

More on a Market Share Theorem

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ABSTRACT

The Market Share Theorem of Bell, Keeney, and Little is extended to some situations where attractiveness ratings of products are not related "linearly" and symmetrically to their market shares. Unique expressions for market share are obtained in these contexts under an axiomatic structure with three postulates, one fewer than used in the original theorem.

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More on a Market Share Theorem

Bell, Keeney, and Little¹ recently proved a market share theorem. They assumed that the sales potential of each product in a market can be summarized by one "attractiveness" number, and postulated several relationships between attractiveness ratings and market share. Then they proceeded to show that, in their model, a seller's proportion of total sales is simply the attractiveness of his product divided by the total attractiveness of all products in the market.

The authors noted that linear-normalized " $u_i/(u_i + u_j)$ " expressions for market share are often hypothesized in the literature, and that one value of their result is its specification of conditions when such formulas are appropriate. But beyond ratifying expressions that have already been assumed, the theorem has a prescriptive value. Suppose one has a set of mathematical functions of product attributes--one for each competitor in a market--that one believes related to the market shares of individual firms by the Bell-Keeney-Little (hereafter BKL) postulates. Then using linear-normalized estimates for percentage distribution of sales is not only permissible under the theorem but is mandatory because of it. Certain broad assumptions about market behavior, when taken together, are surprisingly precise in their implications.

Because of the stringent conditions in the BKL axioms, the result is directly applicable only in limited circumstances. This paper describes an attempt to extend the market share theorem by dropping its assumptions of "symmetry" and "linearity" in favor of more general axioms. Using arguments that rest ultimately on the BKL theorem

itself, we obtain unique expressions for market shares in the wider framework considered. These results, hopefully, may make the market share theorem more accessible to those wishing to use it.

The BKL Model

Associated with the n products in a particular market is an attractiveness vector $\underline{a} = (a_1, \dots, a_n)$, where a_i is the attractiveness of product i . BKL assume that the components of the attractiveness vector are governed and related to those of the market-share vector (m_1, \dots, m_n) by the following rules:

- 1) $a_i \geq 0$ for all i ; $\sum_{i=1}^n a_i > 0$.
- 2) If $a_i = 0$, $m_i = 0$.
- 3) If $a_i = a_j$, $m_i = m_j$.
- 4) Let $\Delta m_{ij}(\delta)_{\underline{a}}$ \equiv the change in market share m_i of product i associated with a change in attractiveness vector from \underline{a} to $(a_1, a_2, \dots, a_j + \delta, a_n)$.

Then for all $j, k \neq i$ and all Δ , $\Delta m_{ij}(\delta)_{\underline{a}} = \Delta m_{ik}(\delta)_{\underline{a}}$.

Under these conditions, BKL proved that

$$\underline{m} = (m_1, m_2, \dots, m_n) \text{ where } m_i = \frac{a_i}{\sum_{j=1}^n a_j} \quad (i=1, \dots, n)$$

is the only possible market-share vector.

To use this theorem in an actual situation the model builder must construct--from vectors about price, advertising, quality, etc.--an attractiveness vector that satisfies the four postulates. In general

this is a nontrivial task. Postulate 4 is in some sense the most "demanding" of the axioms: it specifies that a change δ in another product's attractiveness affects a firm's market-share the same way regardless of which competitor sustained the change. (BKL called this "symmetry"). Furthermore, attractiveness changes enter this axiom only on an absolute scale ("linearity"). Thus, while an increment of δ can mean much greater percentage growth in attractiveness for some products than for others, this is treated as irrelevant. The market analyst may well find that many would-be attractiveness vectors that occur to him are tainted in practice by nonlinearity and asymmetry. He may also have problems with Postulate 3's requirement that equal attractiveness for two different products must mean equal market shares.

Modifying the BKL Postulates

Possible difficulties in using the BKL postulates as they stand lead to an obvious question: can the axiomatic structure be amended so that it is (1) flexible enough to accommodate asymmetry and nonlinearity in the relationship between attractiveness and market shares yet (2) sufficiently rigid to insure a unique market share vector for a given attractiveness vector. We proceed now to investigate this question. We first consider a system of axioms of no great significance in itself, but important because it is a prototype of a large class of asymmetric models. This particular system is highly amenable to mathematical analysis and makes the crucial issues transparent. As will be shown, the results obtained for the model used can be easily generalized.

We assume a market of exactly three products and an attractiveness vector $\underline{a} = (a_1, a_2, a_3)$ assembled from basic marketing variables. It is asserted that the following axioms describe the market's operation:

- 1) $a_i \geq 0, \sum_{i=1}^3 a_i > 0.$
- 2) $a_i = 0 \Rightarrow m_i = 0.$
- 3) $a_i = a_j \Rightarrow m_i = m_j.$
- 4A) For all $\underline{a} = (a_1, a_2, a_3),$
 - i) $\Delta_{12}(\delta)_a = \Delta_{13}(2\delta)_a$
 - ii) $\Delta_{21}(\delta)_a = \Delta_{23}(2\delta)_a$
 - iii) $\Delta_{31}(\delta)_a = \Delta_{32}(\delta)_a$ (in the earlier notation).

In words, statement 4A(i) means that, starting from a given \underline{a} , the impact on m_1 of an attractiveness change for product 2 is the same as that for twice as large a change in product 3's attractiveness. (ii) means that m_2 is in a similar way more "responsive" to changes in a_1 than a_3 , while (iii) indicates that the "elasticity" of m_3 is the same with respect to a_1 and a_2 . (i) and (ii) bring asymmetry into the model; we have not yet, however, introduced nonlinearity.

The question at issue is: can we express m_1, m_2 and m_3 as well-defined functions of (a_1, a_2, a_3) under this set of postulates?

Before proceeding, we might attempt to motivate an expedition into asymmetry. A hypothetical example where the new postulate 4 might conceivably fit is the public transportation market between two nearby cities, served by two bus companies and a railroad line. One might devise an attractiveness function for each carrier based on ticket

prices, frequency of service, amount of advertising etc. Yet in some sense the two bus lines may compete more directly against each other than against the railroad trains, and the asymmetry of axioms such as 4A(i) and 4A(ii) may adequately allow for this. One might argue that the introduction of some "bus or train" variable into the attractiveness functions might be sufficient to put things in the BKL framework. But there is something metaphysical about such variables; it may well be preferable to deal with attractiveness vectors based on well-defined quantities even if they entail asymmetry.

Trouble Develops

While the new postulates above are not fettered by symmetry, there is another problem. The attempts to find market share functions that satisfy axioms 1-4A is doomed to failure, as we shall show presently.

It is important to note that $m_1(a_1, a_2, a_3)$ can be written as $m_1(a_1, 2a_2+a_3)$ (i.e., product 1's market share depends on a_2 and a_3 only through the weighted sum $2a_2+a_3$.) To see this, we consider two consecutive changes in the attractiveness vector: (1) product 2's attractiveness increases by Δ , (2) product 2's attractiveness decreases by Δ . Obviously these two changes have no net affect on m_1 . If the second alteration in the attractiveness vector were instead a decrease of 2Δ in a_3 , axiom 4A tells us that m_1 would likewise sustain no net change. Thus whenever a_2 and a_3 change in such a way that $\Delta a_3 = -2\Delta a_2$ or, equivalently, that $2a_2+a_3$ remains constant, m_1 remains constant unless a_1 varies. Therefore, m_1 can be considered a function of the two variables a_1 and $2a_2+a_3$. Similarly, we can write $m_2(a_2, 2a_1+a_3)$ and $m_3(a_3, a_1+a_2)$.

We now examine five specific attractiveness vectors (a_1, a_2, a_3) and reach an odd conclusion about market shares:

1) $a_1 = y, a_2 = 0, a_3 = y$ for some positive y .

From axiom 2, $m_2 = 0$ here; from 3 and the obvious requirement that $\sum_{i=1}^3 m_i = 1, m_1 = m_3 = .5$. Since $a_1, a_3, 2a_2+a_3$, and a_1+a_2 all equal y , we see that $m_1(y, y) = m_3(y, y) = .5$.

2) $a_1 = y, a_2 = .5y, a_3 = 0$.

$m_1(a_1, 2a_1+a_2) = m_1(y, y) = .5$ from the previous result. Since $m_3 = 0$ here, we can conclude that $m_2(a_2, 2a_1+a_3) = \underline{m_2(.5y, 2y) = .5}$.

In a wholly analogous way, examination of the attractiveness vectors 3: $(0, y, y)$ and 4: $(.5y, y, 0)$ leads to the conclusion $m_1(.5y, 2y) = .5$.

5) $a_1 = .5y, a_2 = .5y, a_3 = y$.

Note that $2a_1+a_3 = 2a_2+a_3 = 2y$ here, thus the facts that

$m_1(.5y, 2y) = m_2(.5y, 2y) = .5$ and $\sum_{i=1}^3 m_i = 1$ require that $m_3(a_3, a_1+a_2) =$

$m_3(y, y) = 0$. But earlier we concluded that $m_3(y, y) = .5$. What a calamity has arisen--the axioms tell us that the third competitor has both none of the sales and half of them! Our attempts to generalize the BKL postulates can hardly at this point be described as successful.

Picking Up the Pieces

A crude, qualitative explanation of what went wrong above is in order. The contradiction we found is similar to that which would arise if one tried calculating the slope of a straight line through three points on a circle. In both problems, the conditions imposed on the solution include some that cannot simultaneously be satisfied. We have noted at length the asymmetric form of postulate 4A, but paid too little attention to the fact that axiom 3 (equal attractiveness implies equal market share) is not only symmetric in structure but a very strong requirement on all would-be market-share functions. The underlying cause of trouble is that axioms 3 and 4A "clash" in their implications in a fashion that cannot be reconciled by single-valued expressions for $m_i(a_1, a_2, a_3)$.

To continue our effort to extend the BKL model, we must weaken or delete one of the conflicting postulates. Since 4A is "cutting edge" of the current attempt to find broader results, it is logical that axiom 3 should be subject to change. The way we will handle postulate 3 is simply to eliminate it. This step removes the immediate cause of stress, but whether the three axioms left specify a well-defined set of $m_i(a_1, a_2, a_3)$ is as yet unclear. With the aid of a surprising result, we will see that they do.

With axiom 3 removed, the problem becomes that of finding market-share functions (m_i) consistent with the rules

$$1) \quad a_i \geq 0, \quad \sum_{i=1}^3 a_i > 0.$$

$$2) \quad a_i = 0 \Rightarrow m_i = 0.$$

$$4A)* \quad i) \quad \Delta_{12a}(\delta) = \Delta_{13a}(2\delta)$$

$$ii) \quad \Delta_{21a}(\delta) = \Delta_{23a}(2\delta)$$

$$iii) \quad \Delta_{31a}(\delta) = \Delta_{32a}(\delta)$$

* We will retain the original numbering of the postulates to avoid confusion.

It is no longer the case that there are no $m_i(a_1, a_2, a_3)$ ($i=1, 2, 3$) satisfying the axioms and the requirement that

$\sum_{i=1}^3 m_i = 1$. Indeed, there is now a linear-normalized solution:

$$m_1 = \frac{2a_1}{2a_1 + 2a_2 + a_3},$$

$$m_2 = \frac{2a_2}{2a_1 + 2a_2 + a_3},$$

$$m_3 = \frac{a_3}{2a_1 + 2a_2 + a_3}.$$

There is a danger, however, that we have so "weakened" the axiomatic structure that there are many, possibly even an infinite number, of well-defined sets of m_i 's consistent with it. It is this possibility we must now consider.

At this point we make a linear change of variables in the components of the attractiveness vector. The new vector $\underline{b} = (b_1, b_2, b_3)$ is related to the original $\underline{a} = (a_1, a_2, a_3)$ by the relationships:
 $b_1 = 2a_1, b_2 = 2a_2, b_3 = a_3$. In the new system, axioms 1, 2 and 4A appear as:

$$1) b_i \geq 0, \quad \sum_{i=1}^3 b_i > 0.$$

$$2) b_i = 0 \Rightarrow m_i = 0.$$

$$4B) \Delta_{12\bar{b}}(\delta) = \Delta_{13\bar{b}}(\delta)$$

$$\Delta_{21\bar{b}}(\delta) = \Delta_{23\bar{b}}(\delta)$$

$$\Delta_{31\bar{b}}(\delta) = \Delta_{32\bar{b}}(\delta).$$

Postulates 1 and 2 are unchanged but 4B, which is equivalent to the former 4A, is no longer asymmetric; 4B is the same as postulate 4 in the original BKL model. With the new attractiveness variables, therefore, three of the four specifications made by BKL are satisfied. This is only a curiosity in the absence of the other axiom (equal attractiveness implies equal market share) UNLESS it could somehow be shown that the three we already have imply the fourth; were that to happen, the Market Share Theorem would be directly relevant to the current problem. Strangely enough, postulates 1, 2 and 4B do imply the remaining axiom, as we will show now.

Theorem: Suppose one has an attractiveness vector $\underline{b} = (b_1, b_2, b_3)$ and market-share functions $m_i(b_1, b_2, b_3)$ satisfying rules 1,2, and 4B. Then if $b_i = b_j$, $m_i = m_j$.

Proof: Let $b_1 = x$, $b_2 = x$ and $b_3 = z$ for any x and/or $z > 0$; we will show here that $m_1(x, x, z) = m_2(x, x, z)$. The argument is wholly parallel when $b_1 = b_3$ or $b_2 = b_3$.

With the same reasoning used earlier (p. 5), we can establish that under postulate 4B, m_1 depends on b_2 and b_3 only through their sum

b_2+b_3 , m_2 depends on b_1 and b_3 only through b_1+b_3 , and m_3 depends on b_1 and b_2 only through b_1+b_2 . (In this argument we will write $m_1(b_1, b_2, b_3)$ rather than $m_1(b_1, b_2+b_3)$ etc. but will utilize the property just described.) We demonstrate that $m_1 = m_2$ by considering some other attractiveness vectors (b_1, b_2, b_3) .

- a) Let $b_1 = x, b_2 = 0, b_3 = x+z$. By 2, $m_2 = 0$; let $m_3(x, 0, x+z) \equiv \gamma$. Since b_1 and b_2+b_3 take the same values as at (x, x, z) , $m_1(x, 0, x+z) = m_1(x, x, z) = 1-\gamma$.
- b) Let $b_1 = 0, b_2 = x, b_3 = x+z$. Here $m_1 = 0$; since b_1+b_2 and b_3 retain their values in (a) $m_3(0, x, x+z) = \gamma$. Thus $m_2(0, x, x+z) = 1-\gamma$ and therefore, since $m_2(0, x, x+z) = m_2(x, x, z)$, it follows that $m_1(x, x, z) = m_2(x, x, z)$. Q.E.D.

The theorem just proved means that 1,2 and 4A imply all four BKL conditions in the new variable system. But then

$$m_i = \frac{b_i}{b_1 + b_2 + b_3} \quad \text{for } i = 1, 2, 3$$

or, transforming back to the original variables,

$$m_1 = \frac{2a_1}{2a_1 + 2a_2 + a_3} ,$$

$$m_2 = \frac{2a_2}{2a_1 + 2a_2 + a_3} ,$$

$$m_3 = \frac{a_3}{2a_1 + 2a_2 + a_3}$$

is the only set of market-share functions consistent with 1, 2 and 3A. Thus the elimination of axiom 3* created an asymmetric set of postulates that did in fact prescribe a unique distribution of percentage sales for a given attractiveness vector. The distribution of percentage sales for a given attractiveness vector. The distribution itself was found through an appropriate application of the Market Share Theorem. In the next section, we use the insights obtained thus far to handle asymmetry and nonlinearity in a more general setting.

More Nonlinearity and Asymmetry

We now consider the following model of market behavior. There are N competing products ($N \geq 3$) and an attractiveness vector (a_1, a_2, \dots, a_N) . Associated with each product i ($i=1, \dots, N$) is a positive mathematical function $h_i(x)$ defined for all $x \geq 0$. The relationships governing market share $m_i(a_1, \dots, a_N)$ ($i=1, \dots, N$) and attractiveness are

- 1) $a_i \geq 0, \sum_{i=1}^N a_i > 0.$
- 2) $a_i = 0 \Rightarrow m_i = 0.$
- 3A) $m_i(a_1, \dots, a_N)$ is a differentiable function of each a_j ($j=1, \dots, n$) and for all $K, L \neq i$, the following equation is satisfied:

$$\frac{\partial m_i}{\partial a_K} = \frac{h_K(a_K)}{h_L(a_L)} \frac{\partial m_i}{\partial a_L}$$

The key features of this model are contained in axion 3A.

* Note that 3 would NOT have become $b_i = b_j \Rightarrow m_i = m_j$ in the new system.

the BKL problem and the example we used earlier, this formulation explicitly assumes each m_i differentiable in the attractiveness variables. This is probably not a great practical restriction, especially since the market share functions obtained in the previous problems turned out to be differentiable. (There is a way to avoid a differentiability assumption in 3A but it is so cumbersome as not to seem worthwhile.) The equation in 3A makes clear that the model is asymmetric since the impact of a change in attractiveness depends in general on which product sustains the change. And the factor

$$\frac{h_K(a_K)}{h_L(a_L)}$$

relates the significance of changing a product's attractiveness directly to its current level, introducing--for the first time--the element of nonlinearity. The functions $h_i(x)$ indicate the "elasticity" of a competitor's market share in product i 's attractiveness level, and allow different assessments of market behavior for different products.

The major result of this paper is the following:

Theorem: Under axioms 1 - 3A,

$$m_i(a_1, \dots, a_N) = \frac{\int_0^{a_i} h_i(x) dx}{\sum_{j=1}^N \int_0^{a_j} h_j(x) dx} \quad \text{uniquely.}$$

Proof: As before, we introduce a change of attractiveness variables.

Let $b_j \equiv \int_0^{a_j} h_j(x) dx$. Then equivalent to postulates 1,2 and 3A are the rules:

- 1) $b_i \geq 0, \quad \sum_{i=1}^N b_i > 0.$
- 2) $b_i = 0 \Rightarrow m_i = 0.$
- 3B) $\frac{\partial m_i}{\partial b_K} = \frac{\partial m_i}{\partial b_\ell}$ for all $K, \ell \neq i.$

Rule 2 follows at once from the previous rule 2 and the definition of b_i ; rule 1 is an immediate consequence of its antecedent in (a_1, \dots, a_N) and the assumption that all $h_i(x)$'s are positive functions. (3B) results from (3A) and the observation that

$$\frac{\partial m_i}{\partial a_j} = \frac{\partial m_i}{\partial b_j} \frac{db_j}{da_j} = h_j(a_j) \frac{\partial m_i}{\partial b_j}.$$

A moment's reflection makes apparent that 3B is the "differentiable" equivalent to BKL postulate 4. Thus BKL axioms 1,2 and 4 are satisfied under the new attractiveness variables; since, as we have seen, 1,2 and 4 imply BKL axiom 3, we can use the Market Share Theorem to state that

$$m_i = \frac{b_i}{\sum_{j=1}^N b_j}.$$

Rewriting m_i in terms of (a_1, \dots, a_N) yields the theorem.

What we have done here is consider a special set of nonlinear and asymmetric situations that can be transformed into linear symmetric problems with relatively simple changes of variables. However, the generality in the functional forms of the $h_i(x)$'s allows a number of potentially practical cases to fall into this set.

Example

Consider a three-product market with a non-negative attractiveness measure for each competitor. Suppose one believes that zero attractiveness implies no sales, and that the sensitivity of market share to attractiveness is described by the differential equations:

$$\frac{\partial m_1}{\partial a_2} = \frac{\sqrt{1+a_2}}{1+a_3} \frac{\partial m_1}{\partial a_3}$$

$$\frac{\partial m_2}{\partial a_1} = \frac{\sqrt{1+a_1}}{1+a_3} \frac{\partial m_2}{\partial a_3}$$

$$\frac{\partial m_3}{\partial a_1} = \sqrt{\frac{1+a_1}{1+a_2}} \frac{\partial m_3}{\partial a_2}$$

Since $h_1(x) = h_2(x) = \sqrt{1+x}$ and $h_3(x) = 1+x$, the result

$$m_i = \frac{\int_0^{a_i} h_i(x) dx}{\sum_{j=1}^3 \int_0^{a_j} h_j(x) dx} \quad \text{just proved}$$

becomes:

$$m_1(a_1, a_2, a_3) = \frac{2/3((1+a_1)^{3/2} - 1)}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

$$m_2(a_1, a_2, a_3) = \frac{2/3((1+a_2)^{3/2} - 1)}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

$$m_3(a_1, a_2, a_3) = \frac{a_3 + a_3^2/2}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

- 1) $b_i \geq 0, \quad \sum_{i=1}^N b_i > 0.$
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Since $h_1(x) = h_2(x) = \sqrt{1+x}$ and $h_3(x) = 1+x$, the result

$$m_i = \frac{\int_0^{a_i} h_i(x) dx}{\sum_{j=1}^3 \int_0^{a_j} h_j(x) dx} \quad \text{just proved}$$

becomes:

$$m_1(a_1, a_2, a_3) = \frac{2/3((1+a_1)^{3/2} - 1)}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

$$m_2(a_1, a_2, a_3) = \frac{2/3((1+a_2)^{3/2} - 1)}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

$$m_3(a_1, a_2, a_3) = \frac{a_3 + a_3^2/2}{2/3((1+a_1)^{3/2} + (1+a_2)^{3/2} - 2) + a_3 + a_3^2/2}$$

In this particular model, the "elasticity" $\partial m_i / \partial a_i$ market shares for products 1 and 2 goes up, roughly speaking, with $a_i^{-3/2}$, while product 3's "elasticity" varies with $(a_3)^{-2}$. When might such a situation actually arise? Suppose current advertising expenditures seem appropriate as major components of the attractiveness numbers. If products 1 and 2 have been on the market somewhat shorter than their competitor, the model-builder might believe a weaker "law of diminishing returns" applies to these products than to 3. In some cases the equations above might quantify that notion adequately. Note that when $a_2 = a_3$

$$\frac{\partial m_1}{\partial a_3} > \frac{\partial m_1}{\partial a_2} \quad (\text{except when } a_2 = a_3 = 0); \text{ similarly}$$

$$\frac{\partial m_2}{\partial a_3} > \frac{\partial m_2}{\partial a_1} \quad \text{when } a_1 = a_3 > 0.$$

This indicates that, other factors being equal, products 1 and 2 compete more directly with product 3 than with each other. We discussed earlier a circumstance where such asymmetric competition might occur (2 bus lines, 1 rail line).

Conclusions

The results here should be viewed not as generalizations of the Market Share Theorem but rather elaborations of it. Direct use of the Bell-Keeney-Little theorem requires an attractiveness vector related linearly and symmetrically to the vector of market-shares. We have obtained results in some asymmetric, nonlinear situations through making variable changes that led to the BKL framework. Our work was greatly facilitated by the finding that the axiom "equal attractiveness implies equal market share" was not essential to reaching the BKL result; this meant that no corresponding postulate was needed in the systems we considered. As we saw, direct imposition of that axiom upon an asymmetric system led to contradictory and meaningless results.

Our findings mean that the Market Share Theorem is of even broader relevance than one might at first realize. Hopefully those trying to understand market behavior will find this information useful.

References

1. Bell, D., Keeney, R., and Little, J., "A Market Share Theorem,"
Journal of Marketing Research, May 1975.