \( \frac{dJ}{dx_i} = 0 \) for all \( i = 1, \ldots, n \). We hence obtain \( n \) equations to be solved in order to find the optimal portfolio.

\[
v^i = \left[ \prod_{k=1}^{i-1} \mathbb{E}[e^{-A^k x^k}] \right] \mathbb{E}[A^i e^{-A^i x^i}] \mathbb{E}[Z^i]
\]
\[
+ \frac{\mathbb{E}[A^i e^{-A^i x^i}] \mathbb{E}[e^{-A^{i+1} x^{i+1}}]}{\mathbb{E}[A^{i+1} e^{-A^{i+1} x^{i+1}}]} v^{i+1} \quad \text{for } i < n,
\]

(7.1)

\[
v^n = \mathbb{E}[Z^n] \left[ \prod_{k=1}^{n-1} \mathbb{E}[e^{-A^k x^k}] \right] \mathbb{E}[A^n e^{-A^n x^n}].
\]

Given this model, we can now solve the problem posed in Sheffi [49]. A high technology company sells medical devices made by a contract manufacturer in Malaysia. The Malaysian supplier delivers the devices at $100 a piece and the devices are sold by the US company at $400 each. Fixed costs, including marketing and setup, have been estimated at $200 per device. The company estimated that there is a 1% probability that the Malaysian supplier will not be able to deliver for an extended period (we model this as a single period). A local supplier can deliver the same devices for $150 each.
One can assume that the final price is $p = 200$, in order to take into account the loaded costs. There is no spot market and thus $S = \infty$ with probability 1. Denote the Malaysian supplier as supplier 1 and the local supplier as supplier 2:

$$v^1 = 100, w^1 = 0, A^1 = \begin{cases} 
0 \text{ w.p. } \alpha \\
1 \text{ w.p. } 1 - \alpha 
\end{cases}$$

$$v^2 = 150, w^2 = 0, A^2 = 1 \text{ w.p. } 1$$

Thus $Z^1 = 0$ and $Z^2 = 200$. We can also assume that the demand is exponentially distributed, with parameter 1. Three cases will occur, depending on the disruption probability $\alpha$, and whether both suppliers are chosen in the optimal portfolio or only one of them.

1. In the first case, $\alpha$ is such that the manufacturer purchases from both suppliers. Thus, first order conditions described in Equation (7.1) are sufficient for optimality:

$$v^2 = 150 = \mathbb{E}[Z^2|E[e^{-A^1 x^1}] E[A^2 e^{-A^2 x^2}]]$$

$$= 200(\alpha + (1 - \alpha)e^{-x^1})e^{-x^2},$$

$$v^1 = 100 = \mathbb{E}[A^1 e^{-A^1 x^1}] \mathbb{E}[Z^1] + \frac{\mathbb{E}[A^1 e^{-A^1 x^1} \mathbb{E}[e^{-A^2 x^2}]]}{\mathbb{E}[e^{-A^2 x^2}]} v^2$$

$$= 150 \frac{(1 - \alpha)e^{-x^1}}{e^{-x^2}} \frac{e^{-x^2}}{\alpha + (1 - \alpha)e^{-x^2}}.$$

The second equation provides the optimal $x^1$, and from there $x^2$ is selected from the first equation. Thus,

$$\frac{2}{3} = \frac{(1 - \alpha)e^{-x^1}}{\alpha + (1 - \alpha)e^{-x^1}}.$$

This is solved in

$$e^{-x^1} = \frac{2\alpha}{1 - \alpha},$$

149
and implies that

\[ e^{-x^2} = \frac{1}{4\alpha}. \]

Of course, this happens when \( \frac{1}{4} \leq \alpha \leq \frac{1}{3} \).

2. In the second case, only supplier 1 is active in the optimal portfolio. This happens when the disruption probability is small enough, i.e. \( \alpha \leq \frac{1}{4} \). In this case, \( x^{2*} = 0, \) and

\[ \frac{dJ}{dx} = -v^1 + E[(Z^1 + Z^2)A^1 e^{-A^1 x^1}] = -100 + 200(1 - \alpha) e^{-x^1}. \]

Clearly

\[ e^{-x^{1*}} = \frac{1}{2(1 - \alpha)}. \]

3. In the third case, only supplier 2 is active in the optimal portfolio. This happens when the disruption probability is too big, i.e. \( \alpha \geq \frac{1}{3} \). In this case, \( x^{1*} = 0, \) and

\[ \frac{dJ}{dx} = -v^2 + E[Z^2 A^2 e^{-A^2 x^2}] = -150 + 200 e^{-x^2}. \]

Thus,

\[ e^{-x^{2*}} = \frac{3}{4}. \]

Figure 7-1 shows the behavior of \( x^{1*}, x^{2*} \) as a function of \( \alpha \). We observe that the optimal portfolio’s structure changes at \( \alpha = 1/4 \) and \( \alpha = 1/3 \). These breakpoints depend on the actual parameters \( p, v^1, v^2 \). Thus one can repeat this analysis for the general case following the same steps used here.

Finally, simulation results can be provided for other distributions, different from the exponential.
Figure 7-1: Optimal portfolio quantities as a function of the disruption probability $\alpha$. 
7.2 Multiple selling channels

In many real situations, a manufacturer that has unsold inventory left at the end of the selling season may obtain additional revenue by selling the inventory at a discount.

Consider the problem of a retailer that sells a fashion product in its Boston downtown branch during the season, and pushes all unsold units to a Wrentham Village outlet shop. Of course, the selling price at the outlet will be cheaper that the one at the downtown branch. In this case, price discrimination can be implemented because the selling times do not overlap.

One can also examine the situation of a manufacturer buying from and selling to a spot market. If the market is perfect, i.e., the buying and selling spot prices are equal, then the financial model analyzed in Section 4.2.1 applies. If, on the other hand, there is a bid-ask spread in the spot market the situation is significantly more complex. We can show that this problem is equivalent to the problem of selling to two different channels, one at full price, and the other at the spot bid price.

Both these situations can be modeled moving from the traditional single-demand newsvendor model to a multi-class setting.

7.2.1 The revenue model

Intuitively, instead of observing an arrival of $Q^1$ customers who are willing to pay $P^1$ dollars per unit of the product, the manufacturer will respond to multiple simultaneous channels $J$. Each one of these represents a stream of $Q^j$ customers who pay $P^j$ per unit. Without loss of generality, we will assume that $P^1 \geq \ldots \geq P^J$ in every outcome, since we can always sort the classes by decreasing order of price. This implies that the optimized revenue of serving a total of $q$ customers is

$$R(q) = \max \sum_{j=1}^{J} P^j q^j$$

subject to

$$\sum_{j=1}^{J} q^j = q$$

$$0 \leq q^j \leq Q^j \quad i = j, \ldots, J$$

152
Since the sorting of the classes is made by decreasing willingness-to-pay, the optimal demand filling policy for a fixed supply \( q \) is greedy: the manufacturer will satisfy class 1 first, and if after serving \( Q^1 \) customers, there is some inventory left, it turns to serving class 2, and so on.

Define
\[
T^j = Q^1 + \ldots + Q^j.
\] (7.2)

The previous arguments yields that
\[
q^j* = \min \left( Q^j, (q - \sum_{k=1}^{j-1} Q^{k-1})^+ \right) = \min(q, T^j) - \min(q, T^{j-1}).
\]

Thus,
\[
R(q) = \sum_{j=1}^{J} P^j \left( \min(q, T^j) - \min(q, T^{j-1}) \right).
\] (7.3)

Such a framework allows both the number of customers of class \( j \), \( Q^j \), and the price paid by this class, \( P^j \), to be random. This modeling approach captures, for instance, the situation where the manufacturer has two different selling channels at fixed prices, such as a business clientele and a leisure clientele for air transportation, a home and office distribution channel for PC retailing, a downtown and outlet retail point for clothing, etc. In addition, it also allows us to model situations in which a manufacturer sells a product at full price during the season and then salves unsold inventory in a spot market at a random price; in this case, \( P^1 \) is fixed, \( Q^1 \) is random, and \( P^2, Q^2 \) are random.

### 7.2.2 Cost structure with spot market

To be consistent with the notation in the thesis, the sources of supply are the following.

- The spot market offers unlimited supply at a unit price \( S \).

- The manufacturer has purchased \( x^i \) units of capacity from the \( i \)-th option contract, with execution price \( w^i \) and reservation price \( v^i, i = 1, \ldots, I - 1 \). Thus, it can request \( 0 \leq q^i \leq x^i \) units from option \( i \).
This specification, together with the definitions

\[ y^i = x^i + \ldots + x^i, \quad i = 1, \ldots, I - 1, \]

\[ y^I = \infty \text{ and } w^I = \infty, \quad v^I = 0, \]

implies that the cost function is, writing \( W^i = \min(w^i, S) \),

\[ C(q) = -\sum_{i=1}^{I} \Delta v^i y^i + \sum_{i=1}^{I} \Delta W^i \min(y^i, q), \quad (7.4) \]

where \( \Delta v^i = v^{i+1} - v^i \) and \( \Delta W^i = W^{i+1} - W^i \geq 0 \).

Evidently, this formulation also assumes that the only control variables from the manufacturer’s point of view are the capacities \( y^1, \ldots, y^{I-1} \), instead of \( y^1, \ldots, y^I \), since here \( y^I = \infty \) always. In this formulation, we will always be able to go back to the no-spot case by defining \( S = \infty \) with probability 1.

### 7.2.3 Profit computation

Given revenue and cost specifications, we can proceed to derive an expression for profit,

\[ \Pi^* = \max_{q \geq 0} \left( R(q) - C(q) \right). \quad (7.5) \]

Clearly, the assumption that the manufacturer does not have control over incoming demand volumes and prices implies that the optimal selling quantity will be such that the marginal revenue is equal to the marginal cost. This is true since the revenue function is concave and the cost function convex.

In addition, note that when there is a spot market, in order to obtain a finite solution, we need to impose the following no-arbitrage condition.

**Assumption 17** For \( j = 1, \ldots, J \), if the demand at price \( P^j \) is infinite, i.e. \( T^j = \infty \), then this price must be no more than the spot market price, i.e. \( P^j \leq S \).

Of course, if \( T^j < \infty \), then there is a finite number of units for which demand exists at a price \( P^j \). The manufacturer can realize a profit on these units, but this
will never allow for infinite profits.

By denoting \( q^* \) a quantity (not necessarily unique) that clears the market, i.e. corresponding to the equilibrium between demand and supply, we can write

\[
\Pi^* = \sum_{i=1}^{I} \Delta v^i y^i + \int_{0}^{q^*} \left( R'(q) - C'(q) \right) dq = \sum_{i=1}^{I} \Delta v^i y^i + \int_{0}^{\infty} \left( R'(q) - C'(q) \right)^+ dq.
\]

We can break this integral into the following parts

\[
\sum_{j=1}^{J} \int_{T_{j-1}}^{T_j} \left( R'(q) - C'(q) \right)^+ dq = \sum_{j=1}^{J} \int_{T_{j-1}}^{T_j} \left( P^j - C'(q) \right)^+ dq,
\]

that we will analyze separately.

For every \( j = 1, \ldots, J \), let \( i_{\text{low}(j)} \) be the first contract that serves class \( j \), i.e. \( y^{i_{\text{low}(j)}-1} \leq T^{j-1} \leq y^{i_{\text{low}(j)}} \); and \( i_{\text{high}(j)} \) be the last one, i.e. \( y^{i_{\text{high}(j)}-1} \leq T^j \leq y^{i_{\text{high}(j)}} \)

\[
\int_{T_{j-1}}^{T_j} \left( P^j - C'(q) \right)^+ dq = \int_{T_{j-1}}^{T_j} \sum_{i=1}^{I} \left( P^j - W^i \right)^+ 1(y^{i-1} \leq q \leq y^i) dq
\]

\[
= \sum_{i=1}^{I} \int_{T_{j-1}}^{T_j} \left( P^j - W^i \right)^+ 1(y^{i-1} \leq q \leq y^i) dq
\]

\[
= \left( P^j - W^{i_{\text{low}(j)}} \right)^+ \left( y^{i_{\text{low}(j)}} - T^{j-1} \right)
\]

\[
+ \sum_{i=i_{\text{low}(j)}+1}^{i_{\text{high}(j)-1}} \left( P^j - W^i \right)^+ \left( y^i - y^{i-1} \right)
\]

\[
+ \left( P^j - W^{i_{\text{high}(j)}} \right)^+ \left( T^j - y^{i_{\text{high}(j)}-1} \right)
\]

\[
= \left( P^j - W^{i_{\text{low}(j)}} \right)^+ \left( \min(y^{i_{\text{low}(j)}}, T^j) - T^{j-1} \right)
\]

\[
+ \sum_{i=i_{\text{low}(j)}+1}^{I} \left( P^j - W^i \right)^+ \left( \min(y^i, T^j) - \min(y^{i-1}, T^j) \right).
\]

The last expression can be reformulated using the Abel sum technique, i.e. for any \( a_k, b_k \ k = 1, \ldots, K \), defined \( a_{K+1} = 0 \) and \( b_0 = 0 \), to get

\[
\sum_{k=1}^{K} a_k (b_k - b_{k-1}) = \sum_{k=1}^{K} (a_k - a_{k+1}) b_k.
\]

155
In our case, we use \( a_k = (P^j - W^k)^+ \), and \( b_k = \min(y^k, T^j) - T^{j-1} \). Hence, using that \( W^{T+1} = \infty \),

\[
\int_{T_{j-1}^j}^{T_j^j} \left( P^j - C'(q) \right)^+ \, dq = \sum_{i=i_{\text{low}}(j)} \left\{ \left( P^j - W^i \right)^+ - \left( P^j - W^{i+1} \right)^+ \right\} \left( \min(y^i, T^j) - T^{j-1} \right).
\]

Finally, since for all \( i \geq i_{\text{low}}(j) \), \( y^i \geq T^{j-1} \), we can write \( T^{j-1} = \min(y^i, T^{j-1}) \). Similarly, for all \( i < i_{\text{low}}(j) \), \( y^i \leq T^{j-1} \), we have that \( \min(y^i, T^j) - \min(y^i, T^{j-1}) = y^i - y^i = 0 \). Hence

\[
\int_{T_{j-1}^j}^{T_j^j} \left( P^j - C'(q) \right)^+ \, dq = \sum_{i=1} \left\{ \left( P^j - W^i \right)^+ - \left( P^j - W^{i+1} \right)^+ \right\} \left( \min(y^i, T^j) - \min(y^i, T^{j-1}) \right).
\]

Hence, by writing

\[
Z^{i,j} = \left( P^j - W^i \right)^+ - \left( P^j - W^{i+1} \right)^+ = \min(W^{i+1}, P^j) - \min(W^i, P^j), \quad (7.6)
\]

we can write the profit \( \Pi^* \) as

\[
\Pi^* = \sum_{i=1}^{I} \Delta v^i y^i + \sum_{i=1}^{I} \sum_{j=1}^{J} Z^{i,j} \left( \min(y^i, T^j) - \min(y^i, T^{j-1}) \right) \quad (7.7)
= \sum_{i=1}^{I} \Delta v^i y^i + \sum_{i=1}^{I} \sum_{j=1}^{J} \left( Z^{i,j} - Z^{i,j+1} \right) \min(y^i, T^j).
\]

Thus, we have been able to write the optimal profit as a random variable that is the sum of products of given random variables times the quantities \( \min\left(y^i, T^j\right) \), \( i = 1, \ldots, I \), \( j = 1, \ldots, J \).

The first immediate conclusion that we can derive is that the expected profit is a concave function of the decision variables \( \{x^i\}_{i=1}^{I-1} \). Of course, this can be extended to a multi-period setting, which is a generalization of Theorem 3.
7.2.4 Example

We are interested in modelling the problem of a manufacturer that sells an item at a fixed price $P^1 = p$. $Q^1$ customers are served by this channel. In addition, there is a spot market where the manufacturer can buy and sell during the selling season. This spot market offers supply at price $S$ and buys excess inventory at price $P^2 = S - \Delta$, where $\Delta$ is the bid-ask spread. The spot market is uncapacitated, so that $Q^2 = \infty$. Hence, in this setting, $J = 2$.

In finance, most of the models assume that the market has no transaction costs, i.e. $\Delta = 0$. However, in most real-world operations, there is a cost involved with reselling to the spot market, which we capture by introducing $\Delta$.

With this notation, the manufacturer secures a portfolio of $I-1$ options in advance, with parameters $y^1, \ldots, y^{I-1}$.

Let's assume, for simplicity, that $I = 2$. That is, the manufacturer purchases a single option contract, at execution price $w \leq p$ and reservation price $v$. We have

\[
\begin{align*}
Z^{1,1} &= (p - \min(S, w))^+ - (p - S)^+ = \left(\min(S, p) - w\right)^+ \\
Z^{1,2} &= (S - \Delta - w)^+ \\
Z^{2,1} &= (p - S)^+ \\
Z^{2,2} &= 0
\end{align*}
\]

Thus,

\[
\Pi^* = -vy + (Z^{1,1} - Z^{1,2}) \min(y, Q^1) + Z^{1,2}y + Z^{2,1}Q^1
\]

\[
= -vy + Z^{1,1} \min(y, Q^1) + Z^{1,2} \left(y - Q^1\right)^+ + Z^{2,1}Q^1,
\]

7.3 Conclusion

In this research, we introduce and analyze models for portfolio strategies in supply contracts. This is important to both buyers and suppliers of components, in the face of demand and spot price uncertainties. In this context, using a portfolio of different contracts with different price/flexibility parameters can be beneficial. Namely, by structuring the right portfolio of options with different reservation and execution
prices, a buyer can improve its profitability.

The models presented focus on procurement of commodity products, where contracting with several suppliers simultaneously may be appropriate. This is true since these are standard components available from multiple suppliers. The only parameters that are important to the buyer are cost and flexibility, and these are the two attributes that we describe, through a capacity reservation fee and an capacity usage fee.

We have developed a general framework where portfolios of options can be analyzed and optimized, from a buyer's point of view. Interestingly, most common contracts in the real work, e.g., fixed-commitment, buy-back or quantity-flexibility contracts, can be included in the analysis, by expressing them through options. The insights derived from such models are of two types. First, in a multi-period setting, we show that portfolios are easy to manage dynamically, since the optimal replenishment policies consist of modified base-stock policies. Of course, finding the base-stock levels can be computationally challenging, since it involves solving a dynamic program, that can have a multi-dimensional state space when demand or spot market prices are correlated through time. Second, we show certain properties of optimal portfolios, both for an expected profit and a profit mean-variance objective. Namely, the expected profit is concave in the installed capacities, i.e., the amounts of contracts to be purchased; the mean-variance objective has connected upper-level sets, which guarantees that the optimum is unique and a greedy search algorithm finds it. In addition, we derive dominance results that characterize effective contracts.

We have then presented a supplier competition model, where the problem of pricing an option is examined, from the suppliers' point of view. This game has been modeled as a single-shot bidding stage, where all suppliers simultaneously submit the reservation and execution price of an option to the buyer. After this, the buyer simply follows its best procurement strategy, as presented in the first part of the thesis. Thus, this is a game à la Stackelberg, where the suppliers are leaders, and the buyer a follower. We analyze the Nash equilibria of the game in pure strategies.

To guarantee that the pay-off functions are well-behaved, we define the concept of
border demand distributions. We show that log-concave distributions, i.e., where the logarithm of the p.d.f. is concave, satisfy this condition. Under log-concave demands, we show existence of equilibria. In any of these equilibria, each supplier bids the same as one of its competitors, and obtains a positive market share and a positive pay-off. We call this behavior cluster competition, since clusters of bids are formed. These clusters of two or three suppliers are always in segments connecting the costs of the suppliers. Finally, we show that in any equilibrium, the profit of the entire supply chain is no less than 75% of the first-best, i.e., the profit of a centralized supply chain.

We now briefly discuss future research.

Within the same multi-period framework, it would be interesting to further explore total-capacity-commitment contracts, analyzed in Section 3.6. The reason why this is relevant is that these are common in industry, and, above all, because their financial counterpart is the American option. Our models show how to purchase and exercise what corresponds to a European option. How should one exercise the American option, i.e., the total-capacity-commitment contract? In particular, how should one coordinate it with other sourcing alternatives, such as a spot market? Results in this direction would allow a buyer to estimate the value of this contract for the buyer, in presence of sourcing alternatives.

In the risk analysis direction, it would be interesting to consider risk measures other than variance. Indeed, variance penalizes for profits above average, which is not appropriate. Thus, other measures such as value-at-risk (VaR) would be more appropriate. Our analysis is a good starting point in this direction, since we provide a closed-form expression of the optimal profit as a function of spot price $S$, demand $D$ and capacity decisions $\{y^j\}_{i=1,...,n}$. As shown in Section 7.2, we can even derive this closed-form expression when the buyer can resell to the spot market, with bid-ask spreads. For example, such expression could be studied analytically, in order to provide loss probabilities. Alternatively, one could perform simulation to study the sensitivity of the profit distribution as a function of the capacity decisions.

The supplier competition model provides the most opportunities for further development. We have studied the situation where 2 parameters – reservation and
execution fees – are important to the buyer, who distributes its orders across all suppliers, since this strategy increases its profits compared to single sourcing. The next challenge is of course to extend this model to larger dimensions, by introducing parameters such as quality or lead-time quotes. What is crucial in such models is to find how the allocation of capacity to each supplier depends on all these parameters. Indeed, in simple price-quality models, one can usually show that what matters to the buyer is not the vector (price, quality), but a quality-adjusted price, such as, for instance price \( \times P[\text{good quality}] \). Thus, competition in this setting is one-dimensional, equivalent to the Bertrand model. A truly multi-dimensional model should make the buyer care about a multi-dimensional vector of parameters, and combining them into a one-dimensional score should not be enough. This extension would open the door to original equilibrium structures such as the cluster competition outcomes described in this thesis.
Appendix A

Proofs

A.1 Chapter 2

A.1.1 Lemma 1

Proof. Let $b_1, b_2 \in Q$ and $\lambda \in (0, 1)$. Define $b_3 = \lambda b_1 + (1 - \lambda) b_2$. Since $b_1, b_2 \in Q$, $P(b_1)$ and $P(b_2)$ are not empty, so we can find $x_1 \in P(b_1)$ and $x_2 \in P(b_2)$. Hence, $x_3 = \lambda x_1 + (1 - \lambda) x_2 \in P(b_3)$ and thus $P(b_3) \neq \emptyset$. Thus, $b_3 \in Q$ which implies that $Q$ is a convex set.

We now show that $g$ is concave on $Q$. For this purpose, consider two different cases depending on whether $g(b_1)$ and $g(b_2)$ are finite, or one of them is infinite.

If $g(b_1) < \infty$ and $g(b_2) < \infty$, take $\epsilon > 0$. We can find close-to-optimum values $x_1 \in P(b_1)$ and $x_2 \in P(b_2)$ such that $f(x_1, b_1) \geq g(b_1) - \epsilon$ and $f(x_2, b_2) \geq g(b_2) - \epsilon$. Clearly, $x_3 = \lambda x_1 + (1 - \lambda) x_2 \in P(b_3)$ and thus $g(b_3) \geq f(x_3, b_3)$. Moreover, the concavity of $f$ implies that $f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2)$. Hence,

$$g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2) \geq \lambda g(b_1) + (1 - \lambda) g(b_2) - \epsilon.$$

Since this is true for every $\epsilon > 0$, we must have $g(b_3) \geq \lambda g(b_1) + (1 - \lambda) g(b_2)$.

Consider the second case in which either $g(b_1)$ or $g(b_2)$ is infinite. Without loss of generality, assume that $g(b_1) = \infty$. Then, for every $M > 0$, we can find $x_1 \in P(b_1)$
such that $f(x_1, b_1) \geq M$. Fix $x_2 \in P(b_2)$. Then again $x_3 = \lambda x_1 + (1 - \lambda)x_2 \in P(b_3)$ and $g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda)f(x_2, b_2)$. Hence,

$$g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda)f(x_2, b_2) \geq \lambda M + (1 - \lambda)f(x_2, b_2).$$

Since this is true for every $M > 0$, and since $\lambda > 0$, we have $g(b_3) = \infty$.

In any case, $g(b_3) \geq \lambda g(b_1) + (1 - \lambda)g(b_2)$, so $g$ is concave over $Q$. 

A.1.2 Proposition 2

Proof. We prove the property by induction on $t$ starting at period $T+1$ and moving backwards to the first period. For $t = T+1$, $V_{T+1}(\cdot, \Phi_{T+1})$ is a linear function of $I_{T+1}$ and thus concave.

Assume now that the statement is true for $t+1$: given $\Phi_{t+1}, V_{t+1}(\cdot, \Phi_{t+1})$ is concave in $I_{t+1}$. We show the inductive proposition for $t$. For this purpose, we assume that $\Phi_t$ is given, and hence the values of $d_t$ and $s_t$ are known. Since $\Phi_{t+1} = (\Phi_t, s_{t+1}, d_{t+1})$ is a random variable that depends only on $\Phi_t$, the distribution of $\Phi_{t+1}$ is also known.

Given $\Phi_{t+1}$, the expected profit-to-go in period $t+1$, $V_{t+1}(\cdot, \Phi_{t+1})$, is concave in $I_{t+1}$. Let $U_{t+1}(I_{t+1}) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1})$ which is also concave in $I_{t+1}$. Thus, we can rewrite $V_t$ as follows:

$$V_t(I_t, \Phi_t) = p_t I_t - h_t(I_t) + \max_{q_t^i, q_t^s} p_t(-I_{t+1} + q_t^i + q_t^s) - r_t(q_t^i) - s_t(q_t^s) + U_{t+1}(I_{t+1})$$

subject to:

$$\begin{align*}
-q_t^i & \leq 0 \\
-q_t^s & \leq 0 \\
-I_{t+1} & \leq 0 \\
-I_{t+1} + q_t^i + q_t^s & \leq I_t \\
I_{t+1} - q_t^i - q_t^s & \leq d_t - I_t
\end{align*}$$

The maximization problem is constrained by linear inequalities involving $(q_t^i, q_t^s, I_{t+1})$. 

162
Also, the objective function,

\[ p_t(-I_{t+1} + q^t_t + q^s_t) - r_t(q^t_t) - s_t(q^s_t) + U_{t+1}(I_{t+1}), \]

is jointly concave in the control variables, since it is the sum of a linear function and three concave functions. Moreover, for any value of \( I_t \geq 0 \), the feasible set is not empty because \( q^t_t = q^s_t = 0 \) and \( I_{t+1} = I_t \geq 0 \) belongs to it. By Lemma 1, this objective is jointly concave in \( d_t - I_t \) and \( I_t \). Hence, it is concave in \( I_t \).

Finally, since \( V_t(I_t, \Phi_t) \) is the sum of a linear function and two functions that are concave in \( I_t \), we have that \( V_t(\cdot, \Phi_t) \) is concave. □

A.1.3 Proposition 3

**Proof.** Consider the replenishment problem defined by Equation (2.5). This problem implies that we are maximizing a concave function over the interval \([\max(0, I_t - d_t), \infty)\). Since the feasible set is linearly constrained, the Karush-Kuhn-Tucker (KKT) conditions hold: the optimal solution is \( I^*_{t+1} \) if and only if there exists dual multiplier \( \mu \geq 0 \) such that

\[ \mu \in \partial C_t(I^*_{t+1} - I_t + d_t) \]

\[ \mu \in \partial U_{t+1}(I^*_{t+1}). \]

Consider \( I^0_t \geq 0 \) and \( I^1_t \geq 0 \) such that \( I^1_t > I^0_t \). Let \( I^0_{t+1} = I^*_{t+1}(I^0_t) \) and let \( \mu^0 \) be an optimal multiplier satisfying the corresponding KKT conditions. Also, let \( I^1_{t+1} = I^*_{t+1}(I^1_t) \) and \( \mu^1 \) be an optimal multiplier.

Assume that \( I^1_{t+1} < I^0_{t+1} \). Then, since \( \mu^1 \) belongs to the sub-gradient of \( C_t \) at \( I^1_{t+1} - I^1_t + d_t \) and \( I^1_{t+1} - I^1_t + d_t < I^0_{t+1} - I^0_t + d_t \), we must have that \( \mu^1 \leq \mu^0 \), because \( C_t \) is convex. Moreover, since \( \mu^1 \in \partial U_{t+1}(I^1_{t+1}) \), \( \mu^0 \in \partial U_{t+1}(I^0_{t+1}) \) and \( I^1_{t+1} < I^0_{t+1} \), since \( U_{t+1} \) is concave, we must have \( \mu^1 \geq \mu^0 \). Hence, \( \mu^1 = \mu^0 \).

Finally, by construction

\[ I^0_{t+1} + d_t - I^0_t > I^1_{t+1} + d_t - I^0_t > I^1_{t+1} + d_t - I^1_t, \]
\[ \mu^0 \in \partial C_t(I_{t+1}^0 - I_t^0 + d_t) \]

and

\[ \mu^1 \in \partial C_t(I_{t+1}^1 - I_t^1 + d_t). \]

Hence, the convexity of \( C_t \), together with \( \mu^1 = \mu^0 \), implies that

\[ \mu^1 \in \partial C_t(I_{t+1}^1 - I_t^0 + d_t). \]

Therefore, \( \mu^1 \in \partial U_{t+1}(I_{t+1}^1) \cap \partial C_t(I_{t+1}^1 - I_t^0 + d_t) \) so that by the KKT conditions \( I_{t+1}^1 \) is an optimal control for the inventory position \( I_t^0 \) too. Since \( I_{t+1}^0 \) was the smallest optimal control at \( I_t^0 \), we must have \( I_{t+1}^1 \geq I_{t+1}^0 \) which is a contradiction. So, \( 0 \leq I_{t+1}^1(I_t^1) - I_{t+1}^1(I_t^0) \).

Assume now that \( I_{t+1}^1 > I_{t+1}^0 + I_t^1 - I_t^0 \). Then \( I_{t+1}^1 - I_t^1 + d_t > I_{t+1}^0 - I_t^0 + d_t \), and since \( C_t \) is convex, we must have \( \mu^1 \geq \mu^0 \). Similarly, since \( I_{t+1}^1 > I_{t+1}^0 \) and \( U_{t+1} \) is concave, \( \mu^1 \leq \mu^0 \). So again \( \mu^1 = \mu^0 \).

But in this case, we have that \( I_{t+1}^0 < I_{t+1}^0 + I_t^1 - I_t^0 < I_{t+1}^1 \). By repeating the previous argument, \( \mu^1 \in \partial U_{t+1}(I_{t+1}^0 + I_t^1 - I_t^0) \cap \partial C_t(I_{t+1}^0 - I_t^0 + d_t) \). Hence, \( I_{t+1}^0 + I_t^1 - I_t^0 \) is an optimal control for the inventory position \( I_t^1 \), which is a contradiction because \( I_{t+1}^0 \) was supposed to be the smallest optimal control for \( I_t^1 \). Consequently, \( I_{t+1}^1(I_t^1) - I_{t+1}^1(I_t^0) \leq I_t^1 - I_t^0 \).

**A.1.4 Proposition 4**

**Proof.** Consider \( I_t \) such that \( \frac{d^2 C_t}{dz_t^2 \mid_{I_{t+1}(I_t)-I_t+d_t}} \) and \( \frac{d^2 U_{t+1}}{dl_{t+1}^2 \mid_{I_{t+1}(I_t)}} \) are well defined, i.e., they are twice differentiable at \( I_t \). Assume too that

\[ \frac{d^2 C_t}{dz_t^2 \mid_{I_{t+1}(I_t)-I_t+d_t}} - \frac{d^2 U_{t+1}}{dl_{t+1}^2 \mid_{I_{t+1}(I_t)}} \neq 0. \]

Since \( U_{t+1} \) and \( C_t \) are twice differentiable at \( I_{t+1}(I_t) \) and \( I_{t+1}(I_t) - I_t + d_t \), respectively, and \( I_{t+1}^* \) is differentiable at \( I_t \), then for \( \epsilon > 0 \) small enough, \( U_{t+1} \) and \( C_t \) are (once) differentiable at \( I_{t+1}^*(I_t + \delta) \) and \( I_{t+1}^*(I_t + \delta) - I_t - \delta + d_t \), respectively, for any
\[ \delta \text{ such that } -\epsilon \leq \delta \leq \epsilon. \]

In the differentiable case, the KKT optimality conditions can be simplified and written as
\[
\frac{dU_{t+1}^{*}}{dI_{t+1}^{*}|_{t+1}^{*}} \frac{dI_{t+1}^{*}}{dt|_{t}} = \frac{dC_{t}}{d\tau|_{t}}|_{t+1}^{*} = \frac{d^{2}C_{t}}{d\tau^{2}|_{t}}|_{t+1}^{*} - \frac{dU_{t+1}^{*}}{dI_{t+1}^{*}|_{t+1}^{*}}|_{t} - 1.\]

This equality holds for \(-\epsilon \leq \delta \leq \epsilon\). Moreover, since we assume \(I_{t+1}^{*}\) to be differentiable at \(I_{t}\), the left-hand side and the right-hand side of the above equation are differentiable at \(\delta = 0\) and the derivative of the left-hand and right-hand terms must be equal at \(\delta = 0\). By the chain rule,
\[
\frac{d^{2}U_{t+1}^{*}}{dI_{t+1}^{*}|_{t+1}^{*}} \frac{dI_{t+1}^{*}}{dt|_{t}} = \frac{d^{2}C_{t}}{d\tau^{2}|_{t}}|_{t+1}^{*} - \frac{d^{2}U_{t+1}^{*}}{dI_{t+1}^{*}|_{t+1}^{*}}|_{t} - 1.\]

We can rearrange this expression into
\[
\frac{dI_{t+1}^{*}}{dt|_{t}} = \frac{\frac{d^{2}C_{t}}{d\tau^{2}|_{t}}|_{t+1}^{*} - \frac{d^{2}U_{t+1}^{*}}{dI_{t+1}^{*}|_{t+1}^{*}}|_{t}}{\frac{d^{2}C_{t}}{d\tau^{2}|_{t}}|_{t+1}^{*}}.\]

A.2 Chapter 3

A.2.1 Theorem 1

Proof. The proof of Proposition 3 tells us that the objective function is concave and the feasible set is convex. As a result, the optimality condition is
\[
\exists \mu \geq 0 \text{ such that } \mu \in \partial C_{t}(I_{t+1} - I_{t} + d_{t}) \text{ and } \mu \in \partial U_{t+1}(I_{t+1}).
\]
Equation (3.1) implies that

\[
\partial C_t(I_{t+1} - I_t + d_t) = \begin{cases} \\
\{w_k\} & \text{when } \sum_{j=1}^{k-1} \bar{x}_j < I_{t+1} - I_t + d_t < \sum_{j=1}^{k} \bar{x}_j \\
[w_k, w_{k+1}] & \text{when } I_{t+1} - I_t + d_t = \sum_{j=1}^{k} \bar{x}_j.
\end{cases}
\]

Hence, we can use this structure and optimality conditions to characterize an optimal replenishment strategy. We can define break-points \( f_0 \leq \ldots \leq f_{2m} \) that satisfy the following description.

- If \( \mu = w_k \) for some \( k = 1, \ldots, n_t \), then \( w_k \in \partial U_{t+1}(I_{t+1}) \) determines a fixed \( I_{t+1} \), because we adopted the convention that we always pick the smallest optimal \( I_{t+1} \). This case happens when \( \sum_{j=1}^{k-1} \bar{x}_j \leq I_{t+1} - I_t + d_t \leq \sum_{j=1}^{k} \bar{x}_j \) which is equivalent to \( I_t \in [f_{2k-1}, f_{2k}] \).

- Otherwise there is \( k = 0, \ldots, n_t - 1 \) such that \( w_k < \mu < w_{k+1} \). In that case, the order quantity \( I_{t+1} - I_t + d_t = \sum_{j=1}^{k} \bar{x}_j \), which is a constant. Observe that \( \mu \) belongs to the interval \( [w_k, w_{k+1}] \) for a given range of \( I_{t+1} \). This is true since \( \mu \in \partial U_{t+1}(I_{t+1}) \). This, together with the fact that \( I_{t+1} - I_t \) is fixed, implies that there also is a range \( [f_{2k}, f_{2k+1}] \) for \( I_t \) for which \( \mu \in [w_k, w_{k+1}] \).

All these intervals are adjacent and alternate. They cover \( \mathbb{R}_+ \), which implies that \( f_0 = 0 \) and \( f_{2m} = \infty \).

We can now derive the optimal replenishment policies from this characterization.

<table>
<thead>
<tr>
<th>Interval ( I_{t+1} )</th>
<th>([f_{2k-1}, f_{2k}])</th>
<th>([f_{2k}, f_{2k+1}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_t )</td>
<td>decreasing with slope -1</td>
<td>constant</td>
</tr>
<tr>
<td></td>
<td>increasing with slope 1</td>
<td>constant</td>
</tr>
</tbody>
</table>

Clearly, the manufacturer will replenish inventory starting with the least expensive source, i.e., suppliers or spot market, and moving to more expensive ones as capacity is exhausted. Thus, for \( i = 1, \ldots, n_t \), let \( b_i^* = f_{2i} \) and observe that \( b_i^* \) is the base-stock level for source \( i \). Moreover, since we start using source \( i + 1 \) when source \( i \) is exhausted, we must have that \( b_i^{*+1} \leq b_i^* - \bar{x}_i \).
A.2.2 Theorem 2

Proof. We prove the property by induction from \( t = T + 1 \) to \( t = 1 \). The proof is similar to the proof of Proposition 2. For \( t = T + 1 \), \( V_{T+1}(I_{T+1}, \Phi_{T+1}, x) = aI_{T+1} \) and is thus concave in \( (I_{T+1}, x) \).

Assume now that the lemma is true for \( t + 1 \): given \( \Phi_{t+1}, V_{t+1}(I_{t+1}, \Phi_{t+1}, x) \) is concave in \( (I_{t+1}, x) \). We show the inductive proposition for \( t \). For this purpose, we assume that \( \Phi_t \) is given, and therefore \( d_t, c_t, \kappa_t \) and the distribution of \( \Phi_{t+1} \) are fixed.

Clearly, since given \( \Phi_{t+1}, V_{t+1}(I_{t+1}, \Phi_{t+1}, x) \) is concave in \( (I_{t+1}, x) \), we must have that \( U_{t+1}(I_{t+1}, x) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}, x) \) is concave in \( (I_{t+1}, x) \).

Combining Equations (2.5) and (3.1) we have,

\[
V_t(I_t, \Phi_t, x) = -h_t(I_t) + p_t d_t + \max_{q_t^1, \ldots, q_t^n, I_{t+1}} - \sum_{k=1}^{\pi_t} \bar{w}_t^k q_t^k + U_{t+1}(I_{t+1}, x)
\]

subject to

\[
\begin{align*}
-I_{t+1} & \leq 0 \\
-q_t^k & \leq 0 & \forall k = 1, \ldots, \pi_t \\
q_t^k & \leq \bar{q}_t^k & \forall k = 1, \ldots, \pi_t \\
\sum_{k=1}^{\pi_t} q_t^k - I_{t+1} & = d_t - I_t
\end{align*}
\]

Observe that there is always a feasible solution to this maximization problem; for any \( I_t \geq 0 \): \( I_{t+1} = I_t, q_t^i = 0 \) for \( i = 1, \ldots, \pi_t \), except for the source of not serving demand, that we set to \( d_t \), is a feasible solution. To apply Lemma 1 note also that the objective of the above maximization problem is concave in \( (q_t^1, \ldots, q_t^n, I_{t+1}, \bar{x}, d_t - I_t, I_t - d_t) \).
In addition, we can rewrite the feasible region of the maximization problem as follows:

\[
A \begin{pmatrix}
q_t^1 \\
\vdots \\
q_t^n \\
I_{t+1}
\end{pmatrix} \leq \begin{pmatrix}
0 \\
\vdots \\
0 \\
\bar{x}
\end{pmatrix}
\begin{pmatrix}
d_t - I_t \\
I_t - d_t
\end{pmatrix}
\]

where \( A \) is a matrix of 0, 1, -1.

Thus, Lemma 1 tells us that the objective of the maximization problem is concave in \((\bar{x}, d_t - I_t, I_t - d_t)\), and hence in \((I_t, x)\). Therefore, since \(V_t(\cdot; \Phi_t, \cdot)\) is the sum of concave functions, it is concave as in \((I_t, x)\). ■

\[\text{A.2.3 Theorem 4}\]

**Proof.** Given \( x \), we define \( y \) by

\[
y^i = \sum_{k=1}^{i} x^k, \quad i = 1, \ldots, n.
\]

The term \( v(x) \) can be written as

\[
\sum_{i=1}^{n} v^i x^i = \sum_{i=1}^{n} (v^i - v^{i+1}) y^i
\]

by defining \( v^{n+1} = 0 \). Define also \( w^{n+1} = p \) and \( y^{n+1} = \infty \).

We analyze the profit for every sample path. If \( w^{jo} \leq S \leq w^{jo+1} \) for some
\[ i_0 = 1, \ldots, n, \text{ the profit that we obtain is} \]
\[
\Pi = - \sum_{i=1}^{n} (v^i - v^{i+1})y^i + \sum_{i=1}^{i_0} (p - w'_{i}) \left[ \min(D - y^{i-1}, x^i) \right]^+ + (p - S) \left[ D - y^{i_0} \right]^+
\]
\[
= - \sum_{i=1}^{n} (v^i - v^{i+1})y^i + \sum_{i=1}^{i_0} (p - w'_{i}) \left[ \min(D, y') - \min(D, y^{i-1}) \right]
\]
\[
+ (p - S) \left[ \min(D, y^{i_0+1}) - \min(D, y^{i_0}) \right]
\]
\[
= - \sum_{i=1}^{n} (v^i - v^{i+1})y^i + \sum_{i=1}^{i_0+1} \left[ p - \min(S, w^i) \right] \left[ \min(D, y') - \min(D, y^{i-1}) \right] ,
\]

by remarking that \( \min(D - y^{i-1}, x^i) = \min(D, y') - y^{i-1} \), and that \[ \min(D, y') - y^{i-1})^+ = \min(D, y') - \min(D, y^{i-1}). \] Using the fact that
\[
p - \min(S, w^i) = \sum_{j=1}^{n} \left( \min(S, w^{i+1}) - \min(S, w^j) \right)
\]

we can rearrange this expression into
\[
\Pi = (p - S)^+ D - \sum_{i=1}^{n} (v^i - v^{i+1})y^i + \sum_{i=1}^{n} \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \min(D, y^i).
\]

The same equation is of course true when \( S > p \). We can now take expectation on \((D, S)\) and the fact that \( \mathbb{E}D < \infty \) guarantees that this expectation is well defined. Hence,
\[
\mathbb{E}_{(D, S)} \Pi = J = (p - S)^+ D - \sum_{i=1}^{n} (v^i - v^{i+1})y^i + \sum_{i=1}^{n} \mathbb{E}\left\{ \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \min(D, y^i) \right\}.
\]

After taking the derivative with respect to \( y^i \), we obtain that,
\[
\frac{dJ}{dy^i} = v^{i+1} - v^i + \mathbb{E}\left\{ 1_{y^i \leq D} \left[ \min(S, w^{i+1}) - \min(S, w^i) \right] \right\}.
\]
A.2.4 Proposition 5

Proof. In the two cases, we compare the sample-path profit of a portfolio containing option \( i_t \) with a portfolio without it. Let \( x > 0 \) be the amount of option \( i_t \) in a given portfolio \( P_0 \).

In case (i), replace the \( x \) units of option \( i_t \) by \( x \) units of option \( k_t \). This forms portfolio \( P_1 \). For every possible outcome \( \omega \), let \( q(\omega) \geq 0 \) be the optimal amount of option \( i_t \) executed when we use portfolio \( P_0 \). The cost of option \( i_t \) is thus, when outcome \( \omega \) happens,

\[
v_t^{i_t} x + w_t^{i_t} q(\omega) = v_t^{i_t}[x - q(\omega)] + (v_t^{i_t} + w_t^{i_t})q(\omega).
\]

Consider now using portfolio \( P_1 \) together with the same replenishment strategy as in \( P_0 \) with one exception. For option \( k_t \), when outcome \( \omega \) is realized, we execute, in addition to what was executed when the manufacturer held portfolio \( P_0 \), \( q(\omega) \) units from the additional \( x \) units of capacity. The cost associated with this modification is

\[
v_t^{k_t} x + w_t^{k_t} q(\omega) = v_t^{k_t}[x - q(\omega)] + (v_t^{k_t} + w_t^{k_t})q(\omega).
\]

Since \( 0 \leq q(\omega) \leq x \) always, the assumption in the proposition clearly implies that portfolio \( P_1 \) with a given replenishment policy yields a smaller cost than portfolio \( P_0 \) with its optimal replenishment policy for every outcome \( \omega \). Also, since \( v_t^{i_t} + w_t^{i_t} > v_t^{k_t} + w_t^{k_t} \) and \( v_t^{i_t} > v_t^{k_t} \) and \( x > 0 \), the difference can never be equal to 0. This implies that \( P_1 \) provides a strictly larger expected profit that \( P_0 \), and hence option \( i_t \) can be excluded from an optimal portfolio.

In case (ii), replace the \( x \) units of option \( i_t \) by \( \lambda x \) units of option \( j_t \) and \( (1 - \lambda)x \) units of option \( k_t \), where \( 0 < \lambda < 1 \) is defined as

\[
\lambda = \frac{w_t^{k_t} - w_t^{j_t}}{w_t^{k_t} - w_t^{i_t}}.
\]

This forms portfolio \( P_2 \). Similarly to the previous case, consider, for portfolio \( P_2 \), the same replenishment policy as for \( P_0 \) except that, for options \( j_t \) and \( k_t \), when outcome
\( \omega \) is realized, we execute, in addition to what was executed when the manufacturer held portfolio \( P_0 \), \( \lambda q(\omega) \) units from option \( j_t \) and \( (1 - \lambda)q(\omega) \) units from option \( k_t \). The cost is in this case

\[
[\lambda w^{i_t}_t + (1 - \lambda)w^{k_t}_t][x - q(\omega)] + [\lambda(v^{i_t}_t + w^{i_t}_t) + (1 - \lambda)(v^{k_t}_t + w^{k_t}_t)]q(\omega).
\]

Similarly to case (i), we have that \( \lambda w^{i_t}_t + (1 - \lambda)w^{k_t}_t < v^{i_t}_t \) and, since \( w^{i_t}_t = \lambda w^{i_t}_t + (1 - \lambda)w^{k_t}_t \), \( \lambda(v^{i_t}_t + w^{i_t}_t) + (1 - \lambda)(v^{k_t}_t + w^{k_t}_t) < v^{i_t}_t + w^{i_t}_t \). Hence, using \( x > 0 \), we see that the cost of \( P_2 \) is strictly smaller than the one of \( P_0 \), for all outcomes. This implies that we can exclude option \( i_t \) from an optimal portfolio. ■

A.2.5 Proposition 6

Proof. To prove the proposition, assume that the portfolio contains \( x > 0 \) units of option \( i_t \). We look at the marginal benefit of decreasing \( x \), the amount of option \( i_t \) purchased. For every outcome \( \omega \) (\( \omega \) describes \( (D_t, S_t) \)), let \( m_t(\omega) \) be an optimal production price, i.e. a price such that the optimal replenishment policy maximizes

\[
m_t(\omega)z - C_t(S_t, D_t, z)
\]

where \( C_t(S_t, D_t, \cdot) \) is defined in Equation (3.1). \( m_t(\omega) \) can be interpreted in the differentiable case as the optimal dual multiplier in the state corresponding to the events described by \( \omega \), i.e.

\[
m_t(\omega) = \frac{dU_{t+1}}{dt+1} (\omega, I_t - d_t + z^*_t) = \frac{dC_t}{dz_t} (\omega, z^*_t).
\]

Clearly, since the spot market is uncapacitated, \( m_t(\omega) \leq S_t \), otherwise we could obtain an infinite profit for some outcome \( \omega \).

For an event \( \omega \), the marginal benefit of option \( i_t \) is equal to

- \( m_t(\omega) - w^{i_t}_t - v^{i_t}_t \) if option \( i_t \) is executed fully, i.e. \( q^{i_t}_t(\omega) = x; \)
- \(-v^{i_t}_t \) if option \( i_t \) is executed partially, i.e. \( 0 < q^{i_t}_t(\omega) < x \), or it is not executed,
i.e. $q_t^i(\omega) = 0$.

Thus, the marginal benefit of increasing $x$ is equal to $\max [m_t(\omega) - w_t^{i^*}, 0] - v_t^{i^*} \leq [S_t - w_t^{i^*}]^+ - v_t^{i^*}$. After taking expectation, the marginal benefit is less than $E[S_t - w_t^{i^*}]^+ - v_t^{i^*}$.

Since by assumption this is non-positive, we can decrease $x$ to 0 while improving the expected profit. ■

### A.2.6 Theorem 5

**Proof.** We prove the theorem in the twice-differentiable case. The proof can be easily extended to the general case. When $U_{t+1}(\Phi_t, \cdot)$ is differentiable at $I_{t+1}^*$ and $C_t(\Phi_t, \cdot)$ is differentiable at $z_t^* = I_{t+1}^* - I_t + d_t$, we have that

$$\frac{dU_{t+1}(\Phi_t, I_{t+1}^*)}{dI_{t+1}} = \frac{dC_t}{dz_t}(\Phi_t, z_t^*). \quad (A.1)$$

This equation can be differentiated with respect to $x_t^k$.

Clearly, when $t' < t$, neither $U_{t+1}(\Phi_t, \cdot)$ nor $C_t(\Phi_t, \cdot)$ depend on $x_t^k$, and therefore, the optimal replenishment policy $I_{t+1}^*$, as a function of $I_t$, is independent of $x_t^k$. This proves the first case of the result.

When $t' = t$, $U_{t+1}(\Phi_t, \cdot)$ is independent of $x_t^k$, so we have that

$$\frac{d^2U_{t+1}(\Phi_t, I_{t+1}^*)}{dI_{t+1}^2} \frac{dI_{t+1}}{dx_t^k} = \frac{d^2C_t}{dz_t}(\Phi_t, I_{t+1}^*-I_t+d_t) + \frac{d^2C_t}{dz_t dx_t^k}(\Phi_t, I_{t+1}^* - I_t + d_t). \quad (A.2)$$

Since

$$\frac{dC_t}{dx_t^k}(\Phi_t, z_t) = -\left[ \frac{dC_t}{dz_t}(\Phi_t, z_t) - w_t^k \right]^+, \quad (A.3)$$

we have that

$$\frac{d^2C_t}{dz_t dx_t^k}(\Phi_t, I_{t+1}^* - I_t + d_t) \leq 0.$$

This, together with the fact that $U_{t+1}(\Phi_t, \cdot)$ is concave, $C_t(\Phi_t, \cdot)$ convex, and Equation (A.2), implies that

$$\frac{dI_{t+1}}{dx_t^k} \geq 0$$

and hence the base-stock level $b_t^i$ increases.
When \( t' > t \), since \( C_t(\Phi_{t'1}, \cdot) \) is independent of \( x_t^{k'} \), differentiating Equation (A.1) yields

\[
\frac{d^2 U_{t+1}}{dt^2}(\Phi_t, I_{t+1}) \frac{dI_{t+1}}{dx_t^k} + \frac{d^2 U_{t+1}}{dt^2}(\Phi_t, I_{t+1}) \frac{dI_{t+1}}{dx_t^k} = \frac{d^2 C_t}{dx_t^2}(\Phi_t, I_{t+1} - I_t + dt) \frac{dI_{t+1}}{dx_t^k}. \tag{A.3}
\]

We prove this case by induction. Consider \( t' = t+1 \). By differentiating \( U_t(\Phi_{t-1}, \cdot) \), see Equation (2.4), we have that

\[
\frac{dU_t}{dt}(\Phi_{t-1}, I_t) = \mathbb{E}_{\Phi_t|\Phi_{t-1}} \frac{d}{dt} \left\{ p_t D_t - h_t(I_t) + \max_{I_{t+1}} [U_{t+1}(\Phi_t, I_{t+1}) - C_t(\Phi_t, I_{t+1} - I_t + D_t)] \right\} = -\frac{dh_t}{dt} + \mathbb{E}_{\Phi_t|\Phi_{t-1}} \left\{ \frac{dC_t}{dx_t}(\Phi_t, I_{t+1} - I_t + D_t) \right\}.
\]

After differentiating with respect to \( x_t^k \), and using Equation (A.2), we establish that

\[
\frac{d^2 U_t}{dt^2}(\Phi_{t-1}, I_t) = \mathbb{E} \left[ \frac{d^2 C_t}{dx_t^2}(\Phi_t, I_{t+1} - I_t + D_t) + \frac{dC_t}{dx_t}(\Phi_t, I_{t+1} - I_t + D_t) \frac{dI_{t+1}}{dx_t^k} \right] = \mathbb{E} \left[ \frac{d^2 U_{t+1}}{dt^2}(\Phi_t, I_{t+1}) \frac{dI_{t+1}}{dx_t^k} \right] \leq 0.
\]

This, together with Equation (A.3), implies \( \frac{dI_t}{dx_t^k} \leq 0 \).

We can thus initiate an induction to show that for all \( m \geq 1 \), we have

\[
\frac{d^2 U_{t+1-m}}{dt^2}(\Phi_t-m, I_{t+1-m}) \leq 0 \text{ and } \frac{dI_{t+1-m}}{dx_t^k} \leq 0.
\]

For this purpose, we again differentiate \( U_{t+1-m}(\Phi_{t-m}, \cdot) \), for \( m > 1 \), to get,

\[
\frac{dU_{t+1-m}}{dt^2}(\Phi_t-m, I_{t+1-m}) = \mathbb{E} \frac{d}{dt} \left\{ p_{t+1-m} D_{t+1-m} - h_{t+1-m}(I_{t+1-m}) + \max_{I_{t+2-m}} \left[ U_{t+2-m}(\Phi_{t+1-m}, I_{t+2-m}) - C_{t+1-m}(\Phi_{t+1-m}, I_{t+2-m} - I_{t+1-m} + D_{t+1-m}) \right] \right\} = -\frac{dh_{t+1-m}}{dt} + \mathbb{E} \left[ \frac{dC_{t+1-m}}{dx_{t+1-m}}(\Phi_{t+1-m}, I_{t+2-m} - I_{t+1-m} + D_{t+1-m}) \right].
\]

173
This implies
\[
\frac{d^2 U_{t+1-m}}{dI_{t+1-m} dx_t} (\Phi_{t-m}, I_{t+1-m})
= \mathbb{E} \left[ \frac{d^2 C_{t+1-m}}{dz_{t+1-m}^2} (\Phi_{t+1-m}, I_{t+2-m} - I_{t+1-m} + D_{t+1-m}) \frac{dI_{t+2-m}}{dx_t} \right] \leq 0.
\]
where we used the induction hypothesis for \(m - 1\) and the convexity of \(C_{t+1-m}\). This, by Equation (A.3), implies that \(\frac{dI_{t+1-m}}{dx_t} \leq 0\), thus concluding the induction. Hence the base-stock level \(b_t^t\) decreases as a function of \(x_t^t\) for every \(t' > t\).  

A.3 Chapter 4

A.3.1 Proposition 7

Proof. We have that for all \(i = 1, \ldots, n\),
\[
\frac{1}{2} \frac{d \text{Var} \Pi^N}{dy_i} = \mathbb{F}_D(y_i) = \begin{pmatrix}
\mathbb{E}[Z_i^2] y_i^2 - \mathbb{E}[Z_i]^2 \int_0^{y_i} \mathcal{F}_D(u) du \\
+ \mathbb{E}[Z_i Z_j] y_i^j + \mathbb{E}[Z_i] \int_0^{y_i} \mathcal{F}_D(u) du - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \int_0^{y_i} \mathcal{F}_D(u) du \\
+ \sum_{j > i} \mathbb{E}[Z_i Z_j] y_i^j - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \int_0^{y_i} \mathcal{F}_D(u) du
\end{pmatrix}
\]
Thus, we can define \( \Phi_i \) where

\[
\frac{1}{2} \Phi_i(y) = \mathbb{E}[Z_i^2]y^i - \mathbb{E}[Z_i]^2 \int_{0}^{y^i} \tilde{F}_D(u)du
+ \mathbb{E}[Z_0Z_i](y^i + \int_{0}^{y^i} \tilde{F}_D(u)du) - \mathbb{E}[Z_0]\mathbb{E}[Z_i]\mathbb{E}[D]
+ \sum_{j>i} \mathbb{E}[Z_iZ_j](y^j + \int_{0}^{y^j} \tilde{F}_D(u)du) - \mathbb{E}[Z_i]\mathbb{E}[Z_j]\int_{0}^{y^j} \tilde{F}_D(u)du
+ \sum_{j<i} \mathbb{E}[Z_iZ_j]y^j - \mathbb{E}[Z_i]\mathbb{E}[Z_j]\int_{0}^{y^j} \tilde{F}_D(u)du
\]

such that \( \frac{d \text{Var} \Pi^N}{dy^i} = \tilde{F}_D(y^i)\Phi_i(y) \).

We can now compute the Hessian of the variance. For \( i, j = 1, \ldots, n \), we have

\[
\frac{1}{2} \frac{d^2 \text{Var} \Pi^N}{(dy^i)^2} = -\frac{1}{2} f_{D}(y^i)\Phi_i(y) + \tilde{F}_D(y^i) \left( \begin{array}{c}
\mathbb{E}[Z_i^2] - \mathbb{E}[Z_i]^2 \tilde{F}_D(y^i) \\
\frac{f_{D}(y^i)\int_{0}^{\infty} \tilde{F}_D(u)du}{\tilde{F}_D(y^i)^2} \\
+ \sum_{j>i} \mathbb{E}[Z_iZ_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j] \tilde{F}_D(y^i) \\
+ \sum_{j<i} \mathbb{E}[Z_iZ_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j] \tilde{F}_D(y^i)
\end{array} \right);
\]

(A.4)

for \( i < j \),

\[
\frac{1}{2} \frac{d^2 \text{Var} \Pi^N}{dy^i dy^j} = \tilde{F}_D(y^i) \left( \mathbb{E}[Z_iZ_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j] \tilde{F}_D(y^i) \right) \left( \mathbb{E}[Z_iZ_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j] \tilde{F}_D(y^j) \right)
\]

(A.5)

\[
= \tilde{F}_D(y^i)\mathbb{E}[Z_iZ_j] - \tilde{F}_D(y^i)\tilde{F}_D(y^j)\mathbb{E}[Z_i]\mathbb{E}[Z_j];
\]

175
and for $i > j$,

$$\frac{1}{2} \frac{d^2 \text{Var} \Pi^N}{dy' dy'} = \bar{F}_D(y') \left( \mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \bar{F}_D(y') \right)$$

(A.6)

$$= \bar{F}_D(y') \mathbb{E}[Z_i Z_j] - \bar{F}_D(y') \bar{F}_D(y') \mathbb{E}[Z_i] \mathbb{E}[Z_j].$$

We claim that the Hessian without the diagonal terms $-f_D(y') \Phi_i(y)$ is a positive definite matrix.

Define $\alpha_{n+1} = 0$, $\alpha_i = \bar{F}_D(y')$ and $\Delta_i = \alpha_i - \alpha_{i+1}$ for $i = 1, \ldots, n$. Clearly, $\Delta_1 + \ldots + \Delta_n = \alpha_1 \leq 1$, we will use this inequality later. Using Equations (A.4), (A.5) and (A.6), together with (using here Assumption 13) the fact that

$$\mathbb{E}[Z_0 Z_i] \frac{f_D(y') \int_{y'}^\infty \bar{F}_D(u) du}{\bar{F}_D(y')^2} + \sum_{j > i} \mathbb{E}[Z_i Z_j] \frac{f_D(y') \int_{y'}^{y_j} \bar{F}_D(u) du}{\bar{F}_D(y')^2} > 0,$$

176
we obtain that

\[
\frac{1}{2} H - \frac{1}{2} \begin{pmatrix}
\cdots & 0 \\
-f_D(y') \Phi_i(y) & \cdots
\end{pmatrix}
\begin{pmatrix}
\alpha_i E[Z_i^2] - \alpha_i^2 E[Z_i]^2 & \cdots & \alpha_j E[Z_i Z_j] - \alpha_i \alpha_j E[Z_i] E[Z_j] \\
\vdots & \ddots & \vdots \\
\alpha_j E[Z_i Z_j] - \alpha_i \alpha_j E[Z_i] E[Z_j] & \cdots & \alpha_j E[Z_j^2] - \alpha_j^2 E[Z_j]^2
\end{pmatrix}
\begin{pmatrix}
\alpha_i Z_i \\
\vdots \\
\alpha_i Z_i \\
\cdots \\
\alpha_j E[Z_j^2] \\
\vdots \\
\alpha_j E[Z_j^2] \\
\vdots \\
\alpha_j E[Z_j^2] \\
0 \\
\vdots \\
0 \\
\cdots \\
E \left( \begin{array}{c}
\alpha_1 Z_1 \\
\vdots \\
\alpha_1 Z_1 \\
\alpha_n Z_n \\
\alpha_n Z_n
\end{array} \right)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 Z_1 \\
\vdots \\
\alpha_1 Z_1 \\
\alpha_n Z_n \\
\alpha_n Z_n
\end{pmatrix},
\]

(A.7)

Define

\[
U_i = \begin{pmatrix}
Z_1 \\
\cdots \\
Z_i \\
0 \\
\cdots \\
0
\end{pmatrix}
\]
We observe that

\[
\begin{pmatrix}
\alpha_1 Z_1^2 & \cdots & \alpha_i Z_i Z_i & \cdots & \alpha_j Z_j Z_j & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\alpha_i Z_i^2 & \cdots & \alpha_j Z_i Z_j & \cdots & \alpha_j Z_j^2 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\alpha_j Z_j^2 & \cdots & \alpha_j Z_j Z_j & \cdots & \alpha_j Z_j^2 & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\end{pmatrix}
= \sum_i \Delta_i 
\begin{pmatrix}
Z_1^2 & \cdots & Z_i Z_i & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
Z_i Z_i & \cdots & Z_i^2 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

= \sum_i \Delta_i U_i U_i'

and also

\[
\begin{pmatrix}
\alpha_1 Z_1 \\
\vdots \\
\alpha_i Z_i \\
\vdots \\
\alpha_n Z_n
\end{pmatrix}
= \sum_i \Delta_i U_i.
\]

We can now express the right hand side of Equation (A.7) as:

\[
\sum_i \Delta_i^2 \mathbb{E}[U_i U_i'] + \sum_i \Delta_i (1 - \Delta_i) \mathbb{E}[U_i U_i']
- \sum_i \Delta_i^2 \mathbb{E}[U_i] \mathbb{E}[U_i']
- \sum_{i,j \neq j} \Delta_i \Delta_j \mathbb{E}[U_i] \mathbb{E}[U_j].
\]

Since a variance-covariance matrix is positive semi-definite, we have that for all \(i\),

\[
\mathbb{E}[U_i U_i'] \succeq \mathbb{E}[U_i] \mathbb{E}[U_i']
\]

This implies that the first term minus the third term is the sum of \(n\) positive semi-definite matrices.
Also, as pointed out before,
\[ 1 - \Delta_i \geq \sum_{j \neq i} \Delta_j. \]

Thus, the second term minus the fourth term is greater than (in the positive semi-definite ordering sense)
\[
\sum_{i,j \neq i} \Delta_i \Delta_j \{ \mathbb{E}[U_i U'_i] + \mathbb{E}[U_j U'_j] - \mathbb{E}[U_i] \mathbb{E}[U'_i] - \mathbb{E}[U_j] \mathbb{E}[U'_j] \}. \tag{A.8}
\]

Observe that for all \( i < j \), we have
\[
\begin{align*}
\mathbb{E}[U_i U'_i] + \mathbb{E}[U_j U'_j] - \mathbb{E}[U_i] \mathbb{E}[U'_i] - \mathbb{E}[U_j] \mathbb{E}[U'_j] \\
&\geq \mathbb{E}[U_i] \mathbb{E}[U'_i] + \mathbb{E}[U_j] \mathbb{E}[U'_j] - \mathbb{E}[U_i] \mathbb{E}[U'_j] - \mathbb{E}[U_j] \mathbb{E}[U'_i] \\
&= \mathbb{E}[U_i - U_j] \mathbb{E}[U'_i - U'_j] \\
&\geq 0.
\end{align*}
\]

In other words, all the terms in the sum in Equation (A.8) are positive semi-definite, and thus the following matrix is positive definite,
\[
H \succ 
\begin{pmatrix}
& & & & 0 \\
& & & -f_D(y')\Phi_i(y) & \\
& & 0 & & \\
& -f_D(y')\Phi_i(y) & & & \\
& & & & \\
& & & & 0
\end{pmatrix}.
\]

\[\Box\]

A.3.2 Proposition 8

**Proof.** We use the results from Proposition 7. Without loss of generality, we can assume that \( I = \{1, \ldots, n\} \), since when \( I \) is smaller, we can conduct the same calculations of Proposition 7 with the subset \( I \) of contracts.

We assume that \( y \) is a critical point of the variance, i.e. \( \frac{d\text{Var} \Pi^N}{y^i}(y) = 0 \) for \( i = 1, \ldots, n \). Proposition 7 implies that \( \Phi_i(y) = 0 \) for all \( i \), and thus the Hessian is
definite positive. Since the variance is twice continuously differentiable, the critical point is a strict minimum.

### A.3.3 Proposition 9

**Proof.** Before starting this proof, define the norm $\| \cdot \|_\infty$ in $\mathbb{R}_+^n$, i.e.

$$\| y \|_\infty = \max_i |y^i|.$$ 

Consider for every $k = 1, \ldots, n$ the portfolio where $y^{k-1} < y^k = y^{k+1} = \ldots = y^n = y^{(k)}$. In other words, the $n-k$ highest inequality constraints of the feasible set are tight. We claim that there exists an $M_k$ such that, for $y^{(k)} > M_k$, the variance is non-decreasing as we increase $y^{(k)}$. To see this, we look at the derivative of the variance as a function of $y^{(k)}$. Since $\text{Covar}[Z_i, Z_j] \geq 0$ for all pairs $(i, j)$ and $y^i \geq \int_0^{y^i} \tilde{F}_D(u)du$, we have

$$\frac{d \text{Var} \Pi^N}{dy^{(k)}} = \tilde{F}_D(y^{(k)}) \left( \begin{array}{l} \mathbb{E}[(Z_k + \ldots + Z_n)^2] y^{(k)} \\
- \mathbb{E}[Z_k + \ldots + Z_n]^2 \int_0^{y^{(k)}} \tilde{F}_D(u)du \\
+ \mathbb{E}[Z_0(Z_k + \ldots + Z_n)](y^{(k)} + \int_0^{y^{(k)}} \tilde{F}_D(u)du) \\
- \mathbb{E}[Z_0]\mathbb{E}[Z_k + \ldots + Z_n]\mathbb{E}[D] \\
+ \sum_{j \leq k-1} \left\{ \mathbb{E}[(Z_k + \ldots + Z_n)Z_j] y^j \\
- \mathbb{E}[Z_k + \ldots + Z_n] \mathbb{E}[Z_j] \int_0^{y^j} \tilde{F}_D(u)du \right\} \\
\mathbb{E}[(Z_k - \ldots + Z_n)^2] y^{(k)} \\
- \mathbb{E}[Z_k + \ldots + Z_n]^2 \int_0^{y^{(k)}} \tilde{F}_D(u)du \\
+ \mathbb{E}[Z_0(Z_k + \ldots + Z_n)](y^{(k)} + \int_0^{y^{(k)}} \tilde{F}_D(u)du) \\
- \mathbb{E}[Z_0]\mathbb{E}[Z_k + \ldots + Z_n]\mathbb{E}[D] \end{array} \right) \right) \geq \tilde{F}_D(y^{(k)}).$$
The expression
\[
E[(Z_0 + \ldots + Z_n)^2]y^{(k)} - E[Z_k + \ldots + Z_n]^2 \int_0^{y^{(k)}} \hat{F}_D(u)du \\
+ E[Z_0(Z_k + \ldots + Z_n)](y^{(k)} + \int_0^{y^{(k)}} \hat{F}_D(u)du) - E[Z_0]E[Z_k + \ldots + Z_n]E[D]
\]
is strictly increasing and tends to $+\infty$ when $y^{(k)} \to +\infty$. Thus, there is $M_k > 0$ such that \( \frac{dVar\Pi^N}{dy^{(k)}} \geq 0 \) for $y^{(k)} > M_k$.

By writing $M = \max\{M_1, \ldots, M_n, y_0, y_1\}$, define the compact set
\[
B = \left\{ y \right| 0 \leq y^1 \leq \ldots \leq y^n \leq M \right\}.
\]
Observe that the set is compact because it is closed and bounded in a finite-dimensional space.

By construction, for every point $y \in F \setminus B$, we can find a point $y' \in B$ with $Var\Pi^N(y) \geq Var\Pi^N(y')$. This can be done by defining $y'_i = M$ for any $i$ such that $y_i > M$.

The function $Var\Pi^N(\cdot)$ is twice differentiable in the bounded set $B$, and this implies that the sets
\[
L_c = \left\{ y \in B \right| Var\Pi^N(y) \leq c \right\}
\]
are compact sets, since the variance is a continuous function. By increasing $c$, the level sets $L_c$ increase in the inclusion sense.

By contradiction, assume that the level sets $L_c$ of the variance are not connected, for some $c = a \geq 0$, i.e. $L_a$ has at least two connected components. Let $C_0$ and $C_1$ be two unconnected components of $L_a$, and take $y_0 \in C_0$ and $y_1 \in C_1$. The variance on $B$ is a continuous function on a closed finite-dimensional set, and hence attains a maximum over $B$. This implies that for some $c$, big enough, $L_c = B$, which connects $y_0$ and $y_1$.

We can now define $b$ the largest $c$ such that $y_0$ and $y_1$ cannot be connected through a continuous path in $L_c$, i.e., they belong to two different connected parts of $L_c$. In
this definition, $b$ is a supremum. We can define for all $n \geq 0$, two sequences in $B$, $p_n$ and $q_n$ that solve

$$
\min \|p - q\|_{\infty}
$$

subject to

1. $p$ and $y_0$ are connected in $L_{b-1/n}$
2. $q$ and $y_1$ are connected in $L_{b-1/n}$

This is possible because $L_{b-1/n}$ is a closed set. These sequences belong to the compact set $L_b$, thus, by the Bolzano-Weierstrass theorem, they have adherence points. We can then extract a subsequence of $(p_n, q_n)$ such that both $p_n$ and $q_n$ converge to a point $\bar{y}$ that belongs to $L_b$, but not to any $L_{b-1/n}$. This implies that $\text{Var}(\bar{y}) = b$.

Define $I$ as the set of indexes such that $\bar{y}_i > \bar{y}_{i-1}$ and $A_I$ as in Equation (4.17). We claim that $\bar{y}$ is a $I$-unconstrained critical point. If this was not the case, then, since the variance is twice differentiable, we could redirect the path that connects $y_0$ to $y_1$ in $L_b$ avoiding the variance level $b$. That is, for some $n$, find a path that connects $y_0$ to $y_1$ in $L_{b-1/n}$. Such a path exists when the first derivative of the variance (as a function of the variables $z_j$ used in Definition 6) at that point is non-zero. We can thus apply Proposition 8 to establish that $\bar{y}$ is a local minimum when the feasible set is $A_I$.

In addition, for $i \notin I$, it must be that $d\text{Var}\Pi^N/\,dy^i \geq 0$, since otherwise we could, again, redirect the path that connects $y_0$ to $y_1$ in $L_b$ avoiding the variance level $b$. This implies that $\bar{y}$ is a local minimum for the variance.

However, the sequence $p_n$ tends to $\bar{y}$ with a variance always strictly smaller that $b$. This is a contradiction, and thus the level sets $L_c$ must be connected. ■

A.3.4 Theorem 9

Proof. We have that for all $i = 1, \ldots, n$, from Equation (4.10)

$$
\frac{d\mathbb{E} \Pi^N}{dy^i} = -a_i + F_D(y^i)\mathbb{E}[Z_i],
$$

where $a_i = v^i - v^{i+1}$. 182
Using Proposition 7, we also know that

$$\frac{d \text{Var} \Pi^N}{dy^i} = \overline{F}_D(y^i) \Phi_i(y).$$

Thus, by writing $\Psi_i(y) = \mathbb{E}[Z_i] - \lambda \Phi_i(y)$, we have that

$$\frac{dU}{dy^i} = -a_i + \overline{F}_D(y^i) \Psi_i(y).$$

Moreover, using again Proposition 7, we know that

$$\left(\frac{d^2U}{dy^i dy^j}\right)_{(i,j)} \preceq \begin{pmatrix} \ddots & 0 \\ -f_D(y^i) \Psi_i(y) & \ddots \\ 0 & \ddots \end{pmatrix}.$$ 

In addition, when for all $i$, $i = 1, \ldots, n$, $a_i \geq 0$, we can use the arguments of Proposition 8 to show that, for any set $I$, whenever some portfolio $y$ is an $I$-unconstrained critical point of the utility, it is a strict local maximum of the utility over $A_I$. Intuitively, for $I = \{1, \ldots, n\}$, $\frac{dU}{dy^i} = 0$ implies $\Psi_i(y) \geq 0$, and hence the Hessian is negative definite.

The proof is completed using the same arguments as that of Proposition 9, and Theorem 8. \qed

### A.4 Chapter 5

#### A.4.1 Proposition 10

**Proof.** To obtain the optimal portfolio we maximize function $V$ over the feasible region $P$ defined in Equation (5.3). From Equation (5.2), we observe that function $V$ is the sum of strictly concave functions of $y^i$, $i = 1, \ldots, n$. Hence, it is strictly concave jointly in $(y^1, \ldots, y^n)$. The feasible region is a polyhedral cone with non-empty interior. This implies that the Slater conditions hold for this problem and that the Karush-Kuhn-Tucker conditions are necessary and sufficient at optimality (see
Define for every constraint $y^{i-1} - y^i \leq 0, i = 1, \ldots, n$, the associate Lagrange multiplier $\lambda_i \geq 0$. The KKT optimality conditions are, for $i = 1, \ldots, n$, assuming $\lambda_{n+1} = 0$:

\[
(v^{i+1} - v^i) + (w^{i+1} - w^i)\overline{F}(y^i) = \lambda_{i+1} - \lambda_i
\]

\[
\lambda_i (y^{i-1} - y^i) = 0
\]

\[
y^{i-1} - y^i \leq 0
\]

\[
\lambda_i \geq 0
\]

(A.9)

Let $\{i_1, \ldots, i_k\}$ be the winning set of $\{(w^1, v^1), \ldots, (w^{n+1}, v^{n+1})\}$. Define $y^{i_1}, \ldots, y^{i_{k-1}}$ such that

\[
\overline{F}(y^{i_j}) = \frac{v^{i_j} - v^{i_{j+1}}}{w^{i_{j+1}} - w^{i_j}}
\]

$\overline{F}(y^{i_k}) = 0$ and for the other variables $y^i = y^{i-1}$ (remember from Equation (5.1) that $y^0 = 0$). Note that it can happen that $y^i = \infty$ for some $i$. Define also $\lambda_{i_1} = \ldots = \lambda_{i_k} = 0$ together with:

(i) for $1 \leq i < i_1$, $\lambda_i = (v^i - v^{i_1}) + (w^i - w^{i_1})$,

(ii) for $j = 1, \ldots, k - 1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v^i - v^{i_j}) + (w^i - w^{i_j})\overline{F}(y^{i_j})$,

(iii) for $i_k < i$, $\lambda_i = (v^i - v^{i_k})$.

It is now sufficient to verify that this solution satisfies the KKT conditions, Equation (A.9). Evidently, the first three requirements in (A.9) are satisfied by construction. It remains to be verified that $\lambda_i \geq 0$ for all $i = 1, \ldots, n$. To see this, we analyze three different cases:

(i) for $1 \leq i < i_1$, $\lambda_i = (v^i - v^{i_1}) + (w^i - w^{i_1}) \geq 0$ from part (b) of Definition 9;

(ii) for $j = 1, \ldots, k - 1$, for $i_j < i < i_{j+1}$, $\lambda_i = (v^i - v^{i_j}) + (w^i - w^{i_j})\overline{F}(y^{i_j}) \geq 0$ from part (c);

(iii) for $i_k < i$, $\lambda_i = (v^i - v^{i_k}) \geq 0$ from part (d).
Finally, we see that no inactive point can be winning since this would imply that one of the inactive points is on the segment joining two other points. This would contradict the minimality of the winning set in Definition 9. ■

A.4.2 Proposition 11

Proof. Assume that the demand is not a border distribution. We show that it cannot satisfy the condition presented in the proposition.

By contradiction, assume we can find \((c, f) \in \mathbb{R}_+^2\) and a region \(A_i\) such that there is no optimum of the profit in the border (we drop the index \(i\)). This implies that there is a strict local maximum inside \(A_i\).

We transform the profit expression from being characterized by \((w, v)\) to \((y_-, y_+),\) as presented in Equation (5.8). Define

\[
\overline{F}(y_m) = \frac{v_l - v_h}{w_h - w_l},
\]

and the feasible region in terms of \((y_-, y_+)\) is \(y_+ \leq y_- \leq \overline{y}_l \leq y_m \leq \overline{y}_h \leq y_+ \leq \overline{y}_h\). The profit function \(J_i\), defined in Equation (5.10), now function of \((y_-, y_+, w^l, v^l, w^h, v^h)\), is

\[
(v^h - f)(y_+ - y_-) + (w^h - c)\int_{y_-}^{y_+} \overline{F}(u)du
\]

\[
- \left[\frac{-(v^l - v^h) + \overline{F}(y_-)(w^l - w^i)}{\overline{F}(y_-) - \overline{F}(y_+)}\right] \int_{y_-}^{y_+} \left(\overline{F}(u) - \overline{F}(y_+)\right)du
\]

and reaches a strict local maximum \((y^*_-, y^*_+)\) such that \(y_+ < y^*_< \overline{y}_l\) and \(y_+ < y^*_+ < \overline{y}_h\). We can now divide this expression by \(w^h - w^l > 0\) and obtain

\[
\frac{v^h - f}{w^h - w^l}(y_+ - y_-) + \frac{w^h - c}{w^h - w^l} \int_{y_-}^{y_+} \overline{F}(u)du - \left[\frac{\overline{F}(y_-) - \overline{F}(y_m)}{\overline{F}(y_-) - \overline{F}(y_+)}\right] \int_{y_-}^{y_+} \left(\overline{F}(u) - \overline{F}(y_+)\right)du.
\]

Define

\[
\alpha_1 = \frac{v^l - f}{c - w^l},
\]

\[
\alpha_2 = \frac{f - v^h}{w^h - c},
\]

185
and observe that
\[ \frac{w^h - c}{w^h - w^l} = \frac{\alpha_1 - \bar{F}(y_m)}{\alpha_1 - \alpha_2}. \]
Of course, when \( c = w^l \), we take \( \alpha_1 = \infty \) and, when \( c = w^h \), \( \alpha_2 = \infty \). Then, we take the limit of \( (\alpha_1 - \bar{F}(y_m))/(\alpha_1 - \alpha_2) \), i.e. 1 or 0 respectively, and the limit of \( \alpha_2(\alpha_1 - \bar{F}(y_m))/(\alpha_1 - \alpha_2) \), i.e. \( \alpha_2 \) or \( -(\alpha_1 - \bar{F}(y_m)) \) respectively.

Thus, the above expression can be expressed as
\[ \left[ \frac{\alpha_1 - \bar{F}(y_m)}{\alpha_1 - \alpha_2} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \alpha_2 \right) du - \left[ \frac{\bar{F}(y_-) - \bar{F}(y_m)}{\bar{F}(y_-) - \bar{F}(y_+)} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \bar{F}(y_+) \right) du. \]
This function has a strict local maximum \((y_*^-, y_*^+)\) such that \( y_i^* < y_*^* < \bar{y}_i \) and \( y_h^* < y_*^* < \bar{y}_h \), and this is a contradiction with the condition of the proposition.

Conversely, if we find a strict local maximum \((y_*^-, y_*^+)\) of the function defined by Equation (5.14), we can select \( v^l, w^l, v^h, w^h \) so that the profit cannot be maximized in the border of \( A^h \), with \( y_i = y_*^* - \epsilon, \bar{y}_i = y_*^* + \epsilon, y_h = y_*^* - \epsilon, \bar{y}_h = y_*^* + \epsilon \).

A.4.3 Proposition 13

Proof. We use Proposition 11 to prove the proposition. It is sufficient to study the case of the exponential distribution, i.e. \( \beta = -\lambda, a = 0, b = \infty, K = 1/\lambda \), since a linear transformation of variables proves the general case. Take \( \alpha_1, \alpha_2 \in \mathbb{R} \) and \( y_m \geq 0 \). We define
\[ \gamma = \frac{\alpha_1 - \bar{F}(y_m)}{\alpha_2 - \alpha_m}, \]
and \( \delta = -\alpha_2 \gamma \).

We show that the function defined in Equation (5.14) becomes
\[ \delta(y_+ - y_-) + \gamma \int_{y_-}^{y_+} \bar{F}(u) du - \left[ \frac{\bar{F}(y_-) - \bar{F}(y_m)}{\bar{F}(y_-) - \bar{F}(y_+)} \right] \int_{y_-}^{y_+} \left( \bar{F}(u) - \bar{F}(y_+) \right) du \]
\[ = \delta(y_+ - y_-) + \gamma \left( \frac{e^{-\lambda y_-} - e^{-\lambda y_+}}{\lambda} \right) \]
\[ - \left( \frac{e^{-\lambda y_-} - e^{-\lambda y_m}}{\lambda} \right) \left( \frac{e^{-\lambda y_-} - e^{-\lambda y_+}}{\lambda} - e^{-\lambda y_+}(y_+ - y_-) \right) \]
186
and does not have a strict local maximum such that $0 < y^* < y_m$ and $y^* > y_m$, which
is sufficient to apply Proposition 11.

For this purpose, we change variables to $(y_-, \Delta)$, where $\Delta = y_+ - y_-$, and hence
the function becomes

$$
\delta \Delta + \gamma \frac{e^{-\lambda y}}{\lambda} \left(1 - e^{-\lambda \Delta}\right) - \frac{1}{\lambda} \left(e^{-\lambda y} - e^{-\lambda y_m}\right) + \left(e^{-\lambda y - \lambda \Delta} - e^{-\lambda \Delta}\right) e^{-\lambda \Delta} \Delta
$$

This is clearly differentiable in $y_-$, and the derivative is

$$
-e^{-\lambda y_-} \gamma \left(1 - e^{-\lambda \Delta}\right) + e^{-\lambda y_-} - \left(e^{-\lambda y_-} - e^{-\lambda \Delta}\right) e^{-\lambda \Delta} \Delta
$$

$$
= e^{-\lambda y_-} \left[1 - \gamma (1 - e^{-\lambda \Delta}) - \frac{\lambda \Delta e^{-\lambda \Delta}}{1 - e^{-\lambda \Delta}}\right]
$$

We see that the sign of the derivative is independent of $y_-$. This implies that if
the derivative is positive, we can increase the profit by increasing $y_-$ and keeping $\Delta$
constant until it hits the border of the region. If it is non-positive, we can obtain at
least the same profit by decreasing $y_-$ and keeping $\Delta$ constant. Therefore, there can
be no interior strict local maximum. Proposition 11 implies that the distribution is
border.

\[\blacksquare\]

A.4.4 Lemma 2

**Proof.** We will prove the log-concavity of $F_1$, the proof for $F_2$ being similar.

The first derivative of $\ln(F_1)$ is

$$
\frac{f(x)}{\int_{x_0}^x f(u)du},
$$

and the second derivative

$$
\left(\frac{f(x)}{\int_{x_0}^x f(u)du}\right) \left(\frac{f'(x)f(x) - f(x)^2}{\int_{x_0}^x f(u)du}\right). \quad (A.10)
$$

187
To show log-concavity, we show next that for $x \geq x_0$

$$
\frac{f'(x)}{f(x)} \leq \frac{f(x)}{\int_{x_0}^{x} f(u) du}.
$$

This is clearly true for $x = x_0$, where the right-hand side goes to infinity. Since $f'(x)/f(x)$ is non-increasing by log-concavity of $f$, it must be that, if the equality happens at some $x_1 > x_0$,

$$
\frac{f(x)}{\int_{x_0}^{x} f(u) du}
$$

is decreasing at $x = x_1$. However, at $x_1$ the derivative of this expression is 0, by Equation (A.10). This is a contradiction, so $F_1$ is log-concave. ■

### A.4.5 Lemma 3

**Proof.** We observe that

$$
\begin{align*}
\frac{dL}{dx} &= -\frac{f(x)}{F(x) - F(y)} R(x, y) \leq 0 \\
\frac{dL}{dy} &= 1 - \frac{f(y)}{F(x) - F(y)} L(x, y) \\
\frac{dR}{dx} &= -1 + \frac{f(x)}{F(x) - F(y)} R(x, y) \\
\frac{dL}{dy} &= \frac{f(y)}{F(x) - F(y)} L(x, y) \geq 0
\end{align*}
$$

This implies that $L(x, y)$ is non-increasing in $x$ and $R(x, y)$ is non-decreasing in $y$, for all demand distributions.

In addition, when $f$ is log-concave, using Lemma 2, we know that, since $\ln(f)$ is concave, $\ln(\bar{F}(x) - \bar{F}(y))$ is concave in $x$ and thus the logarithm of

$$
\int_{x}^{y} (\bar{F}(u) - \bar{F}(y)) du
$$

188
is, again, concave in \( x \). This property implies that the function

\[
\frac{\int_x^y (F(u) - F(y))du}{F(x) - F(y)} = R(x, y)
\]

must be non-increasing in \( x \). Similarly, \( L(x, y) \) is non-decreasing in \( y \). ■

### A.4.6 Lemmas 9 and 10

These lemmas are needed in the proof of Theorem 10.

**Lemma 9** When \( f \) is log-concave, let \( \mu \) be a value for which \( f \) reaches its maximum. There are \( \lambda_A(y) \leq \mu \) and \( \lambda_B(x) \geq \mu \) such that

(i) \( A(\cdot, y) \), as defined in Lemma 4, is non-decreasing in \( [0, \lambda_A(y)] \) and non-increasing in \( (\lambda_A(y), y) \); and

(ii) \( B(x, \cdot) \), as defined in Lemma 4, is non-decreasing in \( (x, \lambda_B(x)) \) and non-increasing in \( (\lambda_B(x), \infty) \).

**Proof.** We can take the derivatives of \( A \) and \( B \),

\[
\frac{1}{A} \frac{dA}{dx} = \frac{f'(x)}{f(x)} - \frac{F(x) - F(y)}{\int_x^y (F(u) - F(y))du} + \frac{2f(x)}{F(x) - F(y)} = \frac{f'(x)}{f(x)} + \left[ \frac{f(x)}{F(x) - F(y)} \right] \left[ 2 - \frac{1}{A} \right],
\]

and similarly

\[
\frac{1}{B} \frac{dB}{dy} = \frac{f'(y)}{f(y)} - \left[ \frac{f(y)}{F(x) - F(y)} \right] \left[ 2 - \frac{1}{B} \right].
\]

When \( \frac{dA}{dx} = 0 \), we have

\[
\frac{1}{A} \frac{d^2A}{dx^2} \leq \left[ 2 - \frac{1}{A} \right] \frac{d}{dx} \left[ \frac{f(x)}{F(x) - F(y)} \right].
\]

Since \( x \to \frac{f(x)}{F(x) - F(y)} \) is non-decreasing when \( f \) is log-concave, then \( A(\cdot, y) \) increases and then decreases to 1/2. This implies that the maximum is reached at a value
greater that 1/2, and thus, when \( \frac{dA}{dx} = 0 \), we must have that \( \frac{f'(x)}{f(x)} \geq 0 \). Hence the value for which \( A(\cdot, y) \) reaches a maximum is such that \( \lambda_A(y) \leq \mu \).

Similarly, when \( \frac{dB}{dy} = 0 \), we have

\[
\frac{1}{B} \frac{d^2B}{dy^2} \leq -\left[ 2 - \frac{1}{B} \frac{d}{dy} \frac{f(y)}{\bar{F}(x) - \bar{F}(y)} \right].
\]

Again, since \( y \to \frac{f(y)}{\bar{F}(x) - \bar{F}(y)} \) is non-increasing when \( f \) is log-concave, then, using the same arguments, we conclude that the value where \( B(x, \cdot) \) reaches a maximum is such that \( \lambda_B(x) \geq \mu \). ■

**Lemma 10** Assume \( f \) is log-concave. Then, for each \( (x, y) \) such that \( x \leq y \),

\[
x \to \frac{A(x, y)}{\bar{F}(x) - \bar{F}(y)} \quad \text{is non-decreasing; and}
\]

\[
y \to \frac{B(x, y)}{\bar{F}(x) - \bar{F}(y)} \quad \text{is non-increasing.}
\]

**Proof.** We use here Lemma 9.

\[
\frac{\bar{F}(x) - \bar{F}(y)}{A(x, y)} \frac{d}{dx} \left\{ \frac{A(x, y)}{\bar{F}(x) - \bar{F}(y)} \right\} = \frac{1}{A} \frac{dA}{dx} + \frac{f(x)}{\bar{F}(x) - \bar{F}(y)}
\]

\[
= \left\{ \frac{f'(x)}{f(y)} + \frac{f(x)}{\bar{F}(x) - \bar{F}(y)}\right\}
\]

\[
+ \left[ \frac{f(x)}{\bar{F}(x) - \bar{F}(y)} \right] [2 - \frac{1}{A}].
\]

Clearly, when \( \frac{1}{A} \frac{dA}{dx} \geq 0 \), i.e. \( x \leq \lambda_A(y) \) as defined in Lemma 9, then the previous expression is non-negative. When \( \lambda_A(y) \leq x \leq y \), \( A(x, y) \geq 1/2 \); since \( \frac{f'(x)}{f(x)} + \frac{f(x)}{\bar{F}(x) - \bar{F}(y)} \geq 0 \) (because \( \frac{f(x)}{\bar{F}(x) - \bar{F}(y)} \) is non-decreasing with \( x \) when \( f \) is log-concave), we have that the expression is again non-negative. Hence \( \frac{A(x, y)}{\bar{F}(x) - \bar{F}(y)} \) is non-decreasing with \( x \).
The same argument shows that \( \frac{B(x, y)}{F(x) - F(y)} \) is non-increasing with \( y \). ■

A.4.7 Theorem 10

Proof. We will use the condition of Proposition 11 in order to prove this result.

Take \( y_m \geq 0, \alpha_1, \alpha_2 \in \mathbb{R} \) and define \( \alpha_m = \overline{F}(y_m) \). Use \( y_l = 0, \overline{y}_l = y_h = y_m \) and \( \overline{y}_h = \infty \). Let \( (y_-, y_+) \) be a strict local maximum of the function

\[
\Phi(y_-, y_+) = \frac{\alpha_1 - \alpha_m}{\alpha_1 - \alpha_2} \int_{y_-}^{y_+} (\overline{F}(u) - \alpha_2) du - \frac{\overline{F}(y_-) - \alpha_m}{\overline{F}(y_-) - \overline{F}(y_+)} \int_{y_-}^{y_+} (\overline{F}(u) - \overline{F}(y_+)) du.
\]

such that \( 0 < y_- < y_m < y_+ \). Define \( \alpha_- = \overline{F}(y_-) \) and \( \alpha_+ = \overline{F}(y_+) \).

Since this is a strict interior maximum, the first order conditions are, after recombining the different terms,

\[
0 = \frac{d\Phi}{dy_-} = \frac{(\alpha_m - \alpha_2)(\alpha_1 - \alpha_-)}{(\alpha_1 - \alpha_2)} + f(y_-) \frac{(\alpha_m - \alpha_+)}{(\alpha_- - \alpha_+)^2} \left\{ \int_{y_-}^{y_+} (\overline{F}(u) - \alpha_+) du \right\} (A.11)
\]

\[
0 = \frac{d\Phi}{dy_+} = \frac{(\alpha_1 - \alpha_m)(\alpha_+ - \alpha_2)}{(\alpha_1 - \alpha_2)} + f(y_+) \frac{(\alpha_- - \alpha_m)}{(\alpha_- - \alpha_+)^2} \left\{ \int_{y_-}^{y_+} (\alpha_- - \overline{F}(u)) du \right\} (A.12)
\]

Let \( A = A(y_-, y_+) \) and \( B = B(y_-, y_+) \), as defined in Equations (5.17) and (5.18). The first order conditions are equivalent to

\[
A = \frac{(\alpha_m - \alpha_2)(\alpha_1 - \alpha_-)}{(\alpha_1 - \alpha_2)(\alpha_m - \alpha_+)}; \quad B = \frac{(\alpha_1 - \alpha_m)(\alpha_+ - \alpha_2)}{(\alpha_1 - \alpha_2)(\alpha_- - \alpha_m)},
\]

or equivalently,

\[
\begin{align*}
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_2} - \frac{1 - A}{A} & \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_2} - \frac{1 - B}{B} + \frac{1}{A} \frac{1}{B} = 1 + \frac{1}{A \alpha_m - \alpha_+}, \\
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_2} - \frac{1 - B}{B} & \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_2} - \frac{1 - A}{A} + \frac{1}{A} \frac{1}{B} = 1 + \frac{1}{B \alpha_- - \alpha_m}. \quad (A.13)
\end{align*}
\]

Lemma 4, shows that \( 0 \leq A \leq 1 \) and \( 0 \leq B \leq 1 \) when \( f \) is log-concave. Thus under this assumption, \( \alpha_- \geq \alpha_m \geq \alpha_+ \) implies that the first order conditions can only be
satisfied when $\alpha_+ - \alpha_2 \geq 0$ and $\alpha_1 - \alpha_- \geq 0$ or $\alpha_+ - \alpha_2 \leq 0$ and $\alpha_1 - \alpha_- \leq 0$.

Let
\[
\begin{align*}
a &= \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1 - A}{A}, \\
b &= \frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1 - B}{B}, \\
c &= \frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_}. \\
\end{align*}
\]

Equation (A.13) can thus be expressed as
\[
a(b + \frac{1}{B}) - 1 = \frac{1}{A} c,
\]
\[
b(a + \frac{1}{A}) - 1 = \frac{1}{B} c.
\]

By multiplying these two equations, one obtains
\[
\left[a(b + \frac{1}{B}) - 1\right] \left[b(a + \frac{1}{A}) - 1\right] = \frac{1}{AB},
\]

or equivalently
\[
\left[ab - 1\right] \left[ab + \frac{a}{B} + \frac{b}{A} + \frac{1}{AB} - 1\right] = 0.
\]

We have two possible cases:

1. In the first case, we have $a, b \geq 0$, $ab = 1$, and thus $\alpha_1 \geq \alpha_- \geq \alpha_m \geq \alpha_+ \geq \alpha_2$.

   Thus Equation (A.13) becomes
\[
\begin{align*}
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1 - A}{A} &= \frac{B}{A} \left[\alpha_- - \alpha_m\right], \\
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1 - B}{B} &= \frac{A}{B} \left[\alpha_m - \alpha_+\right]. \\
\end{align*}
\]

2. In the second case, we have $a, b \leq 0$, and $ab = 1 - \frac{a}{B} - \frac{b}{A} - \frac{1}{AB}$. Thus, Equation (A.13) becomes
\[
\begin{align*}
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - 1 &= \frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}, \\
\frac{\alpha_- - \alpha_+}{\alpha_1 - \alpha_-} - \frac{1}{\alpha_+ - \alpha_2} &= \frac{\alpha_m - \alpha_+}{\alpha_- - \alpha_m}. \\
\end{align*}
\]

192
The second order condition for having an interior local maximum is that the Hessian of $\Phi$ is negative semi-definite. It is straightforward to see that the Hessian being negative semi-definite is equivalent to having that $H$, defined as follows, is negative semi-definite.

$$H = \left( \begin{array}{ccc}
\frac{1}{A} \frac{dA}{dy_-} + \frac{f(y_-)}{\alpha_1 - \alpha_-} & \frac{1}{A} \frac{dA}{dy_+} + \frac{f(y_+)}{\alpha_m - \alpha_+} \\
-\frac{1}{B} \frac{dB}{dy_-} + \frac{f(y_-)}{\alpha_- - \alpha_m} & -\frac{1}{B} \frac{dB}{dy_+} + \frac{f(y_+)}{\alpha_+ - \alpha_2} 
\end{array} \right)$$

We compute the quantities that define $H$ in the following equations, evaluated at $(y_-, y_+)$. We have

$$\frac{1}{A} \frac{dA}{dy_-} = \frac{f'(y_-)}{f(y_-)} + \left[ \frac{f(y_-)}{F(y_-) - F(y_+)} \right] \left[ 2 - \frac{1}{A} \right],$$
$$\frac{1}{A} \frac{dA}{dy_+} = \left[ \frac{f(y_+)}{F(y_-) - F(y_+)} \right] \left[ 1 + \frac{f(y_+)B}{f(y_+)} \right] - 2 \left[ \frac{f(y_+)}{F(y_-) - F(y_+)} \right],$$
$$\frac{1}{B} \frac{dB}{dy_-} = -\left[ \frac{f(y_-)}{F(y_-) - F(y_+)} \right] \left[ 1 + \frac{f(y_-)B}{f(y_-)} \right] + 2 \left[ \frac{f(y_-)}{F(y_-) - F(y_+)} \right],$$
$$\frac{1}{B} \frac{dB}{dy_+} = \frac{f'(y_+)}{f(y_+)} - \left[ \frac{f(y_+)}{F(y_-) - F(y_+)} \right] \left[ 2 - \frac{1}{B} \right].$$

Thus, $H$ can be expressed as

$$H = \left( \begin{array}{ccc}
\frac{f'(y_-)}{f(y_-)} & 0 & \frac{f'(y_+)}{f(y_+)} \\
0 & \frac{f'(y_-)}{f(y_-)} & 0 \\
\frac{f(y_-)}{\alpha_- - \alpha_+} & \frac{f(y_+)}{\alpha_+ - \alpha_2} & \frac{1}{A} \frac{dA}{dy_+} + \frac{f(y_+)}{\alpha_m - \alpha_+} \\
\frac{f(y_-)}{\alpha_- - \alpha_m} & \frac{f(y_+)}{\alpha_+ - \alpha_2} & \frac{1}{B} \frac{dB}{dy_+} + \frac{f(y_+)}{\alpha_+ - \alpha_2} 
\end{array} \right) + \left( \begin{array}{ccc}
\frac{A}{\alpha_- - \alpha_+} & \frac{B}{\alpha_m - \alpha_+} & \frac{B}{\alpha_+ - \alpha_2} \\
\frac{A}{\alpha_- - \alpha_m} & \frac{B}{\alpha_m - \alpha_+} & \frac{B}{\alpha_+ - \alpha_2} 
\end{array} \right)$$

193
In case (2) defined above, we can take a look at $H_{11}$, using Equation (A.15):

\[
\frac{f'(y_-)}{f(y_-)} + \left[ \frac{f(y_-)}{\alpha_- - \alpha_+} \right] \left[ 2 - \frac{1}{A} \alpha_- - \alpha_+ \right] \geq \frac{f'(y_-)}{f(y_-)} + \left[ \frac{f(y_-)}{\alpha_- - \alpha_+} \right] \left[ 3 - \frac{1}{A} \right].
\]

Using Lemma 10, this quantity is the derivative of $A(y_-, y_+)$ with respect to $y_-$, hence non-negative. The same is true for $H_{22}$. Thus, in case (2), the matrix $H$ cannot be negative semi-definite.

In case (1), using that $c = \frac{\alpha_- - \alpha_m}{\alpha_m - \alpha_+}$,

\[
H = \begin{pmatrix}
\frac{f'(y_-)}{f(y_-)} & 0 \\
\frac{f'(y_+)}{f(y_+)} & 0 \\
\end{pmatrix} + \frac{A}{B} \begin{pmatrix}
\frac{f(y_-)}{\alpha_- - \alpha_+} \\
\frac{f(y_+)}{\alpha_- - \alpha_+} \\
\end{pmatrix} \left[ 3 - \frac{2}{A} \frac{Bc}{A} \right] + \frac{B}{A} \begin{pmatrix}
\frac{f(y_-)}{\alpha_- - \alpha_+} \\
\frac{f(y_+)}{\alpha_- - \alpha_+} \\
\end{pmatrix} + \left[ \frac{f(y_+)}{\alpha_- - \alpha_+} \right] \left[ 3 - \frac{2}{B} \frac{A}{Bc} \right].
\]
To get rid of $c$, we examine

\[
\left( \frac{A}{f(y_-)} \frac{B}{f(y_+)} \right) H \left( \frac{A}{f(y_-)} \right)
\]

\[
= \left\{ \frac{f'(y_-) A^2}{f(y_-) f(y_-)^2} \right. \\
\left. + \frac{2A(2A-1)}{f(y_-)(\alpha_- - \alpha_+)} + \frac{2B(2B-1)}{f(y_+(\alpha_- - \alpha_+))} \right. \\
\left. + \frac{f'(y_-)}{f(y_-)} \left\{ \int_{y_-}^{y_+} \frac{F(u) - \alpha_+}{\alpha_- - \alpha_+} du \right\}^2 \right. \\
\left. - \frac{f'(y_+)}{f(y_+)} \left\{ \int_{y_-}^{y_+} \frac{\alpha_- - \bar{F}(u)}{\alpha_- - \alpha_+} du \right\}^2 \right. \\
\left. + 2(2A-1) \left\{ \int_{y_-}^{y_+} \frac{\alpha_- - \bar{F}(u)}{\alpha_- - \alpha_+} du \right\} \right. \\
\left. + 2(2B-1) \left\{ \int_{y_-}^{y_+} \frac{\alpha_- - \bar{F}(u)}{\alpha_- - \alpha_+} du \right\} \right. \\
\left. \right\}
\]

Using that

\[
\Delta = y_+ - y_- = \int_{y_-}^{y_+} \frac{\bar{F}(u) - \alpha_+}{\alpha_- - \alpha_+} du + \int_{y_-}^{y_+} \frac{\alpha_- - \bar{F}(u)}{\alpha_- - \alpha_+} du,
\]

and defining

\[
z = \int_{y_-}^{y_+} \frac{\bar{F}(u) - \alpha_+}{\alpha_- - \alpha_+} du,
\]

we can express the terms in the last bracket as

\[
E = \frac{f'(y_-)}{f(y_-)} z^2 \frac{f'(y_+)(\Delta - z)^2}{f(y_+)} + 4 \frac{f(y_-)}{\alpha_- - \alpha_+} z^2 + 4 \frac{f(y_+)}{\alpha_- - \alpha_+} (\Delta - z)^2 - 2\Delta. \quad (A.16)
\]
By minimizing this expression in terms of \( z \), we obtain a lower bound on this expression, i.e.,

\[
E = \Delta \left\{ \frac{f'(y_-) + 4f(y_-)}{f(y_-)} \left[ \frac{4f(y_+)}{f(y_+)} \alpha_+ - \alpha_- \right] - 2 \right\} 
\]

\[
= \left\{ \frac{1}{f(y_-) + 4f(y_-) - f'(y_+) + 4f(y_+)} \right\} \left\{ \frac{f'(y_-)}{f(y_-)} \Delta + 4 \frac{f(y_-)}{f(y_-)} \Delta - 2 \right\} \left[ - \frac{f'(y_+)}{f(y_+)} \Delta + 4 \frac{f(y_+)}{f(y_+)} \Delta - 2 \right] - 4
\]

We thus focus on the last term of the product,

\[
F = \left[ \frac{f'(y_-)}{f(y_-)} \Delta + 4 \frac{f(y_-)}{f(y_-)} \Delta - 2 \right] \left[ - \frac{f'(y_+)}{f(y_+)} \Delta + 4 \frac{f(y_+)}{f(y_+)} \Delta - 2 \right] - 4. \quad (A.17)
\]

Over all log-concave distribution functions, \( F \) defined in Equation (A.17) is minimized when \( \alpha_- - \alpha_+ \) is maximized. This occurs when, after defining \( \theta \) by \( \beta_- \theta + \beta_+ (1 - \theta) = \beta_0 \),

\[
f(t) = \begin{cases} 
  f(y_-)e^{-\beta_- (t - y_-) / \Delta} & \text{when } y_- \leq t \leq y_- + \theta \Delta \\
  f(y_+)e^{-\beta_+ (y_+ - t) / \Delta} & \text{when } y_+ - (1 - \theta) \Delta \leq t \leq y_+
\end{cases}
\]

We thus know the structure of the worst-case log-concave distribution. By re-scaling the problem, \( F \) can be expressed using only \( \beta_- \), \( \beta_+ \) (with \( \beta_+ \leq \beta_- \)) and \( \theta \in [0, 1] \). To obtain the following expression, we scale \( \Delta \) to 1 and the break-point value of the distribution \( f(y_- + \theta \Delta) \) to 1. After defining, for \( k \geq 0 \),

\[
P_k(z) = \frac{e^z - 1 - z - \ldots - z^{k-1}/(k-1)!}{z^k}, \quad (A.18)
\]

196
we obtained the following scaled quantities

\[ f(y_-) = e^{-\beta_- \theta} = P_0(-\beta_- \theta) \]
\[ f(y_+) = e^{\beta_+ (1 - \theta)} = P_0(\beta_+(1 - \theta)) \]
\[ \alpha_- - \alpha_+ = \frac{e^{-\beta_- \theta} - 1}{-\beta_-} + \frac{e^{\beta_+ (1 - \theta)} - 1}{\beta_+} = \theta P_1(-\beta_- \theta) + (1 - \theta) P_1(\beta_+(1 - \theta)) \].

Define

\[ R_1(\beta_-, \beta_+) = \frac{P_0(-\beta_- \theta)}{\theta P_1(-\beta_- \theta) + (1 - \theta) P_1(\beta_+(1 - \theta))} , \]

\[ G_1(\beta_-, \beta_+) = \beta_- - 2 + 4R_1(\beta_-, \beta_+) \geq 0, \]

and

\[ R_2(\beta_-, \beta_+) = \frac{P_0(\beta_+(1 - \theta))}{\theta P_1(-\beta_- \theta) + (1 - \theta) P_1(\beta_+(1 - \theta))} , \]

\[ G_2(\beta_-, \beta_+) = -\beta_+ - 2 + 4R_2(\beta_-, \beta_+) \geq 0. \]

Thus, \( F \) can be expressed as \( G_1 G_2 - 4 \).

Notice that, since

\[ P_1(z)' = P_1(z) - P_2(z) = P_0(z) P_2(-z) , \]

we have that

\[ \frac{dR_1}{d\beta_-} = R_1 \left( -\theta + \theta^2 P_2(\beta_+ \theta) R_1 \right) \]
\[ \frac{dR_1}{d\beta_+} = R_1 \left( -(1 - \theta)^2 P_2(-\beta_+(1 - \theta)) R_2 \right) , \]

and

\[ \frac{dR_2}{d\beta_-} = R_2 \left( \theta^2 P_2(\beta_+ \theta) R_1 \right) \]
\[ \frac{dR_2}{d\beta_+} = R_2 \left( (1 - \theta) - (1 - \theta)^2 P_2(-\beta_+(1 - \theta)) R_2 \right) . \]

In order to obtain a lower bound on \( F \), we examine two different cases: either the minimal value of \( F \) subject to \( \theta \in [0, 1] \) and \( \beta_- \geq \beta_+ \) is reached in an interior point,
or it is reached at the border of the region, i.e., $\theta = 0$, $\theta = 1$ or $x = y$; in any of the latter cases, the distribution turns out to be an exponential distribution.

To analyze the first case, let's examine the critical points of $F$ with respect to $\theta$, $\beta_-$ and $\beta_+$. That is, assume that

$$
\frac{dF}{d\theta} = 0 = \left[4R_1 \left(\beta_- - (R_2 - R_1)\right)\right] G_2 + G_1 \left[4R_2 \left((R_2 - R_1) - \beta_+\right)\right],
$$

(A.19)

$$
\frac{dF}{d\beta_-} = 0 = \left[1 + 4R_1 \theta \left(-1 + \theta P_2(\beta_\theta)R_1\right)\right] G_2 + G_1 \left[4R_1 R_2 \theta^2 P_2(\beta_\theta)\right],
$$

(A.20)

and

$$
\frac{dF}{d\beta_+} = 0 = \left[-4R_1 R_2 (1-\theta)^2 P_2(-\beta_+(1-\theta))\right] G_2 + G_1 \left[-1 + 4R_2 (1-\theta) \left(1 - (1-\theta)P_2(-\beta_+(1-\theta))R_2\right)\right].
$$

(A.21)

Notice that we can express

$$
R_2 - R_1 = \frac{P_0 \left(\beta_+(1-\theta)\right) - P_0 \left(-\beta_-\theta\right)}{\theta P_1 \left(-\beta_-\theta\right) + (1-\theta) P_1 \left(\beta_+(1-\theta)\right)}
$$

$$
= \beta_+ + \frac{\theta (\beta_- - \beta_+ P_1 \left(-\beta_-\theta\right)}{\theta P_1 \left(-\beta_-\theta\right) + (1-\theta) P_1 \left(\beta_+(1-\theta)\right)}
$$

$$
= \beta_- - \frac{(1-\theta)(\beta_- - \beta_+ P_1 \left(\beta_+(1-\theta)\right)}{\theta P_1 \left(-\beta_-\theta\right) + (1-\theta) P_1 \left(\beta_+(1-\theta)\right)}
$$

Thus, Equation (A.19) can be rewritten as

$$
\frac{R_1/G_1}{R_2/G_2} = \frac{(R_2 - R_1) - \beta_+}{\beta_- - (R_2 - R_1)} = \frac{\theta}{1-\theta} \frac{P_1 \left(-\beta_-\theta\right)}{P_1 \left(\beta_+(1-\theta)\right)}. 
$$

(A.22)
Equation (A.20) can be expressed as

\[
4\theta \frac{R_1}{G_1} = \frac{1}{G_1} + 4\theta^2 P_2(\beta_\theta) \left[ \frac{R_1}{\frac{R_1}{G_1}} + \frac{R_2}{\frac{R_2}{G_2}} \right].
\]

Using the inequality \((a + b)^2 \geq 4ab\) for any \(a, b \in \mathbb{R}\), we obtain

\[
16\theta^2 \left[ \frac{R_1}{G_1} \right]^2 \geq 16\theta^2 P_2(\beta_\theta) \left[ \frac{R_1}{G_1} \left( \frac{R_1}{G_1} + \frac{R_2}{G_2} \right) \right],
\]

and hence

\[
\frac{R_1}{G_1} \geq P_2(\beta_\theta) \left[ \frac{R_1}{G_1} \left( \frac{R_1}{G_1} + \frac{R_2}{G_2} \right) \right],
\]

(A.23)

Similarly, Equation (A.21) yields

\[
\frac{R_2}{G_2} \geq P_2(1 - \beta_\theta) \left[ \frac{R_1}{G_1} \left( \frac{R_1}{G_1} + \frac{R_2}{G_2} \right) \right],
\]

(A.24)

Adding these two last equations, the term \(R_1/G_1 + R_2/G_2\) cancels out, and thus

\[
1 > P_2(\beta_\theta) + P_2(-\beta_+(1 - \theta)).
\]

Notice that \(P_2\) is convex, since \(P_2'' = P_2 - 4P_3 + 6P_4 \geq 0\). We can therefore apply the convexity inequality

\[
\frac{1}{2} P_2(\beta_\theta) + \frac{1}{2} P_2(1 - \beta_\theta) \geq P_2\left( \frac{1}{2} \beta_\theta - \frac{1}{2} \beta_+(1 - \theta) \right).
\]

Thus, since \(P_2(z) \leq 1/2\) if and only if \(z \leq 0\), combining the two inequalities, we obtain that

\[
\beta_\theta \leq \beta_+(1 - \theta)
\]

Also, since \(\beta_- \geq \beta_+\) by construction, we have that

\[
\beta_\theta \leq \beta_+(1 - \theta) \leq \beta_-(1 - \theta).
\]

This implies, that for any critical point, we must have:
either $\beta_- \geq \beta_+ \geq 0$ and $\theta \leq \frac{1}{2}$;

or $0 \geq \beta_- \geq \beta_+$ and $\theta \geq \frac{1}{2}$.

When $\theta < 1/2$, Equation (A.23) yields

$$\frac{R_1}{G_1} \geq \frac{1}{2} \left( \frac{R_1}{G_1} + \frac{R_2}{G_2} \right),$$

and thus

$$\frac{R_1}{G_1} \geq \frac{R_2}{G_2}.$$

On the other hand, Equation (A.22) implies that, since $\theta/(1-\theta) < 1$, $P_1(-\beta_- \theta) \leq P_1(0) = 1$ and $P_1(\beta_+(1-\theta)) \geq P_1(0) = 1$,

$$\frac{R_1/G_1}{R_2/G_2} = \frac{\theta}{1-\theta} \frac{P_1(-\beta_- \theta)}{P_1(\beta_+(1-\theta))} < 1.$$

This is a contradiction.

Similarly, when $\theta > 1/2$, we have again a contradiction using Equation (A.24) to show

$$\frac{R_1}{G_1} \leq \frac{R_2}{G_2},$$

and Equation (A.22) to show

$$\frac{R_1/G_1}{R_2/G_2} > 1.$$

Thus, the only feasible case is $\theta = 1/2$ which implies $\beta_- = \beta_+ = 0$, and in this case $G_1 = G_2 = 2$, so that $F \geq 0$.

Thus, there are no critical points in the interior of $\theta \in [0,1]$ and $\beta_- \geq \beta_+$ such that $F < 0$.

The remaining case is when the minimum of $F$ is reached at the border of the feasible set, i.e., the distribution is exponential, with parameter $\gamma$. In this case, we
can express \( F \) as \( g(\gamma)g(-\gamma) - 4 \), with, for \( z \in \mathbb{R} \),

\[
g(z) = z - 2 + \frac{4}{P_1(z)}.
\]

Note that \( g(z) - g(-z) = -2z \) and

\[
g(z) + g(-z) = 4 \frac{z(1 - e^{-z}) + z(e^z - 1) - (e^z - 1)(e^{-z} - 1)}{(e^z - 1)(e^{-z} - 1)} \equiv 4 \left\{ \frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} - 1 \right\}.
\]

Hence,

\[
g(z)g(-z) - 4 = \frac{1}{4} \left( g(z) + g(-z) \right)^2 - \frac{1}{4} \left( g(z) - g(-z) \right)^2 - 4
\]

\[
= 4 \left\{ \frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} \right\}^2 - 8 \left\{ \frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} \right\} - z^2.
\]

After putting all terms under a common denominator, and observing that

\[
\left\{ \frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} \right\}^2 = 4 \left\{ \frac{P_2(-2z) + P_2(2z)}{P_2(-2z) + P_2(2z)} \right\},
\]

\[
\left\{ \frac{P_1(-z) + P_1(z)}{P_2(-z) + P_2(z)} \right\} \left\{ \frac{P_2(-z) + P_2(z)}{P_2(-z) + P_2(z)} \right\} = 2 \left\{ \frac{P_1(-2z) + P_1(2z) - P_1(-z) - P_1(z)}{z^2} \right\},
\]

\[
\left\{ \frac{P_2(-z) + P_2(z)}{P_2(-z) + P_2(z)} \right\}^2 = 4 \left\{ \frac{P_2(-2z) + P_2(2z) - P_2(-z) - P_2(z)}{z^2} \right\},
\]

the numerator can be expressed as

\[
12 \left\{ \frac{P_2(-2z) + P_2(2z)}{z^2} \right\} + 4 \left\{ \frac{P_2(-z) + P_2(z)}{z^2} \right\} - 16 \left\{ \frac{P_1(-2z) + P_1(2z)}{z^2} \right\} + 16 \left\{ \frac{P_1(-z) + P_1(z)}{z^2} \right\}.
\]

We can use a series expansion to show that this final term is non-negative:

\[
2 \sum_{k=0}^{\infty} \left\{ \frac{12 \cdot 2^k z^{2k}}{(2k + 2)!} + \frac{4z^{2k}}{(2k + 2)!} - \frac{16 \cdot 2^{2k+2} z^{2k}}{(2k + 3)!} + \frac{16z^{2k}}{(2k + 3)!} \right\}
\]

201
The coefficient of $z^{2k}$ in the series is
\[
\frac{12(2k + 3)4^k + 4(2k + 3) - 64 \cdot 4^k + 16}{(2k + 3)!} \geq 0,
\]
for all $k \geq 0$. This shows that $F \geq 0$, in all cases.

This completes the proof, since we have found that the second order maximality condition cannot be satisfied. ■

\section*{A.5 Chapter 6}

\subsection*{A.5.1 Theorem 11}

\textbf{Proof.} Assume that supplier $i$ is not active in an equilibrium of the game. This implies that its profit is 0. Define the function $Z_{(w^{(-i)}, v^{(-i)})}$ as in Equation (5.4), the lower envelope made of all bids except $i$'s. If $f^i < Z_{(w^{(-i)}, v^{(-i)})}(c^i)$, then by bidding $(c^i + \epsilon, f^i + \epsilon)$, supplier $i$ achieves some positive profit for $\epsilon$ small enough. This contradicts the previous hypothesis and therefore we must have $f^i \geq Z_{(w^{(-i)}, v^{(-i)})}(c^i)$.

Construct the lower envelope $C_{(-i)}$ of the costs $(c^1, f^1), \ldots, (c^{i-1}, f^{i-1}), (c^{i+1}, f^{i+1}), \ldots, (c^n, f^n)$, $(p, 0)$. That is, $C_{(-i)} = Z_{(c^{(-i)}, f^{(-i)})}$. Assume that $C_{(-i)}$ is not a lower bound on the function $Z_{(w^{(-i)}, v^{(-i)})}$. This implies that there is an active bid $(w^j, v^j)$ such that $v^j < C_{(-i)}(w^j)$. $j \neq i$ since $i$ is not active and is not defining the function $Z_{(w^{(-i)}, v^{(-i)})}$. We claim that if supplier $j$ bids in some region $A_{(w^{(-i)}, v^{(-i)})}^{th}$, it cannot be at equilibrium. Indeed, we can use Equation (5.13), in particular,

\[
\frac{dJ^h}{dy^i_-} \geq (f^j - v^j) + (c^j - w^j)F(y^j_-),
\]

\[
\frac{dJ^h}{dy^i_+} \leq (v^h - f^j) + (w^h - c^j)F(y^j_+).
\]

If this is an equilibrium, then $j$'s bid in $A_{(w^{(-i)}, v^{(-i)})}^{th}$ must be such that $(f^j - v^j) +
\((c^j - w^j)F(y^j_l) \leq 0 \) and \((v^h - f^j) + (w^h - c^j)F(y^j_h) \geq 0\). This is equivalent to

\[ f^j \leq v^j + \frac{(v^j - v^j)}{w^j - v^j} \cdot \frac{c^j - w^j}{w^j - v^j}, \]

or if \(l = 0\), \(f^j + c^j \leq v^k + w^k\) for some \(k\), and

\[ f^j \leq v^h + \frac{(v^h - v^h)}{w^h - v^h} \cdot \frac{w^h - c^j}{w^h - v^h}. \]

But if all this feasible area is not strictly below \(C_{(k)}\), we can find some other supplier \(k\) bidding next to \(j\) that also satisfies \(v^k < C_{(k)}(w^k)\). By repeating the argument, we must find a third supplier \(l\) satisfying \(v^l < C_{(k)}(w^l)\) that is not \(j\) (so no cycling is possible). When we reach the supplier with the smallest \(w\) or with the biggest \(w\) (the dummy supplier, \(n + 1\)), we reach a contradiction: for the smallest \(w\), we cannot find a different supplier satisfying the condition, for the dummy supplier \(v^{n+1} = f^{n+1} = 0 = C_{(n+1)}(v^{n+1}) = C_{(n+1)}(w^{n+1}) = C_{(n+1)}(p)\). Hence \(j\) cannot be in equilibrium, and this is a contradiction.

Therefore, the function \(C_{(k)}\) lies below the function \(Z_{(w^n, v^n)}\). This implies that \(i\) cannot be efficient, since it is not needed to define the function \(C_{(k)}\), and thus is not a winning point of \{(c^1, f^1), \ldots, (c^n, f^n), (p, 0)\}. \(\blacksquare\)

### A.5.2 Theorem 12

**Proof.** We have explained previously that under the assumptions of the proposition, either \(y^*_1 = y_m\) and \(y^*_2 = y_1\) or \(y^*_1 = y_2\) and \(y^*_2 = y_m\). Otherwise, it would be optimal to bid in some other region \(A^{v_k}\) in addition to \(A^h\). Since this is a contradiction to the hypothesis, it implies that the two possible optimal bids are either \((w^v, v^v)\) or \((w^h, v^h)\).

If \(y_1 > y_m\) or \(y_2 < y_m\), from Equation (5.13) it is clear that it is not optimal for the supplier to bid in this particular region \(A^h\), because it has an incentive to bid in \(A^{V/T}\) instead of \(A^h\). Similarly, if \(y_1 < y_0\) and \(y_2 > y_3\), neither one of the bids is admissible, and therefore there is an optimum outside \(A^h\). We can now partition the
remaining possibilities into the three cases presented in the proposition.

In the two first cases, since only one of the two bids is admissible, it must be optimal. In the third case, it implies that \((c,f) \in A^l\). Bidding \((w^h, v^h)\) is better than \((w^l, v^l)\) when

\[
\Pi_2 = (v^h - f)(y_2 - y_m) + (w^h - c) \int_{y_m}^{y_2} \overline{F}(u)du \\
\geq \Pi_1 = (v^l - f)(y_m - y_1) + (w^l - c) \int_{y_1}^{y_m} \overline{F}(u)du.
\]

Using Equations (6.2) and (6.3), this is equivalent to

\[
(w^h - c) \int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du \geq (c - w^l) \int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du.
\]

But also, we have that, similarly to Equation (5.9),

\[
c = w^h - (w^h - w^l)\frac{\overline{F}(y_1) - \overline{F}(y_m)}{\overline{F}(y_1) - \overline{F}(y_2)} = w^l + (w^h - w^l)\frac{\overline{F}(y_m) - \overline{F}(y_2)}{\overline{F}(y_1) - \overline{F}(y_2)}.
\]

Therefore, we can rewrite the previous condition as

\[
\frac{\overline{F}(y_1) - \overline{F}(y_m)}{\overline{F}(y_1) - \overline{F}(y_2)} \int_{y_m}^{y_2} [\overline{F}(u) - \overline{F}(y_2)]du \geq \frac{\overline{F}(y_m) - \overline{F}(y_2)}{\overline{F}(y_1) - \overline{F}(y_2)} \int_{y_1}^{y_m} [\overline{F}(y_1) - \overline{F}(u)]du.
\]

After simplifying this expression, we obtain Equations (6.4) and (6.5).

\[\boxed{\text{A.5.3 Proposition 14}}\]

\textbf{Proof.} In a Nash equilibrium, if \(i\) and \(j\) submit the same bid and are both active, then \(i\) must bid in some region \(A_{(w^l, v^l)}\) or \(A_{(w^h, v^h)}\); similarly, \(j\) must bid in some region \(A_{(w^l, v^l)}\) or \(A_{(w^h, v^h)}\). Indeed, if \(i\) (resp. \(j\)) bids in a different region, then \(i\) (resp. \(j\)) makes \(j\) (resp. \(i\)) inactive.

In equilibrium, we can apply Theorem 12, for both \(i\) and \(j\). If \(i\) bids in \(A_{(w^l, v^l)}\) then \(j\) must bid in \(A_{(w^l, v^l)}\). Therefore by using the optimality equations (6.2) and (6.3), we have that the slope between \((c^l, f^l)\) and \((w^l, v^l)\), and \((w^l, v^l)\) and \((c^l, f^l)\) must be the same, since this is an equilibrium and \(y_{l+} = y_{l-}\). Similarly, if \(i\) bids in
$A^h_{(w(l-1),v(l-1))}$ then $j$ bids in $A^h_{(w(l-1),v(l-1))}$ and again the slopes $(c^i, f^i)$ and $(w^j, v^j)$, and $(w^j, v^j)$ and $(c^j, f^j)$ must be the same. In both cases, we have that $(w^j, v^j)$ belongs to the segment $[(c^i, f^i); (c^j, f^j)]$. ■

A.5.4 Proposition 15

Proof. Using Theorem 11, we know that every supplier is active.

If the proposition was false, we could find suppliers $i$ and $j$ such that $c^i < c^j$ and $w^j > w^i$. We may furthermore assume without loss of generality that these are consecutive bidders, i.e. there is no bid $(w, v)$ with $w^i < w < w^j$. To see this, assume that the active suppliers are indexed such that $w^1 \leq \ldots \leq w^m$ and in case of a tie, sorted by increasing execution cost $c$.

Select a pair $(i, j)$ such that $i + 1 < j$ with $w^i < w^j$ and $c^i > c^j$. One of the following three cases is possible.

- The pair $(i, i + 1)$ satisfies $w^i < w^{i+1}$ and $c^i > c^{i+1}$ and then $(i, i + 1)$ are consecutive bidders.

- $w^i < w^{i+1}$ and $c^i \leq c^{i+1}$. Then, it is the pair $(i + 1, j)$ that satisfies $c^{i+1} > c^j$ and $w^{i+1} < w^j$. Hence, we can iterate this argument until we find consecutive bidders $i$ and $j$ such that $c^i < c^j$ and $w^i > w^j$.

- $w^i = w^{i+1}$ but then, by construction, $c^i \leq c^{i+1}$. Hence, similarly to the previous case, we iterate the argument with the pair $(i + 1, j)$.

Since $w^i > w^j$ and $i$ and $j$ are consecutive bidders, the bid of supplier $j$ must be in the border of some region $A^j_l$ (where there is no active supplier between $l$ and $i$ because if there was one it would not be active), where supplier $l$ is active. Also, $w^j > w^i$ implies that $w^j = w^l$ and $v^j = v^l$ is optimal, from Theorem 12. But applying Proposition 14 yields that $(w^j, v^l)$ belongs in the segment $[(c^i, f^l); (c^j, f^j)]$.

Similarly, supplier $i$ bids in some region $A^h_{(w(l-1),v(l-1))}$, with supplier $h$ active and no active suppliers between $j$ and $h$. With the same argument as before, we have that $(w^i, v^i) = (w^h, v^h)$ and this bid belongs in the segment $[(c^i, f^h); (c^j, f^h)]$. 205
Define,
\[ F(y_m) = \frac{v^j - v^i}{w^i - w^j}, \]
\[ F(y_1^j) = \frac{v^j - f^j}{c^i - w^j}, \]
\[ F(y_2^j) = \frac{v^j - f^j}{c^j - w^j}, \]
and we have that \( y_1^j < y_m < y_2^j \) because \( i \) and \( j \) are active.

We can also define
\[ F(y_1^i) = \frac{v^j - f^i}{c^i - w^j}, \]
\[ F(y_2^i) = \frac{f^i - v^j}{w^i - c^j}. \]

Since \( c^i < c^j \), and supplier \( i \) is efficient, we must have that \( f^i \leq f^j + (c^j - c^i)F(y_1^j) \) because \( F(y_1^j) \) is the slope of the line joining \((c^i, f^i)\) to \((c^j, f^j)\). Similarly, \( f^j \leq f^i - (c^j - c^i)F(y_2^j) \). This implies that \( y_1^j \leq y_1^j < y_m < y_2^j \leq y_2^j \) as can be seen from Figure A-1.

![Figure A-1: Geometric situation of costs \((c^i, f^i)\) and \((c^j, f^j)\) in region A_1](image)

Finally, we apply Theorem 12. For this purpose, recall the functions \( L(y_1, y_m) \) and
$R(y_m, y_2)$, as defined in Equations (5.15) and (5.16). Lemma 3 shows that $L(\cdot, y_m)$ is non-increasing and $R(y_m, \cdot)$ is non-decreasing.

We now apply the last case of Theorem 12. Since supplier $i$ bids $(w^i, v^i)$ and not $(w^j, v^j)$, we have that $L(y^i_1, y_m) \leq R(y_m, y^i_2)$. Similarly, for $j$, $L(y^j_1, y_m) \geq R(y_m, y^j_2)$. $y^i_1 \leq y^i_2 < y_m < y^j_2 \leq y^j_1$ yields $L(y^i_1, y_m) \geq L(y^i_1, y_m) \geq R(y_m, y^j_2) \geq R(y_m, y^j_2)$, and hence $L(y^i_1, y_m) \leq R(y_m, y^j_2)$ implies that all inequalities are in fact equalities. Therefore $c^i = c^j$ which is a contradiction. ■

A.5.5 Theorem 13

Proof. Consider supplier $1 < i \leq n$. From Theorem 11 and Proposition 15, we know that at equilibrium it will be bidding in region $A^{i-1}i+1$ because otherwise one of the suppliers would be inactive or they would not be sorted in the correct order. Let $l = i - 1$ and $h = i + 1$. Supplier $i$ will in particular bid in the border of this region, with $y^i_- = y_m$ or $y^i_+ = y_m$ as established in Theorem 12. $y^i_- = y_m$ is equivalent to saying that it is bidding $w^i = w^{i+1}$ and $v^i = v^{i+1}$ and $\overline{F}(y^i_+) = \frac{f_i - v^{i+1}}{w^{i+1} - c^i}$. In this case, applying Proposition 14 yields that $(v^i, w^i)$ belongs in the segment $[(c^i, f^i); (c^{i+1}, f^{i+1})]$. Similarly, $y^i_+ = y_m$ implies that $(v^i, w^i)$ belongs in the segment $[(c^{i-1}, f^{i-1}); (c^i, f^i)]$, and this is of course possible only if $i > 1$. For $i = 1$, only the first case can occur, i.e., $w^1 = w^2$, $v^1 = v^2$, $y^1_- = 0$ and $y^1_+$ such that $\overline{F}(y^1_+) = \frac{f^1 - v^2}{w^2 - c^1}$. Again, Proposition 14 implies that $(v^1, w^1)$ belongs in the segment $[(c^1, f^1); (c^2, f^2)]$. ■

A.5.6 Lemma 5

Proof. Since this must be proved for all $t \geq 0$, it is sufficient to show that, when $L(x^0(t), y^0(t)) = R(y^0(t), z)$,

$$
\frac{dL}{dx}(x^0, y^0) \frac{dx^0}{dt} + \frac{dL}{dy}(x^0, y^0) \frac{dy^0}{dt} - \frac{dR}{dx}(y^0, z) \frac{dy_0}{dt} \leq 0. \quad (A.25)
$$

207
We know that
\[
\frac{dL}{dx}(x, y) = -\frac{f(x)}{\overline{F}(x) - \overline{F}(y)} R(x, y),
\]
\[
\frac{dL}{dy}(x, y) = 1 - \frac{f(y)}{\overline{F}(x) - \overline{F}(y)} L(x, y),
\]
\[
\frac{dR}{dx}(x, y) = -1 + \frac{f(x)}{\overline{F}(x) - \overline{F}(y)} R(x, y),
\]
and
\[
-f(x^0) \frac{dx^0}{dt} = \frac{1}{c - w^0} \left(1 - \frac{v^0 - f}{c - w^0} \right) = \frac{1}{c - w^0} \left(1 - \overline{F}(x^0) \right)
\]
\[
-f(y^0) \frac{dy^0}{dt} = \frac{1}{w^2 - w^0} \left(1 - \frac{v^0 - v^2}{w^2 - w^0} \right) = \frac{1}{w^2 - w^0} \left(1 - \overline{F}(y^0) \right).
\]

Notice that \((\overline{F}(x^0) - \overline{F}(y^0))(c - w^0) = (\overline{F}(y^0) - \overline{F}(z))(w^2 - c)\), yields \((\overline{F}(x^0) - \overline{F}(y^0))(c - w^0) = (\overline{F}(y^0) - \overline{F}(z))(w^2 - w^0)\), or equivalently
\[
\frac{1}{w^2 - w^0} = \frac{1}{c - w^0} \left[ \overline{F}(y^0) - \overline{F}(z) \right].
\]

Equation (A.25) is equivalent to
\[
\frac{f(x^0)}{\overline{F}(x^0) - \overline{F}(y^0)} R(x^0, y^0) \left(1 - \overline{F}(x^0) \right) \overline{F}(x^0) - \overline{F}(z) \right) \left( \overline{F}(y^0) - \overline{F}(z) \right)
\]
\[- \left\{ 2 - \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} L(x^0, y^0) - \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} R(y^0, z) \right\} \left(1 - \overline{F}(y^0) \right) \right) \frac{1}{f(y^0)}
\]
\[\leq 0.
\]

or in other words, using that \(L(x^0, y^0) = R(y^0, z)\),
\[
\left(1 - \overline{F}(x^0) \right) \left\{ -2 + \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \overline{F}(x^0) - \overline{F}(y^0) + \overline{F}(x^0) + R(x^0, y^0) \right\}
\]
\[\leq \overline{F}(x^0) \left\{ -2 - \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \overline{F}(x^0) + \overline{F}(y^0) \right\} L(x^0, y^0) \right\}.
\]

(A.26)

Of course, the right-hand side is always non-negative, because \(f\) is log-concave, which
implies, by Lemma 3, that
\[ 2 - \left( \frac{f(y^0)}{F(x^0) - F(y^0)} + \frac{f(y^0)}{F(y^0) - F(z)} \right) L(x^0, y^0) = \frac{dL}{dy}(x^0, y^0) - \frac{dR}{dx}(y^0, z) \geq 0. \]

Thus, when the left-hand side is non-positive, then we are done. We claim that this is the case when \( f'(x^0)/f(x^0) \leq 0 \).

Claim 1 When \( f'(x^0)/f(x^0) \leq 0 \),
\[ -2 + \left( \frac{f(y^0)}{F(x^0) - F(y^0)} + \frac{f(y^0)}{F(y^0) - F(z)} \right) \left( L(x^0, y^0) + R(x^0, y^0) \right) \leq 0. \]

Proof. Indeed, assume that \( f'(x^0)/f(x^0) \leq 0 \). Since \( f \) is log-concave, we have that \( f'(t) \leq 0 \) for \( t \geq x^0 \); equivalently, \( F \) is convex for \( t \geq x^0 \).

After observing that \( L(x^0, y^0) + R(x^0, y^0) = y^0 - x^0 \), we want to show that
\[ f(y^0)(y^0 - x^0) \left( \frac{1}{F(x^0) - F(y^0)} + \frac{1}{F(y^0) - F(z)} \right) \leq 2. \]

It is simple to show that the left-hand side is maximized when \( F(x^0) \) is minimized and \( F(z) \) maximized, given that \( L(x^0, y^0) = R(y^0, z) \). We claim that the worst-case occurs when the distribution \( f \) is reached by a truncated exponential with rate \( f'(y^0)/f(y^0) \).

We start by proving that, in \([y^0, z]\), the worst case is achieved when \( f \) is an exponential of rate \( f'(y^0)/f(y^0) \). Consider \( f \) a worst-case distribution. We have two possible cases: either \( F(z) = 0 \) or not.

If \( F(z) = 0 \), define the distribution equal to \( f \) on \([0, y^0]\) and to the truncated exponential
\[ g_\gamma(t) = \frac{f(y^0)e^{\gamma(t-y^0)}}{\int_{y^0}^z f_\gamma(t) \, dt} \text{ on } [y^0, \infty), \]
for \( \gamma \leq f'(y^0)/f(y^0) \). Since \( f \) is log-concave, then \( f(t) \leq g_{f'(y^0)/f(y^0)}(t) \) for \( y^0 \leq t \leq z \).
Define $\overline{G}_\gamma$ such that
\[
\overline{G}_\gamma(t) = \overline{F}(y^0) - \int_{y^0}^{t} g_\gamma(u) du = \int_{y^0}^{z} f(u) du - \int_{y^0}^{z} g_\gamma(u) du
\]
This is clearly increasing in $\gamma$, since $g_\gamma(\cdot)$ is increasing in $\gamma$. In addition, we have that $\overline{G}_{f'(y^0)/f(y^0)}(z) < 0 < \overline{G}_{-\infty}(z)$. We can thus find $\gamma$ such that $\overline{G}_\gamma(z) = 0$, and hence for this particular $\gamma$,
\[
\overline{F}(y^0) = \overline{F}(y^0) - \overline{F}(z) = \overline{G}_\gamma(y^0) - \overline{G}_\gamma(z).
\]
Moreover, since $\overline{F}(t) \leq \overline{G}_\gamma(t)$ for $y^0 \leq t \leq z$,
\[
L(x^0, y^0) = R(y^0, z) = \frac{\int_{y^0}^{z} [\overline{F}(u) - \overline{F}(z)] du}{\overline{F}(y^0) - \overline{F}(z)} \leq \frac{\int_{y^0}^{z} [\overline{G}_\gamma(u) - \overline{G}_\gamma(z)] du}{G_{\gamma}(y^0) - G_{\gamma}(z)}.
\]
Thus, we can decrease $z$ until we satisfy the constraint. Hence, by using $g_\gamma$ instead of $f$, we are able to increase $\overline{F}(z) = 0$ to some positive number, which means that $f$ cannot be the worst case.

We are left with the case $\overline{F}(z) > 0$, $f(z) > 0$ and $\overline{F}(y^0) > 0$. Assume that $f$ is not exponential. Define, for $f'(y^0)/f(y^0) \geq \gamma \geq f'(z)/f(z)$, the distribution equal to $f$ on $[0, y^0]$, and $[z, \infty)$, and to
\[
g_\gamma(t) = \min \left\{ f(y^0) e^{\gamma(t-y^0)}, f(z) e^{f'(z)/f(z)(t-z)} \right\}
\]
on $[y^0, z]$. This is clearly log-concave. Fix $\gamma$ such that
\[
\overline{F}(y^0) - \overline{F}(z) = \int_{y^0}^{z} g_\gamma(u) du.
\]
This implies that
\[
\overline{G}_\gamma(t) = \overline{F}(z) + \int_{z}^{t} g_\gamma(u) du
\]
is always greater than $\overline{F}(t)$. Since it is strictly smaller for at least some subinterval
of \([y^0, z]\), \(L(x^0, y^0) = R(y^0, z) < \int_y^z \frac{[G^r(u) - G^r(z)]du}{G^r(y^0) - G^r(z)}\). Hence, for the log-concave distribution \(g_r\), we can decrease \(z\) until we satisfy the feasibility constraint, thus increasing \(G^r(z)\) to a larger quantity. Thus \(f\) cannot be the worst-case distribution. The only possibility is that \(f\) is exponential, with rate \(f'(y^0)/f(y^0)\).

We prove now that, in \([x^0, y^0]\), the worst case is achieved for the exponential of rate \(f'(y^0)/f(y^0)\). The arguments that will we use are similar to the ones used previously. Again, we consider \(f\) a worst-case distribution, and study the two possible cases, \(\overline{F}(x^0) = 1\) or not.

When \(\overline{F}(x^0) = 1\), we show that modifying \(f\) to the truncated exponential

\[
g_r(t) = f(y^0) e^{\gamma(t-y^0)} 1_{[x^0, y^0]}
\]

on \([0, y^0]\), for a well-chosen \(\gamma \geq f'(y^0)/f(y^0)\), allows to reduce \(\overline{F}(x^0)\) to a smaller quantity, thus contradicting that \(f\) constitutes a worst case for Equation (A.26).

Similarly, when \(\overline{F}(x^0) < 1\), \(f(x^0) > 0\). We show that a worst case must be an exponential of rate \(f'(y^0)/f(y^0)\) by modifying \(f\) into

\[
g_r(t) = \min \left\{ f(y^0) e^{\gamma(t-y^0)}, f(x^0) e^{f''(x^0)/f(x^0)(t-x^0)} \right\}
\]

on \([x^0, y^0]\), for a well-chosen \(\gamma\). This again yields a contradiction.

Thus, by combining these two steps, we have shown that, in the worst-case, the distribution is a decreasing exponential of parameter \(\beta = -f'(y^0)/f(y^0) \geq 0\). Of course, since \(\beta\) is a scaling factor, we only need to show the result for the standard exponential, with decrease rate of 1.

For this purpose, define the following functions

\[
P_1(t) = \frac{e^t - 1}{t} \quad \text{and} \quad P_2(t) = \frac{e^t - 1 - t}{t^2}.
\]

211
The constraint \( L(x^0, y^0) = R(y^0, z) \) becomes

\[
\frac{1}{P_1(-(y^0 - x^0))} + \frac{1}{P_1(z - y^0)} = 2.
\]

By writing \( \Delta_1 = y^0 - x^0 \) and \( \Delta_2 = z - y^0 \), the constraint becomes

\[
\frac{1}{P_1(-\Delta_1)} + \frac{1}{P_1(\Delta_2)} = 2. \tag{A.27}
\]

Note that, since \( 1/P_1 \) is decreasing and convex, we must have that \( \Delta_2 \geq \Delta_1 \), since otherwise, \( 1/P_1(\Delta_1) + 1/P_1(-\Delta_1) \leq 2/P_1(\Delta_1 - \Delta_1) = 2/P_1(0) = 2 \), which contradicts the convexity of \( 1/P_1 \).

The equation to prove is

\[
f(y^0)(y^0 - x^0)
\left(\frac{1}{F(x^0) - F(y^0)} + \frac{1}{F(y^0) - F(z)}\right) = \frac{1}{P_1(\Delta_1)} \left(1 + \frac{e^{\Delta_1} - 1}{1 - e^{-\Delta_2}}\right) \leq 2.
\]

This is equivalent to

\[e^{\Delta_1} - 1 \leq \left(2P_1(\Delta_1) - 1\right)\left(1 - e^{-\Delta_2}\right)\]

which yields

\[
\left(1 - e^{-\Delta_1}\right)e^{-\Delta_2} \leq \left(2P_1(-\Delta_1) - 1\right)\left(1 - e^{-\Delta_2}\right),
\]

or also

\[\Delta_1 \leq \left(2 - \frac{1}{P_1(-\Delta_1)}\right)\Delta_2 P_1(\Delta_2),\]

By using the constraint, i.e., Equation A.27, this is equivalent to \( \Delta_1 \leq \Delta_2 \), which is always true, as pointed out before.

Thus, we have shown the lemma when \( f'(x^0)/f(x^0) \leq 0 \). We must now examine the remaining case, i.e., \( f'(x^0)/f(x^0) \geq 0 \).

212
Claim 2 When \( f'(x^0)/f(x^0) \geq 0 \),

\[
\left(1 - \overline{F}(x^0)\right) \left\{ -2 + \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \left( L(x^0, y^0) + R(x^0, y^0) \right) \right\} \\
\leq \left( \overline{F}(x^0) - \overline{F}(y^0) \right) \left\{ 2 - \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) L(x^0, y^0) \right\}.
\]

Proof. In this case, since \( f \) is log-concave, we have that \( f'(t)/f(t) \geq f'(x^0)/f(x^0) \) for \( t \leq x^0 \). This implies the following, for \( t \leq x^0 \):

\[
f'(t) \geq \frac{f'(x^0)}{f(x^0)} f(t),
\]

and hence, after integration of the inequality on \((0^-, x^0)\),

\[
f(x^0) \geq \frac{f'(x^0)}{f(x^0)} \left( 1 - \overline{F}(x^0) \right),
\]

Thus,

\[
\frac{1 - \overline{F}(x^0)}{f(x^0)} \leq \frac{f(x^0)}{f'(x^0)}.
\]

It is therefore sufficient to show that, when \( L(x^0, y^0) = R(y^0, z) \),

\[
\frac{f(x^0)^2}{f'(x^0)} \left\{ -2 + \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \left( L(x^0, y^0) + R(x^0, y^0) \right) \right\} \\
\leq \left( \overline{F}(x^0) - \overline{F}(y^0) \right) \left\{ 2 - \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} + \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) L(x^0, y^0) \right\}.
\]

(A.28)

Using that \( L(x^0, y^0) + R(x^0, y^0) = y^0 - x^0 \), this can be expressed as

\[
\left( \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \left\{ \frac{f(x^0)^2}{f'(x^0)} \left( y^0 - x^0 \right) + \left( \overline{F}(x^0) - \overline{F}(y^0) \right) L(x^0, y^0) \right\} \\
\leq \left( \overline{F}(x^0) - \overline{F}(y^0) \right) \left\{ 2 - \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} \right) L(x^0, y^0) \right\} \\
- \frac{f(x^0)^2}{f'(x^0)} \left\{ -2 + \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} \right) \left( y^0 - x^0 \right) \right\}.
\]

It is clear that the worst case occurs for the largest \( \overline{F}(z) \) possible, given the constraint \( L(x^0, y^0) = R(y^0, z) \) and the value of \( f(y^0) \) and \( f'(y^0)/f(y^0) \). As shown in the proof
of the previous claim, this is achieved only by the exponential of rate \( f'(y^0)/f(y^0) \), in \([y^0, z]\).

Equation (A.28) can also be expressed as

\[
\left( \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) \frac{f(x^0)^2}{f'(x^0)} \left( y^0 - x^0 \right) + f(y^0)R(y^0, z) - 2f(x^0)^2 f'(x^0)
\leq \left( \overline{F}(x^0) - \overline{F}(y^0) \right) \left\{ 2 - \left( \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) R(y^0, z) \right\} - \frac{f(x^0)^2}{f'(x^0)} \left( \frac{f(y^0)}{\overline{F}(x^0) - \overline{F}(y^0)} \right) \left( y^0 - x^0 \right).
\]  

(A.29)

Since \( \left( \frac{f(y^0)}{\overline{F}(y^0) - \overline{F}(z)} \right) R(y^0, z) \leq 1 \leq 2 \), it is clear that the worst-case occurs when \( \overline{F}(x^0) \) is minimized given the constraint \( L(x^0, y^0) = R(y^0, z) \), and keeping \( x^0 \) and \( f'(x^0)/f(x^0) \) fixed.

Consider \( f \) that achieves the worst-case for the inequality. Assume that \( f \) is not exponential. Of course, \( \overline{F}(x^0) < 1 \), since otherwise, the claim is immediately true. Define, for \( f'(x^0)/f(x^0) \geq \gamma \geq f'(y^0)/f(y^0) \), the distribution \( g_\gamma \) equal to \( f \) on \([0, x^0]\) and \([y^0, \infty)\), and to

\[
g_\gamma(t) = \min \left\{ f(y^0)e^{\gamma(t-y^0)}, f(x^0)e^{f'(x^0)/f(x^0)(t-x^0)} \right\}
\]
on \([x^0, y^0]\). This is clearly log-concave. Fix \( \gamma \) such that

\[
\overline{F}(x^0) - \overline{F}(y^0) = \int_{x^0}^{y^0} g_\gamma(u)du.
\]

This implies that

\[
\overline{G}_\gamma(t) = \overline{F}(y^0) + \int_t^{y^0} g_\gamma(u)du
\]
is always smaller than \( \overline{F}(t) \). Thus \( R(y^0, z) = L(x^0, y^0) < \frac{\int_{x^0}^{y^0} [\overline{G}_\gamma(x^0) - \overline{G}_\gamma(u)]du}{\overline{G}_\gamma(x^0) - \overline{G}_\gamma(y^0)} \).

Hence, for the log-concave distribution \( g_\gamma \), we can increase \( x^0 \) until we satisfy the
feasibility constraint. The effect of such move is to reduce the difference between right-hand side and left-hand side of Equation (A.29) since $\overline{F}(x^0)$ increases.

Thus, the worst case is achieved when $f$ is exponential, i.e., when $f'(x^0)/f(x^0) = f'(y^0)/f(y^0) \geq 0$. Hence, the worst-case distribution is an increasing exponential, with rate $f'(x^0)/f(x^0)$. By rescaling the problem, it is sufficient to consider the case $f'(x^0)/f(x^0) = 1$. In this case, Equation (A.28) is satisfied when

$$-2 + \frac{1}{P_1(-\Delta_1)} \left(1 + \frac{1}{e^{\Delta_2} - 1} \right) \leq \left(e^{\Delta_1} - 1 \right) \left\{2 - \left(\frac{1}{1 - e^{\Delta_1}} + \frac{1}{e^{\Delta_2} - 1} \right) \left(1 - \frac{1}{P_1(\Delta)} \right) \right\}.$$ 

where $\Delta_1 = y^0 - x^0$ and $\Delta_2 = z - y^0$. We need to show that

$$\Delta_1 \left(\frac{1}{e^{\Delta_2} - 1} \right) + \left(\frac{e^{\Delta_1} - 1}{e^{\Delta_2} - 1} \right) \left(1 - \frac{1}{P_1(\Delta)} \right) = \frac{e^{\Delta_1} - 1}{e^{\Delta_2} - 1} \leq 2 + \left(e^{\Delta_1} - 1 \right) \left\{2 - \left(\frac{1}{1 - e^{\Delta_1}} \right) \left(1 - \frac{1}{P_1(\Delta)} \right) \right\} - \frac{1}{P_1(-\Delta_1)}.$$ 

After simplification, this becomes

$$\frac{1 - e^{-\Delta_1}}{e^{\Delta_2} - 1} \leq 1,$$

or

$$e^{\Delta_2} + e^{-\Delta_1} \geq 2.$$

On the other hand, the constraint $L(x^0, y^0) = R(y^0, z)$ becomes

$$\frac{1}{P_1(\Delta_1)} + \frac{1}{P_1(-\Delta_2)} = 2,$$

which is equivalent to saying

$$(1 - e^{-\Delta_1}) \frac{P_2(\Delta_1)}{P_2(\Delta_1) + P_2(-\Delta_1)} = (e^{\Delta_2} - 1) \frac{P_2(-\Delta_2)}{P_2(-\Delta_2) + P_2(-\Delta_2)}.$$
Hence, since \( P_z \) is increasing, we have that

\[
1 - e^{-\Delta_1} \leq \left(1 - e^{-\Delta_1}\right) \frac{P_z(\Delta_1)}{P_z(\Delta_1) + P_z(-\Delta_1)} \leq e^{\Delta_2} - 1,
\]

and this concludes the proof.

By putting together the results of the two claims, we prove the lemma.

A.5.7 Lemma 6

Proof. This proof follows the lines of the proof of Lemma 5. We outline the main steps of the proof, the details being identical.

It is sufficient to show that, when \( L(x, y^3(t)) = R(y^3(t), z^3(t)) \),

\[
\frac{dL}{dy}(x, y^3) \frac{dy^3}{dt} - \frac{dR}{dx}(y^3, z^3) \frac{dy^3}{dt} - \frac{dR}{dy}(y^3, z^3) \frac{dz^3}{dt} \geq 0. \tag{A.30}
\]

Using that

\[
\begin{align*}
    f(y^3) \frac{dy^3}{dt} &= \frac{1}{w^3 - w^1} \left( \frac{y^3 - y^1}{w^3 - w^1} \right) = \frac{1}{w^3 - w^1} \left( \overline{F}(y^3) \right) \\
    f(z^3) \frac{dz^3}{dt} &= \frac{1}{w^3 - c} \left( \frac{f - v^3}{w^3 - c} \right) = \frac{1}{w^3 - c} \left( \overline{F}(z^3) \right) = \frac{1}{w^3 - w^1} \left( \overline{F}(z^3) \right) \{ \overline{F}(x) - \overline{F}(z^3) \} \left( \frac{\overline{F}(x)}{\overline{F}(x) - \overline{F}(y^3)} \right),
\end{align*}
\]

and the constraint \( L(x, y^3) = R(y^3, z^3) \), Equation (A.30) is equivalent to showing that

\[
\left\{ 2 - \frac{f(y^3)}{\overline{F}(x) - \overline{F}(y^3)} R(y^3, z^3) - \frac{f(y^3)}{\overline{F}(y^3) - \overline{F}(z^3)} R(y^3, z^3) \right\} \overline{F}(y^3)
\]

\[
\geq \frac{\overline{F}(z^3)}{L(y^3, z^3)} \left( 1 + \frac{\overline{F}(y^3) - \overline{F}(z^3)}{\overline{F}(x) - \overline{F}(y^3)} \right),
\]

or alternatively,

\[
\begin{align*}
    \overline{F}(z^3) &\left\{ -2 + \left( \frac{f(y^3)}{\overline{F}(x) - \overline{F}(y^3)} + \frac{f(y^3)}{\overline{F}(y^3) - \overline{F}(z^3)} \right) \left( L(y^3, z^3) + R(y^3, z^3) \right) \right\} \\
    \leq \left( \overline{F}(y^3) - \overline{F}(z^3) \right) \left\{ 2 - \left( \frac{f(y^3)}{\overline{F}(x) - \overline{F}(y^3)} + \frac{f(y^3)}{\overline{F}(y^3) - \overline{F}(z^3)} \right) R(y^3, z^3) \right\}.
\end{align*}
\]

216
This expression is very similar to Equation (A.26), and the proof method is identical to the one presented before. ■

A.5.8 Theorem 15

Proof. As we mentioned before, the loss in welfare occurs for every supplier $i$ when $(w^i, v^i) = (w^{i-1}, v^{i-1})$ and $(w^i, v^i) \neq (w^{i+1}, v^{i+1})$. For all other cases, we have that $y^i = y^{i*}$. We have two different possible cases.

A. The market allocation is such that $y^i < y^{i*}$.

B. The market allocation is such that $y^i > y^{i*}$.

In case (A), Equation (6.4) holds since $(w^i, v^i) = (w^{i-1}, v^{i-1})$. Therefore, using $y_1 = y^{i-1*}$, $y_m = y^i$ and $y_2 \geq y^{i*}$, and using the notation of Equations (5.15) and (5.16),

$$L(0, y^i) \geq L(y^{i-1*}, y^i) = \frac{\int_{y^{i-1*}}^{y^i} [\overline{F}(y^{i-1*}) - \overline{F}(u)] du}{\overline{F}(y^{i-1*}) - \overline{F}(y^i)} \geq \frac{\int_{y^i}^{y^{i*}} [\overline{F}(u) - \overline{F}(y_2)] du}{\overline{F}(y^i) - \overline{F}(y_2)} = R(y^i, y_2) \geq R(y^i, y^{i*}),$$

where we used the fact that the function $R(y^i, \cdot)$ is non-decreasing, $L(\cdot, y_2)$ is non-increasing, $y_2 \geq y^{i*}$ and $0 \leq y^{i-1*}$.

Examine now the loss created by supplier $i$.

$$\int_{y^i}^{y^{i*}} [\overline{F}(u) - \overline{F}(y^{i*})] du = (\overline{F}(y^i) - \overline{F}(y^{i*})) R(y^i, y^{i*})$$

217
Similarly,

\[
\int_{0}^{y^*} (F(u) - \bar{F}(y^*))du = \int_{0}^{y^*} (\bar{F}(u) - \bar{F}(y^*))du + \int_{y^*}^{y^+} (\bar{F}(u) - \bar{F}(y^*))du
\]

\[= (1 - \bar{F}(y^*))y^i - \int_{0}^{y^*} [1 - \bar{F}(u)]du \]

\[+ (\bar{F}(y^i) - \bar{F}(y^*))R(y^i, y^*) \]

\[= (1 - \bar{F}(y^*)) \left( L(0, y^i) + R(0, y^i) \right) \]

\[-(1 - \bar{F}(y^i)) L(0, y^i) \]

\[+ (\bar{F}(y^i) - \bar{F}(y^*)) R(y^i, y^*) \]

\[\geq (\bar{F}(y^i) - \bar{F}(y^*)) \left( L(0, y^i) + R(y^i, y^*) \right) \]

\[\geq (\bar{F}(y^i) - \bar{F}(y^*)) 2R(y^i, y^*) \]

where we used that \(R(y^i, y^*) \leq L(0, y^i)\).

Thus, after multiplying by \(\Delta c^i\), we have that

\[
\Delta c^i \int_{y^i}^{y^+} (F(u) - \bar{F}(y^*))du \leq \frac{1}{2} \Delta c^i \int_{0}^{y^+} (\bar{F}(u) - \bar{F}(y^*))du.
\]

In case (B), it must be that \(i < N\). Since \((w^i, v^i) \neq (w^{i+1}, v^{i+1})\), Theorem 13 implies that \((w^{i+1}, v^{i+1}) = (w^{i+2}, v^{i+2})\), and this means that \(u^i \leq c^i \leq c^{i+1} \leq u^{i+1} \leq c^{i+2}, y^{i+1} = y^{i+1*} \) and \(y^{i} \leq y^{i+1*}\). We can now use Equation (6.5) for supplier \(i + 1\) in order to derive a bound on the loss. Here, \(y_m = y^i, y_2 = y^{i+1*}\) and \(y_1 \leq y^{i*}\).

\[
R(y^i, y^{i+1*}) = \frac{\int_{y^i}^{y^{i+1*}} (F(u) - \bar{F}(y^{i+1*}))du}{\bar{F}(y^i) - \bar{F}(y^{i+1*})} \geq \frac{\int_{y^i}^{y^{i+1*}} (F(y^i) - \bar{F}(u))du}{\bar{F}(y^i) - \bar{F}(y^i)}
\]

\[= L(y^i, y^i) \]

\[\geq L(y^{i*}, y^i) \]

218
The loss created by supplier \(i\) involves

\[
\int_{y^{i*}}^{y^{i}} [\bar{F}(u) - \bar{F}(y^{i*})] du = \int_{y^{i*}}^{y^{i}} [\bar{F}(u) - \bar{F}(y^{i*})] du
\]

\[
= \left( \bar{F}(y^{i*}) - \bar{F}(y^{i}) \right) L(y^{i*}, y^{i})
\]

\[
\leq \left( \bar{F}(y^{i*}) - \bar{F}(y^{i}) \right) R(y^{i}, y^{i+1*}).
\]

Now, note that

\[
(w^{i+1} - w^{i}) \bar{F}(y^{i}) = (c^{i+1} - w^{i}) \bar{F}(y_{1}) + (w^{i+1} - c^{i+1}) \bar{F}(y^{i+1*})
\]

\[
\geq (c^{i+1} - w^{i}) \bar{F}(y^{i*}) + (w^{i+1} - c^{i+1}) \bar{F}(y^{i+1*}),
\]

where the inequality is justified by \(c^{i+1} \geq w^{i}\) and \(y_{1} \leq y^{i*}\). This, together with \(\Delta c^{i} \leq c^{i+1} - w^{i}\) and \(w^{i+1} - c^{i+1} \leq \Delta c^{i+1}\), implies that

\[
\Delta c^{i} [\bar{F}(y^{i*}) - \bar{F}(y^{i})] \leq (c^{i+1} - w^{i}) [\bar{F}(y^{i*}) - \bar{F}(y^{i})]
\]

\[
\leq (w^{i+1} - c^{i+1}) [\bar{F}(y^{i}) - \bar{F}(y^{i+1*})]
\]

\[
\leq \Delta c^{i+1} [\bar{F}(y^{i}) - \bar{F}(y^{i+1*})].
\]

Thus,

\[
\Delta c^{i} \int_{y^{i}}^{y^{i*}} [\bar{F}(u) - \bar{F}(y^{i*})] du \leq \Delta c^{i+1} \left( \bar{F}(y^{i}) - \bar{F}(y^{i+1*}) \right) R(y^{i}, y^{i+1*})
\]

219
Since
\[
\int_{y^i}^{y^i+1} [F(u) - F(y^i+1)] du = \int_0^{y^i} [F(u) - F(y^i+1)] du + \int_{y^i}^{y^i+1} [F(u) - F(y^i+1)] du
\]
\[= \left(1 - F(y^i+1)\right) y^i - \int_0^{y^i} [1 - F(u)] du + \left(\bar{F}(y^i) - F(y^i+1)\right) R(y^i, y^i+1)\]
\[= \left(1 - F(y^i+1)\right) \left(L(0, y^i) + R(0, y^i)\right) - \left(1 - \bar{F}(y^i)\right) L(0, y^i) + \left(\bar{F}(y^i) - F(y^i+1)\right) R(y^i, y^i)\]
\[\geq \left(\bar{F}(y^i) - F(y^i+1)\right) \left(L(0, y^i) + R(0, y^i+1)\right) - \left(1 - \bar{F}(y^i)\right) L(0, y^i)\]
\[\geq \left(\bar{F}(y^i) - F(y^i+1)\right) 2R(y^i, y^i+1),\]
we have that
\[
\Delta c_i \int_{y^i}^{y^i+1} [F(u) - F(y^i+1)] du \leq \frac{1}{2} \Delta c_i + 1 \int_{y^i}^{y^i+1} [F(u) - F(y^i+1)] du
\]

Finally, since \(y_{i+1} = y_{i+1}^\ast\), this completes the proof of the bound for any border distribution.  ■

A.5.9 Lemma 7

Proof. Assume that the condition defined in Equation (6.11) is satisfied for all \(0 \leq x \leq y \leq z\), such that \(L(x, y) \geq R(y, z)\).

The loss in welfare \(U\) occurs when \(y^i \neq y^i\). That happens when a supplier \(i\) bids \((w^i, v^i) = (w^{i-1}, v^{i-1})\) and \((w^i, v^i) \neq (w^{i+1}, v^{i+1})\). When \((w^i, v^i) \neq (w^{i+1}, v^{i+1})\), \(y^i = y^i\).

For the situation when loss is created, since bidding with supplier \(i - 1\) yields the maximum profit for supplier \(i\), then by Theorem 12, we have that
\[
L(y^i, y^i) \geq R(y^i, y_2),
\]
where \(y_2\) is defined by \(\bar{F}(y_2) = \frac{f_i - v^{i+1}}{w^{i+1} - c^i}\). 

\[\text{220}\]
As in the previous proof, we must consider two different possible cases.

(A) The market allocation is such that \( y' < y'^* \).

(B) The market allocation is such that \( y' > y'^* \).

In case (A), since \( y_2 \geq y'^* \) and \( y'^{-1} \geq 0 \), Equation (A.31) yields that \( L(0, y') \geq R(y', y'^*) \), because, as shown in Lemma 3, \( L \) is non-increasing in its first argument and \( R \) non-decreasing in its second argument.

We claim that in this case (A), we have

\[
\int_{y'}^{y'^*} [F(u) - F(y'^*)]du \leq \frac{1}{4} \int_{0}^{y'^*} [F(u) - F(y'^*)]du. \tag{A.32}
\]

This is equivalent to saying that

\[
\left[ F(y') - F(y'^*) \right] R(y', y'^*) \leq \frac{1}{4} \left\{ \begin{array}{c}
[1 - F(y')] R(0, y') \\
+ [F(y') - F(y'^*)] \left[ R(0, y') + L(0, y') \right] \\
+ [F(y') - F(y'^*)] R(y', y'^*)
\end{array} \right\},
\]

or put differently,

\[
\left[ F(y') - F(y'^*) \right] \left[ 3R(y', y'^*) - R(0, y') - L(0, y') \right] \leq \left[ 1 - F(y') \right] R(0, y').
\]

Since \( R(y', y'^*) \leq L(0, y') \), to prove Equation (A.32), it is sufficient to show that

\[
\left[ F(y') - F(y'^*) \right] \left[ 2L(0, y') - R(0, y') \right] \leq \left[ 1 - F(y') \right] R(0, y').
\]

This is exactly the condition that has been assumed true, in Equation 6.11. This concludes the proof that in case (A), i.e.,

\[
\int_{y'}^{y'^*} [F(u) - F(y'^*)]du \leq \frac{1}{4} \int_{0}^{y'^*} [F(u) - F(y'^*)]du.
\]

In case (B), it must be that \( i < n \). Since \( (w_i, v_i) \neq (w_{i+1}, v_{i+1}) \), Theorem 13 implies that \( (w_{i+1}, v_{i+1}) = (w_{i+2}, v_{i+2}) \), and this means that \( w_i \leq c_i \leq c_{i+1} \leq w_{i+1} \leq c_{i+2} \).
$y_{i+1} = y_{i+1}^* \leq y_{i+1}^*$ and $y_i \leq y_i^*$. We can now use the optimality conditions of Theorem 12 for supplier $i + 1$ in order to derive a bound on the loss:

$$R(y^*, y_{i+1}^*) \geq L(y_1, y^*) \geq L(y^*, y^*),$$

where $y_1$ is defined by $F(y_1) = \frac{y^* - f_{i+1}}{c_{i+1} - w^i}$, which implies that $y_1 \leq y^*$, and thus $L(y_1, y^*) \geq L(y^*, y^*)$.

We claim that when $R(y^*, y_{i+1}^*) \geq L(y^*, y^*)$, then

$$\Delta c_i \int_{y_i^*}^{y_i^*} \left[ F(y^*) - F(u) \right] du \leq \frac{1}{4} \Delta c_{i+1} \int_{y_i^*}^{y_{i+1}^*} \left[ F(u) - F(y_{i+1}^*) \right] du. \quad (A.33)$$

Since the right-hand side is non-decreasing in $y_{i+1}^*$, it is sufficient to show that when $R(y^*, y_{i+1}^*) = L(y^*, y^*)$, Equation (A.33) is satisfied.

We must first note that

$$(w^{i+1} - w^i) F(y^*) = (c^{i+1} - w^i) F(y_1) + (w^{i+1} - c^{i+1}) F(y_{i+1}^*)$$

$$\geq (c^{i+1} - w^i) F(y^*) + (w^{i+1} - c^{i+1}) F(y_{i+1}^*),$$

where the inequality is justified by $c^{i+1} \geq w_i^*$ and $y_i \leq y^*$. This, together with $\Delta c_i \leq c^{i+1} - w^i$ and $w^{i+1} - c^{i+1} \leq \Delta c^{i+1}$, implies that

$$\Delta c_i \left[ F(y^*) - F(y^*) \right] \leq (c^{i+1} - w^i) \left[ F(y^*) - F(y^*) \right]$$

$$\leq (w^{i+1} - c^{i+1}) \left[ F(y^*) - F(y_{i+1}^*) \right]$$

$$\leq \Delta c^{i+1} \left[ F(y^*) - F(y_{i+1}^*) \right].$$

Thus, in order to prove Equation (A.33), it is sufficient to show that

$$\int_{y_i^*}^{y_i^*} \frac{F(y^*) - F(u)}{F(y^*) - F(y^*)} du \leq \frac{1}{4} \int_{y_i^*}^{y_{i+1}^*} \frac{F(u) - F(y_{i+1}^*)}{F(y^*) - F(y_{i+1}^*)} du,$$
or equivalently,

\[
\left[ \overline{F}(y') - \overline{F}(y'^{i+1*}) \right] L(y'^{i*}, y') \leq \frac{1}{4} \left\{ \left[ \overline{F}(y') - \overline{F}(y'^{i+1*}) \right] R(y', y'^{i+1*}) + \left[ \overline{F}(y'^{i*}) - \overline{F}(y'^{i+1*}) \right] \left[ L(y'^{i*}, y') + R(y'^{i*}, y') \right] + \left[ \overline{F}(y'^{i*}) - \overline{F}(y') \right] R(y'^{i*}, y') \right\}.
\]

Using that \( L(y'^{i*}, y') = R(y', y'^{i+1*}) \), it is sufficient to show that

\[
\left[ \overline{F}(y') - \overline{F}(y'^{i+1*}) \right] \left[ 2L(y'^{i*}, y') - R(y'^{i*}, y') \right] \leq \left[ \overline{F}(y'^{i*}) - \overline{F}(y') \right] R(y'^{i*}, y').
\]

Again, using the condition defined in Equation (6.11), this is non-positive. Thus, Equation (A.33) is satisfied for all \( y'^{i*}, y', y'^{i+1*} \) such that \( R(y', y'^{i+1*}) \geq L(y'^{i*}, y') \).

Finally, putting together cases (A) and (B), we have

\[
\Delta U = \sum_{i=1}^{n} \Delta c^{i} \int_{y'}^{y'^{i*}} \left[ \overline{F}(u) - \overline{F}(y'^{i*}) \right] du \\
\leq \frac{1}{4} \sum_{i=1}^{n} \Delta c^{i} \int_{0}^{y'^{i*}} \left[ \overline{F}(u) - \overline{F}(y'^{i*}) \right] du \\
= \frac{1}{4} U^{*}.
\]


A.5.10 Lemma 8

**Proof.** Let’s examine the worst-case scenario. For a fixed \( x \), we claim that

\[
\sup_{x \leq y \leq z, F \in \mathcal{F}} \left\{ \left[ \overline{F}(y) - \overline{F}(z) \right] \left[ 2L(x, y) - R(x, y) \right] - \left[ \overline{F}(x) - \overline{F}(y) \right] R(x, y) \right\} \leq 0
\]

s.t. \( L(x, y) \geq R(y, z) \) \hspace{1cm} (A.34)

Examine the worst-case scenario in terms of distribution. Clearly, we only need to examine the case when \( 2L(x, y) \geq R(x, y) \). The objective is then maximized for the largest \( z \) feasible, given \( y \) and \( F \). This implies that at the maximum, \( L(x, y) = R(y, z) \), since \( R(y, z) \) is non-decreasing in \( z \).
Without loss of generality, we can assume that \( x = 0 \), since for any other \( x \geq 0 \) we could prove the lemma with a shifted distribution.

**Claim 3** In the problem posed by Equation (A.34), given optimal \( x = 0, y, z \), we claim that, if there is an optimal solution, there exists an optimal distribution that is truncated exponential in \([y, z]\), with a rate equal to \( f'(y)/f(y) \).

**Proof.** We have two possible cases at an optimum, when an optimal solution \( f \) exists. Either \( F(z) = 0 \) or not.

If \( F(z) = 0 \), \( F(y) > 0 \), otherwise there is nothing to show. Assume that \( f \) is not truncated exponential. Define the distribution equal to \( f \) on \([0, y]\) and to the truncated exponential
\[
g_\gamma(t) = f(y)e^{f'(y)/f(y)(t-y)}1_{[y, \gamma]} \]
on \([y, \infty)\). Since \( f \) is log-concave, then \( f(t) \leq g_\gamma(t) \) for \( y \leq t \leq \gamma \).

Define \( \overline{\gamma} \) such that
\[
\overline{\gamma}(t) = F(y) - \int_y^t g_\gamma(u)du = \int_y^z f(u)du - \int_y^t g_\gamma(u)du
\]
This is clearly increasing in \( \gamma \). We have \( \overline{\gamma}_y(z) > 0 \) and \( \overline{\gamma}_z(z) < 0 \). We can thus find \( \gamma \) such that \( \overline{\gamma}(z) = 0 \), and hence for this particular \( \gamma \),
\[
F(y) - F(z) = \int_y^z g_\gamma(u)du.
\]

Moreover, since \( F(t) \geq \overline{\gamma}_y(t) \) for \( y \leq t \leq z \), \( L(0, y) = R(y, z) \geq \frac{\int_y^z [\overline{\gamma}_y(u) - \overline{\gamma}_z(u)]du}{\overline{\gamma}_y(z) - \overline{\gamma}_z(z)} \).
Thus \( g_\gamma \) is also optimal, if \( f \) is. We can thus find a truncated exponential with rate \( f'(y)/f(y) \), on \([y, \infty)\), that is optimal.

Finally, if \( F(z) > 0 \), \( f(z) > 0 \) and \( F(y) > 0 \). Assume that \( f \) is not exponential. Define, for \( f'(y)/f(y) \geq \gamma \geq f'(z)/f(z) \), the distribution equal to \( f \) on \([0, y]\) and \([z, \infty)\), and to
\[
g_\gamma(t) = \min \left\{ f(y)e^{f'(y)/f(y)(t-y)}, f(z)e^{\gamma(t-z)} \right\}
\]
on [y, z]. This is clearly log-concave. Fix γ such that

$$\bar{F}(y) - \bar{F}(z) = \int_y^z g_\gamma(u) du.$$  

This implies that

$$\bar{G}_\gamma(t) = \bar{F}(z) + \int_t^z g_\gamma(u) du$$

is always smaller than \(\bar{F}(t)\). Since it is strictly smaller for at least some subinterval of \([y, z]\), \(L(0, y) = R(y, z) > \int_y^z \frac{[\bar{G}_\gamma(u) - \bar{G}_\gamma(z)]}{\bar{G}_\gamma(z)} du\). Hence, for the log-concave distribution \(g_\gamma\), we can decrease \(z\) while still satisfying the feasibility constraint, thus increasing \(\bar{G}_\gamma(y) - \bar{G}_\gamma(z)\) to a larger quantity. Thus \(f\) cannot be the worst-case distribution. The only possibility is that \(f\) is exponential, with rate \(f'(y)/f(y)\).

In any case, we have showed that for the worst case distribution must be truncated exponential in \([y, z]\). ■

Claim 4 In the problem posed by Equation (A.34), given optimal \(x, y, z\), we claim that, if there is an optimal solution, there exists an optimal distribution that is truncated exponential in \([x, y]\).

Proof. Equation (A.34) can be rewritten as

$$\sup_{x \leq y \leq z, F \in F} \left\{ \begin{array}{c} \left[\bar{F}(y) - \bar{F}(z)\right] 2R(y, z) \\ -\left[\bar{F}(x) - \bar{F}(z)\right] R(x, y) \end{array} \right\} \leq 0$$

s.t. \(L(x, y) \geq R(y, z)\)

We want to minimize \(\left[\bar{F}(x) - \bar{F}(z)\right] R(x, y)\) while \(L(x, y) = R(y, z)\). The proof is similar to the proof of the previous claim. We have two cases to address: either \(f(x) = 0\) or not.

When \(f(x) = 0\), assume that \(f\) is not truncated exponential. Define the distribution equal to \(f\) on \([y, \infty)\) and to the truncated exponential

$$g_\gamma(t) = f(y)e^{\gamma(t-y)}$$

225
on $[0, y]$, for $\gamma \geq f'(y)/f(y)$. Define $\overline{G}_\gamma$ such that

$$\overline{G}_\gamma(t) = \overline{F}(y) + \int_t^y g_\gamma(u)du.$$ 

This is clearly decreasing in $\gamma$. Since $f$ is log-concave, $f(t) \leq g_{f'(y)/f(y)}(t)$. We have $0 = \overline{G}_{+\infty}(x) < 1$ and $\overline{G}_{f'(y)/f(y)}(x) > 1$. We can thus find $\gamma$ such that $\overline{G}_\gamma(x) = 1$, and hence for this particular $\gamma$,

$$1 - \overline{F}(y) = \overline{F}(x) - \overline{F}(y) = \int_x^y g_\gamma(u)du.$$

Moreover, for this $\gamma$, we clearly have that $\overline{F}(t) \geq \overline{G}_\gamma(t)$, for $x \leq t \leq y$. Thus, since the inequality is strict in at least some subinterval of $[x, y]$, $R(y, z) = L(x, y) < \int_x^y [\overline{G}_\gamma(x) - \overline{G}_\gamma(y)]du$. This implies that for the log-concave distribution $g_\gamma$, we can increase $x$ to $x'$ with $g_\gamma(x') > 0$, while still satisfying the feasibility constraint, thus decreasing $[\overline{F}(x) - \overline{F}(z)] R(x, y)$ to a smaller quantity. This is true because $\overline{F}(x) = 1$ goes down to $\overline{G}_\gamma(x')$ and $R(x, y) = (y - x) - L(x, y) = (y - z) - R(y, z)$ goes down as well. Thus $f$ cannot be the worst-case distribution. The only remaining possibility is that $f$ is truncated exponential on $[x, y]$.

The last case to consider is that $f(x) > 0$. Assume that $f$ is not exponential. Define, for $f'(x)/f(x) \geq \gamma \geq f'(y)/f(y)$, the distribution equal to $f$ on $[0, x]$ and $[y, \infty)$, and to

$$g_\gamma(t) = \min \left\{ f(y)e^{f'(y)/f(y)(t-y)}, f(x)e^{\gamma(t-x)} \right\}$$

on $[x, y]$. This is clearly log-concave. Fix $\gamma$ such that

$$\overline{F}(x) - \overline{F}(y) = \int_x^y g_\gamma(u)du.$$

This implies that

$$\overline{G}_\gamma(t) = \overline{F}(y) + \int_t^y g_\gamma(u)du.$$
is always greater than \( \bar{F}(t) \). Thus \( R(y, z) = L(x, y) > \int_x^y \frac{(G_\gamma(x) - G_\gamma(u))}{G_\gamma(x) - G_\gamma(y)} du \). Hence, for the log-concave distribution \( g_\gamma \), we can decrease \( x \) until we satisfy the feasibility constraint, thus decreasing \( [\bar{F}(x) - \bar{F}(z)]R(x, y) \) to a smaller quantity. Thus \( f \) cannot be the worst-case distribution. The only possibility is that \( f \) is exponential.

In any case, we have showed that for the worst case distribution must be truncated exponential in \([x, y]\). ■

Having proved these two claims, we are ready to complete the proof. The worst-case is obtained for a truncated exponential distribution. We have three different cases to address:

(i) The rate is negative, i.e. \( f(t) = Ke^{-\beta t}1_{[a, b]}(t) \) for some parameters \( K, a, b, \beta \) with \( a < b \) and \( \beta > 0 \).

(ii) The rate is positive, i.e. \( f(t) = Ke^{\beta t}1_{[a, b]}(t) \) for some parameters \( K, a, b, \beta \) with \( a < b \) and \( \beta > 0 \).

(iii) The rate is zero, in which case the distribution is uniform, on \([0, 1]\) without loss of generality.

We will start with the analysis of case (i). Hence, assume that \( f(t) = Ke^{-\beta t}1_{[a, b]}(t) \) for \( a < b \) and \( \beta > 0 \). It is clear that for \( x \leq a < y < b \), \( L(x, y) = L(a, y) \) and that for \( a < x < b \leq y \), \( R(x, y) = R(x, b) \). Thus, we can without loss of generality consider the case where \( a = 0 \leq y_t \leq y \leq b \).

For this distribution, for all \( a \leq x \leq y \leq b \),

\[
L(x, y) = \frac{(y - x)e^{-\beta x}}{e^{-\beta x} - e^{-\beta y}} - \frac{1}{\beta},
\]

and

\[
R(x, y) = \frac{1}{\beta} - \frac{(y - x)e^{-\beta y}}{e^{-\beta x} - e^{-\beta y}}.
\]
Define the following functions

\[ P_1(t) = \frac{e^t - 1}{t} \quad \text{and} \quad P_2(t) = \frac{e^t - 1 - t}{t^2}. \]

It is easy to show that these are analytical functions on \( \mathbb{R} \), infinitely differentiable, increasing and convex. Using this notation, we can express

\[ L(x, y) = \frac{1}{\beta} \left[ \frac{1}{P_1(-\beta(y - x))} - 1 \right], \]

and

\[ R(x, y) = \frac{1}{\beta} \left[ 1 - \frac{1}{P_1(\beta(y - x))} \right], \]

The constraint \( L(x, y) = R(y, z) \) thus becomes

\[ \frac{1}{P_1(-\beta(y - x))} + \frac{1}{P_1(\beta(z - y))} = 2, \]

On the other hand, the objective becomes

\[
\begin{align*}
&\left[ \bar{F}(y) - \bar{F}(z) \right] 2 R(y, z) - \left[ \bar{F}(x) - \bar{F}(z) \right] R(x, y) \\
&= \frac{e^{-\beta y}}{\beta^2} \left[ (1 - e^{-\beta(z-x)}) \left( \frac{1}{P_1(-\beta(y - x))} + \frac{1}{P_1(\beta(y - x))} - 3 \right) \right]
\end{align*}
\]

By writing \( \Delta_1 = \beta(y - x) \) and \( \Delta_2 = \beta(z - y) \), we need to show that for \( \Delta_1, \Delta_2 \geq 0 \) such that

\[ \frac{1}{P_1(-\Delta_1)} + \frac{1}{P_1(\Delta_2)} = 2, \]

we have

\[
\frac{(1 - e^{-\Delta_2})}{(e^{\Delta_1} - 1)} \left( \frac{2}{P_1(-\Delta_1)} + \frac{1}{P_1(\Delta_1)} - 3 \right) - \left( \frac{1}{P_1(\Delta_1)} \right) \leq 0.
\]

Notice first that since \( 1/P_1 \) is convex, we have that \( \Delta_2 \geq \Delta_1 \). Note also that

\[ 1 - \frac{1}{P_1(\Delta_2)} = \frac{\Delta_2 P_2(\Delta_2)}{P_1(\Delta_2)} = \frac{(1 - e^{-\Delta_2}) P_2(\Delta_2)}{P_1(-\Delta_2) P_1(\Delta_2)}, \]

228
and
\[
\frac{1}{P_1(-\Delta_1)} - 1 = \frac{\Delta_1 P_2(-\Delta_1)}{P_1(-\Delta_1)} = \frac{(e^{\Delta_1} - 1)P_2(-\Delta_1)}{P_1(-\Delta_1)P_1(\Delta_1)}.
\]

Finally, we remark that for all \( t \),
\[
\frac{P_2(t)}{P_1(-t)P_1(t)} = \frac{P_2(t)}{P_2(-t)P_2(t)},
\]
which is an increasing function, because \( P_2 \) is increasing.

The constraint on \( \Delta_1, \Delta_2 \), together with \( \Delta_2 \geq \Delta_1 \), thus implies that
\[
\frac{(1 - e^{-\Delta_2})P_2(\Delta_1)}{P_1(-\Delta_1)P_1(\Delta_1)} \leq \frac{(1 - e^{-\Delta_2})P_2(\Delta_2)}{P_1(-\Delta_2)P_1(\Delta_2)} = \frac{(e^{\Delta_1} - 1)P_2(-\Delta_1)}{P_1(-\Delta_1)P_1(\Delta_1)}.
\]

Thus \( (1 - e^{-\Delta_2})/(e^{\Delta_1} - 1) \leq P_2(-\Delta_1)/P_2(\Delta_1) \). Hence it is sufficient to show that for all \( t \geq 0 \),
\[
\frac{P_2(-t)}{P_2(t)} \left( \frac{2}{P_1(-t)} + \frac{1}{P_1(t)} - 3 \right) \leq 1 - \frac{1}{P_1(t)},
\]
or equivalently, using that \( P_1(-t) = e^{-t}P_1(t), e^t = 1 + tP_1(t) \) and \( P_1(t) = 1 + tP_2(t) \),
\[
t^4P_2(-t)\left(2P_1(t) - 3P_2(t)\right) \leq t^4P_2(t)^2. \tag{A.35}
\]

Since
\[
t^4P_2(t)^2 = e^{2t} - 2e^t - 2te^t + 1 + 2t + t^2 = \sum_{k=4}^{\infty} \frac{t^k}{k!}\left[2^k - 2 - 2k\right],
\]
\[
t^4P_2(-t)P_1(t) = t(-e^t - e^{-t} + te^t + 2 - t) = \sum_{k=4}^{\infty} \frac{t^k}{k!}\left[k(k-1) - k(1 - (-1)^k)\right],
\]
\[
t^4P_2(-t)P_2(t) = -e^t - e^{-t} + te^t - te^{-t} + 2 - t^2 = \sum_{k=4}^{\infty} \frac{t^k}{k!}\left[(k-1)(1 + (-1)^k)\right],
\]
we have,
\[
-t^4P_2(-t)\left(2P_1(t) - 3P_2(t)\right) + t^4P_2(t)^2
= \sum_{k=4}^{\infty} \frac{t^k}{k!}\left[2^k - 2 - 2k + 3(k-1)(1 + (-1)^k) - 2k(k-1) + 2k(1 - (-1)^k)\right].
\]

229
The term under brackets is always non-negative for \(k \geq 4\). Indeed, \(1 + (-1)^k \geq 0\) and \(1 - (-1)^k \geq 0\) for all \(k\), and \(2^k - 2 - 2k - 2k(k - 1) = 2^k - 2 - 2k^2 \geq 0\) for \(k \geq 7\). Moreover,

\[
2^k - 2 - 2k^2 + 3(k - 1)(1 + (-1)^k) + 2k(1 - (-1)^k) = \begin{cases} 
0 & \text{for } k = 4, \\
0 & \text{for } k = 5, \\
20 & \text{for } k = 6.
\end{cases}
\]

This shows that Equation (A.35) is satisfied for \(t \geq 0\). Thus, when \(F\) is a truncated exponential with negative rate, i.e. case (i),

\[
\max_{x \leq y \leq z} \begin{bmatrix} [\bar{F}(y) - \bar{F}(z)] & [2L(x, y) - R(x, y)] \\
-\bar{F}(x) - \bar{F}(y) & R(x, y) \end{bmatrix} \leq 0
\]

s.t. \(L(x, y) \geq R(y, z)\)

Case (ii) can be analyzed similarly. In this case, using the same notation, \(\Delta_1 = \beta(y - x)\) and \(\Delta_2 = \beta(z - y)\), where \(\beta\) is now the positive rate of the exponential, the constraint is tight and hence equivalent to

\[
\frac{1}{P_1(\Delta_1)} + \frac{1}{P_1(-\Delta_2)} = 2,
\]

We must show now that

\[
\frac{\bar{F}(y) - \bar{F}(z)}{\beta^2} \left( e^{\beta(y-x)} - 1 \right) \left( 1 - \frac{2}{P_1(\beta(y-x))} - \frac{1}{P_1(-\beta(y-x))} + 3 \right) \\
\leq 0,
\]

or equivalently,

\[
\frac{(e^{\Delta_2} - 1)}{(1 - e^{-\Delta_1})} \left( 2P_1(-\Delta_1) - 3P_2(-\Delta_1) \right) \leq P_2(-\Delta_1).
\]
Now, \( \Delta_2 \leq \Delta_1 \) and this implies that \( (e^\Delta_2 - 1)/(1 - e^{-\Delta_1}) \leq P_2(\Delta_1)/P_2(-\Delta_1) \). Hence, we must show that for all \( t \geq 0 \),

\[
P_2(t) \left( 2P_1(-t) - 3P_2(-t) \right) \leq P_2(-t)^2,
\]

or equivalently, using that \( P_2(-t)e^t = P_1(t) - P_2(t) \),

\[
t^4 e^t P_2(t) \left( 3P_2(t) - P_1(t) \right) \leq t^4 \left( P_1(t) - P_2(t) \right)^2.
\]

Since

\[
t^4 P_2(t)^2 = \sum_{k=4}^\infty \frac{t^k}{k!} \left[ 2^k - 2 - 2k \right],
\]

\[
t^4 e^t P_2(t)^2 = \sum_{k=4}^\infty \frac{t^k}{k!} \left[ 3^k - 2^{k+1} - k2^k + 1 + 2k + k(k-1) \right],
\]

\[
t^4 P_2(t)P_1(t) = \sum_{k=4}^\infty \frac{t^k}{k!} \left[ k2^{k-1} - 2k - k(k-1) \right],
\]

\[
t^4 e^t P_2(t)P_1(t) = \sum_{k=4}^\infty \frac{t^k}{k!} \left[ k3^{k-1} - k2^k - k(k-1)2^{k-2} + k + k(k-1) \right],
\]

\[
t^4 P_1(t)^2 = \sum_{k=4}^\infty \frac{t^k}{k!} \left[ k(k-1)2^{k-2} - 2k(k-1) \right],
\]

we must show that

\[
\sum_{k=4}^\infty \frac{t^k}{k!} \left[ \begin{array}{c} k(k-1) \cdot 2^{k-2} - 2k(k-1) \\ -2k \cdot 2^{k-1} + 4k + 2k(k-1) \\ +2^k - 2 - 2k \\ +k \cdot 3^{k-1} - k \cdot 2^k - k(k-1) \cdot 2^{k-2} + k + k(k-1) \\ -3^{k+1} + 3 \cdot 2^{k+1} + 3k \cdot 2^k - 3 - 6k - 3k(k-1) \end{array} \right] \geq 0.
\]

The coefficients in the brackets are equal to

\[(k - 9)3^{k-1} + (k + 7)2^k + (-2k^2 - k - 5)\]
They are clearly non-negative for $k \geq 9$. For smaller values, we have

$$(k - 9)3^{k-1} + (k + 7)2^k + (-2k^2 - k - 5) = \begin{cases} 
0 & \text{for } k = 4, \\
0 & \text{for } k = 5, \\
20 & \text{for } k = 6, \\
224 & \text{for } k = 7, \\
1512 & \text{for } k = 8.
\end{cases}$$

Hence, for all $t \geq 0$, 

$$t^4e^t P_2(t)\left(3P_2(t) - P_1(t)\right) \leq t^4\left(P_1(t) - P_2(t)\right)^2,$$

and thus, when $F$ is a truncated exponential with positive rate, i.e. case (ii),

$$\max_{z \leq y \leq z} \left\{ \left[\bar{F}(y) - \bar{F}(z)\right]\left[2L(x,y) - R(x,y)\right] - \left[\bar{F}(x) - \bar{F}(y)\right]R(x,y) \right\} \leq 0$$

s.t. $L(x,y) \leq R(y,z)$

Case (iii) is straightforward. When the distribution is uniform, $L(x,y) = R(x,y) = (y-x)/2$ for all $x \leq y$. Also the condition $L(x,y) \geq R(y,z)$ is equivalent to $z - y \leq y - x$. Thus,

$$\left[\bar{F}(y) - \bar{F}(z)\right]\left[2L(x,y) - R(x,y)\right] - \left[\bar{F}(x) - \bar{F}(y)\right]R(x,y) = \frac{1}{2}(y-x)(z+x-2y) \leq 0.$$

\section*{A.5.11 Proposition 16}

\textbf{Proof.} Supplier $i$ is active in this Nash equilibrium. Since the demand follows a border distribution, supplier $i$ bids in the boundary of some region $A_{(w, \nu(-i))}^{lh}$ constructed with all the bids except $i$'s. If the bid $(w^i, \nu^i)$ belongs to more than one region, choose $A_{(w, \nu(-i))}^{l}$ with $l$ and $h$ active. We must consider two cases, either there is no supplier in the lower envelope between the bids of $l$ and $h$, or there is one.
In the first case, there is \(j, j \) being \(l \) or \(h \), such that \(j \) is active, and \((w^i, v^i) = (w^j, v^j)\), from Theorem 12. From Proposition 14, \((w^i, v^i)\) belongs in the segment \([ (c^i, f^i) ; (c^j, f^j) ] \).

In the second case, there is one supplier, \(k \) on the lower envelope between \(l \) and \(h \) such that the bid \((w^i, v^i)\) is in the border of \(A_{(w(-i), v(-i))}^{lh} \) and \(A_{(w(-i), v(-i)))}^{lk} \) or \(A_{(w(-i), v(-i))}^{lh} \) and \(A_{(w(-i), v(-i))}^{kh} \). \(k \) is thus inactive because of supplier \(i \), and either the bids of \(l, k \) and \(i \) are aligned, or those of \(i, k \) and \(h \). Such a situation is depicted in Figure A-2. Hence, we find \(j, j \) being \(l \) or \(h \), active, such that \((w^i, v^i)\) is equal to \((w^j, v^j) + \theta(w^k - w^j, v^k - v^j)\) for some non-negative \(\theta \). □

Figure A-2: Suppliers \(l \) and \(h \) are active and supplier \(k \) is turned inactive by supplier \(i \)'s bid.
Bibliography


235


237


